On the Coincidence of the Shapley Value and the Nucleolus in Queueing Problems

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Given a group of agents to be served in a facility, the *queueing problem* is concerned with finding the order to serve agents and the (positive or negative) monetary compensations they should receive. As shown in Maniquet (2003), the minimal transfer rule coincides with the Shapley value of the game obtained by defining the worth of each coalition to be the minimum total waiting cost incurred by its members under the assumption that they are served before the non-coalitional members. Here, we show that it coincides with the nucleolus of the same game. Thereby, we establish the coincidence of the Shapley value and the nucleolus for queueing problems. We also investigate the relations between the minimal transfer rule and other rules discussed in the literature.

Keywords: Queueing problems, Minimal transfer rule, Shapley value, Nucleolus, Coincidence

JEL Classification: C71, D63, D71

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I. Introduction

Consider a group of agents who must be served in a facility. The facility can handle only one agent at a time and agents incur waiting costs. The *queueing problem* is concerned with finding the order to serve agents and the (positive or negative) monetary compensations they should receive. We assume that an agent's waiting cost per unit of time is constant, but that agents differ in their waiting costs. Each agent's utility is equal to his monetary transfer minus his total waiting cost. This problem has been analyzed extensively from the incentive perspective (Dolan 1978; Suijs 1996; Mitra 2001, 2002; and others).

Maniquet (2003) proposes to solve the queueing problem by applying what is probably the best-known solution for cooperative games, the Shapley value (Shapley 1953). To do this, he defines the worth of a coalition to be the minimum total waiting cost incurred by its members under the optimistic assumption that they are served before the non-coalitional members. The resulting rule, the *minimal transfer rule*, selects an efficient queue and transfers to each agent a half of his unit waiting cost from each of his predecessors minus a half of the unit waiting cost of each of his followers.

In this paper, we apply another well-known solution for cooperative games, the nucleolus (Schmeidler 1969) to the game, and identify the resulting rule. Surprisingly, we obtain the same rule: the Shapley value and the nucleolus coincide for queueing problems. We also investigate the relation between the minimal transfer rule and other rules discussed in the literature, the serial cost sharing rule, the core, the τ-value, and the Dutta-Ray solution.

The paper is organized as follows. Section II contains some preliminaries and introduces the minimal transfer rule. Section III discusses how to solve a queueing problem by applying solutions of cooperative games. Section IV establishes our main result that the minimal transfer rule coincides with the nucleolus. Section V investigates the relations between the minimal transfer rule and other rules, and discusses whether the result can be generalized to a broader class of problems. Concluding remarks are in Section VI.

II. Preliminaries

Let $N \equiv \{1, 2, \cdots, n\}$ be the set of agents. Each agent needs the same amount of time to be served. Agent $i \in N$ is characterized by his *unit waiting cost*, $\theta_i \geq 0$, and is assigned a position $\sigma_i \in \mathbb{N}$ in a queue and a positive or negative transfer $t_i \in \mathbb{R}$. The agent who is served first incurs no waiting cost. If agent $i \in N$ is served in the σ_i^{th} position, his waiting cost is $(\sigma_i - 1)\theta_i$. Each agent $i \in N$ has a quasi-linear utility function: his utility from the assignment (σ_i, t_i) is given by $u(\sigma_i, t_i; \theta_i) = t_i - (\sigma_i - 1)\theta_i$.

A *queueing problem* is defined as a list $q = (N, \theta)$ where N is the set of agents and $\theta \in \mathbb{R}_+^N$ is the vector of unit waiting costs. Let Q^N be the class of all problems for N. An *allocation* for $q \in Q$ is a pair $z = (\sigma, t)$, where for each $i \in N$, σ_i denotes agent i's position in the queue and t_i the monetary transfer to him. An allocation is *feasible* if no two agents are assigned the same position and the sum of transfers is not positive. Thus, the set of feasible allocations $Z(q)$ consists of all pairs $z = (\sigma, t)$ such that for all $i, j \in N$, $i \neq j$ implies $\sigma_i \neq \sigma_j$ and $\sum_{i \in N} t_i \leq 0$.

Given $q = (N, \theta) \in Q^N$, an allocation $z = (\sigma, t) \in Z(q)$ is *queue-efficient* if it minimizes the total waiting costs, that is, for all $z' = (\sigma', t') \in Z(q)$, $\sum_{i \in N}(\sigma_i - 1)\theta_i \leq \sum_{i \in N}(\sigma_i' - 1)\theta_i$. The efficient queue of a problem does not depend on the transfers. Moreover, it is unique except for agents with equal waiting costs. These agents have to be served consecutively but in any order. The set of efficient queues for $q \in Q^N$ is denoted $Eff(q)$. An allocation $z = (\sigma, t) \in Z(q)$ is *budget balanced* if $\sum_{i \in N} t_i = 0$. An allocation rule, or simply a *rule*, is a mapping φ: $Q^N \to Z(q)$, which associates with every problem $q \in Q^N$ a non-empty subset $\varphi(q)$ of feasible allocations. The pair $\varphi_i(q) = (\sigma_i, t_i)$ represents i's position in the queue and his transfer in q. Given $q = (N, \theta) \in Q^N$, $z = (\sigma, t) \in Z(q)$, and $i \in N$, let $P_i(\sigma)$ be the set of agents preceding agent i and $F_i(\sigma)$ the set of agents following him.

Now we introduce an important rule. The minimal transfer rule (Maniquet 2003) selects an efficient queue and transfers to each agent a half of his unit waiting cost from each of his predecessors minus a half of the unit waiting cost of each of his followers.

Minimal transfer rule, φ^M: For each $q \in Q^N$,

$$\varphi^M(q) = \{(\sigma^M, t^M) \in Z(q) \mid \sigma^M \in Eff(q), \text{ and } \forall i \in N, \\ t_i^M = (\sigma_i^M - 1)\frac{\theta_i}{2} - \sum_{j \in F_i(\sigma^M)} \frac{\theta_j}{2}\}.$$

III. Queueing Games

We analyze queueing problems by applying solutions of cooperative games (games, for short). First, we formally describe how queueing problems are mapped into games. Let $N \equiv \{1, 2, \cdots, n\}$ be the set of *players*. A set $S \subseteq N$ is a *coalition*. A *game* is a real-valued function v defined on all coalitions $S \subseteq N$ satisfying $v(\emptyset) = 0$. The number $v(S)$ is the *worth* of S. Let Γ^N be the class of games with player set N. A *solution* is a function ϕ: $\Gamma^N \to \mathbb{R}^N$, which associates with every game $v \in \Gamma^N$ a vector $\phi(v) = (\phi_i(v))_{i \in N} \in \mathbb{R}^N$. The number $\phi_i(v)$ represents the payoff to player i in game v.

Now we introduce two well-known solutions for games, the Shapley value and the nucleolus. The Shapley value assigns to each player a payoff equal to a weighted average of his marginal contributions to all possible coalitions, with weights being determined by the sizes of coalitions. The nucleolus chooses an allocation which minimizes the difference between the worth of a coalition and its payoff (in the lexicographic way).

Shapley value, *Sh*: For each $v \in \Gamma^N$ and each $i \in N$,

$$Sh_i(v) = \sum_{S \subseteq N, S \ni i} \frac{(|S|-1)!|N \setminus S|!}{|N|!}[v(S) - v(S \setminus \{i\})].$$

For each $v \in \Gamma^N$, let $I(v)$ be the set of *imputations* $x \in \mathbb{R}^N$ such that $\sum_{i \in N} x_i = v(N)$ and for each $i \in N$, $x_i \geq v(\{i\})$. For each $x \in I(v)$, its *excess vector* $e(v, x) \in \mathbb{R}^{2^N}$ is defined by setting for each $S \subseteq N$, $e_S(v, x) \equiv v(S) - \sum_{i \in S} x_i$. Its S-coordinate $e_S(v, x)$ measures the amount by which the worth of the coalition S exceeds its payoff at x. For each $y \in \mathbb{R}^{2^N}$, let $\tilde{y} \in \mathbb{R}^{2^{|N|}}$ be obtained by rearranging the coordinates of y in non-increasing order. For each pair y, $z \in \mathbb{R}^{2^N}$, *y is lexicographically smaller than z* if either (i) $\tilde{y}_1 < \tilde{z}_1$ or (ii) there exists $l > 1$ such that

$\tilde{y}_l < \tilde{z}_l$. and for all $k < l$, $\tilde{y}_k = \tilde{z}_k$.

Nucleolus, *Nu*: For each $v \in \Gamma^N$ such that $I(v) \neq \emptyset$,

$$Nu(v) \equiv \left\{ x \in I(v) \;\middle|\; \begin{array}{l} \text{for each } x' \in I(v) \setminus \{x\},\ e(v, x) \text{ is} \\ \text{lexicographically smaller than } e(v, x') \end{array} \right\}$$

For each $v \in \Gamma^N$, the *core* is the set of imputations at which no excess is greater than zero, that is, $Core(v) \equiv \{x \in I(v) \mid \text{for each } S \subset N, \sum_{i \in S} x_i \geq v(S)\}$. A game is *convex* if for each S, $T \subseteq N$, $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$. It is well-known that a convex game has a non-empty core. Moreover, the Shapley value and the nucleolus select allocations in the core.

Maniquet (2003) defines the worth of each coalition $S \subseteq N$ as the minimum total waiting cost incurred by its members under the assumption that they are served before the non-coalitional members. That is, for each $S \subseteq N$, its worth $v_q(S)$ is defined by setting:

$$v_q(S) = -\sum_{i \in S} (\sigma_i^* - 1)\theta_i,$$

where $\sigma^* \in Eff(S, \theta_S)$ and $\theta_S = (\theta_i)_{i \in S}$. By applying the Shapley value to the game $v_q = (v_q(S))_{S \subseteq N}$, he shows that the resulting payoff to each player is the utility assigned to him by the minimal transfer rule.

Since the game v_q is concave (that is, $-v_q$ is convex), $I(v_q)$ may be empty. Here, we define the worth of a coalition to be the negative of Maniquet's, that is, for each $S \subseteq N$, $v_c(S) \equiv -v_q(S)$. We call the game v_c a *queueing cost game*, and for each $S \subseteq N$, $v_c(S)$ is the *cost* of S. Obviously, v_c is a game in Γ^N. Moreover, v_c is convex and its nucleolus is well-defined. If a game theoretic solution is applied, then the resulting payoff to each player is the cost contribution assigned to him. It is obvious that it is the negative of the utility assigned to him by the solution.

As shown in Maniquet (2003), the payoff obtained by applying the Shapley value to the game v_q is the utility assigned by the minimal transfer rule to the corresponding queueing problem.

Theorem M. *For each $q \in Q^N$ and each $i \in N$, $\varphi_i^M(q) = Sh_i(v_q) = -Sh_i(v_c)$.*

IV. Coincidence of the Shapley Value and the Nucleolus

Before we apply another well-known solution for games, the nucleolus, to queueing cost games and investigate what recommendation it makes, we show that the cost of a coalition with more than two members can be expressed as a sum of costs of two-person coalitions.

***Lemma* 1.** *For each $q \in Q^N$, its queueing cost game v_c satisfies*

(i) *for each $i \in N$, $v_c(\{i\})=0$;*

(ii) *for each $S \subseteq N$ with $|S| \geq 2$, $v_c(S)=\sum_{\substack{T\subseteq S\\ |T|=2}} v_c(T)$ and $v_c(S) \geq 0$.*

Since Lemma 1 can easily be proven from the facts that

(i) for each $i \in N$, $v_c(\{i\})=0$ and
(ii) for each pair $i, j \in N$, $v_c(\{i, j\})=\min\{\theta_i, \theta_j\}$,

the detailed proofs are omitted. Instead, we present an example showing how the worth of a coalition is calculated.

Example 1: Let $N \equiv \{1, 2, 3, 4\}$ and $\theta \in \mathbb{R}_+^N$ with $\theta_1 \geq \theta_2 \geq \theta_3 \geq \theta_4$. Then,

$$
\begin{aligned}
v_c(\{1, 2, 3\}) &= \theta_2+2\theta_3 = v_c(\{1, 2\}) + v_c(\{1, 3\}) + v_c(\{2, 3\}),\\
v_c(\{1, 2, 4\}) &= \theta_2+2\theta_4 = v_c(\{1, 2\}) + v_c(\{1, 4\}) + v_c(\{2, 4\}),\\
v_c(\{1, 3, 4\}) &= \theta_3+2\theta_4 = v_c(\{1, 3\}) + v_c(\{1, 4\}) + v_c(\{3, 4\}),\\
v_c(\{2, 3, 4\}) &= \theta_3+2\theta_4 = v_c(\{2, 3\}) + v_c(\{2, 4\}) + v_c(\{3, 4\}),\\
v_c(\{1, 2, 3, 4\}) &= \theta_2+2\theta_3+3\theta_4\\
&= v_c(\{1, 2\})+v_c(\{1, 3\})+v_c(\{1, 4\})\\
&\quad +v_c(\{2, 3\})+v_c(\{2, 4\})+v_c(\{3, 4\}).
\end{aligned}
$$

Let $\tilde{\Gamma}^N$ be the class of games satisfying the two conditions of Lemma 1. That is, $v \in \tilde{\Gamma}^N$ if and only if for each $i \in N$, $v(\{i\})=0$, and for each $S \subseteq N$ with $|S| \geq 2$, $v(S)=\Sigma_{T\subseteq S, |T|=2}\, v(T)$. This class includes, in particular, our queueing cost games, and more. Therefore, as shown in Deng and Papadimitriou (1994) and van den Nouweland *et al.* (1996), the coincidence between the Shapley value and the nucleolus can be established.

For completeness, we present an alternative proof using the Kohlberg's (1971) lemma. First, the following lemma can be easily proven from our previous observations.

Lemma 2. *For each* $v \in \tilde{\Gamma}^N$ *and each* $i \in N$, $Sh_i(v) = \frac{1}{2} \sum_{S \subseteq N, S \ni i, |S|=2} v(S)$.

The Shapley value of the queueing cost game can be calculated by using only the worths of the two-person coalitions: It assigns to each agent a half of the sum of his contributions on all two person coalitions. We note that its computational burden is significantly reduced since we need to know $n(n-1)/2$ numbers instead of $2^n - 1$ numbers.

Now we show that at the Shapley value allocation, the excess of a coalition equals to the excess of its complementary coalition.

Lemma 3. *For each* $v \in \tilde{\Gamma}^N$ *and each* $i \in N$, *if*

$$x_i \equiv \frac{1}{2} \sum_{\substack{S \subseteq N, \\ S \ni i, \\ |S|=2}} v(S),$$

then, for each $S \subseteq N$,

$$v(S) - \sum_{i \in S} x_i = v(N \setminus S) - \sum_{i \in N \setminus S} x_i.$$

Proof: *Let* $v \in \tilde{\Gamma}^N$ *and* $S \subset N$. If $1 < |S| < |N|$, then

$$v(S) - \sum_{i \in S} x_i = \sum_{\substack{T \subseteq S \\ |T|=2}} v(T) - \sum_{i \in S} \frac{1}{2} \sum_{\substack{T \subseteq N, \\ T \ni i \\ |T|=2}} v(T) = -\frac{1}{2} \sum_{\substack{\{i,j\} \subseteq N \\ i \in S \\ j \in N \setminus S}} v(\{i, j\}),$$

and

$$v(N \setminus S) - \sum_{j \in N \setminus S} x_j = \sum_{\substack{T \subseteq N \setminus S \\ |T|=2}} v(T) - \sum_{i \in N \setminus S} \frac{1}{2} \sum_{\substack{T \subseteq N, \\ T \ni j \\ |T|=2}} v(T) = -\frac{1}{2} \sum_{\substack{\{i,j\} \subseteq N \\ i \in S \\ j \in N \setminus S}} v(\{i, j\}).$$

If $S=N\backslash\{j\}$, then

$$v(S)-\sum_{i\in S}x_i=\sum_{\substack{T\subseteq S\\|T|=2}}v(T)-\sum_{i\in S}\frac{1}{2}\sum_{\substack{T\subseteq N,\\T\ni i\\|T|=2}}v(T)$$

$$=-\frac{1}{2}\sum_{\substack{\{i,j\}\subseteq N\\i\in S}}v(\{i,j\})$$

$$=v(\{j\})-x_j,$$

the desired conclusion. ■

For each $v:2^N\to\mathbb{R}$, each $x\in\mathbb{R}^N$ with $\sum_{i\in N}x_i=v(N)$, and each $\alpha\in\mathbb{R}$, let

$$\mathcal{S}_\alpha(v,x)\equiv\{S\in 2^N\,|\,S\neq\emptyset\text{ and }v(S)-\sum_{i\in S}x_i\geq\alpha\}.$$

A collection $\mathcal{B}\subseteq 2^N$ of coalitions is *strictly balanced on N* if there exists a list $(\delta_S)_{S\in\mathcal{B}}$ of positive weights such that for each $i\in N$,

$$\sum_{\substack{S\in\mathcal{B}\\S\ni i}}\delta_S=1.$$

***Lemma* 4.** (Kohlberg 1971) *For each $v\in\Gamma$ and each $x\in I(v)$,*

$$x=Nu(v)\Leftrightarrow\left[\begin{array}{l}\text{for each }\alpha\in\mathbb{R}\text{ with }\mathcal{S}_\alpha(v,x)\neq\emptyset,\\\text{there exists }\mathcal{S}\subseteq\{\{i\}\,|\,i\in N\text{ and }v(\{i\})-x_i=0\}\\\text{such that }\mathcal{S}_\alpha(v,x)\cup\mathcal{S}\text{ is strictly balanced on }N.\end{array}\right]$$

We are ready to state and prove our main result.

***Theorem* 1.** *For each $v\in\tilde{\Gamma}^N$,*

$$Sh(v)=Nu(v).$$

***Proof*:** By Lemma 2, for each $v\in\tilde{\Gamma}^N$ and each $i\in N$,

$$Sh_i(v) = \frac{1}{2} \sum_{\substack{S \subseteq N \\ S \ni i \\ |S|=2}} v(S).$$

Let $\alpha \in \mathbb{R}$ be such that $\mathcal{S}_\alpha(v, Sh(v)) \neq \emptyset$. Let $S \in \mathcal{S}_\alpha(v, Sh(v))$. Since by Lemma 3,

$$v(N \setminus S) - \sum_{i \in N \setminus S} Sh_i(v) = v(S) - \sum_{i \in S} Sh_i(v),$$

$N \setminus S \in \mathcal{S}_\alpha(v, Sh(v))$. Thus, $\mathcal{S}_\alpha(v, Sh(v))$ is strictly balanced on N. The desired conclusion follows from Lemma 4. ■

V. Discussion

In this section, we further investigate the relations between the minimal transfer rule and other rules proposed in the literature. Also, we discuss whether our results can be extended to a broader class of problems.

A. The Serial Cost Sharing Rule

As shown in Moulin (2004), the minimal transfer rule coincides with the serial cost sharing rule for scheduling problems. The same observation can be made for queueing problems. In fact, its proof can be easily obtained by checking our simple formula for the Shapley value given in Lemma 2. To further simplify the argument, let $(N, \theta) \in Q^N$ be such that $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$. From Lemma 2, $Sh_n(v_c) = \{(n-1)/2\}\theta_n$, $Sh_{n-1}(v_c) = \{(n-2)/2\}\theta_{n-1} + (1/2)\theta_n$, and so on.

To calculate the payoff assigned by the serial cost sharing rule, we need to assume that all agents have θ_n. Then, the total cost $\{1 + \cdots + (n-1)\}\theta_n$ is divided equally among all agents, and in particular agent n receives $\{(n-1)/2\}\theta_n$. Now suppose that agent n leaves and the remaining agent have the unit waiting cost θ_{n-1}. Then, the total cost goes up by $\{1 + \cdots + (n-2)\}(\theta_{n-1} - \theta_n)$, and this increase is shared equally among the remaining $(n-1)$ agents, and in particular agent $n-1$ receives $(n-2)/2(\theta_{n-1} - \theta_n)$. Since he was originally assigned $\{(n-1)/2\}\theta_n$, his final assignment is $\{(n-2)/2\}\theta_{n-1} + (1/2)\theta_n$. And so on. It is easy to check that this is exactly the amount assigned by the simple formula of the Shapley value. In our queueing cost problem, the serial cost sharing and the minimal transfer rule make the same recommendation.

B. The Core

For a convex game, both the Shapley value and the nucleolus select an allocation in the core. It is natural to ask about the structure of the core for queueing cost games. In particular, one might conjecture that the coincidence between the two solutions comes from the fact that the core is a singleton. However, as shown in Figure 1 for a 3-agent problem with $N \equiv \{1, 2, 3\}$ and $\theta_1 \geq \theta_2 \geq \theta_3$, this is not the case. Its core is pretty large. However, it has a rather symmetric structure. This is the central reason why the two solutions coincide.

C. The τ-value

For each $v \in \Gamma^N$ and each $i \in N$, let $M_i(v) \equiv v(N) - v(N \backslash \{i\})$ and $m_i(v) \equiv v(\{i\})$. Then, the τ-value (Tijs 1987) selects the maximal feasible allocation on the line connecting $M(v) \equiv (M_i(v))_{i \in N}$ and $m(v) \equiv (m_i(v))_{i \in N}$.

τ-value, τ: For each convex game v,

$$\tau(v) \equiv \lambda M(v) + (1 - \lambda) m(v),$$

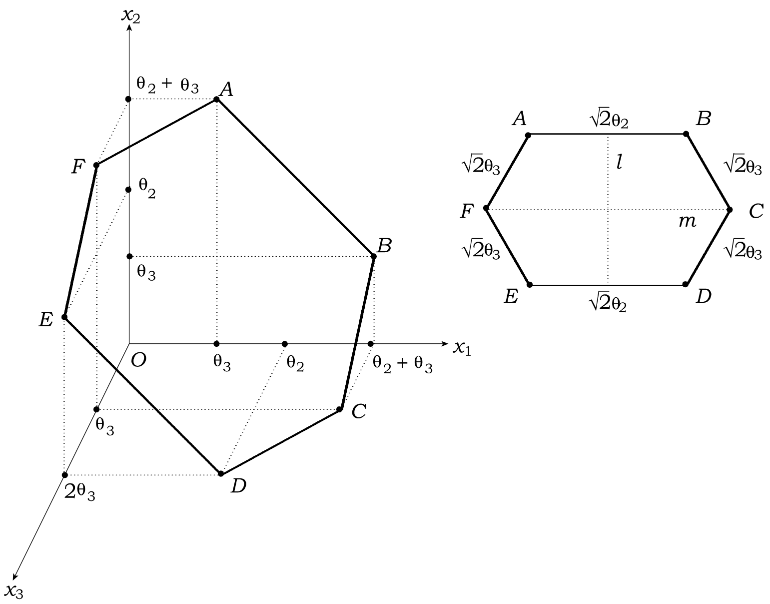
where $\lambda \in [0, 1]$ is chosen so as to satisfy

$$\sum_{j \in N} [\lambda(v(N) - v(N \backslash \{j\})) + (1 - \lambda) v(\{j\})] = v(N).$$

For a queueing cost game v, $m(v) = 0$. Moreover, it is easy to see that for each $j \in N$, $v(N) - v(N \backslash \{j\}) = \sum_{S \ni j, |S| = 2} v(S)$ and that $\lambda = 1/2$. Thus, the τ-value coincides with the Shapley value, and therefore, the nucleolus for queueing problems.

D. The Dutta-Ray Solution

Next we investigate the Dutta-Ray solution (Dutta and Ray 1989). In general, this solution selects a core allocation which maximizes the Lorenz ordering. Since our games are convex, the solution can be



Note: The core of a queueing cost game may not be completely symmetric, but it is sufficiently symmetric to guarantee $Sh(v_c) = Nu(v_c)$. In the figure, the core of a three-agent queueing cost game is the interior (and the boundary) of *ABCDEF*, and it is symmetric with respect to lines *l* and *m*.

FIGURE 1

THE CORE OF A QUEUEING COST GAME

defined as follows.

Dutta-Ray solution, *DR*: For each convex game v for N, $DR(v) \in \mathbb{R}^N$ is the payoff vector derived by the following algorithm:

Step 1. Let $N_1 \equiv N$ and $v_1 \equiv v$. Find the unique coalition $S_1 \in 2^{N_1} \setminus \{\emptyset\}$ such that for each $S \in 2^{N_1} \setminus \{0, S_1\}$, (i) $v_1(S_1)/|S_1| \geq v_1(S)/|S|$ and (ii) if $v_1(S_1)/|S_1| = v_1(S)/|S|$, then $|S_1| > |S|$.[1] For each $i \in S_1$, let $DR_i(v) \equiv v_1(S_1)/|S_1|$. If $S_1 \neq N_1$, proceed to the next step.
Step *k*. Suppose that $N_{k-1} \in 2^N \setminus \{\emptyset\}$, $v_{k-1} \in \Gamma^{N_{k-1}}$ with $v_{k-1}(\emptyset) = 0$, and $S_{k-1} \in 2^{N_{k-1}} \setminus \{\emptyset, N_{k-1}\}$ have been defined. Let $N_k \equiv N_{k-1} \setminus S_{k-1}$ and

[1] The uniqueness of S_1 is guaranteed by the convexity of v_1.

$v_k \in \Gamma^{N_k}$ be defined by setting for each $S \in 2^{N_k}$,

$$v_k(S) \equiv v_{k-1}(S \cup S_{k-1}) - v_{k-1}(S_{k-1}).$$

Find the unique coalition $S_k \in 2^{N_k} \setminus \{\emptyset\}$ such that for each $S \in 2^{N_{k-1}} \setminus \{\emptyset, S_k\}$, (i) $v_k(S_k)/|S_k| \geq v_k(S)/|S|$ and (ii) if $v_k(S_k)/|S_k| = v_k(S)/|S|$, then $|S_k| > |S|$.[2] For each $i \in S_k$, let $DR_i(v) \equiv v_k(S_k)/|S_k|$. If $S_k \neq N_k$, proceed to the next step.

When applied to queueing cost games, the Dutta-Ray solution selects an allocation in the core, different from the Shapley value allocation.

Example 2: Let $N \equiv \{1, 2, 3\}$, $\theta \equiv (6, 5, 1)$, and $q \equiv (N, \theta)$. Then $v_c(\{1, 2\}) = 5$, $v_c(\{1, 3\}) = 1$, $v_c(\{2, 3\}) = 1$, and $v_c(\{1, 2, 3\}) = 7$. Thus, $Sh(v_c) = (3, 3, 1)$ and $DR(v_c) = (5/2, 5/2, 2)$. □

E. The Maximal Transfer Rule

The maximal transfer rule (Chun 2006a, b) selects an efficient queue and transfers to each agent a half of the unit waiting cost of each of his predecessors minus a half of his waiting cost to each of his followers.

Maximal transfer rule, φ^X: For all $q \in Q^N$,

$$\varphi^X(q) = \{(\sigma^X, t^X) \in Z(q) \mid \sigma^X \in Eff(q), \text{ and } \forall i \in N,$$

$$t_i^X = \sum_{j \in P_i(\sigma^X)} \frac{\theta_j}{2} - (|N| - \sigma_i^X) \frac{\theta_i}{2}\}.$$

As shown in Chun (2006a), if the worth of a coalition is defined as the minimum total waiting cost incurred by its members under the assumption that they are served after the non-coalitional members, this rule assigns the same utility as the Shapley value of the corresponding game. Moreover, the coincidence between the Shapley value and the nucleolus can be obtained for this class of games. Although these games do not satisfy two conditions of Lemma 2,

[2] If v is convex, then v_k is convex. Again, the uniqueness of S_k is guaranteed by the convexity of v_k.

their zero-normalized versions obtained by defining for each $S \subseteq N$, $\bar{v}(S) = v(S) - \sum_{i \in S} v(\{i\})$, satisfy them. The proof can be completed by using the fact that both the Shapley value and the nucleolus satisfy *zero-independence*, requiring that adding a constant to the worth of coalitions containing i should affect his payoff by the constant.

Also, it can be shown that the maximal transfer rule coincides with the decreasing serial cost sharing rule (de Frutos 1998) for queueing and scheduling problems.

F. Sequencing Problems

A *sequencing problem* (Suijs 1996)[3] is a list (N, r, θ), where N is the set of agents, $r \equiv (r_i)_{i \in N}$ is the vector representing the service time required by agents, and $\theta \equiv (\theta_i)_{i \in N}$ is the vector of unit waiting costs. A queueing problem is obtained by setting for each $i \in N$, $r_i = 1$, and a *scheduling problem* by setting for each $i \in N$, $\theta_i = 1$. It is interesting to note that if a sequencing problem is transformed to a cost game, then it satisfies two conditions identified in Lemma 2. Therefore, the game belongs to $\tilde{\Gamma}^N$, and therefore, our coincidence result still holds.

VI. Concluding Remarks

In this paper, we show that the Shapley value and the nucleolus coincide on the domain of queueing cost games (also, sequencing cost games). However, our conditions are sufficient, but not necessary. It would be interesting to find a necessary condition to guarantee the coincidence of these two solutions. We leave this as our next research agenda.

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[3] Also, see Curiel, Pederzoli, and Tijs (1989) and Chun (2004) for related problems.

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