이학박사 학위논문

Representations of squares by positive ternary quadratic forms

(삼변수 양 이차형식에 의한 제곱수의 표현)

2017년 8월

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지도교수 오 병 권

이 논문을 이학박사 학위논문으로 제출함

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Representations of squares by positive ternary quadratic forms

A dissertation
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Kyoungmin Kim

Dissertation Director : Professor Byeong-Kweon Oh

Department of Mathematical Sciences
Seoul National University

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Abstract

In this thesis, we study various properties of representations of squares by ternary quadratic forms.

A (positive definite integral) ternary quadratic form is called *strongly S-regular* if it satisfies a regularity property on the number of representations of squares of integers. We explain the relation between the strongly S-regularity and the conjecture given by Cooper and Lam, and we resolve their conjecture completely. We prove that there are only finitely many strongly S-regular ternary forms up to isometry if the minimum of the non zero squares that are represented by the form is fixed. In particular, we show that there are exactly 207 non-classic integral strongly S-regular ternary quadratic forms representing one.

Key words: Representations of ternary quadratic forms, squares
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Contents

Chapter 1

Introduction

A homogeneous quadratic polynomial with three variables

$$f(x_1, x_2, x_3) = \sum_{1 \le i,j \le 3} a_{ij} x_i x_j, \quad (a_{ij} = a_{ji} \in \mathbb{Q})$$

is called a ternary quadratic form over $\mathbb{Q}$. We say that f is positive definite if the corresponding symmetric matrix $M_f := (a_{ij})$ is positive definite. We say f is integral if each coefficient a_{ij} is an integer. The quadratic form f is called non-classic integral if f is an integral polynomial, that is, both a_{ii} and $a_{ij} + a_{ji}$ are all integers for any i, j. Note that every integral form is non-classic integral. Throughout this thesis, we assume that every ternary quadratic form f is positive definite and non-classic integral.

For a (non-classic integral positive definite) ternary quadratic form f and a positive integer n, we define

$$R(n, f) = \{(x_1, x_2, x_3) \in \mathbb{Z}^3 : f(x_1, x_2, x_3) = n\} \quad \text{and} \quad r(n, f) = |R(n, f)|.$$

Since we are assuming that f is positive definite, the set $R(n, f)$ is always finite. Finding a closed formula for $r(n, f)$ or finding all positive integers n such that $r(n, f) \neq 0$ for a given ternary quadratic form f are quite old problems which are still widely open. As one of the simplest cases, Gauss showed that if f is a sum of three squares, that is, $f(x, y, z) = x^2+y^2+z^2$, then $r(n, f)$ is a multiple of the Hurwitz-Kronecker class number of an imaginary quadratic field. In fact, if the class number of f is one, then Minkowski-Siegel formula gives a closed formula for $r(n, f)$. As a natural modification of the Minkowski-Siegel formula, it was proved in [13] and [21] that the weighted sum of the representations of quadratic forms in the spinor genus is also equal to the product of local densities except spinor exceptional integers (see [20] for the definition spinor exceptional integers). Hence, if the spinor class number $g^+(f)$ of f is one, we also have a closed formula for $r(n, f)$. As far as the author knows, there is no known closed formula for $r(n, f)$ except those cases (for some relations between $r(n, f)$'s, see [12]).

Though it seems to be quite difficult to find a closed formula for $r(n, f)$ for any positive integer n, there are some additional closed formulas for $r(n, f)$ if the integer n is contained in some particular proper subset S of the set of integers. For example, for any integer $n \in S$, if $r(n, f) = r(n, f')$ for any $f' \in \text{gen}(f)$, then the Minkowski-Siegel formula gives a closed formula for $r(n, f)$. Note that there is an integer n such that $r(n, f) \neq r(n, f')$ for any $f' \in \text{gen}(f)$ that is not isometric to f by Schiemann's result [18].

In 2013, Cooper and Lam [4] tried to find a closed formula for $r(n^2, f)$, where $f(x, y, z) = x^2 + ay^2 + bz^2$ for some integers a, b. In that article, they proved, by using some q-series identities, that for the quadratic form $f(x, y, z) = x^2 + by^2 + cz^2$ with $(b, c) = (1, 1), (1, 2), (1, 3), (2, 2), (3, 3)$,

$$r(n^2, f) = \prod_{p \mid 2bc} g(b, c, p, \text{ord}_p(n)) \prod_{p \nmid 2bc} h(b, c, p, \text{ord}_p(n)), \tag{1.0.1}$$

where

$$h(b, c, p, \text{ord}_p(n)) = \frac{p^{\text{ord}_p(n)+1} - 1}{p - 1} - \left(\frac{-df}{p}\right) \frac{p^{\text{ord}_p(n)} - 1}{p - 1}$$

and $g(b, c, p, \text{ord}_p(n))$ has to be determined on an individual and case-by-case basis, and they conjectured that the above equality also holds for some 64 pairs of (b, c) (see Table 3.1 in Chapter 3). Recently, Guo, Peng and Qin in [8] verified the conjecture when

$$\begin{aligned}(b, c) = (1, 4), (1, 5), (1, 6), (1, 8), (2, 3), (2, 4),& \\ (2, 6), (3, 6), (4, 4), (4, 8), (5, 5), (6, 6)&\end{aligned}$$

by using theory of modular forms of weight 3/2, and Hürlimann in [9] verified the conjecture when

$$(b, c) = (2, 8), (2, 16), (8, 8), (8, 16)$$

by using some q-series identities on Bell ternary quadratic forms.

In this thesis, we prove Cooper and Lam's conjecture completely. Furthermore, we find all ternary quadratic forms f satisfying the condition (1.0.1) under the assumption that 1 is represented by the form f.

In Chapter 2, we introduce some definitions and well-known results on quadratic spaces and lattices. We adapt the geometric language of quadratic spaces and lattices rather than quadratic forms in the subsequent discussion. The term "lattice" will always refer to a positive definite integral $\mathbb{Z}$-lattice on an n-dimensional positive definite quadratic space over $\mathbb{Q}$. Let $L = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \cdots + \mathbb{Z}x_n$ be a $\mathbb{Z}$-lattice of rank n. We write

$$L \simeq (B(x_i, x_j)).$$

The right hand side matrix is called a *matrix presentation* of L. If $B(x_i, x_j) = 0$ for any $i \neq j$, then we write $L \simeq \langle Q(x_1), Q(x_2), \ldots, Q(x_n)\rangle$, where Q is the quadratic map such that $Q(x) = B(x, x)$ for any $x \in L$. The discriminant dL of the lattice L is defined by the determinant of the corresponding matrix $(B(x_i, x_j))$.

For two $\mathbb{Z}$-lattices M and L, a linear map $\sigma : M \to L$ is called a *representation* from M to L if it preserves the bilinear form, that is,

$$B(\sigma(x), \sigma(y)) = B(x, y) \quad \text{for any } x, y \in M.$$

We define

$$R(M, L) = \{\sigma : M \to L \mid \sigma \text{ is a representation}\}$$

and $r(M, L) = |R(M, L)|$. In particular, $O(L) = R(L, L)$ which is called the isometry group of L, and $o(L) = r(L, L)$. For any $\mathbb{Z}$-lattice L, the isometry class containing L in $\mathrm{gen}(L)$ is denoted by $[L]$. As usual, we define

$$w(L) = \sum_{[M]\in \mathrm{gen}(L)} \frac{1}{o(M)} \quad \text{and} \quad r(K, \mathrm{gen}(L)) = \frac{1}{w(L)} \sum_{[M]\in \mathrm{gen}(L)} \frac{r(K,M)}{o(M)},$$

for any $\mathbb{Z}$-lattice K. Any unexplained notations and terminologies can be found in [16] or [17].

In Chapter 3, we completely resolve Cooper and Lam's conjecture introduced the above. In fact, the condition (1.0.1) in Cooper and Lam's conjecture is a little bit vague, for the function $g(b, c, p, \mathrm{ord}_p(n))$ is not given directly. So, we introduce the notion "strongly S-regularity" of ternary quadratic forms. To be more precise, let f be a ternary quadratic form. For any integer n, let n_1 and n_2 be positive integers such that $P(n_1) \subset P(8df)$, $(n_2, 8df) = 1$ and $n = n_1n_2$. Here $P(n)$ denotes the set of prime factors of n. Then f is called *strongly S-regular* if for any positive integer $n = n_1n_2$,

$$r(n_1^2n_2^2, f) = r(n_1^2, f) \cdot \prod_{p \nmid 8df} h_p(df, \lambda_p), \tag{1.0.2}$$

where $\lambda_p = \mathrm{ord}_p(n)$ for any prime p and

$$h_p(df, \lambda_p) = \frac{p^{\lambda_p+1} - 1}{p-1} - \left(\frac{-df}{p}\right) \frac{p^{\lambda_p} - 1}{p-1}.$$

Clearly, if f does not represent any squares of integers, then f is trivially strongly S-regular. So, we always assume that a strongly S-regular ternary form f represents at least one square of an integer. Note that this condition is equivalent to the condition that f represents one over $\mathbb{Q}$.

Our method is based on the action of Hecke operators on the space of modular forms of weight $\frac{3}{2}$ and the Minkowski-Siegel formula on the weighted sum of the representations by quadratic forms in the same genus.

In Chapter 4, we prove that every strongly S-regular form represents all squares that are represented by its genus, and there are only finitely many strongly S-regular ternary forms up to isometry if

$$m_s(f) = \min_{n\in\mathbb{Z}^+} \{n : r(n^2, f) \neq 0\}$$

is fixed. Furthermore, we show that there are exactly 207 strongly S-regular ternary quadratic forms that represent one (see Tables 4.1 and 4.2). In the proof of Lemma 4.2.1 and Theorem 4.2.2, we extensively use mathematics software MAPLE for large amount of computation. In fact, if a ternary quadratic form f satisfies the condition (1.0.1), then clearly f is strongly S-regular. Furthermore, one may easily show that if

$$r(n^2, f) = r(n^2, \mathrm{gen}(f)) \tag{1.0.3}$$

for any integer n, then f satisfies the condition (1.0.1). In this chapter, we will show that all three conditions (1.0.1), (1.0.2) and (1.0.3) given above are equivalent by showing that every strongly S-regular ternary quadratic form satisfies the condition (1.0.3).

In Chapter 5, we generalize the notion of "strongly S-regularity" of ternary quadratic forms. Let T be a proper subset of positive integers. A ternary quadratic form f is called *strongly T-regular* (*strongly spinor T-regular*) if $r(n,f)=r(n,\mathrm{gen}(f))$ ($r(n,f) = r(n,\mathrm{spn}(f))$, respectively) for any integer $n \in T$, where

$$w_s(f) = \sum_{[g]\in \mathrm{spn}(f)} \frac{1}{o(g)} \qquad \text{and} \qquad r(n, \mathrm{spn}(f)) = \frac{1}{w_s(f)} \sum_{[g]\in \mathrm{spn}(f)} \frac{r(n,g)}{o(g)}.$$

Let t be a positive square free integer. We define

$$S_t = \{tn^2 \mid n \in \mathbb{Z}\}.$$

We prove that there exist only finitely many strongly S_t-regular ternary quadratic forms up to isometry if

$$m_{S_t}(f) = \min_{n\in\mathbb{Z}^+} \{n : r(tn^2, f) \neq 0\}$$

is fixed. We also prove that if any splitting integer for the genus of a ternary quadratic form f is not of the form tn^2, then f is strongly S_t-regular if and only if f is strongly spinor S_t-regular.

Chapter 2

Preliminaries

In this chapter, we introduce some definitions and well-known results which are used in throughout the thesis. Especially, we state the Minkowski-Siegel formula which is important for representation of quadratic forms over $\mathbb{Z}$.

2.1 Definitions

Let $\mathbb{Q}$ be the rational number field. For a prime p (including ∞), we denote the fields of p-adic completions of $\mathbb{Q}$ by $\mathbb{Q}_p$, in particular $\mathbb{Q}_\infty = \mathbb{R}$, field of real number. Let F be a field $\mathbb{Q}$ or $\mathbb{Q}_p$. A *quadratic space* V over F is a finite dimensional vector space over F equipped with a non-degenerate symmetric bilinear form

$$B : V \times V \to F.$$

Then we have the following properties:

$$B(x, y) = B(y, x), \quad B(\alpha x + \beta y, z) = \alpha B(x, z) + \beta B(y, z),$$

for any $x, y, z \in V$ and $\alpha, \beta \in F$. The *quadratic map* Q associated with B is defined by

$$Q(x) = B(x, x),$$

for any $x \in V$. We say that a quadratic space is *unary, binary, ternary, quaternary,*..., *n-ary*, according as its dimension is $1, 2, 3, 4, \ldots, n$.

Let V, W be quadratic spaces over F. If a linear mapping σ from V into W satisfies that

$$Q(\sigma x) = Q(x) \quad \text{for any } x \in V,$$

we call σ a *representation* from V into W and say that V is *represented* by W. Furthermore if σ is a bijective linear map, then we call σ an *isometry* from V onto W. In this case, we say that V and W are *isometric* and write $V \simeq W$. The group of all isometries from V onto itself is denoted by $O(V)$. For $\sigma \in O(V)$, we call σ a *rotation* if $\det\sigma = 1$. We denote the set of all rotations of V by $O^+(V)$.

Let V be a quadratic space over F and let $x_1, x_2, \ldots, x_n$ be a basis of V. The $n \times n$ matrix

$$(B(x_i, x_j))_{1 \le i,j \le n}$$

is called the *matrix* of the quadratic space V in the basis $x_1, x_2, \ldots, x_n$. In this case, we write

$$V \simeq (B(x_i, x_j)).$$

We say that V is *positive definite* if the matrix $(B(x_i, x_j))$ is positive definite. If $B(x_i, x_j) = 0$ for any $i \neq j$, then we write

$$V \simeq \langle Q(x_1), Q(x_2), \ldots, Q(x_n) \rangle.$$

The determinant

$$\det(B(x_i, x_j))$$

of the $n \times n$ matrix $(B(x_i, x_j))$ is called the *discriminant* of V and we denote it by dV. Note that the discriminant of V in $(F^\times/(F^\times)^2) \cup \{0\}$ is independent of the basis of V. If $dV \neq 0$, then we say that V is a *regular* quadratic space.

Let V be a non-zero regular n-ary quadratic space over F. For any $\sigma \in O(V)$, we can express σ as a product of symmetries by Theorem 43:3 in [17], say

$$\sigma = \tau_{v_1} \tau_{v_2} \cdots \tau_{v_r}.$$

We define

$$\theta(\sigma) = Q(v_1) Q(v_2) \cdots Q(v_r) \in F^\times/(F^\times)^2,$$

and call it the *spinor norm* of σ. By Proposition 54:6, the spinor norm of σ

in $F^\times/(F^\times)^2$ is well-defined. Since

$$\theta(\sigma\tau) = \theta(\sigma)\theta(\tau),$$

we have a group homomorphism

$$\theta : O^+(V) \to F^\times/(F^\times)^2.$$

We define

$$O'(V) = \{\sigma \in O^+(V) \mid \theta(\sigma) = 1\}.$$

Clearly this is the kernel of the homomorphism θ.

Let F be a field $\mathbb{Q}_p$ or $\mathbb{Q}_\infty(= \mathbb{R})$. For non-zero elements α, β in F, the *Hibert symbol*

$$\left(\frac{\alpha, \beta}{p}\right),$$

or simply (α, β), is defined to be $+1$ if $\alpha x^2 + \beta y^2 = 1$ has a solution $x, y \in F$; otherwise the symbol is defined to be -1. Let V be a regular n-ary quadratic space over F. If V has a splitting

$$V \simeq \langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle,$$

then we define the Hasse symbol

$$S_p(V) = \prod_{1 \le i \le n} \left(\frac{\alpha_i, d_i}{p}\right),$$

where $d_i = \alpha_1\alpha_2 \cdots \alpha_n$. Note that this is independent of the orthogonal splitting chosen for V.

Let R be the ring of integers $\mathbb{Z}$ or the ring of p-adic integers $\mathbb{Z}_p$. Let F be the quotient field of R and let V be a quadratic space over F. Let L be a subset of V which is an R-module under the laws induced by the vector space structure of V over F. We define

$$FL = \{\alpha x \mid \alpha \in F, x \in L\}.$$

Note that FL is a subspace of V. We call the R-module L a *R-lattice* in V if there is a basis $x_1, x_2, \ldots, x_n$ for V such that

$$L \subseteq Rx_1 + Rx_2 + \cdots + Rx_n.$$

Furthermore if $FL = V$, then we call the R-module L a *R-lattice* on V. We say that a R-lattice is unary, binary, ternary, quaternary,..., n-ary, according as its rank is $1, 2, 3, 4, \ldots, n$.

Let U, V be quadratic spaces over F. Let K, L be R-lattices on the quadratic spaces U, V, respectively. We say that K is *represented* by L if there is a representation $\sigma : FK \to FL$ such that $\sigma K \subseteq L$. We say that K and L are *isometric* if there is a representation $\sigma : FK \to FL$ such that $\sigma K = L$. In this case, we write $K \simeq L$.

Let L be a R-lattice on quadratic space V and let $x_1, x_2, \ldots, x_n$ be a basis of L. As before, the $n \times n$ matrix

$$(B(x_i, x_j))_{1 \le i,j \le n}$$

is called the *matrix* of R-lattice L in the basis $x_1, x_2, \ldots, x_n$. In this case, we write

$$L \simeq (B(x_i, x_j)).$$

We say that L is *positive definite* if the matrix $(B(x_i, x_j))$ is positive definite. If $B(x_i, x_j) = 0$ for any $i \neq j$, then we write

$$L \simeq \langle Q(x_1), Q(x_2), \ldots, Q(x_n) \rangle.$$

The determinant

$$\det(B(x_i, x_j))$$

of the $n \times n$ matrix $(B(x_i, x_j))$ is called the *discriminant* of L and we denote it by dL. Note that the discriminant of L in $(F^\times/(R^\times)^2) \cup \{0\}$ is independent of the basis of L. If $dL \neq 0$, then we say that L is a *regular* R-lattice.

Let L be a regular R-lattice on quadratic space V. We define the *dual lattice* $L^\sharp$ of L by

$$L^\sharp = \{x \in V \mid B(x, L) \subseteq R\}.$$

Let V be a regular non-zero quadratic space and let K, L be R-lattices on the quadratic space V. We say that K and L are in the same *class* if

$$K = \sigma L \quad \text{for some } \sigma \in O(V).$$

This is obviously an equivalence relation on the set of all R-lattices on V and we consequently obtain a partition of this set into equivalence classes. We use

$$\mathrm{cls} L$$

to denote the class of L. The subgroup $O(L)$ of $O(V)$ is defined as follows:

$$O(L) = \{\sigma \in O(V) \mid \sigma L = L\}.$$

We call $O(L)$ the *isometry group* of L and we denote the order $|O(L)|$ of $O(L)$ by $o(L)$. We also definie the subgroup $O^+(L)$ of $O(L)$ by

$$O^+(L) = O(L) \cap O^+(V).$$

Let L be a R-lattice on the quadratic space V over F. We define the *scale* (*norm*) of L by the R-module generated by the subset $B(L, L)$ ($Q(L)$, respectively) of F. Here

$$B(L, L) = \{B(x, y) \mid x, y \in L\} \quad \text{and} \quad Q(L) = \{Q(x) \mid x \in L\}.$$

We denote the scale (norm) of L by $\mathfrak{s}L$ ($\mathfrak{n}L$, respectively). Note that

$$2\mathfrak{s}L \subseteq \mathfrak{n}L \subseteq \mathfrak{s}L.$$

Let L be a $\mathbb{Z}$-lattice on quadratic space V over $\mathbb{Q}$. The *genus* $\mathrm{gen}L$ of $\mathbb{Z}$-lattice L on V is defined by the set of all $\mathbb{Z}$-lattices K on V with the following property: for each finite prime p there exists an isometry $\Sigma_p \in O(V_p)$ such that $K_p = \Sigma_p L_p$. Here L_p is the $\mathbb{Z}_p$-lattice $L \otimes \mathbb{Z}_p$. We say that the $\mathbb{Z}$-lattice on V is in the same *spinor genus* as L if there is an isometry $\sigma \in O(V)$ and a rotation $\Sigma_p \in O'(V)$ at each finite prime such that

$$K_p = \sigma_p \Sigma_p L_p,$$

for every finite prime p. The *spinor genus* spn$L$ of $\mathbb{Z}$-lattice L is defined by the set of all $\mathbb{Z}$-lattices in the same spinor genus as L. Clearly we have cls$L \subseteq$ spn$L \subseteq$ gen$L$. We define $h(L)$ by the number of classes in gen$L$ and call it the *class number* of L. We also define $g(L)$ by the number of spinor genera in gen$L$. Note that $h(L)$ and $g(L)$ is always finite. We define

$$w(L) = \sum_{[M]\in \mathrm{gen}(L)} \frac{1}{o(M)} \quad \text{and} \quad r(K, \mathrm{gen}(L)) = \frac{1}{w(L)} \sum_{[M]\in \mathrm{gen}(L)} \frac{r(K,M)}{o(M)},$$
$$w_s(L) = \sum_{[M]\in \mathrm{spn}(L)} \frac{1}{o(M)} \quad \text{and} \quad r(K, \mathrm{spn}(L)) = \frac{1}{w_s(L)} \sum_{[M]\in \mathrm{spn}(L)} \frac{r(K,M)}{o(M)},$$

for any $\mathbb{Z}$-lattice K.

For a ternary $\mathbb{Z}$-lattice $L = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \mathbb{Z}x_3$, the corresponding ternary quadratic form f_L is defined by

$$f_L = \sum_{1\le i,j\le 3} B(x_i, x_j)x_ix_j.$$

We always assume that unless stated otherwise,

any $\mathbb{Z}$-lattice L is a *positive definite* $\mathbb{Z}$-lattice such that $\mathfrak{n}(L) = \mathbb{Z}$.

Hence $4 \cdot dL$ is an integer. If dL is not an integer, then the Legendre symbol $\left(\frac{dL}{p}\right)$ for an odd prime p is defined as $\left(\frac{4\cdot dL}{p}\right)$. For any odd integer n, we say n (does not) divides dL if n (does not, respectively) divides the integer $4 \cdot dL$.

A binary form $ax^2 + bxy + cy^2$ will be denoted by $[a, b, c]$ and a ternary form $ax^2 + by^2 + cz^2 + dyz + ezx + fxy$ will be denoted by $[a, b, c, d, e, f]$.

If an integer n is represented by L over $\mathbb{Z}_p$ for any prime p including the infinite prime, then we say that n is represented by the genus of L, and we write $n \to \mathrm{gen}(L)$. When n is represented by the lattice L itself, then we write $n \to L$. The class of L in the genus of L will be denoted by $[L]$. For any odd prime p, Δ_p is denoted by a non square unit in $\mathbb{Z}_p^\times$.

2.2 Splitting integers

Let Ω be the set of all primes including ∞. Consider the following multiplicative group

$$\prod_{p\in\Omega} \mathbb{Q}_p^\times.$$

An elements of this group is defined in terms of its p-coordinates, say

$$i = (i_p)_{p\in\Omega} \quad (i_p \in \mathbb{Q}_p^\times).$$

Then the multiplication in the direct product is coordinatewise. We say an element i in the above big group *idèle* if it satisfies the following extra condition:

$$|i_p|_p = 1 \text{ for almost all } p \in \Omega.$$

Then the set of all idèles is a subgroup of the direct product. This subgroup is called the group of idèles, and denoted by $J_{\mathbb{Q}}$. Let $\mathbb{Q}^+$ be the set of all positive rational numbers. Let $P_{\mathbb{Q}^+}$ be the group of principal idèles of the form $(\alpha)_{p\in\Omega}$, where $\alpha \in \mathbb{Q}^+$. For a $\mathbb{Z}$-lattice L, we define the subgroup $J_{\mathbb{Q}}^L$ of $J_{\mathbb{Q}}$ by

$$J_{\mathbb{Q}}^L = \{i \in J_{\mathbb{Q}} \mid i_p \in \theta(O^+(L_p)) \text{ for every finite prime } p\}.$$

Let L be a ternary $\mathbb{Z}$-lattice with discriminant d. Assume that c is a non-zero integer satisfying

$$-cd \notin (\mathbb{Q}^\times)^2.$$

For p in Ω, we define

$$N_c(p) = \{\beta \in \mathbb{Q}_p^\times \mid (\beta, -cd)_p = 1\},$$

where $(\ ,\ )_p$ is the Hilbert symbol. Now we define the subgroup N_c of $J_{\mathbb{Q}}$ by

$$N_c = \{i \in J_{\mathbb{Q}} \mid i_p \in N_c(p) \text{ for all prime } p \in \Omega\}.$$

We call c a *splitting integer* for gen(L) if c is represented by gen(L) and $[J_{\mathbb{Q}} : N_c P_{\mathbb{Q}^+} J_{\mathbb{Q}}^L] = 2$. In this case, gen$(L)$ is split into two half-genera. The half-genus containing the class $[L]$ is denoted by $H_L(c)$ and the other half-genus is denoted by $H_{\tilde{L}}(c)$.

2.3 The Minkowski-Siegel formula

Throughout this section, R and F denote the p-adic integer ring $\mathbb{Z}_p$ and the p-adic number field $\mathbb{Q}_p$, respectively. Here p is a prime number. We denote the set of $m \times n$ matrices with entries in R by $M_{m,n}(R)$ and we put $M_m(R) = M_{m,m}(R)$. For matrices X, Y, we put $Y[X] = X^tYX$ if it is defined. We say that a symmetric matrix S over R is *regular* if the determinant of S is non-zero.

First, we introduce the notion of local density.

Lemma 2.3.1. *Let S and T be regular symmetric matrices over R of degree s and t, respectively. Put*

$$E_t(R) = \{B = B^t \in M_t(R) \mid \textit{all diagonal entries of } B \textit{ is in } 2R\}$$

and take an integer h such that p^hT^{-1} is in $E_t(R)$. For $G \in M_{s,t}(R)$ and nonnegative integers r, e, we put

$$\begin{aligned} &A_{p^r}(T, S; G, p^e) = \\ &\quad \{X \in M_{s,t}(R) \ mod\ p^r \mid S[X] \equiv T \ mod\ p^r E_t(R),\ X \equiv G \ mod\ p^e\}. \end{aligned}$$

If $r \geq h + max\,(e, 1)$, then we have

$$(p^{r+1})^{t(t+1/2-st)} \sharp A_{p^{r+1}}(T, S; G, p^e) = (p^r)^{t(t+1/2-st)} \sharp A_{p^r}(T, S; G, p^e).$$

Proof. See Lemma 5.6.1 in [16]. □

Lemma 2.3.2. *Let S and T be regular symmetric matrices over F of degree s and t, respectively and let $G \in M_{s,t}(R)$. Put*

$$\begin{aligned} &B_{p^r}(T, S; G, p^e) = \\ &\quad \{X \in M_{s,t}(R)\ mod\, p^r S^{-1} M_{s,t}(R) \,|\, S[X] \equiv T\ mod\, p^r E_t(R),\ X \equiv G\ mod\, p^e\}. \end{aligned}$$

Let h' be an integer such that $p^{h'}S^{-1} \in E_s(R)$. Then $B_{p^r}(T, S; G, p^e)$ is well-defined for $r \geq h' + e$. For an integer a and a sufficiently large r, we have

$$\sharp B_{p^{r+a}}(p^a T, p^a S; G, p^e) = \sharp B_{p^r}(T, S; G, p^e).$$

Furthermore if S and T are integral and $r \geq h' + max\,(e, 1)$, then we have

$$\sharp A_{p^r}(T, S; G, p^e) = p^{t ord_p(det S)} \sharp B_{p^r}(T, S; G, p^e).$$

Proof. See Lemma 5.6.3 in [16]. □

Lemma 2.3.3. *Let S and T be regular symmetric matrices over F of degree s and t, respectively and let $G \in M_{s,t}(R)$. Then for a sufficiently large r, $(p^r)^t(t + 1/2 - st) \sharp B_{p^r}(T, S; G, p^e)$ is independent of r.*

Proof. See Lemma 5.6.4 in [16]. □

Lemma 2.3.4. *Let S and T be regular symmetric matrices over R of degree s and t, respectively and let $G \in M_{s,t}(R)$. Put*

$$\begin{aligned} &A'_{p^r}(T, S; G, p^e) = \\ &\quad \{X \in M_{s,t}(R) \ mod\ p^r \mid S[X] \equiv T \ mod\ p^r,\ X \equiv G \ mod\ p^e\}. \end{aligned}$$

Then we have

$$\sharp A'_{p^r}(T, S; G, p^e) = 2^{t\delta_{2,p}} \sharp A_{p^r}(T, S; G, p^e),$$

for a sufficiently large r where δ is Kronecker's delta function.

Proof. See Lemma 5.6.5 in [16]. □

Now we will define the local density. Let K and L be regular R-lattices with rank$K = k$ and rank$L = l$ and let $\{x_i\}$, $\{y_i\}$ be bases of K, L, respectively. Let g be a homomorphism from K to L. Then we have

$$\begin{aligned} &K = Rx_1 + Rx_2 + \cdots + Rx_k \quad \text{and} \quad L = Ry_1 + Ry_2 + \cdots + Ry_l \\ &(g(x_1), g(x_2), \cdots, g(x_k)) = (y_1, y_2, \cdots, y_l)G \quad \text{for some } G \in M_{l,k}(R). \end{aligned}$$

We put

$$T = (B(x_i, x_j))_{1\le i,j\le k} \quad \text{and} \quad S = (B(y_i, y_j))_{1\le i,j\le l}.$$

Then the dual basis of L is given by $(y_1, y_2, \cdots, y_l)S^{-1}$. We put

$$\begin{aligned}\tilde{B}_{p^r}(K, L; g, p^e) = \{\sigma : K \to L/p^r L^\sharp \mid\ & Q(\sigma(x)) \equiv Q(x) \bmod 2p^r,\\ & \sigma(x) \equiv g(x) \bmod p^e L \text{ for } x \in K\}.\end{aligned}$$

Then $\tilde{B}_{p^r}(K, L; g, p^e)$ is canonically identified with $B_{p^r}(T, S; G, p^e)$ through matrix representation. We define

$$\begin{aligned}\beta_p(K, L; g, p^e) &= p^{k\mathrm{ord}_p^{dL}} \lim\nolimits_{r\to\infty}(p^r)^{k(k+1)/2-kl}\sharp\tilde{B}_{p^r}(K, L; g, p^e)\\ &= p^{k\mathrm{ord}_p^{dL}} \lim\nolimits_{r\to\infty}(p^r)^{k(k+1)/2-kl}\sharp B_{p^r}(T, S; G, p^e).\end{aligned}$$

By Lemma 2.3.3, above definition is well-defined. If $e = 0$, then the additional condition $\sigma(x) \equiv g(x) \bmod p^e L$ is clearly satisfied. Hence we put

$$\begin{aligned}A_{p^r}(T, S) &= A_{p^r}(T, S; G, p^0),\\ A'_{p^r}(T, S) &= A'_{p^r}(T, S; G, p^0),\\ \beta_{p^r}(K, L) &= \beta_{p^r}(K, L; g, p^0).\end{aligned}$$

Now we define *local density* as follows:

$$\alpha_p(K, L) = 2^{k\delta_{2,p}-\delta_{k,l}}\beta_{p^r}(K, L),$$

where δ is Kronecker's delta function. If we assume that $\mathfrak{s}(K), \mathfrak{s}(L) \subseteq R$, then by Lemma 2.3.2, we have

$$\beta_{p^r}(K, L) = \lim_{r\to\infty}(p^r)^{k(k+1)/2-kl}\sharp A_{p^r}(T, S).$$

Therefore by Lemma 2.3.4, we also have

$$\alpha_p(K, L) = 2^{-\delta_{k,l}}\lim_{r\to\infty}(p^r)^{k(k+1)/2-kl}\sharp A'_{p^r}(T, S).$$

Actually, this is Siegel's original definition of local density.

For a positive definite $\mathbb{Z}$-lattice M on a positive definite quadratic space over $\mathbb{Q}$, we put

$$w(M) = \sum_{[N]\in \mathrm{gen}(M)} \frac{1}{o(N)},$$

where $[N]$ is the equivalence class containing N in the genus $\mathrm{gen}(M)$.

The following theorem is the famous Minkowski-Siegel formula.

Theorem 2.3.5. *Let K and L be positive definite $\mathbb{Z}$-lattices with $rankK = k$ and $rankL = l$ and put*

$$\epsilon_{k,l} = \begin{cases} 1/2 & \text{if either} \quad l = k+1 \quad \text{or} \quad l = m \geq 1, \\ 1 & \text{otherwise.} \end{cases}$$

Then we have

$$\begin{aligned} &\frac{1}{w(L)} \sum_{[M]\in gen(L)} \frac{r(K,M)}{o(M)} \\ &= \epsilon_{k,l}\pi^{k(2l-k+1)/4} \prod_{i=0}^{k-1} \Gamma((l-i)/2)^{-1} (dL)^{-k/2} (dK)^{(l-k-1)/2} \\ &\quad \times \prod_{p<\infty} \alpha_p(K_p, L_p), \end{aligned}$$

where $r(K,M)$ is the number of the representations from K to M and Γ is the ordinary gamma function.

Proof. See Theorem 6.8.1 in [16]. □

2.4 Calculations of local densities

Let L be a regular $\mathbb{Z}_p$-lattice with $\mathrm{rank}L = l$ and let n be a nonzero p-adic integer. We assume that the norm $\mathfrak{n}L$ is $\mathbb{Z}_p$ and $l > 1$. In this section, we provide an explicit formula for the local density $\alpha_p(n, L)$ of n by L which is defined in previous section. The results in this section were proved by Yang in [22].

First, assume that $p \neq 2$. Then we may assume that

$$L \simeq \langle \epsilon_1 p^{t_1}, \epsilon_2 p^{t_2}, \ldots, \epsilon_l p^{t_l} \rangle,$$

where $\epsilon_i \in \mathbb{Z}_p^\times$ and $t_1 \le t_2 \le \cdots \le t_l$. Here $t_1 = 0$ by the assumption on L. For each positive integer k, we put

$$L(k,1) = \{1 \le i \le l \mid t_i - k < 0 \text{ is odd}\} \quad \text{and} \quad l(k,1) = \sharp L(k.1).$$

Furthermore, we define

$$d(k) = k + \frac{1}{2}\sum_{t_i<k}(t_i - k) \quad \textit{and} \quad \delta_p = \begin{cases} 1 & \text{if} \quad p \equiv 1 \pmod 4, \\ \sqrt{-1} & \text{if} \quad p \equiv 3 \pmod 4). \end{cases}$$

For $n = \beta p^a$ with $\beta \in \mathbb{Z}_p^\times$ and $a \in \mathbb{Z}$, we put

$$f_1(n) = \begin{cases} \dfrac{-1}{p} & \text{if} \quad l(a+1,1) \text{ is even}, \\ \left(\dfrac{\beta}{p}\right)\dfrac{1}{\sqrt{p}} & \text{if} \quad l(a+1,1) \text{ is odd}, \end{cases}$$

and

$$R_1(n,L) = (1-p^{-1}) \sum_{\substack{0<k\le a \\ l(k,1) \text{ is even}}} v_k p^{d(k)} + v_{a+1}p^{d(a+1)}f_1(n).$$

Here

$$v_k = \begin{cases} \delta_p^{3l(k,1)} \displaystyle\prod_{i\in L(k,1)} \left(\frac{\epsilon_i}{p}\right) & \text{if} \quad l(k,1) \text{ is even}, \\ \delta_p^{3l(k,1)+1} \displaystyle\prod_{i\in L(k,1)} \left(\frac{\epsilon_i}{p}\right) & \text{if} \quad l(k,1) \text{ is odd}. \end{cases}$$

Theorem 2.4.1. *Under the same notations given above, we have*

$$\alpha_p(n,L) = 1 + R_1(n,L).$$

Proof. See Theorem 3.1 in [22]. □

Finally, assume that $p = 2$. Then L is isometric to

$$\langle \epsilon_1 2^{t_1}, \epsilon_2 2^{t_2}, \ldots, \epsilon_P 2^{t_P}\rangle \perp \left(\perp_{i=1}^M 2^{m_i}\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}\right) \perp \left(\perp_{j=1}^N 2^{n_j}\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}\right),$$

where $\epsilon_h \in \mathbb{Z}_2^\times, t_h, m_i$ and n_j are all integers. The assumption on L means that the smallest integer among all l_h, m_i and n_j is zero. Note that $P+2M+2N=l$. For each positive integer k, we put

$$\begin{aligned}
&L(k,1) = \{1 \le h \le P \mid t_h - k < 0 \text{ is odd}\}, \quad l(k,1) = \sharp L(k.1),\\
&p(k) = (-1)^{\sum_{n_j<k}(n_j-k)},\\
&\epsilon(k) = \prod_{h \in L(k-1,1)} \epsilon_h,\\
&d(k) = k + \tfrac{1}{2}\sum_{t_h<k-1}(t_h-k+1) + \sum_{m_i<k}(m_i-k) + \sum_{n_j<k}(n_j-k),\\
&\delta(k) = \begin{cases} 0 & \text{if} \quad t_h = k-1 \text{ for some } h,\\ 1 & \text{otherwise.}\end{cases}
\end{aligned}$$

Let $\psi(x) = e^{-2\pi i\lambda(x)}$ be the canonical character of $\mathbb{Q}_p$, where $\lambda : \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow \mathbb{Q}/\mathbb{Z}$. Now we define for $n = \beta 2^a$ with $\beta \in \mathbb{Z}_2^\times$ and $a \in \mathbb{Z}$,

$$\begin{aligned}
R_1(n,L) = &\sum_{\substack{0<k\le a+3\\ l(k-1,1) \text{ is odd}}} \delta(k)p(k)\left(\frac{2}{\mu\epsilon(k)}\right)2^{d(k)-3/2}\\
&+\sum_{\substack{0<k\le a+3\\ l(k-1,1) \text{ is even}}} \delta(k)p(k)\left(\frac{2}{\epsilon(k)}\right)2^{d(k)-1}\psi\left(\frac{\mu}{8}\right)\mathrm{char}(4\mathbb{Z}_2)(\mu),
\end{aligned}$$

where

$$\left(\frac{2}{x}\right) = \begin{cases}(2,x)_2 & \text{if} \quad x \in \mathbb{Z}_2^\times,\\ 0 & \text{otherwise,}\end{cases}$$

and $\mu = \mu_k(n)$ is given by

$$\mu_k(n) = \beta 2^{a-k+3} - \sum_{t_h<k-1}\epsilon_h.$$

Here char(X) is the characteristic function of a set X and $(\ ,\ )_p$ is the Hilbert symbol.

Theorem 2.4.2. *Under the same notations given above, we have*

$$\alpha_2(n, L) = 1 + R_1(n, L).$$

Proof. See Theorem 4.1 in [22]. $\square$

Chapter 3

Representations of squares by ternary forms

In this chapter, we resolve the conjecture given by Cooper and Lam in [4].

3.1 Indistinguishable by squares

In this section, we investigate various properties on the representations of squares by ternary quadratic forms.

Definition 3.1.1. Let L be a ternary $\mathbb{Z}$-lattice. We say the genus of L is *indistinguishable by squares* if $r(n^2, L) = r(n^2, L')$ for any $L' \in \mathrm{gen}(L)$ and any integer n.

Let L be a ternary $\mathbb{Z}$-lattice. It is obvious that if the genus of L does not represent any squares of integers, that is, $r(n^2, L') = 0$ for any integer n and any $L' \in \mathrm{gen}(L)$, or the class number of L is one, then the genus of L is indistinguishable by squares. As pointed out in the above, some genera of ternary $\mathbb{Z}$-lattices are obviously indistinguishable by squares.

Lemma 3.1.2. *Let L be a ternary $\mathbb{Z}$-lattice and let $V = \mathbb{Q} \otimes L$ be the quadratic space. Then $r(n^2, L') = 0$ for any $L' \in gen(L)$ and any positive integer n if and only if $d(V_p) = -1$ and $S_p(V) \neq (-1, -1)_p$ for some prime p. Here $d(V_p)$ is the discriminant of V_p, where $V_p = V \otimes \mathbb{Q}_p$ is the quadratic space over $\mathbb{Q}_p$, and $S_p(V)$ is the Hasse symbol of V over $\mathbb{Q}_p$.*

Proof. The lemma follows directly from the fact that $r(n^2, L') = 0$ for any $L' \in \text{gen}(L)$ and any positive integer n if and only if 1 is not represented by V. $\square$

Remark 3.1.3. In fact, it is possible that 1 is not represented by a genus which is indistinguishable by squares. For example, let $L \simeq \langle 2, 3, 24\rangle$. Then one may easily check that the class number of L is 2 and the other lattice in the genus is $L' \simeq \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \perp \langle 6\rangle$. By checking the local structure at $p = 2$ and 3, one may easily show that $r(n^2, L) = r(n^2, L') = 0$ for any integer n not divisible by 6. Assume that

$$36n^2 = 2x^2 + 3y^2 + 24z^2$$

for some integers n and x, y, z. Then one may easily check that $x \equiv 0 \pmod 6$ and $y \equiv 0 \pmod 2$. Therefore we have $r(36n^2, L) = r(3n^2, \langle 1, 2, 6\rangle)$. Similarly, one may also check that $r(36n^2, L') = r(3n^2, \langle 1, 2, 6\rangle)$. Therefore we have $r(n^2, L) = r(n^2, L')$ for any integer n.

Lemma 3.1.4. *Let L be a ternary $\mathbb{Z}$-lattice and let n be a positive integer. For any prime p, we assume that $ord_p(n) = \lambda_p$. If 1 is represented by the genus of L, then we have*

$$\begin{aligned}\frac{r(n^2, gen(L))}{r(1, gen(L))} &= n \prod_{p \mid n, p \mid 8dL} \frac{\alpha_p(n^2, L)}{\alpha_p(1, L)} \prod_{p \mid n, p \nmid 8dL} \frac{\alpha_p(n^2, L)}{\alpha_p(1, L)} \\ &= \prod_{p \mid n, p \mid 8dL} p^{\lambda_p} \cdot \frac{\alpha_p(n^2, L)}{\alpha_p(1, L)} \prod_{p \mid n, p \nmid 8dL} \left(\frac{p^{\lambda_p+1} - 1}{p-1} - \left(\frac{-dL}{p}\right) \frac{p^{\lambda_p} - 1}{p-1}\right).\end{aligned}$$

In particular, if the lattice L has class number 1, then we have

$$\begin{aligned}r(n^2, L) = \ & r(1, L) \prod_{p \mid n, p \mid 8dL} p^{\lambda_p} \cdot \frac{\alpha_p(n^2, L)}{\alpha_p(1, L)} \\ & \times \prod_{p \mid n, p \nmid 8dL} \left(\frac{p^{\lambda_p+1} - 1}{p-1} - \left(\frac{-dL}{p}\right) \frac{p^{\lambda_p} - 1}{p-1}\right).\end{aligned}$$

Proof. The Minkowski-Siegel formula implies

$$r(n^2, \mathrm{gen}(L)) = \pi^{\frac{3}{2}}\Gamma\left(\frac{3}{2}\right)^{-1} \cdot \frac{1}{\sqrt{dL}} \cdot n \prod_{p<\infty} \alpha_p(n^2, L_p),$$

where α_p is the local density. If p does not divide $8dL$, then by [22],

$$\alpha_p(n^2, L) = \alpha_p(p^{2\lambda_p}, L) = 1 + \frac{1}{p} - \frac{1}{p^{\lambda_p+1}} + \left(\frac{-dL}{p}\right)\frac{1}{p^{\lambda_p+1}}.$$

The lemma follows from this. □

Remark 3.1.5. When the class number of a ternary lattice L is 1, the above lemma gives a closed formula, which is, in principle, a finite product of local densities, for the number of representations of squares by L. This could by extended to other rank cases. Let L be a $\mathbb{Z}$-lattice with $h(L) = 1$. If the rank of the $\mathbb{Z}$-lattice L is an odd (even) integer greater than 1 (0), then we might have a closed formula of $r(n^2, L)$ ($r(n, L)$, respectively) which is essentially given by a finite product of local densities. For example, if the rank of L is 4, then we have

$$r(n, L) = r(1, L) \prod_{p|2dL} p^{\mathrm{ord}_p(n)} \frac{\alpha_p(n, L)}{\alpha_p(1, L)} \prod_{p\nmid 2dL} \frac{p^{\mathrm{ord}_p(n)+1} - \left(\frac{dL}{p}\right)^{\mathrm{ord}_p(n)+1}}{p - \left(\frac{dL}{p}\right)}.$$

There are some articles dealing with this subject by using different methods such as q-series or modular forms. For example, see [2], [3], [5] and [7].

Lemma 3.1.6. *Let L be a ternary $\mathbb{Z}$-lattice. If for any $L' \in gen(L)$, $r(n^2, L) = r(n^2, L')$ for any integer n such that every prime factor of n divides $8dL$, then the genus of L is indistinguishable by squares.*

Proof. The action of Hecke operators $T(p^2)$ for any prime $p \nmid 8dL$ on theta series of the $\mathbb{Z}$-lattice L gives

$$r(p^2 n, L) + \left(\frac{-ndL}{p}\right) r(n, L) + p \cdot r\left(\frac{n}{p^2}, L\right) = \sum_{[L'] \in \mathrm{gen}(L)} \frac{r^*(pL', L)}{o(L')} r(n, L').$$

Here, if $p^2 \nmid n$, then $r\left(\frac{n}{p^2}, L\right) = 0$ and

$$r^*(pL', L) = \begin{cases} r(pL', L) - o(L) & \text{if } L \simeq L', \\ r(pL', L) & \text{otherwise.} \end{cases}$$

For details, see Chapter 3 of [1]. It is well known that

$$\sum_{[L'] \in \text{gen}(L)} \frac{r^*(pL', L)}{o(L')} = p + 1. \tag{3.1.1}$$

Assume that $r(n^2, L) = r(n^2, L')$ for any $L' \in \text{gen}(L)$. Then by (3.1.1), we have

$$r(p^2n^2, K) = \left(p + 1 - \left(\frac{-n^2dL}{p}\right)\right) r(n^2, K) - p \cdot r\left(\frac{n^2}{p^2}, K\right),$$

for any $\mathbb{Z}$-lattice $K \in \text{gen}(L)$. Therefore if n is not divisible by p, then $r(p^2n^2, L) = r(p^2n^2, L')$ for any $L' \in \text{gen}(L)$. The lemma follows from induction on the number of prime factors not dividing $8dL$ counting multiplicity. □

Now we collect some known results on the number of representations of integers by ternary quadratic forms, which are needed later. Let L be a ternary $\mathbb{Z}$-lattice. For any prime p, the λ_p-transformation (or Watson transformation) is defined as follows:

$$\Lambda_p(L) = \{x \in L : Q(x + z) \equiv Q(z) \pmod{p} \text{ for all } z \in L\}.$$

Let $\lambda_p(L)$ be the non-classic integral lattice obtained from $\Lambda_p(L)$ by scaling $V = L \otimes \mathbb{Q}$ by a suitable rational number. For a positive integer $N = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, we also define

$$\lambda_N(L) = \lambda_{p_1}^{e_1}(\lambda_{p_2}^{e_2}(\cdots \lambda_{p_{k-1}}^{e_{k-1}}(\lambda_{p_k}^{e_k}(L)) \cdots)).$$

Note that $\lambda_p(\lambda_q(L)) = \lambda_q(\lambda_p(L))$ for any primes $p \neq q$.

Lemma 3.1.7. *Let L be a ternary $\mathbb{Z}$-lattice and let p be an odd prime. If the* *unimodular component in a Jordan decomposition of L_p is anisotropic, then*

$$r(pn, L) = r(pn, \Lambda_p(L)).$$

Proof. See [6]. $\square$

Now assume that the $\frac{1}{2}\mathbb{Z}_p$-unimodular component in a Jordan decomposition of L_p is nonzero isotropic. Assume that p is a prime dividing $4dL$. Then by Weak Approximation Theorem, there exists a basis $\{x_1, x_2, x_3\}$ for L such that

$$(B(x_i, x_j)) \equiv \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \perp \langle p^{\mathrm{ord}_p(4dL)}\delta \rangle \pmod{p^{\mathrm{ord}_p(4dL)+1}},$$

where δ is an integer not divisible by p. We define

$$\Gamma_{p,1}(L) = \mathbb{Z}px_1 + \mathbb{Z}x_2 + \mathbb{Z}x_3, \qquad \Gamma_{p,2}(L) = \mathbb{Z}x_1 + \mathbb{Z}px_2 + \mathbb{Z}x_3.$$

Note that the lattice $\Gamma_{p,i}(L)$ depends on the choice of basis for L. However the set $\{\Gamma_{p,1}(L), \Gamma_{p,2}(L)\}$ is independent of the choices of the basis for L. There are exactly two sublattices of L with index p whose norm is contained in $p\mathbb{Z}$. They are, in fact, $\Gamma_{p,1}(L)$ and $\Gamma_{p,2}(L)$. For some properties of these sublattices of L, see [12].

Lemma 3.1.8. *Under the same assumptions given above, we have*

$$r(pn, L) = r(pn, \Gamma_{p,1}(L)) + r(pn, \Gamma_{p,2}(L)) - r(pn, \Lambda_p(L)).$$

Proof. See Proposition 4.1 of [12]. $\square$

3.2 The Cooper and Lam's conjecture

Let L be a ternary $\mathbb{Z}$-lattice whose genus is indistinguishable by squares. Then Lemma 3.1.4 gives a closed formula on $r(n^2, L)$. In this section, we resolve the conjecture given by Cooper and Lam in [4] by using this observation.

Table 3.1: Data for Conjecture 3.2.1

b	c
1	1, 2, 3, 4, 5, 6, 8, 9, 12, 21, 24
2	2, 3, 4, 5, 6, 8, 10, 13, 16, 22, 40, 70
3	3, 4, 5, 6, 9, 10, 12, 18, 21, 30, 45
4	4, 6, 8, 12, 24
5	5, 8, 10, 13, 25, 40
6	6, 9, 16, 18, 24
8	8, 10, 13, 16, 40
9	9, 12, 21, 24
10	30
12	12
16	24
21	21
24	24

Conjectrue 3.2.1. *Let b and c be any integers given in Table 1. Let n be a positive integer and let $\lambda_p = ord_p(n)$ for any prime p. Then the number $r(n^2, \langle 1, b, c\rangle)$ of the integer solutions of the diophantine equation*

$$n^2 = x^2 + by^2 + cz^2$$

is given by the formula of the type

$$r(n^2, \langle 1, b, c\rangle) = \left(\prod_{p|2bc} g(b, c, p, \lambda_p)\right)\left(\prod_{p\nmid 2bc} h(b, c, p, \lambda_p)\right), \tag{3.2.1}$$

where

$$h(b, c, p, \lambda_p) = \frac{p^{\lambda_p+1} - 1}{p - 1} - \left(\frac{-bc}{p}\right)\frac{p^{\lambda_p} - 1}{p - 1} \tag{3.2.2}$$

and $g(b, c, p, \lambda_p)$ has to be determined on an individual and case-by-case basis.

Cooper and Lam verified the conjecture when

$$(b,c)=(1,1),(1,2),(1,3),(2,2),(3,3)$$

by using some identities on q-series. In 2014, Guo, Peng and Qin in [8] verified the conjecture when

$$\begin{aligned}(b,c)=(1,4),(1,5),(1,6),(1,8),(2,3),(2,4),&\\ (2,6),(3,6),(4,4),(4,8),(5,5),(6,6)&\end{aligned}$$

by using theory of modular forms of weight 3/2. In 2016, Hürlimann in [9] verified the conjecture when

$$(b,c)=(2,8),(2,16),(8,8),(8,16)$$

by using some q-series identities on Bell ternary quadratic forms.

In fact, by Lemma 3.1.4, the conjecture holds when the class number of $\ell_{b,c}=\langle 1,b,c\rangle$ is one. In this case, we have

$$\prod_{p\mid 2bc} g(b,c,p,\lambda_p)=r(1,\ell_{b,c})\prod_{p\mid 2bc} p^{\lambda_p}\cdot\frac{\alpha_p(n^2,\ell_{b,c})}{\alpha_p(1,\ell_{b,c})}.$$

Therefore we may assume that the class number of $\ell_{b,c}$ is greater than 1, that is,

$$\begin{aligned}(b,c)=(2,13),(2,22),(2,40),(2,70),(3,5),&\\ (3,21),(3,45),(5,13),(8,10),(8,13).&\end{aligned}$$

In fact, one may easily verify that $h(\ell_{b,c})=2$ in all cases given above. In these cases, we define the other $\mathbb{Z}$-lattice in the genus of $\ell_{b,c}$ by $m_{b,c}$.

Definition 3.2.2. Let L be a ternary $\mathbb{Z}$-lattice. For any integer n, let n_1 and n_2 be positive integers such that $P(n_1)\subset P(8dL)$, $(n_2,8dL)=1$ and $n=n_1n_2$. Here $P(n)$ denotes the set of prime factors of n. The lattice L is called *strongly S-regular* if for any positive integer $n=n_1n_2$,

$$r(n_1^2n_2^2,L)=r(n_1^2,L)\cdot\prod_{p\nmid 8dL} h_p(dL,\lambda_p),$$

where $\lambda_p = \mathrm{ord}_p(n)$ for any prime p and

$$h_p(dL, \lambda_p) = \frac{p^{\lambda_p+1}-1}{p-1} - \left(\frac{-dL}{p}\right)\frac{p^{\lambda_p}-1}{p-1}.$$

Lemma 3.2.3. *Let L be a ternary $\mathbb{Z}$-lattice. If L satisfies the condition* (3.2.1) *in Conjecture 3.2.1, then L is strongly S-regular.*

Proof. Suppose that L satisfies the condition (3.2.1) in Conjecture 3.2.1. Note that bc in (3.2.2) should be replaced by dL in general case. Let $n = n_1n_2$, where $P(n_1) \subset P(8dL)$ and $(n_2, 8dL) = 1$. Since

$$r(n_1^2n_2^2, L) = \prod_{p|8dL} g_p(dL, \lambda_p) \prod_{p\nmid 8dL} h_p(dL, \lambda_p), \quad r(n_1^2, L) = \prod_{p|8dL} g_p(dL, \lambda_p),$$

where $\lambda_p = \mathrm{ord}_p(n)$ for any prime p and

$$h_p(dL, \lambda_p) = \frac{p^{\lambda_p+1}-1}{p-1} - \left(\frac{-dL}{p}\right)\frac{p^{\lambda_p}-1}{p-1},$$

we have

$$r(n_1^2n_2^2, L) = r(n_1^2, L) \prod_{p\nmid 8dL} h_p(dL, \lambda_p).$$

Therefore L is strongly S-regular. □

Theorem 3.2.4. *Let L be a ternary $\mathbb{Z}$-lattice. Every $\mathbb{Z}$-lattice in the genus of L is strongly S-regular if and only if gen(L) is indistinguishable by squares.*

Proof. Suppose that $r(n^2, L) = r(n^2, L')$ for any integer n and any $\mathbb{Z}$-lattice L' in the genus of L. Let $n = n_1n_2$, where $P(n_1) \subset P(8dL)$ and $(n_2, 8dL) = 1$. First assume that $r(n_1^2, L) \neq 0$. By Minkowski-Siegel formula, we have

$$\begin{aligned}\frac{r(n_1^2n_2^2, L')}{r(n_1^2, L')} = \frac{r(n_1^2n_2^2, \mathrm{gen}(L))}{r(n_1^2, \mathrm{gen}(L))} &= \prod_{p|8dL} \frac{\alpha_p(n_1^2n_2^2, L'_p)}{\alpha_p(n_1^2, L'_p)} \prod_{p\nmid 8dL} h_p(dL, \lambda_p) \\ &= \prod_{p\nmid 8dL} h_p(dL, \lambda_p)\end{aligned}$$

for any $\mathbb{Z}$-lattice $L' \in \mathrm{gen}(L)$. Hence we have

$$r(n_1^2n_2^2, L') = r(n_1^2, L') \prod_{p \nmid 8dL} h_p(dL, \lambda_p).$$

Now assume that $r(n_1^2, L) = 0$. Then by Lemma 3.1.6, $r(n_1^2n_2^2, L') = 0$ for any $L' \in \mathrm{gen}(L)$. Therefore L' is strongly S-regular for any $L' \in \mathrm{gen}(L)$.

Conversely, suppose that every $\mathbb{Z}$-lattice in the genus of L is strongly S-regular. Let $\mathrm{gen}(L) = \{[L] = [L_1], [L_2], \dots, [L_h]\}$. Let n_1 be any integer such that $P(n_1) \subset P(8dL)$. Note that for any prime $p \nmid 8dL$ and any integer $i \in \{1, 2, \dots, h\}$,

$$r(p^2n_1^2, L_i) + \left(\frac{-dL}{p}\right) r(n_1^2, L_i) = \sum_{[L'] \in \mathrm{gen}(L)} \frac{r^*(pL', L_i)}{o(L')} r(n_1^2, L').$$

Since $r(p^2n_1^2, L_i) = \left(p + 1 - \left(\frac{-dL}{p}\right)\right) r(n_1^2, L_i)$ by the assumption, we have

$$\pi_p(L) \begin{pmatrix} r(n_1^2, L_1) \\ \vdots \\ r(n_1^2, L_h) \end{pmatrix} = (p+1) \begin{pmatrix} r(n_1^2, L_1) \\ \vdots \\ r(n_1^2, L_h) \end{pmatrix},$$

where $\pi_p(L) = \left(\frac{r^*(pL_j, L_i)}{o(L_j)}\right)$ is the transpose of the Eichler's Anzahlmatrix of L at p (see [12]). This implies that $\pi_p(L)$ has an eigenvalue $p+1$ corresponding to the eigenvector $(r(n_1^2, L_1), \dots, r(n_1^2, L_h))$. Assume that

$$r(n_1^2, L_k) = \max(r(n_1^2, L_1), r(n_1^2, L_2), \dots, r(n_1^2, L_h)).$$

Then

$$\begin{aligned}(p+1)r(n_1^2, L_k) &= \sum_{i=1}^{h} \frac{r^*(pL_i, L_k)}{o(L_i)} r(n_1^2, L_i) \\ &\le \sum_{i=1}^{h} \frac{r^*(pL_i, L_k)}{o(L_i)} r(n_1^2, L_k) = (p+1)r(n_1^2, L_k).\end{aligned}$$

This implies that

$$\frac{r^*(pL_i, L_k)}{o(L_i)} r(n_1^2, L_i) = \frac{r^*(pL_i, L_k)}{o(L_i)} r(n_1^2, L_k).$$

Now by class linkage Theorem proved in [10], for any integer $i = 1, 2, \ldots, h$, there is a prime $p \nmid 8dL$ such that $r^*(pL_i, L_k) \neq 0$. This implies that $r(n_1^2, L_i) = r(n_1^2, L_k)$. The theorem follows from this by Lemma 3.1.6. $\square$

Remark 3.2.5. Assume that the class number of a ternary $\mathbb{Z}$-lattice L is two. Then one may easily show that if L is strongly S-regular, then so is the other $\mathbb{Z}$-lattice in the genus of L.

Theorem 3.2.6. *Let L be a ternary $\mathbb{Z}$-lattice representing* 1*. Then L is strongly S-regular if and only if L satisfies the condition* (3.2.1) *in Conjecture 3.2.1.*

Proof. Note that "if" is proved in Lemma 3.2.3. The "only if" is the direct consequence of Theorem 4.3.7 and Lemma 3.1.4. $\square$

Corollary 3.2.7. *Let L be a ternary $\mathbb{Z}$-lattice representing* 1*. Then L satisfies the condition* (3.2.1) *in Conjecture 3.2.1 if and only if gen*(L) *is indistinguishable by squares.*

Proof. The theorem is the direct consequence of Theorem 3.2.4 and Theorem 3.2.6. $\square$

From now on, we prove the conjecture when the class number of the $\mathbb{Z}$-lattice $\ell_{b,c}$ is two. In fact, we will show that each genus of $\ell_{b,c}$ is indistinguishable by squares except the cases when $(b, c) = (3, 5), (3, 21)$ and $(3, 45)$. In the exceptional cases, we show that $r(n^2, \ell_{b,c}) = r(n^2, m_{b,c})$ only when n is odd.

Theorem 3.2.8. *The genera gen*($\ell_{2,40}$)*, gen*($\ell_{2,22}$) *and gen*($\ell_{2,70}$) *are all indistinguishable by squares.*

Proof. Note that

$$\text{gen}(\ell_{2,40}) = \{\ell_{2,40},\ \ell_{8,10}\}, \qquad \text{gen}(\ell_{2,22}) = \left\{\ell_{2,22},\ m_{2,22} = \langle 1\rangle \perp \begin{pmatrix} 6 & 2 \\ 2 & 8 \end{pmatrix}\right\}$$

and

$$\mathrm{gen}(\ell_{2,70}) = \left\{ \ell_{2,70},\ m_{2,70} = \langle 1 \rangle \perp \begin{pmatrix} 8 & 2 \\ 2 & 18 \end{pmatrix} \right\}.$$

Let n be any integer such that $n \equiv 0, 1 \pmod 4$. Define a map

$$\begin{cases} f_n : R(n, \ell_{8,10}) \mapsto R(n, \ell_{2,40}) & \text{by} \quad f_n(x,y,z) = \left(x, 2y, \dfrac{z}{2}\right), \\ f_n : R(n, \ell_{2,22}) \mapsto R\left(n, m_{2,22}\right) & \text{by} \quad f_n(x,y,z) = \left(x, 2z, \dfrac{y-z}{2}\right), \\ f_n : R(n, \ell_{2,70}) \mapsto R\left(n, m_{2,70}\right) & \text{by} \quad f_n(x,y,z) = \left(x, \dfrac{y-z}{2}, 2z\right). \end{cases}$$

One may easily check that f_n is a well defined bijective map. The lemma follows directly from this. $\square$

Lemma 3.2.9. *For any positive integer n such that $n \equiv 1 \pmod 8$,*

$$r(n, \ell_{3,5}) = r\left(n, m_{3,5}\right), \quad r(n, \ell_{3,21}) = r\left(n, m_{3,21} = \langle 1 \rangle \perp \begin{pmatrix} 6 & 3 \\ 3 & 12 \end{pmatrix}\right)$$

and

$$r(n, \ell_{3,45}) = r\left(n, m_{3,45} = \langle 1 \rangle \perp \begin{pmatrix} 12 & 3 \\ 3 & 12 \end{pmatrix}\right).$$

Proof. Since proofs are quite similar to each other, we only provide the proof of the first case. Note that $m_{3,5} = \langle 1 \rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 8 \end{pmatrix}$. Let n be an integer such that $n \equiv 1 \pmod 8$. Define

$$A_1(n) = \{(x,y,z) \in \mathbb{Z}^3 \mid x^2 + 3y^2 + 5z^2 = n,\ x \equiv y \equiv z \equiv 1 \,(\mathrm{mod}\ 2)\},$$

$$A_2(n) = \{(x,y,z) \in \mathbb{Z}^3 \mid x^2 + 3y^2 + 5z^2 = n,\ x \equiv 1 \,(\mathrm{mod}\ 2),\ y \equiv z \equiv 0 \,(\mathrm{mod}\ 2)\},$$

and

$$A_3(n) = \{(x,y,z) \in \mathbb{Z}^3 \mid x^2 + 3y^2 + 5z^2 = n,\ x \equiv y \equiv 0 \,(\mathrm{mod}\ 2),\ z \equiv 1 \,(\mathrm{mod}\ 2)\}.$$

We also define

$B_1(n)=\{(x,y,z)\in\mathbb{Z}^3 \mid x^2+2y^2+8z^2+2yz=n,\ x\equiv y\equiv z\equiv 1 \pmod 2\}$,

$B_2(n)=$
$\{(x,y,z)\in\mathbb{Z}^3 \,|x^2+2y^2+8z^2+2yz=n, x\equiv 1 \pmod 2, y\equiv z\equiv 0 \pmod 2\}$,

and

$B_3(n)=$
$\{(x,y,z)\in\mathbb{Z}^3 \,| \,x^2+2y^2+8z^2+2yz=n, x\equiv z\equiv 1 \pmod 2,\ y\equiv 0 \pmod 4\}$.

Then it is clear that

$R(n,\ell_{3,5})=A_1(n)\cup A_2(n)\cup A_3(n)$ and $R(n,m_{3,5})=B_1(n)\cup B_2(n)\cup B_3(n)$.

One may easily check that $y\equiv z \pmod 4$ for any $(x,y,z)\in A_2(n)$, $x\not\equiv y \pmod 4$ for any $(x,y,z)\in A_3(n)$ and $y\not\equiv z \pmod 4$ for any $(x,y,z)\in B_1(n)$.

We prove that

$$|A_1(n)|+|A_2(n)|=|B_1(n)|+|B_2(n)| \quad \text{and} \quad |A_3(n)|=|B_3(n)|,$$

which implies the assertion. First, we define a map $f : A_1(n)\cup A_2(n)\mapsto B_1(n)\cup B_2(n)$ by

$$f(x,y,z)=\left(x,\frac{y+3z}{2},\frac{y-z}{2}\right).$$

Since

$$x^2+3y^2+5z^2=x^2+2\left(\frac{y+3z}{2}\right)^2+2\left(\frac{y+3z}{2}\right)\left(\frac{y-z}{2}\right)+8\left(\frac{y-z}{2}\right)^2$$

and

$$\frac{y+3z}{2}-\frac{y-z}{2}=2z\equiv 0 \pmod 2,$$

the above map f is well defined. Furthermore one may easily check that f

is bijective.

To show that $|A_3(n)| = |B_3(n)|$, we define

$$A_3^0(n) = \{(x, y, z) \in A_3(n) \mid x - y - 2z \equiv 4 \pmod 8\}$$

and

$$B_3^0(n) = \{(x, y, z) \in B_3(n) \mid 2x - y + 2z \equiv 0 \pmod 8\}.$$

Since $(x, y, z) \in A_3(n) \iff (x, y, -z) \in A_3(n)$, we have $2|A_3^0(n)| = |A_3(n)|$. Similarly, we also have $2|B_3^0(n)| = |B_3(n)|$ from the fact that $(x, y, z) \in B_3(n) \iff (-x, y, z) \in B_3(n)$. Now we define a map $g : A_3^0(n) \mapsto B_3^0(n)$ by

$$g(x, y, z) = \left(\frac{x + 3y}{2}, \frac{x - y + 2z}{2}, \frac{-x + y + 2z}{4}\right)$$

and a map $h : B_3^0(n) \mapsto A_3^0(n)$ by

$$h(x, y, z) = \left(\frac{2x + 3y - 6z}{4}, \frac{2x - y + 2z}{4}, \frac{y + 2z}{2}\right).$$

One may easily check that both g and h are well defined and $h \circ g = \text{id}$, $g \circ h = \text{id}$. $\square$

We put

$$K_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 7 \end{pmatrix}, K_2 = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}, L_1 = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 4 & 2 \\ 1 & 2 & 8 \end{pmatrix}, L_2 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 8 \end{pmatrix},$$

and

$$T = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix}.$$

Theorem 3.2.10. *Both gen($\ell_{2,13}$) and gen($\ell_{8,13}$) are indistinguishable by squares.*

Proof. Note that

$$m_{2,13} = \langle 1 \rangle \perp \begin{pmatrix} 5 & 2 \\ 2 & 6 \end{pmatrix}, \quad m_{8,13} = \langle 1 \rangle \perp \begin{pmatrix} 5 & 1 \\ 1 & 21 \end{pmatrix}.$$

For any integer n,

$$r(4n^2, \ell_{8,13}) = r(n^2, \ell_{2,13}) \quad \text{and} \quad r(4n^2, m_{8,13}) = r(n^2, m_{2,13}).$$

If n is odd, then

$$r(n^2, \ell_{2,13}) = r(n^2, \ell_{8,13}) \quad \text{and} \quad r(n^2, m_{2,13}) = r(n^2, m_{8,13}).$$

Hence gen$(\ell_{2,13})$ is indistinguishable by squares if and only if gen$(\ell_{8,13})$ is indistinguishable by squares. Therefore it suffices to show that gen$(\ell_{2,13})$ is indistinguishable by squares. Note that

$$\begin{aligned} r(4n^2, \ell_{2,13}) &= r(2n^2, K_1) = r(2n^2, L_1), \\ r(4n^2, m_{2,13}) &= r(2n^2, K_2) = r(2n^2, L_2). \end{aligned}$$

Now by Lemma 3.1.8, we have

$$r(8n^2, L_1) = 2r(4n^2, T) - r(2n^2, K_1), \quad r(8n^2, L_2) = 2r(4n^2, T) - r(2n^2, K_2).$$

Also by Lemma 3.1.6 and Lemma 3.1.7, we have $r(2n^2, K_1) = r(2n^2, K_2)$ for any odd integer n. In fact, by Lemma 3.1.8, we have $r(2n^2, K_1) = r(2n^2, K_2)$ for any integer n. By combining all equalities given above, we have

$$r(16n^2, \ell_{2,13}) - r(4n^2, \ell_{2,13}) = r(16n^2, m_{2,13}) - r(4n^2, m_{2,13}).$$

Furthermore, since $r(4n^2, \ell_{2,13}) = r(n^2, \ell_{2,13})$ and $r(4n^2, m_{2,13}) = r(n^2, m_{2,13})$ for any odd integer n, it suffices to show that $r(n^2, \ell_{2,13}) = r(n^2, m_{2,13})$ for any odd integer n. Now by Lemma 3.1.7, we know that $r(13^2 \cdot n^2, \ell_{2,13}) = r(n^2, \ell_{2,13})$ and $r(13^2 \cdot n^2, m_{2,13}) = r(n^2, m_{2,13})$. Therefore the theorem follows directly from Lemma 3.1.6. $\square$

Proposition 3.2.11. *The genus of $\ell_{5,13}$ is indistinguishable by squares.*

Proof. Note that $\text{gen}(\ell_{5,13}) = \left\{\ell_{5,13}, m_{5,13} = \langle 1\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 33 \end{pmatrix}\right\}$. Since $\ell_{5,13}$ is aniso-tropic over $\mathbb{Z}_5$ and $\mathbb{Z}_{13}$, $r(n^2, \ell_{5,13}) = r(n^2, m_{5,13})$ for any odd integer n by Lemma 3.1.6. If $x^2+5y^2+13z^2 = 4n^2$, then x, y, z are all even. Therefore

$$r(4n^2, \ell_{5,13}) = r(n^2, \ell_{5,13}) \quad \text{and} \quad r(4n^2, m_{5,13}) = r(n^2, m_{5,13}).$$

The proposition follows from this. □

Remark 3.2.12. All genera containing $\ell_{3,5}$, $\ell_{3,21}$ or $\ell_{3,45}$ are not indistinguishable by squares. For example, $r(4, \ell_{3,k}) = 6 \neq 2 = r(4, m_{3,k})$ for any $k = 5, 21$ and 45. However by Lemma 4.1.3, $r(n^2, \ell_{3,k}) = r(n^2, m_{3,k})$ for any odd integer n.

Example 3.2.13. *As pointed out earlier, if the genus of a $\mathbb{Z}$-lattice L is indistinguishable by squares, then we may have a closed formula for $r(n^2, L')$ for any $\mathbb{Z}$-lattice $L' \in gen(L)$. For example, one may easily show that*

$$\frac{\alpha_2(n^2, \ell_{2,13})}{\alpha_2(1, \ell_{2,13})} = \frac{2^{\max(1,\lambda_2)+1} - 3}{2^{\lambda_2}} \quad \textit{and} \quad \frac{\alpha_{13}(n^2, \ell_{2,13})}{\alpha_{13}(1, \ell_{2,13})} = \frac{1}{13^{\lambda_{13}}},$$

where $\lambda_p = ord_p(n)$ for any prime p. Therefore by Lemma 3.1.4, we have

$$\begin{aligned} r(n^2, \ell_{2,13}) = r(n^2, m_{2,13}) = &\, 2(2^{\max(1,\lambda_2)+1} - 3) \\ &\times \prod_{\substack{p \mid n, \\ (p,26)=1}} \left(\frac{p^{\lambda_p+1} - 1}{p-1} - \left(\frac{-26}{p}\right) \frac{p^{\lambda_p} - 1}{p-1} \right). \end{aligned}$$

Chapter 4

Strongly S-regular ternary forms

4.1 Some properties of strongly S-regular ternary forms

Let L be a strongly S-regular ternary $\mathbb{Z}$-lattice. Since we are assuming that the genus of L represents at least one square of an integer, by Lemma 3.1.2, we always have

$$d(L \otimes \mathbb{Q}_p) \neq -1 \quad \text{or} \quad S_p(L \otimes \mathbb{Q}_p) = (-1,-1)_p \quad \text{for any prime } p.$$

Lemma 4.1.1. *Any strongly S-regular ternary $\mathbb{Z}$-lattice L represents all squares of integers that are represented by its genus.*

Proof. Let L be a strongly S-regular ternary $\mathbb{Z}$-lattice. Suppose that there is an integer a such that a^2 is represented by the genus of L, whereas it is not represented by L itself. Then for any prime $p \nmid 8dL$, if $a = p^t \cdot b$ for some integer b such that $(b,p) = 1$, then we have

$$r(p^2a^2, L) = r(b^2, L)\left(\frac{p^{t+2}-1}{p-1} - \left(\frac{-dL}{p}\right)\frac{p^{t+1}-1}{p-1}\right) = 0.$$

On the other hand, if we consider the action of the Hecke operator $T(p^2)$ to the theta series given by L for any prime $p \nmid 8dL$, then we have

$$r(p^2a^2, L) + \left(\frac{-a^2dL}{p}\right) r(a^2, L) + p \cdot r\left(\frac{a^2}{p^2}, L\right) = \sum_{[L'] \in \mathrm{gen}(L)} \frac{r^*(pL', L)}{o(L')} r(a^2, L'),$$

where $r^*(pL', L)$ is the number of primitive representations of pL' by L. For details, see Chapter 3 of [1]. Since $r(a^2, L) = r\left(\frac{a^2}{p^2}, L\right) = 0$, we have

$$r(p^2a^2, L) = \sum_{[L'] \in \mathrm{gen}(L)} \frac{r^*(pL', L)}{o(L')} r(a^2, L').$$

From the assumption, there is a $\mathbb{Z}$-lattice $L' \in \mathrm{gen}(L)$ such that $r(a^2, L') \neq 0$. Furthermore, by Class Linkage Theorem given by [10], there is a prime $q \nmid 8dL$ such that $r^*(qL', L) \neq 0$. These imply that $r(q^2a^2, L) \neq 0$, which is a contradiction. □

Corollary 4.1.2. *Let L be a strongly S-regular ternary $\mathbb{Z}$-lattice. Then every integer m such that m^2 is represented by L is a multiple of $m_s(L) = m_s(f_L)$.*

Proof. The corollary follows directly from the fact that for any prime p, $\mathrm{ord}_p(m_s(L))$ is completely determined by L_p by Lemma 4.1.1. □

Proposition 4.1.3. *Let q be an odd prime and let L be a ternary $\mathbb{Z}$-lattice such that L_q does not represent 1. Assume that $L_q \simeq \langle \Delta_q, q^\alpha\epsilon_1, q^\beta\epsilon_2 \rangle$ for $\epsilon_1, \epsilon_2 \in \mathbb{Z}_q^\times$ and $1 \leq \alpha \leq \beta$.*

(i) If $\alpha \geq 2$ and $L_q \not\simeq \langle \Delta_q, q^2\epsilon_1, q^2\epsilon_2 \rangle$ for some $\epsilon_1, \epsilon_2 \in \mathbb{Z}_p^\times$, then L is strongly S-regular if and only if $\lambda_q(L)$ is strongly S-regular. Furthermore, if one of them is true, then $m_s(L) = q \cdot m_s(\lambda_q(L))$.

(ii) If $\alpha = 1$ and $L_q \not\simeq \langle \Delta_q, q, -q \rangle$, then L is strongly S-regular if and only if $\lambda_q^2(L)$ is strongly S-regular. Furthermore, if one of them is true, then $m_s(L) = q \cdot m_s(\lambda_q^2(L))$.

Proof. Since the proof is quite similar to each other, we only provide the proof of the first case. For any positive integer n, let n_1 and n_2 be positive integers such that $P(n_1) \subset P(8dL)$, $(n_2, 8dL) = 1$ and $n = n_1n_2$, where $P(n)$

is the set of primes dividing n. Suppose that L is strongly S-regular. Then we have

$$r(q^2n_1^2n_2^2, L) = r(q^2n_1^2, L) \prod_{p \nmid 8dL} h_p(dL, \lambda_p),$$

where λ_p and $h_p(dL, \lambda_p)$ are defined in the introduction. Since $r(q^2n_1^2n_2^2, L) = r(n_1^2n_2^2, \lambda_q(L))$ and $r(q^2n_1^2, L) = r(n_1^2, \lambda_q(L))$ by Lemma 3.1.7, we have

$$r(n_1^2n_2^2, \lambda_q(L)) = r(n_1^2, \lambda_q(L)) \prod_{p \nmid 8dL} h_p(dL, \lambda_p).$$

Since the set of primes dividing $8dL$ equals to the set of primes dividing $8d(\lambda_q(L))$ from the assumption, the above equation implies that $\lambda_q(L)$ is strongly S-regular.

Conversely, Suppose that $\lambda_q(L)$ is strongly S-regular. Then we have

$$r(n_1^2n_2^2, \lambda_q(L)) = r(n_1^2, \lambda_q(L)) \prod_{p \nmid 8dL} h_p(d\lambda_q(L), \lambda_p).$$

Hence if $\mathrm{ord}_q(n_1) \geq 1$, then

$$r(n_1^2n_2^2, L) = r(n_1^2, L) \prod_{p \nmid 8dL} h_p(dL, \lambda_p).$$

Note that if $\mathrm{ord}_q(n_1) = 0$, then $r(n_1^2n_2^2, L) = r(n_1^2, L) = 0$. Therefore L is a strongly S-regular lattice.

Now assume that L or $\lambda_q(L)$ is strongly S-regular. Since 1 is not represented by L_q, $m_s(L)$ is divisible by q. Furthermore, since $r(q^2n, L) = r(n, \lambda_q(L))$ by Lemma 3.1.7, we have $m_s(L) = q \cdot m_s(\lambda_q(L))$. □

Proposition 4.1.4. *Let L be a ternary $\mathbb{Z}$-lattice such that L_2 does not represent 1. Assume that $L_2 \simeq \langle \epsilon_1 \rangle \perp M$ for $\epsilon_1 \in \mathbb{Z}_2^\times$.*

(i) If M is an improper modular lattice with norm contained in $4\mathbb{Z}_2$ or $M \simeq \langle 2^\alpha \epsilon_2, 2^\beta \epsilon_3 \rangle$ for $\epsilon_2, \epsilon_3 \in \mathbb{Z}_2^\times$ and nonnegative integers α, β such that $\beta \geq \alpha \geq 2$, then L is strongly S-regular if and only if $\lambda_2(L)$ is strongly S-regular. Furthermore, if one of them is true, then $m_s(L) = 2 \cdot m_s(\lambda_2(L))$.

(ii) If $M \simeq \langle 2^\alpha\epsilon_2, 2^\beta\epsilon_3\rangle$ with $\epsilon_2, \epsilon_3 \in \mathbb{Z}_2^\times$ and nonnegative integers α, β ($\beta \geq \alpha$) such that $0 \leq \alpha \leq 1$, then L is strongly S-regular if and only if $\lambda_2^2(L)$ is strongly S-regular. Furthermore, if one of them is true, then $m_s(L) = 2 \cdot m_s(\lambda_2^2(L))$.

Proof. Since the proof is quite similar to the odd case, the proof is left to the reader. $\square$

Theorem 4.1.5. *Let L be a strongly S-regular ternary $\mathbb{Z}$-lattice. Then there is a positive integer N such that*

1. *$\lambda_N(L)$ is a strongly S-regular lattice such that $m_s(\lambda_N(L))$ is odd square free;*
2. *for any odd prime p dividing $m_s(\lambda_N(L))$,*

$$\lambda_N(L)_p \simeq \langle \Delta_p, p^2\epsilon_1, p^2\epsilon_2\rangle \quad or \quad \langle \Delta_p, p, -p\rangle,$$

where $\epsilon_1, \epsilon_2 \in \mathbb{Z}_p^\times$.

Proof. By Propositions 4.1.3 and 4.1.4, if p^2 divides $m_s(L)$ for some prime p, then $\lambda_p(L)$ or $\lambda_p^2(L)$ is also strongly S-regular. Hence by taking λ_p-transformations to L repeatedly, if needed, we may find an integer n such that $\lambda_n(L)$ is a strongly S-regular lattice such that $m_s(\lambda_n(L))$ is odd square free. If $m_s(\lambda_n(L))$ is one, then there is nothing to prove. Assume that $m_s(\lambda_n(L)) = p_1p_2\cdots p_t$ where $p_i \neq p_j$ are primes. Assume that $p = p_i$ for some $i = 1, 2, \ldots, t$. Then 1 is not represented by $\lambda_n(L)_p$ by Lemma 4.1.1. Hence by Proposition 4.1.3, either $\lambda_p^\iota(\lambda_n(L))$ is a strongly S-regular lattice such that

$$p \cdot m_s(\lambda_p^\iota(\lambda_n(L))) = m_s(\lambda_n(L)) \quad \text{and} \quad 1 \to (\lambda_p^\iota(\lambda_n(L)))_p,$$

where $\iota = 1$ or 2 depending on the structure of $(\lambda_n(L))_p$, or

$$L_p \simeq \langle \Delta_p, p^2\epsilon_1, p^2\epsilon_2\rangle \quad \text{or} \quad \langle \Delta_p, p, -p\rangle, \tag{4.1.1}$$

where $\epsilon_1, \epsilon_2 \in \mathbb{Z}_p^\times$. If n' is the product of primes satisfying the first condition, then $N = n \cdot n'$ satisfies all conditions given in the statement of the theorem. $\square$

Definition 4.1.6. A strongly S-regular ternary $\mathbb{Z}$-lattice is called *terminal* if $m_s(L)$ is an odd square free integer, and for any prime p dividing $m_s(L)$, L_p satisfies the above condition (4.1.1).

Note that for any strongly S-regular lattice L, there is an integer N such that $\lambda_N(L)$ is a terminal strongly S-regular lattice by Theorem 4.1.5. Therefore, to classify all strongly S-regular lattices, in some sense, it suffices to find all terminal strongly S-regular lattices.

Remark 4.1.7. In fact, there are infinitely many terminal strongly S-regular ternary $\mathbb{Z}$-lattices. To show this, let q be a prime such that $q \equiv 5 \pmod 8$. We prove that the diagonal ternary lattice $L(q) = \langle 2, q, q\rangle$ is a terminal strongly S-regular $\mathbb{Z}$-lattice for any prime q satisfying the above condition.

If n is not divisible by q, then $r(n^2, L(q)) = 0$. Furthermore

$$r(q^2n^2, L(q)) = r(qn^2, \lambda_q(L(q))),$$

where $\lambda_q(L(q)) = \langle 1, 1, 2q\rangle$. Let $\lambda_q(L(q)) = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \mathbb{Z}x_3$ such that $(B(x_i, x_j)) = \mathrm{diag}(1, 1, 2q)$. Let u be an integer such that $u^2 \equiv -1 \pmod q$. Let $z = ax_1 + bx_2 + cx_3 \in \lambda_q(L(q))$ such that $Q(z) = qn^2$. Then, since

$$Q(z) = a^2 + b^2 + 2qc^2 \equiv a^2 - u^2b^2 \equiv (a - ub)(a + ub) \equiv 0 \pmod q,$$

$z \in L(q, +) := \mathbb{Z}(qx_1) + \mathbb{Z}(ux_1 + x_2) + \mathbb{Z}x_3$ or $z \in L(q, -) := \mathbb{Z}(qx_1) + \mathbb{Z}(-ux_1 + x_2) + \mathbb{Z}x_3$. Note that $L(q, +) \cap L(q, -) = \mathbb{Z}(qx_1) + \mathbb{Z}(qx_2) + \mathbb{Z}x_3$. Furthermore, $d(L(q, \pm)) = 2q^3$ and the scale of each lattice is $q\mathbb{Z}$. Hence, we have

$$r(q^2n^2, L(q)) = r(qn^2, \lambda_q(L(q))) = 2r(n^2, \langle 1, 1, 2\rangle) - r(n^2, L(q)). \tag{4.1.2}$$

Therefore if we use an induction on $\mathrm{ord}_q(n)$, the assertion follows directly from the fact that $\langle 1, 1, 2\rangle$ is strongly S-regular. Furthermore, since every $\mathbb{Z}$-lattice in the genus of $L(q)$ satisfies the equation (4.1.2), the genus of $L(q)$ is indistinguishable by squares.

Theorem 4.1.8. *For any positive integer m, there are only finitely many strongly S-regular ternary $\mathbb{Z}$-lattices L up to isometry such that $m_s(L) = m$.*

Proof. Let L be a strongly S-regular ternary $\mathbb{Z}$-lattice with $m_s(L) = m$. Since for any ternary lattice K and any prime p, there are only finitely many lattices whose λ_p-transformation is isometric to K, it suffices to show that there are only finitely many terminal strongly S-regular lattice L such that $m_s(L) = m$ under the assumption that $m = q_1q_2\cdots q_s$ is an odd square free integer. When $m = 1$, then we let $s = 0$.

Let $\{x_1, x_2, x_3\}$ be a Minkowski reduced basis for L such that

$$(B(x_i,x_j)) \simeq \begin{pmatrix} a & f & e \\ f & b & d \\ e & d & c \end{pmatrix} (0 \le a \le b \le c \ \text{ and } \ 2|f| \le a,\ 2|e| \le a,\ 2|d| \le b).$$

Recall that $a, b, c, 2d, 2e, 2f$ are relatively prime integers and

$$L = [a, b, c, 2d, 2e, 2f].$$

Let p_t be the t-th smallest odd prime so that $p_1 = 3$, $p_2 = 5$ and so on. Define

$$t' = \min\{t \in \mathbb{N} \mid 4m^6p_t^4 < p_1\cdots p_{t-1}\}.$$

Note that such an integer always exists by the Bertrand-Chebyshev Theorem. Let t'' be the smallest integer such that $2p_{t''} > 6\cdot 3^{s+1}$ and $t_0 = \max\{t', t''\}$. Finally, let t_1 be the integer such that $p_1p_2\cdots p_{t_1-1} \mid dL$, but $p_{t_1} \nmid dL$.

First, assume that $t_1 \ge t_0$. Then

$$4m^6p_{t_1}^4 < p_1p_2\cdots p_{t_1-1} < 4dL \le 4abc \le 4ac^2 \le 4m^2c^2.$$

Hence $m^2p_{t_1}^2 < c$ and we have

$$r(m^2p_{t_1}^2, L) = r\left(m^2p_{t_1}^2, \begin{pmatrix} a & f \\ f & b \end{pmatrix}\right) \le 6\cdot 3^{s+1}.$$

However, since $p_{t_1} \nmid 8mdL$, we have

$$r(m^2p_{t_1}^2, L) = r(m^2, L)\left(p_{t_1} + 1 - \left(\frac{-dL}{p_{t_1}}\right)\right) \ge 2p_{t_1} \ge 2p_{t''} > 6\cdot 3^{s+1}.$$

This is a contradiction.

Finally, assume that $t_1 < t_0$. Choose a positive integer λ_0 such that $p_{t_1}^{\lambda_0} > 3^{s+1}(2\lambda_0+1)$. If $c > m^2 p_{t_1}^{2\lambda_0}$, then

$$r(m^2 p_{t_1}^{2\lambda_0}, L) = r\left(m^2 p_{t_1}^{2\lambda_0}, \begin{pmatrix} a & f \\ f & b \end{pmatrix}\right) \le 6 \cdot 3^s (2\lambda_0+1).$$

This is a contradiction for

$$r(m^2 p_{t_1}^{2\lambda_0}, L) = r(m^2, L)\left(\frac{p_{t_1}^{\lambda_0+1}-1}{p_{t_1}-1} - \left(\frac{-dL}{p_{t_1}}\right)\frac{p_{t_1}^{\lambda_0}-1}{p_{t_1}-1}\right) \ge 2p_{t_1}^{\lambda_0}.$$

Therefore we have $c \le m^2 p_{t_1}^{2\lambda_0}$, which implies that the discriminant of L is bounded by a constant depending only on m. This completes the proof. □

4.2 Strongly S-regular ternary forms representing 1

The aim of this section is to find all strongly S-regular ternary lattices L with $m_s(L) = 1$. Recall that we are assuming that the norm $\mathfrak{n}(L)$ of a $\mathbb{Z}$-lattice L is $\mathbb{Z}$. Hence the scale $\mathfrak{s}(L)$ of L is $\mathbb{Z}$ or $\frac{1}{2}\mathbb{Z}$.

Lemma 4.2.1. *Let L be a strongly S-regular ternary $\mathbb{Z}$-lattice with $m_s(L) = 1$. If $\mathfrak{s}(L) = \mathbb{Z}$ ($\mathfrak{s}(L) = \frac{1}{2}\mathbb{Z}$), then dL is not divisible by at least one prime in* $\{3, 5, 7\}$ ($\{3, 5, 7, 11\}$, respectively)*.*

Proof. Let L be a strongly S-regular ternary $\mathbb{Z}$-lattice with $m_s(L) = 1$. First, assume that $\mathfrak{s}(L) = \frac{1}{2}\mathbb{Z}$. Let $\{x_1, x_2, x_3\}$ be a Minkowski reduced basis for L such that

$$(B(x_i, x_j)) \simeq \begin{pmatrix} 1 & e & d \\ e & a & c \\ d & c & b \end{pmatrix} (1 \le a \le b, 0 \le 2e \le 1,\ -1 \le 2d \le 1,\ 0 \le 2c \le a),$$

where $a, b, 2c, 2d, 2e$ are all integers and at least one of $2c, 2d$ and $2e$ is odd. Let p_t be the t-th smallest odd prime. Suppose, on the contrary, that

$p_1p_2\cdots p_t \mid dL$, whereas $p_{t+1} \nmid dL$ for some $t \geq 4$.

First, assume that $t = 4$. Since $3\cdot 5\cdot 7\cdot 11 \mid dL$ and $13 \nmid dL$ by assumption, we have

$$r(13^2, L) = r(1, L)\left(13 + 1 - \left(\frac{-dL}{13}\right)\right) \geq 26. \tag{4.2.1}$$

If $b \geq 13^2 + 1$, then $r(13^2, L) = r(13^2, [1, 2e, a]) \leq 18$. This is a contradiction and hence we have $1 \leq a \leq b \leq 169$. For all possible finite cases, we may check by direct computations that there are no ternary $\mathbb{Z}$-lattice satisfying the equation (4.2.1). The case when $t = 5$ or 6 can be dealt with similar manner to this.

Finally, assume that $t \geq 7$. Since $p_{t+1} \nmid dL$, we have

$$r(p_{t+1}^2, L) = r(1, L)\left(p_{t+1} + 1 - \left(\frac{-dL}{p_{t+1}}\right)\right) \geq 46.$$

If $t \geq 7$, then $4p_{t+1}^4 < p_1\cdots p_t \leq 4dL \leq 4ab \leq 4b^2$ by Bertrand-Chebyshev Theorem. Hence we have $p_{t+1}^2 < b$. Therefore we have

$$r(p_{t+1}^2, L) = r(p_{t+1}^2, [1, 2e, a]) \leq 18,$$

for any positive integer a. This is a contradiction.

Since the proof of the case when $\mathfrak{s}(L) = \mathbb{Z}$ is quite similar to the above, the proof is left to the reader. $\square$

Theorem 4.2.2. *There are exactly* 207 *strongly S-regular ternary $\mathbb{Z}$-lattices L up to isometry such that $m_s(L) = 1$, which are listed in Tables 4.1 and 4.2.*

Proof. Note that all ternary lattices except those with dagger mark and $\langle 1\rangle \perp [4, 4, 9]$, $\langle 1\rangle \perp [4, 4, 25]$ in Table 4.1 are class number one. Hence they are strongly S-regular. There are exactly 12 ternary lattices in Table 4.1 whose class number is 2. The strongly S-regularities of all these lattices with dagger mark were already proved in Chapter 3. Finally, both $\langle 1\rangle \perp [4, 4, 9]$ and $\langle 1\rangle \perp [4, 4, 25]$ highlighted in boldface have class number 3, and the strongly S-regularities of these two lattices will be proved in Proposition 4.3.3.

There are exactly 30 strongly S-regular lattices L such that $\mathfrak{s}(L) = \frac{1}{2}\mathbb{Z}$ and $h(L) = 2$, which are listed in Table 4.2. In fact, the $\mathbb{Z}$-lattice S_i in Table 4.2 has class number two and the other lattice in the genus is T_i, for any $1 \le i \le 15$. The strongly S-regularities of these lattices will be considered in Proposition 4.3.1. Those lattices highlighted in boldface in Table 4.2 has class number 3, and the proof of the strongly S-regularities of these lattices will be given in Proposition 4.3.6.

Let L be a strongly S-regular ternary $\mathbb{Z}$-lattice. First, assume that $\mathfrak{s}(L) = \mathbb{Z}$. Then $L = \langle 1 \rangle \perp \ell$, for some binary lattice ℓ such that

$$\ell = [a, 2b, c] = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \qquad (0 \le 2b \le a \le c).$$

From the above theorem, the discriminant of L, which is $ac - b^2$, is not divisible by at least one prime in $\{3, 5, 7\}$. We will use the fact that if $p \nmid 2dL$, then

$$r(p^{2t}, L) = r(1, L)\left(\frac{p^{t+1}-1}{p-1} - \left(\frac{-dL}{p}\right)\frac{p^t - 1}{p-1}\right).$$

Assume that $L \simeq \langle 1 \rangle \perp [1, 0, s]$ for some positive integer S. If $3 \nmid s$, then

$$r(9, L) = r(1, L) \cdot \left(4 - \left(\frac{-dL}{3}\right)\right) \ge 12.$$

Hence $s = 1, 2, 4, 5$ or 8. Assume that $s = 3s_1$, for some integer s_1 such that $5 \nmid s_1$. Since $r(25, L) > r(25, [1, 0, 1]) = 12$, we have $3s_1 \le 25$. Therefore $s_1 = 1, 2, 3, 4, 7$ or 8. Assume that $s = 15s_2$ for some integer s_2 such that $7 \nmid s_2$. One may apply similar argument to show that there does not exist a strongly S-regular lattice in this case.

From now on, we assume that $r(1, L) = 2$, that is, $a \ge 2$. Assume that $3 \nmid dL$. Then we have $r(3^2, L) = 6$ or 10, and $r(3^4, L) = 18$ or 34. Hence we have $2 \le a \le 9$. If $a = 9$, then $c = 9$. In this case, one may easily show that

$$r\left(3^4, \langle 1 \rangle \perp \begin{pmatrix} 9 & b \\ b & 9 \end{pmatrix}\right) \ne 18, 34,$$

which is a contradiction. Next assume that $a = 8$. If $c \geq 10$, then

$$r(3^2, L) = r(3^2, [1, 0, 8]) = 6 = 2\left(4 - \left(\frac{-dL}{3}\right)\right).$$

Hence $\left(\frac{-dL}{3}\right) = 1$, which implies that $r(3^4, L) = 18$. If $c \geq 82$, then $r(3^4, L) = r(3^4, [1, 0, 8]) = 10$, which is a contradiction. Therefore we have $8 \leq c \leq 81$. For all possible cases, that is, $a = 8$, $0 \leq 2b \leq 8$ and $8 \leq c \leq 81$, one may easily check only when ℓ is isometric to one of

$$\begin{array}{l} [8,0,8],\ [8,0,10]^{\dagger},\ [8,0,13]^{\dagger},\ [8,0,16],\ [8,0,40], \\ [8,4,18]^{\dagger},\ [8,8,12],\ [8,8,24] \quad \text{and} \quad [8,8,72], \end{array}$$

$L = \langle 1 \rangle \perp \ell$ is strongly S-regular. Note that the class number of L is one if ℓ is isometric to one of binary lattices given above, except binary lattices with dagger mark. The lattices ℓ marked with a dagger are strongly S-regular by Chapter 3. The proof of the remaining cases, that is $2 \leq a \leq 7$, is quite similar to this. In particular, the case when $\ell = [4, 4, 9]$, where the class number of $L = \langle 1 \rangle \perp \ell$ is 3 in this case, will be considered in Proposition 4.3.3.

Assume that dL is divisible by 3, but is not divisible by 5. In this case, we have $r(5^2, L) = 10$ or 14, and $r(5^4, L) = 50$ or 74. Since $r(p^2, [1, 0, a]) \leq 6$ for any prime p and any integer $a \geq 2$, we have $2 \leq a \leq c \leq 25$. For all possible cases, one may easily show that L is strongly S-regular if and only if the class number of L is one, except the case when $\ell = [4, 4, 25]$. For the exceptional case, the proof of the strongly S-regularity of L will be given in Proposition 4.3.3.

Finally, assume that $15 \mid dL$ and $7 \nmid dL$. For all possible cases, L is strongly S-regular if and only if the class number of L is one.

Now assume that $\mathfrak{s}(L) = \frac{1}{2}\mathbb{Z}$. Let $\{x_1, x_2, x_3\}$ be a Minkowski reduced basis for L such that

$$(B(x_i, x_j)) \simeq \begin{pmatrix} 1 & e & d \\ e & a & c \\ d & c & b \end{pmatrix} = [1, a, b, 2c, 2d, 2e],$$

where $a, b, 2c, 2d, 2e$ are integers such that $1 \le a \le b$ and $0 \le 2e \le 1$, $-1 \le 2d \le 1$, $0 \le 2c \le a$. Note that at least one of $2c, 2d, 2e$ is odd. In this case, the discriminant of L is not divisible by at least one prime in $\{3,5,7,11\}$ by the above theorem. Assume that $a = 1$ and $b = 1$. Then clearly, $L \simeq [1,1,1,0,0,1]$ or $[1,1,1,1,1,1]$, all of which are strongly S-regular. Next, assume that $a = 1, e = 0$ and $b \ge 2$. Let $p \in \{3,5,7,11\}$ be a prime not dividing dL. Since $r(p^2, L) = 4p$ or $4(p+2)$ and $r(5^2, [1,0,1]) = 12$, $r(p^2, [1,0,1]) = 4$ for any $p \in \{3,7,11\}$, we have $2 \le b \le p^2$. One may easily show that there are exactly 6 strongly S-regular lattices in this case, all of which have class number 1.

Next assume that $a = 1, 2e = 1$ and $b \ge 2$. In this case, since $r(1, L) = 6$, we have $r(p^2, L) = 6p$ or $6(p+2)$. Furthermore, since $r(7^2, [1,1,1]) = 18$ and $r(p^2, [1,1,1]) = 6$ for any $p \in \{3,5,11\}$, we have $2 \le b \le p^2$. One may easily show that there are exactly 12 strongly S-regular lattices in this case, all of which have class number 1.

From now on, we assume that $r(1, L) = 2$, that is, $a \ge 2$. Assume further that $3 \nmid dL$. Since $r(3^2, L) = 6$ or 10, we have $2 \le a \le 9$. Furthermore, since $r(3^4, L) = 18$ or 34, and $r(3^{2n}, [1, 2e, a]) \le 2(2n+1)$ for any positive integer n, we have

$$\begin{cases} 2 \le a \le b \le 9 & \text{if } r(9, [1, 2e, a]) < 6, \\ 2 \le a \le b \le 81 & \text{if } r(9, [1, 2e, a]) = 6. \end{cases}$$

In this case, we have 30 candidates of strongly S-regular lattices. They are listed in the first row of Table 4.2. Among them, there are exactly 18 lattices having class number 1. The remaining 12 lattices $T_1 \sim T_6$ and $S_1 \sim S_6$ in the first row of Table 4.2 have class number 2. The proof of the strongly S-regularities of these lattices will be considered in Proposition 4.3.1.

Next assume that $3 \mid dL$ and $5 \nmid dL$. Since $r(5^2, L) = 10$ or 14, and

$$r(5^2, [1, 2e, a]) \le 6 < 10,$$

we have $2 \le a \le b \le 25$. In this case, we have twenty two candidates with class number 1, twelve lattices with class number 2, and three lattices with class number 3. The proof of the strongly S-regularities of these lattices having class number 2 (class number 3) will be considered in Proposition

	ℓ
$3 \nmid dL$ (47)	$[1,0,1]$, $[1,0,2]$, $[1,0,4]$, $[1,0,5]$, $[1,0,8]$, $[2,0,2]$, $[2,2,3]$, $[2,0,4]$, $[2,0,5]$, $[2,2,6]$, $[2,0,8]$, $[2,0,10]$, $[2,0,13]^\dagger$, $[2,0,16]$, $[2,2,18]$, $[2,0,22]^\dagger$, $[2,2,33]^\dagger$, $[2,0,40]^\dagger$, $[2,0,70]^\dagger$, $[3,2,3]$, $[3,2,5]$, $[3,2,7]$, $[4,0,4]$, $[4,4,5]$, $[4,0,8]$, $[4,4,8]$, **[4,4,9]**, $[5,0,5]$, $[5,4,6]^\dagger$, $[5,0,8]$, $[5,0,10]$, $[5,4,12]$, $[5,0,13]^\dagger$, $[5,2,21]^\dagger$, $[5,0,25]$, $[5,0,40]$, $[6,4,6]$, $[6,4,8]^\dagger$, $[8,0,8]$, $[8,0,10]^\dagger$, $[8,8,12]$, $[8,0,13]^\dagger$, $[8,0,16]$, $[8,4,18]^\dagger$, $[8,8,24]$, $[8,0,40]$, $[8,8,72]$
$3 \mid dL$, $5 \nmid dL$ (45)	$[1,0,3]$, $[1,0,6]$, $[1,0,9]$, $[1,0,12]$, $[1,0,21]$, $[1,0,24]$, $[2,2,2]$, $[2,0,3]$, $[2,2,5]$, $[2,0,6]$, $[3,0,3]$, $[3,0,4]$, $[3,0,6]$, $[3,0,9]$, $[3,0,12]$, $[3,0,18]$, $[4,4,4]$, $[4,0,6]$, $[4,4,7]$, $[4,0,12]$, $[4,4,13]$, $[4,0,24]$, **[4,4,25]**, $[5,2,5]$, $[6,0,6]$, $[6,6,6]$, $[6,0,9]$, $[6,0,16]$, $[6,0,18]$, $[6,6,21]$, $[6,0,24]$, $[8,8,8]$, $[9,0,9]$, $[9,6,9]$, $[9,0,12]$, $[9,0,21]$, $[9,0,24]$, $[10,4,10]$,$[12,0,12]$, $[12,12,21]$, $[16,16,16]$, $[16,0,24]$, $[21,0,21]$, $[24,0,24]$, $[24,24,24]$
$15 \mid dL$, $7 \nmid dL$ (9)	$[3,0,10]$, $[3,0,30]$, $[4,4,16]$, $[6,6,9]$, $[10,10,10]$, $[10,0,30]$, $[12,12,13]$, $[12,12,33]$, $[40,40,40]$

Table 4.1: Strongly S-regular lattices $L = \langle 1 \rangle \perp \ell$

4.3.1 (Proposition 4.3.6, respectively). Recall that all lattices highlighted in boldface in Tables 4.1 and 4.2 have class number 3.

Now assume that dL is divisible by 15, but not divisible by 7. In this case, we have $2 \le a \le b \le 49$. Everything is quite similar to the above cases. In this case, we have twelve candidates with class number 1, two lattices with class number 2, and one lattice with class number 3.

Finally, assume that $105 \mid dL$ and $11 \nmid dL$. In this case, L is isometric to one of 4 lattices listed in fourth line of Table 4.2. The proof of the strongly S-regularities of these 4 ternary lattices will be considered in Proposition 4.3.1. $\square$

	L
$3 \nmid dL$ (37)	$[1,1,1,1,1,1]$, $[1,1,2,0,1,0]$, $[1,1,2,1,1,1]$, $[1,1,3,1,1,0]$, $[1,1,3,1,1,1]$, $[1,1,5,1,1,1]$, $[1,1,7,1,1,1]$, $[1,2,2,1,1,0]$, $[1,2,2,2,1,1]$, $[1,2,3,0,1,0]$, $[1,2,3,1,0,1]$, $T_1=[1,2,4,1,1,1]$, $S_1=[1,2,4,2,1,0]$, $[1,2,7,0,0,1]$, $[1,2,9,0,1,0]$, $S_2=[1,2,23,0,1,0]$, $[1,3,3,2,1,1]$, $[1,3,4,2,0,1]$, $[1,3,5,1,1,1]$, $[1,3,5,3,1,1]$, $S_3=[1,3,9,2,1,1]$, $S_4=[1,3,10,0,0,1]$, $T_2=[1,3,17,2,1,1]$, $[1,3,22,0,0,1]$, $[1,4,4,3,1,1]$, $[1,4,9,3,1,1]$, $T_3=[1,5,5,1,1,0]$, $T_4=[1,5,6,2,0,1]$, $S_5=[1,5,19,5,1,0]$, $S_6=[1,5,49,5,1,0]$, $[1,7,9,7,1,0]$, $[1,9,9,8,1,1]$, $T_5=[1,9,10,0,0,1]$, $[1,9,15,5,0,1]$, $[1,9,21,7,0,1]$, $T_6=[1,9,29,8,1,1]$, $[1,9,70,0,0,1]$
$3 \mid dL$, $5 \nmid dL$ (47)	$[1,1,1,0,0,1]$, $[1,1,2,0,0,1]$, $[1,1,2,1,1,0]$, $[1,1,3,0,0,1]$, $[1,1,4,0,0,1]$, $[1,1,5,1,1,0]$, $[1,1,6,0,0,1]$, $[1,1,11,1,1,0]$, $[1,1,12,0,0,1]$, $[1,1,18,0,0,1]$, $[1,2,2,1,1,1]$, $[1,2,3,1,1,0]$, $[1,2,3,2,1,0]$, $[1,2,4,2,1,1]$, $[1,2,5,1,1,1]$, $S_7=[1,2,7,0,1,0]$, $S_8=[1,2,9,2,1,0]$, $S_9=[1,2,10,1,0,1]$, $[1,3,4,3,1,0]$, $T_7=[1,3,5,1,0,1]$, $T_8=[1,3,6,0,0,1]$, $L_1=\mathbf{[1,3,7,0,1,0]}$, $[1,3,8,2,0,1]$, $[1,4,4,2,1,1]$, $[1,4,5,2,1,0]$, $T_9=[1,4,5,2,1,1]$, $[1,4,6,3,0,1]$, $[1,4,11,2,1,0]$, $[1,4,13,2,1,1]$, $[1,5,5,4,1,1]$, $[1,5,7,1,0,1]$, $[1,5,7,2,1,1]$, $S_{10}=[1,5,13,5,1,1]$, $S_{11}=[1,5,15,3,0,1]$, $[1,6,7,0,1,0]$, $T_{10}=[1,6,11,6,1,0]$, $S_{12}=[1,6,25,0,1,0]$, $[1,7,7,5,1,1]$, $T_{11}=[1,7,11,5,1,0]$, $L_4=\mathbf{[1,7,12,0,0,1]}$, $[1,7,13,5,1,1]$, $[1,7,18,0,0,1]$, $M_6=\mathbf{[1,7,19,5,1,1]}$, $[1,9,13,9,1,0]$, $T_{12}=[1,13,13,8,1,1]$, $[1,13,15,3,0,1]$, $[1,13,23,13,1,0]$
$15 \mid dL$, $7 \nmid dL$ (18)	$[1,1,4,0,1,0]$ $[1,1,10,0,0,1]$, $[1,1,30,0,0,1]$, $[1,2,7,2,1,1]$, $[1,3,3,1,1,1]$, $[1,4,5,4,1,0]$, $[1.4.15,0,0,1]$, $[1,5,9,5,1,0]$, $[1,6,13,6,1,0]$, $[1,7,7,3,1,0]$, $S_{13}=[1,7,10,0,0,1]$, $T_{13}=[1,7,11,5,1,1]$, $L_{10}=\mathbf{[1,7,30,0,0,1]}$, $[1,7,31,5,1,1]$, $[1,10,19,0,1,0]$, $[1,15,19,15,1,0]$, $[1,19,19,8,1,1]$, $[1,19,30,0,0,1]$
$105 \mid dL$, $11 \nmid dL$ (4)	$S_{14}=[1,2,15,0,0,1]$, $T_{14}=[1,4,7,0,0,1]$, $S_{15}=[1,7,17,7,1,0]$, $T_{15}=[1,11,11,7,1,1]$

Table 4.2: Strongly S-regular lattices L with $\mathfrak{s}(L) = \frac{1}{2}\mathbb{Z}$

4.3 Nontrivial strongly S-regular ternary forms

In this section, we prove the strongly S-regularities of ternary lattices with class number greater than 1 in Tables 4.1 and 4.2.

Proposition 4.3.1. *For $i = 1, 2, \dots, 15$, let S_i and T_i be ternary $\mathbb{Z}$-lattices listed in Table 4.2. The genus gen(S_i) is indistinguishable by squares for any i. Therefore S_i and T_i are strongly S-regular for any $1 \le i \le 15$.*

Proof. Note that the class number of S_i is two, and the other lattice in the genus of S_i is T_i for any $i = 1, 2, \dots, 15$.

We only provide the proofs of the cases when $i = 1, 3, 13, 14$ and 15; the other cases being similar. We put

$$
\begin{array}{lll}
P_1 = [2,4,4,2,2,0], & P_2 = [47,47,47,0,47,47], & P_3 = [1,1,10,0,0,1], \\
Q = [4,8,16,2,4,4], & S_{14,1} = [1,2,60,0,0,1], & S_{14,2} = [2,4,60,0,0,2], \\
S_{14,3} = [2,4,15,0,0,2], & T_{14,1} = [1,4,28,0,0,1], & T_{14,2} = [4,4,28,0,0,2], \\
T_{14,3} = [4,4,7,0,0,2]. & &
\end{array}
$$

First, we consider the case when $i = 1$. By Lemma 3.1.7, we see that $r(13^2n^2, S_1) = r(n^2, S_1)$ and $r(13^2n^2, T_1) = r(n^2, T_1)$ for any integer n. Also by Lemma 3.1.8, we have

$$r(4n^2, S_1) = 2r(4n^2, P_1) - r(n^2, S_1) \text{ and } r(4n^2, T_1) = 2r(4n^2, P_1) - r(n^2, T_1),$$

for any integer n. Since $r(1, S_1) = r(1, T_1)$, gen(S_1) is indistinguishable by squares by Lemma 3.1.6.

We consider the case when $i = 3$. Note that $d(S_3) = 2^{-1} \cdot 47$. By Lemma 3.1.8, we have

$$
\begin{aligned}
r(47^2n^2, S_3) &= 2r(47^2n^2, P_2) - r(n^2, S_3), \\
r(47^2n^2, T_3) &= 2r(47^2n^2, P_2) - r(n^2, T_3),
\end{aligned}
$$

for any integer n. If $x^2 + 3y^2 + 9z^2 + xy + 2yz + zx = 4n^2$, then x, y, z are all even. Hence we have

$$r(4n^2, S_3) = r(n^2, S_3) \quad \text{and} \quad r(4n^2, T_3) = r(n^2, T_3).$$

Therefore $\text{gen}(S_3)$ is indistinguishable by squares by Lemma 3.1.6.

Note that $d(S_{13}) = 2^{-1} \cdot 3^3 \cdot 5$. By Lemma 3.1.7, we have $r(3^2n^2, S_{13}) = r(n^2, P_3)$, $r(5^2n^2, S_{13}) = r(n^2, S_{13})$ and $r(3^2n^2, T_{13}) = r(n^2, P_3)$, $r(5^2n^2, T_{13}) = r(n^2, T_{13})$ for any integer n. If $x^2 + 7y^2 + 10z^2 + xy = 4n^2$, then x, y, z are all even. Hence we have

$$r(4n^2, S_{13}) = r(n^2, S_{13}) \quad \text{and} \quad r(4n^2, T_{13}) = r(n^2, T_{13}).$$

Therefore the genus $\text{gen}(S_{13})$ is indistinguishable by squares by Lemma 3.1.6.

Now, we consider the $\mathbb{Z}$-lattice S_{14}, which is one of the most difficult cases. Note that $d(S_{14}) = 2^{-2} \cdot 3 \cdot 5 \cdot 7$. By Lemma 3.1.7, we have

$$r(p^2n^2, S_{14}) = r(n^2, S_{14}) \quad \text{and} \quad r(p^2n^2, T_{14}) = r(n^2, T_{14}),$$

for any prime $p \in \{3, 5, 7\}$. Let $\{x_1, x_2, x_3\}$ be the basis for S_{14} such that $Q(ax_1+bx_2+cx_3) = a^2+ab+2b^2+15c^2$. Assume that $Q(ax_1+bx_2+cx_3) = 4n^2$. Then we have $a \equiv c \pmod 2$, $b \equiv 0 \pmod 2$ or $c \equiv 0 \pmod 2$. This implies that for $z = ax_1 + bx_2 + cx_3$,

$$z \in \mathbb{Z}(2x_1) + \mathbb{Z}(2x_2) + \mathbb{Z}(x_1 + x_3) \quad \text{or} \quad z \in \mathbb{Z}(x_1) + \mathbb{Z}(x_2) + \mathbb{Z}(2x_3).$$

Therefore we have, for any integer n,

$$r(4n^2, S_{14}) = r(4n^2, Q) + r(4n^2, S_{14,1}) - r(n^2, S_{14}).$$

Similarly, we also have

$$r(4n^2, T_{14}) = r(4n^2, Q) + r(4n^2, T_{14,1}) - r(n^2, T_{14}).$$

Furthermore, one may easily show that

$$\begin{aligned} r(4n^2, S_{14,1}) &= 2r(4n^2, S_{14,2}) - r(n^2, S_{14}), \\ r(4n^2, T_{14,1}) &= 2r(4n^2, T_{14,2}) - r(n^2, T_{14}), \end{aligned}$$

and

$$\begin{aligned} r(4n^2, S_{14,3}) = r(4n^2, S_{14,2}) = 2r(n^2, S_{14}) - r(n^2, S_{14,3}), \\ r(4n^2, T_{14,3}) = r(4n^2, T_{14,2}) = 2r(n^2, T_{14}) - r(n^2, T_{14,3}). \end{aligned} \tag{4.3.1}$$

By combining all equalities given above, we have

$$\begin{aligned} r(4n^2, S_{14}) = r(4n^2, Q) + 2r(n^2, S_{14}) - 2r(n^2, S_{14,3}), \\ r(4n^2, T_{14}) = r(4n^2, Q) + 2r(n^2, T_{14}) - 2r(n^2, T_{14,3}). \end{aligned} \tag{4.3.2}$$

Since $r(1, S_{14}) = r(1, T_{14}) = 2$ and $r(1, S_{14,3}) = r(1, T_{14,3}) = 0$, we have $r(2^{2t}, S_{14}) = r(2^{2t}, T_{14})$ for any positive integer t by (4.3.1) and (4.3.2). Therefore the genus of S_{14} is indistinguishable by squares by Lemma 3.1.6, and $r(n^2, S_{14,3}) = r(n^2, T_{14,3})$ for any integer n. Note that the class number of $S_{14,3}$ is 3. In fact, the proof of the strongly S-regularities of S_9 is quite similar to this.

Finally by Lemma 3.1.7, we have

$$r(p^2n^2, S_{15}) = r(n^2, S_{15}) \quad \text{and} \quad r(p^2n^2, T_{15}) = r(n^2, T_{15})$$

for any prime $p \in \{3, 5, 7\}$. Let $\{y_1, y_2, y_3\}$ be the basis for S_{15} such that $Q(ay_1+by_2+cy_3) = a^2+7b^2+17c^2+7bc+ca$. Assume that $Q(ay_1+by_2+cy_3) = 4n^2$. Then we have $a \equiv b \pmod 2$ and $c \equiv 0 \pmod 2$. Therefore we have

$$r(4n^2, S_{15}) = r(4n^2, \mathbb{Z}(2y_1) + \mathbb{Z}(y_1 + y_2) + \mathbb{Z}(2y_3)),$$

which implies that for any integer n,

$$r(4n^2, S_{15}) = r(n^2, S_{14}).$$

Similarly, we also have

$$r(4n^2, T_{15}) = r(n^2, T_{14}).$$

Therefore $\text{gen}(S_{15})$ is indistinguishable by squares by Lemma 3.1.6. $\square$

Now we consider the lattices having class number 3. We define

$$K_{1,t} = \langle 1 \rangle \perp \begin{pmatrix} 4 & 2 \\ 2 & 2^3 3^t + 1 \end{pmatrix}, \quad K_{2,t} = \langle 1, 1, 2^5 3^t \rangle, \quad K_{3,t} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2^3 3^t + 1 \end{pmatrix},$$

for any non negative integer t.

Lemma 4.3.2. *For any nonnegative integer t, the ternary $\mathbb{Z}$-lattices $K_{1,t}$,$K_{2,t}$ and $K_{3,t}$ are in the same genus. Furthermore, we have*

$$2r(n^2, K_{1,t}) = r(n^2, K_{2,t}) + r(n^2, K_{3,t}),$$

for any integer n.

Proof. Note that $d(K_{i,t}) = 2^5 \cdot 3^t$ for any $i = 1, 2, 3$. By checking local structures at $p = 2$ and 3, one may easily show that all ternary $\mathbb{Z}$-lattices $K_{1,t}, K_{2,t}$ and $K_{3,t}$ are in the same genus for any integer $t \geq 0$. Fix a non negative integer t.

Note that for any integer n, one may easily show that

$$r(4n^2, K_{1,t}) = r(4n^2, K_{2,t}) = r(4n^2, K_{3,t}) = r(n^2, \langle 1, 1, 2^3 3^t \rangle).$$

Assume that n is odd. If $x^2 + 4y^2 + 4yz + (2^3 3^t + 1)z^2 = n^2$, then either x or z is odd, but not both. Hence we have

$$r(n^2, K_{1,t}) = r(n^2, \langle 1, 4, 2^5 3^t \rangle) + r\left(n^2, \langle 4 \rangle \perp \begin{pmatrix} 4 & 2 \\ 2 & 2^3 3^t + 1 \end{pmatrix}\right).$$

Similarly, we have $r(n^2, K_{2,t}) = 2r(n^2, \langle 1, 4, 2^5 3^t \rangle)$. Let $K_{3,t} = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \mathbb{Z}x_3$ such that

$$(B(x_i, x_j)) = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2^3 3^t + 1 \end{pmatrix}.$$

Let $v \in K_{3,t}$ such that $Q(v) = n$. Let a, b, c be integers such that $v = a(x_1 - x_2) + b(x_2 + x_3) + cx_3$. Since $Q(v) \equiv b^2 + c^2 \equiv 1 \pmod 2$, either b or

c is odd, but not both. Hence for any odd integer n,

$$\begin{aligned} r(n, K_{3,t}) = \ & r(n, \mathbb{Z}(x_1 - x_2) + \mathbb{Z}(2x_2 + 2x_3) + \mathbb{Z}x_3) \\ & + r(n, \mathbb{Z}(x_1 - x_2) + \mathbb{Z}(x_2 + x_3) + \mathbb{Z}(2x_3)). \end{aligned}$$

Since

$$\begin{aligned} \mathbb{Z}(x_1 - x_2) + \mathbb{Z}(2x_2 + 2x_3) + \mathbb{Z}x_3 & \simeq \mathbb{Z}(x_1 - x_2) + \mathbb{Z}(x_2 + x_3) + \mathbb{Z}(2x_3) \\ & \simeq \langle 4 \rangle \perp \begin{pmatrix} 4 & 2 \\ 2 & 2^3 3^t + 1 \end{pmatrix}, \end{aligned}$$

we have

$$r(n^2, K_{3,t}) = 2r\left(n^2, \langle 4 \rangle \perp \begin{pmatrix} 4 & 2 \\ 2 & 2^3 3^t + 1 \end{pmatrix}\right),$$

for any odd integer n. Consequently, for any integer n,

$$2r(n^2, K_{1,t}) = r(n^2, K_{2,t}) + r(n^2, K_{3,t}).$$

This completes the proof. $\square$

Proposition 4.3.3. *Both ternary $\mathbb{Z}$-lattices $K_{1,0}$ and $K_{1,1}$ defined above are strongly S-regular ternary $\mathbb{Z}$-lattices.*

Proof. Note that

$$\text{gen}(K_{1,0}) = \{[K_{1,0}], [K_{2,0}], [K_{3,0}]\} \quad \text{and} \quad \text{gen}(K_{1,1}) = \{[K_{1,1}], [K_{2,1}], [K_{3,1}]\}.$$

Therefore by Lemma 4.3.2, we have

$$\begin{aligned} r(n^2, \text{gen}(K_{1,i})) & = 4\left(\frac{1}{8} r(n^2, K_{1,i}) + \frac{1}{16} r(n^2, K_{2,i}) + \frac{1}{16} r(n^2, K_{3,i})\right) \\ & = r(n^2, K_{1,i}), \end{aligned}$$

for any integer n and any $i = 0, 1$. Therefore by Lemma 3.1.4, we see that both $K_{1,0}$ and $K_{1,1}$ are strongly S-regular. $\square$

Remark 4.3.4. If a strongly S-regular lattice M has class number two, then the other lattice in the genus of M is also strongly S-regular by Remark 3.2.5.

This is not true in general if the class number of a lattice is greater than two. For example, both ternary $\mathbb{Z}$-lattices $K_{1,0}$ and $K_{1,1}$ are strongly S-regular, however all the other lattices in $\mathrm{gen}(K_{1,0})$ and $\mathrm{gen}(K_{1,1})$ are not strongly S-regular.

For any positive integer t, we define

$$\ell_t = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \perp \langle 3t \rangle, \quad L_t = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 7 \end{pmatrix} \perp \langle 3t \rangle, \quad M_t = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 7 & \frac{5}{2} \\ \frac{1}{2} & \frac{5}{2} & 3t+1 \end{pmatrix}$$

and

$$N_t = \begin{pmatrix} 3 & \frac{3}{2} & 0 \\ \frac{3}{2} & 3 & \frac{3}{2} \\ 0 & \frac{3}{2} & 3t+1 \end{pmatrix}, \quad K_t = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \perp \langle 27t \rangle.$$

Lemma 4.3.5. *Let t be any positive integer. For any positive integer n, we have*

$$r(3n+1, \ell_t) = 3r(3n+1, L_t) = 3r(3n+1, M_t) = 2r(3n+1, N_t) + r(3n+1, K_t).$$

Proof. Let $\{x_1, x_2, x_3\}$ be the basis for ℓ_t whose Gram matrix is given above. Assume that $Q(ax_1 + bx_2 + cx_3) = 3n+1$. Since $(a-b)^2 \equiv 1 \pmod 3$, we have $a \equiv 0 \pmod 3$ or $b \equiv 0 \pmod 3$ or $a+b \equiv 0 \pmod 3$, however any of two cases cannot occur simultaneously. Therefore we have

$$\begin{aligned} r(3n+1, \ell_t) =& r(3n+1, \mathbb{Z}(3x_1) + \mathbb{Z}x_2 + \mathbb{Z}x_3) + r(3n+1, \mathbb{Z}x_1 + \mathbb{Z}(3x_2) + \mathbb{Z}x_3) \\ &+ r(3n+1, \mathbb{Z}(x_1 - x_2) + \mathbb{Z}(3x_2) + \mathbb{Z}x_3), \end{aligned}$$

which implies that $r(3n+1, \ell_t) = 3r(3n+1, L_t)$. Now let $y_1 = x_1, y_2 = x_2$ and $y_3 = x_1 + x_2 + x_3$. Then

$$(B(y_i, y_j)) = \begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & 1 & \frac{3}{2} \\ \frac{3}{2} & \frac{3}{2} & 3t+3 \end{pmatrix}.$$

Assume $Q(ay_1 + by_2 + cy_3) = 3n+1$. Since $(a-b)^2 \equiv 1 \pmod 3$, we have $a \equiv 0 \pmod 3$ or $b \equiv 0 \pmod 3$ or $a+b \equiv 0 \pmod 3$, however any of two cases cannot occur simultaneously. Therefore we have

$$\begin{aligned} r(3n+1,\ell_t) =& r(3n+1,\mathbb{Z}(3x_1)+\mathbb{Z}x_2+\mathbb{Z}x_3)+r(3n+1,\mathbb{Z}x_1+\mathbb{Z}(3x_2)+\mathbb{Z}x_3) \\ &+r(3n+1,\mathbb{Z}(x_1-x_2)+\mathbb{Z}(3x_2)+\mathbb{Z}x_3), \end{aligned}$$

which implies that $r(3n+1,\ell_t) = 3r(3n+1,M_t)$. Finally, if we choose a basis $\{x_1, x_1+x_2, x_1+x_3\}$ for ℓ_t, then we may prove that $r(3n+1,\ell_t) = 2r(3n+1,N_t)+r(3n+1,K_t)$. □

Proposition 4.3.6. *The ternary $\mathbb{Z}$-lattices L_1, L_4, L_{10} and M_6 are all strongly s-regular.*

Proof. First, note that N_t is contained in $\text{gen}(K_t)$ for any positive integer t, $L_t \in \text{gen}(N_t)$ if $t \equiv 1 \pmod 3$, and $M_t \in \text{gen}(N_t)$ if $t \equiv 0 \pmod 3$. For any integer $t \not\equiv 0 \pmod 3$, since $\lambda_3^2(L_t) \simeq \lambda_3^2(N_t) \simeq \lambda_3^2(K_t) \simeq \ell_t$, we have

$$r(9n,L_t) = r(9n,N_t) = r(9n,K_t) = r(n,\ell_t). \tag{4.3.3}$$

If $t \equiv 0 \pmod 3$, then we have

$$r(9n,M_t) = r(9n,N_t) = r(9n,K_t) = r(n,\ell_t).$$

For $t = 1, 4$ or 10, one may easily show that $\text{gen}(L_t) = \{[L_t],[N_t],[K_t]\}$ and $\text{gen}(M_6) = \{[M_6],[N_6],[K_6]\}$. Now by equation (4.3.3) and Lemma 4.3.5, we have

$$r(n^2,L_t) = 4\left(\frac{r(n^2,L_t)}{8} + \frac{r(n^2,N_t)}{12} + \frac{r(n^2,K_t)}{24}\right) = r(n^2,\text{gen}(L_t)),$$

for any $t = 1, 4$ or 10. Furthermore, we have

$$r(n^2,M_6) = \frac{8}{3}\left(\frac{r(n^2,M_6)}{4} + \frac{r(n^2,N_6)}{12} + \frac{r(n^2,K_6)}{24}\right) = r(n^2,\text{gen}(M_6)).$$

This completes the proof. □

Theorem 4.3.7. *Let L be a ternary $\mathbb{Z}$-lattice representing 1. Then L is strongly S-regular if and only if L satisfies $r(n^2,L) = r(n^2, gen(L))$ for any integer n.*

Proof. Note that "if" is trivial. The "only if" is the direct consequence of Theorem 4.2.2 and Propositions 4.3.1, 4.3.3 and 4.3.6. $\square$

Corollary 4.3.8. *Let L be a ternary $\mathbb{Z}$-lattice representing* 1*. Then the followings are all equivalent.*

(1) L satisfies $r(n^2, L) = r(n^2, gen(L))$ for any integer n;

(2) L satisfies the condition (3.2.1) *in Conjecture 3.2.1;*

(3) L is strongly S-regular.

Proof. The corollary is the direct consequence of Theorem 3.2.6 and Theorem 4.3.7. $\square$

Chapter 5

Strongly regularity on square classes

In this chapter, we generalize the notion of "strongly S-regularity" of ternary quadratic forms.

5.1 Strongly S_t-regular ternary forms

Definition 5.1.1. Let L be a ternary $\mathbb{Z}$-lattice and let T be a proper subset of positive integers. The lattice L is called *strongly T-regular* if $r(n, L) = r(n, \text{gen}(L))$ for any integer $n \in T$.

Lemma 5.1.2. *Let L be a ternary $\mathbb{Z}$-lattice. The class number of L is one if and only if L is strongly $\mathbb{Z}^+$-regular.*

Proof. The "only if" part follows directly from the definition. Suppose that $r(n, L) = r(n, \text{gen}(L))$ for any integer n. Then, clearly L is regular. Hence L is isometric to one of 913 candidates of regular ternary forms given [11]. Suppose that the class number of L is bigger than 1. Note that the number of such a lattice is $913 - 794 = 116$. For each case, one may show by a direct computation that there is an integer n such that $r(n, L) \neq r(n, gen(L))$. $\square$

For a positive square-free integer t, we define

$$S_t = \{tn^2 : n \in \mathbb{Z}\}.$$

Lemma 5.1.3. *Let L be a ternary $\mathbb{Z}$-lattice. Let n be a positive integer and let t be a positive square free integer. For any prime p, we assume that $ord_p(n) = \lambda_p$. If t is represented by the genus of L, then we have*

$$\begin{aligned}\frac{r(tn^2, gen(L))}{r(t, gen(L))} &= n \prod_{p \mid 8dL} \frac{\alpha_p(tn^2, L)}{\alpha_p(t, L)} \prod_{p \nmid 8dL} \frac{\alpha_p(tn^2, L)}{\alpha_p(t, L)} \\ &= \prod_{p \mid 8dL} p^{\lambda_p} \cdot \frac{\alpha_p(tn^2, L)}{\alpha_p(t, L)} \prod_{p \nmid 8dL} \left(\frac{p^{\lambda_p+1}-1}{p-1} - \left(\frac{-tdL}{p}\right) \frac{p^{\lambda_p}-1}{p-1} \right).\end{aligned}$$

In particular, if the lattice L has class number 1, then we have

$$\begin{aligned} r(tn^2, L) = \ & r(t, L) \prod_{p \mid 8dL} p^{\lambda_p} \cdot \frac{\alpha_p(tn^2, L)}{\alpha_p(t, L)} \\ & \times \prod_{p \nmid 8dL} \left(\frac{p^{\lambda_p+1}-1}{p-1} - \left(\frac{-tdL}{p}\right) \frac{p^{\lambda_p}-1}{p-1} \right).\end{aligned}$$

Proof. Note that if p does not divide $8dL$, then by [22], we have

$$\alpha_p(tn^2, L) = \begin{cases} 1 + \dfrac{1}{p} - \dfrac{1}{p^{\lambda_p+1}} - \dfrac{1}{p^{\lambda_p+2}} & \text{if} \quad p \mid t, \\ 1 + \dfrac{1}{p} - \dfrac{1}{p^{\lambda_p+1}} + \left(\dfrac{-tdL}{p}\right) \dfrac{1}{p^{\lambda_p+1}} & \text{otherwise.} \end{cases}$$

Hence the lemma follows directly from the Minkowski-Siegel formula. $\square$

Let L be a strongly S_t-regular ternary $\mathbb{Z}$-lattice with a positive square free integer t. For any integer n, let n_1 and n_2 be positive integers such that $P(n_1) \subset P(8dL)$, $(n_2, 8dL) = 1$ and $n = n_1 n_2$. Here $P(n)$ denotes the set of prime factors of n. Then by Lemma 5.1.3, we have

$$r(tn_1^2 n_2^2, L) = r(tn_1^2, L) \prod_{p \nmid 8dL} h_p(tdL, \lambda_p),$$

where $\lambda_p = \mathrm{ord}_p(n)$ for any prime p and

$$h_p(tdL, \lambda_p) = \frac{p^{\lambda_p+1}-1}{p-1} - \left(\frac{-tdL}{p}\right) \frac{p^{\lambda_p}-1}{p-1}.$$

We define

$$m_{S_t}(L) = \min_{n \in \mathbb{Z}^+} \{n : r(tn^2, L) \neq 0\}.$$

Then we also have

$$\{n : r(tn^2, L) \neq 0\} = m_{S_t}(L)\mathbb{Z}.$$

Theorem 5.1.4. *For any positive integer m, there are only finitely many strongly S_t-regular ternary $\mathbb{Z}$-lattice L up to isometry such that $m_{S_t}(L) = m$.*

Proof. The proof is quite similar to Theorem 4.1.8. □

5.2 Strongly spinor S_t-regular ternary forms

Lemma 5.2.1. *Let L be a ternary $\mathbb{Z}$-lattice with $g(L) \geq 2$ and let s be a positive integer such that $\mathfrak{n}(L^\sharp) = s^{-1}\mathbb{Z}$. Let K be a ternary $\mathbb{Z}$-lattice in the genus of L and let t be a positive square free integer. If*

$$r(tn^2, spn(L)) - r(tn^2, spn(K)) \neq 0,$$

for some integer n, then t divides s.

Proof. See Korollar 2 of [20]. □

Let L be a ternary $\mathbb{Z}$-lattice and let s be a positive integer such that $\mathfrak{n}(L^\sharp) = s^{-1}\mathbb{Z}$. Let t be a positive square free integer. Assume that there exists a ternary $\mathbb{Z}$-lattice K in the genus of L such that

$$r(tn^2, \mathrm{spn}(L)) - r(tn^2, \mathrm{spn}(K)) \neq 0,$$

for some integer n. Then by Lemma 5.2.1, t divides s. We define h to be an integer such that $s = tt'h^2$ with t' square free. If $g(L) = 1$, then $r(n, \mathrm{spn}(L)) = r(n, \mathrm{gen}(L))$.

Theorem 5.2.2. *Let L be a ternary $\mathbb{Z}$-lattice with $g(L) \geq 2$ and let n be an integer. Let t be a positive square free integer.*

(i) If any splitting integer for gen(L) is not of the form tn^2, then we have

$$r(tn^2, spn(L)) = r(tn^2, gen(L)).$$

(ii) Assume that there exists a ternary $\mathbb{Z}$-lattice K in the genus of L such that $r(tn^2, spn(L)) - r(tn^2, spn(K)) \neq 0$ for some integer n. If there exists a splitting integer for gen(L), say c, which is of the form tn^2, then we have

$$r(tn^2, spn(L)) = r(tn^2, gen(L)) + \frac{\psi(n)}{2} \cdot n.$$

Here ψ is determined by $r(tn^2, spn(K_1)) - r(tn^2, spn(K_2)) = \psi(n) \cdot n$ modulo h with $K_1 \in H_L(c), K_2 \in H_{\tilde{L}}(c)$.

Proof. Assume that

$$\mathrm{gen}(L) = \{\mathrm{spn}(L) = \mathrm{spn}(L_1), \mathrm{spn}(L_2), \ldots, \mathrm{spn}(L_e)\}.$$

where $\{L_1, L_2, \ldots, L_e\}$ is a complete set of representatives of all spinor genus in gen(L). Note that e is of the form 2^r with positive integer r.

First, we consider the part (i). Since any splitting integer for gen(L) is not of the form tn^2, by Korollar 2 of [20], we have $r(tn^2, \mathrm{spn}(L)) = r(tn^2, \mathrm{spn}(L'))$ for any $\mathbb{Z}$-lattice L' in the genus of L. Hence we have

$$\begin{aligned}
r(tn^2, \mathrm{gen}(L)) &= \frac{1}{w(L)} \sum_{[L'] \in \mathrm{gen}(L)} \frac{r(tn^2, L')}{o(L')} \\
&= \frac{1}{w(L)} (r(tn^2, \mathrm{spn}(L_1)) w_s(L_1) + \cdots + r(tn^2, \mathrm{spn}(L_e)) w_s(L_e)) \\
&= \frac{1}{w(L)} (r(tn^2, \mathrm{spn}(L_1))(w_s(L_1) + \cdot + w_s(L_e))) \\
&= r(tn^2, \mathrm{spn}(L)).
\end{aligned}$$

Finally, we consider the part (ii). Assume that L_2 is contained in $H_{\tilde{L}}(c)$. Then by Korollar 2 of [20], we have

$$r(tn^2, \mathrm{spn}(L_1)) - r(tn^2, \mathrm{spn}(L_2)) = \psi(n) \cdot n. \tag{5.2.1}$$

Here $\psi(n)$ is a function defined by above equation modulo h.

On the other hand, note that $w_s(L) = w_s(L')$ for any $\mathbb{Z}$-lattice L' in the genus of L. Since there exists a splitting integer for $\mathrm{gen}(L)$ which is of the form tn^2, by Korollar 2 of [20], we have

$$\begin{aligned} r(tn^2, \mathrm{gen}(L)) &= \frac{1}{w(L)} \sum_{[L'] \in \mathrm{gen}(L)} \frac{r(tn^2, L')}{o(L')} \\ &= \frac{1}{w(L)} (r(tn^2, \mathrm{spn}(L_1)) w_s(L_1) + \cdots + r(tn^2, \mathrm{spn}(L_e)) w_s(L_e)) \\ &= \frac{1}{2} r(tn^2, \mathrm{spn}(L_1)) + \frac{1}{2} r(tn^2, \mathrm{spn}(L_2)). \end{aligned} \tag{5.2.2}$$

By (5.2.1) and (5.2.2), we have

$$r(tn^2, \mathrm{spn}(L_1)) = r(tn^2, \mathrm{gen}(L_1)) + \frac{\psi(n)}{2} \cdot n.$$

This completes the proof. □

Definition 5.2.3. Let L be a ternary $\mathbb{Z}$-lattice and let T be a proper subset of positive integers. The lattice L is called *strongly spinor T-regular* if $r(n, L) = r(n, \mathrm{spn}(L))$ for any integer $n \in T$.

Corollary 5.2.4. *Let L be a ternary $\mathbb{Z}$-lattice and let t be a positive square free integer. If L is strongly spinor S_t-regular, then we have a closed formula for $r(tn^2, L)$.*

Proof. The corollary is a direct consequence of the definition of strongly spinor S_t-regularity and Theorem 5.2.2. □

Corollary 5.2.5. *Let L be a ternary $\mathbb{Z}$-lattice with $g(L) \geq 2$ and let t be a positive square free integer. If any splitting integer for gen(L) is not of the form tn^2, then L is strongly spinor S_t-regular if and only if L is strongly S_t-regular.*

Proof. The corollary is a direct consequence of Theorem 5.2.2 (i). □

Example 5.2.6. (i) Consider the ternary $\mathbb{Z}$-lattice $L_1 = [1, 2, 64, 0, 0, 0]$. Then the genus of L_1 contains two spinor genera and four classes. More precisely, let

$$L_2 = [1, 8, 18, 8, 0, 0], \quad K_1 = [2, 4, 17, 4, 0, 0], \quad K_2 = [3, 3, 17, -2, 2, 2].$$

Note that the spinor genus of L_1 (K_1) contains the ternary $\mathbb{Z}$-lattice L_2 (K_2, repectively). Then one may easily verify that

$$r(n^2, L_1) = r(n^2, L_2) \quad \text{and} \quad r(n^2, K_1) = r(n^2, K_2),$$

for any integer n. Hence every form in the genus of L_1 is strongly spinor S_1-regular. Furthermore one may easily check that c is a splitting integer for the genus of L_1 if and only if $c = m^2 (m \in \mathbb{Z})$. Therefore we have by Theorem 5.2.2 (ii),

$$r(n^2, L_1) = \Phi(n) \prod_{p \nmid 2} \left(\frac{p^{\lambda_p + 1} - 1}{p - 1} - \left(\frac{-2}{p}\right) \frac{p^{\lambda_p} - 1}{p - 1} \right) + n \cdot \left(\frac{n}{2}\right) \cdot (-1)^{\frac{n-1}{2}},$$
$$r(n^2, K_1) = \Phi(n) \prod_{p \nmid 2} \left(\frac{p^{\lambda_p + 1} - 1}{p - 1} - \left(\frac{-2}{p}\right) \frac{p^{\lambda_p} - 1}{p - 1} \right) - n \cdot \left(\frac{n}{2}\right) \cdot (-1)^{\frac{n-1}{2}},$$

where $\lambda_p = \mathrm{ord}_p(n)$ for any prime p and

$$\Phi(n) = \begin{cases} 2^{\lambda_2} & \text{if} \quad 0 \le \lambda_2 \le 1, \\ 2^{\lambda_2 - 1} & \text{if} \quad 2 \le \lambda_2 \le 3, \\ 12 & \text{if} \quad \lambda_2 \ge 4. \end{cases}$$

(ii) Next consider the ternary $\mathbb{Z}$-lattice $K_{1,0} = [1, 4, 9, 4, 0, 0]$ which is defined in Section 4.3. Note that class number of $K_{1,0}$ is three and the spinor genus of $K_{1,0}$ contains only one class. One may easily check that c is a splitting integer for the genus of $K_{1,0}$ if and only if $c = 2m^2 (m \in \mathbb{Z})$. Since $K_{1,0}$ is strongly spinor S_t-regular for any positive square free integer t, $K_{1,0}$ is strongly S_t-regular for any positive square free integer $t \neq 2$ by Corollary 5.2.5.

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국문초록

이 논문에서는 삼변수 이차형식이 제곱수를 표현할 때, 그 개수에 관한 다양한 성질을 연구한다.

양의 정부호이고 정수계수인 삼변수 이차형식이 제곱수의 표현의 개수에 관한 정규적인 성질을 만족하는 경우, 이를 강력한 S-정규형식이라고 한다. 우리는 강력한 S-정규성과 Cooper와 Lam에 의해 주어진 추측과의 관계를 설명하고 그들의 추측을 완벽하게 해결한다. 또한, 표현되는 양의 제곱수의 최솟값을 고정하면, 강력한 S-정규형식은 유한함을 증명한다. 특별히, 1을 표현하는 강력한 S-정규형식은 정확히 207개 존재함을 증명한다.

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