Condorcet-consistent extension of Simpson rule for weak preference orderings

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Abstract

Condorcet-criterion requires voting rules to choose the Condorcet winner (the Condorcet winner is an alternative which defeats all other alternatives in head-to-head elections) when it exists. When a voting rule satisfies this criterion, we call it a Condorcet-consistent voting rule. When preferences are strict, there are three well-known Condorcet-consistent voting rules, Copeland rule, Simpson rule, and Dodgson's method. When preferences are weak orderings, the Simpson rule (defined on the domain of weak preference orderings in the same way as on the domain of strict orderings, that is, excluding all indifference relations in its vote counting) violates Condorcet-criterion. I extend the Simpson rule by counting not only strict preference relations but indifference relations; different (tie-counting) weights on indifference relations will lead to different extensions. Thus there are numerous extended Simpson rules. I show that there is a unique extension of Simpson rule satisfying Condorcet-consistency; it is the extended Simpson rule with the tie-counting weight equal to 1/2.

Keywords: Condorcet winner, Condorcet-consistency, Voting rule, Copeland rule, Simpson rule, Dodgson's method, Weak preference ordering, Extended Simpson rule

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1 Introduction

In voting theory, the Condorcet-criterion is the most widely accepted criterion for a social choice. This criterion requires that if there exists a single alternative that defeats every other candidate in head-to-head elections, then that alternative should be selected (Condorcet 1785). This alternative is called the Condorcet winner. Some electoral systems are designed to find the Condorcet winner, and we call these Condorcet-consistent voting rules. In the domain of strict preferences, well-known Condorcet-consistent voting rules are Copeland rule, Simpson rule (also called as Minmax rule), and Dodgson's method (Fishburn 1977). On the domain of weak preferences, the Simpson rule loses Condorcet-consistency contrary to other two rules. In particular, the Simpson rule has a scoring system which critically depends on the relative size of votes for each alternative (Levin and Nalebuf 1995), and this is why some distortions could happen when there are indifferent voters, possibly depriving Condorcet-consistency of the Simpson rule. Thus, I introduce *Extended Simpson rules* that count indifferences using a specific (tie-counting) weight unlike the Simpson rule in one-to-one elections. Each Extended Simpson rule is identified by its tie-counting weight.

The purpose of this paper is finding a proper tie-counting weight that leads to a Condorcet-consistent extension of Simpson rule. I find that there is a unique extension of Simpson rule satisfying Condorcet-consistency, which is giving a tie-counting weight equal to one-half. On the other hand, for the Copeland rule and the Dodgson's method, the two rules remain the same whether indifferences are counted or not (whatever tie-counting weights are used) and they are Condorcet-consistent for weak preference too.

Additionally, in this paper, I introduce two novel voting rules; one is derived from the Simpson rule, the *Inverse Simpson rule*, and the other one is derived from the Dodgson's method, which is called *Abstract Dodgson' method.* Similarly to the Simpson rule, Condorcet-consistent extension of *Inverse Simpson rule* will be introduced. For the Abstract Dodgson's method, nice properties including its

simpler computation than Dodgson's method as well as the Condorcet-consistency on the weak domain of preferences, will be shown.

This article is organized as follows. In Section 2, I formally describe five Condorcet-consistent voting rules including two novel Condorcet-consistent voting rules. In Section 3, utilizing Dodgson's matrix, I show that both Simpson rule and Inverse Simpson rule lose Condorcet-consistency on the domain of weak preference orderings. Then, Extended (Inverse) Simpson rules will be introduced. In Section 4, our main results is shown: the Condorcet-consistent extension of (Inverse) Simpson rule for the weak preference orderings. Finally, Section 5 will provide the concluding remarks.

2 Condorcet-consistent voting rules for strict preference orderings

In this section, I explore several voting rules which are well-known for their Condorcet-consistent property in the strict preference orderings: the Copeland rule, the Simpson rule, and the Dodgson's method. In addition, two voting rules are introduced. One is derived from the Simpson rule, Inverse Simpson rule, and the other one is derived from the Dodgson's method, Abstract Dodgson's method.

2.1 Notation and setting

Before defining voting rules, let us verify the setting for an electoral system. Let $\mathcal{A}$ be the class of set of alternatives, and $A = \{1, 2, \cdots\}$ be a set of alternatives. The set of voters is N, and the cardinality of the set is described by $|N|$ (every subset would be applied as well). On the domain of strict preferences, each agent has *complete, transitive, strict(irreflexive and antisymmetric)* preference over A. The binary relation of strict preference is denoted by $P_i \subseteq A \times A$ for each $i \in N$. If a pair (k, l) for some k, $l \in A$, belongs to P_i, we can state either that i prefers k to l or more commonly kP_il (equally applying this to the binary relation of weak

preference). Replacing *strict* to *reflexive and symmetric*, the preference includes indifferent relationships(denoting I_i), belonging to the weak preference orderings, which is written by $R_i \subseteq A \times A$ for each $i \in N$. Let $\mathscr{R}$ be the space of all possible preferences over A (the domain of weak preferences) and $\mathscr{P}$ be the space of all strict preferences over A(the domain of strict preferences). Given a set of alternatives A and admissible preference profiles $\mathscr{D} \subseteq \mathscr{R}^{|N|}$($\mathscr{D} \subseteq \mathscr{P}^{|N|}$ in the case of strict preference orderings), a social choice correspondence on $A \times \mathscr{D}$ is $f : A \times \mathscr{D} \to \mathcal{B}$ where $\mathcal{B}$ is the class of nonempty subsets of A (f is a social choice function if $\mathcal{B}$ is restricted to be single-valued). I will define voting rules by using social choice correspondence.

Now I can define the Condorcet-consistency in this setting. Define $n(P)_{ab} = |\{i \in N : aP_ib\}|$ to be the number of voters who strictly prefer a to b where $a, b \in A$. An alternative c^* is the *Condorcet winner* if $n(P)_{c^*b} > n(P)_{bc^*}$ for every $b \in A\backslash\{c^*\}$. It always beats others in head-to-head matchups.[1] If the Condorcet winner exists, Condorcet-consistent voting rules always choose the Condorcet winner.

Definition. Condorcet-consistent voting rule

Given $R \in \mathscr{D}$ and $A \in \mathcal{A}$, a social choice correspondence f is the Condorcet-consistent voting rule if $f(A, R) = \{c^*\}$ whenever $n(P)_{c^*b} > n(P)_{bc^*}$ $\forall b \in A\backslash\{c^*\}$.

Let's see three representative Condorcet-consistent voting rules.

2.2 Three Condorcet-consistent voting rules

2.2.1 Copeland rule

First one is the Copeland rule. As Copeland rule explicitly depend on majority, I need to clarify the majority scoring to see the Copeland rule's scoring system. The majority scoring is as follows. Given $R \in \mathscr{D}$,[2] compare $a \in A$ with $x \in A\backslash\{a\}$.

[1]Murat and Remzi(2004) provides a weaker version of the definition; there is no other alternative that has more votes supporting it against $c^\star$. This is important when examining the existance of the Condorcet winner in case of the even number of voters. As starting from the situation where the Condorcet winner exists, I just use the definition of the strong version of the Condorcet winner.

[2]Even on the domain of weak preferences, as I assumed complete preference, the binary relation must be in three categories: strict preferring one to the other, the opposite, or indifferent.

Score $+1$ if the voters who prefer a to x is larger than the voters who prefer x to a, -1 in the opposite case, and 0 if they are equal size. Formally writing,

Definition. (Majority scoring)[3] Given $R \in \mathscr{D}$, $A \in \mathcal{A}$,

$$M(R : a, x) = \begin{cases} 1 & \text{if } |\{i \in N : aPx\}| > |\{i \in N : xPa\}| \\ -1 & \text{if } |\{i \in N : xPa\}| > |\{i \in N : aPx\}| \\ 0 & \text{if } |\{i \in N : aPx\}| = |\{i \in N : xPa\}| \end{cases} \quad \textit{for each}\, a \in A,\, x \in \forall N \backslash \{a\}.$$

The Copeland score of a is the sum of the numbers from its every majority matchups. An alternative which gets the highest score (Copeland winner) is selected from the Copeland rule. Condorcet winner, by definition, always gets $|N| - 1$ score which is the maximum possible score, so it must be elected.

Definition. (Copeland rule) Given $P \in \mathscr{D}$ and $A \in \mathcal{A}$, a social choice correspondence f^C is the Copeland rule if $f^C(A, P) = \{a \in A : \max S^C(P : a)\}$ where scoring function $S^C(k : P) = \sum_{l \neq k} M(P : k, l)$ for each $k \in A$.

Note that when counting Copeland score $M(\cdot)$ does not consider the size of indifferent voters. This makes Copeland rule immune to the existence of tie-rank alternatives. That is, tie-rank does not make any distortion to Copeland rule's decision, implying that Copeland rule would be Condorcet-consistent still in the weak preference orderings. I will show the proof using the notion of the tie-counting weight in Section 4.

2.2.2 Simpson rule

Definition. (Simpson rule) For a given $P \in \mathscr{D}$, Simpson score for k $S^S(A, P : k) = \min_{l \neq k} n(P)_{kl}$. The choice set of the Simpson rule is $f^S(A, P) = \{a \in A : S^S(A, P : a) = \max_{k \in A} S^S(A, P : k)\}$.

The Simpson rule compares alternatives on the basis of their lowest supports in head-to-head elections, and selects an alternative which gets the highest supports in

[3] This is borrowed from May(1952).

case of its worst matchup. That is, the Simpson rule uses the "maximin" principle. When there is the Condorcet winner, the Condorcet winner's Simpson score is larger than $\frac{|N|}{2}$. No other alternative can be selected as all of them are defeated by the Condorcet winner, so their Simpson scores must be less than $\frac{|N|}{2}$. Hence, the Simpson rule is Condorcet-consistent rule (in the strict preference orderings).

2.2.3 Dodgson's method

The Dodgson's method relies on the minimum distance between a given preference profile and another preference profile which makes one alternative to be the Condorcet winner. By giving a score based on this distance to each alternative, the Dodgson's method finds an alternative which is the closest to the Condorcet winner. To define the Dodgson's method, I need a concept of measuring the distances between binary relations and preference profiles, respectively. The Kemeny (1959) defines a binary relation distance function on $\mathscr{R}$, using the symmetric differences between preference relations. Let δ be the distance function, which is defined as $\delta : \mathscr{R} \times \mathscr{R} \to \mathbb{R}_+$ such that $\forall R_i, R'_i \in \mathscr{R}$ for every $i \in N$, $\delta(R_i, R'_i) = |(R_i - R'_i) \cup (R'_i - R_i)|/2$.[4] Then, the distance on the set of preference profiles is defined by $d : \mathscr{R}^{|N|} \times \mathscr{R}^{|N|} \to \mathbb{R}_+$ such that $\forall R, R' \in \mathscr{R}^{|N|}$, $d(R, R') = \sum_{i=1}^{|N|} \delta(R_i, R'_i)$. Given $R \in \mathscr{R}^{|N|}$, the score of an alternative $a \in A$ in the Dodgson's method is $S^D(A,\, R\, :\, a) = \min\limits_{R' \in \mathscr{R}_a^{|N|}} d(R, R')$ where $\mathscr{R}_a^{|N|} \subset \mathscr{R}^{|N|}$ is the collection of preference profiles where a is the Condorcet winner.

Definition. (Dodgson's method) Given $A \in \mathcal{A}$ and $P \in \mathscr{D}$, a social choice correspondence f^D is the Dodgson's method if $f^D(A,\, P) = \{a \in A\, :\, S^{DOD}(A,\, P\, :\, a) = \min\limits_{k \in A} S^D(A,\, P\, :\, k)\}$.

The Dodgson's method compares the distances from the preference profiles making one alternative the Condorcet winner. As the Condorcet winner needs a zero distance from the given preference, the rule always chooses the Condorcet winner

[4] This definition is borrowed from Klamler(2004).

when it exists. It is natural that even in the weak domain of preference, the Dodgson's method does not lose its Condorcet-consistency. This is because the indifferent voters do not affect the Condorcet winner's score which is zero; although scores of other alternatives can be changed, but it cannot remotely go below or equal to that of the Condorcet winner, by the uniqueness of the Condorcet winner. I will explain the details in the next section.

Small numerical example for the Dodgson's method

To help your understanding about the Dodgson's method, I give a small numerical example. Given $N = \{1, 2, 3\}$ and $A = \{a, b, c, d\}$, a preference profile is $(aP_1bP_1cP_1d,\ bP_2cP_2aP_2d,\ dP_3cP_3bP_3a)$. By the definition of the Dodgson score, the sum of minimal preference switches for making one alternative the Condorcet winner is each alternative's Dodgson score. Note that as considering the minimal switches, I should exclude every redundant count of switches. Calculating the Dodgson score(total switches without redundancy) for this example, I get the Table 1.[5]

	Election Lost	Margin	Switches	Score
a	cPa	1	1(2)	2(4)
	bPa	1	1(2)	
b	$\emptyset$	0	0(0)	0(0)
c	bPc	1	1(2)	1(2)
d	aPd	1	1(2)	3(6)
	bPd	1	2(4)	
	cPd	1	1(2)	

Table 1. The Dodgson scores in the small numerical example

The Condorcet winner b does not need any switch for being the Condorcet winner, so that its score is zero. As the switches for dPb includes the switches for dPa, I

[5] On the domain of weak preferences, the switch does consider the indifferent steps, so the switch numbers become double like the number in the round bracket.

subtract the redundant switches, 1(2), from the d's total score. As a result, the score of d is 3(6).

2.3 Two additional Condorcet-consistent voting rules

In this subsection, I introduce two novel Condorcet-consistent voting rules which are from the Simpson rule and the Dodgson's method, respectively: *Inverse Simpson rule* and *Abstract Dodgson's method.*

2.3.1 Inverse Simpson rule

First one is the *Inverse Simpson rule.* As the Simpson rule uses the maximin principle, I can imagine a symmetric approach, the minmax principle.

Definition. (Inverse Simpson rule) Given $A \in \mathcal{A}$ and $P \in \mathscr{D}$, the Inverse Simpson score for k $S^{IS}(A, P : k) = \max_{l \neq k} n(P)_{lk}$. The choice set of the Inverse Simpson rule is $f^{IS}(A, P) = \{a \in A : S^{IS}(A, P : a) = \min_{k \in A} S^{IS}(A, P : k)\}$.

An alternative which gets the lowest Inverse Simpson score becomes Inverse Simpson winner and elected. The implication of the Inverse Simpson rule is finding an alternative which is the least severely defeated. If the Condorcet winner exists, its score will be less than $\frac{|N|}{2}$ and others must be larger than $\frac{|N|}{2}$ (at least by the defeat from the Condorcet winner). Accordingly, the inverse Simpson rule is Condorcet-consistent as well (in the strict preference orderings). In particular, the social preference ordering from the Simpson rule and the Inverse Simpson rule is exactly equal.

Proposition 1. *The Inverse Simpson rule coincides with the Simpson rule on the domain of strict preferences.*

Proof. For any $A \in \mathcal{A}$ and $P \in \mathscr{D}$, $S^{IS}(A, P : a) = |N| - S^{S}(A, P : a)$ for all $a \in A$. Thus, the ordering of $S^{S}(A, P : a)$ $\forall a \in A$ in ascending sort is equivalent to the ordering of $S^{IS}(A, P : a)$ $\forall a \in A$ in descending sort. $\square$

2.3.2 Abstract Dodgson's method

The Dodgson's method handles the preference profile to calculate the score of each alternative, and if both the number of voters and the number of alternatives increase, this work can be unwieldily complicated. To be specific, suppose that there are ten alternatives and a hundred voters, calculating each alternative's score(distance) by checking the preference of every person and comparing them would be backbreaking. To resolve this weakness in the Dodgson's method, I introduce an abstract version of the Dodgson's method, the *Abstract Dodgson's method.* The strength of this method is giving a simple path to find the Condorcet winner in the system of the Dodgson's method. That is, the Abstract Dodgson's method provides an equal social ordering to that of the Dodgson's method, and its calculation is much simpler; the Abstract Dodgson's method uses less information than the Dodgson's method.

Given $A \in \mathcal{A}$ and $P \in \mathscr{D}$, let $d(P)_{kl} \in \mathbb{Z}_+ \ \forall k,\, l \in A$ be the number of hypothetical voters to make k tie with l. That is, $d(P)_{kl} = \max\{0,\, n(P)_{lk} - n(P)_{kl}\}$. These hypothetical voters are assumed to participate in only one head-to-head election. Then, the Abstract Dodgson score is the sum of these minimum numbers, $S^{AD}(A,\, P\, :\, k) = \sum_{l \in A} \left(\left[\frac{d(P)_{kl}}{2} \right] + 1 \right)$ where $[x]$ is the biggest integer that is less or equal to x. The Abstract Dodgson score involves an idea that the more votes needed to make an alternative Condorcet winner, the less the alternative is close to the Condorcet winner. Thus, an alternative which has the lowest Abstract Dodgson score is the Abstract Dodgson winner, which would be elected.

Definition. (Abstract Dodgson's method) Given $A \in \mathcal{A}$ and $P \in \mathscr{D}$, social choice correspondence f^{ADOD} is the Abstract Dodgson's method if $f^{AD}(A,\, P) = \{a \in A\, :\, S^{AD}(A,\, P\, :\, a) = \min_{k \in A} S^{AD}(A,\, P\, :\, k)\}$.

Let $m^k(l) \equiv \max\{0,\, n(P)_{lk} - n(P)_{kl}\}$ for every $l \in A \backslash \{k\}$ be the margin for k to be defeated by l.

Lemma 1. The Dodgson score of $k \in A$ is $\sum_{l \in A \backslash \{k\}} \left(\left[\frac{m^k(l)}{2} \right] + 1 \right)$, on the domain of strict preference orderings.

Proof. Given the preference profile P, let P^k be the preference profile which makes an alternative k the Condorcet winner with the minimum switches. Suppose that there is a set of alternatives L which beats k. Let $n : L \to \mathbb{N}$ be a numbering function giving a number to each alternative in L from one to $|L|$. $n(a) < n(b)$ if $\{P_i^{k(a)} - P_i\} \subset \{P_i^{k(b)} - P_i\}$ $\forall i \in N$, $\forall a, b \in L$ where $P^{k(a \in L)}$ is the preference profile which makes k beat a by the minimum number of preference switches. Then, every preference switch at $R^{k(i)}$ $\forall i < |L|$ is included in $R^{k(|L|)}$

One switch brings a two-margin reduction between k and the counterpart.[6] That is, if I do not need to consider any redundancy, the rounded number of half margin between k and $a \in L$ is the minimum number of switches to reverse the social order between k and a. When $a = n^{-1}(1)$, give the first switches to $P^{k(1)}$, resulting $d(P, P^{k(1)}) = \left[\frac{m^k(n^{-1}(1))}{2}\right] + 1$. To avoid recounting $d(P, P^{k(1)})$, I calculate $d(P^{k(1)}, P^{k(2)})$. When changing P to $P^{k(1)}$, the margin $m^k(n^{-1}(x))$ with $x \in \{2, \cdots, |L|\}$ cannot be affected by the definition of $n(a \in L)$, so that there is not redundancy. As a result, $d(P^{k(1)}, P^{k(2)}) = \left[\frac{m^k(l_2)}{2}\right] + 1$. In the same way, I can calculate every $d(P^{k(\ell)}, P^{k(\ell+1)})$ $\forall \ell \in \{1, \cdots, |L| - 1\}$ without redundancy. Therefore, the minimum number of switches for an alternative $k\,(\in A)$ to be the Condorcet winner without redundancy is $d(P, P^{k(1)}) + d(P^{k(1)}, P^{k(2)}) + \cdots + d(P^{k(|L|-1)}, P^{k(|L|)}) = \sum_{l \in A \setminus \{k\}} \left(\left[\frac{m^k(l)}{2}\right] + 1\right)$ □

Proposition 2. *The Dodgson's method coincides with the Abstract Dodgson's method on the domain of strict preferences.*

Proof. As the number of hypothetical voters for k to beat l, d_{kl}, is equal to $m^k(l)$, $S^D(A,\, P : k) = S^{AD}(A,\, P : k)$ $\forall k \in A$ by Lemma 1. □

The Abstract Dodgson's method is obviously Condorcet-consistent, because its scoring is same as that of the Dodgson's method.

[6] In case of the weak preference orderings, two switches are needed for making an interchanging switch within an individual ranking, because of the step for indifferent relation.

3 Condorcet-consistency for weak preference orderings

Although I defined five voting rules, did not check their Condorcet-consistency on the domain of weak preference orderings. Now I look into the Dodgson's matrix, and see that two of them(Simpson rule, Inverse Simpson rule) can lose the Condorcet-consistency in the weak preference orderings.

3.1 Dodgson's matrix

First of all, I present a manner to check Condorcet-consistency with using Dodgson's matrix. Let z be the number of alternatives, $z = |A|$. When assuming a strict preference profile $P \in \mathscr{D}$, $n(P)_{ab} + n(P)_{ba} = |N|$ where $|N|$ is the total number of voters. Then I can make $(z \times z)$ Dodgson's matrix such as

$$\begin{bmatrix} n(P)_{11} & n(P)_{12} & n(P)_{13} & \dots & n(P)_{1z} \\ n(P)_{21} & n(P)_{22} & n(P)_{23} & \dots & n(P)_{2z} \\ n(P)_{31} & n(P)_{32} & n(P)_{33} & \dots & n(P)_{3z} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n(P)_{z1} & n(P)_{z2} & n(P)_{z3} & \dots & n(P)_{zz} \end{bmatrix}.$$

In this matrix, I can easily see how to pick the Condorcet winner. Suppose an alternative a is a Condorcet winner, then $n(P)_{ab}$ should be larger than $\frac{|N|}{2}$ $\forall b \in A \backslash \{a\}$. In particular, this condition can be checked by the pairwise comparison between $n(P)_{ab}$ and $n(P)_{ba}$(the symmetrical point of $n(P)_{ab}$). That is, $n(P)_{ab} > n(P)_{ba}$ $\forall b \in A \backslash \{a\}$ $\Leftrightarrow$ $n(P)_{ab} > n - n(P)_{ab}$ by the definition of $n(P)_{ab}$. Thus, $n(P)_{ab} > \frac{|N|}{2}$ $\forall b \in A \backslash \{a\}$ is the condition for a to be the Condorcet winner.

3.1.1 Numerical examples with a strict preference profile[7]

Let's see how the Copeland rule, the Simpson rule, and the Abstract Dodgson's method is expressed in the Dodgson's matrix.[8] Let $A = \{a, b, c\}$ be the set of alternatives, and cardinality of voters as $|N| = 17$. Given preference of voters

Number of voters	6	3	4	4
	a	c	b	b
Rank of alternatives	b	a	a	c
	c	b	c	a

,

the Dodgson's matrix(ordering alternatives in a, b, c) is represented as

$$\begin{bmatrix} 0 & 9 & 10 \\ 8 & 0 & 14 \\ 7 & 3 & 0 \end{bmatrix}.$$

Note that $n(P)_{ab} = 9 > n(P)_{ba} = 8$ and $n(P)_{ac} = 10 > n(P)_{ca} = 7$, which means a is the Condorcet winner.

First, when making a 3×3 matrix which shows the Copeland scores of the above situation,

$$\begin{bmatrix} 0 & +1 & +1 \\ -1 & 0 & +1 \\ -1 & -1 & 0 \end{bmatrix}.$$

When summing up the scores in the same row, each value indicates the Copeland score of that alternative. For example, the first alternative(a)'s Copeland score is the sum of the first row, which is 2. As 2 is the maximum Copeland score in this voting, a is elected, which is the Condorcet winner.

Second, when writing a 3×3 matrix to see the Simpson scores,

[7] This example is borrowed from Fishburn(1984).

[8] I leave the Inverse Simpson rule and the Dodgson's method out of this example. By Proposition 1, the Inverse Simpson rule is essentially equal to the Simpson rule in strict preference orderings. Similarly, the Dodgson's method is essentially equal to the Abstract Dodgson's method in strict preference orderings as shown in Proposition 2.

$$\begin{bmatrix} 0 & (9) & 10 \\ (8) & 0 & 14 \\ 7 & (3) & 0 \end{bmatrix}.$$

The number in parenthesis in each row is the row numbered alternative's Simpson score. According to the definition of Simpson rule, an alternative which has the highest minimum votes is chosen (maxmin principle). Thus, a is chosen by Simpson winner in this example.

Instead of $n(P)_{ab}$, filling d_{ab}, $\forall a, b \in A$, the Dodgson's matrix shows the (Abstract) Dodgson score.

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 3 & 11 & 0 \end{bmatrix} \quad \begin{array}{l} S^D(A,\, P\, :\, a) = 0 \\ S^D(A,\, P\, :\, b) = 1 = [\frac{1}{2}] + 1 \\ S^D(A,\, P\, :\, c) = 8 = 2 + 6 = \left([\frac{3}{2}] + 1\right) + \left([\frac{11}{2}] + 1\right) \end{array}$$

Zero, one, and eight are the Dodgson score of a, b, and c, respectively, in this example. a is elected by the (Abstract) Dodgson's method as it has the lowest score.

3.1.2 Numerical examples with a weak preference profile

Numerical example for the Simpson rule

This example is for showing that the Simpson rule violates Condorcet-consistency in the weak domain of preference. Giving some ties in the previous example,

Number of voters	6	3	4	4
	a	c	b	b
Rank of alternatives	b	a	a, c	a, c
	c	b	$\vdots$	$\vdots$

where "a, c" denotes that the two options have a tie rank. As the Dodgson's matrix only considers strict orderings, it is

$$\begin{bmatrix} 0 & 9 & 6 \\ 8 & 0 & 14 \\ 3 & 3 & 0 \end{bmatrix}.$$

Note that in this case $n(P)_{ac} + n(P)_{ca} < n$ rather than $n(P)_{ac} + n(P)_{ca} = n$, but still is a the Condorcet winner. Now, let's see each choices of the Copeland rule and the Simpson rule.

$$Copeland: \begin{bmatrix} 0 & +1 & +1 \\ -1 & 0 & +1 \\ -1 & -1 & 0 \end{bmatrix}$$

$$Simpson: \begin{bmatrix} 0 & 9 & (6) \\ (8) & 0 & 14 \\ 3 & (3) & 0 \end{bmatrix}$$

Both the Dodgson's method and the Abstract Dodgson's method require a little consideration about switches in tie-rank alternatives. Let $a, c \in A$, $i \in N$, and $x_{ac} > x_{ca}$. Two switches from aP_ib to bP_ia are needed in weak preference orderings, as the change from aP_ib to aI_ib is counted as one switch, I should consider this indifferent step. Moreover, when there is $|j|$ voters who are indifferent between a and b, $d(R, R^{a(b)}) = \left[\frac{m^a(b)}{2}\right] + 1$ if $m^a(b) \leq |j|$ (denote $\hat{d}(R, R^{a(b)})$ when $m^a(b)$ equals to $|j|$), $d(R, R^{a(b)}) = 2 \cdot (\left[\frac{m^a(b)}{2}\right] + 1) - \hat{d}(R, R^{a(b)})$.

$$Dodgson: \quad \begin{aligned} 0 &= S^D(A,\, R\,:\,a) \\ 2 &= S^D(A,\, R\,:\,b) \\ 4 + 10 &= S^D(A,\, R\,:\,c) \end{aligned}$$

$$Abstract\,Dodgson: \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 3 & 11 & 0 \end{bmatrix} \quad \begin{aligned} 0 &= S^{AD}(A,\, R\,:\,a) \\ 2 &= S^{AD}(A,\, R\,:\,b) \\ 16 &= S^{AD}(A,\, R\,:\,c) \end{aligned}$$

The Copeland rule still pick a, while the Simpson rule changes its choice into b, and (Abstract) Dodgson's method selects a; the Simpson rule is not Condorcet-consistent anymore.

Numerical example for the Inverse Simpson rule

This example is for showing that the Inverse Simpson rule violates Condorcet-consistency in the weak domain of preferences.

Number of voters	6	3	4	4
	a	c	a, b	b
Rank of alternatives	b	a, b	$\vdots$	c
	c	$\vdots$	c	a

The the Dodgson's matrix changes to

$$\begin{bmatrix} 0 & 6 & 10 \\ 4 & 0 & 14 \\ 7 & 3 & 0 \end{bmatrix}.$$

Note that a is still the Condorcet winner. Scores from each voting rule are as follows.

$$Copeland: \begin{bmatrix} 0 & +1 & +1 \\ -1 & 0 & +1 \\ -1 & -1 & 0 \end{bmatrix} \text{(No change from the above example)}$$

$$Inverse\,Simpson: \begin{bmatrix} 0 & 6 & (10) \\ (4) & 0 & 14 \\ 7 & (3) & 0 \end{bmatrix}$$

Each number in the round brackets indicates the Inverse Simpson score of the alternative of each column. $S^{INVSIM}(R : a) = 4$, $S^{INVSIM}(R : b) = 3$, and $S^{INVSIM}(R : c) = 10$; b is the Inverse Simpson winner in this case.

$$
\begin{aligned}
& 0 = S^D(A,\, R\, :\, a) \\
\textit{Dodgson}: \quad & 3 = S^D(A,\, R\, :\, b) \\
& 6 + 10 = S^D(A,\, R\, :\, c)
\end{aligned}
$$

$$
\textit{Abstract Dodgson}: \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 11 & 0 \end{bmatrix} \begin{array}{l} 0 = S^{AD}(A,\, R\, :\, a) \\ 2 = S^{AD}(A,\, R\, :\, b) \\ 16 = S^{AD}(A,\, R\, :\, c) \end{array} \quad \text{(No change from the above example)}
$$

To sum up, only the Inverse Simpson rule loses the Condorcet-consistency; the Inverse Simpson rule elects b.

From these two numerical examples, we confirm that the Simpson rule and the Inverse Simpson rule lose their Condorcet-consistency in weak preference orderings.

3.2 Extended Simpson rules

3.2.1 Tie-counting weight and Extended (Inverse) Simpson rules

The essential reason why both the Simpson rule and the Inverse Simpson rule lose their Condorcet-consistency is that they use scoring systems depending on the size of votes in common. Contrary to the scoring systems of others which depend on margins, Inverse Simpson score and Simpson score are easily affected by the size of indifferent voters. Hence, giving a proper consideration to the indifferences is the very likely way to recover the Condorcet-consistency of the Simpson rule and the Inverse Simpson rule. Based on this understanding, Extended Simpson rules and Extended Inverse Simpson rules are introduced in this subsection.

To consider the indifferent voters, I need to define the tie-counting weight. In the domain of preferences, $n(P)_{ab}$ $\forall a, b \in A$ still indicates the number of voters who prefere a to b, and the number of voters who are indifferent between a and b is denoted by $n(R)_{a\sim b}$ $\forall a, b \in A$. Then, the tie-counting weight can be defined as follows.

Definition. Tie-counting weight

Given $R \in \mathscr{D}$, $A \in \mathcal{A}$, the tie-counting weight is $\phi \in [0, \infty)$, resulting in the valid vote for an alternative a against b as $n(R)_{ab} + \phi \cdot n(R)_{a \sim b}$ $\forall a, b \in A$.

The weight ϕ contains neutrality and anonymity when making tie-counting.[9] All indifferent alternatives are equally treated when getting a valid vote. When the value of ϕ is determined, $\phi \cdot n(R)_{a \sim b}$ is added to both a's valid vote and b's for any a, $b \in A$. This means giving a neutral tie-counting weight, as each indifference is counted only ϕ for any alternative. Second, for anonymity, tie-counting weight ϕ is multiplied by the number of indifferent voters, not by any specific voter's indifference.

Now, using the tie-counting weight, I introduce Extended Simpson rule and Extended Inverse Simpson rule.

Definition. Extended Simpson rule

Given $R \in \mathscr{D}$, $A \in \mathcal{A}$, and $\phi \in [0, \infty)$, an Extended Simpson score for k $S^{ES}(A,\, R,\, \phi\, :\, k) = \min_{l \neq k}(n(R)_{kl} + n(R)_{k \sim l} \cdot \phi)$. The choice set of the Extended Simpson rule is $f^{ES}(A,\, R,\, \phi) = \{a \in A\, :\, S^{ES}(A,\, R,\, \phi\, :\, a) = \max_{k \in A} S^{ES}(A,\, R,\, \phi\, :\, k)\}$.

Definition. Extended Inverse Simpson rule

Given $R \in \mathscr{D}$, $A \in \mathcal{A}$, and $\phi \in [0, \infty)$, an Extended Inverse Simpson score for k $S^{EIS}(A,\, R,\, \phi\, :\, k) = \max_{l \neq k}(n(R)_{lk} + n(R)_{k \sim l} \cdot \phi)$. The choice set of the Extended Inverse Simpson rule is $f^{EIS}(A,\, R,\, \phi) = \{a \in A\, :\, S^{EIS}(A,\, R,\, \phi\, :\, a) = \min_{k \in A} S^{EIS}(A,\, R,\, \phi\, :\, k)\}$.

By determining the tie-counting way, we can select one Extended Simpson rule (Extended Inverse Simpson rule respectively).

[9] Here, the neutrality means that every alternatives are equally treated without reference to their names, and similarly anonymity indicates that no voter is specially treated by its name.

3.2.2 Numerical example: Condorcet-consistent Extended (Inverse) Simpson rule

Now, we will find the proper tie-counting weights to make an Extended Simpson rule (Extended Inverse Simpson rule, respectively) Condorcet-consistent.

Numerical exmaple for the Simpson rule (continued)

When applying a tie-counting function to the Dodgson's matrix,

$$\begin{bmatrix} \phi \cdot 17 & 9 & 6 + 8 \cdot \phi \\ 8 & \phi \cdot 17 & 14 \\ 3 + 8 \cdot \phi & 3 & \phi \cdot 17 \end{bmatrix}.$$

$6 + 8 \cdot \phi > 8$ is the condition that the Extended Simpson score of a is bigger than others. Thus, $\phi \in (\frac{1}{4}, \infty)$ is the range of the proper tie-couting weights. Chosing $\phi \in (\frac{1}{4}, \infty)$ from Extended Simpson rules is the Condorcet-consistent extension of Simpson rule in this example.

Numerical exmaple for the Inverse Simpson rule (continued)

In the same way, when applying a tie-counting function to the Dodgson's matrix,

$$\begin{bmatrix} \phi \cdot 17 & 6 + 7 \cdot \phi & 10 \\ 4 + 7 \cdot \phi & \phi \cdot 17 & 14 \\ 7 & 3 & \phi \cdot 17 \end{bmatrix}.$$

$S^{EIV}(\cdot : b) = (6 + 7 \cdot \phi) > \max\{4 + 7 \cdot \phi, 7\} = S^{EIV}(\cdot : a)$ and $S^{EIV}(\cdot : c) = 14 > \max\{4 + 7 \cdot \phi, 7\} = S^{EIV}(\cdot : a)$ are the conditions for making the Condorcet-consisteny of the Extended Inverse Simpson rule. Then, $6 + 7 \cdot \phi > 7$ and $4 + 7 \cdot \phi < 14$ are the condition for the proper tie-counting weight. Therefore, the Condorcet-consistent extension of Inverse Simpson rule is giving $\phi \in (\frac{1}{7}, \frac{10}{7})$ in this case.

4 Main Results

The Copeland rule and the (Abstract) Dodgson's method depend on the margin rather than the number of votes itself, so any $\phi \in [0, \infty)$ does not affect their outcomes. In other words, these rules are Condorcet-consistent on the domain of weak preferences.

Fact 1. *Copeland rule is always Condorcet-consistent on the domain of weak preference orderings, whatever tie-counting weight we give.*

Proof. Let $\phi \in [0, \infty)$. The Copeland score is determined by $M(A,\, R\, :\, a, x)\ \forall a \in A,\ \forall x \in A \backslash \{a\}$ which only reflects the strict preference and ignores the number of indifferent voters. Thus, ϕ can never affect the decision on the Copeland rule. Any tie-counting weight is allowable for the Condorcet-consistency of Copeland rule. □

Fact 2. *The Simpson rule and Inverse Simpson rule cannot be Condorcet-consistent on the domain of weak preference orderings.*

Proof. The numerical examples in the previous section prove this fact. □

Now, we find the Condorcet-consistent extension of Simpson rule and Extended Simpson rule respectively.

Theorem 1. *On the domain of weak preferences, an Extended Simpson rule is Condorcet-consistent if and only if the tie-counting weight,ϕ is $\frac{1}{2}$.*

Proof. The case of $\phi = \frac{1}{2}$. Let a be the Condorcet winner of the game. Assume that an alternative b is $\arg\max_{i \in A \backslash \{a\}} \left[\min_{j \in A \backslash \{i\}} (n(R)_{ij} + \frac{1}{2} n(R)_{i \sim j})\right]$. As $\min_{j \in A \backslash \{b\}} (n(R)_{bj} + \frac{1}{2} n(R)_{b \sim j}) \leq n(R)_{ba} + \frac{1}{2} n(R)_{b \sim j}$, and $n(R)_{ba} + \frac{1}{2} n(R)_{b \sim j} < \left[\frac{|N|}{2}\right] + 1 \leq n(R)_{ak} + \frac{1}{2} n(R)_{ak}$ for $\forall k \in A \backslash \{a\}$, $\min_{j \in A \backslash \{b\}} (n(R)_{bj} + \frac{1}{2} n(R)_{b \sim j}) < \min_{i \in A \backslash \{a\}} (n(R)_{ai} + \frac{1}{2} n(R)_{a \sim i})$. Hence, the Condorcet winner a is always chosen by Simpson rule with $\phi = \frac{1}{2}$.

Given $\phi \neq \frac{1}{2}$, I will show that we can always find out the case in which the Simpson rule does not choose the Condorcet winner. Accordingly we can see $\phi = \frac{1}{2}$ is the only tie-counting method we look for. First, $0 \leq \phi < \frac{1}{2}$. Suppose there is

only one tie relation, only allowing the existence of some voters who are indifferent between a and b. Then the lowest possible score of a is $1 + \phi \cdot (|N| - 1)$, so put it as the score of a. Let c be a candidate which has $n(R)_{ac} = \left[\frac{|N|}{2}\right] + 1$ and $n(R)_{ci} = |N| - \left[\frac{|N|}{2}\right]$ for $\forall i \in N \backslash \{a, c\}$(especially it must be when $i = b$). Then the score of c is $|N| - \left[\frac{|N|}{2}\right] - 1$, and the score of a is $1 + \phi \cdot (n - 1)$. When $|N|$ is even, comparing $|N| - \left[\frac{|N|}{2}\right] - 1$ and $1 + \phi \cdot (|N| - 1)$ is equivalent to the relation of $\frac{1}{2} - \frac{1}{|N|}$ and $\frac{|N|-1}{|N|}\phi + \frac{1}{|N|}$. As the number of voters, $|N|$, is increasing, each score of the options respectively converges to $\frac{1}{2}$ and ϕ. When $|N|$ is odd, the comparison is $|N| - \frac{|N|}{2} + \frac{1}{2}$ and $1 + \phi \cdot (|N| - 1)$. Similar to even $|N|$ case, the comparison is rewritten as $\frac{1}{2} - \frac{1}{2|N|}$ and $\frac{|N|-1}{|N|}\phi + \frac{1}{|N|}$. When $|N|$ is converging to infinite, the scores converges to $\frac{1}{2}$ and ϕ respectively. By the assumption of $0 \leq \phi < \frac{1}{2}$, the Simpson score of a is less than that of c. Therefore we can always find a situation which Simpson rule does not choose Condorcet winner in this kind of tie-counting weights. The case where $\phi > \frac{1}{2}$. Assume that a has no tie rank with b for every voters' preferences. Then, $n(R)_{ab} \geq \left[\frac{|N|}{2}\right] + 1$, so put b that makes the minimum score of a so that the score of a is$\left[\frac{|N|}{2}\right] + 1$. Put c such that $n(R)_{ac} = 1$, $n(R)_{a\sim c} = |N| - 1$, and $n(R)_{ci} = |N|$ for $\forall i \in N \backslash \{a, c\}$. Then, the score of c is $\phi \cdot (|N| - 1)$, and clearly it is the $\max_{i \in N \backslash \{a\}} \left[\min_{j \in N \backslash \{i\}} (n(R)_{ij} + n(R)x_{i\sim j})\right]$. When $|N|$ is even, we should compare $\phi \cdot (|N| - 1)$ and $\frac{|N|}{2} + 1$. This is equivalent to comparing $\phi \cdot (1 - \frac{1}{|N|})$ and $\frac{1}{2} + \frac{1}{|N|}$. When the number of voters is converges to the infinite, ϕ and $\frac{1}{2}$ remain. Thus, as $\phi > \frac{1}{2}$, c is the choice of Simpson rule. When $|N|$ is odd, the compared formulas are $\phi \cdot (|N| - 1)$ and $\frac{1}{2}(|N| + 1)$. This is equivalent to comparing $\phi \cdot (1 - \frac{1}{|N|})$ and $\frac{1}{2}(1 + \frac{1}{|N|})$. Similarly the limit of the number of voters are ϕ and $\frac{1}{2}$ respectively. Hence, as $\phi > \frac{1}{2}$, c is the choice of Simpson rule again. Therefore, when the number of voter is increasing, it can always happen that the Simpson rule does not choose the Condorcet winner under $\phi \neq \frac{1}{2}$. □

Corollary. *On the domain of weak preferences, an Extended Inverse Simpson rule is Condorcet-consistent if and only if the tie-counting weight,ϕ is $\frac{1}{2}$.*

The proof of the Corollary is almost same as the proof of Theorem 1. The only

difference results from the structure of Inverse Simpson rule's scoring function.

Remark. Even on the domain of weak preferences, when giving the tie-counting weight $\phi = \frac{1}{2}$, social orderings from the Extended Simpson rule and the Extended Inverse Simpson rule coincide. This is because the tie-counting weight makes $n(R)_{ab} + n(R)_{ba} = |N| \ \forall a, b \in A$, which provides the environment equivalent to the strict preference orderings in the Dodgson's matrix. Their equality of ordering, on the domain of strict preference orderings, is already shown in the Proposition 1.

Fact 3. *The Dodgson's method and Abstract Dodgson's method are always Condorcet-consistent on the domain of weak preference orderings, whatever tie-counting weight we give.*

Proof. For any $\phi \in [0, 1]$, ϕ does not affect $n(R)_{ab} - n(R)_{ba}$ for all $a, b \in A$, accordingly $m^a(b)$ and $m^b(a)$ either. Thus, although the number of voters who are indifferent between a and b can change $\hat{d}(R, R^{a(b)})$, resulting in the change in the scores, the value of ϕ itself affect the scoring of neither Dodgson's method nor Abstract Dodgson's method. Therefore, we can freely give any tie-counting weight. Furthermore, the Condorcet winner $c^\star$ is the only one which has $d(R, R^{c^\star}) = 0$ regardless of the appearance of indifferent voters, because $d(R, R^{a(c^\star)}) \geq 2 \ \forall a \in A \backslash \{c^\star\}$. This confirms the Condorcet-consistency of the (Abstract) Dodgson's method in the weak preference orderings. □

5 Concluding remarks

We have seen several voting rules and their Condorcet-consistency on the domain of weak preference orderings. Copeland rule, Dodgson's method, and Abstract Dodgson's method maintain Condorcet-consistency in this preference domain. This is natural as all of them are designed to pick their winners based on the closeness from the Condorcet winner. Both Dodgson's method and Abstract Dodgson's method are defined by the Kemeny's distance function which is a measure for a distance between preference profiles (Ratliff 2001). For the Copeland rule, Klamler (2005)

shows that Copeland rule provides an aggregated preference which is the closest to the Condorcet-ranking built by Kemeny metric. That is, the three voting rules explicitly stand on the Condorcet-criterion, which makes them impervious to the number of indifferent voters. Notable thing is that Simpson rule and Inverse Simpson rule lose their Condorcet-consistency in weak preference orderings. By utilizing tie-counting weights, I introduced Extended Simpson rules and Extended Inverse Simpson rules. As shown in Theorem 1 and Corollary, we can find the Condorcet-consistent extension of Simpson rule and Inverse Simpson rule. In every one-to-one elections, giving valid votes that are half number of indifferent voters to both alternatives is the unique Condorcet-consistent extension of both Simpson rule and Inverse Simpson rule. The intuition of my main result is to construct a Dodgson's matrix that is the sum of valid votes in every one-to-one elections is the total number of voters as if it has been shown in the strict preference orderings. This change prevents a distortion to the (Inverse) Simpson rule's scoring system, consequently guaranteeing the Condorcet-consistency of the Extended (Inverse) Simpson rule.

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