One-Fund Separation and Uniqueness of Equilibrium in Incomplete Markets with Heterogeneous Beliefs

Dong Chul Won

Uniqueness of equilibrium is a relatively unexplored issue in incomplete markets compared with complete markets. This work shows that one-fund separation is sufficient for the uniqueness of equilibrium in a special class of incomplete markets with two agents and two assets. Specifically, it provides a new condition that is necessary and sufficient for equilibrium to exhibit one-fund separation in incomplete market economies, in which two agents have identical homothetic preferences, heterogeneous beliefs, and initial endowments spanned by asset payoffs. The new condition that is jointly imposed on heterogeneous beliefs and asset payoffs is distinct from time-honored conditions, such as gross substitution and restrictions on the Mitjushin–Polterovich coefficient. One-fund separation provides a new perspective into the uniqueness of equilibrium in incomplete markets.

Keywords: Uniqueness of equilibrium, One-fund separation, Incomplete markets, Heterogeneous beliefs, Homothetic preferences

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Dong Chul Won, School of Business, Ajou University, 206 Worldcup-ro, Yeongtong-gu, Suwon 16499, South Korea, Phone: +82-31-219-2708, Email: dcwon@ajou.ac.kr.

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I. Introduction

Uniqueness of competitive equilibrium is the desired property of financial markets in which asset prices are believed to convey dispersed information efficiently. However, uniqueness of equilibrium is a relatively unexplored issue in incomplete markets compared with complete markets. As demonstrated in the work of Hens and Pilgrim (2002), gross substitution and the Mitjushin–Polterovich (MP) restrictions, which are sufficient for complete market economies to achieve unique equilibrium, cannot be extended to incomplete market economies in a straightforward way because equilibrium need not be Pareto optimal in incomplete markets. Thus, a new perspective must be added into the uniqueness issue in incomplete markets to expand conventional wisdom.

This work discusses uniqueness of equilibrium in simple incomplete markets from the viewpoint of fund separation. Specifically, it provides a new condition that is necessary and sufficient for equilibrium to exhibit one-fund separation in a special class of two-period incomplete market economies with two agents and two assets. Agents with identical homothetic preferences and heterogeneous beliefs decide to consume a single good in two periods. They are initially endowed with shares of assets, and thus, the initial endowments are naturally spanned by the asset payoffs. The new condition that is jointly imposed on heterogeneous beliefs and asset payoffs is also sufficient for the economy to achieve a unique equilibrium. Therefore, one-fund separation is sufficient for equilibrium to be unique in the current framework. The joint condition on payoffs and beliefs is distinct from time-honored conditions, such as gross substitution and restrictions on the MP coefficient, which are imposed on demand functions and the curvature of preferences, respectively. One-fund separation provides a new perspective into uniqueness of equilibrium in incomplete markets.

Hens and Pilgrim (2002) provide a comprehensive overview of the results related to equilibrium uniqueness and preference aggregation in two-period incomplete markets. They include an exquisite review of cases in which agents have identical homothetic preferences and spanned endowments. Figure 6.1 in page 175 of Hens and Pilgrim (2002) gives a state-of-the-art summary of the uniqueness results in a single diagram. The diagram includes the literature that verifies the uniqueness result via the monotonicity of demand functions. The monotonicity of demand functions holds if the MP coefficient (relative risk aversion in the expected utility framework) is less than 4 and the individual initial endowments are spanned by the payoff matrix and collinear with the aggregate initial endowments. As shown in Theorem 6.15 of Hens and Pilgrim (2002), the monotonicity of demand functions also holds if agents have relative risk aversion that is less than or equal to 1 in a two-asset economy. The class of economies discussed in the current work does not belong to any category in the aforementioned diagram because the new condition involves neither restrictions on the magnitude of relative risk aversion nor the collinearity between individual and aggregate initial endowments. Detemple and Gottardi (1998) study fund separation and aggregation issues in incomplete markets in which agents have identical locally homothetic preferences and spanned endowments. However, they do not cover cases with heterogeneous beliefs. Bettzüge (1998) provides a sufficient condition for the strict monotonicity of individual demand functions by generalizing the MP theorem to incomplete market economies. He shows that incomplete markets have a unique equilibrium when the MP coefficient is less than 4 and the individual initial endowments are spanned by the payoff matrix and collinear with respect to the aggregate initial endowment. The result of the study is distinct from that of the work by Bettzüge (1998) because the new condition is independent of relative risk aversion. Pilgrim (2002) presents insightful examples in which the uniqueness result for complete markets is not generalized to cases with incomplete markets and vice versa. Geanakoplos and Walsh (2016) provide new sufficient conditions for equilibrium to be unique and stable in two-period economies without uncertainty, which is equivalent to static, two-good economies and discuss the implications to the Diamond–Dyvig literature.

II. Model

We consider a simple two-period finance economy in which two agents $i = 1, 2$ consume a single good in the first period and the second period. The first period is denoted by state 0, and the uncertainty of the second period consists of finitely many states indexed by $s = 1, \ldots, S$ with $S \geq 3$. Let $\mathcal{S}$ denote the set $\{1, \ldots, S\}$ of the second-period states, and let $\mathcal{S}^a$ be the augmented set $\{0\} \cup \mathcal{S}$ of the $\mathcal{S} + 1$ states. As consumption arises in each $s \in \mathcal{S}^a$, a consumption plan is denoted by a point in $\mathbb{R}^{S+1}$.

Two securities $j = 1, 2$ are traded in the first period. As $S \geq 3$, asset markets are incomplete. Asset j pays r_s^j units of the single good in each state $s \in \mathcal{S}^a$. Let r^j denote the payoff vector $(r_0^j, \ldots, r_S^j) \in \mathbb{R}^{S+1}$ of asset j and let $r_s = (r_s^1, r_s^2)$ denote the payoff vectors of the two assets in state $s \in \mathcal{S}^a$. We introduce payoff matrices R^a and R, where R^a is a $(S + 1) \times 2$ matrix with r^j as its jth column and R is a $S \times 2$ matrix defined by

$$R = \begin{bmatrix} r_1^1 & r_1^2 \\ \vdots & \vdots \\ r_S^1 & r_S^2 \end{bmatrix}.$$

Agent i is initially endowed with shares $\bar{\theta}^i = (\bar{\theta}_1^i, \bar{\theta}_2^i)$ of the assets. The outstanding shares of each asset are normalized to 1. Thus, $\bar{\theta}_j^1 + \bar{\theta}_j^2 = 1$ for each $j = 1, 2$ holds.

The preferences of agent i over the consumption set $X \equiv \mathbb{R}_+^{S+1}$ are represented by a utility function

$$u_i(x^i) = v(x_0^i) + \rho \sum_{s \in \mathcal{S}} \pi_s^i v(x_s^i), \qquad x^i \in X.$$

Here, ρ can be considered as a time discount or weight between the current utility and the expected future utility while π_s^i is the subjective probability of agent i that state s occurs in the second period. For an asset price $q = (q^1, q^2)$, agent i faces the following choice problem:

$$\begin{aligned} \max_{(x,\theta) \in X \times \mathbb{R}^2} \quad & u_i(x) \\ \text{s.t.} \quad & x_0 \leq -q \cdot (\theta - \bar{\theta}^i) + r_0 \cdot \bar{\theta}^i, \\ & x_s \leq r_s \cdot \theta \quad \text{for all } s \in \mathcal{S}. \end{aligned} \tag{1}$$

We make the following assumptions.

Assumption 1. The function v is concave and homogenous of degree $k < 1$, *i.e.*, for each $\lambda > 0$ and $y > 0$, $v(\lambda y) = \lambda^k v(y)$.

Assumption 2. The payoff matrix R has a full rank, and R^a satisfies $R^a \cdot \bar{\theta}^i \gg 0$.[1]

[1] For two points x, y in $\mathbb{R}^{S+1}$, $x \geq y$ if $x - y \in \mathbb{R}_+^{S+1}$, $x > y$ if $x \geq y$ and $x \neq y$, and $x \gg y$ if $x - y \in \mathbb{R}_{++}^{S+1}$.

The homogeneity of Assumption 1 implies that $v(y) = y^k v(1)$ for all $y > 0$. By the concavity of v, it holds that if $v(1) > 0$, $0 \leq k < 1$, and if $v(1) < 0$, $k \leq 0$. When we put $k = 1 - \gamma$ for some $\gamma > 0$, γ indicates a constant relative risk aversion. When $k < 1$, v has an infinite marginal utility at 0, which makes agents choose positive consumption in each state. The second assumption ensures that each agent can make positive consumptions in each state under autarky.

Remark 1. The two-period finance economy in which agents are initially endowed with securities can be transformed into a two-period economy of the general equilibrium literature in which agents are initially endowed with consumption goods. This transformation is achieved by rewriting the second-period budget constraints as

$$x_s - r_s \cdot \bar{\theta}^i \leq r_s \cdot (\theta - \bar{\theta}^i), \qquad s \in \mathcal{S}.$$

The second part of Assumption 2 ensures that agents have positive initial endowments $\{r_s \cdot \bar{\theta}^i,\ s \in \mathcal{S}^a\}$ spanned by the asset payoffs. Thus, the second part of Assumption 2 implies a strong survival condition of the general equilibrium literature, which guarantees the existence of equilibrium under standard conditions, such as continuity and convexity of preferences. It also yields

$$R^a \cdot (\bar{\theta}^1 + \bar{\theta}^2) = R^a \cdot 1_2 \gg 0, \tag{2}$$

where 1_n indicates the column vector of n units.

Equilibrium is defined as follows.

Definition 1. A list $(q, (x^1, x^2), (\theta^1, \theta^2)) \in \mathbb{R}^2 \times X^2 \times \mathbb{R}^4$ is an *equilibrium* of the economy if it satisfies the following conditions:

(i) for the given price q, (x^i, θ^i) solves the utility maximization problem for every $i = 1, 2$, and

(ii) $\theta_j^1 + \theta_j^2 = 1$ for each $j = 1, 2$.

Let M denote a set defined by

$$M = \{x \in \mathbb{R} : r_s^1 + x r_s^2 > 0 \qquad \text{for all } s \in \mathcal{S}\}.$$

Note that (2) gives $r_s^1 + r_s^2 > 0$ for all $s \in \mathcal{S}^a$; hence, $1 \in M$. For each $x \in M$ and $i = 1, 2$, we define a function

$$G_i(x) = \frac{\sum_{s\in\mathcal{S}} \pi_s^i \left(r_s^1 + r_s^2 x\right)^{k-1} r_s^1}{\sum_{s\in\mathcal{S}} \pi_s^i \left(r_s^1 + r_s^2 x\right)^{k-1} r_s^2}.$$

As shown later, the function $G_i(x)$ is closely related to the relative price of two assets, which agent i evaluates on the basis of his subjective state prices. For instance, $G_i(1)$ indicates the relative price that makes agent i hold the same shares of the two assets. The main results of this work are built on the following condition jointly imposed on heterogeneous beliefs and payoffs.

Relative price condition (RPC). $G_1(1) = G_2(1)$, *i.e.*,

$$\frac{\sum_{s\in\mathcal{S}} \pi_s^1 \left(r_s^1 + r_s^2\right)^{k-1} r_s^1}{\sum_{s\in\mathcal{S}} \pi_s^1 \left(r_s^1 + r_s^2\right)^{k-1} r_s^2} = \frac{\sum_{s\in\mathcal{S}} \pi_s^2 \left(r_s^1 + r_s^2\right)^{k-1} r_s^1}{\sum_{s\in\mathcal{S}} \pi_s^2 \left(r_s^1 + r_s^2\right)^{k-1} r_s^2}. \tag{3}$$

Condition RPC holds trivially when agents share the same beliefs, *i.e.*, $\pi_s^1 = \pi_s^2$ for all $s \in \mathcal{S}$. As shown later, RPC is realized through the pricing relation imposed by the first-order condition for utility maximization when agent i makes an optimal portfolio choice θ^i of equal size for each $i = 1, 2$, *i.e.*, $\theta_1^1 = \theta_2^1$ and $\theta_1^2 = \theta_2^2$. In this case, agents behave as if they were in the economy with one asset that pays $r^1 + r^2$. That is, one-fund separation holds in equilibrium.

We impose the following sign condition on the terms in (3).

Assumption 3. For each $j = 1, 2$ and $i = 1, 2$,

$$\sum_{s\in\mathcal{S}} \pi_s^i \left(r_s^1 + r_s^2\right)^{k-1} r_s^j > 0. \tag{4}$$

As discussed below, Assumption 3 is slightly weaker than the presence of a "fundamental set of matrices" assumed in page 152 of Hens and Pilgrim (2002).

For analytical convenience, we can transform the payoff matrix R^a into a new one, which generates the same market span as R^a. Let R_1 and R_2 denote the submatrices of dimension 2×2 and $(S-2) \times 2$, respectively. R is decomposed as

$$R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}.$$

According to the full-rank condition of Assumption 2, R has two independent rows. Without loss of generality, we can assume that R_1 has a full rank. (Otherwise, we can rearrange states such that R_1 has a full rank.) We define $\tilde{R}^a$ by

$$\tilde{R}^a \equiv R^a R_1^{-1} = \begin{bmatrix} r_0 R_1^{-1} \\ I_2 \\ R_2 R_1^{-1} \end{bmatrix},$$

where I_2 is the 2×2 identity matrix and R_1^{-1} is the inverse matrix of R_1.

As R and $\tilde{R} = RR_1^{-1}$ have the same span, (q, x, θ) is an equilibrium of the economy with the payoff matrix R^a and the initial endowment $\{\bar{\theta}^i, i = 1, 2\}$ if and only if (q', x, η) is an equilibrium of the new economy with the payoff matrix $\tilde{R}^a$ and the initial endowment $\{\bar{\eta}^i, i = 1, 2\}$, where $\bar{\eta}^i = R_1 \cdot \bar{\theta}^i$, $q' = qR_1^{-1}$, and $\eta^i = R_1 \cdot \theta^i$ for each $i = 1, 2$. Moreover, the second part of Assumption 2 leads to the relation

$$R^a \cdot \bar{\theta}^i = \begin{bmatrix} r_0 R_1^{-1} \\ I_2 \\ R_2 R_1^{-1} \end{bmatrix} \cdot (R_1 \cdot \bar{\theta}^i) = \tilde{R}^a \cdot (R_1 \cdot \bar{\theta}^i) = \tilde{R}^a \cdot \bar{\eta}^i \gg 0. \tag{5}$$

Thus, without loss of generality, we will now assume that $R_1 = I_2$. Then, (5) yields

$$\bar{\theta}^i \gg 0 \quad \text{for each } i = 1, 2. \tag{6}$$

As marginal utility is infinite at 0, agents make an optimal choice of positive consumption in each state (interior solution). In particular, the result combined with $R_1 = I_2$ implies that

$$(x_1^i, x_2^i) = (\theta_2^i, \theta_2^i) \gg 0 \quad \text{for each } i = 1, 2. \tag{7}$$

The submatrix R_1 represents a *fundamental set of matrix R* defined in the work of Hens and Pilgrim (2002) if $R_2 R_1^{-1} \geq 0$. We note that when (4) is expressed in terms of $\tilde{R}^a$, $R_2R_1^{-1} \geq 0$ implies (4).

III. Main Results

This section shows that RPC is necessary and sufficient for one-fund separation, which ensures the unique existence of equilibrium in the economy with identical homothetic preferences and heterogeneous beliefs. The main results of this work are built on the following lemma, which characterizes the property of G_i.

Lemma 1: For any payoff matrix R with a full rank, each $G_i(x)$ is strictly decreasing in M, *i.e.*, $G_i'(x) > 0$ for each $x \in M$.

Proof: By differentiating $G_i(x)$, we obtain

$$G_i{}'(x) = \frac{f_i(x)}{\left(\sum_{s\in\mathcal{S}} \pi_s^i\left(r_s^1 + r_s^2 x\right)^{k-1} r_s^2\right)^2}$$

where

$$\begin{aligned} f_i(x) = &-(k-1)\sum_{s\in\mathcal{S}}\sum_{t\in\mathcal{S}} \pi_s^i (r_s^2)^2 (r_s^1 + xr_s^2)^{k-2} \pi_t^i r_t^1 (r_t^1 + xr_t^2)^{k-1} \\ &+ (k-1)\sum_{s\in\mathcal{S}}\sum_{t\in\mathcal{S}} \pi_s^i r_s^1 r_s^2 (r_s^1 + xr_s^2)^{k-2} \pi_t^i r_t^2 (r_t^1 + xr_t^2)^{k-1}. \end{aligned}$$

We see that

$$f_i(x) = (k-1)\sum_{s\in\mathcal{S}}\sum_{t\in\mathcal{S}} \pi_s^i r_s^2 (r_s^1 + xr_s^2)^{k-2} \pi_t^i (r_t^1 + xr_t^2)^{k-1} (r_s^1 r_t^2 - r_s^2 r_t^1).$$

By exploiting the symmetry of terms in $f_i(x)$, $f_i(x)$ is rearranged as follows.

$$\begin{aligned} f_i(x) &= (k-1)\sum_{s\in\mathcal{S}}\sum_{t\in\mathcal{S}} \pi_s^i r_s^2 (r_s^1 + xr_s^2)^{k-2} \pi_t^i (r_t^1 + xr_t^2)^{k-1} (r_s^1 r_t^2 - r_s^2 r_t^1) \\ &= \frac{k-1}{2}\sum_{s\in\mathcal{S}}\sum_{t\in\mathcal{S}} \Big(\pi_s^i r_s^2 (r_s^1 + xr_s^2)^{k-2} \pi_t^i (r_t^1 + xr_t^2)^{k-1} (r_s^1 r_t^2 - r_s^2 r_t^1) \\ &\quad + \pi_t^i r_t^2 (r_t^1 + xr_t^2)^{k-2} \pi_s^i (r_s^1 + xr_s^2)^{k-1} (r_t^1 r_s^2 - r_t^2 r_s^1)\Big) \\ &= -\frac{k-1}{2}\sum_{s\in\mathcal{S}}\sum_{t\in\mathcal{S}} \pi_s^i \pi_t^i (r_s^1 + xr_s^2)^{k-2} (r_t^1 + xr_t^2)^{k-2} (r_t^1 r_s^2 - r_s^1 r_t^2)^2 > 0. \end{aligned}$$

As the denominator of $G_i'(x)$ is positive, $f_i(x) > 0$ implies that $G_i'(x) > 0$ □

Lemma 1 shows that the monotonicity of G_i holds for any asset structure whose payoff matrix has a full rank. The result enables us to verify that the economy has a unique equilibrium on the basis of the one-fund separation without resorting to any fixed-point arguments.

Theorem 1: Suppose that the economy satisfies Assumptions 1 and 2. Then, the following results are obtained.

1) (One-fund separation) Let (q, x, θ) denote an equilibrium of the economy. RPC is necessary and sufficient for the economy to exhibit one-fund separation in equilibrium, *i.e.*, for each $i = 1, 2$, $\psi^i > 0$ exists such that $\psi^i = \theta_1^i = \theta_2^i$. Thus, for each $s \in S$,

$$x_s^i = \psi^i (r_s^1 + r_s^2). \tag{8}$$

2) If RPC and Assumption 3 hold in addition to Assumptions 1 and 2, the economy has a unique equilibrium.

The one-fund separation result shows that agents make a portfolio choice under RPC as if there were a single asset in the economy, *i.e.*, the market portfolio. The second result of Theorem 1 shows that RPC is sufficient for the unique existence of equilibrium in the case in which Assumptions 1–3 hold. Consequently, Theorem 1 demonstrates that one-fund separation is sufficient for the economy to have a unique equilibrium. The unique existence of Theorem 1 is a new result that has not been covered by the existing literature, such as the works of Hens and Pilgrim (2002) and Bettzüge (1998), which are built on the MP condition and the collinearity between individual and aggregate initial endowments. The MP condition is reduced to

$$-\frac{v''(x_s)x_s}{v'(x_s)} < 4 \quad (i.e., -3 < k < 1)$$

in the current framework. In particular, the MP condition is not implied by the RPC in the case with $k \leq -3$.

The proof of Theorem 1 is given below.

Proof: 1) As mentioned in Remark 1, the second part of Assumption 2 corresponds to the strong survival condition of the general equilibrium literature. Thus, the economy has equilibrium under Assumptions 1 and 2.[2] Let (q, x, θ) be an equilibrium of the economy. The equilibrium profile (q, x, θ) satisfies the system of equations, which consist of the first-order conditions for utility maximization and the market clearing condition.

$$v'(x_0^i) = \lambda_0^i \quad \text{and} \quad \rho\pi_s^i v'(x_s^i) = \lambda_s^i \quad \text{for all } s \in \mathcal{S}, \tag{9}$$

$$\lambda_0^i q = \lambda^i R, \tag{10}$$

$$x_0^i = -q \cdot (\theta^i - \overline{\theta}^i) + r_0 \cdot \overline{\theta}^i, \tag{11}$$

$$x_s^i = r_s^1 \theta_1^i + r_s^2 \theta_2^i \quad \text{for all } s \in \mathcal{S}, \tag{12}$$

$$\theta^1 + \theta^2 = 1_2, \tag{13}$$

where $(\lambda_0^i, \lambda^i) = (\lambda_0^i, \lambda_1^i, \ldots, \lambda_S^i) \in \mathbb{R}^{S+1}$ stands for the Lagrangian multiplier. From (9), (10), and the homogeneity of v, we find that for each $j = 1, 2$ and $i = 1, 2$,

$$(x_0^i)^{k-1} q^j = \sum_{s \in \mathcal{S}} \rho \pi_s^i (x_s^i)^{k-1} r_s^j. \tag{14}$$

By substituting (11) and (12) into the above relation, we obtain

$$\left(-q \cdot (\theta^i - \overline{\theta}^i) + r_0 \cdot \overline{\theta}^i\right)^{k-1} q^j = \sum_{s \in \mathcal{S}} \rho\pi_s^i \left(r_s^1 \theta_1^i + r_s^2 \theta_2^i\right)^{k-1} r_s^j,$$

which leads to the relation

$$\frac{q^1}{q^2} = \frac{\sum_{s \in \mathcal{S}} \rho\pi_s^i \left(r_s^1 \theta_1^i + r_s^2 \theta_2^i\right)^{k-1} r_s^1}{\sum_{s \in \mathcal{S}} \rho\pi_s^i \left(r_s^1 \theta_1^i + r_s^2 \theta_2^i\right)^{k-1} r_s^2} = \frac{\sum_{s \in \mathcal{S}} \pi_s^i \left(r_s^1 + r_s^2 \dfrac{\theta_2^i}{\theta_1^i}\right)^{k-1} r_s^1}{\sum_{s \in \mathcal{S}} \pi_s^i \left(r_s^1 + r_s^2 \dfrac{\theta_2^i}{\theta_1^i}\right)^{k-1} r_s^2}. \tag{15}$$

We set

[2] For existence of equilibrium, see Magill and Quinzii (1996) and Hens and Pilgrim (2002).

$$\phi^1 = \frac{\theta_2^1}{\theta_1^1} \quad \text{and} \quad \phi^2 = \frac{\theta_2^2}{\theta_1^2}.$$

Then, it holds that

$$\frac{q^1}{q^2} = G_1(\phi^1) = G_2(\phi^2). \tag{16}$$

We claim that $\phi^1 = \phi^2$. Suppose otherwise. Then, either $\phi^1 < \phi^2$ or $\phi^1 > \phi^2$. We can focus only on the first case because the same arguments apply to the second case by interchanging the role of agents. The first case is divided into the following three subcases:

a) $1 \leq \phi^1 < \phi^2$
b) $\phi^1 < 1 < \phi^2$
c) $\phi^1 < \phi^2 \leq 1$

Relation b) combined with RPC and the result of Lemma 1 leads to

$$G_1(\phi^1) < G_1(1) = G_2(1) < G_2(\phi^2).$$

This result contradicts (16), and thus, b) is impossible. To check the remaining possibilities a) and c), we recall from (7) that $\theta^i \gg 0$ for each $i = 1, 2$. Then, a) gives

$$1 \leq \frac{\theta_2^1}{\theta_1^1} < \frac{\theta_2^1 + \theta_2^2}{\theta_1^1 + \theta_1^2} < \frac{\theta_2^2}{\theta_1^2}. \tag{17}$$

This result contradicts the market clearing condition (13), which leads to

$$\frac{\theta_2^1 + \theta_2^2}{\theta_1^1 + \theta_1^2} = 1. \tag{18}$$

Thus, a) does not hold. Similarly, we can show that c) is impossible as well. Consequently, we see that $\phi^1 = \phi^2$. This result combined with (18) yields $\phi^1 = \phi^2 = 1$ or

$$\theta_1^1 = \theta_2^1 \quad \text{and} \quad \theta_1^2 = \theta_2^2. \tag{19}$$

Therefore, the economy exhibits one-fund separation in equilibrium.

Conversely, suppose that one-fund separation is fulfilled in (q, x, θ). By (8), we see that

$$\begin{bmatrix} x_1^i \\ x_2^i \end{bmatrix} = R_1 \cdot \theta^i = R_1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \psi^i.$$

As R_1 has a full rank, it gives $\psi^i = \theta_1^i = \theta_2^i$. By substituting the result into (15), we obtain RPC. □

2) To show that the economy has a unique equilibrium, we exploit the one-fund separation property of equilibrium. By substituting the result of (19) into (11) and (12), for each $s \in \mathcal{S}$, we have

$$x_0^i = -(q^1 + q^2)\theta_1^i + (q + r_0) \cdot \bar{\theta}^i \quad \text{and} \quad x_s^i = \theta_1^i (r_s^1 + r_s^2). \tag{20}$$

The following four equations are derived by substituting them into the pricing relation (14).

$$\left(-(q^1 + q^2)\theta_1^1 + (q + r_0) \cdot \bar{\theta}^1\right)(q^1)^{\frac{1}{k-1}} = \theta_1^1 \Big(\sum_{s \in \mathcal{S}} \rho \pi_s^1 (r_s^1 + r_s^2)^{k-1} r_s^1\Big)^{\frac{1}{k-1}} \tag{21}$$

$$\left(-(q^1 + q^2)\theta_1^1 + (q + r_0) \cdot \bar{\theta}^1\right)(q^2)^{\frac{1}{k-1}} = \theta_1^1 \Big(\sum_{s \in \mathcal{S}} \rho \pi_s^1 (r_s^1 + r_s^2)^{k-1} r_s^2\Big)^{\frac{1}{k-1}} \tag{22}$$

$$\left(-(q^1 + q^2)\theta_1^2 + (q + r_0) \cdot \bar{\theta}^2\right)(q^1)^{\frac{1}{k-1}} = \theta_1^2 \Big(\sum_{s \in \mathcal{S}} \rho \pi_s^2 (r_s^1 + r_s^2)^{k-1} r_s^1\Big)^{\frac{1}{k-1}} \tag{23}$$

$$\left(-(q^1 + q^2)\theta_1^2 + (q + r_0) \cdot \bar{\theta}^2\right)(q^2)^{\frac{1}{k-1}} = \theta_1^2 \Big(\sum_{s \in \mathcal{S}} \rho \pi_s^2 (r_s^1 + r_s^2)^{k-1} r_s^2\Big)^{\frac{1}{k-1}} \tag{24}$$

According to Assumption 3, for each $j = 1, 2$, we have

$$A_j \equiv \sum_{s \in \mathcal{S}} \pi_s^1 (r_s^1 + r_s^2)^{k-1} r_s^j > 0 \quad \text{and} \quad B_j \equiv \sum_{s \in \mathcal{S}} \pi_s^2 (r_s^1 + r_s^2)^{k-1} r_s^j > 0.$$

Now, we solve (21) and (23) for θ_1^1 and θ_1^2.

$$\theta_1^1 = \frac{(q^1 + r_0^1)\bar{\theta}_1^1 + (q^2 + r_0^2)\bar{\theta}_2^1}{q^1 + q^2 + (q^1)^{1/(1-k)}(\rho A_1)^{1/(k-1)}} \tag{25}$$

$$\theta_1^2 = \frac{(q^1 + r_0^1)\bar{\theta}_1^2 + (q^2 + r_0^2)\bar{\theta}_2^2}{q^1 + q^2 + (q^1)^{1/(1-k)}(\rho B_1)^{1/(k-1)}} \tag{26}$$

The results combined with the market clearing condition $\theta_1^1 + \theta_1^2 = 1$ lead to the relation

$$\frac{(q^1 + r_0^1)\bar{\theta}_1^1 + (q^2 + r_0^2)\bar{\theta}_2^1}{q^1 + q^2 + (q^1)^{1/(1-k)}(\rho A_1)^{1/(k-1)}} + \frac{(q^1 + r_0^1)\bar{\theta}_1^2 + (q^2 + r_0^2)\bar{\theta}_2^2}{q^1 + q^2 + (q^1)^{1/(1-k)}(\rho B_1)^{1/(k-1)}} - 1 = 0. \tag{27}$$

Relations (21) and (22) yield

$$\frac{q^1}{q^2} = \frac{A_1}{A_2}. \tag{28}$$

We substitute $q^1 = q^2 A_1/A_2$ into (27) and set $y = (q^2)^{1/(k-1)}$. Then the relation (27) is expressed as an equation of y.

$$H(y) \equiv h_1 y^{1-k} - h_2 y^{-k} + h_3 y + h_4 = 0, \tag{29}$$

where

$$\begin{aligned}
h_1 &= A_2^2(\rho A_2)^{\frac{1}{k-1}}\Big(r_0^1(1-\bar{\theta}_1^1) + r_0^2(1-\bar{\theta}_2^1) + \Big(\frac{A_1}{B_1}\Big)^{\frac{1}{1-k}}(r_0^1\bar{\theta}_1^1 + r_0^2\bar{\theta}_2^1)\Big) > 0,\\
h_2 &= A_2^2\Big(\frac{A_1}{A_2}\Big)^{\frac{1}{1-k}}(\rho A_2)^{\frac{1}{k-1}}(\rho B_1)^{\frac{1}{k-1}} > 0,\\
h_3 &= A_2 r_0^1(A_1 + A_2) + A_2 r_0^2(A_1 + A_2) > 0,\\
h_4 &= -\Big(A_2\Big(\frac{A_1}{A_2}\Big)^{\frac{1}{1-k}}(\rho B_1)^{\frac{1}{k-1}}\Big(A_1(1-\bar{\theta}_1^1) + A_2(1-\bar{\theta}_2^1)\Big)\\
&\quad + A_2(\rho A_2)^{\frac{1}{k-1}}(A_1\bar{\theta}_1^1 + A_2\bar{\theta}_2^1)\Big) < 0.
\end{aligned}$$

Note that $h_1 > 0$ comes from , which yields

$$r_0^1(1-\bar{\theta}_1^1) + r_0^2(1-\bar{\theta}_2^1) = r_0^1\bar{\theta}_1^2 + r_0^2\bar{\theta}_2^2 > 0.$$

We then prove that the economy has a unique equilibrium for each

$k < 1$. Hence, we show that $H(y) = 0$ has a unique solution in the two cases, namely, $0 \le k < 1$ and $k < 0$ If $0 \le k < 1$, we have $H'(y) = h_3 + y^{-1-k}(h_2 k + h_1(1 - k)y) > 0$. Thus, $H(y)$ is strictly increasing in $y > 0$ with $H(0) = h_4 < 0$. Moreover, when $0 \le k < 1$, y has the highest order in $H(y)$, which has the coefficient $h_3 > 0$ for $0 < k < 1$ and $h_1 + h_3 > 0$ for $k = 0$, implying that $H(\infty) > 0$. Thus, $H(y) = 0$ has a unique positive solution. If $k < 0$, we have $H''(y) = -ky^{-2-k}(h_2 + h_2 y + h_1 y(1 - k)) > 0$. This result implies that $H(y)$ is strictly convex. In this case, y^{1-k} has the highest order in $H(y)$ with positive coefficient h_1. As $H(0) = h_4 < 0$ and $H(\infty) > 0$, the strict convexity of $H(y)$ implies that $H(y) = 0$ has a unique positive solution.

Consequently, $H(y) = 0$ has a unique positive solution y^* for each $k < 1$. Then, we obtain $q^2 = (y^*)^{k-1}$, which gives $q^1 = (y^*)^{k-1} A_1/A_2$. The asset value (q^1, q^2) determines uniquely the values of θ_1^1 and θ_1^2 through (25) and (26). The result combined with (19) yields the optimal portfolio choice for agents 1 and 2. Consequently, the economy has a unique equilibrium. □

IV. Example

This section illustrates an economy with $S = 3$ in which agents have heterogeneous beliefs. The economy is taken to satisfy Assumptions 1–3 and RPC. The presentation below indicates that one-fund separation holds in the unique equilibrium of the economy.

Agent $i = 1, 2$ has a utility function with relative risk aversion γ.

$$u_i(x) = \frac{x_0^{1-\gamma}}{1-\gamma} + \rho \sum_{s \in S} \pi_s^i \frac{x_s^{1-\gamma}}{1-\gamma},$$

where ρ denotes a time discount and π_s^i indicates agent i's subjective belief that state s occurs in the second period. The asset payoffs are summarized in the following matrix.

$$R = \begin{bmatrix} 1 & 6 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

The first asset is riskless, whereas the second one is risky. Assumption 3 holds in the example:

$$R_2R_1^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \end{bmatrix} \gg 0.$$

Here, the two agents have relative risk aversion $\gamma = 2$ and time discount $\rho = 0.98$, and they are initially endowed with the asset shares $(\bar{\theta}_1^1, \bar{\theta}_2^1) = (1/3, 2/3)$ and $(\bar{\theta}_1^2, \bar{\theta}_2^2) = (2/3, 1/3)$. They have the following heterogeneous beliefs:

$$\pi^1 = (0.4836, 0.2664, 0.25) \text{ and } \pi^2 = (0.1612, 0.0888, 0.75),$$

which satisfy the RPC

$$\frac{\sum_{s\in\mathcal{S}} \pi_s^1 \left(r_s^1 + r_s^2\right)^{k-1} r_s^1}{\sum_{s\in\mathcal{S}} \pi_s^1 \left(r_s^1 + r_s^2\right)^{k-1} r_s^2} = \frac{\sum_{s\in\mathcal{S}} \pi_s^2 \left(r_s^1 + r_s^2\right)^{k-1} r_s^1}{\sum_{s\in\mathcal{S}} \pi_s^2 \left(r_s^1 + r_s^2\right)^{k-1} r_s^2} = \frac{1}{3}.$$

Theorem 1 ensures that the economy has a unique equilibrium (q, x, θ), which is expressed as

$$\begin{aligned} q &= (0.5044, 1.5132), \\ x^1 &= (1.7129, 3.9230, 1.6813, 2.2417), \\ x^2 &= (1.2871, 3.0770, 1.3187, 1.7583), \\ \theta^1 &= (0.5604, 0.5604), \\ \theta^2 &= (0.4396, 0.4396). \end{aligned}$$

The equilibrium displays the one-fund separation property.

V. Conclusion

This work addresses the issues of fund separation and equilibrium uniqueness in a special class of incomplete market economies with two agents and two assets in which the agents have an identical CRRA utility function with heterogeneous beliefs. A unique equilibrium exists if the asset payoffs and beliefs satisfy the RPC. The RPC is necessary and sufficient for one-fund separation to hold in equilibrium. Thus, the one-fund separation property is sufficient for the uniqueness of equilibrium in the class of economies with heterogeneous beliefs at hand.

The RPC, which imposes joint restrictions on the asset payoffs and heterogeneous beliefs, is distinct from the existing conditions for the unique existence of equilibrium, such as gross substitution and the MP restriction. Thus, it provides a new perspective into uniqueness of equilibrium in incomplete markets. The current framework for the RPC is restricted to the case involving two agents and two assets. We consider it interesting to see how the RPC is applied to incomplete markets with more than two assets. We also recognize the challenge of seeking a general version of the RPC in incomplete markets with more than two agents. A far-reaching attempt is to see how far the insight of the RPC can go in verifying the unique existence of equilibrium in a general framework of incomplete markets.

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