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m-step competition graphs of bipartite tournaments

(방향 지어진 완전 이분 그래프의 m-step 경쟁 그래프)

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m-step competition graphs of bipartite tournaments

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Abstract

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In this thesis, we study the m-step competition graphs of bipartite tournaments. We compute the number of distinct bipartite tournaments by Pólya's theory of counting. Then we study the competition indices and competition periods of bipartite tournaments. We characterize the pairs of graphs that can be represented as the m-step competition graphs of bipartite tournaments. Finally, we present the maximum number of edges and the minimum number of edges which the m-step competition graph of a bipartite tournament might have.

Key words: complete bipartite graph; bipartite tournament; m-step competition graph; competition index; competition period; m-step competition-realizable.
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Chapter 1

Introduction

1.1 Basic graph terminology

Given a graph G, and let $S \subset V(G)$ be any subset of vertices of G. Then the *induced subgraph* of $G[S]$ is the graph whose vertex set is S and whose edge set consists of all of the edges in $E(G)$ that have both endpoints in S. The same definition works for undirected graphs and directed graphs. The induced subgraph $G[S]$ may also be called the *subgraph of G induced by S.*

A *clique*, C, of a graph G is a subset of the vertices, $C \subset V(G)$, such that every two distinct vertices are adjacent. This is equivalent to the condition that the induced subgraph of G induced by C is a complete graph. A *maximal clique* is a clique that cannot be extended by including one more adjacent vertex, that is, a clique which does not exist exclusively within the vertex set of a larger clique.

A graph is *bipartite* if its vertex set can be partitioned into two subsets V_1 and V_2 so that every edge has one end in V_1 and the other end in V_2; such a partition (V_1, V_2) is called a *bipartition* of the graph, and V_1 and V_2 are called its *parts*. If a bipartite graph is simple and every vertex in one part is joined to every vertex in the other part, then the graph is called a *complete bipartite graph*. We denote $K_{m,n}$ a complete bipartite graph with bipartition

(V_1, V_2) if $|V_1| = m$ and $|V_2| = n$.

A *walk* in a graph G is a sequence $W := v_0e_1v_1 \ldots v_{l-1}e_lv_l$, whose terms are alternately vertices and edges of G (not necessarily distinct), such that v_{i-1} and v_i are the ends of e_i, $1 \leq i \leq l$. In a simple graph, a walk $v_0e_1v_1 \ldots v_{l-1}e_lv_l$ is determined, and is commonly specified, by the sequence $v_0v_1 \ldots v_l$ of its vertices. If $v_0 = x$ and $v_l = y$, we say that W connects x to y and refer W as an (x, y)-*walk*. The notation xWy is also used simply to signify a (x, y)-walk W. The vertices x and y are called the *ends* of the walk, x being its *initial vertex* and y its *terminal vertex*; the vertices $v_1, \ldots, v_{l-1}$ are its *interval vertices*. The integer l (the number of edge terms) is the *length* of W. A *path* is a walk in which no vertices are repeated. For a path P, if x and y are the initial and terminal vertices of P, we refer P as an (x, y)-*path*. A walk in a graph is *closed* if its initial and terminal vertices are identical. A *cycle* on three or more vertices is a closed walk which no vertices except the initial vertex and the terminal vertex are repeated. The *length* of a path or a cycle is the number of its edges. A path or cycle of length k is called a k-*path* or k-*cycle* and denoted by P_{k+1} or C_k, respectively; the path or cycle is *odd* or *even* according to the parity of k. A 3-cycle is often called a *triangle*. If a graph contains no triangle, then it is *triangle-free*.

Given a digraph D, we denote by $N^+(u)$ (resp. $N^-(u)$) the set of *out-neighbors* (resp. *in-neighbors*) of a vertex u in D. The *out-degree* (resp. *in-degree*) of u in D is defined to be $|N^+(u)|$ (resp. $|N^-(u)|$). A *directed walk* in a digraph D is an alternating sequence of vertices and arcs

$$W := (v_0, a_1, v_1, \ldots, v_{l-1}, a_l, v_l)$$

such that v_{i-1} and v_i are the tail and head of a_i, respectively, $1 \leq i \leq l$. If x and y are the initial and terminal vertices of W, we refer W as a *directed* (x, y)-*walk*. A directed walk in a digraph is *closed* if its initial and terminal vertices are identical. *Directed paths* are directed walks in which no vertices are repeated. *Directed cycles* in digraph are closed directed walks which no vertices except the initial vertex and the terminal vertex are repeated.

For a digraph D, the *underlying graph* of D is the graph G such that $V(G) = V(D)$ and $E(G) = \{uv \mid (u,v) \in A(D)\}$. An *orientation* of a graph G is a digraph having no directed 2-cycles, no loops, and no multiple arcs whose underlying graph is G. An *oriented graph* is a graph with an orientation. A *tournament* is an oriented complete graph. An orientation of a complete bipartite graph is sometimes called a *bipartite tournament* and we use whichever of the two terms is more suitable for a given situation throughout this paper.

1.2 Competition graph and its variants

The *competition graph* $C(D)$ of a digraph D is the (simple undirected) graph G defined by $V(G) = V(D)$ and $E(G) = \{uv \mid u,v \in V(D), u \neq v, N^+(u) \cup N^+(v) \neq \emptyset\}$. Cohen [11] introduced the notion of competition graph while studying predator-prey concepts in ecological food webs. Cohen's empirical observation that real-world competition graphs are usually interval graphs had led to a great deal of research on the structure of competition graphs and on the relation between the structure of digraphs and their corresponding competition graphs. For a comprehensive introduction to competition graphs, see [13, 26]. Competition graphs also have applications in coding, radio transmission, and modeling of complex economic systems (see [30] and [31] for a summary of these applications). For recent work on this topic, see [10, 29, 35, 37].

A variety of generalizations of the notion of competition graph have also been introduced, including the m-step competition graph in [7, 17], the common enemy graph (sometimes called the resource graph) in [27, 34], the competition-common enemy graph (sometimes called the competition-resource graph) in [2, 15, 19, 21, 25, 32, 33], the niche graph in [4, 5, 14] and the p-competition graph in [1, 23, 24].

Lundgren and Maybee [27] introduced the *common enemy graph* of a digraph D which is the graph which has the same vertex set as D and has an edge between two distinct vertices x and y if it holds that both $N^{-}(u) \cup N^{-}(v) \neq \emptyset$. This led Scott [32] to introduce the competition-common enemy graph of D. The *competition-common enemy graph* of a digraph D is the graph which has the same vertex set as D and has an edge between two distinct vertices x and y if it holds that both $N^{+}(u) \cup N^{+}(v) \neq \emptyset$ and $N^{-}(u) \cup N^{-}(v) \neq \emptyset$. This graph is essentially the intersection of the competition graph and the common enemy graph. That is, two vertices are adjacent if and only if they have both a common prey and a common enemy in D. The niche graph is the union of the competition graph and the common enemy graph. If $D = (V, A)$ is a digraph, the *niche graph* corresponding to D is the undirected graph $G = (V, E)$ with an edge between two distinct vertices x and y of V if and only if for some $z \in V$, there are arcs (x, z) and (y, z) in D or there are arcs (z, x) and (z, y) in D. For a digraph D, let $CE(D)$ be the common enemy graph, $CCE(D)$ the competition-common enemy graph, and $N(D)$ the niche graph. Then $CCE(D) \subset C(D) \subset N(D)$. For a digraph $D = (V, A)$, if p is a positive integer, the *p-competition graph* $C_p(D)$ corresponding to D is defined to have vertex set V with an edge between x and y in V if and only if, for some distinct $a_1, \ldots, a_p$ in V, the pairs (x, a_1), (y, a_1), (x, a_2), (y, a_2), $\ldots$, (x, a_p), (y, a_p) are arcs. If D is thought of as a food web whose vertices are species in some ecosystem, with an arc (x, y) if and only if x preys on y, then xy is an edge of $C_p(D)$ if and only if x and y have at least p common prey. Among the variants, the notion of m-step competition graph, which we study here, was introduced by Cho and Kim (2005) [7]. Since its introduction, it has been extensively studied (see for example [3, 6, 17, 18, 20, 28, 36]).

1.3 m-step competition graphs

Given a digraph D and a positive integer m, we define the *m-step digraph* D^m of D as follows: $V(D^m) = V(D)$ and there exists an arc (u, v) in D^m if and only if there exists a directed walk of length m from u to v. If there is a directed walk of length m from a vertex x to a vertex y in D, we call y an *m-step prey* of x, and if a vertex w is an m-step prey of both vertices u and v, then we say that w is an *m-step common prey* of u and v. The *m-step competition graph* of D, denoted by $C^m(D)$, has the same vertex set as D and an edge between x and y if and only if x and y have an m-step common prey in D. From the definition of $C^m(D)$ and D^m, the following proposition immediately follows.

Proposition 1.1 ([7]). *For any digraph D (possibly with loops) and a positive integer m,*

$$C^m(D) = C(D^m).$$

For the two-element Boolean algebra $\mathcal{B} = \{0, 1\}$, $\mathcal{B}_n$ denotes the set of all $n \times n$ (Boolean) matrices over $\mathcal{B}$. Under the Boolean operations, we can define matrix addition and multiplication in $\mathcal{B}_n$. Let D be a digraph with vertex set $\{v_1, v_2, \ldots, v_n\}$, and $A = (a_{ij})$ be the (Boolean) adjacency matrix of D such that

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \text{ is an arc of } D, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that for a positive integer m, the (Boolean) mth power $A^m = (b_{ij})$ of A is a Boolean matrix such that b_{ij} is one if and only if there is a directed walk of length m from v_i to v_j in D. Thus two rows i and i' of A^m have non-zero entry in the jth column if and only if vertex v_j is an m-step common prey of vertices v_i and $v_{i'}$ in D.

For a Boolean matrix A, the *row graph* $\mathcal{R}(A)$ of A is the graph whose vertices are the rows of A, and two vertices in $\mathcal{R}(A)$ are adjacent if and only if their corresponding rows have a non-zero entry in the same column of

A. This notion was studied by Greenberg *et al.* [16]. From the definition of row graphs and m-step competition graphs, the following proposition follows immediately.

Proposition 1.2 ([7]). *A graph G with n vertices is an m-step competition graph if and only if there is a Boolean matrix A in $\mathcal{B}_n$ such that G is the row graph of A^m.*

Kim *et al.* [22] studied the competition graphs of bipartite tournaments and Choi *et al.* [9] studied the $(1,2)$-step competition graphs of bipartite tournaments. In this thesis, we study the m-step competition graphs of bipartite tournaments, which is a natural extension of their results.

1.4 Pólya's theory of counting

This section is written based on Chapter 5 of [12].

Let S be a finite set. A *permutation* of S is a one-to-one mapping of S onto itself. If σ is a permutation, and s is any element of S, then σs denotes the element onto which s is mapped by σ. If σ_1 and σ_2 are permutations, then the product $\sigma_1\sigma_2$ is defined as the composite mapping obtained by applying first σ_2, then σ_1. It is well known that if σ is given, then we can split S in a unique way into *cycles*, that is, subsets of S that are cyclically permuted by σ. If l is the length of such a cycle, and if s is any element of that cycle, then the cycle consists of

$$s, \sigma s, \sigma^2 s, \ldots, \sigma^{l-1} s,$$

where $\sigma^2 = \sigma\sigma$, etc.

If S splits into α_1 cycles of length 1, α_2 cycles of length 2, ..., then the *cycle type* of σ is $z_1^{\alpha_1} z_2^{\alpha_2} \cdots$. Obviously, $\alpha_i = 0$ for all but at most a finite number of $i's$; certainly, $\alpha_i = 0$ for $i > n$, where n stands for the number of elements of S. Furthermore, we clearly have

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 + \cdots = n,$$

the sum of the lengths of the cycles being the total number of elements in S.

We now define the cycle index of a permutation group. Letting G be a group whose elements are the permutations of S, the group operation being the multiplication introduced previously, we define a special polynomial in n variables $z_1, \ldots, z_n$, with nonnegative coefficients as follows. For each $\sigma \in G$, taking the sum of cycle type $z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$ of σ and dividing by the number of elements of G, we get the polynomial

$$P_G(z_1, z_2, \ldots, z_n) = \frac{1}{|G|} \sum_{\sigma \in G} z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$$

which we call the *cycle index* of G.

Let D and R be finite sets. We consider functions defined on D, with values in R; in other words, we consider mappings of D into R. The set D is called the *domain*, and R is called the *range*. The set of all such functions is denoted by R^D. Furthermore, we suppose that we are given a group G of permutations of D. This group introduces an equivalence relation in R^D: Two functions f_1, f_2 (both in R^D) are called *equivalent* (denoted $f_1 \sim f_2$) if there exists an element $\sigma \in G$ such that

$$f_1(\sigma d) = f_2(d) \text{ for all } d \in D$$

This equation can be abbreviated to $f_1\sigma = f_2$ since $f_1\sigma$ is the notation for the composite mapping "first σ, then f_1." Since $\sim$ is an equivalence relation, the set R^D splits into equivalence classes. These equivalence classes will be called *patterns*.

To each element of R we assign a *weight*. This weight may be a number, or a variable, or, more generally, an element of a commutative ring weights, and rational multiples of weights, and these operations satisfy the usual associative, commutative, and distributive laws. The weight assigned to element $r \in R$ will be called $w(r)$. Once these weights have been chosen, we can define the weight $W(f)$ of a function $f \in R^D$ as the product

$$W(f) = \prod_{d \in D} w[f(d)].$$

If f_1 and f_2 are equivalent, that is, if they belong to the same pattern, then they have the same weight. This follows from the fact that if $f_1\sigma = f_2, \sigma \in G$, then we have

$$\prod_{d \in D} w[f_1(d)] = \prod_{d \in D} w[f_1(\sigma d)] = \prod_{d \in D} w[f_2(d)],$$

since the first and the second product have the same factors, in a different order only, and since multiplication of weights is commutative.

Since all functions belonging to one and the same pattern have the same weight, we may define the weight of the pattern as this common value. Thus if F denotes a pattern, we shall denote the weight of F by $W(F)$.

Thinking of R as a set from which we have to choose function values, we call R the *store*. Since the weights can be added, a weight sum exists; this sum is called the *store enumerator*, or the *inventory* of R:

$$\text{inventory of } R = \sum_{r \in R} w(r).$$

The inventory of R^D is just a power of the inventory of R, the exponent being the number of elements of D:

$$\text{inventory of } R^D = \sum_f W(f) = \left[\sum_{r \in R} w(r)\right]^{|D|}.$$

Theorem 1.3 (Pólya's Fundamental Theorem)**.** *The pattern inventory is*

$$\sum_F W(F) = P_G\left\{\sum_{r\in R} w(r), \sum_{r\in R}[w(r)]^2, \sum_{r\in R}[w(r)]^3, \ldots\right\},$$

where P_G is the cycle index. In particular, if all weights are chosen to be equal to unity, then we obtain

$$\textit{the number of patterns} = P_G(|R|, |R|, |R|, \ldots).$$

The following theorem is a simplified version of Theorem 1.3.

Theorem 1.4. *Let S be a set of elements and G be a group of permutations of S that acts to induce an equivalence relation on the colorings of S. The inventory of nonequivalent colorings of S using colors $c_1, c_2, \ldots, c_m$ is given by the generating function*

$$P_G\left(\sum_{j=1}^m c_j, \sum_{j=1}^m c_j^2, \ldots, \sum_{j=1}^m c_j^k\right),$$

where k corresponds to the largest cycle length.

1.5 Competition indices and competition periods

Cho and Kim [8] introduced the notions of the competition index and the competition period of D for a strongly connected digraph D, and Kim [20] extended these notions to a general digraph D.

Consider the competition graph sequence $C^1(D)$, $C^2(D)$, $C^3(D)$, …, $C^m(D)$, …. (Note that for a digraph D and its adjacency matrix A, the graph sequence $C^1(D)$, $C^2(D)$,…,$C^m(D)$,… is equivalent to the row graph sequence $\mathcal{R}(A)$,$\mathcal{R}(A^2)$,…,$\mathcal{R}(A^m)$,… by Proposition 1.1 and Proposition 1.2.)

Since the cardinality of the (Boolean) matrix set $\mathcal{B}_n$ is equal to a finite number 2^{n^2}, there is a smallest positive integer q such that $C^{q+i}(D) = C^{q+r+i}(D)$ (equivalently $\mathcal{R}(A^{q+i}) = \mathcal{R}(A^{q+r+i})$) for some positive integer r and all non-negative integer i. Such an integer q is called the *competition index* of D and is denoted by cindex(D). For $q =$cindex(D), there is also a smallest positive integer p such that $C^q(D) = C^{q+p}(D)$ (equivalently $\mathcal{R}(A^q) = \mathcal{R}(A^{q+p})$). Such an integer p is called the *competition period* of D and is denoted by cperiod(D).

Proposition 1.5 ([8]). *If there is no vertex whose out-degree is zero in a digraph D, we have*

$$\text{cperiod}(D) = 1.$$

Proof. Note that each vertex of D has an outgoing arc. Thus, every edge in $C^m(D)$ is an edge in $C^{m+i}(D)$ for every positive integer i, since any two vertices having an m-step common prey also have an $(m+i)$-step common prey. Therefore, we have cperiod(D)=1. □

1.6 Preview of thesis

In chapter 2, we apply Pólya's theory of counting to compute the number of distinct bipartite tournaments by identifying a bipartite tournament with bipartition (V_1, V_2) with a complete bipartite graph with bipartition (V_1, V_2) whose edges colored with two colors. In chapter 3, we derive the properties of m-step competition graphs of bipartite tournaments. We also study the competition indices and the competition periods of bipartite tournaments when there are no directed cycles. We present the competition indices and the competition periods with the length of the longest directed path in the bipartite tournament. In chapter 4, we characterize the pairs of graphs that can be represented as the m-step competition graphs of bipartite tournaments. In chapter 5, we present the maximum number of edges and the minimum number of edges which the m-step competition graph of a bipartite tournament might have.

Chapter 2

The number of distinct bipartite tournaments

In this chapter, we count the number of distinct bipartite tournaments.

Let S_n denote the symmetric group of order n. The following proposition is a well-known result.

Proposition 2.1. *The number of permutations with cycle type* $z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$ *in* S_n *is*

$$\frac{n!}{1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n} \alpha_1! \alpha_2! \cdots \alpha_n!}.$$

Lemma 2.2. *If the group of symmetries that acts on the vertex set of a complete bipartite graph* K_{n_1,n_2} *is isomorphic to* $S_{n_1} \times S_{n_2}$*, then the cycle index of the group of symmetries that acts on the edge set of a complete bipartite graph* K_{n_1,n_2} *is*

$$\sum \frac{\prod_{(k,l)\in[n_1]\times[n_2]} z_{L(k,l)}^{G(k,l)\alpha_k\beta_l}}{\prod_{p=1}^{n_1} (p^{\alpha_p}\alpha_p!) \prod_{q=1}^{n_2} (q^{\beta_q}\beta_q!)}.$$

where the sum is over all nonnegative integer α_i, β_j *such that*

$$\alpha_1 + 2\alpha_2 + \cdots + n_1\alpha_{n_1} = n_1 \quad \textit{and} \quad \beta_1 + 2\beta_2 + \cdots + n_2\beta_{n_2} = n_2.$$

Proof. An element in $S_{n_1} \times S_{n_2}$ with cycle notation

$$(a_{11}a_{12}\cdots a_{1l_1})\cdots(a_{k1}a_{k2}\cdots a_{kl_k})(b_{11}b_{12}\cdots b_{1l'1})\cdots(b_{k'1}b_{k'2}\cdots b_{k'l'_{k'}}), \quad (2.1)$$

where

$$C_1 := (a_{11}a_{12}\cdots a_{1l_1})\cdots(a_{k1}a_{k2}\cdots a_{kl_k}) \in S_{n_1}$$

and

$$C_2 := (b_{11}b_{12}\cdots b_{1l'_1})\cdots(b_{k'1}b_{k'2}\cdots b_{k'l'_{k'}}) \in S_{n_2},$$

deduces an element C in the automorphism group on the edge set of K_{n_1,n_2} which is isomorphic to $S_{n_1n_2}$ in the following way. We note that each cycle in C_1 independently interacts with each cycle in C_2 when C_1C_2 deduces the element C. Therefore it is sufficient to find the cycle type of an element C' in a group isomorphic to $S_{l_il'_j}$ deduced by $(a_{i1}a_{i2}\cdots a_{il_i})(b_{j1}b_{j2}\cdots b_{jl'_j})$. Let us take a cycle containing $a_{i1}b_{j1}$. Then it is completed when $a_{i1}b_{j1}$ appears again. It is easy to see that it appears after edges as many as the least common multiple $L(l_i, l'_j)$ of l_i and l'_j. Therefore the length of the cycle is $L(l_i, l'_j)$, which is independent of the edge $a_{i1}b_{j1}$. Thus each cycle in C' has length $L(l_i, l'_j)$. Then the number of cycles in C' is $l_il'_j/L(l_i, l'_j)$, which equals the greatest common divisor $G(l_i, l'_j)$. Hence the cycle type of C' is

$$z_{L(l_i,l'_j)}^{G(l_i,l'_j)}.$$

For example, we consider the case $l_i = 6$ and $l'_j = 4$. Let

$$\sigma = (a_{i1}a_{i2}\cdots a_{i6})(b_{j1}b_{j2}b_{j3}b_{j4})$$

and σ deduces σ'. Consider the edge $a_{i1}b_{j1}$. It is mapped to $a_{i2}b_{j2}$, that is

$\sigma'(a_{i1}b_{j1}) = a_{i2}b_{j2}$ and $a_{i3}b_{j3} = \sigma'(a_{i2}b_{j2}) = \sigma'^2(a_{i1}b_{j1})$. We can check that $\sigma^{12}(a_{i1}b_{j1}) = a_{i1}b_{j1}$ but $\sigma^k(a_{i1}b_{j1}) \neq a_{i1}b_{j1}$ for all $k = 1, 2, \ldots, 11$. Therefore $(a_{i1}b_{j1}a_{i2}b_{j2} \ldots a_{i6}b_{j4})$ is a cycle of σ'. Now by the same way, $(a_{i1}b_{j2}a_{i2}b_{j3} \ldots a_{i6}b_{j1})$ is also a cycle of σ'.

As we have observed, the cycle type of C is

$$\prod_{(i,j)\in[k]\times[k']} z_{L(l_i,l'_j)}^{G(l_i,l'_j)} \tag{2.2}$$

where $[n]$ denotes the set $\{1, 2, \ldots, n\}$. The permutation given in (2.1) has the cycle type

$$\prod_{i=1}^{k} z_{l_i} \prod_{j=1}^{k'} z_{l'_j}$$

and we may conclude that a permutation with this cycle type deduces

$$\prod_{(i,j)\in[k]\times[k']} z_{L(l_i,l'_j)}^{G(l_i,l'_j)}$$

in the group of symmetries that acts on the edge set of a complete bipartite graph K_{n_1,n_2}. This implies that

$$z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_{n_1}^{\alpha_{n_1}} z_1^{\beta_1} z_2^{\beta_2} \cdots z_{n_2}^{\beta_{n_2}}$$

in $S_{n_1} \times S_{n_2}$ deduces a permutation with the cycle type

$$\prod_{(k,l)\in[n_1]\times[n_2]} z_{L(k,l)}^{G(k,l)\alpha_k\beta_l}$$

in the group of symmetries that acts on the edge set of a complete bipartite graph K_{n_1,n_2}. By Proposition 2.1, the cycle index of the group of symmetries that acts on the edge set of a complete bipartite graph K_{n_1,n_2} is

$$\frac{1}{n_1! \times n_2!} \sum \left[\frac{n_1! \times n_2!}{\prod_{p=1}^{n_1} (p^{\alpha_p} \alpha_p!) \prod_{q=1}^{n_2} (q^{\beta_q} \beta_q!)} \prod_{(k,l)\in[n_1]\times[n_2]} z_{L(k,l)}^{G(k,l)\alpha_k\beta_l} \right].$$

where the sum is over all nonnegative integer α_i, β_j such that

$$\alpha_1 + 2\alpha_2 + \cdots + n_1\alpha_{n_1} = n_1 \quad \text{and} \quad \beta_1 + 2\beta_2 + \cdots + n_2\beta_{n_2} = n_2.$$

□

Theorem 2.3. *The number of distinct bipartite tournaments with bipartition (V_1, V_2) satisfying $|V_1| > |V_2|$ is*

$$\sum \frac{2^{\sum_{(k,l)\in[n_1]\times[n_2]} G(k,l)\alpha_k\beta_l}}{\prod_{p=1}^{n_1} (p^{\alpha_p} \alpha_p!) \prod_{q=1}^{n_2} (q^{\beta_q} \beta_q!)}$$

where $n_1 = |V_1|$, $n_2 = |V_2|$, and the sum is over all nonnegative integer α_i, β_j such that

$$\alpha_1 + 2\alpha_2 + \cdots + n_1\alpha_{n_1} = n_1 \quad \textit{and} \quad \beta_1 + 2\beta_2 + \cdots + n_2\beta_{n_2} = n_2.$$

Proof. We may identify a bipartite tournament with bipartition (V_1, V_2) with a complete bipartite graph with bipartition (V_1, V_2) whose edges colored with red and blue by regarding an arc (x, y) from V_1 to V_2 as the edge xy colored with red and an arc (x, y) from V_2 to V_1 as the edge xy colored with blue.

It is easy to see that the group of symmetries that acts on the vertex set of a complete bipartite graph K_{n_1,n_2} is isomorphic to $S_{n_1} \times S_{n_2}$. Therefore, by Lemma 2.2 and Theorem 1.4, the number of distinct bipartite tournaments with bipartition (V_1, V_2) is

$$\sum \frac{2^{\sum_{(k,l)\in[n_1]\times[n_2]} G(k,l)\alpha_k\beta_l}}{\prod_{p=1}^{n_1} (p^{\alpha_p} \alpha_p!) \prod_{q=1}^{n_2} (q^{\beta_q} \beta_q!)}$$

$\square$

Theorem 2.4. *The number of distinct bipartite tournaments with bipartition (V_1, V_2), where $|V_1| = |V_2| = n$ is odd, is*

$$\frac{1}{2}\sum \frac{2^{\sum_{(k,l)\in[n]\times[n]} G(k,l)\alpha_k\beta_l}}{\prod_{p=1}^{n}(p^{\alpha_p}\alpha_p!)\prod_{q=1}^{n}(q^{\beta_q}\beta_q!)}$$

where the sum is over all nonnegative integer α_i, β_j such that

$$\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n \quad \text{and} \quad \beta_1 + 2\beta_2 + \cdots + n\beta_n = n.$$

Proof. Let $\mathcal{D}$ be the set of all bipartite tournaments with bipartition (V_1, V_2) where $|V_1| = |V_2| = n$ is odd and all the vertices in $V_1 \cup V_2$ are distinguishable. There are n^2 arcs in every digraph in $\mathcal{D}$. The number of digraphs which have r arcs from V_1 to V_2 and $(n^2 - r)$ arcs from V_2 to V_1 is $\binom{n^2}{r}$ for nonnegative integer $r = 0, 1, \ldots, n^2$. Let $\mathcal{D}_1 \subset \mathcal{D}$ (resp. $\mathcal{D}_2 \subset \mathcal{D}$) be the set of bipartite tournaments which has more arcs from V_1 to V_2 (resp. V_2 to V_1) than the arcs from V_2 to V_1 (resp. V_1 to V_2). Since n is odd, $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$. It is easy to see that there is a one-to-one correspondence between $\mathcal{D}_1$ and $\mathcal{D}_2$.

Now we identify a bipartite tournament with bipartition (V_1, V_2), where the vertices in $V_1 \cup V_2$ are indistinguishable, with a complete bipartite graph with bipartition (V_1, V_2) whose edges colored with red and blue by regarding an arc (x, y) from V_1 to V_2 as the edge xy colored with red and an arc (x, y) from V_2 to V_1 as the edge xy colored with blue. As there is a one-to-one correspondence between $\mathcal{D}_1$ and $\mathcal{D}_2$, the number of a bipartite tournament with bipartition (V_1, V_2) where $|V_1| = |V_2| = n$ is odd is the same as the number of ways to color the edges of $K_{n,n}$ with red and blue so that the red edges are more than the blue edges. Therefore the group of symmetries acting on the vertex set of $K_{n,n}$ is isomorphic to $S_n \times S_n$. Thus, by Lemma 2.2 and Theorem 1.4, the cycle index of the group of symmetries that acts on the edge set of a complete bipartite graph $K_{n,n}$ is

$$\sum \frac{\prod_{(k,l)\in[n]\times[n]} z_{L(k,l)}^{G(k,l)\alpha_k\beta_l}}{\prod_{p=1}^{n}(p^{\alpha_p}\alpha_p!)\prod_{q=1}^{n}(q^{\beta_q}\beta_q!)}.$$

where the sum is over all nonnegative integer α_i, β_j such that

$$\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n \quad \text{and} \quad \beta_1 + 2\beta_2 + \cdots + n\beta_n = n.$$

By Theorem 1.4 again, the number of ways to color the edges of $K_{n,n}$ with red and blue so that the red edges are more than the blue edges is the sum of coefficients of $r^i b^{n^2-i}$ for each integer i, $n^2 \geq i > n^2 - i$, in

$$g(r,b) := \sum \frac{\prod_{(k,l)\in[n]\times[n]} \left(r^{L(k,l)} + b^{L(k,l)}\right)^{G(k,l)\alpha_k\beta_l}}{\prod_{p=1}^{n}(p^{\alpha_p}\alpha_p!)\prod_{q=1}^{n}(q^{\beta_q}\beta_q!)}.$$

where the sum is over all nonnegative integer α_i, β_j such that

$$\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n \quad \text{and} \quad \beta_1 + 2\beta_2 + \cdots + n\beta_n = n.$$

Since $g(r,b)$ is symmetric function of two variables, the desire number is

$$\frac{1}{2}g(1,1) = \frac{1}{2}\sum \frac{2^{\sum_{(k,l)\in[n]\times[n]} G(k,l)\alpha_k\beta_l}}{\prod_{p=1}^{n}(p^{\alpha_p}\alpha_p!)\prod_{q=1}^{n}(q^{\beta_q}\beta_q!)}$$

where the sum is over all nonnegative integer α_i, β_j such that

$$\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n \quad \text{and} \quad \beta_1 + 2\beta_2 + \cdots + n\beta_n = n.$$

□

Chapter 3

Properties of m-step competition graphs of bipartite tournaments

Lemma 3.1. *For a bipartite tournament D, any non-isolated vertex in $C^m(D)$ has out-degree at least one for a positive integer m.*

Proof. It is obvious because it must have an m-step prey. □

Lemma 3.2. *For a bipartite tournament D with bipartition (V_1, V_2), there is no edge joining a vertex in V_1 and a vertex in V_2 in $C^m(D)$ for any positive integer m.*

Proof. For a vertex in V_1, a vertex in V_1 can be only $2k$-step prey for a positive integer k and a vertex in V_2 can be only $(2k' - 1)$-step prey for a positive integer k' while for a vertex in V_2, a vertex in V_1 can be only $(2l - 1)$-step prey for a positive integer l and a vertex in V_2 can be only $2l'$-step prey for a positive integer l'. Therefore a vertex in V_1 and a vertex in V_2 cannot have an m-step common prey for any positive integer m. □

Lemma 3.3. *Let D be a bipartite tournament with bipartition (V_1, V_2). If $C^M(D)$ has an edge for a positive integer M, then so does $C^m(D)$ for any positive integer $m \le M$.*

Proof. Let xy be an edge in $C^M(D)$. Then x and y belong to the same part by Lemma 3.2. Without loss of generality, we may assume that x and y belong to V_1. In addition, x and y have an M-step common prey z in D. Then there exist a directed (x, z)-path P and a directed (y, z)-path Q of length M in D. Let x_1 and y_1 be the vertices immediately following x on P and immediately following y on Q, respectively. If x_1 and y_1 are distinct, then z is an $(M-1)$-step common prey of x_1 and y_1 and so x_1 and y_1 are adjacent in $C^{(M-1)}(D)$. If x_1 and y_1 are the same, then the vertex immediately preceding z is an $(M-1)$-step common prey of x and y and so x and y are adjacent in $C^{(M-1)}(D)$. Therefore there is an edge in $C^{(M-1)}(D)$. We may repeat this argument to show that there is an edge in $C^{(M-2)}(D)$. In this way, we may show that there is an edge in $C^m(D)$ for any positive integer $m \le M$. □

The following corollary is the contrapositive of Lemma 3.3.

Corollary 3.4. *Let D be a bipartite tournament with bipartition (V_1, V_2). If $C^M(D)$ is an edgeless graph for a positive integer M, then so does $C^m(D)$ for any positive integer $m \ge M$.*

Proposition 3.5. *Let D be a bipartite tournament in which every vertex has out-degree at least one. If two vertices are adjacent in $C^M(D)$ for a positive integer M, then they are also adjacent in $C^m(D)$ for any positive integer $m \ge M$.*

Proof. Let x and y are adjacent in $C^M(D)$. Then x and y have an M-step common prey z in D. By the hypothesis, z has an out-neighbor w in D. Then w is an $(M+1)$-step common prey of x and y. Hence x and y are adjacent in $C^{(M+1)}(D)$. We may repeat this argument to show that x and y are adjacent in $C^{(M+2)}(D)$. In this way, we may show that x and y are adjacent in $C^m(D)$ for any positive integer $m \ge M$. □

Corollary 3.6. *Let D be a bipartite tournament in which every vertex has out-degree at least one. Then any vertices having an i-step common prey in D for some positive integer i form a clique in $C^m(D)$ for any positive integer $m \geq i$.*

Proof. Since each vertex has out-degree at least one, any vertices having an i-step common prey in D for some positive integer i have an m-step common prey for any $m \geq i$. □

Corollary 3.7. *Let D be a bipartite tournament in which every vertex has out-degree at least one. If $C^M(D)$ is triangle-free for some positive integer M, then every vertex has in-degree at most two in D.*

Proof. Since the in-neighbors of a vertex in D form a clique in $C^m(D)$ for every positive integer m by Corollary 3.6, the corollary immediately follows. □

Lemma 3.8. *Let D be a bipartite tournament with bipartition (V_1, V_2) which has a directed cycle. Then there is a directed 4-cycle in D.*

Proof. Since D is a bipartite tournament, there is no odd cycle in D. Let $2k$ be a length of a cycle in D for a positive integer $k \geq 2$, and we prove by induction on k. If $k = 2$, it is done. Now suppose that if there is a directed $2(k-1)$-cycle in D, then there is a directed 4-cycle in D. Let $v_1v_2 \dots v_{2k}$ be a directed cycle in D. Then $v_1, v_3, \dots, v_{2k-1}$ and $v_2, v_4, \dots, v_{2k}$ are in different partite sets. Without loss of generality, we may assume $v_1, v_3, \dots, v_{2k-1}$ belong to V_1 and $v_2, v_4, \dots, v_{2k}$ belong to V_2. Since D is a bipartite tournament, there exists an arc between v_1 and v_4. If (v_4, v_1) is an arc in D, then $v_1v_2v_3v_4v_1$ is a directed 4-cycle in D. If (v_1, v_4) is an arc in D, then $v_1v_4v_5 \dots v_{2k}v_1$ is a directed $2(k-1)$-cycle in D and, by the induction hypothesis, there is a directed 4-cycle in D. Hence we have shown that there is a directed 4-cycle in D. □

Proposition 3.9. *Let D be a bipartite tournament with bipartition (V_1, V_2) which has a directed cycle for satisfying $|V_1| \geq |V_2| \geq 2$. There is no edge in $C^m(D)$ for some positive integer m if and only if $|V_1| = |V_2| = 2$. Moreover, if $|V_1| = |V_2| = 2$, then there is no edge in $C^m(D)$ for any positive integer m.*

Proof. Suppose that D_1 is a bipartite tournament with bipartition (V_1, V_2) which has a directed cycle for $|V_1| \geq 3$ and $|V_2| \geq 2$. By Lemma 3.8, there is a directed 4-cycle in D_1. Let $u_1v_1u_2v_2u_1$ be a directed 4-cycle in D_1. Without loss of generality, we may assume $u_1, u_2 \in V_1$ and $v_1, v_2 \in V_2$. Since $|V_1| \geq 3$, there is a vertex u_3 distinct from u_1 and u_2. Since D_1 is a bipartite tournament, there exists an arc between u_3 and v_i for each $i = 1, 2$. If (u_3, v_1) is an arc in D_1, then u_1u_3 is an edge in $C(D_1)$. If (u_3, v_2) is an arc in D_1, then u_2u_3 is an edge in $C(D_1)$. We note that each vertex has out-degree at least one in the subdigraph D' of D_1 induced by $\{u_1, u_2, u_3, v_1, v_2\}$. Then, by Proposition 3.5, $C^m(D')$ has an edge and so $C^m(D)$ has an edge for any positive integer m if (u_3, v_1) or (u_3, v_2) is an arc in D_1. If (v_1, u_3) and (v_2, u_3) are arcs in D_1, then, utilizing the directed 4-cycle $u_1v_1u_2v_2u_1$, u_3 becomes an m-step common prey for u_1 and u_2 if m is even and for v_1 and v_2 if m is odd, and so there is an edge in $C^m(D_1)$ for any positive integer m.

If D_2 is an orientation of $K_{2,2}$ which has a directed cycle, then D_2 itself is a directed 4-cycle and therefore $C^m(D_2)$ is an edgeless graph for any positive integer m. □

Competition index and competition period are defined in terms of m-step competition graphs of a digraph. In this vein, it is interesting to find the competition index and the competition period of a bipartite tournament.

Theorem 3.10. *Let D be a bipartite tournament with bipartition (V_1, V_2) which has no directed cycle and let l be the length of a longest directed path in D. Then* $\text{cperiod}(D) = 1$ *and*

$$l - 1 \leq \text{cindex}(D) \leq l + 1.$$

Proof. Since there is no directed cycle, every directed walk is a directed path. Thus, by the hypothesis that the length of the longest directed path in D is l, any vertex x in D cannot have an m-step prey in D for any integer $m \geq l+1$. Therefore $C^m(D)$ is an edgeless graph for $m \geq l+1$ and we have $\text{cindex}(D) \leq l+1$ and $\text{cperiod}(D) = 1$.

If $l \leq 2$, then $l-1 \leq 1 \leq \text{cindex}(D)$. Now suppose $l \geq 3$. Let $v_1 v_2 \cdots v_{l+1}$ be a longest path in D. Since the underlying graph of D is a bipartite graph, $v_1, v_3, \ldots, v_{2\lfloor \frac{l}{2} \rfloor+1}$ and $v_2, v_4, \ldots, v_{2\lfloor \frac{l+1}{2} \rfloor}$ are in different partite sets. Without loss of generality, we may assume $v_1, v_3, \ldots, v_{2\lfloor \frac{l}{2} \rfloor+1}$ belong to V_1 and $v_2, v_4, \ldots, v_{2\lfloor \frac{l+1}{2} \rfloor}$ belong to V_2. Since D is a bipartite tournament, there exists an arc between v_1 and v_4. Since D has no directed cycle, (v_1, v_4) is an arc of D. Then v_{l+1} is an $(l-2)$-step common prey of v_1 and v_3 and $C^{l-2}(D)$ is not an edgeless graph. Therefore $l-1 \leq \text{cindex}(D)$. As we have shown $l-1 \leq \text{cindex}(D)$ in both cases, we have completed the proof. □

Chapter 4

m-step competition realizable pairs

Definition 4.1. Let G_1 and G_2 be graphs with n_1 vertices and n_2 vertices, respectively. The pair (G_1, G_2) is said to be m-step competition realizable through K_{n_1,n_2} for a positive integer m (in this paper, we only consider orientations of K_{n_1,n_2} and therefore we omit "through K_{n_1,n_2}") if the disjoint union of G_1 and G_2 is the m-step competition graph of a bipartite tournament with bipartition $(V(G_1), V(G_2))$.

Kim *et al.* [22] characterized 1-step competition realizable pair (K_{n_1}, K_{n_2}) as competition realizable is 1-step competition realizable. Now we characterize the pairs (K_{n_1}, K_{n_2}) which are m-step competition realizable for $m \geq 2$.

Lemma 4.2. *Let n_1 and n_2 be positive integers satisfying $n_1 \geq n_2$. If the pair (K_{n_1}, K_{n_2}) is M-step competition realizable for a positive integer M, then the pair (K_{n_1}, K_{n_2}) is also m-step competition realizable for any positive integer $m \geq M$.*

Proof. Suppose that the pair (K_{n_1}, K_{n_2}) is M-step competition realizable for a positive integer M. Then there exists a bipartite tournament D such that $K_{n_1} \cup K_{n_2}$ is an M-step competition graph of D. Since every vertex in

$C^M(D)$, which is $K_{n_1} \cup K_{n_2}$, is non-isolated, every vertex has out-degree at least one in D. Furthermore every pair of vertices in the same part is adjacent in $C^M(D)$. By Proposition 3.5, for any positive integer $m \geq M$, every pair of vertices in the same part is adjacent in $C^m(D)$, which implies that $K_{n_1} \cup K_{n_2}$ is the m-step competition graph of D. Therefore the pair (K_{n_1}, K_{n_2}) is m-step competition realizable for any positive integer $m \geq M$. □

Theorem 4.3. *Let n_1 and n_2 be positive integers satisfying $n_1 \geq n_2$. If the pair (K_{n_1}, K_{n_2}) is m-step competition realizable for some integer $m \geq 2$ then $n_1 = n_2 = 1$ or $n_1 \geq n_2 \geq 3$. Furthermore, if $n_1 = n_2 = 1$ or $n_1 \geq n_2 \geq 3$, then the pair (K_{n_1}, K_{n_2}) is m-step competition realizable for any integer $m \geq 2$.*

Proof. To prove the first part of the statement by contradiction, suppose that (K_{n_1}, K_1) and (K_{n_1}, K_2) are m-step competition realizable for some $m \geq 2$ for some $n_1 \geq 2$. We first consider the case where (K_{n_1}, K_1) is m_1-step competition realizable for some $m_1 \geq 2$. Then there exists an orientation D of $K_{n_1,1}$ such that $K_{n_1} \cup K_1$ is an m_1-step competition graph of D. Since $n_1 \geq 2$, by Lemma 3.1, every vertex in K_{n_1} has out-degree at least one in D and so the vertex in K_1 has no outgoing arc. Thus no vertex has an m_1-step prey in D, and therefore $C^{m_1}(D)$ is an edgeless graph, which is a contradiction. Now consider the case where (K_{n_1}, K_2) is m_2-step competition realizable for some $m_2 \geq 2$. Then there exists an orientation D' of $K_{n_1,2}$ such that $K_{n_1} \cup K_2$ is an m_2-step competition graph of D'. Let $\{u, v\}$ be the vertex set of K_2. Since there is no isolated vertex in $C^{m_2}(D')$, by Lemma 3.1, every vertex has out-degree at least one in D'. Thus $N^+(u) \cap N^+(v) = \emptyset$ and so the vertex set of K_{n_1} is a disjoint union of the following sets:

$$A := N^-(u) \cap N^-(v); B := N^-(u) \cap N^+(v); C := N^+(u) \cap N^-(v).$$

The only possible m_2-step prey of u or v are vertices in B, vertices in C, u, or v. Yet, a vertex in B (resp. C) can be only $4k_1+3$ (resp. $4k_1+1$)-step prey of

u while it is only $4k_2+1$ (resp. $4k_2+3$)-step prey of v, and u (resp. v) can be only $4k_3$ (resp. $4k_3+2$)-step prey of u while it is only $4k_4+2$ (resp. $4k_4$)-step prey of v for nonnegative integers k_1, k_2, k_3 and k_4. Hence there are no m-step common prey of u and v for any integer $m \geq 2$, which is a contradiction. Therefore if the pair (K_{n_1}, K_{n_2}) is m-step competition realizable for some integer $m \geq 2$, then $n_1 = n_2 = 1$ or $n_1 \geq n_2 \geq 3$.

Now we show the second part of the statement (the "furthermore" part). Obviously (K_1, K_1) is m-step competition realizable for any $m \geq 2$. Suppose that $n_1 \geq n_2 \geq 3$. By Lemma 4.2, it is sufficient to consider the case $m = 2$. Let $V_1 = \{u_1, u_2, \ldots, u_{n_1}\}$, $V_2 = \{v_1, v_2, \ldots, v_{n_2}\}$, $V_1' = V_1 \setminus \{u_1, u_2\}$ and $V_2' = V_2 \setminus \{v_1, v_2\}$. Since $n_1 \geq n_2 \geq 3$, any of V_1' and V_2' is not empty. Let D'' be a bipartite tournament with bipartition (V_1, V_2) whose vertex set is

$$V(D'') = V_1 \cup V_2$$

and whose arc set is

$$\begin{aligned} A(D'') &= \{(u_1, v_1), (u_1, v_2), (u_2, v_1), (v_2, u_2)\} \\ &\cup \{(x, y) \mid x \in V_1' \text{ and } y \in V_2'\} \cup \{(v_1, x) \mid x \in V_1'\} \\ &\cup \{(x, v_2) \mid x \in V_1'\} \cup \{(y, u_1) | y \in V_2'\} \cup \{(u_2, y) \mid y \in V_2'\} \end{aligned}$$

(see Figure 4.1). Then Table 4.1 matches a pair of vertices and its 2-step common prey in D''. Therefore the pair (K_{n_1}, K_{n_2}) is 2-step competition realizable and so, by Lemma 4.2, the pair (K_{n_1}, K_{n_2}) is m-step competition realizable for $m \geq 2$. □

Theorem 4.4. *Let n_1 and n_2 be positive integers satisfying $n_1 \geq n_2 \geq 3$. If the pair (C_{n_1}, C_{n_2}) is m-step competition realizable for some integer $m \geq 2$ then $n_1 = n_2 = 3$. Furthermore, if $n_1 = n_2 = 3$, then the pair (C_{n_1}, C_{n_2}) is m-step competition realizable for any integer $m \geq 2$.*

a pair	a 2-step common prey
u_1 and u_2	a vertex in V_1'
u_1 and a vertex in V_1'	u_2
u_2 and a vertex in V_1'	u_1
two vertices in V_1'	u_2
v_1 and v_2	a vertex in V_2'
v_1 and a vertex in V_2'	v_2
v_2 and a vertex in V_2'	v_1
two vertices in V_2'	v_1

Table 4.1: A pair of vertices and its 2-step common prey in D''

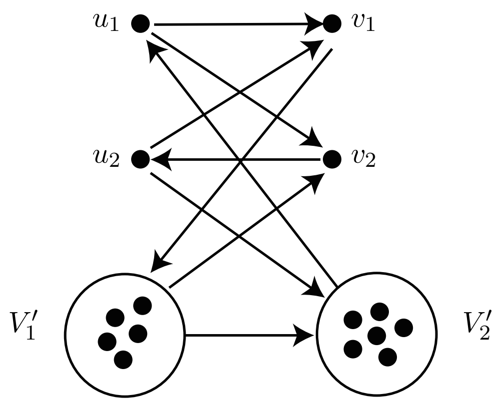

Figure 4.1: D''

Proof. To prove the first part of the statement by contradiction, suppose that the pair (C_{n_1}, C_{n_2}) is m-step competition realizable for some integer $m \geq 2$ for n_1 and n_2 not satisfying $n_1 = n_2 = 3$. Then there exists an orientation D of K_{n_1,n_2} such that $C_{n_1} \cup C_{n_2}$ is an m-step competition graph of D for some integer $m \geq 2$ for $n_1 > n_2 = 3$ or $n_1 \geq n_2 \geq 4$.

We first consider the case $n_1 > n_2 = 3$. Suppose m is odd. Then the end vertices of each edge in C_{n_1} have an m-step common prey in $V(C_{n_2})$. Since each edge in C_{n_1} is a maximal clique, the pairs of vertices u and v, and x and y have distinct m-step common prey if uv and xy are distinct edges in C_{n_1}. Therefore there must be at least n_1 distinct vertices in $V(C_{n_2})$, which is impossible. Suppose m is even. Then each vertex in C_{n_1} has an $(m-1)$-step prey in $V(C_{n_2})$. Since each edge in a cycle is a maximal clique and, by Lemma 3.1, each vertex in $V(C_{n_2})$ has out-degree at least one, we need at least n_1 distinct $(m-1)$-step prey in $V(C_{n_2})$, which is impossible.

Now we consider the case $n_1 \geq n_2 \geq 4$. By Lemma 3.1, the out-degree of each vertex in D is at least one. Since $C_{n_1} \cup C_{n_2}$ is triangle-free, the in-degree of each vertex in D is at most two by Corollary 3.7. Therefore

$$\sum_{v \in V(C_1)} d^+(v) = \sum_{v \in V(C_2)} d^-(v) \leq 2n_2 \quad \text{and} \quad \sum_{v \in V(C_2)} d^+(v) = \sum_{v \in V(C_1)} d^-(v) \leq 2n_1.$$

Thus

$$n_1 n_2 = \sum_{v \in V(C_1)} (d^+(v) + d^-(v)) \leq 2(n_1 + n_2),$$

which implies

$$(n_1 - 2)(n_2 - 2) \leq 4.$$

Since $(n_1 - 2)(n_2 - 2) \geq 4$ by the case assumption, $n_1 = n_2 = 4$ and the out-degree and in-degree of each vertex in D are exactly two. It is tedious to check that D is isomorphic to one of the two digraphs D_1 and D_2 given in Figure 4.2. However, the m-step competition graphs of D_1 and D_2 are $K_2 \cup K_2 \cup K_2 \cup K_2$ and $K_4 \cup K_4$ for any integer $m \geq 2$, respectively, and we

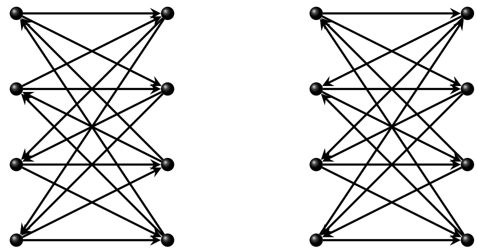

Figure 4.2: Bipartite tournaments D_1 and D_2

reach a contradiction.

Since a 3-cycle is a complete graph, the second part of the statement follows from Theorem 4.3. □

Theorem 4.5. *Let n_1 and n_2 be positive integers satisfying $n_1 \geq n_2$. If the pair (P_{n_1}, P_{n_2}) is m-step competition realizable for some integer $m \geq 2$ then $n_1 = n_2 = 1$ or 3. Furthermore, if $n_1 = n_2 = 1$ or 3, then the pair (P_{n_1}, P_{n_2}) is m-step competition realizable for any integer $m \geq 2$.*

Proof. To prove the first part of the statement by contradiction, suppose that the pair (P_{n_1}, P_{n_2}) is M-step competition realizable for some integer $M \geq 2$ for n_1 and n_2 not satisfying $n_1 = n_2 = 1$ or 3. Since the cases $n_1 > n_2 = 1$ and $n_1 \geq n_2 = 2$ are impossible by the same argument in the proof of Theorem 4.3, we consider the case $n_1 \geq n_2 \geq 3$ except $n_1 = n_2 = 3$. Let D be a bipartite tournament such that the M-step competition graph of D is $P_{n_1} \cup P_{n_2}$. Since each vertex has out-degree at least one in D and $P_{n_1} \cup P_{n_2}$ is triangle-free, the in-degree of each vertex in D is at most two by Corollary 3.7. By the same argument in the proof of Theorem 4.4, we have $(n_1 - 2)(n_2 - 2) \leq 4$. By the case assumption, $(n_1, n_2) = (4, 3), (5, 3), (6, 3)$ or $(4, 4)$. The case $(n_1, n_2) = (4, 4)$ is excluded by the same argument in the proof of Theorem 4.4. Suppose $(n_1, n_2) = (5, 3)$. Since each vertex in P_3 has in-degree at most two and the sum of in-degree and out-degree of each vertex

	v_1	v_2, v_3	u_1, u_2	u_3, u_4
v_1	$4k_1$	$4k_2+2$	$4k_3+1$	$4k_4+3$
v_2	$4k_5+2$	$4k_6$	$4k_7+3$	$4k_8+1$

Table 4.2: The number l in the (i,j)-entry means that the vertex corresponding to the jth column can be only the l-step prey of the vertex corresponding to the ith row where k_i is a positive integer for $i = 1, \ldots, 8$.

in P_3 equals 5, the out-degree of each vertex in P_3 is at least 3. Then, the sum of out-degrees of any two vertices in P_3 is at least 6. Since P_{n_1} has only 5 vertices, any two vertices in P_3 have a common out-neighbor in D by the pigeonhole principle. Therefore $C(D)$ contains a triangle whose vertices are the ones in P_3. By Proposition 3.5, for any integer $m \geq 2$, $C^m(D)$ contains a triangle, a contradiction. We can show that the case $(n_1, n_2) = (6, 3)$ also deduces a contradiction by the same argument. Finally we suppose $(n_1, n_2) = (4, 3)$. Let $V(P_4) = \{u_1, u_2, u_3, u_4\}$ and $P_3 = v_1v_2v_3$. Since the underlying graph of D is a complete bipartite graph, the sum of in-degree and out-degree of each vertex in P_3 equals 4. Since the in-degree of each vertex in D is at most two, the out-degree of each vertex in P_3 is at least 2. Since v_1 and v_3 are not adjacent in P_3, v_1 and v_3 do not have a common out-neighbor. If the out-degree of v_1 or the out-degree of v_3 is at least 3, then they have a common out-neighbor by the pigeonhole principle and so, by Proposition 3.5, v_1 and v_3 are adjacent in $C^m(D)$ for any integer $m \geq 2$, which is impossible. Therefore the out-degrees of v_1 and v_3 are 2. Since v_1 and v_3 do not have a common out-neighbor, we may assume $N^+(v_1) = \{u_1, u_2\}$ and $N^+(v_3) = \{u_3, u_4\}$ without loss of generality. Then $(u_1, v_3), (u_2, v_3), (u_3, v_1)$ and (u_4, v_1) are arcs of D. Now we suppose that v_1 and v_2 do not have a common out-neighbor. Then $N^+(v_2) = N^+(v_3) = \{u_3, u_4\}$ since the out-degree of v_2 is at least 2, so each arc belongs to $[\{v_2, v_3\}, \{u_3, u_4\}] \cup [\{u_3, u_4\}, \{v_1\}] \cup [\{v_1\}, \{u_1, u_2\}] \cup [\{u_1, u_2\}, \{v_2, v_3\}]$ where $[S, T]$ denotes the set of arcs going from the vertex set S to the vertex set T. Therefore we have Table 4.2. Thus there are no

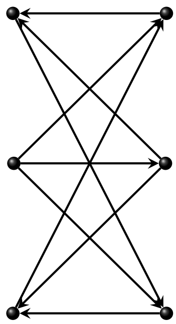

Figure 4.3: A bipartite tournament D' such that $C^m(D') = P_3 \cup P_3$ for any integer $m \geq 2$

m-step common prey of v_1 and v_2 for any integer $m \geq 2$ and we reach a contradiction. Hence v_1 and v_2 have a common out-neighbor. By applying the same argument, we may conclude that v_2 and v_3 have a common out-neighbor. Without loss of generality, we may assume that (v_2, u_2) and (v_2, u_3) are arcs in D. Now it remains to determine the arc joining u_1 and v_2 and the arc joining u_4 and v_2. Up to isomorphism, there are three cases: (i) (u_1, v_2) and (u_4, v_2) are arcs of D; (ii) (u_1, v_2) and (v_2, u_4) are arcs of D; (iii) (v_2, u_1) and (v_2, u_4) are arcs of D. In case (i), v_1 and v_3 have v_2 as a 2-step common prey and so are adjacent in $C^m(D)$ for any integer $m \geq 2$, which is impossible. In case (ii), the subgraph of $C^m(D)$ induced by $\{u_1, u_2, u_3, u_4\}$ is $P_2 \cup P_2$ for any integer $m \geq 2$, which is impossible. In case (iii), u_1, u_2, and u_4 have u_3 as a 2-step common prey and $C^m(D)$ contains a triangle with the vertices u_1, u_2, and u_4 for any integer $m \geq 2$ which is impossible. Thus (n_1, n_2) cannot be $(4, 3)$. Hence if the pair (P_{n_1}, P_{n_2}) is m-step competition realizable for some integer $m \geq 2$ then $n_1 = n_2 = 1$ or 3.

The pair (P_1, P_1) is obviously m-step competition realizable for any integer $m \geq 2$. The m-step competition graphs of the bipartite tournament given in Figure 4.3 are $P_3 \cup P_3$ for any integer $m \geq 2$. □

Chapter 5

Extremal cases

In the following, we compute the maximum number of edges and the minimum number of edges which the m-step competition graph of an orientation of K_{n_1,n_2} might have.

Theorem 5.1. *Let m, n_1, and n_2 be positive integers satisfying $m \geq 2$ and $n_1 \geq n_2$, and $\mathcal{G}^{(m)}_{n_1,n_2}$ be the set of m-step competition graphs of orientations of K_{n_1,n_2}. Then*

$$\min\left\{|E(G)| \mid G \in \mathcal{G}^{(m)}_{n_1,n_2}\right\} = 0$$

and

$$\max\left\{|E(G)| \mid G \in \mathcal{G}^{(m)}_{n_1,n_2}\right\} = \begin{cases} \binom{n_1-1}{2} & n_2 = 1 \text{ and } m = 2; \\ 0 & n_2 = 1 \text{ and } m \geq 3; \\ \binom{n_1}{2} - 1 & n_2 = 2; \\ \binom{n_1}{2} + \binom{n_2}{2} & n_2 \geq 3. \end{cases}$$

Proof. Let D_1 be a bipartite tournament with bipartition (V_1, V_2) with $|V_1| = n_1$, $|V_2| = n_2$, and every arc going from a vertex in V_1 to a vertex in V_2. Then, for any vertex v in D_1, there is no m-step prey of v. Therefore the $C^m(D_1)$ is an edgeless graph and so the $\min\left\{|E(G)| \mid G \in \mathcal{G}^{(m)}_{n_1,n_2}\right\} = 0$. Now, to compute $M := \max\left\{|E(G)| \mid G \in \mathcal{G}^{(m)}_{n_1,n_2}\right\}$, we first consider the case $n_2 = 1$. Then there is no directed path of length greater than or equal to three in any orientation D of $K_{n_1,1}$. Therefore, if $m \geq 3$, $M = 0$. Let $D(i)$ be a bipartite tournament with bipartition (V_1, V_2) satisfying $|V_1| = n_1$, $|V_2| = 1$, and $d^+(v) = i$ for some nonnegative integer $i \leq n_1$. If $i = 0$, there is no 2-step prey of any vertex in V_1 in $D(0)$ and therefore $C^2(D(0))$ is an edgeless graph. Now $C^2(D(i))$ is isomorphic to $K_{n_1-i} \cup I_{i+1}$ where I_{i+1} is the set of $i+1$ isolated vertices for a positive integer $i \leq n_1$. The graph $K_{n_1-i} \cup I_{i+1}$ has maximum number of edges when $i = 1$ and so $M = \binom{n_1-1}{2}$ if $n_2 = 1$ and $m = 2$.

Now consider the case $n_2 = 2$. Let D_2 be a bipartite tournament with bipartition (V_1, V_2) satisfying $|V_1| = n_1$ and $V_2 = \{u, v\}$. Since $n_1 \geq 2$, we can choose two vertices x and y in V_1. Then, let $(x, u), (v, x), (u, y)$, and (y, v) be arcs of D_2 to form a directed 4-cycle and let the remaining arcs of D_2 go from $V_1 \setminus \{x, y\}$ to V_2. Then $C^m(D_2)$ is isomorphic to $(K_{n_1} - xy) \cup I_2$. Thus $M \geq \binom{n_1}{2} - 1$. Now we show that $M \leq \binom{n_1}{2} - 1$. By Theorem 4.3, (K_{n_1}, K_2) is not m-step competition realizable for any integer $m \geq 2$. Therefore $M \leq \binom{n_1}{2}$. Suppose there exists a bipartite tournament D_3 with bipartition (V_1, V_2) satisfying $|V_1| = n_1$ and $V_2 = \{u, v\}$ whose m-step competition graph, say G, has $\binom{n_1}{2}$ edges. Then G is isomorphic to $K_{n_1} \cup I_2$ or $(K_{n_1} - e) \cup K_2$ where $K_{n_1} - e$ means a graph resulting from deleting an edge e from K_{n_1}. Therefore there is no isolated vertex in $G[V_1]$ and so, by Lemma 3.1, every vertex in V_1 has out-degree at least one in D_3. Thus $N^+(u) \cap N^+(v) = \emptyset$ and so V_1 is a disjoint union of the following sets:

$$A := N^-(u) \cap N^-(v); B := N^-(u) \cap N^+(v); C := N^+(u) \cap N^-(v). \quad (5.1)$$

Furthermore, the only possible m-step prey of u or v are vertices in B, vertices in C, u, or v. Yet, a vertex in B (resp. C) can be only $4k_1 + 3$ (resp. $4k_1 + 1$)-step prey of u while it is only $4k_2 + 1$ (resp. $4k_2 + 3$)-step prey of v, and u (resp. v) can be only $4k_3$ (resp. $4k_3 + 2$)-step prey of u while it is only $4k_4 + 2$

(resp. $4k_4$)-step prey of v for nonnegative integers k_1, k_2, k_3 and k_4. Therefore there are no m-step common prey of u and v for any integer $m \geq 2$, so u and v are not adjacent in G. Thus G is isomorphic to $K_{n_1} \cup I_2$. If $B \neq \emptyset$ and $C \neq \emptyset$, then, by applying a similar argument for showing that u and v are not adjacent, one can show that a vertex in B and a vertex in C are not adjacent in G to reach a contradiction. Therefore B or C is an empty set. By symmetry, we may assume that C is an empty set. Then a vertex in B has only 1-step prey. If $B \neq \emptyset$ and $|B| \geq 2$, then $G[B]$ is an edgeless graph. Therefore, if $B \neq \emptyset$, $|B| = 1$. If $A = \emptyset$, $V_1 = B$ by (5.1), which contradicts the hypothesis that $n_1 \geq 2$. Thus $A \neq \emptyset$. If $B \neq \emptyset$, then the vertex in B is not adjacent to any vertex in A in G, a contradiction. Thus $B = \emptyset$. Then $V_1 = A$ and $|A| \geq 2$. Moreover A has only 1-step prey in D_3 and $G[A]$ is an edgeless graph, a contradiction. Hence $M \leq \binom{n_1}{2} - 1$ and we have shown that $M = \binom{n_1}{2} - 1$.

If $n_1 \geq n_2 \geq 3$, by Theorem 4.3, (K_{n_1}, K_{n_2}) is m-step competition realizable. Therefore $M = \binom{n_1}{2} + \binom{n_2}{2}$. □

Lemma 5.2. *Let D be an orientation of K_{n_1,n_2} with bipartition (V_1, V_2) satisfying $|V_1| = n_1 \geq 2$ and $|V_2| = n_2 \geq 2$, in which every vertex has out-degree at least one. For a positive integer m, let $G_{m,i}$ be the subgraph of $C^m(D)$ induced by V_i for each $i = 1, 2$. If $G_{m,1}$ is not a complete graph for a positive integer m, then the vertex set of $G_{m,2}$ can be partitioned into two cliques.*

Proof. Suppose $G_{m^*,1}$ is not a complete graph for a positive integer m^*. For notational convenience, we denote $G_{m^*,i}$ by G_i for each $i = 1, 2$. Then there exist two nonadjacent vertices u_1 and u_2 in G_1. Let $W_1 = N^+(u_1)$ and $W_2 = N^-(u_1)$. Then V_2 is a disjoint union of W_1 and W_2. Since u_1 has out-degree at least one in D, W_1 is not empty. Take a vertex v in W_1. Then v is an out-neighbor of u_1. If v is an out-neighbor of u_2, then u_1 and u_2 are adjacent in $C^m(D)$ by Proposition 3.5, which is a contradiction. Therefore (v, u_2) is an arc in D. Since v is arbitrarily chosen, u_2 is a common out-neighbor of all the vertices in W_1 and thus W_1 is a clique in G_2. On the other hand, by the hypothesis that u_2 has out-degree at least one in D, there is an out-neighbor of u_2 which must belong to W_2. Therefore W_2 is not empty and so $\{W_1, W_2\}$ is a partition of V_2. Furthermore u_1 is a common out-neighbor of all the vertices in W_2 and thus W_2 is a clique in G_2. $\square$

Theorem 5.3. *Let m, n_1, and n_2 be positive integers satisfying $m \geq 2$ and $n_1 \geq n_2 \geq 2$, and $\mathcal{G}^{*(m)}_{n_1,n_2}$ be the set of m-step competition graphs of orientations of K_{n_1,n_2} in which every vertex has out-degree at least one such that each graph in $\mathcal{G}^{*(m)}_{n_1,n_2}$ is a disjoint union of two non-complete graphs. Then*

$$\min\left\{|E(G)| \mid G \in \mathcal{G}^{*(m)}_{n_1,n_2}\right\} = \left\lfloor\left(\frac{n_1-1}{2}\right)\right\rfloor^2 + \left\lfloor\left(\frac{n_2-1}{2}\right)\right\rfloor^2.$$

Proof. Take $G \in \mathcal{G}^{*(m)}_{n_1,n_2}$. Then G is a disjoint union of non-complete graphs G_1 and G_2 satisfying $|V(G_1)| = n_1$ and $|V(G_2)| = n_2$. Since neither G_1 nor G_2 is a complete graph, by Lemma 5.2, $V(G_1)$ (resp. $V(G_2)$) can be partitioned into two subsets X_1 and Y_1 (resp. X_2 and Y_2) which form cliques in G_1 (resp. G_2). Suppose $|X_1| = k$ for a positive integer $1 \leq k \leq n_1 - 1$. Then $|Y_1| = n_1 - k$. Since X_1 and Y_1 form cliques in G_1,

$$\begin{aligned}|E(G_1)| \geq \binom{k}{2} + \binom{n_1-k}{2} &= \frac{k(k-1)}{2} + \frac{(n_1-k)(n-k-1)}{2} \\ &= k^2 - n_1 k + \frac{n_1(n_1-1)}{2} \\ &= \left(k - \frac{n_1}{2}\right)^2 + \frac{{n_1}^2 - 2n_1}{4}.\end{aligned}$$

If n_1 is even, the minimum of $|E(G_1)|$ is $({n_1}^2 - 2n_1)/4$ which is achieved when $k = n_1/2$, and if n_1 is odd, the minimum of $|E(G_1)|$ is $({n_1}^2 - 2n_1 + 1)/4$ which is achieved when $k = (n_1+1)/2$. In other words, the minimum of $|E(G_1)|$ is $\lfloor (n_1-1)/2 \rfloor^2$ which is achieved when $k = \lfloor n_1/2 \rfloor$. Hence G_1 has at least $\lfloor (n_1-1)/2 \rfloor^2$ edges and, by the same argument, we may show that G_2 has at least $\lfloor (n_2-1)/2 \rfloor^2$ edges. Therefore

$$\min\left\{|E(G)| \mid G \in \mathcal{G}^{*(m)}_{n_1,n_2}\right\} \leq \left\lfloor\left(\frac{n_1-1}{2}\right)\right\rfloor^2 + \left\lfloor\left(\frac{n_2-1}{2}\right)\right\rfloor^2. \tag{5.2}$$

Now, let X_1', Y_1', X_2' and Y_2' be disjoint sets such that $|X_1'| = \lfloor n_1/2 \rfloor$, $|Y_1'| = \lceil n_1/2 \rceil$, $|X_2'| = \lfloor n_2/2 \rfloor$, and $|Y_2'| = \lceil n_2/2 \rceil$. Let D be the digraph with $V(D) = X_1' \cup Y_1' \cup X_2' \cup Y_2'$ and $A(D) = \{(u,v) \mid u \in X_1' \text{ and } v \in X_2'\} \cup \{(u,v) \mid u \in X_2' \text{ and } v \in Y_1'\} \cup \{(u,v) \mid u \in Y_1' \text{ and } v \in Y_2'\} \cup \{(u,v) \mid u \in Y_2' \text{ and } v \in X_1'\}$. Then, the underlying graph of D is K_{n_1,n_2} and, for any positive integer m, $C^m(D)$ is the union of four cliques formed by X_1', Y_1', X_2', and Y_2'. Therefore $C^m(D) \in \mathcal{G}^{*(m)}_{n_1,n_2}$ and so

$$\min\left\{|E(G)| \mid G \in \mathcal{G}^{*(m)}_{n_1,n_2}\right\} \geq |E(C^m(D))| = \left\lfloor\left(\frac{n_1-1}{2}\right)\right\rfloor^2 + \left\lfloor\left(\frac{n_2-1}{2}\right)\right\rfloor^2.$$

By this together with (5.2), we have

$$\min\left\{|E(G)| \mid G \in \mathcal{G}^{*(m)}_{n_1,n_2}\right\} = \left\lfloor\left(\frac{n_1-1}{2}\right)\right\rfloor^2 + \left\lfloor\left(\frac{n_2-1}{2}\right)\right\rfloor^2.$$

□

Chapter 6

Concluding remarks

We have obtained the number of distinct bipartite tournaments with bipartition (V_1, V_2) satisfying $|V_1| > |V_2|$ and the number of distinct bipartite tournaments with bipartition (V_1, V_2) where $|V_1| = |V_2| = n$ is odd. We have not yet found the number of distinct bipartite tournaments with bipartition (V_1, V_2) where $|V_1| = |V_2| = n$ is even and leave it as an open problem.

We computed the competition indices and the competition periods of a bipartite tournament when it has no directed cycle. To approach the other case where a bipartite tournament has a directed cycle, we tried bipartite tournaments with a small number of vertices. We computed by hand to show that an orientation with a directed cycle of K_{n_1,n_2} has cperiod(D)=1 or 2 if $2 \leq n_1, n_2 \leq 4$ (see Figure 6.1 for an example). We computed competition periods of orientations with a directed cycle of K_{n_1,n_2} for some larger n_1 and n_2 via MATLAB programming and have not found a counterexample yet. We wish to know if every bipartite tournament with a directed cycle has competition period 1 or 2.

We computed the maximum number of edges and the minimum number of edges which the m-step competition graph of an orientation of K_{n_1,n_2} might have. We also computed the minimum number of edges in the set $\mathcal{G}^{*(m)}_{n_1,n_2}$ of m-step competition graphs of orientations of K_{n_1,n_2} in which every vertex has

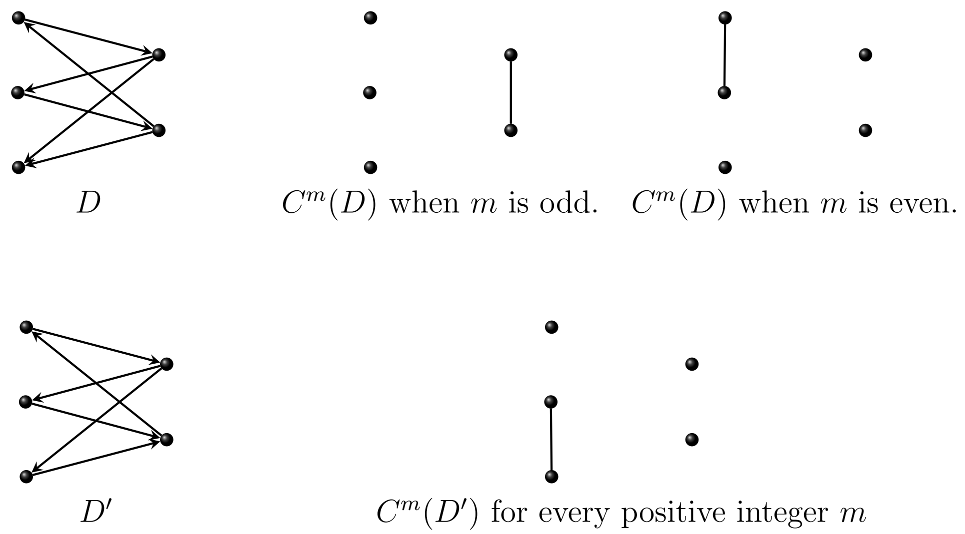

Figure 6.1: Bipartite tournaments D and D' with a directed cycle, cperiod(D)=2, and cperiod(D')=1

out-degree at least one such that each graph in $\mathcal{G}^{*(m)}_{n_1,n_2}$ is a disjoint union of two non-complete graphs. We have a strong belief that the minimum number is still the minimum in a more general case.

Conjecture 6.1. *Let m, n_1, and n_2 be positive integers satisfying $m \geq 2$ and $n_1 \geq n_2 \geq 2$, and $\mathcal{G}^{+(m)}_{n_1,n_2}$ be the set of m-step competition graphs of orientations of K_{n_1,n_2} in which every vertex has out-degree at least one. Then*

$$\min\left\{|E(G)| \mid G \in \mathcal{G}^{+(m)}_{n_1,n_2}\right\} = \left\lfloor\left(\frac{n_1-1}{2}\right)\right\rfloor^2 + \left\lfloor\left(\frac{n_2-1}{2}\right)\right\rfloor^2.$$

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국문초록

이 논문에서는 방향 지어진 완전 이분 그래프의 m-step 경쟁 그래프를 연구했다. 먼저, 포여 열거 정리를 이용하여 방향 지어진 완전 이분 그래프의 개수를 계산했다. 또한 방향 지어진 완전 이분 그래프의 경쟁 지수와 경쟁 주기에 대해 다루었다. 방향 지어진 완전 이분 그래프의 m-step 경쟁 그래프로 나타내어지는 그래프의 쌍을 특징화했다. 마지막으로, 방향 지어진 완전 이분 그래프의 m-step 경쟁 그래프로 나타내어지는 그래프가 가질 수 있는 변의 개수의 최대와 최소에 대해 다루었다.

주요어휘: 완전 이분 그래프, 방향 지어진 완전 이분 그래프, m-step 경쟁 그래프, 경쟁 지수, 경쟁 주기, m-step 경쟁-실현가능한
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