M.S. THESIS

Decoherence due to gravity: Open quantum system approach.

중력에 의한 결잃음: 열린 양자계 접근 방법으로.

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한 정 연

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지도교수 이상민

이 논문을 이학석사 학위논문으로 제출함

2018 년 02 월

서울대학교 대학원

물리천문학부

한정연

한정연의 이학석사 학위논문을 인준함

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위 원 장	제원호
부위원장	이상민
위 원	정현석

Abstract

This thesis addresses various sources of fluctuations and discusses how quantum states are affected by these fluctuation based on the semi-classical gravity theory. Especially, in relation to the macroscopic quantum state.

First, due to the time dilation effect of gravity, decoherence with regarding to the proper time state can arise. By taking a different approach from that taken in previous work (Zych *et al* [1]), we analyze the case of constant-acceleration field using the concept of Rindler spacetime. Furthermore, given that proper acceleration is proportional to the Unruh temperature, we link it to Samuel's experimental proposal of decoherence using the fluctuation in the Unruh temperature [2].

Secondly, in a macroscopic system, the self-localization effect toward the position of the centre of mass arises due to fluctuations in the interaction potential. This leads to decoherence with regard to the position basis (Diósi-Penrose Model [3]). Furthermore, when considering gravity as a classical channel with noise, localization can also be explained by a weak measurement of each mass (Kafri-Taylor-Milburn Model [4]).

Finally, we checked the decoherence effect not merely by examining the quantum characteristics, as is usually done, but with classical fluctuation in the metric field stemming from the dynamics of the field source. In fact, due to the dominance of classical fluctuation, we attempt to analyze this from another perspective. In particular, the energy scale between the gravitational wave which represents the classical field and graviton which indicates quantum field is nearly $3 * 10^{37}$ [5]. Hence, the leading effect of gravitational decoherence is followed by the fluctuation of the classical field. Furthering this analogy, we obtain the justification on using the kernel function in terms of Newtonian interaction in spontaneous localization model proposed by Diósi (SLM) from the fluctuation in the classical field and clarify what this solution indicates.

Keywords: Gravity, Decoherence, Relativisitic quantum information

Student Number: 2015-22600

초록

이 학위논문에서, 우리는 반-고전 중력을 바탕으로 이의 다양한 요동의 원인과 이 요동에 의해서, 특히 거시적 양자상태가 어떻게 영향을 받는 지를 다룰 것입니다.

첫째로, 중력에서의 시간지연효과 때문에 고유시간상태에 대한 결잃음이 일어나는 것입니다. 우리는 원저자들 (Zych 등 [1])과는 다르게 Rindler 시공간으로 일정한 가속도가 있을때를 분석하였습니다. 더 나아가, 고유 가속도가 Unruh 온도에 비례하므로, 이를 Unruh 온도의 요동을 이용한 Samuel의 결잃음 실험모델과 연결시켰습니다 [2].

둘째로, 거시적인 시스템에서는 상호작용되는 퍼텐셜의 요동으로 인해 이 물체의 무게중심을 향하여 자체 국소화가 되는 현상을 확인할 수 있습니다. 이는 위치를 기저로 하여 일어납니다 (Diósi-Penrose 모델 [3]). 더 나아가, 중력을 노이즈가 있는 고전채널로 고려하여 두 질량체의 약한관측으로 국소화를 또한 설명할 수 있습니다 (Kafri-Taylor-Milburn 모델 [4]).

마지막으로, 우리는 앞서 본 양자적 요동이 아닌 메트릭장의 고전적인 요동 때문에 일어나는 결잃음을 확인할 수 있었습니다. 불확정성원리와 관측에서부터 기인하는 양자요동과 달리 고전요동은 장을 만드는 물질의 움직임자체가 만드는 것입니다. 실생활에서는, 고전요동이 훨씬 크기 때문에, 우리는 이를 다른 관점에서 살펴보려고 했습니다. 특히, 고전 장을 대표하는 중력파와 양자 장을 대표하는 중력자의 에너지 크기 차이가 $3 * 10^{37}$입니다 [5]. 그래서, 중력에 의한 결잃음의 주 효과는 중력파의 요동으로부터 올 것 입니다. 이를 이용하여, 우리는 Diósi에 의해서 제안된 동시국소화 모델 (SLM)에 쓰이는 뉴턴 상호작용 형태 커널함수의 정당성을 고전

장의 요동으로 부터 얻었고 이 모델의 결과의 의미를 확실히 할 수 있었습니다.

주요어: 중력, 결잃음, 상대론적 양자정보

학번: 2015-22600

Contents

List of Figures

List of Tables

Chapter 1

Introduction

Quantum Mechanics (QM), a great advance in modern physics, racially changed our perspective on nature from a deterministic stance in Classical Mechanics (CM) to a probabilistic viewpoint. Superposition, one of the most striking features in quantum theory, has led to this interpretation. Heisenberg's uncertainty principle as another feature holds that no localized state exists with the exact value of a position without measurement. Considering these two features in QM, physicist ultimately explained several previously mysterious problems with CM, especially in relation to the microscopic world (i.e., the quantized energy spectrum of a hydrogen atom, and the anomaly of the scattering cross section predicted in CM). In contrast, in our everyday lives (i.e., macroscopic world), quantum features are not be observed. The question thus arises of why quantum nature apparently disappear from the system when it grows in size and the interactions become more complex. One theory can be referred to as '$\hbar$ goes to zero' idea. This idea suggests that we simply 'neglect' small instances of uncertainty in the macro world. It is analogous to a situation in which pixel

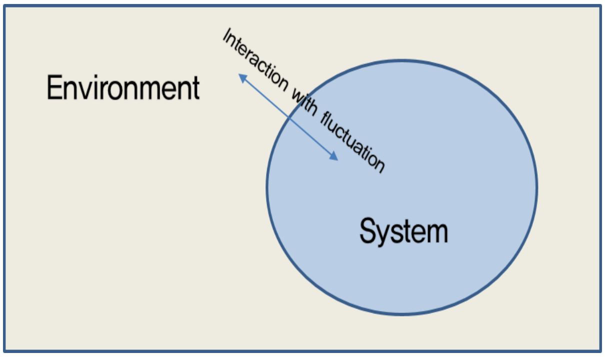

Figure 1.1: Diagram of open system

units on a computer screen appear to arise consecutively when we observe entire picture. The other most prominent role this quantum to classical transition is commonly attributed to **decoherence theory**. It describes the situation that a system and its surrounding environment (fig 1.1). When there exists entanglement due to the interaction between the system and the environment, with additional random fluctuations on the interaction decoherence of the system can be observed by taking an average (partial trace-out) with regarding to the environment. In addition, the selected basis is related to the relative strength level of each Hamiltonian. This type of environment-induced basis choice is known as **einselection**. Decoherence with einselection implies an absence of an approach by which to track each interaction. That is why only the averaged effect is observed (i.e., a reduced state). Thus, we can argue that decoherence originates from a **lack of knowledge**.

General Relativity (GR), another elegant achievement in modern physics, describes particles using equations of motion with general covariance. Because there is no statistical nature - it is instead deterministic in GR - it appears that quantumness of a particle does not need to be considered. However, when a quantum particle is in a fluctuating metric (quantum or classical) or undergoes fluctuations originating from various sources, it becomes necessary to consider the decoherence problem. As a fundamental perspective, due to the inevitability of gravity, information about interaction source is always gained. Through this analogy, it is claimed that **gravity is inherently an open quantum system** [10]. In other words, there is no isolated state in the universe because gravitational interaction is fundamental. Even if it is not dominant in our everyday lives (e.g. thermal fluctuation, electromagnetic radiation), as a primary type of interaction in the early universe, the significance of understanding the gravitational effect on a quantum state increases, as associated effects constitute fundamental decoherence. In such a flow, this topic represents a rising sun for physicists who investigate the open quantum system.

In order to expound the sources of fluctuations, two descriptions have been proposed: fundamental description and effective. **Fundamental description** implies that a quantum metric with quantum fluctuation comes from the nature of quantum gravity. There have been various proposals regarding the source of quantum fluctuations and associated decoherence effects as they pertain to quantum matter. The first originated from the fundamental uncertainty relation, which is $\Delta x \Delta p > \frac{1}{2}$, and this with Lorentz invariance leads to quantum intrinsic decoherence [11, 12, 13, 14]. Fluctuations in the spacetime metric arising from quantum gravity effects become a source of noise, which would necessarily be accompanied by intrinsic decoherence. The second proposal is linked to the imprecise measurement of time given the limitation of temporal resolutions [15, 16, 17, 18, 19]. Due to the fundamental relationship between time and energy, there exists a fundamental time resolution on the Planck scale. Imprecision of time measurement makes decoherence occur. Third, considering time as a statistical variable and introducing time fluctuation, intrinsic decoherence with regarding to time is held to be accompanied, by the probabilistic evolution of the Liouville equation [20].

However, an **effective description** with semi-classical gravity (a classical metric with quantized matter) did not assume the quantum nature of spacetime. It simply starts with the interaction Hamiltonian between quantum matter and classical fluctuating metric $h_{\mu\nu}$ [21, 22].

$$\hat{H} = \int dV h_{\mu\nu} \hat{T}^{\mu\nu} \tag{1.1}$$

In this description, there are two candidates for the fluctuation types. The first is the classical type with the metric fluctuation (see chapter 5-4). The second is the quantum type with quantum interactions or with measurements onto quantum matter. Given the statistical nature of quantum system, it has been explained with two sources of the fluctuation on the quantum interactions. The first of these is known as **proper time**. Because it is no longer a parameter but a variable, as a quantum language, the concept of the proper time state could be devised with the associated Hamiltonian [23, 24, 25, 26, 27, 28]. Therefore, when we do not have precise information about the time of internal particles, it leads to decoherence of the centre of mass of the position basis [1, 29, 30]. The second source, by Diósi [3, 9, 31], Penrose [32] and others [8], disclaimed the use of a fundamental perspective of a single particle and instead posited an effective perspective of many-particle state on account of the **non-linearity of Newton-Schrödinger equation**. In particular, given that non-linearity can only originate from an effective description and inconsistency in a fundamental description allow us to obtain an effective description which becomes meaningful in many-body systems. As a result of this fluctuation, self-focusing occurs, similarly to the case described above.

The above model, the Diósi-Penrose model (the DP model), seemed sufficient but there was a problem such as the super-luminal teleportation originating from the reality interpretation of the wave function. Hence, Kafri *et al* noted that gravitational interaction can be considered as a classical channel between particles because gravity interaction naturally cannot be shielded. In addition, since information about the source always increases due to gravity interaction, entanglement between particles cannot be created[1]. In this channel, decoherence is wholly based on the quantum fluctuation by the weak measurement through the classical channel. In addition, in order to solve the super-luminal teleportation problem in the DP model, noise-induced weak measurements between two massive particles were introduced. As a result, it succeeded in gaining an identical result to that obtained by the DP but without controversies. Furthermore, in their work, Diósi *et al* proposed spontaneous localization model (SLM) using weak measurements by gravitational interaction with white noise [9].

Experimentally, owing to the weakness of gravitational interaction, this macroscopic decoherence issue is challenging to detect without the capability of extreme sensitivity. Despite this obstacle, it is attempted to obtain the experimental result. For instance, Margalit *et al* published there experimental results pertaining to visibility changes due to time dilation using a two-level atom [25]. Samuel also has proposed an experimental method which relied on a double slit in a accelerated frame by which to measure the degree of visibility. Recently,

[1]It is equivalent to LOCC (Local operation and Classical communication) cannot create the entanglement between two systems [33].

the new technical method, quantum opto-mechanics attempts to measure the decoherence effect due to gravity given the sensitive nature of light [34]. This field will become a cornerstone of open quantum system field development.

These types of interesting discussion lead us to undertake research on this topic. In this dissertation, we investigate this topic in detail. The paper is organized as follows. In chapter 2, the cornerstones of decoherence theory and einselection will be introduced. Starting with density matrix theory to defining decoherence, examples of possible channels in which decoherence can occur in a qubit system will be discussed. The last section of this chapter covers why environment-induced decoherence has a preferred basis (einselection) and attempts to define the preferred basis. Chapter 3 is devoted to one as the most important tools of an open quantum system, and here the master equation will be introduced as well. Moreover, in its Born-Markov-Secular approximated version, the Lindblad equation, which works well within a perturbative interaction Hamiltonian, will be introduced. In addition, there are two typical examples with which to apply the Lindblad equation. By solving the equation of spontaneous emission and Schrödinger cat state, the manner in which decoherence arises in two systems can be explained.

In chapter 4, as the first source of decoherence, the lack of knowledge on the proper time of internal particles due to the time dilation effect will be addressed. For simplicity, we will focus the constant gravity only. With reproducing [1] in the Rindler spacetime, we can easily make connections to the decoherence effect using the thermal fluctuation of the Unruh temperature. We create a bridge between aforementioned original paper and the experimental proposal of Samuel [2] in this section as well. The last chapter is devoted to decoherence due to semi-classical gravity. After reviewing DP model in both its classical and quantum field versions, the decoherence of the position basis can be derived due to the lack of knowledge about the interaction potential. As an advanced model, the KTM model will also be introduced with the language of the quantum feedback master equation. In the last section, we suggest decoherence by the classical fluctuations in the metric field. Classical fluctuations, unlike quantum fluctuations by measurements with uncertainty principle, are wholly evoked by the dynamics of the field source. We expound the reason behind the setting of the kernel of the correlation function as a Newtonian potential. In addition, because strength of classical fluctuation is much dominant as compared to the quantum, we justify the main source of decoherence. It is opposite to the measurement-based quantum fluctuation assumption of Diósi but provides a result identical to that provided by the DP model.

Throughout this thesis, we use natural units, $\hbar = c = k_B = G = 1$.

Chapter 2

Decoherence Theory

Decoherence theory has explained the transition from a quantum to a classical well with very small $\hbar$ limit explanation in quantum field theory. In a closed system, a quantum state undergoes unitary evolution only with regarding to the given Hamiltonian. On the contrary, when the system interacts with its surrounding, the dynamics of state becomes non-trivial. Specific observer in the system is only able to observe averaged effect of the environment information. Resulting process provides non-unitary dynamics onto a system state, and coherence of it disappears. Hence, reduced system state obtained by tracing out the degree of an environment becomes the outcome of decoherence. In this chapter, as a proper language to explain decoherence mathematically, we introduce the density matrix formula and the following map of evolution. It is the so-called Kraus operator map. Moreover, basic parameters to describe decoherence time scale factor are introduced in this chapter. In the last section, as the environment-induced effect, the concept of einselection is presented.

2.1 Density matrix formula

The density matrix is not only just a representation of quantum state but crucial to explain decoherence of a quantum state. Density matrix is defined by

$$\rho = \sum_{a=0}^{d-1} P_a |a\rangle\langle a| \tag{2.1}$$

where d is the number of dimensions of the state which is in d - dimensional Hilbert subspace. The set $\{|a\rangle\}$ forms a complete basis

$$\sum_{a=0}^{d-1} |a\rangle\langle a| = I. \tag{2.2}$$

P_a also yields the information about the state in a as a probability with these axioms

$$\sum_{a=0}^{d-1} P_a = 1, \quad P_a \geq 0 \quad ({}^{\forall}a : 0 \leq a \leq d-1), \quad \sum_{a=0}^{d-1} aP_a = \bar{a}. \tag{2.3}$$

In case that ρ can be represented by the only summation on one state, it is a pure state.

$$\rho = |\Psi\rangle\langle\Psi| \tag{2.4}$$

Otherwise,

$$\rho = \sum_{a=0}^{d-1} P_a |a\rangle\langle a| \quad (P_a < 1, {}^{\forall} a) \tag{2.5}$$

is a mixed state. To sum up, provided that we can represent ρ as the outer product of a single state, it is a pure state. If not, a mixed state.

2.1.1 Properties of the density matrix

Consider several properties of the density matrix. First, total probability is unity,

$$Tr(\rho) = 1. \tag{2.6}$$

Secondly, for any operator Q, expectation value is

$$\langle Q \rangle = Tr(\rho Q). \tag{2.7}$$

and Thirdly,

$$Tr(\rho^2) = \sum_{a=0}^{d-1} P_a^2 \leq 1. \tag{2.8}$$

For Tr(ρ^2) = 1, this state is a **Pure state**, Tr (ρ^2) < 1 is a **Mixed state**. Because P_a for an arbitrary index a is smaller than 1 in a mixed state, its square sum must be less than 1. Tr (ρ^2) is also the so-called purity factor since it determines whether a state is mixed or not.

2.1.2 Parameters of decoherence in the density matrix

Let us consider a two dimensional, spin 1/2 system. The eigenstate of Pauli-X operator in Pauli-Z basis up and down is

$$|+\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle). \tag{2.9}$$

In terms of the density matrix, it becomes

$$\rho = |+\rangle\langle+| = \frac{1}{2}(|\uparrow\rangle\langle\uparrow| + |\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{2.10}$$

Coefficients of terms $|\uparrow\rangle\langle\downarrow|$ and $|\downarrow\rangle\langle\uparrow|$ indicate coherence between $|\uparrow\rangle$ and $|\downarrow\rangle$. In this notation, we can define decoherence as the off-diagonal components of the density matrix tending to zero. [1]

Following complete decoherence occurs, the density matrix is thus

$$\rho' = \frac{1}{2}(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|). \tag{2.11}$$

[1] Notice that we just treat very simplified examination to explain what decoherence is. Sometimes, the diagonal term can be also decayed. Moreover, such as Schrödinger cat state, coherence does not completely reach zero. We shall deal with this kind of process in the next section

Thus, decoherence can transform a pure state into a mixed. Because ρ' is I/2 $\mathrm{Tr}(\rho'^2) = \frac{1}{2} < 1$. As can be seen above, decoherence is the crucial process to describe quantum to classical transitions.

2.1.3 Wigner function

Wigner function is usually used to observe the quantumness in a graphical way using a (quasi) phase space (x,p). Wigner function is defined by

$$W(x,p) = \frac{1}{\pi} \int du \exp(-2ipu) \left\langle x + \frac{u}{2} |\rho| x - \frac{u}{2} \right\rangle . \tag{2.12}$$

Wigner function is a real distribution on a phase space (equivalently, in a complex space), whose knowledge is equivalent to that of ρ. This function is the so-called "Quasi-Probability distribution function". It is very similar to classical probability distribution function but the difference is evoked by quantum property as the negative probability. For example, in an infinite potential well, there are n-1 nodes for the nth excited state (n $\geq$ 1). This is exactly a quantum phenomenon. Wigner function for the nth excited state represents negative values where nodes are created in the original real space.

When decoherence occurs, the negativity of Wigner distribution disappears (In every region, it has a semi-positive value.) It means quantum phenomena are weaker as decoherence occurs. We can graphically explain the quantum to classical transition as the time evolution of Wigner distribution[2].

There also exists a similarity with classical Liouville equation in terms of "flow in the phase space". With the help of Wigner function, establishing the quan-

[2]See the example in section 3.6

tum Liouville equation can be feasible.

$$\frac{\partial W(x,p,t)}{\partial t} = -i[W(x,p,t), H] \tag{2.13}$$

We can easily check resemblance with the classical one by substituting $W(x,p,t)$ for a probability distribution in the phase space $\rho(x,p,t)$ and $i[,]_{QM}$ into $\{,\}_{PB}$

2.2 Kraus operator sum method

2.2.1 Definition of Kraus operator sum method

From now on, we consider methods which describe maps between an original quantum state to the decohered one. Introduce this kind of maps using super-operator [33]

$$\varepsilon(\rho) = \rho'. \tag{2.14}$$

Where ρ is an original quantum state and ρ' is the output state after operating super-operator ε. To describe the exact form of super-operator, let us consider an original quantum state which is not entangled yet. Also, the environment state is a pure one. Under unitary evolution which acts on $H_s \otimes H_\epsilon$, the total quantum system can be written as

$$\rho' = \hat{U}_{s\epsilon}(\rho_s \otimes |e\rangle_\varepsilon \langle e|)\hat{U}^\dagger_{s\epsilon}. \tag{2.15}$$

When we trace-out all the information of environment since we only want to consider system, the reduced state of system becomes

$$\rho'_s = tr_\epsilon(\rho') = \sum_a \ {}_\varepsilon\langle a|\hat{U}_{s\epsilon}|e\rangle_\varepsilon \ \rho_s \ {}_\varepsilon\langle e|\hat{U}^\dagger_{s\epsilon}|a\rangle_\varepsilon \equiv \sum_a \hat{E}_a \rho_s \hat{E}^\dagger_a \tag{2.16}$$

where

$$\sum_a \hat{E}_a \hat{E}^\dagger_a = \sum_a \ {}_\varepsilon\langle a|\hat{U}_{s\epsilon}|e\rangle_\varepsilon \langle e|\hat{U}^\dagger_{s\epsilon}|a\rangle_\varepsilon = \ {}_\varepsilon\langle e|\hat{U}^\dagger_{s\epsilon}\hat{U}_{s\epsilon}|e\rangle_\varepsilon = \hat{I}_s. \tag{2.17}$$

This method to describe the map using super-operator is known as **Kraus operator sum representation method**.

Moreover, there is an important property in Kraus super-operator: **Completely Positivity**. The definition is following. For a linear map ε, considering it maps a positive operator in H_s into a positive operator in H_s (positive map) and also even in the case of given our state as a total state including environment which lives in $H_s \otimes H_\epsilon$ positivity must not be broken. This map is the so-called completely positive map. This property is often convinced as the postulate of quantum operation.

2.2.2 POVM measurement

POVM, positive operator value measurement, is a kind of general measurement. In open quantum system, the system is used to attach to an ancillary state (environment). Consequently, given that projection measurement on ancillary is operated, because system and environment are connected by a total unitary operator, "indirect" knowledge about the system's state can be provided without destroying it. It is quite different with projection measurement which is the familiar concept in the closed quantum system language. Since POVM says measurement not on the system directly and also does not make any operations onto the system, conjecture is only induced by a probability distribution.

As utilizing Kraus sum method, we are able to describe POVM. First, let us consider the positive operator $\hat{E}_i = \hat{M}_i^\dagger \hat{M}_i$. Where $\hat{M}_i$ is a measurement operator with $\sum_i \hat{E}_i = \hat{I}_s$ comes from the property of Kraus operator. Following this definition, ρ after measurement is therefore

$$\rho \to \rho' = \frac{\hat{M}_i \rho \hat{M}_i^\dagger}{p_i} \tag{2.18}$$

where

$$p_i = Tr_s(\rho \hat{E}_i). \tag{2.19}$$

Positivity of $\hat{E}_i$ must be imposed for each probability must larger than zero. Notice that for the special case, provided that $\hat{M}_i$ is projection operator, there is nothing different with projection measurement. Conversely, in general, unlike projection measurement, $\hat{M}_i \hat{M}_j \neq \hat{M}_i \delta_{ij}$. In other words, the quantum state after measurement does not have to remain same even if there were repeated measurements. Also, since there is no preferred basis unlike projection measurement (measurement operator satisfies $\sum_i \hat{M}_i^\dagger \hat{M}_i = \hat{I}_s$ only), there are degrees of freedom to choose each measurement operator. If POVM operator also satisfies sufficient condition $[\hat{E}_i, \rho] = 0$, this operator does not destroy the state (It can be changed for repeated operation). It is known as quantum non-demolition measurement (QND-measurement). Let us provide an example to do optimized measurement by controlling POVM operators in appendix A.

2.3 Quantum error process and decoherence channels

From now on, using Kraus representation method, not only measurement but also several kinds of error processes with decoherence channels are able to possible. This method is really convenient when treating the finite dimensional Hamiltonian and associated states. In this section, as elementary examples, we categorize errors and decoherence channels for dealing with a single qubit state using Kraus method.

2.3.1 Quantum errors

In general, given probability to occur error p, corresponding Kraus operator is defined by

$$\hat{E}_0 \equiv \sqrt{1-p}\hat{I}_2, \qquad \hat{E}_1 \equiv \sqrt{p}\hat{A}. \tag{2.20}$$

For instance, three kinds of errors can occur in a single qubit state.

1) Bit flip error $|0\rangle \rightarrow |1\rangle$ *and* $|1\rangle \rightarrow |0\rangle$. Following Kraus operator $\hat{A} = \hat{\sigma}_x$.

2) Phase flip error $|0\rangle \rightarrow |0\rangle$ *and* $|1\rangle \rightarrow -|1\rangle$ and $\hat{A} = \hat{\sigma}_z$.

3) Both errors with phase i, $|0\rangle \rightarrow i|1\rangle$ *and* $|1\rangle \rightarrow -i|0\rangle$ and $\hat{A} = \hat{\sigma}_y$.

Needless to say, all errors can happen simultaneously. As an important process in quantum information, quantum error correction is the method to notice and correct such kinds of errors.

2.3.2 Depolarizing channel

Combining all types of errors and assuming equal probability of each error p/3, the state after operation can be written as

$$\rho' = (1-p)\rho + \frac{p}{3}(\hat{\sigma}_x\rho\hat{\sigma}_x + \hat{\sigma}_y\rho\hat{\sigma}_y + \hat{\sigma}_z\rho\hat{\sigma}_z). \tag{2.21}$$

Hence, using the identity

$$\frac{\hat{I}_2}{2} = \frac{\rho + \hat{\sigma}_x\rho\hat{\sigma}_x + \hat{\sigma}_y\rho\hat{\sigma}_y + \hat{\sigma}_z\rho\hat{\sigma}_z}{4}, \tag{2.22}$$

Kraus super-operator is often represented as

$$\varepsilon(\rho) = \frac{p\hat{I}_2}{2} + (1-p)\rho. \tag{2.23}$$

In an arbitrary d - dimensional state, it is represented by just substituting $\frac{p\hat{I}_2}{2}$ for $\frac{p\hat{I}_d}{d}$. Its physical meaning becomes obvious, with a probability p, the state undergoes the transformation into a completely mixed state. Consequently, after depolarizing occurs, the amplitude of a Bloch vector is decreased as the state undergo a change pure into mixed. Considering a Bloch vector represents the polarization of the photon, it loses its quantity. Channel with this characteristic property is known as depolarizing channel.

2.3.3 Amplitude damping

Amplitude damping channels is a description of energy dissipation. That is, an effect due to loss of energy occurs through this channel. Spontaneous emission (we shall deal with more details using Lindblad equation in chap 3) is one example of this processes [35]. As a description of this channel, let us consider the state is in Hilbert space $H_{sys} \otimes H_{\varepsilon}$. Given system is a two-level atom and assume an environment state as a vacuum and boson. Through amplitude damping channel with transition probability p, the state is then

$$\begin{aligned} &|0\rangle_s|0\rangle_\varepsilon \rightarrow |0\rangle_s|0\rangle_\varepsilon \\ &|1\rangle_s|0\rangle_\varepsilon \rightarrow \sqrt{1-p}|1\rangle_s|0\rangle_\varepsilon + \sqrt{p}|0\rangle_s|1\rangle_\varepsilon. \end{aligned} \tag{2.24}$$

In terms of Kraus operator, its matrix representation is

$$\hat{E}_0 = {}_\varepsilon\langle 0|\hat{U}_{s\varepsilon}|0\rangle_\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \quad \hat{E}_1 = {}_\varepsilon\langle 1|\hat{U}_{s\varepsilon}|0\rangle_\varepsilon = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}. \tag{2.25}$$

Let assume we set the initial state $\rho_{00} = a, \rho_{01} = x, \rho_{10} = \bar{x}$ and $Tr\rho = 1$. State after propagating this channel becomes

$$\varepsilon(\rho) = \begin{pmatrix} a + p(1-a) & \sqrt{1-p}\; x \\ \sqrt{1-p}\; \bar{x} & (1-p)(1-a) \end{pmatrix}. \tag{2.26}$$

Consider applying N consecutive events. Given that each time interval is δt and total time t, resulting N yields $t/\delta t$. Following p is replaced by $\gamma\delta t$ where γ is the rate to occur transition in δt. For sufficiently small time δt, $(1-p)^N = (1-\gamma\delta t)^{t/\delta t} \approx e^{-\gamma t}$. Using this formula, state after traverse the channel N times is therefore

$$\varepsilon(\rho) = \begin{pmatrix} 1 - \exp(-\gamma t) + a\exp(-\gamma t) & x\exp(-\gamma t/2) \\ \bar{x}\exp(-\gamma t/2) & (1-a)\exp(-\gamma t) \end{pmatrix}. \tag{2.27}$$

This can be easily demonstrated that diagonal and off-diagonal terms decay simultaneously. Finally, as the result of applying amplitude damping channel, every atom in a vacuum is going to submerged into the ground state. In a Bloch representation, its entire sphere shrunk towards the certain pole.

Some authors call the decay of diagonal term as 'Relaxation' and the off-diagonal term as 'Dephasing'. The ratio between defined decay rates of each term as $1/T_1$ (diagonal) and $1/T_2$ (off-diagonal) can be easily checked as

$$2T_1 = T_2. \tag{2.28}$$

In general, a ratio between Dephasing and Relaxation time (e.g. Nuclear Magnetic Resonance experiment) is given by $T_2 \geq 2T_1$ [33]. A searching T_2/T_1 ratio is also the kind of research field in open quantum system.

2.3.4 Phase dephasing

In contrast to amplitude damping channel, through this channel, only the loss of information occurs without loss of energy. For instance, Consider Hamiltonian $\hat{H}_{int} = \chi \hat{a}^\dagger \hat{a}(\hat{b} + \hat{b}^\dagger)$, where $\hat{a}$ is the annihilation operator of system and $\hat{b}$ for the environment. Namely, Hamiltonian represents number operator of system couples with a quasi-position operator in an environment. For small operational time δt, the unitary operator is represented by $U = exp(-i\hat{H}_{int}\delta t)$. Like above, given that we assume the environment is in a vacuum, corresponding Kraus operator is thus

$$\hat{E}_0 = {}_{\varepsilon}\langle 0|\hat{U}_{s\varepsilon}|0\rangle_{\varepsilon} = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \quad \hat{E}_1 = {}_{\varepsilon}\langle 1|\hat{U}_{s\varepsilon}|0\rangle_{\varepsilon} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{p} \end{pmatrix}. \tag{2.29}$$

Where $p = sin^2(\chi\delta t)$ is probability to be scattered by photons. Following they traverse this channel N times during time t with very small δt, the state becomes

$$\varepsilon(\rho) = \begin{pmatrix} a & x\exp(-t/T_2) \\ \bar{x}\exp(-t/T_2) & 1-a \end{pmatrix}. \tag{2.30}$$

T_2 is dephasing time defined in the above section. It is obvious that decay of off-diagonal terms only occurs in this channel. Only coherence between two ket states disappear as time goes on[3]. This kind of process is also known as 'Pure dephasing'.

In the continuum basis used in a continuous measurement, from the fact interaction Hamiltonian includes $\hat{x}$, off-diagonal component with regarding to the position basis of the state by scattering is well known as [36]

$$\rho(x, x', t) \propto \rho_0 \exp(-(x - x')^2 t). \tag{2.31}$$

As a result of the continuous measurement, dephasing process is progressed. In details, consider that quantum atom are spread such as a cloud initially. Following scattering process happened, this atom becomes localized one with designated position[4].

2.4 Environment induced einselection

In the previous section, we explored the cases of decoherence through the noisy channel. However, the most curious phenomena in quantum mechanics 'Post state selection' seems not to be clear in our language. Because in the above section, which basis should be selected when decoherence occurs is not dealt

[3]Some physicists distinct between reversible process and irreversible process. Former one comes from when off-diagonal terms depend on sinusoidal function, components coming back. On the contrary, later one is gone forever as time goes on

[4]For more details, read chapter 3 of [36]

with. Zurek *et al* [37] insinuate question, why nature has the preferred basis and How? To answer it, let us define the pointer state $|E\rangle$ indicates the state of the environment. For provided distinct system's states $|\psi_1\rangle$ and $|\psi_2\rangle$, given that our states evolve such as

$$\begin{aligned} |\psi_1\rangle \otimes |E\rangle &\rightarrow |\psi_1\rangle \otimes |E_1\rangle, \\ |\psi_2\rangle \otimes |E\rangle &\rightarrow |\psi_2\rangle \otimes |E_2\rangle. \end{aligned} \tag{2.32}$$

For the state $|\psi_\pm\rangle$ equally even and odd superposed, it evolves as (skipping $\otimes$)

$$|\psi_\pm\rangle = \frac{1}{\sqrt{2}}\left(|\psi_1\rangle \pm |\psi_2\rangle\right) \rightarrow \frac{1}{\sqrt{2}}\left(|\psi_1\rangle|E_1\rangle \pm |\psi_2\rangle|E_2\rangle\right). \tag{2.33}$$

In distinction of pointer states, total states become entangled. In addition, as fidelity $(|\langle E1|E_2\rangle|)$ of pointer states goes to zero, reduced density matrix becomes a completely mixed state. That is the result of decoherence in $\{|\psi_1\rangle, |\psi_2\rangle\}$ basis. In general, this kind of entanglement with system and environment is often considered as the cue of decoherence.

Conversely, $|\psi_1\rangle$ in terms of the basis $|\psi_\pm\rangle$ does not be entangled with a pointer state. In $\{|\psi_\pm\rangle\}$ basis, it is the most robust to the interaction with environment. As a result, it enables us to define the **preferred basis** in this system as $\{|\psi_1\rangle, |\psi_2\rangle\}$. The resulting environmental dynamical selection of a preferred state is **Environment induced einselection**. This process is key to explain 'Why we are not able to observe specific superposed state?' For example,

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|owl - alive\rangle + |owl - dead\rangle\right) \tag{2.34}$$

in a realistic situation. Only alive or dead state of owl can be detected in reality. Its preferred basis becomes alive or dead state not its superposed one.

From now on, the question can be raised. For the given total Hamiltonian

$\hat{H}_{tot} = \hat{H}_{Sys} + \hat{H}_{Env} + \hat{H}_{Int}$, what is the preferred and pointer states in this system? Briefly speaking, there are three cases related to the relative strength of the system and interaction Hamiltonian for the system state [36].

1. **The quantum measurement limit**

When interaction is dominant with regarding to system Hamiltonian, preferred states is the eigenstate of $\hat{H}_{int}$. Usually, system part of interaction Hamiltonian is the function of relative distance, it commutes with position operator. Thus, localization due to decoherence in the position basis is observed.

2. **The intermediate regime**

When strength between interaction and system Hamiltonian is similar, interesting story evolves in this region. For instance, Quantum Brownian motion like Hamiltonian leads to localization of both position and momentum with dissipation [36, 6].

3. **The quantum limit of decoherence**

For the perturbative strength of interaction with regarding to system Hamiltonian, for the well-thermalized and slowly varying bath, a preferred state is then eigenstates of system Hamiltonian. Maybe, almost dynamics on nature can be approximately considered in this limit (e.g. spontaneous emission etc). As a dynamical equation of reduced density matrix on the system, we deal with Quantum Master Equation in next section. This is a powerful tool to explain decoherence dynamics in system basis.

Chapter 3

Quantum Master Equation

The specified equation to describe the decoherence process in continuous time within the quantum limit of decoherence is known as the **quantum master equation (QME)**. Needless to say, this equation is the one example of quantum operation. A difference with Kraus representation method originates from the separation of an unitary and non-unitary term by considering the system - environment model. It is also significant to describe the dynamics of quantum to classical transition in the perturbative limit.

In order to derive the QME, we start with Liouville equation which describes the unitary evolution in a closed system, and use the interaction picture with the assumption on perturbative interaction Hamiltonian. In addition, after tracing out the information of the environment one, additional non-unitary term appears. This term yields the decoherence effect and we are able to explain why quantum to classical transition occurs. Methodologically, this resulted equation makes it easier to regarding the decoherence term separately and more convenient to understand what kind of operator yields the decoherence effects. In this chapter, we derive the QME and treat basic examples of decoherence using this equation.

3.1 Interaction picture

Interaction picture is one of the important pictures derived by Richard Feynman [38]. It is really useful for treating the perturbative system. When the interaction Hamiltonian is not small compared with the original Hamiltonian, there are problems to reconstruct Hilbert space in terms of original Hilbert space. Consequently, in this derivation, we assume interaction Hamiltonian can be dealt with perturbative term.

First, let us begin by decomposing the total Hamiltonian into the system, the environment and the interaction.

$$\hat{H} = \hat{H}_s + \hat{H}_\epsilon + \hat{H}_{int} \tag{3.1}$$

where $\hat{H}_s$ and $\hat{H}_\epsilon$ are Hamiltonian only act on Hilbert space of their own. Whereas $\hat{H}_{int}$ acts on both system and environment. To switch our Hamiltonian into the interaction picture, it is convenient to rewrite Hamiltonian as

$$\hat{H}_0 = \hat{H}_s + \hat{H}_\epsilon. \tag{3.2}$$

It denotes the total free Hamiltonian and we treat $\hat{H}_{int}$ perturbatively.

We now transform the interaction Hamiltonian and total density matrix into the interaction picture. These transformations are

$$\hat{H}^{(I)}_{int}(t) = e^{i\hat{H}_0 t}\hat{H}_{int}e^{-i\hat{H}_0 t}, \tag{3.3}$$

$$\hat{\rho}^{(I)}(t) = e^{i\hat{H}_0 t} e^{i\hat{H}t} \hat{\rho} e^{i\hat{H}t} e^{-i\hat{H}_0 t}. \tag{3.4}$$

Starting from Liouville-Von Neumann equation for Schrödinger picture,

$$\frac{d\hat{\rho}}{dt}(t) = -i[\hat{H}, \hat{\rho}(t)] \tag{3.5}$$

we obtain

$$\begin{aligned} \frac{d\rho^{(I)}}{dt}(t) &= i[\hat{H}_0, \hat{\rho}^{(I)}(t)] + e^{i\hat{H}_0 t} \left(\frac{d\hat{\rho}}{dt} \right) e^{-i\hat{H}_0 t} \\ &= -i[\hat{H}^{(I)}_{int}, \hat{\rho}^{(I)}(t)]. \end{aligned} \tag{3.6}$$

It is another version of Liouville-Von Neumann equation in the interaction picture [36].

3.2 Derivation of the quantum master equation

Now when we formally integrate (3.6), it is obtained that

$$\hat{\rho}^{(I)}(t) = \hat{\rho}^{(I)}(0) - i \int_0^t dt' [\hat{H}^{(I)}_{int}(t'), \hat{\rho}^{(I)}(t')]. \tag{3.7}$$

Because $\hat{\rho}^{(I)}(0) = \hat{\rho}(0)$ (insert 0 into (3.4)), (3.7) becomes

$$\hat{\rho}^{(I)}(t) = \hat{\rho}(0) - i \int_0^t dt' [\hat{H}^{(I)}_{int}(t'), \hat{\rho}^{(I)}(t')]. \tag{3.8}$$

Let us insert this expression for $\hat{\rho}^{(I)}$ into (3.6) again. This yields

$$\begin{aligned} \frac{d\rho^{(I)}}{dt}(t) &= -i[\hat{H}^{(I)}_{int}, \hat{\rho}^{(I)}(t)] \\ &= -i[\hat{H}^{(I)}_{int}(t), \hat{\rho}(0)] - \int_0^t dt' [\hat{H}^{(I)}_{int}(t), [\hat{H}^{(I)}_{int}(t'), \hat{\rho}^{(I)}(t')]]. \end{aligned} \tag{3.9}$$

Notice that for treating $\hat{H}_{int}$ perturbatively, more than the 2nd order of $\hat{H}_{int}$ is neglected. Eq (3.9) is the integro-differential equation for the total quantum state $\rho(t)$ in time. In succession, because we only hope to obtain the quantum state of the system, let us extract it from the total quantum state by partial trace-out.

$$
\begin{aligned}
Tr_{\epsilon}[\hat{\rho}^{(I)}(t)] &= e^{i\hat{H}_s t} Tr_{\epsilon}[e^{i\hat{H}_{\epsilon} t}\hat{\rho}(t)e^{-i\hat{H}_{\epsilon} t}]e^{-i\hat{H}_s t} \\
&= \hat{\rho}_s^{(I)}(t)
\end{aligned}
\tag{3.10}
$$

with using the fact

$$
[\hat{H}_s, \hat{H}_{\epsilon}] = 0. \tag{3.11}
$$

Using the result of (3.10), the equation for reduced density operator $\hat{\rho}_s^{(I)}(t)$ is obtained as

$$
\frac{d}{dt}\hat{\rho}_s^{(I)}(t) = -i\, Tr_{\epsilon}[\hat{H}_{int}^{(I)}(t), \hat{\rho}(0)] - \int_0^t dt'\; Tr_{\epsilon}[\hat{H}_{int}^{(I)}(t), [\hat{H}_{int}^{(I)}(t'), \hat{\rho}^{(I)}(t')]]. \tag{3.12}
$$

We also can make $Tr_{\epsilon}[\hat{H}_{int}^{(I)}(t), \hat{\rho}(0)] = 0$ by presuming interaction Hamiltonian turned on at t = 0 and it cannot yield dynamics on the initial state. In addition, it is assumed that the initial state is such that the interaction does not generate any (first-order) dynamics in the bath. That is, $[\hat{H}_{int}^{(I)}(t), \hat{\rho}(0)] = 0$. On the other side, set the initial state as the eigenstate of interaction Hamiltonian also legitimate. It also leads to identical relation. Anyway, as a result, the expectation value of interaction Hamiltonian with regarding to the environment at the initial time can be zero. There exists another explanation for it using random phase approximation. We introduce it in the approximation section. Let set it zero from now on.

Finally, simplified equation is obtained by

$$
\frac{d}{dt}\hat{\rho}_s^{(I)}(t) = -\int_0^t dt'\; Tr_{\epsilon}[\hat{H}_{int}^{(I)}(t), [\hat{H}_{int}^{(I)}(t'), \hat{\rho}^{(I)}(t')]]. \tag{3.13}
$$

Eq (3.13) is the so-called **"Quantum master equation"** or **"Redfield equation"** for the quantum state on the system. It describes the general time tendency of the system. You may also check, unlike Liouville-Von Neumann equation, there exists non-unitary property in this equation. This property yields crucial process in decoherence. Anyway, how we can assure non-unitarity occurs decoherence by solving this equation? In general, for the arbitrary time dependence interaction, it is terrible to solve. Nevertheless, it is possible to overcome this situation using several approximations and derive most simplified form. In the next section, let us introduce approximation schemes to reach the solvable equation.

3.3 Approximations

In equation (3.13), the integro-differential equation for $\hat{\rho}_s^{(I)}$ is obtained. But unfortunately, right-hand side of eq (3.13) still depends on $\hat{\rho}^{(I)}$. In addition, time non - locality of this equation yields obstacles to solving this. In order to make this as an analytic equation, we introduce several approximations which are reasonable for usual physical systems.

1. Initial state has been un-correlated

$$\hat{\rho}_{tot}(0) \equiv \hat{\rho}_s(0) \otimes \hat{\rho}_\epsilon(0) \tag{3.14}$$

This condition represents system has been insulated from Environment by designing at t = 0. In general, there are degrees of freedom to pick up this initial state by choosing a system and an environment properly.

2. Born approximation with initially thermalized bath

By assuming coupling between system and the bath is weak, perturbative description on interaction Hamiltonian is legitimate to write. Thus, a quantum state at t from t = 0 becomes

$$\hat{\rho}_{tot}(t) \approx \hat{\rho}_s(t) \otimes \hat{\rho}_\epsilon(t) + \hat{\rho}_{correlation}(t). \tag{3.15}$$

As can be seen above, when interaction is operated on t = 0, there exist freedom to set $\hat{\rho}_{corr}(0) = 0$ by first approximation. After correlation time ($t > \tau_{corr}$), correlated quantum state decays. Again, it goes to zero. When τ_{corr} is short enough with regarding to t (this assumption is also used in the Markov approximation.) $\hat{\rho}_{corr}(t) \approx 0$. Consequently, the total quantum state is approximately

$$\hat{\rho}_{tot}(t) \approx \hat{\rho}_s(t) \otimes \hat{\rho}_\epsilon(t) \tag{3.16}$$

Quantum state of the environment at t is represented as small variation from t = 0

$$\hat{\rho}_\epsilon(t) = \hat{\rho}_\epsilon(0) + \delta\hat{\rho}_\epsilon(t) + O((\delta\hat{\rho}_\epsilon(t))^2) \tag{3.17}$$

On account of the assumption τ_{corr} is short enough, the fluctuation becomes small and random for $t >> \tau_{corr}$. In other words, since the huge environment was initially thermalized, this kind of expansion is quite reasonable. Therefore, the ensemble average of the environment system is nearly zero. Furthermore, it is similar to the initial state of the environment so that

$$\langle\hat{\rho}_\epsilon(t)\rangle \approx \langle\hat{\rho}_\epsilon(0)\rangle \approx \hat{\rho}_\epsilon(0). \tag{3.18}$$

Finally, Born approximation says

$$\hat{\rho}_{tot}(t) \approx \hat{\rho}_s(t) \otimes \hat{\rho}_\epsilon(0) \tag{3.19}$$

It looks like a terrible condition. Nonetheless, it works well in the weak coupling regime between system and environment. Like the most fundamental assumption in the equilibrium statistical mechanics also, the physics of bath imposes "the quantum state of bath do not vary in time"

Eq (3.13) requires knowledge of $\hat{\rho}_s^{(I)}$ at all previous time t' < t. In order to we transform eq (3.13) into a time-local master equation, let us assume this property

$$\hat{H}_{int} = \sum_a \hat{S}_a \otimes \hat{E}_a. \tag{3.20}$$

One can demonstrate that it is always feasible to write interaction Hamiltonian in form of a diagonal decomposition of system and environment operators $\hat{S}_a$, $\hat{E}_a$ respectively,

$$\hat{H}^{(I)}_{int}(t) = \sum_a \hat{S}^{(I)}_a \otimes \hat{E}^{(I)}_a. \tag{3.21}$$

Then, eq (3.13) can be rewritten as

$$-\int_0^t dt' \sum_{a,b} Tr_\epsilon[\hat{S}^{(I)}_a(t) \otimes \hat{E}^{(I)}_a(t), [\hat{S}^{(I)}_b(t') \otimes \hat{E}^{(I)}_b(t'), \hat{\rho}^{(I)}_s \otimes \hat{\rho}_\epsilon(0)]]. \tag{3.22}$$

Now, let us define correlation function of two operators

$$C_{ab}(t,t') \equiv Tr_\epsilon \left\{ \hat{E}^{(I)}_a(t)\hat{E}^{(I)}_b(t')\hat{\rho}_\epsilon \right\}. \tag{3.23}$$

Based on the assumption $\hat{\rho}_\epsilon$ does not vary in time, it is possible to reconstruct it as an interval function ($\tau = t - t'$). Define the correlation function

$$C_{ab}(\tau) \equiv Tr_\epsilon \left\{ \hat{E}^{(I)}_a(\tau)\hat{E}^{(I)}_b(0)\hat{\rho}_\epsilon \right\}. \tag{3.24}$$

From the fact $[\hat{\rho}_\epsilon, e^{-i\hat{H}_s t}] = 0$, eq (3.22) can be rewritten as

$$\begin{aligned}\frac{d}{dt}\hat{\rho}^{(I)}_s(t) = &-\int_0^t d\tau \sum_{a,b} C_{ab}(\tau) \left[\hat{S}^{(I)}_a(t)\hat{S}^{(I)}_b(t-\tau)\hat{\rho}^{(I)}_s(t) - \hat{S}^{(I)}_b(t-\tau)\hat{\rho}^{(I)}_s(t)\hat{S}^{(I)}_a(t)\right] \\ &-\int_0^t d\tau \sum_{a,b} C_{ba}(-\tau) \left[\hat{\rho}^{(I)}_s(t)\hat{S}^{(I)}_b(t-\tau)\hat{S}^{(I)}_a(t) - \hat{S}^{(I)}_a(t)\hat{\rho}^{(I)}_s(t)\hat{S}^{(I)}_b(t-\tau)\right].\end{aligned} \tag{3.25}$$

3. Markov approximation

Let us consider the density of state variation in dt (dt $\ll 1$) from t. In general,

$\hat{\rho}(t+dt)$ depends on not only $\hat{\rho}(t)$ but every states $\hat{\rho}(t')$ where every $t' < t$. In Markov approximation, we assume that correlation time(τ_{corr}) which determines how long time effects on $\hat{\rho}(t+dt)$ is very short compared with system characteristic time(τ_s). This time is related to intrinsic evolution time or damping of system's quantum state. Our coarse-graining time scale (integral time scale t we want to observe) is also much larger than τ_{corr} but much smaller than τ_s (this assumption is related to Born approximation).

In other words, the memory of bath is "forget" very quickly with regarding to system's characteristic time(τ_s). For instance, scattering with many particles in the environment leads memory to lose faster. It also means the reservoir is at a sufficiently high temperature. Namely, we are able to use this assumption as $t >> \tau_{corr}$.

Furthermore, correlation function which is defined in eq (3.24) is sharply peak (such as delta function) at t = t'. This condition imposes the possibility that t in integral range can be expanded to ∞. As a result, time non-local equation such as

$$\begin{aligned}\dot{\hat{\rho}}_s^{(I)}(t) = &-\int_0^\infty d\tau \sum_{a,b} C_{ab}(\tau)\left[\hat{S}_a^{(I)}(t)\hat{S}_b^{(I)}(t-\tau)\hat{\rho}_s^{(I)}(t) - \hat{S}_b^{(I)}(t-\tau)\hat{\rho}_s^{(I)}(t)\hat{S}_a^{(I)}(t)\right] \\ &-\int_0^\infty d\tau \sum_{a,b} C_{ba}(-\tau)\left[\hat{\rho}_s^{(I)}(t)\hat{S}_b^{(I)}(t-\tau)\hat{S}_a^{(I)}(t) - \hat{S}_a^{(I)}(t)\hat{\rho}_s^{(I)}(t)\hat{S}_b^{(I)}(t-\tau)\right]\end{aligned} \tag{3.26}$$

is obtained. Where $\dot{\hat{\rho}}_s^{(I)}(t) = \frac{d}{dt}\hat{\rho}_s^{(I)}(t)$.

4. Random phase approximation

Let us see more details why $Tr_\epsilon[\hat{H}_{int}^{(I)}(t), \hat{\rho}(0)]$ vanishes. Starting from assumption 1 and 3,

$$Tr_\epsilon\left[\sum_a \hat{S}_a\otimes\hat{E}_a, \hat{\rho}_s(0)\otimes\hat{\rho}_\epsilon(0)\right] = C_a * [\hat{S}_a, \hat{\rho}_s(0)]. \tag{3.27}$$

Where $C_a = Tr_\epsilon[\hat{E}_a\hat{\rho}_\epsilon(0)]$. It implies the average of environment part of interaction acts on the initial state. Generally, $Tr_\epsilon[\hat{E}_a\hat{\rho}_\epsilon(0)]$ is not zero. Whereas in the interaction picture,

$$Tr_\epsilon[\hat{E}_a^{(I)}(t)\hat{\rho}_\epsilon(0)] = Tr_\epsilon[e^{i\hat{H}_\epsilon t}\hat{E}_a e^{-i\hat{H}_\epsilon t}\hat{\rho}_\epsilon(0)]. \tag{3.28}$$

Integrate this state in t and if the integral region is large enough with regarding to correlation time, we are able to treat it as a summation of random phases sufficiently. Therefore, initial state part of the solution becomes approximately zero. In other words,

$$\int_0^t dt' Tr_\epsilon[e^{i\hat{H}_\epsilon t'}\hat{E}_a e^{-i\hat{H}_\epsilon t'}\hat{\rho}_\epsilon(0)] \approx 0 \tag{3.29}$$

and $Tr_\epsilon[\hat{H}_{int}^{(I)}(t), \hat{\rho}(0)]$ can be neglected.

5. Secular approximation

Since closed system imposes energy conservation, the probability to occur transition in the bath becomes maximize and dominant when the frequency is near at natural frequency related to the bath's energy spacing [6]. Namely, it discards fast oscillating term. First, supposing the spectrum of $\hat{H}_s$ to be discrete. This is achieved as defining the operator

$$\hat{S}_a(\omega) \equiv \sum_{\varepsilon'-\varepsilon\approx\omega} \Pi(\varepsilon)\hat{S}_a\Pi(\varepsilon'). \tag{3.30}$$

Where $\Pi(\varepsilon)$ denotes projection onto the eigenspace belonging to the eigenvalue ε of $\hat{H}_s$. As summing over all energy eigenvalues with a fixed energy difference of ω, the associated operator in the interaction picture is

$$\begin{aligned} e^{i\hat{H}_s t}\hat{S}_a(\omega)e^{-i\hat{H}_s t} &= e^{-i\omega t}\hat{S}_a(\omega), \\ e^{i\hat{H}_s t}\hat{S}_a^\dagger(\omega)e^{-i\hat{H}_s t} &= e^{i\omega t}\hat{S}_a^\dagger(\omega). \end{aligned} \tag{3.31}$$

In addition,

$$\begin{aligned} &\hat{S}_a^\dagger(\omega) = \hat{S}_a(-\omega), \\ &\sum_\omega \hat{S}_a(\omega) = \sum_\omega \hat{S}_a^\dagger(\omega) = \hat{S}_a. \end{aligned} \tag{3.32}$$

In this analogy, interaction Hamiltonian can be expanded as

$$\begin{aligned} \hat{H}_{int} &= \sum_{a,\omega} \hat{S}_a(\omega) \otimes \hat{E}_a = \sum_{a,\omega} \hat{S}_a^\dagger(\omega) \otimes \hat{E}_a^\dagger, \\ \hat{H}_{int}^{(I)}(t) &= \sum_{a,\omega} e^{-i\omega t}\hat{S}_a(\omega) \otimes \hat{E}_a^{(I)}(t) = \sum_{a,\omega} e^{i\omega t}\hat{S}_a^\dagger(\omega) \otimes \hat{E}_a^{\dagger(I)}(t). \end{aligned} \tag{3.33}$$

Using two identical expressions and after some algebra, (3.26) is thus

$$\frac{d}{dt}\hat{\rho}_s^{(I)}(t) = \sum_{\omega,\omega'}\sum_{a,b} e^{i(\omega'-\omega)t}\Gamma_{ab}(\omega)\left(\hat{S}_b(\omega)\hat{\rho}_s^{(I)}(t)\hat{S}_a^\dagger(\omega') - \hat{S}_a^\dagger(\omega')\hat{S}_b(\omega)\hat{\rho}_s^{(I)}(t) + h.c\right) \tag{3.34}$$

where

$$\Gamma_{ab}(\omega) \equiv \int_0^\infty d\tau \exp(i\omega\tau)C_{ab}(\tau). \tag{3.35}$$

We are also able to decompose Γ since it does not guarantee real.

$$\Gamma_{ab}(\omega) = \frac{1}{2}\gamma_{ab}(\omega) + i\varsigma_{ab}(\omega) \tag{3.36}$$

and the real part can be represented as

$$\gamma_{ab} = \Gamma_{ab}(\omega) + \Gamma_{ba}^*(\omega) = \int_{-\infty}^\infty d\tau \exp(i\omega\tau)C_{ab}(\tau) \equiv \tilde{C}_{ab}. \tag{3.37}$$

By applying Markov approximation, $\omega' \neq \omega$ can be neglected. That is $\tau_s \sim |\omega' - \omega|^{-1} \ll \tau_R$. Where τ_R is the relaxation time of the system. Finally, eq (3.34) becomes

$$\frac{d}{dt}\hat{\rho}_s^{(I)}(t) = -i[\hat{H}_{ls}, \hat{\rho}_s(t)] + \sum_{\omega}\sum_{a,b}\gamma_{ab}(\omega)\left(\hat{S}_b(\omega)\hat{\rho}_s^{(I)}(t)\hat{S}_a^\dagger(\omega) - \frac{1}{2}\{\hat{S}_a^\dagger(\omega)\hat{S}_b(\omega), \hat{\rho}_s^{(I)}(t)\}\right). \tag{3.38}$$

Where $\{A, B\} = AB + BA$ and $\hat{H}_{ls}$ is Lamb-shift Hamiltonian,

$$\hat{H}_{ls} = \sum_{\omega}\sum_{a,b}\varsigma_{ab}(\omega)\hat{S}_a^\dagger(\omega)\hat{S}_b(\omega). \tag{3.39}$$

Lamb-shift Hamiltonian and unperturbed system Hamiltonian are commute. That is,

$$[\hat{H}_s, \hat{S}_a^\dagger(\omega)\hat{S}_b(\omega)] = 0. \tag{3.40}$$

It is easily derived from (3.31)

3.4 Lindblad equation

In order to solve the original equation for a quantum state function of time, we have to return it to the Schrödinger picture. Inverse transformation is therefore

$$\hat{\rho}_s^{(I)}(t) = e^{i\hat{H}_s t}\hat{\rho}_s(t)e^{-i\hat{H}_s t}. \tag{3.41}$$

Differentiation on the quantum state of system in Schrödinger picture becomes

$$\frac{d}{dt}\hat{\rho}_s(t) = -i\ [\hat{H}_s, \hat{\rho}_s(t)] + e^{-i\hat{H}_s t}\left(\frac{d}{dt}\hat{\rho}_s^{(I)}(t)\right)e^{i\hat{H}_s t}. \tag{3.42}$$

Insert eq (3.26) into (3.42) with some algebra using

$$\begin{aligned} &e^{-i\hat{H}_s t}\hat{S}_a^{(I)}(t)\hat{S}_b^{(I)}(t-\tau)\hat{\rho}_s^{(I)}(t)e^{i\hat{H}_s t} \\ &= e^{-i\hat{H}_s t}e^{i\hat{H}_s t}\hat{S}_a e^{-i\hat{H}_s t}e^{i\hat{H}_s(t-\tau)}\hat{S}_b e^{-i\hat{H}_s(t-\tau)}e^{i\hat{H}_s t}\hat{\rho}_s(t)e^{-i\hat{H}_s t}e^{i\hat{H}_s t} \\ &= \hat{S}_a e^{-i\hat{H}\tau}\hat{S}_b e^{i\hat{H}\tau}\hat{\rho}_s(t) = \hat{S}_a\hat{S}_b(-\tau)\hat{\rho}_s(t) \end{aligned} \tag{3.43}$$

and apply secular approximation yields

$$\frac{d}{dt}\hat{\rho}_s(t) = -i\ [\hat{H}_s + \hat{H}_{ls}, \hat{\rho}_s(t)] + \sum_{a,b}\gamma_{ab}\left(\hat{S}_b\hat{\rho}_s(t)\hat{S}_a^\dagger - \frac{1}{2}\{\hat{S}_a^\dagger\hat{S}_b, \hat{\rho}_s(t)\}\right). \tag{3.44}$$

In particular, if correlation function is nearly sharp as mentioned above $\tilde{C}_{ab} \simeq \gamma_{ab}\delta_{ab}$ with assuming no Lamb-shift effect , $\varsigma_{ab} = 0$, (3.44) becomes

$$\frac{d}{dt}\hat{\rho}_s(t) = -i\ [\hat{H}_s, \hat{\rho}_s(t)] + \sum_a \gamma_a \left(\hat{S}_a\hat{\rho}_s(t)\hat{S}_a^\dagger - \frac{1}{2}\{\hat{S}_a^\dagger\hat{S}_a, \hat{\rho}_s(t)\} \right). \tag{3.45}$$

The representative equation (3.45) is **Lindblad equation** which is commonly used to solve Quantum optics problem also Open quantum system. In addition, $\hat{S}_a$ is called as Lindblad or quantum jump operator. Notice that there is also a probability conservation even if there exists non - unitary process. Because

$$\begin{aligned} &Tr[\hat{S}^\dagger\hat{S}\hat{\rho}] = Tr[\hat{S}\hat{\rho}\hat{S}^\dagger] = Tr[\hat{\rho}\hat{S}^\dagger\hat{S}], \\ &Tr\left[\frac{1}{2}\sum_a[\hat{S}_a, [\hat{S}_a, \hat{\rho}_s(t)]]\right] = 0. \end{aligned} \tag{3.46}$$

In the following section, we deal with spontaneous emission where it is useful to utilize Lindblad formula as a typical example of this equation.

3.5 Example 1. Spontaneous emission

Let suppose a two - level atom in a bosonic bath (Classically, “Dipole coupling” in electrodynamics). For instance, a photon or a phonon can exist in a bosonic bath. Suggesting Hamiltonian is such that

$$\begin{aligned} &\hat{H}_\epsilon = \sum_k \omega_k \hat{b}_k^\dagger\hat{b}_k \ \&\ \hat{H}_s = \frac{\omega_s\sigma_z}{2}, \\ &\hat{H}_{int} = \sum_k (g_k\hat{b}_k + g_k\hat{b}_k^\dagger)(\sigma_+ + \sigma_-). \end{aligned} \tag{3.47}$$

Notice that each term of interaction Hamiltonian should not necessarily Hermitian [39]. For the bosonic bath, $\hat{b}$ satisfies following commutation relation,

$$\left[\hat{b}_k, \hat{b}_l^\dagger\right] = \delta_{kl}. \tag{3.48}$$

To apply rotating wave approximation, let us change our interaction Hamiltonian into the interaction picture such as

$$\hat{H}_{int}^{(I)} = \sum_k (g_k \hat{b}_k e^{-i\omega_k t} + g_k \hat{b}_k^\dagger e^{+i\omega_k t})(\sigma_+ e^{+i\omega_s t} + \sigma_- e^{-i\omega_s t}). \tag{3.49}$$

Apply rotating wave approximation, namely, we neglect fast oscillating terms which are proportional to $e^{\omega_k + \omega_s}$ $(\omega_k + \omega_s >> \omega_k - \omega_s)$. Resulting simplified interaction equation can be obtained as

$$\hat{H}_{int}^{(I)}(t) = \sum_k \left(g_k \hat{b}_k \sigma_+ e^{-i(\omega_k - \omega_s)t} + g_k \hat{b}_k^\dagger \sigma_- e^{i(\omega_k - \omega_s)t} \right). \tag{3.50}$$

This interaction equation describes the situation that atom initially at the excited state goes to the ground state (σ_-), while its surrounding bath gains one boson particle $(\hat{b}^\dagger)$. Oppositely, given atom at the ground state absorbs a boson particle from the bath$(\hat{b})$, the atom goes to the excited state(σ_+). In the physical sense, this approximated Hamiltonian describes such processes, spontaneous emission and absorption. From now on, let us see what happen through this Hamiltonian. Suppose our initial state of the bath is a vacuum. Namely,

$$\hat{\rho}_{s\epsilon}(0) = \hat{\rho}(0) \otimes |0\rangle\langle 0|. \tag{3.51}$$

Now, recall the equation (3.13)

$$\frac{d}{dt}\hat{\rho}_s^{(I)}(t) = -\int_0^t dt' \; Tr_\epsilon[\hat{H}_{int}^{(I)}(t), [\hat{H}_{int}^{(I)}(t'), \hat{\rho}^{(I)}(t')]]. \tag{3.52}$$

From the fact that $\langle 0|\hat{b}^\dagger \; = \; \hat{b}|0\rangle = 0$, resulted expression after little algebra becomes

$$\frac{d}{dt}\hat{\rho}_s^{(I)}(t) = \sum_{k,l} \left(g_k g_l e^{-i(\omega_k - \omega_s)t} e^{i(\omega_k - \omega_s)t'} \sigma_+ \sigma_- \hat{\rho}_s(t') \right) \langle 0|\hat{b}_k \hat{b}_l^\dagger|0\rangle. \tag{3.53}$$

$\langle 0|\hat{b}_k \hat{b}_l^\dagger|0\rangle = \delta_{kl}$ also leaves

$$\frac{d}{dt}\hat{\rho}_s^{(I)}(t) = \sum_k g_k^2 e^{i(\omega_k - \omega_s)(t' - t)} \sigma_+ \sigma_- \hat{\rho}_s(t'). \tag{3.54}$$

Finally, we are left with

$$\frac{d}{dt}\hat{\rho}_s^{(I)}(t) = -\int_0^t dt' \left\{\Gamma(t-t')\left[\sigma_+\sigma_-\hat{\rho}_s(t') - \sigma_-\hat{\rho}_s(t')\sigma_+\right] \ + \ H.C.\right\} \tag{3.55}$$

where

$$\Gamma(\tau) = \sum_k g_k^2 e^{-i(\omega_k-\omega_s)\tau}. \tag{3.56}$$

Let us apply Markov approximation on here (Coupling coefficients is infinitesimal). Replace summation into integral in the nearly continuous frequency space such as

$$\Gamma(\tau) = \int_0^\infty d\omega_k d(\omega_k) g^2(\omega_k) e^{-i(\omega_k-\omega_s)\tau}. \tag{3.57}$$

Where $d(\omega)$ is the density of state in $\omega \sim \omega + d\omega$. In addition, assume $d(\omega)$ smooth (but nearly delta function) and $d(\omega)g^2(\omega)$ is finite with slowly varying over $\omega \simeq \omega_s$. Anyway, what happens in $\Gamma(\tau)$ after setting properties of Γ? Let us define response function $\chi(\omega)$,

$$\Theta(\omega)d(\omega)g^2(\omega) \equiv \chi(\omega). \tag{3.58}$$

Step function $\Theta(\omega)$ comes from the fact frequency must be positive. Furthermore, by assumption, it's Fourier transformation becomes nearly delta function. Then,

$$\Gamma(\tau) = \Gamma_0 \int_{-\infty}^{\infty} d\omega_k e^{-i(\omega_k-\omega_s)\tau}\chi(\omega_k) = \Gamma_0\hat{\chi}(\tau)e^{-i\omega_s\tau} \simeq \Gamma_0\delta(\tau)e^{-i\omega_s\tau}. \tag{3.59}$$

Look back at our equation (3.55) of $\hat{\rho}_s(t)$ and set $\Gamma_0 = \frac{\gamma}{2} + i\Delta\omega_s$. γ becomes the variable of decoherence time factor. Hence, $\Delta\omega_s$ represents frequency shift and it simply yields faster unitary evolution. Then, eq (3.55) becomes

$$\begin{aligned}
&\frac{d}{dt}\hat{\rho}_s^{(I)}(t) = -i\left(\frac{\Delta\omega_s}{2}\right)[\sigma_+\sigma_-, \hat{\rho}_s^{(I)}(t)] + \gamma D[\sigma_-](\hat{\rho}_s^{(I)}(t)), \\
&D[\sigma_-](\hat{\rho}) = \sigma_-\hat{\rho}\sigma_+ - \frac{1}{2}(\sigma_+\sigma_-\hat{\rho} + \hat{\rho}\sigma_+\sigma_-).
\end{aligned} \tag{3.60}$$

Notice, the total trace of this part remains same as a positive value from the fact $Tr(D[\sigma](\hat{\rho})) = 0$. Each term of eq (3.60) (Master equation) leads to different physical situations. The first part of right-hand side is the unitary part. It comes from the imaginary part of Γ_0 and also drives fast unitary evolution. Last part, non - unitary part, describes the decoherence effect of the atom. For more details, let us solve this equation and see what happen in this quantum state.

For simplicity, set $\Delta\omega_s = 0$ and represent $\hat{\rho}_s^{(I)}$ as a 2×2 matrix because sigma Pauli matrix is a member of SU(2) group. By setting the trace of quantum state as 1 and off-diagonal term as randomly, eq (3.60) in matrix form becomes

$$\frac{d}{dt}\begin{pmatrix} a & x \\ \bar{x} & 1-a \end{pmatrix} = \gamma \begin{pmatrix} 1-a & -\frac{1}{2}x \\ -\frac{1}{2}\bar{x} & a-1 \end{pmatrix}. \tag{3.61}$$

Comparing each component in matrix yields coupled equations such as

$$\frac{da}{dt} = \gamma(1-a)\ ,\ \frac{dx}{dt} = -\frac{\gamma}{2}x\ ,\ \frac{d\bar{x}}{dt} = -\frac{\gamma}{2}\bar{x}. \tag{3.62}$$

Therefore, $\hat{\rho}_s(t)$ becomes

$$\hat{\rho}_s^{(I)}(t) = \begin{bmatrix} 1 - e^{-\gamma t}(1-a(0)) & \bar{x}(0)e^{-\frac{\gamma}{2}t} \\ x(0)e^{-\frac{\gamma}{2}t} & e^{-\gamma t}(1-a(0)) \end{bmatrix}. \tag{3.63}$$

As t goes to infinity, the final state of $\hat{\rho}_s^{(I)}$ only remains the ground state. In addition, it is equivalent to the Schrödinger picture also.

$$\lim_{t\to\infty} \hat{\rho}_s^{(I)}(t) = |0\rangle\langle 0| = e^{-i\frac{\omega_s}{2}t}|0\rangle\langle 0|e^{+i\frac{\omega_s}{2}t} = e^{-i\frac{\omega_s\sigma_z}{2}t}|0\rangle\langle 0|e^{+i\frac{\omega_s\sigma_z}{2}t} = \hat{\rho}_s(\infty) \tag{3.64}$$

Finally, we can conclude this decoherence effect leads to relaxation (Decay of diagonal component) with dephasing (Decay of off-diagonal component). As can be seen in the chap 2, all atoms in the excited state fall into the ground state due to the interaction with a bosonic bath. In the interaction and Schrödinger picture yield congruent results because the final state goes to the eigenstate of system Hamiltonian.

By the way, for this Hamiltonian only contains a quadratic order, we are able to solve it exactly without using approximations. Unruh [40] and Palma *et al* [41] also solved it exactly. Summarizing their results [6], we are able to describe the decoherence function which determines the decoherence time scale. Potion of off-diagonal terms decay proportional to $\exp(\Gamma(t))$ where $\Gamma(t)$ is

$$\Gamma(t) = \Gamma_{vac}(t) + \Gamma_{th}(t) = -\frac{1}{2}\ln(1 + (\Omega t)^2) - \ln\left[\frac{\sinh(t/\tau_{ther})}{t/\tau_{ther}}\right]. \tag{3.65}$$

Where $\tau_{ther} = \frac{1}{\pi T}$ and Ω is the cutoff frequency of spectral density (Related to characteristic time scale). Furthermore, we are able to classify the time scale by dividing decoherence function into three regions with regarding to t,Ω,τ_{ther}.

(1) Short time region $t \ll (\Omega)^{-1}$,

$$\Gamma(t) \approx -\frac{1}{2}(\Omega t)^2. \tag{3.66}$$

Decoherence function follows non-Markovian property because there is no enough time for relaxation in the view of the bath. In addition, the system is hard to obtain an effect from the bath.

(2) Intermediate region $(\Omega)^{-1} \ll t \ll \tau_{ther}$,

$$\Gamma(t) \approx -\ln \Omega t. \tag{3.67}$$

In this region, decoherence effects are mainly due to the vacuum fluctuation of the field.

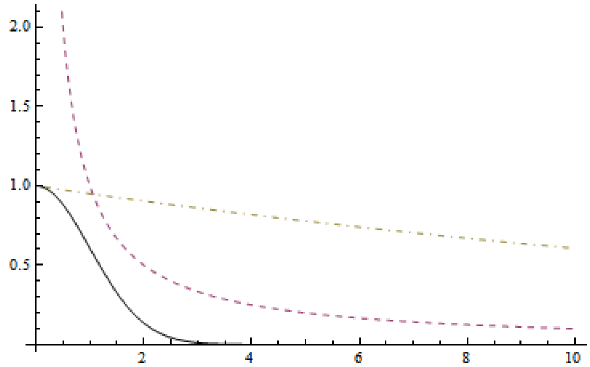

Figure 3.1: Potion of off-diagonal term. Short (Solid), Intermediate (Dashed), Long (Dot-Dashed) [6]

(3) Long time region $\tau_{ther} \ll t$,

$$\Gamma(t) \approx -\frac{t}{\tau_{ther}}. \tag{3.68}$$

Long time means the bath is nearly at the thermal equilibrium state. Namely, time flows enough to stable in the view of bath and system both. As can be seen, this resulted equation is equivalent to the result using the Markovian approximation. You may refer the decoherence rate from the following picture 3.1.

3.6 Example 2. Decoherence of Schrödinger cat state

As the modern topic in Quantum optics, let us consider how Schrödinger cat state in cavity undergoes decoherence. Schrödinger cat state is defined by the linearly equal superposition of two coherent states. Coherent state in number state expansion yields

$$|\alpha\rangle = \exp(-|\alpha|^2/2) \sum_{n=0}^{\infty} \frac{(\alpha)^n}{n!} |n\rangle\langle n| \tag{3.69}$$

It is easy to comprehend the fact $|\alpha|^2 = \bar{n}$ by simple algebra. Now, Schrödinger cat state in terms of coherent state is therefore

$$|\Psi\rangle = \frac{1}{\sqrt{2[1 \pm \exp(-|\alpha|^2/2)]}}(|\alpha\rangle \pm |-\alpha\rangle). \tag{3.70}$$

This state indicates superposition between two distinct quasi positions with same average photon number. In addition, the normalization factor seems to be different with ordinary equal superposition state due to $\langle\alpha|{-\alpha}\rangle = \exp(-|\alpha|^2/2)$. Superposed with relatively same sign is 'even cat', otherwise, 'odd cat'.

To control this macroscopic superposition state, it is necessary to be confined to the cavity. However, cat state interacting with cavity undergoes decoherence by surrounding. By assuming linear coupling between cavity and cat, it is legitimate to describe decoherence quantitatively using Lindblad equation.

Rewind (3.45), let us introduce Lindblad operations which are associated with creation and annihilation operator of a cavity. With its system Hamiltonian,

$$\hat{H}_s = \omega_s \hat{a}^\dagger \hat{a}, \qquad \hat{L}_- \equiv \sqrt{\kappa_-}\hat{a}, \qquad \hat{L}_+ \equiv \sqrt{\kappa_+}\hat{a}^\dagger. \tag{3.71}$$

Where κ_-, κ_+ are related to the thermodynamical argument. Assuming that cavity (environment) is in the thermal equilibrium state and energy level spacing between eigenstates of atoms are equal to ω_s. Therefore, we can formulate the relation between rates as $\kappa_+ = \kappa_- e^{-\omega_s/T}$. Since the distribution of photons in canonical ensemble is given by Planck distribution, the average number n of thermal photons per mode at frequency ω_s becomes

$$n = \frac{1}{e^{\omega_s/T} - 1}, \qquad \frac{\kappa_-}{\kappa_+} = \frac{1+n}{n}. \tag{3.72}$$

It leads us to define the unique cavity rate κ, each rate requires

$$\kappa_- = \kappa(1+n), \qquad \kappa_+ = \kappa(n). \tag{3.73}$$

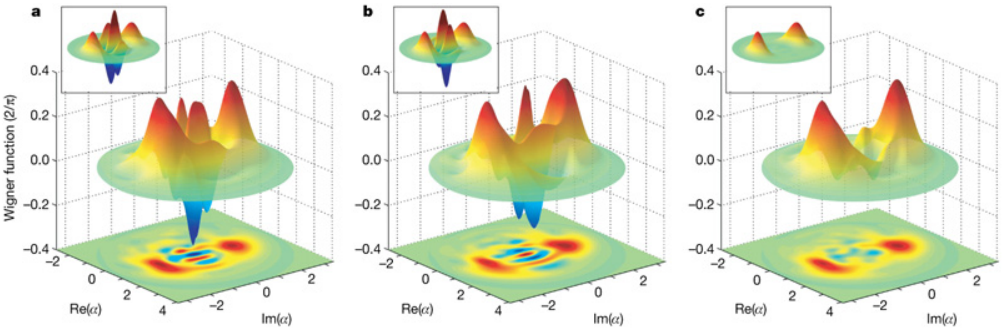

Figure 3.2: Wigner functions of Schrödinger cat : Negativity decreases as t larger [7].

Finally, by substituting (3.71) into (3.45), in terms of $\hat{L}_{-}$, $\hat{L}_{+}$ respectively,

$$
\begin{aligned}
\frac{d}{dt}\hat{\rho}_s(t) = -i\omega_s[\hat{a}^\dagger\hat{a}, \hat{\rho}_s(t)] - \frac{\kappa(1+n)}{2}\left[\hat{a}^\dagger\hat{a}\hat{\rho}_s(t) + \hat{\rho}_s(t)\hat{a}^\dagger\hat{a} - 2\hat{a}\hat{\rho}_s(t)\hat{a}^\dagger\right] - \\
\frac{\kappa n}{2}\left[\hat{a}\hat{a}^\dagger\hat{\rho}_s(t) + \hat{\rho}_s(t)\hat{a}\hat{a}^\dagger - 2\hat{a}^\dagger\hat{\rho}_s(t)\hat{a}\right].
\end{aligned}
\tag{3.74}
$$

For T = 0 (Vacuum), master equation in the interaction picture becomes

$$
\frac{d}{dt}\hat{\rho}_s(t) = -\frac{\kappa}{2}\left[\hat{a}^\dagger\hat{a}\hat{\rho}_s(t) + \hat{\rho}_s(t)\hat{a}^\dagger\hat{a} - 2\hat{a}\hat{\rho}_s(t)\hat{a}^\dagger\right]. \tag{3.75}
$$

Solution for $\hat{\rho}_s(t) = |\Psi(t)\rangle\langle\Psi(t)|$ is

$$
\begin{aligned}
\rho(t) =& \frac{1}{2}(|\alpha(t)\rangle\langle\alpha(t)| + |-\alpha(t)\rangle\langle-\alpha(t)| \\
& \pm \exp(-2\bar{n}[1-e^{-\kappa t}])(|\alpha(t)\rangle\langle-\alpha(t)| + \langle|-\alpha(t)\rangle\langle\alpha(t)|)).
\end{aligned}
\tag{3.76}
$$

It obtains the exactly identical formula for using Bayes law in [42]. For the short time $\kappa t << 1$, off-diagonal terms are decayed proportional to $exp(-2n\kappa t)$. In the fast decay limit $\kappa t >> 1$, decay rate depends on the average number of the photon in thermal equilibrium state.

Chapter 4

Decoherence due to time dilation

According to the papers of Zych *et al* [1, 23, 26, 29, 43], **proper time** can be described as a quantum state in Hilbert space of interaction Hamiltonian between centre of mass and internal particles. Because proper time depends on the position with respect to ground originating from the **time dilation effect** by a gravitational field. Thus, it is possible to claim universal decoherence occurs due to time dilation by gravity. Their results are significant for several reasons: 1) even if there exists constant gravitational field only, there is proper time differences in each internal particle; 2) it explains localization of the position toward centre of mass (c.o.m) under the interaction between a c.o.m Hamiltonian and an internal one. In this chapter, we reproduce the situation of a constant acceleration using Rindler spacetime based on the **equivalence principle**. By redefining the proper time state in this spacetime using the internal Hamiltonian, we check the glimpse of decoherence from the variation on visibility. In addition, obtaining the Hamiltonian, we analyze the master equation in the view of c.o.m and discuss how decoherence on c.o.m occurs in a constant acceleration field. It is elucidated that it gives the equivalent result with the original one.

4.1 Hamiltonian approach

Let us start from the metric with affine parameter τ.

$$ds = \sqrt{-g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}d\tau \tag{4.1}$$

where $\dot{x}^{\mu} = \frac{dx^{\mu}}{d\tau}$. With Rindler metric, which describes a constant acceleration in the flat spacetime, interval becomes [44, 45]

$$ds^2 = -(1+\mathbf{a}X)^2 dT^2 + dX^2 + dY^2 + dZ^2. \tag{4.2}$$

Where our new coordinate (T,X,Y,Z) in terms of Minkowski coordinate (t,x,y,z) is obtained by using this non-linear transformation

$$t = (1+\mathbf{a}X)\sinh T, \qquad x = (1+\mathbf{a}X)\cosh T, \qquad y = Y, \qquad z = Z. \tag{4.3}$$

Given $\mathbf{a}$ denotes the proper acceleration in Rindler spacetime. Then, action of a particle which follows the geodetic line is

$$S = -m\int dT\sqrt{(1+\mathbf{a}X)^2 - \dot{X}^2} = \int L(X,\dot{X})dT. \tag{4.4}$$

Following momentum and Hamiltonian are driven from Euler-Lagrangian equation

$$\frac{d}{dT}\left(\frac{m\dot{X}}{\sqrt{(1+\mathbf{a}X)^2 - \dot{X}^2}}\right) - \frac{m\mathbf{a}(1+\mathbf{a}X)}{\sqrt{(1+\mathbf{a}X)^2 - \dot{X}^2}} = 0, \tag{4.5}$$

$$\begin{aligned} P &= \frac{\partial L}{\partial \dot{X}} = \frac{m\dot{X}}{\sqrt{(1+\mathbf{a}X)^2 - \dot{X}^2}}, \\ H &= P\dot{X} - L = \frac{m(1+\mathbf{a}X)^2}{\sqrt{(1+\mathbf{a}X)^2 - \dot{X}^2}}. \end{aligned} \tag{4.6}$$

Hamiltonian in terms of momentum is obtained with help of this relation

$$-g_{00}(P^0)^2 + g_{ij}P^iP^j = -m^2. \tag{4.7}$$

Here, we applied the diagonalized metric tensor such that $g_{\mu\nu} = diag(-(1+\mathbf{a}X)^2, 1, 1, 1)$. Following little algebra, Hamiltonian becomes

$$-g_{00}P^0 = H(P,X) = \sqrt{-g_{00}(m^2+\vec{P}^2)} \equiv \sqrt{-g_{00}(H_{rest}^2+\vec{P}^2)}. \tag{4.8}$$

When we consider internal dynamics also, it simply yields shift of rest mass. Consequently, rest Hamiltonian can be written in

$$H_{rest} = m + H_{int} \tag{4.9}$$

where the dynamical part for the internal energy is described by H_{int}. From now on, let us apply first quantization $[\hat{x},\hat{p}] = i$. And provided that mgX, $\frac{P^2}{2m} \ll m$, we can expand Hamiltonian using Taylor expansion of the square root function,

$$\hat{H} = m\sqrt{(1+\mathbf{a}\hat{X})\left(\left(\hat{I}+\frac{\hat{H}_{int}}{m}\right)^2 + \left(\frac{\hat{P}}{m}\right)^2\right)} \approx \hat{H}_{cm} + \left(1+\mathbf{a}\hat{X}-\frac{\hat{P}^2}{2m^2}\right)\hat{H}_{int}. \tag{4.10}$$

Notice that $m\mathbf{a}X$, $\frac{P^2}{2m}$ are same energy order. Following $\hat{H}_{cm}$ up to the order of energy square times mass is therefore

$$\begin{aligned}\hat{H}_{cm} \approx{}& m\hat{I} + \frac{\hat{P}^2}{2m} + m\hat{X} - \frac{\hat{P}^4}{8m^3} + \frac{m\hat{X}^2}{2} \\ &+ \frac{3\mathbf{a}}{2m}\left(\hat{X}\hat{P}^2 + \hat{P}\hat{X}\hat{P} + \frac{1}{2}\hat{P}^2\hat{X}\right).\end{aligned} \tag{4.11}$$

Coupling term between internal Hamiltonian and c.o.m degrees of freedom in equation (4.10) becomes the key to make decoherence by gravity. It would be obvious when we deal with master equation approach.

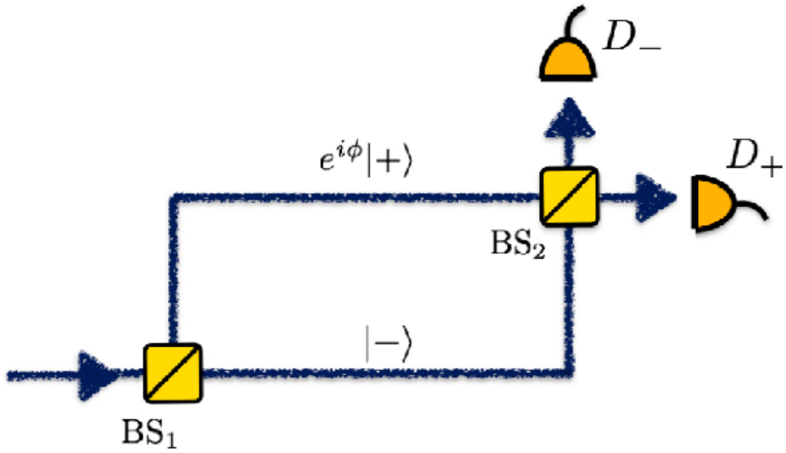

Figure 4.1: Mach-Zehnder interferometry[1]

4.2 Gravity effects in quantum phase

Before we concern the quantum master equation, a notion of interaction should be necessary to searching newly constructed entanglement. By introducing Hilbert space of internal particles by the proper time state, we are able to describe interaction Hamiltonian which yields non-unitary evolution.

4.2.1 Mach-Zehnder interferometry and Hong-Ou-Mandel effect

Let us consider Mach-Zehnder interferometry in a Euclidean spacetime, such as fig 4.1. An incident beam state which lives in $H_{inc} \equiv H_0 \otimes H_1$ where 0 represents the horizontal line with regarding to the photon source, 1 along vertical one. Then, incidence of a single photon state in fig 4.1 can be described as

$$|inc\rangle = |1\rangle_0 \otimes |0\rangle_1 \equiv |10\rangle. \tag{4.12}$$

In addition, beam splitter as a mathematically unitary operator maps $H_{pho} \equiv$

$H_0 \otimes H_1 \mapsto H_0 \otimes H_1$, general beam splitter operator is following: [46]

$$\hat{U}(\theta) = \exp[i\theta(\hat{a}_1^\dagger \hat{a}_0 + \hat{a}_0^\dagger \hat{a}_1)]. \tag{4.13}$$

In particular, for 50:50 beam splitter where $\theta = \pi/4$, state after traverse it would be divided such as

$$|mid\rangle = \hat{U}|inc\rangle = \exp[i\pi(\hat{a}_0^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_0)/4]|inc\rangle = \frac{1}{\sqrt{2}}(|10\rangle + i|01\rangle)_{01}. \tag{4.14}$$

Finally, after two states coincidently pass the last splitter, outcome state becomes

$$\begin{aligned} |out\rangle = \hat{U}|mid\rangle &= \exp[i\pi(\hat{a}_1^\dagger \hat{a}_0 + \hat{a}_0^\dagger \hat{a}_1)/4] \\ &= \frac{1}{2}(|10\rangle + i|01\rangle + i|01\rangle - |10\rangle)_{01} = i|01\rangle_{01}. \end{aligned} \tag{4.15}$$

Notice that in the view of the last splitter, modes are exchanged: mode 0 becomes vertical one and vice versa. Given that there is no interruption in this state (e.g. measurement), the photon is detected only in the detector D+. It suggests quite different intuition with classical approach! This kind effect due to the quantum interference is known as **Hong-Ou-Mandel effect** [47].

For general θ, consider phase shift of the vertical one, the middle state can be written as

$$|mid\rangle = a|10\rangle + be^{i\phi}|01\rangle. \tag{4.16}$$

Where $a, b \in \mathbb{R}$ and $a^2 + b^2 = 1$. To quantifying interference effect, we introduce visibility (V) and predictability (P). First of all, detecting probability for each detector D+, D- is after doing little algebra

$$p_\pm = \frac{1}{2} \pm ab\cos\phi. \tag{4.17}$$

In addition, visibility is defined by

$$V \equiv \frac{max(p_\pm) - min(p_\pm)}{max(p_\pm) + min(p_\pm)} = 2ab. \tag{4.18}$$

In the previous case, Hong-Ou-Mandel effect, for $a = b = 1/\sqrt{2}$ and $\phi = 0$, visibility was 1. Consequently, we are able to interpret visibility represents 'How quantum states interfere?'. Namely, it quantifies **quantum interference**.

Predictability is defined by difference between probabilities to exist in each mode,

$$P \equiv |a^2 - b^2|. \tag{4.19}$$

It means when $a = b$, there is no way to predict which path the photon takes. That is, predictability quantifies **which way information**. Furthermore, there exists inequality between visibility and predictability such that

$$V^2 + P^2 \leq 1. \tag{4.20}$$

Equal unity when given state is in a pure state. Less in a mixed state. The proof is simple. For a general mixed state

$$\rho_{mid} = a^2|10\rangle\langle 10| + b^2|01\rangle\langle 01| + ce^{i\phi}|01\rangle\langle 10| + c^*e^{-i\phi}|10\rangle\langle 01| \tag{4.21}$$

where $Tr\rho^2 = 1 - 2((ab)^2 - |c|^4) \leq 1$, predictability seems not be changed. While visibility varies because

$$\begin{aligned}\rho_{out} &= \hat{U}\rho\hat{U}^\dagger \\ &= \frac{1}{2}[(1 - 2\Re(ce^{i\phi}))|10\rangle\langle 10| + (ib^2 - ia^2 + 2\Re(ce^{i\phi}))|10\rangle\langle 01| \\ &+ (ia^2 - ib^2 + 2\Re(ce^{i\phi}))|01\rangle\langle 10| + (1 + 2\Re(ce^{i\phi}))|01\rangle\langle 01|].\end{aligned} \tag{4.22}$$

Therefore, $p_\pm$ becomes

$$\begin{aligned}p_+ &= Tr(|01\rangle\langle 01|\rho_{out}|01\rangle\langle 01|), \\ p_- &= Tr(|10\rangle\langle 10|\rho_{out}|10\rangle\langle 10|), \\ p_\pm &= \frac{1}{2} \pm |c|cos(\phi + \alpha).\end{aligned} \tag{4.23}$$

Where $c = |c|e^{i\alpha}$. Consequently,

$$V^2 + P^2 = a^4 + b^4 - 2a^2b^2 + 4|c|^2 \leq 1. \tag{4.24}$$

Equal unity if and only if $|c| = ab$. In other words, it is maximum for the pure state.

4.2.2 Which-way detector and the proper time state.

Two quantities to explain interference effect has been introduced; visibility and predictability. In this section, we introduce additional internal state as a new degree of freedom by expanding Hilbert space as $H_{tot} = H_{pho} \otimes H_{int}$. Where H_{pho} is a Hilbert space of photon states, H_{int} is newly introduced Hilbert space of internal Hamiltonian. For example, our state with internal state after traverse arbitrary beam splitter becomes

$$\begin{aligned} |inc\rangle &= |10\rangle \otimes |W_0\rangle \rightarrow \\ |mid\rangle &= a|10\rangle \otimes |W_-\rangle + be^{i\phi}|01\rangle \otimes |W_+\rangle. \end{aligned} \tag{4.25}$$

Now, tracing out internal degrees of freedom as counting all possible ways to track before passing through the last beam splitter. Initially, the middle state was the entangled state but it becomes the mixed such as

$$\rho_{mid} = a^2|10\rangle\langle 10| + abe^{-i\phi}\langle W_+|W_-\rangle|10\rangle\langle 01| + abe^{+i\phi}\langle W_-|W_+\rangle|01\rangle\langle 10| + b^2|01\rangle\langle 01|. \tag{4.26}$$

As a result, after traverse last beam splitter, probabilities to be detected in each detector are

$$p_{\pm} = \frac{1}{2} \pm ab|\langle W_+|W_-\rangle|cos(\phi + \delta). \tag{4.27}$$

Where $\langle W_+|W_-\rangle = |\langle W_+|W_-\rangle|e^{i\delta}$. As can be seen, visibility factor would be changed to

$$V = 2ab|\langle W_+|W_-\rangle|. \tag{4.28}$$

That means interference is limited by the overlap of internal states. Even if there is no measurement onto the photon, naturally we can distinguish which way the photon takes. For the specific case, $\langle W_+|W_-\rangle = 0$, interference becomes zero. It implies obtaining the which way information is completely possible. To quantifying this, introduce distinguish-ability (D) which is defined by

$$D \equiv \sqrt{1 - |\langle W_+|W_-\rangle|^2}. \tag{4.29}$$

When two internal states are orthogonal, distinguish-ability becomes maximize and gives the full information on which path the photon takes. From the relation between visibility and predictability, additional construction on the relation between visibility and distinguish-ability gives

$$V^2+D^2 = 4(ab)^2|\langle W_+|W_-\rangle|^2+D^2 = (1-P^2)|\langle W_+|W_-\rangle|^2+D^2 = 1-(1-D^2)P^2 \leq 1. \tag{4.30}$$

Notice that for 50:50 beam splitter, $P = 0$, $V^2 + D^2 = 1$.

Provided that we set Mach-Zehnder interferometry vertically with regarding to the ground, each photon in two distinct paths has different proper time. Expanding our assertion, since the final state of its internal degrees of freedom depends on the which path taken, it allows us to introduce "Quantum clock". To achieve the effect due to proper time, let us set internal state of time as $|W_\pm\rangle = |\tau_\pm\rangle$ by introducing 'the proper time state' which is

$$|\tau_\pm\rangle = \exp\left(-i\int_l d\tau_\pm H_{int}\right)|\tau_0\rangle. \tag{4.31}$$

Proper time state is introduced as a time evolution with regarding to internal Hamiltonian by proper time of each path starting from the initial state. In Rindler spacetime with proper acceleration (fig 4.2), $\tau_\pm = (1+\mathbf{a}X_\pm)T$. Needless to say, X and T are Rindler coordinates.

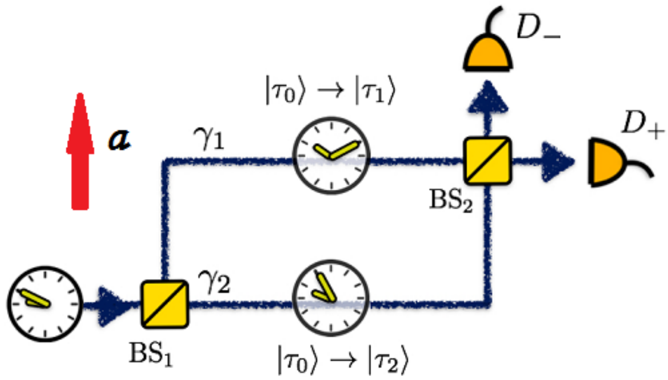

Figure 4.2: Gedankenexperiment in Rindler spacetime with acceleration a [1].

By assuming internal Hamiltonian itself does not depend on path and also does not explicitly depend on time, we can rewrite distinguish-ability in terms of newly defined internal states

$$D = \sqrt{1 - |\langle\tau_0| \exp(-iH_{int}\Delta\tau)|\tau_0\rangle|^2}. \tag{4.32}$$

In addition, visibility is obtained by

$$V = 2ab|\langle\tau_0| \exp(-iH_{int}\Delta\tau)|\tau_0\rangle|. \tag{4.33}$$

When Beam splitter is characterized as 50:50, for small variation $H_{int}\Delta\tau \ll 1$, visibility can be expanded as

$$V = \left|\sum_{n=0}^{\infty}(-i\Delta\tau)^n\langle H_{int}^n\rangle\right|. \tag{4.34}$$

This expression yields intuition that the decoherence effect can fully be described by an internal energy distribution and time dilation factor because visibility depends on this variables. In particular, given expressed terms up to 2nd order of $H_{int}\Delta\tau$,

$$V \approx |1 - i\Delta\tau\langle H_{int}\rangle - \frac{1}{2}(\Delta\tau)^2\langle H_{int}^2\rangle| \approx \sqrt{1 - (\Delta\tau\Delta H_{int})^2} \approx 1 - \frac{1}{2}(\Delta\tau\Delta H_{int})^2. \tag{4.35}$$

When we consider N particles system, visibility of large N particles approximately

$$V = |\langle\tau_0|\exp(-iH_{int}\Delta\tau)|\tau_0\rangle|^N \approx (1 - \frac{1}{2}(\Delta\tau\Delta H_{int})^2)^N \approx \exp(-t/\tau_{dec})^2 \tag{4.36}$$

Where $\tau_{dec} = \sqrt{\frac{2}{N}}\frac{1}{\Delta H_{int}}$. As can be seen, visibility decays as a Gaussian form for small time variation. Despite the fact that we must notice that this decay is **not irreversible**. Because original function form was periodic of the proper time difference. As it becomes larger, interference comes back to original at $\Delta\tau = \pi/H_{int}$. A cautionary remark is that there exists a revival of coherence (not decoherence process exactly) for the few particles. Otherwise, for huge particles we are able to obtain exact process on decoherence without revival. It completely originates from the **lack of knowledge** onto internal particles. We make gradients on the analysis of decoherence due to proper time difference of numerous internal particles in the next section.

4.3 Prepared state in proposal

From now on, to specify system and its internal energy in this analysis, let us assume macroscopic object and its approximately numerous composing particles do not interact with each other. Then, it is reasonable to consider internal particles with nearly infinite degrees of freedom as a boson field. Resulting energy of field becomes the eigenvalue of internal Hamiltonian. From the action of Klein-Gordon with corresponding massive real scalar field $\phi(x)$

$$S = -\frac{1}{2}\int d^4x\sqrt{-g}\left[g^{\mu\nu}(\partial_\mu\phi(x))(\partial_\nu\phi(x)) + m^2\phi^2(x)\right], \tag{4.37}$$

with the low energy approximation associate with Minkowski spacetime, it becomes

$$S = -\frac{1}{2}\int d^4x\sqrt{-g_{00}}\left[g^{00}(\partial_t\phi(x))^2 + (\nabla\phi(x))^2 + m^2\phi^2(x)\right]. \tag{4.38}$$

Following Hamiltonian using conserved momentum which is driven from the time killing vector is obtained by

$$H = \frac{1}{2}\int d^3x\sqrt{-g_{00}}\left[\pi^2(x) + (\nabla\phi(x))^2 + m^2\phi^2(x)\right] \equiv \sqrt{-g_{00}}H_0. \qquad (4.39)$$

$\phi(x)$ becomes the solution of Klein-Gordon equation; $(\Box - m^2)\phi(x) = 0$. As a consequence, let us express the solution of Klein-Gordon equation $\phi(x)$ as a combination of Fourier basis,

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3 2E_p}\left(a_p e^{-i\mathbf{px}} + a_p^* e^{i\mathbf{px}}\right). \qquad (4.40)$$

Where $E_p = \sqrt{p^2 + m^2}$. As a second quantization mapping N-particles Hilbert space to Fock space of number state, using equal time commutation relation

$$[\hat{a}_p(t), \hat{a}^\dagger_{p'}(t)] = (2\pi)^3\delta^{(3)}(\vec{p}\,' - \vec{p}) \qquad (4.41)$$

leads to relation between fields and diagonalized Hamiltonian neglect the vacuum energy

$$[\hat{\phi}, \hat{\phi}^\dagger] = 1, \qquad \hat{H}_{int} = \sum_k E_k \hat{a}^\dagger_k \hat{a}_k = \sum_i E_i \hat{n}_i. \qquad (4.42)$$

Combine with coupling term, we obtain

$$\hat{H}'_{int} = \left(1 + \mathbf{a}X - \frac{P^2}{2m^2}\right)\sum_i E_i \hat{n}_i. \qquad (4.43)$$

Eq (4.43) demonstrates the coupling between position and also momentum with internal energy.

Now, we are left with two questions: 1. what is the corresponding quantum state of the internal one? 2. how can we describe the coupling between a centre of mass state and an internal one? To demonstrate the decoherence effect, let us consider the case particle is at rest ($P = 0$) in the gravitational field. As a typical situation, assume that 2 particles are at distinct positions X_1, X_2. Using the definition of centre of mass, position coherence between particles can be described by

$$|\Psi_{cm}(0)\rangle = \frac{1}{\sqrt{2}}(|X_1\rangle + |X_2\rangle). \tag{4.44}$$

In addition, as a hypothesis, internal states are in thermal equilibrium at T. Namely, the internal quantum state becomes a thermal state. A state of ith particle can be written by from $\beta \equiv 1/T$,

$$\rho_i = \frac{\exp(-\beta \hat{H}_i)}{Tr(\exp(-\beta \hat{H}_i))}. \tag{4.45}$$

Since Hamiltonian is hermitian operator, this state can be decomposed by infinite summation of diagonalized number states with weighting factor such that

$$\rho_i = \sum_{n=0}^{\infty} \frac{\exp(-n\beta\omega_i)}{Z_i} |n\rangle\langle n|. \tag{4.46}$$

Where $Z_i = (1 - \exp(\beta\omega_i))^{-1}$. Now, in order to confirm time dilation effect by time evolution with different frequency, it is convenient to transform this state into coherent state basis. Using the fact average number $\bar{n}_i = (\exp(\beta\omega) - 1)^{-1}$, state becomes

$$\rho_i = \frac{1}{\pi \bar{n}_i} \int d^2\alpha_i \exp(-|\alpha_i|^2/\bar{n}_i) |\alpha_i\rangle\langle\alpha_i|. \tag{4.47}$$

Suppose that initial state $\rho(0) = \rho_{cm}(0) \otimes \prod_{i=1}^{N} \rho_i$. The off-diagonal term $\rho_{12}(t) = \langle X_1|\rho|X_2\rangle$ which denotes position coherence of this system becomes

$$\begin{aligned}
\rho_{12}(t) &= \langle X_1| \prod_{i=1}^{N} \frac{\exp(im\mathbf{a}\Delta Xt)}{2\pi\bar{n}_i} \int d^2\alpha_i \exp(-|\alpha_i|^2/\bar{n}_i)(|X_1\rangle \otimes |\alpha_i(t)\rangle)(\langle X_2| \otimes \langle\alpha_i(t)|)|X_2\rangle \\
&\equiv \langle X_1| \prod_{i=1}^{N} \frac{\exp(im\mathbf{a}\Delta Xt)}{2\pi\bar{n}_i} \int d^2\alpha_i \exp(-|\alpha_i|^2/\bar{n}_i)(|\alpha_i^1(t)\rangle\langle\alpha_i^2(t)|)|X_2\rangle \\
&= \frac{\exp(im\mathbf{a}\Delta Xt)}{2\pi\bar{n}_i} \prod_{i=1}^{N} \int d^2\alpha_i \exp(-|\alpha_i|^2/\bar{n}_i)|\alpha_i e^{-i\omega_i(X_1)t}\rangle\langle\alpha_i e^{-i\omega_i(X_2)t}|
\end{aligned} \tag{4.48}$$

where $\Delta X = |X_1 - X_2|$ & $\omega_i(X) = \omega_i(1 + aX)$. We conclude using this definition $|\alpha_i^j(t)\rangle = |X_j\rangle \otimes |\alpha_i e^{-i\omega_i(X_j)t}\rangle$, time evolution in distinct positions is completely different.

Following interferometer visibility as a measure of interference becomes

$$V(t) = 2\prod_{i=1}^{N} |Tr_i[\rho_{12}(t)]| = \prod_{i=1}^{N} |[1 + \bar{n}_i(1 - \exp(-i\omega_i \mathbf{a} t \Delta X))]^{-1}|. \tag{4.49}$$

For short time region $t \ll \sqrt{N}\tau_{dec}$, where $\tau_{dec} = \sqrt{\frac{2}{N}\frac{\beta}{\mathbf{a}\Delta X}}$ visibility can be approximated as

$$V(t) \approx \exp(-(t/\tau_{dec})^2). \tag{4.50}$$

Like the previous section, we simply estimate the decoherence factor using short time approximation. Notice that this process is also reversible. Namely, visibility oscillates. Through sections 4.2 and 4.3, we have found the glimpse that proper time is a variable in the lack of knowledge also becomes a source of decoherence. In the last section, we use our powerful weapon to check decoherence process by proper time difference exactly: Master equation.

4.4 Master equation approach

From Rindler Hamiltonian written in eq (4.10), we are able to decompose total Hamiltonian into

$$\hat{H} = \hat{H}_{cm} + \hat{H}_{int} + \hat{H}_{coup} \equiv \hat{H}_0 + \hat{H}_{coup}. \tag{4.51}$$

Where $\hat{H}_{coup} = \left(\mathbf{a}\hat{X} - \frac{\hat{P}^2}{2m^2}\right), \hat{H}_{int} \equiv K(X, P)\hat{H}_{int}$. In addition, define the quantum state in the specific rotating frame (in the interaction picture) as

$$\hat{\rho}'(t) \equiv e^{it(\hat{H}_0 + h(t))}\hat{\rho}(t)e^{-it(\hat{H}_0 + h(t))} \tag{4.52}$$

where $h(t) \equiv \prod_{i=1}^{N} Tr_i[\hat{H}_{coup}\hat{\rho}(t)] = K(X,P)\bar{E}_{int}(t)$. $\bar{E}_{int}(t)$ is the average internal energy of N-particles in time. In this frame, it becomes possible to average out the internal degree of freedom. Resulting Von-Neumann equation becomes with skipping hat notation,

$$\dot{\rho}'(t) = ie^{it(H_0+h(t))}[H_0 + h(t), \rho(t)]\rho(t)e^{-it(H_0+h(t))} + e^{it(H_0+h(t))}\dot{\rho}(t)e^{-it(H_0+h(t))}$$
$$= ie^{it(H_0+h(t))}[H_0 + h(t), \rho(t)]\rho(t)e^{-it(H_0+h(t))} - ie^{it(H_0+h(t))}[H, \rho(t)]e^{-it(H_0+h(t))}$$
$$= -ie^{it(H_0+h(t))}[H_{coup} - h(t), \rho(t)]e^{-it(H_0+h(t))} = -i[H'_{coup} - h'(t), \rho'(t)]. \tag{4.53}$$

Using identical techniques in chap 3, $\rho(t)$ becomes

$$\dot{\rho}'(t) = -i[H'_{coup}(t) - h'(t), \rho'(0)] - \int_0^t ds[H'_{coup}(t) - h'(t), [H'_{coup}(s) - h'(s)\rho'(s)]]. \tag{4.54}$$

As usually did, assume that initial state has been settled as $\rho = \rho_{cm} \otimes \Pi_1^N \rho_i$. In addition, apply Born approximation. Now we focus on the centre of mass degree of freedom. It is achieved by tracing out the information of total N-particles,

$$\begin{aligned}
\dot{\rho}'_{cm}(t) &= \prod_{i=1}^{N} Tr_i[\dot{\rho}'(t)] \\
&= -\prod_{i=1}^{N} \int_0^t ds\ Tr_i[H'_{coup}(t) - h'(t), [H'_{coup}(s) - h'(s)\rho'(s)]] \\
&= -\prod_{i=1}^{N} \int_0^t ds\ Tr_i\{(H_{int} - \bar{E}_{int})^2\ [K'(t), [K'(s), \rho'(s)]]\} \\
&= -(\Delta E_{int})^2 \int_0^t ds[K'(t), [K'(s), \rho'_{cm}(s)]].
\end{aligned} \tag{4.55}$$

Variance of internal energy yields

$$(\Delta E_{int})^2 = \prod_{i=1}^{N} Tr_i\{(H_{int} - \bar{E}_{int})^2\} = \langle H_{int}^2 \rangle - \langle H_{int} \rangle^2. \tag{4.56}$$

Now, let us return to the original picture, using eq (3.42). Master equation in the Schrödinger picture is obtained by

$$\begin{aligned}\dot{\rho}_{cm}(t) &= -i[H_{cm} + \bar{E}_{int}K(X,P)], \rho_{cm}(t)] \\ &\quad - (\Delta E_{int})^2 \int_0^t ds[K(X,P), [e^{-iH_{cm}s}K(X,P)e^{iH_{cm}s}, \rho_{cm}(t)]].\end{aligned} \tag{4.57}$$

In a thermal equilibrium, $(\Delta E_{int})^2 \approx NT^2$ and $\bar{E}_{int} \approx NT$. For stationary particles, $P = 0$, eq (4.57) becomes

$$\begin{aligned}\dot{\rho}_{cm}(t) &= -i[H_{cm} + \mathbf{a}\bar{E}_{int}X], \rho_{cm}(t)] - N(\mathbf{a}T)^2 \int_0^t ds[X, [X, \rho_{cm}(t)]] \\ &= -i[H_{cm} + \mathbf{a}\bar{E}_{int}X], \rho_{cm}(t)] - N(\mathbf{a}T)^2 \; t \; [X, [X, \rho_{cm}(t)]].\end{aligned} \tag{4.58}$$

The first unitary term in right-hand side simply imposes the mass shift by internal energy $m \to m + \bar{E}_{int}$. Second, non-unitary term finally becomes independent with accumulate time s. As a result, we are able to obtain the simplified equation eq (4.58).

For short time t in a time interval $[0, \tau_{dec}]$, this kind of stationary approximation is valid for the case of large, many particles case. On account of small enough τ_{dec}, momentum is nearly zero in this interval. Oppositely, for the small and few particles case, resulting master equation yields quantum Brownian motion like dynamics [1].

During the calculation, you may have the desire to do Markov approximation onto that part. However, we must grave this fact: The evolution in the presence of gravitational time dilation is inherently **'non-Markovian'** since **overall acquired proper time difference is crucial** [29].

As a solution of eq (4.58), off-diagonal term of the centre of mass position state in time becomes

$$\rho_{cm}^{12}(t) \equiv 2\prod_{i=1}^{N} |Tr_i[\rho_{12}(t)]| \approx e^{-(t/\tau_{dec})^2}. \tag{4.59}$$

Its decay property is Gaussian and differs with the linear decay of Markovian. In addition, this decoherence process is **pure dephasing**. The interesting point is this result is already predicted before using visibility only!

Furthermore, proper decay characteristic time (τ_{dec}) is proportional to

$$\tau_{dec} \propto \frac{\sqrt{2}\beta}{\sqrt{N}\mathbf{a}\Delta X}. \tag{4.60}$$

ΔX means the position difference between two internal particles. In addition, since the proper time difference in different position is

$$\Delta\tau = \Delta(1+\mathbf{a}X)\ T \approx \mathbf{a}\Delta X\ T. \tag{4.61}$$

Coherence in terms of the proper time difference becomes

$$\rho_{cm}^{12}(t) \approx e^{-(\Delta\tau)^2}. \tag{4.62}$$

As a conclusion, when there exists the proper time difference, dephasing occurs with Non-Markovian Gaussian decay. Notice that, when $\mathbf{a} = g$ we are able to get the identical result with original paper [1]. In reality in the constant gravitational field, estimated decoherence time of 1 mol of molecule by inserting $N \sim 10^{23}$, $\Delta X \sim 10^{-6}m$ is

$$\tau_{dec} \approx 10^{-3} sec. \tag{4.63}$$

This decoherence time scale looks too small with regarding to another the decoherence effect (e.g. Electromagnetic radiation) [1]. Nevertheless, it is also significant result because even if a particle is at rest, decoherence behavior occurs due to gravity as **Fundamental decoherence** at the constant gravity.

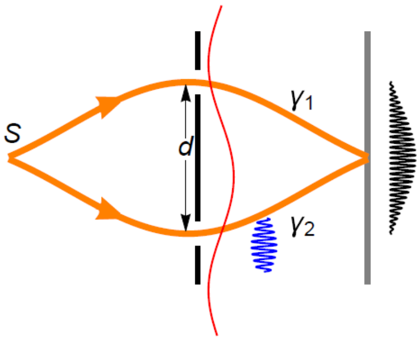

Figure 4.3: Double-slit experiment under the thermal environment [2].

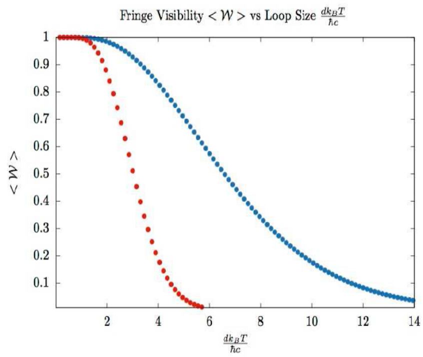

Figure 4.4: Visibility with regarding to temperature [2].

4.5 How does interference fall in Rindler spacetime?

Recently, Samuel [2] and Orlando *et al* [48] have suggested proposals about "How does interference fall in Rindler spacetime?" Orlando used path integral method and Samuel treated double slit experiment under thermal equilibrium condition. In Rindler space time, due to the existence of Unruh effect, description on thermal bath corresponding to system's proper acceleration becomes possible. Consequently, when we are able to obtain the visibility related a thermal bath, also possible to describe it in terms of a proper acceleration equivalently.

In this section, we review the idea of Samuel. Let us consider double slit experiment of a electron. Visibility for wave functions ψ_1 and ψ_2 corresponding each path γ_1 and γ_2 is

$$V = \langle W \rangle (\psi_1^* \psi_2 + \psi_1 \psi_2^*). \tag{4.64}$$

Where $\langle W \rangle$ is the average value of Wilson loop in fig 4.3. On account of Aharanov-Bohm effect, $\langle W \rangle$ becomes $\exp(ie \oint_\gamma \vec{A} \cdot \vec{dx})$. When system is in a thermal equilibrium, by quantization of electromagnetic vector potential and using Bosonic statistic, visibility is obtained by [2]

$$\langle W \rangle = \exp\left(-\frac{e^2}{2} \sum_l |\alpha_l|^2 \coth\left(\frac{\omega_l}{2T} \right) \right). \tag{4.65}$$

l is the index of momentum and polarization of the photon field and α_l is the Fourier transformed coefficient of Wilson loop. In our original Rindler spacetime, substitute real temperature for the Unruh temperature [49]

$$T = \frac{a}{2\pi}, \tag{4.66}$$

visibility of the electron in a proper acceleration a becomes [2]

$$\langle W \rangle = \exp\left(-\frac{e^2}{2} \sum_l |\alpha_l|^2 \coth\left(\frac{\pi\omega_l}{a} \right) \right). \tag{4.67}$$

As a result, as a proper acceleration bigger, visibility drops as shown in the fig 4.4. In a constant gravity $g = 9.8m/s^2$, the Unruh temperature $T \approx 4 * 10^{-20}K$ [2]. It looks tiny but we have to notice that visibility totally depends on the frame of observer. In that sense, observers feel various decoherence effect depending on their acceleration with regarding to reference frame! This result would be the key idea of the development of **Relativistic quantum information**.

Experimentally, by comparing double slit in vertically and horizontally set with regarding to the gravity field, we can do an experiment to check the decoherence effect by gravity. Even if decoherence due to constant gravity is tiny for the electron, but with cutting-edge field "Quantum opto-mechanics", it is attempted to do experiment with extreme sensitivity [34, 50].

Chapter 5

Quantum leaks of spacetime: Decoherence through semi-classical gravity

Historically, development of a standard model using gauge field theory has been successful [51]. As a result, every fundamental interaction except gravity can be considered as a quantum field. Experimentally, discovering neutrinos, quarks and Higgs boson confirmed the theoretical predictions. As the last part of the grand unification model, attempts are made to quantize gravity in a manner similar to other fundamental interactions.

Like other interactions, this idea starts with **1**) faith that the universe is intrinsically quantum, and **2**) the belief that the uncertainty principle in the quantum sector is violated on the short time scale. Mathematically, this appears to have been completed (e.g. string theory and loop quantum gravity) for the quantization gravitation field, and the theory predicts the existence of gravitons. On the other hand, in experiments, even gravitational waves are difficult to detect, and this must be done using LIGO. The energy density of gravitational wave is expressed as follows (with c, $\hbar$) [52]:

$$E_{wave} = (c^2/32\pi G)\omega^2 f^2 \tag{5.1}$$

where ω is the angular frequency of the wave and f is the amplitude. With $\omega \sim$ 1KHz, f $\sim 10^{-21}$, the density is approximately $10^{-10}/cm^3$. Otherwise, the energy density of a graviton is identical to that of a photon [52],

$$E_{par} = \frac{\hbar\omega^4}{c^3}. \tag{5.2}$$

Its energy is roughly $3 \cdot 10^{-47} ergs/cm^3$. The energy scale of a gravitational wave is approximately $3 \cdot 10^{37}$ times that of a graviton in terms of quantum gravity! To measure a single graviton in LIGO, variation of at least $\sim \sqrt{32\pi} l_p$ is necessary [5]. Here, l_p denotes the Planck length scale. In other words, verifying the existence of quantum fluctuation in a gravitational field appears to be an unrealistic exercise.

This kind of super-weakness of gravitons and the incomplete interpretation of quantized spacetime has led to criticisms of quantized gravity. These critiques are based on the following facts: **1**) the world is intrinsically geometric; **2**) when there is noise, the problem of uncertainty in the quantum sector is irrelevant; **3**) most importantly, in contrast to other interactions, gravity cannot be shielded. Thus, every massive particle follows the geodesic congruence according to the geodesic equation [10]; **4**) if the world is fundamentally quantum, there is no way to describe the origin of non-linearity with which to describe the wave function collapse in the world, as the state assumed by quantum mechanics fundamentally impose linearity.

In the quantum language, particles in the fundamental interactions is “gravity perpetually ‘measures ’massive particles and gravity is perpetually ‘measured ’by particles” [10]. As a result, one can conceive of this fundamental question: **Is the universe inherently an open quantum system due to gravity channels?** [10] In other words, is there no closed quantum system except if setting the entire universe as a closed system?

This type of fundamental debate about quantized gravity continues, and it is also a very interesting topic in physics. In this chapter, we review the semi-classical treatment of gravity. Thus, we explain decoherence process without a quantizing gravity field via the Schrödinger-Newton equation. **That is, we do not consider a fundamental description but only an effective description**. First, the Diósi-Penrose model and the Schrödinger-Newton equation based on the non-linearity in the equation and non-conservation terms are considered. Secondly, in the successive DP model, the Kafri-Milburn-Taylor model which treats gravity as a classical channel is introduced. As a result, we check for inconsistencies in the decoherence process given the fundamental description. Thus, we find that gravity need not be quantized in such a sense.

5.1 Newtonian gravity mechanisms on the localization and decoherence

In 2007, Diósi claimed the non-linear term of the Schrödinger equation and additional non-unitary term in the Von-Neumann equation ensure the localization resulting from the decoherence process [3]. He also showed the connection between $\hbar$ and G in classical Newtonian interaction using the effective description. Now, let us follow more details of it.

First, let us consider Newtonian interaction

$$V(X,Y) = -\iint d^3r d^3r' \frac{m(r|X)m(r'|Y)}{|r-r'|}. \tag{5.3}$$

Where $m(r|X)$ is a mass density function located in r from the reference frame and its centre of mass X. Therefore, Schrödinger equation for massive wave function $\psi(X)$ is following.

$$i\frac{\partial\psi(X,t)}{\partial t} = \hat{H}_0\psi(X,t) + \left(\int dY\ V(X,Y)|\psi(Y,t)|^2\right)\psi(X,t) \tag{5.4}$$

$\hat{H}_0$ is free Hamiltonian without interaction. This one is Newton-Schrödinger equation and it is non-linear about ψ. To describe this effect, let me suggest an example. In spherical distributed mass case, density is described by the function of difference $r-X$. Interaction also depends on the only distance between centre of masses. In addition, assume the degree of de-localization does not vicious. In other words $X - Y << 1$. Consequently, in the stationary state, potential can be expanded as

$$V(X-Y) = V_0 + \frac{1}{2}M\omega^2(X-Y)^2. \tag{5.5}$$

Where

$$\begin{aligned} M\omega^2 &= \nabla_X^2 V(X,Y)|_{Y=X} = \nabla_r^2 V(X,Y)|_{Y=X} \\ &= 4\pi\iint m(r)m(r')\delta^3(r-r')d^3r d^3r' = 4\pi\int m^2(r)d^3r. \end{aligned} \tag{5.6}$$

Suppose that the constant mass density around sphere with radius R, ω becomes

$$\omega^2 = \frac{3M}{R^3}. \tag{5.7}$$

Following Newton-Schrödinger equation after averaging is

$$i\frac{\partial\psi(X,t)}{\partial t} = \hat{H}_0\psi(X,t) + \frac{1}{2}M\omega^2(X-\langle X\rangle)^2\psi(X,t). \tag{5.8}$$

Without any external potential, **self-localization** is happened by effective harmonic potential by gravity interaction. Likewise, Von-Neumann equation represented in different basis X, Y is written by

$$\frac{d\rho(X,Y,t)}{dt} = \frac{i}{2M}(\nabla_X^2 - \nabla_Y^2)\rho(X,Y,t) - \frac{M\omega^2}{2}(X-Y)^2\rho(X,Y,t). \tag{5.9}$$

Its solution without given unitary part is the well-known formula of continuous measurement inducing localization due to decoherence such that

$$\rho(X,Y,t) \propto \rho(X,Y,0)\exp(-M\omega^2(X-Y)^2t/2). \tag{5.10}$$

As separation gets bigger, the decoherence effect becomes larger. It implies **wave function is localized by gravitational interaction**! Following decoherence time scale is given by (with $\hbar$ and G) [3]

$$\tau_{dec} \equiv \frac{2\hbar}{M\omega^2(X-Y)^2} = \frac{2\hbar R^3}{3GM^2(X-Y)^2}. \tag{5.11}$$

As can be seen, we are able to check as the scale of mass and separation are smaller, decoherence time scale becomes larger. It means coherence of heavy object and far from each other decays faster.

5.2 Schrödinger-Newton equation from Semi-classical gravity: Wave function collapse model

As mentioned at the beginning, it is often assumed that gravity can be quantized. It means nature is fundamentally quantum. In the quantum world, linearity is the most interesting feature. Hilbert space is defined by infinity dimensional complete vector space which quantum states live. And most fundamental equation for a state; Schrödinger equation is also linear. In addition, the primary theorem in quantum information, No-cloning theorem, is also based on linearity. Briefly, no-cloning theorem imposes that it is possible to clone two bases but cannot do that for an arbitrary linearly superposed state.

Therefore, provided that we quantize gravity such as other fundamental forces, it is possible to find linearity again. However, in order to describe the decoherence effect, we have to acquire non-linear term which leads to information non-preserving. In this spirit, Bahrami *et al* [8, 53] claimed Schrödinger-Newton equation assumed that only matter is quantized but gravity remains as purely classical.

The most natural dynamics in quantum-classical hybrid theory starts from semi-classical Einstein equation such that [8]

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi\langle\psi_N|\hat{T}_{\mu\nu}|\psi_N\rangle. \tag{5.12}$$

The expectation value of stress tensor is the ensemble averaged value with regarding to the N-particles state $|\psi_N\rangle$. Needless to say, in order to obtain validity of this equation, local-energy conservation must be satisfied such that

$$\nabla^{\mu}\langle\psi_N|\hat{T}_{\mu\nu}|\psi_N\rangle = 0. \tag{5.13}$$

Where ∇ means covariant derivative. From now on, for consistency in non-relativistic regime, we should take linearized gravity limit. Considering metric field decomposition. It is legitimate to decompose metric into the reference part and fluctuation part as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{5.14}$$

In case we assume $|h_{\mu\nu}| \ll 1$ (weak field), using identical methodology above,

$$\begin{aligned} \Box\bar{h}_{\mu\nu} &= -16\pi\ \langle\psi_N|\hat{T}_{\mu\nu}|\psi_N\rangle, \\ \Box h_{\mu\nu} &= -16\pi\left(\langle\psi_N|\hat{T}_{\mu\nu}|\psi_N\rangle - \frac{1}{2}\eta_{\mu\nu}\langle\psi_N|\eta^{ab}\hat{T}_{ab}|\psi_N\rangle\right). \end{aligned} \tag{5.15}$$

De Donder gauge (Lorentz gauge) and associated definitions are

$$\begin{aligned}&\partial^{\mu}h_{\mu\nu}=\partial^{\mu}\bar{h}_{\mu\nu}=0,\\&h\equiv\eta^{\mu\nu}h_{\mu\nu},\qquad\bar{h}_{\mu\nu}\equiv h_{\mu\nu}-\eta_{\mu\nu}\frac{h}{2},\qquad\bar{h}\equiv\bar{h}^{a}_{a}=-h.\end{aligned}\tag{5.16}$$

After taking a Newtonian limit, $|T_{00}|\gg|T_{0i}|\gg|T_{ij}|$, (where i,j $\neq 0$)

$$\nabla^2\Phi=4\pi\ \langle\psi_N|\hat{T}_{00}|\psi_N\rangle\equiv4\pi\ \langle\psi_N|\hat{\varrho}|\psi_N\rangle.\tag{5.17}$$

It is simply equivalent to $\bar{h}_{00}=-4\Phi$ with the fact ϱ is equal to the mass density. In Newtonian limit, Einstein equation implies matter distribution in gravitational potential but treats the matter as a quantized one. Because gravitational potential yields effect on the matter, following action can be driven. Its matter-field interaction action is given by [54]

$$\hat{S}_{int}=-\frac{1}{2}\int d^4r h_{\mu\nu}\hat{T}^{\mu\nu}.\tag{5.18}$$

Before we turn to Newtonian limit, notice that such action should be gauge independent. Let us consider an infinitesimal shift of position which is generated by the flow of the vector field χ^{μ} such that

$$x^{\mu}\to x^{\mu}+\varepsilon\chi^{\mu}\equiv x^{\mu}+\zeta^{\mu}.\tag{5.19}$$

Where $\varepsilon\ll1$. In this shift, perturbative metric h is transformed into [44, 54]

$$h_{\mu\nu}\to h'_{\mu\nu}=h_{\mu\nu}-\partial_{\mu}\zeta_{\nu}-\partial_{\nu}\zeta_{\mu}.\tag{5.20}$$

In this gauge transformation, action is obtained by

$$\hat{S}'_{int}=-\frac{1}{2}\int d^4r h'_{\mu\nu}\hat{T}^{\mu\nu}=-\frac{1}{2}\int d^4r\left[h_{\mu\nu}-\partial_{\mu}\zeta_{\nu}-\partial_{\nu}\zeta_{\mu}\right]\hat{T}^{\mu\nu}.\tag{5.21}$$

According to local conservation law with perturbative metric, $\partial_{\nu}T^{\mu\nu}=0$, and invariance of action under additional total differentiation, it is possible to neglect the second and third terms using intergration by parts. As a result, it is legitimate to use it as an action. Following Hamiltonian is therefore [21, 22],

$$\hat{H}_{int} = -\frac{1}{2}\int d^3 r h_{\mu\nu}\hat{T}^{\mu\nu} \tag{5.22}$$

In Newtonian limit, from Poisson equation eq (5.17), equation is obtained by

$$\hat{H}_{int} = -\frac{1}{2}\int d^3 r h_{00}\hat{T}^{00} = -\iint d^3 r d^3 r' \frac{\langle\psi_N|\hat{\varrho}(r')|\psi_N\rangle}{|r-r'|}\ \hat{\varrho}(r). \tag{5.23}$$

In order to clarify the meaning of $\hat{\rho}$, let us consider Schrödinger field ψ with originating Lagrangian such that

$$\mathcal{L} = i\psi(\vec{x},t)\partial_t\psi(\vec{x},t) - \frac{|\vec{\nabla}\psi(\vec{x},t)|^2}{2m} - V(\vec{x})\psi(\vec{x},t). \tag{5.24}$$

Following from second quantization rule of the boson field

$$\pi(\vec{x},t) = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = i\psi^*, \qquad \left[\hat{\psi}(\vec{x},t), i\hat{\psi}^\dagger(\vec{y},t)\right] = i\delta^{(3)}(\vec{x}-\vec{y}). \tag{5.25}$$

Because $\hat{\psi}^\dagger\hat{\psi}$ acts as Number density operator, $\hat{\varrho}$ becomes $M\hat{\psi}^\dagger\hat{\psi}$. With this quantized field, resulting Schrödinger equation in Fock space with the N-particles state $|\psi_N\rangle$ is therefore

$$\begin{aligned} i\partial_t|\psi_N\rangle = &-\int d^3 r\hat{\psi}^\dagger(r)\left(\frac{\nabla^2}{2M}\right)\hat{\psi}(r)|\psi_N\rangle \\ &- M^2\iint d^3 r d^3 r' \frac{\langle\psi_N|\hat{\psi}^\dagger(r')\hat{\psi}(r')|\psi_N\rangle}{|r-r'|}\hat{\psi}^\dagger(r)\hat{\psi}(r)|\psi_N\rangle. \end{aligned} \tag{5.26}$$

We can easily check whether this equation equivalent to Schrödinger equation or not by taking $\langle r|$ in the left side. Thus, the N-particles state in terms of field operator and associated expectation value are

$$\begin{aligned} |\psi_N\rangle &= \frac{1}{\sqrt{N!}}\left(\int d^3 r_i \prod_{i=1}^{N}\right)\phi_N(r_1,...,r_N)\hat{\psi}^\dagger(r_1)...\hat{\psi}^\dagger(r_N)|0\rangle, \\ \langle\psi_N|\hat{\varrho}(r)|\psi_N\rangle &= \sum_{j=1}^{N}\left(\int d^3 r_i \prod_{i=1\neq j}^{N}\right)|\phi(r_1,...,r_{j-1},r,r_{j+1},...,r_N)|^2. \end{aligned} \tag{5.27}$$

$\phi_N(r_1, ..., r_N)$ is wave function for N-particles in Schrödinger equation. Finally, Schrödinger-Newton equation becomes

$$i\partial_t \phi_N(t, \{r'\}) = \left[\sum_{i=1} \frac{\nabla_i^2}{2M} - M^2 \sum_{i,j=1} \left(\int d^3 r'_k \prod_{k=1}^{N} \right) \frac{|\phi_N(t, \{r'\})|^2}{r'_i - r'_j} \right] \phi_N(t, \{r'\}). \tag{5.28}$$

Where $\{r'\} = r_1, ..., r_N$ and we can achieve congruent formula with (5.4) for one particle case. Notice that it is obtained from the effective description on the gravitational field. However, what if we use quantized gravity? As substituting $h_{\mu\nu}$ into $\hat{h}_{\mu\nu}$, obtaining equation is transformed as [8]

$$\begin{aligned} i\partial_t |\psi_N\rangle = & - \int d^3 r \hat{\psi}^\dagger(r) \left(\frac{\nabla^2}{2M} \right) \hat{\psi}(r) |\psi_N\rangle \\ & - M^2 \iint d^3 r d^3 r' \frac{\hat{\psi}^\dagger(r') \hat{\psi}(r')}{|r - r'|} \hat{\psi}^\dagger(r) \hat{\psi}(r) |\psi_N\rangle. \end{aligned} \tag{5.29}$$

In order to obtain the identical result with above, it is necessary to apply Hartree-Fock approximation in $N \to \infty$ with the mean-field limit. Notice that this is effective description again! Therefore, we can check that there exists inconsistency in the fundamental description.

From this equation, it is claimed that Schrödinger-Newton equation is the proper explanation about wave function collapse with deterministic quantum-mechanics. Consider linearly superposed the N-particles state $\psi_N(r) = (\psi_1(r) + \psi_2(r))/\sqrt{2}$ where ψ_i are well localized near r_i. Using massive pointer(Φ) to find where the particle is, the total state becomes entangled by

$$\Psi_N(r, R) = \frac{1}{\sqrt{2}} (\psi_1(r)\Phi_1(R) + \psi_2(r)\Phi_2(R)) \tag{5.30}$$

In the standard wave function collapse model, the particle is always measured at $R = R_1$ or $R = R_2$. On the contrary, as can be seen in eq (5.10), on account of Schrödinger-Newton equation, self-localization toward centre-of-mass occurs

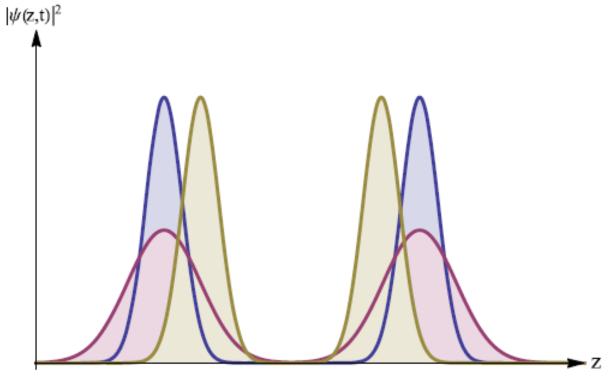

Figure 5.1: Wave packet in the initial state (Blue line) originally spread out one as time goes on (Purple line). But on account of localization effect and self-localization effect, wave function focused toward the middle (Yellow line) [8].

and pointer's wave function is collapsed toward at $R = (R_1 + R_2)/2$ (Refer the fig 5.1) [32, 55]. It is completely different from standard prediction of QM!

Unfortunately, this model has some problems. First, in numerical calculation, there is a small portion of infinity during the collapse. Second, it cannot explain Born rule in standard QM. Third, because we treat wave function as a real object, there is a possibility of super-luminal teleportation. Whereas in this equation, there is no way to block it. In order to solve the last problem, it is attempted to impose stochastic noise and succeeded in solving the problem [4, 8]. In the next section, we treat model on semi-classical gravity using stochastic noise induced on gravity channel by weak measurement.

5.3 Gravity as classical channel

Recently, Kafri *et al* has suggested the interesting model about the classical channel on gravity from the fact it cannot be shielded. In this setup, the gravitational source and the source of noise also exist in the measurement channel between two particles. In addition, from the argument on possibility of gaining information about the gravitational potential source, it is argued gaining information process can be modeled with the weak measurement channel with inducing decoherence.

From this analogy, key point follows "classicality from the unavailability of creating entanglement". Like identical methodology with Diósi and Penrose model, effective description was used on the gravity field without quantized it. Hence, there is no entanglement gain through the classical gravity interaction in this process since LOCC(Local operation with classical communication) cannot create entanglement [33, 56, 57].

Pushing this analogy, gauge field interaction in this description becomes **non-unitary and Markovian**. On the other side, quantized gravity as a fundamental description interaction becomes **unitary and non-Markovian**. Consequently, it is legitimate to describe localization process due to continuous measurement through the gravity channel using effective description. In addition, in order to solve the problem on super-luminal teleportation, it is presumed that noise exists in this channel during the measurement. Namely, weak measurement on each particle which induce noise leads to decoherence (localization).

Let us now attempt to estimate how large decoherence rate is, we explore their model of the continuous measurement by the classical channel. Following the idea of Kafri *et al*, let assume there are two simple harmonic oscillators (SHOs) and both are weakly coupled to each other. Following Hamiltonian is

$$\hat{H} = \hat{H}_1 + \hat{H}_2 + K\hat{x}_1\hat{x}_2 \equiv \hat{H}_0 + \hat{H}_1. \tag{5.31}$$

Where

$$\begin{aligned} \hat{H}_i &= \frac{\hat{p}_i^2}{2m_i} + \frac{1}{2}m_i\Omega_i^2\hat{x}_i^2, \\ \Omega_i^2 &= \omega_i^2 - K/m_i, \qquad K = \frac{2m_1m_2}{d^3}. \end{aligned} \tag{5.32}$$

Coupling term in (5.31) only leads to the leakage on each information through the channel. Consider two masses do the measurement on each other by gravitational interaction channel. That is, each particle's centre of mass position is continuously measured by opposite particle. Each measurement through the channel during dt yields recording variable dJ including an average position and the leakage as a stochastic noise. Therefore, it is legitimate to write corresponding interaction Hamiltonian with feedback (recording affects interaction Hamiltonian) as

$$\hat{H}_{int} = \chi_1\frac{dJ_1}{dt}\hat{x}_2 + \chi_2\frac{dJ_2}{dt}\hat{x}_1. \tag{5.33}$$

Each χ_i represents measurement rate between two particles. Following recording of each mass 1 and 2 is defined as in terms of It$\hat{o}$ stochastic differential calculus, $dJ_i, i = 1, 2$ where

$$dJ_i(t) = \langle\hat{x}_i\rangle dt + \frac{1}{2}\sqrt{\frac{1}{\Gamma_i}}dW_i(t). \tag{5.34}$$

Where $\langle\hat{x}_i\rangle$ is the conditional average value of the records up to time t. Γ_i is the decoherence factor in the stochastic quantum master equation with Wiener increment $dW(t)$. In addition, in order to obtain master equation for continuous measurement, introduce continuous measurement operator in time Δt for ith mass, $\hat{M}(\alpha_i)$ as [58]

$$\hat{M}(\alpha_i) = \left(\frac{2\Gamma_i\Delta t}{\pi}\right)^{1/4}\exp(-\Gamma_i\Delta t(\hat{x}_i - \alpha_i)^2). \tag{5.35}$$

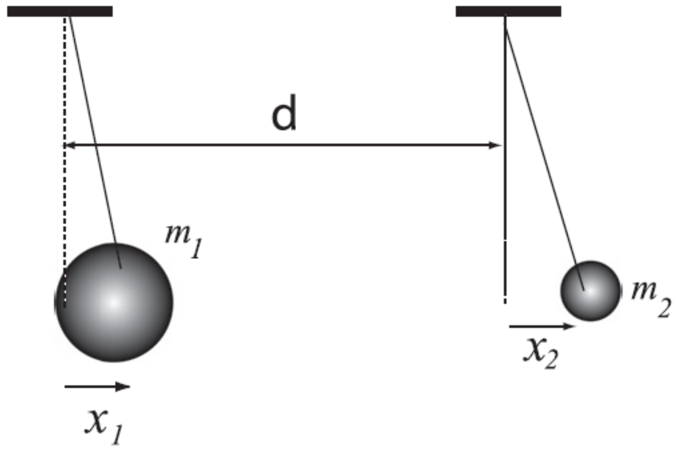

Figure 5.2: Two weakly coupled harmonic oscillators [4]

α now means the measured value of each measurement with variance $1/2\Gamma$. As a stochastic quantity, $\alpha \equiv \langle \hat{x} \rangle + \frac{\Delta W}{\sqrt{4\Gamma \Delta t}}$. Now, the meaning of α becomes clear when we deal with it in a short time limit, $\Delta t \to dt$ and $\Delta W \to dW$. α is going to be represented in terms of Stratonovish calculus such that

$$\alpha = \frac{dJ_i}{dt} = \langle \hat{x}_i \rangle + \frac{1}{2}\sqrt{\frac{1}{\Gamma_i}}\zeta(t). \tag{5.36}$$

Where $dW(t) = \zeta(t)dt$. Furthermore, the quantum state at initially time t after dt, it evolves such as

$$\begin{aligned} |\psi(t+\Delta t)\rangle &= \hat{M}(\alpha)|\psi(t)\rangle \\ &\propto \exp\left(-\Gamma_i(\hat{x}_i - \langle \hat{x}_i \rangle)^2 dt + \sqrt{\Gamma_i}(\hat{x}_i - \langle \hat{x}_i \rangle)dW(t) + \frac{1}{4}\right)|\psi(t)\rangle. \end{aligned} \tag{5.37}$$

Obtaining the fact in Itô calculus, $dW_i(t)^2 \to dt$ in the infinitesimal region. Given that we expand using Taylor expansion and neglect the term order of higher than $dtdW_i$ by presuming $dt \ll 1$,

$$|\psi(t+\Delta t)\rangle \propto \left(1 - \frac{\Gamma_i}{2}(\hat{x}_i - \langle \hat{x}_i \rangle)^2 dt + \sqrt{\Gamma_i}(\hat{x}_i - \langle \hat{x}_i \rangle)dW_i(t)\right)|\psi(t)\rangle. \tag{5.38}$$

Define $d|\psi\rangle(t) \equiv |\psi(t+dt)\rangle - |\psi(t)\rangle$, following variation of density matrix can be obtained from

$$d\rho = (d|\psi\rangle)\langle\psi| + |\psi\rangle(d\langle\psi|) + (d|\psi\rangle)(d\langle\psi|). \tag{5.39}$$

As a result, stochastic master equation is obtained by

$$\frac{d}{dt}\rho_c(t) = -\frac{\Gamma_i}{2}[\hat{x}_i, [\hat{x}_i, \rho_c]] + \sqrt{\Gamma_i}(\hat{x}_i\rho_c + \rho_c\hat{x}_i - 2\langle\hat{x}_i\rangle\rho_c)dW_i(t). \tag{5.40}$$

ρ_c indicates conditional evolution state (without evolution by Hamiltonian). Since time evolution with regarding to Hamiltonian during the measurement occurs, we have to consider additional factor also. During the measurement time scale dt, variation of this quantum state is

$$d|\psi\rangle \approx (-i\hat{H}_0 dt - i\hat{H}_{int}dt - \hat{H}_{int}^2(dt)^2/2)|\psi\rangle \equiv \hat{V}|\psi\rangle. \tag{5.41}$$

The reason we keep up to the term $\hat{H}_{int}^2$ is $dW^2 = dt$ can yield additional contribute. Thus, total evolution of state becomes

$$\rho_c + d\rho_c \to (1+\hat{V})(\rho_c + d\rho_c)(1+\hat{V}^\dagger). \tag{5.42}$$

Now, take the ensemble average to consider whole information at once using Wiener increment with $\langle\langle dW(t)^2\rangle\rangle = dt$, $\langle\langle dW(t)dW(t')\rangle\rangle = 0$ for $t \neq t'$, where $\langle\langle ...\rangle\rangle$ is the ensemble averaged value. In addition, use unconditioned dynamics which means the ensemble averaged value of noise by It$\bar{o}$ calculus $\langle\langle dW(t)\rangle\rangle = 0$. Finally, obtained Stochastic feedback master equation in unconditional dynamics accompanied by stochastic process and time evolution with regarding to feedback Hamiltonian becomes [39]

$$\begin{aligned}\frac{d\rho}{dt} &= -i[\hat{H}_0, \rho] - \frac{i}{2}(\chi_2[\hat{x}_2, \{\hat{x}_1, \rho\}] + \chi_1[\hat{x}_1, \{\hat{x}_2, \rho\}]) \\ &\quad - \sum_{i=1}^{2}\frac{\Gamma_i}{2}[\hat{x}_i, [\hat{x}_i, \rho]] - \frac{\chi_1^2}{8\Gamma_1}[\hat{x}_2, [\hat{x}_2, \rho]] - \frac{\chi_2^2}{8\Gamma_2}[\hat{x}_1, [\hat{x}_1, \rho]].\end{aligned} \tag{5.43}$$

The first line indicates unitary evolution by original one(H_0) and the additional one with the response of feedback. The second line represents non-unitary terms due to interaction Hamiltonian. In particular, the term proportional to Γ indicates how system response to feedback by measuring $\hat{x}_i$. For identical pendulums case $m_1 = m_2$ & $\chi_1 = \chi_2 = K$, feedback term (2nd line) becomes minimum when $\Gamma_i = K/2$ (Without any additional interaction). Simplified equation is expressed by

$$\frac{d\rho_i}{dt} = -i[\hat{H}_i, \rho_i] - iK[\hat{x}_1\hat{x}_2, \rho_i] - \sum_{i=1}^{2} \frac{K}{2}[\hat{x}_i, [\hat{x}_i, \rho_i]]. \tag{5.44}$$

The resulting equation is familiar one we have treated. One point to focus is the decoherence rate equivalent to Diósi's prediction [3, 31]! Its solution is

$$\rho_i(x, x', t) = \rho_i(x, x', 0) \exp(-K(x - x')^2 t/2). \tag{5.45}$$

Its solution represents localization process by continuous measurement on each mass through the classical gravity channel. Because decoherence occurs for off-diagonal basis (x $\neq$ x') only, this process is **dephasing with irreversible process**.

Provided that two pendulums are in the ground state, decoherence rate is able to be calculated with reviving unit $\hbar$ [4],

$$\Lambda = \frac{K}{2\hbar}(\Delta x)^2 = \frac{K}{m\omega}. \tag{5.46}$$

Since position localization leads to momentum diffusion and heating of mechanical resonators, it is convenient to set heating rate as the average mechanical energy divided by energy of a photon [4]. For uranium, its mechanical frequency is nearly one hertz with following the decoherence rate $\Lambda \sim 10^{-7}s^{-1}$.

By the way, we also can exactly solve coupled harmonic oscillator problem itself. As a result, we derive the equivalent description of Λ. First, introduce normal mode coordinate

$$\hat{q}_+ \equiv (\hat{x}_1 + \hat{x}_2)/\sqrt{2}, \qquad \hat{q}_- \equiv (\hat{x}_1 - \hat{x}_2)/\sqrt{2}. \tag{5.47}$$

For the symmetric case ($m_1 = m_2 = m$, $\Omega_1 = \Omega_2 = \Omega$), frequencies in normal modes becomes

$$\begin{aligned} \omega_\pm &= \Omega^2 \pm \frac{K}{m}, \\ \omega_+ &= \omega, \qquad \omega_- = \omega\sqrt{1 - \frac{2K}{m\omega^2}}. \end{aligned} \tag{5.48}$$

For the weak gravity, $m\omega^2 \gg K$, the difference between two normal mode frequencies is

$$\Delta\omega = \omega_+ - \omega_- \simeq \frac{K}{m\omega} = \Lambda. \tag{5.49}$$

Through the calculation, we can also check the decoherence rate is proportional to the difference of two frequencies.

By comparing to relative temperature relation of quantum Brownian motion, dissipation rate Γ can be estimated with associated quality factor Q. Effective temperature is given by [4]

$$T_{grav} = \frac{\hbar K}{2m\gamma k_B} = \frac{\hbar Q \Lambda}{2k_B}. \tag{5.50}$$

With assuming high quality factor Q $\equiv \frac{\omega}{\gamma} \sim 10^9$, T is thus roughly $10^{-9}K$. In the laboratory, using the Bose-Einstein condensate (BEC) atom, we may simulate the decoherence effect with the challenge.

5.4 Classical fluctuation and Decoherence

In the previous section, we have checked the effective description of gravity is valid for explain self-focusing and have dealt with it as a classical communication channel including stochastic noise. Now, let us take a closer look at "What if physical observable fluctuates with uncertainty?'. As we consider an environment as a quantum system, quantum fluctuation should be included with uncertainty relation. As can be seen above, for the effective description which we try to describe, fluctuation should be represented as a classical as we have taken an ensemble average on that though. In this sense, the classical fluctuation is now induced by noise from average. Especially in Quantum-Classical hybrid equation, provided that stress-energy tensor fluctuates with random signal, Einstein tensor in the terms of stress tensor is granted by [9]

$$G_{\mu\nu} = 8\pi\langle\psi_N|\hat{T}_{\mu\nu}|\psi_N\rangle + \delta T_{\mu\nu}. \tag{5.51}$$

Depending on the property of $\delta T_{\mu\nu}$, induced character of decoherence is decided. Diósi explained continuous spontaneous localization (CSL) using the fluctuation with Markovian noise. Let us take a closer look at his proposal. As an extended version of the DP and the KTM model, given that each centre of mass acting as a detector localized in a lattice (refer fig 5.3). Each mass detects the density of other mass through the gravitational channel (interaction) with stochastic noise. In other words, in Newtonian limit, measured value on mass density operator at position $\vec{r}$, time t becomes

$$\varrho(\vec{r})_t = \langle\hat{\varrho}(\vec{r})_\sigma\rangle_t + \delta\varrho(\vec{r})_t. \tag{5.52}$$

As we assume the mass distribution in each position as a Gaussian, it is possible to represent σ as a Gaussian width. In addition, suppose that noise is a spatially correlated Gaussian white-noise such that

$$\langle\langle\delta\varrho(\vec{r},t_1)\delta\varrho(\vec{s},t_2)\rangle\rangle \equiv \gamma_{\mathbf{rs}}\delta(t_1-t_2) \equiv \gamma\delta(\vec{r}-\vec{s})\delta(t_1-t_2). \tag{5.53}$$

As a back-action potential due to the quantum fluctuation originating from a measurement, insert the stochastic semi-classical interaction as a form

$$\hat{V}_\sigma = \int d\vec{r} \varrho(\vec{r})_t \hat{\Phi}_\sigma. \tag{5.54}$$

V is a gravitational potential energy, Φ is a potential density. As a result, following the stochastic feedback master equation by the gravitational interaction becomes

$$\begin{aligned} \frac{d}{dt}\hat{\rho} =& -i[\hat{H} + \hat{V}_{G,\sigma} + \underbrace{\int d\vec{r} \delta\varrho(\vec{r}) \hat{\Phi}_\sigma}_{\text{Back-action}}, \hat{\rho}] - \int d\vec{r} \underbrace{\left(\frac{\gamma}{8}[\hat{\varrho}_\sigma(\vec{r}), [\hat{\varrho}_\sigma(\vec{r}), \hat{\rho}]]\right)}_{\text{Decoherence}} \\ & - \int d\vec{r} \left(\underbrace{\frac{\gamma^{-1}}{8}[\hat{\Phi}_\sigma(\vec{r}), [\hat{\Phi}_\sigma(\vec{r}), \hat{\rho}]]}_{\text{Decoherence}} + \underbrace{\frac{\gamma}{2}\mathcal{H}[\hat{\varrho}_\sigma(\vec{r})](\hat{\rho})\delta\varrho(\vec{r})}_{\text{Feedback}} \right). \end{aligned} \tag{5.55}$$

Decoherence term in mass density implies continuous spontaneous localization on mass density. Back-action originating decoherence also leads to similar result from the relation

$$\hat{V}_\sigma = -\frac{1}{2} \int d\vec{r} d\vec{s} \, \frac{\hat{\varrho}_\sigma(\vec{r}) \hat{\varrho}_\sigma(\vec{s})}{|\vec{r} - \vec{s}|} \tag{5.56}$$

Furthermore, suppose the correlation function on eq (5.53) as

$$\gamma_{\mathbf{rs}}^{-1} = \frac{1}{|\vec{r} - \vec{s}|} \tag{5.57}$$

with following inverse kernel function [9]

$$\gamma_{\mathbf{rs}} = -\frac{1}{4\pi} \nabla^2 \delta(\vec{r} - \vec{s}). \tag{5.58}$$

Diósi derived gravity induced spontaneous collapse with this kernel function and its related results [9].

Now, in contrast to the Diósi's proposal, "**How about we deal with the fluctuation of the classical field itself not the quantum fluctuation of**

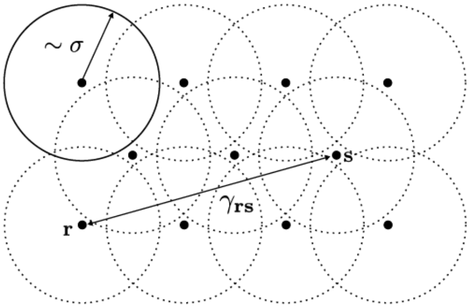

Figure 5.3: Intuitive representation of the detectors. Each detector is represented by black dots on a lattice [9].

stress-energy tensor?" becomes our question. Physically, we change the perspective of "measurement" into "natural decoherence" by taking an ensemble average of a system due to disturbing background field by quantized matter. Namely,

$$\hat{H} = \langle \hat{H} \rangle + \delta \hat{H} = \left\langle \int dV h_{\mu\nu} \hat{T}^{\mu\nu} \right\rangle + \delta h_{\mu\nu} \hat{T}^{\mu\nu}. \tag{5.59}$$

From the eq (3.13), retrieve quantization property of a metric fluctuation with considering the environment as a metric field. Following equation is

$$\begin{aligned} \frac{d}{dt}\rho(t) &= -\int_0^t dt'\ Tr_\epsilon[\hat{H}(t), [\hat{H}(t'), \hat{\rho}(t')]] \\ &\equiv -\int_0^t dt' \int dh_{\mu\nu} P(h_{\mu\nu})[\hat{H}(t), [\hat{H}(t'), \hat{\rho}(t')]] \\ &\approx -\int_0^t dt' \iint d\vec{r} d\vec{r}' \langle\langle h_{\mu\nu}(\mathbf{x}) h_{\rho\sigma}(\mathbf{x}')\rangle\rangle [\hat{T}^{\mu\nu}(\mathbf{x}'), [\hat{T}^{\rho\sigma}(\mathbf{x}), \rho(t)]. \end{aligned} \tag{5.60}$$

x,**x**' are relevant four vectors, $(t, \vec{r})$ and $(t', \vec{r}')$. Notice that there is no necessity to consider convolution problem given that we assume the field as Markovian or time independent field[1]. In addition, $\langle\langle hh \rangle\rangle$ now represents the correlation function of perturbative metric h[2]. Correlation function used in Diósi paper,

[1]For more details, please refer Appendix B

[2]As can be seen above, we have to assume that $|h_{\mu\nu}| \ll 1$

unlike that we used the source of the fluctuation as a classical such as thermal or internal dynamics. In addition, we assume this classical fluctuation as Markovian. In Newtonian limit, only 00 components dominant, our correlation function is therefore

$$\langle\langle h_{00}(\mathbf{x})h_{00}(\mathbf{x}')\rangle\rangle = \gamma_{rr'}\delta(t-t'). \tag{5.61}$$

In the DP model, $\gamma_{rr'}$ is granted by Newtonian interaction between matters [9]

$$\gamma_{rr'} = \gamma\frac{1}{|\vec{r}-\vec{r}'|}. \tag{5.62}$$

This kernel represents the classical interaction between matters in the metric field. Finally, resulting master equation can be simplified as

$$\frac{d}{dt}\rho(t) = -\iint d\vec{r}d\vec{r}'\gamma_{rr'}[\hat{\varrho}(\vec{r}'),[\hat{\varrho}(\vec{r}),\rho(t)]] = -\gamma\iint\frac{d\vec{r}d\vec{r}'}{|\vec{r}-\vec{r}'|}[\hat{\varrho}(\vec{r}'),[\hat{\varrho}(\vec{r}),\rho(t)]]. \tag{5.63}$$

Similar to Diósi's result, we can also check gravity related spontaneous collapse. Its decoherence rate is proportional to the difference between position vectors; $\rho(\vec{r}-\vec{r}') \propto \exp(-|\vec{r}-\vec{r}'|t)$ [9].

Whereas since we do not deal with measurement, description on the back-action effect induced by a stochastic noise is impossible. Another difference originates from the ensemble averaged state. Notice that we have to interpret this result as not just a statistical average but the decoherence effect for we have no exact information about the fluctuation of the field in each interaction. I address that this decoherence effect in a macroscopic state is **not fake decoherence at all**. In this sense, especially in the macroscopic body interaction, this effect becomes significant due to the dominant energy scale.

Appendix A

Example of POVM measurement

This example suggest the optimized measurement with the minimum error [33, 59]. To explain it, we have to remind the concepts of probability operator measures (POM) in physics or positive operator valued measurement (POVM) in mathematics. It can depict quantum teleportation situation constructed by POVM operator. In addition, we have to remember POVM measurement is a **non-selective measurement**. Namely, each measurement operator is decided not by observer's will but within specific probability with regarding to the system. Let us assume Alice and Bob do quantum teleportation using Non-orthogonal states. Alice sends her qubit $|0\rangle$ or $|+\rangle$ to Bob where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. In general, there is no way to distinguish non-orthogonal states completely because there fidelity is larger than zero ($|\langle 0|+\rangle| > 0$). However, when Bob measures her message using the set of three POVM operators $\{\hat{E}_1, \hat{E}_2, \hat{E}_3\}$ there exists the method to *do not make mistakes at all on distinction between two states* (even it is not a perfect discrimination in each time).

Alice	Bob	Probability
$\|0\rangle$	$\hat{E}_1$	0
$\|0\rangle$	$\hat{E}_2$	C_2
$\|0\rangle$	$\hat{E}_3$	$1 - C_2$
$\|+\rangle$	$\hat{E}_1$	C_1
$\|+\rangle$	$\hat{E}_2$	0
$\|+\rangle$	$\hat{E}_3$	$1 - C_1$

Table A.1: Probabilities for each POVM cases

Operators are obtained by

$$\begin{aligned} \hat{E}_1 &= C_1(|1\rangle\langle 1|), \\ \hat{E}_2 &= C_2(|-\rangle\langle -|), \\ \hat{E}_3 &= \hat{I} - \hat{E}_1 - \hat{E}_2. \end{aligned} \tag{A.1}$$

Where $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. Notice that these 3 operators satisfy POVM conditions as supposing each constant C is less than 1 ($\sum_i \hat{E}_i = \hat{I}$). In this setting, Bob can verify when he measures the result of $\hat{E}_1$ he can conclude that she sent $|+\rangle$ with safety. Since there is no probability to be detected $|0\rangle$ with $\hat{E}_1$. In addition, if he measures the result of $\hat{E}_2$ he also can verify she sent $|0\rangle$ for the same reason.

On the contrary, when Bob measures the result of $\hat{E}_3$, there is no way to Bob decide a message. That is because there are probabilities to be detected in both cases. As an interpretation on $\hat{E}_3$, it indicates "error rate". Then, during the quantum teleportation, there is the necessity of reducing the probability to be detected by $\hat{E}_3$ to the minimum value. This constants C_1, C_2 are decided to minimize eigenvalue of $\hat{E}_3$ which is related to probability. In other words, it is settled to make maximize success rate. While $\hat{E}_3$ must positive operator also, eigenvalues of $\hat{E}_3$ cannot go to zero. Constant is granted for an arbitrary Alice's message $|a\rangle$, $|b\rangle$

$$C_1 = C_2 = C_{max} = \frac{1}{1+|\langle a|b\rangle|}. \tag{A.2}$$

There are maybe various methods to prove it. But we think this method is the simplest one. Since each operator is positive, we can use a Bloch sphere representation of density matrix with regarding to $\vec{a}^{\perp}$ (Notice: Density matrix is positive matrix). In addition, assuming maximum entropy principle, so constant of E_1 and E_2 are equivalent with C ($C_1 = C_2 = C$).

In order to C_{max}, let us represent $|a\rangle$ and $|b\rangle$ using a Bloch sphere representation

$$|a\rangle\langle a| = \frac{I + (\vec{a}\cdot\vec{\sigma})}{2}, \qquad |b\rangle\langle b| = \frac{I + (\vec{b}\cdot\vec{\sigma})}{2}. \tag{A.3}$$

Now, we use the orthogonal set of state $|a\rangle$, $|a^{\perp}\rangle$ corresponds to a Bloch vector $-\vec{a}$. Consequently, $\hat{E}_1, \hat{E}_2$ and $\hat{E}_3$ in terms of a Bloch sphere representation is then

$$\begin{aligned}
\hat{E}_1 &= C\frac{I - (\vec{a}\cdot\vec{\sigma})}{2}, \\
\hat{E}_2 &= C\frac{I - (\vec{b}\cdot\vec{\sigma})}{2}, \\
\hat{E}_3 &= I - C\frac{I - (\vec{a}\cdot\vec{\sigma})}{2} - C\frac{I - (\vec{b}\cdot\vec{\sigma})}{2}.
\end{aligned} \tag{A.4}$$

Arrangement on E_3 becomes

$$E_3 = (1-C)I + C\frac{((\vec{a}+\vec{b})\cdot\vec{\sigma})}{2}. \tag{A.5}$$

Its eigenvalues are

$$\begin{aligned}
\lambda_1 &= (1-C) + \frac{C|\vec{a}+\vec{b}|}{2}, \\
\lambda_2 &= (1-C) - \frac{C|\vec{a}+\vec{b}|}{2}
\end{aligned}. \tag{A.6}$$

From $|b\rangle$ in terms of $|a\rangle$ in a Bloch sphere can be represented by

$$|b\rangle = cos\left(\frac{\theta}{2}\right)|a\rangle + e^{i\phi}sin\left(\frac{\theta}{2}\right)|a^{\perp}\rangle. \tag{A.7}$$

In addition, $\vec{a} \cdot \vec{b} = |ab|cos\theta$. Use cosine law, $|\vec{a} + \vec{b}|$ becomes

$$|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2(\vec{a} \cdot \vec{b}) = 2 + 2(2|\langle a|b\rangle|^2 - 1) = 4|\langle a|b\rangle|^2. \tag{A.8}$$

As we require that E_3 is also positive. Its eigenvalues should be equal or larger than 0. Since λ_1 is larger than λ_2, when

$$\lambda_2 = 1 - C - C|\langle a|b\rangle| \geq 0, \tag{A.9}$$

E_3 is positive with certain. Consequently, C is thus

$$C \leq \frac{1}{1 + |\langle a|b\rangle|}. \tag{A.10}$$

C_{max} becomes the maximum value for minimize error rate. In our example, since $\langle 0|+\rangle = \frac{1}{\sqrt{2}}$, $C_{max} = \frac{\sqrt{2}}{1+\sqrt{2}}$. You can check the identical constant in book [33].

Appendix B

Classical fluctuation and decoherence example: Magnetic field fluctuation.

Let us consider entangled bell state of two distinguishable spin 1/2 particles in sigma z basis which is separated left and right in the middle of our coordinates. In addition, each particle state leaves in Hilbert space $H_L \otimes H_R$ such that

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(| \uparrow\downarrow\rangle_{LR} + | \downarrow\uparrow\rangle_{LR}\right). \tag{B.1}$$

After they fly away from each other, let us apply the constant magnetic field in z-direction -z on the left, +z on the right simultaneously. When we designate t as passage time of particles in the magnetic field, this state after time evolution is

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}} \left(e^{i\mu Bt}| \uparrow\downarrow\rangle_{LR} + e^{-i\mu Bt}| \downarrow\uparrow\rangle_{LR}\right). \tag{B.2}$$

Where $\mu = \frac{e\hbar}{mc}$. From now on, we assume that the magnetic field has Gaussian distribution with classical noise by the environment of the field applier. In

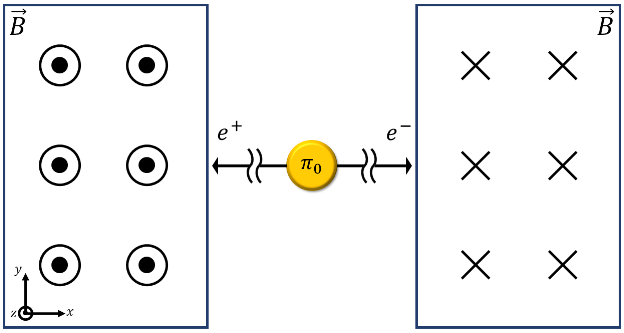

Figure B.1: Applying magnetic field onto bell state

addition, its strength is near to the ensemble-averaged value of the magnetic field $\langle B \rangle$ like a formal situation in Lab.

$$p(B) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp^{-\frac{(B-\langle B \rangle)^2}{2\sigma^2}} \tag{B.3}$$

Using the density matrix, its time evolution is written by

$$\begin{aligned} \rho(t) &= \int_{-\infty}^{\infty} dB \; p(B)\hat{U}(t)\hat{\rho}(0)\hat{U}^\dagger(t) \\ &= |\uparrow\downarrow\rangle_{LR}\langle\uparrow\downarrow| + |\downarrow\uparrow\rangle_{LR}\langle\downarrow\uparrow| + e^{-2\mu^2 t^2 \sigma^2}\left(|\uparrow\downarrow\rangle_{LR}\langle\downarrow\uparrow| + |\downarrow\uparrow\rangle_{LR}\langle\uparrow\downarrow|\right). \end{aligned} \tag{B.4}$$

As can be seen, off-diagonal terms decay as t larger. This decoherence[1] appears because of the fluctuation of the magnetic field. In other words, we do not have exact information about the surround. This decoherence implies the existence of interaction between the bell state and magnetic field with a fluctuation.

In general, decoherence property of an arbitrary mixed state can be described using master equation. Let us recall eq (5.60). With parameter B using the technique written in Diósi's paper [61],

[1] Notice that in the microscopic picture, it is **not** decoherence exactly. Sometimes, it is the so-called "fake decoherence"[36, 60]. However, when given that macroscopic picture describing numerous interactions, it contains **"Lack of knowledge"** also. Consequently, it is valid for describe decoherence due to the classical field fluctuation.

$$
\begin{aligned}
\frac{d}{dt}\rho(t) &= -\int_0^t dt' \int_{-\infty}^{\infty} dBP(B)[\hat{\mu}\cdot\vec{B}(t),[\hat{\mu}\cdot\vec{B}(t),\rho(t)]] \\
&= -\int_0^t dt' \langle\langle B(t)B(t')\rangle\rangle[\hat{\mu}_{z1}\otimes\hat{I}_2-\hat{I}_1\otimes\hat{\mu}_{z2},[\hat{\mu}_{z1}\otimes\hat{I}_2-\hat{I}_1\otimes\hat{\mu}_{z2},\rho(t)]].
\end{aligned}
\tag{B.5}
$$

Because we have assumed the magnetic field is time-independent[2], correlation function of the magnetic field is simply obtained by the variance σ^2. As a result, for the general mixed state in the basis of 2 qubits state, equation becomes

$$
\frac{d}{dt}\rho(t) = \frac{d}{dt}\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} = -\frac{\sigma^2\mu^2 t}{4}\begin{pmatrix} 0 & 4b & 4c & 0 \\ 4e & 0 & 16g & 4h \\ 4i & 16j & 0 & 4l \\ 0 & 4n & 4o & p \end{pmatrix}. \tag{B.6}
$$

It can be easily checked that there is no amplitude damping. In addition, our bell state case, the decoherence factor is 4 times larger than other terms and decoherence time is nearly $7 * 10^{-7}$ sec. Finally, general two qubits state is obtained by

$$
\rho(t) = \begin{pmatrix} a(0) & b(0)*e^{-\mu^2\sigma^2t^2/2} & c(0)*e^{-\mu^2\sigma^2t^2/2} & d(0) \\ e(0)*e^{-\mu^2\sigma^2t^2/2} & f(0) & g(0)*e^{-2\mu^2\sigma^2t^2} & h(0)*e^{-\mu^2\sigma^2t^2/2} \\ i(0)*e^{-\mu^2\sigma^2t^2/2} & j(0)*e^{-2\mu^2\sigma^2t^2} & k(0) & l(0)*e^{-\mu^2\sigma^2t^2/2} \\ m(0) & n(0)*e^{-\mu^2\sigma^2t^2/2} & o(0)*e^{-\mu^2\sigma^2t^2/2} & p(0) \end{pmatrix}. \tag{B.7}
$$

Notice that this kind of Gaussian decay occurs for we have taken the ensemble average. As mentioned above, it can be check if and only if we take the ensemble average. That is why someone does not accept this decoherence idea. However, notice that we have to understand this phenomenon as a simulation the influence of the environment on the macroscopic system. In that sense, it is decoherence in the macroscopic picture by the classical field fluctuation!

[2]If you want to treat AC-field, we ask for more complicated one, Floquet type master equation.

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