이학석사학위논문

# Beyond General Relativity: Modified Newtonian Dynamics and Loop Quantum Gravity

## 일반상대성 이론 넘어, 수정 뉴턴 역학과 루프 양자중력

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지도교수 이상민

이 논문을 이학석사 학위논문으로 제출함

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| 위 원 장 | 김석 | (인) |
| 부위원장 | 이상민 | (인) |
| 위 원 | 김형도 | (인) |

# Abstract

## Beyond General Relativity: Modified Newtonian Dynamics and Loop Quantum Gravity

Youngsub Yoon
Department of Physics and Astronomy
The Graduate School
Seoul National University

We review theories beyond general relativity. In particular, we focus on Mannheim's conformal gravity program and corrections to Bekenstein-Hawking entropy in the framework of loop quantum gravity.

**Keywords**: Kaluza-Klein theory, string theory, loop quantum gravity, Modified Newtonian Dynamics

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# Chapter 1

# Introduction

It has been almost 100 years since Einstein's theory of general relativity was published. It has seen a remarkable success, but found its own limits. As itself, there is a consensus that general relativity is more or less a finished field. Beginning with the deflection of light due to Sun's gravity in 1919, the theory was numerously tested, but no disagreement found. Moreover, general relativity was mathematically re-casted in many different ways, and many mathematical theorems on it were proved. However, it has its own limits. Physicists encountered difficulties in quantizing the theory. Also, some claim that the deviation of galaxy rotation curve is not due to the presence of dark matter, but believe that Newtonian Dynamics as well as general relativity is not a final theory. In this thesis, focusing on author's own work [1] [2], we will briefly review theories that try to go beyond the limits the general relativity poses.

The organization of this thesis is as follows. In chapter 2, we introduce how various theories have gone beyond the general relativity. The aim is to provide the backgrounds for later sections. In chapter 3, we show that a type of

MOND (Modified Newtonian Dynamics), Mannheim's conformal gravity program, whose potential has a term proportional to $1/r$ and another term proportional to $r$, does not reduce to Newtonian gravity at short distances, unless one assumes undesirable singularities of the mass density of the proton. Therefore, despite the claim that it successfully explains galaxy rotation curves, unless one assumes the singularities, it seems to be falsified by numerous Cavendish-type experiments performed at laboratories on Earth whose work have not found any deviations from Newton's theory. Moreover, it can be shown that as long as the total mass of the proton is positive, Mannheim's conformal gravity program leads to negative linear potential, which is problematic from the point of view of fitting galaxy rotation curves, which necessarily requires positive linear potential.

In chapter 4, using Rovelli's suggestion in 1996 which connects black hole entropy and the area spectrum and a theorem we prove in this paper, we briefly show the procedure to calculate the quantum corrections to the Bekenstein-Hawking entropy. One can do this by two steps. First, one can calculate the "naive" black hole degeneracy without the projection constraint (in case of the $U(1)$ symmetry reduced framework) or the $SU(2)$ invariant subspace constraint (in case of the fully $SU(2)$ framework). Second, then one can impose the projection constraint or the $SU(2)$ invariant subspace constraint, obtaining logarithmic corrections to the Bekenstein-Hawking entropy. The exact meanings of these constraints are not important; in this thesis, we focus on the first step and show that we obtain infinite relations between the area spectrum and the naive black hole degeneracy. Promoting the naive black hole degeneracy into its approximation, we obtain the full solution to the infinite relations.

# Chapter 2

# Theoretical background

## 2.1 Kaluza-Klein theory

In 1921, Kaluza published his paper [3] in which he successfully provided a unified framework of gravitation and electromagnetim by assuming that our universe is not 4-dimensional, but 5-dimensional. The rough idea is that the extra degree of freedom living in the metric of 5-dimensional spacetime provides the degree of freedom for electromagnetic potential and one can show that the Einstein-Hilbert action for 5-dimensional metric reproduces that of 4-dimensional one and Maxwell action.

In 1926, Klein extended Kaluza's idea to put it onto the quantum framework then recently discovered by Heisenberg and Schrödinger. In particular, he assumed that the fifth dimension is periodic, having a finite size and calculated it from the fundamental constants such as $e$ the charge of the electron, $G$ Newton's constant, $c$ the speed of light and $h$ the Planck's constant. In particular, he showed that the electric charge must be given by integer multiples of a unit charge if the fifth dimension is periodic.

Kaluza and Klein's idea, now called Kaluza-Klein theory, has received more attention that deserved as an extra-dimension scenario when string theorists discovered that our universe must have extra-dimensions if string theory is the correct theory that describes our nature.

Let us review Kaluza Klein theory. Imagine that we are in five dimensions, with metric components $g^{(5)}_{MN}$, $M, N = 0, 1, 2, 3, 4$ and that the spacetime is actually of topology $\mathbf{R}^4 \times S^1$, and so has one compact direction ($S^1$ denotes a circle). So we will have the usual four dimensional coordinates on $\mathbf{R}^4$, $x^\mu$ ($\mu, \nu = 0, 1, 2, 3$) and a periodic coordinate:

$$x^4 = x^4 + 2\pi R \tag{2.1}$$

where $2\pi R$ is the size of extra dimension.

Now, under the five-dimensional coordinate transformation $x'^M = x^M + \epsilon^M(x)$, the five-dimensional metric transforms as follows:

$$g^{(5)'}_{MN} = g^{(5)}_{MN} - \partial_M \epsilon_N - \partial_N \epsilon_M \tag{2.2}$$

Given this, let us assume that the metric doesn't depend on the periodic coordinate, $x^4$. Then, we immediately see the followings:

$$\epsilon^\nu = \epsilon^\nu(x^\mu) \tag{2.3}$$

$$\epsilon^4 = \epsilon^4(x^\mu) \tag{2.4}$$

which means,

$$x^{\mu'} = \psi^\mu(x^0, x^1, x^2, x^3) \tag{2.5}$$

$$x^{4'} = x^4 + \epsilon^4(x^0, x^1, x^2, x^3) \tag{2.6}$$

They have obvious physical interpretations. The first one is the usual four-dimensional diffeomorphism invariance. The second one is an $x^\mu$-dependant isometry(rotation) of the circle; one has a complete freedom of choosing which point on the circle is $x^4 = 0$.

Then from (2.2), $G^{(5)}_{44}$ is invariant, and from (2.2) and (2.6) we also have:

$$g^{(5)'}_{\mu 4} = g^{(5)}_{\mu 4} - \partial_\mu \epsilon_4 \tag{2.7}$$

However, from the four dimensional point of view, $g^{(5)}_{\mu 4}$ is a vector, proportional to what we will call $A_\mu$, and so the above equation is simply a $U(1)$ gauge transformation for the electromagnetic potential: $A'_\mu = A_\mu - \partial_\mu \Lambda$. So the $U(1)$ of electromagnetism can be thought of as resulting from compactifying gravity, the gauge field being an internal component of metric. We also see that (2.6) implies $U(1)$ gauge freedom. Assuming $g^{(5)}_{44} = e^{2\phi}$ we get the following metric:

$$ds^2 = g^{(5)}_{MN} dx^M dx^N = g^{(4)}_{\mu\nu} dx^\mu dx^\nu + e^{2\phi}(dx^4 + kA_\mu dx^\mu)^2 \tag{2.8}$$

for some $k$ that will be determined later. In other words,

$$g^{(5)}_{\mu\nu} = g^{(4)}_{\mu\nu} + e^{2\phi} k^2 A_\mu A_\nu, \qquad g^{(5)}_{\mu 4} = g^{(5)}_{4\mu} = k e^{2\phi} A_\mu, \qquad g^{(5)}_{44} = e^{2\phi} \tag{2.9}$$

Now, one can easily check that the inverse metric is given as follows:

$$g^{\mu\nu}_{(5)} = g^{\mu\nu}_{(4)}, \qquad g^{\mu 4}_{(5)} = g^{4\mu}_{(5)} = -kA^\mu, \qquad g^{44}_{(5)} = e^{-2\phi} + k^2 A_\alpha A^\alpha \tag{2.10}$$

Given this, the five-dimensional Ricci scalar can be re-expressed as the four-dimensional one and the electromagnetic field tensor as follows:

$$R^{(5)} = R^{(4)} - 2e^{-\phi}\nabla^2 e^\phi - \frac{1}{4}k^2 e^{2\phi} F_{\mu\nu} F^{\mu\nu} \tag{2.11}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ as usual. One can also check that the determinant of $g^{(5)}_{MN}$ is given by the determinant of $g^{(4)}_{\mu\nu}$ multiplied by $e^{2\phi}$.

Now, if we denote $G^N_{(4)}$ as 4-dimensional Newton's constant and $G^N_{(5)}$ as its 5-dimensional counterpart, the Einstein-Hilbert action in 5d becomes:

$$S = \frac{1}{16\pi G^N_{(5)}} \int d^5x(-g_{(5)})^{1/2} R^{(5)} \tag{2.12}$$

$$= \frac{2\pi R}{16\pi G^N_{(5)}} \int d^4x(-g_{(4)})^{1/2}(e^\phi R^{(4)} - 2\nabla^2 e^\phi - \frac{1}{4}k^2 e^{3\phi} F_{\mu\nu}F^{\mu\nu}) \tag{2.13}$$

Therefore, up to some normalization factors and up to a dilaton term, Einstein-Hilbert action in five-dimensional Kaluza-Klein theory reproduces Einstein-Hilbert action in four-dimensional theory and Maxwell-Lagrangian, which means the unification of gravity and electromagnetism. Let's look at this more closely. Upon Weyl transformation $\tilde{g}_{\mu\nu} = e^{-\phi} g_{\mu\nu}$, up to total derivative, the above action becomes:

$$S = \frac{2\pi R}{16\pi G^N_{(5)}} \int d^4x(-g_{(4)})^{1/2}(\tilde{R}^{(4)} + \frac{3}{2}e^\phi \nabla\phi\nabla\phi - \frac{1}{4}k^2 e^{3\phi} F_{\mu\nu}F^{\mu\nu}) \tag{2.14}$$

Now, let's determine the normalization factors. The first and the third term in the above action must be equal to

$$S = \int d^4x(-g_{(4)})^{1/2}(\frac{1}{16\pi G^N_{(4)}}\tilde{R}^{(4)} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}) \tag{2.15}$$

Therefore, we conclude

$$\frac{2\pi R}{G^N_{(5)}} = \frac{1}{G^N_{(4)}}, \qquad k^2 e^{3\phi} = 16\pi G^N_{(4)} \tag{2.16}$$

where $2\pi R$ is the size of extra-dimension, as stated before.

Now comes Klein's work. After quantum mechanics was formulated, Klein showed in 1926 that one could determine the size of extra-dimension in Kaluza's scenario from quantum mechanics. This is something that we will show in the rest of the article.

To this end, we need to first find an explicit geodesic equation. It turns out that extremizing the square of the line element instead of line element is more convenient. After all, the former is equivalent to the latter, when we parametrize the path by the proper time. If we denote the proper time by $\tau$, and $g^{(4)}_{\mu\nu}$ by $g_{\mu\nu}$, and normalize $k$ in such a way that the vacuum expectation value of $\phi$ to be zero, what we want to extremize is the following:

$$L = \frac{1}{2}m\left((\frac{dx^4}{d\tau} + kA_\mu \frac{dx^\mu}{d\tau})^2 + g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right) \tag{2.17}$$

where $m$ is the mass of the particle concerned and the overall factor $\frac{1}{2}m$ is for future convenience. Now, let's obtain the "momentum" and its equation of motion for this "Lagrangian." We have:

$$p_4 = \frac{\partial L}{\partial(dx^4/d\tau)} = m(\frac{dx^4}{d\tau} + kA_\mu \frac{dx^\mu}{d\tau}) \tag{2.18}$$

$$\frac{dp_4}{d\tau} = \frac{\partial L}{\partial x^4} = 0 \tag{2.19}$$

$$p_\mu = \frac{\partial L}{\partial(dx^\mu/d\tau)} = mg_{\mu\nu}\frac{dx^\nu}{d\tau} + kp_4 A_\mu \tag{2.20}$$

$$\frac{dp_\mu}{d\tau} = \frac{\partial L}{\partial x^\mu} = \frac{1}{2}m\partial_\mu g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} + kp_4\frac{\partial A_\nu}{\partial x^\mu}\frac{dx^\nu}{d\tau} \tag{2.21}$$

(2.19) shows that $p_4$ is a conserved quantity. Recall

$$p_x = m\ddot{x} + qA_x, \qquad p_x = m\ddot{y} + qA_y, \qquad p_z = m\ddot{z} + qA_z \tag{2.22}$$

(2.20) turns out to be the momentum, provided

$$p_4 = \frac{q}{k} \tag{2.23}$$

Therefore, we indeed see that the charge conservation implies that $p_4$ is a conserved quantity. One can also check that this choice of $p_4$ yields the correct equation of motion in the presence of electromagnetic field. Plugging (2.20) to (2.21) yields:

$$\frac{dp_\mu}{d\tau} = m\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right) + kp_4\frac{\partial A_\mu}{\partial x^\nu}\frac{dx^\nu}{d\tau} \tag{2.24}$$

Equating this with the right-hand side of (2.21) yields the following:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = kp_4 F_{\nu\mu}\frac{dx^\nu}{d\tau} \tag{2.25}$$

With the identification of $q = kp_4$, this equation exactly becomes the equation of motion in the presence of electromagnetic field $F_{\nu\mu}$.

Given all these, let's obtain the size of the extra dimension. The wave function of a particle with momentum $\vec{p}$ is given as follows

$$\psi(x) = Ae^{i\vec{p}\cdot\vec{x}/\hbar} = e^{i(p_1x^1+p_2x^2+p_3x^3)/\hbar}e^{ip_4x^4/\hbar} \tag{2.26}$$

Focusing on the last factor of the above equation and using (2.1), we must have:

$$e^{ip_4x^4/\hbar} = e^{ip_4(x^4+2\pi R)/\hbar} \tag{2.27}$$

which implies:

$$p_4(2\pi R)/\hbar = 2\pi N, \qquad \rightarrow \qquad p_4 = N\frac{h}{2\pi R} \tag{2.28}$$

for an integer $N$. Using $q = kp_4$, we have:

$$q = N\frac{kh}{2\pi R} \tag{2.29}$$

So, we derived the fact that an electric charge must be the integer multiples of the fundamental charge $h/(2\pi Rk)$. According to quantum chromodynamics, the charge of quark is $\pm e/3$ or $\pm 2e/3$ where $-e$ is the charge of the electron. Therefore, the fundamental charge seems to be $e/3$. Therefore, we obtain:

$$2\pi R = \frac{3h\sqrt{16\pi G^N_{(4)}}}{e} \approx 2.5 \times 10^{-32}\text{meter} \tag{2.30}$$

In terms of Planck length $l_p$ and the fine structure constant $\alpha$ defined as follows,

$$l_p = \sqrt{\frac{\hbar G}{c^3}}, \qquad \alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137.03\cdots} \tag{2.31}$$

we have:

$$2\pi R = \frac{3(4\pi)^{3/2}}{\sqrt{\alpha}} l_p \tag{2.32}$$

All this may seem nice, but it is easy to show that the simplest Kaluza-Klein model is phenomenologically excluded. Suppose we start with a massless field in five-dimension. Then, from Klein-Gordon equation, we must have:

$$\left(\frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_4^2}\right)\psi = 0 \tag{2.33}$$

Plugging (2.26) and (2.28) to the above equation, we get:

$$\left(\frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} + \frac{N^2}{R^2}\right)\psi = 0 \tag{2.34}$$

Therefore, we see that the mass in 4d is given by $N/R$. For $N = 1$ or 2, we should have quarks and for $N = 3$ electrons. However, $N/R$ which is of order Planck mass multiplied by the square root of fine structure constant, is many orders higher than the masses of quarks or electrons. Therefore, the simplest Kaluza-Klein model is excluded.

## 2.2 Dark Matter and MOND

In 1932, a Dutch astronomer Jan Hendrik Oort noted that stars rotating around the center of our galaxy have a much greater speed than that predicted by either

Newton's theory or Einstein's theory given the distribution of the observed mass inside our galaxy. To account for this perceived discrepancy, he suggested that there is "dark matter" across our galaxy that, due to its mass, affects the rotation speed of stars in our galaxy yet remains unobserved and therefore neglected during mass measurements and the speed calculations based on these measurements.

In 1933, a Swiss astronomer Fritz Zwicky also realized that the rotation speed of galaxies in clusters also didn't match with the visible mass either. In 1960s and 70s, an American astronomer Vera Rubin came up with more accurate measurements of rotation speeds of many galaxies, which re-confirmed the discrepancy between the rotation speed and the visible mass. Although this result was initially met with skepticism, it was re-confirmed by other measurements over the decades. The Dark matter theory became a mainstream.

However, a small minority of physicists still believe that there is no such thing as dark matter and believe instead that Newton's theory and Einstein's theory simply fail regarding a system that is both massive and on as large a scale as a galaxy. Such theories are collectively called MOND (Modified Newtonian Dynamics). One such theory is that of "conformal gravity," proposed by Philip D. Mannheim at the University of Connecticut in 1989, which, Mannheim claims, successfully predicts the observed galaxy rotation speed. As we will see later, Mannheim's conformal gravity program is wrong.

In 2006, a group of astronomers came up with a harder evidence to dark matter theory in their paper "a direct empirical proof of the existence of dark matter" [5]. They examined gravitational lensing of the Bullet cluster that consists of two colliding clusters of galaxies and found out that the lensing was strongest not at the part where the visible mass was present as MOND suggested, but at the invisible part where the dark matter should be. However, a

Korean astronomer Jounghun Lee and a Japanese astronomer Eiichiro Komatsu suggested in 2010 that the measured velocities of the collision are incompatible with the prediction of a ΛCDM model [7]. (More on ΛCDM model soon)

Then, what is dark matter made out of? There are three evidences that the majority of dark matter is nonbaryonic. (Astronomers include electrons and neutrinos when they refer to baryonic matter.) First, the theory of big bang nucleosynthesis, which predicts the observed amount of the chemical elements very accurately, predicts that baryonic matter accounts for about 4-5 percent of the critical density of the Universe, while observation of large-scale structure suggests that the total matter density is about 30 percent of the critical density. Second, astronomical searches for gravitational microlensing have suggested that only a small fraction of the dark matter in our galaxy can be in the form of dark compact objects. Third, detailed analysis of cosmic microwave background anisotropy suggests that the majority of the matter should not interact much with ordinary matter or photons except through gravity.

Therefore, a nonbaryonic matter is a dark matter candidate. As nonbaryonic matter such as supersymmetric particles or axion has never been detected before, physicists are still looking for colliders to find dark matter candidates. Good candidates are weakly interacting massive particles (WIMP). Obtaining the correct amount of dark matter today via thermal production in the early universe requires the dark matter to have a cross section $3 \times 10^{-26} \mathrm{cm}^3 \mathrm{s}^{-1}$ which coincides with the cross section expected for a new particle in the 100 GeV mass range interacting through weak force. This is the reason why they are named WIMP.

Also, dark matter candidates can be classified into three categories. Cold dark matter has speeds much less than light. On the other hand, hot dark matter has speeds close to light. A neutrino can be an example. Warm dark matter is placed somewhere between cold dark matter and hot dark matter. As hot dark matter doesn't seem to be viable for galaxy formation, most astronomers agree that dark matter should be cold. Therefore, ΛCDM, where Λ means the cosmological constant, and CDM stands for cold dark matter, is accepted as mainstream theory for cosmology.

On the other hand, I personally believe in MOND in light of an empirical relation found by Tully and Fisher in 1977 [6].

Let us explain what it is. The rotation velocity of star "$v$" is roughly given by the function of the distance from the center, "$r$". Surprisingly, for large $r$, $v$ is not proportional to $1/\sqrt{r}$, as would be the case of Newtonian dynamics, but is constant. Moreover, this asymptotic velocity $v_c$ ($v$ when $r$ approached the infinity) is not proportional to $\sqrt{M_0}$ where $M_0$ is the total mass of the galaxy (i.e. visible mass, not including the invisible dark matter), but proportional to $M_0^{1/4}$. In particular,

$$M_0 = \frac{v_c^4}{1.3 G a_M} \tag{2.35}$$

This relation is known as Tully-Fisher relation. The factor 1.3 comes from the fact that the shape of galaxy is a disk rather than sphere, and $a_M$ called "Milgrom's constant" is about $1.15 \times 10^{-10}\mathrm{m/s^2}$.

For a galaxy rotation curve to satisfy Tully-Fisher relation, the amount of dark matter must be distributed in galaxy in such a way that is a complicated function of the amount of visible mass. On the other hand, Tully-Fisher relation is very simple. One will be further surprised in the simplicity if one considers Milgrom's law.

In 1983, to explain Tully-Fisher relation, Milgrom suggested a law that modifies Newton's second law as follows [8]:

$$F = m\mu(\frac{a}{a_M})a \tag{2.36}$$

where $\mu$ is a function that depends on $a/a_M$. Agreement with Newton's law requires

$$\mu(\frac{a}{a_M}) = 1 \text{ for a} \gg \mathrm{a_M} \tag{2.37}$$

while agreement with Tully-Fisher relation requires

$$\mu(\frac{a}{a_M}) = \frac{a}{a_M} \text{ for a} \ll \mathrm{a_M} \tag{2.38}$$

One can easily check this, as

$$F = m\frac{a^2}{a_M} = m\frac{(v^2/r)^2}{a_M} = \frac{G(1.3M_0)m}{r^2} \tag{2.39}$$

Of course, Milgrom's law cannot be a self-contained, final law itself, but has to be explained by a more fundamental theory. After all, it tells nothing about the function $\mu$ except for the two different limits.

## 2.3 Semi-classical approach to quantum gravity: black hole thermodynamics

In 1974, by considering quantum effects, Hawking showed that a black hole can radiate light with its spectrum being Planck's radiation spectrum. In particular, Hawking showed that the temperature of a black hole corresponds to $\hbar c^3/(8\pi GMk)$ where $M$ is the mass of the black hole. We will derive this using a modern treatment by closely following pages 562∼563 of *String theory and M-theory* by Becker, Becker and Schwarz.

Recall that the partition function in statistical mechanics is given as follows:

$$Z = \mathrm{Tr}(e^{-\beta H}) \tag{2.40}$$

where $\beta$ is given by $1/kT$. In the natural units in which we set $k = 1$, this is simply given by $1/T$.

In quantum field theory, the partition function for the Euclideanized theory is given in terms of $\tau = it$, the Euclideanized time, which has period $\beta$. Therefore, if we find the period for $\tau$ from the metric, our job is done. In this article, we will consider the metric for Schwarzschild black hole, since it is the simplest black hole. Putting Schwarzschild radius $r_s = 2GM/c^2 = 2GM$ (we use $c = 1$), let's define $\rho$ by the following formula:

$$r = r_s(1 + \rho^2) \tag{2.41}$$

So, when $\rho$ is small, we can examine the vicinity of the black hole horizon. In this limit Schwarzschild metric becomes

$$ds^2 \approx r_s^2 \left( d\rho^2 + \rho^2 \left( \frac{d\tau}{2r_s} \right)^2 + \frac{1}{4}(d\theta^2 + \sin^2\theta d\phi^2) \right) \tag{2.42}$$

The first two terms have the form of the metric for a flat plane in polar coordinates, if we identify $\frac{d\tau}{2r_s}$ with $d\Theta$, where $\Theta$ is the angular coordinate. As $\Theta$ has period $2\pi$, $\tau$ has period $4\pi r_s$ which, in turn, is equal to $8\pi GM$. Since this is $\beta$, the Hawking temperature $T = \beta^{-1}$ is equal to $1/(8\pi GM)$. If we don't set $\hbar = k = c = 1$, this is exactly $\hbar c^3/(8\pi GMk)$ as mentioned before.

Given this, we will now present a derivation for the black hole entropy in case of a Schwarzschild black hole again, as it is the simplest case and can be generalized easily. To this end, consider the well-known following thermodynamic identity:

$$dQ = TdS \tag{2.43}$$

Since $Q$ is the energy, it corresponds to the mass $M$. Also, from the Schwarzschild black hole solution, we know that $M = r/(2G)$ where $r$ is the radius of the black hole. Therefore, we have:

$$dQ = dMc^2 = \frac{dr}{2G} = \frac{dS}{8\pi GM} = \frac{dS}{4\pi r} \tag{2.44}$$

$$S = \pi r^2 \tag{2.45}$$

However, $A$, the black hole horizon area is given by $4\pi r^2$, since the black hole has spherical shape. Therefore, we conclude:

$$S = \frac{A}{4G} = \frac{kA}{4l_p^2} \tag{2.46}$$

where $l_p \equiv \sqrt{G\hbar/c^3}$ is the Planck length. This relation is known as the "Bekenstein-Hawking entropy" and holds for other types of black holes as well, at least to the leading order in $A$.

Let us conclude this subsection with remark on Unruh effect. In 1976, Unruh mathematically discovered that an observer accelerating can see a vacuum in an inertial frame to have particles with a thermal spectrum. In other words, the notion of vacuum and the notion of particles are observer-dependent, if the observer is accelerating. Notice also that Unruh effect is a quantum effect, as the notion of vacuum and the notion of particles depend on the raising and lowering operator in second-quantized field; the commutator between the raising operator and the lowering operator is quantum origin. Given this, if you apply Einstein's equivalence principle, gravitational force is equivalent to acceleration, which makes Hawking radiation be understood as a variant of Unruh effect.

## 2.4 String theory

At the beginning of the 70's 't Hooft and Veltman among many others found evidences that general relativity was not renormalizable. Therefore, the naive way of quantizing general relativity using field theory method was out of question. However, string theory, discovered in early 70's and received enormous attention since 1984, has shown a possibility to describe gravity without any infinite divergences. Roughly speaking, the finite size of string smears out the infinities, which a point particle, having zero size, could never do.

Let's briefly review the basics of the basics of string theory. String theory assumes only two simple principles. The first is Nambu-Goto action and the second is supersymmetry. Nambu-Goto action is stated as the action of string is proportional to the area swept by the string. Therefore, let's see how the area can be written. To this end, recall that volume is given as follows:

$$V = \int d^3x \sqrt{\det g} \tag{2.47}$$

In our case, what we need is not volume, but area, which is two-dimensional. Let's say that $\tau$ and $\sigma$ are the two-dimensional coordinates that parametrize the area, and let's use $\alpha$ and $\beta$ to denote them. In other words, $\alpha = 0$ is $\tau$ and $\alpha = 1$ is $\sigma$. Then, the analogous equation to (2.47) is:

$$A = \int d\tau d\sigma \sqrt{-\det G_{\alpha\beta}} \tag{2.48}$$

Now, let's calculate the metric $G_{\alpha\beta}$. The line element in the "target space" must match with the line element in $\tau$ and $\sigma$ variables known as "world-sheet." This implies:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} \frac{\partial x^\mu}{\partial \alpha} \frac{\partial x^\nu}{\partial \beta} d\alpha d\beta = G_{\alpha\beta} d\alpha d\beta \tag{2.49}$$

which implies:

$$G_{\alpha\beta} = \frac{\partial x^\mu}{\partial \alpha} \frac{\partial x_\mu}{\partial \beta} \tag{2.50}$$

Therefore, (2.48) can be re-written as:

$$A = \int d\tau d\sigma \sqrt{-\det\left[\frac{\partial x^a}{\partial \alpha}\frac{\partial x_a}{\partial \beta}\right]} \tag{2.51}$$

$$= \int d\tau d\sigma \left( -\det \begin{bmatrix} \frac{\partial x^\mu}{\partial \tau}\frac{\partial x_\mu}{\partial \tau} & \frac{\partial x_\nu}{\partial \tau}\frac{\partial x^\nu}{\partial \sigma} \\ \frac{\partial x_\mu}{\partial \sigma}\frac{\partial x^\mu}{\partial \tau} & \frac{\partial x^\nu}{\partial \sigma}\frac{\partial x_\nu}{\partial \sigma} \end{bmatrix} \right)^{1/2} \tag{2.52}$$

Therefore, the string action can be written as:

$$S = -T \int d\tau d\sigma \left( -\det \begin{bmatrix} \frac{\partial x^\mu}{\partial \tau}\frac{\partial x_\mu}{\partial \tau} & \frac{\partial x_\nu}{\partial \tau}\frac{\partial x^\nu}{\partial \sigma} \\ \frac{\partial x_\mu}{\partial \sigma}\frac{\partial x^\mu}{\partial \tau} & \frac{\partial x^\nu}{\partial \sigma}\frac{\partial x_\nu}{\partial \sigma} \end{bmatrix} \right)^{1/2} \tag{2.53}$$

where the proportionality constant $T$ is called “string tension.”

However, Nambu-Goto action is not easy to quantize because of the square root in it. Therefore, to quantize string action, Polyakov used Polyakov action, which is equivalent to Nambu-Goto action, but much easier to quantize, as it is free of square root. After gauge-fixing, Polyakov action is given by

$$S = -\frac{T}{2} \int d\tau d\sigma (-\partial_\tau X^\mu \partial_\tau X_\mu + \partial_\sigma X^\mu \partial_\sigma X_\mu) \tag{2.54}$$

To quantize the action, one should express $X^\mu$ in the above equation in terms of Fourier-mode. In case of open string (i.e. string with two ends) we have:

$$X^\mu = x^\mu + l_s^2 p^\mu \tau + i l_s \sum_{m \neq 0} \frac{1}{m} \alpha_m^\mu e^{-im\tau} \cos(m\tau) \tag{2.55}$$

where $l_s = 1/\sqrt{\pi T}$ and $\alpha_m^\mu$s are Fourier components. Also, that $X^\mu$ is real implies $\alpha_{-n}^\mu = \alpha_n^{\mu\dagger}$

In case of closed string (i.e. string with no ends) we have:

$$X_R^\mu = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2(\tau - \sigma) + \frac{i}{2} l_s \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-2in(\tau - \sigma)} \tag{2.56}$$

$$X_L^\mu = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2(\tau+\sigma) + \frac{i}{2}l_s \sum_{n\neq 0} \frac{1}{n}\tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)} \tag{2.57}$$

where we have $X^\mu = X_R^\mu + X_L^\mu$ and $X_R^\mu$ is called right-movers, as it only depends on $(\tau - \sigma)$ while $X_L^\mu$ is called left-movers, as it only depends on $(\tau + \sigma)$. Notice that in case of closed string we have two different Fourier components for right-moving mode and left-moving mode.

Then, the quantization yields:

$$[\alpha_m^\mu, \alpha_n^\nu] = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\eta^{\mu\nu}\delta_{m,-n} \tag{2.58}$$

Defining

$$a_m^\mu = \frac{1}{\sqrt{m}}\alpha_m^\mu, \qquad a_m^{\mu\dagger} = \frac{1}{\sqrt{m}}\alpha_{-m}^\mu, \qquad \text{for} \qquad m > 0 \tag{2.59}$$

we can re-express (2.58) as

$$[a_m^\mu, a_n^{\nu\dagger}] = [\tilde{a}_m^\mu, \tilde{a}_n^{\nu\dagger}] = \eta^{\mu\nu}\delta_{m,n}, \qquad \text{for} \qquad m, n > 0 \tag{2.60}$$

In other words, $a_m^\mu$ and $a_m^{\mu\dagger}$ can be interpreted as lowering operators and raising operators, except for $a_m^0$ and $a_m^{0\dagger}$ which satisfies $[a_m^0, a_m^{0\dagger}] = -1$.

Therefore, apart from the subtlety of the time component of $a_m$, the ground state is defined by

$$a_m^\mu|0>= 0 \qquad \text{for} \qquad m > 0 \tag{2.61}$$

Then, we can construct a general state $|\phi>$ by multiplying raising operators to the ground state as follows:

$$|\phi>= a_{m_1}^{\mu_1\dagger}a_{m_1}^{\mu_1\dagger}\cdots a_{m_n}^{\mu_n\dagger}|0;k^\mu> \tag{2.62}$$

where $|0;k>$ is the ground state with momentum $k^\mu$. As an aside, the subtlety of the time component of $a_m$ is not a simple matter, and it would take some pages to explain how to resolve this, which we do not do in this review.

In any case, we see here how a generic particle (i.e. a general state $|\phi >$) can be constructed in string theory. They have certain occupation numbers for Fourier-modes of string. We also see that there can be infinite numbers of kinds of particle as the occupation numbers and the number of modes (i.e. $m$ in (2.58)) can be infinite.

However, this picture is incomplete since there is no way to accommodate fermions in this picture. Certainly (2.60) are commutators, not anti-commutators. To this end, we introduce an additional term to the Polyakov action (2.54) as follows:

$$S = -\frac{T}{2}\int d\tau d\sigma(-\partial_\tau X^\mu \partial_\tau X_\mu + \partial_\sigma X^\mu \partial_\sigma X_\mu - i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu) \tag{2.63}$$

where $\rho^\alpha$s are two-dimensional Dirac matrices satisfying

$$\{\rho^\alpha, \rho^\beta\} = -2\eta^{\alpha\beta} \tag{2.64}$$

In 1980s, five anomaly-free string theories "Type I," "Type IIA," "TypeIIB," "Heterotic $SO(32)$," "Heterotic $E_8 \times E_8$" were discovered. This discovery is called "first superstring theory revolution. These five theories are consistent only in 10 spacetime dimension and are supersymmetric. Type I theory has both open and closed strings and the gauge group is $SO(32)$. Type IIA theory has only closed strings and massless fermions are non-chiral. Type IIB theory has only closed strings and massless fermions are chiral. Heterotic theories have only closed strings and their left-moving mode is bosonic while right-moving mode is supersymmetric. Heterotic $SO(32)$ theory has $SO(32)$ as gauge group while heterotic $E_8 \times E_8$ theory has $E_8 \times E_8$ as gauge group.

In 1990s, it was found out that these five string theories are intricately related to one another by something called S-duality and T-duality. S-duality means that a theory is related to another theory in such a way that the former theory with the coupling constant $g$ is equivalent to the latter theory with the coupling constant $1/g$. Similarly, T-duality means that a theory with strings in a circle of some radius $R$ inside the extra-dimensional compact manifold is equivalent to another theory with strings in a circle of some radius $1/R$. Type I and Heterotic $SO(32)$ are related to one another by S-duality. Type IIB is related to itself by S-duality. Type IIA and Type IIB are related to one another by T-dualtiy. Heterotic $SO(32)$ and heterotic $E_8 \times E_8$ are related to one another by T-duality. Also, there is evidence that Type IIA and M-theory are related to one another by S-duality, and heterotic $E_8 \times E_8$ and M-theory are related to one another by S-duality. M-theory is an eleven-dimensional theory that unifies all five different string theories. The five different string theories are conjectured to be the different limits of M-theory. M-theory has not been discovered yet, but many string theorists are working hard to find it. It is truly the best candidate for theory of everything. These discoveries of dualities and evidence for M-theory in 1990s is called "second superstring revolution."

## 2.5 Loop Quantum Gravity

Unlike string theory, loop quantum gravity smears out the infinities by proposing Wilson line and Wilson loop as the basic building block of space-time, as Rovelli and Smolin have done in 1990; naturally Wilson loop has no zero size. They used Ashtekar's connection reformulation of general relativity in 1986 for this purpose; they considered the Wilson loop of Ashtekar connection. In 1994, they went on to show that spacetime is not continuous but discrete as long as loop quantum gravity is correct; the eigenvalues of the area and the volume operators admit only discrete values.

Then, an area of any object has many partitions, each of which is given by the eigenvalues of the area operator. In other words, a generic area is given by the sum of the eigenvalues of the area operator.

There have been some criticisms on loop quantum gravity [9], but there has also been a response [10]. We believe that loop quantum gravity is on a right track unlike other theories that have ignored the fact that the area eigenvalues are discrete, which is a key prediction, not an assumption, of loop quantum gravity; as much as the angular momentum treated in quantum mechanics only allows discrete values, so does the area operator. Theories that ignore this fact is incomplete, and could lead to a possibly incorrect result; it would be like trying to solve hydrogen atom problem without considering the fact that angular momentum is quantized.

In this section, we will show how the area eigenvalues are calculated in the case in which the concerned geometry is given by the Wilson Loop, which is the simplest case. First, let me mention that in loop quantum gravity the area operator is given as follows:

$$E_i(S) \equiv \frac{1}{2} E_i^a \tilde{\epsilon}_{abc} dx^b \wedge dx^c \tag{2.65}$$

where $E_i^a$ is called "gravitational electric field," and satisfies $gg^{ab} = \sum_i E_i^a E_i^b$. Here, $a$,$b$, and $c$ are spacetime variables while $i$,$j$,and $k$ are Lorentz indices.

According to Ashtekar formulation of general relativity, Ashtekar connection $A_a^i$ and gravitational electric field $E_j^b$ satisfies the following commutation relation

$$[A_a^i(\overrightarrow{\tau}), E_j^b(\overrightarrow{\tau}')] = i\gamma \delta_j^i \delta_a^b \delta^3(\overrightarrow{\tau}, \overrightarrow{\tau}') \tag{2.66}$$

where $\gamma$ is known as Immirizi parameter, a free parameter. This implies $A$s and $E$s acts on Hilbert space as follows.

$$A_i^a \Psi_s(A) = A_a^i \Psi(A) \tag{2.67}$$

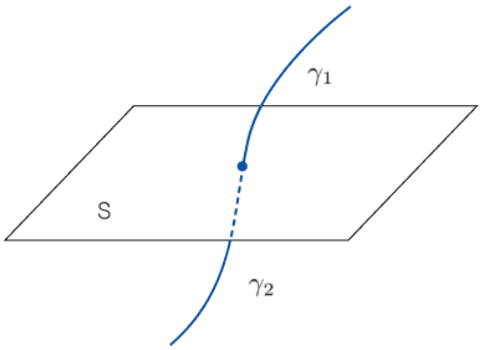


Figure 2.1 area for a simple holonomy

$$E_i^a \Psi_s(A) = -i\gamma \frac{\delta}{\delta A_a^i} \Psi(A) \tag{2.68}$$

The basis for Hilbert space $\Psi[A]$ is known to be a spin network state composed of Wilson line and Wilson loop. The simplest spin network state is a holonomy of $A$.

$$U = P \int_\gamma \exp(iA) = P \int_\gamma \exp(iA^i \tau^i) = P \left[ \exp\left( i \int_\gamma ds \frac{d\gamma^a(s)}{ds} A_a^i(\gamma(s)) \tau_i \right) \right] \tag{2.69}$$

If we take a derivative with respect to $A$, we get:

$$\frac{\delta}{\delta A_a^i(x)} U(A, \gamma) = i \int ds \frac{d\gamma^a(s)}{ds} \delta^3(\gamma(s), x) [U(A, \gamma_1) \tau_i U(A, \gamma_2)] \tag{2.70}$$

where $\gamma_1$ and $\gamma_2$ are denoted in Fig.1. This allows us to apply area operator to the holonomy as follows:

$$E_i(S) \equiv \frac{1}{2} E_i^a \tilde{\epsilon}_{abc} dx^b \wedge dx^c = -i\gamma \int_S \frac{1}{2} \tilde{\epsilon}_{abc} dx^b \wedge dx^c \frac{\delta}{\delta A_a^i} \tag{2.71}$$

Parametrizing the plane $S$ in terms of $\vec{\sigma} = (\sigma^1, \sigma^2)$, we have:

$$
\begin{aligned}
&E_i(S)U(A,\gamma) = -i\gamma \int_S \frac{1}{2}\tilde{\epsilon}_{abc} dx^b \wedge dx^c \frac{\delta U(A,\gamma)}{\delta A_a^i} \\
&= \gamma \int_S d\sigma^1 d\sigma^2 \tilde{\epsilon}_{abc} \frac{\partial x^b(\vec{\sigma})}{\partial \sigma^1} \frac{\partial x^c(\vec{\sigma})}{\partial \sigma^2} \int_\gamma ds \frac{\partial \gamma^a}{\partial s} \delta^3(x(\overrightarrow{\sigma}), \gamma(s)) U(A,\gamma_1)\tau_i U(A,\gamma_2) \\
&= \gamma \left( \int_S \int_\gamma d\sigma^1 d\sigma^2 ds \tilde{\epsilon}_{abc} \frac{\partial x^b(\vec{\sigma})}{\partial \sigma^1} \frac{\partial x^c(\vec{\sigma})}{\partial \sigma^2} \frac{\partial \gamma^a(s)}{\partial s} \delta^3(x(\overrightarrow{\sigma}), \gamma(s)) \right) \\
&\quad \times \left[ U(A,\gamma_1)\tau_i U(A,\gamma_2) \right]
\end{aligned}
\tag{2.72}
$$

Now, we will assume that there is only single intersection between $\gamma$ and $S$. (This condition allows the delta function in the above expression to be non-zero only for a single point. Similarly, if there is no intersection at all the delta function always vanishes.) Then, consider the following map from the integration domain $(\sigma^1, \sigma^2, s)$ to $(x'^1, x'^2, x'^3)$:

$$
x'^a(\sigma^1, \sigma^2, s) = x^a(\sigma^1, \sigma^2) - \gamma^a(s) \tag{2.73}
$$

Then, the Jacobian of this map is given by:

$$
J = \frac{\partial(x'^1, x'^2, x'^3)}{\partial(\sigma^1, \sigma^2, s)} = -\tilde{\epsilon}_{abc} \frac{\partial x^b(\vec{\sigma})}{\partial \sigma^1} \frac{\partial x^c(\vec{\sigma})}{\partial \sigma^2} \frac{\partial \gamma^a(s)}{\partial s} \tag{2.74}
$$

Therefore, the term in the parenthesis in (2.72) becomes:

$$
\pm \int\int\int (-) dx'^1 dx'^2 dx'^3 \delta^3(x') = \mp 1 \tag{2.75}
$$

where the $\pm$ sign in the front depends on how we orient $\gamma$ and $\sigma$s. For example, if we parametrized the path $\gamma$ in the opposite direction to the original one, we will get an extra negative sign. In conclusion, we have:

$$
E_i(S)U(A,\gamma) = \mp\gamma U(A,\gamma_1)\tau_i U(A,\gamma_2) \tag{2.76}
$$

Now, let's apply the area operator twice to get the exact area spectrum as follows

$$
\begin{aligned}
E_i(S)E_i(S)U(A,\gamma) &= \gamma^2 U(A,\gamma_1)\tau_i^2 U(A,\gamma_2) \\
&= \gamma^2 U(A,\gamma_1)(j(j+1))U(A,\gamma_2) \\
&= j(j+1)\gamma^2 U(A,\gamma) \qquad (2.77)
\end{aligned}
$$

So, we obtain the following:

$$
\mid E \mid = \gamma\sqrt{j(j+1)} \qquad (2.78)
$$

where $j$ is a positive half-integer. In the generic case, the area spectrum is given as follows:

$$
\mid E \mid = \frac{1}{2}\gamma\sqrt{2j^u(j^u+1)+2j^d(j^d+1)-j^t(j^t+1)} \qquad (2.79)
$$

where $j$s are half-integers or integers and satisfy triangle inequalities, and their sum is an integer.

# Chapter 3

# Mannheim's conformal gravity program

## 3.1 Introduction

Recently, Mannheim's conformal gravity program has attracted much attention as an alternative to dark matter and dark energy [11–13]. However, so far, the only way its validity could be tested was through cosmological considerations. In this section, we suggest that Mannheim's conformal gravity program seems problematic according to numerous Cavendish-type experiments on Earth. One of our ideas is that Mannheim's conformal gravity program predicts that the gravitational force due to an object depends on its mass distribution even in the case that in which it has a spherical symmetric mass distribution. (i.e., the mass density only depends on the distance from the center of the body.) For example, Newtonian gravity predicts that we can calculate the gravitational force due to the Earth as if all the mass of Earth were at its center. This is not true in the case of conformal gravity; the gravitational force *heavily* depends on the mass distribution.

In Secs. 3.2 and 3.3, we introduce and review conformal gravity. In Secs. 3.4 and 3.5, we give a couple of arguments why conformal gravity is problematic. All these sections are based on our paper [1]. In Sec. 3.6, we present Mannheim's criticism on our paper, and our stance on this criticism.

## 3.2 Mannheim's conformal gravity

Instead of Einstein-Hilbert action, in conformal gravity, we have the following action.

$$\begin{aligned} S &= -\alpha_g \int d^4x\sqrt{-g}C_{\lambda\mu\nu\kappa}C^{\lambda\mu\nu\kappa} \\ &= -2\alpha_g \int d^4x\sqrt{-g}[R^{\mu\nu}R_{\mu\nu} - \frac{1}{3}R^2] \end{aligned} \tag{3.1}$$

where $C_{\lambda\mu\nu\kappa}$ is the conformal Weyl tensor, and $\alpha_g$ is a purely dimensionless coefficient. By adding this to the action of matter and varying it with respect to the metric, one can obtain the conformal gravity version of the Einstein equation.

## 3.3 The metric solution in conformal gravity

The following derivation closely follows Mannheim and Kazanas's in Ref. [13]. [See in particular Eqs. (9), (13), (14) and (16) in their paper.] In case there is a spherical symmetry in the distribution of the mass, one can write the metric as follows:

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega_2 \tag{3.2}$$

Plugging this into the conformal gravity version of the Einstein equation, and assuming that all the matter is inside the radius $r_0$, Mannheim and Kazanas obtain

$$B(r > r_0) = 1 - \frac{2\beta}{r} + \gamma r \tag{3.3}$$

$$\nabla^4 B(r) = f(r) \tag{3.4}$$

The solution is given by

$$2\beta = \frac{1}{6}\int_0^{r_0} dr' r'^4 f(r') \tag{3.5}$$

$$\gamma = -\frac{1}{2}\int_0^{r_0} dr' r'^2 f(r') \tag{3.6}$$

where

$$f(r) \equiv \frac{3}{4\alpha_g B(r)}(T_0^0 - T_r^r) \tag{3.7}$$

without any approximation whatsoever.

## 3.4 Non-Newtonian potential

If we ignore $T_r^r$ in the above equation, as it is small, set $T_0^0 = \rho$, and use $B(r) \approx 1$, we get:

$$\frac{2\beta}{r} = \frac{1}{r}(\frac{1}{8\alpha_g}\int_0^{r_0} dr' r'^4 \rho) \tag{3.8}$$

Compare this with the Newtonian case, which is the following:

$$\frac{2\beta}{r} = \frac{2G}{rc^2}\int_0^{r_0} dr' 4\pi r'^2 \rho = \frac{2GM_{total}}{rc^2} \tag{3.9}$$

Thus, unlike in the Newtonian case, we see that in Mannheim's conformal gravity, the gravitational attraction depends not only on the total mass, but also on the mass distribution. Therefore, if two spherically symmetric objects with the same mass but different density distributions yield the same strength of gravitational forces, then conformal gravity is troublesome. On the other hand, if they yield different strengths of gravitational force, in precisely the manner that conformal gravity predicts, then conformal gravity will be verified. Notice that the difference of the gravitational force would be big; it would be in leading order, not in next-to-leading order. For example, if the mass of two objects is the same, but the first one's size is double that of the second one, the former will exert quadruple the amount of gravitational force. Conformal gravity seems troublesome, as many Cavendish-type experiments have been performed, and none of them has detected that gravity depends on the density distribution [14]. We introduce Mannheim and Kazanas's circumvention of this dilemma in the next section.

## 3.5 The wrong sign of the linear potential term

Mannheim compares the gravitational potential in Newtonian gravity and conformal gravity in Ref. [11]. He considers the case in which all the matter is inside the region ($r < R$), and the mass distribution only depends on $r$ (i.e., spherically symmetric). In the case of Newtonian gravity, the potential is given by

$$\nabla^2\phi(\vec{r}) = g(\vec{r}) \tag{3.10}$$

The solution is given by

$$\phi(\vec{r}) = -\frac{1}{4\pi}\int d^3\vec{r'}\frac{g(\vec{r'})}{|\vec{r}-\vec{r'}|} \tag{3.11}$$

$$\phi(r > R) = -\frac{1}{r}\int_0^R dr' r'^2 g(r') \tag{3.12}$$

$$\phi(r < R) = -\frac{1}{r}\int_0^r dr'r'^2 g(r') - \int_r^R dr'r'g(r') \tag{3.13}$$

In the case of Mannheim's conformal gravity, we have

$$\nabla^4\phi(\vec{r}) = h(\vec{r}) \tag{3.14}$$

The solution is given by

$$\phi(\vec{r}) = -\frac{1}{8\pi}\int d^3\vec{r'}h(\vec{r'})|\vec{r}-\vec{r'}| \tag{3.15}$$

$$\phi(r > R) = -\frac{1}{6r}\int_0^R dr'r'^4 h(r') - \frac{r}{2}\int_0^R dr'r'^2 h(r') \tag{3.16}$$

$$\phi(r < R) = -\frac{1}{6r}\int_0^r dr'r'^4 h(r') - \frac{1}{2}\int_r^R dr'r'^3 h(r') - \frac{r}{2}\int_0^r dr'r'^2 h(r') - \frac{r^2}{6}\int_r^R dr'r'h(r') \tag{3.17}$$

Then, Mannheim notes that the following $h(r)$:

$$h(r < R) = -\gamma c^2 \sum_{i=1}^{N} \frac{\delta(r - r^i)}{r^2} - \frac{3\beta c^2}{2}\sum_{i=1}^{N}\left[\nabla^2 - \frac{r^2}{12}\nabla^4\right]\left[\frac{\delta(r-r_i)}{r^2}\right] \tag{3.18}$$

yields the following gravitational potential:

$$\phi(r > R) = -\frac{N\beta c^2}{r} + \frac{N\gamma c^2 r}{2} \tag{3.19}$$

Thus, by assuming the singularities of the mass density of the proton, Mannheim tries to circumvent the problem we raised in Sec. 3.4: we can arbitrarily make $1/r$ potential from a single proton, and if we add them up, they would reproduce Newton's law.

However, a closer look at the last term of Eq. (3.16) shows that this circumvention will not work. Notice that $h$ is the mass density up to a certain positive coefficient. Therefore,

$$\int hr'^2 dr' \tag{3.20}$$

should be equal to the *total* mass of the proton divided by the positive coefficient. This is obvious from the following elementary formula:

$$M = \int 4\pi r^2 \rho dr \tag{3.21}$$

Therefore, the last term of Eq. (3.16) yields the negative linear gravitational potential. However, we know that we want the positive linear gravitational potential to fit the galaxy rotation curve; what we need there is not extra repulsion but extra attraction. Therefore, Mannheim's conformal gravity program seems problematic, unless someone can come up with an argument that $\alpha_g$ in Eq. (3.1) can take a negative value.

We want to note that Mannheim's conformal gravity program, with which we have dealt in this paper, should not be confused with Anderson-Barbour-Foster-Murchadha conformal gravity [15].

## 3.6 Mannheim's criticism

About two years after my paper "Problems with Mannheim's conformal gravity program" was published, Mannheim uploaded his own paper to the arXiv criticizing my paper. He noted $T^\mu_\mu = 0$, which should be satisfied for a conformal theory, is not satisfied for my approximation $T^0_0 \approx \rho$ and $T^r_r \approx 0$.

Let me first explain why the trace of the energy momentum must vanish in a conformal theory (i.e. a theory which has a conformal symmetry which means that the action is invariant under conformal transformation). Let's write the total action of conformal theory as the sum of the gravitational action and the matter action as follows:

$$S = \int d^4x \sqrt{-g}(\mathcal{L}_g + \mathcal{L}_m) \tag{3.22}$$

Now, for a small variation of the metric, the action is invariant as follows:

$$0 = \delta S = \int d^4x \left( \frac{\delta(\sqrt{-g}\mathcal{L}_g)}{\delta g^{ab}} \delta g^{ab} + \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{ab}} \delta g^{ab} \right) \tag{3.23}$$

Given this, notice that an infinitesimal conformal transformation can be written as follows:

$$\delta g_{ab}(x) = 2\phi(x) g_{ab}(x) \tag{3.24}$$

If we plug this into (3.23), the first term must be zero since the gravitational action of conformal theory doesn't change under conformal transformation. Therefore, the second term must be zero as well. Recall now,

$$T_{ab} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{ab}} \tag{3.25}$$

Plugging this in, we get:

$$0 = \int d^4x \sqrt{-g} \phi(x) g^{ab}(x) T_{ab}(x) \tag{3.26}$$

Since this has to be true for any $\phi(x)$, we conclude $T^a_a = 0$.

To make the trace of the energy-momentum tensor vanish, Mannheim assumes that the Higgs field is distributed within galaxies in such a manner that it compensates the trace of the energy-momentum tensor of ordinary matter (i.e. stars, etc.). However, it is also very difficult to propose a mechanism that would distribute the Higgs field in such a manner. It is simply unlikely.

# Chapter 4

# Approximation of the naive black hole degeneracy

## 4.1 Introduction

It is well known that the entropy of a black hole is given by the quarter of its area (i.e. $A/4$), regardless of the type of black hole considered [16, 17]. However, as it is so only in the leading order, many have calculated the corrections to it [18, 19].

In this section, we will consider the connection between the black hole entropy and the area spectrum suggested by Rovelli in 1996 [21] as loop quantum gravity predicts that the area spectrum is quantized [22–24]. To apply this connection, we will consider the formula proposed by Domagala, Lewandowski and Meissner which can check whether Bekenstein-Hawking entropy is consistent with a given area spectrum [19, 25]. Stepping further, we will use the mathematics of "compositions," to prove a theorem that shows that Bekenstein-Hawking entropy is reproduced, if their formula is satisfied. Then, by basing on this formalism, we calculate the "naive" degeneracy of black hole. We call it "naive" as we calculated it without the consideration of the projection constraint (in case of the $U(1)$ symmetry reduced framework) or the $SU(2)$ invariant subspace constraint (in case of the fully $SU(2)$ framework). During the process, we obtain infinite relations between the area spectrum and the naive black hole degeneracy. Then, we "continutize" or "approximate" the naive black hole degeneracy to the smooth function of area and obtain the full solution to the infinite relations. This is the main result and objective of this paper.

As an aside, we would also like to mention that Ashoke Sen has criticized loop quantum gravity, as his calculation of logarithmic corrections to Bekenstein-Hawking entropy didn't match the one predicted by loop quantum gravity [20]. However, theories that didn't take account the fact that areas are discrete, which is a key prediction of loop quantum gravity, cannot be complete, as argued in Sec. 2.5.

The organization of this section is as follows. In Sec. 4.2, we introduce the relation between black hole entropy and the area spectrum proposed by Rovelli. In Sec. 4.3, we introduce Domagala-Lewandowksi-Meissner formula. In Sec 4.4, we introduce the mathematics of "compositions." In Sec 4.5, we prove that the Bekenstein-Hawking entropy is reproduced, if the area spectrum satisfies Domagala-Lewandowski-Meissner formula. In Sec. 4.6, we apply this formalism to calculate the "naive" black hole degeneracy. We will also obtain the infinite relations just advertised. In Sec. 4.7, we will obtain the full solution to the infinite relations. In Sec. 4.8, we show, as an example, how one can obtain logarithmic correction; the result of this section is nothing new. We consider the $U(1)$ symmetry reduced framework and show how the projection constraint $\sum_i m_i = 0$ yields the logarithmic corrections. All these sections are based on our paper [2]

## 4.2 Black hole entropy and the area spectrum

According to loop quantum gravity, the eigenvalues of the area operator are discrete. Let's say that we have the following area eigenvalues, or the unit areas:

$$A_1, A_2, A_3, A_4, A_5, A_6.... \tag{4.1}$$

Here, we used the notation that the $i$th unit area is $A_i$. Then, a generic area should be partial sum of them, including the case in which the same area eigenvalues are repeated in the sum. In other words, a generic area has many partitions, each of which must be $A_i$ for some $i$.

Given this, an interesting proposal was made by Rovelli [21]. As black hole entropy is given by $A/4$, the degeneracy of black hole is given by $e^{A/4}$. Rovelli proposed that the black hole degeneracy is obtained by counting the number of ways in which the area of black hole can be expressed as the sum of unit areas. In other words, $N(A)$ the degeneracy of the black hole with area $A$ is given by following.

$$N(A) := \left\{ (i_1, i_2, i_3 \cdots), \sum_x A_{i_x} = A \right\} \tag{4.2}$$

Here, I want to note an important point. For the parenthesis in the above formula $(\cdots, a, \cdots, b, \cdots)$ should be regarded different from $(\cdots, b, \cdots, a, \cdots)$. In other words, the order in the summation is important.

## 4.3 Domagala-Lewandowski-Meissner trick

This section closely follows [26] which explains Domagala-Lewandowski-Meissner trick in an easier way. To understand their formula which gives a necessary condition for the black hole entropy to be $A/4$, we reconsider the "simplified area spectrum" or "isolated horizon" as follows [27]. In this case, $A_i = 8\pi\gamma\sqrt{j_i(j_i+1)}$. Then, (4.2) becomes the following.

$$N(A) := \left\{ (j_1, \cdots, j_n) | 0 \neq j_i \in \frac{\mathbb{N}}{2}, \sum_i \sqrt{j_i(j_i+1)} = \frac{A}{8\pi\gamma} \right\} \tag{4.3}$$

We derive a recursion relation to obtain the value of $N(A)$. When we consider $(j_1, \cdots, j_n) \in N(A - a_{1/2})$ we obtain $(j_1, \cdots, j_n, \frac{1}{2}) \in N(A)$, where $a_{1/2}$ is the minimum area where only one $j = 1/2$ edge contributes to the area eigenvalue. i.e., $a_{1/2} = 8\pi\gamma\sqrt{\frac{1}{2}(\frac{1}{2}+1)} = 4\pi\gamma\sqrt{3}$. Likewise, for any eigenvalue $a_{j_x} (0 < a_{j_x} \leq A)$ of the area operator, we have

$$(j_1, \cdots, j_n) \in N(A - a_{j_x}) \Longrightarrow (j_1, \cdots, j_n, j_x) \in N(A). \tag{4.4}$$

Then, important point is that if we consider all $0 < a_{j_x} \leq A$ and $(j_1, \cdots, j_n) \in N(A - a_{j_x})$, $(j_1, \cdots, j_n, j_x)$ form the entire set $N(A)$. Thus, we obtain

$$N(A) = \sum_j N(A - 8\pi\gamma\sqrt{j(j+1)}) \tag{4.5}$$

By plugging $N(A) = \exp(A/4)$, one can determine whether the above formula satisfies Bekenstein-Hawking entropy formula. If the Bekenstein-Hawking entropy is satisfied, from the above formula, we have [19, 25]:

$$1 = \sum_j \exp(-8\pi\gamma\sqrt{j(j+1)}/4) \tag{4.6}$$

In other words,

$$1 = \sum_i e^{-A_i/4} \tag{4.7}$$

## 4.4 Compositions

This section closely follows [28, 29]. A composition is an integer partition in which order is taken into account. For example, there are eight compositions of 4: $4, 3+1, 1+3, 2+2, 2+1+1, 1+2+1, 1+1+2$ and $1+1+1+1$. $c(n)$ denotes the number of compositions of $n$, and $c_m(n)$ is the number of compositions into exactly $m$ parts. For example: $c(4) = 8$, $c_3(4) = 3$.

It is easy to understand that $c_m(n)$ is given by the coefficient of $x^n$ in the expansion of

$$(x + x^2 + x^3 + \cdots)^m \tag{4.8}$$

for writing the function as a product of $m$ factors and performing the multiplication by picking out the terms $x^{p_1}, x^{p_2}, \cdots, x^{p_m}$, where

$$p_1 + p_2 + \cdots + p_m = n \tag{4.9}$$

in succession, we obtain for this particular selection the term $x^{p_1+p_2+\cdots+p_m}$ of the product, where $(p_1, p_2, \cdots, p_m)$ is one composition of $n$ into exactly $m$ parts.

In other words,

$$(x + x^2 + x^3 + \cdots)^m = \sum_{n=1}^{\infty} c_m(n)x^n \tag{4.10}$$

Then, as

$$c(n) = \sum_{m=1}^{\infty} c_m(n) \tag{4.11}$$

we have

$$\sum_{m=1}^{\infty}(x + x^2 + x^3 + \cdots)^m = \sum_{n=1}^{\infty} c(n)x^n \tag{4.12}$$

Now, we can explicitly calculate $c(n)$. The above formula is equal to:

$$\frac{(x + x^2 + x^3 + \cdots)}{1 - (x + x^2 + x^3 + \cdots)} = \frac{x}{1 - 2x} \tag{4.13}$$

Therefore, we obtain $c(n) = 2^{n-1}$

## 4.5 Our theorem

Now, let's apply the lesson from our earlier section to our case, namely, Bekenstein-Hawking entropy. The fact that $\{A_i, A_j\}$ should be regarded different from $\{A_j, A_i\}$ suggests that the calculation of black hole entropy has a similar structure to "compositions" in which the order is taken into account. Considering this, (4.2) can be translated into

$$\sum_{m=1}^{\infty}(e^{-sA_1}+e^{-sA_2}+\cdots)^m = \frac{e^{-sA_1}+e^{-sA_2}+\cdots}{1-(e^{-sA_1}+e^{-sA_2}+\cdots)} = \sum_{A} N(A)e^{-sA} \tag{4.14}$$

where $s$ is an arbitrary parameter. It is easy to see that the above formula converges for $s$ such that

$$e^{-sA_1}+e^{-sA_2}+\cdots < 1 \tag{4.15}$$

and diverges for $s$ such that

$$e^{-sA_1}+e^{-sA_2}+\cdots \geq 1 \tag{4.16}$$

However, from Domagala-Lewandowski-Meissner formula (4.7), we have:

$$e^{-A_1/4}+e^{-A_2/4}+\cdots = 1 \tag{4.17}$$

Therefore, by examining (4.15) and (4.16), we can see that (4.14) converges for $s > \frac{1}{4}$, and diverges for $s \leq \frac{1}{4}$. Given this, if we closely examine the right-hand side of (4.14), the only conclusion that we can draw is that (4.17) implies

$$N(A) \sim P(A)e^{A/4} \tag{4.18}$$

for large enough $A$, and for $P(A)$ which does not increase or decrease faster than an exponential function.

## 4.6 The naive black hole degeneracy

Let's focus on the behavior of (4.14), when $s$ is slightly bigger than $\frac{1}{4}$. We write:

$$s = \frac{1}{4} + \alpha \tag{4.19}$$

And, let's use following notation, which should be familiar from statistical mechanics.

$$< P(A) >\equiv \sum_i P(A_i) e^{-A_i/4} \tag{4.20}$$

Notice that the above formula is correctly normalized, as $< 1 >= 1$. Then, by Taylor expansion, we have:

$$\sum_i e^{-(1/4+\alpha)A_i} = 1 - \alpha < A > + \frac{\alpha^2}{2} < A^2 > - \frac{\alpha^3}{6} < A^3 > + \cdots \tag{4.21}$$

Plugging the above formula to (4.14), we obtain:

$$\sum_A N(A) e^{-(1/4+\alpha)A} = \frac{1}{\alpha < A >} + (\frac{< A^2 >}{2 < A >^2} - 1) + (-\frac{1}{3}\frac{< A^3 >}{< A >^2} + \frac{1}{4}\frac{< A^2 >^2}{< A >^3})\alpha + \cdots \tag{4.22}$$

Now, let's reexpress the left-hand side of the above formula. Using the following notation,

$$P(A) \equiv N(A) e^{-A/4} \tag{4.23}$$

and considering the fact that one can approximate summation in terms of integration in the limit $A_{cut} \to \infty$, we can write:

$$\sum_A N(A) e^{-(1/4+\alpha)A} = \lim_{A_{cut}\to\infty} \{ \sum_{A<A_{cut}} N(A) e^{-(1/4+\alpha)A} + \int_{A_{cut}}^{\infty} N(A) e^{-(1/4+\alpha)A} dA \}$$

$$= \lim_{A_{cut}\to\infty} \lim_{\alpha A_{cut}\ll 1} \{ \sum_{A<A_{cut}} P(A)(1 - \alpha A + \frac{\alpha^2 A^2}{2} + \cdots) + \int_{A_{cut}}^{\infty} P(A) e^{-\alpha A} dA \} \tag{4.24}$$

This separation of $N(A)$ into the case when $A$ is small and the case when $A$ is big is useful, as when $A$ is too small, the "fluctuation" or the "randomness" of $N(A)$ is so big that it cannot be approximated by a well-behaving function of $A$. Moreover, it will turn out soon that the last term in the above formula would diverge, if we didn't do the separation and took the whole range of $A$ into the consideration. (i.e. if $A_{cut}$=0) Therefore, the separation is essential. Now, we must compare the above formula with (4.22). We easily see the following:

$$\lim_{A_{cut}\to\infty}\lim_{\alpha A_{cut}\ll 1}\int_{A_{cut}}^{\infty} P(A)e^{-\alpha A}dA = \frac{1}{\alpha < A >} + O(1) + O(\alpha) + \cdots \quad (4.25)$$

which suggests the following approximation for large $A$:

$$P(A) \approx P_0 + \frac{P_1}{A} + \frac{P_2}{A^2} + \cdots \quad (4.26)$$

as

$$\lim_{A_{cut}\to\infty}\lim_{\alpha A_{cut}\ll 1}\int_{A_{cut}}^{\infty} (P_0 + \frac{P_1}{A} + \frac{P_2}{A^2} + \cdots)e^{-\alpha A}dA$$
$$= \frac{P_0}{\alpha} + O(1) + O(\alpha) + \cdots \quad (4.27)$$

In other words, in order that (4.25) and (4.27) match each other order by order, the terms proportional to the positive powers of $A$ are absent in (4.26). This implies:

$$P_0 = \frac{1}{< A >} \quad (4.28)$$

Now, let's explicitly consider the matching for the higher-order terms. If we take a derivative of (4.27) with respect to $\alpha$, we get:

$$
\begin{aligned}
\lim_{A_{cut}\to\infty}\lim_{\alpha A_{cut}\ll 1} \quad & \int_{A_{cut}}^{\infty} -(P_0 A + P_1 + \frac{P_2}{A} + \cdots)e^{-\alpha A} dA \\
&= -P_0 \frac{e^{-\alpha A_{cut}}}{\alpha^2} - P_0 \frac{A_{cut}}{\alpha} e^{-\alpha A_{cut}} - \frac{P_1}{\alpha} e^{-\alpha A_{cut}} + \cdots \\
&= -\frac{P_0}{\alpha^2} - \frac{P_1}{\alpha} + \cdots \qquad (4.29)
\end{aligned}
$$

where we have Taylor expanded $e^{-\alpha A_{cut}}$ in the last step.

Given this, notice that the above formula must be equal to the following:

$$
\lim_{A_{cut}\to\infty}\lim_{\alpha A_{cut}\ll 1} \int_{A_{cut}}^{\infty} -(P_0 A + P_1 + \frac{P_2}{A} + \cdots)e^{-\alpha A} dA = -\frac{P_0}{\alpha^2} + O(1) + O(\alpha) + \cdot \qquad (4.30)
$$

which is the derivative of the right-hand side of (4.27) with respect to $\alpha$. In other words, the term proportional to $1/\alpha$ is absent in the above formula. This suggests:

$$
P_1 = 0 \qquad (4.31)
$$

Let us give you some interpretations for this result. A non-zero $P_1$ suggests that (4.29) implies the presence of the term $P_1 \ln \alpha$ in (4.27). However, a term proportional to $\ln \alpha$ is absent in (4.25). So, we conclude $P_1 = 0$.

Similarly, by considering the higher derivatives of (4.27) with respect to $\alpha$, we conclude:

$$
P_1 = P_2 = \cdots = 0 \qquad (4.32)
$$

Let us briefly sketch how this is done. Assume that we have a non-zero $P_k$. Then we would have

$$
\begin{aligned}
&\frac{\partial^k}{\partial \alpha^k}\int_{A_{cut}}^{\infty}\frac{P_k}{A^k}e^{-\alpha A}dA \\
&= (-1)^k\int_{A_{cut}}^{\infty}P_k e^{-\alpha A}dA && (4.33)\\
&= (-1)^k\frac{P_k}{\alpha}e^{-\alpha A_{cut}} = (-1)^k\frac{P_k}{\alpha} + O(1) + O(\alpha) + \cdots && (4.34)
\end{aligned}
$$

Integrating (4.34) by $\alpha$, $k$ times, we have:

$$
\int_{A_{cut}}^{\infty}\frac{P_k}{A^k}e^{-\alpha A}dA = (-1)^k\frac{P_k}{(k-1)!}\alpha^{k-1}\ln\alpha + \cdots \qquad (4.35)
$$

As the term proportional to $\alpha^{k-1}\ln\alpha$ is absent in (4.25), we obtain $P_k = 0$ for $k > 0$.

Plugging these values and (4.28) to (4.26), we conclude:

$$
\lim_{A\to\infty} P(A) = \frac{1}{< A >} \qquad (4.36)
$$

$$
\lim_{A\to\infty} N(A) = \frac{1}{< A >}e^{A/4} \qquad (4.37)
$$

However, this $N(A)$ is a naive one without the projection constraint or the $SU(2)$ invariant subspace constraint. We will correct this in the next section

Now, let's plug (4.36) to the formula (4.24) and equate it with (4.22). By matching order by order, we obtain followings:

$$
\frac{< A^2 >}{2 < A >^2} - 1 = \lim_{A_{cut}\to\infty}\{-\frac{A_{cut}}{< A >} + \sum_{A<A_{cut}} P(A)\} \qquad (4.38)
$$

$$
-\frac{1}{3}\frac{< A^3 >}{< A >^2} + \frac{1}{4}\frac{< A^2 >^2}{< A >^3} = \lim_{A_{cut}\to\infty}\{\frac{A_{cut}^2}{2 < A >} - \sum_{A<A_{cut}} P(A)A\} \qquad (4.39)
$$

and so on. In other words, we can obtain the value for the following formula

$$
\lim_{A_{cut}\to\infty}\{-\frac{A_{cut}^{n+1}}{(n+1) < A >} + \sum_{A<A_{cut}} P(A)A^n\} \qquad (4.40)
$$

which is convergent. In other words, (4.38) and (4.39) are the cases when $n = 0, 1$ in the above formula.

Given this, I want to note that $P(A)(\equiv N(A)\exp(-A/4))$ is zero for most of the values, as it would be a big coincidence if a given random $A$ is a sum of the area eigenvalues. In other words, $P(A)$ is non-zero only for the set which is measure zero. We can fix this by introducing $P'(A)$ as the "continutization" of $P(A)$ as follows:

$$\frac{< A^2 >}{2 < A >^2} - 1 = \lim_{A_{cut}\to\infty}\{-\frac{A_{cut}}{< A >} + \int_0^{A_{cut}} P'(A)dA\} \tag{4.41}$$

$$-\frac{1}{3}\frac{< A^3 >}{< A >^2} + \frac{1}{4}\frac{< A^2 >^2}{< A >^3} = \lim_{A_{cut}\to\infty}\{\frac{A_{cut}^2}{2 < A >} - \int_0^{A_{cut}} P'(A)AdA\} \tag{4.42}$$

and so on. In other words, we have certain non-diverging values for the following formula

$$\lim_{A_{cut}\to\infty}\{-\frac{A_{cut}^{n+1}}{(n+1) < A >} + \int_0^{A_{cut}} P'(A)A^n dA\} \tag{4.43}$$

Furthermore, even though it may sound redundant, we want to note that this convergence implies that $P_n = 0$ for $n > 0$. To see this, let's consider a non-zero $P_n$. Then, for large $A$ we have the following:

$$P'(A)A^n = (\frac{1}{< A >} + \frac{P_n}{A^n})A^n = \frac{A^n}{< A >} + P_n \tag{4.44}$$

When the above term is plugged into the right-hand side of (4.43) and integrated, the potential divergence of the term proportional to $A_{cut}^{n+1}$ is removed, but the integration of $P_n$ survives, which yields roughly $P_n A_{cut}$, which is divergent in the limit $A_{cut}$ goes to infinity.

At this point, it may seem odd that $P'(A)$ doesn't receive any Laurent series corrections, but nevertheless still some corrections so that the left-hand sides of (4.41) and (4.42) are not zero. One may wonder such a function exists at all. An example of such function is following.

$$P'(A) = \frac{1}{< A >} + e^{-\beta A} \tag{4.45}$$

for some positive $\beta$. We can clearly see that the above expression cannot be written in terms of Laurent series expansion exact in the limit in which $A$ is large, but that it is clearly different from $\frac{1}{<A>}$. Of course, this function does not satisfy (4.41) and (4.42), but one can guess that a suitable form for $P'(A)$ should be something of this kind. In the next section, we obtain an explicit solution for $P'(A)$

## 4.7 Solution

We suggest the following:

$$P'(A) = \frac{1}{< A >} + B(A)e^{-A} \tag{4.46}$$

where $B(A)$ is a suitable polynomial.

Plugging this to (4.43), we obtain:

$$\int_0^\infty B(x)x^n e^{-x} dx = C_n \tag{4.47}$$

for a suitable $C_n$. For example, from (4.41) and from (4.42), we obtain:

$$C_0 = \frac{< A^2 >}{2 < A >^2} - 1 \tag{4.48}$$

$$C_1 = \frac{1}{3}\frac{< A^3 >}{< A >^2} - \frac{1}{4}\frac{< A^2 >^2}{< A >^3} \tag{4.49}$$

Now recall Laguerre polynomial:

$$L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!}\binom{n}{k} x^k \tag{4.50}$$

Then, we have:

$$\int_0^\infty B(x)L_n(x)e^{-x}dx = \sum_{k=0}^{n} \frac{(-1)^k}{k!}\binom{n}{k}C_k \tag{4.51}$$

Since

$$\int_0^\infty e^{-x}L_m(x)L_n(x)dx = \delta_{mn} \tag{4.52}$$

We have:

$$B(x) = \sum_{n=0}^{\infty} d_n L_n(x) \tag{4.53}$$

where

$$d_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!}\binom{n}{k}C_k \tag{4.54}$$

Therefore, the solution is:

$$P'(A) = \frac{1}{< A >} + \sum_{n=0}^{\infty} d_n L_n(A)e^{-A} \tag{4.55}$$

## 4.8 Corrections

In case of the $U(1)$ symmetry reduced framework, if we consider the extra condition, the so-called "projection constraint" $\sum_i m_i = 0$ [19], the black hole degeneracy (4.37) will be reduced. We will calculate the reduced black hole degeneracy by multiplying the probability that this condition is satisfied to our earlier naive black hole degeneracy. Before doing so, let us explain what $m_i$s are. $m_i$ are half integers which satisfy:

$$-j_i \leq m_i \leq j_i \tag{4.56}$$

where $j_i$s are given in Sec. 4.3. In other words, it has the same structure as 3d-angular momentum in quantum mechanics.

Given this, let's define $x \equiv \sum_i m_i$. Then, we have:

$$\Delta x^2 = \sum_i \Delta m_i^2 \tag{4.57}$$

Of course, we can express $\Delta m_i^2$ as in terms of $j_i$, as

$$\Delta m_i^2 = (\sum_{m_i=-j_i}^{j_i} m_i^2)/(2j_i+1) \tag{4.58}$$

Then, (4.57) becomes

$$\Delta x^2 = \sum_i \Delta m_i^2(j_i) \tag{4.59}$$

Now, we need to calculate $F_{j_i}$, the number of times given $j_i$ appears on the right-hand side of the above equation. From thermodynamics consideration or observation from Domagala-Lewandowski-Meissner trick [19, 25], it is obvious that this frequency is proportional to $e^{-A_{j_i}/4}$, where we remind the reader that $A_{j_i}$ is given by:

$$A_{j_i} = 8\pi\gamma\sqrt{j_i(j_i+1)} \tag{4.60}$$

in isolated horizon case. Also, taking into account the fact that the total sum of area of each segment in the black hole horizon is $A$, we obtain:

$$A = \sum_j A_j(\frac{A}{<A>}e^{-A_j/4}) = \sum A_j F_j$$
$$F_j = \frac{A}{<A>}e^{-A_j/4} \tag{4.61}$$

For a macroscopic black hole, (4.59) can be written as:

$$\Delta x^2 = \sum_j F_j \Delta m^2(j) = AC \tag{4.62}$$

where $C$ is an unimportant constant which one can calculate from the area spectrum and $\Delta m^2(j)$.

Now, noticing that the distribution of $x$ reaches Gaussian for macroscopic black hole by the well-known theorem in statistics, we can write $p(0)$, the probability that $x = 0$ as follows:

$$p(0) \approx \int_{x=-1/4}^{x=1/4} \frac{1}{\sqrt{2\pi CA}} e^{-x^2/(2CA)} \approx \frac{1}{2\sqrt{2\pi CA}} \tag{4.63}$$

Therefore, the correct degeneracy is given by:

$$N_{cor}(A) = \frac{1}{2 < A > \sqrt{2\pi CA}} e^{A/4} = \frac{1}{D\sqrt{A}} e^{A/4} \tag{4.64}$$

where D is an unimportant constant. (Remember that $< A >$ is merely a constant which one can calculate from the area spectrum and which doesn't depend on the black hole area $A$.) Therefore, we conclude that the black hole entropy is given by:

$$S = \ln N_{cor}(A) = \frac{A}{4} - \frac{1}{2} \ln A + O(1) \tag{4.65}$$

The logarithmic corrections to the Bekenstein-Hawking entropy in case of the fully $SU(2)$ framework can be obtained similarly. For a detailed discussion, please read [30]. See also, [31–33].

# Chapter 5

# Conclusion

We have briefly reviewed theories beyond general relativity focusing on author's work on Mannheim's conformal gravity program and loop quantum gravity. We hope they see a new era in physics, as general relativity did in last century.

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# 요약

이 논문에서는 만하임의 등각 중력 이론과 루프 양자중력 이론에서의 베켄슈타인-호킹 엔트로피의 보정을 중심으로 일반 상대성 이론을 뛰어넘는 이론들을 소개했다.



**주요어**: 칼루자-클라인 이론, 끈 이론, 루프 양자중력, 수정 뉴턴 역학

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