이학박사 학위논문

Higher rank vector bundles in Fukaya category

(푸카야 범주의 고차 벡터 다발)

2017 년 8 월

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Higher rank vector bundles in Fukaya category

A dissertation
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Abstract

In this thesis, we study higher rank vector bundles in Fukaya category and its relation with twisted complex. We enlarge the class of objects of Fukaya category so that they contain not only flat line bundles, but also any finite rank flat vector bundles. For this purpose we use the de Rham version of Fukaya category and verify that it has some nice features, which other versions do not share.

We prove that every flat vector bundle on a Lagrangian submanifold can be realized as a twisted complex of flat line bundles when the fundamental group of the Lagrangian submanifold is abelian. Futhermore, we also prove that every flat vector bundle whose holonomy reprsentation is triangularizable is quasi-isomorphic to a certain twisted complex of flat line bundles.

Key words: Lagrangian Floer theory, Fukaya category, flat vector bundles, twisted complex

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Chapter 1

Introduction

To understand the intersection of Lagrangian submanifolds in a given symplectic manifold is one of the main concerns in symplectic geometry. In [13], Floer introduced the Lagrangian Floer cohomology, which measures the minimal intersection number of two Lagrangian submanifolds. It has served as a main tool for understanding the nature of the intersection of Lagrangian submanifolds.

In [14], Fukaya introduced the notion of A_∞-category and associate an A_∞-category to each symplectic manifold, which is said to have Lagrangian submanifolds as objects and is now called *Fukaya category*. Later, it was proposed by Kontsevich that the objects of Fukaya category should be pairs of a Lagrangian submanifold and a local system on it in aspect of homological mirror symmetry. [25]

One example was given by Polishchuk and Zaslow in the case of elliptic curve in [32]. The authors proved that a Fukaya category of two-torus is equivalent to the derived category of coherent sheaves on elliptic curve. In their paper, the objects of the Fukaya category are flat vector bundles on closed Lagrangian submanifolds in two-torus. Continuing the result of Polishchuk and Zaslow, Haug proved that the derived category of Fukaya category of two-torus is split-closed in [22].

Recently, Konstantinov used higher rank local systems to show that Chiang Lagrangian of $\mathbb{C}P^3$ is non-displaceable from $\mathbb{R}P^3$ via a Hamiltonian isotopy.[26]

In this thesis, we use the notion of flat vector bundles instead of that of local systems. This is mainly because we use de Rham complexes when defining endomorphism spaces and it turns out to be more natural to use the notion of flat connection.

The de Rham version of Fukaya category was developed by Fukaya-Oh-Ohta-Ono [19, 20]. But, only trivial flat line bundles are allowed as objects in their works. We develop a de Rham version of Fukaya category that includes flat vector bundles of any finite rank as objects. We follow the idea of Fukaya-Oh-Ohta-Ono and Fukaya [16], that is, we use the notion of Kuranishi structure and continuous family of multisections to define A_∞-structure.

Then we prove the following theorem.

Theorem 1.1. *Let M be a symplectic manifold. There exists an A_∞-category such that objects are pairs (L, E, ∇) of a Lagrangian submanifold L and a flat vector bundle (E, ∇) on it and that the morphism space between (L_0, E_0, ∇_0) and (L_1, E_1, ∇_1) is given by*

$$\Omega(L, Hom(E_0, E_1)).$$

We also introduce the notion of weakly unobstructedness of higher rank vector bundles, which was also introduced by Konstantinov in [26]. Furthermore, we show that our version of Fukaya category has the strict unitality, which other versions may not enjoy.

On the other hand, the notion of twisted complex of differential graded category was introduced by Bondal and Kapranov [6] and it was generalized by Kontsevich [25] to the case of A_∞-category. It is well-known that twisted complexes of a given A_∞-category form a triangulated A_∞-category.

We prove that a flat vector bundle on a Lagrangian submanifold can be realized as a twisted complex of flat line bundles under a specific condition on holonomy representation of fundamental group, that is, the holonomy representation is lower-triangularizable. Showing that every representation of an finitely generated abelian group is lower-triangularizable, we prove the following theorem.

Theorem 1.2. *Suppose that the fundamental group of a Lagrangian submanifold L is abelian. Every flat vector bundle on L is isomorphic to a twisted complex of flat line bundles in Fukaya category.*

This theorem will be stated in Theorem 5.8 more precisely. This result implies that the triangulated closure of our Fukaya category of flat vector bundles is essentially generated by a full subcategory which consists of flat line bundles. We also prove

Theorem 1.3. *Every flat vector bundle whose holonomy representation is lower-triangular is quasi-isomorphic to a twisted complex of flat line bundles in Fukaya category.*

We will outline the contents of the thesis here.

In Chapter 2, we review the Lagrangian Floer theory and some basic notions of A_∞-category. In Chapter 3, we introduce the integration along the fiber of vector-valued differential forms and check various properties. In Chapter 4, we construct a de Rham version Fukaya category and prove that it enjoys nice properties such as strict unitality. In Chapter 5, we finally prove that some flat vector bundle is realized as a twisted complex of flat line bundles under certain conditions in Fukaya category.

Chapter 2

Preliminaries

2.1 Symplectic Geoemtry

Symlectic geometry is the study of smooth manifolds M equipped with a closed, nondegenerate 2 form ω, called a *symplectic form*. Such a pair (M, ω) is called a *symplectic manifold*. The existence of a symplectic form forces M to be even dimensional.

Symplectic geometry originated from the Hamiltonian mechanics. The standard example in classical mechanics deals with the phase space $T^*\mathbb{R}^n \cong \mathbb{R}^{2n} = \{(q_1, ..., q_n, p_1, ..., p_n) | q_i, p_i \in \mathbb{R}\}$, where the first n-coordinates $(q_1, ..., q_n)$ stand for the position and and the last n-coordinates $(p_1, ..., p_n)$ stand for the momentum. Furthermore, the even dimensional Euclidean space $\mathbb{R}^{2n}$ is equipped with the standard symplectic form $\omega_{std} = \sum_{i=1}^{n} dq_i \wedge dp_i$.

More generally, the cotangent bundle of a given smooth manifold has a symplectic structure because the position and momentum of a system is described by the points of the cotangent bundle.

A remarkable theorem due to Darboux says that every symplectic manifold is locally equivalent to an open set in the Euclidean space $(\mathbb{R}^{2n}, \omega_{std})$.

Theorem 2.1 (Darboux). *Let (M, ω) be a symplectic manifold. For every point $p \in M$, there exist an open neighborhood U of p and a diffeomorphism φ from U onto an open set in $\mathbb{R}^{2n}$ such that $\varphi^*\omega_{std} = \omega$.*

Darboux Theorem implies that there are no interesting local invariants in symplectic geometry.

One of global invariants of symplectic manifolds arises from a specific family of submanifolds in symplectic manifolds, which is called Lagrangian. Indeed, a *Lagrangian submanifold* L of a symplectic manifold (M^{2n}, ω) is an n-dimensional submanifold such that the restriction of ω to L is zero. The half-dimensional Euclidean space

$$\mathbb{R}^n \cong \{(q_1, ..., q_n, 0, ..., 0) \in \mathbb{R}^{2n} | q_i \in \mathbb{R}\}$$

is an example of a Lagrangian submanifold in the Euclidean space $(\mathbb{R}^{2n}, \omega_{std})$.

The Lagrangian Floer theory is an intersection theory of Lagrangian submanifolds. More precisely, given two Lagrangian submanifolds L_0 and L_1 of a symplectic manifold M that intersect transversely, we consider a vector space $CF(L_0, L_1)$ generated by intersection points of two Lagrangian submanifolds and define a differential on it by counting pseudo-holomorphic strips connecting two intersections points. Finally the associated cohomology is called the Lagrangian Floer cohomology between L_0 and L_1. More details will be reviewed in the next section.

One of the most important properties of Lagrangian Floer cohomology is that it is invariant under Hamiltonian isotopy. As a consequence, the study of Lagrangian Floer cohomology has answered the following conjecture in several cases [13, 29, 19].

Conjecture (Arnold-Givental)**.** *Let L be a Lagrangian submanifold of a symplectic manifold (M, ω). Let H be a Hamiltonian function on M and let ϕ_H be the time-1 flow of the corresponding Hamiltonian vector field.*

Suppose L and $\phi_H(L)$ intersect transversely. Then the number of intersection points of L and $\phi_H(L)$ is bounded below by the sum of $\mathbb{Z}_2$ Betti numbers of L, i.e.

$$|L \cap \phi_H(L)| \geq \sum_{k=0}^{n} b_k(L, \mathbb{Z}_2).$$

Furthermore, it enables us to prove many non-displaceablity of Lagrangian submanifolds, which is one of the most intrinsic questions in symplectic geometry. [8, 11, 12, 26]. Indeed, if the Lagrangian Floer cohomology $HF(L_0, L_1)$ between two Lagrangian submanifolds L_0 and L_1 is shown to be nonzero, then one of those two Lagrangian submanifolds cannot be displaced from the other one via a Hamiltonian isotopy.

2.2 Lagrangian Floer theory

2.2.1 Maslov Index

Let us consider a symplectic vector space $(\mathbb{R}^{2n}, \omega_{std})$. The space of all Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega_{std})$ will be denoted by $\Lambda(\mathbb{R}^{2n})$.

There is a transitive action of $U(n)$ on $\Lambda(\mathbb{R}^{2n})$ with its the stabilizer group isomorphic to $O(n)$. We briefly review the action. Let us identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ by sending $(q_1, ..., q_n, p_1, ..., p_n)$ to $(q_1 + p_1 i, ..., q_n + p_n i)$. Then every element of $U(n)$ naturally maps a subspace of $\mathbb{R}^{2n}$ to a subspace of $\mathbb{R}^{2n}$. In particular, every element of $U(n)$ sends a Lagrangian subspace to a Lagrangian subspace because the standard symplectic form is invariant under the action of $U(n)$.

Every Lagrangian subspace of $\mathbb{R}^{2n}$ is obtained as $A \cdot \mathbb{R}^n$ for some $A \in U(n)$ where $\mathbb{R}^n$ is the Lagrangian subspace introduced in the previous section. Furthermore, the stabilizer of this action is the subgroup $O(n)$ of $U(n)$. We summarize this result as follows.

Proposition 2.2. *The Lagrangian Grassmannian space $\Lambda(\mathbb{R}^{2n})$ is homeomorphic to $U(n)/O(n)$.* □

Let us denote this homeomorphism by $B : \Lambda(\mathbb{R}^{2n}) \to U(n)/O(n)$.

On the other hand, the fundamental group of $U(n)/O(n)$ is isomorphic to $\mathbb{Z}$. Indeed, the continuous map $U(n)/O(n) \to S^1$ defined by $[A] \mapsto (\det A)^2$ induces an isomorphism on the fundamental groups.

Given a loop $\alpha : S^1 \to \Lambda(\mathbb{R}^{2n}) \cong U(n)/O(n)$, we define the *Maslov index* of α by the degree of the map $(\det)^2 \circ B \circ \alpha$ from S^1 to itself. The Maslov index will be denoted by $\mu(\alpha)$.

Maslov indices in Lagrangian Floer theory

We define two different Maslov-type indices used in the Lagrangian Floer theory.

Let (M, ω) be a compact symplectic manifold of dimension $2n$ and let L be a Lagrangian submanifold of M.

For a continuous map $w : (D^2, \partial D^2) \to (M, L)$ that bounds the Lagrangian submanifold L, we define the Maslov index $\mu(w)$ as follows.

Since D^2 is contractible, there is a symplectic trivialization $\Phi : w^*TM \cong D^2 \times \mathbb{R}^{2n}$. Consider a loop of Lagrangian subspaces $\alpha : S^1 \to \Lambda(\mathbb{R}^{2n})$ defined by

$$\alpha(e^{it}) = \Phi(T_{w(e^{it})}L),$$

where ∂D^2 is identified with S^1.

The Maslov index $\mu(w)$ is defined by the Maslov index $\mu(\alpha)$ of this loop. Because the Maslov index only depends on the homotopy class of the loop α, it is possible to define $\mu : \pi_2(M, L) \to \mathbb{Z}$ by, for each $\beta \in \pi_2(M, L)$,

$$\mu(\beta) = \mu(w)$$

for a map $w : (D^2, \partial D^2) \to (M, L)$ with $[w] = \beta \in \pi_2(M.L)$.

We are ready to define the notion of monotone Lagrangian submanifolds.

Definition 2.3. A Lagrangian submanifold L is called *monotone* if there exists $\lambda > 0$ such that

$$\mu(\beta) = \lambda\omega(\beta) := \lambda \int_{D^2} w^*\omega.$$

for every $\beta \in \pi_2(M, L)$ and $w : (D^2, \partial D^2) \to (M, L)$ with $[w] = \beta$.

Next we consider the case of two Lagrangian submanifolds L_0 and L_1 of M that intersect transversely. Let p, q be two intersections points of L_0 and L_1.

Suppose we have a map $u : [0,1] \times [0,1] \to M$ satisfying

$$u(\tau, 0) \in L_0,\ u(\tau, 1) \in L_1$$
$$u(0,t) = p,\ u(1,t) = q$$

for all $\tau, t \in [0,1]$.

For such a map u, consider a pullback bundle u^*TM. Since $[0,1] \times [0,1]$ is contractible, one can find a symplectic trivialization $\Phi : u^*TM \to ([0,1] \times [0,1]) \times \mathbb{R}^{2n}$ such that Φ is constant on $0 \times [0,1]$ and $1 \times [0,1]$ and such that

$$\Phi(T_pL_1) = i\Phi(T_pL_0),$$
$$\Phi(T_qL_0) = i\Phi(T_qL_1).$$

where i is the complex multiplication on $\mathbb{R}^{2n}$, which is identified with $\mathbb{C}^n$ by a map $(q_1, ..., q_n, p_1, ..., p_n) \mapsto (q_1 + ip_1, ..., q_n + p_n)$.

Let us define a loop of Lagrangian subspace of $\mathbb{R}^{2n}$ along the boundary of $[0,1] \times [0,1]$ as follows:

$$(\tau, 0) \mapsto \Phi(T_{u(\tau,0)}L_0),$$
$$(1, t) \mapsto e^{\frac{i\pi t}{2}}\Phi(T_pL_0),$$
$$(\tau, 1) \mapsto \Phi(T_{u(\tau,1)}L_1),$$
$$(0, t) \mapsto e^{\frac{i\pi(1-t)}{2}}\Phi(T_qL_1).$$

The *Maslov-Viterbo index* $\mu_u(p,q)$ is defined as the Maslov index of this loop in $\Lambda(\mathbb{R}^{2n})$.

2.2.2 Review of Lagrangian Floer theory

We review the Floer cohomology of Lagrangian intersections. We follow the paper of Yong-Guen Oh [29] and the book of Fukaya-Oh-Ohta-Ono [19].

Let (M, ω) be a compact symplectic manifold of dimension $2n$. Consider two closed Lagrangian submanifolds L_0 and L_1 that intersect with each other transversely.

We consider the space of paths from L_0 to L_1,

$$\Omega = \Omega(L_0, L_1) = \{l : [0,1] \to M | l(0) \in L_0, l(1) \in L_1\}.$$

Let us choose a based point $l_0 \in \Omega$ and consider the component Ω_0 of Ω containing l_0. The universal covering of Ω_0 is the set of homotopy class of the paths $\tilde{w} : [0,1] \to \Omega_0$ with $\tilde{w}(0) = l_0$ or equivalently the maps $w : [0,1] \times [0,1] \to M$ such that $w(0,t) = l_0(t)$ and $w(s,.) : [0,1] \to M, t \mapsto w(s,t)$ defines a path from L_0 to L_1 for each $s \in [0,1]$. For each point of the universal cover, let us choose a representative (l, w) where $w : [0,1] \times [0,1] \to M$ is a map described above and $l \in \Omega_0$ is the ending point of w, i.e. $l(t) = w(1,t)$.

Let us consider an equivalence relation on the universal cover. For two representatives (l, w) and (l', w') with $l = l'$, consider a map $\bar{w}\#w' : (\mathbb{R}/2\mathbb{Z}) \times [0,1] = S^1 \times [0,1] \to M$ defined by

$$\bar{w}\#w'(s,t) = \begin{cases} w(-s,t) & \text{if } -1 \leq s \leq 0 \\ w'(s,t) & \text{if } 0 \leq s \leq 1 \end{cases}.$$

We consider its symplectic area defined as

$$I_\omega(\bar{w}\#w') = \iint_{S^1 \times [0,1]} (\bar{w}\#w')^* \omega.$$

It is also possible to define its Maslov index. Indeed, as the symplectic linear group $Sp(2n)$ is connected, one can find a symplectic trivialization of $(\bar{w}\#w')^*TM$ on $S^1 \times [0,1]$. Let us denote such a trivialization by $\Psi : (\bar{w}\#w')^*TM \to (S^1 \times [0,1]) \times \mathbb{R}^{2n}$.

Consider a loop $\alpha_i : S^1 \to M$ defined by $\alpha_i(s) = \bar{w}\#w'(s,i)$ for each $i = 0, 1$. Along each of these loops, we consider a loop of Lagrangian spaces $\tilde{\alpha}_i : S^1 \to \Lambda(\mathbb{R}^{2n})$ defined by

$$\tilde{\alpha}_i(s) = \Psi(T_{\alpha_i(s)} L_i) \in \Lambda(\mathbb{R}^{2n}).$$

We define the Maslov index of $\bar{w}\#w'$ by

$$I_\mu(\bar{w}\#w') = \mu(\tilde{\alpha}_1) - \mu(\tilde{\alpha}_0).$$

Definition 2.4. We define an equivalence relation on the universal cover of Ω_0 by $(l, w) \sim (l, w')$ if and only if $I_\omega(\bar{w}\#w') = 0 = I_\mu(\bar{w}\#w')$.

The space of equivalence classes is also a covering space of Ω_0. This covering space is called *Novikov covering space* and will be denoted by $\tilde{\Omega}_0$.

We consider an action functional $\mathcal{A} : \tilde{\Omega}_0 \to \mathbb{R}$ by

$$\mathcal{A}([l, \omega]) = \iint_{[0,1]\times[0,1]} w^*\omega.$$

By definition of the Novikov covering space, this action functional is well-defined. The Lagrangian Floer cohomology is an infinite dimensional Morse cohomology with respect to this action functional $\mathcal{A}$.

Every critical point of the functional is just an intersection point of L_0 and L_1. Indeed, we observe

$$d\mathcal{A}(l, w)(\xi) = \int_0^1 \omega(\xi(t), l'(t))dt, \tag{2.1}$$

where $\xi \in T_{[l,w]}\tilde{\Omega}_0$ is regarded as a vector field along the image of l.

One can deduce that $[l, w]$ is a critical point of this functional if and only if $l'(t) = 0$ for all $t \in [0, 1]$ or equivalently, l is a constant path from L_0 to L_1. From now on, for each $p \in L_0 \cap L_1$, let us denote the constant path at p by $l_p : [0, 1] \to M$.

In order to define a Riemannian metric on $\tilde{\Omega}_0$, we introduce a family of almost complex structures on M. Let us consider

$$\begin{aligned} j_\omega &:= \text{ the space of } \omega\text{-compatible almost complex structures on } M, \\ \mathcal{J}_\omega &:= C^\infty([0, 1], j_\omega). \end{aligned}$$

Whenever we are given $J = \{J_t\}_{t=0}^1 \in \mathcal{J}_\omega$, we define a L^2 metric on $\tilde{\Omega}_0$

by

$$\langle \xi_1, \xi_2 \rangle := \int_0^1 \omega(\xi_1(t), J_t \xi_2(t)) dt. \tag{2.2}$$

By definition of compatible almost complex structures, we deduce that this indeed defines a Riemannian metric on the space $\tilde{\Omega}_0$.

From the equation (2.1) and (2.2), we observe that the L^2-gradient flow equation of the action functional is given by a map $u : \mathbb{R} \times [0,1] \to M$ satisfying

$$\begin{cases} \frac{\partial u}{\partial s} + J_t \frac{\partial u}{\partial t} = 0 \\ u(s,0) \in L_0, u(s,1) \in L_1 \end{cases} \tag{2.3}$$

This implies that the gradient flow equation is equivalent to the Cauchy-Riemann equation corresponding to the parametrized almost complex structures $J = \{J_t\}$.

We introduce the notion of *energy* of a map $u : \mathbb{R} \times [0,1] \to M$

$$E(u) = \frac{1}{2} \iint_{\mathbb{R}\times[0,1]} \left| \frac{\partial u}{\partial s} \right|^2 + \left| \frac{\partial u}{\partial t} \right|^2 dsdt. \tag{2.4}$$

Then we consider the moduli space of gradient flow trajectories with finite energy

$$\widetilde{\mathcal{M}}(L_0, L_1; J) = \{u : \mathbb{R} \times [0,1] \to M | u \text{ satisfies } (2.3), E(u) < \infty\}. \tag{2.5}$$

The boundedness condition $E(u) < \infty$ implies that this moduli space decomposes into

$$\widetilde{\mathcal{M}}([l_p, w], [l_q, w']; J) = \{u \in \widetilde{\mathcal{M}}(L_0, L_1; J) | \lim_{s \to -\infty} u(s,t) = p, \lim_{s \to \infty} u(s,t) = q, \\ w \# u = w'\}$$

where p, q run over all intersection points of L_0 and L_1.

Each trajectory $u \in \widetilde{\mathcal{M}}([l_p, w], [l_q, w']; J)$ is assigned the Maslov-Viterbo index $\mu_u(p,q) := \mu_{\tilde{u}}(p,q)$ where $\tilde{u}$ is defined as follows.

Since the image of u is compact, one can find $\tilde{u} : [0,1] \times [0,1]$ such that $\tilde{u}(\tau(s), t) = u(s,t)$ for some $\tau : \mathbb{R} \to [0,1]$ such that $\lim_{s\to-\infty} \tau(s) = 0, \lim_{s\to\infty} \tau(s) = 1$ and τ is strictly increasing.

We will abbreviate our notation as

$$\widetilde{\mathcal{M}}(p,q;J) := \widetilde{\mathcal{M}}([l_p,w],[l_q,w'];J)$$

whenever it does not make any confusion.

The space $\widetilde{\mathcal{M}}(p,q;J)$ carries an $\mathbb{R}$-action given by

$$(\tau \cdot u)(\cdot,\cdot) \mapsto u(\cdot - \tau, \cdot),$$

where $u \in \widetilde{\mathcal{M}}(L_0, L_1; J)$ and $\tau \in \mathbb{R}$.

We consider the quotient space by this action

$$\mathcal{M}(p,q;J) = \widetilde{\mathcal{M}}(p,q;J)/\mathbb{R}.$$

Theorem 2.5 ([13, 29])**.** *There exists a Baire subset $\mathcal{J}_\omega^{reg} \subset \mathcal{J}_\omega$ such that the component of u in $\mathcal{M}(p,q;J)$ becomes a smooth manifold of dimension $\mu_u(p,q) - 1$ for all $p, q \in L_0 \cap L_1$ and $J \in \mathcal{J}_\omega^{reg}$.*

Here we need the monotonicity on Lagrangian submanifolds to get a compactness statement. Indeed, we have

Theorem 2.6 ([29])**.** *Suppose L_0 and L_1 are monotone Lagrangian submanifolds. There exists a Baire subset $\mathcal{J}_\omega^0 \subset \mathcal{J}_\omega$ such that the zero dimensional component in $\mathcal{M}(p,q;J)$ is compact for all $J \in \mathcal{J}_\omega^0 \cap \mathcal{J}_\omega^{reg}$.*

For each $p, q \in L_0, L_1$, let us define an integer $n(p,q)$ by counting zero dimensional component of the moduli space $\mathcal{M}(p,q;J)$, i.e.

$$n(p,q) := \#\mathcal{M}^0(p,q;J) \mod 2.$$

Here we only count the number modulo 2 because we do not want to concentrate on the orientation of the moduli spaces. So we use the field $\mathbb{F}_2$ of two elements as the coefficient ring.

Definition 2.7. The Floer cochain complex $CF(L_0, L_1)$ is defined by

$$CF(L_0, L_1) = \bigoplus_{p \in L_0 \cap L_1} \mathbb{F}_2 \langle p \rangle.$$

The differential $\delta : CF(L_0, L_1) \to CF(L_0, L_1)$ is defined by

$$\delta(p) = \sum_{p} n(p, q) q. \tag{2.6}$$

In monotone Lagrangian case, the map δ indeed becomes a differential if both of the minimal Maslov indices of L_0 and L_1 are greater than or equal to 3.

This follows from the following theorem.

Theorem 2.8 ([29])**.** *Suppose L_0 and L_1 are monotone Lagrangian submanifolds with minimal Maslov indices ≥ 3. There exists a Baire subset $\mathcal{J}^1_\omega \subset \mathcal{J}_\omega$ such that the boundary of an one-dimensional component in $\mathcal{M}(p, q; J)$ come from the splittings into two isolated holomorphic strips, i.e. the boundary of an one-dimensional component is given by a union of*

$$\bigcup_{r \in L_0 \cap L_1} \mathcal{M}^0(p, r; J) \times \mathcal{M}^0(r, q; J)$$

for all $p, q \in L_0 \cap L_1$ and $J \in \mathcal{J}^1_\omega \cap \mathcal{J}^{reg}_\omega$.

Under the condition in Theorem 2.8, we have $\delta \circ \delta = 0$. The Lagrangian Floer cohomology $HF(L_0, L_1)$ is the cohomology of this cochain complex

$$HF(L_0, L_1) = \ker \delta / \mathrm{im} \delta.$$

If the minimal Maslov indices of the Lagrangian submanifolds are equal to 2, then the map δ may not square to zero due to the existence of Maslov index 2 discs.

In order to understand this case, we first introduce the moduli space of pseudo-holomorphic discs bounding a Lagrangian submanifold. Indeed, for a Lagrangian submanifold L and an almost complex structure J on M, consider

$$\mathcal{M}(L, \beta; J) = \{u : (D^2, \partial D^2) \to (M, L) | u \text{ is } J\text{-holomorphic and } [u] = \beta\}.$$

This moduli space can be realized as a smooth manifold of dimension $n + \mu(\beta)$ for a generic choice of almost complex structure J as above.

Furthermore, we consider the moduli space of pseudo-holomorphic discs with one boundary marked points. Indeed, the group G of all bi-holomorphisms from D^2 to itself acts on $\mathcal{M}(L, \beta; J) \times_G \partial D^2$ by

$$\varphi \cdot (u, z) = (u \circ \varphi^{-1}, \varphi(z)).$$

We denote its quotient space by

$$\mathcal{M}_1(L, \beta; J) := \mathcal{M}(L, \beta; J) \times_G \partial D^2.$$

We consider the evaluation map $ev : \mathcal{M}_1(L, \beta; J) \to L$ defined by $(u, z) \mapsto u(z)$. Then we have

Theorem 2.9 ([30])**.** *Suppose L_0 and L_1 are monotone Lagrangian submanifolds with minimal Maslov indices ≥ 2. There exists a Baire subset $\mathcal{J}^2_\omega \subset \mathcal{J}_\omega$ such that the boundary of an one-dimensional component in $\mathcal{M}(p, p; J)$ is given by a union of*

$$\bigcup_{r \in L_0 \cap L_1} \mathcal{M}^0(p, p; J) \times \mathcal{M}^0(r, p; J)$$

and

$$\bigcup_{i=0,1} \bigcup_{\substack{\beta_i \in \pi_2(M, L_i) \\ \mu(\beta_i)=2}} \widetilde{\mathcal{M}}^0(p, p; J) \times_{L_i} \mathcal{M}_1(L_i, \beta_i; J_i)$$

for all $p \in L_0 \cap L_1$ and $J \in \mathcal{J}^2_\omega \cap \mathcal{J}^{reg}_\omega$.

The boundary component $\bigcup_{i=0,1} \bigcup_{\substack{\beta_i \in \pi_2(M, L_i) \\ \mu(\beta_i)=2}} \widetilde{\mathcal{M}}^0(p, p; J) \times_{L_i} \mathcal{M}_1(L_i, \beta_i; J_i)$ describes disc bubbles attached to the constant map.

The moduli space $\widetilde{\mathcal{M}}^0(p,p;J)$ of the constant map at p is regular for any choice of almost complex structures. Furthermore, the set $\mathcal{J}^2_\omega$ can be chosen so that the evaluation maps $ev : \mathcal{M}_1(L_i, \beta_i; J_i) \to L_i$ are transverse to all intersection points of L_0 and L_1. See [30].

We define the *obstruction* $m_{L_i,0}(p)$ as the number of J_i-holomorphic discs of Maslov index 2 that bound L_i and pass through p, i.e. the number of points in $ev^{-1}(p)$ of the evaluation map $ev : M_1(L_i, \beta_i; J_i) \to L_i$.

It follows from Theorem 2.9 that

$$\delta \circ \delta(p) = (m_{L_0,0}(p) + m_{L_1,0}(p))p.$$

for a generic choice $J \in \mathcal{J}^2_\omega$.

Thus the square of the map δ may not be zero due to the existence of Maslov index 2 discs bounding L_0 or L_1. Therefore the term $m_{L_i,0}$ is called an obstruction class in Lagrangian Floer theory. Furthermore, this notion will be extended to the Fukaya category, which will be introduced in Remark 4.4.

Remark 2.10 (Independence under several choices). We have observed that for generic choice of almost complex structure J, the moduli spaces of Floer trajectories have smooth structures and its boundaries are well-described. Hence the Lagrangian Floer cohomology is well-defined under specific conditions as above.

One of the most important properties of Lagrangian Floer cohomology is that the Floer cohomogy is independent of the choice of such a generic almost complex structure.

Furthermore, if we perturb one of Lagrangian submanifolds, say L_0, via a generic Hamiltonian isotopy ϕ, then one can find a chain homotopy equivalence between $CF(L_0, L_1)$ and $CF(\phi(L_0), L_1)$ so that the corresponding Floer cohomologies are isomorphic.

Special Case $L_0 = L = L_1$

It is possible to extend the construction of Floer cohomology between L_0 and L_1 even in the case when two Lagrangian submanifolds L_0 and L_1 do not intersect transversely. The most extreme case occurs when two Lagrangian submanifolds L_0 and L_1 are equal. The idea is to choose a Hamiltonian isotopy ϕ such that L and $\phi(L)$ intersect transversely.

Let L be a closed, oriented, relatively spin Lagrangian submanifold in a symplectic manifold (M, ω). The Floer cochain complex $CF(L, \phi(L))$ can be defined as in Definition 2.7. Moreover, Fukaya-Oh-Ohta-Ono proved that the moduli space of pseudo-holomorphic strips $\mathcal{M}(L, \phi(L); J)$ has an oriented Kuranishi structure(see Appendix A) and that it is possible to construct a family of compatible multisections on the moduli spaces. They defined a differential δ via a virtual count of the moduli space of pseudo-holomorphic strips and defined a Floer cohomology $HF(L, \phi(L))$ as the cohomology with respect to the differential δ.

But there exist several different approaches which give cohomology groups isomorphic to $HF(L, \phi(L))$. Indeed, Fukaya-Oh-Ohta-Ono constructed a Bott-Morse version of Floer cohomology in [19, 20] and a de Rham version of Floer cohomology in [18, 20, 16].

We give a brief description of the de Rham version of Floer cohomology here. The Floer cochain complex is given by the de Rham complex $\Omega^\bullet(L)$ on L. Let us denote the differential on this complex by m_1.

Then the differential m_1 is given by a summation

$$m_1 = \sum_{\beta \in \pi_2(M,L)} m_{1,\beta} T^{\omega(\beta)},$$

in such a way that

$$m_{1,0}(a) = (-1)^{\deg a + 1} da$$

for $a \in \Omega^\bullet(L)$.

The de Rham version Floer cohomology is defined as the cohomology with respect to the differential m_1

$$HF(L) := H^*(\Omega^\bullet(L), m_1).$$

Theorem 2.11 ([18, 19]). *There exist chain maps*

$$\begin{aligned} h &: (CF(L, \phi(L)), \delta) \to (\Omega^\bullet(L), m_1), \\ g &: (\Omega^\bullet(L), m_1) \to (CF(L, \phi(L)), \delta) \end{aligned}$$

such that $h \circ g$ and $g \circ h$ are chain homotopic to the identity. In particular, the Floer coholomology $HF(L, \phi(L))$ is isomorphic to the de Rham version Floer cohomology $HF(L)$.

One advantage of the de Rham version is that it becomes more natural to define higher product structures. Indeed, the authors constructed a family of higher products $m_k : \Omega(L)^{\otimes k} \to \Omega(L)$ for $k \in \mathbb{Z}_{\geq 0}$, which satisfies quadratic relations called A_∞-relation.

As a consequence, the Floer cochain complex $\Omega(L)$ has an additional structure, which is called an A_∞-algebra structure. The notion of A_∞-algebra will be generalized to A_∞-category and the construction of A_∞-structures will be introduced in Chapter 4.

2.3 Preliminaries on A_∞-categories

The purpose of this section is to review some basic notions on A_∞-category.

Let $\mathbb{K}$ be a field. An A_∞-category $\mathcal{A}$ over $\mathbb{K}$ consists of

- a set of objects,
- a morphism space $\mathrm{Hom}_{\mathcal{A}}(X_0, X_1)$ for each pair (X_0, X_1) of objects, which is a $\mathbb{Z}$ or $\mathbb{Z}/2$ graded $\mathbb{K}$-vector space,
- a family of compositions of morphisms $\{m_k\}_{k \geq 0}$ such that

$$m_k : \mathrm{Hom}_{\mathcal{A}}(X_0, X_1) \otimes ... \otimes \mathrm{Hom}_{\mathcal{A}}(X_{k-1}, X_k) \to \mathrm{Hom}_{\mathcal{A}}(X_0, X_k)$$

is a $\mathbb{K}$-linear map of degree $2-k$ for each $k \in \mathbb{Z}_{\geq 0}$ and these satisfy the A_∞-equations

$$\sum_{k_1+k_2=k+1} \sum_i (-1)^* m_{k_1}(x_1, ..., x_{i-1}, m_{k_2}(x_i, ..., x_{i+k_2-1}), x_{i+k_2}, ..., x_k) = 0 \tag{2.7}$$

for any k-tuple of morphisms $x_i \in \mathrm{Hom}_{\mathcal{A}}(X_{i-1}, X_i)$. Here the sign $*$ is given by $* = (\deg x_1 + 1) + ... + (\deg x_{i-1} + 1)$

Obstruction

Note that the A_∞-structure constant m_0 is given by a summation $\sum_{X \in obj(\mathcal{A})} m_0(X)$ where $m_0(X)$ is an element in $\mathrm{Hom}^{even}_{\mathcal{A}}(X, X)$ for each object X.

The term $m_0(X)$ is called an *obstruction* for X. Indeed, $m_0(X) \in \mathrm{Hom}_{\mathcal{A}}(X, X)$ is an obstruction for $m_1 : \mathrm{Hom}_{\mathcal{A}}(X, X) \to \mathrm{Hom}_{\mathcal{A}}(X, X)$ to be a differential. Consider the simplest equation among A_∞-equations (2.7)

$$m_1(m_1(x)) + m_2(m_0(X), x) + (-1)^{\deg x + 1} m_2(x, m_0(X)) = 0. \tag{2.8}$$

where $x \in \mathrm{Hom}_{\mathcal{A}}(X, X)$. This implies that if $m_0(X) = 0$, then we have $m_1(m_1(x)) = 0$ for all $x \in \mathrm{Hom}_{\mathcal{A}}(X, X)$. We summarize this observation as follows.

Definition 2.12 (Obstruction)**.** An object X is called *unobstructed* if $m_0(X)$ is zero.

There exists a slightly weaker condition on an object X that ensures m_1 to be a differential, which will be called weakly unobstructedness. We first introduce the notion of a strict unit.

Definition 2.13 (Strict Unit)**.** Let X be an object of $\mathcal{A}$. A *strict unit* e_X of X is an element of $\mathrm{Hom}^0_{\mathcal{A}}(X, X)$ satisfying

$$m_2(e_X, a) = a$$
$$m_2(b, e_X) = (-1)^{\deg b} b.$$

for any $a \in \mathrm{Hom}_{\mathcal{A}}(X,Y)$ and $b \in \mathrm{Hom}_{\mathcal{A}}(Z,X)$ and

$$m_k(..., e_X, ...) = 0$$

for all $k \neq 2$.

If every object of $\mathcal{A}$ has a strict unit, then the category $\mathcal{A}$ is called *strict unital*.

Definition 2.14 (Weakly Unobstructedness)**.** Let X be an object of $\mathcal{A}$ with a strict unit e_X. Then X is said to be *weakly unobstructed* if

$$m_0(X) = \lambda e_X,$$

for some $\lambda \in \mathbb{K}$.

Again from the equation (2.8) and the definition of the strict unit, one can deduce that if an object X is weakly unobstructed, then $m_1 \circ m_1 = 0$. This justifies the terminology "weakly unobstructedness".

For a given strict unital A_∞-category $\mathcal{A}$, we define $\mathcal{A}^\lambda$ of $\mathcal{A}$ as a full subcategory which consists of all weakly unobstructed objects X with $m_0(X) = \lambda e_X$.

Definition 2.15 (Isomorphism)**.** Let $\mathcal{A}$ be a strict unital A_∞-category.

Two objects X, Y of $\mathcal{A}$ are said to be *isomorphic* if there exists $a \in \mathrm{Hom}^0_{\mathcal{A}}(X,Y)$ and $b \in \mathrm{Hom}^0_{\mathcal{A}}(Y,X)$ such that

$$m_1(a) = 0,\ m_1(b) = 0$$
$$m_2(a,b) = e_X,\ m_2(b,a) = e_Y.$$

The notion of isomorphism defines an equivalence relation on the set of objects of $\mathcal{A}$. Indeed, the reflexivity and symmetry are obvious. The transitivity follows from the A_∞-equation (2.7).

Homology category

We are ready to introduce the notion of homology category of an A_∞-category. The homology category will be a genuine category rather than A_∞-category.

Given an A_∞-category $\mathcal{A}$, we define the *homology category* $H(\mathcal{A})$ as follows.

- The objects of $H(\mathcal{A})$ are the weakly unobstructed objects of $\mathcal{A}$.
- For weakly unobstructed objects of X, Y of $\mathcal{A}$ such that

  $$m_0(X) = \lambda e_X, \ m_0(Y) = \mu e_Y,$$

  the morphism space from X to Y is defined by

  $$\mathrm{Hom}_{H(\mathcal{A})}(X, Y) = H^*(\mathrm{Hom}_{\mathcal{A}}(X, Y), m_1).$$

  Note that $m_1 \circ m_1 = 0$ if and only if $\lambda = \mu$.
- For $a \in \mathrm{Hom}_{H(\mathcal{A})}(X, Y)$ and $b \in \mathrm{Hom}_{H(\mathcal{A})}(Y, Z)$, the composition $b \circ a \in \mathrm{Hom}_{H(\mathcal{A})}(X, Z)$ is given by

  $$b \circ a = (-1)^{\deg a}[m_2(a, b)],$$

  where [ ] denote the m_1-cocycle class. The associativity of this composition follows from the A_∞-equation (2.7).

Definition 2.16. Two objects X, Y of $\mathcal{A}$ are said to be *quasi-isomorphic* if there exists $a \in \mathrm{Hom}^0_{\mathcal{A}}(X, Y)$ such that

- $m_1(a) = 0$
- $m_2(-, a)$ induces an isomorphism $\mathrm{Hom}_{H(\mathcal{A})}(Z, X) \cong \mathrm{Hom}_{H(\mathcal{A})}(Z, Y)$ for every object Z of $\mathcal{A}$ and the isomorphism is natural in Z in the homology category $H(\mathcal{A})$.

Twisted complex

Every A_∞-category can be extended to an A_∞-category, which is closed under shifts and extensions.

We first construct an A_∞-category $\mathbb{Z}\mathcal{A}$ which is closed under shifts. Indeed, the objects of $\mathbb{Z}\mathcal{A}$ have the form $X[m]$ where X is an object of $\mathcal{A}$ and $[m]$ denotes a formal shift by $m \in \mathbb{Z}$. Then the morphism between $X[m]$ and $Y[n]$ in $\mathbb{Z}\mathcal{A}$ is defined by

$$\mathrm{Hom}^*_{\mathbb{Z}\mathcal{A}}(X[m], Y[n]) = \mathrm{Hom}^{*+n-m}_{\mathcal{A}}(X, Y).$$

Because morphism spaces of $\mathbb{Z}\mathcal{A}$ come from those of $\mathcal{A}$, the A_∞-structures of $\mathcal{A}$ can be used to define new A_∞-structures for $\mathbb{Z}\mathcal{A}$.

Now we introduce the notion of a twisted complex.

Definition 2.17 ([34])**.** A *twisted complex with internal curvature* λ *over* $\mathcal{A}$ consists of

- finitely many objects $\{X_i\}_{1\leq i\leq n}$ of $\mathbb{Z}\mathcal{A}$, each of which has a strict unit,
- a family of morphisms $\{\delta_{ij} \in \mathrm{Hom}^1_{\mathbb{Z}\mathcal{A}}(X_j, X_i)\}_{1\leq j<i\leq n}$ such that the following *Maurer-Cartan equation* holds

  $$m_0(X_i) = \lambda e_{X_i} \tag{2.9}$$

  $$\sum_{d\geq 1} \sum_{j=i_0<\ldots<i_d=i} m_d(\delta_{i_0,i_1}, ..., \delta_{i_{d-1},i_d}) = 0 \tag{2.10}$$

  for any pair $1 \leq j < i \leq n$.

If we regard $\delta := (\delta_{ij})$ as a matrix of morphisms, then δ is a strictly lower-triangular matrix of morphisms. We find an alternative expression for the above two equations (2.9) and (2.10) using the matrix representation:

$$\sum_{d\geq 0} m_d(\underbrace{\delta, ..., \delta}_{d}) = \lambda id. \tag{2.11}$$

Introducing the notation $e^{\delta} := \sum_{k\geq 0} \delta^{\otimes k}$, the above equation can be rewritten as follows

$$m(e^{\delta}) = \lambda id.$$

We call $\delta = (\delta_{ij})$ a *Maurer-Cartan element* and denote by $(\oplus X_i, \delta)$ such a twisted complex of internal curvature λ.

For each $\lambda \in \mathbb{K}$, twisted complexes of internal curvature λ over $\mathcal{A}$ forms an A_∞-category $Tw^{\lambda}\mathcal{A}$, which will be described as follows

The morphism space between two twisted complexes $X^0 = (\oplus X_a^0, \delta^0 = (\delta_{ab}^0))$ and $X^1 = (\oplus X_c^1, \delta^1 = (\delta_{cd}^1))$ is given by

$$\mathrm{Hom}_{Tw^{\lambda}\mathcal{A}}(X, Y) = \bigoplus_{a,c} \mathrm{Hom}_{\mathbb{Z}\mathcal{A}}(X_a^0, X_c^1),$$

and the A_∞-operation m_k^{tw} is modified by Maurer-Cartan elements in the following way:

$$\begin{aligned} m_k^{tw}(x_1, .., x_k) &:= m(e^{\delta^0}, x_1, e^{\delta^1}, ..., e^{\delta^{k-1}}, x_k, e^{\delta^k}) \\ &:= \sum_{d_0,\ldots,d_k} m(\underbrace{\delta^0, ..., \delta^0}_{d_0}, x_1, \underbrace{\delta^1, ..., \delta^1}_{d_1}, ..., \underbrace{\delta^{k-1}, ..., \delta^{k-1}}_{d_{k-1}}, x_k, \underbrace{\delta^k, ..., \delta^k}_{d_k}). \end{aligned}$$

The notation m denotes $m_{k+d_0+\ldots+d_k}$ in the last equation.

In order to show that $Tw^{\lambda}\mathcal{A}$ is indeed an A_∞-category, we need to check if the A_∞-equation holds for these A_∞-operations.

Let $X^i = (\oplus X_j^i, \delta^i)$ be twisted complexes of internal curvature λ for $i = 0, ..., k$ and let $x_i \in \mathrm{Hom}_{Tw^{\lambda}\mathcal{A}}(X^{i-1}, X^i)$ for each $i = 1, ..., k$.

We have

$$
\begin{aligned}
&\sum_{k_1+k_2=k+1,i}(-1)^* m^{tw}_{k_1}(x_1,...,x_{i-1},m^{tw}_{k_2}(x_i,...,x_{i+k_2-1}),x_{i+k_2},...,x_k)\\
=&\sum_{k_1+k_2=k+1,i}(-1)^* m(e^{\delta^0},x_1,e^{\delta^1},...,x_{i-1},e^{\delta^i},m(e^{\delta^i},x_i,...,x_{i+k_2-1},e^{\delta^{i+k_2-1}})\\
&e^{\delta^{i+k_2-1}},x_{i+k_2},....,x_k,e^{\delta^k})\\
=&(-1)^*(m(e^{\delta^0},m(e^{\delta^0}),e^{\delta^0},x_1,e^{\delta^1},...,x_k,e^{\delta^k})+...+\\
&m(e^{\delta^0},x_1,...,e^{\delta^i}m(e^{\delta^i}),e^{\delta^i},...,x_k,e^{\delta^k})+...+m(e^{\delta^0},x_1,e^{\delta^1},...,x_k,e^{\delta^k,m(e^{\delta^k}),\delta^k}))\\
=&0.
\end{aligned}
$$

Here the sign $(-1)^*$ does not change from that in the equation (2.7). This is because the shifted degree $\deg\delta^i+1$ is even for all i. The second equality follows from the original A_∞-equation for $\mathcal{A}$ and the last equality follows from the fact that $m(e^{\delta^i})=\lambda id$.

Hence we indeed get an A_∞-category $Tw^\lambda\mathcal{A}$. Also we define an $\mathcal{A}_\infty$-category $Tw\mathcal{A}$ by

$$Tw\mathcal{A}=\bigoplus_\lambda Tw^\lambda\mathcal{A}.$$

The full subcategory $\mathcal{A}^\lambda$ of $\mathcal{A}$ can be regarded as a full subcategory of $Tw^\lambda\mathcal{A}$ by identifying X with a twisted complex $(X,0)$.

Remark 2.18. If an A_∞-category $\mathcal{A}$ is strict-unital, then the corresponding A_∞-category of twisted complexes $Tw\mathcal{A}$ is strict-unital. Indeed, for a given twisted complex $(\oplus_i X_i,\delta)$, the strict unit is given by $\oplus_i e_{X_i}$.

One of the most important features of $Tw^\lambda\mathcal{A}$ is that it is closed under taking extension. Indeed, let $X^0=(\oplus X^0_i,\delta^0)$ and $X^1=(\oplus X^1_j,\delta^1)$ be two twisted complexes with internal curvature λ and let $a\in\mathrm{Hom}^1_{Tw^\lambda\mathcal{A}}(X^1,X^0)$ be a m^{tw}_1-closed morphism of degree 1.

We construct the extension X by

$$X=\left(\bigoplus X^0_i\oplus\bigoplus X^1_j,\delta:=\begin{pmatrix}\delta^0 & 0\\ a & \delta^1\end{pmatrix}\right).$$

It follows from the m_1^{tw}-closedness of a that X becomes a twisted complex. Furthermore, there exists a short exact sequence

$$0 \longrightarrow X^0 \xrightarrow{i} X \xrightarrow{p} X^1 \longrightarrow 0$$

in the sense that

$$0 \longrightarrow \mathrm{Hom}_{H(Tw^\lambda\mathcal{A})}(Y, X^0) \xrightarrow{m_2(.,i)} \mathrm{Hom}_{H(Tw^\lambda\mathcal{A})}(Y, X) \xrightarrow{m_2(.,p)} \mathrm{Hom}_{H(Tw^\lambda\mathcal{A})}(Y, X^1) \longrightarrow 0$$

forms a short exact sequence for all Y and that this isomorphism is natural in Y.

Theorem 2.19. *The A_∞-category $Tw^\lambda\mathcal{A}$ is closed under the extension. In particular, it is a triangulated A_∞-category. (See [33]).* □

As a corollary, we also have

Corollary 2.20. *The homology category $H(Tw^\lambda\mathcal{A})$ is closed under the extension.* □

Chapter 3

Integration along the fiber

We review the definition of the integration along the fiber in order to fix our convention. Actually we define an integration along the fiber of vector valued differential forms. For this purpose, we follow the construction given in [7].

Let M and N be smooth manifolds of dimension m and n respectively. Suppose N has no boundary. Furthermore suppose that there is a submersion $f : M \to N$ and a vector bundle E on N.

Under this assumption, we define the integration along the fiber for vector valued differential forms, i.e. we define

$$f_! : \Omega_c^d(M, f^*E) \to \Omega^{d-m+n}(N, E).$$

Let $\eta \in \Omega_c^d(M, f^*E)$ be a form of degree d with compact support K.

Let $q \in N$. Then there exist finitely many local charts U_α of M and $V_\alpha \ni q$ of N such that $f^{-1}(q) \cap K$ is covered by U_α's, E is trivialized over V_α and locally $f|_{U_\alpha} : U_\alpha \to V_\alpha$ is written as

$$f(x_\alpha^1, ..., x_\alpha^n, x_\alpha^{n+1}, ..., x_\alpha^m) = (x_\alpha^1, ..., x_\alpha^n).$$

Thus $\eta|_{U_\alpha}$ is a linear combination of

$$(a)\ (f^*\phi)\mu(x_\alpha)dx_\alpha^{i_1}...dx_\alpha^{i_r}\otimes f^*e,\ r<m-n$$
$$(b)\ (f^*\phi)\mu(x_\alpha)dx_\alpha^{n+1}...dx_\alpha^{m}\otimes f^*e,$$

where ϕ is a local differential form on N,μ is a smooth function on U_α and e is a local section of $E|_{V_\alpha}$.

Choose a small neighborhood $V\subset\bigcap_\alpha V_\alpha$ of q such that $f^{-1}(V)\cap K$ is also covered by U_α's. Then $f_!(\eta|_{U_\alpha})_p\in\wedge T_p^*N\otimes E|_p$ is defined by sending

$$(a)\ (f^*\phi)\mu(x_\alpha)dx_\alpha^{i_1}...dx_\alpha^{i_r}\otimes f^*e\mapsto 0 \text{ if } r<m-n$$
$$(b)\ (f^*\phi)\mu(x_\alpha)dx_\alpha^{n+1}...dx_\alpha^{m}\otimes f^*e\mapsto \phi_p\int_{f^{-1}(p)\cap U_\alpha}\mu(x_\alpha)dx_\alpha^{n+1}...dx_\alpha^{m}\otimes e_p$$

for $p\in V$.

Finally, choosing functions $\{\rho_\alpha : M\to[0,1]\}_\alpha$ such that the support of ρ_α is contained in U_α and $\sum_\alpha\rho_\alpha(x)=1$ whenever $x\in f^{-1}(V)\cap K$, we define $(f_!\eta)_p$ by

$$\sum_\alpha f_!(\rho_\alpha\eta|_{U_\alpha})_p.$$

for $p\in V$.

Note that $\{\rho_\alpha|_{f^{-1}(p)}\}_\alpha$ becomes a partition of unity subordinate to $\{f^{-1}(p)\cap U_\alpha\}_\alpha$ for $p\in V$.

As usual, the result of this construction is independent of the choice of local charts and functions $\{\rho_\alpha\}$. Hence we get a vector valued differential form $f_!\eta\in\Omega^{d-m+n}(N,E)$.

Now we state a version of Stokes's theorem relating $f_!$ and $(f|_{\partial M})_!$. Here the manifold M is oriented by $f^*\Omega_N\wedge\Omega_F$ locally, where Ω_N is a local orientation form on N and Ω_F is a local orientation form of the fiber.

Theorem 3.1 (Stokes)**.** *Suppose N has no boundary. For any $\eta\in\Omega_c(M,f^*E)$, we have*

$$f_!\tilde{\nabla}\eta-\nabla f_!\eta=(-1)^{m+\deg\eta+1}(f|_{\partial M})_!\eta$$

where ∇ *is any connection of* E *and* $\tilde{\nabla} = f^*\nabla$*.*

Proof. Since the integration is defined locally, it is enough to consider the case when the support of η is contained in a coordinate chart U of M. We may identify

$$U \subseteq \{(x^1, ..., x^n, x^{n+1}, ..., x^m) | x_m \geq 0\}$$

and

$$\partial U := U \cap \{(x^1, ..., x^n, x^{n+1}, ..., x^m) | x^{n+1} = 0\} \subseteq \partial M.$$

There is a coordinate chart V of N where $f|_U : U \to f(U) \cap V$ is written as

$$f(x^1, ..., x^n, x^{n+1}, ..., x^m) = (x^1, ..., x^n).$$

Choosing V sufficiently small, we may assume that $E|_V$ is trivial and hence we find a bundle isomorphism

$$\psi : V \times \mathbb{C}^r \to E|_V.$$

Let $\{v_i = \psi(e_i)\}_{i=1}^r$ be a local frame of $E|_V$.

Using this trivialization, we may write

$$\nabla = d + B$$

for some $B = (B_{ij}) \in \Omega^1(V) \otimes_{\mathbb{R}} \mathrm{Hom}(\Lambda^r, \Lambda^r)$.

Consider $\eta_1 = (f^*\phi_1)g_1(x)dx^{n+2}...dx^m \otimes f^*v_k$ where ϕ_1 is a local differential form on N and g_1 is a function on U. Then we have

$$f_!\eta_1 = 0$$

by definition of $f_!$. But, we have

$$\begin{aligned}\tilde{\nabla}\eta_1 &= d(f^*\phi_1\mu_1(x)dx^{n+1}...dx^{m-1}) \otimes f^*v_k \\ &\quad + \sum_{j=1}^{r} f^*B_{jk}f^*\phi_1\mu_1(x)dx^{n+1}...dx^{m-1} \otimes f^*v_j\end{aligned}$$

$$
\begin{aligned}
&= (f^* d\phi_1 \mu_1(x) dx^{n+1} \dots dx^{m-1} \\
&+ (-1)^{\deg \eta_1 - (m-n-1)} f^* \phi_1 \frac{\partial \mu_1(x)}{\partial x^{n+1}} dx^{n+1} dx^{n+2} \dots dx^m) \otimes f^* v_k \\
&+ \sum_{j=1}^{r} f^* B_{jk} f^* \phi_1 \mu_1(x) dx^{n+1} \dots dx^{m-1} \otimes f^* v_j.
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
f_! \tilde{\nabla} \eta_1 &= (-1)^{\deg \eta_1 - (m-n-1)} f_! (f^* \phi_1 \frac{\partial \mu_1(x)}{\partial x^{n+1}} dx^{n+1} dx^{n+2} \dots dx^m \otimes f^* v_k) \\
&= (-1)^{\deg \eta_1 - (m-n-1)} \phi_1 \int_{\partial U} -\mu_1(x', 0, x'') dx^{n+2} \dots dx^m \otimes v_k \\
&= (-1)^{n + \deg \eta_1 - (m-n-1)} (f|_{\partial M})_! \eta_1 \\
&= (-1)^{m + \deg \eta_1 + 1} (f|_{\partial M})_! \eta_1.
\end{aligned}
$$

where $(x', 0, x'') = (x^1, \dots, x^n, 0, x^{n+2}, \dots, x^m)$.

Here, $(-1)^n$ is multiplied in the third equality because the orientations of the boundary of M and the boundary of fiber differ by factor $(-1)^{\dim N} = (-1)^n$.

For any η_2 of the form $(f^* \phi_2) \mu_2(x) dx^{i_1} \dots dx^{i_{m-n-1}} \otimes f^* v_k$ with $i_1 = n+1$, we see that all of $(f|_{\partial M})_! \eta_2$, $f_! \tilde{\nabla} \eta_2$, $\nabla f_! \eta_2$ are zero by similar arguments above.

Finally consider $\eta_3 = (f^* \phi_3) \mu_3(x) dx^{n+1} \dots dx^m \otimes f^* v_k$. By similar arguments above, we have

$$(f|_{\partial M})_! \eta_3 = 0$$

and

$$
\begin{aligned}
f_! \tilde{\nabla} \eta_3 &= d\phi_3 \int_U \mu_3(x) dx^{n+1} \dots dx^m \otimes v_k + \sum_{j=1}^{r} B_{jk} \phi_3 \int_U \mu_3(x) dx^{n+1} \dots dx^m \otimes v_j. \\
&= \nabla f_! \eta_3.
\end{aligned}
$$

We have observed that the equality holds for three types of the vector valued forms η_1, η_2, η_3. Because these are everything we need to consider in the equality, the assertion follows. $\square$

We also have the following properties of the integration along the fiber. We follow the Lino Amorim's papers [2, 3] closely. But there are some differences in sign because the definition of the integration along the fiber used here is different from that in [2].

Proposition 3.2. *1. Suppose there are submersions $f : M \to N, g : N \to L$. Then we have*

$$(g \circ f)_! = g_! \circ f_!.$$

2. Suppose $f : M \to L$ is a submersion. Then we have

$$f_!(f^*\beta \wedge \alpha) = \beta \wedge f_!\alpha$$

where $\alpha \in \Omega(M, Hom(f^*E_0, f^*E_1))$ and $\beta \in \Omega(L, Hom(E_1, E_2))$.

3. Suppose there exist a smooth map $g_1 : M_1 \to L$ and a submersion $f_2 : M_2 \to L$. We consider the corresponding fiber product $M_1 \times_L M_2$.

$$\begin{tikzcd} M_1 \times_L M_2 \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & M_1 \arrow[d, "g_1"] \\ M_2 \arrow[r, "f_2"'] & L \end{tikzcd}$$

Then we have

$$(\pi_1)_! \pi_2^* \alpha = g_1^*(f_2)_! \alpha,$$

where $\alpha \in \Omega(M_2, f_2^* E)$.

Proposition 3.3 ([2])**.** *Suppose there exist a smooth map $g_1 : M_1 \to L$ and a submersion $f_2 : M_2 \to L$. Suppose further that there exists a submersion*

$f_2 : M_2 \to L$.

$$\begin{tikzcd} M_1 \times_L M_2 \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & M_1 \arrow[r, "f_1"] \arrow[d, "g_1"] & L \\ M_2 \arrow[r, "f_2"'] & L & \end{tikzcd}$$

Then we have the following equality.

$$(f_1 \circ \pi_1)_!(\pi_1^*\alpha_3 \wedge \pi_2^*\alpha_2 \wedge \pi_1^*\alpha_1) = (-1)^{(\dim M_2 - \dim L)|\alpha_1|}(f_1)_!(\alpha_3 \wedge {g_1}^*(f_2)_!\alpha_2 \wedge \alpha_1)$$

where $\alpha_1 \in \Omega(M_1, Hom((f_1)^*E_0, (g_1)^*E_1)), \alpha_2 \in \Omega(M_2, Hom((f_2)^*E_1, (f_2)^*E_2))$ *and* $\alpha_3 \in \Omega(M_1, Hom((g_1)^*E_2, (f_1)^*E_3))$ *and* E_0, E_1, E_2, E_3 *are vector bundles on* L.

Proof. We have the following equalities.

$$\begin{aligned}
(f_1 \circ \pi_1)_!(\pi_1^*\alpha_3\pi_2^*\alpha_2\pi_1^*\alpha_1) &= (-1)^{|\alpha_1||\alpha_2|}(f_1 \circ \pi_1)_!(\pi_1^*\alpha_3\pi_1^*\alpha_1\pi_2^*\alpha_2) \\
&= (-1)^{|\alpha_1||\alpha_2|}(f_1)_!(\pi_1)_!(\pi_1^*(\alpha_3\alpha_1)\pi_2^*(\alpha_2)) \\
&= (-1)^{|\alpha_1||\alpha_2|}(f_1)_!(\alpha_3\alpha_1(\pi_1)_!\pi_2^*(\alpha_2)) \\
&= (-1)^{|\alpha_1||\alpha_2|}(f_1)_!(\alpha_3\alpha_1 g_1^*(f_2)_!(\alpha_2)) \\
&= (-1)^{|\alpha_1||\alpha_2|+|\alpha_1||(f_2)_!\alpha_2|}(f_1)_!(\alpha_3 g_1^*(f_2)_!(\alpha_2)\alpha_1) \\
&= (-1)^{(\dim M_2 - \dim L)|\alpha_1|}(f_1)_!(\alpha_3 g_1^*(f_2)_!(\alpha_2)\alpha_1)
\end{aligned}$$

The second(respectively the third, the fourth) equality follows from Proposition 3.2 (1)(respectively (2), (3)). ☐

Proposition 3.4 ([2]). *Suppose there exist a smooth map* $h : M \to N$ *and submersions* $f : M \to L, g : N \to L$ *such that* $f = g \circ h$, *i.e. we have the following commuting diagram.*

$$\begin{tikzcd} M \arrow[d, "h"'] \arrow[dr, "f"] & \\ N \arrow[r, "g"'] & L \end{tikzcd}$$

If $\dim M > \dim N$*, then we have*

$$(f)_!(h^*\alpha_2 \wedge \beta \wedge h^*\alpha_1) = 0$$

where $\alpha_1 \in \Omega(N, Hom(g^*E_0, F_1))$*,* $\alpha_2 \in \Omega(N, Hom(F_2, g^*E_3))$ *and* $\beta \in \Gamma(M, Hom(h^*F_1, h^*F_2)) = \Omega^0(M, Hom(h^*F_1, h^*F_2))$*.*

Proof. Locally, we may find the local coordinates $(z^1, ..., z^l, x^{l+1}, ...x^m)$ for M, $(z^1, ..., z^l, y^{l+1}, ..., y^n)$ for N and $(z^1, ..., z^l)$ for L such that

$$\begin{aligned} f(z^1, ..., z^l, x^{l+1}, ...x^m) &= (z^1, ..., z^l) \\ g(z^1, ..., z^l, y^{l+1}, ..., y^n) &= (z^1, ..., z^l) \\ h(z^1, ..., z^l, x^{l+1}, ..., x^m) &= (z^1, ..., z^l, f^{l+1}, .., f^n) \end{aligned}$$

where $f^{l+1}, ..., f^n$ are local functions of $(z^1, ..., z^l, x^{l+1}, ..., x^m)$. We deduce that $h^*\alpha_2 \wedge \beta \wedge h^*\alpha_1$ does not contain a term involving $dx^{l+1} \wedge ... \wedge dx^m$ in these coordinates because $m > n$ and $\deg \beta = 0$. Recalling that the integration along the fiber is defined by using local charts, we conclude that $(f)_!(h^*\alpha_2 \wedge \beta \wedge h^*\alpha_1)$ is zero. $\square$

Chapter 4

Higher rank vector bundles in Fukaya category

In this chapter, we construct a de Rham version of Fukaya category. As explained above, the objects of the Fukaya category will be Lagrangian submanifolds equipped with flat vector bundles.

The ground field of our Fukaya category is a *Novikov field* Λ which is defined by

$$\Lambda = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \middle| a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}, \lambda_i < \lambda_{i+1}, \lim_{i \to \infty} \lambda_i = \infty \right\}, \tag{4.1}$$

where T is a formal variable of degree 0.

The Novikov field is endowed with a non-Archimedean norm defined by

$$\Big| \sum_{i=0}^{\infty} a_i T^{\lambda_i} \Big| = \begin{cases} 0 & \text{if } a_i = 0, \ \forall i \\ e^{-\lambda_0} & \text{otherwise} \end{cases} \tag{4.2}$$

Let $a = \sum_{i=0}^{\infty} a_i T^{\lambda_i} \in \Lambda$ be an element with $|a| = 1$, i.e. $\lambda_0 = 0$. Then its logarithm can be obtained by

$$\log a = \mathrm{Log}\ a_0 + \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} \left(\frac{a}{a_0} - 1 \right)^j . \tag{4.3}$$

where Log is a branch of logarithm function on $\mathbb{C}$.

Let us return to the construction of Fukaya category. Let (M, ω) be a compact symplectic manifold. We construct an A_∞-category associated to the symplectic manifold M, which will be denoted by $Fuk(M)$.

The objects of $Fuk(M)$ are weakly unobstructed triples $\mathcal{L} = (L, E, \nabla)$ of a closed, oriented, relatively spin Lagrangian submanifold L, Λ-vector bundle E on L and a flat connection ∇ of E such that all eigenvalues of the holonomy have norm 1 with respect to (4.2).

By its definition, the Fukaya category $Fuk(M)$ decomposes as

$$Fuk(M) = \bigoplus_{\lambda \in \Lambda} Fuk_\lambda(M),$$

where $Fuk_\lambda(M)$ is the full subcategory of $Fuk(M)$ whose objects are weakly-ubobstructed objects $\mathcal{L}$ with $m_0(\mathcal{L}) = \lambda e_{\mathcal{L}}$.

In the next section we will show that our Fukaya category is strict unital. Hence it makes sense to use the notion of weakly unobstructed objects here.

Given two objects $\mathcal{L}_0 = (L_0, E_0, \nabla_0)$ and $\mathcal{L}_1 = (L_1, E_1, \nabla_1)$, the morphism space is described as follows.

If two Lagrangians L_0 and L_1 intersect transversely, then we define the morphism space by

$$CF(\mathcal{L}_0, \mathcal{L}_1) = \bigoplus_{p \in L_0 \cap L_1} \Lambda p \otimes \mathrm{Hom}_\Lambda(E_0|_p, E_1|_p).$$

Else if two Lagrangians L_0 and L_1 are equal, say equal to L, then we define the morphism space by

$$CF(\mathcal{L}_0, \mathcal{L}_1) = \Omega(L, \mathrm{Hom}(E_0, E_1)) \otimes_{\mathbb{R}} \Lambda,$$

where $\Omega(L, \mathrm{Hom}(E_0, E_1))$ means the space of $\mathrm{Hom}(E_0, E_1)$-valued differential forms on L.

We will denote the object (L, E, ∇) by (E, ∇) when the base Lagrangian submanifold is clear from the context.

Remark 4.1. Even though two Lagrangian submanifolds L_0 and L_1 may not be equal nor intersect transversely, this definition makes sense due to the following reason: for any pair L_0, L_1 of Lagrangians, it is possible to perturb L_1 by a Hamiltonian isotopy ϕ so that either L_0 and $\phi(L_1)$ transversely intersect, or L_0 and $\phi(L_1)$ coincide.

Furthermore, it will be noted later in this thesis that the Floer cohomology is independent under Hamiltonian isotopy. See Remark 4.10.

Remark 4.2 (Grading). Our Fukaya category is $\mathbb{Z}/2$-graded. Indeed, every transversal intersection point can be graded by Maslov-Viterbo index (See Subsection 2.2.1), but this index is well-defined only modulo 2.

When two Lagrangian submanifolds coincide, every element of morphism space comes from a vector valued differential form and hence has a natural degree (as a differential form). The degree modulo 2 is taken as its degree as an element of morphism space.

4.1 A-infinity structure

In this section, we define an A_∞-structure of the Fukaya category.

We consider two separated cases

- Special case : The base Lagrangian submanifolds of objects (L_i, E_i, ∇_i) are all equal to a single Lagrangian L.
- General case : The base Lagrangian submanifolds of objects (L_i, E_i, ∇_i) may not be equal, i.e. some of them are not in the same Hamiltonian isotopy class.

For each case, we first describe an idea how to define A_∞-structure maps with the following assumptions:

Assumtion. (1) The moduli spaces of holomorphic curves are compact smooth manifolds with corners.

(2) Lower dimensional stratum of such moduli spaces consist of semistable curves possibly with disc bubbles or sphere bubbles. (Gromov compactness) Furthermore, the smooth structures of lower dimensional stratum are compatible with those of higher dimensional stratum.

(3) The evaluation map ev_0 at the zeroth marked point is submersive.

The study of pseudo-holomorphic curves with boundary on a single Lagrangian submanifold has well developed. (See [1, 17, 19, 20, 16]) So we give a rigorous definition by using the Kuranishi structures on the moduli spaces for the special case. However, we will not give a rigorous definition for the general case. This is mainly due to the lack of the construction of Kuranishi structures on the moduli spaces of general pseudo-holomorphic curves.

4.1.1 Special case

In this subsection, we consider the case of a single Lagrangian L, i.e. we only consider objects (L_i, E_i, ∇_i) such that $L_i = L$ for all $i = 0, ..., k$.

Consider a space $\widetilde{\mathcal{M}}_{k+1}(L)$ that consists of all pairs $((D, \vec{z}), u)$ where

1. D is a two-disc and $\vec{z} = \{z_0, ..., z_k\}$ are marked points on ∂D, which respect the cyclic order.
2. $u : D \to M$ is a pseudo-holomorphic map such that $u(\partial D) \subseteq L$.

Then $\widetilde{\mathcal{M}}_{k+1}(L)$ carries a $Aut(D) \cong PSL(2, \mathbb{R})$-action defined by

$$\varphi \cdot ((D, (z_0, ..., z_k), u) = ((D, (\varphi(z_0), ..., \varphi(z_k))), u \circ \varphi^{-1}).$$

First we quotient $\widetilde{\mathcal{M}}_{k+1}(L)$ by $PSL(2, R)$ and then compactify it to the moduli space $\mathcal{M}_{k+1}(L)$ by adding semistable curves.

As usual we consider the evaluation maps. The j-th evaluation map ev_j is defined by

$$ev_j([(\Sigma, \vec{z}), u]) = u(z_j).$$

In fact, the moduli space $\mathcal{M}_{k+1}(L)$ has a Kuranishi structure. But we will pretend this moduli space is a smooth manifold and the evaluation map at 0-th marked point is submersive as mentioned in Assumption 4.1.

The moduli space $\mathcal{M}_{k+1}(L)$ is a union of some components, each of which is classified by the homotopy class $[u] =: \beta \in \pi_2(M, L)$ of the map u, i.e.

$$\mathcal{M}_{k+1}(L) = \bigsqcup_{\beta \in \pi_2(M,L)} \mathcal{M}_{k+1}(L; \beta). \tag{4.4}$$

Correspondingly, the A_∞-structure map m_k also decomposes into a summation

$$m_k = \sum_{\beta \in \pi_2(M,L)} m_{k,\beta} T^{\omega(\beta)}.$$

First consider the case $\beta = 0$. We define $m_{1,0}$ by

$$m_{1,0}(a) = (-1)^{\deg a+1} \nabla a := (-1)^{\deg a+1} \nabla_1 \circ a + a \circ \nabla_0, \tag{4.5}$$

where ∇ is the flat connection of $\mathrm{Hom}(E_0, E_1)$ induced from ∇_0 and ∇_1.

Furthermore, we define

$$\begin{aligned} m_{2,0}(a, b) &= (-1)^{\deg a} b \circ a, \\ m_{k,0} &= 0 \text{ for } k \neq 1, 2. \end{aligned} \tag{4.6}$$

Now let us consider the case when $\beta \neq 0 \in \pi_2(M, L)$.

We consider parallel transports parametrized by $\mathcal{M} = \mathcal{M}_{k+1}(L, \beta)$. Namely, for each $w = [(D, \vec{z}), u] \in \mathcal{M}_{k+1}(L, \beta)$, let $P_j(w)$ be the parallel transport of $u^* E_j$ from j-th marked point to $(j+1)$-th marked point with respect to

$u^*\nabla_j$. It follows that P_j defines an element of

$$\Gamma(\mathcal{M}, \mathrm{Hom}(ev_j^* E_j, ev_{j+1}^* E_j)).$$

Now we define

$$\eta \in \Omega(\mathcal{M}, \mathrm{Hom}(ev_0^* E_0, ev_0^* E_k))$$

by

$$\eta = P_k \circ ev_k^* x_k \circ P_{k-1} \circ ... \circ P_1 \circ ev_1^* x_1 \circ P_0, \tag{4.7}$$

where $x_j \in CF(\mathcal{L}_{j-1}, \mathcal{L}_j) = \Omega(L, \mathrm{Hom}(E_{j-1}, E_j))$.

Then we define the A_∞-operation by

$$m_{k,\beta}(x_1, ..., x_k) = (-1)^{*_k}(ev_0)_! \eta \in \Omega(L, \mathrm{Hom}(E_0, E_k)), \tag{4.8}$$

where $*_k = 1 + \sum_{i=1}^{k}(k-i)|x_i|$.

Rigorous Definition

We will use the Kuranishi structure on $\mathcal{M}_{k+1}(\beta)$ given in Proposition A.9 and we choose a W_α-parametrized family S_α of multisections given in Proposition A.11 on each Kuranishi chart $(V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha)_{\alpha \in I}$.

Suppose there are $(k+1)$-flat vector bundles $E_0, ..., E_k$ on L

We first define $m_{1,0}$ by

$$m_{1,0}(x) = (-1)^{|x|+1}\nabla x \tag{4.9}$$

for $x \in \Omega(L, \mathrm{Hom}(E_0, E_1))$.

For the remaining A_∞-operations, we define $P_{\alpha,j} \in \Gamma(V_\alpha \times W_\alpha, \mathrm{Hom}(ev_{\alpha,i} * E_i, ev_{\alpha,i+1}^* E_{i+1}))$ by letting $P_{\alpha,j}((\Sigma, \vec{z}), u)$ be the parallel transport $u^* E_j$ form j-th marked point to $(j+1)$-marked point with respect to the connection $u^*\nabla_j$.

Let $x_j \in \Omega(L, \mathrm{Hom}(E_{j-1}, E_j))$ for each $j \in \{1, ..., k\}$.

$$\eta_\alpha = \chi_\alpha(P_{\alpha,k} \circ ev^*_{\alpha,k} x_k \circ ... \circ ev^*_{\alpha,1} x_1 \circ P_{\alpha,0}) \tag{4.10}$$

where χ_α is an auxiliary partition of unity $\chi_\alpha : V_\alpha \to \mathbb{R}$ subordinated to $\{V_\alpha\}_\alpha$ with some compatibility condition [18, Definition 12.10].

Now we are able to define $\sum_{\alpha \in I}(V_\alpha, S_\alpha, ev_{\alpha,0})_! \eta_\alpha$ by using the equation (A.1).

Finally we define our A_∞-operation $m_{k,\beta}$ for $\beta \neq 0$ by

$$m_{k,\beta}(x_1, ..., x_k) = (-1)^{1+\sum_{i=1}^k (k-i)|x_i|} \sum_{\alpha \in I} (V_\alpha, S_\alpha, ev_{\alpha,0})_! \eta_\alpha. \tag{4.11}$$

We prove that our A_∞-operations satisfy the A_∞-equation.

Proposition 4.3. *The operations $\{m_k\}_{k \geq 0}$ satisfy the A_∞-equation (2.7).*

Proof. We will show that the A_∞-equation holds on each zero set $S^{-1}_{\alpha,i,j}(0)$ of the W_α-parametrized multisections.

Because $S^{-1}_{\alpha,i,j}(0)$ is a smooth manifold, it is able to apply theorems for the integration along the fiber constructed in Chapter 3.

We first apply Theorem 3.1 to the case when $M = S^{-1}_{\alpha,i,j}(0)$, $N = L$, $f = ev_{\alpha,0}$ and η_α given in the equation (4.10).

Let us just denote $S^{-1}_{\alpha,i,j}(0)$ by $\mathcal{M}$, $ev_{\alpha,0}$ by ev_0 and η_α by η.

Then we have

$$(ev_0)_! \tilde{\nabla} \eta - \nabla (ev_0)_!(\eta) + (-1)^{n+k+\sum_{i=1}^k |x_i|} (ev_0|_{\partial \mathcal{M}})_! \eta = 0. \tag{4.12}$$

For the first term of the left hand side, we observe from the equations (4.9), (4.11) that

$$(ev_0)_! \tilde{\nabla} \eta = \sum_{i=1}^k (-1)^{\sum_{j=i}^k |x_j| + \sum_{i=1}^k (k-i)|x_i| + k - i} m_{k,\beta}(x_1, ..., m_{1,0}(x_i), ... x_k) \tag{4.13}$$

Similarly for the second them, we observe that

$$-\nabla(ev_0)_!\eta = (-1)^{\sum_{j=1}^k |x_j| + \sum_{i=1}^k (k-i)|x_i| + k+1} m_{1,0} m_{k,\beta}(x_1 ... x_k) \tag{4.14}$$

For the third term, we use Proposition 3.3 and Proposition A.6. Because we chose W_α in such a way that the dimension of W_α is even, we are able to use these two propositions without sign change. Then we have

$$(-1)^{n+k+\sum_{i=1}^k |x_i|}(ev_0|_{\partial\mathcal{M}})_!\eta = \sum_{\substack{\beta_1+\beta_2=\beta \\ 0\le j\le d \\ 1\le i\le d-j+2}} (-1)^{\sum_{j=i}^k |x_j| + \sum_{i=1}^k (k-i)|x_i| + k+i} m_{k-j+1,\beta_1}(x_1, ..., m_{j,\beta_2}(x_i, ..., x_{i+j-1}), ..., x_k). \tag{4.15}$$

By plugging (4.13),(4.14) and (4.15) into the equality (4.12) and dividing the equation by a common factor $(-1)^{\sum_{j=1}^k |x_j| + \sum_{j=1}^k (k-j)|x_j| + k+1}$, we observe the A_∞-equation (2.7) holds. □

Remark 4.4. (cf. [26, Definition 2.2]) Let $\mathcal{L} = (L, E, \nabla)$ be a flat vector bundle. Recall that we constructed a section P_0 of $(ev_0)^*\mathrm{Hom}(E, E)$ on $\mathcal{M}_1(\beta)$ in Subsection 4.1.1, which is defined by using the parallel transport of E.

We remark that the obstruction class $m_0(\mathcal{L})$ is given by

$$m_0(\mathcal{L}) = \sum_{\beta\in\pi_2(M,L)} m_{0,\beta}(\mathcal{L}) T^{\omega(\beta)}$$

where $m_{0,\beta}(\mathcal{L}) \in \Omega(L, \mathrm{Hom}(E, E))$ is defined by

$$(ev_0|_{\mathcal{M}_1(\beta)})_! P_0.$$

4.1.2 General case

Let us return to general case. As mentioned above, we only describe a rough idea how to define A_∞-structure maps in this case.

After applying Hamiltonian perturbations if necessary, there exist Lagrangian submanifolds $L_{(0)}, ..., L_{(m)}$ for some m and integers $d_0, ..., d_m$ such that $L_0 = ... = L_{d_0-1} = L_{(0)}$, $L_{d_0} = ... = L_{d_0+d_1-1} = L_{(1)}$, ... , $L_{d_0+...+d_{m-1}} = ... = L_{d_0+...+d_m-1} = L_{(m)}$, $d_0 + ... + d_m - 1 = k$ and $L_{(i)} \pitchfork L_{(i+1)}$ for $0 \leq i < m$.

There are two possible cases for $L_{(0)}$ and $L_{(m)}$.

Case (a) : $L_{(0)} \pitchfork L_{(m)}$

We choose $p_{(i)} \in L_{(i-1)} \cap L_{(i)}$ for $0 \leq i \leq m$ where $L_{(-1)} := L_{(m)}$.

Let $\widetilde{\mathcal{M}}_{d_0,...,d_m}(L_{(0)}, ..., L_{(m)}; p_{(0)}; p_{(1)}, ..., p_{(m)})$ be the set of all $((D, \vec{z}), u)$ such that

1. D is a two-disc and $\vec{z} = \{z_0, ..., z_k\}$ are marked points on ∂D, which respect the cyclic order.

2. $u : D \to M$ is a pseudo-holomorphic map such that $u(\partial_i D) \subseteq L_i$.

3. $u(z_j) = p_{(i)}$ if $L_{j-1} = L_{(i-1)}$ and $L_j = L_{(i)}$, i.e. $j = 0$ or $j = d_0 + ... + d_i$.

As before, we quotient this space by $PSL(2, \mathbb{R})$ and then compactify it to the moduli space $\mathcal{M}_{d_0,...,d_m}(L_{(0)}, ..., L_{(m)}; p_{(0)}; p_{(1)}, ..., p_{(m)})$. Let us denote this space by $\mathcal{M}$.

We need to define

$$m_k(x_1, ..., x_k)$$

for any choices $x_i \in CF(\mathcal{L}_{i-1}, \mathcal{L}_i)$.

If $L_{(i-1)} = L_{j-1} \pitchfork L_j = L_{(i)}$ for some $1 \leq i \leq m$, then

$$x_j = p_{(i)} \otimes a_{(i)}$$

for some $p_{(i)} \in L_{j-1} \cap L_j$ and $a_{(i)} \in \mathrm{Hom}(E_{j-1}|_{p_{(i)}}, E_j|_{p_{(i)}})$.

If $L_{j-1} = L_{(i)} = L_j$ for some $1 \leq i \leq m$, then

$$x_j \in \Omega(L_{(i)}, \mathrm{Hom}(E_{j-1}, E_j)).$$

Just as before, we follow the construction (4.7) to construct

$$\eta \in \Omega(\mathcal{M}) \otimes_{\mathbb{R}} \mathrm{Hom}(E_0|_{p_{(0)}}, E_1|_{p_{(0)}})$$

where $\mathcal{M} = \mathcal{M}_{d_0,\ldots,d_m}(L_{(0)}, \ldots, L_{(m)}; p_{(0)}; p_{(1)}, \ldots, p_{(m)})$.

We define the A_∞-operation by

$$m_k(x_1, \ldots, x_k) = (-1)^{*_k} \sum_{p_{(0)} \in L_{(0)} \cap L_{(m)}} p_{(0)} \otimes \left(\int_{\mathcal{M}} T^{\omega(u)} \eta \right) \tag{4.16}$$

where $*_k = 1 + \sum_{i=1}^{k} (k-i)|x_i|$.

Case (b) : $L_{(0)} = L_{(m)}$

As in the case (a), choose $p_{(i)} \in L_{(i-1)} \cap L_{(i)}$ for $1 \le i \le m$. Note that we do not have to choose $p_{(0)} \in L_{(0)} \cap L_{(m)}$.

Let $\widetilde{\mathcal{M}}_{d_0,\ldots,d_m}(L_{(0)}, \ldots, L_{(m)}; p_{(1)}, \ldots, p_{(m)})$ be the set of all $((D, \vec{z}), u)$ such that

1. D is a two-disc and $\vec{z} = \{z_0, \ldots, z_k\}$ are marked points on ∂D, which respect the cyclic order.
2. $u : D \to M$ is a pseudo-holomorphic map such that $u(\partial_i D) \subseteq L_i$.
3. $u(z_j) = p_{(i)}$ if $L_{j-1} = L_{(i-1)}$ and $L_j = L_{(i)}$, i.e. $j = d_0 + \ldots + d_i$.

As before, we quotient this space by $PSL(2, \mathbb{R})$ and then compactify it to the moduli space $\mathcal{M}_{d_0,\ldots,d_m}(L_{(0)}, \ldots, L_{(m)}; p_{(1)}, \ldots, p_{(m)})$. Let us denote this space by $\mathcal{M}$.

We follow the construction (4.7) to define

$$\eta \in \Omega(\mathcal{M}, \mathrm{Hom}(ev_0^* E_0, ev_0^* E_k))$$

where $\mathcal{M} = \mathcal{M}_{d_0,\ldots,d_m}(L_{(0)}, \ldots, L_{(m)}; p_{(1)}, \ldots, p_{(m)})$.

We define the A_∞-operation by

$$m_k(x_1, \ldots, x_k) = (-1)^{*_k} (ev_0)_! (\eta T^{\omega(u)}) \in \Omega(L_{(0)}, \mathrm{Hom}(E_0, E_k)). \tag{4.17}$$

where $*_k = 1 + \sum_{i=1}^{k}(k-i)|x_i|$.

4.2 Properties of Fukaya category

Let us state some useful properties of our Fukaya category.

Lemma 4.5. *Let L be a Lagrangian submanifold of M. Let $k \geq 1$ be an integer and $\mathcal{L}_j = (L, E_j, \nabla_j)$ for $j \in \{1, ..., k\}$.*

Let us choose

$$x_j \in CF(\mathcal{L}_{j-1}, \mathcal{L}_j) = \Omega(L, Hom(E_{j-1}, E_j))$$

for each $j \in \{1, ..., k\}$. Suppose that $\deg(x_l) = 0$ *for some $l \in \{1, ..., k\}$, i.e. $x_l \in \Omega^0(L, Hom(E_{j-1}.E_j)) = \Gamma(L, Hom(E_{j-1}, E_j))$.*

Then we have

$$m_{k,\beta}(x_1, ..., x_l, ..., x_k) = 0$$

for $k \neq 1, 2$ or $\beta \neq 0 \in \pi_2(M, L)$.

Proof. Consider the following commutative diagram

$$\begin{tikzcd} \mathcal{M}_{k+1}(\beta) \arrow[d, "forg_l"'] \arrow[dr, "ev_0"] & \\ \mathcal{M}_k(\beta) \arrow[r, "ev_0"'] & L \end{tikzcd} \tag{4.18}$$

where $forg_l$ is the map forgetting the l-th marked point.

By compatibility of continuous family of multisections(See Definition A.10), there is a corresponding diagram

$$\begin{tikzcd} S^{-1}_{\alpha,i,j}(0) \arrow[d, "forg_l"'] \arrow[dr, "ev_{\alpha,0}"] & \\ S^{-1}_{\alpha',i,j}(0) \arrow[r, "ev_{\beta(\alpha),0}"'] & L \end{tikzcd} \quad . \tag{4.19}$$

We apply Proposition 3.4 to this case in such a way that

$$
\begin{aligned}
& f = ev_{\alpha,0},\ g = ev_{\alpha',0},\ h = forg_l, \\
& \alpha_1 = (ev_{\alpha',i-1})^* x_{i-1} \circ \ldots \circ (ev_{\alpha'),1})^* x_1 \circ P_{\alpha',0}, \\
& \alpha_2 = P_{\alpha',k} \circ (ev_{\alpha',k})^* x_k \circ \ldots \circ (ev_{\alpha',i+1})^* x_{i+1}, \\
& \beta = P_{\alpha,i} \circ ev^*_{\alpha,i} x_i \circ P_{\alpha,i-1}.
\end{aligned}
$$

Then we have

$$(V_\alpha, S_\alpha, ev_{\alpha,0})_!(h^*\alpha_2 \wedge \beta \wedge h^*\alpha_1) = 0.$$

This implies that $m_k(x_1, ...x_l, ..., x_k) = 0$ as desired. □

Lemma 4.6. *Let L_0, L_1 be Lagrangian submanifolds that transversely intersect each other. Let $\mathcal{L}_0 = (L_0, E_0, \nabla_0)$, $\mathcal{L}_1 = (L_0, E_1, \nabla_1)$ be flat vector bundles on L_0 and $\mathcal{L} = (L_1, F, \nabla_2)$ be a flat vector bundle on L_1.*

Consider the following elements in morphism spaces

$$
\begin{aligned}
x &\in CF(\mathcal{L}_0, \mathcal{L}_1) = \Omega^0(L_0, Hom(E_0, E_1)), \\
y = p \otimes a &\in CF(\mathcal{L}_1, \mathcal{L}) \text{ where } p \in L_0 \cap L_1 \text{ and } a \in Hom_\Lambda(E_1|_p, F|_p), \\
z = q \otimes b &\in CF(\mathcal{L}, \mathcal{L}_0) \text{ where } q \in L_1 \cap L_0 \text{ and } b \in Hom_\Lambda(F|_q, E_0|_q).
\end{aligned}
$$

Then we have

$$m_2(x, y) = p \otimes (a \circ x|_p).$$

$$m_2(z, x) = (-1)^{|z|} q \otimes (x|_q \circ b).$$

The proof of Lemma 4.6 can be done just as above with the following assumption: There exist continuous families of multisections on the Kuranishi strucutres of $\mathcal{M}_{k_0,k_1}(L_0, L_1; p : p)$ and $\mathcal{M}_{k_0,k_1}(L_1, L_0; q; q)$ which are compatible with forgetful map. This kind of construction has not been developed yet. However it is expected that the construction of such families of multisections can be done as in [16]. □

Assuming Lemma 4.6, we prove

Proposition 4.7 (Strict Unitality)**.** *The Fukaya category is strict unital.*

Proof. Let $\mathcal{L} = (L, E, \nabla)$ be an object of our Fukaya category.

We claim that the identity section $id \in \Gamma(L, \mathrm{Hom}(E, E)) \subset \Omega(L, \mathrm{Hom}(E, E)) = CF(\mathcal{L}, \mathcal{L})$ is a strict unit $e_{\mathcal{L}}$ for $\mathcal{L}$. Indeed, Lemma 4.5 implies that only $m_{1,0}$ contributes to $m_1(id)$. Thus we have

$$m_1(id) = m_{1,0}(id) = -\nabla id = 0.$$

Again from Lemma 4.5 and Lemma 4.6 we deduce that

$$m_2(id, a) = m_{2,0}(id, a) = (-1)^{\deg id} a \wedge id = a,$$

for any $a \in CF(\mathcal{L}, \mathcal{L}')$ and

$$m_2(a, id) = m_{2,0}(a, id) = (-1)^{\deg a} id \wedge a = (-1)^{\deg a} a.$$

for any $a \in CF(\mathcal{L}', \mathcal{L})$. □

Remark 4.8. The proof of Proposition 4.7 uses both Lemma 4.5 and Lemma 4.6. Because the proof of Lemma 4.6 has not been completed yet, the Proposition 4.7 can be proven only when a single Lagrangian submanifold is involved so that only Lemma 4.5 is necessary.

We may consider a Fukaya category associated to a single Lagrangian, that is, an A_∞-category whose objects are flat vector bundles on a single Lagrangian submanifold and whose morphisms and A_∞-structure maps are described as above. The Proposition 4.7 can be applied to such a case without any assumption.

Proposition 4.9. *Two gauge equivalent vector bundles are isomorphic in the Fukaya category.*

Proof. Suppose $\mathcal{L}_0 = (L, E_0, \nabla_0)$ and $\mathcal{L}_1 = (L, E_1, \nabla_1)$ are gauge equivalent, i.e. there are bundle isomorphisms $\phi : E_0 \to E_1$ and $\psi = \phi^{-1} : E_1 \to E_0$ that intertwine the connections, i.e. $\phi \circ \nabla_0 = \nabla_1 \circ \phi$ and $\psi \circ \nabla_1 = \nabla_0 \circ \psi$.

In terms of A_∞-structures, Lemma 4.5 implies that $\phi \in CF(\mathcal{L}_1, \mathcal{L}_2)$ satisfies

$$m_1(\phi) = m_{1,0}(\phi) = -\nabla(\phi) = -\nabla_1 \circ \phi + \phi \circ \nabla_0 = 0.$$

Similarly, $\psi \in CF(\mathcal{L}_2, \mathcal{L}_1)$ satisfies

$$m_1(\psi) = 0.$$

We use Lemma 4.5 again to have

$$m_2(\phi, \psi) = m_{2,0}(\phi, \psi) = \psi \circ \phi = id,$$

$$m_2(\psi, \phi) = m_{2,0}(\psi, \phi) = \phi \circ \psi = id.$$

This proves that $\mathcal{L}_0$ and $\mathcal{L}_1$ are isomorphic in Fukaya category. □

Remark 4.10 (Independence under Hamiltonian isotopy, cf. Remark 2.10). As mentioned in Remark 4.1, we show that the Floer cohomology does not change even after perturbing one Lagrangian submanifold via a Hamiltonian isotopy.

In [19, Theorem 4.15], the following fact was shown: Let L_0 and L_1 be two Lagrangian submanifolds such that L_0 and L_1 intersect transversely and $m_0(L_0) = \lambda e_{L_0}$ and $m_0(L_1) = \lambda e_{L_1}$. If $\{\phi^s\}_{s\in[0,1]}$ is a Hamiltonian isotopy of M with $\phi^0 = id_M$ and $\phi^1 =: \phi$ such that L_0 and $\phi(L_1)$ intersect transversely, then there is an isomorphism

$$HF(L_0, L_1; \Lambda) \cong HF(L_0, \phi(L_1); \Lambda),$$

where $HF(L_0, L_1; \Lambda)$ is the cohomology of the cochain complex $CF(L_0, L_1; \Lambda)$ with respect to m_1.

Here, for each $i = 0, 1$, L_i can be regarded as an object (L_i, E_i, ∇_i) where (E_i, ∇_i) is the trivial flat line bundle on L_i. In this aspect, the theorem can be generalized to the case of general objects $\mathcal{L}_0 = (L_0, E_0, \nabla_0)$ and $\mathcal{L}_1 = (L_1, E_1, \nabla_1)$ by using the same idea.

Regarding the case $L_0 = L = L_1$, we consider $\mathcal{L}_0 = (L, E, \nabla)$ and $\mathcal{L}_1 =$

$(\phi(L), (\phi^{-1})^*E, (\phi^{-1})^*\nabla)$ there exist chain maps

$$f : CF(\mathcal{L}_0, \mathcal{L}_0) \to CF(\mathcal{L}_0, \mathcal{L}_1) \text{ and } g : CF(\mathcal{L}_0, \mathcal{L}_1) \to CF(\mathcal{L}_0, \mathcal{L}_0),$$

such that $g \circ f$ is chain homotopic to the identity. As a conclusion, the Floer cohomology is invariant under Hamiltonian isotopy even in this case.

The proof of [18, Proposition 8.24] can be applied to our situation again. The only difference is that here the A_∞-operation $m_{1,0}$ is not just the exterior derivative, but the flat connection. This reflects the difference between the Stokes theorem 3.1 and that in [18].

Chapter 5

Equivalence of vector bundles and twisted complexes

5.1 Classificiation of flat vector bundles

One of the most important facts used in this section is that every representation of finitely generated abelian group is simultaneously triangularizable.

Lemma 5.1. *Let $\mathbb{K}$ be an algebraically closed field. Let G be a finitely generated abelian group and let $\rho : G \to GL(r, \mathbb{K})$ be any representation. Then there exists $P \in GL(r, \mathbb{K})$ such that $P^{-1}\rho(a)P$ is lower-triangular for all $a \in G$.*

The proof of this lemma is postponed to Appendix B.

From now on, we consider the case that the ground field is the Novikov field Λ. It is proven in [17] that the Novikov field is algebraically closed and hence it is possible to apply Lemma 5.1 to our case.

Let $A_1, \ldots, A_k$ be k-commuting lower-triangular matrices in $GL(r, \Lambda)$, which are factorized as

$$A_i = \lambda_i N_i, \tag{5.1}$$

where N_i is the unipotent matrix defined by $N_i = (\lambda_i)^{-1} A_i$. Then we easily find their matrix logarithms.

A logarithm of $\lambda_i I$ is given by a matrix $(\log \lambda_i)I$ where log is as in the equation (4.3).

Furthermore, a logarithm of N_i is obtained by

$$\sum_{j=1}^{r}(-1)^{j-1}\frac{(N_i - I)^j}{j}. \tag{5.2}$$

Finally we define $\log A_i = (\log \lambda_i)I + \sum_{j=1}^{r}(-1)^{j-1}\frac{(N_i-I)^j}{j}$. Because $(\log \lambda_i)I$ are scalar multiples of the identity matrices and $\sum_{j=1}^{r}(-1)^{j-1}\frac{(N_i-I)^j}{j}$ are polynomials of A_i, we deduce that $\log A_1, ..., \log A_k$ commute with each others.

Lemma 5.2. *There exist logarithms* $\log A_1, ..., \log A_k$ *of matrices* $A_1, ..., A_k$, *which commute with each others.* □

Using the lemma above, we classify the isomorphism classes (as vector bundles) of indecomposable flat vector bundles on a given compact manifold whose fundamental group is abelian.

Lemma 5.3. *Let L be a compact manifold with abelian fundamental group. Then there is a bijection between the isomorphism classes as vector bundles of indecomposable flat vector bundles of a fixed rank on L and the torsion part of the fundamental group of L.*

Proof. By Lemma 5.1, we may assume that the holonomy representation $\rho : \pi_1(L) \cong \bigoplus_{i=1}^{k}\mathbb{Z} \oplus \bigoplus_{j=1}^{l}\mathbb{Z}/a_i\mathbb{Z} \to GL(r, \Lambda)$ is lower-triangular.

Let $e_i \in \pi_1(L)$ be a generator of the summand $\mathbb{Z}$ for each $i = 1, ..., k$ and let $f_j \in \pi_1(L)$ be a generator of the summand $\mathbb{Z}/a_j\mathbb{Z}$ for each $j = 1, ..., l$

Because the representation ρ is indecomposable and lower-triangular by assumption, we observe that $\rho(f_j) = e^{\frac{2\pi b_j\sqrt{-1}}{a_j}} I$ for some $b_j \in \{0, ..., a_j - 1\}$.

We construct a $[0, 1]$-family of flat vector bundles: for each $t \in [0, 1]$, consider a flat vector bundle (E_t, ∇_t) corresponding to a holonomy representation ρ_t such that

$$\rho_t(e_i) = \exp(t\log(\rho(e_i))),\ \rho_t(f_j) = \rho(f_j).$$

Here the logarithms are defined as in Lemma 5.2.

We observe that $\rho_0(e_i) = 1$ and $\rho_0(f_j) = \rho(f_j) = e^{\frac{2\pi b_j \sqrt{-1}}{a_j}} I$. From this observation, we deduce that the number of component of conjugacy classes of representation of $\pi_1(L)$ is given by $\prod_{j=1}^{k} a_j$. The assertion follows from the fact that there is a bijection between the gauge equivalence classes of flat vector bundles on L and the conjugacy classes of the representation $\pi_1(L) \to GL(r, \Lambda)$. □

5.2 Equivalence

Suppose L is a Lagrangian submanifold whose fundamental group is abelian. We will prove that every weakly unobstructed flat vector bundle on L is isomorphic to a twisted complex of flat line bundles in Fukaya category.

Because every flat vector bundle is given as a direct sum of indecomposable flat vector bundles and the A_∞-category of twisted complexes is closed under direct sum, we only need to consider indecomposable flat vector bundles.

Let $\mathcal{L} = (E, \nabla)$ be an indecomposable flat vector bundle of rank r whose holonomy representation is conjugate to a lower-triangular representation $\rho : \pi_1(L) \to GL(r, \Lambda)$.

Suppose $\rho(e_i) = A_i = \lambda_i N_i$ and $\rho(f_j) = e^{\frac{2\pi b_j \sqrt{-1}}{a_j}} I$ using the notation given in the previous section.

Let (E', ∇') be a flat line bundle on L whose holonomy representation $hol_{\nabla'} : \pi_1(L) \to GL(1, \Lambda)$ is given by

$$hol_{\nabla'}(e_i) = \lambda_i \text{ and } hol_{\nabla'}(f_j) = e^{\frac{2\pi b_j \sqrt{-1}}{a_j}}.$$

We deduce from the proof of Lemma 5.3 that two vector bundles E and $\bigoplus_{j=1}^{r} E'$ are isomorphic as vector bundles. We identify these two by choosing an isomorphism as vector bundles.

Now we construct a twisted complex, which is isomorphic to $\mathcal{L}$.

First we consider r-flat line bundles by putting $\mathcal{L}_j = (L, E_j, \nabla_j) := (L, E', \nabla')$ for $j = 1, ..., r$.

For a Maurer-Cartan element, we define a matrix of morphisms $\delta = (\delta_{ij} \in CF^1(\mathcal{L}_j, \mathcal{L}_i))$ by putting

$$\delta = \sum_{i=1}^{n} (\log N_j) e^j,$$

where $e^j \in \Omega^1(L) \cong \Omega^1(L, \mathrm{Hom}(\mathcal{L}', \mathcal{L}'))$ are closed one forms chosen in such a way that

$$\langle e_i, e^j \rangle = \delta_{ij}.$$

This is possible because $H^1_{\mathrm{dR}}(L) \cong \mathrm{Hom}(\pi_1(L), \mathbb{R})$ if $\pi_1(L)$ is abelian.

We will prove that $\mathcal{L}$ and $\tilde{\mathcal{L}} := (\oplus_{j=1}^{r} \mathcal{L}_j, \delta)$ are equivalent in the A_∞-category of twisted complexes over Fukaya category.

Before proving the equivalence, we should check if $\tilde{\mathcal{L}}$ is a twisted complex of some internal curvature or not. The following theorem tells us that $\tilde{\mathcal{L}}$ becomes a twisted complex of internal curvature λ if and only if $m_0(\mathcal{L}) = \lambda e_{\mathcal{L}}$. This shows that the obstructions for two objects are equivalent.

Theorem 5.4 (Obstruction)**.** *The object $\mathcal{L}$ is weakly unobstructed with $m_0(\mathcal{L}) = \lambda e_{\mathcal{L}}$ if and only if $\tilde{\mathcal{L}}$ is a twisted complex of internal curvature λ.*

Before proving this theorem, we recall a theorem which is analogous to the divisor equation for Gromov-Witten invariant.

Theorem 5.5 (K. Fukaya, [16] Lemma 13.1)**.** *Suppose L is a closed Lagrangian submanifold and $\mathcal{L} = (L, E, \nabla)$ is a flat line bundle.*

Let $b \in CF^1(\mathcal{L}, \mathcal{L})$. Then we have

$$\sum_{m_0 + ... + m_d = m} m_{d+m,\beta}(b^{\otimes m_0}, x_1, , ..., x_d, b^{\otimes m_d}) = \frac{1}{m!} \langle \partial\beta, b \rangle^m m_{d,\beta}(x_1, ..., x_d).$$

for any $x_1, ... x_d \in CF(\mathcal{L}, \mathcal{L})$.

Remark 5.6. Theorem 5.5 continues to hold for our A_∞-operations since we use compatible families of multisections to define our A_∞-operations. (See Proposition A.11.)

Corollary 5.7. *Let* $b^1, ..., b^l \in CF^1(\mathcal{L}, \mathcal{L})$. *Let* $d_1, ..., d_l$ *be nonnegative integers such that* $d_1 + ... + d_l = d$. *Let us define*

$$[d_1, ..., d_l] := \{(k_1, ..., k_d) \in \mathbb{Z}^d_{\geq 0} | \#\{j | k_j = i\} = d_i \text{ for } 1 \leq i \leq l\}. \tag{5.3}$$

Then we have

$$\sum_{(k_1,...,k_d) \in [d_1,...,d_l]} m_{d,\beta}(b^{k_1}, ..., b^{k_d}) = \frac{\langle \partial\beta, b^1 \rangle^{d_1}}{(d_1)!} ... \frac{\langle \partial\beta, b^l \rangle^{d_l}}{(d_l)!} m_{0,\beta}(\mathcal{L}). \tag{5.4}$$

Proof of Theorem 5.4. We use the notation introduced in the proof of Lemma 5.3.

We just assume that the holonomy representation hol_∇ with respect to the flat connection ∇ is equal to ρ by choosing a suitable trivialization of the vector bundle E.

We first observe

$$\begin{aligned} hol_\nabla(e_i) &= \rho(e_i) \\ &= \lambda_i N_i \\ &= N_i hol_{\nabla'}(e_i). \end{aligned} \tag{5.5}$$

for $i = 1, ..., k$.

Further, we also have

$$hol_\nabla(f_j) = e^{\frac{2\pi b_j \sqrt{-1}}{a_j}} I = hol_{\nabla'}(f_j) I. \tag{5.6}$$

for $j = 1, ..., l$.

Let $\beta \in \pi_2(M, L)$. We denote the Maurer-Cartan equation $\sum_{d \geq 0} m_{d,\beta}(\delta, ..., \delta)$ by $m_\beta(e^\delta)$. We compute $m_\beta(e^\delta)$ and then relate it with $m_{0,\beta}(\mathcal{L})$.

Indeed we have

$$\begin{aligned}
m_\beta(e^\delta) &= \sum_{d\geq 0}\sum_{k_1,\dots,k_d} m_{d,\beta}(\log N_{k_1}e^{k_1}, \log N_{k_2}e^{k_2}, \dots, \log N_{k_d}e^{k_d}) \\
&= \sum_{d\geq 0}\sum_{d_1+\dots+d_n=d}\sum_{(k_1,\dots,k_d)\in[d_1,\dots,d_n]} (\log N_{k_d}\dots\log N_{k_1}) m_{d,\beta}(e^{k_1},\dots,e^{k_d}) \\
&= \sum_{d\geq 0}\sum_{d_1+\dots+d_n=d} ((\log N_1)^{d_1}\dots(\log N_k)^{d_n})\frac{\langle\partial\beta, e^1\rangle^{d_1}}{(d_1)!}\dots\frac{\langle\partial\beta, e^n\rangle^{d_n}}{(d_n)!} m_{0,\beta}(\mathcal{L}') \\
&= \exp\left(\sum_{i=1}^k \log N_i\langle\partial\beta, e^i\rangle\right) m_{0,\beta}(\mathcal{L}') \\
&= \exp\left(\sum_{i=1}^k \log N_i\langle\partial\beta, e^i\rangle\right) (ev_0|_{\mathcal{M}_1(\beta)})_! hol_{\nabla'}(\partial\beta) \\
&= (ev_0|_{\mathcal{M}_1(\beta)})_!\left(\exp\left(\sum_{i=1}^k \log N_i\langle\partial\beta, e^i\rangle\right) hol_{\nabla'}(\partial\beta)\right) \\
&= (ev_0|_{\mathcal{M}_1(\beta)})_! hol_{\nabla}(\partial\beta) \\
&= m_{0,\beta}(\mathcal{L}).
\end{aligned}$$

We use Corollary 5.7 for the third equality. Also, the seventh equality follows from the equation (5.5) and (5.6).

The assertion follows from this equality and the fact that $m_0(\mathcal{L}) = \sum_{\beta\in\pi_2(M,L)} m_{0,\beta}(\mathcal{L})T^{\omega(\beta)}$ and $m(e^\delta) = \sum_{\beta\in\pi_2(M.L)} m_\beta(e^\delta)T^{\omega(\beta)}$. $\square$

Now we verify that $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are equivalent when the conditions of Theorem 5.4 are satisfied.

Theorem 5.8 (Equivalence)**.** *Two objects $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are isomorphic in the twisted Fukaya category $Tw^\lambda Fuk(M)$.*

Proof. We first show that the connection ∇ on E is gauge-equivalent to

$$\begin{aligned}\tilde{\nabla} &= \bigoplus_{j=1}^{r} \nabla_j - \sum_{i=1}^{k} \log N_i e^i \\ &= \bigoplus_{j=1}^{r} \nabla' - \sum_{i=1}^{k} \log N_i e^i.\end{aligned}$$

Indeed, we have

$$\begin{aligned} hol_{\tilde{\nabla}}(e_i) &= hol_{\oplus_{j=1}^{r}\nabla'}(e_i)\exp(\log N_i) \\ &= \lambda_i N_i \\ &= A_i. \end{aligned}$$

and

$$\begin{aligned} hol_{\tilde{\nabla}}(f_j) &= hol_{\oplus_{j=1}^{r}\nabla'}(f_j) \\ &= e^{\frac{2\pi b_j\sqrt{-1}}{a_j}} I. \end{aligned}$$

By Proposition 4.9, we may assume $\nabla = \tilde{\nabla}$.

Recall that two vector bundles E and $\bigoplus_{j=1}^{r} E'$ are isomorphic and so these two can be identified. Let $e_{12} \in \Gamma(L, \mathrm{Hom}(E, \oplus_{j=1}^{r} E_j)) \subset CF(\mathcal{L}, \tilde{\mathcal{L}})$ be the element corresponding to the identity section of $\mathrm{Hom}(E, \oplus_{j=1}^{r} E_j)$ and conversely let $e_{21} \in CF(\tilde{\mathcal{L}}, \mathcal{L})$ be the element corresponding to the identity section in of $\mathrm{Hom}(\oplus_{j=1}^{r} E_j, E)$.

Now we check that both e_{12} and e_{21} are m_1^{tw} closed. Indeed, we have

$$m_1^{tw}(e_{12}) = m(e_{12}, e^{\delta}) = m_1(e_{12}) + m_2(e_{12}, \delta) + \ldots.$$

But by Lemma 4.5, we have $m_d(e_{12}, \underbrace{\delta, ..., \delta}_{d-1}) = 0$ for $d \geq 3$. Also we have

$$\begin{aligned} m_1(e_{12}) = m_{1,0}(e_{12}) &= (-1)^{|e_{12}|+1}\left((\bigoplus_{i=1}^{r} \nabla_i) \circ e_{12} - e_{12} \circ \nabla \right) \\ &= -\left((\bigoplus_{i=1}^{r} \nabla_i) \circ e_{12} - e_{12} \circ (\bigoplus_{j=1}^{r} \nabla_j - \sum_{i=1}^{k} \log N_i e^i) \right) \\ &= -\sum_{i=1}^{k} \log N_i e^i \end{aligned}$$

and

$$m_2(e_{12}, \delta) = m_{2,0}(e_{12}, \delta) = (-1)^{|e_{12}|}\delta = \sum_{i=1}^{k} \log N_i e^i.$$

From these observations, it follows that $m_1^{tw}(e_{12}) = 0$.

The proof that e_{21} is m_1^{tw} closed is similar.

Again Lemma 4.5 implies

$$m_2^{tw}(e_{12}, e_{21}) = m_{2,0}(e_{12}, e_{21}) = e_{\mathcal{L}}$$

and

$$m_2^{tw}(e_{21}, e_{12}) = m_{2,0}(e_{21}, e_{12}) = e_{\tilde{\mathcal{L}}}.$$

The assertion follows. □

As a consequence of Theorem 5.8, we have the following corollary.

Corollary 5.9. *Let L be a Lagrangian submanifold which is diffeomorphic to n-torus T^n. Then every flat vector bundle on L is isomorphic to a twisted complex of flat line bundles in Fukaya category.* □

5.2.1 Generalization

The idea used in the proof of Theorem 5.8 can be generalized to the following Theorem 6.10.

Before we state and prove the theorem, we first introduce a subcategory of *Fuk*(M), which will be denoted by *Fuk*$^*(M)$. Indeed, we define *Fuk*$^*(M)$ as the full subcategory of *Fuk*(M) whose objects are weakly-unobstructed flat line bundles on Lagrangian submanifolds rather than flat vector bundles of arbitrary rank.

Consequently, we get a full subcategory Tw^λ*Fuk*$^*(M)$ of Tw^λ*Fuk*(M) of twisted complexes over *Fuk*$^*(M)$. It follows from Theorem 2.19 and 2.20 that Tw^λ*Fuk*$^*(M)$ is closed under extension and so is its homology category.

Theorem 5.10. *Let (E, ∇) be a flat vector bundle on L associated to a holonomy representation is $\rho : \pi_1(L) \to GL(r, \Lambda)$.*

Suppose that the representation is lower-triangularizable, i.e. there exists $P \in GL(r, \Lambda)$ such that $P^{-1}\rho(a)P$ is lower-triangular for all $a \in \pi_1(L)$ and that the object $\mathcal{L} = (E, \nabla)$ is weakly-ubobstructed with $m_0(\mathcal{L}) = \lambda e_{\mathcal{L}}$.

The object (E, ∇) is quasi-isomorphic to a twisted complex of flat line bundles with internal curvature λ in Tw^λFuk(M).

Proof. As previously, we may assume that the flat vector bundle $\mathcal{L} = (E, \nabla)$ is indecomposable.

We use an induction on the rank r of the bundle E.

There is nothing to prove when $r = 1$. Suppose we have shown the assertion $r = k-1$ for some $k \geq 2$. Let $\mathcal{L} = (E, \nabla)$ be a flat vector bundle of rank k on L such that its holonomy representation ρ satisfies the condition in the theorem.

The subspace $\{0\}^{k-1} \times \Lambda \subset \Lambda^k$ is preserved by the image of ρ and hence we get a sub-representation $\rho_1 : \pi_1(L) \to GL(1, \Lambda) = \Lambda^*$ of ρ. Let $\mathcal{L}_1 := (E_1, \nabla_1)$ be the corresponding flat line bundle on L.

The quotient bundle $E_2 = E/E_1$ naturally has a flat connection ∇_2 induced from the flat connection ∇ on E. Let us denote the object (E_2, ∇_2) by $\mathcal{L}_2$.

Here note that both the inclusion map $i : \mathcal{L}_1 \to \mathcal{L}$ and the quotient map $q : \mathcal{L} \to \mathcal{L}_2$ can be regarded as morphisms in the Fukaya category and furthermore m_1-closed. Indeed, Lemma 4.5 implies that

$$m_1(i) = m_{1,0}(i) = -\nabla(i) = 0$$

and similarly that

$$m_1(q) = m_{1,0}(q) = -\nabla(q) = 0.$$

On the other hand, by induction hypothesis, there exists a quasi-isomorphism

$$g \in \mathrm{Hom}^0_{Tw^\lambda \mathit{Fuk}(M)}(\mathcal{L}_2, Y)$$

for some twisted complex Y.

It follows from Lemma 4.5 again that

$$(-1)^{\deg q} m_2^{tw}(q, g) = (-1)^{\deg q + 1} m_{2,0}(q, g) = g \circ q.$$

Hence the morphism $g \circ q \in \mathrm{Hom}_{Tw^\lambda \mathit{Fuk}(M)}(\mathcal{L}, Y)$ is m_1^{tw}-closed

Now we get an exact sequence in $H(Tw^\lambda \mathit{Fuk}(M))$

$$0 \longrightarrow \mathcal{L}_1 \xrightarrow{i} \mathcal{L} \xrightarrow{g \circ q} Y \longrightarrow 0$$

Since the full subcategory $H(Tw^\lambda \mathit{Fuk}^*(M))$ is closed under extension and both $\mathcal{L}_1$ and Y are contained in that full subcategory, we deduce that $\mathcal{L}$ is quasi-isomorphic to an object in the full subcategory $H(Tw^\lambda \mathit{Fuk}^*(M))$. This proves the assertion.

□

5.3 Application

Recall that the objects of $\mathit{Fuk}(M)$ are pairs of a Lagrangian submanifold of M and a flat Λ-vector bundle on it such that every eigenvalue of the holonomy representation has norm 1.

For an A_∞-category $\mathcal{A}$, one may construct its derived category $D\mathcal{A}$ by embedding $\mathcal{A}$ into the category of A_∞-modules over $\mathcal{A}$ and the taking the localization with respect to the quasi-isomorphism. It is also well-known that the derived category $D\mathcal{A}$ is realized by $H^0 Tw\mathcal{A}$. [33]

In Luis Haug's paper [22], two Fukaya categories are introduced and those two are essentially the same as $Fuk(M)$ and $Fuk^*(M)$ above. The flat Λ-vector bundles are interpreted as Λ-local systems in [22]. It was proven in [**?**, Corollary 7.4 and Corollary 7.5] that

Theorem 5.11 (Haug)**.** *Every object of $D^\pi Fuk(T^2)$ is a direct sum of objects of $Fuk(T^2)$. Furthermore, $DFuk(T^2)$ is already split-closed.*

Here D^π means the split-closure of the given derived category.

More precisely, it was shown that every object of $D^\pi Fuk^\pi(T^2)$ is a direct sum of objects of the form (L, E, ∇) where L is a Lagrangian submanifold of T^2 and (E, ∇) is an indecomposable flat vector bundle on L.

Let L be a closed, oriented Lagrangian submanifold of T^2. Then L is diffeomorphic to 1-torus T^1. We introduce an (angle) coordinate θ on L such that its differential $d\theta$ is a volume form on L with $\int_L d\theta = 1$.

Let $\mathcal{E}^r_\lambda = (E, \nabla^r_\lambda)$ denote the indecomposable flat vector bundle of rank r on L whose holonomy representation has λ as the only eigenvalue. Note that such an indecomposable flat vector bundle is unique up to gauge equivalence. This fact follows from the Jordan decomposition theorem in linear algebra.

Indeed the flat vector bundle $\mathcal{E}^r_\lambda = (E, \nabla^r_\lambda)$ is gauge-equivalent to a flat vector bundle whose holonomy representation $\rho : \pi_1(L) \cong \mathbb{Z} \to GL(r, \Lambda)$ is given by

$$\rho(1) = \begin{pmatrix} \lambda & 0 & ... & 0 & 0 \\ 1 & \lambda & ... & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & \lambda & 0 \\ 0 & 0 & ... & 1 & \lambda \end{pmatrix} = \lambda \begin{pmatrix} 1 & 0 & ... & 0 & 0 \\ \frac{1}{\lambda} & 1 & ... & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & 1 & 0 \\ 0 & 0 & ... & \frac{1}{\lambda} & 1 \end{pmatrix},$$

i.e.

$$(i, j)\text{-entry of } \rho(1) = \begin{cases} \lambda \text{ if } i = j \\ 1 \text{ if } i = j + 1 \\ 0 \text{ otherwise} \end{cases}.$$

Therefore we observe the corresponding connection is given by

$$\nabla = d - \begin{pmatrix} \log\lambda & 0 & ... & 0 & 0 \\ 0 & \log\lambda & ... & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & \log\lambda & 0 \\ 0 & 0 & ... & 0 & \log\lambda \end{pmatrix} d\theta - \sum_{k=1}^{r} \frac{(-1)^{k-1}}{k} \begin{pmatrix} 0 & 0 & ... & 0 & 0 \\ \frac{1}{\lambda} & 0 & ... & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & 0 & 0 \\ 0 & 0 & ... & \frac{1}{\lambda} & 0 \end{pmatrix}^{k} d\theta.$$

The proof of Theorem 5.8 implies that this flat vector bundle is isomorphic to the twisted complex $(\oplus_{i=1}^{r}\mathcal{E}_{\lambda}^{1}, (\delta_{ij}))$ where

$$\delta_{ij} = \begin{cases} \frac{(-1)^{i-j-1}}{(i-j)\lambda^{i-j}} d\theta \text{ if } j < i \\ 0 \text{ otherwise.} \end{cases}$$

This shows that every object of $DFuk(T^2)$ is a direct sum of twisted complexes of $Fuk^*(T^2)$. As a consequence, we have

Theorem 5.12. *The derived Fukaya category $DFuk^*(T^2)$ is split-closed.* □

In fact, we have seen that $H^0 Tw Fuk^*(T^2)$ is split-closed.

Appendix A

Kuranishi structure

We review some basic notions on Kuranishi spaces in Section A.1. Then we will see how to define an integration along the fiber on Kuranishi spaces in Section A.2.

A.1 Kuranishi space

Let X be a compact topological space. A *Kuranishi chart* on X consists of (V, E, Γ, ψ, s) that satisfies the following:

1. V is a smooth manifold and Γ is a finite group acting on V effectively.

2. E is a vector bundle on V and Γ also acts on E in such a way that the projection map $E \to V$ is Γ-equivariant.

3. $s : V \to E$ is a Γ-equivariant section.

4. $\psi : s^{-1}(0)/\Gamma \to X$ is a homeomorphism onto its image.

Let $(V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha)$ and $(V_\beta, E_\beta, \Gamma_\beta, \psi_\beta, s_\beta)$ be Kuranishi charts on X. Suppose $\psi_\alpha(s_\alpha^{-1}(0)/\Gamma_\alpha) \cap \psi_\beta(s_\beta^{-1}(0)/\Gamma_\beta) \neq \emptyset$. A *coordinate transformation* from $(V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha)$ to $(V_\beta, E_\beta, \Gamma_\beta, \psi_\beta, s_\beta)$ consists of a triple $(\hat{\phi}_{\beta\alpha}, \phi_{\beta\alpha}, h_{\beta\alpha})$ that satisfies the following:

1. $h_{\beta\alpha} : \Gamma_\alpha \to \Gamma_\beta$ is an injective group homomorphism.

2. There is a Γ_α-invariant open set $V_{\beta\alpha}$ of V_α. Then $\phi_{\beta\alpha} : V_{\beta\alpha} \to V_\beta$ is an $h_{\beta\alpha}$-equivariant smooth embedding such that the induced map $\overline{\phi_{\beta\alpha}} : V_{\alpha\beta}/\Gamma_\alpha \to V_\beta/\Gamma_\beta$ is injective.

3. The map $h_{\beta\alpha}$ induces an isomorphism from $(\Gamma_\alpha)_x$ to $(\Gamma_\beta)_{\phi_{\beta\alpha}(x)}$ for any $x \in V_{\beta\alpha}$.

4. $\hat{\phi}_{\beta\alpha} : E_\alpha|_{V_{\beta\alpha}} \to E_\beta$ is an $h_{\beta\alpha}$-equivariant injective bundle map.

5. $\hat{\phi}_{\beta\alpha} \circ s_\alpha = s_\beta \circ \phi_{\beta\alpha}$.

6. $\psi_\alpha = \psi_\beta \circ \overline{\phi_{\beta\alpha}}$ on $s_\alpha^{-1}(0) \cap V_{\beta\alpha}/\Gamma_\alpha$.

7. $\psi_\alpha(s_\alpha^{-1}(0)/\Gamma_\alpha) \cap \psi_\beta(s_\beta^{-1}(0)/\Gamma_\beta) = \psi_\alpha((s_\alpha^{-1}(0) \cap V_{\beta\alpha})/\Gamma_\alpha)$.

Definition A.1. A compact topological space X is said to have a *Kuranishi structure* if the following holds:

There exists a Kuranishi neighborhood $(V_p, E_p, \Gamma_p, \psi_p, s_p)$ for each point $p \in X$ such that

1. $\dim V_p -$ rank E_p is constant

2. For each $q \in \psi_p(s_p^{-1}(0)/\Gamma_p)$ there exists a coordinate transformation $(\hat{\phi}_{pq}, \phi_{pq}, h_{pq})$ from $(V_q, E_q, \Gamma_q, \psi_q, s_q)$ to $(V_p, E_p, \Gamma_p, \psi_p, s_p)$.

3. If $q \in \psi_p(s_p^{-1}(0)/\Gamma_p)$ and $r \in \psi_q((V_{pq} \cap s_q^{-1}(0))/\Gamma_q)$, then there exists $\gamma_{pqr} \in \Gamma_p$ such that

$$h_{pq} \circ h_{qr} = \gamma_{pqr} \cdot h_{pr} \cdot \gamma_{pqr}^{-1},\ \phi_{pq} \circ \phi_{qr} = \gamma_{pqr} \cdot \phi_{pr},\ \hat{\phi}_{pq} \circ \hat{\phi}_{qr} = \gamma_{pqr} \cdot \hat{\phi}_{pr}.$$

A compact topological space X is said to be a *Kuranishi space* if it has a Kuranishi structure.

Using the notation given above, we may identify a neighborhood of $\phi_{pq}(V_{pq})$ in V_p with a neighborhood of the zero section in the normal bundle $N_{V_{pq}}V_p :=$

$N_{\phi_{pq}(V_{pq})}V_p$ on V_p. We say that the Kuranishi space X with Kuranishi neighborhoods $(V_p, E_p, \Gamma_p, \psi_p, s_p)$ around each $p \in X$ has a *tangent bundle* if the differential of s_p along the fiber direction

$$d_{fiber}s_p : N_{V_{pq}}V_p \to \frac{E_p|_{V_{pq}}}{\hat{\phi}_{pq}(E_q)}$$

induces an isomorphism.

Furthermore, we say that a Kuranishi structure with tangent bundle is *oriented* if the bundle $\wedge^{top}E_p^* \otimes \wedge^{top}TV_p$ can be trivialized on each Kuranishi chart in such a way that the trivializations are compatible with $d_{fiber}s_p$ and Γ_p-action preserves the orientation.

From now on we assume that our Kuranishi space has tangent bundle and is oriented.

Now we introduce the notion of good coordinate system.

Definition A.2. A family of Kuranishi charts $\{\mathcal{U}_\alpha = (V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha)\}_{\alpha\in I}$ is a *good coordinate system* if the following condition holds:

There is a partial order $<$ on the index set I, where either $\alpha_1 \leq \alpha_2$ or $\alpha_2 \leq \alpha_1$ holds for all $\alpha_1, \alpha_2 \in I$ such that

$$\psi_{\alpha_1}(s_{\alpha_1}^{-1}(0)/\Gamma_{\alpha_1}) \cap \psi_{\alpha_2}(s_{\alpha_2}^{-1}(0)/\Gamma_{\alpha_2}) \neq \emptyset.$$

Moreover, we require that there exists a coordinate transformations $(\hat{\phi}_{\beta\alpha}, \phi_{\beta\alpha}, h_{\beta\alpha})$ from $(V_{\alpha_1}, E_{\alpha_1}, \Gamma_{\alpha_1}, \psi_{\alpha_1}, s_{\alpha_1})$ to $(V_{\alpha_2}, E_{\alpha_2}, \Gamma_{\alpha_2}, \psi_{\alpha_2}, s_{\alpha_2})$ for each pair $\alpha_1 \leq \alpha_2$ and that these coordinate transformations satisfy the following compatibility conditions:

If $\alpha_1 < \alpha_2 < \alpha_3$, then we require $\phi_{\alpha_3,\alpha_2} \circ \phi_{\alpha_2,\alpha_1} = \phi_{\alpha_3,\alpha_1}$. Similarly, we require further that $\hat{\phi}_{\alpha_3,\alpha_2} \circ \hat{\phi}_{\alpha_2,\alpha_1} = \hat{\phi}_{\alpha_3,\alpha_1}$ and $h_{\alpha_3,\alpha_2} \circ h_{\alpha_2,\alpha_1} = h_{\alpha_3,\alpha_1}$.

Next we define the notion of a continuous map from a Kuranishi space to another topological space. We will use the notation used in Definition A.1.

Definition A.3. Let X be a Kuranishi space and Y be any topological space. A family of Γ_p-invariant continuous maps $f_p : V_p \to Y$ is called a *strongly* *continuous map* if

$$f_p \circ \phi_{pq} = f_q$$

on V_{pq}. Such a family $\{f_p\}$ induces a continuous map $f : X \to Y$.

When Y is a smooth manifold, a strongly continuous map $f = \{f_p\} : X \to Y$ is smooth if each $f_p : V_p \to Y$ is smooth. Further, f is called *weakly submersive* if each of f_p is a submersion.

A.2 Integration along the fiber on a Kuranishi chart

In Chapter 3, we construct the integration along the fiber on smooth manifolds. For our purpose, we need to generalize this notion to a Kuranishi space because the moduli space of pseudo-holomorphic discs will be realized by a Kuranishi space. We will review how to define the integration along the fiber on a Kuranishi chart.

Let $\mathcal{M}$ be a space with oriented Kuranishi structure with Kuranishi charts $(V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha)_{\alpha\in I}$. Suppose that we have a weakly submersive map $f = \{f_\alpha\}_{\alpha\in I} : \mathcal{M} \to L$ for some smooth manifold L without boundary.

Furthermore., let E be a flat vector bundle on L. We want to define an integration along the fiber $f_!$, which sends f^*E-valued differential forms on $\mathcal{M}$ to E-valued differential forms on L.

One might notice that each f_α cannot be submersive whenever $\dim \mathcal{M} = \dim V_\alpha -$ rank $E_\alpha < \dim L$. In order to resolve this problem, we need to use the notion of a continuous family of multisections.

We first introduce the notion of multisections.

For a given topological space E and a positive integer l, we denote by

$$S^l(E) = \underbrace{E \times .. \times E}_{l} / S_l$$

the quotient of $\prod_{i=1}^{l} E$ by the natural action of the permutation group S_l.

There are maps

$$t_m : S^l(E) \to S^{lm}(E),\ [a_1, ..., a_l] \mapsto [\underbrace{a_1, ..., a_1}_{m}, ..., \underbrace{a_l, ..., a_l}_{m}].$$

Definition A.4. A multisection s of $E_\alpha \to V_\alpha$ consists of an open covering $V_\alpha = \bigcup_{i \in I} U_{\alpha,i}$ and $s_{\alpha,i} : U_{\alpha,i} \to S^{l_i}(E_\alpha|_{U_{\alpha,i}})$ a section of $S^{l_i}(E_\alpha)|_{U_{\alpha,i}} \to U_{\alpha,i}$ with some compatibility conditions:

1. $U_{\alpha,i}$ is Γ_α-invariant and $s_{\alpha,i}$ is Γ_α-equivariant.
2. if $x \in U_{\alpha,i} \subseteq U_{\alpha,j}$, then $t_{l_j}(s_{\alpha,i}(x)) = t_{l_i}(s_{\alpha,j}(x))$.
3. $s_{\alpha,i}$ is liftable and smooth in the following sense. For each x, there is a smooth section $\tilde{s}_{\alpha,i} = (\tilde{s}_{\alpha,i,1}, ..., \tilde{s}_{\alpha,i,l_i})$ of $E_\alpha^{\oplus l_i}$ on a neighborhood of x that represents s_i, i.e.

$$[\tilde{s}_{\alpha,i}(y)] = [(\tilde{s}_{\alpha,i,1}(y), ..., \tilde{s}_{\alpha,i,l_i}(y))] = s_{\alpha,i}(y).$$

Next we introduce the notion of a continuous family of multisections.

Recall that the space $\mathcal{M}$ is a Kuranishi space with Kuranishi charts $(V_\alpha, , E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha)_{\alpha \in I}$. For each $\alpha \in I$, let W_α be a smooth oriented manifold of even dimension and let θ_α be a compactly supported volume form on W_α such that $\int_{W_\alpha} \theta_\alpha = 1$. Then we consider the pullback orbibundle $\pi_\alpha^* E_\alpha$ under the projection $\pi_\alpha : V_\alpha \times W_\alpha \to V_\alpha$. Here Γ_α acts on W_α trivially.

A *W_α-parametrized family S_α of multisections* is a multisection of $\pi_\alpha^* E_\alpha$, i.e. there exist an open covering $\bigcup_i U_{\alpha,i}$ of V_α and a section $S_{\alpha,i} = \{S_{\alpha,i,j}\}$: $U_{\alpha,i} \times W_\alpha \to S^{l_i}(\pi_\alpha^* E_\alpha)$.

Suppose there is a W_α-parametrized family of multisections $S_\alpha = \{S_{\alpha,i,j}\}_{i,j}$ of $E_\alpha \to V_\alpha$ satisfying the following conditions.

Condition A.5. (1) The multisection S_α is ϵ-close to the given section s_α: For $\epsilon > 0$ and for $(x, w) \in V_\alpha \times W_\alpha$, there is an open neighborhood U of x

such that each branch $S_{\alpha,i,j}$ satisfies

$$dist(S_{\alpha,i,j}(y,w), s_\alpha(y)) < \epsilon$$

for all $w \in W_\alpha$ and $y \in U$.

(2) The multisection S_α is transversal to 0 in the sense that each branch $S_{\alpha,i,j}$ is transversal to 0.

(3) The evaluation map $f_\alpha|_{S_\alpha^{-1}(0)}$ is submersive in the sense that the restriction of $f_\alpha \circ \pi_\alpha : V_\alpha \times W_\alpha \to L$ to $S^{-1}_{\alpha,i,j}(0)$ is submersive for each branch.

Now we define the integration along the fiber on a Kuranishi chart $(V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha)_\alpha$ using the W_α-parametrized family of multisections S_α.

Let $\eta \in \Omega_c(V_\alpha \times W_\alpha, \pi_\alpha^* f_\alpha^* E)$ be given. First take a partition of unity τ_i subordinate to $U_{\alpha,i}$. Then we define $(V_\alpha, S_\alpha, f_\alpha)_! \eta$ by

$$\frac{1}{|\Gamma_\alpha|} \sum_i \sum_{j=1}^{l_i} \frac{1}{l_i} (f_\alpha \circ \pi_\alpha|_{S^{-1}_{\alpha,i,j}(0)})_! (\tau_i \eta \wedge \theta_\alpha). \tag{A.1}$$

Because $S^{-1}_{\alpha,i,j}(0)$ is a manifold by transversality and $f_\alpha \circ \pi_\alpha|_{S^{-1}_{\alpha,i,j}(0)}$ is submersive, $(f_\alpha \circ \pi_\alpha|_{S^{-1}_{\alpha,i,j}(0)})_!(\tau_i \eta \wedge \theta_\alpha)$ is well-defined by using the integration along the fiber in manifold case. (See Chapter 3.)

Note that it is enough to require η to be defined on $S^{-1}_{\alpha,i,j}(0)$ for each i, j.

A.3 Kuranishi structure on moduli space $\mathcal{M}_{k+1}(L,\beta)$.

In this section, we review a Kuranishi structure on the moduli space of pseudo-holomorphic discs bounding a Lagrangian submanifold.

Let us consider the moduli space $\mathcal{M}_{k+1}(L,\beta) = \mathcal{M}_{k+1}(L,\beta;J)$ which consists of pairs $((D,\vec{z}),u)$ of a two disc D with $(k+1)$-boundary marked points $\vec{z} = \{z_0, ..., z_k\}$ which respect the cyclic order and a J-holomorphic map $u : (D, \partial D) \to (M, L)$ such that $[u] = \beta \in \pi_2(M,L)$. We will denote its compactification by the same notation and will abbreviate our notation as $\mathcal{M}_{k+1}(\beta) = \mathcal{M}_{k+1}(L, \beta; J)$ from now on.

We give a rough idea to construct a Kuranishi chart (V, E, Γ, ψ, s) for $\mathcal{M}_{k+1}(\beta)$. Refer to [20, 16] for more detailed construction of Kuranishi charts.

Let $((D, \vec{z}), u)$ be an element in (the interior of) $\mathcal{M}_{k+1}(\beta)$. Consider the Banach completion $W^{1,p}(u^*TM)$ of the pullback bundle u^*TM on D^2 for sufficiently large p and the linearized Cauchy-Riemann operator

$$D\bar{\partial}_J : W^{1,p}(u^*TM) \to L^p(\Lambda^{0,1} \otimes u^*TM).$$

Choose a finite-dimensional subspace E of $L^p(\Lambda^{0,1} \otimes u^*TM)$ large enough so that

$$\mathrm{Im}\ D\bar{\partial}_J + E = L^p(\Lambda^{0,1} \otimes u^*TM).$$

In order to define the space V, we use the Deligne-Mumford-Stasheff compactification $\mathcal{M}_{k+1}$ of the space of all two discs with $(k+1)$-boundary marked points. (See [20, 33])

We define V as a space of pairs $((D', \vec{z}'), u')$ of $(D', \vec{z}')$ which is close to $(D, \vec{z})$ in $\mathcal{M}_{k+1}$ and a map $u' : (D', \partial D') \to (M, L)$ such that $u' = \exp_u(\xi)$ for some sufficiently small $\xi \in W^{1,p}(u^*TM)$ and $D\bar{\partial}_J(\xi) \in E$.

The section $s : V \to E$ is just the linearized Cauchy-Riemann operator $D\bar{\partial}_J$. Furthermore Γ is the isotopy group of $((D, \vec{z}), u)$ in $\mathrm{Aut}(D)$. The group Γ clearly acts effectively both on V and E in such a way that the projection $E \to V$ respects the action of Γ.

Then there is a homeomorphism Ψ from the quotient orbifold $s^{-1}(0)/\Gamma$ onto a subset of $\mathcal{M}_{k+1}(\beta)$. This gives a neighborhood of $((D, \vec{z}), u)$ in $\mathcal{M}_{k+1}(\beta)$.

Naturally there are evaluation maps from our Kuranishi chart $ev_i : V \to L$ at i-th marked points for $0 \le i \le k$. Indeed, the evaluation map is defined by

$$ev_i((D, \vec{z}), u) = u(z_i)$$

where $((D, \vec{z}), u)$ is an element in V. These evaluation maps glue together to define an evaluation map $ev_i : \mathcal{M}_{k+1}(\beta) \to L$

$$ev_i([(D, \vec{z}), u]) = u(z_i)$$

where $((D, \vec{z}), u)$ is a pseudo-holomorphic disc.

It is well-known that the compactification of moduli space $\mathcal{M}_{k+1}(\beta)$ is obtained by adding semistable curves. Indeed we have

Proposition A.6. *[1] As spaces with oriented Kuranishi structures, we have*

$$\partial \mathcal{M}_{k+1}(\beta) = \bigcup_{\substack{\beta_1+\beta_2=\beta \\ 0\le j\le k \\ 1\le i\le k-j+2}} (-1)^{n+i(1+j)+jk} \mathcal{M}_{k-j+2}(\beta_1)_{ev_i} \times_{ev_0} \mathcal{M}_{j+1}(\beta_2).$$

Note that the right hand side of the above equality does not include any term involving $\mathcal{M}_2(0)$ because holomorphic curves in $\mathcal{M}_2(0)$ are not semistable.

Next we review the notion of forgetful map and forgetful map compatible Kuranishi structures on $\mathcal{M}_{k+1}(\beta)$. We follow the paper of Lino Amorim [2]. There is a natural map

$$forg : \mathcal{M}_{k+1}(\beta) \to \mathcal{M}_1(\beta),$$

for each $k \ge 0$, which forgets the last k-marked points and ignore every irreducible components that become unstable after forgetting marked points.

Definition A.7. We say that Kuranishi charts $(V_\alpha, E_\alpha, \Gamma_\alpha, s_\alpha, \psi_\alpha)$ for $\mathcal{M}_{k+1}(\beta)$ and $(V'_{\alpha'}, E'_{\alpha'}, \Gamma'_{\alpha'}, s'_{\alpha'}, \psi'_{\alpha'})$ for $\mathcal{M}_1(\beta)$ are *compatible with forgetful map* if

1. $V_\alpha = V'_{\alpha'} \times [0,1)^m \times (0,1)^{k-m}$ (for some m),
2. $E_\alpha = \pi^* E'_{\alpha'}$ where $\pi : V_\alpha \to V'_{\alpha'}$ is the projection map,
3. $\Gamma_\alpha = \Gamma'_{\alpha'}$,
4. $s_\alpha = \pi^* s'_{\alpha'}$,

5. $forg \circ \psi_\alpha = \psi'_{\alpha'} \circ \pi$.

Definition A.8. We say that Kuranishi structures on $\mathcal{M}_{k+1}(\beta)$ and $\mathcal{M}_1(\beta)$ are *compatible with forgetful map* if for every $p \in \mathcal{M}_{k+1}(\beta)$ and $q = forg(p)$, there exist Kuranishi neighborhoods $(V_p, E_p, \Gamma_p, s_p, \psi_p)$ and $(V_q, E_q, \Gamma_q, s_q, \psi_q)$, which are compatible with forgetful map in the sense of Definition A.7.

Proposition A.9. *[16] There exist Kuranishi structures on $\mathcal{M}_{k+1}(\beta)$ such that they are compatible with respect to $forg : \mathcal{M}_{k+1}(\beta) \to \mathcal{M}_1(\beta)$ for every $k \geq 0$ and the map*

$$ev_0 : \mathcal{M}_{k+1}(\beta) \to L$$

is weakly submersive, i.e. $ev_{\alpha,0} : V_\alpha \to L$ is submersive for each $\alpha \in I$.

Definition A.10. Suppose there exist good coordinate systems $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ on $\mathcal{M}_{k+1}(\beta)$ and $\{\mathcal{U}_{\alpha'}\}_{\alpha' \in I'}$ on $\mathcal{M}_1(\beta)$ which are compatible with $forg$, i.e. there is an order-preserving map $I \to I'$, $\alpha \mapsto \alpha'$ and a map $forg : V_\alpha \to V_{\alpha'}$ which satisfy the axioms of Definition A.7.

We say that family of multisections $(W_\alpha, S_\alpha)_{\alpha \in I}$ on $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ and $(W_{\alpha'}, S_{\alpha'})_{\alpha' \in I'}$ on $\{\mathcal{U}_{\alpha'}\}_{\alpha' \in I'}$ are *compatible* if

1. $U_{\alpha,i} = U'_{\alpha',i} \times [0,1)^m \times (0,1)^{k-m}$,

2. $W_\alpha = W_{\alpha'}$ and $\theta_\alpha = \theta_{\alpha'}$

3. $S_\alpha = \pi^* S_{\alpha'}$ where $\pi : U_{\alpha,i} \times W_\alpha \to U'_{\alpha',i} \times W_{\alpha'}$ is the projection map

Proposition A.11. *[16, 2] For each $\epsilon > 0$ and $E > 0$, there exist compatible systems of families of multisections on $\mathcal{M}_{k+1}(\beta)$ for $k \geq 0$ and $\omega(\beta) \leq E$ which satisfy Condition A.5 for $f = ev_0$ and are compatible with $forg$ in the sense of Definition A.10.*

Moreover, the restriction of the multisections to the boundary agrees with the fiber product of multisections of the right hand side of the equality in Proposition A.6.

Appendix B

Linear algebra

Let $\mathbb{K}$ be an algebraically closed field and let V be a $\mathbb{K}$-vector space of dimension r. Let us denote the set of all eigenvalues of T by $\sigma(T)$ for any $T \in \mathrm{End}(V)$.

Let $T \in \mathrm{End}(V)$ with $\sigma(T) = \{\lambda_1, ..., \lambda_l\}$. We consider its generalized eigenspaces:

$$E_i = \ker(T - \lambda_i id)^{m_i} \leq V,$$

where m_i is the multiplicity of $x - \lambda_i$ in the minimal polynomial of T.

Then the following lemma is well known.

Lemma B.1. *1.* $V = \bigoplus_{i=1}^{l} E_i$.

2. There exist polynomials $g_i(x) \in \mathbb{K}[x]$ *such that* $p_i = g_i(T)$ *is the projection onto* E_i *with respect to the decomposition (1).*

Let us consider the case when there are several commuting linear operators. Let $T^1, ..., T^n \in \mathrm{End}(V)$ be commuting n-linear operators. Let us write

$$\sigma(T^k) = \{\lambda_1^k, ..., \lambda_{l_k}^k\}$$

for $k \in \{1, ..., n\}$.

As usual, we consider the generalized eigenspace

$$E_i^k = \ker(T^k - \lambda_i^k id)^{m_i^k}$$

for each $k \in \{1, ..., n\}$ and $1 \leq i \leq l_k$. Here m_i^k is the multiplicity of $x - \lambda_i^k$ in the minimal polynomial of T^k.

For each n-tuple $I = (i_1, ..., i_n)$ with $1 \leq i_k \leq l_k$, we define

$$E_I = \bigcap_{k=1}^{n} E_{i_k}^k.$$

Then we observe from Lemma B.1 that the vector space V decomposes into

$$V = \bigoplus_I E_I,$$

where I runs over all possible n-tuples. Also note that each T^k preserves E_I for every I because any pair of $\{T^k\}$ commute with each other.

We summarize our observation in the following lemma.

Lemma B.2. *There is a decomposition of the vector space V into a direct sum $\bigoplus_I E_I$ such that each E_I is invariant under T^k for every $k \in \{1, ..., n\}$.* $\square$

Therefore, if we find a basis S_I for E_I for every I, then the union $\cup_I S_I$ will be a basis for V. So it is enough to consider the case that $l_k = 1$ for every $k \in \{1, ..., n\}$, i.e. T^k has only one eigenvalue λ^k for each $k \in \{1, ..., n\}$.

Then Lemma 5.1 is a corollary of the following proposition.

Proposition B.3. *Let $T^1, ..., T^n \in End(V)$ be n-commuting linear operators such that each T^k has only one eigenvalue λ^k.*

There is an ordered basis in which every T^k has a lower-triangular matrix representation.

Proof. Let us assume that $n = 2$ for simplicity.

For each $k = 1, 2$, we find a filtration $0 \subseteq F_1^k \subseteq F_2^k \subseteq ... \subseteq F_{m_k}^k = V$ of V, where $F_j^k = \ker(T^k - \lambda^k id)^j$.

Then we have the following diagram.

$$
\begin{array}{ccccccc}
F_1^1 \cap F_1^2 & \subseteq & F_1^1 \cap F_2^2 & \subseteq & ... & \subseteq & F_1^1 \cap F_{m_2}^2 \\
\rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\
F_2^1 \cap F_1^2 & \subseteq & F_2^1 \cap F_2^2 & \subseteq & ... & \subseteq & F_2^1 \cap F_{m_2}^2 \\
\rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\
... & \subseteq & ... & \subseteq & ... & \subseteq & ... \\
\rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\
F_{m_1}^1 \cap F_1^2 & \subseteq & F_{m_1}^1 \cap F_2^2 & \subseteq & ... & \subseteq & F_{m_1}^1 \cap F_{m_2}^2.
\end{array}
$$

Let us give a lexicographic order on $\{(i_1, i_2)|1 \leq i_1 \leq m_1, 1 \leq i_2 \leq m_2\}$, i.e. $(i_1, i_2) < (j_1, j_2)$ if and only if $i_1 < j_1$ or ($i_1 = j_1$ and $i_2 < j_2$).

Let us define F_{i_1,i_2} to be the subspace of V spanned by all $F_{j_1}^1 \cap F_{j_2}^2$ for $(j_1, j_2) \leq (i_1, i_2)$ with respect to the lexicographic order.

We use an induction on the lexicographic order to find a good ordered basis for V. We first find an ordered basis $S_{1,1}$ for $F_{1,1} = F_1^1 \cap F_1^2$. Now suppose we have extended this ordered basis into an ordered basis S_{i_1,i_2} for F_{i_1,i_2}.

If $i_2 < m_2$, then we extend the ordered basis S_{i_1,i_2} into an ordered basis S_{i_1,i_2+1} for F_{i_1,i_2+1} in such a way that

$$S_{i_1,i_2+1} = \{v_1, ..., v_{m_{i_1,i_2+1}}\} \cup S_{i_1,i_2},$$

where $\{v_1, ..., v_{m_{i_1,i_2+1}}\}$ in $F_{i_1,i_2+1} \setminus F_{i_1,i_2}$ is chosen so that $\{v_1, ..., v_{m_{i_1,i_2+1}}\} \cup S_{i_1,i_2}$ forms a basis for F_{i_1,i_2+1}.

Here, we observe that

$$\begin{aligned} &T^2 v_k = (T^2 - \lambda^2)v_k + \lambda^2 v_k \quad \text{and} \\ &(T^2 - \lambda^2)v_k \in F_{i_1,i_2} \text{ for all } k \in \{1, ..., m_{i_1,i_2+1}\} \end{aligned} \tag{B.1}$$

and

$$\begin{aligned} &T^1 v_k = (T^1 - \lambda^1) v_k + \lambda^1 v_k \ \text{ and} \\ &(T^1 - \lambda^1) v_k \in F_{i_1 - 1, i_2 + 1} \text{ for all } k \in \{1, ..., m_{i_1, i_2 + 1}\}. \end{aligned} \tag{B.2}$$

Else if $i_2 = m_2$, then we extend the ordered basis S_{i_1,i_2} into an ordered basis $S_{i_1+1,1}$ for $F_{i_1+1,1}$ in such a way that

$$S_{i_1+1,1} = \{v_1, ..., v_{m_{i_1+1,1}}\} \cup S_{i_1,m_2},$$

where $\{v_1, ..., v_{m_{i_1+1,1}}\}$ in $F_{i_1+1,1} \setminus F_{i_1,i_2}$ is chosen so that $\{v_1, ..., v_{m_{i_1+1,1}}\} \cup S_{i_1,m_2}$ forms a basis for $F_{i_1+1,1}$.

As before, we observe that

$$\begin{aligned} &T^1 v_k = (T^1 - \lambda^1) v_k + \lambda^1 v_k \ \text{ and} \\ &(T^1 - \lambda^1) v_k \in F_{i_1, i_2} \text{ for all } k \in \{1, ..., m_{i_1+1,1}\} \end{aligned} \tag{B.3}$$

and

$$T^2 v_k = \lambda^2 v_k \ \text{ and } \ \text{for all } k \in \{1, ..., m_{i_1+1,1}\}. \tag{B.4}$$

Finally we get an ordered basis $S := S_{m_1,m_2}$ for $V = F_{m_1,m_2}$. The observations (B.1), (B.2), (B.3), (B.4) imply that each T^k has a lower-triangular matrix representation in the ordered basis S. $\square$

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국문초록

이 학위 논문에서는 푸카야 범주의 고차 벡터 다발과 뒤틀린 복합체 사이의 관계를 연구한다. 이를 위하여 먼저 드람 버전의 푸카야 범주를 사용하여 푸카야 범주의 대상을 일차 벡터 다발뿐 아니라 고차 벡터 다발을 포함하도록 확장한다. 또한 특별히 드람 버전의 푸카야 범주가 가지는 성질들을 연구한다.

결과적으로 라그랑지안 부분다양체의 기본군이 가환일때 그 위의 모든 평탄한 벡터 다발이 특정한 뒤틀린 복합체와 동형임을 보인다. 더 나아가 삼각형화 가능한 홀로노미 표현에 대응하는 모든 평탄한 벡터 다발이 특정한 뒤틀린 복합체와 준동형임을 보인다.

주요어휘 : 라그랑지안 플로어 이론, 푸카야 범주, 평탄한 벡터 다발, 뒤틀린 복합체

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