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Measuring Inequalities on Social Networks: A Network-based Generalization of the Gini Index

사회 관계망에서의 불평등도 측정: 네트워크에 기반한 지니 지수 일반화

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Abstract

Measuring Inequalities on Social Networks: A Network-based Generalization of the Gini Index

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We construct a novel inequality index that incorporates a social network structure, and name it the *network-based Gini index*, as it generalizes the Gini index. We also provide a basic axiomatization of our index, following the approach of Sen(1954): Any social welfare function satisfying Weighting Equity, Independent Monotonicity, Zero Weight on Top, and Ordinal Information must rank income distributions of a given mean link income in precisely the same way as the negative of the network-based Gini indexes of the respective distributions. We also show that the network-based Gini index satisfies Transfer Monotonicity, Scale Invariance, and Comparability. Finally, we construct the *weighted network-based Gini index* on weighted networks, and verify that the index has a similar axiomatization result and satisfies almost the same properties with the network-based Gini index.

Key Words: Income inequality, Gini index, Social network

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1 Introduction

Why do economists care about inequality? May be they believe there is some intrinsic badness in inequality itself. British philosopher Parfit(1995) calls such a view *Teleological egalitarianism*, and explains their belief by an example what he named *the Divided World*. This world contains two groups of people, each unaware of the other's existence. Consider the next two possible states of the world where the figures are in utility terms:[1]

(1)	Half at 100	Half at 200
(2)	Half at 140	Half at 140

Teleological Egalitarians might believe (1) is worse than (2), as they could argue that, although (1) is in one way better as people are on average better off, those benefits are outweighed by the badness of its inequality. Since inequality is in itself bad, there is no need to explain why the inequality in (1) matters while no one actually perceives it.

However, it might be more natural to think that we have no reason to care about the inequality in the Divided World example as it does no harm on anyone in the world. This may lead us to the *Non-Intrinsic egalitarianism* defined by O'Neill(2008). Non-Intrinsic egalitarians do not believe inequality is bad in itself, but they are concerned with the badness of states of affairs related to inequality. For example, they might believe inequality creates stigmatizing differences in status, leads to unacceptable forms of power and domination, or weakens self-respect of the worst-off. Since Non-Intrinsic egalitarians are concerned with those kinds of badness, they "can also take the view that the degree of interaction between different individuals, peoples, or societies can determine

[1]This example occurs at Parfit(1995), pp.6-8.

the extent to which a distributive inequality between them is objectionable. When there is minimal interaction between two groups, an inequality between those groups will be less significant than an equivalent inequality that exists between two groups that are in intimately close contact."[2]

Regarding the Divided World example, most of economists would take the Non-Intrinsic view instead of the Teleological view. They would not consider it bad if there is inequality between people from totally unrelated groups like people from Europe and America before the Atlantic has been crossed yet. Sen(1997) has also demonstrated a Non-Intrinsic view in his statement that the relevance of our(economists') ideas on economic inequality must be judged by their ability to relate to the economic and political preoccupations of our time. However, the analysis on economic inequality has not successfully incorporated the Non-Intrinsic view so far. There are not many tools that coincide with the Divided World example as we respond to it.

This paper concentrates on developing an inequality measure that incorporates the Non-Intrinsic egalitarianism. As O'Neill(2008) has pointed out, it is important to take into account the degree of interaction between different individuals. We especially focus on the badness coming from the worse off people interacting with the better off people, which we call relative deprivation, as we assume that relative deprivation plays a greater roll than the opposite direction of perception when it comes to bad effects of inequality.

In chapter 2, we employ the graph theory and introduce a network structure to capture the social network of inequality perception. A network consists of nodes and links that connects the nodes. In our model, a node represents each individual in a society and when there is a link between two nodes, the interpretation would be that the two individuals represented by the nodes are

[2]O'Neill(2008), p.138.

aware of each other's existence. The construction of our relative deprivation index, the network-based Gini index, would be a natural network-based generalization of the Gini index. In chapter 3, we provide an axiomatization of the network-based Gini index, following the axiomatization of the Gini index by Sen(1974). In chapter 4, we introduce some other desirable properties that the network-based Gini index satisfies. In chapter 5, we consider a weighted network and construct the weighted network-based Gini index. The weight on each link indicates how closely the two nodes are connected, and hence this model allows us to analyze relative deprivation in a more detailed way. In the last chapter, we will discuss some limitations and possible extensions of our model.

Although the perception of inequality has not received a great attention among economists, Van Praag(1977) is one of the few economists who examined the issue. He questioned how individual inequality perceptions are reflected in several well-known inequality indexes. However, his notion of perception depends only on income levels, and not on individual social networks. The work of Yitzhaki(1979) is more related to ours as he relates the Gini index to the theory of relative deprivation. He suggests to measure the level of relative deprivation in a society by multiplying the Gini index to the mean income. However, his interpretation does not have an individual social network analysis, either. While our axiomatization of the network-based Gini index is analogous to that of the Gini index by Sen(1974), we also owe to Atkinson(1970), Kakwani(1980), Weymark(1981), and Thon(1982).

2 Network-based Gini index

Let $N = \{1, 2, \ldots, n\}, n \geq 2$ be the finite set of *nodes*, or *individuals*. The *adjacency matrix*, e, is an $n \times n$ symmetric nonzero binary matrix with all diagonal entries equal to 0. We say there is a *link* between nodes i and j if and only if $e_{ij} = 1$. The set of nodes N and the adjacency matrix e characterizes the *network*, which is the ordered pair (N, e). The assumptions on e indicate that the network is undirected, unweighted, and there exists at least one link in the network. An *income distribution* of network (N, e) is a vector $y \in \mathbb{Y} = \{y \in \mathbb{R}^n_+ : \exists i, j \in N \text{ s.t. } e_{ij} = 1, y_i + y_j > 0\}$, where y_i represents the income level of node $i \in N$. Let $N_i = \{j : e_{ij} = 1\}$ be the *neighborhood* of node i and $d_i = |N_i|$ be the *degree* of node i. Given y, let $U_i(y) = \{j : e_{ij} = 1, j > i\} \cup \{j : e_{ij} = 1, y_j = y_i, j > i\}$ be the *upper-neighborhood* of node i and $u_i(y) = |U_i(y)|$ be the *upper-degree* of node i. The upper-degree of a node is a key concept of our model, as it mainly determines each node's perception of deprivation: the sense of relative deprivation of an individual becomes greater as her upper-degree increases.

Before the construction of the network-based Gini index, let us review the construction of the usual Gini index. Given an income distribution y, let $m(y) = (y_1 + y_2 + \cdots + y_n)/n$ be the mean income of y. Another expression of $m(y)$ would be

$$m(y) = \frac{\sum_{i\in N} \sum_{j\in N} \frac{y_i+y_j}{2}}{n^2}. \tag{1}$$

The Gini index is often considered as the relative mean income difference, which leads to the following expression:

$$G(y) = \frac{\sum_{i\in N} \sum_{j\in N} |y_i - y_j|}{2m(y)n^2}. \tag{2}$$

Since $m(y) > 0$ by the definition of $\mathbb{Y}$, the above definition of the Gini index is well-defined. There is a concern in the literature that the Gini index has a small sample bias, which means the difficulty of comparing the Gini index of two different samples with different but small sizes as the index ranges between zero and $\frac{n-1}{n}$. The adjusted Gini index $\hat{G}(y)$ is introduced to deal with the problem:

$$\hat{G}(y) = \frac{\sum_{i\in N}\sum_{j\in N}|y_i - y_j|}{2m(y)n(n-1)}. \tag{3}$$

Although the adjusted Gini index does not correspond to the famous Lorenz curve interpretation, it satisfies some desirable properties that the Gini index does not.[3] Our network-based Gini index would be a network-based generalization of the adjusted Gini index.

In order to construct the network-based Gini index as a natural generalization of the Gini index, we incorporate the network structure to the definitions of $m(y)$ and $G(y)$. Given an income distribution y, define the mean link income, $\mu(y)$, as follows:

$$\mu(y) = \frac{\sum_{i\in N}\sum_{j\in N} e_{ij}\frac{y_i+y_j}{2}}{\sum_{i\in N}\sum_{j\in N} e_{ij}}. \tag{4}$$

The definition of mean link income can be interpreted as the average income of the links where each link had an income level of $\frac{y_i+y_j}{2}$. Based on the concept of $\mu(y)$, we define the network-based Gini index, $\Gamma(y)$, as follows:

$$\Gamma(y) = \frac{\sum_{i\in N}\sum_{j\in N} e_{ij}|y_i - y_j|}{2\mu(y)\sum_{i\in N}\sum_{j\in N} e_{ij}}. \tag{5}$$

The network-based Gini index is a natural network-based generalization of the adjusted Gini index, since $\Gamma(y) = \tilde{G}(y)$ for any y when the network is complete. If, contrary to our assumptions, e is a matrix having all entries(even the diagonal ones) equal to 1, then $\Gamma(y) = G(y)$ for any y, i.e., the network-based Gini index becomes a generalization of the Gini index. We will also show in chapter 5 that the definition can be extended to an environment with weighted networks.

[3] See, for example, Allison (1978), Jasso (1979) and Deltas (2003) for more details.

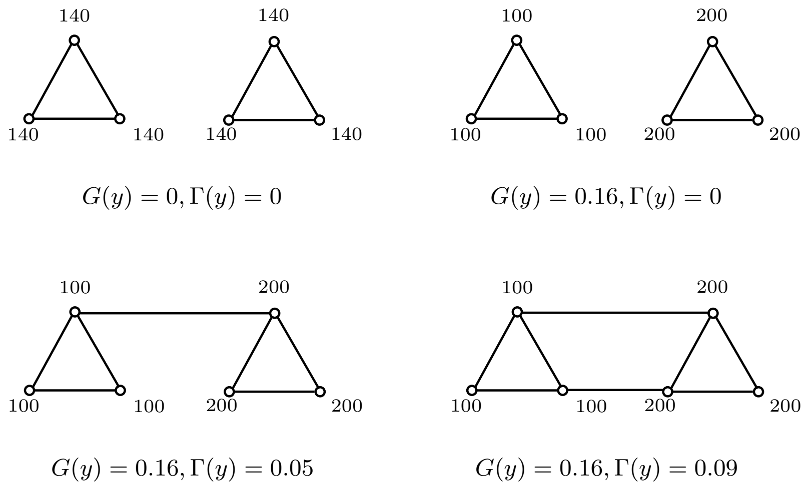

Figure 1: The Divided World

The network-based Gini index, unlike the Gini index, coincides with our response to the Divided World example. In the first two networks in Figure 1, we can observe that the network-based Gini index values the relative deprivation level in both societies to be equally zero. Moreover, the network-based Gini index captures the increase in relative deprivation when the Divided World becomes connected, while the Gini index does not(See the last two networks in Figure 1).

3 Axiomatization of the network-based Gini index

Our axiomatization of the network-based Gini index is based on the axiomatization of the Gini index by Sen(1974). Following his approach, we introduce a social welfare function $W : \mathbb{R}^n_+ \to \mathbb{R}$, assigning a social welfare level to any income distribution vector, and then particularly choose the following weighted sum form:

$$W(y) = \sum_{i \in N} v_i(y) y_i. \tag{6}$$

Since the 'weight' on y_i, given by $v_i(y)$, is a function of the income distribution vector, there is no loss of generality at this stage. Note that any function $W(y)$ can be expressed in the form of (6) in view of the fact that with each income distribution vector y, a different weight vector v can be chosen if necessary.[4]

A set of four axioms constraining the form of $W(.)$ further will now be proposed. The position of being in individual i's shoes in the network given a distribution y is represented by (y, i).

Axiom 1. (Weighting Equity) If everyone prefers (y, i) to (y, j), then $v_j(y) > v_i(y)$.

Axiom 2. (Independent Monotonicity) For all y, everyone regards (y, i) to be at least as good as (y, j) if and only if $u_i(y) \leq u_j(y)$.

Axiom 3. (Zero Weight on Top) If everyone regards (y, i) to be at least as good as any other (y, j), then $v_i(y) = 0$.

Axiom 4. (Ordinal Information) If everyone prefers (y, i) to (y, j) and also (y, m) to (y, n), and there is neither a person k such that someone prefers (y, i) to (y, k) and (y, k) to (y, j), nor a person p such that someone prefers (y, m) to (y, p) and (y, p) to (y, n), then $v_j(y) - v_i(y) = v_n(y) - v_m(y)$.

[4] Sen(1974), p.398

Weighing Equity recommends a higher weight on the income of a person who is regarded as worse off by all. Independent Monotonicity makes each individual decide her preference on the basis of each node's upper-degree alone and it is assumed that she prefers less to more. Zero Weight on Top wants an individual envied by everyone else to have zero weights. Ordinal Information bases weights on ordering information only without any regard for intensities of preference except what is revealed by the number of actual positions that lie in between two alternatives, which is of course, part of the ranking information itself.

These axioms are based on the axioms introduced by Sen(1974): Weighing Equity and Ordinal Information are exactly the same with the original setting, Independent Monotonicity differs only with the use of upper-degree, $u_i(y)$, instead of individual income, y_i, and Zero Weight on Top is the counterpart of Limit Equality. However, these axioms are open to criticism as the original ones. In particular, Independent Monotonicity assumes that every individuals concentrate only on the upper-degree regardless of other information, and Ordinal Information makes the weighting system independent of the magnitude of the gap between two positions in the network.

The set of four axioms characterize the network-based Gini index when the mean link income is fixed.

Theorem 1. A social welfare function $W(.)$ satisfying Weighting Equity, Independent Monotonicity, Zero Weight on Top, and Ordinal Information must rank distributions of a given mean link income in precisely the same way as the negative of the network-based Gini indexes of the respective distributions.

Proof. We will first change the form of the network-based Gini index for computational advantages.

$$\begin{aligned}
\Gamma(y) &= \frac{\sum_{i\in N}\sum_{j\in N} e_{ij}|y_i - y_j|}{2\mu(y)\sum_{i\in N}\sum_{j\in N} e_{ij}} && (7)\\
&= \frac{\sum_{i\in N}\sum_{j\in N} e_{ij}|y_i - y_j|}{\sum_{i\in N}\sum_{j\in N} e_{ij}(y_i + y_j)} && (8)\\
&= 1 - \frac{\sum_{i\in N}\sum_{j\in N} e_{ij}(y_i + y_j - |y_i - y_j|)}{\sum_{i\in N}\sum_{j\in N} e_{ij}(y_i + y_j)} && (9)\\
&= 1 - \frac{\sum_{i\in N}\sum_{j\in N} e_{ij}\cdot 2\min\{y_i, y_j\}}{\sum_{i\in N}\sum_{j\in N} e_{ij}(y_i + y_j)} && (10)\\
&= 1 - \frac{2\sum_{i\in N} u_i(y) y_i}{\sum_{i\in N} d_i y_i} && (11)\\
&= 1 - \frac{1}{m\mu(y)}\sum_{i\in N} u_i(y) y_i. && (12)
\end{aligned}$$

Now, we show that any social welfare function satisfying the four axioms must rank distributions in the same way as the negative of (12) when $\mu(y)$ is fixed. Given any distribution y, define $N_k = \{i \in N : u_i(y) = k\}$ for any $k = 0, 1, \ldots, K$ where $K = \max_{i\in N} u_i(y)$. By Independent Monotonicity, all individuals prefer (y, i) to (y, j) where $i \in N_k$, $j \in N_{k+1}$ for any $k \leq K - 1$. By Weighting Equity and Ordinal Information, for all $k, p \leq K - 1$,

$$v_j(y) - v_i(y) = v_n(y) - v_m(y) = t > 0, \tag{13}$$

for any $i \in N_k, j \in N_{k+1}, m \in N_p, n \in N_{p+1}$. By Zero Weight on Top,

$$v_i(y) = 0 \text{ for any } i \in N_0, \text{ and} \tag{14}$$

$$v_i(y) = kt \text{ for any } i \in N_k. \tag{15}$$

Since $k = u_i(y)$ for any $i \in N_k$,

$$v_i(y) = u_i(y)t. \tag{16}$$

This yields

$$W(y) = \sum_{i \in N} v_i(y) y_i = t \sum_{i \in N} u_i(y) y_i. \tag{17}$$

It is clear from equation (12) that $W(y)$ ranks distributions exactly the opposite order given by the network-based Gini indexes when $\mu(y)$ is fixed. $\square$

4 Other properties

One of the drawbacks of the previous axiomatization is that it requires $\mu(y)$ to be fixed. The meaning of fixing the mean link income may not be as intuitive as fixing the usual mean income. In this chapter, we provide some properties that do not require the mean link income to be fixed. The first one is a network-based generalization of the famous Pigou-Dalton transfer principle. The principle requires an inequality index to capture the fall in inequality level in response to an order-preserving income transfer from a richer person to a poorer person. In our network environment, we focus on each individual's relative poorness among their neighborhoods. Hence, instead of merely concentrating on income levels, we consider the degrees and upper-degrees of individuals when defining the transfer from a 'richer' to a 'poorer'. Given y, let the *proper upper-neighborhood* and the *proper lower-neighborhood* of node i be $\bar{U}_i(y) = \{j : e_{ij} = 1, y_j > y_i\}$ and $\bar{L}_i(y) = \{j : e_{ij} = 1, y_j < y_i\}$, respectively.

Definition 1. An income distribution z is said to be a *network-preserving transfer of y from a to b* if it satisfies the following conditions:

(i) $z_a = y_a - r, z_b = y_b + r$, and $z_k = y_k$ for all $k \in N \backslash \{a, b\}$, where $0 < r < \min\{y_a - y_{\max \bar{L}_a}, y_{\min \bar{U}_b} - y_b\}$,

(ii) $d_a \geq d_b$, and

(iii) $u_a(z) \leq u_b(z)$.

A network-preserving transfer is a network-based order-preserving income transfer from a more influential and relatively less deprived individual to a less influential and more deprived one. It is worth noting that, when the network is complete, the order-preserving transfer related to the Pigou-Dalton

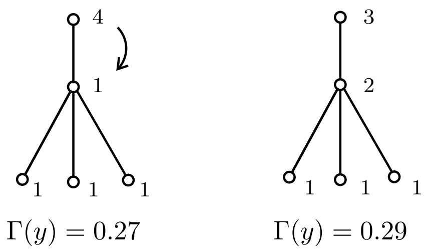

Figure 2: Not a network-preserving transfer

transfer principle becomes a network-preserving transfer, because (i) an order-preserving transfer is also network-preserving, (ii) every individual has the same degree, and (iii) the richer always has a smaller upper-neighborhood than the poorer in a complete network. However, the network-preserving transfer does not coincide with the order-preserving transfer in general. In Figure 2, the transfer from the top node to the middle node is order-preserving in the sense that their relative order does not change, but is not network-preserving because the transfer violates condition (ii). We can also observe an increase in the network-based Gini value after the transfer, even though the income is transfered from a richer to a poorer.

The second property is Scale Invariance, requiring an index to be independent of the choice of an income scale.

The third property requires the range of an index to be the same over income distributions from different population size and network structure. This property allows an index to compare two different income distributions from networks of different sizes. Let $N' = \{1, \ldots, n'\}$ be any finite set of nodes, e' be any corresponding adjacency matrix, and $\mathbb{Y}' = \{y \in \mathbb{R}_+^{n'} : \exists i, j \in N' \text{ s.t. } e'_{ij} = 1, y_i + y_j > 0\}$. For any $y \in \mathbb{Y}'$, define

$$\Gamma'(y) = \frac{\sum_{i \in N'} \sum_{j \in N'} e'_{ij} |y_i - y_j|}{\sum_{i \in N'} \sum_{j \in N'} e'_{ij} (y_i + y_j)}. \tag{18}$$

Proposition 1. The network-based Gini index has the following properties:

(i) (Transfer Monotonicity) If z is a network-preserving transfer of y from a to b, then $\Gamma(z) \leq \Gamma(y)$.

(ii) (Scale Invariance) For any $\lambda > 0$, $\Gamma(\lambda y) = \Gamma(y)$.

(iii) (Comparability) $\min\{\Gamma(y) : y \in \mathbb{Y}\} = \min\{\Gamma'(y) : y \in \mathbb{Y}'\}$ and $\max\{\Gamma(y) : y \in \mathbb{Y}\} = \max\{\Gamma'(y) : y \in \mathbb{Y}'\}$.

Proof. It is easy to show that the network-based Gini index has Scale Invariance.

$$\Gamma(\lambda y) = \frac{\sum_{i\in N}\sum_{j\in N} e_{ij}|\lambda(y_i - y_j)|}{\sum_{i\in N}\sum_{j\in N} e_{ij}\lambda(y_i + y_j)} \tag{19}$$

$$= \frac{\sum_{i\in N}\sum_{j\in N} e_{ij}|y_i - y_j|}{\sum_{i\in N}\sum_{j\in N} e_{ij}(y_i + y_j)} \tag{20}$$

$$= \Gamma(y). \tag{21}$$

To prove Comparability, suppose $y \in \mathbb{Y}$ and $y' \in \mathbb{Y}'$ are vectors with all components equal to 1. Then $\Gamma(y) = 0 = \Gamma'(y')$. Since both indexes have nonnegative values, we have shown that $\min\{\Gamma(y) : y \in \mathbb{Y}\} = \min\{\Gamma'(y) : y \in \mathbb{Y}'\}$. Now suppose that $y \in \mathbb{Y}$ and $y' \in \mathbb{Y}'$ are vectors with all components equal to zero except for one component whose node is not isolated. Then $\Gamma(y) = 1 = \Gamma'(y')$. It is obvious that both indexes cannot exceed 1.

Next, we verify that the network-based Gini index has Transfer Monotonicity. Let z be a network-preserving transfer of y from a to b. We first show that $\mu(z) \leq \mu(y)$. Let $\tilde{E} = \{(i,j) \in N \times N : e_{ij} = 1, y_i < y_j\} \cup \{(i,j) \in N \times N : e_{ij} = 1, y_i = y_j, i < j\}$, $m = |\tilde{E}|$, and $E = \{1, 2, \ldots, m\}$. Fix a bijective function $\varphi : E \to \tilde{E}$. Note that if $\varphi(k) = (i,j)$, then $\varphi_1(k) = i$ and $\varphi_2(k) = j$. For any $i \in N$, let $E_i = \{k \in E : \varphi_1(k) = i\}$ and $E^i = \{k \in E : \varphi_2(k) = i\}$. E_i and E^i are the sets of links that node i is the relatively poorer and the relatively richer, respectively. Note that $E_i \cap E^i = \varnothing$ for any $i \in N$, and that $E^i \cap E^j = \varnothing$ and $E_i \cap E_j = \varnothing$ for any $i \neq j$. Hence, the sets $E_a \cap E^b, E_a \backslash E^b, E^a \cap E_b, E^a \backslash E_b, E_b \backslash E^a, E^b \backslash E_a$, and $E \backslash (E_a \cup E^a \cup E_b \cup E^b)$ form a partition of E. When $k \in E_a \cap E^b$, for example, $\varphi_1(k) = a$ and $\varphi_2(k) = b$. Note also that $|E_i \cup E^i| = d_i$. Now we are prepared to verify our first result.

$$
\begin{aligned}
m\mu(z) &= \sum_{k \in E} \left(z_{\varphi_2(k)} + z_{\varphi_1(k)} \right) && (22)\\
&= \sum_{k \in E_a \cap E^b} (z_b + z_a) + \sum_{k \in E_a \backslash E^b} (z_{\varphi_2(k)} + z_a) + \sum_{k \in E^a \cap E_b} (z_a + z_b) \\
&\quad + \sum_{k \in E^a \backslash E_b} \left(z_a + z_{\varphi_1(k)} \right) + \sum_{k \in E_b \backslash E^a} \left(z_{\varphi_2(k)} + z_b \right) + \sum_{k \in E^b \backslash E_a} \left(z_b + z_{\varphi_1(k)} \right) \\
&\quad + \sum_{k \in E \backslash (E_a \cup E^a \cup E_b \cup E^b)} \left(z_{\varphi_2(k)} + z_{\varphi_1(k)} \right) && (23)\\
&= \sum_{k \in E_a \cap E^b} (y_b + r + y_a - r) + \sum_{k \in E_a \backslash E^b} (y_{\varphi_2(k)} + y_a - r) \\
&\quad + \sum_{k \in E^a \cap E_b} (y_a - r + y_b + r) + \sum_{k \in E^a \backslash E_b} \left(y_a - r + y_{\varphi_1(k)} \right) \\
&\quad + \sum_{k \in E_b \backslash E^a} \left(y_{\varphi_2(k)} + y_b + r \right) + \sum_{k \in E^b \backslash E_a} \left(y_b + r + y_{\varphi_1(k)} \right) \\
&\quad + \sum_{k \in E \backslash (E_a \cup E^a \cup E_b \cup E^b)} \left(y_{\varphi_2(k)} + y_{\varphi_1(k)} \right) && (24)\\
&= \sum_{k \in E_a \cap E^b} (y_b + y_a) + \sum_{k \in E_a \backslash E^b} (y_{\varphi_2(k)} + y_a) + \sum_{k \in E^a \cap E_b} (y_a + y_b) \\
&\quad + \sum_{k \in E^a \backslash E_b} \left(y_a + y_{\varphi_1(k)} \right) + \sum_{k \in E_b \backslash E^a} \left(y_{\varphi_2(k)} + y_b \right) + \sum_{k \in E^b \backslash E_a} \left(y_b + y_{\varphi_1(k)} \right) \\
&\quad + \sum_{k \in E \backslash (E_a \cup E^a \cup E_b \cup E^b)} \left(y_{\varphi_2(k)} + y_{\varphi_1(k)} \right) + r\,(d_b - d_a) && (25)\\
&= \sum_{k \in E} \left(y_{\varphi_2(k)} + y_{\varphi_1(k)} \right) + r\,(d_b - d_a) && (26)\\
&= m\mu(y) + r\,(d_b - d_a) && (27)\\
&\leq m\mu(y). && (28)
\end{aligned}
$$

Inequality (28) comes from condition (ii).

Next, we show that $\sum_{i\in N} u_i(z)z_i \geq \sum_{i\in N} u_i(y)y_i$. Let $\hat{I}_i(y) = \{j \in N : j > i, y_j = y_i\}$, $\tilde{I}_i(y) = \{j \in N : j < i, y_j = y_i\}$, $\hat{\imath}_i(y) = |\hat{I}_i(y)|$, and $\tilde{\imath}_i(y) = |\tilde{I}_i(y)|$. Note that $\hat{I}_i(y) \subseteq U_i(y)$. Since $z_a = y_a - r$, $z_b = y_a + r$, and $0 < r < \min\{y_a - y_{\max \hat{L}_a}, y_{\min \hat{U}_b} - y_b\}$, we know that $U_a(z) = U_a(y) \cup \tilde{I}_a(y)$, and $U_b(z) = U_b(y) \backslash \hat{I}_b(y)$. Hence,

$$\begin{aligned}
\sum_{i\in N} u_i(z)z_i &= u_a(z)z_a + u_b(z)z_b + \sum_{i\in\tilde{I}_a(y)} u_i(z)z_i + \sum_{i\in\hat{I}_b(y)} u_i(z)z_i \\
&\quad + \sum_{i\in N\backslash(\{a,b\}\cup\tilde{I}_a(y)\cup\hat{I}_b(y))} u_i(z)z_i \qquad (29)\\
&= (u_a(y) + \tilde{\imath}_a(y))(y_a - r) + (u_b(y) - \hat{\imath}_b(y))(y_b + r) \\
&\quad + \sum_{i\in\tilde{I}_a(y)} (u_i(y) - 1)y_i + \sum_{i\in\hat{I}_b(y)} (u_i(y) + 1)y_i \\
&\quad + \sum_{i\in N\backslash(\{a,b\}\cup\tilde{I}_a(y)\cup\hat{I}_b(y))} u_i(y)y_i \qquad (30)\\
&= (u_a(y) + \tilde{\imath}_a(y))\, y_a - (u_a(y) + \tilde{\imath}_a(y))\, r + (u_b(y) - \hat{\imath}_b(y))\, y_b \\
&\quad + (u_b(y) - \hat{\imath}_b(y))\, r + \sum_{i\in\tilde{I}_a(y)} u_i(y)y_a - \tilde{\imath}_a(y)y_a \\
&\quad + \sum_{i\in\hat{I}_b(y)} u_i(y)y_b + \hat{\imath}_b(y)y_b + \sum_{i\in N\backslash(\{a,b\}\cup\tilde{I}_a(y)\cup\hat{I}_b(y))} u_i(y)y_i (31)\\
&= \sum_{i\in N} u_i(y)y_i + r\left((u_b(y) - \hat{\imath}_b(y)) - (u_a(y) + \tilde{\imath}_a(y))\right) \qquad (32)\\
&= \sum_{i\in N} u_i(y)y_i + r(u_b(z) - u_b(z)) \qquad (33)\\
&\geq \sum_{i\in N} u_i(y)y_i \qquad (34)
\end{aligned}$$

Inequality (34) comes from condition (iii).

In conclusion,

$$\Gamma(z) = 1 - \frac{1}{m\mu(z)} \sum_{i\in N} u_i(z)z_i \leq 1 - \frac{1}{m\mu(y)} \sum_{i\in N} u_i(y)y_i = \Gamma(y). \qquad (35)$$

□

5 Weighted network-based Gini index

Until now, we have assumed that the links of the network does not have any weights. However, there are plenty of situations where weighted networks are of interest. For example, a link in an international trade network may have a weight equal to the trade volume between the two adjacent countries. In this chapter, we incorporate such weights into the definition of the network-based Gini index and axiomatize our new index in an analogous way to the original index.

Let w be the $n \times n$ *weight matrix* with nonnegative entries. There is a link between nodes i and j if and only if $w_{ij} > 0$ and its value represents the weight on the link. We also define the weighted mean link income, $\mu^w(y)$, as follows:

$$\mu^w(y) = \frac{\sum_{i \in N} \sum_{j \in N} w_{ij} \frac{y_i + y_j}{2}}{\sum_{i \in N} \sum_{j \in N} e_{ij}}. \tag{36}$$

The weighted network-based Gini index, $\Gamma^w(y)$ is defined as

$$\Gamma^w(y) = \frac{\sum_{i \in N} \sum_{j \in N} w_{ij} |y_i - y_j|}{2\mu^w(y) \sum_{i \in N} \sum_{j \in N} e_{ij}}. \tag{37}$$

In the non-weighted environment, the upper-degree of each node was the main source of relative deprivation. We define the counterpart of the upper-degree, the *weighted upper-degree* of node i as $u_i^w(y) = \sum_{j \in U_i(y)} w_{ij}$. The weighted network-based Gini index can be easily axiomatized using the axioms Weighting Equity, Zero Weight on Top, Ordinal Information, and a weighted version of Independent Monotonicity.

Axiom 5. (Independent Monotonicity with Weights) For all y, everyone regards (y, i) to be at least as good as (y, j) if and only if $u_i^w(y) \leq u_j^w(y)$.

Independent Monotonicity with Weights requires individuals to not only count the size of their upper-neighborhoods but also consider the weights of each links.Hence, by carefully choosing the weight matrix, we could extend our analysis in various directions. For example, a weight might reflect the income gap between two nodes so that we could incorporate cardinal information to the model. Instead, one might use the concept of harmonic distance of Marchiori and Latora(2000) as a weight, since it appeals to many network models. However, we leave a more careful analysis on weights for future research.

Theorem 2. A social welfare function $W(.)$ satisfying Weighting Equity, Independent Monotonicity with Weights, Zero Weight on Top, and Ordinal Information must rank distributions of a given weighted mean link income in precisely the same way as the negative of the weighted network-based Gini indexes of the respective distributions.

Proof. Let $\tilde{E} = \{(i,j) : i < j, e_{ij} = e_{ji} = 1\}$, $m = |\tilde{E}|$, and $E = \{1, 2, \ldots, m\}$. Fix a bijective function $\varphi : E \to \tilde{E}$, and let $w(k) = w_{\varphi_2(k)\varphi_1(k)}$ for any $k \in E$. Then,

$$\begin{aligned}
\Gamma^w(y) &= \frac{\sum_{i\in N}\sum_{j\in N} w_{ij}|y_i - y_j|}{2\mu^w(y)\sum_{i\in N}\sum_{j\in N} e_{ij}} && (38)\\
&= \frac{\sum_{i\in N}\sum_{j\in N} w_{ij}|y_i - y_j|}{\sum_{i\in N}\sum_{j\in N} w_{ij}(y_i + y_j)} && (39)\\
&= 1 - \frac{\sum_{i\in N}\sum_{j\in N} w_{ij}(y_i + y_j - |y_i - y_j|)}{\sum_{i\in N}\sum_{j\in N} w_{ij}(y_i + y_j)} && (40)\\
&= 1 - \frac{\sum_{i\in N}\sum_{j\in N} w_{ij}\cdot 2\min\{y_i, y_j\}}{\sum_{i\in N}\sum_{j\in N} w_{ij}(y_i + y_j)} && (41)\\
&= 1 - \frac{1}{m\mu^w(y)}\sum_{i\in N} u_i^w(y)y_i. && (42)
\end{aligned}$$

We omit the rest of the proof as it replicates the proof of Theorem 1. □

The weighted network-based Gini index also satisfies the weighted versions of the desirable properties of the network-based Gini index.

Definition 2. An income distribution z is said to be a *weighted network-preserving transfer of y from a to b* if it satisfies the following conditions:

(i) $z_a = y_a - r, z_b = y_b + r$, and $z_k = y_k$ for all $k \in N\backslash\{a, b\}$, where $0 < r < \min\{y_a - y_{\max \bar{L}_a}, y_{\min \bar{U}_b} - y_b\}$,

(ii) $d_a \geq d_b$, and

(iii) $u_a^w(z) \leq u_b^w(z)$.

Also let w' be a corresponding weight matrix to e', and define

$$\Gamma^{w'}(y) = \frac{\sum_{i \in N'} \sum_{j \in N'} w'_{ij} |y_i - y_j|}{\sum_{i \in N'} \sum_{j \in N'} w'_{ij} (y_i + y_j)}. \tag{43}$$

Proposition 2. The weighted network-based Gini index has the following properties:

(i) (Transfer Monotonicity*) If z is a weighted network-preserving transfer of y from a to b, then $\Gamma^w(z) \leq \Gamma^w(y)$.

(ii) (Scale Invariance*) For any $\lambda > 0$, $\Gamma^w(\lambda y) = \Gamma^w(y)$.

(iii) (Comparability*) $\min\{\Gamma^w(y) : y \in \mathbb{Y}\} = \min\{\Gamma^{w'}(y) : y \in \mathbb{Y}'\}$ and $\max\{\Gamma^w(y) : y \in \mathbb{Y}\} = \max\{\Gamma^{w'}(y) : y \in \mathbb{Y}'\}$.

Proof. It is obvious that the weighted network-based Gini index has Scale Invariance*.

To prove Comparability*, suppose $y \in \mathbb{Y}$ and $y' \in \mathbb{Y}'$ are vectors with all components equal to 1. Then $\Gamma^w(y) = 0 = \Gamma^{w'}(y')$. Obviously, both indexes have nonnegative values. Now suppose that $y \in \mathbb{Y}$ and $y' \in \mathbb{Y}'$ are vectors with all components equal to zero except for one component whose node is not isolated. Then $\Gamma^w(y) = 1 = \Gamma^{w'}(y')$. It is also obvious that both indexes cannot exceed 1.

Next, we verify that the network-based Gini index has Transfer Monotonicity. Let z be a weighted network-preserving transfer of y from a to b. From the network-preserving order case, we know that $\mu(z) \leq \mu(y)$, since $d_a \geq d_b$.

It remains to show that $\sum_{i\in N} u_i^w(z)z_i \geq \sum_{i\in N} u_i^w(y)y_i$. Let $\hat{I}_i(y) = \{j \in N : j > i, y_j = y_i\}$, $\tilde{I}_i(y) = \{j \in N : j < i, y_j = y_i\}$, $\hat{\imath}_i(y) = |\hat{I}_i(y)|$, and $\tilde{\imath}_i(y) = |\tilde{I}_i(y)|$. Note that $\hat{I}_i(y) \subseteq U_i(y)$. Since $z_a = y_a - r$, $z_b = y_a + r$, and $0 < r < \min\{y_a - y_{\max \hat{L}_a}, y_{\min \hat{U}_b} - y_b\}$, we know that $U_a(z) = U_a(y) \cup \tilde{I}_a(y)$, and $U_b(z) = U_b(y) \backslash \hat{I}_b(y)$. Hence, the same proof of showing $\sum_{i\in N} u_i(z)z_i \geq \sum_{i\in N} u_i(y)y_i$ in the network-preserving order case can be applied to this weighted case.

In conclusion,

$$\Gamma^w(z) = 1 - \frac{1}{m\mu(z)} \sum_{i\in N} u_i^w(z)z_i \leq 1 - \frac{1}{m\mu(y)} \sum_{i\in N} u_i^w(y)y_i = \Gamma^w(y). \quad (44)$$

□

6 Concluding remarks

We have constructed an index coinciding with the Non-Intrinsic view on inequality, and examined its characteristics by providing an axiomatization and three other properties. Our analysis, however, is limited as the axiomatization assumes a fixed mean link income. One extension of our work would be giving an alternative characterization using more general axioms.

We can also think of more generalized network structures. In chapters 2 to 4, we have only considered a simple network without weights, and in chapter 5, we have introduced a weighted network. However, the definition of the network-based Gini index can be easily extended to any kind of networks—a directed network or a network with loops and multiple edges. Therefore, examining the properties of the index with more general network structures might be interesting.

Along with theoretical works, it is also important to apply the index to real world examples. Some economic works have focused on measuring global inequality.[5] It is easy to imagine that the difference in perception of inequality would be more important when dealing with a larger society.

[5] See, for example, Bouguignon and Morrison (2002), and Milanovic (2011).

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국문 초록

사회 관계망에서의 불평등도 측정: 네트워크에 기반한 지니 지수 일반화

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이 논문에서 우리는 경제적 불평등을 측정하는 데에 널리 사용되는 지니 지수(Gini index)를 네트워크 구조에 맞추어 일반화한 *네트워크 기반 지니 지수 (Network-based Gini index)*를 개발하여, 사회 관계망(Social Network)이 주어진 상황에서 노드(node)사이의 관계를 고려하여 불평등도를 측정하는 새로운 방법을 제시한다. 또한 네트워크 기반 지니 지수의 성질을 연구하기 위하여 센(Sen, 1974)의 방법론을 도입하여 지수를 공리화한다. 구체적으로 우리는 불평등 측정, 독립 단조성, 최상층에 무가중치, 서수적 정보의 네 가지 공리를 만족하는 모든 사회 후생 함수는 동일한 평균 링크 소득(mean link income)을 가진 소득 분포들을 네트워크 기반 지니 지수의 음수값과 정확히 동일한 순서로 평가함을 보인다. 또한 네트워크 기반 지니 지수가 이전 단조성, 단위 불변성, 비교가능성을 만족함을 보인다. 마지막으로 가중 네트워크(weighted network)를 도입하여 *가중 네트워크 기반 지니 지수(weighted network-based Gini index)*를 정의하고, 이 지수 또한 유사한 공리화가 가능하며 네트워크 기반 지니 지수와 거의 동일한 속성을 만족함을 확인한다.

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