이학석사 학위논문

Representations by a binary quadratic form with class number 3

(류수가 3인 이변수 이차형식의 표현)

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최 재 민

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이 논문을 이학석사 학위논문으로 제출함

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Representations by a binary quadratic form with class number 3

by

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A DISSERTATION

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Abstract

For integers a, b, c, the homogeneous quadratic polynomial $f(x, y) = ax^2 + bxy + cy^2$ is called a binary quadratic form. In this thesis, we will give a closed formula on the number of integral solutions of $x^2 + 27y^2 = n$ for any integer n. To do this, we follow the framework given by Min and Oh in [4], where they considered the binary form $x^2 + 32y^2$. In section 2, we briefly survey on the theory of binary quadratic forms and give some lemmas which are needed in the later. In section 3, we consider the case when n is a prime power. In section 4, we consider the case when n is any integer relatively prime to 6. In section 5, we summarize everything and give a complete formula on the number of solutions of the above equation.

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1 Introduction

For an integral binary quadratic form $f(x,y) = ax^2 + bxy + cy^2$, it is quite an old problem to find the number of integer solutions of the equation $f(x,y) = n$ for some integer n. If the class number of f is one, then the answer of this problem is completely known. However there is no known general method on this problem when the class number of f is arbitrary.

In 2006, Sun and Williams [5] considered this problem when the class number of f is less than or equal to 4. Recently, Min and Oh [4] introduced a different method on the case when $f(x,y) = x^2 + 32y^2$ and gave a closed formula on the number of solutions under the assumption that the set of primes that are represented by each form having same discriminant is completely known.

In this thesis, we consider the case when $f(x,y) = x^2 + 27y^2$. Note that the class number of $x^2 + 32y^2(x^2 + 27y^2)$ is 4(3, respectively). The method adopted in this thesis is quite similar to [4].

In section 2, we introduce some notations and terminologies. Unexplained notations and terminologies will follow that of Hua's book [2].

In section 3, we consider the case when n is a prime power. In most cases, we consider the form $x^2 + 3y^2$ instead of $x^2 + 27y^2$. Since the class number of $x^2 + 3y^2$ is one, everything is quite well known. Furthermore $x^2 + 3y^2 = n$ has an integer solution such that y is divisible by 3 if and only if $x^2 + 27y^2 = n$ has an integer solution.

In section 4, we consider the case when n is any integer relatively prime to the discriminant. In many cases, we use an induction and all results obtained from Section 3 will be crucially used in this section as a first step of the induction.

Finally in section 5, we summarize everything and provide a complete closed formula on the number of solutions of the above equation.

Throughout this thesis, we assume that the set of primes that are represented by each form of discriminant -108 is completely known.

2 Some Technical Lemmas

In this section, we briefly survey on the theory of binary quadratic forms and give some lemmas which are needed in the later.

Let $f(x,y) = ax^2+bxy+cy^2$ be a positive definite integral binary primitive form of discriminant d, that is, $\gcd(a,b,c) = 1$, $a > 0$, $d = b^2 - 4ac < 0$. Then we briefly define $f = \{a,b,c\}$.

We define $h(d)$ the number of equivalence classes of primitive forms with discriminant d. For a precise definition of the class number, see [2].

From each class of primitive positive definite forms we select a representative giving a representative system which we denote by

$$F_1, \cdots, F_{h(d)}.$$

Theorem 2.1. *Let k be a positive integer such that $(k,d) = 1$, and denote by $\psi(k)$ the total number of solutions to*

$$k = F_1(x,y), \cdots, k = F_{h(d)}(x,y).$$

Then

$$\psi(k) = \omega \sum_{n|k} \left(\frac{d}{n}\right),$$

where

$$\omega = \begin{cases} 2 & \text{if } d < -4, \\ 4 & \text{if } d = -4, \\ 6 & \text{if } d = -3. \end{cases}$$

Proof. See [[2], 12.4.1].

If $h(d)$=1, then we may give an exact formula for the number of solutions of the binary form with discriminant d.

Corollary 2.2. *Let m be any positive integer relatively prime to 6. Then, for any non negative integers a and b, we have*

$$\sharp\{(x,y) \in \mathbb{Z}^2 \mid x^2 + 3y^2 = 2^a 3^b m\} = t \sum_{k|m} \left(\frac{-3}{k}\right),$$

where

$$t = \begin{cases} 2 & \text{if } a \text{ is } 0, \\ 0 & \text{if } a \text{ is odd}, \\ 6 & \text{otherwise}. \end{cases}$$

Proof. See [[3], 2.2.17].

For any integers n, we define

$$R(n,f) := \{(x,y) \in \mathbb{Z}^2 \mid f(x,y) = n\}, \text{ and } r(n,f) = |R(n,f)|.$$

Clearly, $d(\{1,0,27\}) = -108$ and one may easily check that $h(-108) = 3$. Furthermore if $\gcd(n,-108) = 1$, then we have

$$2\sum_{k|n}\left(\frac{-108}{k}\right) = r(n,f_1) + r(n,f_2) + r(n,f_3)$$

by Theorem 2.1. Here

$$f_1 = x^2 + 27y^2, f_2 = 4x^2 + 2xy + 7y^2, f_3 = 4x^2 - 2xy + 7y^2.$$

We first consider the case when $\gcd(n,-108) \neq 1$.

Lemma 2.3. *Let m be any positive integer relatively prime to 6. For any positive integer a or b, we have*

$$r(2^a3^bm, x^2+27y^2) = t\sum_{k|m}\left(\frac{-3}{k}\right),$$

where

$$t = \begin{cases} 2 & \text{if } a = 0 \text{ and } b \geqslant 2 \text{ or } a \geqslant 2 \text{ and } b = 0, \\ 6 & \text{if } a \text{ is a positive even integer and } b \geqslant 2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If $b = 1$, then we can easily check that there is no integer solution of this equation.

Suppose that $x^2 + 27y^2 = 2^a3^bm$ for $b \geqslant 2$. Then x should be divisible by 3. If we put $x = 3x_1$, then we obtain an equation $x_1^2 + 3y^2 = 2^a3^{b-2}m$. Hence

$$r(2^a3^bm, x^2+27y^2) = \sharp\{(x,y) \in \mathbb{Z}^2 \mid x^2+3y^2 = 2^a3^{b-2}m\} = t\sum_{k|m}\left(\frac{-3}{k}\right),$$

where

$$t = \begin{cases} 2 & \text{if } a \text{ is } 0, \\ 0 & \text{if } a \text{ is odd}, \\ 6 & \text{otherwise.} \end{cases}$$

by Corollary 2.2.

Now suppose that $b = 0$. If $a = 1$, then we can easily show that there is no solution of this equation.

Next, suppose that $x^2 + 27y^2 = 2^a m$ for $a \geqslant 2$. Then clearly $x - y$ is even. If we put $y = x - 2y_1$, then we obtain an equation $7x^2 - 27xy_1 + 27y_1^2 = 2^{a-2}m$. Note that there is a bijection from $\{(x,y) \in \mathbb{Z}^2 \mid 7x^2 - 27xy_1 + 27y_1^2 = 2^{a-2}m\}$ to $\{(x,y) \in \mathbb{Z}^2 \mid 7x^2 + xy + y^2 = 2^{a-2}m\}$ given by $(\alpha, \beta) \longmapsto (\alpha - 2\beta, \beta)$. Clearly, $d(\{7,1,1\}) = -27$ and one may easily check that $h(-27) = 1$. Since $(-27, 2^{a-2}m) = 1$,

$$\sharp\{(x,y) \in \mathbb{Z}^2 \mid x^2 + 27y^2 = 2^a m\} = \sharp\{(x,y) \in \mathbb{Z}^2 \mid 7x^2 + xy + y^2 = 2^{a-2}m\}.$$

Hence we have

$$\sharp\{(x,y) \in \mathbb{Z}^2 \mid 7x^2 + xy + y^2 = 2^{a-2}m\} = 2\sum_{k|m} \left(\frac{-3}{k}\right).$$

The lemma follows from this. $\square$

Now we consider the case when $\gcd(n, -108) = 1$ for any integer n.

Lemma 2.4. *Let $n = p_1^{e_1} \cdots p_r^{e_r} r_1^{g_1} \cdots r_t^{g_t}$ be any positive integer relatively prime to 6, where p_i, r_k are primes such that $p_i \equiv 1 \pmod 3$ and $r_k \equiv 2 \pmod 3$ for any i, j. Then we have*

$$r(n, x^2 + 27y^2) = \begin{cases} 0 & \text{if } g_k \text{ is odd for some } k, \\ r(p_1^{e_1} \cdots p_r^{e_r}, x^2 + 27y^2) & \text{if } g_1, \cdots, g_t \text{ are all even.} \end{cases}$$

Proof. First, we assume that $x^2 + 27y^2 \equiv 0 \pmod p$ for some prime p such that $p \equiv 2 \pmod 3$. Note that

$$\left(\frac{-3}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 3, \\ -1 & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

If $x \not\equiv 0 \pmod p$, then $y \not\equiv 0 \pmod p$ and $-3 \equiv \left(\frac{x}{3y}\right)^2 \pmod p$. This is a contradiction. Hence $x \equiv y \equiv 0 \pmod p$. This implies that $ord_p(x^2 + 27y^2) \equiv 0 \pmod 2$. Therefore, if (x,y) is an integer solution of this equation such that $x, y \not\equiv 0 \pmod p$, then $p \equiv 1 \pmod 3$.

Assume that $g_k \equiv 1 \pmod 2$. Since $r_k \equiv 2 \pmod 3$, $x, y \equiv 0 \pmod{r_k}$. Furthermore since $x^2 + 27y^2 \not\equiv 2 \pmod 3$, for any $x, y \in Z$,

$$\sharp\{(x,y) \in \mathbb{Z}^2 \mid x^2 + 27y^2 = n\} = \sharp\{(x,y) \in \mathbb{Z}^2 \mid x^2 + 27y^2 = p_1^{e_1} \cdots r_k^1 \cdots r_t^{g_t}\} = 0.$$

Assume that $g_k \equiv 0 \pmod 2$ for any k. Then we have

$$\sharp\{(x,y) \in \mathbb{Z}^2 \mid x^2 + 27y^2 = n\} = \sharp\{(x,y) \in \mathbb{Z}^2 \mid x^2 + 27y^2 = p_1^{e_1} \cdots p_r^{e_r}\}.$$

The lemma follows from this. □

From now on, we consider the case when $x^2 + 27y^2 = p_1^{e_1} \cdots p_r^{e_r}$, where p_i is a prime such that $p_i \equiv 1 \pmod 3$ for any i.

Lemma 2.5. *For any prime p, the equation $x^2 + 27y^2 = p$ has an integer solution if and only if $p \equiv 1 \pmod 3$ and 2 is a cubic residue modulo p.*

Proof. See [[1], 1.D.22].

From now on,

$$P := \{p : \text{ prime } \mid p \equiv 1 \pmod 3, \ \ r(p, x^2 + 27y^2) \neq 0\}$$

and

$$Q := \{p : \text{ prime } \mid p \equiv 1 \pmod 3, \ \ r(p, x^2 + 27y^2) = 0\}.$$

Example 2.6. By Lemma 2.5, we have $P = \{31, 43, 109, 127, 157, \cdots\}$.

Definition 2.7. For two solutions (x_1, y_1), (x_2, y_2) of the equation $x^2 + ky^2 = n$ for $k \geqslant 2$, we say they are *essentially different* if

$$(x_2, y_2) \neq (x_1, y_1), (x_1, -y_1), (-x_1, y_1) \text{ and } (-x_1, -y_1).$$

If $x^2 + ky^2 = n$ has no pair of essentially different solutions, then the number of solutions is zero, two or four.

Theorem 2.8. *Let $(x_1, y_1), (x_2, y_2)$ be solutions of $x^2 + ky^2 = n$ and (s_1, t_1), (s_2, t_2) be solutions of $x^2 + ky^2 = m$ for $k \geqslant 2$. Assume that $(k, nm) = 1$, $(n, m) = 1$, $y_1 t_1 s_1 \neq 0$. If at least one pair of two equations is essentially different, then both*

$$(x_1 s_1 + k y_1 t_1, x_1 t_1 - y_1 s_1), \ (x_1 s_1 - k y_1 t_1, x_1 t_1 + y_1 s_1)$$

and

$$(x_1 s_1 \pm k y_1 t_1, x_1 t_1 \mp y_1 s_1), \ (x_2 s_2 \pm k y_2 t_2, x_2 t_2 \mp y_2 t_2)$$

are all essentially different solutions of the equation $x^2 + ky^2 = nm$.

Proof. See [4].

3 Prime Power Cases

In this section, we consider the case when n is a prime power.

Lemma 3.1. *Let n be an any positive integer relatively prime to 6. Then*

$$\sharp\{(x,y) \in \mathbb{Z}^2 \mid x^2+27y^2 = n\} = \sharp\{(x,y) \in \mathbb{Z}^2 \mid x^2+3y^2 = n,\ y \equiv 0 \pmod 3\}.$$

Proof. One may easily show that the map form $\{(x,y) \in \mathbb{Z}^2 \mid x^2+27y^2 = n\}$ to $\{(x,y) \in \mathbb{Z}^2 \mid x^2+3y^2 = n,\ y \equiv 0 \pmod 3\}$ given by $(\alpha,\beta) \longmapsto (\alpha, 3\beta)$ is bijective. □

Lemma 3.2. *Assume that p is a prime such that $p \equiv 1 \pmod 3$ and e is a positive integer. Then the equation $x^2+3y^2 = p^e$ has a solution (x,y) such that $(xy,p) = 1$.*

Proof. We will use an induction on e.

When $e = 1$, we can easily check that every integer solution (x,y) of $x^2+3y^2 = p$ satisfies $(xy,p) = 1$. Let (a,b) be an integer solution of $x^2+3y^2 = p$ such that $(ab,p) = 1$. Suppose that the equation $x^2+3y^2 = p^{e-1}$ has a solution (s,t) such that $(st,p) = 1$. Then

$$(as \pm 3bt, at \mp bs)$$

are solutions of $x^2+3y^2 = p^e$. Assume that $at \mp bs \equiv 0 \pmod p$. Then $(at-bs)+(at+bs) \equiv 2at \equiv 0 \pmod p$, which is a contradiction. So p doesn't divide one of $at+bs$ and $at-bs$. Therefore there is a solution (x,y) of $x^2+3y^2 = p^e$ such that $(xy,p) = 1$. By induction, the lemma is proved. □

Lemma 3.3. *Let p be a prime in P and e be any non negative integer. For any solution (x,y) of $x^2+3y^2 = p^e$, y is divisible by 3. Furthermore, $r(p^e, x^2+27y^2) = 2(e+1)$.*

Proof. Since $r(p^e, x^2+3y^2) = 2(e+1)$, it suffices to show that for any integers x, y such that $x^2+3y^2 = p^e$, y is divisible by 3.

We will use an induction on e. If $e = 1$, then the lemma follows from the fact $p \in P$. Assume that the lemma holds for $e-1$. Suppose that there are integers c, d such that $c^2+3d^2 = p^e$ and $d \not\equiv 0 \pmod 3$. Let a, b be integers such that $a^2+3b^2 = p$. Note that $b \equiv 0 \pmod 3$. Then $(ac \pm 3bd)^2 + 3(ad \mp bc)^2 = p^{e+1}$. Since

$$(ad-bc)(ad+bc) \equiv a^2d^2 - b^2c^2 \equiv -3b^2d^2 - b^2(-3d^2) \equiv 0 \pmod p,$$

we may assume, without loss of generality, that $ac + 3bd \equiv ad - bc \equiv 0 \pmod p$. Hence

$$\left(\frac{ac+3bd}{p}\right)^2 + 3\left(\frac{ad-bc}{p}\right)^2 = p^{e-1}.$$

Note that $\dfrac{ad-bc}{p} \equiv \dfrac{ad}{p} \not\equiv 0 \pmod 3$. This is a contradiction to the induction by hypothesis. Therefore $d \equiv 0 \pmod 3$, and

$$r(p^e, x^2 + 27y^2) = r(p^e, x^2 + 3y^2) = 2(e+1).$$

The lemma follows from this. □

Lemma 3.4. *Let q be a prime in Q. For any positive integer e, there is a unique essentially different solution (a, b) of $x^2 + 3y^2 = q^f$ such that $(ab, q) = 1$. Furthermore $b \equiv 0 \pmod 3$ if and only if $f \equiv 0 \pmod 3$.*

Proof. Assume that (a_i, b_i) be an integer solution of $x^2 + 3y^2 = q^i$ such that $(a_i b_i, q) = 1$. Note that such an integer solution always exists (see. Lemma 3.2). If $f = 2k$, then

$$(a_0 q^k, 0), (a_2 q^{k-1}, b_2 q^{k-1}), \cdots, (a_{2k}, b_{2k})$$

are all essentially different solutions of $x^2 + 3y^2 = q^f$. If $f = 2k + 1$, then

$$(a_1 q^k, b_1 q^k), (a_3 q^{k-1}, b_3 q^{k-1}), \cdots, (a_{2k+1}, b_{2k+1})$$

are all essentially different solutions of $x^2 + 3y^2 = q^f$. Therefore only (a_{2k}, b_{2k}) or (a_{2k+1}, b_{2k+1}) is the solution satisfying the condition. This proves the first statement. Assume that $a_1^2 + 3b_1^2 = q$. Then $(a_1 b_1, q) = 1$ and $b_1 \not\equiv 0 \pmod 3$ for $q \in Q$. Note that $(a_1^2 - 3b_1^2, 2a_1 b_1)$, $(a_1^2 + 3b_1^2, 0)$ are all essentially different solutions of $x^2 + 3y^2 = q^2$. Hence we may let $(a_2, b_2) = (a_1^2 - 3b_1^2, 2a_1 b_1)$, by changing the sign suitably. Furthermore

$$((a_1^2 - 3b_1^2)a_1 \pm 3(2a_1 b_1)b_1, (a_1^2 - 3b_1^2)b_1 \mp 2a_1^2 b_1),$$

are all essentially different solutions of $x^2 + 3y^2 = q^3$. Note that

$$(a_1^2 - 3b_1^2)b_1 + 2a_1^2 b_1 \equiv 3b_1(a_1^2 - b_1^2) \equiv 0 \pmod 3$$

and

$$(a_1^2 - 3b_1^2)b_1 + 2a_1^2 b_1 \equiv 3b_1(a_1^2 - b_1^2) \equiv 3b_1(-4b_1^2) \not\equiv 0 \pmod q.$$

Therefore we may let $a_3 = (a_1^2 - 3b_1^2)a_1 - 3(2a_1b_1)b_1$, $b_3 = (a_1^2 - 3b_1^2)b_1 + 2a_1^2b_1$. More generally,

$$a_{n+3} = a_na_3 + 3b_nb_3, \ b_{n+3} = a_nb_3 - b_na_3$$

or

$$a_{n+3} = a_na_3 - 3b_nb_3, \ b_{n+3} = a_nb_3 + b_na_3.$$

Therefore

$$b_{n+3}^2 \equiv b_n^2 \pmod 3.$$

This completes the proof. □

Corollary 3.5. *Let q be a prime in Q.*

$$r(q^f, x^2+27y^2) = \begin{cases} \frac{2}{3}(f-1) & \text{if } f \equiv 1 \pmod 3, \\ \frac{2}{3}(f+1) & \text{if } f \equiv 2 \pmod 3, \\ \frac{2}{3}(f+3) & \text{if } f \equiv 0 \pmod 3. \end{cases}$$

Proof. The corollary follows directly from the above lemma. □

4 General Cases

In this section, we consider the general case. We will prove a complete formula for the number of solutions of $x^2 + 27y^2 = n$ for any positive integer n.

Lemma 4.1. *Let A, B be positive integers such that $(A, B) = 1$ and $(AB, 6) = 1$. For any integers a, b such that $a^2 + 3b^2 = AB$, there are integers c, d, e, f such that $c^2 + 3d^2 = A$, $e^2 + 3f^2 = B$ and*

$$(a, b) = \pm(ce + 3df, cf - de) \quad or \quad \pm(ce - 3df, cf + de).$$

Proof. Since all the other cases can be done in similar manner, we only provide the proof of the case when A is a perfect square and B is not a perfect square. Let

$$(a_0, 0), (a_1, b_1), \cdots, (a_s, b_s)$$

be all essentially different solutions of $x^2 + 3y^2 = A$ and

$$(c_1, d_1), \cdots, (c_t, d_t)$$

be all essentially different solutions of $x^2+3y^2=B$. Since B is not a perfect square, d_j is not zero for any j. Then

$$(ac_l, ad_l), (a_kc_l \pm 3b_kd_l, a_kd_l \mp b_kc_l)$$

are all essentially different solutions of $x^2+3y^2=AB$ by Theorem 2.8. Hence the number of solutions that are not essentially different from the above is $4t+8st$. Furthermore the total number of solutions of $x^2+3y^2=N$ is $2\sum_{k|N}\left(\frac{-3}{k}\right)$. for any integer N such that $(N,6)=1$. Since $(A,B)=1$, we have

$$2\sum_{k|AB}\left(\frac{-3}{k}\right)=2\sum_{k|A}\left(\frac{-3}{k}\right)\sum_{k|B}\left(\frac{-3}{k}\right).$$

Hence

$$2\sum_{k|AB}\left(\frac{-3}{k}\right)=\frac{1}{2}(2+4s)(4t)=4t+8st.$$

Therefore every solution of $x^2+3y^2=AB$ is not essentially different from one of solutions given above. The lemma follows from this. □

Lemma 4.2. *Let $p_1, \cdots, p_r$ be primes in P. Then for any non negative integers e_i,*

$$r(p_1^{e_1}\cdots p_r^{e_r}, x^2+27y^2)=2\prod_{i=1}^{r}(e_i+1).$$

Proof. The proof of this lemma is almost same to that of Lemma 3.3. Hence we omit the proof. □

Lemma 4.3. *Let $q_1,\ldots,q_s$ be primes in Q. If $n=q_1^{f_1}\cdots q_s^{f_s}$ for some non negative integers f_j, then*

$$r(n,x^2+27y^2)=\begin{cases}\frac{2}{3}\prod_{j=1}^{s}(f_j+1) & \textit{if } \prod_{j=1}^{s}(f_j+1)\equiv 0 \pmod 3,\\ \frac{2}{3}(\prod_{j=1}^{s}(f_j+1)+2) & \textit{if } \prod_{j=1}^{s}(f_j+1)\equiv 1 \pmod 3,\\ \frac{2}{3}(\prod_{j=1}^{s}(f_j+1)-2) & \textit{if } \prod_{j=1}^{s}(f_j+1)\equiv 2 \pmod 3.\end{cases}$$

Proof. We will use an induction on s. We already proved the case when $s = 1$. Assume that $s \geqslant 2$. Let $A = \prod_{j=1}^{s-1}(f_j + 1)$ and $B = f_s + 1$.

Case 1 If $A \equiv 0 \pmod 3$ and $B \equiv 0 \pmod 3$, then $AB = \prod_{j=1}^{s}(f_j + 1) \equiv 0 \pmod 3$. So it suffices to show that

$$r(n, x^2 + 27y^2) = \frac{2}{3}AB.$$

First, assume that $f_1, \cdots, f_s$ are all even. Let

$$(a, 0), (a_1, b_1), \cdots, (a_\alpha, b_\alpha)$$

be all essentially different solutions of $x^2 + 3y^2 = q_1^{f_1} \cdots q_{s-1}^{f_{s-1}}$ such that $b_k \equiv 0 \pmod 3$ and

$$(c, 0), (c_1, d_1), \cdots, (c_\beta, d_\beta)$$

be all essentially different solutions of $x^2 + 3y^2 = q_s^{f_s}$ such that $d_l \equiv 0 \pmod 3$. Here $2 + 4\alpha = \frac{2}{3}A$, $2 + 4\beta = \frac{2}{3}B$ by induction hypothesis. Then by Theorem 2.8,

$$(ac, 0), (ac_l, ad_l), (a_kc, b_kc), (a_kc_l \pm 3b_kd_l, a_kd_l \mp b_kc_l)$$

are all essentially different solutions of $x^2 + 3y^2 = q_1^{f_1} \cdots q_s^{f_s}$ such that

$$0, ad_l, b_kc, a_kd_l \mp b_kc_l \equiv 0 \pmod 3.$$

If (a', b') is a solution of $x^2 + 3y^2 = q_1^{f_1} \cdots q_{s-1}^{f_{s-1}}$ and (c', d') is a solution of $x^2 + 3y^2 = q_s^{f_s}$ such that $b', d' \not\equiv 0 \pmod 3$, then 3 divides exactly one of $a'd' \mp b'c'$. Let

$$(a'_1, b'_1), \cdots, (a'_\gamma, b'_\gamma)$$

be all essentially different solutions of $x^2 + 3y^2 = q_1^{f_1} \cdots q_{s-1}^{f_{s-1}}$ such that $b'_m \not\equiv 0 \pmod 3$ and

$$(c'_1, d'_1), \cdots, (c'_\delta, d'_\delta)$$

be all essentially different solutions of $x^2 + 3y^2 = q_s^{f_s}$ such that $d'_n \not\equiv 0 \pmod 3$. Here $4\gamma = 2A - \frac{2}{3}A$ and $4\delta = 2B - \frac{2}{3}B$. Then

$$(a'_mc'_n \pm 3b'_md'_n, a'_md'_n \mp b'_mc'_n)$$

are all essentially different solutions of $x^2 + 3y^2 = q_1^{f_1} \cdots q_s^{f_s}$. Here 3 divides exactly one of $a_m d_n \mp b_m c_n$. Therefore

$$r(n, x^2 + 27y^2) = 2 + 4\alpha + 4\beta + 8\alpha\beta + 4\gamma\delta = \frac{2}{3}AB.$$

Assume that f_j is odd for some j and f_s is even. Then every solution of $x^2 + 3y^2 = q_1^{f_1} \cdots q_{s-1}^{f_{s-1}}$ is not of the form $(a, 0)$. Hence $4\alpha = \frac{2}{3}A$. Therefore

$$r(n, x^2 + 27y^2) = 4\alpha + 8\alpha\beta + 4\gamma\delta = \frac{2}{3}AB.$$

Assume that $f_1, \cdots, f_{s-1}$ are all even and f_s is odd. Then every solution of $x^2 + 3y^2 = q_s^{f_s}$ is not of the form $(c, 0)$. Hence $4\beta = \frac{2}{3}B$. Therefore

$$r(n, x^2 + 27y^2) = 4\beta + 8\alpha\beta + 4\gamma\delta = \frac{2}{3}AB.$$

Finally, assume that f_j is odd for some j and f_s is odd. Then every solution of $x^2 + 3y^2 = q_1^{f_1} \cdots q_{s-1}^{f_{s-1}}$ is not of the form $(a, 0)$ and every solution of $x^2 + 3y^2 = q_s^{f_s}$ is also not of the form $(c, 0)$. Hence $4\alpha = \frac{2}{3}A$ and $4\beta = \frac{2}{3}B$. Furthermore

$$r(n, x^2 + 27y^2) = 8\alpha\beta + 4\gamma\delta = \frac{2}{3}AB.$$

Case 2 If $A \equiv 0 \pmod 3$ and $B \equiv 1 \pmod 3$, then $AB = \prod_{j=1}^{s}(f_j + 1) \equiv 0 \pmod 3$. So it suffices to show that

$$r(n, x^2 + 27y^2) = \frac{2}{3}AB.$$

Similarly to the case 1, we divide this case into 4 subcases and prove the lemma separately. First, assume that $f_1, \cdots, f_s$ are all even. Similarly, $2 + 4\alpha = \frac{2}{3}A$ and $2 + 4\beta = \frac{2}{3}(B + 2)$, $4\gamma = 2A - \frac{2}{3}A$, $4\delta = 2B - \frac{2}{3}(B + 2)$ by induction hypothesis. Hence

$$r(n, x^2 + 27y^2) = 2 + 4\alpha + 4\beta + 8\alpha\beta + 4\gamma\delta = \frac{2}{3}AB.$$

Assume that f_j is odd for some j and f_s is even. Similarly, $4\alpha = \frac{2}{3}A$. Hence

$$r(n, x^2 + 27y^2) = 4\alpha + 8\alpha\beta + 4\gamma\delta = \frac{2}{3}AB.$$

Assume that $f_1, \cdots, f_{s-1}$ are all even and f_s is odd. Similarly, $4\beta = \frac{2}{3}(B+2)$. Hence

$$r(n, x^2 + 27y^2) = 4\beta + 8\alpha\beta + 4\gamma\delta = \frac{2}{3}AB.$$

Finally, assume that f_j is odd for some j and f_s is odd. Similarly, $4\alpha = \frac{2}{3}A$ and $4\beta = \frac{2}{3}(B+2)$. Hence

$$r(n, x^2 + 27y^2) = 8\alpha\beta + 4\gamma\delta = \frac{2}{3}AB.$$

Case 3 If $A \equiv 0 \pmod 3$ and $B \equiv 2 \pmod 3$, then $AB = \prod_{j=1}^{s}(f_j + 1) \equiv 0 \pmod 3$. So it suffices to show that

$$r(n, x^2 + 27y^2) = \frac{2}{3}AB.$$

Similarly to the case 1, we divide this case into 4 subcases and prove the lemma separately. First, assume that $f_1, \cdots, f_s$ are all even. Similarly, $2 + 4\alpha = \frac{2}{3}A$ and $2 + 4\beta = \frac{2}{3}(B-2)$, $4\gamma = 2A - \frac{2}{3}A$, $4\delta = 2B - \frac{2}{3}(B-2)$ by induction hypothesis. Hence

$$r(n, x^2 + 27y^2) = 2 + 4\alpha + 4\beta + 8\alpha\beta + 4\gamma\delta = \frac{2}{3}AB.$$

Assume that f_j is odd for some j and f_s is even. Similarly, $4\alpha = \frac{2}{3}A$. Hence

$$r(n, x^2 + 27y^2) = 4\alpha + 8\alpha\beta + 4\gamma\delta = \frac{2}{3}AB.$$

Assume that $f_1, \cdots, f_{s-1}$ are all even and f_s is odd. Similarly, $4\beta = \frac{2}{3}(B-2)$. Hence

$$r(n, x^2 + 27y^2) = 4\beta + 8\alpha\beta + 4\gamma\delta = \frac{2}{3}AB.$$

Finally, assume that f_j is odd for some j and f_s is odd. Similarly, $4\alpha = \frac{2}{3}A$ and $4\beta = \frac{2}{3}(B-2)$. Hence

$$r(n, x^2 + 27y^2) = 8\alpha\beta + 4\gamma\delta = \frac{2}{3}AB.$$

Case 4 If $A \equiv 1 \pmod 3$ and $B \equiv 0 \pmod 3$, then $AB = \prod_{j=1}^{s}(f_j + 1) \equiv 0 \pmod 3$. So it suffices to show that

$$r(n, x^2 + 27y^2) = \frac{2}{3}AB.$$

Similarly to the case 1, we divide this case into 4 subcases and prove the lemma separately. First, assume that $f_1, \cdots, f_s$ are all even. Similarly, $2 + 4\alpha = \frac{2}{3}(A+2)$ and $2 + 4\beta = \frac{2}{3}B$, $4\gamma = 2A - \frac{2}{3}(A+2)$, $4\delta = 2B - \frac{2}{3}B$ by induction hypothesis. Hence

$$r(n, x^2 + 27y^2) = 2 + 4\alpha + 4\beta + 8\alpha\beta + 4\gamma\delta = \frac{2}{3}AB.$$

Assume that f_j is odd for j and f_s is even. Similarly, $4\alpha = \frac{2}{3}(A+2)$. Hence

$$r(n, x^2 + 27y^2) = 4\alpha + 8\alpha\beta + 4\gamma\delta = \frac{2}{3}AB.$$

Assume that $f_1, \cdots, f_{s-1}$ are all even and f_s is odd. Similarly, $4\beta = \frac{2}{3}B$. Hence

$$r(n, x^2 + 27y^2) = 4\beta + 8\alpha\beta + 4\gamma\delta = \frac{2}{3}AB.$$

Finally, assume that f_j is odd for j and f_s is odd. Similarly, $4\alpha = \frac{2}{3}(A+2)$ and $4\beta = \frac{2}{3}B$. Hence

$$r(n, x^2 + 27y^2) = 8\alpha\beta + 4\gamma\delta = \frac{2}{3}AB.$$

Case 5 If $A \equiv 2 \pmod 3$ and $B \equiv 0 \pmod 3$, then $AB = \prod_{j=1}^{s}(f_j + 1) \equiv 0 \pmod 3$. So it suffices to show that

$$r(n, x^2 + 27y^2) = \frac{2}{3}AB.$$

Similarly to the case 1, we divide this case into 4 subcases and prove the lemma separately. First, assume that $f_1, \cdots, f_s$ are all even. Similarly, $2 +$

$4\alpha = \dfrac{2}{3}(A-2)$ and $2+4\beta = \dfrac{2}{3}B$, $4\gamma = 2A - \dfrac{2}{3}(A-2)$, $4\delta = 2B - \dfrac{2}{3}B$ by induction hypothesis. Hence

$$r(n, x^2+27y^2) = 2 + 4\alpha + 4\beta + 8\alpha\beta + 4\gamma\delta = \frac{2}{3}AB.$$

Assume that f_j is odd for some j and f_s is even. Similarly, $4\alpha = \frac{2}{3}(A-2)$. Hence

$$r(n, x^2+27y^2) = 4\alpha + 8\alpha\beta + 4\gamma\delta = \frac{2}{3}AB.$$

Assume that $f_1, \cdots, f_{s-1}$ are all even and f_s is odd. Similarly, $4\beta = \dfrac{2}{3}B$. Hence

$$r(n, x^2+27y^2) = 4\beta + 8\alpha\beta + 4\gamma\delta = \frac{2}{3}AB.$$

Finally, assume that f_j is odd for some j and f_s is odd. Similarly, $4\alpha = \dfrac{2}{3}(A-2)$ and $4\beta = \dfrac{2}{3}B$. Hence

$$r(n, x^2+27y^2) = 8\alpha\beta + 4\gamma\delta = \frac{2}{3}AB.$$

Case 6 If $A \equiv 1 \pmod 3$ and $B \equiv 1 \pmod 3$, then $AB = \prod_{j=1}^{s}(f_j+1) \equiv 1 \pmod 3$. So we it suffices to show that

$$r(n, x^2+27y^2) = \frac{2}{3}(AB+2).$$

Similarly to the case 1, we divide this case into 4 subcases and prove the lemma separately. First, assume that $f_1, \cdots, f_s$ are all even. Similarly, $2 + 4\alpha = \dfrac{2}{3}(A+2)$ and $2+4\beta = \dfrac{2}{3}(B+2)$, $4\gamma = 2A - \dfrac{2}{3}(A+2)$, $4\delta = 2B - \dfrac{2}{3}(B+2)$ by induction hypothesis. Hence

$$r(n, x^2+27y^2) = 2 + 4\alpha + 4\beta + 8\alpha\beta + 4\gamma\delta = \frac{2}{3}(AB+2).$$

Assume that f_j is odd for some j and f_s is even. Similarly, $4\alpha = \dfrac{2}{3}(A+2)$. Hence

$$r(n, x^2+27y^2) = 4\alpha + 8\alpha\beta + 4\gamma\delta = \frac{2}{3}(AB+2).$$

Assume that $f_1, \cdots, f_{s-1}$ are all even and f_s is odd. Similarly, $4\beta = \frac{2}{3}(B+2)$. Hence

$$r(n, x^2+27y^2) = 4\beta + 8\alpha\beta + 4\gamma\delta = \frac{2}{3}(AB+2).$$

Finally, assume that f_j is odd for some j and f_s is odd. Similarly, $4\alpha = \frac{2}{3}(A+2)$ and $4\beta = \frac{2}{3}(B+2)$. Hence

$$r(n, x^2+27y^2) = 8\alpha\beta + 4\gamma\delta = \frac{2}{3}(AB+2).$$

Case 7 If $A \equiv 2 \pmod 3$ and $B \equiv 2 \pmod 3$, then $AB = \prod_{j=1}^{s}(f_j+1) \equiv 1 \pmod 3$. So it suffices to show that

$$r(n, x^2+27y^2) = \frac{2}{3}(AB+2).$$

Similarly to the case 1, we divide this case into 4 subcases and prove the lemma separately. First, assume that $f_1, \cdots, f_s$ are all even. Similarly, $2 + 4\alpha = \frac{2}{3}(A-2)$ and $2+4\beta = \frac{2}{3}(B-2)$, $4\gamma = 2A - \frac{2}{3}(A-2)$, $4\delta = 2B - \frac{2}{3}(B-2)$ by induction hypothesis. Hence

$$r(n, x^2+27y^2) = 2 + 4\alpha + 4\beta + 8\alpha\beta + 4\gamma\delta = \frac{2}{3}(AB+2).$$

Assume that f_j is odd for some j and f_s is even. Similarly, $4\alpha = \frac{2}{3}(A-2)$. Hence

$$r(n, x^2+27y^2) = 4\alpha + 8\alpha\beta + 4\gamma\delta = \frac{2}{3}(AB+2).$$

Assume that $f_1, \cdots, f_{s-1}$ are all even and f_s is odd. Similarly, $4\beta = \frac{2}{3}(B-2)$. Hence

$$r(n, x^2+27y^2) = 4\beta + 8\alpha\beta + 4\gamma\delta = \frac{2}{3}(AB+2).$$

Finally, assume that f_j is odd for some j and f_s is odd. Similarly, $4\alpha = \frac{2}{3}(A-2)$ and $4\beta = \frac{2}{3}(B-2)$. Hence

$$r(n, x^2+27y^2) = 8\alpha\beta + 4\gamma\delta = \frac{2}{3}(AB+2).$$

Case 8 If $A \equiv 1 \pmod 3$ and $B \equiv 2 \pmod 3$, then $AB = \prod_{j=1}^{s}(f_j+1) \equiv 2 \pmod 3$. So it suffices to show that

$$r(n, x^2+27y^2) = \frac{2}{3}(AB-2).$$

Similarly to the case 1, we divide this case into 4 subcases and prove the lemma separately. First, assume that $f_1, \cdots, f_s$ are all even. Similarly, $2+4\alpha = \frac{2}{3}(A+2)$ and $2+4\beta = \frac{2}{3}(B-2)$, $4\gamma = 2A-\frac{2}{3}(A+2)$, $4\delta = 2B-\frac{2}{3}(B-2)$ by induction hypothesis. Hence

$$r(n, x^2+27y^2) = 2+4\alpha+4\beta+8\alpha\beta+4\gamma\delta = \frac{2}{3}(AB-2).$$

Assume that f_j is odd for some j and f_s is even. Similarly, $4\alpha = \frac{2}{3}(A+2)$. Hence

$$r(n, x^2+27y^2) = 4\alpha+8\alpha\beta+4\gamma\delta = \frac{2}{3}(AB-2).$$

Assume that $f_1, \cdots, f_{s-1}$ are all even and f_s is odd. Similarly, $4\alpha = \frac{2}{3}(A+2)$. Hence

$$r(n, x^2+27y^2) = 4\beta+8\alpha\beta+4\gamma\delta = \frac{2}{3}(AB-2).$$

Finally, assume that f_j is odd for some j and f_s is odd. Similarly, $4\alpha = \frac{2}{3}(A+2)$ and $4\beta = \frac{2}{3}(B-2)$. Hence

$$r(n, x^2+27y^2) = 8\alpha\beta+4\gamma\delta = \frac{2}{3}(AB-2).$$

Case 9 If $A \equiv 2 \pmod 3$ and $B \equiv 1 \pmod 3$, then $AB = \prod_{j=1}^{s}(f_j+1) \equiv 2 \pmod 3$. So it suffices to show that

$$r(n, x^2+27y^2) = \frac{2}{3}(AB-2).$$

Similarly to the case 1, we divide this case into 4 subcases and prove the lemma separately. First, assume that $f_1, \cdots, f_s$ are all even. Similarly, $2+$

$4\alpha = \frac{2}{3}(A-2)$ and $2+4\beta = \frac{2}{3}(B+2)$, $4\gamma = 2A-\frac{2}{3}(A-2)$, $4\delta = 2B-\frac{2}{3}(B+2)$ by induction hypothesis. Hence

$$r(n, x^2+27y^2) = 2 + 4\alpha + 4\beta + 8\alpha\beta + 4\gamma\delta = \frac{2}{3}(AB-2).$$

Assume that f_j is odd for some j and f_s is even. Similarly, $4\alpha = \frac{2}{3}(A-2)$. Hence

$$r(n, x^2+27y^2) = 4\alpha + 8\alpha\beta + 4\gamma\delta = \frac{2}{3}(AB-2).$$

Assume that $f_1, \cdots, f_{s-1}$ are all even and f_s is odd. Similarly, $4\beta = \frac{2}{3}(B+2)$. Hence

$$r(n, x^2+27y^2) = 4\beta + 8\alpha\beta + 4\gamma\delta = \frac{2}{3}(AB-2).$$

Finally, assume that f_j is odd for some j and f_s is odd. Similarly, $4\alpha = \frac{2}{3}(A-2)$ and $4\beta = \frac{2}{3}(B+2)$. Hence

$$r(n, x^2+27y^2) = 8\alpha\beta + 4\gamma\delta = \frac{2}{3}(AB-2).$$

Hence the lemma is proved. □

Theorem 4.4. *Let $p_1, \cdots, p_r$ be primes in P and $q_1, \cdots, q_s$ be primes in Q. If $n = p_1^{e_1} \cdots p_r^{e_r} q_1^{f_1} \cdots q_s^{f_s}$, then for any non negative integers $e_i's$, $f_j's$,*

$$r(n, x^2+27y^2) = \begin{cases} \frac{2}{3}\prod_{i=1}^{r}(e_i+1)\prod_{j=1}^{s}(f_j+1) & \text{if } \prod_{j=1}^{s}(f_j+1) \equiv 0 \pmod 3, \\ \frac{2}{3}\prod_{i=1}^{r}(e_i+1)(\prod_{j=1}^{s}(f_j+1)+2) & \text{if } \prod_{j=1}^{s}(f_j+1) \equiv 1 \pmod 3, \\ \frac{2}{3}\prod_{i=1}^{r}(e_i+1)(\prod_{j=1}^{s}(f_j+1)-2) & \text{if } \prod_{j=1}^{s}(f_j+1) \equiv 2 \pmod 3. \end{cases}$$

.

Proof. By lemma 4.2, we have

$$r\left(\prod_{i=1}^{r} p_i^{e_i}, x^2+27y^2\right) = 2\prod_{i=1}^{r}(e_i+1).$$

Also by lemma 4.3, we have

$$r\left(\prod_{j=1}^{s} q_j^{f_j}, x^2+27y^2\right) = \frac{2}{3}(\prod_{j=1}^{s}(f_j+1)+a),$$

where $a = 0, 2, -2$. Let $2\prod_{i=1}^{r}(e_i+1) = E_r$ and $\frac{2}{3}(\prod_{j=1}^{s}(f_j+1)+a) = F_s$.

Case 1 Assume that $e_1,\cdots,e_r$ and $f_1, \cdots, f_s$ are all even. Let

$$(a,0),(a_1,b_1),\cdots,(a_s,b_s)$$

be all essentially different solutions of $x^2+3y^2 = \prod_{i=1}^{r} p_i^{e_i}$ such that $b_k \equiv 0 \pmod 3$ and

$$(c,0),(c_1,d_1),\cdots,(c_t,d_t)$$

be all essentially different solutions of $x^2+3y^2 = \prod_{j=1}^{s} q_j^{f_j}$ such that $d_l \equiv 0 \pmod 3$, where $2+4s = E_r$ and $2+4t = F_s$. Then by Theorem 2.8,

$$(ac,0),(ac_l,ad_l),(a_kc,b_kc),(a_kc_l \pm 3b_kd_l, a_kd_l \mp b_kc_l)$$

are all essentially different solutions of $x^2+3y^2 = n$ such that

$$0, ad_l, b_kc, a_kd_l \mp b_kc_l \equiv 0 \pmod 3.$$

Note that for every solution (a',b') of $x^2+3y^2 = \prod_{i=1}^{r} p_i^{e_i}$ such that $b' \equiv 0 \pmod 3$. If (c',d') is a solution of $x^2+3y^2 = \prod_{j=1}^{s} q_j^{f_j}$ such that $d' \not\equiv 0 \pmod 3$, then $a'd' \mp b'c' \not\equiv 0 \pmod 3$. Therefore

$$r(n, x^2+27y^2) = 2+4s+4t+8st = \frac{1}{2}E_rF_s.$$

Case 2 Assume that $f_1,\cdots,f_s$ are all even and e_i is odd for some i. Then every solution of $x^2+3y^2 = \prod_{i=1}^{r} p_i^{e_i}$ is not of the form $(a,0)$. Hence $4s = E_r$. Therefore we have

$$r(n, x^2+27y^2) = 4s+8st = \frac{1}{2}E_rF_s.$$

Case 3 Assume that $e_1,\cdots,e_r$ are all even and f_j is odd for some j. Then every solution of $x^2+3y^2=\prod_{j=1}^{s} q_j^{f_j}$ is not of the form $(c,0)$. Hence $4t=F_s$. Therefore we have

$$r(n, x^2+27y^2) = 4t + 8st = \frac{1}{2}E_rF_s.$$

Case 4 Assume that e_i is odd for some i and f_j is odd for some j. Then every solution of $x^2+3y^2=\prod_{i=1}^{r} p_i^{e_i}$ is not of the form $(a,0)$ and also every solution of $x^2+3y^2=\prod_{j=1}^{s} q_j^{f_j}$ is not of the form $(c,0)$. Hence $4t=F_s$ and $4s=E_r$. Therefore we have

$$r(n, x^2+27y^2) = 8st = \frac{1}{2}E_rF_s.$$

Thus the theorem is proved. □

5 Summary

In this section, we summarize all results proved in the previous sections. We give a closed formula for the number of representations of the equation $x^2+27y^2=n$.

Let n be a positive integer. Assume that $\gcd(n,-108)\neq 1$. Let $n=2^a3^bk$ for some integers k, a and b such that $(k,6)=1$ and some $a\geqslant 1$ or $b\geqslant 1$.

$$r(n, x^2+27y^2) = t\sum_{m|k}\left(\frac{-3}{m}\right),$$

where

$$t=\begin{cases} 2 & \text{if } a=0 \text{ and } b\geqslant 2 \text{ or } a\geqslant 2 \text{ and } b=0,\\ 6 & \text{if } a \text{ is a positive even integer and } b\geqslant 2,\\ 0 & \text{otherwise.}\end{cases}$$

Assume that $\gcd(n,-108)=1$. Let $n=p_1^{e_1}\cdots p_r^{e_r}q_1^{f_1}\cdots q_s^{f_s}r_1^{g_1}\cdots r_t^{g_t}$, where p_i,q_j,r_k are all primes such that $p_i\in P$ and $q_j\in Q$, $r_k\equiv 2 \pmod 3$

and e_i, f_j, g_k are all positive integers. If g_k is odd for some k, then

$$r(n, x^2 + 27y^2) = 0.$$

If g_k is even for any k, then

$$r(n, x^2+27y^2) = \begin{cases} \frac{2}{3}\prod_{i=1}^{r}(e_i+1)\prod_{j=1}^{s}(f_j+1) & \text{if } \prod_{j=1}^{s}(f_j+1) \equiv 0 \pmod 3, \\ \frac{2}{3}\prod_{i=1}^{r}(e_i+1)(\prod_{j=1}^{s}(f_j+1)+2) & \text{if } \prod_{j=1}^{s}(f_j+1) \equiv 1 \pmod 3, \\ \frac{2}{3}\prod_{i=1}^{r}(e_i+1)(\prod_{j=1}^{s}(f_j+1)-2) & \text{if } \prod_{j=1}^{s}(f_j+1) \equiv 2 \pmod 3. \end{cases}$$

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국문초록

정수 a, b, c에 대하여 $f(x, y) = ax^2 + bxy + cy^2$을 이변수 이차형식이라 한다. 이 논문에서는 정수 n에 대하여 $x^2 + 27y^2 = n$의 정수해의 개수를 구하는 공식을 제공한다. 이를 위해, 이차형식 x^2+32y^2에 대해 Min과 Oh가 제시한 체계를 따른다. 제2절에서는 이변수 이차형식에 대해 간략하게 살펴보고, 후에 필요한 보조정리를 제시한다. 제3절에서는 n이 한 소수에 대한 제곱인 경우에 대해 살펴본다. 제4절에서는 6과 서로소인 n인 경우에 대해 살펴본다. 제5절에서는 앞에서 살펴본 모든 경우에 대해 요약하고 $x^2 + 27y^2 = n$의 정수해의 개수를 구하는 완벽한 공식을 제공한다.

주요 어휘 : 이변수 이차형식
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