Ph.D. DISSERTATION

오비폴드 특이점 및 매듭에 관한 정수체의 p진법 제타함수

p-adic zeta function for number field associated with orbifold singularity and knot.

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이 논문을 이학박사 학위논문으로 제출함

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서울대학교 대학원

물리천문학부

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Abstract

In this thesis, we consider **p**-adic zeta function of real quadratic number field and relate it to string partition function on irregular Sasaki-Einstein CY orbifold. With this we have additional sign from conductor. Then we obtain algebraicity of **p**-adic zeta function of totally real number field F and AdS dual 4d SCFT index(Hilbert series) of Sasaki-Einstein manifold, by Stark-Heegner unit in Hilbert class field K or F.

We analyze the extremal metric from the Heun equation(Painleve 6-th), and relate to integrable system for elliptic surface with torsion Mordell Weil group. We recover integrable system from the cluster transformation of Poissin algebra(path algebra of Sasaki-Einstein quiver) with symplectic double. For Sasaki-Einstein CY, we have **p**-adic Galois representation which is the torsion globa Galois representation in supersingular locus over sufficiently ramified field from pro-**p** covering of Bianchi manifold.

By Dehn twist with punctured torus, we consider exotic 4 manifold by Lens space surgery, and conjecture that Lens space realizing knot as ramification knot for irrational parameter of Painleve 6-th equation. By Hitchin moduli space of rank 2 vector bundle on $\mathbb{P}^1/\{0, 1, \hat{a}, \infty\}$, with parabolic structure for $\hat{a} \in K$, we obtain moduli space of stability condition on mirror of orbifold Fano base of Sasaki-Einstein manifold.

Keywords: **p**-adic zeta function, Arason invariant, spin CS invariant

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Contents

Chapter 1

Introduction

In this paper, we consider arithmetic geometry behind the **p**-adic zeta function of real quadratic number field,and analysis the algebraicity. There exist associated string background(CY motive) which realize such **p**-adic Galois representation of number field. We are particularly interested in nonrigid varieties − Sasaki-Einstein CY manifold and its noncommutative resolution and hidden modularity of them[1]. For irregular Sasaki-Einstein manifold, the N=1 SCFT index is computed with irrational parameter in non-compact dilogarithm whose limit is extremal volume of CY.

Due to extremality of the singular Kahler Einstein metic, we have a notion of non-maximal Swan conductor for ordinary locus in Shimura varieties which is generalization of additional $\mathbb{Q}$ point(maximal conductor) which exceed Hasse bound for supersingular curve with **p** rank 0, associated with imaginary quadratic field. For irregular Sasaki-Einstein manifold associated with totally

[1] We consider Serre modularity conjecture for **p**-adic Galois representation which come from geometric origin.

real number field, we consider ordinary locus of Shimura varieties with non-zero $\mathbf{p}$ rank $f > 0$. The associated non-commutative motive in ordinary locus in Shimura varieties can be seen as supersingular motive over sufficiently ramified field with non-maximal conductor.

We consider infinite family of $N = 1$ Sasaki-Einstein CY string background. For that we need to consider hyperbolic reflection group action on Leech lattice from N=4 string vacua.

Note that,there are only finite arithmetic reflection group(ex. Bianchi group) of hyperbolic lattice(Leech lattice) classified in [30]. And also there exist non-arithmetic reflection group. Our Sasaki-Einstein manifold arise as pro-$\mathbf{p}$ covering of arithmetic reflection group which is non-arithmetic which is infinite. So, for infinite family of Sasaki-Einstein CY, we have the torsion Galois representation from torsion cohomology of hyperbolic 3 manifold with Jacquet Langlands correspondence. Such torsion global Galois representation arise from supersingular locus of Shimura varieties over sufficiently ramified field.

In this paper,we consider three object. One is singular Kahler Einstein metric with extremal volume and irrational Reeb vector from Painleve 6-th differential equation and cluster algebra, and the others are the $\mathbf{p}$-adic zeta function and Stark-Heegner unit of real quadratic number field and the hyperbolic double bridge knot complement with pseudo-Anosov diffiomorphism.

We also consider CS invariant from $\mathbf{p}$-adic zeta function of real quadratic number field, and $\mathbf{p}$-adic Landau Ginzburg model.

1.1 Sasaki-Einstsein manifold with real multiplication on torus

In string theory,there is AdS-CFT duality which is powerful enough to determine all Galois $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ invariant quantities from dual CFT. For example,we can obtain CY manifold with modularity arise from specific Galois representation.

Here we analysis simplest but nontrivial example arising from $N = 1$ string background called Sasaki-Einstein CY. The Sasaki-Einstein CY manifold is cone over Sasaki-Einstein 5 manifold $Y^{p,q}$ which is contact 5 space with Reeb foliation.

$Y^{p,q}$ is the class of Sasakian manifold which admit Kahler Einstein metric. It is the $U(1)$ bundle over Fano orbifold with vanishing Futaki invariant, whose real cone is CY. Such that $Y^{p,q}$ can be the non-trivial Black-Hole horizon with entropy determined from extremal metric. The irregular extremal Reeb foliation on $Y^{p,q}$ project to pseudo-Anosov mapping class foliation on 1-punctured mapping torus where torus with dimer tiling, both are constructed by S-duality along 5 brane (p, q) web, $T_{p,q} = \begin{pmatrix} 1-pq & q^2 \\ -p^2 & 1+pq \end{pmatrix}$. $T_{p,q}$ is the Dehn twist of punctured torus which gives the exotic smooth structure by Lens space $L(p^2, pq-1)$ surgery. This is the exotic monopole for 3-dimensional domain wall from N=1 4d SCFT dual to $Y^{p,q}$. We consider pseudo-Anosov flow which is the Teichmuller geodesic flow as real multiplication on torus by real quadratic field F. Such that it is the example of real multiplication problem.

The CFT dual is 4d quiver gauge theory with quiver Q associated with toric diagram of $Y^{p,q}$. The Seiberg duality of dual gauge theory enable us to determine Galois invariant quantities - extremal R charge(anomalous dimension) which is dual to extremal volume of $C(Y^{p,q})$.

There exist map between cluster algebra of $Y^{p,q}$ quiver with periodic mutation and cluster algebra of Lens space knot complement with triangulation. The mutation(Seiberg duality) on tiling maps to tetrahedron triangulation and gluing. Given tiling, by performing mutation, we obtain dual tiling. By gluing two, we obtain triangulation of hyperbolic 3 manifold to tetrahedrons realizing the solution of Yang Baxter equation. By pseudo-Anosov mapping class for punctured surface, we obtain mapping torus for Lens space two bridged knot complement whose projection is Teichmuller disc with real multiplication.

For generic $Y^{p,q}$ with $q \neq 0, p$, the extremal volume is the irrationality which is the algebraic integral(Stark-Heegner unit) in real quadratic number field, $F = \mathbb{Q}(\sqrt{4p^2 - 3q^2})$.

As a result, we obtain the algebraicity of string partition function with asymptotic Black Hole entropy from the algebraicity of $\mathbf{p}$-adic zeta function of real quadratic number field with Stark-Heegner unit by Kronecker limit formula.

So on one hands, we have $Y^{p,q}$ with $L(p^2, pq-1)$ two bridge knot in one hands(string theory construction) and on the other hands, we have $L(p,q)$ boundary of Hilbert modular surface of real quadratic $F = \mathbb{Q}(\sqrt{4p^2 - 3q^2})$ over characteristic $\mathbf{p} = 1 \mod p$ and pro-$\mathbf{p}$ covering of the Bianchi manifold with vanishing first Betti number.

From string theory construction we have 4d index with irrational spectral parameter in F. And from number field side, we have torsion Galois representation from pro-$\mathbf{p}$ covering of Bianchi manifold. From the Bianchi manifold, the real quadratic field F is not visible. But from the pro-$\mathbf{p}$ covering of Bianchi manifold, we have torsion Galois representation in supersingular locus in Siegel modular varieties over sufficiently ramified field after theta lift such that we have totally real number field F for irregular Sasaki-Einstein CY manifold. And also from metaplectic lift, we have sign refined spin CS partition function of 3d manifold providing extremal volume for Sasaki-Einstein manifold.

There is additional duality in supersingular extremal elliptic surface which realize Leech lattice at characteristic 2 as Mordell Weil lattice of elliptic fibration for F-theory background for Sasaki-Einstein manifold.The duality exchange finite automorphism of Nef cone to automorphism of Leech lattice.

And $N = 1$ Sasaki-Einstein manifold comes from non-arithmetic hyperbolic reflection group of Leech lattice for $N = 4$ string vacua over sufficiently ramified field.

Our approach to real multiplication is one from **p**-adic zeta function of real quadratic number field $F = \mathbb{Q}(\sqrt{4p^2 - 3q^2})$,and the other from Sasaki-Einstein CY $C(Y^{p,q})$ manifold with finite order Seiberg duality on dual 4d CFT via AdS-CFT.

Now we explain why the irregular Sasaki-Einstein manifold is relevant to real multiplication problem. The Sasaki-Einstein CY manifold is obtained from real cone of $U(1)$ twisted fiber of $S^2 \times S^2$,such that we have infinite family $Y^{p,q}$ of Kahler-Einstein metric on $S^2 \times S^3$. We have in mind the twistor construction for Sasaki-Einstein CY with self-dual Yang-Mills on $\mathbb{CP}^2 \# m\overline{\mathbb{CP}^2}$ with exotic smooth structure by Lens space surgery[39, 38]. For each Sasaki-Einstein manifold $Y^{p,q}$, there exist $\# m\overline{\mathbb{CP}^2}$ with Lens space $L(p^2, pq-1)$ boundary. By analytic continuation, we consider hyperbolic 3 manifold which is Lens space knot complement,and consider hyperbolic mapping torus which share the same Haken covering. Then the two bridge knot for $L(p^2, pq-1)$ realize the pseudo-Anosov flow on punctured torus with mapping torus whose projection is Teichmuller disc. We consider Teichmuller disc which is orbi disc with $\mathbb{Z}/p$ action. By pro-**p** covering, we realize Teichmuller disc with real multiplication by $F = \mathbb{Q}(\sqrt{4p^2 - 3q^2})$, by pro-**p** lift of real multiplication by Arakelov class group $\mathbb{Q}(\sqrt{\pm})$ associated with pro-**p** lifting of Bianchi manifold with $\mathbb{Q}(\sqrt{-p})$. [2].

So the infinite families come from the infinite exotic smooth structure of

[2]Over Teichmiller disc we have Hodge bundle of N=1 mirror curve. With pseudo-Anosov flow, we have the Lyapunov exponent.

$\mathbb{CP}^2\#m\overline{\mathbb{CP}^2}$ by Lens space $L(p^2, pq-1)$ surgery. Associated with the exotic $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$, we have hyperbolic knot complement by two bridge knot $\mathcal{K}$[3].

We have non-arithmetic hyperbolic 3 manifold(Lens space $L(p^2, pq-1)$ knot quotient) and mapping torus with pseudo-Anosov flow from irregular Reeb foliation in Sasaki-Einstein manifold.

Later we will see that by Painleve 6-th differential equation for Kahler Einstein metric on $Y^{p,q}$ provide a singularity on base $\mathbb{P}^1\backslash\{0,1,\hat{a},\infty\}$ with irrational $\hat{a}$ in Hilbert class field K , such that the pro-$\mathbf{p}$ covering $L(p,q)$ arise from Seifert fiber of punctured disc with parabolic structure from $\hat{a}$ by standard Hitchin's construction.

So the real multiplication need to considered in pro-$\mathbf{p}$ geometry such that, on elliptic curve, we have generalized CM by Hilbert class field K of totally real number field $F = \mathbb{Q}(\sqrt{4p^2-3q^2})$. Due to pro-$\mathbf{p}$ structure, we also need Bianchi manifold with CM by imaginary quadratic $\mathbb{Q}(\sqrt{-p})$. So the definition of real multiplication involve not only action of F but also Arakelov class field $\mathbb{Q}(\sqrt{\pm p})$ of imaginary quadratic field $\mathbb{Q}(\sqrt{-p})$.

The string theory realization is following. Given Sasaki-Einstein quiver, we have 2-dimensional dimer Brane tiling in torus with irrational slope(R-charge/anomalous dimension) from irrational Reeb vector(pseudo-Anosov flow in torus). So we consider the two dimensional tiling as torus at cusp of Lens space knot complement, with specific pseudo-Anosov flow.

We consider torus with pseudo-Anosov flow as supersingular elliptic curve over sufficiently ramified field with CM lift by Hilbert class field K of totally real number field F with $[K : F] = N$ with N step mutation on Sasaki-Einstein quiver,where elliptic curve arise from elliptic factor of Jacobian of

[3]We consider $L(p^2, pq-1)$ as ramified covering of S^3 along two bridged knot $\mathcal{K}$,with non-arithmetic lattice $\Gamma_{\mathcal{K}}$ for $S^3/\mathcal{K} = \mathcal{H}^3/\Gamma_{\mathcal{K}}$.

mirror curve/Fermat quotient over finite field.

The elliptic curve have Stark-Heenger point of infinite order which is big $\mathbb{Q}$ point of infinite order associated with generalized CM point by Hilbert class field K of totally real number field F on moduli space of CY as generalized attractor point associated with irregular $N = 1$ vacua.

This realize the Brane tilting on elliptic curve with Fermion realizing supersingular $\mathbb{Q}$ points from $\mathbf{p}$-adic origin for higher ramification group realizing sign from extremal field configuration.

After considering 3d-3d correspondence, we have 3d super CS quiver gauge theory and 4d SCFT at IR fixed point. Then by AdS-CFT correspondence,we recover Sasaki-Einstein CY manifold.

We recover the extremal R-charge(anomalous dimension) which is extremal volume of Sasaki-Einstein CY from Stark-Heegner unit of real quadratic number field F. From algebraicity and integrality of $\mathbf{p}$-adic zeta function of real quadratic number field F, we recover algebraicity and integrality of string partition function for irregular Sasaki-Einstein CY with $\mathbf{p}$-adic convergence(pro-$\mathbf{p}$ asymptotic). So we have modularity for string partition function with $\mathbf{p}$-adic reciprocity, such that by mod $\mathbf{p}$ reduction,we have sign refined algebraicity and finiteness.

Now we state the speciality arising from real multiplication problem. First,given $Y^{p,q}$ quiver, the Seiberg duality cascade is finite order N with irrational R charge in Hilbert class field F of totally real number field F with $[K : F] = N$,and there is dense wall nearby generalized CM point by K from Teichmuller geodesic flow with real multiplication by F. So we need to consider $\mathbf{p}$-adic zeta function of real quadratic number field with Atkin-Lehner operation on $\mathbf{p}$-adic measure. There is additional sign in functional equation of $\mathbf{p}$-adic zeta function of real quadratic number field from conductor from intrinsic Higgs state of quiver which is real

GW invariant from higher ramification group.

So we need to consider infinite dimensional Hilbert space of irrational CFT, which arise from supersingular locus of Shimura varieties over sufficiently ramified field(Drinfeld modular scheme with level structure).

Second, associated with $N = 1$ string vacua, the Sasaki-Einstein CY with extremal metric determine a generalized CM point in moduli space of CY(string vacua) with dense wall nearby it. The dense wall comes from Teichmuller disc with real multiplication by $F = \mathbb{Q}(\sqrt{4p^2 - 3q^2})$ realising pro-**p** lift of real multiplication by Arakelov class field $\mathbb{Q}(\sqrt{\pm p})$. The Teichmuller geodesic flow(pseudo-Anosov flow) determining $SL(2, \mathbb{R})$ cocycle as rational homology sphere which is Bianchi manifold of $\mathbb{Q}(\sqrt{-p})$ whose projection is Teichmuller disc. The Teichmuller geodesic flow is pro-**p** homology sphere such that Teichmuller disc has pro-**p** structure by **p**-adic uniformazation.

The dense walls realizing Teichmuller disc there exist orbifold point which is the generalized CM point by Hilbert class field K of totally real number field F. The Seiberg duality for $Y^{p,q}$ is the mutation with finite order N producing resonance by Swan conductor N. We have N period of Markov triple for wild Sasaki-Einstein quiver.

The Swan conductor measure the period of the mutation for resonance of quiver which is is the additional data for wild quotient singularity(irregular Sasaki-Einstine CY) from orbifold(arithmetic) Riemann-Roch. For the wild Sasaki-Einstein quiver having the path algebra with more than 3 variable, there exist wild Nagata automorphism of first Weyl algebra realising non-maximal Swan conductor for supersingular elliptic curve over sufficiently ramified field.

Third, we have a automorphism of Leech lattice in $N = 4$ string background which is hyperbolic reflection group for $N = 1$ vacua as $Y^{p,q}$ Sasaki-Einstein CY manifold, which do the role for kernel of Langlands duality from two bridge knot $L(p^2, pq-1)$.

For that, we need to consider supersingular extremal elliptic surface with vanishing Mordell Weil group[4]. Such that we obtain $GSp_4(\mathbb{Q})$ representation for pro-$\mathbf{p}$ covering of (hyperbolic)[5] Bianchi 3 manifold which is Lens space $L(p^2, pq-1)$ knot complement with additional sign from metaplectic $\mathbb{Z}/2$ kernel.

With this,we have totally real number field $F = \mathbb{Q}(\sqrt{4p^2 - 3q^2})$ for irregular Sasaki-Einstein CY which is total space of canonical bundle over irregular toric Fano surface, such that we have Mordell-Weil torsion group with vanishing Mordell-Weil rank on Selmer group for elliptic factor Jacobian Fermat quotient over finite field which is Galois cohomology of F, such that $\mathbf{p}$-adic zeta function has vanishing order 0. Then we have elliptic fibered CY manifold with Mordell-Weil torsion with vanishing Mordell-Weil rank with a section from big $\mathbb{Q}$-point of infinite order associated from Stark-Heegner unit of F[6]. We can see additional sign from Swan conductor N from real GW invariant by counting self-dual representation of the symplectic double(metaplectic lift) of path algebra of wild Sasaki-Einstein quiver. We have orthogonal degeneracy locus in Flag bundle over Shimura varieties in ordinary locus with non-maximal conductor measuring non-transversality of Frobenius and Frobenius pull-back filtration in irregular Hodge bundle(non-commutative motive). Fourth,there exist a Bianchi manifold of $\mathbb{Q}(\sqrt{-p})$of with vanishing first Betti number whose pro-$\mathbf{p}$ covering gives the specific global torsion Galois representation in supersingular locus of

[4]There is duality between automorphism of MW lattice(Leech lattice) and finite automorphism of Nef cone realising cone conjecture from generalized CM by Hilbert class field K of totally real number field F. Associated with this we have Borcherds automorphic form of orthogonal group.

[5]We have hyperbolic Bianchi manifold of $\mathbb{Q}(\sqrt{-p}$ except $p = 2$.

[6]For totally real number field with Mordell Weil rank, we have irregular non-toric Fano surface associated with determinantal varieties with Picard jumping locus realising Pfaffian-Grassmannian derived pair. The associated CY is elliptic fibered CY with multi section by product of two elliptic surface realising Picard jumping locus.

Shimura varieties over sufficiently ramified field associated with Sasaki-Einstein CY manifold with totally real number field $F = \mathbb{Q}(\sqrt{4p^2 - 3q^2})$. And $L(p, q)$ over characteristic $\mathbf{p} = 1 \mod p$ for cusp of Hilbert modular surface of F.

By pro-$\mathbf{p}$ tower of arithmetic Bianchi orbifold of $\mathbb{Q}(\sqrt{-p})$, we have $GSp_4(\mathbb{Q})$ representation of Bianchi group[7] which realize extremal elliptic fibered CY with vanishing Mordell-Weil(MW) rank from vanishing first Betti number of pro-$\mathbf{p}$ covering. But we have big $\mathbb{Q}$ point of infinite order from Stark-Heegner unit of totally real number field F in Hilbert class field K realising Haken covering.

With this we have torsion Galois representation from supersingular locus of Siegel modular varieties. Due to duality[8] for supersinguler extremal elliptic surface at characteristic 2, such that Bianchi group of $\mathbb{Q}(\sqrt{-2})$ is essential with Haken covering from Siegel unit. Note that the Stark-Heegner unit of totally real number field F $12N$-th power of Siegel unit in Hilbert class field K with $[K : F] = N$.

The direct relation between pro-$\mathbf{p}$ covering of Bianchi manifold(Galois representation with $\mathbb{Z}/2$ metaplectic lift) and Sasaki-Einstein quiver comes from considering elliptic fibered CY, with specific singular fiber with Mordell Weil group which realize integrable system with Painleve 6-th differential equation. The ramification knot($L(p^2, pq-1)$ double bridge knot) determine the parabolic structure from irrational parameter in K.

Due to irrational resonant parameter for 6-th Painleve equation from finite

[7] There exist no $SL_2(\mathbb{C})$ Galois representation for Bianchi group since Bianchi manifold is not algebraic.The non-existence of two dimensional $SL_2(\mathbb{C})$ Galois representation is from Serre conjecture, so that we need $GSp_4(\mathbb{Q})$ representation with sign from additional conductor associate with Galois representation of totally real number field F. This means that, we do not have central extension of $PSL(2, \mathbb{C})$ representation to $SL(2, \mathbb{C})$. So, we need to consider $Sp(4, \mathbb{C})$ representation with metaplectic lift with additional sign as global torsion Galois representation over supersingular locus over sufficiently ramified field. For that, we need to consider pro-$\mathbf{p}$ covering of Bianchi manifold.

[8] This is the duality between automorphisn of Leech lattice and automorphicm of Nef cone relating supersingular locus and ordinary locus from Level-Rank duality.

order Seiberg mutation, we have additional sign from conductor by metaplectic lift. For some Bianchi manifold with vanishing first Betti number, we have no 2-dimensional Galois representation but only $GSp_4(\mathbb{Q})$ Galois representation from pro-**p** covering. This is the torsion Galois representation for Sasaki-Einstein CY arising from supersingular locus of Siegel modular varieties over sufficiently ramified field.

So we have the dimer Brane tiling(bipartite graph) on elliptic curve with irrational slope with train track model- mapping torus with pseudo-Anosov flow(Teichmuller geodesic flow).

1.2 N=4 vacua in string theory, and elliptic fibered CY

We start from $N = 4$ quiver, which is the quiver with 1 vertex with 3 loops. There exist elliptic curve with ABC triple for ABC conjecture fixed by j invariant(Belyi map) which is mirror curve. On Leech lattice we have S-duality.By considered as elliptic K3 surface over characteristic 2,the Mordell Weil group is equal to Leech lattice.

For $N = 2$ Seiberg Witten quiver,we have S-duality which exchange UV-IR. For $N = 1$ Sasaki-Einstein quiver,we also have S-duality as Seiberg duality with finite order such that UV-IR flow is obtained with Seiberg duality cascade. The order of Seiberg duality gives the conductor by correspondence(resonance). We have Painleve 6-th with resonant parameter. As simple factor of Jacobian of mirror curve as elliptic fibration of elliptic surface, Mordell-Weil group is torsion of rank 0. This is supersingular extremal elliptic surface.

At characteristic 2,we have duality[36] on Mordell-Weil group(Leech lattice) which exchange automorphism of lattice to finite order automorphism of Nef cone from Level-Rank duality. By supersingular extremal elliptic surface, the irregular Sasaki-Einstein CY is realized with an non-arithmetic hyperbolic automorphism of Leech lattice, such that we have pro **p** covering of Bianchi group.

$N = 4$ vacua in string theory is equivalent to having supersingular locus on Shimura varieties over sufficiently ramified field(perfectoid field). Then over perfectoid field,the Shimura varieties has integral model with level raising and lowering operator providing pro-**p** asymptotic of string partition function and **p**-adic zeta function of totally real number field. And there is **p**-torsion in cohomology of Shimura varieties from pro-**p** lift of Bianchi manifold in supersingular locus over sufficiently ramified field. The Galois representation associated with torsion class is for Sasaki-Einstein manifold, arise from pro-**p** covering of Bianchi manifold which gives $GSp_4(\mathbb{Q})$ representation.

By F-theory dual, we have the extremal elliptic surface with MW rank 0 for irregular Sasaki-Einstein CY motive in supersingular locus over sufficiently ramified field. With MW rank 0, we have finite polyhedral Nef cone(Cone conjecture) with generalized CM by Hilbert class field K of totally real number field F with $[K : F] = N$, by mod **p** completion. We reduce Galois group of sufficiently ramified field to Absolute Galois group on moduli space of CY realising N number of connected component from finite order Seiberg duality of derived category at characteristic 0 by generalized CM by Hilbert class field K[**?**].

Here in class field theory of real quadratic field(totally real field), there exist reciprocity from Adelic Galois group which gives the Seiberg duality cascade with additional metaplectic lift for additional sign. This Adelic Galois group contain the Fourier Mukai transform(twistor transform) which is birational. But Fourier Mukai transform also include derived equivalence, such that for non-toric irregular Fano surface associated with pfaffian-Grassmannian pair with non-vanishing Mordell-Weil rank, we have Picard rank jump locus from inverting of motive of affine line $\mathbb{A}^1_{\mathbb{C}}$, realizing sign from wild monodromy along $\infty \in \mathbb{A}^1_{\mathbb{C}}$.

So on moduli space of CY we have the action of Adelic Galois group, one is birational transformation and the other is derived equivalence which is not birational. Two type of walls are different in moduli space of CY, the type of wall of second kind which corresponds to jumping of MW rank(increase a point of infinite order).

1.3 String theory on Calabi Yau cone over Sasaki-Einstein manifold

The string partition function is known to be the convergence series in formal parameter with particular asymptotic. Given a string orbifold background,the string partition function has the pro-$\mathbf{p}$ asymptotic expansion with the integral coefficient by mod $\mathbf{p}$ reduction. We prove the algebraicity of string partition function from the algebraicity of Stark unit in the Hilbert class field K of real quadratic field F. The asymptotic of the string partition function is determined by extremal Reeb vector(foliation) which extremizing volume function. The extremizing volume(inverse of R-charge) satisfy the certain algebraic equation. It determine Killing spinor(from extremal Reeb vector) of Sasaki-Einstein CY manifold. We consider Killing spinor as Galois invariant object.

The string background(orbifold) with Killing spinor determine a totally real number field F with its Hilbert class field K, and string partition function lies in Hilbert class field K. With this we prove the algebraicity of String partition function from the algebraicity of Shintani zeta function of totally real number field F with Hilbert class field K.It provides a particular example of Stark conjecture and Langlands conjecture from global torsion Galois representation.

The extremal Reeb vector determine the Hilbert class field of the real quadratic number field F, and we consider Lens space $L(p,q)$ as link of the quotient singularity in boundary of Hilbert modular surface of F with chosen ideal class.

For Lens 3-space $L(p,q)$ with Shintani cocycle with $\mathbf{p}$-adic lift from the level in CS partition function on $L(p,q)$, we consider the CY cone over a Sasaki-Einstein 5-space geometry.

There are embedding of CY 3 fold cone over $Y^{p,q}$ to $\mathbb{C}^3/\mathbb{Z}_{p+1} \times \mathbb{Z}_{p+1}$ with $p+1 = \mathbf{p}$. There are three kind of special Lagrangian $S^3/\mathbb{Z}_p$, $S^3/\mathbb{Z}_{p+q}$ and $S^3/\mathbb{Z}_{p-q}$. We choose real quadratic field F with discriminant $4p^2-3q^2$ associated with $Y^{p,q}$. Associated with this, we choose Shintani cocycle with $-p/q$ for cusp $L(p,q)$ by periodic continued fraction with pro-$\mathbf{p}$ covering $L(p^2-pq-1)$ for totally real number field $\mathbb{Q}(\sqrt{4p^2-3q^2})$ realising $\mathbf{p}$-adic lift of Shintani cocycle. Such that, we consider pro-$\mathbf{p}$ geometry of $L(p,q)$ in cusp of Hilbert modular surface of F.

The choice of $L(p,q)$ for $Y^{p,q}$ is from wild ramification from $\mathbf{p}$-adic lifting in mind.we consider $L(p,q)$ as the link of wild quotient singularity. We will be used in connection with $\mathbf{p}$-adic Landau Ginzburg model associated with Fermat quotient $C = \{y^p = x^q(1-x)^r(-1)^s, q+r+s = 0 \mod p\}$ over $\mathbf{p} = 1 \mod p$. Since we consider wild quotient singularity, there exist 3d-3d duality exchanging Lens space index p and level $\mathbf{p}$. This is the Level-Rank duality on supersingular locus,exchanging meridian(inertia) and longitude(Frobenius).

This is used to construct nearby ordinary Galois representation associated with pro-$\mathbf{p}$ covering of Bianchi orbifold[29].

The toric diagram of $Y^{p,q}$ is not parallelogram but the toric diagram of $L(p,q)$ is parallelogram. We obtain $Y^{p,q}$ from $L(p,q)$ by pro-$\mathbf{p}$ covering $L(p^2, pq-1)$ such that we have $\mathbf{p}$-torsion Galois representation in supersingular locus of

Siegel modular varieties $\mathcal{A}_g$ over sufficiently ramified field realising wild quotient $L(p,q)$ singularity[9].

This can be obtained by considering $L(p^2, pq-1)$ with exotic smooth structure by Dehn surgery. The Seiberg duality is toric duality by N step mutation realising pro-$\mathbf{p}$ geometry determining sign from lowest homothety class. There exist other supersingular transition in supersingular locus over sufficiently ramified field by choice of higher homothety class in $\mathbf{p}$-adic Hecke character in subsection.3.4. The latter class of transition is adding matter field and integrate out change the level $q \to q-1$ of 4d dual $SU(2p)$ gauge theory.

We use $\mathbf{p}$-adic lift for proving algebraicity and integrality of Black Hole entropy and string partition function. From the point of view of extremising volume the algebraicity comes from extremising volume functional- it satisfies algebraic equation from the existence of Killing spinor(extremal Reeb vector).

In our consideration,the Sasaki-Einstein manifold arise from hyperbolic Knot complement. The mirror curve of Sasaki-Einstein manifold has Mordell Weil group with rank 0[10].

Mordell Weil group for mirror curve of Sasaki-Einstein manifold determine a supersingular extremal elliptic surface over characteristic 2, and there exist duality in supersingular elliptic surface over sufficiently ramified field from mirror symmetry.

The automorphism of Mordell Weil lattice in supersingular locus is dual to finite automorphism group of Nef cone of extremal elliptic surface. The duality comes from Borcherds product for automorphic form associated with the automorphism of the MW lattice. For that we have restriction for lattice to be

[9] By supersingular decomposition, we only need modular curve with $\mathbf{p}$ power conductor in Shimura varieties over sufficiently ramified field.

[10] For general totally real number field, we have Mordell-Weil rank for associated elliptic curve over sufficiently ramified field.

reflexive lattice(the reflection subgroup is finite in orthogonal group) with rank by 22 of signature $(1, r-1)$ for $0 \leq r \leq 20$ or $r = 22$. So at supersingular locus we obtain all **p** torsion class in Shimura varieties from pro-**p** covering of Bianchi manifold, each of which is the kernel of Langlands duality, and all $N = 1$ CY string vacua has $(0, 2)$ worldsheet supersymmetry with sign from higher ramification group.

As a result, we have Tate conjecture for supersingular elliptic K3 surface over characteristic 2 where we need Clifford lift and metaplectic kernel $\mathbb{Z}/2$. So this duality comes from additional $\mathbb{Z}/2$ at infinity from pro-**p** covering which gives the twistor spinor which inherit from Milnor homomorphism on Galois cohomology of totally real number field with coefficient in sufficiently ramified field. We first consider Leech lattice($N = 4$ lattice), then there exist mirror symmetry (S-duality) which exchange meridian and longitude of knot.

We need to see transcendental lattice for MW torsion which is skelepton of mirror curve. By Belyi map with j invariant, this is the transcendental part of **p**-adic regulator/inhomogeneous period map(inhomogeneous part of Picard Fuchs equation) taking care of the pro-**p** part with metaplectic $Z/2$.

We need duality on supersingular locus over sufficiently ramified field which exchange MW torsion which is **p**-torsion from Bianchi manifold providing hyperbolic automorphism of Leech lattice to MW rank which is Level-Rank duality. With this we realize all non-trivial higher dimensional irregular geometry associated with nilpotent cone from versal deformation of ADE singularity with Mordell-Weil rank too.

The Sasaki-Einstein manifold and its mirror curve(skelepton) with torsion MW group arise from pro-**p** covering of Bianchi group arise from **p**-torsion in supersingular locus over sufficiently ramified field[11]. This is the duality in

[11]There is no transcendental cycle in supersingular locus(MW torsion), instead there is **p**

supersingular locus over sufficiently ramified field.

Since we consider pro-**p** geometry of Hilbert modular surface, we can consider hyperbolic reflection group(Bianchi group) by lifting to Siegel modular varieties. Then from the pro-**p** covering of Bianchi group, thin monodromy group(non-arithmetic lattice) in $Sp_4(\mathbb{R})$ representation is obtained. At supersingular locus, we have Galois representation for Bianchi orbifold with sign from $\mathbb{Z}/2$. Then from imaginary quadatic field $L = \mathbb{Q}(\sqrt{-p})$ of Bianchi group with pro-**p** covering of it, we have supersingular elliptic curve over sufficiently ramified field with CM by L for singular fiber for elliptic fibered CY, which can be lift of CM by Hilbert class field K of totally real number field $F = \mathbb{Q}(\sqrt{4p^2 - 3q^2})$. This is compatible with Teichmuller disc have real multiplication by F and also Arakelov class field $\mathbb{Q}(\sqrt{\pm p})$ of imaginary quadratic field L,

Since the $GSp_4(\mathbb{Q})$ Galois representation can be obtained from pro-**p** covering of Bianchi orbifold, this gives the direct relation between **p**-adic Galois representation(torsion Galois representation) and Killing(twistor) spinor. This is the torsion global Galois representation with **p**-adic epsilon factor associated with Killing spinor on Sasaki-Einstein manifold which extremize the volume. The extremal Einstein Kahler metric is the arithmetic data- Stark-Heegner unit in Mordell Weil group(unit group) in Hilbert class field K of totally real number field F. Such that it provide the **p**-adic Grothendeick section. The Stark-Heegner unit is logarithmic derivative of **p**-adic zeta function of F(the Stark Heegner unit) with Kronecker limit formula.

By $Sp_4(\mathbb{C})$ representation of Bianchi group, we have refined Hodge conjecture for elliptic fibered CY with sign from metaplectic kernel. So we have Langlands duality in ordinary locus of Shimura varieties which can be seen as superingular locus over sufficiently ramified field for the global Galois represen-

torsion.The duality exchanges MW torsion to **p** torsion in supersingular locus.

tation for Sasaki-Einstein CY with from pro-$\mathbf{p}$ covering of Bianchi manifold.

Then we have Langlands duality from non-Abelian Fourier transform along Heegaard torus(tubular neighborhood of knot) by exchanging meridian(inertia) and longitude(Frobenius),where knot is the Lens space knot for exotic smooth structure.

For general non-parallelogram toric diagram, we have dual Penrose tiling with perfect matching condition having the irrational slope in Hilbert class field K of the real quadratic field F. The irrational slope comes from extremal Reeb foliation on $Y^{p,q}$. The perfect matching condition for free fermion in tiling is the consistency condition comes with $\mathbb{Z}/2$ sign from global reciprocity. Counting free fermion in tiling realize the string partition function on Sasaki-Einstein CY.

We are interested in CY cone $C(Y^{p,q})$ with irrational Reeb vector(foliation) in K and Heterotic F-theory elliptic CY from the 1-cusped Bianchi manifold from Lens space knot surgery $L(p,q)$. String theory on $C(Y^{p,q})$ and $T^*L(p,q)$ is different,but $C(Y^{p,q})$ is obtained from pro $\mathbf{p}$ covering of $L(p,q)$ in supersingular locus over sufficiently ramified field from wild ramification, by exotic smooth structure by Lens space $L(p^2, pq-1)$ surgery.

For that, we need to use global reciprocity in supersingular locus once (automorphism of the reflexive lattice,for Borcherds automorphic form with theta lift). The extremal elliptic surface with torsion MW group for $Y^{p,q}$ is dual to $\mathbf{p}$-torsion from pro-$\mathbf{p}$ covering of Bianchi manifold containing $L(p^2, pq-1)$ knot $\mathcal{K}$. In this way $C(Y^{p,q})$ is obtained from normal bundle over hyperbolic two bridge knot of $L(p^2, pq-1)$.

We need integral model of Shimura varieties from which we have level raising/lowering from reciprocity of perfectoid field. At the supersingular locus, we have Sasaki-Einstein CY motive with sign by $\mathbf{p}$-rank stratification.

We remark that our global Galois representation associated with Bianchi manifold realising Sasaki-Einstein CY motive come from $GSp_4(\mathbb{Q})$ representation and not $SL_2(\mathbb{C})$. We have central extension as $Sp_4(\mathbb{C})$ representation of Bianchi group(not $SL_2(\mathbb{C})$ representation), so we have additional sign from metaplectic $\mathbb{Z}/2$ kernel. This $\mathbb{Z}/2$ provide sign for Sasaki-Einstein CY motive. In our **p**-adic zeta function of real quadratic field, we obtain $\mathbb{Z}/2$ from Hida's ordinary limit at arithmetic infinity. The sign of functional equation of **p**-adic zeta function is obtained from additional conductor N.

So from supersingular locus, we obtain non-parallelogram $Y^{p,q}$ toric diagram with additional sign from intrinsic Higgs state(real GW invariant).

Then we obtain Shintani zeta function for totally real number field F by quasi Fuchsian group action on upper half plane(Shintani cocycle) with specific Hilbert class field K and additional conductor N. The **p**-adic zeta function of number field F with sign provide the Kahler potential of moduli of CY space. The trace field L of Bianchi manifold is $\mathbb{Q}\sqrt{-p}$ which is the CM field of Fermat curve of degree p, over characteristic $\mathbf{p} = 1 \mod p$.

Each of Bianchi manifold,by pro-**p** tower, we have maximal order in central simple algebra realizing quiver from **p**-torsion in supersingular locus providing global Langlands correspondence. So each quiver with quiver mutation, we obtain Galois representation of number field from $Sp_4(\mathbb{C})$ representation arising from **p**-torsion in supersingular locus.

We use **p**-adic analysis with conductor of number field F which replace plurisubharmonic function for non-archimedean Kahler manifold. The Shilov boundary for boundary of Einstein Kahler metric(extremal metric) is recovered by Eisenstein series with **p**-adic lift and the choice of conductor.

The asymptotic limit of string partition function,R-charge of Baryonic operator,determine the Hilbert class field K. The string partition function gives the generation series of zeta value of the number field F. We can see the zeta value as the Euler volume of moduli space of orbifold stable map.

The Hilbert series of mesonic moduli space which is the resolution of singularity with link $Y^{p,q}$[9] has limit as extremal R charge. So we can consider mesonic moduli space which is smooth CY as the Hilbert scheme which is the resolution of singularity.

By Seiberg duality cascade, we obtain extremal R charge with conductor in IR limit which we obtain from **p**-adic Shintani cocycle for real quadratic field F. The structure of **p**-adic measure with Atkin-Lehner operator gives the Seiberg duality cascade by **p**-adic uniformization (Harmonic cocycle for Bruhat-Tits building of $SL(2, F)$).

$Y^{p,q}$ family is realized in $\mathbb{R}^4 \times S^1$ 5 dimensional $SU(p)$ CS theory at level q. We can change the level q by add matter field(impurity) and integrate out realizing blow up and down in toric diagram of $Y^{p,q}$, which can be seen as tropical transition to different Sasaki-Einstein manifold. This process can be seen as **p**-adic transition in supersingular locus(Wall of second kind in moduli space of stability condition). The quiver for Sasaki-Einstein manifold $Y^{p,q}$ with $q \neq 0, p$ is the tachyonic quiver with more then 1 collapsing 4 cycle, comes from supersingular locus[**?**]. In this case, the there exist certain duality for Markov triple for exotic Lagrangian tori(and Sasaki-Eisntein manifold) and it gives the cocycle of Bruhat-Tits building. For totally real number field with non-zero Mordell-Weil rank associated with irregular non-toric Fano surface,We also have the Level-Rank duality for not birational derived equivalence which is the automorphism of reflexive polygon for quiver. This realize pfaffian-Grassmannian correspondence realizing non-trivial $\mathbb{Z}/2$ sign by **p**-adic Ceresa cycle with Mordell-Weil rank realizing divisorial contraction for Picard jump locus.[12].

[12]There is Shintani zeta function(multiple Dirichlet series) associated with quiver(XXZ model) counting spin configuration of lattice by bump et al[43], but we adapt **p**-adic zeta function for number field associated with torsion Galois representation in supersingular locus.

By considering cyclic quotient singularity with link $L(p,q)$ as wild quotient singularity, we obtain $Y^{p,q}$ singularity. The Swan conductor is the additional data of $L(p,q)$ which is the intrinsic Higgs state of quiver $Y^{p,q}$. With Swan conductor, we have additional sign for spinor(correspondence).

We have following theorem from formality of Fano orbifold [45][44].

Theorem 1 *For given Sasaki-Einstein CY with orbifold (irregular) Fano base V , we have mirror Landau Ginzburg model by sum over all vertex in toric diagram of V. Due to orbifold, we have additional sign from Seiberg duality of Sasaki-Einstein quiver.*

We have formality and unobstruction of mirror LG family, but with sign from compactification divisor of moduli space of mirror LG model,considered as $\mathcal{D}$-module over affine line with wild monodromy.

The boundary divisor of moduli space of mirror LG model is obtained from embedding of mirror curve to its Jacobian at infinity. For that we need compute intersection of torsion point in mirror curve and its Jacobian, computing τ function for spectral curve.

By considered as Fermat curve over finite field, we obtain the sign from splitting constant of Fermat curve by Swan conductor.

We have singular Hermitian metric in Arakelov geometry and we have lower bound $\log\log$ form(plurisubharmonic form) for extremal metric determining extremal volume determining shape, where real GW invariant which determining shape contributing extremal volume(entropy). Since class number of class group of totally real number field exceed class number of class group of imaginary quadratic which proportional to $\log\log d$ where d is discriminant of field.

Additional sign allow us to interpolate Lens space(real quadratic field) and Bianchi manifold(with imaginary quadratic field $L = \mathbb{Q}(\sqrt{-p})$) with Arakelov class group $\mathbb{Q}(\sqrt{\pm p})$ providing varieties over function field. For that we need to determine totally real number field from pro-$\mathbf{p}$ covering of Bianchi manifold. As the function field with coefficient in finite field can be obtained by sufficiently ramified cyclotomic field(perfectoid field) we have totally real number field F

and Hilbert class field of F, and we have motivic Galois group action with sign.

The Mahler measure(Hasse Weil zeta function of character varieties of knot) is the L-value of elliptic curve with CM by imaginary number field of Bianchi group of the knot. So Bianchi modular form by L function of imaginary number field, gives the Mahler measure of character varieties. By hither ramification group of character varieties, we have **p**-adic Bianchi modular form with sign from higher ramification group determined by **p**-adic zeta function of totally real number field.

The **p**-adic Galois representation for **p**-adic zeta function of real quadratic field and extremal volume satisfies reciprocity. We have congruence relation on **p**-torsion Galois representation from pro-**p** tower of Bianchi manifold. With this we have big CM point with averaged Faltings height and also equidistribution of CM point which again constraint averaged Faltings height(finiteness).

So associated with big CM point from Hilbert class field K of real quadratic field F which provide UV completion by Galois completion, we have following structure of averaged Faltings Height, which can be compared with our construction for Stark Heegner unit of F.

$$\frac{1}{2}^{d}\sum_{\Phi} h_{K/\Phi} = \frac{-1}{2}\sum_{\Phi}\frac{\zeta_K'(0,\chi)}{\zeta_K(0,\chi)} + \frac{-1}{4}log\mid\frac{D_K}{D_F}\mid + \frac{-d}{2}log(2\pi)$$

Here K is big CM field, and F is maximal totally real subfield. We consider Abelian varieties $\mathbb{A}$ of dimension d, and $\chi : \mathbb{A}_F{}^{\times} \to \pm 1$ is the quadratic Hecke character for extension K/F. We need to sum over all CM type $\Phi \in Hom(K, \mathbb{C})$ with $[K : F] = N$ where N is Swan conductor. In our computation based on **p**-adic zeta function of F, we have additional conductor N for sign. In averaged Faltings height, by almost real structure from sum over all CM type, we have **p**-adic Ceresa cycle with sign.

This can be obtained from spinor lift to Siegel modular varieties with additional sign. And also can recovered from **p**-adic zeta function of real quadratic number field by choosing Hilbert class field to be big CM field. So the congruence of **p**-adic zeta function has additional $\mathbb{Z}/2$ from Hilbert reciprocity for compactification boundary of Jacobian of singular plane curve by knot, as we can see from pro-**p**-tower of Bianchi orbifold(congruence of Bianchi modular form with theta lift).

The Stark-Heegner unit-entropy do the role for stability condition as pro-**p** structure in moduli space of CY. So with changing Black Hole entropy by topology transition in sting theory, we obtain wall crossing which change (real) Donaldson Thomas invariant, changing totally real number field. We have Leech lattice for N=4 vacua and by considering automorphism of Leech lattice by Bianchi orbifold, we can obtain N=1 vacua. In supersingular locus over sufficiently ramified field,we have S-duality of Leech lattice between automorphism of the finite nef cone and automorphism of Mordell Weil lattice. By pro-**p** covering of Bianchi manifold, we have non-arithmetic automorphism of Leech lattice where we have Seiberg duality acts on pro-**p** covering. The change of totally real number field(cascade of number field) is obtained in supersingular locus over sufficiently ramified field with Level-Rank duality. We have cascade of number filed realizing cascade of nilpotent cone in Hasse diagram realising Ekedahl-Oort stratification in ordinary locus in Shimura varieties considered as supersingular locus over sufficiently ramified field.

Theorem 2 *Sasaki-Einstein CY are non-rigid varieties associated with non-arithmetic lattice(thin monodromy group) parametrized by Shimura varieties in supersingular locus over sufficiently ramified field. By Seiberg duality correspondence with Higgsing, the derived category of Fano orbifold base has quasi-phantom factor. Such that we have fake projective surface which has additional cycle from correspondence and is higher dimensional analog of maximal curve in supersingular locus. Each irregular Sasaki-Einstein CY with totally real number field, we have Beauville surface with non-vanishing* K_0 *with vanishing*

Hochschild homology, realizing building cocycle of $SL(2, F)$.

By UV-IR flow with IR limit realising irregular Sasaki-Einstein CY,by UV completion, we have building at infinity in moduli space of Higgs bundle by boundary divisor of compactification of Jacobian of singular plane curve by Lens space realizing knot at discriminant locus. By τ function of mirror curve, we recover Stark Heegner point on its Jacobian.

The supersingular locus over sufficiently ramified field parametize motive of Sasaki-Einstein CY. And by **p**-rank(Ekedahl-Oort) stratification with different root number, we have cascade of totally real number field.The root number determine sign for functional equation, and comes from real GW invariant at the wall supporting perverse coherent sheaves. By wall crossing in moduli space in supersingular locus over sufficiently ramified field, we can change totally real number field, realizing Level-Rank duality.

The intrinsic Higgs state is mirror to non-torsion mod 2 algebraic (p, p) Hodge class over sufficiently ramified field for compactification of vector multiplet moduli space. This is the Hodge class with no algebraic origin at characteristic 0,from correspondence(global reciprocity/Seiberg duality). We have non-torsion mod 2 algebraic class by **p**-adic Ceresa cycle.

The resonance state(intrinsic Higgs state) do not need to lies on middle comology (mirror of (p, p) class),we only need Hodge class over sufficiently ramified field. For Sasaki-Einstein manifold, we have non-middle cohomology class from higher K group(non-trivial homotopy class) in quasi-phantom factor but can be considered as mirror of Hodge class over sufficiently ramified field. Consequently, after finite mutation by arithmetic Minimal Model Program, we have fake projective space(arithmetic Mori dream space which is supersingular over sufficiently ramified field) as base of non-commutative resolution of Sasaki-

Einstein manifold. Since fake projective space[41] is analog of maximal curve, we need to consider non-commutative resolution of Sasaki-Einstein manifold as motive over finite field in supersingular locus, with F-theory supersingular extremal elliptic fibered CY. This gives the opers locus in compactification of moduli space of Higgs bundle of plane curve singularity for knot(two bridged knot).

For irregular Sasaki-Einstein, we have integrable system - XXZ relativistic spin chain, where we have Bruhat-Tits building $SL(2, F)$ with Seiberg duality cascade with Higgsing from integral structure of Shimura varieties over sufficiently ramified field, and we need **p**-adic measure with **p**-adic reciprocity for purely quantum state. The $SL(2, F)$ Bruhat-Tits building cocycle is recovered by **p**-adic Ceresa cycle associated with fake projective plane.

The associated CFT has additional level structure and is defined with **p**-adic uniformization. [13]. In 4d gauge theory side, the extremal volume is computed with Seiberg duality cascade realizing cocycle for Bruhat Tits building. So having Bruhat-Tits building with embedding of mirror curve(Fermat curve over finite field) to its Jacobian as divisor at infinity of moduli space of mirror curve(moduli of Higgs bundle), we obtain compactification at infinity for Fano orbifold,from the non-commutative resolution. So by UV-IR duality(UV completion) we obtain Fake projective plane. By $\mathbb{Q}$ point of infinite order from $SL(2, F)$ Bruhat-Tits building cocycle,we have wall crossing at infinity of Sasaki-Einstein quiver, which is elliptic divisor system satisfying **p**-adic modularity by Markov triple determining sign from real GW invariant.

So we have compactification of Bruhat Tits building from Sasaki-Einstein manifold with Seiberg dualtiy. Stark Heegner point is the point at infinity. Due

[13] The algebraicity of Stark Heegner unit($12N$ power of Siegel unit) can be seen from the K_2 torsion of sufficiently ramified field, satisfying global reciprocity for Steinberg elements. We have three term relation for modular symbol from cup product of two Siegel unit.

to the correspondence from torsion in K_2(for non-Abelian central extension) we have non-existence of 2-dimensional Galois representation for Sasaki-Einstein manifold, we need to consider metaplectic lift by theta lift with additional sign from conductor. We need symplectic double, which is not split due to correspondence(spinor).

From the gauge theory point of view the additional sign indicate the arithmetic data (Swan conductor) for singular Einstein Kahler metric which is limit Higgs bundle in boundary of moduli of Higgs bundle containing $Spec(K)$ for CM Hilbert class field K.

For that, we consider **p**-adic lift of it and obtain **p**-adic Shintani cocycle with **p**-adic zeta function. By the choice of conductor set(conductor system), we obtain $Y^{p,q}$ with non-parallelogram 2d toric diagram after projection from **p**-adic lift of the Shintani cocycle for $L(p,q)$ with parallelogram 2d toric diagram. So we can consider $Y^{p,q}$ with extremal volume as the geometric realization of a Galois representation associated with the conductor system for **p**-adic Shintani cocycle. The reciprocity from conductors translate to the algebraicity for Stark unit and extremal volume of $Y^{p,q}$.

After **p**-adic lift, we obtain the spin refinement of CS partition function of $L(p,q)$,which means that, we have the additional $\mathbb{Z}/2$ Galois symmetry from pro-**p** covering $L(p^2, pq-1)$. By considering $L(p,q)$ as the wild quotient singularity from **p**-adic lift, we can recover 3d toric diagram for $Y^{p,q}$,from **p**-adic lift of $L(p,q)$ 2d toric diagram by pro**p** covering in supersingular locus over sufficiently ramified field. We need additional sign from Swan conductor from choice of conductor for Galois representation. From $L(p,q)$ CS partition function, the additional sign comes from $\mathbb{Z}/2$ from interpolating p and arithmetic infinity(spin CS) realizing localization of CS partition function of $L(p,q)$ at p-th roots of unity which is not well defined(wild singularity).

The spin CS partition function is obtained by evaluating even roots of unity, which have additional $\mathbb{Z}/2$ Galois group action from $Gal(\mathbb{C}/\mathbb{R}) = Gal(\overline{\mathbb{Q}_\infty}/\mathbb{Q}_\infty) = \mathbb{Z}/2$. This comes from metaplectic lift of Clifford group for sign refined spin CS invariant.

By localizing at $\mathbf{p}^n$-th root of unity,we have additional Galois action from $Gal(\overline{\mathbb{Q}_\mathbf{p}}/\mathbb{Q}_\mathbf{p})$, and CS partition function comes from the $\mathbf{p}$-adic zeta function. Then $Gal(\mathbb{C}/\mathbb{R}) = Gal(\overline{\mathbb{Q}_\infty}/\mathbb{Q}_\infty) = \mathbb{Z}/2$ action is recovered by Hida's ordinary limit of modular tower, by generalized CM by Hilbert class field K of totally real number field F with $[K : F] = N$. Such that we have almost real structure with pro-$\mathbf{p}$ asymptotic from Galois group of sufficiently ramified field.

$$\lim_{\leftarrow} Gal(\overline{\mathbb{Q}}(\mu_{N\mathbf{p}^n})/\mathbb{Q}(\mu_{N\mathbf{p}^n})) = (\mathbb{Z}/N\mathbf{p})^\times/\{\pm\} \rtimes \lim_{\leftarrow}(1 + \mathbf{p}\mathbb{Z}/\mathbf{p}^n)$$

From this global reciprocity, we obtain the fat/global point if infinite order - $\mathbb{Q}$ point of the elliptic curve from the Stark Heegner unit, where the elliptic curve is the factor of Jacobian of the Fermat curve $C = \{y^p = x^q(1-x)^r(-1)^s, q + r + s = 0 \mod p\}$ with CM multiplication by $L = \mathbb{Q}(\sqrt{-p})$.

From the Stark Heegner unit, which is the modular Height of Stark Heegner point, we obtain extremal volume of CY manifold(extremal R-charge)-the entropy of the black hole with horizon $Y^{p,q}$. The Stark Heegner unit satisfy the Deligne's period conjecture. We consider the entropy as the period of mirror CY of Sasaki-Einstein cone $C(Y^{p,q})$ satisfying sign refined algebraicity.

There exist corresponding F-theory CY background by extremal elliptic fibered CY(torsion MW group with MW rank 0), where we have wild ramification by Tate form. Since wild ramification is controlled by reciprocity of Galois group, we obtain F-theory anomaly cancellation condition(consistency condition) for Killing spinor which comes from additional $\mathbb{Z}/2$ from arithmetic infinity. So with anomaly cancellation condition(consistency condition) for extremal elliptic fibered CY , we can indicate additional sign of a point of infinite order restoring algebraicity by conductor. The anomaly cancellation condition comes from global(Hilbert) reciprocity of sufficiently ramified field, with this we can determine higher ramification group.

In general, We have MW rank jumping locus for Galois representation associate with non-torsion class of infinite order Abel Jabobi Kernel in Griffiths group.

Chapter 2

Sasaki-Einstein manifold

We consider the infinite family $Y^{p,q}$ of Kahler Einstein metric $S^3 \times S^2$ - Sasaki-Einstein manifold from the infinite exotic smooth structure on 4 manifold $S^2 \times S^2$ with Lens space $L(p^2, pq-1)$ gluing. So we have infinite family of homeomorphic but not diffomorphic 4 manifold with different Seiberg-Witten invariant, which means that we have exotic monopole associated with Lens space knot with pseudo-Anosov flow which comes from Sasaki-Einstein quiver.

2.1 Kahler-Einstesin metric with Heun equation

The Kahler Einstein metric on Sasaki-Einstein 5 manifold and its cone CY is obtained by Heun equation(Painleve 6-th equation)[35] with irregular monodromy for $Y^{p,q}$ with $q \neq 0, p$, due to vanishing of Futaki invariant insuring K stability. We have finite nef cone for base of the Sasaki-Einstein manifold. The vanishing of Futaki invariant ensure that the existence of integrable system(Painleve 6-th equation),such that the cone is Calabi-Yau with unique Kahler Einstein metric.

Heun equation is the isomonodromy differential equation from 'non-hyperelliptic' mirror curve of Sasaki-Einstein CY associated with Mordell Weil lattice of rank 0.

It is the differential equation associated with elliptic surface with elliptic curve fibration with real multiplication and the Mordell Weil torsion group associated with integrable system for $N = 1$ Sasaki-Einstein CY. Unlike $N = 2$ Seiberg-Witten integrable system with non-relativistic Toda equation with trigonometric solution, we need elliptic solution for relativistic Toda equation by relativistic Gauss hypergeometric function at irrational parameter. We need hyperbolic Gamma function which can be seen as Carlitz Gamma function for Drinfeld module.

The Painleve 6-th equation gives the solution of Riemann-Hilbert correspondence for moduli space of parabolic connection with irregular monodromy which come from hyperbolic knot. So we have the Hitchin integrable system for singular plane curve associated with knot.

Even if Futaki invariant dones not vanish, the Sasaki-Einstein manifold $Y^{p,q}$ have extremal metric by airithmetic Minimal Model Program and can be the Black Hole horizon with negative scalar curvature. By AdS-CFT duality, the area of horizon proportional to extremal volume of CY cone is the anomalous dimension of dual SCFT. From the metric point of view, we need to solve the Heun equation associated with the deformation of relativistic Toda equation. The metric has specific coordinate ranges.

$$ds^2 = \frac{1-y}{6}(d\theta^2 \sin^2\theta d\psi^2) + \frac{1}{a(y)b(y)}dy^2 + \frac{b(y)}{9}(d\phi - \cos\theta d\psi)^2 + a(y)[d\alpha + c(y)(d\phi - \cos\theta d\psi)]^2$$

with

$$a(y) = \frac{2(\mathbf{b} - y^2)}{1-y},\ b(y) = \frac{\mathbf{b} - 3y^2 + 2y^3}{\mathbf{b} - y^2},\ c(y) = \frac{\mathbf{b} - 2y + y^2}{6(\mathbf{b} - y^2)},\ \mathbf{b} = \frac{1}{2} - \frac{p^2 - 3q^2}{4p^3}\sqrt{4p^2 - 3q^2}$$

The coordinate ranges are,

$$0 \leq \phi, \psi \leq 2\pi, 0 \leq \theta \leq \pi, y_1 \leq y \leq y_2,\ 0 \leq \alpha \leq 2\pi l$$

with boundary for y is $y_{1,2} = \frac{1}{4p}(2p \mp 3q - \sqrt{4p^2 - 3q^2})$ and y_3 with $\sum_{1\leq i\leq 3} y_i = \frac{3}{2}$ is cubic roots of $\mathbf{b} - 3y^2 + 2y^3$ and the boundary for α is $l = \frac{q}{3p^2 - 2q^2 + p\sqrt{4p^2 - 3q^2}}$.

The Heun equation is determined by the Laplacian spectrum on $Y^{p,q}$. With this we have zeta function for Sasaki-Einstein manifold.

$$\Box\Phi = -E\Phi, \quad E = 4\lambda(\lambda + 2)$$

$$\Box = \frac{1}{1-y}\frac{\partial}{\partial y}(1-y)a(y)b(y)\frac{\partial}{\partial y} + (\frac{3}{2}\hat{Q}_R)^2 + \frac{1}{a(y)b(y)}(\frac{\partial}{\partial \alpha} + 3y\hat{Q}_R)^2 + \frac{6}{1-y}[\hat{(}K) - (\frac{\partial}{\partial \phi})^2]$$

where $\hat{Q}_R = 2\frac{\partial}{\partial \phi} - \frac{1}{3}\frac{\partial}{\partial \alpha}$ is the Reeb Killing vector field for R symmetry of dual gauge theory. $\hat{K}$ is second Casimir of $SU(2)$ for angular momentum for $SU(2) \times U(1)^2$ isometry. We can decompose the angular part and radial part, the radial part reduce to Heun equation.

$$\Phi(y, \theta, \phi, \psi, \alpha) = \exp[i(n_\phi \phi + n_\psi \psi + n_\alpha \frac{\alpha}{l})]R(y)\Theta(\theta),\ n_{\phi,\psi,\alpha} \in \mathbb{Z}$$

The angular part is,

$$\hat{K}\Theta(\theta) = -J(J+1)\Theta(\theta)$$

The radial part is,

$$[\frac{\partial^2}{\partial y^2} + (\sum_{1\leq i\leq 3} \frac{1}{y - y_i})\frac{\partial}{\partial y} + d(y)]R = 0$$

with

$$d(y) = \frac{1}{H(y)}[\mu - \frac{y}{4}E \sum_{1\leq i\leq 3} \frac{{\alpha_i}^2 H^{'}(y_i)}{y - y_i}], \quad H(y) = \prod_{1\leq i\leq 3}(y - y_i)$$

$\mu = \frac{E}{4} - \frac{3}{2}J(J+1) + \frac{3}{32}(\frac{2n_\alpha}{3l} - Q_R)$ with angular momentum J and R-charge $Q_R = 2n_\phi - \frac{n_\alpha}{2l}$. The critical exponents are,

$$\alpha_1 = \pm\frac{1}{4}[n_\alpha(p+q-\frac{1}{3l}) - Q_R], \quad \alpha_2 = \pm\frac{1}{4}[n_\alpha(p-q-\frac{1}{3l}) + Q_R],$$
$$\alpha_3 = \pm\frac{1}{4}[n_\alpha(\frac{-p^2+q^2+p\sqrt{4p^2-3q^2}}{q} - \frac{1}{3l}) - Q_R]$$

This is regular singular at $y = y_1, y_2, y_3, \infty$ with critical exponents $\pm\alpha_i$ at y_i and $-\lambda, \lambda+2$ at ∞ with $E = 4\lambda(\lambda+2)$.

After suitable coordinate transform, the Heun equation is transform to the standard form which can be viewed as the Painleve 6-th differential equation. We consider the transformation of singularity

$$\{y_1, y_2, y_3, \infty\} \mapsto \{0, 1, \hat{a} = \frac{y_1 - y_3}{y_1 - y_2}, \infty\}, \ x = \frac{y - y_1}{y_2 - y_1}, \ R(x) = x^{|\alpha_1|}(1-x)^{|\alpha_2|}(\hat{a}-x)^{|\alpha_3|}h(x)$$
$$[\frac{d^2}{dx^2} + (\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-\hat{a}})\frac{d}{dx} + \frac{\alpha\beta x - k}{x(x-1)(x-\hat{a})}]h(x) = 0$$

$\alpha = -\lambda + \sum_{1\le i\le 3}|\alpha_i|$, $\beta = 2 + \lambda + \sum_{1\le i\le 3}|\alpha_i|$, $\gamma, \delta, \epsilon = 1 + 2|\alpha_{1,2,3}|$.

The accessory parameter k is $k = (|\alpha_1| + |\alpha_3|)(|\alpha_1| + |\alpha_3| + 1) - |\alpha_2|^2 + p[(|\alpha_1| + |\alpha_2|)(|\alpha_1| + |\alpha_2| + 1) - |\alpha_3|^2] - \hat{\mu}$.

$\hat{\mu} = \frac{1}{y_1 - y_2}(\mu - y_1\frac{E}{4}) = \frac{p}{q}[\frac{1}{6}(1-y_1)E - J(J+1) + \frac{1}{16}(\frac{2n_\alpha}{2l} - Q_R)^2]$ and from the cross ratio of roots of $\mathbf{b} - 3y^2 + 3y^3$,

$$\hat{a} = \frac{y_1 - y_3}{y_1 - y_2} = \frac{1}{2}(1 + \frac{\sqrt{4p^2 - 3q^2}}{q})$$

which is irrational for $q \neq 1, p$.

By the parameter of Painleve equation and the solution of it, we can see the quantities for CY extremal metric and dual SCFT.

From the radial part,we can see extremal R-charge of mesons in terms of exponents α_i,

$Q_R = p \pm q \mp \frac{3q^2 - 2p^2 + p\sqrt{4p^2 - 3q^2}}{3q}$ which corresponds to angular momentum $J = \frac{1}{2}(p \pm q)$ from angular part corresponds to special Lagrangian $S^3/\mathbb{Z}_{p\pm q}$ in

$Y^{p,q}$. This determine the ground state energy $E = 4\lambda(\lambda+2)$ for Heun equation by $\frac{2}{3}Q_R = -2 + \sqrt{4+E}$, from which we have the extremal volume of CY.

We can consider cross ratio coordinate $x = \frac{y-y_1}{y_2-y_1}$ to be Weierstrass $\mathfrak{p}$ function associated with K_2 torsion of sufficiently ramified field, then Heun equation is the Lame equation for elliptic soliton of Treibich-Verdier elliptic potential.

$$[\frac{d^2}{dx^2} + \sum_{1\le i\le 3}(2\alpha_i - \frac{1}{2})(2\alpha_i + \frac{1}{2})\mathfrak{p}(x+\omega_i)]\Phi = k\Phi$$

with $\mathfrak{p}$ with period $(2\omega_1, 2\omega_2)$,$\omega_1 = 1/2, \omega_2 = \tau/2, \omega_3 = -\omega_1 - \omega_2 = -(\tau+1)/2$.

In this form, Heun equation is the Hamiltonian for Painleve 6-th equation. We have $y_i = \omega_i$ for all $i = 1, 2, 3$ such that

$$\hat{a} = \frac{y_3 - y_1}{y_2 - y_1} = \frac{\omega_3 - \omega_1}{\omega_2 - \omega_1}$$

So for irrational $\hat{a}$, we have transcendental period τ. There is inversion formula between $\hat{a}$ and τ.

$$\frac{d\tau}{d\hat{a}} = \frac{\pi i}{\alpha(\alpha-1)(y_2-y_1)}, \quad y_i = \omega_i$$

Then we have elliptic parametrization of Painleve 6-th with Picard-Fuchs equation $L_{\hat{a}}$ for elliptic curve with parameter $\hat{a}$.

$$\hat{a}(1-\hat{a})L_{\hat{a}}\int_{\infty}^{\lambda}\frac{dx}{\sqrt{x(x-1)(x-\hat{a})}} = \sqrt{\lambda(\lambda-1)(\lambda-\hat{a})}[\alpha+\frac{\beta\hat{a}}{\lambda^2}+\frac{\gamma(\hat{a}-1)}{(\lambda-1)^2}+(\delta-\frac{1}{2})\frac{\hat{a}(\hat{a}-1)}{(\lambda-\hat{a})^2}]$$

$$z^2 = x(x-1)(x-\hat{a}), \quad L_{\hat{a}} = \hat{a}(1-\hat{a})\frac{d^2}{d\hat{a}^2} + (1-2\hat{a})\frac{d}{d\hat{a}} - \frac{1}{4}$$

With the solution of Painleve 6-th equation with irrational parameter[32, 33], we have elliptic relativistic Calosaro Moser integrable system(q-deformed relativistic Toda) with non-compact quantum dilogarithm from Stark-Heegner unit of totally real number field. This provide the irrational Hodge filtration in compactification of Landau Ginzburg model[44, **?**].

Note that, for irregular Sasaki-Einstein CY manifold, the non-compact quantum dilogarithm is evaluated at irrational parameter. There exist a XXZ integrable Yang Baxter model associated with Sasaki-Einstein quiver.

For the Painleve 6-th with irrational parameter, we need to consider either **p**-adic differential equation or q-difference equation.

For q-difference equation, the cluster transform(mapping class group of punctured surface) do the role for Seiberg duality for dual SCFT. And for **p**-adic differential equation, there exist Atkin-Lehner operation for resurgence. They come from K_2 torsion in Bloch group and K_2 torsion in sufficiently ramified field.

Definition 1 *Given quiver $Q = (Q_v, Q_e)$ with path algebra A which is the Poisson algebra with poisson bracket $\langle i,j \rangle$ by arrow of quiver $e_{i\to j} \in Q_e$,we have quantized Poisson algebra. Let Q_v be the vertex set and Q_e be the edge set.*

$$x_i x_j = q^{\frac{1}{2}\langle i,j\rangle} x_j x_i, \quad i,j \in Q_v, e_{i\to j} \in Q_e$$

The cluster transformation T_j on quantized Poisson algebra from path algebra A of quiver is the Poisson automorphism determined by K_2 torsion condition.

By K_2 torsion $x_j \wedge (1-x_j)^{\langle i,j\rangle} = 0$ associated with Q, we have,

$$T_j : x_i \mapsto (1-x_j)^{\langle i,j\rangle} x_i, \quad i,j \in Q_v$$

$$x_j \wedge x_i \mapsto x_j \wedge (1-x_j)^{\langle i,j\rangle} x_i = \langle i,j\rangle x_j \wedge (1-x_j) + x_j \wedge x_i = x_j \wedge x_i$$

This coincides with Seiberg duality in dual gauge theory. We have additional sign from conductor by finite order mutation.

This Theorem is the sign refinement of [19]. With K_2 torsion associated with Q which is the Bloch group element, we have a hyperbolic 3 manifold which is the double bridge knot complement with pseudo-Anosov mapping class[40]. Having cluster mutation, we can define discrete Backlund transformation for q-difference Painleve equation.

Note that the quiver of Sasaki-Einstein manifold is degenerate type(non-symmetric quiver) - the bilinear form $\langle i,j \rangle$ is degenerate for $Y^{p,q}$ and essentially degenerate for $q \neq 0, p$ non-symmetric quiver with impurity. With impurity, the role of sympletic double is essential to recover Galois invariance. We have additional $\mathbb{Z}/2$ sign form functional equation which can be obtained by integral transform(Legandre transform) and **p**-adic lift as conductor determining shape with extremum volume by real Gromov Witten invariant controlled by reciprocity.

So we need to consider symplectic double of Poisson algebra which is the Cliffold double with metaplectic $\mathbb{Z}/2$ kernel for sign.

The sign comes from exist purely quantum state which is the Swan conductor for wild ramification.

By symplectic double, we have non-degenerate bilinear form in $Q_v \oplus {Q_v}^{\vee}$,by introducing dual variable ${x_i}^{\vee} = y_i$ with,

$$x_i x_j = q^{\frac{1}{2}\langle i,j\rangle} x_j x_i, \quad y_i y_j = y_j y_i, \quad x_i y_j = q^{\frac{1}{2}\delta_{ij}} y_j x_i$$

This is underlying symplectic algebra for degenerate Poisson algebra.

$$\langle (x_i, y_i), (x_j, y_j)\rangle = \langle x_i, x_j\rangle + y_j(x_i) - y_i(x_j)$$

We require that the symplectic automorphism of symplectic double reduce to Poisson automorphism. For that we need to consider submanifold of symplectic double which is preserved by sympletomorphism,

$$x_i \prod_j {y_i}^{\langle ij\rangle} = -1, \quad i, j \in Q_v\,,$$

$$x_k \wedge x_i \prod_j {y_i}^{\langle ij\rangle} = x_k \wedge x_i + \langle i,j\rangle x_k \wedge y_i = -\langle i,k\rangle + \langle i,k\rangle = 0\,.$$

Theorem 3 *With symplectic double, we have Legandre/Laplace transform of Poisson algebra- path algebra of quiver for analytic continuation of q(formal parameter) in generating series.*

Then we have map $q \to q + q^{-1}$ such that we can allow irrational spectral parameter $\hbar \in K$ with $q = e^{2i\pi/\hbar}$, which define $SO(3)$/spin (Chern-Simons) invariant,where K is Hilbert class field of totally real number field F.

Also we have dual partition function with functional equation with sign from epsilon factor at arithmetic infinity.

With Legendre transform with dual variable in symplectic double, we can recover the Painleve 6-th differential equation with irrational parameter.

For that we need to consider center $Z(\mathcal{A})$ of dg algebra $\mathcal{A}$ of Poisson algebra A by potential

$$W = \sum_{P \in Q_2^+} w_P - \sum_{P \in Q_2^-} w_P, \quad \partial W = \{\partial_e W \mid e \in Q_e\}$$

of quiver defined by sum over all oriented cycle w_P in face of quiver $Q_f = Q_f^+ + Q_f^-$.

$Z(\mathcal{A})$ is coordinate algebra of Sasaki-Einstein CY.

$$Z(\mathcal{A}) = \mathcal{A}/[\mathcal{A}, \mathcal{A}] \in HH^0(\mathcal{A})$$

After that we can define Kahler potential $F(x)$ with metric $\sum_{1 \leq i,j \leq 3} \frac{\partial^2 F(x)}{\partial x_i \partial x_j} dx_i dx_j$. So now x_i is variable in center $Z(\mathcal{A})$ of Poisson algebra and y_i be Legendre dual of it.

$$y_i = \frac{\partial F(x)}{\partial x_i}, \quad x_i = \frac{\partial G}{\partial y_i} + a_i, \quad F(x) = \sum_{1 \leq i \leq 3} y_i \frac{\partial G}{\partial y_i} - G(y)$$

Then we have the generating series- Hilbert polynomial of resolution of irregular Sasaki-Einstein CY which is 4d SCFT index of CFT dual of Sasaki-Einstein manifold which count BPS mesonic operator. The constant term is the extremal volume with sign from pro-$\mathbf{p}$ asymptotic.

Theorem 4 *We have Hilbert series from orbifold Riemann-Roch of power of ample line bundle on del Pezzo toric base V of Sasaki-Einstein manifold $Y^{p,q}$.*

$$\begin{aligned} S(\mathbf{b}, t) &= \sum_{k=0}^{\infty} e^{-kt} dim H^0(L^k) = \sum_{k=0}^{\infty} e^{-kt} (a_0(\mathbf{b}) k^3 + (a_1(\mathbf{b}) + c) k^2 + \cdots \\ &= \frac{a_0(\mathbf{b}) 3!}{t^4} + \frac{a_1(\mathbf{b}) 2!}{t^3} + \cdots, \end{aligned}$$

where k is the $U(1)$ charge of Reeb vector, and c is periodic function determined by Todd class, $\mathbf{b}$ is Reeb vector. Let $q = e^{-\tau}$ and with $t = c_1(L)$ of ample line bundle of V. Then $S(\mathbf{b}, \tau)$ is $q \to 1$ limit of N=1 4d SCFT index.

$$I = \int_V \frac{Td(V)}{1 - qe^{c_1(L)}} = \sum_{k \geq 0} \int_V q^k e^{kc_1(L)} Td(V) = \sum_{k \geq 0} q^k \chi(V, L^k)$$

The Todd class comes from Euler characteristic of Cyclic cohomology by trace map with secondary class(Cheeger-Simons).

This theorem is not new, proven by various method. But we have sign refinement from non-maximal Swan conductor by mod $\mathbf{p}$ reduction of $\mathbf{p}$-adic zeta function. The Todd class indicate that the N=1 4d SCFT index I satisfy global reciprocity. The constant term of $S(\mathbf{b}, t)$ is extremal volume of Sasaki-Einstein manifold.

With HKR isomorphism via square root of the Todd class in Hochschild homology, Euler characteristic of cyclic homology (Serre polynomial) is obtained. So by computing Euler characteristic, we define formality morphism for Poisson algebra A of path algebra of quiver[48].

Theorem 5 *We have formality morphism for Poisson algebra - the path algebra of quiver from N=1 SCFT index I - the Euler characteristic of cyclic homology.*

With wild automorphism of Poisson algebra from irregular Sasaki-Einstein quiver, we have additional data- Swan conductor which measure wild ramification.

For that we need refined motivic measure(Tits motivic measure) for Serre polynomial. Then we have sign for functional equation for motivic zeta function from torsion Galois representation(epsilon factor) with additional pole in motivic zeta function.

The proof of theorem comes from algebraicity of Stark-Heegner unit and computation of Euler characteristic of orbifold. We need sign refined partition function/$\mathbf{p}$-adic zeta function by conductor(correspondence).

Then we can define resurgence for any toric base with Atkin-Lehner operator.With this we have dual partition with Mobius map.

For irregular Sasaki-Einstein quiver, due to degenerate Poisson structure, we have finite order Seiberg duality with conductor.In its phase space of integrable we have pseudo-Anosov map with positive entropy [1] which indicate the irrational Reeb foliation.

The Sasaki-Einstein manifold is has path algebra(Poisson algebra) with more than 3 variable. So it can have wild Poisson automorphism of Nagata introducing conductor. The cluster mutation for degenerate(non-symmetric) Poisson structure determine Somes recurrence.

We identify the cluster transformation of quantized Poisson algebra with the Atkin-Lehner operation in **p**-adic differential equation. From the **p**-adic differential equation, the sign from additional quantum state in symplectic double is obtained by Swan conductor of wild ramification.

So for Sasaki-Einstein quiver with associated CY, there is wall crossing cocycle $\prod_{j\rightarrow} {T_j}^{\Omega(j)}$ with motivic Donaldson Thomas invariant $\Omega(j)$. Ω comes from Casson invariant of underlying 3 manifold- mapping torus with pseudo-Anosov map for two bridge Lens space knot. Here we need to consider the restriction of range of $j \in Z^{-1}(V) \cap \Gamma$ in convex from the K-stability Z of Sasaki-Einstein CY, where Γ is the Poisson torus of Poisson algebra.

2.2 Mapping torus with pseudo-Anosov map and Bianchi manifold

We have isomonodromy deformation with moduli space of rank 2 parabolic connection with parabolic structure from irrational $\hat{a}$ of Painleve 6-th equation.

[1]The entropy of Heron map measure the failure of Liouville Arnold notion of the integrability. After higgsing,having 4-vertex toric diagram,non-parallelogram diagram is of this kind(irregular Sasaki-Einstein quiver).

With irrational $\hat{a}$ we have plane curve singularity associated with double bridge knot for $L(p^2, pq-1)$.Such Lens space arise from exotic smooth structure on 4 manifold $\mathbb{CP}^2 \# m\overline{\mathbb{CP}^2}$ by Lens space surgery.

Given Sasaki-Einstein $Y^{p,q}$ quiver, as 5 brane (p,q) monodromy $T_{p,q} = \begin{pmatrix} -1+pq & q^2 \\ p^2 & 1+pq \end{pmatrix}$, we can consider Dehn twist[39, 38] for exotic 4 manifold,with Lens space $L(p^2, pq-1)$ boundary. The Lens space boundary can be seen as punctured mapping torus with monodromy $T_{p,q}$,with pseudo-Anosov foliation(Teichmuller geodesic flow)[40] whose 2-dimensional projection is Teichmuller disc with real multiplication by $\mathcal{O}_F$. From periodic mutation(Seiberg duality) of $Y^{p,q}$ quiver, we can obtain tetrahedron decomposition of Lens space knot complement. The irrational R charge determine the shape parameter of tetrahedron. The R charge is the modular object which is the eigenvalue of Heun equation(Painleve 6-th), so it depends on irrational parameter $\hat{a}$. By elliptic parametrization of Heun equation, we need inversion of modular function once which relate period τ of elliptic curve and $\hat{a}$ in Hilbert class field K of totally real number field F. For that we need to choose the spectral parameter of non-compact quantum dilogarithm whose evaluation at spectral parameter is the extremal R-charge(Stark-Heegner unit) in Hilbert class field K. This is the parameter from modular origin which need to be fixed as the limit of supersymmetric index/DT partition function. By the functional equation of DT partition function, we obtain modularity of spectral parameter.[2]

Having pseudo-Anosov map, by Perron-Frobenius theorem, we can define entropy of the map with ergodic measure. For Sasaki-Einstein quiver, we have

[2] In DT partition function computing for a given quiver, the importance of extremal R charge is suppressed. By spectral parameter, we have a number in Hilbert class field K from partition function defining actual deformation with pro-$\mathbf{p}$ asymptotic. And the spectral parameter is obtained as constant term (limit of partition function)- with non perturbative(pro-$\mathbf{p}$) asymptotic expansion.

two bridge knot from Lens space, with this we have specific hyperbolic knot complement by $L(p^2, pq-1)$ realising knot which is non-arithmetic.

As a Galois representation there exist arithmetic orbifold-Bianchi manifold associated with Sasaki-Einstein CY manifold From pro-$\mathbf{p}$ covering of Bianchi manfold, we have global torsion Galois representation from metaplectic lift. This is coincide with Galois representation for Artin-Mazur zeta function.

Now we present the connection to torsion Galois representation in supersingular locus to pro-$\mathbf{p}$ covering of Bianchi manifold. We need the theorem for non-existence of certain Galois representation such that we need to consider pro-$\mathbf{p}$-covering of Bianchi manifold $\mathcal{H}^3/PSL(2, \mathcal{O}_L)$ quadratic imaginary L[47].

Following theorem realize Bianchi manifold of $\mathbb{Q}(\sqrt{-2})$ associated with pro-$\mathbf{p}$ covering of $L(2,1)$ realising Siegel unit from B_n field counting. We will see that Siegel unit also obtained from By supersingular decomposition,we can reduce Bianchi manifold for $\mathbb{Q}(\sqrt{-p})$ with $\mathbf{p} = 1 \mod p$ to Bianchi manifold $\mathbb{Q}(\sqrt{-3}), \mathbb{Q}(-i)$ with $\mathbf{p} = 1 \mod 3, 4$ associated with elliptic Fermat quotient by Weierstrass $\mathfrak{p}$ function. Then by Level-Rank dualtiy, we have Bianchi manifold of $\mathbb{Q}(\sqrt{-2})$ from Bianchi manifold of $\mathbb{Q}(\sqrt{-3})$.

Theorem 6 *For given $L = \mathbb{Q}(\sqrt{-2})$ with $\mathbf{p} = \omega\omega'$ with $\mathbf{p} = 1 \mod 2$(we set $\mathbf{p} = 3$),there exist a quaternion algebra D ramified at ω, ω'.Let B is maximal order in D,and m be a maximal bi-ideal of B away from ω, ω'. Let B_n be the complex imbedding of $B^\times/(1+m^n)$ to $PSL(2,\mathbb{C})$. Let $M_n = \mathcal{H}^3/B_n$. M_n has Haken covering with first Betti number 0 by 4-dimensional Galois representation, and M_n is pro-$\mathbf{p}$ covering of Bianchi manifold of link complement.*

There exist a 3-adic Galois representation from pro $\mathbf{p}$ covering M_n of M_0 which is 4 dimensional Galois representation. The vanishing of first Betti number imply that it comes from non-existence of 2 dimensional Galois representation corresponds to M_0. Associated with this we have Siegel unit provide modularity of pro-$\mathbf{p}$ covering of Bianchi manifold realising Haken covering of M_n.

This 3-adic Galois representation is for Sasaki-Einstein CY of $Y^{2,1}$ associated with pro-$\mathbf{p}$ covering $L(4,1)$ of $L(2,1)$ providing additional sign from metaplectic lift(pro-$\mathbf{p}$ tower). $L(4,1)$ lies on first covering M_1. With order of mutation to be 1, we have Stark-Heegner unit of totally real number field $F = \mathbb{Q}(\sqrt{13})$ as Siegel unit. In this theorem, M_n has same Haken covering with $L(4,1)$ knot complement from big $\mathbb{Q}$ point of infinite order from Stark-Heegner unit(Siegel unit) of totally real number field $\mathbb{Q}(\sqrt{13})$.

There exist duality in Mordell-Weil group of supersingular (extremal) elliptic surface at characteristic 2 from metaplectic double[3], which exchange finite automorphism of Nef cone with automorphism of Leech lattice.

By Level-Rank duality we can reduce degree p singularity to degree 2 singularity,the above result for small prime is enough for torsion Galois representation from Sasaki-Einstein orbifold of arbitrary degree p.

Also by supersingular decomposition, we can reduce degree p Fermat quotient over $\mathbf{p} = 1 \mod p$ with Bianchi manifold of $\mathbb{Q}(\sqrt{-p})$ to degree $3,4$ elliptic Fermat quotient over $\mathbf{p} = 1 \mod 3,4$ with Bianchi manifold of $\mathbb{Q}(\sqrt{-3}), \mathbb{Q}(i)$.

Both provide Seigel unit with Galois group of sufficiently ramified field with sign of $\mathbf{p}$-adic Hecke character depending in $\mathbf{p}$, realising supersingular transition by choice of higher homothety class.

The relation to two-bridged Lens space knot $L(p^2, pq-1)$ comes from following fact from [34] which relate hyperbolic two bridge knot to order 2 automorphsim. We have pro-$\mathbf{p}$ covering from figure 8 knot, $M_n/\mathcal{K} = \mathcal{H}^3/\text{figure 8}$ where $\mathcal{K}$ is double bridled knot of pro-$\mathbf{p}$ covering $L(p^2, pq-1)$ and M_n being pro-$\mathbf{p}$ tower of Bianchi manifold of $\mathbb{Q}(\sqrt{-p})$.

Theorem 7 *Let $\mathcal{K}$ be the hyperbolic two bridge knot. For given $S^3/\mathcal{K} = \mathcal{H}^3/\Gamma_{\mathcal{K}}$ with normal subgroup $N(\Gamma_{\mathcal{K}})$, $Isom^+(S^3/\mathcal{K}) = N(\Gamma_{\mathcal{K}})/\Gamma_{\mathcal{K}}$ is either $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ or Dihedral group of 8 element D_4.*

Let $N = \mathcal{H}^3/N(\Gamma_{\mathcal{K}})$. Then $N(\Gamma_{\mathcal{K}})$ is generated by element of order 2. And there exist a unique two fold covering $N' = H^3/\Gamma$ of N such that abelianization of Γ has only torsion element of order 2.

[3]For Tate conjecture of supersingular K3 surface, we need such results.

With this theorem, we can reduce for double bridged knot complement to above theorem from Dunfield and Calegary via Level-Rank duality between $p = 2$ and $p = 3$ such that we have $\mathbb{Q}(\sqrt{-3})$ over $\mathbf{p} = 1 \mod 3$. By supersingular decomposition from $\mathbb{Q}(\sqrt{-p})$ to $\mathbb{Q}(\sqrt{-3}), \mathbb{Q}(i)$ with Level-Rank duality to $\mathbb{Q}(\sqrt{-2})$, we can reduce to figure 8 knot complement with non-trivial $\mathbb{Z}/2$ automorphism from arithmetic infinity.

Then we have 4-dimensional Galois representation associated with generalized CM point by Hilbert class field K of totally real number field with $[K : F] = N$ by $12N$ power of Siegel unit.

From the supersingular decomposition, associated with arbitrary cusp $-q/p$, we have Siegel unit from Weierstrass $\mathfrak{p}$ function of elliptic Fermat quotient of degree $3, 4$, such that by elliptic identity(3-term relation) of Weirerstrass $\mathfrak{p}$ function, we have modularity of pro-$\mathbf{p}$ covering of Bianchi manifold with Siegel unit realising Haken covering in Section.??. Note that small prime $p = 2, 3$ is special from Mordell-Weil torsion point of view over $\mathbb{Q}$, which provide supersingular transition to other totally real number field.

From the above reason following Theorem.8 do the meta role which can be bypass to Bianchi manifold of $\mathbb{Q}(\sqrt{-3}), \mathbb{Q}(i)$ from the reciprocity of sufficiently ramified field(perfectoid field) by supersingular decomposition to supersingular elliptic curve, with Thm.7 and Level-Rank duality of Leech lattice in supersingular locus[36] at characteristic 2. By integral structure of Shimura varieties over sufficiently ramified field, from B_n Mass formula, we have pro-$\mathbf{p}$ $L(2, 1)$ orientifold geometry for Bianchi manifold of $\mathbb{Q}(\sqrt{-2})$ over $\mathbf{p} = 3$ which is Level-Rank dual to Bianchi manifold $\mathbb{Q}(\sqrt{-3})$ over $\mathbf{p} = 1 \mod 3$ with Siegel unit for non-trivial $\mathbb{Z}/2$ at infinity.

Theorem 8 *In general, for $Y^{p,q}$ with $L(p^2, pq - 1)$,we have Bianchi manifold for $L = \mathbb{Q}(\sqrt{-p})$ associated with of degree p Fermat curve with $\mathbf{p} = 1$*

mod p*,with a quaternion algebra* D *ramified at splitting prime of* $\mathbf{p} = \omega\omega'$. *Let* B *the maximal order in* D*,and* m *be the maximal bi-ideal of* B *away from* ω, ω'. *Let* B_n *is the complex imbedding of* $B^\times/(1+m^n)$ *to* $PSL(2,\mathbb{C})$.

Again, $M_n = \mathcal{H}^3/B_n$ *is pro-*$\mathbf{p}$*-covering of* M_0 *which has Haken covering with vanishing first Betti number by 4-dimensional Galois representation. Then we have torsion Galois representation for pro-*$\mathbf{p}$ *covering for Sasaki-Einstein CY* $C(Y^{p,q})$.

We have Haken covering of M_n *from 4-dimensional Galois representation with Stark-Heegner unit in Hilbert class field* K *of of totally real number field* $\mathbb{Q}(\sqrt{4p^2-3q^2})$ *with* $[K:F] = N$.

The associated $\mathbf{p}$-adic Galois representation has the Bianchi new form.

By $L(p^2, pq-1)$ two bridge knot, we have two bridge knot complement which define pseudo-Anosov flow,which differ from link defining M_0 but we expect that both has the same Haken covering. Note that the 4-dimensional Galois representation for pro-$\mathbf{p}$ covering of Bianchi manifold of $L = \mathbb{Q}(\sqrt{-p})$ with $\mathbf{p} = 1 \mod p$ comes from Fermat quotient of degree p over $\mathbf{p} = 1 \mod p$ with CM lift by Hilbert class field K of totally real number field $F = \mathbb{Q}(\sqrt{4p^2-3q^2})$. So this comes from pro-$\mathbf{p}$ covering of $L(p,q)$ singularity over field of characteristic $\mathbf{p} = 1 \mod p$ in cusp in Hilbert modular surface of F.

So by considering $L(p,q)$ as wild quotient singularity, $L(p^2, pq-1)$ lines on pro-$\mathbf{p}$ cover of Bianchi manifold. So conjecturally , $L(p^2, pq-1)$ lies on pro-$\mathbf{p}$ congruence covering Bianchi manifold M_n of $\mathbb{Q}(\sqrt{-p})$. By pro-$\mathbf{p}$ congruence covering for M_n with vanishing Betti number, we have Haken covering from 4-dimensional Galois representation from Stark-Heegner unit of totally real number field F.

We can consider Lens space $L(p^2, pq-1)$ as Seifert fibration over singular curve $\mathbb{P}^1 \setminus 0$ encoding Sasaki-Einstein quiver singularity as parabolic structure. We have pro-$\mathbf{p}$ covering of Bianchi manifold and finite covering Haken manifold which is the mapping torus with pseudo-Anosov irrational foliation associated with Stark-Heegner unit of totally real number field. The irrational Reeb vector of Sasaki-Einstein manifold maps to pseudo-Anosov irrational foliation on mapping torus(quasi-conformal self map on torus) which is Haken covering of pro-$\mathbf{p}$ covering of Bianchi manifold. We can consider pseudo-Anosov diffiomorphism of torus as real multiplication by $\mathcal{O}_F$ with $F = \mathbb{Q}(\sqrt{4p^2 - 3q^2})$.

It gives the Hitchin twistor construction for Sasaki-Einstein CY metric by Yang-Mills connection which is almost real configuration on $\mathbb{CP}^2 \# m\overline{\mathbb{CP}^2}$ with exotic smooth structure from $L(p^2, pq - 1)$ monopole. Its half is $\mathbb{CP}^2$ with Lens space boundary.

Such pseudo-Anosov irrational foliation is obtained from exotic smooth structure on $\mathbb{CP}^2 \# m\overline{\mathbb{CP}^2}$ by Lens space $L(p^2, pq-1)$ surgery. Since the mapping torus with pseudo-Anosov flow is Galois complete object, we can consider this as knot realizing problem by $L(p^2, pq - 1)$ associated with non-trivial Sasaki-Einstein CY $C(Y^{p,q})$. Associated with pseudo-Anosov flow, we have Artin-Mazur spectral zeta function which compute sign refined spin CS partition function of mapping torus Γ arise from Lens space $L(p^2, pq - 1)$ realising knot quotient such that we have normal function of $\mathbf{p}$-adic Ceresa cycle $C - C^-$ from mapping as sign refined CS invariant which is Stark-Heegner unit of totally real number field F. This can be realized by third Milnor homomorphism of Galois cohomology of totally real number field with Brauer Severi varieties containing mapping torus Γ with $\partial\Gamma = C - C^-$ from almost real projection.

Associated with Heun equation (Painleve 6-th), we have elliptic surface with 4 singular fiber with specific ramification data by Picard Fuchs equation for mirror curve. The elliptic curve is factor of Jacobian of mirror curve where mirror curve is defined as sum over all vertex inside of the toric diagram of Sasaki-Einstein quiver.

So it determine mirror period and also period of Jacobian by modular embedding, by inhomogeneous Picard Fuchs equation which is the Painleve 6-th with irrational parameter in Hilbert class field K of F.

So from Kahler Einstein metric on Sasaki-Einstein CY as absolute Galois invariant(with Seiberg duality RG flow), we obtain mirror transcendental period. In this way, we determine string vacua with absolute Galois action, which determine in-homogenius period with sign as Stark-Heegner unit for generalized CM point.

So having Kahler Einstein metric which is extremal, we have Picard Fuchs equation for CY motive which is elliptic motive. We obtain metric from 4d SCFT, with Yang-Mills equation in $Y^{p,q}/U(1)$ surface with extremal field configuration realising pro-$\mathbf{p}$ asymptotic with almost real structure. On the other hands, we have elliptic surface with specific wild singular fiber determined by Galois completion of N=1 mirror curve of $Y^{p,q}$ by logarithmic compactification of Jacobian, with sign from wild monodromy group as irregular $\mathcal{D}$ module over affine line. The Mordell-Weil torsion is determined by Skelepton of mirror curve and the Mordell-Weil rank is determined by poles of meromorphic differential which is 0 for Sasaki-Einstein CY.

Then by Heun equation(Painleve 6-th equation) with 4 singularity $\mathbb{P}^1 \setminus \{0, 1, \hat{a}, \infty\}$, we can consider compactification of moduli space of rank 2 flat connection on $\mathbb{P}^1 \setminus \{0, 1, \hat{a}, \infty\}$, such that we have compactification of the moduli space of pro-$\mathbf{p}$ Higgs bundle over orbicurve by pro$\mathbf{p}$ Hurwitz covering from $SL(2, F)$ Bruhat-Tits cocycle as big $\mathbb{Q}$ point of infinite order.

This moduli space of rank 2 Higgs bundle(solution space of Painleve 6-th equation) is the moduli space of stability condition for Sasaki-Einstein quiver. We have a slope stability for quiver from orbifold Riemann-Roch which can be written in terms of volume of CY. On the compactification of boundary, the wall crossing group acts(Seiberg duality), and from that the extremal metric is obtained as K stability.

Given (irregular) Sasaki-Einstein quiver, with Painleve 6-th differential equation with (irrational) parameter,the space of stability condition of quiver is obtained by the space of initial condition (solution space) which is moduli of rank 2 Higgs bundle over 4 punctured sphere. This is character varieties of 4 punctured sphere with specific ramification(a hyperbolic knot) at a point $\hat{a}$.

We can consider ramification at $\hat{a}$ as plane curve singularity from Lens space realizing knot, and by the compactification of Jacobian fibration of plane curve singularity, we can solve Painleve 6-th differential equation with divisors at infinity of compactification over ramification locus realising generalized CM point by Hilbert class field K of totally real number field F with $[K : F] = N$. Local geometry at infinity of compactification of moduli space of stability condition is by Milnor fiber of Sasaki-Einstein singularity which is pro-**p** conifold geometry with Galois completion of N=1 mirror curve of Sasaki-Einstein manifold as compactification divisor realising generalized CM.

This is self mirror pro-**p** geometry associated with Galois cohomology of totally real number field. We can see the Fake Fano surface for Sasaki-Einstein manifold from compactification of Hitchin moduli space by $SL(2, F)$ Bruhat-Tits building cocycle.

We obtain period(K-stability) as Galois invariant object with action of motivic Galois group. Under the supersingular transition(wall crossing) which change sign by higher homothety class, The K-stability can be jump which can be seen as Higgsing(change quiver and totally real number field). This is the arithmetic MMP(arithmetic Minimal Model Program) transition from motivic Galois group from **p**-adic reciproicity from integral structure of Shimura varieties over sufficiently ramified field with level raising and lowering operation . The motivic Galois group is defined by pro-**p** group, such that by mod

p reduction , we have generalized CM by Hilbert class field K of totally real number field F with $[K : F] = N$ for higher ramification group by N formal Lie group of graded Hamiltonian vector field with lower central series by filtration of Hamiltonian vector field realising modular embedding with wild singular fiber. Then, we have almost real structure from wild singular fiber having higher ramification group.

Let $C_0 = \mathbb{P}^1/\{0, 1, t, \infty\}$ with T^*C_0. By choosing meromorphic 1-form(Higgs field) with singularity at 4 points, the moduli space of rank 2 Higgs bundle is locally T^*C_0 with spectral(mirror) curve of Sasaki-Einstein CY as boundary divisor. The Betti realization of moduli space is cubic surface, and de Rham realization is the del Pezzo surface with irrational parameter $\hat{a}$ and can be seen as elliptic surface with wild singular fiber.

The moduli space of Higgs bundle is the moduli space of stability condition which is cotangent bundle over moduli space of spectral curve with Jacobian fiber. The boundary divisor at infinity is obtained by modular embedding spectral curve to its Jacobian with sign from higher ramification group of wild singular fiber. In this way the point of infinite order on Jacobian of spectral curve is obtained by τ function(section of determinant line bundle which is intersection of torsion point on curve and its Jacobian).

Then by moduli space of rank 2 Higgs bundle which is elliptic surface, we have Mordell Weil group by singularity of Painleve 6-th. On our case, we have only 1 non-trivial irrational singularity $\hat{a}$ realising a section from big $\mathbb{Q}$ point of infinite order even if we have torsion Mordell-Weil group.

Due to irrational(resonance) singularity,for Sasaki-Einstein manifold,we have supersingular extremal elliptic surface whose Mordell Weil lattice is Leech lattice at characteristic 2.Then we can determine sign from wild ramification. By considering $SL(2, F)$ Bruhat-Tits building cocycle for the compactification divisor of by spectral curve(image of Harmonic map),we have Geometric Langlands with Langlands duality kernel from pro-$\mathbf{p}$ covering of 3 manifold from link of singularity with sign from mod $\mathbf{p}$ reduction, where 3 manifold comes from $L(p,q)$ with pro $\mathbf{p}$ covering $L(p^2, pq-1)$ for exotic smooth structure associated with the Sasaki-Einstein link.

For Sasaki-Einstein quiver, cluster mutation(Seiberg duality) do the role for generalized Weyl group(Backlund transformation) of $SL(2,F)$.

For Sasaki-Einstein quiver, due to parameter of Painleve 6-th equation is non-generic(irrational/resonance), we need to consider big $\mathbb{Q}$ point with complexity(entropy) on its phase space(moduli space of Higgs bundle) realising non-commutative resolution of wild singularity on phase space. Since this is the codimension 2 irregular singularity, we have link of Sasaki-Einstein manifold with higher ramification group on discriminant locus in Hitchin base for the compactification of Jacobian fibration by Milnor fiber. The point of infinite order on Jacobian fibration(building at infinity) determine the section of determinant line bundle with sign.

Associated with Painleve 6-th equation with covering of punctured torus of $\mathbb{P}^1 \setminus \{0,1,\hat{a},\infty\}$, we have pseudo-Anosov (ergodic) mapping class group of (punctured) torus with irrational slope(real multiplication by F).[4] We have a solution of Painleve 6-th from character varieties of $\mathbb{P}^1 \setminus \{0,1,\hat{a},\infty\}$. Note that $\pi_1(\mathbb{P}^1 \setminus \{0,1,\hat{a},\infty\})$ acts faithfully on the mapping class group of profinite covering. After replacing by punctured torus covering, we have Grothendieck section associated with elliptic/non-commutative motive. We will consider punctured genus 1 surface as Heegaard surface on Lens space, with non-Abelian T-duality along torus as signed S-duality.

[4]We have $\mathbf{p}$-adic zeta function of totally real field associated with pseudo-Anosov irrational foliation.

Having this, by Hitchin's construction, we have 3 manifold $L(a^2, ab-1)$ from singular point $\hat{a} \in \mathbb{P}^1$.

By twistor realization,with Painleve 6-th equation,from the underlying extremal Yang-Mills field configuration over Fake Fano surface with exotic smooth structure, we have **p**-adic modularity from twistor spinor. We consider Sasaki-Einstein CY as twistor space(complex cone) over irregular toric Fano orbifold.By extremal Yang-Mills field configuration with exotic monopole, with twistor construction, the extremal Sasaki-Einstein CY metric with extremal volume is obtained by Stark-Heegner unit. We can consider Fano orbifold from Dehn twist $\mathbb{P}^1 \times \mathbb{P}^1$ by $\begin{pmatrix} -pq+1 & -q^2 \\ p^2 & pq+1 \end{pmatrix}$ considered as punctured mapping class action on Jacobian of torus. By exotic structure,the Yang-Mills extremal field configuration can be obtained by irrational Painleve equation, reflecting AdS dual irregular Sasaki-Einstein geometry. The relation between toric irregular Fano base and exotic smooth 4 manifold done by Dehn surgery of blow up of $\mathbb{CP}^2$ with arithmetic MMP realising **p**-adic uniformazation. Later we will see extremal Yang-Mill equation from extremal Sasaki-Einstein metric with Stark-Heegner unit of real quadratic number field which is elliptic function. With exotic smooth structure,we have Mordell-Weil torsion group with supersinuglar extremal elliptic surface.

Then, by duality in supersingular elliptic curve, we have Leech lattice, with automorphism from non-arithmetic lattice from $L(p^2, pq-1)$ two bridge knot.

The irrational boundary condition for Heun equation for irregular Sasaki-Einstein manifold gives the irrational parameter for Painleve 6-th equation.Then by Painleve 6-th differential equation, we have Riemann Hilbert correspondence for moduli space of the parabolic connection on $\mathbb{P}^1 \setminus \{0, 1, t, \infty\}$.

Then the elliptic surface arise as moduli space of rank 2 local system as wild character varieties.

But the Sasaki-Einstein CY is marginally integrable in a sense that it is the cone of singular Fano manifold(irregular toric Fano surface). So we have the integrable system associated with singular Fano surface whose twistor space is Sasaki-Einstein CY. We need analogous of unobstruction Theorem(Tian-Todolov Theorem) of compact CY manifold, by compactification of moduli of mirror Landau-Ginzburg model. As seen before we need compatification of moduli of Higgs bundle associated with Painleve 6-th equation by Harmonic embedding of spectral curve with $SL(2,F)$ Bruhat-Tits building at infinity(boundary divisor).

Since Fano is rigid, we need additional formal parameter(quantization parameter) from motive of affine line with wild monodromy at infinity, such that we have irrational conformal field theory with level structure.

We need LG model over Witt ring for compactification on moduli space of LG model and unobstruction Theorem.Since the mirror LG polynomial is degree $p > 1$ for Fano orbifold of $Y^{p,q}$, we have additional automorphism $\mathbb{Z}/p$ on potential other than Frobenius from base field of charactersitic $\mathbf{p}$. There exist Level-Rank duality which exchange p and $\mathbf{p}$ on p^2 Weyl alcove.The compactification of moduli space of opers(whose support is spectral curve) with Harder-Narasimhan filtration has moduli space of Frobenius destabilising sheaves whose dimension is the real GW invariant realising mod $\mathbf{p}$ reduction. On the compactification, the Seiberg duality acts as automorphism such that we have $SL(2,F)$ Bruhat-Tits building.

The formality and boundary compactification by harmonic embedding of spectral curve is obtained from additional sign with integral structure of underlying Drinfeld modular scheme(Shimura varieties in supersingular locus over sufficiently ramified field).

From the additional orbifold structure, we have vanishing of p-curvature realising real GW invariant by counting Frobenius destabilising sheaves measuring non-transversal Frobenius and Frobenius pull-back filtration on Hodge bundle of N=1 mirror curve. The boundary cocycle defining arithmetic varieties with Swan conductor from wild singular fiber. With this we have additional sign for irregular Fano orbifold for Sasaki-Einstein CY.

There is another geometric transition(supersingular transition) from 'adding matter and Higgsing'(motivic Galois group) which acts on level. So we need to consider all level at the same time from integral structure with level raising and lowering operation from reciprocity of Drinfeld modular scheme with level structure/Shimura varieties over sufficiently ramified field[5]. By different choice of homothety class, we can change sign realising supersingular transition from **p**-adic reciprocity. The changing Sasaki-Einstein quiver by adding matter and Higgsing(motivic Galois group) can be realised from level rank duality in supersingular locus over sufficiently ramified field.Such that we have supersingular transition(Higgsing) in moduli space CY with each N=1 strung vacua as generalized CM point.By action of Galois group of sufficiently ramified field, we have super(sign refined) matrix integral measure with equidistribution property of N=1 vacua[16].

[5]This is the reciprocity in Shimura varieties over Perfectoid field.

Chapter 3

Volume of Sasaki Einstein manifold from Stark unit of p-adic Shintani cocycle.

In the string partition function on the local CY 3 fold, we have dibaryon operator which wrap special Lagrangian $Y^{p,q}$. There is the dual t'Hooft operator which anticommute with dibaryon operator.

The dibaryon operator(Wilson loop) on this string background has R-charge which is inverse of the volume of Sasaki Einstein manifold $Y^{p,q}$.

We have the extremal R-charge of baryonic operator as the algebraic number w(inverse of extremal volume) in Hilbert class field K of F. The extremal R-charge satisfies the algebraic equation which is equivalent to the extremizing volume functional.The extremal Reeb vector is defined as the extremal point(generalized attractor point-Stark Heegner point of real quadratic number field F) in moduli of Kahler metric. In other words, the extremal Reeb foliation in Sasaki-Einstein manifold determine a Teichmuller geodesic flow in moduli space of CY cone associated with real quadratic number field F. Then by extremal Reeb vector, the generalized CM point is defined on Teichmuller curve which is totally geodesic submanifold on moduli space of CY.

For general $Y^{p,q}$ with $q \neq 0, p$,the projection of the non-compact Reeb vector induce the irrational volume w valued in Hilbert class field K of F. K is CM field which is extension of F by modular unit w.

The Sasaki-Einstein matrix is determined with Heun equation(Painleve equation),associated with integrable system with irregular(wild) monodromy.

The w is not the fundamental unit of F but the Stark unit, obtained from **p**-adic zeta function.

Let ξ be a Reeb vector on Sasaki Einstein manifold whose metric cone CY (X, g_X) is Kahler. It is the Killing vector in the center of Lie algebra of the isometry group.

Under momentum map of cone CY,with angle basis ϕ_i and radial coordinate r, given complex structure J with $\nabla^X J = 0$,

$$\xi = J(r\frac{\partial}{\partial r}) = \sum_{1 \leq i \leq 3} b_i \phi_i \ , \ \mathcal{L}_\xi r = 0$$

whose dual contact 1 form is,

$$\eta = J(\frac{dr}{r}) = \frac{1}{r^2} g_X(\xi, \cdot)$$

The Kahler two form on X is,$\omega = \frac{1}{2} d(r^2 \eta) = \frac{i}{2} \partial\overline{\partial} r^2$.

The mesonic moduli space M is the conjectural smooth moduli space for given singularity with quiver. The extremal volume is computed as limit value(constant term) of the Hilbert series of local(non compact) CY 3 fold known as Mesonic moduli space M which is the resolution of singularity with link as $Y^{p,q}$, $M = \mu^{-1}(\epsilon) =: (\mathbb{C}^3 // Q_F) // Q_D$[9] where Q_F, Q_D is F-term, D-ferm equation determining momentum map. By the constant term in Hilbert series on M which is the local model of the compactification of Higgs branch moduli space(hypermultiplet) at infinity, we obtain algebraic Black Hole entropy with Hodge Tate state(mirror to intrinsic Higgs) in quasi-phantom category.

Such torsion object comes from conductor(resonance from Seiberg duality). It comes from correspondence -Frobenius destabilising sheaves counting real GW invariant.

Given a momentum polytope $C = \mu(X)$ for toric varieties $\mu : X \to \mathbb{R}^3$,determined by $\langle y, v_k\rangle \geq \lambda_k$ with $1 \leq k \leq 3$. Let $l_k(y) = \langle y, v_k\rangle - \lambda_k$, $l_\infty = \sum_{1\leq k\leq 3}\langle y, v_k\rangle$ such that $y \in C^\circ$ for $l_k(y) > 0$. The Kahler two form is

$$i\partial\overline{\partial}\mu^*(\sum_{1\leq k\leq 3} \lambda_i \log l_k + l_\infty)$$

we have the symplectic Kahler potential F which induce the Kahler Einstein metric $F_{i,j}(x) = \frac{\partial^2 F(x)}{\partial x_i \partial x_j}$ and Kahler two form $\omega = \partial\overline{\partial}F$ on Fano surface and CY cone of it. By Legandre transform, we have $y = \frac{\partial F}{\partial x} = \mu$.

$$Vol(Y^{p,q}) = \pi^3 \frac{q^2(2p + \sqrt{4p^2 - 3q^2})}{3p^2 4q^2 - p(2p + \sqrt{4p^2 - 3q^2})} = \frac{8\pi^3}{27}\lim_{\tau\to 0} \tau^3 g(q = e^{-\tau} : M)$$

where M(mesonic moduli space) is the smooth resolution of CY cone $X = C(Y^{p,q})$ can be seen as the total space of $L \to V$ with Fano surface V.

$$g(q = e^{-\tau} : M) = \int_V \frac{Td(V)}{1 - qe^{-c_i(L)}} = \sum_{k\geq 0} \int_V q^k e^{kc_1(L)} Td(V)$$

This is the Hilbert series of the noncommutative smooth CY M with constant term in Hilbert class field of $F = \mathbb{Q}(\sqrt{4p^2 - 3q^2})$ .It is the algebraic period of Landau Ginzburg mirror of local CY X.

We will consider $g(q = e^{-\tau} : M)$ as the lattice point counting - the index formula for counting Mesonic operator.

By CY condition,the volume of $Y^{p,q}$ is the sum of the volume of special Lagrangian submanifold.

By the CY condition (total space of line bundle on Fano surface),we reduce to two dimensional lattice problem- Penrose tiling(Dimers) for quiver with irrational slope in Hilbert class field K. Then the irrational extremal volume is obtained by lattice counting problem.

Then by ideal class group of totally real number field F whose ideal number exceed bound from totally real number field,we need to consider pro-$\mathbf{p}$ geometry by $L(p^2, qq-1)$ which is pro-$\mathbf{p}$ covering of $L(p,q)$ in characteristic $\mathbf{p} = 1 \mod p$. So as $\mathbf{p}$-adic modular surface of totally real number field $F = \mathbb{Q}(\sqrt{4p^2 - 3q^2})$, we have pro-$\mathbf{p}$ cusp geometry by $L(p,q)$. To recover the extremal volume(constant term of Hilbert series of M), we need to consider $\mathbf{p}$-adic lift of Shintani cone associated with $L(p,q)$ and Kronecker limit formula.

Each of the smooth resolution as total space of Fano surface,the Hilbert series compute a virtual Euler number of moduli space of conformal block. At limit, it is the Euler number the moduli space of ribbon graph genus 0 with n marked points where n is the external legs of dual graph of toric diagram.It is a component of Hurwitz tower.

Note that toric diagram of $Y^{p,q}$ is not parallelogram for $1 \leq q \leq p-1$. The sequence of adding matter(blow up) Higgsing(blow down) which is the tropical procedure on dual 4d CFT , we obtain $Y^{p,0}$ with parallelogram toric diagram realising cascade of totally real number field. At $Y^{p,p}$ and $Y^{p,0}$, we have relativistic Toda integrable system with rational solution[46] with imaginary quadratic number field $\mathbb{Q}(\sqrt{-p})$. In general, we need elliptic solution for relativistic deformed Toda system with where we need $\mathbf{p}$-adic structure for approximate irrational parameter in Painleve 6-th equation,with finite Seiberg duality.

By the cusp of $\mathbf{p}$-adic Hilbert modular surface, we have Lens space $L(p,q)$ over characteristic $\mathbf{p} = 1 \mod p$ realising exotic smooth structure by $L(p^2, pq-1)$ surgery. Then by parallelogram cone, we define the Shintani cone decomposition for $L(p,q)$.

Each extremal volume of special Lagrangian in local CY fold is computed from the extremal Reeb vector in $Y^{p,q}$, the Killing spinor in $X = C(Y^{p,q})$. So the Killing spinor see the arithmetic information of Black Hole entropy. The reciprocity for algebraicity of entropy reduce to spinor reciprocity(Gamma function type global reciprocity), which is special case of reciprocity of Stark unit.

By solving Monge Ampere equation, we find the Hilbert class field K of F containing $F(w)$ for $Y^{p,q}$ where w is the irrational volume from Reeb vector. We will construct irrationality w from Stark Heegner unit of number field F. The toric diagrams of $Y^{p,q}$ spans the general 2d toric diagram which is not parallelogram which comes from the projection of 3 dimensional parallelogram. So, we may to consider totally real cubic field for 3 dimensional toric diagram and zeta function of it. But the cyclic action of $Y^{p,q}$ is special in a sense that $Y^{p,q}$ is topologically $S^2 \times S^3$ with winding number p for first S^2 and q for second S^2 in S^3. As CY cone, this is of the form of action $\frac{1}{p}(p,q,-q)$ so that there is no net p-th roots of unity action.

For this reason(no net root of unit action), we could recover the extremal metric from zeta function of real quadratic number field with $\mathbf{p}$-adic lift(real multiplication problem). The Hilbert class field of the real quadratic number field is determined from extremal volume(irrationality).

For each $Y^{p,q}$ with irrational Reeb vector $\mathbf{b}$ which extremize the volume, there exist an induced Reeb vector on special Lagrangian $L(n,1), n = p+q, p-q, p, p$ and vice versa(CY condition). So by 5 dimensional extremal Reeb foliation, we can obtain the extremal contact 1 form η on X (dual of Reeb vector $\xi = \mathbf{b}$). We consider family of $Y^{p,q}$ as the the wild ramification phenomenon(from exotic smooth structure) with embedding in $\mathbb{Z}_{p+1} \times \mathbb{Z}_{p+1}$ realising p^2 Weyl alcove.

But due to irrationality(algebraic number), we need to consider $\mathbf{p}$-adic lift and the wild ramification phenomenon. The wild ramification phenomenon can be realized as wall crossing of the Shintani cone. After the $\mathbf{p}$-adic lift, the $\mathbf{p}$-adic Frobenius reciprociy provide the algebraicity of extremal volume by $\mathbf{p}$-adic approximation.

Since the volume functional satisfies cocycle properties, we call it as the volume cocycle. Given CY cone for local CY $C(Y^{p,q})$,we have following volume cocycle.

Fix the toric Gorenstein singularity with polyhedral cone $C \subset \mathbb{R}^3$, with a Reeb vector $\xi = \mathbf{b}$ in dual cone with norm 1 at CY cone X of radius 1, $1 = 2(\mathbf{b}, y)$.

Then we have the base of the cone X at radius 1, define a hyperplane which is $Y^{p,q}$,

$$\{y \in \mathbb{R}^3 | (\mathbf{b}, y) = 1/2\}$$

The $Y^{p,q}$ is the T^3 fibration on $H = \{y \in \mathbb{R}^3 | (\mathbf{b}, y) = 1/2\} \cap C$. The moment polytope is of CY cone at radius 1 is $\mu(X_{rad\leq 1}) = \Delta$ where $\mu : \mathbb{C}^3 \to \mathbb{R}^3$ is the moment map. The $Y^{p,q}$ is called quasi regular iff Reeb vector is rational $\mathbf{b} \in \mathbb{Q}^3$. We are mainly interested in the non quasi regular case with irrational Reeb vector $\mathbf{b}$. The extremal Reeb vector is obtained by solving Monge Ampere equation. It extremizes the volume(Mabuchi) functional.

Let $\omega = \omega_0 + i\partial\overline{\partial}u$ be the Kahler form. Monte Ampere equation is,for smooth positive function f in $\mathbb{R}^4$ and bounded function u,

$$det(\frac{\partial^2 u}{\partial_i \partial_j}) = f(u, du)$$

We define volume(Mabuchi) functional with Monge Ampere measure measure MA.

$$MA(u) = \frac{\omega^3}{3!}, \quad dVol|_u = MA(u)$$

$$Vol(u) = \int_X MA(u) = \frac{1}{8}\int_X e^{r^2/2} e^{\omega}$$

we have

$$\frac{d}{dt}|_{t=0}(Vol(u+tv)) = \int_X MA(u)v$$

We avoid the use of the extremal Reeb vector, instead we use the Shintani cocycle and **p**-adic lift which determine $SL(2, F)$ Bruhat-Tits building cocycle. We obtain zeta value of real quadratic number field F with Hilbert class field K. The logarithmic derivative of zeta function do the role for potential function for Kahler metric. For that, we will construct **p**-adic measure.

The volume of CY cone at radius 1 is defined with Kahler form ω,

$$Vol(X_{r\leq 1}) = \int_{\mu^{-1}(\Delta)} \frac{1}{3!}\omega^3 = \frac{1}{6}Vol(Y^{p,q}) = (2\pi)^3 Vol(\Delta)$$

Now consider toric divisor D_k in X which is the inverse image of μ the facets F_k in polyhedral cone $C \in \mathbb{R}^3$, $\{Z_k = 0\} \in \mathbb{C}^3, 1 \leq k \leq d$.

Then from CY condition, the Reeb vector $\mathbf{b}$ on $C(Y^{p,q})$ induced from the Reeb vector on special Lagrangian, so that, we have the volume of the special Lagrangian 3 manifold $\Sigma_k = L(n, 1), n = p \pm q, p, p$ with $D_k = Cone(\Sigma_k)$ for $k = 1, ..., 4$ spanning the edges of toric diagram of $Y^{p,q}$. We define facets $F_k = \{l_k(y) = 0\} \cap \{r \geq 1\}$ where r is radius.

$$Vol(\Sigma_k = \mu^{-1}(F_k)_{r=1}) = 4(2\pi)^2 \frac{1}{v_k} Vol(F_k)$$

where v_k is the primitive normal vector.

The extremal $Vol(Y^{p,q})$ and $Vol(\Sigma_k)$ only depends on extremal Reeb vector $\mathbf{b}$. In other words, the $\mathbf{b}$ is the unique critical point of volume(Mabuchi) functional for CY.

By Ricci flatness of CY manifold, we can reduce to two dimensional lattice.

$$0 = \int_\Delta R_X dy_1 dy_2 dy_3 = (\sum_k \frac{2}{v_k} Vol(F_k)) - Vol(\Delta)$$

which lead

$$\pi \sum_k Vol(\Sigma_k) = 6Vol(Y^{p,q})$$

The volume functional is the strictly convex function with unique minimum in each chamber. The extremum value is the irrationality in number field K for irrational Reeb vector $\mathbf{b}$. This is the higher dimensional geometric realization of irrationality determined by the extremal Reeb foliation.

Given extremal Reeb vector $\mathbf{b} = \{b_1, b_2, b_3\} \in \mathbb{R}^3$, we have following volume cocycle. Let $\{v_k\}, 1 \leq k \leq d$ be the primitive normal vector which define facet of polyhedral cone C.

$$Vol(\Delta_{\mathbf{b}}) = \frac{1}{6b_1} \sum_k \frac{1}{|v_k|} Vol(F_k) = \frac{1}{6b_1} \sum_k \frac{(v_{k-1}, v_k, v_{k+2})}{(\mathbf{b}, v_{k-1}, v_k)(\mathbf{b}, v_k, v_{k+1})}$$

where (v, w, z) is the determinant of 3×3 matrix with rows v, w, z.

By considering exotic smooth structure from Sasaki-Einstein manifold by Lens space surgery $L(p^2, pq-1)$ to recover Sasaki-Einstein extremal volume, we need to consider $\mathbf{p}$-adic lift of the Shintani cocycle of $L(p,q)$ of real quadratic number field. So we need to consider evaluate partition function at specific spectral parameter(extremal volume/R-charge),for that we need to consider continued fraction of $p^2/pq-1$ approximating $\sqrt{4p^2-3q^2}$ to determine the number field.

So we have parallelogram 2d toric diagram for Shintani cocycle. After **p**-adic lift, we recover the Sasaki-Einstein volume from Stark-Heegner unit.

Now we consider Shintani cocycle with pro**p** lifting for real quadratic field F associated with pro-**p** cusp $L(p,q)$ in Hilbert modular surface of F. Let $\{u_k\} \in \mathbb{C}^2, k = 1, ..., 4$ be the two dimensional vector for Shintani cone before **p**-adic lift which is the projection of $\{v_k\} = \{(1, u_k)\}$.

The Shintani cones can defined from the parallelogram by considering subdivision. The analogous volume functional is,

$$f(u_k)(x) = \frac{(u_k, u_{k+1})}{\langle x, u_k \rangle \langle x, u_{k+1} \rangle}$$

where (u, v) is determinant of 2×2 matrix with rows u, v and $\langle u, v \rangle$ is inner product. The role of the extremal Reeb vector $\mathbf{b}$(subtle data from extremal metric) will be recovered from **p**-adic lift.

f satisfies cocycle condition,

$$f(a, b) - f(c, a) + f(b, c) = 0, \quad \text{with} \quad f(a, b) =: f(a)$$

which is the area cocycle for Reidermacher ϕ function. With before **p**-adic lift, the volume of $L(p,q)$ is proportional to cone of $L(p,q)$ which is Hilbert modular surface $H = \overline{(\mathcal{H} \times \mathcal{H})}/SL(2, \mathcal{O}_F)$ of real quadratic number field $F = \mathbb{Q}(\sqrt{4p^2 - 3q^2})$ with ring of integer $\mathcal{O}_F$.

Since $Y^{p,q}$ arise in supersingular locus of Shimura varieties as torsion Galois representation from pro-**p** covering of the Bianchi manifold associated with hyperbolic knot quotient,we have additional sign from metaplectic lift. The additional sign is determined by the number N of the intrinsic Higgs state(conductor) which is the resonance for $Y^{p,q}$ quiver by periodic Seiberg duality. N is the additional conductor for p-adic zeta function for real quadratic number field.

Note that we can choose real quadratic number field F with a ideal class such that the Hilbert modular surface has Lens space $L(p,q)$ as the boundary compactification. By the Stark unit of F in K, we obtain the extremal volume of $C(Y^{p,q})$ with extremal Reeb vector. So we intertwine the use of 5d CS theory on $Y^{p,q}$ which gives the extremal Reeb vector, by using $\mathbf{p}$-adic lift of Shintani cocycle associated with 3-dimensional CS theory on $L(p,q)$ with Galois group action on its level.Due to additional conductor N in level of CS theory on $L(p,q)$ from intrinsic Higgs state, by $\mathbf{p}$-adic zeta function of real quadratic number field,5-dimensional $Y^{p,q}$ theory is realized. It determine additional sign from metaplectic lift of Bianchi manifold M by pro-$\mathbf{p}$ covering associated with $L(p^2, pq-1)$ realizing knot. So in our realization, the exotic smooth structure on $\mathbb{CP}^2 \times \#\overline{\mathbb{CP}^2}$ by Lens space $L(p^2, pq-1)$ is important.For that we need $M_n/\mathcal{K} = H^3/$(figure 8) where M_n is the pro-$\mathbf{p}$ covering of the rational hyperbolic homology sphere(Bianchi manifold) and $\mathcal{K}$ is two bridge Lens space knot. We realize the pseudo-Anosov flow from two bridge knot complement of $L(p^2, pq-1)$ from pro-$\mathbf{p}$ covering of arithmetic Bianchi manifold with $L = \mathbb{Q}(\sqrt{-p})$ with vanishing first Betti number.

Then by pro-$\mathbf{p}$ covering of Bianchi manifold which is Haken ,the torsion Galois representation in supersingular locus over sufficiently ramified field for Sasaki-Einstein CY is obtained.

On cusp torus of Lens space knot complement,by Lens space knot, we have pseudo-Anosov flow which realize real multiplication on cusp torus by $\mathcal{O}_F$ realising pseudo-Anosov mapping torus. The Bianchi manifold and Mapping torus share the same Haken manifold by pro-$\mathbf{p}$ covering of Bianchi manifold, so associated Galois representation is the same. So we can equate Lens space knot complement of pro-$\mathbf{p}$ covering $L(p^2, pq-1)$ and pro-$\mathbf{p}$ covering of Bianchi manifold.

The zeta value of F for the negative argument compute the volume(Euler characteristic) of moduli space of stable map for $\mathbb{Z}/p$ orbifold CY for orbifold Hurwitz partition function See Section.3.3 [11].

We will consider Eisenstein measure which gives the zeta value of negative argument.For negative zeta value,the associated geometry is $\overline{M_{g,1}}(B\mathbb{Z}/p\mathbb{Z})$. As we know, the Euler characteristic $e(\overline{M_{g,1}}(B\mathbb{Z}/p\mathbb{Z}))$ is obtained recursively by Hurwitz recursion which is the part of reciprocity of Eisenstein measure. The $\mathbb{Z}/p\mathbb{Z}$ orbifold Hurwitz theory has sign from Swan conductor(Real GW). And the sign comes from real curve counting with additional Clifford torus special Lagrangian. With this sign, we can have $Y^{p,q}$ real Gromov Witten invariant, from arithmetic invariant -extremal volume(entropy).

Note that after two dimensional projection, the determinant in denominator change to inner product by projecting 3-dimensional rotation symmetry. The restoring 3 dimensional rotational symmetry is quite subtle from two dimensional point of view. It can be obtained as the reciprocity after elliptic/$\mathbf{p}$-adic lifting or the additional symmetry of continued fraction with additional $\mathbb{Z}/2$ sign for sign for 3 dimensional spinor. In two dimensional case, we need to introduce elliptic measure (Eisenstein measure) for totally real field which corresponds to the measure from 3 dimensional partition in 3d case.

Our Eisenstein measure do the role for ergodic measure for irrational map from continued fraction.

After projecting to two dimensional cone, the problem for construction of the string partition function reduce to the construction of the Eisenstein/$\mathbf{p}$-adic measure in totally real field for zeta value.

Then we need to construct the Shintani cocycle from the area cocycle[5]. This is the generalized Dedekind sum associated with two dimensional lattice. It's the special form of the Witten's zeta function which is the zeta function root lattice.

We can choose $u_k = (p,q), u_{k+1} = (1,0)$. By continued fraction, we have Shintani cocycle.

$$D_{u_k,u_{k+1}} = (2\pi i)^2 \sum_{x\in L} f(u_k,x)e^{2i\pi\langle x,v\rangle} = \frac{1}{(u_k,u_k+1)} \sum_{z\in L/pL} B_1(z/p)B_1(zq/p) = \frac{1}{p}s(q,p)$$

Here we consider arbitrary $x \in L =: \mathbb{Z}^2 - \{0\}$ and sum over all x.

For the extremizing process by Reeb vector $\mathbf{b}$ as before, we need to consider infinitesimal deformation by infinitesimal derivation with higher ramification group from $\mathbf{p}$-adic lift. By mod $\mathbf{p}$ reduction we determine higher ramification group with Swan conductor realising extremal Reeb vector which is not holomorphic foliation. We have almost real foliation for pro-$\mathbf{p}$ asymptotic of extremal volume.

We introduce partial differential operator, $P(-\partial_{x_1}, -\partial_{x_2})$ on area cocycle $f(u_k)(x)$,

$$f(u_k)(P,x) =: P(-\partial_{x_1}, -\partial_{x_2})f(u_k)(x) = (u_k,u_{k+1}) \sum_{r=(r_1,r_2)} P_r(u)\frac{1}{\langle x,u_k\rangle^{1+r_1}\langle x,u_{k+1}\rangle^{1+r_2}}$$

where $P_r(u)$ is the homogeneous polynomial in 2×2 matrix u with rows u_k, u_{k+1}.

$$P({x_i}^t) = \sum_r P_r(u)\frac{{x_1}^{r_1}{x_2}^{r_2}}{r_1!r_2!}$$

The Eisenstein cocycle is given as follows,we choose a vector v in dual lattice,

$$\mathcal{F}(u_k,P,Q,v) = (2\pi i)^{-2-deg(P)} \sum_{x\in L=:\mathbb{Z}^2-\{0\}} e^{2i\pi\langle x,v\rangle} f(u_k,P)(x)$$

$$=: \lim_{t\to\infty}(2\pi i)^{-2-deg(P)} \sum_{x\in L,|Q(x)|<t} e^{2i\pi\langle x,v\rangle} f(u_k,P)(x)$$

$Q = (Q_1, Q_2)$ is the linear form -2 $\times$ 2 matrix with rows Q_i- in $\mathbb{R}^2$,nonvanishing on $\mathbb{Q}^2 - \{0\}$. For any vector $x \in L =: \mathbb{R}^2 - \{0\}$,

$$Q(x) = \prod_{1 \le i \le 2} Q_i(x), \quad Q_i(x) = Q_i x^t$$

With this Q rearrangement by 2×2 matrix, under the condition $Q(x) < t$,the sum to be converge and the limit exist.

The Q arrangement comes with partial derivation by P.After infinitesimal perturbing by P, we do resummation by rearrangement Q. P acts on two dimensional lattice, and Q is the dual of it get convergence.

The $\mathcal{F}(u_k, P, v)$ is the 1-cocycle for $\begin{pmatrix} -q & r \\ p & s \end{pmatrix} \in \Gamma = GL(2, \mathbb{Q})$ associated with $L(p, q)$ which we pro-$\mathbf{p}$ lift to $\begin{pmatrix} -q & r \\ \mathbf{N}p & s \end{pmatrix} \in SL(2, F)$ with totally real number field $F = \mathbb{Q}(\sqrt{4p^2 - 3q^2})$ with $\mathbf{N} = N\mathbf{p}^n$ with $\mathbf{p} = 1 \mod p$.

It is the boundary cohomology class in $H^1(\Gamma, M_{\mathbb{Q}})$, where $M_{\mathbb{Q}}$ is the $\mathbb{Q}$ vector space of function ϕ which satisfy distribution relation(reciprocity) $\phi : \mathcal{P} \times \mathcal{Q} \times \mathbb{Q}/\mathbb{Z}^2 \to \mathbb{Q}$, where $\mathcal{P} = \mathbb{Q}[x_1, x_2]$ is the space of the partial derivation operator P and $\mathcal{Q}$ is the space of linear form Q in $\mathbb{R}^2$ for average(rearrangement) operator.

Both $\mathcal{P}, \mathcal{Q}$ endowed with left action of $GL(2, \mathbb{Q})$. The distribution (reciprocity) relation for $\lambda \in \mathbb{Z}_+$ is ,

$$\phi(P, Q, v) = sign(\lambda)^n \sum_{w \in (\mathbb{Q}/\mathbb{Z})^n, \lambda w = v} \phi(\lambda^{degP} P, \lambda^{-1} Q, w)$$

So $\mathcal{F}(u_k, P, Q, v)$ is in $H^1(GL(2, \mathbb{Q}, M_{\mathbb{Q}})$ giving a cusp form by Eichler integral, for determining asymptotic which is pro-$\mathbf{p}$ asymptotic, we will consider $\mathbf{p}$-adic Shintani cocycle.

With this Eisenstein cocycle by specialization and **p**-adic lift, we obtain a zeta value at negative argument for real quadratic field $F = \mathbb{Q}(\sqrt{4p^2 - 3q^2})$ with sign of functional equation of zeta function[6].

For that, we need to define dual cycle in $H_1(\Gamma, M_{\mathbb{Q}}{}^*)$.

The dual cycle is determine with additional conductors of integral ideal $\mathfrak{f}$ and $\mathfrak{a}$ in $\mathcal{O}_F$ forming fractional ideal $\mathfrak{a}^{-1}\mathfrak{f}$,such that we have degree 2 polynomial from the norm $P(x_1, x_2) = N(\mathfrak{a})N(w_1x_1 + w_2x_2)$ where $\{w_1, w_2\}$ is $\mathbb{Z}$ basis of $\mathfrak{a}^{-1}\mathfrak{f}$ in $\mathcal{O}_F$. With conductor system which satisfies **p**-adic reciprocity, we will have higher ramification group for **p**-adic lift of Shintani cocycle by pro-**p** reduction. With **p**-adic reciprocity on conductor, we have dual cycle in modular curve with **p**-power conductor with pro-**p** geometry of cusp. Then we have irregular Fano surface with non-parallelogram toric diagram for irregular Sasaki-Einstein manifold $Y^{p,q}$ associated with totally real number field F.

$Q = (Q_1, Q_2)$ is the linear forms by $Q_i = \tau_i(w_1{}^*)x_1 + \tau_i(w_2{}^*)x_2$ where $\{w_1{}^*, w_2{}^*\}$ is dual basis with respect trace on F and τ_i is the embedding $F \hookrightarrow \mathbb{R}$. We choose $v = \{Tr(w_1{}^*), Tr(w_2{}^*)\}$. Then define dual cycle as follows. Define an element in $M_{\mathbb{Q}}{}^*$, $\phi_s =: \phi(P^s, Q, v)$, then we have a 1-cycle,

$$\mathcal{F}^*{}_{\mathfrak{f},\mathfrak{a},s} = A \otimes \phi_s \in C_1(\Gamma, M_{\mathbb{Q}}^*)$$

for 1-chain $A \in \mathbb{Z}[\Gamma^2]$.

Then we can determine the zeta value at negative argument using Poincare duality(Tate duality) for modular curve,

$$[\cdot,\cdot] : H^1(\Gamma, M_{\mathbb{Q}}) \times H_1(\Gamma, M_{\mathbb{Q}}{}^*) \to \mathbb{Q}$$

$$\begin{aligned}\zeta_{F,\mathfrak{f}}(\mathfrak{a}, -s) &= \sum_{\mathfrak{n} \in \mathcal{O}_F, \mathfrak{n} \equiv \mathfrak{a} \mod \mathfrak{f}} \frac{1}{(N\mathfrak{n})^s} \\ &= [\mathcal{F}(u_k, P, Q, v), \mathcal{F}^*{}_{\mathfrak{f},\mathfrak{a},s}] \in \mathbb{Q}\end{aligned}$$

Note that the zeta function is well defined in pro-$\mathbf{p}$ sense with evaluation at conductor which satisfying $\mathbf{p}$-adic reciprocity. The choice of s corresponds to s-th shift and each special value of negative argument determined by Euler characteristic of orbifold Hurwitz space,where we have recursion on s. In paper [11] we construct it from genus s orbifold Gromov Witten partition function for CY $\mathbb{Z}_p$ orbifold with pro-$\mathbf{p}$ asymptotic with sign for functional equation.

The comparison with orbifold GW invariant with sign from pro-$\mathbf{p}$ asymptotic (real GW invariant) suggest that hidden integrality of $\zeta_{F,\mathfrak{f}}(\mathfrak{a}, -s)$ at negative argument.

We will see the integrality by using $\mathbf{p}$-adic lifting. For that we need to have the elliptic/$\mathbf{p}$-adic measure having congruence properties. In the real Gromov-Witten computation, the orbifold multiple covering is the obstruction for integrality. By considering higher ramification group, we kill such multiple covering. For that we use $\mathbf{p}$-adic lift and Frobenius congruence.

Finally, the zeta value at negative argument address the counting problem including non-BPS stable configuration, the state arise from wall crossing phenomenon(Seiberg duality cascade)[1].

Such state only exist by Galois reason from $\mathbf{p}$-adic origin, which means that, it comes from global symmetry(Hilbert reciprocity) containing $\mathbb{Z}_2$ involution in orientifold. By breaking global $U(1)$ symmetry, we can obtain discrete(finite) group action with higher ramification group. Then we can obtain Artin representation for non-BPS stable state which comes from Hecke correspondence.

For the Sasaki Einstein manifold with irrational Reeb vector, the AdS dual has irrational R-charge(anomalous dimension) at IR fixed point of 4d SCFT with infinite dimensional Hilbert space. So we recover the infinite dimensionality

[1] Note that GW on $T^*(L(p,q))$ do not have wall crossing phenomenon.After $\mathbf{p}$-adic lift, it has wall crossing by wild ramification by Galois group action with additional conductor.

of Hilbert space from **p**-adic lift of Shintani cocycle - the supersingular locus of Shimura varieties over sufficiently ramified field(Drinfeld modular scheme) for CFT with level structure.

The construction of mirror should address weak modularity with **p**-adic Eisenstein measure for **p**-adic convergence and sign refined integrality from higher ramification group. The string partition function is the weak Maass form(**p**-adic Bianchi modular form) associated with rational homology sphere Bianchi orbifold M and pro-**p** covering M_n with $M_n/\mathcal{K} = S^3/\text{figure8}$ with $\mathcal{K}$ be the Lens space $L(p^2, pq-1)$ realizing knot. Due to hidden symmetry of figure 8 knot,we have reciprocity of perfectoid field(sufficiently ramified field) with additional $\mathbb{Z}/2$ sign at infinity. The shadow(non-holomorphic part) of weak Maass form is the Jacobi form, realizing non-holomorphic Teichmuller geodesic flow by pro-**p** covering of rational homology sphere Bianchi manifold. With this we have signed pro-**p** asymptotic **p**-adic modular form(weak Maass form with non-holomorpic geodesic flow), realizing sign refined algebraicity.

For quotient geometry, associated with general 3 manifold is hyperbolic quotient by Bianchi group with quadratic imaginary field $L = \mathbb{Q}(\sqrt{-p})$ associated with Hilbert class field K of totally real number field $F = \mathbb{Q}(\sqrt{4p^2 - 3q^2})$, which provide torsion global Galois representation.

The real quadratic field F and imaginary quadratic field L is related by realize pro-**p** covering of Lens space as covering along two component knot. The knot complement of $L(p^2, pq-1)$ realization knot from pro-**p** covering of $L(p, q)$ is the pro-**p** covering of Bianchi 3 manifold with thin monodromy group as fundamental group of CY.

3.1 p-adic measure

Now we describe the Eisenstein measure. The zeta function $\zeta_{F,\mathfrak{f}}(\mathfrak{a}, -s)$ can be expressed as following form. Let $\mathbf{Q}$ be the quadratic form for real quadratic number field $F = \mathbb{Q}(\sqrt{4p^2 - 3q^2})$ on $\mathbb{R}^2 - \{0\}$.

$$\zeta_{\mathbf{Q},\sigma}(-s) = (-1)^s s!\{(Td_\sigma)_{(2s+2)}(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}) - \delta_{s,0}\frac{p}{2}\frac{\partial^2}{\partial x_1 \partial x_2}\} \cdot \int_{\sigma(x)} e^{-\mathbf{Q}(u)} du$$

where $\sigma(x)$ is the neighborhood of Shintani cone $\sigma = (1,0), (q,p) \in \mathbb{Z}^2$ from two dimensional local projection of polyhedral cone C for a special Lagrangian $L(p,q)$ by $x = (x_1, x_2) \in \mathbb{R}^2$. $(Td_\sigma)_{(2s+2)}$ is the $2s+2$-th order term in power series expansion for infinitesimal derivation P.

We use the Todd power series for resolution of singularity whose link is $L(p,q)$ by continued fraction,for asymptotic, we need to consider pro-$\mathbf{p}$ geometry of $L(p,q)$ cusp in Hilbert modular surface of totally real number field F. We fix two dimensional lattice $\mathbf{M} =: \mathbb{Z} + v\mathbb{Z}$ of real quadratic field F, with quadratic form $\mathbf{Q}(1,v) = 0$. Let $v = [[\overline{t_1, ..., t_m}]]$ is the quadratic irrationality which approximated by infinite continued fraction $-p/q = [[t_1, ..., t_n]]$. Then we have v is the irrationality in F with adjustment of continued fraction of $-p/q$ from $\mathbf{p}$-adic reciprocity of sufficiently ramified field. Note that we do not need exact continued fraction due higher ramification group with pro-$\mathbf{p}$ asymptotic with non-holomorpic foliation which is almost real.

$$Td_{\sigma(p,q)}(x,y) = pxy \sum_{i \in \check{\sigma} \cap \mathbf{M}} e^{(\langle i,\rho_1\rangle x + \langle i,\rho_n\rangle y)} = \sum_{\eta^p = 1} \frac{xy}{(1-\eta^{-q}e^x)(1-\eta e^{-y})}$$

$$= \frac{pxy}{(1-e^{-px})(1-e^{-py})} \sum_{i \mod p} e^{-p(\frac{ix}{p} + \lfloor \frac{iq}{p} \rfloor y)}$$

where $\rho_1 = (1,0), \rho_n = (p,q)$ determine the two final side of Shintani cone generated by continued fraction.

$\rho_1 = (1,0), \rho_1 = (1,0) \ \ \rho_2 = (t_1,1) \ \ \rho_i = (h_i,k_i) \ \ \rho_n = (p,q), \quad \rho_{i+1}+\rho_{i-1} = t_i\rho_i$

For last equality, we sum over Shintani dual cone decomposition by $(\frac{ix}{p}+\lfloor\frac{iq}{p}\rfloor y)$. Note that for $(x,y) = (1,1)$, this is analytic torsion of $L(p,q)$. The sign of Shintani cocyle will be determined by **p**-adic lift which can determined by boundary condition of Shintani cone by considering all wall crossing group action as wall crossing cocycle with higher ramification group. So we need to consider **p**-adic reciprocity of wall crossing group(motivic Galois group) which containing Adelic Galois group,then we have is $SL(2,F)$ Bruhat-Tits cocycle.

$Td_{\sigma(p,q)}$ gives the generalized Dedekind sum associated with resolution of singularity satisfying cocycle condition,

$$Td_{\sigma(q,p)}(x,y) = Td_{\sigma(-q,p)}(-x,y) + \frac{pxy}{1-e^{py}}$$

The coefficient of x^iy^j of $Td_\sigma(x,y)$ is,

$$\hat{D_{i,j}}(q,p) = p^{i-1}\sum_{1\leq k\leq p}\hat{B}_i(k/p)\hat{B}_j(kq/p)$$

with $\frac{e^{tx}}{e^t-1} = \sum_n \frac{B_n(x)}{n!}t^{n-1}$, $\hat{B}_i(x) = B_i(x-[x])$ for $i\neq 1$ and $x \not\subseteq \mathbb{Z}$.

Let $D_{i,j}(q,p) := \frac{\hat{D_{i,j}}(a,b)}{ij}$ with $D_{1,1}(p,q) = \hat{D_{1,1}}(q,p) = s(q,p) - 1/4$.

With this, we have the Eisenstein measure.

p-adic Eisenstein measure

Let **p** be the inert prime in totally real number field F. Choose the conductor $\mathbf{N} = N\mathbf{p}^n$. We consider following sum, which is the real part of period integral of modular form $F_k(z)$ of weight k on $\Gamma(\mathbf{N})$ with $\gamma = \begin{pmatrix} q & b \\ \mathbf{N}p & d \end{pmatrix}$ in $\Gamma_0(\mathbf{N})$.

$$Re(\int_{\frac{q}{\mathbf{N}p}}^{i\infty} z^m F_k(z)dz) = 12\sum_{0\leq l\leq m}\begin{pmatrix} m \\ l \end{pmatrix}\frac{q}{\mathbf{N}p}^{m-l}(-1)^l\sum_{d|\mathbf{N}} n_d d^{-l} D_{k-l-1,l+1}(q,\mathbf{N}p/d)$$

$$F_k(z) = -24\sum_{d|\mathbf{N}} n_d \cdot d \cdot E_k(dz) = -24\sum_n \sigma_{k-1}\sum_{d|\mathbf{N}} n_d d e^{(2i\pi)\tau n d}$$

with $\tau = z \mod 2i\pi$ with $z = it + q/\mathbf{N}p$. It has $\mathbf{p}$-adic reciprocity

$$F^*{}_k(z) = F_k(z) - \mathbf{p}^{k-1}F_k(\mathbf{p}z)$$

which provide globalization of Eisenstein series providing 1-form on modular surface with $\mathbf{p}$ power conductor, $E_k(\tau) = \sum_n \sigma_k(n)e^{(2i\pi)\tau n} = \sum_n \frac{n^k e^{(2i\pi)\tau n}}{1-e^{2i\pi\tau n}}$ with $z = it + q/\mathbf{N}p$, $z = \tau, \mod 2i\pi$.

So we have the weight 2 modular form for the Dedekind sum.

$$\int_{q/\mathbf{N}p}^{i\infty} F_2(z)dz = 12\sum_{d|N} n_d D_{1,1}(q, \mathbf{N}p/d) =: 12D^\delta(q, \mathbf{N}p)$$

Then we have pair of period and $\mathbf{p}$-adic period of modular curve of $\Gamma_0(\mathbf{N})$-modular symbol. For $\mathbf{p}$-adic period with sign from higher ramification group, we need to consider higher weight modular form F_k and congruence relation between them. We consider modification F_k^* to have sign refined algebraicity.

For period,let $\gamma = \begin{pmatrix} q & b \\ \mathbf{N}p & d \end{pmatrix}$ with $p \neq 0$ for cusp $L(p,q)$ of Hilbert modular surface of totally real number field.

$$\int_r^{s=\gamma r} F_2(z)dz = \frac{1}{2\pi i}\int_r^{s=\gamma r} d\log\alpha = 12\mathrm{sign}(p)D^\delta(q, \mathbf{N} \mid p \mid)$$

For $\mathbf{p}$-adic period of $F_k(z)$ with $k = i+j+2$,let $\xi = q/\mathbf{N}p \in \Gamma(\infty)$ with $\mathbb{Z}_\mathbf{p}$ measure μ on $\mathbb{Z}_\mathbf{p} \times \mathbb{Z}_\mathbf{p}$. We choose homogeneous polynomial of degree $m = k-2 = i+j$, x^iy^j with $i,j \geq 0$.

The $\mathbf{p}$-adic Eisenstein measure μ is defined as,

$$\int_{\mathbb{Z}_\mathbf{p}\times\mathbb{Z}_\mathbf{p}} x^iy^j d\mu(x,y) = Re((1-\mathbf{p}^{i+j})\int_{q/p\mathbf{N}}^{i\infty} z^m F_{i+j+2}(z)dz)$$

$$= 12(1-\mathbf{p}^{i+j}) \sum_{0\le l\le m} \begin{pmatrix} m \\ l \end{pmatrix} \frac{q}{\mathbf{N}p}^{m-l} (-1)^l \sum_{d|N} n_d d^{-l} D_{i+j-l+1,l+1}(q, \mathbf{N}p/d)$$

Then we have the $\mathbf{p}$-adic reciprocity of modified ${F_k}^*$ from $\lim_{j\to\infty} y^{(\mathbf{p}-1)\mathbf{p}^j}$ in second coordinate of $\mathbb{Z}_\mathbf{p} \times {\mathbb{Z}_\mathbf{p}}^\times$ for pro-$\mathbf{p}$ cusp geometry. With this we can describe higher ramification group of totally real number field with non-maximal Swan conductor.

So we need to consider modified Dedekind sum determining Hida's ordinary limit.

$$D^{mod}_{k-l-1,l+1} = \lim_{j\to\infty} D_{k+(\mathbf{p}-1)\mathbf{p}^j-l-1,l+1}(x) =: D_{k-l-1,l+1}(x) - \mathbf{p}^{k-l-2} D_{k-l-1,l+1}(\mathbf{p}x)$$

Then we have the $\mathbf{p}$-adic lifting of Eisenstein measure.

$$Re[\int_{q/p\mathbf{N}}^{i\infty} z^m {F^*}_k(z)dz] = 12 \sum_{0\le l\le m} \begin{pmatrix} m \\ l \end{pmatrix} \frac{q}{\mathbf{N}p}^{m-l} (-1)^l \sum_{d|\mathbf{N}} n_d d^{-l} D^{mod}_{k-l-1,l+1}(q, \mathbf{N}p/d)$$

$F^*_k(z)$ is the modular form of weight k with $\gamma = \begin{pmatrix} q & b \\ \mathbf{N}p & d \end{pmatrix}$ with $\mathbf{p}$ power conductor associated with with homogeneous polynomial of degree $m = k-2$ in x, y.

The $\mathbf{p}$ power conductor $N\mathbf{p}^n$ is realized with $F^*_k(z)$ with $\gamma = \begin{pmatrix} q & b \\ \mathbf{N}p & d \end{pmatrix}$, which determine $\mathbf{p}$-adic zeta function of totally real number field with Stark-Heegner unit realizing conductor N. This is the local-global compatibility of Galois representation with integral structure of Shimura varieties over sufficiently ramified field. By $\mathbf{p}$-adic modularity of F^*_k which is $\mathbf{p}$-adic reciprocity from modification of Eisenstein series,the conductor N in $\mathbf{N} = N\mathbf{p}^n$ has meaning as Swan conductor $N = \mathfrak{f}$ from higher ramification group of Galois representation of totally real number field. We have Stark-Heegner unit by $12N$-th power of Siegel unit.

It gives us the additional characteristic from $t_k(\mathfrak{N} : p, ..., p)$ for mod p partition of degree $\mathfrak{N} = d = pk$ $\mathbb{Z}/p$ orbifold partition function in Section.3.3 for the orbifold Hurwitz partition function which constructed in [11].

The two generalization of the Dedekind sum $t_k(\mathfrak{N}, p, ..., p)$ and $D_{i,j}(p, q), i+j = k - 2$ gives same characteristic class. One from k-th iterated cotangent sum and the other from k-th Eisenstein series. We use $t_k(\mathfrak{N} : p, ..., p)$ to give characteristic class on $\overline{M_{g,1}}(B\mathbb{Z}/p\mathbb{Z})$, the Euler characteristic class. Here we get **p**-adic period of modular curve.

The the **p**-adic period $F^*{}_k(z)$ reduce to period of **p**-adic Hilbert modular form with sign from higher ramification group.From the geodesic is $\{q/\mathbf{N}p \to \infty\}$, we have pro-**p** geometry of $L(p, q)$ cusp in Hilbert modular surface which realize pro-**p** cusp p/q over modular curve with $N\mathbf{p}^n$ power conductor as ideal class group element of totally real field $F = \mathbb{Q}(\sqrt{4p^2 - 3q^2}$ with additional conductor N.We have non-holomorphic geodesic at pro-**p** cusp with almost real structure realizing big $\mathbb{Q}$ point of infinite order for **p**-adic Ceresa cycle whose normal function is Stark-Heegner unit of totally real number field.

The reciprocity of modified Eisenstein measure can viewed as the recursion in orbifold string partition function with pro-**p** asymptotic by non-holomorphic geodesic.

3.2 p-adic cocycle

Now we can lift our cocycle $\mathcal{F}(u_k, P, Q, v)$ as the **p**-adic cocycle. We construct **p**-adic cocycle to encode the structure of **p**-adic measure of previous Section.3.1. Then, by the inner product with cycle in logarithmic homology class of modular curve, we obtain **p**-adic zeta function. In paper [13], we construct integrality of sign refined spin CS partition function of Seifert 3 manifold(Lens space) by using Mahler expansion.

Here we construct integral cocycle using integral **p**-adic Hodge theory. It comes from the reciprocity of Witt ring. Such cocycle at fixed negative argument s corresponds to virtual Euler class in orbifold Hurwitz space [11]. The wall crossing formula(Seiberg duality) is the another form of Frobenius reciprocity for the sign refined integrality with higher ramification group.

We first define the refined cocycle by congruence. The refined cocycle has integral expression which is the direct analog of Euler Maclaurin formula for **p**-adic expansion.

For integral expression, we define refined Shintani cocycle with reciprocity(**p**-adic congruence) as follows [4][3].

The following modified zeta function has value in integral. This is the direct analog of the integrality of CS partition function at root of unity.

$$\zeta_{F,\mathfrak{f}}(\mathfrak{a},s)^l =: \zeta_{F,\mathfrak{f}}(\mathfrak{ac},s) - (N\mathfrak{c})^{1-s}\zeta_{F,\mathfrak{f}}(\mathfrak{a},s) \in \mathbb{Z}[1/l]$$

for integral ideal $\mathfrak{c}$ has prime norm $N\mathfrak{c} = l$ relatively prime to conductor $N\mathfrak{f}$. We take l to be the splitting prime on F. For inert prime,we constructed the measure in previous Section.3.1.

The associated refined Eisenstein cocycle is defined by using two dimensional lattice automorphism $Aut(\Gamma_0[l])$ with $N\mathfrak{c} = l$. We can shift the vector $u_k = (p,q)$ and $u_{k+1} = (1,0)$ by $u_k{}' = A_l u_k A_l{}^{-1}, u_{k+1}{}' = A_l u_{k+1} A_l{}^{-1}$ with action of 2×2 matrix $A_l = \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}$, which corresponds to change ideal class $\mathfrak{a}$ to $\mathfrak{ac}$. For $P \in \mathcal{P}$ of degree s, $P' = A_l{}^{-1}P$ with $P'(x_1,x_2) = P(x_1/l,x_2)$ and $v' = A_l v = (lv_1,v_2)$, $v \in (\mathbb{Q}^2 - (\frac{1}{l}\mathbb{Z}\times\mathbb{Z}))/\mathbb{Z}^2$.

The modified cocycle is,

$$\mathcal{F}(u_k,P,Q,v)^l =: l^s\mathcal{F}(u_k{}',P',Q,v') - l\mathcal{F}(u_k,P,Q,v) \in Z_1(\Gamma[l],M_l)$$

where $M_l = M_{\mathbb{Q}} \times_{\mathbb{Z}} \mathbb{Z}[1/l]$. We choose the modified dual 1-cycle ${\mathcal{F}^*}_{\mathfrak{f},\mathfrak{a},s}{}^l$ to lies in $C_1(\Gamma[l], M_l)$ by shifting with A_l. We shift 1 chain A by $A^{'} = A_l A$ with $A = \{0, \gamma\}$.

$$\mathcal{F}^*{}_{\mathfrak{f},\mathfrak{a},s}{}^l =: A^{'} \times \phi(P^{'s}) \in C(\Gamma[l], M_l)$$

With this refined cocycle and cycle, we obtain modified zeta function by the Tate pairing.

$$\zeta_{F,\mathfrak{f}}(\mathfrak{a}, s)^l = [\mathcal{F}(u_k, P, Q, v)^l, \mathcal{F}^*{}_{\mathfrak{f},\mathfrak{a},s}{}^l]_l \in \mathbb{Z}[1/l]$$

$$[\cdot,\cdot]_l : H^1(\Gamma[l], M_l) \times H_1(\Gamma[l], M_l{}^*) \to \mathbb{Z}[1/l]$$

For integral expression, we introduce modified Eisenstein series. Fix $v = (v_1, v_2) \in (\mathbb{Q}/\mathbb{Z})^2$,the modified Eisenstein series is following with q-expansion.

$$E_{k,v}(z) = \sum_{m,n \neq \{0,0\}} \frac{e^{2i\pi(mv_1+nv_2)}}{(mz+n)^k}$$

For $v = (a/\mathbf{p}, b/\mathbf{p})$, the q-expansion is,

$$E_{k,v} = \frac{2(2i\pi)^k}{(k-1)!}[\frac{\hat{B}_k(b/\mathbf{p})}{2k} + \frac{1}{l^{k-1}} \sum_{1 \leq n \leq \infty} \sigma_{k-1,v}(n) q_l^n], \quad q = e^{2i\pi(z/\mathbf{p})}$$

with

$$\sigma_{k-1,v}(n) = \frac{1}{2}[\sum_{d|n, d\equiv b \mod \mathbf{p}} d^{k-1} e^{2i\pi\frac{-nv_1}{d}} + (-1)^k \sum_{d|n, d\equiv -b \mod \mathbf{p}} d^{k-1} e^{2i\pi\frac{nv_1}{d}}]$$

For example,we have sum over isogeny $\sum_{1\leq i\leq \mathbf{p}-1} E_{2,(i/\mathbf{p},0)}(z) = -E_2{}^*(z)$ where $E_k{}^*(z) = E_k(z) - \mathbf{p}^{k-1}E_k(\mathbf{p}z)$.

The modified Eisenstein series for splitting prime $l \in F$ is defined analogously for $\mathbf{p}$-adic measure. We choose shifted convention with not shifted degree of polynomial $P(z, 1)$ which is essentially the same.

$$E_{s,v}{}^l(z) =: l^{s-2} E_{s,(lv_1,v_2)}(lz) - E_{l,v}(z)$$

Then, the integral expression for zeta value is,

$$\zeta_{F,\mathfrak{f}}(\mathfrak{a},s)^l = \frac{(s-1)!}{(2\pi i)^{s+2}}\int_{\tau}^{\tau\gamma} P(z,1){E_{s+2,v}}^l(z)dz \subset \mathbb{Z}[1/l]$$

The mod l integrality comes from the fact that ${E_{s,v}}^l$ is the modular form of weight s with mod l integral coefficient on $\Gamma_0[l] \cap \Gamma(N)$ where N is the denominator of v.

We define modified $\mathbf{p}$-adic Eisenstein measure as follows,it is defined with $\mathbf{p}$-adic topology as before. Choose $x \in {\mathbb{Z}_{\mathbf{p}}}^2$,

$$\frac{(s-1)!}{(2\pi i)^{s+2}}\int_{\tau}^{\tau\gamma} {E_{s+2,v}}^l(z)dz = \int_{{\mathbb{Z}_{\mathbf{p}}}^2-\mathbf{p}{\mathbb{Z}_{\mathbf{p}}}^2} d\mu_l(1,\gamma)(v)(x) \in \frac{1}{2}\mathbb{Z}[1/l]$$

with $\mathbf{p}$-adic congruence on $\mu_l(v)(x)$ by

$$\mu_l(v)(v+a+\mathbf{p}^n{\mathbb{Z}_{\mathbf{p}}}^2) = \frac{(s-1)!}{(2\pi i)^{s+2}}\int_{\tau}^{\tau\gamma} E^l_{s+2,\frac{v+a}{\mathbf{p}^n}}(z)dz \ \in \frac{1}{2}\mathbb{Z}[1/l] \subset \frac{1}{2}\mathbb{Z}_{\mathbf{p}}$$

We have the following limit. Let $Y =: \cup_{1\le i\le y_n}(a_{i,n}+\mathbf{p}^n{\mathbb{Z}_{\mathbf{p}}}^2)$ be the compact open subset of ${\mathbb{Q}_{\mathbf{p}}}^2$,

$$\int_Y f(x)d\mu_l(v)(x) = \lim_{n\to\infty}\sum_{1\le i\le y_n} f(a_i)\mu(a_i+p^n{\mathbb{Z}_{\mathbf{p}}}^2)$$

The Cauchy limit does not depends on specific choice of $a_{i,n} \in Y$.

Now we can obtain Stark unit u_l^σ of the real quadratic field in F. $\mathfrak{f}$ is the integral ideal and $\mathfrak{a},\mathfrak{c}$ be the fractional ideal. Let $\mathbf{p}$ be a inert prime in F and l be the split prime with $l \nmid \mathbf{p}$ in F, and K be the Hilbert class field of F.Choose $\sigma \in G =: Gal(K/F)$. We can compare imaginary quadratic field extension for Hilbert class field as the Metaplectic lift in automorphic counterpart.

We need three mutually coprime conductor $N\mathfrak{f} = N, N\mathfrak{a}, N\mathfrak{c} = l$ with basis (w_1,w_2) for $\mathfrak{a}^{-1}N$ and $(\frac{1}{l}w_1,w_2)$ for $\mathfrak{c}^{-1}\mathfrak{a}^{-1}\mathfrak{f}$, with $N\mathfrak{f} = N$ for denominator of (v_1,v_2) with $v_1w_1+v_2w_2 = 1$.

The Stark Heegner unit is, from the derivation of the $\mathbf{p}$-adic logarithm of zeta function at $s = 0$.

$$d \log_{\mathbf{p}} \zeta_{F,\mathbf{p}}{}^{l}(\sigma, 0) = \log_{\mathbf{p}}(u_l{}^{\sigma}) = \int_{\mathbb{Z}_{\mathbf{p}}{}^2 - \mathbf{p}\mathbb{Z}_{\mathbf{p}}{}^2} \log_{\mathbf{p}}(x_1 w_1 + x_2 w_2) d\mu_l(1,\gamma)(v)(x) \in O_{F,\mathbf{p}}$$

with the integral expression of $\mathbf{p}$-adic zeta function is,

$$\zeta_{F,\mathbf{p}}{}^{l}(\sigma, s) = \int_{\mathbb{Z}_{\mathbf{p}}{}^2 - \mathbf{p}\mathbb{Z}_{\mathbf{p}}{}^2} P(x)^{-s} d\mu_l(1,\gamma)(v)(x) = \int_{\mathbb{Z}_{\mathbf{p}}{}^2 - \mathbf{p}\mathbb{Z}_{\mathbf{p}}{}^2} (N_{F_{\mathbf{p}}/\mathbb{Q}_{\mathbf{p}}}(x_1 w_1 + x_2 w_2))^{-s} d\mu_l(1,\gamma)(v$$

The Stark unit $u_l{}^{\sigma}$ from the $\mathbf{p}$-adic lift the Shintani cocycle has limit as the the irrational extremal volume of from extremal metric of Sasaki-Einstein CY. The extremal metric can be written as the section of determinant line bundle which is the logarithmic derivative of $\mathbf{p}$-adic zeta function, valued in Hilbert class field K.

It is power of the Siegel unit which is the elliptic function. So the Stark unit has the modular origin. This is the height of the Stark Heegner point in elliptic curve with conductor N which we is the extremal volume of CY. So it is the Black Hole entropy with non-spherical horizon $Y^{p,q}$.

So we recover irrational entropy ($\simeq$ extremal volume) as Stark-Heegner unit of totally real number field. Here,by the specialization of Stark-Heegner unit, we have zeta value of cusp form which is $\mathbf{p}$-adic zeta function of totally real number field.

By $\mathbf{p}$-adic lifting with Hilbert reciprocity, we obtain almost totally real sub field in Hilbert class field with one complex place by $\mathbf{p}$-adic Ceresa cycle whose normal function of Stark-Heegner unit which explain inverting 2 in the definition of modified Eisenstein measure. We have Bianchi manifold M and pro-$\mathbf{p}$ covering M_n with imaginary quadratic field $L = \mathbb{Q}(\sqrt{-p})$ determined by $M_n/\mathcal{K} = S^3/figure8$, where $\mathcal{K}$ is the $L(p^2, pq-1)$ double bridge knot.

With this complex place, we have Stark Heegner point for real quadratic number field with sign for functional equation from $\mathbf{p}$-adic reciprocity. The sign comes from real GW invariant from spin refined CS invariant of $L(p,q)$ by non-Abelian central extension with additional conductor.

Stark-Heegner point is the big $\mathbb{Q}$ point of infinite order on elliptic curve E over sufficiently ramified field with CM lift by Hilbert class field K of totally real number field. The height(Black hole entropy) of Stark Heegner point is expressed from extremal volume of $Y^{p,q}$ CY which is the in-homogeneous period(stability condition) of mirror CY(Brauer-Severi varieties) with splitting field K.

The algebraicity of the Stark-Heegner unit gives the algebraicity of String partition function. By the asymptotic algebraicity,we have the extremal volume as irrationality for algebraic equation(volume extremising equation). The importance of extremising volume functional in number theory is that we can obtain modular quantities without reciprocity of class field tower. The extremal volume(singular Kahler Einstein metric) is the Arakelov invariant and comes from underlying integral structure of $\overline{Spec\mathbb{Z}}$ with periodicity of K theory on it. This is the averaged Faltings height of CM field K of F which is Stark-Heegner unit of F. By sum over all CM type(averaging) we obtain almost real structure(singular Hermitian metric with sign from higher ramification group), and can compare $\mathbf{p}$-adic Stark Heegner unit with $\mathbf{p}$-adic reciprocity(sign).

This lies on modularity of string partition function and that of the Stark unit with functional equation. We have torsion global Galois representation from modular origin by $SP(4)$ representation from metaplectic lift.The Stark Heegner point is big $\mathbb{Q}$ algebraic point of elliptic curve over sufficiently ramified field with CM lift by Hilbert class field K of totally real number field F. This is the point of infinite order in elliptic factor of Jacobian of Fermat quotient of degree $\mathbf{p}$ over finite field of characteristic $\mathbf{p} = 1 \mod p$ which is pro-$\mathbf{p}$ mirror to $T^*L(p,q)$ with pro-$\mathbf{p}$ fundamental group. Over finite field, we can consider mirror curve as Fermat quotient $C = \{y^p = x^q(1-x)^r\}$ of degree $\mathbf{p}$, since we only consider the singularity structure of it. By viewing $L(p,q)$ as link of wild quotient singularity,we consider $L(p^2, pq-2)$ from Dehn twist along punctured torus with exotic smooth structure satisfying Markov triple reciprocity determining higher ramification group for irregular singularity $C(Y^{p,q})$.

We have Mordell Weil rank 0 elliptic curve for irregular Sasaki-Einstein manifold associated with Bianchi manifold with vanishing first Betti number. We have non-existence of of rank 2 Galois representation,so we need metaplectic lift to $Sp_4(\mathbb{C})$ representation with sign. This is the big $\mathbb{Q}$ point of infinite order with dense wall nearby it associated with real multiplication problem in supersingular locus of Shimira varieties over sufficiently ramified field.

So we recover the sign refined algebraicity from pro-$\mathbf{p}$ reciprocity for pro-$\mathbf{p}$ asymptotic expansion with integral coefficient by mod $\mathbf{p}$ reduction and modularity for Sasaki-Einstein CY from non-trivial analytic continuation(functional equation) which is twistor transformation with sign from higher ramification group. The Stark Heegner unit is the epsilon factor for functional equation of $\mathbf{p}$-adic zeta function with Swan conductor N.

By the modular parametrization $X_0(N\mathbf{p}) \to E$,the $\mathbb{Q}$ point(Stark-Heegner point) on modular curve gives the global $\mathbb{Q}$ point(Stark-Heegner point) on elliptic curve with absolute Galois group action which comes from $\mathbf{p}$-adic origin. By modular Jacobian with crystalline cohomology with Poincare duality with higher ramification group from Atkin-Lehner operation, we can recover the $\mathbf{p}$-adic zeta function of real quadratic number field.

Up to rational multiple, the Stark Heegner unit which is the logarithmic derivative of $\mathbf{p}$-adic zeta function of F, proportional to $\mathbf{p}$-adic period of elliptic curve E over sufficiently ramified field with CM lift by Hilbert class field K of totally real number field F. This gives an example for Deligne period conjecture with generalized CM by Hilbert class field K of totally real number field F associated with the string partition function with entropy for non-trivial Horizon topology with irrational(non-holomorphic) foliation and sign by averaging CM type.

Since, we have modified cocycle with splitting prime l,this is compatible with additional $\mathbb{Z}/2$ at arithmetic infinity(metaplectic kernel), such that after lifting to Siegel modular form, it can directly related to lift of Bianchi modular form from imaginary quadratic field(See Sect.**??** for Bianchi manifold).[2]. We have Mordell Weil rank 0 for elliptic curve E due to corresponding Galois representation comes from pro-$\mathbf{p}$ covering of Bianchi manifold with first Betti number 0 which comes from exotic smooth structure of 4 manifold by $L(p^2, pq-1)$ surgery.

Such big $\mathbb{Q}$ point on elliptic curve E is the purely quantum object associated with Selmer sheaves of Galois cohomology of totally real number field from Stark-Heegner unit of totally real number field. We consider E as the simple factor in Jacobian of Fermat quotient over finite field pro-$\mathbf{p}$ mirror to pro$\mathbf{p}$ covering of $L(p, q)$.

We will consider it as Fermat curve over finite field, such that we obtain big $\mathbb{Q}$ point on its Jacobian with additional data(sign).

$$C = \{y^p = x^q(1-x)^r(-1)^s, q+r+s = 0 \mod p\}$$

over char $\mathbf{p}$. The big $\mathbb{Q}$ point in elliptic factor on its Jacobian comes from Galois/modular completion of cusp point at infinity which has the automorphism of infinite order from $\mathbf{p}$-adic tau function for C.

By modular parametrization, such big $\mathbb{Q}$ point is in modular Jacobian

[2]See Thm.8,conjecturally, Bianchi manifold of $L = \mathbb{Q}(\sqrt{-p})$ from $L(p^2, pq-1)$ Lens space knot complement. It might be different L with small discriminant due to reciprocity of sufficiently ramified field. Note that our field F is exotic field with discriminant $4p^2 - 3q^2$.

$J(X_0(N\mathbf{p}^n))$.It is the generalized attractor point(gen. CM point) in the moduli space of CY.

By the Shafarevich-Tate group E with Mordell Weil rank 0, the elliptic fibered CY is define with section from Stark Heegner $\mathbb{Q}$ point with extremal elliptic surface. The $\mathbb{Q}$ point is the big point with dense walls nearby it from pseudo-Anosov Teichmuller geomesic flow from irrational Reeb vector foliation. It comes from $\mathbf{p}$ power torsion at supersingular locus,as the point of infinite order with sign.

We consider E with Mordell Weil rank 0 embedded in irregular toric Fano surface for Sasaki-Einstein $Y^{p,q}$. By considering noncommutative of $C(Y^{p,q})$ as the twistor space of the smooth compactification at infinity of the moduli of rank 2 Higgs bundle which is Selmer sheaves (Hitchin system) on $\mathbb{P}^1/\{0, 1, \hat{a}, \infty\}$, we can see Stark-Heegner point on E by irrational extremal volume and sign(realising real multiplication on E). Note that for Sasaki-Einstein quiver, we have Mordell Weil rank 0 for E which is supersingular over sufficiently ramified field in Mori dream space(Fake projective space) from arithmetic MMP of irregular toric surface, consequently rank 0 Mordell Weil group for elliptic fibered CY(F-theory) which is the modular/elliptic completion of Sasaki-Einstein cone with elliptic curve over sufficiently ramified field. So we obtain elliptic fibered CY with modular Jacobian as the universal torsor for irregular toric Fano base of Sasaki-Einstein manifold by action of elliptic curve E with real multiplication.

Over the generalized attractor point in modular curve(Teichmuller curve) for Stark Heegner point in E,the associated varieties(CY motive with sign) has faithful action of absolute Galois group with complex multiplication from CM field K.

Though they are generalized CM points in moduli space,due to the CM field K comes from real quadratic field,the varieties over generalized CM point need not be rigid varieties due to non-Abelian Galois group of Witt ring(We need to average over all CM type to obtain almost real structure for sign from metaplectic $\mathbb{Z}/2$ kernel). So this is differ by CM point in moduli space,but generalized CM point(big CM point of infinite order) with dense wall nearby it arise from real quadratic field F with discriminant $4p^2 - 3q^2$. There exist Teichmuller curve which is projection of the pseudo-Anosov $SL(2,\mathbb{R})$ orbit. On Teichmuller curve, we have an orbifold point which is the generalized CM point from the Stark Heegner big $\mathbb{Q}$ point. With this we can have strong result for equidistribution of gen.CM point. ABC type argument for Masser-Wustoltz subgroup Theorem comes from reciprocity(duality) in supersingular locus over sufficiently ramified field for irregular Sasaki-Einstein motive(non-commutative motive) and F-theory extremal elliptic fibered CY with supersingular elliptic surface at characteristic 2 by Level-Rank duality.

Associated with Mordell-Weil group of supersingular extremal elliptic surface, where Leech lattice is the Mordell-Weil lattice at characteristic 2 ,the automorphism of Leech lattice from rational homology sphere Bianchi manifold M associated with $L(p^2, pq-1)$ two bridge knot $\mathcal{K}$ with $M_n/\mathcal{K} = S^3/figure8$ where M_n is some pro-$\mathbf{p}$ covering. So depending on the choice of automorphism of Leech lattice we obtain different N=1 SCFT vacua from irregular Sasaki-Einstein CYs with Borcherds automorphic form which is the theta lift of the Bianchi form for Bianchi manifold with vanishing first Betti number.

The $12N$ power of Stark Heegner unit is the Siegel unit which is elliptic function. The Siegel unit with conductor N is obtained as follows. Choose $(a_1, b_2) \in \mathbb{Z}/N\mathbb{Z}^2 - \{0,0\}$,with the fixed representative $0 \leq \bar{a_1} < N$.Let ζ_N be the primitive N-th roots of unity. Using q-expansion, for $\mathbf{p}$-th roots of unity q,

$$g(a_1, a_2) = \prod_{m \geq 0} (1 - q^m q^{\bar{a_1}/N} \zeta_N{}^{a_2}) \prod_{m \geq 1} (1 - q^m q^{-\bar{a_1}/N} \zeta_N{}^{-a_2})$$

$g(a, b)$ is essentially the same as Weierstrass σ function, whose specialization is the eta function.

$$u_l{}^{\sigma} = g(a_1, a_2)^{12N} \quad N = N\mathfrak{f}$$

By specialization of Siegel unit, we obtain orbifold Hurwitz 1-point partition function of arbitrary genus from the Massey product of Siegel unit. The specialization has the form of $\hat{A}$ genus so that we directly compare with Hurwitz partition function of degree k as the k-th iterated Massey product See Section.3.3.

The conductor N is the Serre conductor for torsion global Galois representation with higher ramification group. Swan conductor of elliptic surface which is the self intersection number of the wild singular fiber, and is the count of real GW invariant(intrinsic Higgs state) which is non-maximal conductor for supersingular elliptic curve over sufficiently ramified field by higher ramification group of Shintani cocycle ($SL(2, F)$ Bruhat-Tits building cocycle), which is analogous of rational points(maximal conductor) exceeding Hasse bound for supersingular curve. We expect that our approach meet to two known approach for real multiplication problem,one from Manin's approach from quantum Theta function(non-compact quantum dilogarithm) and the other from **p**-adic zeta function of real quadratic number field with Stark Heegner point on elliptic curve.

Given arbitrary non parallelogram toric diagram (ex. for $Y^{p,q}$),there exist the construction of refined topological vertex by gluing quantum dilogarithm with framing match. Here we obtain the string partition function for toric diagram for $Y^{p,q}$ from **p**-adic zeta function of real quadratic number field, where framing match comes with additional conductor for sign. Two approach for real multiplication coincide -using refined topological vertex from gluing quantum dilogarithm with framing match and **p**-adic zeta function of totally real number field.

Theorem 9 *The* $\mathbf{p}$*-adic zeta function and Stark Heegner unit* $u_l{}^{\sigma}$ *of real quadratic number field with the choice of conductor* N *gives* $\mathbf{p}$*-adic Galois representation associated with irregular Sasaki-Einstein* $Y^{p,q}$ *CY manifold. It is the torsion global Galois representation in supersingular locus of Siegel modular varieties over sufficiently ramified field from pro-*$\mathbf{p}$ *covering of Bianchi manifold with vanishing Betti number of Bianchi manifold with additional sign from higher ramification group for metaplectic kernel.*

So the refined Hodge conjecture holds for Sasaki-Einstein maniold with additional sign determined by conductor N*.*

We have associated F theory extremal elliptic fibered CY which come from Painleve 6-th with irrational(resonant) parameter(degenerate Hamiltonian system). The elliptic curve in extremal elliptic fibered CY has Stark-Heegner point of real quadratic number field realising irrational parameter.

Here, we need to consider imaginary quadratic number field L for Bianchi manifold M, such that our $\mathbf{p}$-adic Galois representation in supersingular locus over sufficiently ramified field from pro-$\mathbf{p}$ covering of Bianchi group of L.

We need to consider Lens space $L(p,q)$ in Hilbert modular surface as link of wild quotient singularity. This enters in specific choice of conductor N. Since we are in supersingular locus over sufficiently ramified field,from reciprocity and duality of supersingular locus over sufficiently ramified field realising sign refined S-duality, and we have additional sign from metaplectic lift for pro-$\mathbf{p}$ covering of the Bianchi manifold with conductor which can seen from non-arithmetic automorphism of Leech lattice.So the Sasaki-Einstein manifold comes from supersingular locus of Siegel modular varieties, and carries additional sign from conductor N.

By considering Lens space as wild quotient singularity, we also need to consider hyperbolic counterpart(Bianchi manifold with vanishing first Betti number), so that associated $\mathbf{p}$-adic Galois representation comes from torsion Galois representation from pro-$\mathbf{p}$ covering of Bianchi manifold .

There is no two dimensional Galois representation of number field L associated with Bianchi manifold with vanishing first Betti number(no essential surface), which means that there is no lifting of PSL_2 representation to SL_2 representation. So we need to consider non-Abelian central extension with metaplectic covering . After metaplectic covering,obtained from additional $\mathbb{Z}/2$ from the limit of pro-$\mathbf{p}$ congruence tower of Bianchi manifold, we have torsion Galois representation from $Sp_4(\mathbb{C})$ metaplectic lifting which realize non-Abelian central extension. This gives the 4-dimensional $\mathbf{p}$-adic Galois representation for Sasaki-Einstein CY manifold and associated extremal elliptic fibered CY(F theory background) from supersingular locus of Siegel modular varieties over sufficiently ramified field. We have additional $\mathbb{Z}/2$ sign from limit of pro-$\mathbf{p}$ tower determined by choice of conductor N.

Then ,we have global Galois representation from torsion Galois representation from Bianchi manifold which is hyperbolic counterpart of Lens space in Hilbert modular surface considered as wild quotient singularity, with $\mathbf{p}$-adic epsilon factor with sign from higher ramification group with Swan conductor N. This provide an example of $\mathbf{p}$-adic Langlands correspondence from torsion Galois representation from geometric origin(Sasaki-Einstein CY).

Correspondence from p-adic lift

There exist two class of Hodge class which violate Hodge conjecture,torsion class and non-torsion class.

To Galois complete such class to have the algebraic cycle,we need to find modular completion of intermediate Jacobian as a good moduli space/stack over $\mathbb{Z}$. This is done with representation theory by automorphic form with epsilon factor from (non-)arithmetic hyperbolic orbifold with spinor representation.

Consider Hamiltonian formal vector field with additional $\mathbb{Z}/2$ from symplectic double. This is the group of automorphism of Poisson algebra. Conjecturally, this coincide with the group of automorphism of Weyl algebra. Due to existence of wild automorphism from wild singularity(Sasaki-Einstein quiver), to restore algebraicity(linearity) we need to consider quantum/pro-$\mathbf{p}$ lift. The realization of such resonant state is from mutation(correspondence) of Sasaki-Einstein quiver. So the order of mutation is the number of correspondence. From $\mathbf{p}$-adic lift, such correspondence(spinor) is the rational point which exceed Hasse bound which is the conductor N. With correspondence, we have additional sign of N. The Frobenius destabilising sheaves is of such kinds.With such state in quantum lift, we can restore algebraicity with additional sign.

Note that the correspondence in category of motive comes from K_2 torsion, so we need to consider sympletic double by enlarge phase space. In enlarged phase space,the correspondence exist as quantum state with no classical limit. This is a symmetry of big phase space which is the Seiberg duality(S-duality). With correspondence, the RG running can be controlled connecting UV and IR without solving differential equation.

The duality in big phase space has number theory counterpart- Atkin-Lehner operation which acts on $\mathbf{p}$-adic phase space. Atkin-Lehner operation is the Poincare duality on modular curve with $\mathbf{p}^n$ conductor, provide the integrality of $\mathbf{p}$-adic Gamma function by Mahler expansion with reciprocity of the binomial number. With this we have Siegel unit from K_2 torsion of number field with $\mathbf{p}$ power ramification, also we have $\mathbf{p}$-adic measure for $\mathbf{p}$-adic zeta function. We have additional conductor N for sign of functional equation and reciprocity of $\mathbf{p}$-adic zeta function which reflect $\mathbb{Z}/2$ at infinity.

Restoring algebraicity of Hodge class can be understood as correspondence(mutation), from the symmetry of big phase space,we can restore algebraicity. So for given Hodge class which violate Hodge conjecture, we can restore algebraicity from metaplectic $\mathbb{Z}/2$ lift with sign from conductor. Having Stark-Heegner unit,we achive this from by considering modular form with additional conductor N.

From String theory side, associated with orbifold(totally real number field) we have Seiberg duality(correpondence), providing modular form with conductor to be N.

Such quantum object/resonence(spinor) satisfy reciprocity(ex.Gamma function reciprocity).

Conductor system for higher ramification group

By choice of conductor for **p**-adic zeta function of totally real number field, which satisfying **p**-adic reciprocity we have conductor system. Associated with this we have pro-**p** geometry form cusp of Hilbert modular surface of totally real number field. The pro-**p** geometry satisfy Markov triple which is **p**-adic reciprocity. With this we have complete control of conductor system by reciprocity determining higher ramification from real GW invariant of pro-**p** geometry.

For **p**-adic zeta function, we need to choose integral ideals $\mathfrak{a}, \mathfrak{c}, \mathfrak{f}$ in ideal class group for the basis of modular curve of $SL(2, F)$ with lattice $\mathcal{O}_F$. For **p**-adic zeta function we need to choose conductor system $N\mathfrak{a}, N\mathfrak{c} = l, N\mathfrak{f} = N$ with **p**-adic reciprocity,

$$E_{s,v}{}^{l}(z) =: l^{s-2}E_{s,(lv_1,v_2)}(lz) - E_{l,v}(z)$$
$$\zeta_{F,\mathfrak{f}}(\mathfrak{a}, s)^l = \frac{(s-1)!}{(2\pi i)^{s+2}} \int_{\tau}^{\tau\gamma} P(z,1)E_{s+2,v}{}^{l}(z)dz \subset \mathbb{Z}[1/l]$$

which defined to satisfy **p**-adic reciprocity,where $\zeta_{F,\mathfrak{f}}$ come from Shintani cocycle before **p**-adic lift.

$$\zeta_{F,\mathfrak{f}}(\mathfrak{a}, s)^l =: \zeta_{F,\mathfrak{f}}(\mathfrak{a}\mathfrak{c}, s) - (N\mathfrak{c})^{1-s}\zeta_{F,\mathfrak{f}}(\mathfrak{a}, s) \in \mathbb{Z}[1/l]$$

So by Galois coinvariant cocycle for $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, we have **p**-adic zeta function of totally real number field.

We need three mutually coprime conductor $N\mathfrak{f} = N, N\mathfrak{a}, N\mathfrak{c} = l$ with basis (w_1, w_2) for $\mathfrak{a}^{-1}N$ and $(\frac{1}{l}w_1, w_2)$ for $\mathfrak{c}^{-1}\mathfrak{a}^{-1}\mathfrak{f}$, with $N\mathfrak{f} = N$ for basis of (v_1, v_2) with $v_1w_1 + v_2w_2 = 1$. N arise as additional characteristic conductor in pro-$\mathbf{p}$ geometry of cusp of Hilbert modular surface by conductor N for modular tower with $N\mathbf{p}^n$ conductor, providing generalized CM with average of all isogenies by $[K : F] = N$. Then we have $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ coinvariant Shintani cocycle for $\mathbf{p}$-adic zeta function with conductor N from higher ramification group.

The Stark Heegner unit is, from the derivation of the $\mathbf{p}$-adic logarithm of zeta function at $s = 0$.

$$d\log_{\mathbf{p}} \zeta_{F,\mathbf{p}}{}^{l}(\sigma, 0) = log_{\mathbf{p}}(u_l{}^{\sigma}) = \int_{\mathbb{Z}_{\mathbf{p}}{}^2 - \mathbf{p}\mathbb{Z}_{\mathbf{p}}{}^2} \log_{\mathbf{p}}(x_1w_1 + x_2w_2) d\mu_l(1, \gamma)(v)(x) \in O_{F,\mathbf{p}}$$

Associated with this, we have pro-$\mathbf{p}$ geometry of cusp of Hilbert modular surface which is reciprocal geometry reflecting $\mathbf{p}$-adic reciprocity. We have Markov triple for reciprocal geometry realizing $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ coinvariant Shintani cocycle with N mutation realizing generalized CM by Hilbert class field K of totally real number field F.

We consider pro-$\mathbf{p}$ covering $L(p^2, pq - 1)$ of $L(p, q)$ which is cusp in Hilbert modular surface of totally real number field.Then we have pro-$\mathbf{p}$ lifting of reciprocal geometry of $L(p, q)$ satisfying Markov triple. We need to remove quadratic reciprocity relation(anomalous term) from associativity relation of Dedekind sum for $L(p, q)$ by pro-$\mathbf{p}$ structure with N step mutation from Atkin-Lehner operation(Seiberg duality).

$$\triangle(L(p^2, pq - 1), L(p_1^2, p_1q_1 - 1), L(p_2^2, p_2q_2 - 1)) \text{ with } L(p, q), L(p_1, q_1), L(p_2, q_2)$$

with non-toric Lagrangian fibration in center of $\triangle$, realising N step mutation by real GW invariant which come from correspondence, satisfying Markov triple

$$p^2 + p_1^2 + p_2^2 = 3pp_1p_2$$

$$(p,p_1,p_2)\to(p_1,p_2,p_3=p_1p_2-3p)\to\cdots\to(p_N,p_{N+1},p_{N+2})=(p,p_1,p_2)$$

The real GW invariant from correspondence is computed with mod $\mathbf{p}$ reduction, which capturing higher ramification group. So mod $\mathbf{p}$ reduction, we have realization of conductor system for $\mathbf{p}$-adic zeta function of totally real number field. This is the reciprocity of modified Eisenstein series.

The $\mathbf{p}$-adic reciprocity come from $\mathbf{p}$-adic lift of reciprocity of Dedekind sum. By pro-$\mathbf{p}$ structure, we remove relation in quadratic reciprocity describing anomalous term for quantization of $L(p,q)$.

This the $\mathbf{p}$-adic lift of convolution product(Dedekind sum) which associativity.For $f(x)=((x/p))$ and $t_2(p:p_1,p_2)=s(p,p_1/p_2)$ to be Legandre symbol, we choose $p_1=q,p_2=r$ for $L(p,q)$.

$$(f(x)*_{p_1}f(x))*_{p_2}f(x)=f(x)*_{p_1}(f(x)*_{p_2}f(x))$$

providing

$$p^2+p_1^2+p_2^2-3pp_1p_2=-(p_1p_2\cdot t_2(p:p_1,p_2)+p_2p\cdot t_2(p_1:p_2,p)+pp_1\cdot t_2(p_2:p,p_1))$$

to remove quadratic reciprocity relation(anomalous term) from associativity relation, we need to be $\mathbf{p}$-adic lift to have N step mutation with N profinite structure, by modified Eisenstein series for $\mathbf{p}$-adic measure satisfying $\mathbf{p}$-adic reciprocity. After $\mathbf{p}$-adic lift $p_1\neq q$ and representing mutated profinite structure. Then we have the metaplectic quantization of $L(p,q)$ by pro-$\mathbf{p}$ covering.

Then we have pro-$\mathbf{p}$ structure which replace Frobenius manifold structure, such that we have Galois complete description with sign from higher ramification group(Swan conductor). Note that real GW invariant satisfy rigid analytic Frobenius manifold structure by Galois group of sufficiently ramified field. We have almost linear geometry(almost profinite) but with higher ramification group by metaplectic quantization which is pro-$\mathbf{p}$ geometry. Then we have $K(\pi_1,1)$ space with $\mathbf{p}$-adic Etale class which is non-torsion mod 2 algebraic.

As a result, we have reciprocity of K_2 torsion(Siegel unit) of sufficiently ramified field providing Markov triple condition. By cup

So by real GW invariant satisfying **p**-adic reciprocity, we have higher ramification group for Galois representation of totally real number field with Stark-Heegner unit by $12N$-th power of Siegel unit.

With this we have pro-**p** geometry realization of **p**-adic uniformazation of $SL(2, F)$ Bruhat-Tits building cocycle which is **p**-adic lift of Shintani cocycle. By real GW invariant, we have **p**-adic modularity determining compactification at infinity of Hitchin moduli space for mirror of irregular Sasaki-Einstein CY by big $\mathbb{Q}$ point of infinite order from higher ramification group. This big $\mathbb{Q}$ point has generalized Faltings height as Stark-Heenger unit of totally real number field, determining pro-**p** asymptotic for compactification at infinity of Hitchin moduli space realizing wild monodromy at infinity(higher ramification group), by real GW invariant of pro-**p** geometry.

3.3 Relation to the orbifold Hurwitz number

The zeta values at s is the dimension of space of conformal block on Riemann surface of genus $g = s$. The integrality of zeta value reduce to counting parabolic vector bundle on Riemann surface, as the Euler characteristic by Riemann Roch formula. Let L be the line bundle on $\mathcal{M}$ where $\mathcal{M}$ is the space of flat connection in Riemann surface.

$$\int_{\mathcal{M}} e^{kc_1(L)} Td(\mathcal{M})$$

For $\mathcal{M}$ being V be irregular Fano surface for mesonic moduli space M(resolution of Sasaki-Einstein CY) whose dual graph of toric diagram of V has n leg with no loop, we have the conformal block for the genus 0 with n marked point Riemann surface. The Hilbert series of M is written by conformal block of irrational CFT with higher ramification group(sign) where we need Hitchin map over sufficiently ramified field from supersingular splitting over sufficiently ramified field. For irregular CY,the Hitchin map is well defined over sufficiently ramified field(at least over finite field), between Hilbert scheme over arithmetic surface and Hitchin moduli space with Jacobian fibration with mod $\mathbf{p}$ compactification with sign from higher ramification group. Then we have non-commutative Hilbert scheme of arithmetic surface as moduli space of rank 2 perverse coherent sheaves/torsion sheaves on arithmetic surface providing Mori dream space which is supersingular over sufficiently ramified field with Hitchin map to Hitchin moduli space with Jacobian fibration with mod $\mathbf{p}$(log) compactification over discriminant locus. Then we have $\mathbf{p}$-adic Ceresa cycle associated with discriminant locus(Noether Lefschets locus) with normal function as Stark-Heegner unit which compute extremal volume of cone over irregular Fano surface. The $\mathbf{p}$-adic Ceresa cycle is non-torsion mod 2 algebraic with sign determined by higher ramification group.

There are Harder Narasimhan recursion in different s which suggest that we have whole tower of Hurwitz moduli space with genus recursion. Harder Narasimhan recursion is mod $\mathbf{p}$ recursion from $\mathbf{p}$-adic reciprocity by mod $\mathbf{p}$ reduction, such that we have Verlinde formula from reciprocity of supersingular locus over sufficiently ramified field.

Then, our $\mathbf{p}$-adic modified zeta value with $\mathbf{p}$-adic modified Shintani cocycle with mod $\mathbf{p}$ integrality fit to Harder Narasimhan recursion with the boundary/degenerating contribution terms(for integrality) in recursion. Such that we have discriminant divisor(providing compactification of Jacobian for Hitchin fibration) in Hurwitz space with sign from wild monodromy of discriminant divisor by higher ramification group of singular fiber.Then we have Mordell-Weil group of singular fiber with Mordell Weil torsion determining higher ramification group. Even if Mordell-Weil rank is 0, we have $\mathbb{Q}$ point of infinite order which is non-torsion mod 2 algebraic from $\mathbf{p}$-adic origin by $\mathbf{p}$-adic Ceresa cycle with $\mathbf{p}$-adic uniformazation.

From $\mathbf{p}$-adic lift with Frobenius reciprocity, we have the orthogonal Galois representation(Hecke cycle) which is the stable but non BPS configuration-twistor spinor- with infinite wall nearby big $\mathbb{Q}$ point of infinite order. For totally real number field with non-zero Mordell-Weil rank, we have pfaffian - Grassmannian type derived equivalence providing Picard jumping locus which realise inverting motive of affine line. The class of motive of affine line have higher ramification group and come from big $\mathbb{Q}$ point of infinite order for $\mathbf{p}$-adic Ceresa cycle whose normal function is Stark-Heenger unit with sign from higher ramification group. The purely $\mathbf{p}$-adic states(real GW invariant) for higher ramification group live on the wall(discriminant locus) with Jacobian fiber with log compactification by higher ramification group. By purely $\mathbf{p}$-adic states which exceed Hasse bound,we restore algebraicity by sign.

We have moduli space of Hecke cycle(Frobenius destabilising sheaves) with orbital integral for compactification of Jacobian with higher ramification group. The stringy E polynomial of such moduli space is not polynomial indicating higher ramification group with non-maximal conductor such that we have fake projective space as moduli space.

Here we consider $\mathbb{Z}/p$ orbifold Hurwitz space. Let $\mathfrak{N}$ be integer for mod p orbifold partition with orbifold index p. We consider orbifold partition for degree $\mathfrak{N} = kp$ for $\mathbb{Z}/p$ orbifold partition. The only mod p partition $d = 0$ mod p contribute.

This is the other way to obtain pro-$\mathbf{p}$ structure with higher ramification group associated with irregular $Y^{p,q}$ singularity other than $\mathbf{p}$-adic zeta function of totally real number field. The pro-$\mathbf{p}$ structure with $K(\pi_1, 1)$ over sufficiently ramified field $\lim_{\leftarrow} \mathbb{Q}(\mu_{N\mathbf{p}^n}$ with higher ramification group is recovered from orbifold Hurwitz partition function by mod p partition where sign is obtained from orbifold recursion by embedding pro-$\mathbf{p}$ covering of $L(p,q)$ with choice of equivariant character $e^{2i\pi q/p}$ for first Chern class λ_1 of Hodge bundle on $M_{g,1}(B\mathbb{Z}/p)$.

Note that $\mathfrak{N}$ is degree of orbifold partition.Let $f(x) = ((x/\mathfrak{N})) = x/\mathfrak{N} - [x/\mathfrak{N}]-1/2$ be periodic function with $[\cdot]$ being integral part. $f(x) = (-1)^{x/\mathfrak{N}}$, if $x/\mathfrak{N} \in \mathbb{Z}$.

The k-th iterated convolution product is defined as k-th cotangent sum which is higher generalized Dedekind sum.

$$t_k(\mathfrak{N}: p,...,p) = f(x) *_p f(x) *_p \cdots *_p f(x) = \sum_{0\le i\le k} \sum_{1\le px_i \le \mathfrak{N}-1} f(x_0 + p\sum_{1\le i\le k} x_i) \tag{3.1}$$

$$= \frac{(-1)^{(k-1)/2}}{\mathfrak{N}} \sum_{i \mod 2\mathfrak{N}, i \text{ odd}} \cot(\frac{\pi i}{2\mathfrak{N}})\cot(\frac{\pi p i}{2\mathfrak{N}})...\cot(\frac{\pi p i}{2\mathfrak{N}})$$

Then $t_k(\mathfrak{N}: p,...,p)$ is the generating function of characteristic class $l_i(1,p_1,...,p_k)$ with $p_j = p$ for all $1\le j\le k$,

$$t_k(\mathfrak{N}: p,...,p) = 1 - \sum_i \frac{1}{p^k} \frac{2^{2i}(2^{2i}-1)}{(2i)!} B_{2i}\mathfrak{N}^{2i-1} l_{k-2i+1}(1,p_1,..p_k),\ \ p_i = p \text{ for all } 1\le i\le k$$

By using $\tanh(\mathfrak{N}x) = \sum_{k\ge 1}\frac{2^{2i}(2^{2i}-1)}{(2i)!}B_{2i}(\mathfrak{N}x)^{2i-1}$, and $l_i(1,p,\cdot,p)$ begin polynomial in p, we have the L genus,

$$(\frac{px}{\tanh(px)})^k(\frac{x}{\tanh(x)}) = \sum_{i\ge 0} l_i(1,p_1,...,p_k)x^i,\ p_j = p \ for\ all\ 1\le j\le k$$

Here we need evaluate infinite sum to be cotangent sum.

$$f(p/\mathfrak{N}) = ((p/\mathfrak{N})) = \frac{-1}{2\pi}\sum_{i \mod \mathfrak{N}} \sin(\frac{i\pi p}{\mathfrak{N}}) \sum_{j=-\infty, j=i \mod \mathfrak{N}}^{\infty} \frac{1}{j} = \frac{p}{\mathfrak{N}} \sum_{i \mod N} \sin(\frac{2i\pi p}{\mathfrak{N}})\cot(\frac{\pi i}{\mathfrak{N}})$$

from $\sum_{i=-\infty}^{\infty}\frac{1}{i+x} = \pi\cot(\pi x)$.

Note that convolution product satisfy associativity which can lift to Markov triple(**p**-adic reciprocity). For $f(x) = ((x/p))$

$$(f(x) *_{p_1} f(x)) *_{p_2} f(x) = f(x) *_{p_1} (f(x) *_{p_2} f(x))$$

providing

$$p^2+p_1^2+p_2^2-3pp_1p_2 = -(p_1p_2\cdot t_2(p:p_1,p_2)+p_2p\cdot t_2(p_1:p_2,p)+pp_1\cdot t_2(p_2:p,p_1))$$

Markov triple $p^2 + p_1^2 + p_2^2 - 3pp_1p_2 = 0$ with vanishing of right hand side form dedekind sum, we need to lift to modified Eisenstein series with **p**-adic reciprocity to N step mutation for reciprocal pro-**p** geometry of cusp in Hilbert modular surface over finite field in subsection.**??**.Note that the Dedekind sum represent reciprocal geometry of Lens space but we need **p**-adic lift by modified Eisenstein series for pro-**p** geometry of Lens space. As pro-**p** geometry, reciprocal geometry of Lens space lift to pro-**p** structure with additional conductor from higher ramification group determining sign.

Then we have Galois complete description with sign refined spin CS invariant of $L(p, q)$ from pro-**p** covering $L(p^2, pq-1)$ by Stark-Heegner unit of totally real number field.

This govern real GW invariant by reciprocity which come from higher ramification group of pro-**p** geometry. We recover the **p**-adic lift of Dedekind sum from higher generalized Dedekind sum,such that we have modified Eisenstsein series $E_k^*(\tau)$ and associated weight k modular form $F_k^*(\tau)$ which can further reduce to weight 2 modular form by Stark-Heegner unit.

$l_k(1, p_1, ..., p_k) =: l_k(p_1, ..., p_k)$ with $p_j = p$ for all $1 \leq j \leq k$ is L-class of the homogenous space in symmetric polynomial(Pontryagin class).

Other $l_i(1, p_1, ..., p_k) =: l_i(p_1, ..., p_k)$ with $p_j = p$ for all $1 \leq j \leq k$ with $i \neq k$ has the enumeration meaning in the (orbifold) Gromov Witten theory.

We fix notation $l_i(1, p_1, ..., p_k)$ with $p_j = p$ for all $1 \leq j \leq k$ to be $l_i(p, ..., p)$.We also fix the length of argument of $l_i(p, ..., p)$ to be k.

$l_k(p, ..., p) = L_{k/2}(\mathbf{p}_1, ..., \mathbf{p}_{k/2})$ with $\mathbf{p}_i$ is the i-th symmetric polynomial(Pontryagin class) in variable p^2. Since we get the polynomial $l_i(p, ..., p)$ in p^2,so l_i has mod p structure(it is the power series in p^2).Note that only $d = 0 \mod p$ partition contribute in $\mathbb{Z}/p$ orbifold Hurwitz covering.

$l_i(p, \cdots, p)$ can be considered as the motivic characteristic class in orbifold Hurwitz space.It can be the almost real class which is is not algebraic but non-torsion mod 2 algebraic by sign from higher ramification group. For that,we need to consider Galois coinvariance to recover algebraicity by metaplectic spin cover with the Hilbert class group.

We will consider the elliptic lift of L-class, to compare a class in orbifold Hurwitz space. Here the elliptic lift means the two parameter Elliptic genus with modularity. The modularity comes from the genus expansion of the Gromov Witten theory. This is done by considering equivariant compactification of $M_{g,n}(B\mathbb{Z}/p\mathbb{Z})$.

Let the orbifold ψ class be first Chern class line bundle on universal curve where line bundle is the the pull back of the dualizing sheaf on the $\overline{M_{g,n}}(B\mathbb{Z}/p\mathbb{Z})$ to the universal curve. ϕ class is the cycle class in $M_{g,n}(B\mathbb{Z}/p\mathbb{Z})$. There are the κ class on the $\overline{M_{g,n}}(B\mathbb{Z}/p\mathbb{Z})$ from the orbifold ψ class (Witten's p-spin class). For given forgetting the $n+1$-th marking map $\pi : M_{g,n+1} \to M_{g,n}$,

$$\kappa_j = \pi_*({\psi_{n+1}}^{j+1}) \in CH(\overline{M_{g,n}}(B\mathbb{Z}/p\mathbb{Z}))$$

With $n = 1$, κ_j class is degree $2g-1$ Hodge class on $CH(\overline{M_{g,n}}(B\mathbb{Z}/p\mathbb{Z})$ pullback of ϕ class on Hodge bundle. The ratio of the equivariant Euler characteristic of the Hodge bundle and the line bundle is the integrand of the Hodge integral.The Hodge integral gives the orbifold genus $\hat{A}$ on $\overline{M_{g,n}}(B\mathbb{Z}/p\mathbb{Z})$.

After the elliptic lifting of the $\hat{A}$ class, we get the Witten genus with the elliptic function(the Weierstrass $\mathcal{P}$ function).Then there are huge room to embed the wild p-torsion in supersingular locus over sufficiently ramified field by specialization of Siegel unit.

In this way we can compare two class, L class from the $t_k(\mathfrak{N} : p, ..., p)$ and $\hat{A}$ class from the orbifold Hurwitz partition function. They are particular two limits of an elliptic genus,representing wild p-torsion class. In the paper [11], we relate $t_k(\mathfrak{N} : p, ..., p)$ to Massey product of the Stark-Heegner unit of totally real number field by k-th Milnor homomorphism of totally real number field(associated with wild $\mathbb{Z}/p$ singularity) with coefficient in μ_p or sufficiently ramified field.

For the comparison, consider $t_k(\mathfrak{N} : p, ..., p)$, which corresponds to degree $\mathfrak{N} = d$ with orbifold 1-point Hurwitz partition function. With the orbifold ψ class which is first Chern class of pull back line bundle on universal curve with the equivariant Chern class on the Hodge bundle $\lambda_i = c_i(\mathcal{E}) \in \overline{M_{g,1}}(B\mathbb{Z}/p\mathbb{Z})$. Here we first consider trivial equivariant character $e^{2i\pi/p}$,associated with $\mathbb{Z}/p$ orbifold singularity with simple higher ramification group with maximal conductor.

$$\frac{1}{p}+\sum_{g>0}\sum_{0\leq i\leq g} x^{2g}k^i \int_{\overline{M_{g,1}}(B\mathbb{Z}/p\mathbb{Z})} {\psi_1}^{2g-2+i}\lambda_{g-i} = \frac{1}{p}(\frac{px}{\sinh(px)})^k(\frac{x}{\sinh(x)}) =: \frac{1}{p}\hat{A}(p^2) \tag{3.2}$$

We use notation for $\hat{A}(p^2) = (\frac{px}{\sinh(px)})^k(\frac{x}{\sinh(x)})$ for the orbifold $\hat{A}$ genus. Its i-th coefficient is the $\hat{A}(p^2)$ is the polynomial in p^2 variable. This compute the equivariant index of $\overline{M_{g,1}}(B\mathbb{Z}/p\mathbb{Z})$.

$$e(\overline{M_{g,1}}(B\mathbb{Z}/p\mathbb{Z})) = \sum_{2g\leq k\leq 6g-3} \frac{(-1)^{k+1}}{2k} h_g(k) = \zeta_{F_0}(1-2g), \quad F_0 = \mathbb{Q}(\sqrt{-p})$$

with

$$l_g(k) = \frac{(2k)!}{(k+1)!(k-2g)!} l_{2g}(p,...p), \ \ l_{2g}(p,...p) = \text{coeff. of } x^{2g} \ in \ ((\frac{xp}{\tanh(xp)})^k(\frac{x}{\tanh(x)}))$$

Two classes are related by

$$l_g(k) = \sum_{i \geq 0} \begin{pmatrix} 2k \\ i \end{pmatrix} \sum_{j \geq 0} \begin{pmatrix} k-i \\ j \end{pmatrix} h_g(k-i-j)$$

Here due to choice of trivial character $e^{2i\pi/p}$ of equivariant Chern class of Hodge bundle,$F_0 = \mathbb{Q}(\sqrt{-p})$ by imaginary quadratic field with higher ramification group for maximal conductor. For general character $e^{2i\pi q/p}$ we have $F = \mathbb{Q}(\sqrt{4p^2 - 3q^2})$ with higher ramification group for non-maximal conductor determining real GW invariant. Then the orbifold GW invariant(real GW invariant) of non-trivial equivariant character is expressed as **p**-adic zeta function of totally real number field F with sign from higher ramification group.

By the $\mathbb{Z}/p$ orbifold Hurwitz covering with sign from higher ramification group, we have pro-**p** asymptotic which is equivalent to pro-**p** Hurwitz covering of totally real number field $F = \mathbb{Q}(\sqrt{4p^2 - 3q^2})$ with Galois group of sufficiently ramified field. The pro-**p** Hurwitz covering is defined by pro-**p** fundamental group of $\mathbb{Z}/p$ with higher ramification group, which compute **p**-adic zeta function of totally real number field by counting of field extension of F which corresponds to p-spin curve counting.

Then we have pro-**p** geometry of $L(p, q)$ for pro-**p** conifold geometry with real GW invariant which we consider from Galois cohomology of totally real number field with coefficient in μ_p or sufficiently ramified field. By local-global compatibility of Galois representation, we only need to consider real GW invariant from higher ramification group determining Swan conductor.

The expansion with the binomial coefficient is the particular use of the Mahler expansion which we use to show the sign refined integrality of the real

Gromov Witten invariant by mod $\mathbf{p}$ expansion. As seen we need whole tower of ϕ orbifold class and κ orbifold class, which comes from pro-$\mathbf{p}$ structure from orbifold recursion providing pro-$\mathbf{p}$ asymptotic with almost real structure.

We will consider Eisenstein measure with $\mathbf{p}$-adic reciprocity which give the integral zeta value of totally real number field at negative argument. For negative zeta value,the associated geometry is $\overline{M_{g,1}}(B\mathbb{Z}/p\mathbb{Z})$, where we need 1-puncture for $\mathbf{p}$-adic reciprocity. $e(\overline{M_{g,1}}(B\mathbb{Z}/p\mathbb{Z}))$ is obtained recursively by orbifold Hurwitz recursion which is part of $\mathbf{p}$-adic reciprocity of Eisenstein measure which we constructed before. The map between modified Eisenstein series and orbifold Hurwitz partition function is as follows. We equate orbifold Hurwitz partition function with sign with Stark-Heegner unit(Kronecker limit) of $\mathbf{p}$-adic zeta function of totally real number field, where $\mathbf{p}$-adic zeta function is obtained from modified Eisenstein series.

The direct map between two is from $t_k(\mathfrak{N}, p, \cdots, p)$ with characteristic class $l_i(p, \cdots, p)$ with elliptic genus whose limit is $(\frac{px}{\tanh(px)})^k \frac{x}{\tanh(x)}$ from k-th cotangent sum(higher generalized Dedekind sum) by k convolution product which differ from generalized Dedekind sym $D_{i,j}$ for modified weight k modular form F_k^* from Eisenstein series. But both provide same motivic characteristic class $l_i(p, \cdots, p)$. The relation between two generalized Dedekind sum from eq.(3.2) computing Euler characteristic of $\overline{M_{g,1}}(B\mathbb{Z}/p\mathbb{Z})$ which realize the reciprocity in supersingular locus over sufficiently ramified field. For totally real number field F we need to choose non-trivial equivariant character of orbifold Hodge bundle.

By supersingular decomposition, we can reduce weight 2 modular form associated with Stark-Heegner unit, reflecting recursive construction of Milnor homomorphism of Galois cohomology over sufficiently ramified field by successive cup product of Stark-Heegner unit. This is equivalent to $t_k(\mathfrak{N}, p, \cdot, p)$ is obtained recursively with k-th cotangent sum(higher generalzed Dedekind sum) by

k convolution product. The convolution product represent cup product of Stark-Heegner unit with specialization by elliptic genus. Note that Stark-Heegner unit represent class of motive of affine line with higher ramification group, which is global zero divisor from $\mathbf{p}$-adic origin. [3] It govern all Galois complete global invariant like all Milnor homomorphism.

We remark that for Hurwitz partition function, we do not need auxiliary prime $\mathbf{p}$, instead we have q expansion. But for proof of the integrality of GW invariant(zeta values), we need mod $\mathbf{p}$ structure(with $\mathbf{p} = 1 \mod p$) and Mahler expansion. This means that,there are spinor class(correspondence) from higher ramification group for $\mathbb{Z}/2$ sign at arithmetic infinity by open(real) GW invariant which cancel divergence with remaining Swan conductor from $\mathbf{p}$-adic reciprocity.With such orientifold contribution(spinor class), the sign refined Hodge conjecture for $\overline{M_{g,1}}(B\mathbb{Z}/p\mathbb{Z})$ can be proven by spinor class(correspondence) with $\mathbf{p}$-adic uniformazation providing pro-$\mathbf{p}$ limit[11]. Such spinor class is non-torsion mod 2 algebraic class lies in $CH^2(\overline{M_{g,1}}(B\mathbb{Z}/p\mathbb{Z}))/2$ and vanishes on $CH^2(\overline{M_{g,1}}(B\mathbb{Z}/p\mathbb{Z}))$ corresponds to Arason invariant for $\mathbf{p}$-adic Ceresa cycle. Note that we consider $\overline{M_{g,1}}(B\mathbb{Z}/p\mathbb{Z})$ as Brauer Severi varieties over sufficiently ramified field for Galois cohomology of totally real number field F. So we have Arason invariant(non-torsion mod 2 algebraic class) which is the spin refined Chern-Simons invariant. We have additional sign from correspondence(conductor N).

They are $\mathbf{p}$-adic Hodge class lives in $\mathbf{p}$-adic Etale covering of $\overline{M_{g,1}}(B\mathbb{Z}/p\mathbb{Z})$ parametrizing $\mathbf{p}$-adic Hurwitz covering map($\mathbf{p} = 1 \mod p$) for $\mathbf{p}$-adic Simpson correspondence. Due to discrete torsion from Seiberg duality of finite order on

[3]By inverting motive of affine line with higher ramification group from Mordell-Weil torsion,with Mordell-Weil rank 0, we have sign of pro-$\mathbf{p}$ limit of orbifold partition function. With non-vanishing Mordell-Weil rank, we can have pfaffian-Grassmanian derived equivalent pair for totally real number field with Mordell-Weil rank with Mordell-Weil torsion.

$\mathbb{Z}/p$ orbifold(considered as wild singularity with higher ramification group), we can capture such **p**-adic Hodge class which is not algebraic(spinor class), but non-torsion $\mod 2$ algebraic.

3.4 Invariants of $L(p,q)$ and p-adic Landau Ginzburg model mirror

We remark that $\mathcal{F}(u_k, P, Q, v)$ is constructed by generalized Dedekind cocycle associated with $s(q,p)$ and we obtain Stark unit. By the cup product of two modular(Stark) unit, we can recover **p**-adic epsilon factor(Chern-Simons invariant with sign) as regulator map.

In our model, there are mirror curve of Sasaki-Einstein manifold $C(Y^{p,q})$, mirror curve of Lens space $L(p,q)$ and the character varieties of $L(p,q)$ double bridge knot. Due to the irrationality of Reeb vector for irregular Sasaki-Einstein manifold,we need to consider metaplectic lift to have additional sign. The sign can be seen from mirror curve(character varieties) of Lens space.

Because we are in characteristic $\mathbf{p} = 1 \mod p$, we only consider singularity structure at degeneration limit. So we do not need continued fraction expansion, but with $\mathbb{Z}/2$ sign from conductor.

$$P(x,y) = (x^p y^q - 1)(x-1) + d_p + \sum_{i<p} d_i x^i y^{q-[(p-i)q/p]}$$

where $[\cdot]$ means the integral part. By setting $d_i = 0$ we obtain singular curve at conifold point.

For double bridge knot complement, the character varieties is, at the degenerate limit which is double twisted knot,

$$P(x^1, y) = y^p x^q - y^p x^q,$$

in quantum torus $k(x^{\pm}, y^{\pm})/(xy - q^2 yx)$ with involution $\sigma : (x^p y^q) \mapsto (x^{-p} y^{-q})$ with root of unity q.

We consider a Fermat curve over finite field which do the role mirror curve of Lens space.

$$C = \{x^q(1-x)^r(-1)^s = y^p, q+r+s = 0 \mod p\}$$

over characteriatic $\mathbf{p} = 1 \mod p$

This is possible, since at conifold limit,in mirror curve(character varieties), we have only singular curve which do not depends on d_i. And the $\mathbf{p}$-adic tau function for Fermat curve C also only depends in (p,q) over finite field $\mathbf{p} = 1 \mod p$. We only see singularity structure, but it is enough to obtain additional $\mathbb{Q}$ point at arithmetic infinity with sign(root number) of Gauss sum.

Now we consider $\mathbf{p}$-adic τ function for fermat curve C. The splitting constant of $\mathbf{p}$-adic τ function determine the conductor N and the sign of wave function.

This cocycle can be considered as elliptic lifting of quadratic Legandre symbol

$$(\frac{q}{p})_2 = (-1)^{lk_2(q,p)}$$

with p-th Legandre symbol We determine sign of quadratic Legendre symbol, by wild ramification at $\mathbf{p} = 1 \mod p$. $e = N$ determine sign of quadratic Legandre symbol by $(-1)^e$,such that we have critical value of CS invariant $L(p,q)$ with sign Hecke character with sign from root number. $e = \begin{pmatrix} -q/p \\ (\mathbf{p}-1)/p \end{pmatrix}$, $\mathbf{p} = 1 \mod p$ are two $\mod p$ prime, and ζ_p is the primitive p-th roots of unity.

p-th Legandre symbol is obtained by considering mirror in supersingular locus over sufficiently ramified field, such that, it is the multiplicative character for Gauss sum for mirror curve. By localizing Lens space $L(p,q)$ partition function at p-th roots of unity, we recover CS critical point from p-th Legandre symbol.

The choice of e comes from splitting constant of Fermat quotient $C =$

$\{x^q(1-x)^r(-1)^s = y^p, q+r+s = 0 \mod p\}$ over characteriatic $\mathbf{p} = 1 \mod p$. The e is the splitting $\mathbf{p}$-unit for the construction of Dwork operator for $\mathbf{p}$-adic τ function of Fermat quotient C[7].

Dwork operator is follows,fix $\pi^{\mathbf{p}-1} = -\mathbf{p}$ for local parameter T of curve C.

$$h(T) = e^{\pi((\epsilon T)-(\epsilon T)^{\mathbf{p}})}$$

with local parametrization of curve by T whose $\mathbf{p}$-th power only can split with splitting constant e- $\mathbf{p}$-adic Birkoff factorization work only for $h^{\mathbf{p}}$ with $h = e^{\pi(T-T^{\mathbf{p}})}$. For local parametrization of curve by t by expansion of x,

$$T^{\mathbf{p}-1} = (-1)^{\frac{(\mathbf{p}-1)q}{p}} t^{\frac{(\mathbf{p}-1)}{p}} (1-1/t)^{\frac{(\mathbf{p}-1)q}{p}} = te_+(t) + e + e_-(t) \in \mathbb{Z}[t, t^{-1}]$$

where $(\epsilon)^{\mathbf{p}-1} = 1/e$.

We identify the splitting constant e with conductor N which determine sign for $\mathbf{p}$-adic zeta function.

$(\frac{q}{p})_2 = (-1)^{lk_2(q,p)}$ determine the sign of analytic torsion τ_n of $L(p,q)$. For spin CS invariant, we need to consider metaplectic double of it which is recovered by splitting constant e or $\mathbf{p}$-adic τ function with almost real structure by sum over generalized CM type by Hilbert class field K of totally real number field F.

$$CS_n = n^2(\frac{q^*}{p}), \quad qq^* = 1 \mod p$$

$$sign(\tau_n) = 1,\ for\, nq^* < p/2,\ \ sign(\tau_n) = 1,\ for\, nq^* < p/2,\ \ \tau_n = \frac{32}{p}|\sin(\frac{2\pi i q^* n}{p})\sin(\frac{2\pi i n}{p})|$$

With this, we can define $\mathbf{p}$-adic Landau Ginzburg model by pro-$\mathbf{p}$ geometry over sufficiently ramified field.

$$G_p(\omega) = \sum_{n \mod \mathbf{p}} (\frac{n}{\omega})_p \exp(2i\pi\frac{n}{\mathbf{p}}) = \sum_{n \mod \mathbf{p}} \exp(2i\pi\frac{n^p}{\mathbf{p}}) \quad \in \mathbb{Q}(\mu_{p\mathbf{p}}) \qquad (3.3)$$

where we use the splitting of prime $\mathbf{p} = 1 \mod p$ with $\omega\omega^{'} = \mathbf{p}$ in $\mathbb{Z}[\mu_p]$, $\omega = 1 \mod p$.The splitting condition for $\mathbf{p}$ comes from Complex multiplication by $\mathbb{Q}(\sqrt{-p}$ on Fermat Jacobian. We need this for Bianchi manifold with $L = \mathbb{Q}(\sqrt{-p})$ with $\mathbf{p} = 1 \mod p$ with $\omega\omega^{'} = \mathbf{p}$. The Hecke character $(\frac{n}{\omega})_p$ has additional sign from global reciprocity of $\mathbf{p}$-adic Gamma function reciprocity. We consider Hecke character which is p-Legandre symbol as modular critical value which satisfy $\mathbf{p}$-adic reciprocity. The sign of Hecke character is determined by splitting constant $e = N$, root number of Fermat quotient.

With $\mathbf{p}$-adic Landau Ginzburg model, we can describe the spin CS invariant with additional sign from root number of Fermat curve. It is the CS invariant at level $r \mid p$ where usual quadratic Gauss sum is not well defined. The additional sign capture the wild ramification data from root number. We consider Fermat curve as Heegner curve in Hilbert modular surface over finite field of characteristic $\mathbf{p} = 1 \mod p$ which capture singularity structure only. Note that as wild singularity, only the singularity structure(rigid analytic structure for degeneration stratification of Heegner curve/mirror curve of $L(p, q)$) is important. The degeneration strata of Heegner curve/mirror curve of $L(p, q)$ is determined by homothety class of $\mathbf{p}$-torsion point on Jacobian on Fermat curve C, determining mod $\mathbf{p}$ reduction of rigid analytic structure, by splitting constant $e = N$.

$G_p(\omega)$ is the period of elliptic curve E in the Jacobian of Fermat quotient C of degree p over characteristic $\mathbf{p} = 1 \mod p$. This is the product of $\mathbf{p}$-adic Gamma function[10] satisfying reciprocity.

This provides $\mathbf{p}$-adic Landau Ginzburg model global mirror to pro-$\mathbf{p}$ geometry of local CY $T^*L(p, q)$ with sign considered as resolution of the wild singularity, which is supersingular geometry over sufficiently ramified field.

At characteristic $\mathbf{p}$, we only see the ramification structure which is $e = N, \mathbf{p}$ with sufficiently ramified field $\lim_{\leftarrow} \mathbb{Q}(\mu_{N\mathbf{p}^n})$ for spin CS invariant of $L(p, q)$

with sign. The sign is determined by Hilbert reciprocity relating finite place $\mathbf{p}$ with $\mathbb{Z}/(\mathbf{p}-1)$ Galois action with $p \mid \mathbf{p}-1$ and infinite place ∞ with $\mathbb{Z}/2$ Galois group action for sum over generalize CM type by Hilbert class field K of totally real number field F.

The sign comes from $\mathbf{p}$-adic correspondence(which is equivariant analytic torsion which is non-trivial by higher ramification group with non-maximal conductor) which we can recover from conductor $e = N$ which is the splitting constant $e = N$ which do the role for uniforming constant of $\mathbf{p}$-adic τ function other than $\mathbf{p}$.

From this we obtain the additional pole in motivic zeta function by $\mathbf{p}$-adic epsilon factor. Note that usual quadratic Gauss sum compute sign by analytic torsion with CS critical value away from p-th root of unity. At p-th root of unity,from $\mathbf{p}$-adic Gauss sum, we can recover the sign from root number which is the splitting constant.

The sign can be obtained by Hida's ordinary limit by $\mathbb{Z}/2$ at arithmetic infinity from tower of modular curve by Hilbert reciprocity of sufficiently ramified field. Likewise,in motivic zeta function of cyclic quotient singularity, it arise as $\mathbf{p}$-adic epsilon factor for functional equation as Greenberg $\mathcal{L}$ invariant. Additional $\mathbb{Z}/2$ at infinity in pro-$\mathbf{p}$ limit implies that we have additional rational point($\mathbf{p}$-adic state) which exceed Hasse bound,from which we determine the sign. This is the phenomenon in supersingular locus over sufficiently ramified field with non-maximal conductor(for totally real field).

But unlike usual CS invariant,we have sign refined spin CS invariant with sign from Swan conductor[20]. For that, we use the definition of Arason invariant for sign refined spin CS invariant.

The $\mathbf{p}$-adic Landau Ginzburg(LG) model is the finite analog of the complex

LG path integral.For monomial $f(x)$, two LG model is defined by,

$$\int \exp(2i\pi t f(x))dx, \qquad \sum_{x\in\mathbb{F}_{\mathbf{p}}} \exp(2i\pi f(x)/\mathbf{p})$$

Note that for complex path integral, we consider 1-parameter family of $f_t(x) = tf(x)$ satisfying differential equation with respect the variable t. The $1/t$ do the role for $\mathbf{p}$.

We are interested in $\hbar = 1/t$ and $\mathbf{p}$ torsion for wild ramification,in 1 parameter family along $\hbar$ and the $\mathbf{p}$-adic direction, with wild monodromy at infinity of $\hbar \in \mathbb{A}^1$ for sign. The wall crossing(Picard-Lefschetz) transformation for complex LG model is realized as wall crossing from wild ramification for $\mathbf{p}$-adic LG model, by wild monodromy at infinity,realizing $\mathbf{p}$-adic uniformazation. It has the Stokes phenomenon in complex LG model and the ideal class group action in $\mathbf{p}$-adic LG model determining $\mathbf{p}$-adic Ceresa cycle by Stark-Heegner unit of totally real number field. By ideal class group action, we have realization of Seiberg duality which is Atkin-Lehner operation. By Stokes phenomenon, we have realization non-holomorphic foliation determining pro-$\mathbf{p}$ asymptotic from Seiberg duality(Atkin-Lehner operation). For general totally real number field, we have generalized Seiberg duality which is not birational but derived equivalence.

The matrix factorization of complex LG model is realized $\mathbf{p}$-adic LG model at arithmetic infinity with $\mathbb{Z}/2$ Galois action realizing spinor state(correspondence) which is supersingular $\mathbb{Q}$ points over sufficiently ramified field.

In p-th Gauss sum, wild ramification can seen from intersection of p-torsion and $\mathbf{p}$ torsion in Jacobian of Fermat quotient C. With this we obtain $\mathbf{p}$-unit e of C. e is the splitting uniformizing parameter for $\mathbf{p}$-adic τ function of curve C. By $\mathbf{p}$-adic τ function of C $h(T)$, we obtain a point of infinite order on Jacobian which can be completed to $\mathbb{Q}$ point in elliptic curve E with complex multiplication by CM field K. By previous construction,with matching with conductor $N\mathfrak{f} = N = e$, the point of infinite order on $J(C)$ is equal to Stark-Heegner point on E-the simple factor of Jacobian with complex multiplication by Hilbert class field(CM field) K. We consider E as supersingular elliptic curve over sufficiently ramified field with modular parametrization by modular curve with $\mathbf{p}$ power conductor. Then by Stark-Heegner point, we have $\mathbf{p}$-adic Shintani cocycle on modular curve with $\mathbf{p}$ power conductor as $SL(2, F)$ Bruhat-Tits building cocycle.

Note that from the Fermat curve C over finite field, we obtain averaged Faltings height of CM field by Hilbert class field of totally real number field. For the comparison with $\mathbf{p}$-adic zeta function of real quadratic number field, we need to consider average over CM type to have almost real structure. After that we can define additional sign from non-maximal conductor. We expect that this additional sign comes from conductor N which we equate with splitting constant of $\mathbf{p}$-adic τ function of C,since to obtain $\mathbf{p}$-adic τ function with associated Stark-Heegner unit, we average over all CM type of Hilbert class field of totally real number field from the mod $\mathbf{p}$ partition by reciprocity of $\mathbf{p}$-adic Gamma function.

So we have following Theorem.

Theorem 10 *We have Stark-Heegner unit of* $\mathbf{p}$*-adic zeta function of real quadratic number field* F *and Hilbert class field* K *with sign from conductor* N*, which coincide with averaged Faltings Height of CM field* K*.*

The sign for both come from additional conductor N *obtained by* $\mathbf{p}$*-adic* τ *function of Fermat curve* C *by splitting constant* e*. By the Stark-Heegner unit with sign from non-maximal Swan conductor,we have the almost real structure.*

The point of infinite order(Stark Heegner point) in modular Jacobian of Fermat quotient is a particular wild ramification phenomenon for wild quotient singularity,because it encode the CS invariant by p-th linking number of the ramification prime $e = N, \mathbf{p}$. This can be seen as the non-Abelian central extension by conductor $N, \mathbf{p}$ for CS partition function on $L(p, q)$ which we equate the pro-$\mathbf{p}$ Hurwitz covering totally ramified over $\mathbf{p}$ and tamely ramified over N for $\mathbf{p}$-adic Galois representation.

We first start from cyclic quotient singularity whose link is $L(p, q)$ with pro-$\mathbf{p}$ covering $L(p^2, pq-1)$ for real quadratic field F and Hilbert class field K of order $[K : F]$ for averaged CM type for generalized Faltings height.

There are associated $\mathbf{p}$-adic cycle in triple product of modular curve of elliptic curve E with conductor N from triple product of $\mathbf{p}$-adic L function of real quadratic number field.This is the $\mathbf{p}$-adic Ceresa cycle which is the Arason invariant of quadratic form with $GL(\mathbf{p})$ at Hida's ordinary limit.At tropical limit with maximal conductor,we have supersingular Artin Schreire curve of special type (Hermitian curve) $C_H = \{x^{\mathbf{p}} - x = x^{\mathbf{p}+1}\}$ over characteristic $\mathbf{p}$ which is equivalent to Fermat curve $x^{\mathbf{p}+1} + y^{\mathbf{p}+1} = 1$ of degree $\mathbf{p}+1$ over field $\mathbb{F}_{\mathbf{p}^2}$. The dual graph quiver of singular fiber of arithmetic surface $tot(C_H \to C_H/G \simeq \mathbb{P}^1)$ with $G = Aut(C)$ contains $\mathbf{p}$ number of $L(\mathbf{p}, \mathbf{p}-1)$ graph with 1 wild ramification vertex[20]. This curve is the rank 2 Deligne-Lustigz curve with Swan conductor $\mathbf{p}^2 - 1 = \mathbf{p}(\mathbf{p}-1) + \frac{1}{\mathbf{p}}\mathbf{p}(\mathbf{p}-1)$ from $\mathbf{p}^2 - 1$ 1-dimensional Frobenius eigenspace. For maximal conductor, we have imaginary quadratic field by $\mathbb{Q}(\sqrt{-p})$.

For general Fermat quotient, we have supersingular Abelian varieties over sufficiently ramified field with non-maximal conductor associated with totally real number field $\mathbb{Q}(\sqrt{4p^2 - 3q^2})$.

The infinite generation of Griffiths group by codimension 2 cycle in Chow group of the triple product of modular curves with $\mathbf{p}$ power conductor is generated by $\mathbf{p}$-adic Ceresa cycle from the Bloch Ogus map for K_2 whose image lies in mod 2 second Chow group quotient by p realising Cliffold bundle for quadratic form of degree p. There exist the specific $\mathbf{p}$-adic Etale class from the Etale path in $\mathbf{p}$-adic fundamental group which comes from non-Abelian Galois representation from $\mathbf{p}$-adic correspondence. We have $\mathbb{Z}/2$ sign at infinity from Stark-Heegner unit from K_2 torsion of sufficiently ramified field (Siegel unit). For characteristic 0(by passing to the Hida's ordinary limit), this reduce to orthogonal Galois representation(spinor representation) which is almost real cycle with $\mathbb{Z}/2$ sign.

We obtain the irrational Reeb vector with irrational volume from the specialization of Stark Heegner unit in K. The irrational extremal volume w satisfying algebraic equation and extremize the Mabuchi volume functional. As the algebraicity of the Height function(Stark Heegner unit) by global reciprocity, we can see the sign refined algebraicity of period of elliptic curve with CM lift by Hilbert class field of totally real number field by averaging CM type $[K : F]$ - Deligne period conjecture[4]. We have sign refined algebraicity of supersingular elliptic curve over sufficiently ramified field in supersingular locus in Shimura varieties over sufficiently ramified field.

This is the algebraitity for generating series for the motivic DT invariant for quiver known from [18],[19] which comes from the reciprocity of the Stark unit from K_2 torsion of cyclotomic number field $\lim_{\leftarrow} \mathbb{Q}(\mu_{N\mathbf{p}^n})$.

We also obtain integral zeta value of totally real field at negative argument from the expansion of the the orbifold Hurwitz partition function. We equate $\mathbf{p}$-adic expansion of zeta value with integral coefficient from expansion orbifold Hurwitz partition function computing real GW invariant. The sign of orbifold Hurwitz partition function of functional equation is determined by

[4]There exist Painleve 6-th differential operator(Picard-Fuchs equation) with irrational parameter(not algebaic) - which annihilate the period of mirror curve of irregular Sasaki-Einstein CY and Jacobian of it. We write the period as the product of indefinite Gamma function. We obtain this from $\mathbf{p}$-adic zeta function of real quadratic number field and Stark-Heegner unit of it.

pro-**p** asymptotic with conductor from mod **p** structure of Hurwitz recursion.

The Eisenstein cocycle using continued fraction and Shintani cone decomposition gives the motivic measure for moduli of quiver representation from toric diagram of $Y^{p,q}$. Then it gives a way to construct Eisenstein cocycle and constant term of it(Stark unit) of real quadratic number field using derived category of representation of the Sasaki-Einstein quiver[15], such that the **p**-adic reciprocity from **p**-adic measure provide the Seiberg duality cascade for $Y^{p,q}$ quiver. The Seiberg duality is essentially the Atkin-Lehner operation in tower of modular curve with **p** power conductor, by N-step mutation defining pro-**p** asymptotic with additional conductor N. It gives the Poincare duality on modular curve with **p** power conductor, and **p**-adic reciprocity determining reciprocal geometry which is pro-**p** asymptotic determined by N-step mutation/Seiberg duality determined by mod **p** structure of orbifold recursion.

By the **p**-adic epsilon factor for functional equation formed by conductor, we obtain the sign for spin refined CS invariant from pro-**p** covering of Lens space and Bianchi manifold as the sign of wild ramification(Swan conductor).

3.5 Swan conductor of curve and p-adic epsilon factor.

Now we consider Stark unit from the constant term in Eisenstein series(Kronecker limit formula) which realize **p**-adic epsilon factor. In our construction of Stark unit with chosen Shintani cocycle(cusp of modular curve) for $L(p,q)$, we need the condition $\mathbf{p} = 1 \mod p$.

p-homothety class of **p**-torsion gives the p different Hecke character, which do the role for Dirichlet character ϵ which is the restricted of $\theta : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) = \lim_{\leftarrow} \mathbb{Z}/\mathbf{p}^n \times \mathbb{Z}/(\mathbf{p}-1) \to \mathbb{Q}$ on $\mathbb{Z}/(\mathbf{p}-1)$.We choose the additional conductor N to be splitting constant e. There exist p distinct Hecke character $\chi(i)$.Later for Bianchi manifold which is hyperbolic 3 manifold with knot complement(of

Lens space knot), we need non-algebraic Hecke character with additional $\mathbb{Z}/2$ for sign in supersingular locus in Siegel modular varieties over sufficiently ramified field(metaplectic lift). The non-algeraic Hecke character come from Tits motivic measure for irregular Hodge filtration with irrational Frobenius eigenvalue.

The Hecke character of Fermat curve for pro-$\mathbf{p}$ mirror dual for Lens space with pro-$\mathbf{p}$ fundamental group(pro$\mathbf{p}$ covering of Lens space) is defined as follows. We can associate pro-$\mathbf{p}$ covering of Lens space $L(p,q)$ to Fermat quotient over finite field over characteristic $\mathbf{p} = 1 \mod p$ as mirror curve(as rigid analytic curve with log completion), since we consider wild ramification.

Due to mirror(S-duality),in our model, degree of Fermat curve arise as Lens space index p which is Langlands dual of usual construction of Iwasawa polynomial(Alexander polynomial) whose support is character varieties for 3 manifold as Galois deformation space.In supersingular locus of Shimura varieties over sufficiently ramified field, we have 3d-3d duality from Level-Rank duality.

We have duality between inertia(Weil-Deligne group) and absolute Galois(Frobenius) providing duality in supersingular locus over sufficiently ramified field[5].

For the period of Fermat quotient $C = \{y^p = x^q(1-x)^r(-1)^s, q+r+s = 0 \mod p\}$, we have the Jacobi sum $J_{q,r}$ which is the finite analog of beta function for the Steinberg element in K_2.

$$G_{\mathbf{p},q} = \sum_{n \mod \mathbf{p}} (\frac{n}{\mathbf{p}})_p{}^q \exp(2i\pi\frac{n}{\mathbf{p}}) \ \in \mathbb{Q}(\mu_{p\mathbf{p}})$$

$$J_{q,r} = \frac{1}{N(\mathbf{p})} G_{\mathbf{p},q} G_{\mathbf{p},r} = \sum_{n \mod \mathbf{p}} (\frac{n}{\mathbf{p}})_p{}^q (\frac{1-n}{\mathbf{p}})_p{}^r \ \in \mathbb{Q}(\mu_p)$$

[5] $Gal(M/\overline{F}) \to Gal(M/F) \to Gal(\overline{F}/F) = Fr_{\mathbf{p}}$ with $F = \mathbb{F}_{\mathbf{p}}(t)$ and M be number field whose Galois group is the fundamental group of 3 manifold. By S-duality we can exchange $Gal(M/\overline{F})$ of order p and $Fr_{\mathbf{p}}$.

We have Hecke character $(\frac{n}{\mathbf{p}})_p{}^q \in \mu_p$ which we need for Lens space $L(p,q)$ sign refined CS critical value.

We introduce another prime N, the conductor of elliptic curve E in Jacobian of C from splitting constant e which is prime to $\mathbf{p}$($\mathbf{p}$-adic unit).

With the Hecke character of modulus $\mathbf{p} = 1 \mod p$ $\chi : (\mathbb{Z}/(\mathbf{p}-1)) \to \mathbb{C}^\times$ and the Teichmuller character $\Theta : \mathbb{F}_\mathbf{p} \to \mu_\mathbf{p}$, we define Gauss sum for Fermat curve C as product of $\mathbf{p}$-adic Gamma function.

$$G(\chi) = \sum_{n \mod \mathbf{p}} \chi(n)\Theta(n) = -\pi^{s_\mathbf{p}(r)} \prod_{0 \le i < f} \Gamma_\mathbf{p}(\frac{r^{(i)}}{\mathbf{p}-1}) \in \mathbb{Q}(\mu_{p\mathbf{p}}) \tag{3.4}$$

$$G_{\mathbf{p},p} = \sum_{1 \le i \le p-1} G(\chi^i)$$

where $\Gamma_\mathbf{p}(n) = (-1)^n \prod_{0<i<n, \mathbf{p}|i} i$ is the $\mathbf{p}$-adic Gamma function satisfying cocycle condition $\Gamma_\mathbf{p}(x)\Gamma_\mathbf{p}(1-x) = -(-1)^{\mu_x}$ for $\mathbf{p} \mid x + \mu_x$. $r^{(i)}$ is the integral whose $\mathbf{p}$ expansion is a cyclic permutation of the digits(shift) of r by i position, where r is an integral $0 \le r < \mathbf{p}-1$, $s_\mathbf{p}(r)$ is the sum of the digits of r in $\mathbf{p}$ expansion.

Associated with Fermat curve C over characteristic $\mathbf{p} = 1 \mod p$, we have two $\mathbf{p}$-adic units $\mathbf{p}$ and $e = \begin{pmatrix} -q/p \\ \mathbf{p}-1/p \end{pmatrix}$ giving uniformazing parameters for $\mathbf{p}$-adic τ function of C. It gives the $\mathbf{p}$-adic period of Fermat curve and the root number which is sign of Swan conductor[22][23].

For the root number,we consider elliptic Gauss sum with the Hecke character of modulus β where β is the multiple of $\mathbf{p}$. With this, we can consider homothety class of $\mathbf{p}$-torsion point on Jacobian on C.

For that we can reduce $p = 3, 4$ by supersingular decomposition over sufficiently ramified field. Then we have elliptic Gauss sum. So given Fermat quotient of degree $p > 4$ over sufficiently ramified field, the Jacobian decompose to supersingular elliptic curve over sufficiently ramified field representing elliptic Gauss sum. $\chi_1 : (\mathbb{Z}/\beta)^\times \to \mathbb{C}^\times$ and $S(\lambda) = a$ for $\lambda = a + b\sqrt[p]{1}$, $2i\pi S(\lambda) = \frac{2}{\sqrt{p}}\pi(\lambda\sqrt[p]{1} - \overline{\lambda\sqrt[p]{1}})$,

$$G(\chi_1) = \sum_{n \mod \beta} \chi_1(n)e^{2i\pi S(n/p)} = (sign)G_{\mathbf{p},p}$$

with $\chi_1(n) = (\frac{n}{\mathbf{p}})_p\chi_0(n)$ where $\chi_0(n) \in \pm\mu_p$ is $\mod p$ character. $G(\chi_1)$ has p different values depending on $\mathbf{p}$.

For general p, we have p different sign of Hecke character depending on p homothety class of intersection of $\mathbf{p}$-torsion point and p torsino point. For even p we have p different sign for $\mathbf{p} = 1, p+1, 2p+1, ..., (p-1)p+1 \mod p^2$ Weyl alcove. For odd p we have p different sign values for $\mathbf{p} = 1, 2p+1, 4p+1, ..., 2p(p-1)+1 \mod 2p^2$. For $p = 3, 4$,this realize the roots number for modular representation theory for supersingular elliptic curve over sufficiently ramified field.

So,we can reduce to supersingular elliptic curve over sufficiently ramified field, which is elliptic Fermat quotient of degree $3, 4$ over characterstic $\mathbf{p} = 1 \mod 3, 4$ with $3, 4$ different root number depending on $\mathbf{p}$.

Then we can realize supersingular transition over sufficiently ramified field with supersingular $\mathbb{Q}$ point at arithmetic infinity realizing $\mathbb{Z}/2$ and $\mathbb{Z}/3$ from Mordell-Weil torsion for elliptic curve over $\mathbb{Q}$,which is bounded by mod 12 by Mazur theorem. The higher ramification group by Swan conductor N is governed by mod 12 Mordell-Weil torsion over $\mathbb{Q}$ such that $12N$ power of Siegel unit is Stark-Heegner unit which is weight 2 modular form. So we have Mordell-Weil torsion over $\mathbb{Q}$ with supersingular decomposition over sufficiently ramified field, such that we have supersingular extremal elliptic surface over characteristic 2 for Sasaki-Einstein CY.

Note that $\mathbb{Z}/2, \mathbb{Z}/3$ arise from reciprocity on torsion point in elliptic factor over $\mathbb{Q}$ in Jacobian of Fermat quotient of degree p over finite field of characteristic $\mathbf{p} = 1 \mod p$. Then by supersingular decomposition to elliptic Fermat curve over sufficiently ramified field, we have $\mathbb{Z}/2, \mathbb{Z}/3$ from Mordell-Weil torsion over $\mathbb{Q}$ at arithmetic infinity realizing big $\mathbb{Q}$ point of infinite order(generalized CM point) on supersingular elliptic curve over sufficiently ramified field.

Note that the root number determine Mordell-Weil torsion over $\mathbb{Q}$ and is determined by higher ramification group over sufficiently ramified field, such that we have $\mathbb{Z}/2$,$\mathbb{Z}/3$ structure over $\mathbb{Q}$ on p homothety class of intersection of $\mathbf{p}$ torsion point of and p torsion point by p-different root number.

So both $\mathbb{Z}/2$ and $\mathbb{Z}/3$ Mordell-Weil torsion over $\mathbb{Q}$ realize big $\mathbb{Q}$ point of infinite order which is non-torsion mod 2 algebraic class, such that we have supersingular transition by different root number depending on homothety class by choice of $\mathbf{p}$, from reciprocity of supersingular $\mathbb{Q}$ point on elliptic Fermat quotient over sufficiently ramified field. In this way, we can determine sign of elliptic Gauss sum of supersingular elliptic curve over sufficiently ramified field representing Ekedahl-Oort stratification in supersingular locus over sufficiently ramified field, for cascade of totally real number field.

In supersingular locus of Shimura varieties over sufficiently ramified field, we have Level-Rank duality from supersingular decomposition to supersingular elliptic curve over sufficiently ramified field, such that we reduce to elliptic Fermat quotient case with degree $3, 4$ over $\mathbf{p} = 1 \mod 3$ and $\mathbf{p} = 1 \mod 4$ resp. which can be considered as elliptic factor in Jacobian of Fermat quotient of arbitrary degree over sufficiently ramified field. The Level-Rank duality is realized from supersingular $\mathbb{Q}$ point, and we have Twistor plane(Fake Fano surface) with twistor transformation realizing non-Abelian Fourier transform realizing Level-Rank duality.

By weight 2 motivic complex by Selmer sheave, we have realization of

Twistor plane(Fake Fano surface) with Bloch-Ogus map realizing non-Abelian Galois representation with Arason invariant. The Selmer sheave has Mordell Weil torsion with a big $\mathbb{Q}$ point from $\mathbf{p}$-adic Ceresa cycle even of Mordell-Weil rank 0.

We have $\mathbf{p}^2$ Weyl alcove realizing Fake Fano surface which birational to $\mathbb{P}^2$ over sufficiently ramified field. We have realization of Mordell-Weil torsion by non-vanishing Grothendieck group of Fake Fano surface, realising $\mathbb{Z}/2, \mathbb{Z}/3$ at arithmetic infinity by non-torsion mod 2 algebraic class. For supersingular transition, we consider supersingular extremal elliptic surface over charactersitic $2, 3$, with duality on Leech lattice over sufficiently ramified field.

From Fake Fano surface with sign from Mordell-Weil torsion, the change of singular fiber with Mordell-Weil torsion(cascade) is realized by $\mathbb{Z}/2$ and $\mathbb{Z}/3$ arithmetic infinity by different sign at different characteristic $\mathbf{p}$, from different homothety class of $\mathbf{p}$ torsion point and p torsion point with supersingular decomposition. So by different choice of homothety class we have different sign, such that for not lowest homothety class we can describe supersingular transition over sufficiently ramified field which is Ekedahl-Oort stratification. Note that we have Galois completion of $\mathbf{p}$-adic Ceresa cycle for lowest homothety class. We have $\mathbf{p}$-adic reciprocity for homothety class for intersection of $\mathbf{p}$ torsion and p torsion point realizing conductor system. By $12N$ power of Siegel unit with mod 12 Mordell-Weil torsion over $\mathbb{Q}$, we have Galois complete invariant by Stark-Heegner unit realising $\mathbb{Z}/2$ and $\mathbb{Z}/3$ at arithmetic infinity from Mordell-Weil torsion over $\mathbb{Q}$. Then with Mordell-Weil torsion with conductor N we have complete description for real GW invariant satisfying $\mathbf{p}$-adic reciprocity. The different sign from different choice of homothety class provide supersingular transition from Mordell-Weil torsion over $\mathbb{Q}$. Then we have right integral structure on Selmer group of Galois cohomology of totally real number field with valued in $\mathbf{p}$-adic Tate module which is motive of supersingular elliptic curve over sufficiently ramified field.

So we can distinguish homothety class, The lowest homothety class(which can be defined over $\mathbb{Q}$) which Galois complete by mod $\mathbf{p}$ reduction with sign from Swan conductor(counting supersingular $\mathbb{Q}$ point) and higher homothety class for supersingular transition by considering supersingular $\mathbb{Q}$ point as saddle connection which exist over sufficiently ramified field. By supersingular $\mathbb{Q}$ point over sufficiently ramified field, we have pro-$\mathbf{p}$ asymptotic with right integral structure compatible with $\mathbf{p}$-adic reciprocity.

The two description of supersingular $\mathbb{Q}$ point as Swan conductor for Galois completion by mod $\mathbf{p}$ reduction, or as saddle connection in recursion process for supersingular transition,provide two role for supersingular $\mathbb{Q}$ point. Due to the determination of integral structure, there is ambiguity from $\mathbf{p}$-adic reciprocity whose mod $\mathbf{p}$ reduction provide real GW invariant which can defined over $\mathbb{Q}$,for lowest homothety class. So with higher homothety class, we have level lowering and raising operation by integral structure on Shimura variation over sufficiently ramified field.

Note that conductor N is $\mathbf{p}$-adic integer depending on choice of $\mathbf{p}$ satisfying $\mathbf{p}$-adic reciprocity. For Galois completion, we choose N to be number of connected component in moduli space over $\mathbb{Q}$ from mod $\mathbf{p}$ reduction which do not depends on $\mathbf{p}$ and correspond to lowest homothety class of intersection of $\mathbf{p}$ torsion point and p torsion point. This conductor(sign) is determined by limiting of zeta function of totally real number field with non-Abelian Fourier transform. From Hurwitz covering counting for zeta function of totally real number field, we only see the Swan conductor from lowest homothety class for Galois completion by mod $\mathbf{p}$ completion. Other homothety class for supersingular transition can be seen from Frobenius reciprocity in generating series. So we need the integral structure of measure for supersingular locus over sufficiently ramified field with level raising and lowering operation, which is super(sign refined) matrix integral measure for Adelic Grassmannian.

In this sense, the Mordell-Weil torsion over $\mathbb{Q}$ provide big $\mathbb{Q}$ point of infinite order from Galois completion by mod $\mathbf{p}$ reduction with sign(realising $\mathbf{p}$-adic uniformazation) and also provide supersingular transition(saddle connection) by $\mathbf{p}$-adic reciprocity over sufficiently ramified field. For supersingular(tropical) transition, we have hyperelliptic curve which has no global $\mathbb{Q}$ point having only local point as rigid analytic curve in recursion for pro-$\mathbf{p}$ asymptotic. Note that most of such curve for supersingular transition from $\mathbf{p}$-adic reciprocity cancelled with only remaining mod $\mathbf{p}$ reduction part. Note that by mod $\mathbf{p}$ reduction with pro-$\mathbf{p}$ asymptotic, we have a rank-2 Selmer sheave associated with big $\mathbb{Q}$ point of infinite order even if we do not have global point.

So from local global compatibility, locally we consider supersingular $\mathbb{Q}$ point as Swan conductor from mod $\mathbf{p}$ reduction providing big $\mathbb{Q}$ point of infinite order, and globally we can consider supersingular $\mathbb{Q}$ point satisfying $\mathbf{p}$-adic reciprocity as saddle connection for curve without global $\mathbb{Q}$ point providing pro-$\mathbf{p}$ asymptotic with recursion.

So we have super(sign refined) matrix integral measure for Galois completion locally by mod $\mathbf{p}$ reduction and providing pro-$\mathbf{p}$ asymptotic globally with cancelling curve with no global point from $\mathbf{p}$-adic reciprocity.

Both are realization of Mordell-Weil torsion on elliptic factor in Jacobian over $\mathbb{Q}$, one for big $\mathbb{Q}$ point of infinite order and other for supersingular transition.

The $p = 3, 4$, p different values of $G(\chi_1)$(sign) depending on choice of $\mathbf{p}$,realising supersingular transition over sufficiently ramified field. So for Ekedahl-Oort stratification ,we need to factor through supersingular locus over sufficiently ramified field with supersingular decomposition.

Now let p to be arbitrary. The sign in $G(\chi_1)$ from elliptic Gauss sum is determined by the sign of CS critical value after supersingular decomposition. At each i-th critical point, there exist a sign constructed from spectral flow. The spectral flow realize supersingular reduction over sufficiently ramified field by mod 8 reciprocity. We have sign from mod 2 Casson invariant. For $L(p, q)$, we determine character for sign which is 1 for $q^*n < p/2$ and -1 for $q^*n > p/2$ for p-critical points $0 \leq n < p$ of CS partition function. It determine the sign for epsilon factor of the **p**-adic Galois representation by spin CS invariant.

By mod 8 reciprocity of spectral flow, we have Galois completion with pro-**p** reciprocity realizing mod 2 Casson invariant, which determine higher ramification group from the class number of totally real number field which is extremal and exceed class number of imaginary quadratic number field. Note that mod 2 Casson invariant is sign of spin CS invariant with valued in Hilbert class field K of totally real number field F, realizing irrational spectral parameter $\hbar \in \mathbb{C}$ associated with metaplectic quantization of $L(p, q)$ with pro-**p** covering.

For the hyperelliptic Seiberg-Witten curve over characteristic 0, the associated geometry is $N = 2$ with $\mathbb{Z}/2$ involution without sign.

For N=1 vacua, we need to consider Seiberg-Witten curve over finite field. Also for hyperelliptic Seiberg-Witten curve over finite field, we have higher ramification group from wild Frobenius such that we have totally real number field, with orthogonal degeneracy locus for self-dual representation of symmetric quiver.

For non-hyperelliptic Seiberg-Witten curve,with non symmetric quiver, we need either hyperelliptic covering with orthogonal degeneracy locus or need to consider curve as rigid analytic curve over finite field and consider pro-**p** limit to obtain $\mathbb{Z}/2$, associated with N=1 CY vacua with totally real number field.

So as N=1 vacua we have Galois completion by higher ramification group over sufficiently ramified field,having sign refined algebraicity.

In the procedure to recover almost real structure with $\mathbb{Z}/2$ at arithmetic infinite, we have additional sign from Swan conductor which capture wild ramification.

Then depending on $N = 2 \to N = 1$ super symmetry breaking, depending on $N = 1$ gauge theory, we have confinement index, depending on degeneration of SW curve(via singular curve). Then we have Ekedahl-Oort stratification in moduli space of $N = 1$ vacua with sign from Swan conductor, considered as supersingular N=4 vacua Leech lattice over sufficiently ramified field. So sign comes from Galois completion. In geometry, by Ekedahl-Oort stratification of moduli space of $N = 1$ vacua,we have Higgsing(cascade of nilpotent cone) with different sign from Swan conductor, depending on different Levi subgroup of nilpotent cone.

Over supersingular locus of Shimura varieties over sufficiently ramified field, we can interpolate N=2 vacua $L(p, q)$ and N=1 vacua $Y^{p,q}$, considered as N=2 vacua as N=1 vacua by pro-$\mathbf{p}$ covering of $L(p, q)$. So we can compare Seiberg Witten curve $Y^{p,q}$ and Fermat curve of degree p over finite field of characteristic $\mathbf{p} = 1 \mod p$ at supersingular locus.

So over sufficiently ramified field, we have all N=1 vacua from N=4 Leech lattice Level-Rank duality realized on supersingular locus.

On super singular locus the $N = 1$ vacua is realized with singular curve(opers). There is so called infinite framing limit in 3d partition function in string theory which can be considered as the Artin Schreire curve(Maximal curve) for compactification with wall crossing correspondence. In general, we can consider N=1 curve with $\mathbf{p}$-adic uniformazation as supersingular curve over sufficiently ramified field realizing compatification of Hitchin moduli space by $SL(2, F)$

Bruhat-Tits cocycle which is **p**-adic lift of Shintani cocycle for **p**-adic zeta function of totally real number field F.

This is at UV limit in supersingular locus over sufficiently ramified field(N=4 vacua). With **p**-adic τ function with conductor, we can realize the Absolute Galois action on moduli space by number of connected component from mod **p** reduction of Galois group of sufficiently ramified field. We have representation of pro-**p** mapping class group(pro**p** fundamental group of $L(p,q)$) by pro-**p** Grothendieck section(signed section of determinant line bundle/CS partition function). With Absolute Galois action on moduli space, we have Swan conductor N which determined sign, and with Galois action of sufficiently ramified field, we have pro-**p** structure which can also describe supersingular transition to other N=1 vacua as generalized CM point by Hilbert class field K of totally real number field F, by changing root number of different homothety class of **p**-torsion point and p torsion point. By super(sign refined) matrix integral measure, we have full structure of moduli space of N=1 vacua(as generalized CM point) with equidistribution property with mod **p** reduction for each vacua(generalized CM point as big $\mathbb{Q}$ point of infinite order). This is the local global compatibility. For defining pro-**p** asymptotic we need full structure of N=1 vacua whose sign depends on **p**, and by locally mod-**p** completion, we see only lowest homothety class which Galois complete by absolute Galois action which determine sign of spin CS invariant and $[K : F] = N$ as order of mutation of derived category at characteristic 0. But over characteristic **p**(over sufficiently ramified field) with **p**-adic reciprocity, we have fine structure which can also describe supersingular transition between vacua.

By metaplectic lift, we have additional $\mathbb{Z}/2$ at arithmetic infinity from Hilbert reciprocity of pro-**p** tower. There exist intermediate perfectoid space - the Shimura varieties over sufficiently ramified field(perfectoid field) of mixed characteristic. Over sufficiently ramified field(perfectoid field), we have integral model for Shimura varieties with level lowering and raising operation, and consequently all Hodge type conjecture with sign, since it provide global torsion Galois representation from **p**-adic modular origin for non-torsion mod 2 algebraic class(**p**-adic Ceresa cycle) from metaplectic lift. We have Stark-Heegner unit of totally real number field realizing normal function of **p**-adic Ceresa cycle.

As seen before for Sasaki-Einstein CY manifold has intrinsic Higgs state from correspondence which possibly come from non-Hodge Tate class in Hypermultiplit moduli space. By using sufficiently ramified field we can restore Hodge-Tate but with sign. Note that over splitting field (Hilbert class field of totally real number field) we can restore rationality by supersingular varieties over sufficiently ramified field(Mori dream space). Then we have algebraicity for rationally connected but not unirational varieties by higher ramification group with non-maximal Swan conductor.

So the non-arithmetic non-torsion class from non-Hodge Tate class can be seen as arithmetic Hodge class over perfectoid field in supersingular locus which is non-torsion mod 2 algebraic class.

By using reciprocity of class field theory of sufficiently ramified field, which is the wall crossing/Seiberg duality for Sasaki-Einstein CY from **p**-torsion, we restore algebraicity by Hodge-Tate representation over sufficiently ramified field(sign refined purity). So by wall crossing(Seiberg duality), we have **p**-adic uniformazation over sufficiently ramified field(perfectoid field). From that we can obtain refined Hodge conjecture and Tate conjecture. Over perfectoid field, we obtain UV completion by compactification of building with $SL(2, F)$ Bruhat-Tits cocycle by **p**-adic lift of Shintani cocycle. Then we have Fake projective plane(Mori dream space) for non-parallelogram toric orbifold by arithmetic MMP realising **p**-adic uniformazation.

This is why we have supersingular elliptic curve over sufficiently ramified field with slope 0 rank 2 sheaves(Selmer sheaves) over perfectoid curve(modular curve over sufficiently ramified field). Because we have **p**-adic uniformazation, there is no Frobenius destabilising sheaves, and we have $K(\pi_1, 1)$ space -the space determined by pro-Etale algebraic fundamental group. We realize **p**-adic uniformazation by arithmetic MMP(successive wall crossing).

This is compatible with sign refined spin CS partition function, since it only depends on pro-**p** fundamental group of 3 manifold with sign. The perfectoid field comes from super string measure for super worldsheet(moduli space of super Riemann surface) realising irratinoal CFT.

Theorem 11 *From perfectoid space, we have the pro-***p** *tower of non-algebraic class from Bianchi manifold with vanishing first Betti number with additional* $\mathbb{Z}/2$ *at arithmetic infinity.*

Then by metaplectic lift by $\mathbb{Z}/2$*, we obtain 4-dimensional Galois representation in supersingular locus over sufficiently ramified field with additional sign from* $\mathbb{Z}/2$ *by Swan conductor. The Galois representation with sign from metaplectic spinor lift solve refined Hodge conjecture by recovering algebraicity of non-algebraic class by sign(spinor) of* $\mathbb{Z}/2$*, such that we have non-torsion mod 2 algebraic class by* **p***-adic Ceresa cycle whose normal function is Stark-Heegner unit of totally real number field* F *with sign.*

The non-algebraic class from Lens space knot complement are of this class. For each knot, there exist arithmetic lattice of number field which is not algebraic class with no central extension to $SL(2, \mathbb{Z})$. The algebraicity is recovered from metaplectic lift with additional $\mathbb{Z}/2$ from pro-**p** of Bianchi manifold.The resulting Galois representation from metaplectic lift has additional sign.

The string theory resolution of the non-existence of Galois representation is the refined partition function $3D$ partition function for additional framing anomaly(framing match)-there exist conductor from resonance state from reciprocity of incomplete Gamma function as Stokes phenomenon(resurgence). As a result at UV limit, we have limiting shape(crystal limit) from 3d partition function, and is purely combinatorial.

So we can compare torsion class from non-arithmetic quotient(non-Hodge Tate) and non-torsion class from pro **p** covering of arithmetic quotient(Hodge Tate) in supersingular locus with perfectoid field. For Sasaki-Einstein CY, the non-arithmetic quotient comes from $L(p^2, pq-1)$ two bridge knot and the arithmetic quotient comes from Bianchi manifold $\mathbb{Q}(\sqrt{-p})$[6].

Now we state our level rank duality in supersingular locus in Shimura varieties over sufficiently ramified field.

Given number field $L = \mathbb{Q}(\sqrt{-p})$ over $\mathbb{Q}$, we have following short exact sequence, let M be Bianchi manifold associated with imaginary quadratic L.

$$\pi_1(M_{\overline{\mathbb{Q}}}) \to \pi_1(M_{\mathbb{Q}}) \to Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \tag{3.5}$$

For L being imaginary quadratic number field, there is no SL_2 representation of Galois group $Gal(L/\overline{\mathbb{Q}})$ which is expected to be the central extension of PSL_2 representation of $\pi_1(\mathcal{H}^3/PSL_2(\mathcal{O}_L)$.

The eq.(3.5) split only after metaplectic lift with additional sign. Then we have 4-dimensional Galois representation with sign. For that we need to determine minimal conductor for global Galois representation from higher ramification group from mod **p** reduction, which we obtained from **p**-adic uniformazation over supersingular locus over sufficiently ramified field as Swan conductor from lowest homothety class.

To have the Galois representation from splitting of eq.(3.5) associated with pro-**p** covering of Bianchi manifold $M = \mathcal{H}^3/PSL_2(\mathcal{O}_L)$, we need to consider metaplectic lift $Sp_4(\mathbb{C})$ and Galois induction. This is obtained from pro-**p** tower

[6] The choice of Bianchi manifold is not unique due to we can use Level-Rank duality with supersingular decomposition over sufficiently ramified field, which enable us to reduce $\mathbb{Q}(\sqrt{-p})$ to $\mathbb{Q}(\sqrt{-2})$ by supersingular decomposition using class field tower of sufficiently ramified field.

of covering of Bianchi manifold by additional $\mathbb{Z}/2$ at arithmetic infinity.

By pro-$\mathbf{p}$ tower, we have following tower of exact sequence,

$$\pi_1(M_{n\overline{\mathbb{F}_\mathbf{p}}(t)}) \to \pi_1(M_{n\mathbb{F}_\mathbf{p}(t)}) \to \overline{\mathbb{F}_\mathbf{p}}(t)/\mathbb{F}_\mathbf{p}(t) = Fr_\mathbf{p} \tag{3.6}$$

Now we can consider Level-Rank duality which exchange inertia(meridian) and Frobenius(longitude). This arise in supersingular locus over sufficiently ramified field by torsion(global) Galois representation.For that we need to choose $\mathbf{p} = 1 \mod p$.

By ramification knot for Bianchi manifold, we have a prime p by inertia of number field L, and from Frobenius we have prime $\mathbf{p}$ from level($\mathbf{p}$ power level structure). The exchange of this two, the meridean representation of inertia and longitude representation of Frobenius, we have Level-Rank duality which is the non-Abelian Fourier transformation punctured torus along tubular neighborhood of knot(Heegaard torus). For two component knot for Lens space, this is done by non-Abelian Fourier transformation with additional sign, We need metaplectic lift.

Then we also have signed S-duality which is Langlands duality from Galois representation from Grothendieck section with modularity come from Level-Rank duality.

In geometric term, this non-Abelian transform is the Fourier Mukai(twistor) transformation along complexified Jacobian.And additional sign is obtained from Swan conductor. Then we have global Langlands correspondence with torsion Galois representation as $\mathbf{p}$-adic Grothendieck section of eq.(3.6) with sign. So sign refined Langlands duality(mirror symmetry) is realized in supersingular locus over sufficiently ramified field as S-duality with sign. This gives the mirror symmetry for G/B for any reductive group G by embedding to affine Grassmannian of Langlands dual group G^{dual}, which is the $\mathbf{p}$-adic uniformization. For globalization of mirror symmetry, we need compactification of Jacobian fiber along discriminant locus determined by the ramification knot, with compactification is done by mod $\mathbf{p}$ reduction. Each of such globalization, we have global torsion Galois representation which constructed before with metaplectic lift by pro-$\mathbf{p}$ covering of local conifold geometry of discriminant locus.

This is done with level-rank duality on complexified Cartan torus as Twistor plane(Fake Fano surface) in supersingular locus over sufficiently ramified field. By Leve-Rank duality with supersingular $\mathbb{Q}$ point we can realize versal deformation of ADE singularity as Kloostermann sum of reductive group G with Swan conductor from wild monodromy at infintiy with higher ramification group.

Associated with this we have unique $\mathcal{D}_{\mathbb{A}^1_{\mathbb{C}}}$-module(quantized A-polynomial) determined by knot for Bianchi manifold whose support is character varieties in $\mathbb{C}^* \times \mathbb{C}^*$ determining a vector in Hilbert space with framing (sign). As $\mathcal{D}$-module over affine line, the Fourier transform can be seen as Seiberg trace formula by considering $\hbar \in \mathbb{A}^1_{\mathbb{C}}$ represent modulus of Heegaard torus by $q^{2i\pi/\hbar}$. Then we have functional equation for $\hbar \to 1/\hbar$ realizing signed S-duality(fourier transform) by inverting motive $\mathbb{L}$ of $\mathbb{A}_{\mathbb{C}}$. On supersingular locus over sufficiently ramified field, the wave function has limiting(extremal) shape and lies in L_p class, and the limit shape measured by additional sign(conductor). They lie in real algebraic geometry with $\mathbf{p}$-adic topology(almost real structure). Such spinor wave function with sign can be obtained from metaplectic lift associated with almost real cycle. And we have correspondence of motive over finite field which is again non-commutative motive, such that,it satisfy reciprocity from sign refined motivic measure(Tits motivic measure).

This can be seen as Rees $\mathcal{D}$-module over $\mathbb{C}^* \times \mathbb{C}^*$ supported on spectral curve(Character varieties) where $\hbar$ do the role for formal completion parameter for formal neighborhood of spectral curve considered as 1-parameter family of spectral curve along affine line, such that we have $\hbar$ flat connection along affine line. By Galois completion, we determine $\hbar$ as spectral parameter in Hilbert class field in totally real number field. It is formal parameter realizing motive of affine line with non-trivial higher ramification for Galois completion. This realize Galois completion of formal deformation to actual deformation such that we have spinor wave function.

The non-Abelian Fourier transform acts on $\mathbb{C}^* \times \mathbb{C}^*$ as derived duality(birational transform) along quantum plane,then we can see non-Abelian Fourier dual wave function with sign. The self dual wave function is realized as wild ramification in supersingular locus over sufficiently ramified field.We consider ramification locus (by wild singular fiber at infinity) with the non-Abelian Fourier transform action on them by rearrangement of them so that we can consider arbitrary degeneration of spectral curve(arbitrary configuration of ramification point), then we have global structure of moduli space as number of connected component which realize mod $\mathbf{p}$ reduction from wild monodromy at infinity of singular fiber.

We need to consider linking number over characteristic $\mathbf{p}$. This realize the string duality by hyperplane arrangement-(non-Abelian Fourier transform) along $q^{2i\pi/\hbar}$ torus. Consider simplest ramification by Prym varieties, they the non-Abelian Fourier transform realize the abelianization(for arbitrary degeneration) by modular representation with 1-dimensional summand of Frobenius eigenspace in $\mathbf{p}^k$ Weyl alcove which realize higher ramification group by correspondence(intrinsig Higgs state for conductor). This can be seen as pro-$\mathbf{p}$ Galois section with wild Frobenius(Frobenius pull-back) such that we have pro-$\mathbf{p}$ etale fundamental group with additional Swan conductor from higher ramification group($\mathbf{p}$-adic Etale path in $\mathbf{p}$-adic non-Abelian Hodge theroy). Then we have the reciprocity of $\mathbf{p}$-adic Gauss sum with sign from higher ramification group.

With this we have modular representation on $\mathbf{p}^k$ Weyl alcove realising complexified cartan of SL_2 by quantum plane for $SL(2,F)$ Bruhat-Tits cocycle.

On the supersingular locus over sufficiently ramified field, we have Schur-Frobenius duality from representation theory, and Level-Rank duality in $\mathbf{p}^k$ Weyl alcove from $\mathbf{p}$-adic reciprocity. The non-Abelian Fourier transform with wild monodromy along compactification infinity of Jacobian fibration over discriminant locus, is done with non-Abelian Fourier transform along toroidal compactification on cusp in Hilbert modular surface(mirror of discriminant locus under period map) realising boundary divisor from conductor system of $\mathbf{p}$-adic zeta function of F satisfying $\mathbf{p}$-adic reciprocity. We have UV degree of Freedom from conductor system of pro-$\mathbf{p}$ conifold geometry with Lens space local geometry for boundary divisor.

In our setting we do not need to distinguish $\mathbf{p} = 2$ case since it can obtained from pro-$\mathbf{p}$ tower at arithmetic infinity which provide generating function of B_n field counting by Siegel unit[7]. The equality of Galois group $\lim_{\leftarrow} Gal(\overline{\mathbb{Q}}(\mu_{N\mathbf{p}^n})/\mathbb{Q}(\mu_{N\mathbf{p}^n}))$ enable us to recover almost real structure with Level-Rank duality with $\mathbb{Z}/2$ sign at arithmetic infinity from supersingular locus over sufficiently ramified field.

Note that we have $\mathbb{Z}/2$ from Hida's ordinary limit by

$$\lim_{\leftarrow} Gal(\overline{\mathbb{Q}}(\mu_{\mathbf{p}^n})/\mathbb{Q}(\mu_{\mathbf{p}^n}) = Gal(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2$$

This suggest we can avoid $\mathbf{p}$-adic tower but need only to consider signed spinor (2-adic) at arithmetic infinity. Due to over sufficiently ramified field, we have twistor spinor which have sign from higher ramification group. With spinor algebra with Pauli matrix(composition of spinor), the reciprocity of 2-adic spinor compute sign by Boltzmann weight-sum over all spin configuration.

[7]Note that all motive can be seen as B_n field B_n cover over $\mathbb{F}_q(t)$.

This is the twistor spinor(signed spinor) partition function whose almost perfect cancellation reduce to averaged Faltings Height by sum over all CM type $[K : F]$ of Hilbert class field K of totally real number field F. So with twistor spinor, we have equidistribution of generalized CM point(Hilbert class field).

The signed spinor(on brane tiling) is the object from **p**-adic tower which restore signed algebraicity from $\mathbb{Z}/2$ sign at infinity from Galois group of sufficiently ramified field. The use of sufficiently ramified field is the intermediate step, since Galois group is composition of $\mathbb{Z}/2$ reflection which can be obtained from supersingular splitting over sufficiently ramified field from which we have sign refined finiteness. With this we can realize supersingular transition with Mordell-Weil torsion over $\mathbb{Q}$ by supersingular decomposition.

By extremal metric(optimizing) of CY with twistor spinor, without using **p**-adic tower, we can have same result by averaged Falting Height which is Stark Heegner unit of **p**-adic zeta function of totally real number field. So from supersingular decomposition, with only composition of reflection realising Mordell-Weil torsion over $\mathbb{Q}$, we can realize extremal metric(opimizing) with Monge Ampere equation with matrix measure for Hessian from twistor spinor.

At even root of unity,we have sign refined spin CS invariant for Lens space and $Sp_4(\mathbb{C})$ CS invariant for Bianchi manifold, which can be recovered from pro-**p** covering of Bianchi manifold which provide Sasaki-Einstein CY with Seiberg duality cascade at supersigular locus over sufficiently ramified field with $\mathbb{Z}/2$ sign from order of Seiberg duality. This additional $\mathbb{Z}/2$ sign define sign form spin CS invariant at even root of unit from Hilbert reciprocity.

In principle, any Artin L function with modular unit(generalized Faltings height) by Kronecker limit formula can be obtained from twistor spinor of corresponding motive over finite field(orthogonal/non-Abelian Galois representation). So the twistor spinor represent global torsion Galois representation from irregular toric Fano surface with elliptic fibered CY motive with Mordell-Weil rank 0 and Sasaki-Einstein CY. For non-vanishing Mordell-Weil rank, we have twistor spinor associated with higher dimensional determinantal varieties associated with non-toric irregular Fano surface.

Chapter 4

CS partition function and p-adic L function

CS secondary characteristic can be considered as $\log_{\mathbf{p}}$ of the **p**-adic regulator, and sign refined spin CS partition function of $L(p, q)$ as Gauss sum(with sign from root number) is **p**-adic epsilon factor for **p**-adic zeta function of totally real number with sign from functional equation.

For that we consider **p**-adic L function of sufficiently ramified field. Then by local-global compatibility we have **p**-adic zeta function of totally real number field, with specific higher ramification group can be obtained from **p**-adic L function of sufficiently ramified field.

p-adic L function of sufficiently ramified field has Kronecker limit as Siegel unit. The $12N$-th power of Siegel unit is weight 2 modular form. By choosing specific cusp $-q/p$ in modular curve, we have specific Lens space $L(p, q)$ boundary in Hilbert modular surface of totally real number field F. Over sufficiently ramified field, the cusp has congruence relation.

Let consider modular curve $X_1(N\mathbf{p}^n)$ with $Y_1(N\mathbf{p}^n)$ over $\mathbb{C}$ with cusp

$C_1^n(N) = X_1(N\mathbf{p}^n) - Y_1(N\mathbf{p}^n)$. Then we have action of modular Hecke algebra $\mathfrak{H}_n$ of weight 2 with level $N\mathbf{p}^n$ over $\mathbb{Z}_{\mathbf{p}}$ in $H_1(X_1(N\mathbf{p}^n), C_1^n(N); \mathbb{Z}_{\mathbf{p}})$, and cuspical Hecke algebra $\mathfrak{h}_n$ over $\mathbb{Z}_{\mathbf{p}}$ acts on $H_1(X_1(N\mathbf{p}^n); \mathbb{Z}_{\mathbf{p}})$. We consider cusp which do not lie in 0 cusp in $X_0(N\mathbf{p}^n)$. Then we have canonical Manin-Drinfeld splitting over $\mathbb{Q}_{\mathbf{p}}$,

$$s_n : H_1(X_1(N\mathbf{p}^n), C_1^n(N); \mathbb{Q}_{\mathbf{p}}) \to H_1(X_1(N\mathbf{p}^n; \mathbb{Q}_p))$$

Associated with cusp $\frac{-q}{N\mathbf{p}^n p} \in C_1^n(N)$ with $\frac{-q}{p} \in \mathbb{P}^1(\mathbb{Q})$ with $p, q \in \mathbb{Z}$, we have class in $(-q, p)_n \in H_0(C_1^n(N) : \mathbb{Z}_{\mathbf{p}})$. We have mod $N\mathbf{p}^n$ congruence relation in cusp. Consider coprime $(p, q) = (p^{'}, q^{'}) = 1$,

$$(-q, p)_n = (-q^{'} + jp^{'}, p^{'})_n, \quad a = a^{'} \mod N\mathbf{p}^n, b = b^{'} \mod N\mathbf{p}^n,$$

Let $\{\alpha, \beta\}_n \in H_1(X_1(N\mathbf{p}^n), C_1^n(N); \mathbb{Z}_{\mathbf{p}})$ be the geodesic between two cusp $\alpha, \beta \in \mathbb{P}^1(\mathbb{Q})$. Then we have following relation from Markov triple for $\alpha, \beta, \gamma \in PP^1(\mathbb{Q})$, which is triple relation from Weierstrass σ function. This realize Mazur torsion theorem over $\mathbb{Q}$ genus 0 properties for $X_1(n)$ with $n = 1, \cdots 10$ and $n = 12$ from Weierstrass $\mathfrak{p}$ function. Let $\mathfrak{p}_a(z) = \mathfrak{p}(z, \frac{\tilde{a}_1 z + \tilde{a}_2}{N})$ for $a = (a_1, a_2) \in (\mathbb{Z}/N)^2/\{0, 0\}$ with $\tilde{a}_1, \tilde{a}_2$ being representative in $\mathbb{Z}$ with $z \in \mathcal{H}$. We have Steinberg relation for

$$\frac{\mathfrak{p}_a - \mathfrak{p}_c}{\mathfrak{p}_a - \mathfrak{p}_c} + \frac{\mathfrak{p}_b - \mathfrak{p}_c}{\mathfrak{p}_a - \mathfrak{p}_c} = 1, \quad \frac{\mathfrak{p}_a - \mathfrak{p}_c}{\mathfrak{p}_a - \mathfrak{p}_b} \wedge (1 - \frac{\mathfrak{p}_b - \mathfrak{p}_c}{\mathfrak{p}_a - \mathfrak{p}_c}) = 0$$

where $\frac{\mathfrak{p}_a - \mathfrak{p}_b}{\mathfrak{p}_c - \mathfrak{p}_d}$ is the Weierstrass unit which is torsion in K_2 as element in $\mathcal{O}_{Y_N}$.

We have following relation to Siegel unit and Weierstrass σ function,with $g_a = g_{a_1, a_b}$ with $(a_1, a_2) \in (\mathbb{Z}/N)^2/\{0, 0\}$.Let q_N be the primitive N-th root of unity.

$$\frac{\mathfrak{p}_a - \mathfrak{p}_b}{\mathfrak{p}_c - \mathfrak{p}_d} = q_N^{b_2 - d_2} \frac{g_{a+b} g_{a-b}}{g_a^2 g_b^2} \frac{g_c^2 g_d^2}{g_{c+d} g_{c-d}} = \frac{\sigma_{a+b} \sigma_{a-b}}{\sigma_a^2 \sigma b^2} \frac{\sigma_c^2 \sigma d^2}{\sigma_{c+d} \sigma_{c-d}}$$

Then we have relation on regulator from three term relation of modular symbol realizing Euler system.

$$\{\alpha, \beta\}_n + \{\beta, \gamma\}_n + \{\gamma, \alpha\}_n = 0 \tag{4.1}$$

Then we consider geodesic,

$$\{\alpha, \beta\}_n = \{\frac{-q}{N\mathbf{p}^n p}, \frac{-r}{N\mathbf{p}^n s}\}_n$$

associated with $\gamma = \begin{pmatrix} -q & -r \\ p & s \end{pmatrix}$ with $qs - rp = 1$.

Then we consider projection of modular Hecke algebra $\mathfrak{H}_n$ and cuspidal Hecke algebra $\mathfrak{h}_n$ to ordinary part by idempotent.We have pro-$\mathbf{p}$ asymptotic from mod $\mathbf{p}$ reduction with choice of $\mathbb{Z}/2$ sign of spin structure of modular curve which is Hida's ordinary limit $\mathfrak{H}_n^{ord}$ with almost real structure. Let e_n be the Hida's idemponent projector which can be defined from $\mathbf{p}$-typical curve in algebraic K theory of modular curve over sufficiently ramified field, satisfying $\mathbf{p}$-adic reciprocity. Let $T_{\mathbf{p}} : X_1(\mathbf{p}) \to X_1(\mathbf{p})$ by sum over all isogenies in conductor $\mathbf{p}$. The Hida's projector is defined as

$$e = lim_{\leftarrow} e_n = \lim_{n \leftarrow} T_{\mathbf{p}}^{n!}$$

The tower of modular curve realize motivic complex over sufficiently ramified field from Milnor homomorphism by cup product of Siegel unit by 3-term relation eq.(4.1).

$$\mathfrak{H} =: \lim_{\leftarrow} \mathfrak{H}_n^{ord}, \ \mathfrak{h} = \lim_{\leftarrow} \mathfrak{h}_n^{ord}$$

$$H_1(N) = \lim_{\leftarrow} H_1(X_1(N\mathbf{p}^n); \mathbb{Z}_{\mathbf{p}})^{ord}, \ \mathcal{H}_1(N) = \lim_{\leftarrow} s_n H_1(X_1(N\mathbf{p}^n), C_1^n(N); \mathbb{Z}_{\mathbf{p}})^{ord}$$

$$e_n : \mathfrak{H}_n \to \mathfrak{H}_n^{ord}, \ e_n : \mathfrak{h}_n \to \mathfrak{h}_n$$

$$e_n\{\alpha, \beta\} \in H_1(X_1(N\mathbf{p}^n); \mathbb{Z}_{\mathbf{p}})^{ord}$$

Then associated with $\gamma \in SL(2,\mathbb{Z})$ we have,

$$\xi_n\{\alpha,\beta\}_n = e_n \cdot s_n\{\alpha,\beta\}$$

Note that s_n splitting is trivial for α,β without $\mathbf{p}$ power denominator such that we can discard pro-$\mathbf{p}$ structure. In this case, we have realization of generalized CM by Hilbert class field K of totally real number field F with $[K:F]=N$. We have N specialization of of Siegel unit associated with higher ramification group for pro-$\mathbf{p}$ covering of $-q/p$ cusp in modular curve. Note that specialization of Siegel unit realize mod $\mathbf{p}$ reduction for higher ramification group, such that we have sign refined finiteness at characteristic 0 realizing derived equivalence by N mutation. With mod $\mathbf{p}$ reduction, we have equidistribution for generalized CM points.

Then we have $\xi(\alpha,\beta) \in \mathcal{H}_1(N)$ in Hida's ordinary limit.Let $\mathcal{H}_1(N)^+$ be plus part of Hida's ordinary projection, then we have element $\overline{\xi}(\alpha,\beta) \in \mathcal{H}_1(N)^+$ by cup product of specialization of two Siegel unit as K_2 torsion of sufficiently ramified field.

$$\log_{\mathbf{p}} reg_{\mathbf{p}}(g_{0,a_2} \cup g_{0,b_2}) = \overline{\xi}(\alpha,\beta)$$

with Poincare paring $[\cdot,\cdot]_N$ with reciprocity on N conductor from pro $\mathbf{p}$ asymptotic, and logarithmic of $\mathbf{p}$-adic cyclotomic unit which is logarithmic 1-form η (Dieudonne module) in $H^1(X_1(N\mathbf{p}^n)$. Then we have special value of $\mathbf{p}$-adic L function of sufficiently ramified field.

$$[log_{\mathbf{p}} reg_{\mathbf{p}}(g_{0,a_2} \cup g_{0,b_2}),\eta] = L(\epsilon,-1)$$

evaluated at ϵ is the Dirichlet character $Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Q}$ localized on $\mathbb{Z}/\mathbf{p}-1^{\times}$. depending on choice ϵ with choice of η associated with cusp $-q/p$ we have $\mathbf{p}$-adic zeta function of totally real number field. The choice of ϵ determine homothety class of $\mathbf{p}$-torsion point and p torsion point in Jacobian of Fermat quotient of degree p over finite field of characteristic $\mathbf{p} = 1 \mod p$, determining root number.

$\overline{\xi}(\alpha, \beta)$ is defined with cup product of Siegel unit which is K_2 torsion of sufficiently ramified field satisfying Markov type relation. By specialization of Siegel unit, we have N cyclotomic unit g_{0,b_2} with $1 \leq b_2 \leq N$ realising pro-$\mathbf{p}$ structure from N profinite structure of higher ramification group with non-maximal conductor. We have N step mutation realizing $\mathbf{p}$-adic reciprocity of elliptic function(Siegel unit) which is essentially the same as Weierstrass $\mathfrak{p}$ function. The reciprocity of Siegel unit provide N specialization corresponds to averaging generalized CM type of Hilbert class field K of totally real number field F with $[K : F] = N$.

As elliptic function, N specialization represent Markov triple type reciprocity realising Mordell-Weil torsion over $\mathbb{Q}$ with higher ramification group. We have modular embedding by Weierstrass $\mathfrak{p}$ function with sign(root number) from $\mathbf{p}$-adic reciprocity of Hecke character, realizing supersingular elliptic curve over sufficiently ramified field. Then by Markov type relation for K_2 torsion $\{\alpha, \beta\}$ we have complete description for higher ramification group as reciprocity of N specialization by N step mutation from mod $\mathbf{p}$ reduction.

So from cup product of two the Siegel unit $g(a_1, a_2)$ whose $12N$-th power is Stark Heegner unit of totally real number field F with conductor $N\mathfrak{f} = N$, we have the second Milnor homomorphism for Galois cohomology of sufficiently ramified field $m_2 : K_2(\mathbb{Q}(\mu_{N\mathbf{p}^n})) \to H^2(\mathbb{Q}(\mu_{N\mathbf{p}^n}), (\mu_{N\mathbf{p}^n})^{\otimes 2})$. Also by choosing cusp, we have Milnor homomorphism for totally real number field.

$$g(0, a_2) \cup g(0, b_2) =: m_2[a, b]$$

where $[\cdot, \cdot]_N$ is the inner product from Poincare duality(Tate duality),compatible with $\mathbf{p}$-adic measure from Section.3.1. η is the element in $H^1(X_0(\mathbf{p}^n))$ with $\mathbf{p}$ power conductor from $\log_{\mathbf{p}}$ of the $\mathbf{p}$-adic cyclotomic character(Tate object).

We define volume CS invariant from $\mathbf{p}$-adic regulator. It has the sign from $\mathbf{p}$-adic epsilon factor. So it can be considered as sign refined spin CS invariant. If we do not choose cusp but only consider $\mathbb{Z}/2$ involution of modular curve from Hida's ordinary limit, we have sign refined spin CS partition function of $L(2,1)$ which we define as modular CS invariant, associated with pro-$\mathbf{p}$ covering of Bianchi manifold of $\mathbb{Q}(\sqrt{-2})$.

$$CS^{\mathbb{C}}_{L(2,1)}(A) = [\log_{\mathbf{p}} reg_{\mathbf{p}}(g_{0,a_2} \cup g_{0,b_2}), \eta]_N \tag{4.2}$$

Due to Hilbert reciprocity of $\mathbf{p}^n$ tower of modular curve, we have $\mathbb{Z}/2$ at infinite such that $CS^{\mathbb{C}}_{L(2,1)}(A)$ is the spin CS invariant(Arason invariant). We choose η to be Dieudonne 1-form on modular curve with cusp from $\mathbb{Z}/2$ involution on modular curve from sign structure. We have Seigel unit by modular CS invariant for $L(2,1)$ such that we add to big $\mathbb{Q}$ point on $X_1(13)$ from modular origin, where a priori we have no rational point in $X_1(13)$ due to Mordell-Weil rank vanish for Galois cohomology of $\mathbb{Q}(\sqrt{13})$.

Now we choose cusp $(-q/N\mathbf{p}^n p)$ at modular curve $X_1(N\mathbf{p}^n)$ by torsion in $K_2(\mathbb{Q}(\mu_{N\mathbf{p}^n}))$. Then with obtain spin CS invariant for $L(p,q)$ from cusp form as special value of $\mathbf{p}$-adic L function localized on specific geodesic between cusps $-q/p$ and and infinity.

After determining $N = N\mathfrak{f}$ for integral ideal in totally real number field F from $\mathbf{p}$-adic reciprocity on conductor system, we have spin CS invariant for $L(p,q)$ with specific choice of cusp from supersingular decomposition on supersingular locus over sufficiently ramified field. By choosing specific cusp, we have sign refined spin CS invariant for $L(p,q)$ from cup product of two

Siegel unit associated with $\gamma = \begin{pmatrix} -q & -r \\ p & s \end{pmatrix} \in SL(2,\mathbb{Z}), qs - rp = 1$, with $b_1 = \frac{-q}{\mathbf{p}^n p}, b_2 = \frac{-r}{\mathbf{p}^n s}$. .

$$CS^{\mathbb{C}}_{L(p,q)}(A) = [\log_{\mathbf{p}} reg_{\mathbf{p}}(g_{0,a_2} \cup g_{0,b_2}), \eta]_N \tag{4.3}$$

where η is Dieudonne 1-form on modular curve with $-p/q$ cusp.

So we obtain sign refined spin CS invariant as $\mathbf{p}$-adic regulator image of cup product of two Siegel unit as Stark-Heegner unit. We have extremal field configuration from Siegel unit satisfying self dual Yang Mills equation on Fake Fano surface $\mathbb{CP}^2 \times \overline{\mathbb{CP}^2}$(Mori dream space) with exotic smooth structure with $L(p^2, pq-1)$ surgery. We have Fake Fano surface(Beauville surface) which is locally birational to $\mathbb{P}^2$ with wild singular fiber with higher ramification group, as supersingular surface over sufficiently ramified field realizing rank 2 Selmer sheaves of Galois cohomology of totally real number field from $\mathbf{p}$-adic Ceresa cycle providing big $\mathbb{Q}$ point of infinite order, even of we have Mordell-Weil rank 0 for totally real number field $F = \mathbb{Q}(\sqrt{4p^2 - 3q^2})$.

The field configuration from Stark-Heegner unit determine sign from Fermion state of $\mathbf{p}$-adic origin(higher ramification group) for square root from $\mathbf{p}$-adic reciprocity. With spin CS invariant, we have sign for square root of geometry which is sign of spin structure from higher ramification group by Hilbert reciprocity - $\mathbf{p}$-adic Gamma function reciprocity at arithmetic infinity. This is obtained by double Fourier transform for $\mathbf{p}$-adic zeta function of real quadratic field [4].

Then we have integral representation with extremal field configuration from Stark-Heegner unit which is weight 2 modular form. We choose $A = d\log x$ with $x = q_N^{b_2 - d_2} \frac{g_{a+b} g_{a-b}}{g_a^2 g_b^2} \frac{g_c^2 g_d^2}{g_{c+d} g_{c-d}}$ satisfying $x \wedge (1-x) = 0$ from K_2 torsion of sufficiently ramified field. Let M be a 3 manifold of Lens space type, we consider

M as pro-$\mathbf{p}$ space with rigid analytic structure.

$$CS(A)^{\mathbb{C}}_M = \int_M Tr(AdA + \frac{1}{3}A(A \wedge A)) \tag{4.4}$$

$$= \int_D d\log x \wedge d\log(1-x) = [\log_{\mathbf{p}} reg_{\mathbf{p}}(g_{0,b_1}, g_{0,b_2}), \eta]_N$$

The integrals are performed in rigid analytic sense.The first integral formally represents triple product of two weight 2 modular form from Siegel unit and 1 weight 1 modular form from $\mathbf{p}$-adic cyclotomic unit η on modular curve with $-q/p$ cusp such that we have Stark-Heegner unit of totally real number field F. The second integral is performed in $\mathbf{p}$-adic domain D with rigid analytic topology after transgression by Bloch-Ogus differential d_2 for motivic complex realising Milnor homomorphism.

Note that all the rigid analytic integration, we only need to consider reciprocity from conductor N, by specialization of Stark-Heegner unit. We can consider them as averaging CM type of Hilbert class field K of totally real number field F with $[K : F] = N$. Then after averaging CM type which represent integral over pro-$\mathbf{p}$ 3 manifold in eq.(4.4), we have Stark-Heegner unit which is $12N$ power of Siegel unit.

This realize the pro-$\mathbf{p}$ completion of fundamental group of 3 manifold with ramification prime/knot of pro-$\mathbf{p}$ covering of M. With the $\mathbf{p}$-torsion in pro-$\mathbf{p}$ tower of Bianchi manifold of $L = \mathbb{Q}(\sqrt{-p})$ which is the completion of fundamental group of 3 manifold M,the Galois representation for irregular Sasaki-Einstein CY is realized. For that we need to take M to be Bianchi manifold with vanishing first Betti number which is not Haken, but Haken pro-$\mathbf{p}$ covering with modularity from $GSp_4(\mathbb{Q})$ Galois representation by metaplectic lift.

By supersingular decomposition over sufficiently ramified field we can reduce to supersingular elliptic curve over sufficiently ramified field associated with pro-

$\mathbf{p}$ covering of Bianchi manifold of $L = \mathbb{Q}(\sqrt{-i}), \mathbb{Q}(\sqrt{-3})$. Each of then we have elliptic Fermat quotient with degree $4,3$ over $\mathbf{p} = 1 \mod 4,3$.

By second Milnor homomorphism, we have sign refined spin CS invariant of Lens space. Associated with this we need to have weight 2 motivic complex realizing rank 2 Selmer sheaves associated with $\mathbf{p}$-adic Ceresa cycle, which can also obtained from third Milnor homomorphism. Note that the Milnor homomorphism is determined recursively with cup product of Siegel unit.

We remark that the equivalence between second Milnor homomorphism and third Milnor homomorphism provide global mirror symmetry for pro-$\mathbf{p}$ conifold geometry associated with pro-$\mathbf{p}$ covering of $L(p,q)$ by Galois cohomology of totally real number field $\mathbb{Q}(\sqrt{4p^2 - 3q^2})$ by Stark-Heegner unit.

By weight 2 motivic complex with Bloch Ogus map d_2 in Section.4.1, we obtain class CH^2 of moduli space of flat connection on Heegaard surface of pro-$\mathbf{p}$ covering of M.We consider moduli space of flat connection on pro-$\mathbf{p}$ covering of $L(p,q)$(as $\mathbf{p}$-adic 3 manifold) which is equivalent to moduli space of representation of pro-$\mathbf{p}$ fundamental group of $L(p,q)$ as $\mathbf{p}$-adic uniformizing space which is moduli space of Galois represntation for comparison with the Arason invariant.

We have Arason invariant of quadratic form of degree p over Hilbert class field K of totally real number field F from third Milnor homomorphism of Galois cohomology of F (obtained recursively from second Milnor homomorphism -by transgression with Bloch-Ogus differential) in Section.4.1. We bypass the use of path integral by using the extremal field configuration from Seigel unit which is weight 2 modular form[1].

For given quadratic form of degree p over Hilbert class field K(CM field),

[1]This can be considered as saddle point approximation but with global data from reciprocity of Siegel unit from specialization. For higher ramification group, by specialization to g_{0,b_2}, we can see N reciprocity from sum over generalized CM type with $[K : F] = N$.

we have Brauer-Severi varieties X with splitting field as Hilbert class field K associated with Galois cohomology of totally real number field F. On X, Bloch Ogus map is defined as,

$$d_2 : H^0(X, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2))) \to H^2(X, \mathcal{H}^2(\mathbb{Q}/\mathbb{Z})) = CH^2(X)/p$$

realizing rank 2 Selmer sheaves by weight 2 motivic complex. Over sufficiently ramified field, we have higher ramification group from twistor transformation along twistor line as wild monodromy at infinity associated with weight 2 motivic complex. With this, we have sign for Arason invariant which is orthogonal Galois representation, then we have Galois representation of totally real number field with higher ramification group(non-maximal conductor N). The third Milnor homomorphism is realizing with almost holomorphic CS invariant of Brauer Severi varieties of degree p quadratic form over Hilbert class field K of totally real number field F which is the splitting field.By third Milnor homomorphism, we have subvarieties X^3 in Brauer Severi varieties X of degree p quadratic form,which is classifying space of $Cl(2p)$ with splitting field by Hilbert class field K of totally real number field F.

$$X^3 = \lim_{\leftarrow} X_0(N\mathbf{p}) \times X_1(\mathbf{N}\mathbf{p}^n)^{\times 2}$$

Since higher Milnor homomorphism is obtained recursively, we have realization of global mirror geometry by pro-**p** geometry of Lens space with sign from higher ramification group. We have global mirror between cotangent bundle of Lens space $L(p, q)$ with pro-**p** covering $L(p^2, pq - 1)$ and Brauer Severi varieties for Galois cohomology of totally real number field with higher ramification group. Then by almost real projection by Galois coinvariance, we have **p**-adic Ceresa cycle(almost real cycle) from third Milnor homomorphism which is non-torsion mod 2 algebraic realizing rank 2 Selmer sheaves for big $\mathbb{Q}$ point from **p**-adic origin.

Since we choose the extremal field configuration $A = d\log x$ with $x \wedge (1-x) = 0$, it satisfy the global reciprocity which can be reduce to reciprocity of N choice of specialization of g_{0,b_1} with $1 \leq b_1 \leq N$ realizing averaging generalized CM type of Hilbert class field K of totally real number field F with $[K : F] = N$. So we have extremal field configuration A from torsion in $K_2(\mathbb{Q}(\mu_{N\mathbf{p}^n})$ satisfying Steinberg relation. The specialization of $K_2(\mathbb{Q}(\mu_{N\mathbf{p}^n})$ torsion(Siegel unit) provide modular symbol satisfying 3 term relation which is relation for elliptic motive. With this we have complete control of mod $\mathbf{p}$ reciprocity of higher ramification group.

Using the Stark Heegner unit from K_2 of maximal unramified extension(containing Hilbert class field) of F totally ramified at $\mathbf{p}$ and ramified at N, we recover spin CS invariant for Lens space with non-Abelian central extension with additional ramification prime $N = e$ prime to $\mathbf{p}$ from Section.3.4 which determine higher ramification group.

So for the Lens space, by Stark unit,we have the $CS^{\mathbb{C}}(A)_{L(p,q)}$ with $A = d\log_{\mathbf{p}} x$ such that the Stark Heegner unit ${u_l}^{\sigma}$ equal to modular CS invariant. We have reciprocity of p-th Legandre symbol in Section.3.4 of $\mathbf{p}$-adic Gauss sum $(\frac{n}{\mathbf{p}})_p$ of Fermat quotient with sign from root number.

$CS^{\mathbb{C}}(A)_{L(p,q)}$ recover sign refined spin CS invariant from $\mathbf{p}$-adic Landau Ginzburg model($\mathbf{p}$-adic Gauss sum) for Lens space with sign from wild ramification. We recover extremal volume of $Y^{p,q}$ in Hilbert class field K as $CS^{\mathbb{C}}(A)_{L(p,q)}$ from the cup product of Siegel unit which is the $\mathbf{p}$-adic epsilon factor of $\mathbf{p}$-adic zeta function of totally real number field. We have the Greenberg $\mathcal{L}$ invariant from $\mathbf{p}$-adic period $q_E = G_{\mathbf{p},p}$ of elliptic curve E -the $\mathbf{p}$-adic Gauss sum of elliptic curve. We consider elliptic curve as simple factor in Jacobian of Fermat curve of degree p, realizing supersingular decomposition over sufficiently ramified field. Note that $\mathbf{p}$-adic Hecke character satisfy supersingular reciprocity which enable is to express as $\mathbf{p}$-adic period of elliptic curve which is supersingular over sufficiently ramified field. We can express $\mathbf{p}$-adic Hecke character from Weierstrass σ function which is essentially the same as Siegel unit.

$$\mathcal{L} = \frac{\log_{\mathbf{p}} G_{\mathbf{p},p}}{ord_{\mathbf{p}} G_{\mathbf{p},p}} = \frac{\log_{\mathbf{p}} u_l{}^{\sigma}(\chi)}{ord_{\mathbf{p}} u_l{}^{\sigma}(\chi)}$$

with Stark-Heegner unit $u_l{}^{\sigma}(\chi)$ for real quadratic number field F. The character χ only need for comparison due to reciprocity of choice of character of sufficiently ramified field. For each character, we have a specific cyclotomic unit from specialization of Siegel unit. The comparison comes from the χ to be the mod $\mathbf{p}$ Hecke character of Fermat quotient C and splitting constant e of $\mathbf{p}$-adic τ function to be the conductor $N\mathfrak{f} = N$ with the cusp $-q/p$ for modular symbol realising discriminant of field $F = \mathbb{Q}(\sqrt{4p^2 - 3q^2})$ from pro-$\mathbf{p}$ covering of cusp. So Greenberg invariant $\mathcal{L}$ from Stark-Heegner unit determine constant term on Eisenstein series for Eisenstein cocycle with specified cusp.

Finally we only need to introduce elliptic curve with modular parametrization $X_0(N) \to E$ for higher ramification group. There exist universal torsor of E realizing extremal elliptic fibered CY which is the F-theory background whose singular fiber depending on the conductor N realising generalized CM lift by Hilbert class field K of totally real number field F with $[K : F] = N$. So even if Mordell-Weil rank 0 over $\mathbb{Q}$ we have has a section of extremal fibered CY from big $\mathbb{Q}$ point of infinite order with Galois action over sufficiently ramified field which realise generalized CM point. We have supersingular transition of extremal elliptic fibered CY(changing singular fiber) from p homothety class of $\mathbf{p}$-torsion point with intersection of $\mathbf{p}$ torsion point.Even if the Mordell-Weil rank 0 over $\mathbb{Q}$ for totally real number field $F = \mathbb{Q}(\sqrt{4p^2 - 3q^2})$ for irregular Sasaki-Einstein CY(total space of canonical bundle over toric irregular Fano surface), The extremal elliptic fibered CY corresponds has a section from big

$\mathbb{Q}$ point of infinite order. On triple product of modular curve, we have $\mathbf{p}$-adic Ceresa cycle realizing non-rigid representation with non-vanishing symmetric differential.

The specialization for $N = e$ enable us to recover $\mathbf{p}$-torsion data for Fermat quotient C. After supersingular decomposition to elliptic curve, the $\mathbf{p}$-adic Hecke character which is $\mathbf{p}$-adic Gamma function can be expressed as Weierstrass σ function. The Hecke character determine specific parameter x, y of $\sigma(x, y)$ such that we have $3, 4$ choice of roots number depending on odd, even p. This is the modular structure of Modular symbol relation as K_2 torsion of sufficiently ramified field.

This realize the Weierstrass gap sequence of Fermat curve, which can be seen as real multiplication type by arbitrary degeneration. The supersingular transition over sufficiently ramified field which change Fermat quotient of degree p over finite field of characteristic $\mathbf{p} = 1 \mod p$ in moduli space of rigid analytic curve with sign, is realized by different choice of root number depending on $\mathbf{p}$. By supersingular decomposition, this is realized by sign of elliptic Hecke character by modular symbol relation with $\mathbb{Z}/2$ involution in modular curve. At arithmetic infinity we have $\mathbb{Z}/3$ and $\mathbb{Z}/2$ from Mordell-Weil torsion realising supersingular transition with choice of 3 modulus(resp 2 modulus) sign.

By supersingular decomposition over sufficiently ramified field, we can reduce elliptic Fermat quotient of degree $p = 3, 4$ over finite field $\mathbf{p} = 1 \mod 3$ and $\mathbf{p} = 1 \mod 4$ resp. Then by modular embedding with Weierstrass $\mathfrak{p}$ function we have Siegel unit as Weierstrass σ function. Then the root number computation reduce to elliptic Fermat quotient case with $p = 3, 4$ choice of root number,from elliptic function identity realizing Ekedahl-Oort stratification in supersingular locus over sufficiently ramified field, which provide totally real number field cascade. By the reciprocity of Siegel unit, $CS^{\mathbb{C}}(A)$ satisfies the global reciprocity.

By choosing specific ansatz(field configuration) for CS characteristic class by Siegel unit, we obtain CS invariant by **p**-adic regulator map. The extremal field configuration solving nonlinear differential equation(self dual Yang Mill equation with exotic monopole from Sasaki-Einstein singularity) which we recovered from Stark unit as solution of Monge-Ampere equation by extremal Kahler Einstein metric.

For suitable choice,the additional sign from intrinsic/pure Higgs state mirror to almost Hodge Tate which is Hodge Tate over sufficiently ramified field, which is determined with irregular Killing spinor on irregular Sasaki Einstein manifold. The intrinsic/pure Higgs state counting is real GW counting which is the counting of the maximal destabilising subsheaves which destabilised by Frobenius pull-back. Note that over sufficiently ramified field, we can replace almost Hodge Tate state to sign refined Hodge Tate which is supersingular Hodge class over sufficiently ramified field[2], such that we have sign restored algebraicy. And we consider the sign and the (irrational) extremal volume as the wild ramification datum from plurisubharmonic function satisfying Monge Ampere equation. The extremal volume is the constant term in Eisenstein cocycle(series) by Stark-Heegner unit and it determine sign by **p**-adic epsilon factor ($\mathcal{L}$ invariant) in functional equation with global Galois representation. The sign is determined by conductor which satisfy **p**-adic reciprocity for modified Shintani cocycle.

As we seen before, the singular Kahler Einstein metric on irregular Sasaki-Einstein CY has extremal metric with irrational K-stability which is not algebraic but restore algebraicity by sign of conductor for wild ramification(higher

[2]We call it supersingular Hodge class over sufficiently ramified field as arithmetic supersingular Hodge class[16].

ramification group). The sign restored algebraicity comes from metaplectic lift of Bianchi manifold by pro-**p** covering with 4-dimensional Galois representation realising Haken covering.

For the determination of sign from CS partition function, we need to evaluate at even roots of unity(spin CS). By the Galois action on level of CS action functional, we have the non-Abelian Galois representation from correspondence for Galois action on level where sign is determined wild p group action at inverting prime realising wild Frobenius on level.We also consider the fractional level with **p**-adic interpolation for analytic continuation on level away from root of unity by wild Frobenius.Then by **p**-adic reciprocity we restore sign refined algebraicity and sign refined finiteness.

By considering the Hilbert class group $Gal(K/\mathbb{Q})$ for $[K : F] = N$ with F from arbitrary 2d toric diagram by the projection of symmetric 3d diagram, we have 3 dimensional lattice symmetry and recover ideal class group for cubic field. The cubic field embedding is done by considering sufficiently ramified field containing Hilbert class field K. By sum over generalized CM type $[K : F] = N$ we have pro-**p** structure realizing symmetric 3d partition function associated with pro-**p** covering of Lens space in Hilbert modular surface. So Hilbert class field K which is extension of F by Stark-Heegner unit represent averaging all cubic field extension with measure from Shimura varieties over sufficiently ramified field.

The CS partition function is the section of the determinant line bundle on moduli space of flat connection on 3 manifold M. By considering pro **p** lifting of $\pi_1(M)$, The flat connection corresponds to the **p**-adic Galois representation of number field which comes from geometry(modular Galois representation which is global Galois representation). The spin refinement gives the specific Galois representation from arithmetic infinity non-Abelian(orthogonal) Galois group,from global reciprocity $\Gamma(1-x)\Gamma(x) = \frac{\pi}{\sin(\pi x)}$ representing Steinberg element $x \wedge (1-x)$ in $K_2(\mathbb{C})$. The Seiberg dualtiy(N step cluster mutation by N Markov triple) realize the loop/deloop functor in algebraic K group over sufficiently ramified field which gives the integral structure with almost real structure by pro-$\mathbf{p}$ structure. Such that we have $\mathbf{p}$-adic uniformazation.

Note that elliptic genus from Weierstrass σ function and its orbifold limit $\frac{\sin(\pi x)}{\pi}$ is the simplest non-trivial modular unit which provide the twistor spinor class(Gamma class with sign) as non-torsion mod 2 algebraic class, from Steinberg element(Siegel unit) which is K_2 torsion class of sufficiently ramified field.

By using power residue symbol(p-th Legadre symbol) from second Milnor homomorphism of Galois cohomology number field with coefficient in p-th root of unity,satisfying Hilbert reciprocity, the regulator map was defined by cup product of two Siegel unit. So our definition of CS invariant $CS^{\mathbb{C}}(A)$ satisfy the global reciprocity with $\mathbb{Z}/2$ sign at arithmetic infinity.The global(Hilbert) reciprocity is obtained by connecting finite place $\mathbf{p}$ and the arithmetic infinite ∞.

We note that for the Sasakian geometry with extremal Reeb foliation, we have irrational Reeb vector by twistor spinor(Killing spinor). The the cone of irregular Sasaki-Einstein 5 space is the example of rationally connected non-rigid CY 3 fold. We have associated supersingular extremal elliptic surface at characteristic 2 which is unirational with Mordell-Weil rank 0.

For elliptic fibered CY with non-vanishing Mordell-Weil rank, we have infinite generation of Griffith group from non-torsion class in Abel-Jacobi Kernel inducing CY with multi section representing pfaffian-Grassmannian derived equivalent pair.There exist associated Galois representation of totally real number field but not of Sasaki-Einstein type and arise from Horrocks-Mumford bundle by intersection of two surface with same base. This also can realised from automorphism of Leech lattice over supersingular locus over sufficiently ramified field. Such that, we have the string theory $N = 1$ vacua associated with generalized CM point by Hilbert class field K of totally real number field F with non-zero vanishing of $\mathbf{p}$-adic zeta function by Mordell-Weil rank in Shafarevich-Tate group. Then we have higher dimensional determinantal varieties with secondary polytope with exceptional collection in derived category. Note that in Sasaki-Einstein case with vanishing Mordell-Weil rank, the derived category does not have exceptional collection and have only quasi-phantom factor.

For irregular Sasaki-Einstein manifold, we considered the non trivial central extension for CS partition function with mutually coprime conductors $N, \mathbf{p}$ for wild ramification. With non-trivial sign(spin structure) from Swan conductor of higher ramification group, we need a complex place at arithmetic infinity from $\mathbf{p}$-adic Etale path in $\mathbf{p}$-adic non-Abelian Hodge theory. Such that, we obtain Hilbert class field as with one complex place at infinity from big $\mathbb{Q}$ point of infinite order. So we recover p-adic non-Abelian Hodge theory from our modular CS theory via $\mathbf{p}$-adic zeta function of totally real number field. We have wall crossing on boundary of the compactification of the moduli of Higgs bundle from mod $\mathbf{p}$ reduction with $\mathbf{p}$-adic reciprocity providing Hitchin-Mochizuki map over sufficiently ramified field. Then we have the moduli of Higgs bundle(semi-stable sheaves) as perfectoid space having sign refined purity from $\mathbf{p}$-adic uniformazation for Galois completion[3] and signed determinant line bundle for opers.

[3] The perfectoid space encode the Hilbert reciprocity of sufficiently ramified field,which is the functorial structure(formal scheme) on moduli space of Higgs bundle with rank n and degree d on curve with level structure. As supersingular Shimura curve over sufficiently ramified field , we have Drinfeld modular curve for irrational/logarithmic super Conformal Field theory.

4.1 Arason invariant

Here we construct number theory analog of spin refined CS invariant called Arason invariant using use Galois cohomology and motivic complex,[8]. We consider third Milnor homomorphism. Let F be the number field containing p-th root of unity and X is Brauer Severi varieties with splitting filed K containing $spec(K)$ where K is the Hilbert class field of totally real number field F. There exist the determinant line bundle on X from motivic complex of X.

The third Milnor homomorphism for Galois cohomology in coefficient in μ_p is follows. Here the p is the prime for coefficient system. p is the Dynkin index of $GL(p)$. Here we consider degree p quadratic form with $GL(p)$ action with Clifford lift $Cl(2p)$ with splitting field as Hilbert class field K of totally real number field F.

$$a_3 : K_3(F) \simeq H^3(F, {\mu_p}^3)$$

Consider the complex.

$$K_3(F(X)) \to \coprod_{x\in X, codim(x)=1} K_2(F(x)) \to \coprod_{x\in X, codim(x)=2} K_1(F(x)) \to \coprod_{x\in X, codim(x)=3} K_0(F(x))$$

Let $\mathcal{K}_2$ is Zarizki sheaf of $U \to K_2(U)$. The cohomology of above complex compute,

$$K_3(F(X))_{ind} \to H^0(X, \mathcal{K}_2) \to H^1(X, \mathcal{K}_2) \to H^2(X, \mathcal{K}_2)$$

We consider the weight 2 motivic complex, $\Gamma(2)$ whose hypercohomology gives above complex.

With $\alpha : X_{et} \to X_{zar}$, $R^i\alpha_*\Gamma(2)$ has nontrivial term for $i = 2, 4$ with $R^2\alpha_*\Gamma(2) = \mathcal{K}_2$ and $R^4\alpha_*\Gamma(2) = \mathcal{H}^{i-1}(\mathbb{Q}/\mathbb{Z}(2))$.

$$\mathcal{H}^i{}_{Zar}(X,\Gamma(2)) = \begin{cases} K_3(F(X))_{ind}, & \text{i=1;} \\ H^i{}_{Zar}(X,\mathcal{K}_2), & \text{i=2,3,4;} \\ 0, & \text{otherwise.} \end{cases}$$

Then we have associated etale version with extension by $a_3 : K_3(F) \simeq H^3(F, {\mu_p}^3)$.

Let $\mathcal{H}^i(\mathcal{F})$ be the Zariski sheaf associated with presheaf $U \to {H_{et}}^i(U,\mathcal{F})$.

$\mathbb{Q}/\mathbb{Z}(2) = \lim_{l\to\infty,(p,charF)=1} {\mu_l}^{\otimes 2} \oplus \lim_r W_r{\Omega_{\log}}^2[-2]$ is the Etale sheaf where $W_r{\Omega_{\log}}^2$ is the sheaf of logarithmic Witt de Rham differential of X. Note that X can be the Brauer Severi varieties over F with splitting field K(Hilbert class field), with sign from higher ramification group of totally real number field.

$$CH_2(X) = \mathcal{H}^4{}_{Zar}(X,\Gamma(2)) \to \mathcal{H}^4{}_{et}(X,\Gamma(2)) \to H^0{}_{Zar}(X,\mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2))) \quad (4.5)$$

$$\mathcal{H}^i{}_{et}(X,\Gamma(2)) = \begin{cases} K_3(F(X))_{ind}, & \text{i=1;} \\ H^i{}_{Zar}(X,\mathcal{K}_2), & \text{i=2,3;} \\ \mathcal{H}^4{}_{et}(X,\Gamma(2)), & 4; \\ 0, & \text{otherwise.} \end{cases}$$

Using ${H_{Zar}}^0(X,\mathcal{H}^3({\mu_p}^{\otimes 2})) \to {H_{Zar}}^0(X,\mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2)))$, for $(p, charF) = 1$, applying to eq.(4.5) with multiplying p map, we get the structure from the Witt ring, the Galois group of field extension of F ramified at p.

Then it gives Bloch-Ogus differential d_2.

$$d_2 : H^0{}_{Zar}(X,\mathcal{H}^3({\mu_p}^{\otimes 2})) \to CH^2(X)/p \simeq H^2{}_{Zar}(X,\mathcal{H}^2({\mu_p}^{\otimes 2}))$$

From this, we get p-torsion class a_3 on X whose image is $d_2(a_3) = c_2(V) = 0 \in CH^2(X)/p$.

There is map $\alpha : X_{et} \to X_{Zar}$. With the resolution $\Gamma(2) \simeq I^{\cdot}$,with differential d, consider with multiply p map on $\alpha_* I^{\cdot}$ with cokernel quasi-isomorphic to ${\mu_p}^{\otimes 2}$.

Any element of p-torsion class $e \in H^0(X, R^4\alpha_*\Gamma(2)) = {H^0}_{Zar}(X, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2)))$ in Zariski covering $X_\bullet$ with $e_i \in \Gamma(X_i, \alpha_* I^4)$, $de_i = 0, pe_i = df_i$ has a lifting to a class in ${\mathbb{H}^4}_{et}(X, \Gamma(2)) = {H_{Zar}}^2(X, \mathcal{H}^2({\mu_p}^{\otimes 2}))$.

Using Cech cocycle with Cech differental δ, $e = (e_{i_0 \ldots i_4}, e_{i_0 \ldots i_3}, \ldots, e_{i_0})$,

$$pe = (pe_{i_0 \ldots i_4}, \ldots, pe_{i_0}) = (pe_{i_0 \ldots i_4}, \ldots, pe_{i_0 i_1} - \delta f_{i_0 i_1}, 0)$$

This is the construction of the secondary characteristic class $df_{i_0} = \delta f_{i_0 i_1}$.

We can do on step further.

$$pe = (pe_{i_0 \ldots i_4}, \ldots, pe_{i_0 i_1 i_2} - \delta f_{i_0 i_1 i_2}, 0, 0)$$

representing $df_{i_0 i_1} = \delta f_{i_0 i_1 i_3}$. Then the class $pe_{i_0 i_1 i_2} - \delta f_{i_0 i_1 i_2} \in CH^2(X)$ is d_2 image of p-torsion e.

If the image $d_2(e) = 0 \in CH^2(X)/2$ then we get secondary characteristic class called Arason class (which is Chern Simons class with $\mathbb{Z}/2$ refinement) with sign from higher ramification group. The sign of Arason invariant is determined by twistor transformation along twistor line by non-Abelian Fourier transformation with higher ramification group determining **p**-adic epsilon factor for global Galois representation, from weight 2 motivic complex over Witt ring/over finite field with wild ramification prime.

The construction recover BRST construction with spin refinement.

By Etale covering of X with Clifford bundle $Cl_p\ \bar{V}$,

$$[\bar{V}]^* : {\mathcal{H}^{2i}}_{et}(BCl_p, \Gamma(i)) \to {\mathcal{H}^{2i}}_{et}(X, \Gamma(i))$$

Then the class $e = [\bar{V}]^* \gamma_2 \in {\mathcal{H}^4}_{et}(X, \Gamma(2))$ with $\gamma_i \in {\mathcal{H}^{2i}}_{et}(BCl(p), \Gamma(i))$ is not the Zariski cohomology class but Etale cohomology class which correspond to spin refined CS characteristic class, unlike the universal $GL(p)$ bundle where we have the Chern class in the Zariski cohomology class.

So the class e depends on the Etale topology of X. For example the class $e = [\bar{V}]^* \gamma_2$ does not need to be algebraic class, so that $d_2(e) \nsubseteq CH^2(X)$ but lies in $CH^2(X)/2$.

If image of $d_2(e) = 0$,it lies in $CH^2(X)/2$(note that it already in $CH^2(X)/p$ by construction). This is the Arason invariant of $\bar{V}$ for $\mathbb{Z}_2$-torsion class in Etale topology. Over Witt vector, we obtain **p**-adic Ceresa cycle from p-adic Etale cohomology class $d_2(e)$ of X, which gives the example of codimension 2 cycle comes from codimension 2 singularity with infinite image(infinite order) in Griffiths group. Such class is non-rigid representation which is non-torsion from wild singularity(arithmetic/non-arithmetic lattice for Bianchi manifold).

In this paper, the **p**-adic lift of Maslov class corresponds to Arason invariant over field of characteristic **p** provide infinite generation of the Griffiths group by $CH^3(Y)$ where Y is elliptic fibered CY from modular completion of Sasaki-Einstein orbifold CY.

Chapter 5

Conclusion

We consider **p**-adic zeta function of real quadratic number field F with Hilbert class field K associate with singular CY(Sasaki-Einstein CY) in string theory. From Lens space and associated Bianchi manifold, we obtain torsion Galois representation with choice of conductor for we Sasaki-Einstein CY manifold. For that we need to consider supersingular locus.

At negative argument, we obtain integral zeta value from **p**-adic congruence which is higher genus partition function(integral GW invariant). At $s = 0$ we obtain the Stark unit whose specialization(limit) is the Black Hole entropy in Hilbert class field as the extremal volume of Sasaki-Einstein manifold. As the consequence of the algebraicity of Stark unit from **p**-adic reciprocity, we recover the algebraicity of Black hole entropy.

So we have the Stark Heegner $\mathbb{Q}$ point associated with generalized CM point on moduli space of CY for Sasaki Einstein CY. Such point is the generalized attractor point with complexity(height) as black hole entropy with Sasaki-Einstein manifold as horizon. Due to Seiberg duality cascade of finite order toward IR limit,we have conductor with sign. We also have dense wall near big CM point realizing Teichmuller curve in moduli space of CY. This can be seen as dense irrational CFT nearby rational CFT(big CM point by Hilbert class field K of totally real number field F) in moduli space of 2d CFT realizing non-commutative geometry by irrational CFT. With this we have complete description of moduli space of $N = 1$ vacua by equidistribution of generalized CM point with torsion global Galois representation in Shimura varieties over sufficiently ramified field for irregular Sasaki-Einstein CY associated with toric irregular Fano surface.

From Seiberg duality cascade, we obtain boundary divisor for toric orbifold base of Sasaki-Einstein manifold as Bruhat Tits building(moduli space of Higgs bundle) at infinity. Fake projective space has quasi-phantom factor from correspondence(Seiberg duality).

We have associated extremal elliptic fibered CY realising absolute Galois group action on moduli space of irregular Sasaki-Einstein CY giving modular realization with sign from number of connected component of mirror N=1 curve.

In our formulation, we start from the Shintani cocycle associated with Lens space. By considering the **p**-adic lifting of Shintani cocycle with choice of conductor and Hilbert class field, we obtain integrality of zeta value, by integral expression, for Sasaki-Einstein CY.

We show the algebraicity of Stark Heegner unit and the algebraicity of string partition function from reciprocity of **p**-adic measure by sign.

We also consider torsion Galois representation arise from pro-**p** covering of Bianchi manifold with vanishing first Betti number in supersingular locus in Siegel modular varieties over sufficiently ramified field. As the non-arithmetic automorphism of Leech lattice(N=4 string vacua) in supersingular locus over sufficiently ramified field, we pro-**p** covering of Bianchi manifold of $\mathbb{Q}(\sqrt{-p})$

and Lens space $L(p^2, pq-1)$ knot from exotic smooth structure with Haken covering from big $\mathbb{Q}$ point of infinite order having Stark-Heegner unit of totally real number field $\mathbb{Q}(\sqrt{4p^2-3q^2})$.

So we can compare $\mathbf{p}$-adic Galois representation in supersingular locus over sufficiently ramified field from pro-$\mathbf{p}$ geometry of boundary of Hilbert modular surface $L(p,q)$ Lens space of real quadratic field $F = \mathbb{Q}(\sqrt{4p^2-3q^2})$ and pro-$\mathbf{p}$ covering of Bianchi manifold associated with string theory on $Y^{p,q}$ CY and associated exotic smooth structure from Dehn twist surgery by $L(p^2, pq-1)$.

We interpret the exotic smooth structure from Painleve 6-th equation with irrational parameter $\hat{a} \in K$ for Sasaki-Einstein manifold as parabolic structure on singularity. With this we have rank 2 Higgs bundle over $\mathbb{P}^1/\{0,1,\hat{a},\infty\}$-moduli space of stability condition for mirror LG model for toric orbifold base of Sasaki-Einstein manifold. We obtain the boundary divisor from Harmonic embedding of mirror curve to Jacobian by τ function, as compactification of $SL(2,F)$ Bruhat-Tits building by Building cocycle. From this, we obtain extremal elliptic fibered CY with higher ramification group from Mordell-Weil torsion for non-maximal conductor of wild singular fiber(not ADE type) for $SL(2,F)$ Bruhat-Tits building cocycle with for F-theory with supersingular extremal elliptic curve over characteristic 2 realising supergeometry.

We also introduce p-th Legandre symbol which recover sign refined CS invariant Lens space by $\mathbf{p}$-adic Hecke character with sign from root number. Using this, we lead $\mathbf{p}$-adic Landau Ginzburg model associated with Fermat quotient for wild singularity with Lens space link. From sign of $\mathbf{p}$-adic Hecke character with $\mathbf{p}$-adic reciprocity, we describe supersingular transition between N=1 vacua realizing Higgsing by choice of higher homothety class.

We obtain 5 dimensional irregular Reeb dynamics from non-Abelian central extension by conductor system of pro-$\mathbf{p}$ covering of Lens space $L(p,q)$ realising $\mathbf{p}$-adic reciprocity conductor for $\mathbf{p}$-adic zeta function. Note that by mod $\mathbf{p}$ reduction, we determine generalized CM by K with $[K : F] = N$ pro-$\mathbf{p}$ asymptotic defined over $\mathbb{Q}$ with Absolute Galois action by lowest homothety class from choice of Hecke character $(\mathbb{Z}/\mathbf{p}-1)^{\times}$. With this we obtain real GW invarinat for sign for irregular geometry.

We recover CS invariant of 3 manifold(Lens space and Bianchi manifold) as the $\mathbf{p}$-adic regulator map which is the cup product of two Stark-Heegner unit. The unit do the role for limiting configuration. From our sign refined spin CS invariant $CS^{\mathbb{C}}(A)_{L(p,q)}$, where we obtain sign from Swan conductor with analytic continuation from $\mathbf{p}$-adic reciprocity.

By the gluing construction of 3 manifold and Poincare conjecture with arithmetic MMP and gluing from refined topological vertex by cup product of specialization of Siegel unit, we can extend our consideration for arbitrary 3 manifold associated with totally real number field with non-vanishing Mordell-Weil rank for irregular non-toric Fano surface with higher dimensional determinantal varieties realization. our $\mathbf{p}$-adic analysis gives the Ricci flow reaching supersingular S^3 topology over sufficiently ramified field with conductor from higher ramification group.

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국문초록

[[다음 문단들을 한국어로 번역하시오]]

In this thesis, we consider **p**-adic zeta function of real quadratic number field and relate it to string partition function on irregular Sasaki-Einstein CY orbifold. With this we have additional sign from conductor. Then we obtain algebraicity of **p**-adic zeta function of totally real number field F and AdS dual 4d SCFT index(Hilbert series) of Sasaki-Einstein manifold, by Stark-Heegner unit in Hilbert class field K or F.

We analyze the extremal metric from the Heun equation(Painleve 6-th), and relate to integrable system for elliptic surface with torsion Mordell Weil group. We recover integrable system from the cluster transformation of Poissin algebra(path algebra of Sasaki-Einstein quiver) with symplectic double. For Sasaki-Einstein CY, we have **p**-adic Galois representation which is the torsion globa Galois representation in supersingular locus over sufficiently ramified field from pro-**p** covering of Bianchi manifold.

By Dehn twist with punctured torus, we consider exotic 4 manifold by Lens space surgery, and conjecture that Lens space realizing knot as ramification knot for irrational parameter of Painleve 6-th equation. By Hitchin moduli space of rank 2 vector bundle on $\mathbb{P}^1/\{0, 1, \hat{a}, \infty\}$, with parabolic structure for $\hat{a} \in K$, we obtain moduli space of stability condition on mirror of orbifold Fano base of Sasaki-Einstein manifold.

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