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A Survey of the McKay Correspondence

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Abstract

A Survey of the McKay Correspondence

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The McKay correspondence gives two constructions of simply laced Dynkin diagrams from a finite subgroup $G \subset SU(2)$–the resolution graph of the quotient singularity of $\mathbb{A}^2/G$, and the McKay graph constructed from the nontrivial irreducible representations of G. We review the computation of the classical McKay correspondence for the binary dihedral group BD_8, and discuss its generalizations. We also review the construction of tautological sheaves for the cyclic group as an example of a generalization to K-theory.

Keywords : McKay correspondence, McKay graph, Dynkin diagrams
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Contents

Introduction

The McKay correspondence describes two different ways to arrive at simply laced Dynkin diagrams from a finite subgroup G of $SU(2)$–one algebro-geometric and one representation theoretic.

The algebro-geometric construction arises when considering the composition of two natural algebraic constructions of singularities–the quotient and the resolution.

Firstly, given a finite group action on a variety $G \circlearrowright X$, the geometric quotient $q : X \to X/G$ is obtained by describing X/G as the set-theoretic orbit space (with orbit space projection map from X), with structure sheaf consisting of the G-invariant regular functions of X (with inclusion map). This is a way to *create* singularities.

Secondly, given a singular algebraic variety X, we can take a resolution: a smooth variety Y and a proper birational map $\pi : Y \to X$–an isomorphism outside the singular locus. The prime example of a resolution is the *blowup*, where the singular point is 'replaced' by all the lines passing through it (the *exceptional curve*).[1] This is a way to *eliminate* singularities.

We can then ask what kind of structure arises when we compose these two constructions, first taking the quotient and then the resolution.

A finite subgroup G of $SU(2)$ has an obvious action on the affine plane, and the resulting quotient singularity has been well studied by DuVal [DuV34].

[1]This is only a heuristic description, for a more formal definition, see [Har77].

$$
\begin{tikzcd}
Y \arrow[dr, "\pi"'] & \sim ? & X \arrow[dl, "q"] \\
& X/G &
\end{tikzcd}
$$

Figure 1: The resolution π and the quotient q

DuVal showed that the equivalent condition for a resolution of a surface singularity to be *minimal*[2] was for the singularity to be *rational*[3] with multiplicity 2. Moreover, from the consistency requirements of intersection numbers with the *fundamental cycle*[4] of the resolution, DuVal argued that the irreducible components of the exceptional divisor of the minimal resolution could only have incidence relations described by the simply laced Dynkin diagrams. DuVal showed that the *resolution graph*–consisting of a vertex for every irreducible component of the exceptional curve and an edge whenever two components intersect–could only be a Dynkin diagram[5] of type A_n, D_n, or E_6, E_7, E_8.[6]

Furthermore, DuVal showed that rational double singularities can be realized by the quotient of $\mathbb{A}^2$ by the action of a finite subgroup of $SU(2)$.[7] Thus in these cases, (the so called *Kleinian, DuVal, simple, or ADE-type singularities*), we a rather complete description: corresponding to the choice of G, after a quotient and a minimal resolution, the intersection data of the irreducible components of the exceptional divisor is described by a simply laced Dynkin diagram.

[2] It does not factor nontrivially through another resolution of singularities, or in this case, without exceptional curves with self intersection number 1.

[3] For complete surfaces, π preserves the arithmetic genus.

[4] a unique positive cycle whose intersection number with each irreducible component of the exceptional divisor is nonpositive, see [Art66].

[5] via an obvious identification of graphs

[6] the converse statement, that resolution graphs that correspond to simply laced Dynkin diagrams only arise from double points has been shown by Artin [Art66].

[7] These had been completely classified by Klein, and can be verified case by case.

In 1980, McKay [McK80] gave an alternate construction of the resolution graph of G in purely representation theoretic terms. He introduced the *McKay graph (or McKay quiver)* corresponding to a representation of the group (G, ρ). The McKay graph [8] is defined by having vertices corresponding to the irreducible representations $\{\rho_i\}$ of G, and with m_{ij} edges from ρ_i to ρ_j if and only if $\rho \otimes_{\mathbb{C}} \rho_i = \bigoplus_j \rho_j^{\oplus m_{ij}}$.[9]

McKay claimed that for the finite subgroups G of $SU(2)$, the McKay graph is described by simply laced *extended* Dynkin diagrams, which differ from the original diagrams only by an additional edge and vertex. This identification between resolution graph and the McKay graph is known as the *classical McKay correspondence*, and it has successfully been generalized to various wider settings. The guiding principle for generalization has been given in the following statement of M. Reid:

Principle. *The geometry of Y is equivalent to the G-equivariant geometry of X.*

What was meant by *geometry* was deliberately left vague. It has been known to hold for betti numbers [Bat98], bases of the cohomology [IR96], for the Grothendieck group of vector bundles [GSV83], [BKR01], and the derived category of coherent sheaves [KV00], [BKR01], for Hodge structures and physicists' Euler numbers [Bat98]. It has been known to hold for surfaces and threefolds other than the affine space [BKR01].

The groups under consideration have expanded from finite subgroups of $SU(2)$ [McK80] (or equivalently, of $SL(2, \mathbb{C})$), to finite abelian subgroups of $SL(3, \mathbb{C})$ [IR96]

[8] Here we have defined it with directed edges, however, $m_{ij} = \langle \chi_{\rho_i}, \chi_\rho \chi_{\rho_j} \rangle = \langle \chi_{\rho_i} \chi_\rho, \chi_{\rho_j} \rangle = m_{ji}$ shows it is undirected.

[9] The tensor product of two representations is another representation, and by Maschke's Theorem such a decomposition always exists.

to all finite subgroups of $SL(3,\mathbb{C})$ [BKR01]. More recently, it has been known to hold for finite subgroups of $\mathrm{Sp}(n,\mathbb{C})$ as well [Kal02].

There is an important point to note in the background, which is that minimal (or *crepant*[10]) resolutions do not always exist. They exist and are unique in dimension 2, exist but are not unique in dimension 3, and may not exist in dimensions greater than 3. Thus we cannot expect even a proper formulation of the statement for higher dimensions. [11]

In this paper, we want to give a (by no means comprehensive) review of these results. In Chapter 1, we give an explicit example of the classical McKay correspondence for the binary dihedral group as an illustration. For the proof we refer the reader to [Sun10] for a clean proof of the representation theory side, and [DuV34] for the proof of the resolution side. In Chapter 2, we examine some ways in which the McKay correspondence has been generalized and reinterpreted: to cohomology, to K-theory, and to looking at resolutions as Hilbert schemes.

[10] $K_Y \cong \pi^* K_X$

[11] In 3 dimensions, however, there is an interesting wall crossing phenomenon, where, the equivalence of derived categories still holds, with a nontrivial identification between them, see [CI04], [ABC$^+$09].

1 An Example: the Binary Dihedral Group BD_8

The classical McKay correspondence can be summarized by the following table:

Conjugacy Class	Defining Equation	Resolution Graph	McKay Graph
Z_{n+1}	$x^2 + y^2 + z^{n+1}$	A_n	$\widetilde{A_n}$
$BD_{2n}, (n \geqslant 4)$	$x^2 + y^2 z + z^{n-1}$	D_n	$\widetilde{D_n}$
BT_{24}	$x^2 + y^3 + z^4$	E_6	$\widetilde{E_6}$
BO_{48}	$x^2 + y^3 + yz^3$	E_7	$\widetilde{E_7}$
BI_{120}	$x^2 + y^3 + z^5$	E_8	$\widetilde{E_8}$

Table 1: The classical McKay correspondence.

The first three columns can be seen as results from classical algebraic geometry, and the last column as McKay's new observation in 1980.

The rows of the table describe Klein's classification of finite subgroups of $SU(2)$ [12] into five different classes up to conjugacy–Z_{n+1} the cyclic group of order $n+1$, BD_{2n} the binary dihedral group of order $2n$, BD_{24} the binary tetrahedral group of order 24, BO_{48} the binary octahedral group of order 48, and BI_{120} the binary icosahedral group of order 120. [13]

In this chapter, we illustrate the McKay correspondence for the case of the binary dihedral group of order 8. The group has the following presentation:

$$G := BD_8 = \langle \sigma, \tau | \sigma^4 = e, \tau^2 = \sigma^2, \tau^{-1}\sigma\tau = \sigma^{-1} \rangle.$$

[12] In fact, to $SL(2, \mathbb{C})$, since any finite subgroup $G \subset SL(2, \mathbb{C})$ is conjugate to a finite subgroup of $SU(2)$. We can define a new hermitian inner product by averaging the standard inner product in $\mathbb{C}^2$ over the elements of g. Then any action of $g \in G$ preserves this hermitian inner product, so that it can be considered unitary in this new hermitian product.

[13] We can understand the binary groups in the following way. The finite subgroups of $SO(3, \mathbb{R})$ are easy to see as the symmetry groups of Platonic solids, and the cyclic and dihedral groups up to conjugacy. The binary groups in $SU(2)$ are the preimages of these groups under the double cover $\pi : SU(2) \twoheadrightarrow SO(3, \mathbb{R})$.

1.1 The Resolution graph of BD_8

In this section, we compute the resolution graph of BD_8. We follow the computation from [lew08].

Computation 1.1. *The resolution graph of BD_8 is described by the Dynkin diagram D_4.*

The inclusion map $G \hookrightarrow SU(2)$ is given by

$$\sigma \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \tau \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It induces an action on $\mathbb{C}[\mathbb{A}^2] = \mathbb{C}[u, v]$, the ring of regular functions of $\mathbb{A}^2$, by

$$\sigma \cdot (u, v) = (iu, -iv), \qquad \tau \cdot (u, v) = (-v, u).$$

Then the ring of G-invariant regular functions is [14]

$$C[u, v]^G = \mathbb{C}[(u^4 - v^4)uv, u^4 + v^4, u^2v^2],$$

and from the change of coordinates:

$$x(u, v) := (u^4 - v^4)uv \qquad y(u, v) := u^4 + v^4 \qquad z(u, v) := u^2v^2,$$

[14] The zeros of the generators correspond to the summits of the dihedron, the midpoints between the summits, and the poles. One can see $SU(2)$-$PSL(2, \mathbb{C})$ (or $SO(3, \mathbb{R})$) correspondence as the 2-to-1 correspondence for each orbit $G \cdot (u, v) \to G \cdot [u : v]$. Moreover, the orbits correspond exactly to the invariant homogeneous polynomials in $\mathbb{C}[u, v]$.

we get $\mathbb{C}[u,v]^G \cong \mathbb{C}[x,y,z]/(x^2+y^2z+4z^3)$. Thus the orbit space $\mathbb{A}^2/G$ is isomorphic to a hypersurface in $\mathbb{A}^3$ defined by the polynomial $x^2+y^2z+4z^3$ (under another suitable change of coordinates, we can see that the hypersurface is isomorphic to the one defined by the polynomial $x^2+y^3+z^3$ which we will use from now on).

The polynomial $x^2+y^3+z^3$ has Jacobian matrix $\begin{pmatrix} 2x & 3y^2 & 3z^2 \end{pmatrix}$, which has an isolated singularity at $(0,0,0)$. Then we can the describe the blowup[15] as:

$$\begin{aligned} \mathrm{Bl}_0\, X/G &\cong \mathrm{Bl}_0 \operatorname{Spec} \mathbb{C}[x,y,z]/(x^2+y^3+z^3) \\ &= \{((x,y,z),[a:b:c]) \in \mathbb{A}^3 \times \mathbb{P}^2 | x^2+y^3+z^3=0, \\ &\qquad az-cx = bz-cy = ay-bx = 0\}. \end{aligned}$$

In the chart $c \neq 0$ (formally, we substitute $c=1$), we get

$$\begin{aligned} \mathrm{Bl}_0 \operatorname{Spec} \mathbb{C}[x,y,z]/(x^2+y^3+z^3)\Big|_{\{c\neq 0\}} &\cong \{((az,bz,z),(a/1,b/1)) \in \mathbb{A}^3 \times \mathbb{A}^2 | \\ &\qquad z^2(a^2+z(b^3+1))=0\}. \end{aligned}$$

When $z^2=0$, the exceptional locus is given by $\{((0,0,0),[a:b:c])\} \cong \mathbb{P}^2$. Its intersection with $a^2+z(b^3+1)=0$ is $\{((0,0,0),[0:b:1]) \in \mathbb{A}^3 \times \mathbb{P}^2\} \cong \mathbb{P}^1$, the exceptional curve of the blowup.

The Jacobian matrix of $a^2+z(b^3+1)$ is $\begin{pmatrix} 2a & 3b^2z & b^3+1 \end{pmatrix}$, so the blowup has three remaining singularities at $\{((0,0,0),[0:\zeta:1])\}_{\zeta^3=1}$, lying on the exceptional

[15] set-theoretically

curve. [16]

We pick a cube root of unity ζ and give another change of coordinates:

$$X(a,b,z) := a \quad Y(a,b,z) := b+\zeta \quad Z(a,b,z) := z$$

This gives a hypersurface in $\mathbb{A}^3$ defined as the zero set: $X^2+Z(Y^3-3\zeta Y^2+3\zeta^2 Y)=0$, with an isolated singularity at $(0,0,0)$. Similarly as before, we take the blowup at the origin, and consider it in the chart $c'=1$.

We have $Z^2(a'^2+b'^3Z^2-3\zeta b'^2 Z+3\zeta^2 b')$. The intersection with the exceptional locus is given by $\{((0,0,0),[a':b':1]) \in \mathbb{A}^3 \times \mathbb{P}^2 | a'^2+3\zeta^2 b' = 0\} \cong \mathbb{P}^1$. Its intersection with the exceptional curve of the previous blowup is $((0,0,0),[0:0:1])$.

The latter term has Jacobian

$$\begin{pmatrix} 2a' & 3b'^2Z^2-6\zeta b' Z+6\zeta^2 & 2b'^3 Z-3\zeta b'^2 \end{pmatrix}$$

which has rank 1 everywhere. So the component complement of the exceptional locus is nonsingular, and the resolution obtained after two blowups is smooth.

The same argument applies for the other choices of cube roots of unity. So the exceptional locus of the minimal resolution has resolution graph corresponding to the Dynkin diagram D_4. ■

[16]On the other charts $a=1$, $b=1$, we get no additional singularities. For $a=1$, a similar procedure gives the polynomial $c^2+x(b^3+c^3)$ whose Jacobian is $\begin{pmatrix} 2c+3c^2x & 3b^2x & b^3+c^3 \end{pmatrix}$, and the $b=1$ chart gives $a^2+y(c^3+1)$ with Jacobian $\begin{pmatrix} 2a & 3c^2y & c^3+1 \end{pmatrix}$. The coordinates of singularities in these charts only differ by multiplication of scalars.

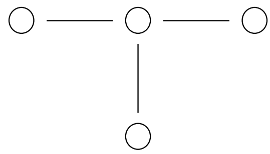

Figure 2: The Dynkin diagram D_4

1.2 The McKay graph of BD_8

In this section, we carry out the rather straightfoward computation:

Computation 1.2. *The McKay graph of BD_8 is described by the extended Dynkin diagram $\widetilde{D_4}$*

There are four irreducible representations of BD_8 in dimension 1. They can be described by the action of the generators of BD_8:

$$\rho_0 : \sigma \mapsto (+1), \tau \mapsto (+1) \qquad \rho_1 : \sigma \mapsto (+1), \tau \mapsto (-1)$$

$$\rho_2 : \sigma \mapsto (-1), \tau \mapsto (+1) \qquad \rho_3 : \sigma \mapsto (-1), \tau \mapsto (-1).$$

In addition, the inclusion $\rho : BD_8 \to SU(2) \subset GL(\mathbb{C}^2)$ gives an irreducible representation in dimension 2, and since $|BD_8| = 1^2 + 1^2 + 1^2 + 1^2 + 2^2 = 8$, these are all the irreducible representations of BD_8.

We compute some terms for the linear representation corresponding to the tensor product:

$$
\begin{aligned}
(\rho \otimes \rho)(\sigma)(e_1 \otimes e_1 + e_2 \otimes e_2) &= (ie_1) \otimes (ie_1) + (-ie_2) \otimes (-ie_2) \\
&= -(e_1 \otimes e_1 + e_2 \otimes e_2) \\
(\rho \otimes \rho)(\tau)(e_1 \otimes e_1 + e_2 \otimes e_2) &= (-e_2) \otimes (-e_2) + (e_1 \otimes e_1) \\
&= e_1 \otimes e_1 + e_2 \otimes e_2.
\end{aligned}
$$

Hence $\langle e_1 \otimes e_1 + e_2 \otimes e_2 \rangle_{\rho \otimes \rho} \cong \rho_2$.

In the same way, we have the following isomorphisms:

$$
\begin{aligned}
\langle e_1 \otimes e_2 - e_2 \otimes e_1 \rangle_{\rho \otimes \rho} &\cong \rho_0 \\
\langle e_1 \otimes e_2 + e_2 \otimes e_1 \rangle_{\rho \otimes \rho} &\cong \rho_1 \\
\langle e_1 \otimes e_1 + e_2 \otimes e_2 \rangle_{\rho \otimes \rho} &\cong \rho_2 \\
\langle e_1 \otimes e_1 - e_1 \otimes e_1 \rangle_{\rho \otimes \rho} &\cong \rho_3.
\end{aligned}
$$

Thus $\rho \otimes \rho \cong \rho_0 \oplus \rho_1 \oplus \rho_2 \oplus \rho_3$. So the McKay graph of (BD_8, ρ) has a directed edge from ρ to each of the ρ_i.

On the other hand, we can see that $\rho \otimes \rho_j \cong \rho_j$. For example, for $j = 2$

$$
\begin{aligned}
(\rho \otimes \rho_2)(\sigma)(e_1 \otimes e_1 - e_2 \otimes e_1) &= (ie_1) \otimes (-e_1) - (-ie_2) \otimes (-e_1) \\
&= -i(e_1 \otimes e_1 + e_2 \otimes e_1) \\
(\rho \otimes \rho_2)(\tau)(e_1 \otimes e_1 - e_2 \otimes e_1) &= (-e_2) \otimes (e_1) - (e_1) \otimes (e_1) \\
&= -(e_1 \otimes e_1 - e_2 \otimes e_1) \\
(\rho \otimes \rho_2)(\sigma)(e_1 \otimes e_1 + e_2 \otimes e_1) &= (ie_1) \otimes (-e_1) + (-ie_2) \otimes (-e_1) \\
&= i(e_1 \otimes e_1 - e_2 \otimes e_1) \\
(\rho \otimes \rho_2)(\tau)(e_1 \otimes e_1 + e_2 \otimes e_1) &= (-e_2) \otimes (e_1) + (e_1) \otimes (e_1) \\
&= e_1 \otimes e_1 - e_2 \otimes e_1
\end{aligned}
$$

so that $(\rho \otimes \rho_2)(\sigma) \cong i\rho(\tau)$, $(\rho \otimes \rho_2)(\tau) \cong -i\rho(\sigma)$ after change of basis. Then the McKay graph of (BD_8, ρ) has a directed edge from ρ_2 to ρ. A similar computation gives the rest of the direct edges from ρ_i to ρ.

Thus the McKay graph of (G, ρ) is given by the extended Dynkin diagram $\widetilde{D_4}$. ■

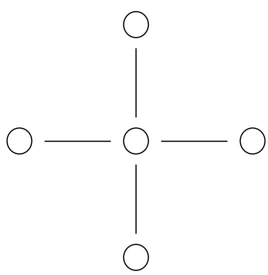

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Figure 3: The extended Dynkin diagram $\widetilde{D_4}$

2 Generalizations

In this chapter, we survey the generalizations of the McKay correspondence. First, we explain Ito and Reid's *strong McKay correspondence* from [IR96]. Next, we review Gonzalez-Sprinberg and Verdier's construction of tautological sheaves from [GSV83] and compute an explicit example for a cyclic subgroup of $SU(2)$ from [Cra01]. Then, we motivate Ito and Nakamura's introduction of G-Hilbert schemes as crepant resolutions [IN96a], and lastly, we make some remarks on the generalization of the McKay correspondence to the derived category of coherent sheaves in [KV00], [BKR01].

2.1 To the basis of cohomology

One may wonder about the appearance of the additional vertex corresponding to the trivial representation in the McKay graph of (G, ρ)–i.e. why on the resolution side we just have simply laced Dynkin diagrams and on the quotient side we have *extended* simply laced Dynkin diagrams.

The additional vertex has a natural interpretation when we view the exceptional divisors as the basis of $H^2(Y, \mathbb{Z})$ (or $H_2(Y, \mathbb{Z})$). The 'exceptional divisor' corresponding to the trivial representation can be viewed as the dual of the point class in $H^0(Y, \mathbb{Z})$.[17] Then the McKay correspondence [18] can be viewed as the following bijection:

$$\{\text{irreducible representations of } G\} \leftrightarrow \text{basis of } H^*(Y, \mathbb{Z}).$$

[17] Alternatively, in [IN96a], it is interpreted as the fundamental cycle.
[18] forgetting the incidence relations

which allows us to formulate it interpretation in higher dimensions.

Before this statement makes sense for $G \subset SL(n, \mathbb{C})$, there are some arguments to be made about the vanishing of odd dimensional cohomology, the representation of cohomology classes by algebraic cycles, and what kind of grading structure is to be expected on the set of irreducible representations of G. We discuss the last point here, for which [IR96] introduced the notion of an *age grading* on G from toric geometry.

We consider the following: For G a finite subgroup of $SL(n, \mathbb{C})$, $\rho : G \hookrightarrow SU(n) \subset SL(n, \mathbb{C})$ and $g \in G$, the eigenvalues of the linear transformation $\rho(g)$ have to be a $|g|$th root of unity ζ. Moreover, we can find an eigenbasis of $\mathbb{C}^n$ in which $\rho(g)$ can be represented by $\mathrm{diag}(\zeta^{a_1}, \ldots, \zeta^{a_n})$, with $0 \leqslant a_i < |g|$. Then the condition for $\rho(g) \in SL(n, \mathbb{C})$ gives the condition that $\sum a_i/|g|$ is a positive integer, which is called the *age of* g.

The age is well defined on conjugacy classes of g, since the matrix representations are similar, but it is not well defined in general for the group G. This is because there are ambiguities in the choice of the root of unity for each conjugacy class, and can be resolved by making a consistent choice of root of unity over all conjugacy classes. Thus the age grading is only well defined on $\Gamma := \mathrm{Hom}(\mu_R, G)$ where R is any common multiple of all the orders of $g \in G$. A homomorphism $\mu_R \to G$ formalizes the idea of choosing a root of unity, and the age grading can now defined on Γ. The conjugacy classes with minimal age 1 called *junior elements* of Γ.

With this notion, [IR96] proved the following:

Theorem 2.1 (Ito-Reid, 1996). *There is a canonical one-to-one correspondence between junior conjugacy classes in Γ and crepant discrete valuations*[19] *of G.*

Moreover, [IR96] showed that there is indeed a bijection between the irreducible representations of finite subgroups G of $SL(3, \mathbb{C})$, and the basis of the cohomology $H^*(Y/\mathbb{Q})$ for a crepant resolution $\pi : Y \to \mathbb{A}^3/G$. They conjectured the *strong McKay correspondence* or *cohomological McKay correspondence*, which claims that given any finite subgroup $G \hookrightarrow SL(n, \mathbb{C})$, and a crepant resolution $\pi : Y \to \mathbb{A}^n/G$, then there is an bijection between the basis of $H^{2i}(Y, \mathbb{Z})$ and the irreducible representations of G of age i.

The strong McKay correspondence was proved in [Bat98] using motivic integration.

2.2 To K-Theory

In [GSV83], the classical McKay correspondence was interpreted as giving isomorphisms of K-theories $K(\mathcal{Y}) \cong K^G(\mathcal{X})$[20] for finite subgroups $G \subset SL(2, \mathbb{C})$, where $\mathcal{Y}$ is the resolution of the germ of X/G at the origin, and $\mathcal{X}$ is the germ of $\mathbb{A}^2$ at the origin.[21] In [IN00] the isomorphism in K-theories was shown for finite abelian subgroups $G \subset SL(3, \mathbb{C})$. Further isomorphisms in K-theory were obtained by [BKR01] for finite subgroups $G \subset SL(3, \mathbb{C})$, induced from an equivalence of derived categories

[19] From a primitive vector encoding the powers of ζ, Ito and Reid first define a monomial valuation on the function field. The center of these valuation define an exceptional prime divisor. Then Ito and Reid show that the condition for the divisor to be crepant exactly corresponds to having an age grading of 1.

[20] as $\mathbb{Z}$-modules

[21] The identification of $K^G(\mathcal{X})$ with the representation ring of $R(G)$ is apparent, as an irreducible representation can be viewed as a G-equivariant vector bundle at the origin, and generate the K-theory. We can also see Reid's guiding principle as the 'G-equivariant geometry' of A^2.

of coherent sheaves. In this section, we introduce [GSV83]'s original construction, referencing the treatment in [Rei07].

[GSV83] defined what are now called *tautological sheaves* $\mathcal{F}_{\rho_i}$ on Y for every irreducible representation ρ_i of the finite subgroups $G \subset SL(2, \mathbb{C})$. The tautological sheaves have the property that, when restricted to the exceptional curves E_j, have degrees δ_{ij}. This identifies their first Chern classes[22] as a basis of $H^*(Y, \mathbb{Z})$ dual to the homology classes of the exceptional curves. Moreover, from here they showed that they are the tautological sheaves generate $K(\mathcal{Y})$.

We now proceed to define these sheaves. One can observe that on $\mathbb{A}^2 \backslash \{0\}$ we have the isomorphism $\mathbb{C}(\mathbb{A}^2)^G \cong \mathbb{C}(\mathbb{A}^2/G)$ so that $\mathbb{C}(\mathbb{A}^2/G) \subset \mathbb{C}(\mathbb{A}^2)$ is a Galois extension. Thus there is a primitive element, whose G-action on its G-orbit is given by the composition rules of G. In other words, $\mathbb{C}(\mathbb{A}^2) \cong \mathbb{C}(\mathbb{A}^2/G)[G]$, the regular representation.

Then we can think of decomposing $\mathbb{C}(\mathbb{A}^2)$ into irreducible $\mathbb{C}(\mathbb{A}^2/G)[G]$-modules. In fact, Maschke's theorem goes through for $\mathbb{C}[\mathbb{A}^2]$ as well, and there is a canonical decomposition of $\mathbb{C}[\mathbb{A}^2]$ into simple $\mathbb{C}[\mathbb{A}^2/G][G]$-modules:

$$\mathbb{C}[\mathbb{A}^2] \cong \bigoplus_i \mathrm{Hom}^G(V_{\rho_i}, \mathbb{C}[\mathbb{A}^2]) \otimes V_{\rho_i}.$$

Now the module that appears in the coefficient $M_i := \mathrm{Hom}^G(V_{\rho_i}, \mathbb{C}[\mathbb{A}^2])$ can be viewed either as the set of G-equivariant linear maps satisfying $f(\rho_i(g) \cdot v) = \rho_{\mathrm{reg}}(g) \cdot f(v)$, or the G-fixed points of the action $(g \cdot f)(v) := \rho_{\mathrm{reg}}(g) \cdot (f(\rho_i(g) \cdot v)$. Being

[22] or more generally, the Chern character

G-fixed, they have the structure of $\mathcal{O}_{\mathbb{A}^2/G}$-modules on $\mathbb{A}^2/G$.

The tautological sheaves $\mathcal{F}_{\rho_i}$ are defined as $\pi^* M_i/\mathrm{Tor}_{\mathcal{O}_Y}$ where $\pi : \mathcal{Y} \to \mathcal{X}/G$ is a minimal resolution of $\mathcal{X}/G$.

We now compute the tautological sheaves for the quotient singularity of type A_n, following [Cra01].

Computation 2.2. *For $G := Z_{n+1}$, the tautological sheaves $\mathcal{F}_{\rho_i}$ have first chern classes dual to the homology classes of irreducible components of the exceptional divisor of the resolution $\pi : \mathcal{Y} \to \mathcal{X}/G$.*

For ζ a fixed choice of primitive $(n+1)$th root of unity, we have the following representation:

$$G \hookrightarrow SU(2) : 1_{Z_{n+1}} \mapsto \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^n \end{pmatrix}.$$

The G-invariant polynomials of $\mathbb{C}[\mathbb{A}^2]$ are then given by:[23]

$$\mathbb{C}[u,v]^G = \mathbb{C}[u^{n+1}, uv, v^{n+1}] \cong \mathbb{C}[\mathbb{A}^2/G].$$

Let $V_{\rho_i} = \langle \alpha_i \rangle$ for some generator α_i, for $i = 0, \cdots n$. The $n+1$ irreducible representations of Z_{n+1} are given by

$$\rho_i : Z_{n+1} \to GL(V_{\rho_i}) : 1_{Z_{n+1}} \mapsto (\alpha_i \mapsto \zeta^i \alpha_i).$$

[23] the generators corresponding to the poles of S^2

The G-equivariant maps are generated by $\alpha_i \mapsto u^i, \alpha_i \mapsto v^{(n+1)-i}$, so that:

$$M_i := \mathrm{Hom}^G(V_{\rho_i}, \mathbb{C}[u,v]) = \mathbb{C}[u^{n+1}, uv, v^{n+1}][\alpha_i \mapsto u^i, \alpha_i \mapsto v^{n+1-i}].$$

To get the charts on Y, it is convenient to use the *Hirzebruch-Jung resolution* of $\mathbb{A}^2/G$ as a toric variety.

Take the lattice $L := \mathbb{Z}^2 + \mathbb{Z} \cdot \frac{1}{n+1}(1,n)$, and the fan $\Sigma := \langle e_1, e_2 \rangle$. Then the dual lattice can be described by the following generators:

$$\mathrm{Hom}(L, \mathbb{Z}) = \mathbb{Z}\langle (n+1)e_1^*, (n+1)e_2^*, e_1^* + e_2^* \rangle,$$

and the toric variety generated by the fan $X_\Sigma := \operatorname{Spec} \mathbb{C}[u^{n+1}, v^{n+1}, uv]$ can be identified with $\mathbb{A}^2/G$.

The Hirzebruch-Jung continued fraction is given by:

$$\frac{n+1}{n} = 2 - \cfrac{1}{2 - \cfrac{1}{\cdots - \cfrac{1}{2}}}$$

so letting $a_0 := e_2$, $a_1 := \frac{1}{n+1}e_1 + \frac{n}{n+1}e_2$, and $a_{j+1} := 2 \cdot a_j - v_{j-1}$ we get the fan

$$\Sigma' := \left\langle e_1, \frac{1}{n+1}\left(e_1 + ne_2\right), \ldots, \frac{1}{n+1}\left(ne_1 + e_2\right), e_2 \right\rangle.$$

Then the toric variety generated by it $X_{\Sigma'}$ is covered by affine charts $U_0, \ldots, U_n$, where

$$\mathbb{C}[U_j] = \mathbb{C}\left[x_j = \frac{u^{j+1}}{v^{(n+1)-(j+1)}}, y_j = \frac{v^{(n+1)-j}}{u^j}\right].$$

Indeed, $x_i y_i = uv$, $(x_i y_i)^{(n+1)-(i+1)} \cdot x_i = u^{n+1}$, and $(x_i y_i)^i \cdot y_i = v^{n+1}$, so from the inclusion map of rings we get the chartwise resolution map $\pi|_{U_i} : \operatorname{Spec}\mathbb{C}[U_i] \to \operatorname{Spec}\mathbb{C}[\mathbb{A}^2/G]$.

The singularity $u = v = 0$ in $\mathbb{A}^2/G$ has exceptional divisors D_j charted by $\{x_{j-1} = 0\} \subset U_{j-1}$ and $\{y_j = 0\} \subset U_j$. (We can verify the incidence relations described by the Dynkin diagram A_n as well).

Then the tautological sheaf $\mathcal{F}_{\rho_i}$ is described chartwise:

$$\mathcal{F}_{\rho_i}|_{U_j} = \begin{cases} \langle u^i \rangle & \text{when } j \geqslant i \\ \langle v^{(n+1)-i} \rangle & \text{when } j < i \end{cases}$$

So the transition functions of $\mathcal{F}_{\rho_i}$ are trivial when $i \neq j$, and when $i = j$, we have $v^{(n+1)-i} = y_i \cdot u^i$.

Hence

$$c_1(\mathcal{F}_{\rho_i}) \cdot [D_j] = \deg \mathcal{F}_{\rho_i}|_{U_j} = \delta_{ij}$$

and the tautological sheaves have Chern classes dual to the exceptional curves of the resolution. ■

2.3 To resolutions by G-Hilbert schemes

The proofs of the resolution of simple singularities are mainly computational, and one may wonder if there is a more uniform approach to the problem. This is given by Nakamura's G-Hilbert scheme. In this section we follow [IN96b].

Let n be a positive integer.[24] We can consider the following *symmetric product* of the affine plane:

$$\mathrm{Symm}^n(\mathbb{A}^2) := \prod_{i=1}^{n} (\mathbb{A}^2)_i / \Sigma_n$$

where the action of the permutation group Σ_n is given by permutations on the coordinate functions:

$$\Sigma_n \circlearrowright \prod_{i=1}^{n} (\mathbb{A}^2)_i : \sigma \cdot (x_1, \ldots, x_n) := (x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

This space is singular, and has a natural resolution: *the Hilbert scheme of n-points in $\mathbb{A}^2$* $\mathrm{Hilb}^n_{\mathbb{A}^2}$. This can can be roughly thought of as the space of n-points of $\mathbb{A}^2$.[25] It is set-theoretically described as: [26]

$$\begin{aligned} \mathrm{Hilb}^n_{\mathbb{A}^2} &:= \{\text{closed subschemes } Z \subset \mathbb{A}^2 | \dim(\mathcal{O}_Z) = n\} \\ &\cong \{\text{ideals } I \subset \mathcal{O}_{\mathbb{A}^2} | \dim(\mathcal{O}_{\mathbb{A}^2}/I) = n\}. \end{aligned}$$

There is a natural *Hilbert-Chow morphism* of $\mathbb{A}^2$ from $\mathrm{Hilb}^n_{\mathbb{A}^2} \to \mathrm{Symm}^n(\mathbb{A}^2)$ given

[24] Keeping Cayley's Theorem in mind, we secretly think $n := |G|$.

[25] In fact an open dense subset of it is the space of n distinct points.

[26] We note how this description defies the need for charts.

by:

$$\pi : Z \mapsto \sum_{p \in \mathrm{Supp}\, Z} \dim(\mathcal{O}_{Z,p}) \cdot p.$$

The following is an illuminating result about $\mathrm{Hilb}^n_{\mathbb{A}^2}$:

Theorem 2.3 (Fogarty)**.** $\mathrm{Hilb}^n_{\mathbb{A}^2}$ *is a smooth quasiprojective scheme,*[27] *and the Hilbert-Chow morphism* $\pi : \mathrm{Hilb}^n_{\mathbb{A}^2} \to \mathrm{Symm}^n(\mathbb{A}^2)$ *is a resolution of singularities.*

Now if we recall Cayley's theorem, we can consider the points of $\mathrm{Symm}^n \mathbb{A}^2$ fixed under the action of G acting as a subgroup of $\Sigma_{n=|G|}$. The Σ_n-orbits of n-tuples of points that are fixed by the G-action are those whose coordinates are given by the G-locus of some point, so that in fact we recover the G-orbits of a single point. In other words, $\mathbb{A}^2/G \cong (\mathrm{Symm}^{|G|} \mathbb{A}^2)^G$.

On the other hand, we can ask what are the G-fixed points of $\mathrm{Hilb}^n_{\mathbb{A}^2}$, and whether the Hilbert-Chow morphism restricts well to a resolution of $\mathbb{A}^2/G$. It turns out that $(\mathrm{Hilb}^n_{\mathbb{A}^2})^G$ has many components, but there is one component that dominates $\mathbb{A}^2/G$, which is called the *G-Hilbert scheme* denoted $\mathrm{Hilb}^G_{\mathbb{A}^2}$.

The G-Hilbert scheme also has another set-theoretic description, as the moduli space of *G-clusters*, or 0-dimensional subschemes Z such that:

- the length of Z is $|G|$
- Z is invariant under the G-action
- $\Gamma(\mathcal{O}_Z)$ is the regular representation of G.

[27] Hilb^n_X is not smooth for general X.

Theorem 2.4 (Ginzburg-Kapranov, Ito-Nakamura, 1996)**.** *If $G \subset SL(2,\mathbb{C})$, the restriction of the Hilbert-Chow morphism to $\mathrm{Hilb}^G_{\mathbb{A}^2}$ is the minimal resolution of singularities of $\mathbb{A}^2/G \cong (\mathrm{Symm}^{|G|}\mathbb{A}^2)^G$.*

Ito and Nakamura verified the classical McKay correspondence using this G-Hilbert scheme as the resolution. Moreover, as the Hilbert scheme is a more uniform construction, there has been successful results to reformulate the McKay correspondence in 3 dimensions in terms of G-Hilbert schemes.

Theorem 2.5 (Nakamura, 2001)**.** *If G is a finite abelian subgroup of $SL(3,\mathbb{C})$, then $\mathrm{Hilb}^G_{\mathbb{A}^3}$ is smooth, and the restriction of the Hilbert-Chow morphism to $\mathrm{Hilb}^G_{\mathbb{A}^3}$ is a crepant resolution of singularities of $\mathbb{A}^3/G$.*

Theorem 2.6 (Bridgeland-King-Reid, 2001)**.** *If G is a finite subgroup of $SL(3,\mathbb{C})$, then $\mathrm{Hilb}^G_{\mathbb{A}^3}$ is smooth, and the restriction of the Hilbert-Chow morphism to $\mathrm{Hilb}^G_{\mathbb{A}^3}$ is a crepant resolution of singularities of $\mathbb{A}^3/G$.*

However, for finite subgroups of $SL(4,\mathbb{C})$ there are known examples where $\mathrm{Hilb}^G_{\mathbb{A}^4}$ fails to be smooth, and examples where it fails to be a crepant resolution.

2.4 Further remarks

In this section we try to give a very rough[28] sketch on how the McKay correspondence generalizes to the derived category of coherent sheaves.

[28] very very very rough

One can interpret the equivalence in K-theory[29]

$$K(\mathcal{Y}) \cong K^G(\mathcal{X})$$

in the following way. We have a basis of $K(\mathcal{Y})$ given by the tautological sheaves, and the representation ring $R(G) \cong K^G(\mathcal{X})$ parametrizing the basis set. We can consider this as an analogous construction to Fourier series, where we have a basis $\{\theta \to e^{in\theta}\}$ of $L^2(S^1)$, and the group $\mathbb{Z}$ parametrizing the set of generators. We can think of the situation as a 'change of basis' from the basis one side to the delta functions on the parametrizing set, inducing an invertible transformation.

The Fourier series can be expressed in the following push-pull construction:

$$\begin{tikzcd} & \cdot e^{int} \\ & L^2(S^1 \times \mathbb{Z}) \arrow[dl, "\pi"'] \arrow[dr, "\int"] & \\ L^2(S^1) & & Ł^2(\mathbb{Z}). \end{tikzcd}$$

We first pull back a function to the product space by the canonical projection map of the first factor, multiply it by a designated exponential function, and push it forward by the canonical projection map of the second factor.

It can be applied in the same spirit to the following diagram of projective schemes:

$$\begin{tikzcd} & \bigotimes_L \mathcal{F} \\ & Y \times X \arrow[dl, "\pi_Y"'] \arrow[dr, "\pi_X"] & \\ Y & & X. \end{tikzcd}$$

[29] as $\mathbb{Z}$-modules

More precisely, for a designated complex of sheaves $\mathcal{F} \in \mathrm{D^b}\operatorname{Coh}(Y \times X)$, we can construct an *integral functor* of triangulated categories:

$$\begin{aligned} \Phi_{\mathcal{F}} : \mathrm{D^b}\operatorname{Coh} Y &\to \mathrm{D^b}\operatorname{Coh} X \\ (-) &\mapsto \mathbf{R}\pi_{X*}(\mathcal{F} \underset{L}{\otimes} \mathbf{L}\pi_X^*(-)). \end{aligned}$$

Analogously to the situation in Fourier series, $\mathcal{F}$ is called the *kernel* and when the integral functor induces an equivalence of triangulated categories, it is called a *Fourier-Mukai transform.*[30]

Although we do not delve into the details of triangulated categories and Fourier-Mukai theory here, we still point out that when either Y or X is a moduli space of the other, we can interpret the basis of delta functions on one space as 'transforming' to the basis on the other.[31]

Thus the introduction of G-Hilbert schemes as the moduli space of G-clusters on A^2 motivates the use of Fourier-Mukai transforms, and indeed it was used to prove the following general versions of the McKay correspondence:

Theorem 2.7 (Kapranov-Vasserot, 2000). *If G is a finite subgroup of $SL(2, \mathbb{C})$ then*

$$\mathrm{D^b}\operatorname{Coh}(Y) \cong \mathrm{D^b}\operatorname{Coh}(X)^G.$$

Theorem 2.8 (Bridgeland-King-Reid, 2001). *If G is a finite subgroup of $SL(3, \mathbb{C})$,*

[30]Fourier-Mukai transforms can be seen as generalizations of morphisms, have nice composition and adjunction properties.

[31]In fact, Mukai's original application of Fourier-Mukai transforms was also in this situation, where Y was the Picard variety of line bundles for an elliptic curve X.

then

$$\mathrm{D}^{\mathrm{b}}\,\mathrm{Coh}(Y) \cong \mathrm{D}^{\mathrm{b}}\,\mathrm{Coh}(X)^G.$$

References

[ABC+09] Paul S. Aspinwall, Tom Bridgeland, Alastair Craw, Michael R. Douglas, Mark Gross, Anton Kapustin, Gregory W. Moore, Graeme Segal, Balázs Szendrői, and P.M.H. Wilson, *Dirichlet Branes and Mirror Symmetry*, Clay Mathematics Institute Monograph Series, vol. 4, American Mathematical Society, Providence, RI, 2009.

[Art66] Michael Artin, *On Isolated Rational Singularities of Surfaces*, American Journal of Mathematics **88** (1966), no. 1, 129–136.

[Bat98] Viktor V. Batyrev, *Non-Archimedean integrals and stringy Euler numbers of log-terminal pairs*, Journal of the European Math Society **1** (1998), 5–33.

[BD96] Viktor V. Batyrev and Dimitrios I. Dais, *Strong McKay Correspondence, String-Theoretic Hodge Numbers and Mirror Symmetry*, Topology **35** (1996), no. 4, 901–929.

[BKR01] Tom Bridgeland, Alastair King, and Miles Reid, *The McKay correspondence as an equivalence of derived categories*, Journal of the American Mathematics Society **14** (2001), no. 3, 535–554.

[Bri99] Tom Bridgeland, *Equivalences of Triangulated Categories and Fourier-Mukai Transforms*, Bulletin of the London Mathematical Society **31** (1999), 25–34.

[CI04] Alastair Craw and Akira Ishii, *Flops of G-Hilb and equivalences of derived categories by variation of GIT quotient*, Duke Mathematical Journal **124** (2004), no. 2, 259–307.

[Cra01] Alastair Craw, *The McKay Correspondence and representations of the McKay Quiver*, Ph.D. thesis, University of Warwick, June 2001.

[Dol09] Igor V. Dolgachev, *McKay Correspondence*, October 2009.

[DuV34] Patrick DuVal, *On isolated singularities of surfaces which do not affect the conditions of adjunction I, II, III*, Mathematical Proceedings of the Cambridge Philosophical Society **30** (1934), no. 4, 453–459, 483–491.

[GSV83] G. Gonzalez-Sprinberg and J.-L Verdier, *Construction géométrique de la correspondance de McKay*, Annales Scientifiques de L'É.N.S. **16** (1983), no. 3, 409–449.

[Har77] Robin Harthshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, vol. 52, Springer, 1977.

[Huy06] Daniel Huybrechts, *Fourier-Mukai Transforms in Algebraic Geometry*, Oxford Mathematical Monographs, Oxford Science Publications, 2006.

[IN96a] Yukari Ito and Iku Nakamura, *Hilbert schemes and simple singularities*, New Trends in Algebraic Geometry (K. Hulek, M. Reid, C. Peters, and F. Catanese, eds.), London Mathematical Society Lecture Note Series, no. 264, Cambridge University Press, July 1996, pp. 151–234.

[IN96b] ———, *McKay Correspondence and Hilbert Schemes*, Proceedings of the Japan Academy, Series A, Mathematical Sciences **72** (1996), no. 7, 135–138.

[IN00] Yukari Ito and Hiraku Nakajima, *McKay Correspondence and Hilbert Schemes in Dimension Three*, Topology **39** (2000), 1155–1191.

[IR96] Yukari Ito and Miles Reid, *The McKay Correspondence for Finite Subgroups of* $\mathrm{SL}(3,\mathbb{C})$, De Gruyter, Mar 1996, pp. 221–240.

[Kal02] D. Kaledin, *McKay correspondence for symplectic quotient singularities*, Inventiones mathematicae **148** (2002), 151–175.

[Kle56] F. Klein, *Lectures on the icosahedron and the solution of equations of the fifth degree*, Dover Publications, Inc., 1956.

[Kol07] János Kollár, *Lectures on Resolution of Singularities*, Annals of Mathematics Studies, no. 166, Princeton University Press, 2007.

[KV00] M. Kapranov and E. Vasserot, *Kleinian singularities, derived categories and Hall algebras*, Mathematische Annalen **316** (2000), no. 3, 565–576.

[lew08] lewallen, *Surface Singularities and Dynkin Diagrams*, November 2008.

[McK80] John McKay, *Graphs, Singularities, and Finite Groups*, The Santa Cruz Conference on Finite Groups (Providence, RI) (B. Cooperstein and G. Mason, eds.), Proceedings of the Symposium in Pure Mathematics of the American Mathematical Society, vol. 37, American Mathematical Society, 1980, pp. 183–186.

[Nak99] Hiraku Nakajima, *Lectures on Hilbert Schemes of Points on Surfaces*, University Lecture Series, vol. 18, American Mathematical Society, 1999.

[Nak01] Iku Nakamura, *Hilbert schemes and abelian group orbits*, Journal of Algebraic Geometry **10** (2001).

[Rei07] Miles Reid, *McKay correspondence*, Feb 2007.

[Sha88] Igor R. Shafarevich, *Basic Algebraic Geometry: Varieties in Projective Space*, 2 ed., vol. I, Springer-Verlag, 1988.

[Slo80] Peter Slodowy, *Simple Singularities and Simple Algebraic Groups*, Mathematisches Institut Universität Bonn, no. 815, Springer-Verlag, 1980.

[Slo83] ———, *Platonic Solids, Kleinian Singularities, and Lie Groups*, Algebraic Geometry (I. Dolgachev, ed.), Lecture Notes in Mathematics, vol. 1008, Springer Berlin Heidelberg, 1983, pp. 102–138.

[Sun10] Yi Sun, *The McKay correspondence*, December 2010.

초 록

$SU(2)$의 부분군 G 로부터 두 가지 방법으로 ADE-딘킨 도표를 얻을 수 있다. 첫째는 $\mathbb{A}^2/G$의 단순특이점의 분해도표 (resolution graph)로 구할 수 있고, 둘째는 G의 기약표현들로부터 계산하는 멕케이 도표를 통해서인데, 이 대응관계를 멕케이 대응이라고 한다. 본 논문에서는 이항 정이면체 군 BD_8 에서의 고전 멕케이 대응을 복습하고 일반화를 의논한다. 또한 그로텐디크 군으로의 일반화의 예로 순환군에 대한 동의중복층 (tautological sheaf) 을 복습한다.

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