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이학 박사 학위논문

Geometry of moduli spaces of rational curves on Fano varieties

(파노 대수다양체 상의 유리곡선들의 모듈라이 공간의 기하학)

2018년 8월

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Geometry of moduli spaces of rational curves on Fano varieties

A dissertation
submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
to the faculty of the Graduate School of
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by

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Abstract

Geometry of moduli spaces of rational curves on Fano varieties

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In this thesis, we consider two families of Fano varieties as main objects. One is the moduli space $\mathcal{N}$ of rank 2 stable vector bundles over a smooth projective curve X over $\mathbb{C}$ with a fixed determinant line bundle $\mathcal{O}_X(-x)$ for a fixed point $x \in X$, and the others are hyperplane sections of the Grassmannian $\mathrm{Gr}(2,5)$. We study the moduli spaces of smooth rational curves and their various compactifications as well as their geometric structures. For the Fano variety $\mathcal{N}$, we mainly consider the compactifications of the moduli space of degree 3 smooth rational curves as a stable map space and discuss topological types of stable maps contained in the boundary of the compactified space. For hyperplane sections of a Grassmannian $\mathrm{Gr}(2,5)$, we discuss rationality of moduli space of smooth rational curves of degree ≤ 3, and then we consider compactifications of the moduli space of smooth conics by the Hilbert scheme. We further discuss smoothness of these compactified spaces using birational models of the compactified spaces.

Key words: Moduli space, Rational curves, Fano varieties, Hilbert scheme, Stable map space, Grassmannian, Moduli space of vector bundles
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Contents

Chapter 1

Introduction

In this thesis, we deal with two families of Fano varieties as main objects. One is the moduli space $\mathcal{N}$ of rank 2 stable vector bundles over a smooth projective curve X, of genus $g(X) \geq 4$, with fixed determinant line bundle $\mathcal{O}_X(-x)$ for a fixed point $x \in X$, and the others are hyperplane sections of the Grassmannian $\mathrm{Gr}(2,5)$. We denote Y^m the intersection of the image of $\mathrm{Gr}(2,5)$ under the Plücker embedding into $\mathbb{P}^9$ with $6-m$ general hyperplanes in $\mathbb{P}^9$. Then Y^m is a smooth Fano variety with dimension m. These Fano varieties have been studied for a long time. The moduli space $\mathcal{N}$ was first constructed by Seshadri [92], in the 1960s, and its properties have been studied in numerous works including [29, 79, 88, 7, 81]. The study of the hyperplane sections of the Grassmannian $\mathrm{Gr}(2,5)$ dates back to the 1890s. For instance, Castelnuovo studied Y^3 on his work [11]. From a more general viewpoint, Piontkowski and Van de Ven studied the automorphisms group of hyperplane sections of $\mathrm{Gr}(2,n)$ and its orbits in [85], and also Cheltsov and Shramov studied the Fano threefold Y^3 from the birational geometry viewpoint [14]. We summarize these results in Chapter 2, Section 2.6.

In this thesis, we study moduli space of smooth rational curves in these Fano varieties and their various compactifications. We consider the moduli space of degree d smooth rational curves on a smooth projective variety

V with a fixed polarization $\mathcal{O}_V(1)$, as an open subscheme of the degree d map space $\mathrm{Hom}_d(\mathbb{P}^1, V)$, which is defined in [61] as an open subscheme of a Hilbert scheme of curves in $\mathbb{P}^1 \times V$. From now on, we denote the moduli space of degree d smooth rational curves on the projective variety V by $R_d(V)$.

The study of rational curves and moduli space of rational curves in Fano varieties has led to useful results in many cases. First, there has been a close connection between constructions of holomorphic symplectic manifolds and moduli spaces of rational curves in Fano varieties. Beauville and Donagi in [3] considered the Fano variety of lines in a cubic 4-fold $X \subset \mathbb{P}^5$, denoted by $F_1(X)$. The authors showed that $F_1(X)$ is a holomorphic symplectic manifold. The holomorphic 2-form is constructed as follows. Consider a universal family of lines F over $F_1(X) \times X$. Choose a generater α of $(3,1)$-forms $H^{3,1}(X) \cong \mathbb{C}$. Then, using the projections p_1, p_2 from $F_1(X) \times X$ to $F_1(X)$ and X respectively, they obtained a holomorphic 2-form $w := (p_1)_*(p_2)^*\alpha$.

Iliev and Manivel in [50] considered the Hilbert scheme of conics in a Fano 4-fold $Z := \mathrm{Gr}(2,5) \cap H \cap Q$, where H is a general hyperplane in the Plücker embedding space $\mathbb{P}^9$, and Q is a general quadric hypersurface. The authors denoted the Hilbert scheme of conics in Z by $F_g(Z)$, which is a smooth 5-fold. From the space $F_g(Z)$ the authors constructed a holomorphic symplectic 4-fold, denoted by $\widetilde{Y}_Z^\vee$. Moreover, the authors showed that $\widetilde{Y}_Z^\vee$ coincide with the an EPW sextic, which is a double cover of a sextic hypersurface in $\mathbb{P}^5$, constructed by O'Grady [83].

Lehn, Lehn, Sorger and van Straten [67] considered the Hilbert scheme of twisted cubics in a cubic 4-fold $Y \subset \mathbb{P}^5$. The authors denoted this space by $M_3(Y)$. Then the authors showed that $M_3(Y)$ is a smooth 8-dimensional variety and there is a contraction $M_3(Y) \to Z$ where Z is a holomorphic symplectic 8-fold.

On the other hand, in [24], Clemens and Griffith considered the Fano variety of lines in a smooth cubic threefold $V \subset \mathbb{P}^4$, denoted by S. Using its

Albanese variety $\mathrm{Alb}(S)$, the authors proved that the smooth cubic threefold V is not rational. The authors used the result of Gherardelli [37] that the Albanese variety $\mathrm{Alb}(S)$ and the intermediate Jacobian $J(V)$ of V are isogenous. In fact, the authors considered a more general setting. When V is a smooth algebraic threefold and S is a smooth parameter space of a family of algebraic curves in V, then there exists a map called Abel-Jacobi map :

$$\mathrm{Alb}(S) \to J(V).$$

In addition, Takkagi and Zucconi in [95] proved the existence of a Scorza quartic by studying the geometry of the Hilbert scheme of conics in the blow-up space of a smooth Fano threefold Y^3. Also, in [15, 86], the geometry of rational curves and the moduli of rational curves in Fano varieties was used.

Moreover, of course, the study on the geometry of moduli spaces of rational curves on Fano varieties also helps virtual curve counts on Fano varieties. Munoz [74] studied the quantum cohomology of the moduli space $\mathcal{N}$ of rank 2 stable vector bundles on the smooth projective curve X over $\mathbb{C}$ with genus $g \geq 1$ with fixed odd degree line bundle. For this, he studied moduli space of genus 0, degree 1 stable map space $\mathbf{M}_0(\mathcal{N}, 1) := \overline{M}_{0,0}(\mathcal{N}, 1)$ with target space $\mathcal{N}$.

1.1 Moduli spaces of smooth rational curves in Fano varieties

The results presented in Chapter 3 are based on the results obtained joint with Kiryong Chung and Jaehyun Hong in [19], the results of Castravet [12, 13] and the results of Kiem [54].

The moduli space $R_d(\mathcal{N})$ of degree d smooth rational curves on $\mathcal{N}$ has been studied for a long time. Brosius studied rank 2 vector bundles on a ruled surface [9, 10]. Since a regular map $\mathbb{P}^1 \to \mathcal{N}$ corresponds to a rank 2 bundle on the ruled surface $\mathbb{P}^1 \times X$, Castravet classified all irreducible components of $R_d(\mathcal{N})$ for all degree d based on the result of Brosius. Furthermore, Castravet gave a geometric interpretation for the elements of each irreducible components. Also, Kiem [54] independenty classified all maps $\mathbb{P}^1 \to \mathcal{N}$ for degree $d \leq 4$ cases based on the Brosius result. On the other hand, Kilaru [57] classified all maps $\mathbb{P}^1 \to \mathcal{N}$ for degree $d = 1, 2$ cases independently from Brosius and Castravet's work.

To study the moduli space $R_d(Y^m)$ for degree $d \leq 3$, we first classify all smooth rational curves $\mathbb{P}^1 \to Y^m$, with degree ≤ 3. For this purpose, we first classify all smooth rational curves $\mathbb{P}^1 \to \mathrm{Gr}(2, n) =: G$ with degree ≤ 3. Using this classification, we define the following rational morphisms(cf. Proposition 3.2.3) :

1. A *vertex map* $\zeta_1 : R_1(G) \to \mathbb{P}^{n-1}$ which maps each projective lines in G to its vertex.

2. An *envelope map* $\zeta_2 : R_2(G) \dashrightarrow \mathrm{Gr}(4, n)$ which maps each smooth conic in G to its envelope.

3. A *axis map* $\zeta_3 : R_3(G)) \dashrightarrow \mathrm{Gr}(2, n)$ which maps each twisted cubic curve in $G = \mathrm{Gr}(2, 5)$ to its axis.

This classification of smooth rational curves and construction of rational morphisms were already studied in the literature. For the degree 1 case, there is a corresponding result in Harris' book [40, Excercise 6.9].

For degree 2 case, the classification of conics in the Grassmannian $\mathrm{Gr}(2, n)$ can be found in [48], [26] and [80]. Our classification may look different from theirs but we can easily check that smooth conics obtained from a rational normal scroll $S(p_0, C_0)$ of a point p_0 and a smooth conic C_0 in the projective space $\mathbb{P}^{n-1}$ (See Proposition 3.2.3) correspond to σ-conics in [48, 26], and smooth conics obtained from a rational normal scroll $S(\ell_0, \ell_1)$ of two lines ℓ_0 and ℓ_1 in $\mathbb{P}^{n-1}$ correspond to τ-conics and ρ-conics in [48, 26]. Moreover, the idea of assigning an envelope $\mathbb{P}^3 \subset \mathbb{P}^{n-1}$ for each conic in $\mathrm{Gr}(2, n)$ also appeared in [48, 26].

For the degree 3 case, we could not find a former reference about the classification of twisted cubics in $\mathrm{Gr}(2, n)$ and the axis map. But this construction may be classical since its construction is very simple.

In addition, we exactly describe general fibers of these morphisms. These rational morphisms ζ_i restrict to the moduli space of rational curves $R_i(Y^m)$ in $Y^m \subset \mathrm{Gr}(2, 5)$. Then we can also exactly describe the general fiber of these restricted morphisms. Moreover, we show that these morphisms are birationally equivalent to Grassmannian bundles. Using these properties, we show that $R_i(Y^m)$ are rational varieties for $1 \leq i \leq 3$ and $1 \leq m \leq 6$, which is the main result of Chapter 3.

Main Theorem 1 (Theorem 3.3.1)**.** *Each moduli space $R_d(Y^m)$ of degree d smooth rational curves on Y^m is a rational variety for $2 \leq m \leq 6$ and $1 \leq d \leq 3$.*

Next, we consider various compactifications of these moduli spaces in Chapter 4 and 5.

1.2 Compactifications of the moduli spaces of smooth rational curves in Y^m

The results presented in Chapter 4 are based on the results obtained joint with Chung and Hong in [19].

In this chapter, we consider compactifications of the moduli spaces $R_3(Y_m)$ of smooth rational curves of degree $d \leq 3$ in $Y^m \subset \mathrm{Gr}(2, 5)$.

For $m = 6$ case, i.e. $Y^m = \mathrm{Gr}(2, 5) = G$, G is a homogeneous variety. In this case, we can use a result of Chung, Hong, and Kiem [18], which deals with the birational geometry of the Simpson compactifications and the Hilbert compactifications of moduli spaces of smooth conics and moduli spaces of twisted cubics in homogeneous varieties. As a result, we obtain Theorem 4.2.3 and 4.2.4. Furthermore, we will check that we can apply the methods in [18] for the homogeneous space $\mathrm{Gr}(2,2n)\cap H$ in Chapter 6.

On the other hand, we construct the following blow-up and blow-down diagram :

$$\begin{tikzcd}
 & H_2(\mathrm{Gr}(2,\mathcal{U})) \arrow[dl, "\Xi"'] \arrow[dr, "\Phi"] \arrow[dd, Rightarrow, "\tilde{\zeta}_2"] & \\
\mathrm{Gr}(2,\wedge^2\mathcal{U}) \arrow[dr, Rightarrow] & & H_2(G) \arrow[dl, dashed, "\zeta_2"'] \\
 & \mathrm{Gr}(4,5), &
\end{tikzcd} \tag{1.1}$$

where $\mathcal{U}$ is the tautological rank 4 bundle over the Grassmannian $\mathrm{Gr}(4,5)$, $H_2(G)$ and $H_2(\mathsf{Gr}(2,\mathcal{U}))$ are Hilbert scheme compactification of $R_2(G)$ and $R_2(\mathsf{Gr}(2,\mathcal{U}))$ respectively, ζ_2 is a rational map induced from the envelope map $R_2(G) \dashrightarrow \mathrm{Gr}(4,5)$. The blow-up morphisms Φ and Ξ were constructed by Iliev-Manivel in [50].

The blow-up locus of the map Ξ can be identified with a set consisting of pairs (P,V_4), where P is a $\sigma_{2,2}$-type plane or $\sigma_{3,1}$-type plane in G, and V_4 corresponds to a linear space $\mathbb{P}^3 \subset \mathbb{P}^4 = \mathbb{P}(\mathbb{C}^5)$ enveloping the plane P. We denote this blow-up locus by $T(G)$.

The above diagram also plays a key role in studying the Hilbert scheme of conics $H_2(Y^m)$ in Y^m, for the $m=4,5$ cases. For this purpose, we want to 'restrict' the above diagram to the Y^m case. So we need to know how the blow-up loci of Ξ and Φ change for the $m=4,5$ cases. So we study the spaces of lines and planes in Y^m in this chapter. For $m=4$, the result on the spaces of lines and planes are due to Todd [97]. For $m=6$, the result on the space of lines and planes appeared in [26, Section 3.1].

If we let $S(Y^m) = \{V_2 \in \mathrm{Gr}(2,\wedge^2\mathcal{U}) \mid V_2 \subset Y^m\}$, we should check that

$\mathsf{T}(\mathsf{G})$ and $\mathsf{S}(\mathsf{Y}^m)$ cleanly intersect in $\mathrm{Gr}(3, \wedge^2\mathcal{U})$. We first compute the intersection locus in Chapter 4, Section 4.3.3. We check the clean intersection in two ways (Chapter 4, Lemma 4.3.7 and Subsection 4.3.4). As a conclusion, we succeed to restrict the above diagram 1.1 to Y^m cases, and obtain the following main result of Chapter 4.

Main Theorem 2 (Theorem 4.3.9, 4.4.7, 4.5.2)**.** *The Hilbert scheme* $\mathsf{H}_2(\mathsf{Y}^m)$ *smooth conics in* Y^m *for* $m = 3, 4, 5$ *is a blow-down of* $\widetilde{\mathsf{S}}(\mathsf{Y}^m)$*, which is a blow-up of* $\mathsf{S}(\mathsf{Y}^m) := \mathrm{Gr}(3, \mathcal{K})$ *:*

$$\begin{tikzcd} & \widetilde{\mathsf{S}}(\mathsf{Y}^m) \arrow[dl, "\Xi"'] \arrow[dr, "\Phi"] & \\ \mathsf{S}(\mathsf{Y}^m) & & \mathsf{H}_2(\mathsf{Y}^m), \end{tikzcd} \tag{1.2}$$

where Ξ *is the blow-up along* $\mathsf{T}(\mathsf{Y}^m)$ *and* Φ *is the blow-up along the locus of conics lying on* $\sigma_{2,2}$*-type planes. Furthermore,* $\mathsf{H}_2(\mathsf{Y}^m)$ *is an irreducible smooth variety for* $m = 3, 4, 5$.

We also note that blow-up and blow-down diagrams like (1.2) are usually helpful for computing Poincare polynomials (cf. [18, Chapter 5]) and Chow rings (cf. [22]).

1.3 Compactifications of the moduli spaces of degree 3 smooth rational curves in $\mathcal{N}$

The results presented in Chapter 5 are based on the results obtained joint with Chung in [20].

Independently of the compactification story, the moduli space of stable maps $\mathbf{M}_0(\mathcal{N}, d) := \overline{\mathsf{M}}_{0,0}(\mathcal{N}, d)$ in $\mathcal{N}$ has been studied for low degree cases.

For the $d = 1$ case, Munoz [74] showed that $\mathbf{M}_0(\mathcal{N}, 1)$ is a fibration over $\mathrm{Pic}^0(\mathsf{X})$, with fiber $\mathrm{Gr}(2, g(\mathsf{X}))$. Since the virtual counts on the stable map space are related to quantum cohomology, the authors studied the quantum cohomology of the space $\mathcal{N}$.

For the $d = 2$ case, Kiem [54] showed that $\mathbf{M}_0(\mathcal{N}, 2)$ has two irreducible components. One parametrizes Hecke curves and the other one parametrizes the rational curves of extension type. Furthermore, the two irreducible components intersect transversally and both components can be obtained by the partial desingularizations of GIT(Geometric Invariant Theory) quotients of projective varieties. Furthermore, the author also studied the Hilbert scheme $\mathrm{Hilb}^{2m+1}_{\mathcal{N}}$ of conics in $\mathcal{N}$, which the author denotes it by $\mathbf{H}$. The author related this Hilbert scheme $\mathbf{H}$ with the stable map space $\mathbf{M}_0(\mathcal{N}, 2)$ by a composition of a blow-up and a contraction. The author also showed that the two irreducible components of $\mathbf{H}$ are smooth.

In this thesis, we deal with $d = 3$ case. A big difference arises as there exists an irreducible component in $\mathbf{M}_0(\mathcal{N}, 3)$, whose general elements has nodal domain curves, and whose dimension is much bigger than the expected dimension. In fact, there are 4 irreducible components in $\mathbf{M}_0(\mathcal{N}, 3)$. Only two of them comes from compactifying of the moduli space $\mathsf{R}_3(\mathcal{N})$ of smooth rational curves. We can easily observe that one of them is easily described. So we concentrate on the other component in our thesis. We denote this component by Λ_1. We study which topological types of nodal curves are contained in the boundary of Λ_1. We classify all stable maps in $\mathbf{M}_0(\mathcal{N}, 3)$ in Lemma 5.3.1 into five types, and study which types of stable maps are contained in the component Λ_1.

In Section 5.3.1, we consider a conjectural morphism :

$$\widetilde{\Psi} : \widetilde{\mathbf{P}} \to \mathcal{N}$$

where $\widetilde{\mathbf{P}}$ is a relative blow-up space, which is a fibration over $\mathrm{Pic}^1(X)$, whose fiber over a line bundle $L \in \mathrm{Pic}^1(X)$ is isomorphic to $\widetilde{\mathbf{P}}_L := \mathrm{Bl}_X \mathbb{P}\mathrm{Ext}^1(L, L^{-1}(-x))$, where the blow-up locus X is embedded in $\mathbb{P}\mathrm{Ext}^1(L, L^{-1}(-x))$ by the complete linear system $|K_X \otimes L^2(x)|$. Then, the conjectural morphism $\widetilde{\Psi}$ will induces a morphism between stable map spaces :

$$\mathbf{M}_0(\widetilde{\mathbf{P}}_L, \beta) \xrightarrow{i} \mathbf{M}_0(\widetilde{\mathbf{P}}, \beta) \xrightarrow{j} \mathbf{M}_0(\mathcal{N}, 3)$$

where the homology class β is the l.c.i pull-back π^*[line], for the blow-up morphism $\pi : \widetilde{\mathbf{P}}_L \to \mathbb{P}\mathrm{Ext}^1(L, L^{-1}(-x))$ where [line] is the homology class of line in the projective space $\mathbb{P}\mathrm{Ext}^1(L, L^{-1}(-x))$.

Then we can observe that the component Λ_1 is contained in the image of the morphism j. Therefore, it is enough to classify topological types of nodal curves in the boundary of $\mathbf{M}_0(\widetilde{\mathbf{P}}, \beta)$, under this conjectural picture. For a non-trisecant line bundle $L \in \mathrm{Pic}^1(X)$ (see Definition 5.2.2), we proved that $\widetilde{\Psi}_L : \widetilde{\mathbf{P}}_L \to \mathcal{N}$ is a closed embedding(see Proposition 5.2.4). Therefore the induced morphism of stable maps $\mathbf{M}_0(\widetilde{\mathbf{P}}_L, \beta) \to \mathbf{M}_0(\mathcal{N}, 3)$ is also a closed embedding.

On the other hand, we also conjecture that there is a morphism $p : \Lambda_1 \to \mathrm{Pic}^1(X)$ which is compatible with the morphism j and the projection $q : \mathbf{M}_0(\widetilde{\mathbf{P}}, \beta) \to \mathrm{Pic}^1(X)$. Then over the non-trisecant line bundle $L \in \mathrm{Pic}^1(X)$, we expect that the fiber $p^{-1}(\Lambda_1)$ isomorphic to an irreducible component of the stable map space $q^{-1}(L) = \mathbf{M}_0(\widetilde{\mathbf{P}}_L, \beta)$(We also conjecture that the fiber of the projection q over L is equal to $\mathbf{M}_0(\widetilde{\mathbf{P}}_L, \beta)$). Based on this conjectural picture, we focused on the stable map space $\mathbf{M}_0(\widetilde{\mathbf{P}}_L, \beta)$ in this thesis.

In this chapter, we classify all stable maps which are element of $\mathbf{M}_0(\widetilde{\mathbf{P}}_L, \beta)$. It is the main theorem of this chapter.

Main Theorem 3 (Theorem 5.3.2)**.** *The stable map space* $\mathbf{M}_0(\widetilde{\mathbf{P}}_L, \beta)$ *is a union of two irreducible components* $\overline{B}_1$ *and* $\overline{B}_2$ *which satisfies the following*

1. B_1 *parametrizes projective lines in* $\mathbb{P}^{g+1}_L \setminus X$*. Moreover,* $\overline{B}_1$ *consists of stable maps of types* $(1), (2), (3), (5)$ *in Lemma 5.3.1.*

2. B_2 *parametrizes the union of a smooth conic in the exceptional divisor*

of $\widetilde{\mathbf{P}}$ and a proper transformation of a projective line ℓ where ℓ is a projective line which intersects the curve X with multiplicity 2 (so that ℓ can be a tangent line of X), intersecting with the smooth conic at a point. Moreover, $\overline{B}_2$ consists of stable maps of types $(4), (5)$ in Lemma 5.3.1.

In particular, closed points of the intersection $\overline{B}_1 \cap \overline{B}_2$ correspond to type (5) stable maps of Lemma 5.3.1.

We also note that the two irreducible components $\overline{B}_1$ and $\overline{B}_2$ have dimension $3g$, which exactly coincide with the expected dimension of the moduli space $\mathbf{M}_0(\widetilde{\mathbf{P}}_L, \beta)$. Here, we expect that the irreducible $\overline{B}_1$ maps to the Λ_1 via the morphism of stable map spaces Ψ_L.

The existence of the conjectural morphisms $\widetilde{\Psi} : \widetilde{\mathbf{P}} \to \mathcal{N}$, $p : \Lambda_1 \to \mathrm{Pic}^1(X)$ is not proven yet. Moreover, the statement that the fiber $q^{-1}(L)$ is isomorphic to $\mathbf{M}_0(\widetilde{\mathbf{P}}_L, \beta)$ is not clear yet. Also, over the trisecant line bundle $L \in \mathrm{Pic}^1(X)$, we do not know the topological types of the stable maps which are elements of $\mathbf{M}_0(\widetilde{\mathbf{P}}_L, \beta)$. So, there are still many obstacles remains for figuring out all topological types of all nodal curves in the boundary of $\mathbf{M}_0(\widetilde{\mathbf{P}}, \beta)$. We conclude with the following questions.

Question.

1. Classify all stable maps in the component Λ_1 of the moduli space $\mathbf{M}_0(\mathcal{N}, 3)$.
2. Let $U \subset \mathrm{Pic}^1(X)$ be an open sublocus of non-trisecant line bundles. Let us assume that there is a conjectural morphism $p : \Lambda_1 \to \mathrm{Pic}^1(X)$. Then, elements of $\Lambda_1 \times_{\mathrm{Pic}^1(X)} U$ consists of stable maps of types $(1), (2), (3), (5)$ in Lemma 5.3.1?

Chapter 2

Preliminaries

2.1 Moduli problems

Throughout this chapter, we fix k to be an algebraically closed field with characteristic 0.

Moduli problem arises in many areas in algebraic geometry. First, we consider a class of object we want to collect, i.e. algebraic curves, vector bundles, closed subschemes in projective spaces, etc. Then, roughly speaking, moduli problem is to find a family of these object over some parameter space. Further, in many cases, we want to view objects up to isomorphisms. For examples, degree d-hypersurfaces in $\mathbb{P}^n$ up to PGL$(n+1)$-action, algebraic curves up to isomorphisms, etc. So we also consider equivalences between families.

In summary, a moduli problem consists of three components : (1) a parameter space scheme S, (2) a flat morphism $\phi : F \to S$ such that each fiber F_s over any closed points $s \in S$ are objects what we want to collect, i.e. algebraic curves of genus g, algebraic surfaces, etc. (3) an equivalence relations between families: For example, When $\phi_1 : F_1 \to S$ and $\phi_2 : F_2 \to S$ are two flat families of genus g curves on S, then equivalence relation is an isomorphism $\psi : F_1 \to F_2$ such that $\psi \circ \phi_2 = \phi_1$. Flatness condition is important since it preserves many topological invariants of the fibers, i.e. degrees, genus, Hilbert polynomials.

Sometimes, we want to consider a family with extra structures, there are some examples :

Example 2.1.1 (Family of conics in $\mathbb{P}^2$). Let $S = k^m$ and $F = c_0(t_1, ..., t_m)x_0^2 + c_1(t_1, ..., t_m)x_1^2 = c_0(t_1, ..., t_m)x_2^2 + c_3(t_1, ..., t_m)x_0x_1 + c_4(t_1, ..., t_m)x_0x_2 + c_5(t_1, ..., t_m)x_0^2$, be a polynomial which is homogeneous in coordinate x_0, x_1, x_2 with degree 2, such that $c_0, ..., c_5$ does not commonly vanish in k^m. Then $\{F = 0\} \subset \mathbb{P}^2 \times k^m$ is a flat family over k^m, with a natural projection $\pi : \{F = 0\} \to k^m$. In this case this flat family naturally has an additional structure, an embedding to the ambient space $\mathbb{P}^2 \times k^m$. This kind of addition structure leads to the definition of Hilbert scheme which will be introduced later.

Example 2.1.2. (Family of maps) Consider a flat family of nodal curves $\phi : C \to S$ with genus g over a parameter space S. Furthermore, Consider a map $f : C \to \mathbb{P}^n$. We define equivalence between this pairs $(\phi : C \to S, f : C \to \mathbb{P}^n)$ and $(\phi' : C' \to S, f' : C' \to \mathbb{P}^n)$ if there is an isomorphism $F : C \to C'$ such that $f = f' \circ F$. This kind of additional structure leads to the definition of Stable map space which will be introduced later.

2.1.1 Moduli functors

In various kind of moduli problems, we define moduli functors as a correspondence corresponding to a parameter space S to a set of equivalence class of a flat family. i.e. it is a functor :

$$\begin{aligned} F : (\mathrm{Sch}/k)^{op} &\longmapsto \mathrm{Sets} \\ S &\longmapsto \{\text{equivalence class of flat families over } S\} \end{aligned}$$

If there exists a classifying space of this functor, we call it a fine moduli space

Definition 2.1.1 (Representable functor, fine moduli space, universal family)**.** If there is a scheme $X \in \mathrm{Sch}/k$ such that its Yoneda embedding $h_X(-) := \mathrm{Hom}_{\mathrm{Sch}/k}(-, X)$ is isomorphic to F, we call our moduli functor F representable and we call X a find moduli space of our moduli problem. Furthermore, we call the family on X corresponding to an element $\mathrm{id}_X \in \mathrm{Hom}_{\mathrm{Sch}/k}(X, X)$ a universal family.

In many moduli problems, fine moduli space does not exist. Instead, we have a weaker form of moduli space, called coarse moduli space.

Definition 2.1.2. For a moduli functor F, a coarse moduli space is a pair of a scheme $X \in \mathrm{Sch}/k$ and a natural transform $u : F \to h_M$ such that

(i) $u(\mathrm{Spec}(k)) : F(\mathrm{Spec}(k)) \to \mathrm{Hom}(\mathrm{Spec}(k), M) = \{\text{Set of closed points of M}\}$ is bijective.

(ii) (M, u) is initial among this kind of pairs, i.e. if there are another pair (M', u'), $u' : F \to h_{M'}$. Then there exists a unique natural transform $T : h_M \to h'_M$ makes the following diagram commutes:

$$\begin{tikzcd} F \arrow[rr, "u"] \arrow[dr, "u'"'] & \arrow[d, phantom, "\circlearrowleft"] & h_M \arrow[dl, dashed, "T", "\exists!"'] \\ & h_{M'} & \end{tikzcd}$$

2.2 Hilbert schemes

2.2.1 Hilbert functor and Quot functor

The Hilbert scheme is the moduli space parametrizing subschemes in the projective spaces $\mathbb{P}^n$ with some fixed Hilbert polynomial. A very simple example about family of conics $\mathbb{P}^2$, which has Hilbert polynomial $2t + 1$, was already appeared in the previous section in example 2.1.1.

More explicitly, moduli functor is given by the following. Fix $\mathcal{O}_{\mathbb{P}^n}(1)$ a very ample line bundle of $\mathbb{P}^n$. For a coherent sheaf $\mathcal{E}$ on $\mathbb{P}^n$ we define $\chi(\mathcal{E}) := \sum_{i\geq 0}(-1)^i h^i(\mathcal{E})$. Furthermore, we define a Hilbert polynomial $\mathsf{HP}(\mathcal{E})$ of $\mathcal{E}$ to be $\mathsf{HP}(\mathcal{E})(m) := \chi(\mathcal{E}\otimes\mathcal{O}_{\mathbb{P}^n}(1)^m)$. When we fix a Hilbert polynomial F, then we define a Hilbert functor to be :

$$\begin{aligned}\mathfrak{Hilb}^F_{\mathbb{P}^n} : (\mathrm{Sch}/k)^{op} &\longmapsto \mathrm{Sets}\\ S &\longmapsto \{Z\subset S\times\mathbb{P}^n \mid Z\to S \text{ is flat, } \mathsf{HP}(\mathcal{O}_{Z\times_S\{s\}}) = F\ \forall s\in S\}.\end{aligned}$$

When we replace $\mathbb{P}^n$ by a general projective variety X over k and fix a very ample line bundle L, and define a Hilbert characteristic of a coherent sheaf $\mathcal{E}$ on X to be $\mathsf{HP}(\mathcal{E})(m) := \chi(\mathcal{E}\otimes L^n)$, we have a definition of Hilbert functor $\mathfrak{Hilb}^{F,L}_X$, which parametrizes closed subschemes in the projective variety X with the Hilbert polynomial F.

Sometimes, it is more convenient to consider a slight generalization of Hilbert functors, called Quot functor. Let X be a projective variety and L be a very ample line bundle on X. Then For any coherent sheaf $\mathcal{E}$ on X, we define its Hilbert polynomial $\mathsf{HP}(\mathcal{E})$ to be $\mathsf{HP}(\mathcal{E})(m) := \chi(\mathcal{E}\otimes L^n)$. We fix a coherent sheaf $\mathcal{V}$ on X.

Then the Quot functor $\mathfrak{Quot}^{F,L}_{\mathcal{V}/X}$ is a functor corresponding to each scheme $S\in \mathrm{Sch}/k$ to the set of isomorphism class of pairs $(\mathcal{E}, p)$ where $\mathcal{E}$ is a coherent sheaf on $X\times S$ with Hilbert polynomial $\mathsf{HP}(\mathcal{E}) = F$, and $p : \mathcal{V}\twoheadrightarrow\mathcal{E}$ is a surjection. We define an isomorphism between two pairs $(\mathcal{E}, p)$, $(\mathcal{E}', p')$ as an isomorphism $q : \mathcal{E}\to\mathcal{E}'$ of coherent sheaves such that $q'\circ f = q$.

$$\begin{tikzcd} & \mathcal{E} \arrow[d, "q", "\cong"'] \\ \mathcal{V} \arrow[ur, "p"] \arrow[dr, "p'"'] \arrow[r, phantom, "\circlearrowleft"] & \mathcal{E}' \end{tikzcd}$$

When in the case $X = \mathbb{P}^n$, $L = \mathcal{O}_{\mathbb{P}^n}(1)$, $E = \mathcal{O}_{\mathbb{P}^n}$, we can easily observe that the Quot functor is isomorphic to the original Hilbert functor by a natural transformation. We sometimes abbreviate a Quot functor by $\mathfrak{Quot}^{F}_{\mathcal{V}/X}$ if there was no confusion for the choice of very ample line bundle L.

2.2.2 Existence of Quot scheme and Hilbert scheme

Contents in this section mostly follow [31, Part 2, Chapter 5]. In this section, we briefly explain the existence of a fine moduli space of a Quot functor. For this purpose, we use following two theorems and one lemma without proofs.

Let $\mathcal{E}$ be a coherent sheaf on the projective space $\mathbb{P}^n$. For an integer m, The $\mathcal{E}$ is called m-regular if satisfies the following :

$$H^i(\mathbb{P}^n, \mathcal{E}(m-i)) = 0 \text{ for all } i \geq 1.$$

Then we have the following theorem. According to Mumford, it is due to Castelnuovo [72].

Theorem 2.2.1 (Castelnuovo-Mumford regularity)**.** [31, Lemma 5.1] Let $\mathcal{E}$ be a m-regular coherent sheaf on the projective space $\mathbb{P}^n$. Then $\mathcal{E}$ satisfies the following properties:

(i) The natural morphism $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \otimes_k H^0(\mathbb{P}^n, \mathcal{E}(k)) \to H^0(\mathbb{P}^n, \mathcal{E}(k+1))$ is surjective for every $k \geq m$.

(ii) $H^i(\mathbb{P}^n, \mathcal{E}(r)) = 0$ for every $i \geq 1$ and $k \geq m-i$. Or equivalently, we can also say that If $\mathcal{E}$ is m-regular, then $\mathcal{E}$ is m' regular for every $m' \geq m$.

(iii) $\mathcal{E}(m')$ is globally generated and $H^i(\mathbb{P}^n, E(m')) = 0$ for every $m' \geq m$.

The following lemma is a weaker form of the much powerful theorem of Mumford.

Lemma 2.2.2. [31, Theorem 5.3] Consider a following short exact sequence of coherent sheaves :

$$0 \to \mathcal{F} \to \mathcal{O}_{\mathbb{P}^n}^{\oplus m} \to \mathcal{E} \to 0$$

with Hilbert polynomial $\mathsf{HP}(\mathcal{E}) = \mathsf{F}$

Then there exists an integer m_0 which only depends on m, n and the polynomial F such that $\mathcal{F}$ and $\mathcal{E}$ are m_0-regular.

Theorem 2.2.3 (Flattening stratification)**.** [31, Theorem 5.13]

Let X be a noetherian scheme over k and $\mathcal{E}$ be a coherent sheaf on $X \times \mathbb{P}^n$. Then there exists a finite set $\mathcal{I}$ of Hilbert polynomials and for each $\mathsf{F} \in \mathcal{I}$, and there exist a locally closed subschemes $X_\mathsf{F} \subset X$ of X which satisfies the followings :

(i) The set of closed points $|X_\mathsf{F}|$ of X_F is the set of all closed points $x \in X$ such that over its fiber $\mathbb{P}^n_x := \mathbb{P}^n \times \{x\}$, Hilbert polynomial $\mathsf{HP}(\mathcal{F}|_{\mathbb{P}^n_x})$ is equal to F.

(ii) Let $\widehat{X} := \coprod X_\mathsf{F}$. Consider a morphism $\iota : \widehat{X} \to X$ induced from the inclusion morphisms $\iota_\mathsf{F} : X_\mathsf{F} \hookrightarrow X$ by a universal property of the coproduct. Then $\iota^*(\mathcal{E})$ is flat over $\widehat{X}$. Furthermore, the morphism ι has the following universal property: Consider an arbitrary morphism $u : Y \to X$. Then $u^*\mathcal{E}$ is flat over Y if and only if the morphism u factors through the morphism $\iota : \widehat{X} \to X$.

(iii) Consider a total order on $\mathcal{I}$ defined by the relation $\mathsf{F} < \mathsf{G}$ if and only if $\mathsf{F}(t) < \mathsf{G}(t)$ for all $t \gg 0$. Then the closure of $|X_\mathsf{F}|$ is contained in the union of all $|X_\mathsf{G}|$ for all polynomials $\mathsf{G} \geq \mathsf{F}$, i.e.

$$\overline{|X_\mathsf{F}|} \subset \bigcup_{\mathsf{G} \geq \mathsf{F}} |X_\mathsf{G}|$$

We call each X_F a stratum of X corresponding to a Hilbert polynomial F appeared in this theorem.

Before explaining the existence of Hilbert schemes, we should explain one simple and interesting moduli functor called Grassmannian functor, whose fine moduli space is a Grassmannian variety.

Definition 2.2.1 (Grassmannian functor)**.** For an integer $m \leq n$, the Grassmannian functor is the following moduli functor :

$$\begin{aligned} \mathsf{Grass}(n,m) : (\mathrm{Sch}/k)^{op} &\longmapsto \mathrm{Sets} \\ S &\longrightarrow \{S \times k^n \twoheadrightarrow E \mid E \text{ is a rank } m \text{ bundle on } S\} \end{aligned}$$

We note that $\mathsf{Grass}(n,m)$ is representable by a fine moduli space $\mathrm{Gr}(n-m,n)$, a space of $(n-m)$-dimensional sub-vector spaces of k^n. Now we are ready to sketch the proof of the existence of the Quot scheme, which is the slight generalization of the Hilbert scheme.

First, we construct a natural transform from Quot functor to Grassmannian functor. For a scheme $S \in \mathrm{Sch}/k$, consider a family $p : \mathcal{O}_{\mathbb{P}^n \times S}^{\oplus m} \twoheadrightarrow \mathcal{E}$ on S which is an element of $\mathfrak{Quot}^F_{\mathcal{O}_{\mathbb{P}^n}^{\oplus m}}(S)$. Let $\mathcal{F} := \ker(p)$. Then, by lemma 2.2.2, we can find an integer m_0 Such that $\mathcal{E}_s$ and $\mathcal{F}_s$ are all m_0 regular for all closed points $s \in S$. Then $\mathrm{H}^i(\mathbb{P}^n_s, \mathcal{F}_s(k))$ and $\mathrm{H}^i(\mathbb{P}^n_s, \mathcal{E}_s(k))$ are all 0 for $i > 0$ and $k \geq m_0$. For the projection $\mathbb{P}^n \times S \to S$, m_0-regularity guarantees that $\pi_{S*}\mathcal{E}(k)$ and $\pi_{S*}\mathcal{F}(k)$ are locally free sheaves and globally generated for $k \geq m_0$.

For $k \geq m_0$, we have a short exact sequence of locally free sheaves :

$$0 \to \pi_{S*}\mathcal{F}(k) \to S \times (\mathrm{Sym}^k(k^n))^{\oplus m} \to \pi_{S*}\mathcal{E}(k) \to 0$$

such that the Hilbert polynomial $\mathsf{HP}(\pi_{S*}\mathcal{E}(k))$ is equal to $F(k)$. Now, when we fix an integer $k \geq m_0$, from the surjection $S \times (\mathrm{Sym}^k(k^n))^{\oplus m} \twoheadrightarrow$

$\pi_{S\to *}\mathcal{E}(k)$ we obtain a natural transformation of functors :

$$\begin{aligned} T_k : \mathfrak{Quot}^F_{\mathcal{O}^{\oplus m}_{\mathbb{P}^n}} &\longmapsto \text{Grass}\left(\binom{n+k-1}{k} m, F(k)\right) \\ \left[p : \mathcal{O}^{\oplus m}_{\mathbb{P}^n\times S} \twoheadrightarrow \mathcal{E}\right] &\longmapsto \left[S \times (\text{Sym}^k(k^n))^{\oplus m} \twoheadrightarrow \pi_{S*}\mathcal{E}(k)\right] \end{aligned}$$

Roughly speaking, we can show injectiveness of this functor since $\pi_{S*}\mathcal{E}(k)$ and $\pi_{S*}\mathcal{F}(k)$ are generated by their global sections.

Next, we construct a coherent sheaf $\mathcal{G}$ over $\text{Gr}(\binom{n+k-1}{k}m - F(k), \binom{n+k-1}{k}m) \times \mathbb{P}^n$. Then we can show that the Quot functor is representable by the strata $\text{Gr}(\binom{n+k-1}{k}m - F(k), \binom{n+k-1}{k}m)_F$ corresponds to the Hilbert polynomial F. So we denote it by $\text{Quot}^F_{\mathcal{O}^{\oplus m}_{\mathbb{P}^n}}$, a fine moduli space of the Quot functor. When $m = 1$, Quot functor is isomorphic to the Hilbert functor by a natural transformation so we denote it by $\text{Hilb}^F_{\mathbb{P}^n}$, a Hilbert scheme.

We introduce a general existence result for a Quot scheme by Grothendieck. The original theorem covers the case when X is a projective scheme over a noetherian scheme S but we omit here.

Theorem 2.2.4 (Grothendieck)**.** [31, Theorem 5.14]
The Quot functor $\mathfrak{Quot}^{F,L}_{\mathcal{V}/X}$ is representable by a projective scheme $\text{Quot}^{F,L}_{\mathcal{V}/X} \in \text{Sch/k}$ for any coherent sheaf $\mathcal{G}$ and a Hilbert polynomial F.

So, for the case when $\mathcal{V} = \mathcal{O}_X$, since Quot functor is isomorphic to the Hilbert functor by natural transformation, we obtain that $\mathfrak{Hilb}^{F,L}_X$ is representable by a projective scheme for any projective variety X over k and any Hilbert polynomial F. As a special case, we obtain the projectiveness of the Hilbert scheme $\text{Hilb}^F_{\mathbb{P}^n}$.

2.2.3 Tangent-obstruction theories of Quot schemes

Contents in this section mostly follow [31, Part 3, Chapter 6]. In this section we study tangent spaces of Hilbert schemes and smoothness condition of Hilbert schemes on certain points. For this, we need some elementary deformation theory. We start with defining a deformation functor.

Consider the category (Art/k) of local Artinian k-algebras, which have residue field k.

Definition 2.2.2. A deformation functor D is a following covariant functor

$$\mathsf{D} : (\mathrm{Art/k}) \xrightarrow{\varphi} (\mathrm{Sets})$$

such that $\mathsf{D}(\mathrm{k})$ is a one point set. For an local Artinian ring $(\mathsf{R}, \mathfrak{m}_\mathsf{R})$, we can consider a canonical morphism $\mathsf{D}(\mathrm{k}) = \mathsf{D}(\mathsf{R}/\mathfrak{m}_\mathsf{R}) \to \mathsf{D}(\mathsf{R})$, and therefore we can consider a distinguished element in $\mathsf{D}(\mathsf{R})$, which is an image of $\mathsf{D}(\mathrm{k})$. We denote this element by $0 \in \mathsf{D}(\mathsf{R})$.

Definition 2.2.3 (Small extension)**.** A small extension is a following exact sequence of R-modules :

$$0 \to \mathsf{K} \to \mathsf{R} \to \mathsf{S} \to 0$$

where $\varphi : \mathsf{R} \to \mathsf{S}$ is a surjective homomorphism of Artinian rings over k, $\mathsf{K} = \mathrm{Ker}\varphi$ and $\mathsf{K} \cdot \mathfrak{m}_\mathsf{R} = 0$ ($\mathfrak{m}_\mathsf{R}$ is a maximal ideal of R). Therefore R acts on K just as a scalar multiplication of its residue field $\mathsf{R}/\mathfrak{m}_\mathsf{R} \cong \mathrm{k}$.

The type of deformation functor we usually want is a deformation functor with the following properties :

Definition 2.2.4. [31, Definition 6.1.21] We said that a deformation functor D have a tangent-obstruction theory when there exist a k-vector space T_1, called tangent space for D, and T_2, called obstruction space for D, which satisfies the following properties :

1. For any small extension $0 \to \mathsf{K} \to \mathsf{R} \to \mathsf{S} \to 0$, there is a corresponding exact sequence of sets

$$\mathsf{T}_1 \otimes_\mathrm{k} \mathsf{K} \to \mathsf{D}(\mathsf{R}) \to \mathsf{D}(\mathsf{S}) \xrightarrow{\mathrm{ob}} \mathsf{T}_2 \otimes_\mathrm{k} \mathsf{K}.$$

We note that we can consider exact sequence of sets since $D(R)$ and $D(S)$ contains distinguished elements $0 \in D(R)$ and $0 \in D(S)$.

2. When $S = k$, then the above sequence is left exact :

$$0 \to T_1 \otimes_k K \to D(R) \to D(k) \xrightarrow{\text{ob}} T_2 \otimes_k K$$

3. The exact sequence of sets in (1) and (2) are functorial for small extensions. i.e. for a commutative diagram :

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, "f"] & R \arrow[r] \arrow[d, "g"] & S \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & K' \arrow[r] & R' \arrow[r] & S' \arrow[r] & 0 \end{tikzcd}$$

where horizontal rows are small extensions, g, h are morphisms in Art/k, f is a morphism of k-vector spaces, we have a corresponding commutative diagram of sets :

$$\begin{tikzcd} T_1 \otimes_k K \arrow[r] \arrow[d, "\mathrm{id}\otimes f"] & D(R) \arrow[r] \arrow[d, "D(g)"] & D(S) \arrow[r, "\text{ob}"] \arrow[d, "D(h)"] & T_2 \otimes_k K \arrow[d, "\mathrm{id}\otimes f"] \\ T_1 \otimes_k K' \arrow[r] & D(R') \arrow[r] & D(S') \arrow[r, "\text{ob}"] & T_2 \otimes_k K' \end{tikzcd}$$

or when $S = k$, we have commutative diagram where each horizontal rows are left exact :

$$\begin{tikzcd} 0 \arrow[r] & T_1 \otimes_k K \arrow[r] \arrow[d, "\mathrm{id}\otimes f"] & D(R) \arrow[r] \arrow[d, "D(g)"] & D(S) \arrow[r, "\text{ob}"] \arrow[d, "D(h)"] & T_2 \otimes_k K \arrow[d, "\mathrm{id}\otimes f"] \\ 0 \arrow[r] & T_1 \otimes_k K' \arrow[r] & D(R') \arrow[r] & D(S') \arrow[r, "\text{ob}"] & T_2 \otimes_k K'. \end{tikzcd}$$

Now, consider the Quot functor $\mathfrak{Quot}^{F,L}_{\mathcal{V}/X}$ for a projective variety X over k, very ample line bundle L on X, a coherent sheaf $\mathcal{V}$ on X and a Hilbert polynomial F. We want to restrict our Quot functor $\mathfrak{Quot}^{F,L}_{\mathcal{V}/X}$ which represents the local neighborhood of the Quot scheme $\mathrm{Quot}^{F,L}_{\mathcal{V}/X}$ around the point $\mathcal{Q}$, which is a quotient sheaf $\mathcal{Q} = \mathcal{V}/\mathcal{E}$ such that Hilbert polynomial $\mathsf{HP}(\mathcal{Q}) = F$. So we define our deformation functor $D^{\mathrm{Quot}^{F,L}_{\mathcal{V}/X}}_{[\mathcal{Q}]}$: Art/k $\to$ Sets to be, for an Artinian local ring R, $D^{\mathrm{Quot}^{F,L}_{\mathcal{V}/X}}_{[\mathcal{Q}]}(R)$ is a set is a quotient sheaf $\widehat{\mathcal{Q}} = (\mathcal{V} \otimes R)/\widehat{\mathcal{E}}$ which is flat over R, and whose restriction to $X \times \mathrm{Spec}(R/\mathfrak{m}_R)$ is the quotient sheaf $\mathcal{Q}$.

Then this deformation functor admits a tangent-obstruction theory with the tangent space $T_1 = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{Q})$ and the obstruction space $T_2 = \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}, \mathcal{Q})$. For the proof of this statement, we need some information about the extension about quotient sheaf. First, we consider a small extension $0 \to K \to R \to S \to 0$. Then we consider an extension of the following short exact sequence along the small extension.

$$0 \to \widehat{\mathcal{E}} \to \mathcal{V} \otimes S \to \widehat{\mathcal{Q}} \to 0. \tag{2.1}$$

Its extension over Spec$R$ is defined to be the following coherent sheaf $\widehat{\mathcal{Q}}'$:

$$0 \to \widehat{\mathcal{E}}' \to \mathcal{V} \otimes R \to \widehat{\mathcal{Q}}' \to 0 \tag{2.2}$$

such that $\widehat{\mathcal{Q}}'$ is flat over S, $\widehat{\mathcal{E}}' = \widehat{\mathcal{E}} \otimes_R S$. Then we have the following proposition on the extension of the above short exact sequence associated to flat quotient sheaves.

Proposition 2.2.5. [31, Theorem 6.4.5, Proposition 6.4.7]

For the short exact sequence 2.1, we can assign an obstruction class $\mathrm{ob}(e) \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}, \mathcal{Q}) \otimes_k K$. Then, an extension of the short exact sequence 2.1 for the small extension $0 \to K \to R \to S \to 0$ in the form of 2.2 satisfies the conditions stated above exists if and only if the obstruction class $\mathrm{ob}(e)$=0. Moreover, if an extension exists, the set of such extensions is a torsor under $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{Q}) \otimes_k K$.

Using this result, we get the following result which we want to prove from the beginning of this subsection.

Proposition 2.2.6. [31, Theorem 6.4.9] The deformation functor $\mathsf{D}_{[\mathcal{Q}]}^{\mathrm{Quot}_{\mathcal{V}/\mathsf{X}}^{\mathsf{F},\mathsf{L}}}$ admits a generalized tangent-obstruction theory with its tangent space $\mathsf{T}_1 = \mathrm{Hom}_{\mathcal{O}_\mathsf{X}}(\mathcal{E}, \mathcal{Q})$ and the obstruction space $\mathsf{T}_2 = \mathrm{Ext}^1_{\mathcal{O}_\mathsf{X}}(\mathcal{E}, \mathcal{Q})$.

Proof. By Proposition 2.2.5, axiom (1) and (2) of tangent-obstruction theory, Definition 2.2.4 is automatically satisfied. We can check axiom (3) by direct diagram chasing so we omit here. □

When $\mathcal{V} = \mathcal{O}_\mathsf{X}$, we have $\mathfrak{Quot}^{\mathsf{F},\mathsf{L}}_{\mathcal{O}_\mathsf{X}/\mathsf{X}} = \mathfrak{Hilb}^{\mathsf{F},\mathsf{L}}_\mathsf{X}$. In a similar manner as we constructed the deformation functor $\mathsf{D}_{[\mathcal{Q}]}^{\mathrm{Quot}_{\mathcal{V}/\mathsf{X}}^{\mathsf{F},\mathsf{L}}}$, for a closed subscheme $\mathsf{Z} \subset \mathsf{X}$ with Hilbert polynomial $\mathsf{HP}(\mathcal{O}_\mathsf{Z}) = \mathsf{F}$, we can define a deformation functor $\mathsf{D}_{[\mathsf{Z}]}^{\mathrm{Hilb}_\mathsf{X}^{\mathsf{F},\mathsf{L}}}$. Then as a result of Proposition 2.2.6, $\mathsf{D}_{[\mathsf{Z}]}^{\mathrm{Hilb}_\mathsf{Z}^{\mathsf{F},\mathsf{L}}}$ is a deformation functor which has a tangent-obstruction theory with the tangent space $\mathsf{T}_1 = \mathrm{Hom}_{\mathcal{O}_\mathsf{X}}(\mathcal{I}_\mathsf{Z}, \mathcal{O}_\mathsf{Z})$ and the obstruction space $\mathsf{T}_2 = \mathrm{Ext}^1_{\mathcal{O}_\mathsf{X}}(\mathsf{I}_\mathsf{Z}, \mathcal{O}_\mathsf{Z})$.

Then, using the results of pro-representable functors [31, Theorem 6.2.4, Corollary 6.2.6], we have the following results on the local geometry of Hilbert schemes.

Proposition 2.2.7. [42, Corollary 2.5] Let $\mathsf{Z} \subset \mathsf{X}$ be a closed subscheme of a projective variety X over k polarized by a very ample line bundle L, with a Hilbert polynomial F, which is a closed point in a Hilbert scheme $\mathrm{Hilb}_\mathsf{X}^{\mathsf{F},\mathsf{L}}$. Then the Zariski tangent space $\mathsf{T}_{[\mathsf{Z}]}\mathrm{Hilb}_\mathsf{X}^{\mathsf{F},\mathsf{L}}$ of $\mathrm{Hilb}_\mathsf{X}^{\mathsf{F},\mathsf{L}}$ at the point $[\mathsf{Z}]$ is isomorphic to the vector space $\mathrm{Hom}_{\mathcal{O}_\mathsf{X}}(\mathcal{I}_\mathsf{Z}, \mathcal{O}_\mathsf{Z})$.

Proposition 2.2.8. [31, Corollary 6.4.11] For a closed subscheme $\mathsf{Z} \subset \mathsf{X}$ in a projective variety X over k polarized by a very ample line bundle L with a Hilbert polynomial F, which is a closed point in a Hilbert scheme $\mathrm{Hilb}_\mathsf{X}^{\mathsf{F},\mathsf{L}}$. Let $\mathsf{d}_1 := \dim(\mathrm{Hom}_{\mathcal{O}_\mathsf{X}}(\mathcal{I}_\mathsf{Z}, \mathcal{O}_\mathsf{Z}))$ and $\mathsf{d}_2 := \dim(\mathrm{Ext}^1_{\mathcal{O}_\mathsf{X}}(\mathcal{I}_\mathsf{Z}, \mathcal{O}_\mathsf{Z}))$. Then we have $\mathsf{d}_1 \geq \dim_{[\mathsf{Z}]}\mathrm{Hilb}_\mathsf{X}^{\mathsf{F},\mathsf{L}} \geq \mathsf{d}_1 - \mathsf{d}_2$. Furthermore, if $\dim_{[\mathsf{Z}]}\mathrm{Hilb}_\mathsf{X}^{\mathsf{F},\mathsf{L}} = \mathsf{d}_1 - \mathsf{d}_2$, then the Hilbert scheme $\mathrm{Hilb}_X^{F,L}$ is locally a complete intersection around the point $[Z]$. In particular, if $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{I}_Z, \mathcal{O}_Z) = 0$, then the Hilbert scheme $\mathrm{Hilb}_X^{F,L}$ is smooth at the point $[Z]$.

2.3 Geometric invariant theory

We use [82, 28] as main references of this section. In this section, we study how to construct a quotient of a variety via a group action. Let X be a variety and G be an algebraic group acting on X. The group action $G \times X \to X$ is algebraic.

We start by classifying the notion of quotients. There are three notions of quotients: Categorical, good, and geometric.

Definition 2.3.1 (Categorical quotient)**.** Let X is a variety equipped with a G-action. Then consider a pair (Y, p) where Y is a variety and $p : X \to Y$ is a G-invariant morphism. Then we call the pair (Y, p) a categorical quotient if is satisfies the following universal property. If there is another G-invariant morphism $f : X \to Z$, then there exists a unique morphism $\bar{f} : Y \to Z$ such that $f = p \circ \bar{f}$.

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "f"] \arrow[d, phantom, shift left=3ex, "\circlearrowleft"] & \\ Y \arrow[r, dashed, "\bar{f}", "\exists!"'] & Z \end{tikzcd}$$

We note that the categorical quotient (Y, p) is unique up to isomorphism by universal property.

Definition 2.3.2 (Good quotient)**.** Let X be a variety equipped with a G-action. Then a pair (Y, p) of a variety Y and a G-invariant morphism $p : X \to Y$ is called a good quotient of X if it satisfies the following properties :

1. The morphism p is surjective and affine.

2. The image of a closed G-invariant subspace of X is again closed in Y, and two closed disjoint G-invariant subspaces of X has disjoint images in Y.

3. For an affine open set $U \subset Y$, we have $\Gamma(p^{-1}(U), \mathcal{O}_X)^G = \Gamma(U, \mathcal{O}_Y)$, which is equivalent to say that the sections of the structure sheaf $\mathcal{O}_Y$ is the G-invariant sections of the structure sheaf $\mathcal{O}_X$. In this case, we emphasize that $p^{-1}(U)$ is also affine since the morphism p is affine.

The following proposition says that good quotient is a stronger condition than categorical quotient, but it is not an orbit space in general.

Proposition 2.3.1. [82, Proposition 3.11] Let X be a variety equipped with G-action and let a pair $(Y, p : X \to Y)$ be a good quotient. Then we have the followings

1. The pair (Y, p) is a categorical quotient.

2. For $x, y \in X$, $p(x) = p(y)$ if and only if two orbit closures intersects, i.e. $\overline{Gx} \cap \overline{Gy} \neq \varnothing$.

Therefore, even if two orbits Gx and Gy are disjoint, they may intersect in their closures. So if we want to make quotient to an orbit space, we need the condition that every orbit is closed. This condition leads to the definition of the geometric quotient in the following.

Definition 2.3.3 (Geometric quotient)**.** Let X be a variety equipped with G-action and let a pair $(Y, p : X \to Y)$ be a good quotient. Then we call (Y, p) a geometric quotient if all G-orbits in X are closed.

In summary, a geometric quotient is a good quotient, and a good quotient is a categorical quotient. We note that the notions of the good quotient and the geometric quotient are local.

Proposition 2.3.2. [82, Proposition 3.10] Let X be a variety equipped with G-action. Consider a pair (Y, p) of variety Y and a G-invariant morphism $p : X \to Y$.

Then the pair (Y, p) is a good(resp. geometric) quotient if and only if There is an open cover $\{U_i\}_{i \in \mathcal{I}}$ of Y such that G-invariant morphisms $p|_{p^{-1}(U_i)} : p^{-1}(U_i) \to U_i$ are good(resp. geometric) quotients.

Next, we start from the case when X is affine variety.

2.3.1 Affine quotient

Let X be a affine variety $X \cong \mathrm{Spec} R$. Since there is an algebraic action on X by an algebraic group G, there is also an induced algebraic action on the ring of functions $R = \Gamma(X, \mathcal{O}_X)$ by the group G. Let R^G be its invariant subring.

From now on, we further assume that G is a reductive group. We will not explain about the definition of linearly reductive groups. But we note that general linear groups $\mathrm{GL}(n, \mathrm{k})$, special linear groups $\mathrm{SL}(n, \mathrm{k})$, projective linear groups $\mathrm{PGL}(n, \mathrm{k})$ are all reductive groups. There is a following famous theorem of Nagata [75] for linear reductive groups. For state Nagata's theorem, we first define the notion of rational group action by a group G on a ring R.

Definition 2.3.4 (Rational actions)**.** [73, Definition 1.2],[82, Definition on p. 47] Let G be a affine linear algebraic algebraic group acting on a ring R. Let $S = \Gamma(G, \mathcal{O}_G)$ be the function ring of the group G. Then the group action $G \times R \to R$ induces a morphism of rings $\widehat{a} : R \to S \otimes_{\mathrm{k}} R$ (If we fix an element $r \in R$, then it induces a function $G \to R$ given by $g \mapsto g \cdot r$. Then the function $G \to R$ induces an element of the ring $S \otimes_{\mathrm{k}} R$).

On the other hand, multiplication on the group G, $G \times G \to G$ induces a dual multiplication $\widehat{m} : S \to S \otimes_{\mathrm{k}} S$ and an identity map $id : \mathrm{Spec}\,\mathrm{k} \to G$ induces a dual identity map $\widehat{id} : S \to \mathrm{k}$.

Next, we define the notion of dual actions for the class of morphism of rings $\varphi : \mathsf{R} \to \mathsf{S} \otimes_k \mathsf{R}$. This is a dual notion of the definition of group actions on algebraic varieties. A morphism of rings $\varphi : \mathsf{R} \to \mathsf{S} \otimes_k \mathsf{R}$ is said to be a dual action if and only if it satisfies the following axioms.

(1) (Associativity axiom)

$$\begin{tikzcd} & \mathsf{S} \otimes_k \mathsf{R} \arrow[dr, "\widehat{m} \otimes \mathrm{id}_\mathsf{R}"] & \\ \mathsf{S} \arrow[ur, "\varphi"] \arrow[dr, "\varphi"'] & & \mathsf{S} \otimes_k \mathsf{S} \otimes_k \mathsf{R} \\ & \mathsf{S} \otimes_k \mathsf{R} \arrow[ur, "\mathrm{id}_\mathsf{S} \otimes \varphi"'] & \end{tikzcd}$$

The above diagram commutes.

(2) (Identity axiom)

$$\mathsf{R} \xrightarrow{\varphi} \mathsf{S} \otimes_k \mathsf{R} \xrightarrow{\widehat{\mathrm{id}} \otimes \mathrm{id}_\mathsf{R}} \mathsf{R}$$

We have the composition $(\widehat{\mathrm{id}} \otimes \mathrm{id}_\mathsf{R}) \circ \varphi$ is equal to the identity id_R.

Then we call the group action is rational if the induced morphism of groups $\widehat{a} : \mathsf{R} \to \mathsf{S} \otimes_k \mathsf{R}$ is a dual action.

Remark 2.3.3. To extend the notion of rational action to the linear reductive group action of a group G which is not affine, it is enough to consider a sheaf of rings $\mathcal{O}_\mathsf{G}$ instead of the function ring $\mathsf{S} = \Gamma(\mathsf{G}, \mathcal{O}_\mathsf{G})$. It need some technical justification but we omit here.

Theorem 2.3.4 (Nagata)**.** [75] Let G be a linearly reductive group and let R be a finitely generated k-algebra where the group G rationally acts on it. Then the invariant ring R^G is finitely generated.

Remark 2.3.5. In 1975, Haboush [39] proved that every reductive group is linearly reductive. Therefore we can use above theorem in the assumption that the group G is reductive.

Let R be a finitely generated k-algebra equipped with a G-action. Then the inclusion of rings $\mathsf{R}^{\mathsf{G}} \hookrightarrow \mathsf{R}$ induces a morphism of affine schemes $p : X = \mathrm{Spec}\mathsf{R} \to Y = \mathrm{Spec}\mathsf{R}^{\mathsf{G}}$. Then it is natural to define a quotient of the affine variety $X = \mathrm{Spec}\mathsf{R}$ to be the pair (Y, p). The following proposition says that it is the right way.

Proposition 2.3.6 (Affine quotients)**.** [28, Theorem 6.1, p. 97] In the above setting, the G-invariant morphism $p : X \to Y$ is a good quotient.

2.3.2 Projective quotients

Consider a projective variety X which is embedded in $\mathbb{P}^n$ as a closed embedding $\iota : X \hookrightarrow \mathbb{P}^n$, equipped with an algebraic group action given by a linearly reductive group G.

In this case, we further assume that the group action G extends to lifts to the general linear action of the affine cone $\mathbb{A}^{n+1}$ of the projective space $\mathbb{P}^n$. More explicitly, this means that there is a homomorphism $\mathsf{G} \to \mathrm{GL}(n+1)$ and the affine cone $\widehat{X} \subset \mathbb{A}^{n+1}$ of X is $\mathrm{GL}(n+1)$-invariant, and G-action on X is induced from the $\mathrm{GL}(n+1)$-action on $\widehat{X}$. In this case, we say that the group G acts on X linearly.

Since there is a $\mathrm{GL}(n+1)$-action on $\mathbb{A}^{n+1}$, we claim that there is also a canonically induced action on the global section space $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. Let $(x_0, ..., x_n)$ be coordinate functions of $\mathbb{A}^{n+1}$. Then we can see $x_0, ..., x_1$ as generators of $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. Consider a tautological family on $\mathbb{P}^n$:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1} \cong \mathbb{P}^n \times k^{n+1} \to \mathcal{Q} \to 0$$

where $\mathcal{O}_{\mathbb{P}^n}(-1)$ is a sub-line bundle of the trivial bundle $\mathbb{P}^n \times k^{n+1}$ whose fiber over a point $[t_0 : t_1 : \cdots : t_n]$ is a 1-dimensional sub-vector space generated by the vector $(t_0, t_1, \ldots, t_n) \in k^{n+1}$, and $\mathcal{Q}$ is a tautological quotient bundle.

Next, consider an isomorphism $k^{n+1} \xrightarrow{x_0 \oplus \cdots \oplus x_n} \underbrace{k \oplus k \oplus \cdots \oplus k}_{n+1}$. Finally, consider the following composition :

$$\mathcal{O}_{\mathbb{P}^n}(-1) \hookrightarrow \mathbb{P}^n \times k^{n+1} \xrightarrow{\cong} \mathbb{P}^n \times k^{\oplus n+1} \xrightarrow{p_i} \mathbb{P}^n \times k = \mathcal{O}_{\mathbb{P}^n}$$

where p_i is a projection to the i-th summand. Then we observe that on the fiber, the morphism acts exactly the same as the coordinate function x_i, and by definition, this is an element of the global section $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. So we call element as $x_i \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. So we constructed the correspondence between coordinate functions $x_0, ..., x_n$ of k^{n+1} and the global sections $x_0, ..., x_{n+1}$ of $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

So we have the induced G-action on the space of sections $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. Therefore we also have the induced G-action on $\Gamma(X, L)$ where $L := \mathcal{O}_{\mathbb{P}^n}(1)|_X$. In a similar manner, we can define a G-action on the graded ring of sections $\bigoplus_{d \geq 0} \Gamma(X, L^d)$. Then we can again consider its invariant ring, and consider its proj, $\mathrm{Proj}(\bigoplus_{d \geq 0} \Gamma(X, L^d)^G)$.

Unfortunately, in projective case, $\mathrm{Proj}(\bigoplus_{d \geq 0} \Gamma(X, L^d)^G)$ is not a good quotient or even a categorical quotient of the projective variety X in general. To solve this problem. We should discard some bad locus of X for the G-action. For this, we need a notion of stable and semi-stable points.

2.3.3 Stable and semi-stable points

Again we consider the projective variety $X \subset \mathbb{P}^n$ where the reductive group G acts linearly.

Definition 2.3.5. 1. A point $x \in X$ is called semi-stable if there exist a nonconstant G-invariant homogeneous polynomial $f \in (\bigoplus_{d \geq 0} \Gamma(X, L^d))^G$ such that $f(x) \neq 0$. We write $X^{ss} \subset X$ be a subset of semi-stable points.

2. A semi-stable point $x \in X$ is called stable if its G-orbit Gx is closed in X and the dimension of Gx equals to the dimension of the group G.

We write $X^s \subset X$ be a subset of stable points.

3. A semi-stable point $x \in X$ is called strictly semi-stable if it is not stable.

4. A point $x \in X$ is called unstable if x is not semi-stable.

We can easily observe that X^{ss} and X^s are open subsets of X.

Proposition 2.3.7. [82, Theorem 3.14],[28, Proposition 8.1] Consider the morphism $\phi : X \to Y := \mathrm{Proj}(\bigoplus_{d \geq 0} \Gamma(X, L^d))$ induced from the inclusion of graded rings $(\bigoplus_{d \geq 0} \Gamma(X, L^d))^G \hookrightarrow \bigoplus_{d \geq 0} \Gamma(X, L^d)$. Its restrictions to X^{ss} and X^s have the following properties.

1. $\phi|_{X^{ss}} : X^{ss} \to Y$ is a good quotient.

2. There exist an open subset $Y^s \subset Y$ such that $\phi^{-1}(Y^s) = X^s$ and $\phi|_{X^s} : X^s \to Y^s$ is a geometric quotient.

So, by the above proposition, if we discard bad locus for G-actions, which means unstable locus, from X then we can construct good quotients in a similar manner as in the case of affine quotients.

2.3.4 Linearization

In the previous subsection, we constructed the good quotient of the semi-stable locus of the projective variety. In this section, we consider a more general case.

Let X be a variety and G be a reductive group act on X. Consider a line bundle L on the variety X. We first define a notion of linearization of the group action with respect to the line bundle L.

Definition 2.3.6 (Linearization)**.** Let L be a line bundle on the variety X. Then a linearization of the group action with respect to the line bundle L

is an action on the total space of the line bundle L, which is compatible with the action on X, which is linear on the fiber. On the other word, for each group element $g \in G$ and a point $x \in X$, we correspond linear maps $L_x \to L_{g \cdot x}$ on fibers of the line bundle.

If there is a linearization of the group action G with respect to the line bundle L, then there is a group action on the section space $\Gamma(X, L^d)$ for all $d \geq 0$. Then we call each section $s \in \Gamma(X, L^d)^G$ a homogeneous invariant section.

Then in a similar manner as we defined semi-stable points and stable points in the previous subsection, we can define semi-stable and stable points

Definition 2.3.7. We classify points in the variety X as follows.

1. A point $x \in X$ is called semi-stable if there exists a nonzero homogeneous invariant section $s \in \Gamma(X, L^d)^G$ for some $d \geq 1$ such that $s(x) \neq 0$ and $X_s = \{x \in X | s(x) \neq 0\}$ is affine. We write $X^{ss}(L)$ as the set of semi-stable points of X.

2. A semi-stable point $x \in X$ is called stable if $\dim Gx = \dim G$ and G-action on X_s is affine, and all G-orbits in X_s are closed. We write $X^s(L)$ as the set of stable points of X.

3. A semi-stable point x is called strictly semi-stable if it is not stable.

4. A point x is called unstable if it is not semi-stable. We write $X^s(L)$ as the set of stable points of X.

We easily observe that $X^{ss}(L)$ and $X^s(L)$ are open G-invariant subsets of X.

Then, we can find a good quotient for semi-stable locus, and geometric quotient for stable locus, which is exactly the same as the projective quotient case. But in this case, $\mathrm{Proj}(\bigoplus_{d \geq 0} \Gamma(X, L^d)^G)$ is not the answer for the quotient. It is a big difference.

Proposition 2.3.8. [28, Theorem 8.1] Let X be a variety equipped with a reductive group G-action. Let L be a line bundle on X and G has a linearization respect to the line bundle L. Then we have the followings :

1. There exist a good quotient $p : X^{ss}(L) \to X^{ss} /\!\!/_L G$, where $X^{ss} /\!\!/_L G$ is a quasi-projective variety. In this case, we call $X^{ss} /\!\!/_L G$ a GIT quotient of X by G.

2. There is an open subset $X^s /\!\!/_L G \subset X^{ss} /\!\!/_L G$ such that $p^{-1}(X^s /\!\!/_L G) = X^s$ and $p|_{X^s} : X^s \to X^s /\!\!/_L G$ is a geometric quotient.

The quotients $p : X^{ss}(L) \to X^{ss}(L) /\!\!/_L G$ and its restriction $p|_{X^s} : X^s \to X^s /\!\!/_L G$ satisfies the following properties.

Proposition 2.3.9. [82, Theorem 3.21]

1. For semistable points $x_1, x_2 \in X^{ss}(L)$, $p(x_1) = p(x_2)$ if and only if the closure of two orbits meet on the semi-stable locus. i.e. $(\overline{G(x_1)} \cap \overline{G(x_2)}) \cap X^{ss}(L) \neq \varnothing$.

2. A semi-stable point x is stable if and only if $\dim Gx = \dim G$ and the orbit Gx is closed in the semi-stable locus $X^{ss}(L)$.

2.3.5 Hilbert-Mumford criterion

In this subsection, we introduce practical way how to determine each point is semi-stable or stable, or unstable.

First we define 1-parameter subgroups of the group G.

Definition 2.3.8 (1-parameter subgroups)**.** A 1-parameter subgroup λ is an injective group homomorphism $k^* \overset{\lambda}{\hookrightarrow} G$.

The following proposition is a small part of a much powerful theorem of Borel on the diagonalizable groups, which says that every 1-parameter subgroup actions on the affine space k^{n+1} can be diagonalized.

Proposition 2.3.10. [5, Chapter III, §8 Proposition, 114p] Let λ be a 1-parameter subgroup. Since the induced action on k^{n+1} is algebraic, there is a basis $(v_0, \ldots, v_n)$ of k^{n+1} where the 1-parameter subgroup λ acts diagonally. When we write $x_0, \ldots, x_n$ as a coordinate functions of k^{n+1} for the basis $(v_0, \ldots v_n)$, λ acts on the point $(x_0, \ldots, x_n) \in k^{n+1}$ as $t \cdot (x_0, \ldots, x_n) = (t^{d_0}x_0, \ldots, t_n^{d_n}x_n)$ such that $d_0 \leq d_1 \leq \cdots \leq d_n$. We call this number $d_0, \ldots, d_n$ weights of this 1-parameter group action.

Furthermore, this series of numbers $d_0, ..., d_n$ are unique, i.e. invariant up to the choice of a basis of k^{n+1}.

The following criterion enables us to compute semistable and unstable locus exactly in many examples.

Proposition 2.3.11 (Hilbert-Mumford criterion)**.** [82, Proposition 4.8, Theorem 4.9] For a point $x \in X$, we define $\mu(x, \lambda) := -\min(d_i : x_i \neq 0)$. Then we have

1. The point x is semistable if and only if $\mu(x, \lambda) \geq 0$ for all 1-parameter subgroups λ.

2. The point x is stable if and only if $\mu(x, \lambda) > 0$ for all 1-parameter subgroups λ.

2.3.6 Examples

Example 2.3.12 (Projective space)**.** It is well known that projective space $\mathbb{P}^n$ is a good quotient of a variety $k^{n+1} \setminus \{0\}$ for an action of a group k^*. We explain about this good quotient $k^{n+1} \setminus \{0\} \to \mathbb{P}^n$ here. A group element $t \in k^*$ act on $(x_0, \ldots, x_n) \in k^{n+1} \setminus \{0\}$ as $t \cdot (x_0, \ldots, x_n) = (tx_0, \ldots, tx_n)$. So, the problem is that there is no invariant functions. To solve this problem, we embed $k^{n+1} \setminus \{0\}$ in $\mathbb{P}^{n+1}$ as follows:

$$\begin{aligned} k^{n+1} \setminus \{0\} &\hookrightarrow \mathbb{P}^{n+1} \\ (x_0, \ldots, x_{n+1}) &\longmapsto [x_0 : \cdots : x_{n+1} : 1]. \end{aligned}$$

Then we extend a k^*-action on $\mathbb{P}^{n+1}$ to be $t \cdot [x_0 : \cdots : x_{n+1}] = [tx_0 : \cdots : tx_n : t^{-1}x_{n+1}]$. Then invariant homogeneous polynomials are $f(x_0, \ldots, x_n)x_{n+1}^d$ where $f(x_0, \ldots, x_n)$ is a homogeneous polynomial in $x_0, \ldots, x_n$ with degree d. Therefore, we observe that semi-stable points of this k^*-action on $\mathbb{P}^{n+1}$ is exactly equal to $k^{n+1} \setminus \{0\}$.

Furthermore, there is a natural graded ring isomorphism from the graded invariant ring $\bigoplus_{d \geq 0}\{f(x_0, \ldots, x_n)x_{n+1}^d \mid f$ is degree d homogeneous.$\}$ to the graded ring $\bigoplus_{d \geq 0}\{f(x_0, \ldots, x_n) | f$ is degree d homogeneous.$\}$. Therefore we have a good quotient :

$$k^{n+1} \setminus \{0\} \to \mathrm{Proj}(\bigoplus_{d \geq 0}\{f(x_0, \ldots, x_n) | f \text{ is degree } d \text{ homogeneous.}\}) = \mathbb{P}^n$$

by Proposition 2.3.7.

On the other hand, let $L := \mathcal{O}_{\mathbb{P}^{n+1}}(1)$. By definition, k^*-action on $\mathbb{P}^{n+1}$ already has linearization with respect to the line bundle $\mathcal{O}_{\mathbb{P}^{n+1}}(1)$. Therefore, k^* on $k^{n+1} \setminus \{0\}$ has also linearization respect to the line bundle L. Then we can write $(k^{n+1} \setminus \{0\}) /\!\!/_L \, k^* = \mathbb{P}^n$.

Example 2.3.13 (Multiple projective lines)**.** Let $X = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 = (\mathbb{P}^1)^n$ and $G = \mathrm{SL}(2, k)$. Each element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, k)$ acts on an element $[u : v] \in \mathbb{P}^1$ as left multiplication $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$. Consider the Segre embedding as follows :

$$\begin{array}{rcl} X = (\mathbb{P}^1)^n & \xhookrightarrow{\quad\quad\quad\quad f \quad\quad\quad\quad} & \mathbb{P}^{2^n-1} \\ ([x_1 : y_1], \dots, [x_n : y_n]) & \longmapsto & [x_1x_2, \dots, x_n : x_1, \dots, x_{n-1}y_n : \dots : y_1y_2, \dots, y_n], \end{array}$$

and a very ample line bundle $f^*\mathcal{O}_{\mathbb{P}^{2^n-1}}(n)$ on X. Then there is a natural linearization on $f^*\mathcal{O}_{\mathbb{P}^{2^n-1}}(n)$.

Next, consider a 1-parameter subgroup λ of $\mathrm{SL}(2, k)$ given by :

$$\lambda(t) = \begin{pmatrix} t^w & 0 \\ 0 & t^{-w} \end{pmatrix}.$$

Then $t \cdot x_{i_1}x_{i_2} \dots x_{i_{n-a}}y_{j_1}y_{j_2} \dots y_{j_a} = t^w(n-2a) \cdot x_{i_1}x_{i_2} \dots x_{i_{n-a}}y_{j_1}y_{j_2} \dots y_{j_a}$. Therefore, for a point $([x_1 : y_1], \dots, [x_n : y_n])$ with a-indices where $y_i = 0$, homogeneous polynomial which has minimal weight for λ-action has weight $w(n-2a)$. Since $y_i = 0$ is equivalent to say that $[x_i : y_i] = [1 : 0]$, we can also say that a multiple point $x = ([x_1 : y_1], \dots, [x_n : y_n])$ has $[1 : 0]$ with multiplicity a then $\mu(x, \lambda) = -w(n-2a)$.

By the base change, we can take similar 1-parameter subgroup. So we can observe that if a multiple point $x = ([x_1 : y_1], \dots, [x_n : y_n])$ has any point $p \in \mathbb{P}^1$ with multiplicity a, we can find a 1-parameter subgroup λ' such that $\mu(x, \lambda') = -w(n-2a)$. Therefore we have the following description of the semi-stable and stable points in the multiple line $(\mathbb{P}^1)^n$

In summary, we obtain the following criterion for semi-stable and stable locus of multiple projective lines under the $\mathrm{SL}(2, k)$-action.

Proposition 2.3.14. [82, Proposition 4.16]

1. A multiple point $p = (p_1, ..., p_n) \in (\mathbb{P}^1)^n$ is semi-stable for $\mathrm{SL}(2, k)$-action if it all points $p_i \in \mathbb{P}^1$ has multiplicity $\leq \frac{n}{2}$.

2. A multiple point $p = (p_1, ..., p_n) \in (\mathbb{P}^1)^n$ is semi-stable for $\mathrm{SL}(2, k)$-action if it all points $p_i \in \mathbb{P}^1$ has multiplicity $< \frac{n}{2}$.

So we note that the integer n is odd, then semi-stable conditions and stable conditions are equal so there are no strictly semi-stable points.

Example 2.3.15 ($\mathrm{Gr}(r, V \otimes W)$ with $\mathrm{SL}(V)$-action)**.** Here we make a key observation which is crucially used in the construction of moduli space of vector bundles on a curve in the next section.

Consider a vector space V, W with dimension n, m and the Grassmannian $\mathrm{Gr}(r, V \otimes W)$. Consider an $\mathrm{SL}(V)$-action act on it. Consider a one-parameter subgroup λ of $\mathrm{SL}(V)$. Let $d_0 \le \cdots \le d_\ell$ be a series of numbers obtained from the series appeared in Proposition 2.3.10 by removing duplicated values. Then, by Proposition 2.3.10, we have a weight decomposition $V = V_0 \oplus \cdots \oplus V_\ell$ such that for each vector $v_i \in V_i$, λ act on v_i as weight d_i i.e. for $t \in k^*$, $t \cdot v_i = t^{d_i} v_i$.

For an r-dimensional subspace $K \in \mathrm{Gr}(r, V \otimes W)$, we let $K_i := (V_i \otimes W) \cap K$ and define $u_i := \mathrm{dim} K_i$. Therefore, when we take a Plücker embedding $\mathrm{Gr}(r, V \otimes W) \hookrightarrow \mathbb{P}(\wedge^r V \otimes W)$, we observe that $t \cdot \wedge^r K = t^{(\sum_i d_i u_i)} \cdot \wedge^r K$ so we deduce that $\mu(K, \lambda) = -(\sum_i d_i u_i)$.

Therefore, we have the following description of semi-stable and stable points.

Proposition 2.3.16. [66, Proposition 6.6.1] Consider an r-dimensional subvector space $K \subset V \otimes W$, an element of $\mathrm{Gr}(r, V \otimes W)$. Then K is semi-stable with respect to the $\mathrm{SL}(V)$-action if and only if for any proper nonzero subvector space $V' \subset V$, it satisfies the following equation :

$$\frac{\mathrm{dim} K'}{\mathrm{dim} V'} \le \frac{\mathrm{dim} K}{\mathrm{dim} V} \quad (K' := (V' \otimes W) \cap K)\,.$$

Proof. We first prove inverse direction. We use Hilbert-Mumford criterion. For a 1-parameter subgroup λ, we already calculated that $\mu(\lambda, K) = \sum_{i=0}^{\ell} d_i u_i$. Then by Abel's summation formula, we have $\mu(\lambda, K) = d_\ell r - \sum_{i=0}^{\ell-1} (d_{i+1} -$

$d_i)\left(\sum_{j=0}^{i} u_j\right)$. We note that when we define $V'_i := \bigoplus_{j=0}^{i} V_i \subset V$ and $K'_i := (V'_i \otimes W) \cap K$ then we have $\sum_{j=0}^{i} u_j = \dim K'_i$. Therefore we can write $\mu(\lambda, K) = d_\ell r - \sum_{i=0}^{\ell-1} (d_{i+1} - d_i)\dim K'_i$.

Now, assume that we have $\frac{\dim K'}{\dim V'} \leq \frac{\dim K}{\dim V}$ for any nonzero proper subspace $V' \subset V$. We apply this inequality to each $\dim K'_i$, then we have :

$$\mu(\lambda, k) \geq d_\ell r - \sum_{i=0}^{\ell-1} (d_{i+1} - d_i) r \frac{\dim V'_i}{\dim V} = -\frac{r}{n}\left(\sum_{i=0}^{\ell-1} (d_i - d_{i+1})\dim V'_i + d_\ell n\right).$$
$$= \sum_{i=0}^{\ell} d_i(\dim V'_i - \dim V'_{i-1}) = \sum_{i=0}^{\ell} d_i \dim V_i = 0$$

where the last equality comes from the fact that total weight of an action of any 1-parameter subgroup of an SL(V)-action must be zero. Therefore, by Hilbert-Mumford, $K \in \mathrm{Gr}(r, V \otimes W)$ is a semi-stable point. Now, assume that K is a semi-stable point and $V_0 \subset V$ be a nonzero proper sub-vector space of V. of dimension $n_0 < n = \dim V$. Then by the basis extension theorem, we can find a $n - n_0$-dimensional sub-vector space $V_1 \subset V$ such that $V = V_0 \oplus V_1$ and we can construct a 1-parameter subgroup λ-action on V as follows. For an element $t \in k^*$, t act on V by the matrix :

$$\begin{pmatrix} t^{-(n-n_0)} \mathrm{id}_{V_0} & 0 \\ 0 & t^{n_0} \mathrm{id}_{V_1} \end{pmatrix}.$$

Similarly we define $K_0 := (V_0 \otimes W) \cap K$. Then as we calculated above, we have $\mu(\lambda, K) = -(n - n_0)\dim K_0 + n_0(\dim K - \dim K_0) = n \dim K_0 - n_0 \dim K \geq 0$. Thus we have :

$$\frac{\dim K_0}{\dim V'} \leq \frac{\dim K}{\dim V} .$$

□

2.4 Vector bundles and coherent sheaves on a smooth projective curve

Contents in this section mostly follow [66, Part I, Chapter 1,2,5]. In this section, we study various properties on coherent sheaves on a smooth projective curve C.

2.4.1 Basic properties

In this section, we introduces some basic properties on coherent sheaves and vector bundles on the smooth projective curve C. We start by defining a notion of rank of coherent sheaves.

Lemma 2.4.1. [66, Lemma 2.6.1] Let $\mathcal{F}$ be a coherent sheaf on the curve C. Then there is an open dense subset $U \subset C$ of C such that $\mathcal{F}|_U \cong \mathcal{O}_u^{\oplus r}$.

Proof. By [43, II, Chapter 5, ex 5.8], there is an open dense subset $U \subset C$ such that a stalk $\mathcal{F}_x$ has a maximal rank r for a point $x \in U$. Then again by [43, II, Chapter 5, ex 5.8], $\mathcal{F}|_U$ is locally free. Therefore, we can choose a smaller dense open subset $V \subset U$ such that $\mathcal{F}|_V \cong \mathcal{O}_V^{\oplus r}$. □

Using this lemma, we can define a notion of rank of a coherent sheaf $\mathcal{F}$ on a curve C

Definition 2.4.1 (Rank)**.** We define a rank of $\mathcal{F}$ to be $\operatorname{rank} \mathcal{F} := r$.

We can easily check that this number r is uniquely defined since the c urve C is connected. So we do not check it here.

The following definition is, in fact, equivalent to famous 'Riemann-Roch theorem' when $\mathcal{F}$ is a line bundle. So it uses the Riemann-Roch theorem in the line bundle case to define a notion of degree of coherent sheaves.

Definition 2.4.2 (Degree)**.** [66, Chapter 2.6] Let $\mathcal{F}$ be a coherent sheaf on C. Then we define a degree F to be:

$$\deg\mathcal{F} := \chi(\mathcal{F}) - r \cdot \chi(\mathcal{O}_C)$$

where $r = \mathrm{rank}(\mathcal{F})$ and $\chi(\mathcal{E}) := \sum_i (-1)^i h^i(\mathcal{E})$ is the Euler characteristic for the coherent sheaf $\mathcal{E}$.

Example 2.4.2. (a) For a coherent sheaf $\mathcal{O}_C(x_1 + \cdots + x_\ell)$, $\deg\mathcal{O}_C(x_1 + \cdots + x_\ell) = \ell$

(b) For a subscheme $Z \subset C$, its structure sheaf $\mathcal{O}_Z$ has degree equal to $\mathrm{length}(Z)$.

We note that the degree of coherent sheaves is additive in short exact sequences

Lemma 2.4.3. [66, Chapter 2.6, p. 30] Consider a short exact of coherent sheaves :

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

Then we have $\deg\mathcal{F} = \deg\mathcal{E} + \deg\mathcal{G}$.

Proof. From long exact sequences of cohomologies, we can show that Euler characteristic χ is additive in short exact sequence. Also, it is trivial that rank of coherent sheaves is additive in short exact sequence. □

2.4.2 Grothendieck group

In this section, we define the Grothendieck group $K(C)$ of coherent sheaves and Grothendieck group $K_0(C)$ of locally free sheaves. We compare these two definitions and we prove the degree of determinant line bundle of a coherent sheaf is equal to the degree of the original coherent sheaf using the structure of Grothendieck groups.

Definition 2.4.3 (Grothendieck group)**.** Grothendieck group $K(C)$(resp. $K_0(C)$) of coherent sheaves on the curve C is defined by the following. Let M be a free abelian group generated by coherent(resp. locally free) sheaves on the curve C and let N be a subgroup of N generated by elements of the forms $\mathcal{F}-\mathcal{F}'-\mathcal{F}''$ comes from all short exact sequences $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ of coherent(resp. locally free) sheaves. Then we define the Grothendieck group $K(C)$(resp. $K_0(C)$) to be $K(C) := M/N$.

To Compare $K(C)$ and $K_0(C)$ we want to find a two-term resolution $0 \to E_1 \to E_0 \to \mathcal{F} \to 0$ for any coherent sheaf $\mathcal{F}$ on C.

Since C is projective, by [43, Theorem II.5.17], there is an integer n such that $\mathcal{F}(n)$ is globally generated, i.e. there is a surjection $E_0 = \mathcal{O}_C^{\oplus m}(-n) \overset{p}{\twoheadrightarrow} \mathcal{F}$ from a locally free sheaf E_0. Thus, it is enough to show that $\ker(p)$ is locally free. To show this, we need the following lemma.

Lemma 2.4.4. [66, Lemma 2.3.3] Consider a commutative local ring (R, m) and its residue field $k = R/m$. Let N be a finitely generated R-module. Then N is a free R-module if and only if $\mathrm{Tor}_1^R(N, k) = 0$.

Proof. First, assume that $\mathrm{Tor}_1^R(N, k) = 0$ and choose a basis $\{\overline{\nu_1}, ..., \overline{\nu_d}\}$ of the vector space $N \otimes_R k = N/mN$. Then we can find an element $\nu_1, ..., \nu_d$ which are lifts of elements $\overline{\nu_1}, ..., \overline{\nu_d}$. Consider a surjective morphism $R^{\oplus d} \to N$ sending each generators of i-th components to ν_i. Then we have the following short exact sequence :

$$0 \to \mathrm{Ker} \to R^{\oplus d} \to N \to 0$$

By taking the functor $(- \otimes k)$ to the above short exact sequence, we obtain the following long exact sequence :

$$0 \to \mathrm{Tor}_1^R(N, k) \to \mathrm{Ker} \otimes_R k \to k^{\oplus d} \to N \otimes_R k$$

Since we choose $(\overline{\nu_1}, ..., \overline{\nu_d})$ to be basis, the last morphism in the above long exact sequence is injective. Furthermore since $\mathrm{Tor}_1^R(N, k) = 0$, we have $\mathrm{Ker} \otimes_R k = 0$. By Nakayama's lemma, we have $\mathrm{Ker} = 0$. Therefore we have $R^{\oplus d} \cong N$. The opposite direction is trivial so we omit it here. □

Now, using the above lemma, we show that $\ker(p)$ is locally free. Since $\ker(p)$ is clearly a coherent sheaf, it is enough to show that its stalk $\ker(p)_x$ is a free $\mathcal{O}_{C,x}$ module for every point $x \in k$. Consider the short exact sequence of stalks $0 \to \ker(p)_x \to (E_0)_x \to \mathcal{F}_x \to 0$. By applying the functor $(- \otimes_{\mathcal{O}_x} k)$, we obtain the long exact sequence and using the result in the lemma 2.4.4, we have $\mathrm{Tor}_2^{\mathcal{O}_{C,x}}(\mathcal{F}_x, k) \cong \mathrm{Tor}_1^{\mathcal{O}_{C,x}}(\ker(p)_x, k)$. But since C is a smooth projective curve, $\mathcal{O}_{C,x}$ is a principal ideal domain. Therefore maximal ideal is generated by a single element. Therefore, we have a free resolution $0 \to \mathcal{O}_{C,x} \to \mathcal{O}_{C,x} \to k \to 0$, and by taking a functor $(\mathcal{F}_x \otimes_{\mathcal{O}_{C,x}} -)$, we obtain a long exact sequence and using Lemma 2.4.4 again we conclude that $\mathrm{Tor}_i^{\mathcal{O}_{C,x}}(\mathcal{F}_x, k) = 0$ for all $i > 1$. Thus we obtain $\mathrm{Tor}_1^{\mathcal{O}_{C,x}}(\ker(p)_x, k) = 0$ and therefore $\ker(p)$ is locally free.

In summary, for all coherent sheaf $\mathcal{F}$ on the smooth projective curve C, we can find a two-term locally free resolution :

$$0 \to E_1 \to E_0 \to \mathcal{F} \to 0. \tag{2.3}$$

Using this resolution, we can compare $K(C)$ and $K_0(C)$. We define a morphism $\phi : K(C) \to K_0(C)$ to be $\phi(\mathcal{F}) := [E_0] - [E_1]$. First, we should check well-definedness of this morphism. Assume that there is another locally free resolution of $\mathcal{F}$, $0 \to E_1' \to E_0' \to \mathcal{F} \to 0$. Then we can construct the third locally free resolution, $0 \to \mathrm{Ker} \to E_0 \oplus E_0' \to \mathcal{F} \to 0$. We can check that Ker is locally free in the same manner as we used above. It is enough to show that $[E_0] - [E_1] = [E_0 \oplus E_0'] - [\mathrm{Ker}]$. Since $[E_0 \oplus E_0'] = [E_0] + [E_0']$, it is equivalent to show that $[\mathrm{Ker}] = [E_0'] + [E_1]$. Consider the following commutative diagram :

$$\begin{tikzcd} & 0 \arrow[r] \arrow[d] & E_0 \oplus E_0' \arrow[r, equal] \arrow[d, "pr_1"] & E_0 \oplus E_0' \arrow[r] \arrow[d, two heads] & 0 \\ 0 \arrow[r] & E_1 \arrow[r] & E_0 \arrow[r] & \mathcal{F} \arrow[r] & 0 \end{tikzcd}$$

By the snake lemma, we obtain the short exact sequence $0 \to E_0' \to \mathrm{Ker} \to E_1 \to 0$. Therefore we have $[\mathrm{Ker}] = [E_0'] + [E_1]$. Therefore ϕ is well-defined. It is easy to prove that ϕ is a homomorphism so we omit it. Let $\psi : K(C) \to K_0(C)$ be a trivial homomorphism sending a locally free sheaf to itself. Then it is clear that ψ and ϕ are inverse to each other. Therefore $K(C)$ and $K_0(C)$ are isomorphic. In summary, we prove the following.

Proposition 2.4.5. [66, Proposition 2.6.6] Grothendieck groups $K(C)$ and $K_0(C)$ are isomorphic to each other. More explicitly, $\phi : K(C) \to K_0(C)$ and $\psi : K_0(C) \to K(C)$ are inverse to each other.

Remark 2.4.6. There is a natural ring structure on $K_0(X)$ given by a tensor product of locally free sheaves since the tensor product of locally free sheaves preserves short exact sequence. Therefore, we give a ring structure on $K(X)$ transferred from $K_0(X)$ via isomorphisms ϕ and ψ.

Definition 2.4.4 (Determinant line bundle)**.** Let E be a locally free sheaf of rank r on a smooth projective curve C. Then by taking a top wedge $\wedge^r E$, we obtain a line bundle and we call it a determinant line bundle $\det E$ of E.

We note that for a short exact sequence of locally free sheaves $0 \to E \to F \to G \to 0$ with $\operatorname{rank} E = r_1, \operatorname{rank} F = r_2, \operatorname{rank} G = r_3$, we can easily observe that $\wedge^{r_2} F \cong \wedge^{r_1} E \otimes \wedge^{r_3} G$ from linear algebra. Therefore, taking a determinant is a functor $K_0(C) \to \mathrm{Pic}(C)$. Therefore, composing with isomorphism ϕ, we have a functor $\det : K(C) \to \mathrm{Pic}(C)$.

Lemma 2.4.7. [66, Corollary 2.6.8] For an effective divisor D on the curve C, and a structure sheaf $\mathcal{O}_D$ we have $\det(\mathcal{O}_D) = \mathcal{O}_C(D)$. In particular, we have $\deg(\det(\mathcal{O}_D)) = \deg(\mathcal{O}_D)$.

Proof. This is clear from the following resolution of the structure sheaf $\mathcal{O}_D$:

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(D) \to \mathcal{O}_D \to 0$$

□

Next, we state the following structure theorem of the Grothendieck group $K(C)$.

Lemma 2.4.8. [66, Lemma 2.6.10] Let $S(C)$ be a subgroup of $K(C)$ generated by skyscraper sheaves $\mathcal{O}_x$ for all point $x \in C$. Then we have :

$$K(C) \cong S(C) \oplus \mathbb{Z}$$

By the structure theorem of the Grothendieck group $K(C)$ and Lemma 2.4.7, finally we obtain the following :

Proposition 2.4.9. [66, Corollary 2.6.7] For a coherent sheaf $\mathcal{F}$ on the curve $\mathcal{F}$, we have $\deg\mathcal{F} = \deg(\det\mathcal{F})$.

(sketch of the proof). By Lemma 2.4.8, for any coherent sheaf $\mathcal{F}$ its class $[\mathcal{F}]$ equals to the sum of classes of skyscraper sheaves, i.e. $[\mathcal{F}] = \sum_i [\mathcal{O}_{p_i}]$. Therefore, by Lemma 2.4.7 we have $[\det\mathcal{F}] = \sum_i [\mathcal{O}(p_i)]$. Therefore we have $\deg(\det\mathcal{F}) = \deg\mathcal{F}$. □

2.4.3 Semi-stability

In many moduli problems, we define a notion of stability or semi-stability of isomorphism class of objects, we want to collect. There are many reasons we define this notion. One of them is that almost all cases we cannot find moduli space which parametrizes all objects. Many cases this problem solved by defining suitable notion of semi-stability and collect only semi-stable objects.

In this section we define stable and semi-stable vector bundles on the curve C and study how it works in the category of vector bundles on the curve C.

Definition 2.4.5 (slope). Let $\mathcal{F}$ be a coherent sheaf on the curve C. Then we define a rational number called slope of $\mathcal{F}$ to be

$$\mu(\mathcal{F}) := \frac{\deg \mathcal{F}}{\operatorname{rank} \mathcal{F}}$$

Definition 2.4.6 (Stable, Semi-stable bundles). Let E be a vector bundle(locally free sheaf) on the curve C. Then we call E stable(resp. semi-stable) if every nontrivial coherent subsheaf $\mathcal{F} \subset \mathsf{E}$ satisfies the following condition

$$\mu(\mathcal{F}) < (\text{resp.} \leq)\ \mu(\mathsf{E}).$$

Fortunately, since we are working on the smooth projective curve case, we can make above definition more simple. For this, we need the following lemma.

Lemma 2.4.10. [66, Chapter 5.3, p. 73] Let E be a vector bundle on the curve C and $\mathcal{F} \subset \mathsf{E}$ be a coherent subsheaf. Then there exists a vector bundle F between $\mathcal{F}$ and E, $\mathcal{F} \subset \mathsf{F} \subset \mathsf{E}$, i.e. F is a sub-vector bundle of E and contains $\mathcal{F}$ as a subsheaf, satisfying :

$$\mu(\mathcal{F}) \leq \mu(\mathsf{F}).$$

Proof. Consider a quotient sheaf $\mathsf{E}/\mathcal{F}$. Then we can decompose it as a direct sum of torsion free part and torsion part. Therefore we can write $\mathsf{E}/\mathcal{F} \cong \widetilde{\mathsf{E}} \oplus \mathrm{Tor}$, where $\widetilde{\mathsf{E}}$ is the torsion free part and Tor is the torsion part.

Since every local ring $\mathcal{O}_{\mathsf{C},x}$ at a point $x \in \mathsf{C}$ are principal ideal domains torsion free $\mathcal{O}_{\mathsf{C},x}$-modules are free modules [70, Excercise 11.10]. Therefore the torsion free part $\widetilde{\mathsf{E}}$ is locally free. Then consider the projection $p : \mathsf{E} \to \widetilde{\mathsf{E}}$

and let $\mathsf{F} := \ker(\mathfrak{p})$. Then F is clearly a sub-vector bundle of E containing $\mathcal{F}$. Furthermore, by construction, $\mathsf{F}/\mathcal{F}$ is a torsion sheaf. Thus we obtain $\operatorname{rank} \mathsf{F}/\mathcal{F} = 0$. Therefore, since we have the following short exact sequence :

$$0 \to \mathcal{F} \to \mathsf{F} \to \mathsf{F}/\mathcal{F} \to 0$$

$\operatorname{rank}(\mathcal{F}) = \operatorname{rank} \mathsf{F}$ and $\deg \mathcal{F} \leq \deg \mathsf{F}$. So we conclude that $\mu(\mathcal{F}) \leq \mu(\mathsf{F})$. $\square$

By the above lemma, we can make the equivalent definition of stable and semi-stable bundles, which is more simple.

Definition 2.4.7 (Stable, Semi-stable bundles)**.** Let E be a vector bundle(locally free sheaf) on the curve C. Then we call E is stable(resp. semi-stable) if every nontrivial vector bundle $\mathsf{F} \subset \mathsf{E}$ satisfies the following condition condition :

$$\mu(\mathsf{F}) < (\text{resp. } \leq)\ \mu(\mathsf{E})$$

The next proposition says that semi-stability determines the 'direction' of morphisms in the category of semi-stable vector bundles on the curve C. Morphisms always arise in the direction of increasing slopes of semi-stable vector bundles.

Proposition 2.4.11. [66, Proposition 5.3.3] When there is a non-zero morphism $\varphi : \mathsf{E} \to \mathsf{F}$ between semi-stable vector bundle E and F on the curve C, we obtain $\mu(\mathsf{E}) \leq \mu(\mathsf{F})$.

Proof. First consider the image sheaf $\operatorname{Im}(\varphi) \subset \mathsf{F}$ of the morphism φ. Then, applying Lemma 2.4.10 to $\operatorname{Im}(\varphi)$, we obtain the sub-vector bundle $\mathrm{I} \subset \mathsf{F}$ containing $\operatorname{Im}(\varphi)$ having the same rank with $\operatorname{Im}(\varphi)$. Since F is semi-stable, we have $\mu(\mathrm{I}) \leq \mu(\mathsf{F})$. By definition, we have the following long exact sequence :

$$0 \to \ker(\varphi) \to \mathsf{E} \xrightarrow{\varphi} \mathrm{I} \to \mathrm{Tor} \to 0$$

Where Tor is a torsion sheaf. We can observe that $\ker(\varphi)$ is a locally free sheaf in the same manner as we showed the existence of a 2-term resolution in the equation (2.3). Since E is semi-stable, we have $\mu(\ker(\varphi)) \leq \mu(\mathsf{E})$ and since Tor is torsion free, we obtain $\mathrm{rank}(\mathrm{Tor}) = 0$. Therefore, by direct calculation, we have $\mu(\mathsf{E}) \leq \mu(\mathrm{I})$. Thus we showed $\mu(\mathsf{E}) \leq \mu(\mathsf{F})$. □

Furthermore, for morphisms between stable vector bundles, we obtain more powerful result.

Proposition 2.4.12. [66, Proposition 5.3.3] Let $\varphi : \mathsf{E} \to \mathsf{F}$ be a nonzero morphism between stable vector bundles E and F on the curve C where $\mu(\mathsf{E}) = \mu(\mathsf{F})$. Then φ is an isomorphism.

Proof. Recall the proof of the previous lemma. If $\mathrm{I} \neq \mathsf{F}$, then since F is stable we have $\mu(\mathsf{E}) \leq \mu(\mathrm{I}) < \mu(\mathsf{F})$, which contradicts to the fact that $\mu(\mathsf{E}) = \mu(\mathsf{F})$. Therefore we have $\mathrm{I} = \mathsf{F}$. Next, recall the following long exact sequence :

$$0 \to \ker(\varphi) \to \mathsf{E} \to \mathrm{I} \to \mathrm{Tor} \to 0$$

If $\ker(\varphi) \neq 0$, then $\mu(\ker(\varphi)) < 0$. If $\mathrm{Tor} \neq 0$, then $\deg \mathrm{Tor} > 0$. Therefore if $\ker(\varphi) \neq 0$ or $\mathrm{Tor} \neq 0$ then we have $\mu(\mathsf{E}) < \mu(\mathrm{I})$, which leads to a contradiction. Thus we have $\ker(\varphi) = \mathrm{Tor} = 0$ so we conclude that $\varphi : \mathsf{E} \to \mathsf{F}$ is an isomorphism. □

Corollary 2.4.13. [66, Corollary 5.3.4] Let $\varphi : \mathsf{E} \to \mathsf{E}$ be a nonzero endomorphism of stable vector bundle E on the curve C. Then φ is a scalar multiplication.

Proof. Consider an algebra of endomorphisms $k[\varphi]$ generated by the endomorphism φ. By the above proposition, this algebra is a field. But since the field k is algebraically closed, $k[\varphi] = k$. Therefore φ must be equal to one of the scalar multiplication. □

The next lemma follows from the direct calculations.

Lemma 2.4.14. [66, Proposition 5.3.6] Consider the following short exact of vector bundles :

$$0 \to E \to F \to G \to 0$$

If two of three vector bundles have the same slope μ, then the third one have also same slope μ.

Next, we introduce an important structure result about semi-stable vector bundles. A filtration called Jordan-Hölder filtration suggest us a way to analyze a semi-stable bundle via stable bundles with same slopes.

Definition 2.4.8 (Jordan-Hölder filtration)**.** Let E be a semi-stable vector bundle on the curve C with a slope μ. Then a Jordan-Hölder filtration is the following increasing filtration of sub-vector bundles with the same slope μ :

$$0 \subset E_0 \subset E_1 \subset \cdots \subset E_m = E$$

such that each successive quotient $\mathrm{gr}_i := E_i/E_{i-1}$ are stable vector bundles. We call $\bigoplus_i \mathrm{gr}_i$ the associated grading of the Jordan-Hölder filtration.

Proposition 2.4.15. [66, Proposition 5.3.7] For a semi-stable vector bundle E on the curve C, every Jordan-Hölder filtration of E has the same length and the associated gradings $\bigoplus_i \mathrm{gr}_i$ are isomorphic as a vector bundle.

Proof. Consider two different Jordan-Hölder filtrations

$$0 \subset E_0 \subset E_1 \subset \cdots \subset E_m = E$$

and

$$0 \subset E'_0 \subset E_1 \subset \cdots \subset E'_n = E.$$

Then there exist an integer i such that $E_0 \subset E'_i$ and $E_0 \subsetneq E'_{i-1}$. Since nonzero morphisms between stable vector bundles are isomorphisms, we have $E_0 \cong$

E'_i/E'_{i-1}. Now we obtain two Jordan-Hölder filtrations of E/E_0

$$0 \subset E_1/E_0 \subset \cdots \subset E_m/E_0 = E/E_0$$

and

$$0 \subset E'_0/(E_0 \cap E'_0) \subset E_1/(E_0 \cap E_1) \subset \cdots \subset E'_{i-1}/(E_0 \cap E'_{i-1}) \subset E_{i+1}/E_0 \subset \ldots$$
$$\subset E'_n/E_0 = E/E_0.$$

But using induction on the rank of E, we obtain that associated grading of these two filtrations are isomorphic. Since associated grading of original two filtrations are just obtained by the direct sum of E_0 with this associated gradings, we completed the proof by induction on the rank of E. $\square$

Definition 2.4.9 (S-equivalence class)**.** For two semi-stable vector bundles E and E', we call E and E' are S-equivalent if their associated gradings are isomorphic.

Similar to Jordan-Hölder filtration, we introduce a structure result about vector bundles, called Harder-Narasimhan filtration. It suggests us a way to analyze a vector bundle via semi-stable vector bundles.

Definition-Proposition 2.4.16 (Harder-Narasimhan filtration)**.** [66, Proposition 5.4.2] Let E be a vector bundle on the curve C. Then there exist a unique increasing filtration called Harder-Narasimhan filtration

$$0 \subset E_0 \subset E_1 \subset \cdots \subset E_m = E$$

satisfies the following properties

1. the i-the grading $\mathrm{gr}_i = E_i/E_{i-1}$ is a semi-stable vector bundle
2. Slopes $\mu(\mathrm{gr}_i)$ are strictly decreasing.

2.5 Moduli space of vector bundles on a smooth projective curves

2.5.1 Construction of Moduli spaces of vector bundles

Contents in this section mostly follow [66, Part I, Chapter 7]. In this section we define the moduli space of isomorphism classes of vector bundles on the smooth projective curve C with a fixed rank r and fixed degree d, as a GIT quotient of a Hilbert scheme. We observe a quite surprising phenomenon that the GIT stable and semi-stable conditions coincide with the stable and semi-stable condition of vector bundles on the curve C, what we defined in the previous section.

The construction of moduli space based on the following observation. Let $S(r,d)$ be an isomorphism class of semi-stable vector bundles on C with rank r and degree d. Consider a vector bundle class $[E] \in S(r,d)$. Fix a point $x \in C$, and choose a large integer $N > 2g-1-\mu(E)$. Then by Proposition 2.4.11, we obtain $\mathrm{Hom}(F(Nx), w_C) = 0$, where w_C is a dualizing sheaf of the curve C. Therefore, by Serre duality, we have $H^1(C, E(Nx)) = 0$. By Riemann-Roch formula, we obtain $H^0(C, E(Nx)) = d+r(N+1-g)$. Again by Riemann-Roch, we can check that $H^0(C, E(Nx-1)) = H^0(C, E(Nx)) - 1$. Therefore, $E(Nx)$ is generated by its global sections. Thus, If we fix a $d+r(N+1-g)$-dimensional k-vector space W, we have a surjection of vector bundles :

$$W \otimes \mathcal{O}_C(-Nx) \twoheadrightarrow E$$

Therefore this surjection correspond to a closed point in $\mathrm{QuotF}^{rt+d+r(1-g)}_{W\otimes\mathcal{O}_C(-Nx)/C}$. The reason that we choose Hilbert polynomial as $rt+d+r(1-g)$ is the following. Consider a rank r and degree d vector bundle E, and for a fixed point $x \in C$, $\mathcal{O}_C(x)$ is an ample line bundle. Then For a sufficiently large integer $t >> 0$, we have $H^0(\mathcal{F}\otimes\mathcal{O}_C(x)^t) = r(\mu+1+t-g) = rt+d+r(1-g)$ by Riemann-Roch. Therefore we choose the Hilbert polynomial to be $rt+$

$d + r(1-g)$. We usually write $d + r(1-g) := \chi$. Therefore we have the Hilbert polynomial $rt + \chi$.

Now we have natural SL(W)-action on this Quot scheme $\mathrm{Quot}^{rt+\chi}_{W\otimes\mathcal{O}_C(-Nx)/C}$ by definition. We will obtain the moduli space of vector bundles on the curve C with rank r, degree d as a good quotient of this Quot scheme for SL(W)-action. For this goal. We should show that GIT semi-stable locus in the Quot scheme coincides with the locus of points comes from semi-stable vector bundles. The following proposition helps us to compare these two loci.

Proposition 2.5.1. [66, Proposition 7.1.1, Proposition 7.1.3] We can choose a sufficiently large integer $N(r, d)$ satisfies the following:

1. For any $N' \geq N(r, d)$, and a coherent sheaf $\mathcal{E}$ with rank r and degree d and a fixed point $x \in C$, $\mathcal{E}$ is locally free and semi-stable if and only if all coherent subsheaves of $\mathcal{E}' \subset \mathcal{E}$ with rank r' satisfies :
$$h^0(C, \mathcal{E}'(N'x)) \leq \frac{r'}{r} h^0(C, \mathcal{E}(N'x)).$$
2. For any integer $N' \geq N(r, d)$ and a vector bundle E with rank r and degree d, and for any nonzero subsheaf $\mathcal{E} \subset E$ of rank r' and a fixed point $x \in C$, $\mu(F') = \mu(F)$ if and only if
$$\frac{h^0(C, \mathcal{E}(N'x))}{r'} = \frac{h^0(C, E(N'x))}{r}.$$

Now we recall the construction of Quot scheme in Section 2.2.2. For sufficiently large integer N', there is an embedding of functors :

$$\begin{aligned} T_{N'} : \mathfrak{Quot}^{rt+\chi}_{W\otimes\mathcal{O}_C(-Nx)/C} &\longmapsto \mathsf{Grass}\,(H \otimes \Gamma(C, \mathcal{O}_C((N'-N)x), rN' + \chi) \\ [p : W \otimes \mathcal{O}_C(-Nx) \twoheadrightarrow E] &\longmapsto \left[W \otimes H^0(C, \mathcal{O}_C((N'-N)x) \twoheadrightarrow H^0(C, E(N'x))\right]. \end{aligned}$$

When we write $D_{N'-N} := H^0(C, \mathcal{O}_C((N'-N)x)$, then we can rewrite $\mathsf{Grass}\,(W \otimes \Gamma(C, \mathcal{O}_C((N'-N)x), rN'+\chi)$ by $\mathsf{Grass}\,(W \otimes D_{N'-N}, rN'+\chi)$.

Based on Proposition 2.3.16, we can show the following lemma, which gives us the description about stable and semi-stable locus in the $T_{N'}$-embedding under SL(W)-action.

Lemma 2.5.2. [66, Lemma 7.2.2] For a point $x = [W \otimes \mathcal{O}_C(-Nx) \twoheadrightarrow \mathcal{E}] \in \mathfrak{Quot}^{rt+\chi}_{W\otimes\mathcal{O}_C(-Nx)/C}$ and a sufficiently large integer $n >> 0$, the following statements are equivalent

1. The point x is semi-stable(resp. stable) under the functor T_n and the SL(W)-action.

2. For any nonzero proper sub-vector space $W' \subset W$, and its image sub-coherent sheaf $\mathcal{E}' \subset \mathcal{E}$, it satisfies the following equation :

$$\frac{\mathsf{HP}(\mathcal{E}')}{\dim W'} \geq (\text{resp.}>) \frac{rt+\chi}{\dim W}$$

   where $\mathsf{HP}(\mathcal{E}')$ is the Hilbert polynomial of the sub-coherent sheaf of $\mathcal{E}'$.

(Sketch of the proof). Assume that the point x is semi-stable. Using [66, Lemma 7.2.3] for boundedness, we can show that there is an integer $N'(r,d)$ such that $H^1(C, \mathcal{E}(n)) = H^1(C, \mathcal{E}'(n)) = 0$ and the canonical morphism $W' \otimes D_{n-N} \twoheadrightarrow \mathcal{E}'(n)$ is surjective for any $n > N'(r,d)$.

Then from Proposition 2.3.16, we can directly induce that :

$$\frac{h^0(C, \mathcal{E}'(nx))}{\dim W'} \geq \frac{h^0(C, \mathcal{E}(nx))}{\dim W}$$

for all $n > N'(r,d)$. Therefore we obtain

$$\frac{\mathsf{HP}(\mathcal{E}')}{\dim W'} \geq \frac{rt+\chi}{\dim W}.$$

The inverse direction is straightforward. □

We can induce another description about stable and SL(W)-semistable locus.

Lemma 2.5.3. [66, Lemma 7.2.4] For a point $x = [W \otimes \mathcal{O}_C(-Nx) \twoheadrightarrow \mathcal{E}] \in \mathrm{Quot}^{rt+\chi}_{W\otimes\mathcal{O}_C(-Nx)/C}$, there exist an integer $N''(r,d)$ such that for every $n > N''(r,d)$, x is semi-stable with respect to the SL(W)-action if and only if $\mathcal{E}$ is semi-stable coherent sheaf and the natural morphism :

$$W \otimes D_{n-N} \to \mathrm{H}^0(C, \mathcal{E}(nx))$$

is an isomorphism.

Furthermore, we can find 1-1 correspondence between sub-coherent sheaves of semi-stable coherent sheaves with same slope and sub-vector spaces of sections.

Lemma 2.5.4. [66, Lemma 7.2.5] Consider a point $x = [W \otimes \mathcal{O}_C(-Nx) \twoheadrightarrow \mathcal{E}] \in \mathrm{Quot}^{rt+\chi}_{W\otimes\mathcal{O}_C(-Nx)/C}$. We note that $\mathcal{E}$ is a semi-stable coherent sheaf by Lemma 2.5.3. Let $\mathcal{E}' \subset \mathcal{E}$ be a coherent subsheaf with $\mathrm{rank}\,\mathcal{E}' > 0$ and $\mu(\mathcal{E}') = \mu(\mathcal{E})$. Then $\mathcal{E}'$ is generated by a vector subspace $W' \subset W$ which satisfies :

$$\frac{\mathsf{HP}(\mathcal{E}')}{\mathrm{dim}W'} = \frac{\mathsf{HP}(\mathcal{E})}{\mathrm{dim}W}.$$

Explicitly, we have $W' = \mathrm{H}^0(C, \mathcal{E}'(Nx))$ for such $W' \subset W$.

Finally, we define the projective space $M(r,d) := \left(\mathrm{Quot}^{rt+\chi}_{W\otimes\mathcal{O}_C(-Nx)/C}\right)^{ss} /\!\!/ \mathrm{SL}(W)$. Consider a point $x = [W \otimes \mathcal{O}_C(-Nx) \twoheadrightarrow \mathcal{E}] \in \left(\mathrm{Quot}^{rt+\chi}_{W\otimes\mathcal{O}_C(-Nx)/C}\right)^{ss}$. Then by Lemma 2.5.3, 2.5.3, 2.5.4, We can show that $\mathcal{E}$ is a semi-stable vector bundle with rank r and degree d. Similarly, we can also show that if $x \in \left(\mathrm{Quot}^{rt+\chi}_{W\otimes\mathcal{O}_C(-Nx)/C}\right)^{s}$, $\mathcal{E}$ is a stable vector bundle with rank r and degree d. Furthermore, we have the following theorem.

Theorem 2.5.5. [66, Theorem 7.2.1] Consider a functor $\mathcal{M}(r,d) : \mathrm{Sch}/k \to \mathrm{Sets}$ which correspond a parameter scheme S to a set of isomorphism class of vector bundle E on $S \times C$ such that each fiber $E|_x$ over $x \in C$ are semi-stable bundle of degree d.

Then $M(r,d)$ is a coarse moduli space of the functor $\mathcal{M}(r,d)$. A closed point of $M(r,d)$ correspond to an S-equivalence class of semi-stable bundles of rank r and degree d. Furthermore, its stable locus
$M^s(r,d) = \left(\mathrm{Quot}^{rt+\chi}_{W\otimes\mathcal{O}_C(-Nx)/C}\right)^s /\!\!/ \mathrm{SL}(W) \subset M(r,d)$ parametrizes isomorphism classes of stable vector bundles.

We note that if two semi-stable bundle E and E' with rank r and degree d are S-equivalent, then we can check that closures of their $\mathrm{SL}(W)$-orbit in some embedding in Grassmannian has nonzero intersection. So by Proposition 2.3.9, E and E' induce the same point in $M(r,d)$.

2.5.2 Smoothness of $M^s(r,d)$

Contents in this section mostly follow [66, Part I, Chapter 8]. In this section, we show the smoothness of the stable locus

$$M^s(r,d) \overset{\text{open}}{\subset} M^s(r,d).$$

Let us introduce the main result of this subsection

Proposition 2.5.6. [66, Theorem 8.3.2] If $M^s(r,d)$ is nonempty, then it is a smooth $(r^2(g-1)+1)$-dimensional variety where $g = g(C)$. Furthermore, for a closed point $[F] \in M^s(r,d)$, represented by a stable bundle F, we have a natural identification for the tangent space of $M^s(r,d)$ at the point $[F]$ as follows

$$T_{[F]}M^s(r,d) \cong \mathrm{Ext}^1(F,F).$$

For a reader who is familiar with the deformation theory of the vector bundle, the above result on tangent space looks clear. It seems that the result directly follows from the result on the extension of vector bundle [42, Theorem 7.3]. But in fact, it is not so clear since $M^s(r,d)$ is not always a fine moduli. In fact, $M^s(r,d)$ is a fine moduli space if and only if $(r,d)=1$ [82, Theorem 5.12], [88].

So we prove the theorem from the GIT quotient construction of $M^s(r,d)$. For a point $[E] \in M^s(r,d)$, we can represent it as a point in the Quot scheme $\mathrm{Quot}^{rt+\chi}_{W\otimes\mathcal{O}_C(-Nx)/C}$:

$$W \otimes \mathcal{O}_C(-Nx) \twoheadrightarrow E.$$

Let K be the kernel of the above surjection so we have the short exact sequence $0 \to K \to W \otimes \mathcal{O}_C(-Nx) \to E \to 0$. Then by taking the functor $\mathrm{Hom}(-,E)$, we have the following long exact sequence :

$$\begin{aligned} &0 \to \mathrm{Hom}(E,E) \cong \mathrm{k} \to \mathrm{Hom}(W \otimes \mathcal{O}_C(-Nx), E) \cong \mathrm{End}(W) \to \mathrm{Hom}(E,K) \\ &\to \mathrm{Ext}^1(E,E) \to W \otimes_{\mathrm{k}} H^1(E(Nx)) = 0 \end{aligned} \tag{2.4}$$

Then by the result in section 2.2, Proposition 2.2.7 on the tangent space of Quot schemes, we have $T_{[E]}\mathrm{Quot}^{rt+\chi}_{W\otimes\mathcal{O}_C(-Nx)/C} = \mathrm{Hom}_{\mathcal{O}_C}(K,E)$.

Since $\mathrm{End}(W)/\mathrm{k}$ is isomorphic to the lie algebra of $\mathrm{SL}(W)$, which we use in the GIT quotient construction $M^s(r,d) := \left(\mathrm{Quot}^{rt+\chi}_{W\otimes\mathcal{O}_C(-Nx)/C}\right)^s /\!/ \mathrm{SL}(W)$, we need to know the information about the stabilizer subgroup of $\mathrm{GL}(W)$ at the point $[E]$. We state the following result :

Proposition 2.5.7. [66, Lemma 8.3.1] The stabilizer subgroup of $\mathrm{GL}(W)$-action on $\mathrm{Quot}^{rt+\chi}_{W\otimes\mathcal{O}_C(-Nx)/C}$ at the point $[W \otimes \mathcal{O}_C(-Nx) \twoheadrightarrow E]$ is the automorphism group $\mathrm{Aut}_{\mathcal{O}_C}(E,E)$ of the sheaf E.

Proof of Proposition 2.5.6. Let E is a stable bundle, then the stabilizer subgroup is isomorphic to k^*. Therefore, the stabilizer subgroup of $\mathrm{SL}(W)$-subgroup at the point $[E]$ where E is a stable bundle, is $\mathbb{Z}_2$, which has endomorphism group 0. Therefore, from the sequence (2.4) and the above argument, we conclude that $T_{[E]}M^s(r,d) \cong T_{[E]}\mathrm{Quot}^{rt+\chi}_{W\otimes\mathcal{O}_C(-Nx)/C}/\mathrm{End}(\mathrm{SL}(W)) = \mathrm{Ext}^1(E,E)$.

By using the local-to-global spectral sequence, we have $\dim \mathrm{Ext}^1(E, E) = H^1(C, \mathcal{H}om(E, E))$. Since endomorphism bundle $\mathcal{H}om(E, E)$ has rank r^2 and degree 0, and since $H^0(\mathcal{H}om(E, E)) = \mathrm{Hom}(E, E) = k$, by Riemann-Roch formula, we have $\dim \mathrm{Ext}^1(E, E) = r^2(g - 1) + 1$. Therefore, $M^s(r, d)$ has same tangent bundle dimension at every point. Therefore, by generic smoothness, $M^s(r, d)$ is smooth with dimension $r^2(g - 1) + 1$. □

2.5.3 Various properties

In this subsection, we introduce various geometric properties of the moduli space $M(r, d)$. In addition to the smoothness result of $M^s(r, d)$ in the previous subsection, the following proposition figures out exactly what is the singularity of the moduli space $M(r, d)$.

Proposition 2.5.8. [79, Theorem 1] Except for $n = 2, g = 2$ case, singular locus of $M(r, d)$ is exactly the locus of strictly semi-stable bundles.

Fortunately, regardless of (r, d), we have the irreducibility of the moduli space $M(r, d)$.

Proposition 2.5.9. [66, Theorem 8.5.2] The moduli space $M(r, d)$ is an irreducible variety.

We often overlook that non-emptiness of $M(r, d)$ is not a trivial fact. In fact, moduli space $M(r, d)$ is empty if the curve C is the projective line $\mathbb{P}^1$, $r \geq 2$ and if r does not divides d.

Proposition 2.5.10. [66, Theorem 8.6.1, Theorem 8.6.2],[2, Theorem 7]

1. If the curve C has genus $g(C) \geq 2$, stable locus $M^s(r, d)$ is non-empty for all (r, d).

2. If the curve C is elliptic, then $M(r, d)$ is non-empty for all (r, d).

As we commented above, stable locus $M^s(r,d)$ is a fine moduli space, i.e. it carries universal family if and only if $(r,d)=1$.

Proposition 2.5.11. [82, Theorem 5.12],[88, Theorem 2] If the curve C has genus $g(C) \geq 2$, then the open sublocus $M^s(r,d)$ is a fine moduli space if and only if (r,d)=1

2.6 Examples of Fano varieties

In this section, we introduce past results on the geometry of Fano varieties which will be the main objects of our paper.

2.6.1 Moduli space $\mathcal{N}$ of rank 2 stable vector bundle on a curve with fixed determinant

Let C be a smooth projective curve with genus $g \geq 4$ over $\mathbb{C}$ and $M(r,d)$ be the moduli space of stable rank r and degree d vector bundles over C, which we constructed in Section 2.5. Then there is a determinant map $\mathtt{det}: K_0(C) \to \mathrm{Pic}(C)$ induces a morphism $M(r,d) \to \mathrm{Pic}^d(C)$ [66, Chapter 8.6] from the moduli space of vector bundles to the Picard group of degree d line bundles.

For a moduli functor $\mathcal{M}_L(r,d) : \mathrm{Sch/k} \to \mathrm{Sets}$ which assigns a parameter scheme S to a set of isomorphism class of vector bundle E on $S \times C$ such that its determinant $\wedge^r E$ is isomorphic to $\pi_2^* L \otimes \pi_1^* M$ where $\pi_1, \pi_2 : S \times C \to S, C$ are projections and M is a line bundle on S. Seshadri [92] showed that there exist a coarse moduli space $M_L(r,d)$ of the moduli functor $\mathcal{M}_L(r,d)$ which parametrizes S-equivalence class of semi-stable bundles with fixed determinant L.

Proposition 2.6.1. [92, Theorem 8.1] A moduli functor $\mathcal{M}_L(r,d)$ has a coarse moduli space $M_L(r,d)$ which is a normal projective variety parametrizing S-equivalence class of semi-stable rank r, degree d vector bundles which has fixed determinant line bundle $L \in \mathrm{Pic}^d(C)$.

By construction in [92], we have $M(r,d) \times_{\mathrm{Pic}^d(C)} [L] = M_L(r,d)$. Since $M(r,d)$ is a fine moduli space i.e. carries a universal family for a pair $(r,d) = 1$, $M_L(r,d)$ is also a fine moduli, which carries a universal family which is a pull-back of a universal family on $M(r,d)$.

Moreover, $M(r,d) \to \mathrm{Pic}^d(C)$ induces a map between tangent spaces $\mathrm{Ext}^1(E,E) \cong H^1(C, \mathcal{H}om(E,E)) \to H^1(C, \mathcal{O}_C)$ which is obtained from the trace map $\mathcal{H}om(E,E) \xrightarrow{\mathrm{tr}} \mathcal{O}_C$. Since this map is surjective with constant codimension, we conclude that the map $M(r,d) \to \mathrm{Pic}^d(C)$ is smooth [43, Chapter III, Proposition 10.4], therefore we conclude that $M_L(r,d)$ is smooth with dimension $r^2(g-1)+1-g = (r^2-1)(g-1)$.

Furthermore, in [29], Drezet and Narasimhan found out the Picard group of $M_L(r,d)$ is isomorphic to $\mathbb{Z}$, where its generator is a divisor called generalized theta divisor.

Proposition 2.6.2. [29, Theorem B] Let, the curve C has genus $g(C) \geq 2$. Then we have the following.

1. We define the generalized theta divisor Θ to be the following Brill-Noether type divisor :

$$\Theta = \{E \in \mathcal{N} | H^0(C, E \otimes L) \neq 0\}$$

   for any degree g line bundle L, i.e. it does not depend on the choice of the degree g line bundle L.

2. The Picard group of the moduli space $M_L(r,d)$ is given by $\mathrm{Pic}(M_L(r,d)) \cong \mathbb{Z} = \langle \Theta \rangle$.

Moreover, for $(r,d) = 1$, Ramanan [88] figured out that canonical class of the smooth variety $M_L(r,d)$ is equal to -2Θ, which says that $M_L(r,d)$ is a Fano variety.

Proposition 2.6.3. [88, Theorem 1] The canonical class $K_{M_L(r,d)}$ is equal to -2Θ.

Since we know that $M_L(r, d)$ is projective by Seshadri [92, Theorem 8.1], it is natural to curious about the very ampleness about the divisor $k\Theta$. We have an answer for the case $r = 2$ and d is odd by Brivio and Verra.

Proposition 2.6.4. [7, Theorem 1] For $r = 2$, d is odd, the curve C has genus $g(C) \geq 2$, then the generalized theta divisor Θ is very ample.

For small (r, d), we have explicit information about $M_L(r, d)$. The following results are due to Newstead and Narasiman-Ramanan.

Proposition 2.6.5. [79, Theorem 3, Remarks 1], [81, Theorem 1], [78] For the $r = 2$ case, we have the following results:

1. If $g(C) = 2$, $M_L(2, 0) \cong \mathbb{P}^3$.
2. If $g(C) = 2$, $M_L(2, 1)$ is a smooth complete intersection of two quadric hypersurfaces in $\mathbb{P}^5$.
3. If $g(C) = 3$ and C is not hyperelliptic, then $M_L(2, 0)$ is a coble quartic [25] in $\mathbb{P}^7$, which is singular along the Kummer variety $K_C \subset M_L(2, 0)$.

On the other hand, we have a result on the rationality of the moduli space $M_L(r, d)$ by King and Schofield.

Proposition 2.6.6. [60, Theorem 1, Theorem 2] When $g(C) \geq 2$, we have the following.

1. The moduli space $M_L(r, d)$ and the product $M(m, 0) \times \mathbb{P}^{(r^2-m^2)(g-1)}$ are birational when $m = (r, d)$.
2. If $(r, d) = 1$, then the moduli space $M_L(r, d)$ is rational.

Finally, we define a space $\mathcal{N} := M_{\mathcal{O}_X(-x)}(2,-1)$, which is a moduli of rank 2 stable vector bundles on the smooth projective curve X with determinant $g(X) \geq 4$ over $\mathbb{C}$ with a fixed determinant line bundle $\mathcal{O}_X(-x)$, where $x \in X$ is a fixed point. This space will be one of the main object of our story.

By the results we summarized above, $\mathcal{N}$ is a smooth projective Fano variety with Picard group Pic$\mathcal{N} \cong \mathbb{Z} = \langle \Theta \rangle$, where Θ is a generalized theta divisor, which is very ample. We also have $K_{\mathcal{N}} = -2\Theta$. Moreover, $\mathcal{N}$ is a fine moduli space and $\mathcal{N}$ is a rational variety.

As we finish this subsection, we strongly recommend the lecture note, 'Vector Bundles on Algebraic Curves' by P.E. Newstead which is good for review the result from past to the present on the study of moduli space of vector bundles on algebraic curves even though it is not published.

2.6.2 Hyperplane sections of the Grassmannian $\mathrm{Gr}(2,5)$

Let $G = \mathrm{Gr}(2,5)$ and we denote by Y^m the intersection of the Grassmannian $\mathrm{Gr}(2,5) \subset \mathbb{P}^{\binom{5}{2}-1} = \mathbb{P}^9$ and $6-m$ general hyperplanes. Then Y^m is a smooth Fano variety of dimension m.

We note that for any choice of general $6-m$ hyperplane sections, the smooth Fano variety Y^m the intersection of $\mathrm{Gr}(2,5)$ with these $6-m$ hyperplane sections does not depend on the choice of $6-m$ hyperplanes up to the projective equivalence given by $\mathrm{PGL}(\mathbb{C}^5)$-action. For $m = 3$ case, the proof is appeared in [91, Lemma 2.1] and [52, Chapter II, theorem 1.1].

Then, hyperplane sections of Grassmannian $\mathrm{Gr}(2,5)$ has many interesting properties. One of them is about their automorphism groups. More generally, Piontkowski and Van de Ven [85] studied automorphism groups of hyperplane sections of $\mathrm{Gr}(2,n)$. In particular, for $Y^3 = \mathrm{Gr}(2,5) \cap H^1 \cap H^2 \cap H^3$ for general hyperplanes $H^1, H^2, H^3 \subset \mathbb{P}^9$, its geometry well explained in the book of Cheltsov and Shramov, 'Cremona Groups and the Icosahedron' [14]. We introduce part of these results in this section.

For hyperplane sections of Grassmannian $\mathrm{Gr}(2,n)$, we have the following results for their automorphism groups by Piontkowski and Van de Ven.

Proposition 2.6.7. [85, Proposition 2.1, Theorem 3.5, Theorem 6.6]

1. The automorphism group $\mathrm{Aut}(\mathrm{Gr}(2,2n)\cap H)$ where H is a general hyperplane section of the Grassmannian under the Plüker embedding is $\mathrm{Sp}(2n,\mathbb{C})/\mathbb{Z}_2$. Furthermore, its action on $\mathrm{Gr}(2,2n)\cap H$ is homogeneous.

2. When $n \geq 3$, the automorphism group $\mathrm{Aut}(\mathrm{Gr}(2,2n)\cap H_1 \cap H_2)$ where H_1, H_2 are 2 general hyperplane sections of the Grassmannian under the Plüker embedding has $\mathrm{SL}(2,\mathbb{C})^n/\mathbb{Z}_2$ as a normal subgroup. Moreover, its quotient group $\mathrm{Aut}(\mathrm{Gr}(2,2n)\cap H_1\cap H_2)/\left(\mathrm{SL}(2,\mathbb{C})^n/\mathbb{Z}_2\right)$ is isomorphic to the symmetric group S_3 for $n=3$, isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ for $n=4$, and trivial for $n \geq 5$.

Proposition 2.6.8. [85, Proposition 5.2, Theorem 6.6]

1. The automorphism group $\mathrm{Aut}(\mathrm{Gr}(2,2n+1)\cap H)$ where H is a general hyperplane section of the Grassmannian under the Plücker embedding is isomorphic to an extension of $\mathrm{Sp}(2n,\mathbb{C}) \times \mathbb{C}^*/\mathbb{Z}_2$ by $\mathbb{C}^{2n}$, which is also isomorphic to the group :

$$\left\{ \left(\begin{array}{ccc|c} & & & 0 \\ & T & & \vdots \\ & & & 0 \\ \hline a_0 & \dots & a_{2n-1} & b \end{array} \right) \;\middle|\; \begin{array}{c} T \in \mathrm{Sp}(2n,\mathbb{C}) \\ a_i \in \mathbb{C} \\ b \in \mathbb{C}^* \end{array} \right\} / \{1,-1\}.$$

2. The automorphism group $\mathrm{Aut}(\mathrm{Gr}(2,2n+1)\cap H_1\cap H_2)$ where H_1, H_2 are a 2 general hyperplane sections of the Grassmannian under the Plücker embedding is isomorphic to an extension of $\mathrm{PGL}(2,\mathbb{C})$ by the semi-direct product $\mathbb{C}^{2n} \ltimes \mathbb{C}^*$. More precisely, the automorphism group is

isomorphic to the subgroup of $\mathrm{PGL}(2n+1,\mathbb{C})$, whose elements are of the following form :

$$\begin{pmatrix} cM_n & 0 \\ N & M_{n+1} \end{pmatrix} \begin{pmatrix} {}^t t_n^{-1} & 0 \\ 0 & t_{n+1} \end{pmatrix}$$

where $c \in \mathbb{C}^*$, $N \in M_{n+1,n}(\mathbb{C})$ such that its entry satisfies $n_{ij} = n_{kl}$ whenever $i+j = k+l$, and t_n are transformation induced from the $\mathrm{PGL}(2,\mathbb{C})$-action on the standard rational normal curve in $\mathbb{P}^{n-1}$, t_{n+1} is defined in the same manner.

For a general hyperplane section H of the Grassmannian $\mathrm{Gr}(2, 2n+1)$ under the Plücker embedding, H is defined by the linear equation $\Omega_H \in (\wedge^2 \mathbb{C}^{2n+1})^\vee$, which is a skew-symmetric 2-form. Since every skew-symmetric 2-form has even rank, Ω_H should have a kernel $0 \neq c_H \in \mathbb{C}^{2n+1}$. Since we choose general hyperplane H, $\operatorname{rank}\Omega_H = 2n$ and c_H is unique up to scaling. So we call the unique point $[c_H] \in \mathbb{P}^{2n}$ the center of H.

The following proposition says that the center point plays a key role in the geometry of $\mathrm{Gr}(2, 2n+1) \cap H$.

Proposition 2.6.9. [85, Proposition 5.3] The automorphism group $\mathrm{Aut}(\mathrm{Gr}(2, 2n+1) \cap H)$ acts on $\mathrm{Gr}(2, 2n+1) \cap H$, which is a subspace of the space of projective lines in $\mathbb{P}^{2n}$, with two orbits :

1. lines passing through the center point $[c_H]$.

2. lines which do not pass through the center point.

Moreover, if we consider two general hyperplane sections H_1, H_2 in the Grassmannian $\mathrm{Gr}(2, 2n+1)$, we can also consider $\mathbb{P}^1$-parameter $[s:t] \in \mathbb{P}^1$, and for each $[s:t] \in \mathbb{P}^1$, we can assign a center point $c_{[s:t]} := c_{[sH_1 - tH_2]}$. Since H_1, H_2 are general hyperplane sections, $sH_1 - tH_2$ are has rank $2n$ so $c_{[sH_1+tH_2]}$ is well-defined. So, we have an assignment from $\mathbb{P}^1$ to the point in $\mathbb{P}^{2n}$. Moreover, it is known that it is a rational normal curve of degree n.

Proposition 2.6.10. [85, Proposition 6.3] The map defined above :

$$\begin{aligned} c : \mathbb{P}^1 &\longrightarrow \mathbb{P}^{2n} \\ [s:t] &\longmapsto [c_{sH_1-tH_2}] := [\ker(sH_1 - tH_2)]. \end{aligned}$$

is equal to the standard rational normal curve of degree n up to a linear coordinate change.

We call this rational normal curve the center curve. This center plays a key role in the geometry of $\mathrm{Gr}(2, 2n+1) \cap H_1 \cap H_2$. This is clear by [85, Remark 6.7]. The following proposition is also an example.

Proposition 2.6.11. [85, Proposition 6.8] The automorphism group $\mathrm{Aut}(\mathrm{Gr}(2,5) \cap H_1 \cap H_2)$ acts $\mathrm{Gr}(2,5) \cap H_1 \cap H_2$, which is a subspace of the space projective lines in $\mathbb{P}^4$, with four orbits :

1. Projective tangent lines of the center conics in $\mathbb{P}^4$
2. Projective lines joining two distinct points on the center conics
3. Projective lines passing through the center conics and do not lie on the plane which is spanned by the center conic
4. Projective lines do not intersect with the plane which is spanned by the center conic.

When we consider three general hyperplane sections H_1, H_2, H_3 in the Grassmannian $\mathrm{Gr}(2, 2n+1)$, we can consider a $\mathbb{P}^2$-parameter $[s:t:u] \in \mathbb{P}^2$ and for each $[s:t:u] \in \mathbb{P}^2$, we can assign a center point $c_{[s:t:u]} := v_{[sH_1+tH_2+uH_3]}$. So, we have an assignment from $\mathbb{P}^2$ to the point in $\mathbb{P}^{2n}$. Moreover, for $n = 2$ case, is known that it is a degree 2 embedding with its image isomorphic to the Veronese surface.

Proposition 2.6.12. [85, Proposition 7.2] The map defined above :

$$\begin{aligned} \mathbb{P}^2 &\longrightarrow \mathbb{P}^4 \\ [s:t:u] &\longmapsto [c_{sH_1+tH_2+uH_3}] := [\ker(sH_1+tH_2+uH_3)]. \end{aligned}$$

is a degree 2 embedding, and its image $c(\mathbb{P}^2)$ is a Veronese surface.

We can view $\mathrm{Gr}(2,5)$ as the space of projective lines in the projective space $\mathbb{P}^4$. For a closed subvariety $Z \subset \mathbb{P}^N$, we can define the trisecant variety $\mathrm{Tri}(Z)$ to be the following :

$$\mathrm{Tri}(Z) := \overline{\{\ell \in \mathrm{Gr}(2, N+1) \mid \#(Z \cap \ell) \geq 3\}} \subset \mathrm{Gr}(2, N+1)$$

where $\#(Z \cap \ell)$ is the scheme-theoretic intersection number of Z and the projective line ℓ. Then, we have the following description of the smooth Fano threefold $Y^3 = \mathrm{Gr}(2,5) \cap H_1 \cap H_2 \cap H_3$ due to Castelnuovo.

Proposition 2.6.13. [85, Corollary 7.4],[11] The smooth Fano threefold Y^3 is the trisecant variety $\mathrm{Tri}(c(\mathbb{P}^2))$ of the Veronese surface $c(\mathbb{P}^2) \subset \mathbb{P}^4$.

As before, the image of the center map c, the Veronese surface $c(\mathbb{P}^2)$ plays a key role in the geometry of Y^3 as follows. We introduce the following result on the automorphism group of Y^3.

Proposition 2.6.14. [85, Theorem 7.5] The automorphism group of the smooth Fano threefold Y^3 is isomorphic to the projective linear group $\mathrm{PGL}(2, \mathbb{C})$.

Then we can describe the orbit of this automorphism group action in Y^3 via the geometry of the Veronese surface.

Proposition 2.6.15. [85, Proposition 7.6] The automorphism group $\mathrm{Aut}(Y^3)$, which is isomorphic to $\mathrm{PGL}(2, \mathbb{C})$ acts on Y^3, which is a subspace of the space of projective lines in $\mathbb{P}^4$ with three orbits :

1. Projective lines intersect with the Veronese surface $c(\mathbb{P}^2)$ at 3 distinct points.

2. Projective lines tangent to the Veronese surface $c(\mathbb{P}^2)$ at one point and intersect to it at another point.

3. Projective lines intersect with the Veronese surface $c(\mathbb{P}^2)$ at only one point with multiplicity 3.

Moreover, we have another geometric interpretation about this three $\mathrm{PSL}(2,\mathbb{C})$-orbits.

Proposition 2.6.16. [71, Lemma 1.5], [51, Remark 3.4.6 and p.61], [91, Proposition 2.13] The smooth Fano threefold Y^3 has three $\mathrm{Aut}(Y^3)$-orbits :

1. Degree 6 rational normal sextic curve $C \subset Y^3 \subset \mathbb{P}^6$. This orbit matches to the orbit 3 in Proposition 2.6.15.

2. $\mathcal{S} \setminus C$ where $\mathcal{S}$ is some general quadric surface containing C in the linear system $\mathcal{O}_{\mathbb{P}^9}(2)$ in the Plücker embedding $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$. This orbit matches to the orbit 2 in Proposition 2.6.15.

3. $Y^3 \setminus \mathcal{S}$ is a single orbit, isomorphic to $\mathrm{PSL}(2,\mathbb{C})/S_4$. This orbit matches to the orbit 1 in Proposition 2.6.15.

On the other hand, Cheltsov and Shramov [14] concentrated on the icosahedral group A_5 embedded in the automorphism group $\mathrm{Aut}(Y^3) \cong \mathrm{PSL}(S,\mathbb{C})$. They found the following important result in the viewpoint of the birational geometry.

Proposition 2.6.17. [14, Theorem 1.4.1] The smooth Fano threefold Y^3 is A_5-birationally rigid, and the group of A_5-invariant birational selfmaps on Y^3, $\mathrm{Bir}^{A_5}(Y^3)$ is isomorphic to the group $S_5 \times \mathbb{Z}_2$.

Furthermore, using the A_5-action on Y^3, Cheltsov and Shramov revealed many interesting geometric structures in the smooth Fano threefold Y^3. For example, they classified invariant cubic hypersurfaces and invariant low degree curves under A_5-actions in [14, Chapter 12, Chapter 13]. However, the research in this direction is somewhat distant from the subject of our paper, so we will stop here, despite interest of their study.

Chapter 3

Moduli spaces of smooth rational curves in Fano varieties

The results presented in this chapter are based on the results obtained joint with Kiryong Chung and Jaehyun Hong in [19], the results of Castravet [12, 13] and the results of Kiem [54]. In this chapter, we study moduli space of degree d smooth rational curves $R_d(V)$ where V is a Fano variety $\mathcal{N}$ or Y^m we studied in Subsection 2.6. Here, $R_d(V)$ is defined to be an open subscheme of the degree d map space $\mathrm{Hom}_d(\mathbb{P}^1, V)$, where $\mathrm{Hom}_d(\mathbb{P}^1, V)$ is constructed as an open subscheme in the Hilbert scheme of the graph space in $\mathbb{P} \times V$ [61, Chapter I, Theorem 1.10]. Before we study the moduli space of rational curves in the smooth Fano variety $\mathcal{N}$, we note that by the result on local geometry of Hilbert schemes in Chapter 2, Proposition 2.2.8, all irreducible components of $R_d(\mathcal{N})$ should have the dimension greater or equal than $2d + 3g(X) - 3$.

3.1 Moduli space $\mathsf{R}_\mathsf{d}(\mathcal{N})$ of smooth rational curves in $\mathcal{N}$

Contents in this section based on the results of Castravet [12, 13] and the results of Kiem [54]. In the work [13], Castravet classified all irreducible components of the moduli space $\mathsf{R}_\mathsf{d}(\mathcal{N})$ which parametrizes degree d smooth rational curves $\mathbb{P}^1 \to \mathcal{N}$. Castravet's work based on the classification of the rank 2 vector bundles on the ruled surface by Brosius [9], [10]. But there is a slight difference that Castravet considered a moduli space $\mathcal{N}_1$ of stable rank 2 bundles on a smooth projective curve X with fixed determinant line N bundle of degree 1. But we can easily compensate this difference by taking dual. We first introduce Castravet's result and we can obtain the result in our setting by taking dual.

Before we introduce the result of Castravet, we introduce an important notion called elementary modification.

Definition-Proposition 3.1.1. [54] Consider a rank 2 vector bundle E on the curve X and let fix a point $\mathsf{p} \in \mathsf{X}$. Then we define $\mathsf{E}^{\nu_\mathsf{p}}$ to be the kernel of the surjective map ν_p in the following short exact sequence :

$$0 \longrightarrow \mathsf{E}^{\nu_\mathsf{p}} \longrightarrow \mathsf{E} \xrightarrow{\nu_\mathrm{p}} \mathrm{k}_\mathrm{p} \longrightarrow 0. \tag{3.1}$$

Then we can easily see that $\mathsf{E}^{\nu_\mathsf{p}}$ is again a vector bundle such that $\det(\mathsf{E}^{\nu_\mathsf{p}}) = \det(\mathsf{E})(-\mathsf{p})$. We call $\mathsf{E}^{\nu_\mathsf{p}}$ an *elementary modification* of the vector bundle E at the point p.

We have $\mathrm{Hom}(\mathsf{E}, \mathrm{k}_\mathrm{p}) = \mathrm{Hom}(\mathsf{E}|_\mathrm{p}, \mathrm{k}_\mathrm{p}) = \mathrm{k}^2$ and $\mathrm{Ker}(\nu_\mathrm{p}) = \mathrm{Ker}(\lambda \cdot \nu_\mathrm{p})$ for all $\lambda \in \mathrm{k}^*$. So we may choose ν_p in the equivalence class under k^*, $[\nu_\mathrm{p}] \in (\mathrm{k}^2 \setminus \{0\})/\mathrm{k}^* \cong \mathbb{P}^1$. If we choose E to be a rank 2 stable bundle with determinant $\mathcal{O}(\mathsf{p} - \mathsf{x})$, then we can observe that there is an induced map $\mathbb{P}^1 \to \mathcal{N}$. It is well known that this is a degree two smooth conic by Narasiman-Ramanan [77] this is called Hecke curve.

In the work of Kiem [54], he introduced a generalized notion of the Hecke curve, the generalized Hecke curve. We explain it in a slightly different way. First, consider a degree D effective divisor on X and consider a short exact sequence :

$$0 \longrightarrow E^{\nu_D} \longrightarrow E \xrightarrow{\nu_D} \mathcal{O}_D \longrightarrow 0.$$

Then we can also show that E^{ν_D} is a vector bundle with determinant $\det(E^{\nu_D}) = \det(E)(-D)$. We call E^{ν_D} an elementary transform of the vector bundle E at the divisor D.

Then, consider a projectivized extension group $\mathbb{P}\mathrm{Ext}^1(F, \mathcal{O}_D)$, where F is a rank 2 stable vector bundle such that its determinant line bundle is $\mathcal{O}(-D-x)$. An elements of this space is an equivalence class of the following extension sequence :

$$0 \to \mathcal{O}_D \to F' \to F \to 0.$$

We can check that F' becomes a rank 2 vector bundle with determinant $\mathcal{O}(-x)$. In fact, we can observe that $(F')^\vee$ can be obtained by an elementary transform of $F^\vee$ at the exceptional divisor D.

Since vector bundles F' appeared in the extension space can be unstable, we consider stable locus $\mathbb{P}(\mathrm{Ext}^1(F, \mathcal{O}_D))^s$. Next, consider the middle term of the universal extension sequence [47, Example 2.1.12] on the projectivized extension group $P := \mathbb{P}(\mathrm{Ext}^1(F, \mathcal{O}_D))$

$$0 \to p_1^*\mathcal{O}_D \to \mathcal{F} \to p_1^*F \otimes p_2^*\mathcal{O}_P(1) \to 0$$

on $X \times P$ where p_1, p_2 are projections to X and P, is a rank 2 vector bundle on $X \times P$. Then, its restriction $\mathcal{F}|_{X \times P^s}$ induces a morphism

$$\mathbb{P}(\mathrm{Ext}^1(F, \mathcal{O}_D))^s \xrightarrow{H_{D,F}} \mathcal{N}.$$

since $\mathcal{N}$ is a fine moduli space. The degree of this curve will be turned out to be 2deg$D$ since the $H_{E,D}$ has degree 2deg$D$.

Now we introduce the following result of Castravet, which classifies irreducible components of the moduli space of degree d smooth rational curves $R_d(\mathcal{N}_1)$ in the moduli space of rank 2 vector bundles $\mathcal{N}_1$.

Proposition 3.1.2 (Castravet)**.** [13, Theorem 1.5, Lemma 4.10, Lemma 4.12] For all pair of integers (a,b) in the following :

$$\left\{(a,b) \mid d \geq a > d/2,\ \frac{d-a}{2a-d} \geq b > 0\right\} \cup \{(d,0)\}$$

There exist irreducible subvarieties $R(a,b)$ of $R_d(\mathcal{N}_1)$ whose elements are rational curves $f : \mathbb{P}^1 \to \mathcal{N}_1$ obtained by the completion of the following rational map $f' : \mathbb{P}^1 \dashrightarrow \mathcal{N}_1$ obtained by the composition :

$$f' : \mathbb{P}^1 \dashrightarrow \mathbb{P}(\mathrm{Ext}^1_{\mathcal{O}_X}(L^{-1} \otimes N, L))^s \xrightarrow{\Psi_L} \mathcal{N}_1$$

where $L \in \mathrm{Pic}^{-b}(X)$ is a degree $-b$ line bundle, Ψ_L is a morphism induced by extension which has degree $2b+1$, taking a class of rank 2 vector bundle in the middle term of an extension as its value and $(\mathrm{Ext}^1_{\mathcal{O}_X}(L^{-1} \otimes N, L))^s$ means a stable part in the extension group. Here, if $r = (d-a) - b(2a-d) = 0$, then f' is a regular map with degree $2a - d = \frac{d}{2b+1}$.

If $d = 2k$ is even, there is an irreducible subvariety $R_E \subset R_d(\mathbb{P}^1, \mathcal{N}_1)$ whose element is a rational curve obtained by the following composition :

$$f : \mathbb{P}^1 \xrightarrow{\deg 1} \mathbb{P}\mathrm{Ext}^1((E, \mathcal{O}_D))^s \xrightarrow{H'_{D,E}} \mathcal{N}_1$$

where the first arrow is a degree 1 regular map, D is a degree k divisor on the curve X, E is a rank 2 stable vector bundle with the determinant line bundle $L(-D)$ and $H'_{D,E}$ is a morphism induced by the extension, which exactly coincide with the dual notion of the elementary modification.

Furthermore, we have the followings :

1. For odd d, closures $R(a, b) \subset R_d(\mathcal{N}_1)$ such that $\dim R(a, b) \geq 2d+3g-3$ are all irreducible components of $R_d(\mathcal{N}_1)$.

2. For even d, $R(a, b)$ satisfies the same dimension condition and R_E are all irreducible components of $R_d(\mathcal{N}_1)$.

We note that since the map $H'_{E,D} : \mathrm{Ext}^1((E, \mathcal{O}_D))^s \to \mathcal{N}_1$ has degree $2k = 2\deg D$, we conclude that the map $H_{E,D}$ we defined at the beginning has also degree $2\deg D$ as we announced since it is the dual notion of $H'_{E,D}$.

As we previously announced, by taking dual, we obtain the result in our setting, the information on irreducible components of $R_d(\mathcal{N})$. On the other hand, for $d \leq 4$ case, there is an independent result of Kiem [54, Proposition 3.6, Proposition 3.9] on the classification of the smooth rational curves $\mathbb{P}^1 \to \mathcal{N}$. We summarize it to the following.

Proposition 3.1.3. [54, Proposition 3.6, Proposition 3.9], [13, Theorem 1.5, Lemma 4.10, Lemma 4.12] We denote $\mathcal{N}$ the moduli space of rank 2 stable vector bundles on the curve X whose determinant are fixed line bundle $\mathcal{O}(-x)$. For $d \leq 4$, we have the following results on $R_d(\mathcal{N})$

1. For $d = 1$, $R_1(\mathcal{N})$ is irreducible, which parametrizes degree 1 rational curves obtained from the following composition :

$$f : \mathbb{P}^1 \xrightarrow{\deg 1} \mathbb{P}\mathrm{Ext}^1(L, L^{-1}(-x)) \xrightarrow{\Psi_L} \mathcal{N}$$

   where $L \in \mathrm{Pic}^0(X)$ is a degree 0 line bundle.

2. For $d = 2$, $R_2(\mathcal{N})$ has two irreducible component $R_2(0)$ and $R_{2,E}$. Here, $R_2(0)$ parametrizes degree 2 rational curves obtained from the following composition :

$$f : \mathbb{P}^1 \xrightarrow{\deg 2} \mathbb{P}\mathrm{Ext}^1(L, L^{-1}(-x)) \xrightarrow{\Psi_L} \mathcal{N}$$

where $L \in \mathrm{Pic}^0(X)$ is a degree 0 line bundle, and $R_{2,E}$ parametrizes degree 2 rational curves which is a form of Hecke curves

3. For $d = 3$, $R_3(\mathcal{N})$ has two irreducible component $R_3(0)$ and $R_3(1)$. Here, $R_3(0)$ parametrizes degree 3 rational curves obtained from the following composition :

$$f : \mathbb{P}^1 \xrightarrow{\deg 3} \mathbb{P}\mathrm{Ext}^1(L, L^{-1}(-x)) \xrightarrow{\Psi_L} \mathcal{N}$$

where $L \in \mathrm{Pic}^0(X)$ is a degree 0 line bundle, and $R_3(1)$ parametrizes degree 3 rational curves obtained from the following composition :

$$f : \mathbb{P}^1 \xrightarrow{\deg 1} \mathbb{P}(\mathrm{Ext}^1(L, L^{-1}(-x)))^s \xrightarrow{\Psi_L} \mathcal{N}$$

where $L \in \mathrm{Pic}^1(X)$ is a degree 1 L line bundle.

4. For $d = 4$, $R_4(\mathcal{N})$ has two irreducible component $R_4(0)$ and $R_{4,E}$. Here, $R_4(0)$ parametrizes degree 4 rational curves obtained from the following composition :

$$f : \mathbb{P}^1 \xrightarrow{\deg 4} \mathbb{P}\mathrm{Ext}^1(L, L^{-1}(-x)) \xrightarrow{\Psi_L} \mathcal{N}$$

where $L \in \mathrm{Pic}^0(X)$ is a degree 0 line bundle, and $R_{4,E}$ parametrizes rational curves which is a form of generalized Hecke curves of degree 4.

Here, Ψ_L are morphisms induced from the middle terms of the universal extension sequence [47, Exmaple 2.1.12] of the projectivized extension groups $\mathbb{P}\mathrm{Ext}^1(L, L^{-1}(-1))$, which takes isomorphism class of a rank 2 vector bundle in the middle term of the extension as a value. It has degree $\deg\Psi_L = 2\deg L + 1$. Moreover, when $L \in \mathrm{Pic}^0(X)$, Ψ_L is a closed embedding.

3.2 Rational curves in $\mathrm{Gr}(2, n)$

Contents in this section based on the results obtained joint with Chung and Hong [19]. In this section, we describe all degree ≤ 3 rational curves in the Grassmannian $\mathrm{Gr}(2, n)$ explicitly. We can consider the space $\mathrm{Gr}(2, n)(n \geq 4)$ as the moduli space of lines in $\mathbb{P}^{n-1}$ and keep in mind the following Plücker embedding :

$$G := \mathrm{Gr}(2, n) \hookrightarrow \mathbb{P}(\wedge^2 \mathbb{C}^n) = \mathbb{P}^{\binom{n}{2}-1}.$$

For fixed vector subspaces $V_1 \subset V_2 \subset \mathbb{C}^n$, we define $\sigma_{c_1,c_2} = \{[L] \in G | \dim(L \cap V_i) \geq i\}$ by the Schubert variety where $c_i := n + i - \dim(V_i) - 2$, $i = 1, 2$.

To describe rational curves in the Grassmannian $\mathrm{Gr}(2, n)$, we need to consider the Schubert varieties in $\mathrm{Gr}(2, n)$ in the following.

Definition 3.2.1. Consider a point $\ell \in \mathrm{Gr}(2, n)$, which correspond to a projective line in $\mathbb{P}^{n-1}$, and choose a flag $p \in \mathbb{P}^1 \subset \mathbb{P}^2 \subset \mathbb{P}^3 \subset \mathbb{P}^{n-1}$. Then we define the following Schubert varieties :

- $\sigma_{n-4,0} = \{\ell \,|\, \ell \cap \mathbb{P}^2 \neq \emptyset\}$ $\quad(\dim n, \deg n(n-3)/2)$
- $\sigma_{n-3,0} = \{\ell \,|\, \ell \cap \mathbb{P}^1 \neq \emptyset\}$ $\quad(\dim n-1, \deg n-2)$
- $\sigma_{n-4,n-4} = \{\ell \,|\, \ell \subset \mathbb{P}^3\}$ $\quad(\dim 4, \deg 2)$
- $\sigma_{n-3,n-4} = \{\ell \,|\, \ell \cap \mathbb{P}^1 \neq \emptyset, \ell \subset \mathbb{P}^3\}$ $\quad(\dim 3, \deg 2)$
- $\sigma_{n-2,0} = \{\ell \,|\, p \in \ell\}$ $\quad(\dim n-2, \deg 1)$
- $\sigma_{n-3,n-3} = \{\ell \,|\, \ell \subset \mathbb{P}^2\}$ $\quad(\dim 2, \deg 1)$
- $\sigma_{n-2,n-4} = \{\ell \,|\, p \in \ell \subset \mathbb{P}^3\}$ $\quad(\dim 2, \deg 1)$
- $\sigma_{n-2,n-3} = \{\ell \,|\, p \in \ell \subset \mathbb{P}^2\}$ $\quad(\dim 1, \deg 1)$.

Remark 3.2.1. We note that family of Schubert varieties $\sigma_{n-2,n-4}$ and $\sigma_{n-3,n-3}$ are all planes in $\mathrm{Gr}(2,n)$.

Remark 3.2.2. For a point p, a line ℓ, a plane P, a 3-dimensional linear space $\mathbb{P}^3 \subset \mathbb{P}^{n-1}$, we sometimes use a notation $\sigma_{i,j}(p), \sigma_{i,j}(\ell), \sigma_{i,j}(P), \sigma_{i,j}(\mathbb{P}^3)$, $\sigma_{i,j}(p, \mathbb{P}^3)$ which denotes a Schubert variety correspond to a flag containing p, ℓ, P, $\mathbb{P}^3, \{p\} \subset \mathbb{P}^3$ at each cases.

We can find the degrees and dimensions of the Schubert varieties from [38, Page 196] and [34, Example 14.7.11]. When $n = 5$ case, these varieties are free generators of the homology group $H_*(\mathrm{Gr}(2,5), \mathbb{Z})$.

Next, we write $S(C_0, C_1)$ to denote the *rational normal scroll* induced from two smooth rational curves C_0 and C_1 (The curve C_0 can be a point).

Proposition 3.2.3. Consider a degree d smooth rational curve $C : \mathbb{P}^1 \to \mathrm{Gr}(2,n)$ in the Grassmannian $\mathrm{Gr}(2,n)$ where the degree is defined via the Plücker embedding.

1. If $d = 1$, then the image of C is equal to the the Schubert variety $\sigma_{n-2,n-3}(p_0, P)$, which is a family of projective lines contained in a fixed plane $P \subset \mathbb{P}^{n-1}$ containing the fixed point $p_0 \in P$.

2. If $d = 2$, then the image of C is either the family of projective lines in the ruling of the rational normal scroll $S(\ell_0, \ell_1)$ of two projective lines ℓ_0 and ℓ_1, or the family of projective lines in the ruling of the rational normal scroll $S(p_0, C_0)$ for a fixed point p_0 and a smooth conic C_0 in $\mathbb{P}^{n-1}$.

3. If $d = 3$, then the image of C is either the family of projective lines in the ruling of $S(\ell, C_0)$ for a projective line ℓ and a smooth conic C_0, or the family of projective lines in the ruling of the rational normal scroll $S(p_0, C_1)$ for a fixed point p_0 and a twisted cubic C_1 in $\mathbb{P}^{n-1}$.

Proof. Let ℓ be a projective line in $\mathbb{P}^{n-1}$. Then hyperplanes in $\mathbb{P}^{n-1}$ which contains the line ℓ forms a sublocus in the dual projective space $(\mathbb{P}^{n-1})^*$ which is isomorphic to $\mathbb{P}^{n-3}$. Thus, the sublocus of hyperplanes in $\mathbb{P}^{n-1}$ which contains one of a projective line in the family correspond to the curve C with dimension $\leq n-2$. Therefore we can choose a point $[\Lambda] \in (\mathbb{P}^{n-1})^*$ of the complement of this sublocus. Thus $\Lambda \subset \mathbb{P}^{n-1}$ intersects each projective line in the family correspond to C transversely by construction ([43, Chapter I, Theorem 7.1]). We denote $C \leftarrow F \xrightarrow{\phi} \mathbb{P}^{n-1}$ the family of projective lines correspond to C, where $\pi : F \to C$ is a $\mathbb{P}^1$-bundle and ϕ is the morphism such that it is the natural embedding when restricted on each fiber of π. As a result, we obtain a following fiber diagram :

$$\begin{tikzcd}
f^{-1}(\Lambda) \arrow[r] \arrow[d] & \Lambda \arrow[d, hook] \\
F \arrow[r, "\phi"] \arrow[d, "\pi"] & \mathbb{P}^{n-1} \\
C &
\end{tikzcd}$$

Since each fibers $\pi^{-1}(x)$ intersect with the hyperplane Λ transversely, locally we can describe the bijection $\phi^{-1}(\Lambda) \to F \to C$ by the following :

$$\{(z_1, z_2) \,|\, z_2 = g(z_1)\} \subset \mathbb{C}^2 \to \mathbb{C}, \quad (z_1, z_2) \mapsto z_1.$$

When we let $C_0 := \phi^{-1}(\Lambda)$, then it is the image of a section $s_0 : C \to F$. Consider a normal bundle $N_{C_0/F}$ of C_0 in F. Then we can observe that F has the projective bundle structure $F = \mathbb{P}(\mathcal{O}_C \oplus N)$ where $N = s_0^* N_{C_0/F}$. Let $s_1 : C \cong \mathbb{P}N \hookrightarrow \mathbb{P}(\mathcal{O}_C \oplus N) = F$ be the canonical section. Let $L_0 = (\phi \circ s_0)^* \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ and $L_1 = (\phi \circ s_1)^* \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ so the induced morphism $\phi \circ s_0 : C \to \mathbb{P}^{n-1}$ becomes $(a_0 : a_1 : \cdots : a_{n-1})$ where $a_i \in H^0(C, L_0)$ and $\phi \circ s_1 : C \to \mathbb{P}^{n-1}$ becomes $(b_0 : b_1 : \cdots : b_{n-1})$ where $b_i \in H^0(C, L_1)$. In summary, projective lines in the family F can be described by two-dimensional vector subspaces of $\mathbb{C}^n$ the row space of the following matrix :

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots & a_{n-1} \\ b_0 & b_1 & b_2 & b_3 & \cdots & b_{n-1} \end{pmatrix}$$

where $a_i \in H^0(C, L_0)$ and $b_i \in H^0(C, L_1)$ are sections of the line bundles L_0 and L_1. Thus we can write Plücker coordinates of $C \subset \mathrm{Gr}(2, n) \subset \mathbb{P}^{\binom{n}{2}-1}$ as $a_i b_j - a_j b_i \in H^0(C, L_0 \otimes L_1)$. Hence, we observe that the degree of the curve C is equal to :

$$\deg C = d_0 + d_1, \quad \text{where } d_0 = \deg L_0 \text{ and } d_1 = \deg L_1.$$

We recall the fact that $C \cong \mathbb{P}^1$ since C is a smooth rational curve. If $d = 1$, without loss of generality, it should be $L_0 = \mathcal{O}_{\mathbb{P}^1}$ and $L_1 = \mathcal{O}_{\mathbb{P}^1}(1)$. Therefore, If we let $p_0 = (a_0 : a_1 : \cdots : a_{n-1})$ and P be the projective plane spanned by three vectors p_0, $(b_0(0) : \cdots : b_{n-1}(0))$ and $(b_0(1) : \cdots : b_{n-1}(1))$, we prove the case (1)

Next, consider the case $d = 2$. If $d_0 = 0$ and $d_1 = 2$, we may write $p_0 = (a_0 : a_1 : \cdots : a_{n-1})$ and $C_0 = \{(b_0(t) : \cdots : b_{n-1}(t)) \,|\, t \in \mathbb{P}^1\}$ and C is the family of projective lines in the ruling of the rational normal scroll $S(p_0, C_0)$. If $d_0 = 1$ and $d_1 = 1$, the images $\ell_0 := f \circ s_0(C)$ and $\ell_1 := f \circ s_1(C)$ are both projective lines and therefore C parametrizes lines passing through a pair of points, one of them moving in the line ℓ_0 and the other one moving in the line ℓ_1.

We can also show the $d = 3$ case in a similar manner so we omit here. $\square$

Remark 3.2.4. We can also prove Proposition 3.2.3 through Grothendieck's theorem that every vector bundle over the projective line $\mathbb{P}^1$ can be decompose to a direct sum of line bundles.

For a degree d curve $C : \mathbb{P}^1 \xrightarrow{f} \mathrm{Gr}(2, n)$, consider the pull-back of the rank 2 tautological bundle $\mathcal{U} \overset{\varphi}{\hookrightarrow} \mathcal{O}_G^{\oplus n}$. Then we have a splitting of a vector bundle :

$$f^*\mathcal{U} = \mathcal{O}_{\mathbb{P}^1}(-d_1) \oplus \mathcal{O}_{\mathbb{P}^1}(-d_2)$$

where $d_1 + d_2 = 2d$.

Moreover, let us write the induced morphism $\phi^*\mathcal{U} = \mathcal{O}_{\mathbb{P}^1}(-d_1) \oplus \mathcal{O}_{\mathbb{P}^1}(-d_2) \overset{f^*\varphi}{\hookrightarrow} \mathcal{O}_{\mathbb{P}^1}^{\oplus n}$ as the following $n \times 2$ matrix :

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ b_0 & b_1 & b_2 & \cdots & b_{n-1} \end{pmatrix}$$

where $a_i \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d_1))$ and $b_i \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d_2))$.

Then the sections $a_0, a_1, \ldots, a_{n-1}$ defines a degree d_1 rational curve C_1 and the sections $b_0, b_1, \ldots, b_{n-1}$ defines a degree d_2 curve C_2.

For a point $x \in \mathbb{P}^1$, the image $f(x) \in \mathrm{Gr}(2, n)$ is the row space of the matrix :

$$\begin{pmatrix} a_0(x) & a_1(x) & a_2(x) & \cdots & a_{n-1}(x) \\ b_0(x) & b_1(x) & b_2(x) & \cdots & b_{n-1}(x) \end{pmatrix},$$

which correspond to the projective line in $\mathbb{P}^{n-1}$ joining two points $[a_0(x) : a_1(x) : \cdots : a_{n-1}(x)]$ and $[b_0(x) : b_1(x) : \cdots : b_{n-1}(x)]$. Therefore the curve the curve C is a family of lines in the ruling of the rational normal scroll $S(C_1, C_2)$. Especially, for $d = 1$ case, family of lines in the ruling $S(p_0, \ell)$ of the point p_0 and a line ℓ, is equivalent to the family of lines in P which pass through the point $p_0 \in P$ where P is the plane spanned by p_0 and the line ℓ.

We note that even distribution types are general types among the splitting types. This says which types are general types in $d = 2$ and $d = 3$ case. In $d = 2$ case, the curve C which is a family of lines in the ruling of rational normal scroll $S(\ell_0, \ell_1)$ for two lines $\ell_0, \ell_1 \subset \mathbb{P}^{n-1}$, is the general type. In $d = 3$ case, the curve C which is a family of lines in the ruling of rational normal scroll $S(\ell, C_0)$ for a line ℓ and a smooth conic C_0 in $\mathbb{P}^{n-1}$ is the general type.

By the Part 1 of Proposition 3.2.3, we can prove the following result on the planes in the Grassmannian $\mathrm{Gr}(2,n)$, which we already mentioned in Remark 3.2.1. When $n=5$ case, the result in the following corollary already appeared in [26], and we think that this result may be classical since it is very simple. But we provide the proof for readers convenience.

Corollary 3.2.5. [26, Section 3.1] Every plane in the Grassmannian $\mathrm{Gr}(2,n)$ arises in one of the following forms :

1. A family of projective lines in a fixed plane $P \in \mathbb{P}^{n-1}$. This family of lines is equal to the Schubert variety $\sigma_{n-3,n-3}(P) \subset \mathrm{Gr}(2,n)$.
2. A family of lines in a fixed three-dimensional space $\mathbb{P}^3 \in \mathbb{P}^{n-1}$, passing through a fixed point $p \in \mathbb{P}^3$. This family of lines is equal to the Schubert variety $\sigma_{n-2,n-4}(p,\mathbb{P}^3) \subset \mathrm{Gr}(2,n)$(See Remark 3.2.2 about the definition of this Schubert variety).

Proof. Let Λ be a plane in $\mathrm{Gr}(2,n)$. Consider two different lines $\ell_0,\ell_1 \subset \Lambda$. Then by Proposition 3.2.3, the line ℓ_0 is a set of lines contained in a plane $P_0 \subset \mathbb{P}^{n-1}$ which pass through a fixed point $p_0 \in P_0$ and the line ℓ_1 is a set of lines in $\mathbb{P}^{n-1}$ in a projective plane $P_1 \subset \mathbb{P}^{n-1}$ which pass through a fixed point $p_1 \in P_1$. Let $x := \ell_0 \cap \ell_1$ be the intersection point of two lines.

If $p_0 = p_1 = p$, then we can observe that the planes P_0 and P_1 intersects along the line which corresponds to the point $x \in \mathrm{Gr}(2,n)$. Therefore, P_0 and P_1 spans the three-dimensional space $\mathbb{P}^3 \subset \mathbb{P}^{n-1}$. Therefore, we can observe that the lines ℓ_0 and ℓ_1 contained in the Schubert variety $\sigma_{n-2,n-4}(p,\mathbb{P}^3)$, which is a plane in $\mathrm{Gr}(2,n)$. Therefore we have $\Lambda = \sigma_{n-2,n-4}(p,\mathbb{P}^3)$.

If $p_0 \neq p_1$, then we can observe that x is a line joining two points p_0 and p_1, so we can write $x = \overline{p_0p_1}$. Therefore, planes P_0 and P_1 intersects along the line $\overline{p_0p_1}$. If $P_0 \neq P_1$, then we can choose two lines $\overline{p_0a} \in \ell_0$ for a point $a \in P_0$ and $\overline{p_1b} \in \ell_1$ for a point $b \in P_1$, such that $\overline{p_0a} \cap \overline{p_1b} = \varnothing$. Then by Proposition 3.2.3, $\overline{p_0a}, \overline{p_1b} \in \mathrm{Gr}(2,n)$ cannot lie on a line in $\mathrm{Gr}(2,n)$.

Therefore we have $P_0 = P_1 = P$. Then the lines ℓ_0 and ℓ_1 contained in the Schubert variety $\sigma_{n-3,n-3}(P)$, which is a plane in $\mathrm{Gr}(2, n)$. Therefore we have $\Lambda = \sigma_{n-3,n-3}(P)$. □

Also, as a result of Proposition 3.2.3, we have the following geometric descriptions for smooth rational curves in the Grassmannian $\mathrm{Gr}(2, n)$.

Proposition 3.2.6. (1) ([40, Exercise 6.9]) The variety $R_1(\mathrm{Gr}(2, n))$ of projective lines in $G = \mathrm{Gr}(2, n)$ is isomorphic to the flag variety $\mathrm{Gr}(1, 3, n)$, which parametrizes flags $V_1 \subset V_3 \subset \mathbb{C}^n$ of $\mathbb{C}^n$ where $\dim V_i = i$.

(2) For a smooth conic curve $C \subset \mathrm{Gr}(2, n) \subset \mathbb{P}^{\binom{n}{2}-1}$, there exists a three dimensional sub-linear space $\mathbb{P}^3 \subset \mathbb{P}^{\binom{n}{2}-1}$ which contains every projective lines in $\mathbb{P}^{n-1}$ parametrized by the curve C.

(3) For a twisted cubic curve $C \subset \mathrm{Gr}(2, n) \subset \mathbb{P}^{\binom{n}{2}-1}$, there exist a projective line $\ell \subset \mathbb{P}^{n-1}$ which intersects all projective lines parametrized by C transversally in $\mathbb{P}^{n-1}$.

Proof. By Proposition 3.2.3 (1), each projective line in $\mathrm{Gr}(2, n)$ corresponds to the family of projective lines in a plane $\mathbb{P}^2 \subset \mathbb{P}^{n-1}$ which contains a fixed point $p \in \mathbb{P}^2$. On the contrary, such family of projective lines in $\mathbb{P}^{n-1}$ determines a projective line in $\mathrm{Gr}(2, n)$.

By Proposition 3.2.3 (2), a conic C in the Grassmannian $\mathrm{Gr}(2, n)$ is the family of lines in the ruling of a rational normal scroll $S(\ell_0, \ell_1)$ for projective lines ℓ_0, ℓ_1 in $\mathbb{P}^{n-1}$ or a rational normal scroll $S(p_0, C_0)$ for a point p_0 and a smooth conic C_0 in $\mathbb{P}^{n-1}$. Hence, if we choose a $\mathbb{P}^3$(may not unique) containing p_0 and C_0 in the former case or $\mathbb{P}^3$ containing ℓ_0 and ℓ_1 in the latter case, then all lines of the family parametrized by the curve C should be contained in the linear space $\mathbb{P}^3$.

By Proposition 3.2.3 (3), a twisted cubic C in the Grassmannian $\mathrm{Gr}(2, n)$ is the family of lines in the ruling of a rational normal scroll $S(\ell, C_0)$ for a projective lines ℓ and a smooth conic C_0 in $\mathbb{P}^{n-1}$ or a rational normal scroll $S(p_0, C_1)$ for a single point p_0 and a twisted cubic C_1 in $\mathbb{P}^{n-1}$. If we choose the projective line ℓ in the former case or any projective line ℓ passing through the point p_0 in the latter case, it is clear that every line of the family parametrized by C should intersect with the line ℓ. □

The above Proposition leads to the following definition.

Definition 3.2.2. (1) We call the point p in Proposition 3.2.3 (1), the *vertex* of the projective line in $\mathrm{Gr}(2, n)$.

(2) We call the three-dimensional linear subspace $\mathbb{P}^3$ in Proposition 3.2.6 (2), an *envelope* of the conic $C \subset \mathrm{Gr}(2, n)$.

(3) We call the line ℓ in Proposition 3.2.6 (3), an *axis* of the twisted cubic $C \subset \mathrm{Gr}(2, n)$.

Corollary 3.2.7. (cf. [40, Exercise 6.9] and [16]) We denote $R_d(\mathrm{Gr}(2, n))$ the moduli of degree d smooth rational curves in $\mathrm{Gr}(2, n)$ where $d \leq 3$, $n \geq 4$. Then we have the followings :

1. We have a regular map $\zeta_1 : R_1(\mathrm{Gr}(2, n)) \to \mathbb{P}^{n-1} = \mathrm{Gr}(1, n)$ that sends each projective lines in G to its vertex. Then, each fiber of ζ_1 over $V_1 \in \mathrm{Gr}(1, n)$ is isomorphic to $\mathrm{Gr}(2, \mathbb{C}^n/V_1)$.

2. We have a rational map $\zeta_2 : R_2(\mathrm{Gr}(2, n)) \dashrightarrow \mathrm{Gr}(4, n)$ that sends each smooth conic in $\mathrm{Gr}(2, n)$ to its envelopes. A fiber of the rational map ζ_2 over a point $V_4 \in \mathrm{Gr}(4, n)$ is isomorphic to the moduli space $R_2(\mathrm{Gr}(2, V_4))$ of smooth conics in the Grassmannian $\mathrm{Gr}(2, V_4) \cong \mathrm{Gr}(2, 4)$.

3. We have a rational map $\zeta_3 : R_3(\mathrm{Gr}(2, n)) \dashrightarrow \mathrm{Gr}(2, n)$ that sends each twisted cubic in G to its axis. A fiber of the map ζ_3 over a point $\ell \in \mathrm{Gr}(2, n)$ is the moduli space $R_3(\sigma_{n-3,0}(\ell))$ of twisted cubic curves in the Schubert variety $\sigma_{n-3,0}(\ell)$ in Remark 3.2.2.

Proof. (1) By Proposition 3.2.6 (1), the map ζ_1 is, in fact, the forgetful map $R_1(\mathrm{Gr}(2, n)) = \mathrm{Gr}(1, 3, n) \to \mathrm{Gr}(1, n)$ which is given by $(V_1, V_3) \to V_1$. The choice of the vector space V_3 which contains V_1 is clearly parametrized by $\mathrm{Gr}(2, \mathbb{C}^n/V_1)$.

(2) General smooth conic C is a family of lines of the ruling of a rational normal scroll $S(\ell_0, \ell_1)$ of two lines ℓ_0, ℓ_1 in the projective space $\mathbb{P}^{n-1}$ by Proposition 3.2.3. Then since C is general, lines ℓ_0 and, ℓ_1 span a three-dimensional linear subspace $\mathbb{P}^3 = \mathbb{P}(V_4) \subset \mathbb{P}^{n-1}$. Thus, the smooth conic C should be contained in the Grassmannian $\mathrm{Gr}(2, V_4)$, which is a space of lines in $\mathbb{P}_4$, a fiber of the map ζ_2 over $V_4 \in \mathrm{Gr}(4, n)$ is isomorphic to $R_2(\mathrm{Gr}(2, V_4))$.

(3) General twisted cubic curve C is determined by (ℓ, C_1) in the notation of the part (3) of the Proposition 3.2.3. The locus of projective lines intersecting ℓ is the Schubert variety $\sigma_{n-3,0}(\ell)$. Thus the curve C should be contained in the Schubert variety $\sigma_{n-3,0}(\ell)$ and therefore a fiber of the map ζ_3 over $\ell \in \mathrm{Gr}(2, n)$ is isomorphic to $R_3(\sigma_{n-3,0}(\ell))$. □

3.3 Moduli space $R_d(Y^m)$ of smooth rational curves in Y^m

Every scheme in this section is defined over $\mathbb{C}$ and the Grassmannian $\mathrm{Gr}(\ell, n)$ means the moduli space of ℓ-dimensional subspaces of the vector space $\mathbb{C}^n$.

We write $\{e_0, e_1, \cdots, e_{n-1}\}$ as the standard basis of the n-dimensional vector space $\mathbb{C}^n$ unless we mention it otherwise. We denote $p_{i_1 i_2 \cdots i_\ell}$ the projective coordinates of the Plücker embedding $\mathrm{Gr}(\ell, n) \hookrightarrow \mathbb{P}(\wedge^\ell \mathbb{C}^n)$, which is called Plücker coordinates.

Before we start to study the birational models of moduli space of rational curves in linear sections of Grassmannians, we need to clarify their birational types. In this section, we prove rationality results of the moduli spaces in the following.

From now on, we adopt the following notations. We let $G := \mathrm{Gr}(2, 5)$

and Y^m be the intersection of the Grassmannian $\mathrm{Gr}(2,5) \subset \mathbb{P}^{\binom{5}{2}-1} = \mathbb{P}^5$ with $6-m$ general hyperplanes in $\mathbb{P}^5$. For example, $Y^6 = \mathrm{Gr}(2,5) = G$, $Y^5 = \mathrm{Gr}(2,5) \cap H$ and $Y^4 = \mathrm{Gr}(2,5 \cap H_1 \cap H_2)$, where H, H_1, H_2 are general hyperplanes in $\mathbb{P}^5$. Then Y^m are smooth Fano varieties. We first introduce the main result of this Chapter.

Theorem 3.3.1. The moduli space $R_d(Y^m)$ of degree d smooth rational curves in Y^m are all rational varieties for $1 \le d \le 3$ and $2 \le m \le 6$.

We note that if $m = 0$, then Y^0 is a five point set since the degree of G in $\mathbb{P}^9$ is 5. If $m = 1$, then Y^1 is a degree 5 smooth elliptic curve so that there exist no rational curve in Y^1.

Lemma 3.3.2.

1. The moduli of lines $R_1(Y^2)$ is a variety of 10 disjoint reduced points.
2. The moduli of conics $R_2(Y^2)$ is the disjoint union of five $\mathbb{P}^1 - \{0,1,\infty\}$.
3. The moduli of cubics $R_3(Y^2)$ is isomorphic to the disjoint union of four $\mathbb{P}^2 - \mathbb{P}^1$ and $\mathbb{P}^2 - \{4$ projective lines$\}$.
4. There does not exist any planes in Y^2.

Proof. We can observe that Y^2 is a degree 5 del Pezzo surface and therefore it is isomorphic to the blow-up of $\mathbb{P}^2$ at 4 general points. Hence, it is obvious that there is no plain contained in Y^2.

By adjunction, we can observe that a projective line in Y^2 is equivalent to a rational curve in Y^2 which has self-intersection number -1. Since Y^2 is isomorphic to the blow-up of $\mathbb{P}^2$ at 4 general points, there are 4 exceptional curves and strict transformations of 6 projective lines in $\mathbb{P}^2$ joining 2 out of the 4 blow-up points. Thus, there are exactly 10 projective lines in the del Pezzo surface Y^2.

Again by adjunction, we can observe that a smooth conic in Y^2 is equivalent to a rational curve which has self-intersection number 0. They are strict transformations of projective lines passing through one of the 4 blow-up points or conics which pass through every blow-up point, minus projective lines passing through 2 out of the 4 blow-up points and the three singular conics passing through the 4 points.

Again by adjunction, we can observe that a twisted cubic curve in Y^2 is equivalent to a rational curve which has self-intersection number 1. They are strict transformations of projective lines in $\mathbb{P}^2$ which does not pass through any of the blow-up center points, or conics which pass through 3 out of the 4 blow-up points. The first family parametrized by $\mathbb{P}^2$ minus four projective lines and the second family is parametrized by disjoint union of four $\mathbb{P}^2 \setminus \mathbb{P}^1$. $\square$

Proposition 3.3.3. ([30, 36, 48, 91]) For $d = 1, 2, 3$, the Hilbert schemes $H_d(Y^3)$ with Hilbert polynomials $dt + 1$ in the Fano variety Y^3 are equal to the following :

$$H_1(Y^3) \cong \mathbb{P}^2, \quad H_2(Y^3) \cong \mathbb{P}^4, \quad \text{and} \quad H_3(Y^3) \cong \mathrm{Gr}(2, 5). \tag{3.2}$$

In particular, moduli of smooth rational curves $R_d(Y^3)$ for $d \leq 3$ are rational.

We will re-prove the same result on the moduli space of lines and the moduli space of conics in Y^3 in Section 4.5 through our own method.

Remark 3.3.4. The isomorphisms in (3.2) are defined by the composition map $\zeta_d \circ \iota$ where $\iota : R_d(Y^3) \subset R_d(G)$ is the inclusion and the map ζ_d defined in Corollary 3.2.7. We will geometrically describe the generic fibers of the map $\zeta_d \circ \iota$ for $d = 2, 3$ (cf. [1, §1] and [91, Remark 2.47]) in the remainder of this section. Since the Schubert variety $\sigma_{1,1}(\mathbb{P}^3) \cong \mathrm{Gr}(2, 4)$ has degree two (Definition 3.2.1) in the Plücker embedding, the intersection $\sigma_{1,1}(\mathbb{P}^3) \cap H_1 \cap H_2 \cap H_3$ is generically a conic in Y^3.

In a similar way, we can show that a Schubert variety $\sigma_{2,0}(\mathbb{P}^1)$ is contained in some $\mathbb{P}^6 \subset \mathbb{P}^9$ and $\sigma_{2,0}(\mathbb{P}^1)$ is determined by three quadric equations using an explicit coordinate computation.

Hence the intersection $\sigma_{2,0}(\mathbb{P}^1)$ of hyperplanes $H_1 \cap H_2 \cap H_3$ with $\mathbb{P}^6 \subset \mathbb{P}^9$ is generically a twisted cubic, and this is the generic fiber of the map $\zeta_3 \circ \iota$ (cf. [49, Proposition 4.5]).

Corollary 3.3.5. $R_d(Y^m)$ are all irreducible for $d \leq 3$ and $m \geq 3$.

Proof. We recall that the spaces Y^m does not depend on the generic choice of the hyperplane sections H_i. We define

$$\mathfrak{I} = \{(C, H) \in R_d(Y^m) \times \mathrm{Gr}(13 - m, 10) | C \subset H\}$$

as the incidence variety of pairs of a curve C and a linear subspace $H \subset \mathbb{P}^9$ with codimension $m-3$. We observe that the second projection map $p_2 : \mathfrak{I} \to \mathrm{Gr}(13 - m, 10)$ is dominant. Also, since the Grassmannian $\mathrm{Gr}(13 - m, 10)$ is irreducible and the generic fiber $p_2^{-1}(H) = R_d(Y^3)$ is irreducible for the general linear subspace H (Proposition 3.3.3), the incidence variety $\mathfrak{I}$ is irreducible. Next, we observe that the first projection $p_1 : \mathfrak{I} \to R_d(Y^m)$ is dominant since each degree $d \leq 3$ smooth rational curve in $Y^m \subset \mathrm{Gr}(2, 5) \subset \mathbb{P}^9$ is contained in some three dimensional linear subspace $\mathbb{P}^3$ and also contained in some linear subspace $H \subset \mathbb{P}^9$ of codimension $m-3$. Since dominant image of the irreducible space is irreducible, we prove the claim. □

Combined with irreducibility result (Corollary 3.3.5), we prove the rationality of moduli spaces $R_d(Y^m)$ for $1 \leq d \leq 3$ and $4 \leq m \leq 6$ in the following lemmas.

Lemma 3.3.6. (cf. [59, Theorem 3] and [63, Theorem 4.9]) Recall that $G = \mathrm{Gr}(2, 5)$. Then we have the following :

1. $R_1(G) = F(1, 3, 5)$ is isomorphic to a $\mathrm{Gr}(2, 4)$-bundle on the projective space $\mathbb{P}^4$;

2. $R_2(G)$ is birational to a Grassmannian $\mathrm{Gr}(3,6)$-bundle on the Grassmannian $\mathrm{Gr}(4,5) = \mathbb{P}^4$;

3. $R_3(G)$ is birational to a Grassmannian $\mathrm{Gr}(4,7)$-bundle on the Grassmannian $\mathrm{Gr}(2,5)$;

4. The Fano variety of projective planes in G is isomorphic to the disjoint union $F(1,4,5) \sqcup \mathrm{Gr}(3,5)$.

Proof. By Proposition 3.2.3, a line in G is determined by a pair of a vertex point $p \in \mathbb{P}^4$ and a projective plane P which contains the point p. So we proved the Part 1.

By Corollary 3.2.7 (2), there is the rational map (envelope) $\zeta_2 : R_2(G) \dashrightarrow \mathrm{Gr}(4,5) \cong \mathbb{P}^4$ where its fiber over $V_4 \in \mathrm{Gr}(4,5)$ is $R_2(\mathrm{Gr}(2,V_4))$. Via the Plücker embedding, we have an identification $\mathrm{Gr}(2,V_4) \subset \mathbb{P}^5$ with the quadric hypersurface in $\mathbb{P}^5$. Therefore a general plane $P \subset \mathbb{P}^5$ determines a smooth conic $\mathrm{Gr}(2,V_4) \cap P$. Conversely, a smooth conic in $\mathrm{Gr}(2,V_4)$ spans the plain $P \subset \mathbb{P}^5$. Thus, $\zeta_2^{-1}(V_4)$ is birational to $\mathrm{Gr}(3,\wedge^2 V_4)$, which is a moduli space of planes in the Plücker embedding $\mathbb{P}^5 = \mathbb{P}(\wedge^2 V_4)$. Then, if we consider $\mathcal{U} \to \mathrm{Gr}(4,5)$, the tautological rank 4 vector bundle, a fiber of the relative Grassmannian bundle $\mathrm{Gr}(3,\wedge^2\mathcal{U})$ over $V_4 \in \mathrm{Gr}(4,n)$ equals to $\mathrm{Gr}(3,\wedge^2 V_4)$. Therefore, $R_2(G)$ is birational to this Grassmannian bundle, so we proved Part 2.

By Corollary 3.2.7 (3), there is the rational map (axis) $\zeta_3 : R_3(G) \dashrightarrow \mathrm{Gr}(2,5)$ where its fiber over $\ell \in \mathrm{Gr}(2,5)$ is $R_3(\sigma_{2,0}(\ell))$. Let $\ell = \mathbb{P}(V_2)$ for a 2-dimensional subspace $V_2 \subset \mathbb{C}^5$, consider a kernel of the morphism $K_L := \ker(\wedge^2\mathbb{C}^5 \to \wedge^2(\mathbb{C}^5/V_2))$. Then we can check that the Schubert variety $\sigma_{2,0}(\ell)$ is a 4-dimensional space contained in $\mathbb{P}^6 \cong \mathbb{P}(K_L) \subset \mathbb{P}^9$ and $\sigma_{2,0}(\ell)$ is determined by three(hence linearly dependent) Plücker quadric equations. As we explained in Remark 3.3.4, a general linear subspace $\mathbb{P}^3$ in $\mathbb{P}^6$ intersects with the Schubert variety $\sigma_{2,0}(\ell)$ along with a twisted cubic curve. Therefore, $\zeta_3^{-1}(\ell)$ is birational to $\mathrm{Gr}(4,K_L)$.

We denote $\mathcal{Q}$ the universal quotient bundle of the Grassmannian $\mathrm{Gr}(2,5)$. We define $\mathcal{K}_7$ to be the kernel of the natural surjection $\wedge^2\mathcal{O}_{\mathrm{Gr}(2,5)}^{\oplus 5} \twoheadrightarrow \wedge^2\mathcal{Q}$. Then a fiber of the relative Grassmannian bundle $\mathrm{Gr}(4,\mathcal{K}_7)$ over $\ell \in \mathrm{Gr}(2,5)$ over $\ell = \mathbb{P}(L) \in \mathrm{Gr}(2,5)$ is $\mathrm{Gr}(4,K_L)$. Therefore $R_3(G)$ is birational to this Grassmannian bundle, so we proved the Part 3.

By Corollary 3.2.5, a plane in the Grassmannian $\mathrm{Gr}(2,5)$ is either the family of projective lines in $\mathbb{P}^3 \subset \mathbb{P}^4$ which pass through a fixed point $p \in \mathbb{P}^3$ or the family of lines in a projective plane $\mathbb{P} \cong P \subset \mathbb{P}^4$. The former type of planes are parametrized by the flag variety $F(1,4,5)$ and the latter type of planes are parametrized by $\mathrm{Gr}(3,5)$. □

Lemma 3.3.7.
1. $R_1(Y^5)$ is birational to a Grassmannian $\mathrm{Gr}(2,3) \cong \mathbb{P}^2$-bundle on the projective space $\mathbb{P}^4$;
2. $R_2(Y^5)$ is birational to a Grassmannian $\mathrm{Gr}(3,5)$-bundle on the Grassmannian $\mathrm{Gr}(4,5) \cong \mathbb{P}^4$;
3. $R_3(Y^5)$ is birational to a Grassmannian $\mathrm{Gr}(4,6)$-bundle on the Grassmannian $\mathrm{Gr}(2,5)$.

Proof. Recall that $Y^5 = \mathrm{Gr}(2,5) \cap H_1 \subset \mathbb{P}^9$ where H_1 is a general hyperplane. Then the inclusion $Y^5 \subset G = \mathrm{Gr}(2,5)$ naturally induces the inclusion between moduli of smooth rational curves :

$$\imath_d : R_d(Y^5) \hookrightarrow R_d(G).$$

For a point $p = \mathbb{P}(V_1) \in \mathbb{P}^4$, consider the Schubert variety $\sigma_{3,0}(p) = \{\ell \in \mathrm{Gr}(2,5) = G \,|\, p \in \ell\} \cong \mathbb{P}^3$ which is embedded in the projective space $\mathbb{P}^9$ as a linear subspace. Then, for a general point p, the Schubert variety $\sigma_{3,0}(p)$ intersects the hyperplane H_1 cleanly along a projective plane $\mathbb{P}^2$. Thus, a general fiber of the rational map $\zeta_1 \circ \imath_1 : R_1(Y^5) \to \mathbb{P}^4$ is isomorphic to $R_1(H_1 \cap \sigma_{3,0}(p)) \cong \mathrm{Gr}(2,3) \cong \mathbb{P}^2$.

A general fiber of the rational map $\zeta_2 \circ \iota_2 : \mathsf{R}_2(\mathsf{Y}^5) \dashrightarrow \mathrm{Gr}(4,5) \cong \mathbb{P}^4$ is isomorphic to $\mathsf{R}_2(\sigma_{1,1} \cap \mathsf{H}_1)$ for the restricted hyperplane H_1 of $\sigma_{1,1} = \mathrm{Gr}(2,4) \subset \mathbb{P}^5$. Since the Schubert variety $\sigma_{1,1}$ is isomorphic to a quadric hypersurface in the projective space $\mathbb{P}^5$, the fiber is birational to $\mathrm{Gr}(3,5)$ which parametrizes projective planes in $\mathsf{H}_1 \cap \mathbb{P}^5$. Let $\mathcal{U}$ be the tautological rank 4 vector bundle on the Grassmannian $\mathrm{Gr}(4,5)$ and we define $\mathcal{K}_5 := \ker\{\wedge^2 \mathcal{U} \to \wedge^2 \mathcal{O}^{\oplus 5} \to \mathcal{O}\}$ as the kernel of the above composition morphism where the second arrow in the sequence is induced from the linear equation of the hyperplane H_1. Then general points of the relative Grassmannian bundle $\mathrm{Gr}(3, \mathcal{K}_5)$ on the Grassmannian $\mathrm{Gr}(4,5)$ determine conics in Y^5.

The general fiber of the rational map $\zeta_3 \circ \iota_3 : \mathsf{R}_3(\mathsf{Y}^5) \dashrightarrow \mathrm{Gr}(2,5)$ is isomorphic to $\mathsf{R}_3(\sigma_{2,0}(\mathbb{P}^1) \cap \mathsf{H}_1)$ for the restricted hyperplane H_1 of $\mathbb{P}^6$. Through the proof of Lemma 3.3.6, we know that a general linear subspace $\mathbb{P}^3$ in $\mathsf{H}_1 \cap \mathbb{P}^6$ determines a twisted cubic $\sigma_{2,0}(\mathbb{P}^1) \cap \mathbb{P}^3$ and therefore the general fiber is isomorphic to $\mathrm{Gr}(4,6)$. Thus, when we recall $\mathcal{K}_7$ the rank 7 bundle defined in Part (3) of Lemma 3.3.6, then we define $\mathcal{K}_6$ as the kernel of the following composition morphism $\mathcal{K}_7 \hookrightarrow \wedge^2 \mathcal{O}^{\oplus 5}_{\mathrm{Gr}(2,5)} \to \mathcal{O}_{\mathrm{Gr}(2,5)}$ where the second arrow is induced by the linear equation of the hyperplane H_1. Thus the relative Grassmannian bundle $\mathrm{Gr}(4, \mathcal{K}_6)$ over the Grassmannian $\mathrm{Gr}(2,5)$ becomes the birational model for the moduli space of cubics $\mathsf{R}^3(\mathsf{Y}^4)$. □

Lemma 3.3.8. 1. $\mathsf{R}_1(\mathsf{Y}^4)$ is birational to the projective space $\mathbb{P}^4$;

2. $\mathsf{R}_2(\mathsf{Y}^4)$ is birational to a Grassmannian $\mathrm{Gr}(3,4) = \mathbb{P}^3$-bundle on the Grassmannian $\mathrm{Gr}(4,5) = \mathbb{P}^4$;

3. $\mathsf{R}_3(\mathsf{Y}^4)$ is birational to a Grassmannian $\mathrm{Gr}(4,5) = \mathbb{P}^4$-bundle on the Grassmannian $\mathrm{Gr}(2,5)$.

Proof. The proof proceeds in the same manner as the proof for Lemma 3.3.7. If we replace H_1 with $\mathsf{H}_1 \cap \mathsf{H}_2$ and replace Y^5 with Y^4 where H_1 and H_2 are general hyperplanes, then the rest of proof proceeds in the same manner. □

Combining Corollary 3.3.5 and the above lemmas, we finally obtain the proof of Theorem 3.3.1 since the Grassmannian varieties $\mathrm{Gr}(\ell, n)$ are clearly rational.

Chapter 4

Compactifications of the moduli spaces of smooth rational curves in Y^m

4.1 Various compactifications

The results presented in this chapter are based on the results obtained joint with Chung and Hong in [19]. Let us start this chapter by introducing some typical compactifications of moduli space of smooth rational curves in the case of $\mathsf{G} = \mathrm{Gr}(2,5)$.

(1) **Hilbert compactification**: Since $\mathsf{G} = \mathrm{Gr}(2,5) \subset \mathbb{P}^9$ is a projective variety, Grothendieck's existence theorem 2.2.4 guarantees us the existence of the Hilbert scheme $\mathrm{Hilb}^{dt+1}(\mathsf{G})$ of closed subschemes of the Grassmannian G which have the Hilbert polynomial $\mathsf{HP}(t) = dt+1$. We denote the closure of $\mathsf{R}_d(\mathsf{G})$ in $\mathrm{Hilb}^{dt+1}(\mathsf{G})$ by $\mathsf{H}_d(\mathsf{G})$ and we call it the *Hilbert compactification* of $\mathsf{R}_d(\mathsf{G})$.

Before we introduce the Kontsevich compactification, we briefly introduce the definition of the stable map space.

Definition 4.1.1 (Stable map space)**.** [35] Let X be a smooth projective vareity over $\mathbb{C}$, and $\beta \in H_2(X)$ be a homology class. Then, we call a projective genus g, n-pointed connected reduced nodal curve $(C, p_1, p_2, \dots, p_n)$ where $p_1, \dots, p_n$ are all distinct *quasi-stable curve*. Then we call a morphism $f : C \to X$ a *stable map* with homology class β if automorphism group of f preserving marked points $p_1, \dots, p_n$ is finite and $f_*[C] = \beta$.

Then we consider a family of maps. For a scheme S, a family of quasi-stable n-pointed genus g stable maps consist of the following data :

- Flat family of nodal curves $\pi : \mathcal{C} \to S$.
- Disjoint n-sections $p_1, \dots, p_n : S \to \mathcal{C}$.
- Family morphism $F : \mathcal{C} \to X$

which satisfies for each closed point $s \in S$, the fiber $(\mathcal{C}_s, p_1(s), \dots, p_n(s))$ and $F_s : \mathcal{C}_s \to X$ is a stable map with homology class β. We define an isomorphisms between families, as an isomorphisms between families of curves, which commutes with family morphisms and sections.

Then there exists a fine moduli space of this moduli problem, as a proper Deligne-Mumford stack, we denote it by $\overline{M}_{g,n}(X, \beta)$. If the Picard group of X is generated by the very ample line bundle on X, we use notation $\overline{M}_{g,n}(X, d)$, where d means the homology class correspond to d times of the Poincare dual of the very ample divisor.

From now on, we use notation $\overline{M}_{0,0}(X, \beta) =: \mathbf{M}_0(X, \beta)$.

(2) **Kontsevich compactification**: We denote the closure of $R_d(G)$ in the stable map space $\mathbf{M}_0(G, d)$ by $M_d(G)$ and we call it the *Kontsevich compactification* of $R_d(G)$.

(3) **Simpson compactification**: An arbitrary coherent sheaf $\mathcal{E}$ over the Grassmannian G is called *pure* if for any nonzero subsheaf $\mathcal{E}' \subset \mathcal{E}$ of $\mathcal{E}$, its support $\mathrm{Supp}(\mathcal{E}')$ and $\mathrm{Supp}(\mathcal{E})$ have same dimension.

An arbitrary pure sheaf $\mathcal{E}$ is said to be *semi-stable*(resp. stable) if

$$\frac{\mathsf{H}_T(\mathcal{E}')}{\mathsf{r}(\mathcal{E}')} \leq \text{(resp. } <) \frac{\mathsf{H}_T(\mathcal{E})}{\mathsf{r}(\mathcal{E})} \qquad \text{for } t >> 0$$

for any nontrivial subsheaf $\mathcal{E}'$, and the leading coefficient $\mathsf{r}(\mathcal{E})$ of the Hilbert polynomial $\mathsf{HP}(\mathcal{E})(t) = \chi(\mathcal{E} \otimes \mathcal{O}_G(t))$. By replacing the subsheaves $\mathcal{E}'$ by quotient sheaves $\mathcal{E}'$ and inverting the inequality, we have the equivalent definition of the (semi-)stability.

Next, under this stability condition, we can define a projective moduli space $\mathcal{S}im^P(G)$ of semi-stable sheaves on G which have Hilbert polynomial P, called Simpson moduli space [65, 47, 94]. There is a natural embedding $\mathsf{R}_d(G) \hookrightarrow \mathcal{S}im^{dt+1}(G)$ which assigns a smooth rational curve C on G to its structure sheaf $\mathcal{O}_C$. We note that $\mathcal{O}_C$ is a stable pure sheaf. We denote the closure of $\mathsf{R}_d(G)$ in the Simpson moduli space $\mathcal{S}im^{dt+1}(G)$ by $\mathrm{Sim}_d(G)$ and call it the *Simpson compactification* of $\mathsf{R}_d(G)$.

In the remaining sections, we deal with various compactifications of moduli of smooth rational curves $\mathsf{R}_d(\mathsf{Y}^m)$. We mainly study Hilbert compactifications $\mathsf{H}_d(\mathsf{Y}^m)$ and their birational models in this Chapter.

4.2 Fano 6-fold $G = \mathrm{Gr}(2,5) = G$

Throughout this section, we fix notation $G = \mathrm{Gr}(2,5) = \mathsf{Y}^6$ and we consider various compactification of moduli spaces of smooth rational curves $\mathsf{R}_d(G)$ of degree $1 \leq d \leq 3$ in G.

We first note that $\mathsf{R}_1(G) = \mathsf{F}(1,3,5)$ is already compact and therefore $\mathsf{H}_1(G) = \mathsf{M}_1(G) = \mathsf{P}_1(G) = \mathsf{R}_1(G) = \mathrm{Gr}(1,3,5)$. So we have nothing to do with for compactification of $\mathsf{R}_1(G)$.

4.2.1 Hilbert scheme of conics $\mathsf{H}_2(\mathsf{G})$ in $\mathsf{G} = \mathrm{Gr}(2,5)$

We start by discussing the birational geometry of the Hilbert scheme $\mathsf{H}_2(\mathsf{G})$ via the envelope map, which we defined in Corollary 3.2.7 (2) :

$$\zeta_2 : \mathsf{R}_2(\mathsf{G}) \dashrightarrow \mathbb{P}^4 = \mathrm{Gr}(4,5).$$

Therefore there is also a rational map $\zeta_2 : \mathsf{H}_2(\mathsf{G}) \to \mathrm{Gr}(4,5)$. So it is natural to blow up the base locus of the rational map ζ_2, to complete it as a regular map. Then we should know what is the base locus of the map ζ_2.

First, we can easily observe that the base locus of the rational map $\zeta_2 : \mathsf{R}_2(\mathsf{G}) \dashrightarrow \mathbb{P}^4 = \mathrm{Gr}(4,5)$ consists of the following types of conics : (1) A conic which is the family of lines in the ruling of a rational normal scroll $\mathsf{S}(\ell_0, \ell_1)$ of two projective lines $\ell_0, \ell_1 \subset \mathbb{P}^{n-1}$ such that ℓ_0 and ℓ_1 lies in a same projective plane P. (2) A conic which is the family of lines in the ruling of a rational normal scroll $\mathsf{S}(p_0, \mathsf{C}_0)$ for a fixed point p_0 and a smooth conic $\mathsf{C}_0 \subset \mathbb{P}^{n-1}$, such that the point p_0 lies in the plane P spanned by the conic C_0.

We can easily observe that both cases happen if and only if a smooth conic lies in a $\sigma_{2,2}$-plane, which correspond to the projective plane $\mathsf{P} \subset \mathbb{P}^{n-1}$. Therefore, we can guess that the base locus of the extended map $\zeta_2 : \mathsf{H}_2(\mathsf{G}) \dashrightarrow \mathrm{Gr}(4,5)$ is the locus of conics in the $\sigma_{2,2}$-planes. We denote this locus as $\Gamma_{2,2}$.

On the other hand, consider the relative Grassmannian $\mathrm{Gr}(2,\mathcal{U})$ on the Grassmannian $\mathrm{Gr}(4,5)$ where $\mathcal{U}$ is the tautological bundle over $\mathrm{Gr}(4,5)$. Then it is known by [61, Theorem 1.4], that we have a relative Hilbert scheme of conics

$$\widetilde{\zeta}_2 : \mathsf{H}_2(\mathrm{Gr}(2,\mathcal{U})) \to \mathrm{Gr}(4,5)$$

with the natural projection map $\widetilde{\zeta}_2$. In this viewpoint, it is natural to guess that $\mathrm{Bl}_{\Gamma_{2,2}} \mathsf{H}_2(\mathsf{G})$ is isomorphic to $\mathrm{Gr}(2,\mathcal{U})$, and it is true by the following theorem of Iliev and Manivel.

Proposition 4.2.1. [50, Section 3.1, p. 9] Under the above definitions and notations, there is a natural birational morphism

$$\Phi : H_2(\mathrm{Gr}(2,\mathcal{U})) \longrightarrow H_2(G)$$

which is a *smooth* blow-up along the sub-locus $\Gamma_{2,2}$ consists of conics lying on the $\sigma_{2,2}$-planes.

Proof. The exceptional divisor of the blow-up is the $\mathbb{P}^5$-bundle on the flag variety $F(3,4,5)$. By its construction, the flag variety $F(3,4,5)$ is canonically isomorphic to the $\mathrm{Gr}(1,2) \cong \mathbb{P}^1$-bundle on the Grassmannian $\mathrm{Gr}(3,5)$ where $\mathrm{Gr}(1,2)$ parametrizes linear subspace $\mathbb{P}^3$ in $\mathbb{P}^4$ containing a fixed projective plane $\mathbb{P}^2 \subset \mathbb{P}^4$. Next, to show that Φ is the smooth blow-up, we compute the normal space of the blow-up locus $\Gamma_{2,2}$ in $H_2(G)$ at arbitrary conic C. From the following canonical exact sequence of normal bundles $0 \to N_{C/\mathbb{P}^2} \to N_{C/G} \to N_{\mathbb{P}^2/G}|_C \to 0$ and the the structure sequence $0 \to N_{\mathbb{P}^2/G}(-2) \to N_{\mathbb{P}^2/G} \to N_{\mathbb{P}^2/G}|_C \to 0$, we compute the normal space as follows

$$N_{\Gamma_{2,2}/H_2(G),C} \cong H^1(N_{\mathbb{P}^2/G}(-2)).$$

By diagram chasing, we can check that $N_{\mathbb{P}^2/G} \cong \mathcal{Q} \otimes \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}$ for a $\sigma_{2,2}$-type plane $\mathbb{P}^2 \subset G$, where $\mathcal{Q}$ is the universal quotient bundle restricted on $\mathbb{P}^2$. So we conclude that the later space $H^1(N_{\mathbb{P}^2/G}(-2))$ is isomorphic to $H^0(\mathcal{O}_{\mathbb{P}^2}^{\oplus 2})^\vee$. Furthermore, this space has a 1-1 correspondence with the choice of linear subspace $\mathbb{P}^3$ in $\mathbb{P}^4$ which contains the fixed projective plane $\mathbb{P}^2$. $\square$

In Iliev-Manivel [50], the authors also explained blow-down of $H_2(\mathrm{Gr}(2,\mathcal{U}))$ which contracts conics lies in $\sigma_{3,1}$-type planes. We state it as follows.

Proposition 4.2.2. [50, Section 3.1, p. 9] We denote $S(G) = \mathrm{Gr}(3, \wedge^2\mathcal{U})$ the relative Grassmannian of the wedge product tautological bundle $\mathcal{U}$ over the Grassmannian $\mathrm{Gr}(4,5)$. Then there is a blow-up morphism

$$\Xi : \mathsf{H}_2(\mathrm{Gr}(2,\mathcal{U})) \longrightarrow \mathsf{S}(\mathsf{G})$$

along the smooth blow-up center $\mathsf{T}(\mathsf{G})$, which equal to the relative Orthogonal Grassmannian $\mathrm{OG}(3,\wedge^2\mathcal{U})$ over the Grassmannian $\mathrm{Gr}(4,5) \subset \mathrm{Gr}(3,\wedge^2\mathcal{U})$. Here, orthogonal means orthogonal via the canonical symmetric 2-form on $\wedge^2\mathcal{U}$.

Proof. Since blow-up morphisms satisfify the base change property, it is enough to show the claim up to fiber. Then the fiberwise construction has been studied in [16, Lemma 3.9]. It should be noted that $\mathsf{T}(\mathsf{G})$ can be identified with the disjoint union of two flag varieties, i.e. $\mathsf{T}(\mathsf{G}) \cong \mathsf{F}(1,4,5) \sqcup \mathsf{F}(3,4,5)$ ([46, Proposition 4.16]). $\square$

Combining Proposition 4.2.1 and 4.2.2, we have the blow-up and blow-down diagram in the following

$$\begin{tikzcd}
 & \mathsf{H}_2(\mathrm{Gr}(2,\mathcal{U})) \arrow[dl, "\Xi"'] \arrow[dr, "\Phi"] \arrow[dd, Rightarrow, "\tilde{\zeta}_2"] & \\
\mathsf{S}(\mathrm{Gr}(2,5)) \arrow[dr, Rightarrow] & & \mathsf{H}_2(\mathrm{Gr}(2,5)) \arrow[dl, dashed, "\zeta_2"'] \\
 & \mathrm{Gr}(4,5) &
\end{tikzcd} \tag{4.1}$$

where $\mathcal{U}$ is the tautological rank 4 vector bundle over $\mathrm{Gr}(4,5)$. This diagram (4.1) plays a key role when we show smoothness of $\mathsf{H}_2(\mathsf{Y}^4), \mathsf{H}_2(\mathsf{Y}^5)$ later.

Furthermore, it turned out that there is a similar blow-up blow-down diagram Kontsevich compactification $\mathsf{M}_2(\mathsf{G})$ of $\mathsf{R}_2(\mathsf{G})$ [16]. Since this contents does not appear again in the remaining parts of this thesis, we only explain the results briefly. We denote $\mathsf{M}_2(\mathrm{Gr}(2,\mathcal{U}))$ the moduli space of relative stable maps with genus zero and degree two. Let $\mathsf{M}_2(\mathrm{Gr}(2,\mathcal{U})) \to \mathsf{M}_2(\mathrm{Gr}(2,5))$

be the map induced by the natural inclusion $\mathcal{U} \hookrightarrow \mathcal{O}_G^{\oplus 5}$. Furthermore we denote by $N(\mathrm{Gr}(2,5))$ the moduli space of relative Kronecker quiver representations $N(\mathcal{U};2,2)$, whose fibers are isomorphic to $N(4;2,2)$ (for the definition of the moduli space of Kronecker quiver representations, see [21]). Then there is a map obtained by divisorial contraction $M_2(\mathrm{Gr}(2,\mathcal{U})) \to N(\mathrm{Gr}(2,5))$, which contracts the locus of stable maps such that their images are planar ([21]). In summary, we have the following diagram :

$$\begin{tikzcd}
& M_2(\mathrm{Gr}(2,\mathcal{U})) \arrow[dl] \arrow[dr] \arrow[dd, Rightarrow] & \\
N(\mathrm{Gr}(2,5)) \arrow[dr, Rightarrow] & & M_2(\mathrm{Gr}(2,5)) \arrow[dl, dashed] \\
& \mathrm{Gr}(4,5). &
\end{tikzcd}$$

Let $U \hookrightarrow \mathcal{O}_G^{\oplus 5}$ be the tautological rank 2 subbundle over the Grassmannian $G = \mathrm{Gr}(2,5)$ and let $\mathcal{O}_Y^{\oplus 5} \twoheadrightarrow U^\vee$ its dual bundle. For a general smooth conic $\mathbb{P}^1 \cong C \hookrightarrow \mathrm{Gr}(2,5)$, the restriction of the bundle $U^\vee$ on C splits in the form of $U^\vee|_C \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. Therefore, the dual map $\mathcal{O}_Y^{\oplus 5} \twoheadrightarrow U^\vee$ restricted to C as $\mathcal{O}_{\mathbb{P}^1}^{\oplus 5} \twoheadrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. Hence general conics are parametrized by an open subset of the following GIT quotient :

$$\mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}(1)) \otimes \mathbb{C}^2 \otimes \mathbb{C}^5) // \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$$

where the first $\mathrm{SL}_2(\mathbb{C})$ acts on $H^0(\mathbb{P}^1, \mathcal{O}(1))$ in the canonical way and the second $\mathrm{SL}_2(\mathbb{C})$ acts on $\mathbb{C}^2$ by canonical matrix multiplication. This GIT quotient is in fact isomorphic to the moduli of quiver representations $N(5;2,2)$ correspond to the quiver which has two vertices equipped with 2-dimensional vector spaces on each of them and five edges between the two vertices. The geometry of the stable map space $M_2(\mathrm{Gr}(2,5))$ in the viewpoint of the minimal model program was studied in [23].

On the other hand, since the Grassmannian G is a homogeneous variety, there is the following results of Chung, Hong, and Kiem [18].

Proposition 4.2.3. [18, Theorem 3.7]

1. $\mathrm{Sim}_2(G) \cong H_2(G)$.
2. The blow-up of the Kontsevich compactification $M_2(G)$ along the sub-locus of stable maps whose image is a projective line in the Plücker embedding $G \hookrightarrow \mathbb{P}^9$ and the smooth blow-up of the Simpson compactification $\mathrm{Sim}_2(G)$ along the sub-locus of semi-stable pure sheaves whose support is a projective line in the Plücker embedding $G \hookrightarrow \mathbb{P}^9$. In summary, we have a blow-up and blow-down diagram :

$$\begin{tikzcd} & \widetilde{M}_2(G) \arrow[dl] \arrow[dr] & \\ M_2(G) & & \mathrm{Sim}_2(G) \end{tikzcd}$$

4.2.2 Hilbert scheme of twisted cubics $H_3(G)$ in $G = \mathrm{Gr}(2,5)$

Because the Grassmannian $\mathrm{Gr}(k,n)$ can be represented by a quotient of a Matrix group $M_{k+n,k+n}(\mathbb{C})$ by a parabolic subgroup of block upper triangular matrices of the form :

$$\begin{pmatrix} M_k & * \\ 0 & M_n \end{pmatrix}$$

where M_k is a $k\times k$-matrix and M_n is an $n\times n$-matrix. So the Grassmannian $\mathrm{Gr}(k,n)$ is a homogeneous variety. Therefore we again use the results of [18] on $G = \mathrm{Gr}(2,5)$.

Proposition 4.2.4. [18, §4]

1. The Hilbert compactification $H_3(G)$ is obtained by the smooth blow-up of the Simpson compactification $\mathrm{Sim}_3(G)$ along the sublocus $\Lambda(G)$ consists of planar stable pure sheaves.

2. $\mathrm{Sim}_3(G)$ is obtained from the Kontsevich compactification $M_3(G)$ by three times weighted blow-ups which is followed by three times weighted blow-downs. More precisely, the blow-up centers are Γ_0^1, Γ_1^2, Γ_2^3 and the blow-down is taken along the loci Γ_3^2, Γ_4^3, Γ_5^1. Here, Γ_i^j is the proper transformation of Γ_{i-1}^j if Γ_{i-1}^j is neither the blow-up/-down center nor the image/preimage of Γ_{i-1}^j. Furthermore, Γ_0^1 is the locus consists of stable maps such that their images are projective lines in $G \subset \mathbb{P}^9$. Γ_1^2 is the locus consists of stable maps such that their images are unions of two projective lines. Γ_2^3 is the sublocus of Γ_1^1, and it is a fiber bundle via the morphism $\Gamma_2^3 \subset \Gamma_1^1 \to \Gamma_0^1$. Its fiber over a stable map $f \in \Gamma_0^1$ which has projective line $L \subset G$ as its image is isomorphic to

$$\mathbb{P}\mathrm{Hom}_1(\mathbb{C}^2, \mathrm{Ext}^1_G(\mathcal{O}_L, \mathcal{O}_L(-1))) \cong \mathbb{P}^1 \times \mathbb{P}\mathrm{Ext}^1_G(\mathcal{O}_L, \mathcal{O}_L(-1))$$

where Hom_1 is the locus of rank 1 linear maps.

$$\begin{tikzcd}
 & & & M_3 \arrow[dl, "\Gamma_2^3"'] \arrow[dr, "\Gamma_3^2"] & & & & \\
 & & M_2 \arrow[dl, "\Gamma_1^2"'] & & M_4 \arrow[dr, "\Gamma_4^3"] & & & \\
 & M_1 \arrow[dl, "\Gamma_0^1"'] & & & & M_5 \arrow[dr, "\Gamma_5^1"] & & H_3(G) \arrow[d, "\Lambda(G)"] \\
M_3(G) & & & & & & M_6 \arrow[r, "\cong"] & \mathrm{Sim}_3(G).
\end{tikzcd}$$

4.3 Fano 5-fold Y^5

In this section, we denote by $Y = Y^5$ the intersection $Y = Y^5 = \mathrm{Gr}(2, 5) \cap H$ where H is a general hyperplane in $\mathbb{P}^9$. As we commented in Chapter 2, Section 2.6.2, the Fano 5-fold Y^5 does not depend on the choice of H up to projective equivalence given by PGL($\mathbb{C}^5$)-action. So we may assume that

$$H = \{p_{12} - p_{03} = 0\}$$

where p_{ij} are Plücker coordinates, for explicit calculation.

We want to restrict the key birational model (4.1) for the Hilbert compactification $H_2(Y^5) \subset H_2(G)$. For this, we first need to know how the blow-up and blow-down locus of the map Ξ and Φ in (4.1) changes. Since these loci are induced from the loci of Fano variety of $\sigma_{2,2}$-planes and Fano variety of $\sigma_{3,1}$-planes, we first study Fano variety of projective planes in Y.

4.3.1 Fano varieties of lines and planes in Y^5

In this section, we precisely describe the Fano variety of projective lines $F_1(Y)$ in Y and the Fano variety of projective planes $F_2(Y)$ in Y. The next two propositions summarize the contents of this subsection.

Proposition 4.3.1. $F_1(Y) = H_1(Y) = S_1(Y) = M_1(Y)$ is isomorphic to the blow-up of Gr$(3,5)$ along the smooth quadric threefold Σ.

Proof. Each projective line in G can be uniquely written by $\{\ell \in G \,|\, p \in \ell \subset P\} \subset G$ for a point $p \in \mathbb{P}^4$ and a projective plane $P \subset \mathbb{P}^4$ which contains a point p. Then we have the following forgetful map :

$$\psi : F_1(Y) \hookrightarrow F_1(G) = \mathrm{Gr}(1,3,5) \longrightarrow \mathrm{Gr}(3,5), \quad (p, P) \mapsto P.$$

Consider a projective plane $P \in \mathrm{Gr}(3,5)$ which is represented by the row span of the following matrix :

$$\begin{pmatrix} 1 & 0 & 0 & a_3 & a_4 \\ 0 & 1 & 0 & b_3 & b_4 \\ 0 & 0 & 1 & c_3 & c_4 \end{pmatrix},$$

and we consider projective lines in P which is represented by the row span of the following matrix :

$$\begin{pmatrix} 1 & 0 & \alpha & a_3+\alpha c_3 & a_4+\alpha c_4 \\ 0 & 1 & \beta & b_3+\beta c_3 & b_4+\beta c_4 \end{pmatrix}. \tag{4.2}$$

Then the equation $p_{12}=p_{03}$ induces the equation $\alpha+b_3+\beta c_3=0$ so it determines a unique line $\mathbb{L}$ in Y. Furthermore, we can easily check that projective lines which have types different from (4.2) cannot satisfy the equation $p_{12}=p_{03}$. Thus we conclude that $\psi^{-1}(P)$ is a unique point $\mathbb{L}$.

Consider a projective plane $P \in \mathrm{Gr}(3,5)$ which is represented by the row span of the following matrix :

$$\begin{pmatrix} 1 & 0 & a_2 & a_3 & 0 \\ 0 & 1 & b_2 & b_3 & 0 \\ 0 & 0 & c_2 & c_3 & 1 \end{pmatrix}, \tag{4.3}$$

and we consider projective lines in P which are represented by the row span of the following matrix :

$$\begin{pmatrix} 1 & 0 & a_2+\alpha c_2 & a_3+\alpha c_2 & \alpha \\ 0 & 1 & b_2+\beta c_2 & b_3+\beta c_3 & \beta \end{pmatrix}.$$

Then the equation $p_{12}=p_{03}$ induces the equation $a_2+b_3+\alpha c_2+\beta c_3=0$ so it determines a unique line in Y unless $c_2=c_3=a_2+b_3=0$. Furthermore, we can easily check that projective lines which has types different from (4.3) cannot satisfy the equation $p_{12}=p_{03}$. Thus we conclude that $\psi^{-1}(P)$ is a single point unless $c_2=c_3=a_2+b_3=0$. When $c_2=c_3=a_2+b_3=0$, we have $\psi^{-1}(P)=P^{\vee}\cong\mathbb{P}^2$ is the set of all projective lines contained in the plane P.

By applying the same process to all other affine charts, we can observe that if we consider the following smooth quadric threefold

$$\Sigma = \mathrm{Gr}(2, W_4) \cap \overline{H} \subset \overline{H} \cong \mathbb{P}^4$$

where $W_4 = \langle e_0, e_1, e_2, e_3 \rangle$ is the vector subspace of $\mathbb{C}^5$ and $\mathrm{Gr}(2, W_4) \subset \mathrm{Gr}(3, 5)$ is the linear embedding, which assigns a 2-dimensional subspace A to $A + \langle e_4 \rangle$, and $\overline{H} = \mathrm{zero}(p_{12} - p_{03})$ is the hyperplane in $\mathbb{P}(\wedge^2 W_4) \cong \mathbb{P}^5$, which is the restriction of the hyperplane H, then $\psi^{-1}(P)$ is a single point if $P \notin \Sigma$ and $\psi^{-1}(P)$ is the set of all lines represented by pairs $(p, P), p \in P$ if $P \in \Sigma$. By local chart computation, we can directly check that ψ is the blow-up along the smooth quadric threefold Σ. For example, consider the local chart $(a_2, b_2, c_2, a_3, b_3, c_3, \lambda, \mu)$ of the flag variety $F(1, 3, 5)$ represented following matrix :

$$\begin{pmatrix} 1 & \lambda & a_2 + \lambda b_2 + \mu c_2 & a_3 + \lambda b_3 + \mu c_3 & \mu \\ 0 & 1 & b_2 & b_3 & 0 \\ 0 & 0 & c_2 & c_3 & 1 \end{pmatrix}$$

where its first row corresponds to the one-dimensional subspace V_1 and the row span of all three rows corresponds to the three-dimensional subspace V_3. Then projective lines in the projective plane $\mathbb{P}V_3$ which pass through the point $\mathbb{P}V_1$ are represented by the following matrix :

$$\begin{pmatrix} 1 & \lambda & a_2 + \lambda b_2 + \mu c_2 & a_3 + \lambda b_3 + \mu c_3 & \mu \\ 0 & \alpha & \alpha b_2 + \beta c_2 & \alpha b_3 + \beta c_3 & \beta \end{pmatrix}.$$

The equation $p_{12} = p_{03}$ induces the equations $a_2 + b_3 = -\mu c_2$ and $c_3 = \lambda c_2$, which determines a family of lines in lines in Y, parametrized by λ and μ. Clearly, this is the blow-up map $(c_2, \lambda, \mu) \mapsto (c_2, c_3, a_2 + b_3)$ along the locus $\Sigma = \mathrm{zero}(c_2, c_3, a_2 + b_3)$ in this local coordinates. $\square$

Lemma 4.3.2. The moduli space of lines $F_1(Y)$ is smooth.

Proof. As it is appeared in Proposition 4.3.1, the reduced induced scheme $F_1(Y)_{\mathrm{red}}$ is the smooth blow-up of the irreducible variety so it is again irreducible. So it is enough to show that $F_1(Y)$ is a reduced scheme. We note that any line L in Y is locally a complete intersection. Consider any line $L \subset Y$. Then, from the following natural exact sequence of normal bundles $0 \to N_{L/Y} \to N_{L/G} \to N_{Y/G}|_L = \mathcal{O}_L(1) \to 0$, we compute the expected dimension of $F_1(Y)$ as $h^0(N_{L/Y}) - h^1(N_{L/Y}) = 6$. But the moduli space $F_1(Y)$ has dimension 6 at the closed point L by Proposition 4.3.1. Therefore, we obtain that $F_1(Y)$ is locally a complete intersection from [61, Theorem 2.15]. Hence we obtain that $F_1(Y)$ is a Cohen-Macaulay scheme. Furthermore we can use the following fact that any Cohen-Macaulay and generically reduced scheme is reduced ([69, page 49-51]). Therefore, it is enough to show that $h^1(N_{L/Y}) = 0$ for a certain projective line L in Y, which is represented by the row span of the following matrix :

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & s & t & 0 & 0 \end{pmatrix}.$$

We can check this by direct calculation. It implies that $F_1(Y)$ is smooth at the point L and therefore smooth at the open set containing the point L, therefore generically smooth, hence reduced. In summary, we proved that the Fano variety of lines $F_1(Y) = F_1(Y)_{\mathrm{red}} \cong \mathrm{bl}_\Sigma \mathrm{Gr}(3,5)$ is smooth. $\square$

Proposition 4.3.3. ([50, Section 4.4]) We can write the Fano variety of projective planes $F_2(Y)$ as a disjoint union $F_2^{3,1}(Y) \sqcup F_2^{2,2}(Y)$, where $F_2^{3,1}(Y)$ parametrizes $\sigma_{3,1}$-type planes in Y and $F_2^{2,2}(Y)$ parametrizes $\sigma_{2,2}$-type planes in Y. The first component $F_2^{3,1}(Y)$ is isomorphic to the blow-up of the projective space $\mathbb{P}^4$ at the point y_0 and $F_2^{2,2}(Y)$ is isomorphic to the smooth quadric threefold Σ.

Proof. First, we can observe that the sub-locus $F_2^{2,2}(Y)$ of $\sigma_{2,2}$-planes in $\mathrm{Gr}(3,5)$ is equal to the quadric threefold Σ from the proof of Lemma 4.3.2.

On the other hand, consider the morphism $\psi : F_2^{3,1}(Y) \to \mathbb{P}^4$ assigning the vertex. Let $y = [1 : a_1 : a_2 : a_3 : a_4]$ a point in $\mathbb{P}^4$ and a projective line $\ell \in G$ passing through the point y is represented by the following matrix :

$$\begin{pmatrix} 1 & a_1 & a_2 & a_3 & a_4 \\ 0 & b_1 & b_2 & b_3 & b_4 \end{pmatrix}.$$

Then the equation $p_{12} - p_{03}$ induces a linear equation $b_3 = a_1 b_2 - a_2 b_1$, and therefore, defines a unique three-dimensional linear space $\Lambda = \mathbb{P}^3 \subset \mathbb{P}^4$. Then the pair (y, Λ) determines a unique plane in $F_2^{3,1}$, which is the inverse image $\psi^{-1}(-y)$ of y.

By calculating over all affine charts, one can check that $\psi^{-1}(y)$ is a single point if $y \neq [0 : 0 : 0 : 0 : 1] =: y_0$ and $\psi^{-1}(y_0)$ is the set of planes represented by pairs $(y, \mathrm{zero}(y_4)), y \in \mathrm{zero}(y_4) \cong \mathbb{P}^3$.

We can directly check that ψ is the blow-up of the projective space $\mathbb{P}^4$ at the point y_0 by explicit local chart computation. For example, let us consider the following local chart :

$$\{([a_0 : a_1 : a_2 : a_3 : 1], [c_0 : c_1 : c_2 : c_3 : c_4]) | a_0 c_0 + a_1 c_1 + a_2 c_2 + a_3 c_3 + c_4 = 0\}$$

of $F(1,4,5) \subset \mathrm{Gr}(1,5) \times \mathrm{Gr}(4,5) \cong \mathbb{P}^4 \times (\mathbb{P}^4)^*$.

On the other hand, for a $\sigma_{3,1}$-plane in Y represented by a pair (y, Λ) where λ is defined by a linear equation $c_0 x_0 + \cdots + c_4 x_4 = 0$, the equation $p_{12} - p_{03} = 0$ induces the equation $a_1 c_2 - a_2 c_1 - a_0 c_3 + a_3 c_0 = 0$. In summary, the equation for $F_2^{3,1}(Y)$ in this local chart is equivalent to the matrix equation :

$$\mathrm{rank}\begin{pmatrix} a_3 & -a_2 & a_1 & -a_0 & 0 \\ c_0 & c_1 & c_2 & c_3 & c_4 \end{pmatrix} = \mathrm{rank}\begin{pmatrix} a_3 & -a_2 & a_1 & -a_0 \\ c_0 & c_1 & c_2 & c_3 \end{pmatrix} = 1.$$

This clearly implies that $F_2^{3,1}(Y) \cong \mathrm{bl}_0 \mathbb{C}^4$. By the same argument as in the proof of Lemma 4.3.2, we can show that the moduli space $F_2(Y)$ of projective planes is reduced and thus we complete the proof. $\square$

4.3.2 Hilbert scheme of conics $\mathsf{H}_2(\mathsf{Y}^5)$ in Y^5

By using the geometry of projective lines and planes in Y^5, we construct birational morphisms connecting $\mathsf{H}_2(\mathsf{Y}^5)$ and its projective models in a similar manner as in the diagram (4.1). The technically important point in our argument is the description of blow-up space under clean intersection condition. [18, Definition-Proposition 3.4].

We denote $\mathcal{U} \hookrightarrow \mathcal{O}^{\oplus 5}$ the tautological sub-bundle on the Grassmannian $\mathrm{Gr}(4,5)$. We define

$$\mathcal{K} := \ker\{\wedge^2\mathcal{U} \hookrightarrow \wedge^2\mathcal{O}^{\oplus 5} \to \mathcal{O}\} \tag{4.4}$$

the bundle $\mathcal{K}$ as the kernel of the composition of the above sequence where the second map in the above sequence is induced from the equation $p_{12}-p_{03}$ (cf. [62, Proposition B.6.1]). We can check that $\mathcal{K}$ is locally free by direct rank computation of the composition map. We define $\mathsf{S}(\mathsf{Y}) := \mathrm{Gr}(3,\mathcal{K})$ and then we have $\mathsf{S}(\mathsf{Y}) \subset \mathsf{S}(\mathsf{G}) = \mathrm{Gr}(3,\wedge^2\mathcal{U})$ by definition.

Next, we recall that $\mathsf{T}(\mathsf{G}) = \mathrm{OG}(3,\mathcal{U}) \subset \mathsf{S}(\mathsf{G})$ in Proposition 4.2.2, is isomorphic to the disjoint union $\mathsf{F}(1,4,5) \sqcup \mathsf{F}(3,4,5)$ of the two flag varieties. Then we define $\mathsf{T}^{3,1}(\mathsf{G}) := \mathsf{F}(1,4,5)$ and $\mathsf{T}^{2,2}(\mathsf{G}) := \mathsf{F}(3,4,5)$. We observe that the space $\mathsf{S}(\mathsf{G})$ can be written by the following incidence variety :

$$\mathsf{S}(\mathsf{G}) = \{(\mathsf{U},\mathsf{V}_4) \,|\, \mathsf{U} \subset \wedge^2\mathsf{V}_4\} \subset \mathrm{Gr}(3,\wedge^2\mathbb{C}^5) \times \mathrm{Gr}(4,\mathbb{C}^5).$$

Then we have the natural embedding $\mathsf{T}^{3,1}(\mathsf{G}) \sqcup \mathsf{T}^{2,2}(\mathsf{G}) \hookrightarrow \mathsf{S}(\mathsf{G})$ constructed in the following way.

(1) For a pair $(\mathsf{V}_1,\mathsf{V}_4) \in \mathsf{T}^{3,1}(\mathsf{G})$ (V_1 is a 1-dimensional vector space rep-

resenting a vertex point of $\sigma_{3,1}$-plane),

$$(V_1, V_4) \mapsto (W, V_4)$$

where $W = \ker(\wedge^2 V_4 \twoheadrightarrow \wedge^2(V_4/V_1))(= V_1 \wedge V_4)$ is the 3-dimensional vector space. In this case, (V_1, V_4) determines a $\sigma_{3,1}$-type plane.

(2) For a pair $(V_3, V_4) \in T^{2,2}(G)$,

$$(V_3, V_4) \mapsto (\wedge^2 V_3, V_4).$$

In this case, V_3 determines a $\sigma_{2,2}$-type plane.

Above embedding $T(G) \hookrightarrow S(G)$ induces an isomorphism $F(1,4,5) \sqcup F(3,4,5) =: T(G)^{3,1} \sqcup T^{2,2}(G) \xrightarrow{\cong} \mathrm{OG}(3, \mathcal{U}) = T(G)$. From now on, we identify the blow-up locus $T(G)$ with $T(G)^{3,1} \sqcup T^{2,2}(G)$ via this isomorphism. We define the intersection $T(Y) := S(Y) \cap T(G)$ in $S(G)$.

Proposition 4.3.4. When we define $T^{3,1}(Y) := T^{3,1}(G) \cap T(Y)$ and $T^{2,2}(Y) := T^{2,2}(G) \cap T(Y)$, then $T(Y)$ is the disjoint union of irreducible connected components $T^{3,1}(Y) \sqcup T^{2,2}(Y)$ such that

1. $T^{3,1}(Y) \cong F^{3,1}(Y)$ and
2. $T^{2,2}(Y)$ is isomorphic to a fiber bundle on the smooth quadric threefold $\Sigma(= F^{2,2}(Y))$ with fibers isomorphic to $\mathbb{P}^1$.

Proof. The first part just comes from the definition. The second part obtained by some direct calculation via the following composition map

$$T^{2,2}(Y) \overset{\iota}{\hookrightarrow} F(3,4,5) \xrightarrow{p} \mathrm{Gr}(3,5).$$

We can show that the image $(p \circ \iota)(T^{2,2}(Y)) = \Sigma$ which is, in fact, equal to the smooth quadric threefold $\mathrm{Gr}(2, V_4^0) \cap \overline{H}$ appeared in Proposition 4.3.3

by some direct calculation. Here we let $V_4^0 = \mathrm{span}\langle e_0, e_1, e_2, e_3\rangle$ and $\overline{H} = \mathrm{zero}(p_{12} - p_{03})$ is the restriction of H in $\mathbb{P}(\wedge^2 V_4^0)$. We can prove this by direct computation for each affine chart. For example, consider an element $(V_3, V_4) \in F(3,4,5)$ and $P(V_3) \in \mathrm{Gr}(3,5)$ be the plane represented by the row span of the following matrix :

$$\begin{pmatrix} 1 & 0 & a_2 & a_3 & 0 \\ 0 & 1 & b_2 & b_3 & 0 \\ 0 & 0 & c_2 & c_3 & 1 \end{pmatrix}.$$

Then, by direct calculation, we can check that $(\wedge^2 V_3, V_4) \in \mathcal{K}|_{V_4}$ if and only if it satisfies $c_2 = c_3 = a_2 + b_3 = 0$. We can do the same computation in other affine charts, so we have the conclusion. $\square$

Remark 4.3.5. Part 2 of the above proposition can also be explained in this way. Elements of $T^{2,2}(Y)$ are pairs $(V_3, V_4) \in F(3,4,5)$ such that the $\sigma_{2,2}$-plane determined by V_3 is contained in Y. Therefore, $T^{2,2}(Y)$ is fibered over Σ, whose fiber over $V_3 \in \Sigma$ is $\mathrm{Gr}(1, \mathbb{C}^5/V_3)$. Therefore, it is a $\mathbb{P}^1$-fibration over Σ.

On the other hand, we can check that the natural projection from the intersection part $T(Y) \subset T(G) = F(1,4,5) \sqcup F(3,4,5) \to \mathrm{Gr}(4,5)$ is a fiber bundle on the image of the projection. We will focus on this fiber bundle structure in subsection 4.3.3. Here we introduce the following result we will use now.

Proposition 4.3.6 (Proposition 4.3.11)**.** The intersection part $T(Y) = S(Y) \cap T(G)$ is isomorphic to a $\mathbb{P}^1 \sqcup \mathbb{P}^1$-bundle over $\mathrm{Gr}(3,4)$ which is linearly embedded in the Grassmannian $\mathrm{Gr}(4,5)$.

Lemma 4.3.7. We have :

$$T_{T(Y),P} = T_{S(Y),P} \cap T_{T(G),P}$$

for all $P \in T(Y)$.

Proof. Consider the following exact sequences of tangent bundles :

$$\begin{tikzcd} 0 \arrow[r] & T_{T(Y),P} \arrow[r] \arrow[d, hook] & T_{T(G),P} \arrow[r] \arrow[d, hook] & N_{T(Y)/T(G),P} \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & T_{S(Y),P} \arrow[r] & T_{S(G),P} \arrow[r] & N_{S(Y)/S(G),P} \arrow[r] & 0. \end{tikzcd} \tag{4.5}$$

To prove the lemma, it is enough to show that the horizontal map f naturally induced from the diagram (4.5) is an isomorphism. First, we assume that $P \in T^{3,1}(Y)$. Then, by direct computation, we can observe that there is a following commutative diagram :

$$\begin{tikzcd} N_{T(Y)/T(G),P} \arrow[r, "\cong"] \arrow[d, "f"] & \mathrm{H}^0(\mathcal{O}_{\mathbb{H}}(1)) \arrow[d, equal] \\ N_{S(Y)/S(G),P} \arrow[r, "\cong"] & \mathrm{Hom}(V_3, \mathbb{C}). \end{tikzcd} \tag{4.6}$$

where the plane P is written by $P = (V_1, V_4) \in T^{3,1}(Y) = F^{3,1}(Y)$, $V_3 := \ker(\wedge^2 V_4 \to \wedge^2(V_4/V_1))$, and $\mathbb{H} := \mathbb{P}(V_3)$ is the projective plane in $\mathbb{P}^4 = \mathbb{P}V$. The first horizontal isomorphism in the diagram (4.6) induced from the normal bundle sequence

$$0 \to N_{\mathbb{H}/Y} \to N_{\mathbb{H}/G} \to N_{Y/G}|_{\mathbb{H}} \cong \mathcal{O}_{\mathbb{H}}(1) \to 0,$$

and the fact that $\mathrm{h}^1(N_{\mathbb{H}/Y}) = 0$ by Proposition 4.3.3. The second horizontal isomorphism in the diagram obtained from the following correspondence :

$$\begin{aligned} N_{S(Y)/S(G),P} = N_{\mathrm{Gr}(3,5)/\mathrm{Gr}(3,6),P} &\cong \mathrm{Hom}(V_3, \wedge^2 V_4/V_3)/\mathrm{Hom}(V_3, \mathcal{K}|_{V_4}/V_3) \\ &\cong \mathrm{Hom}(V_3, \mathbb{C}) \end{aligned}$$

which is induced by the equation (4.4). In summary, we obtain the proof of the lemma. $\square$

Remark 4.3.8. We can check the above lemma 4.3.7 in a different way. By [68, Lemma 5.1] and since $\mathsf{T}(\mathsf{G})$, $\mathsf{S}(\mathsf{Y})$ and $\mathsf{T}(\mathsf{Y})$ are smooth, which will be checked later, the clean intersection is equivalent to the scheme-theoretic intersection, $I_{\mathsf{T}(\mathsf{Y})} = I_{\mathsf{S}(\mathsf{Y})} + I_{\mathsf{T}(\mathsf{G})}$. We compute the locus $\mathsf{T}(\mathsf{Y})$ in the following section 4.3.3, and scheme-theoretic intersection in the following sections 4.3.4 by direct local chart computation, which accompanies lots of linear algebra and brute force. So we obtain a new proof of Lemma 4.3.7.

By Lemma 4.3.7 and Fujiki-Nakano criterion [76, Main Theorem], [32], we obtain the following main theorem of this Chapter.

Theorem 4.3.9. Recall the space $\mathsf{H}_2(\mathsf{Y})$, Hilbert scheme of conics in $\mathsf{Y} = \mathsf{Y}^5$. Then $\mathsf{H}_2(\mathsf{Y})$ is a blow-down of $\widetilde{\mathsf{S}}(\mathsf{Y})$, which is a blow-up of $\mathsf{S}(\mathsf{Y}) := \mathrm{Gr}(3, \mathcal{K})$:

$$\begin{tikzcd}[column sep=small] & \widetilde{\mathsf{S}}(\mathsf{Y}) \arrow[dl, "\Xi"'] \arrow[dr, "\Phi"] & \\ \mathsf{S}(\mathsf{Y}) & & \mathsf{H}_2(\mathsf{Y}), \end{tikzcd} \tag{4.7}$$

where Ξ is the blow-up along $\mathsf{T}(\mathsf{Y})$ and Φ is the blow-up along the locus of conics contained in $\sigma_{2,2}$-type planes. Furthermore, $\mathsf{H}_2(\mathsf{Y})$ is irreducible, smooth variety and has dimension 10.

Proof. By Lemma 4.3.7, the blow-up space $\widetilde{\mathsf{S}}(\mathsf{Y})$ is isomorphic to the strict transform of $\mathsf{S}(\mathsf{Y})$ along the blow-up $\Xi : \mathsf{H}_2(\mathrm{Gr}(2, \mathcal{U})) \longrightarrow \mathsf{S}(\mathrm{Gr}(2, 5))$ defined in Proposition 4.2.2 ([68, Lemma 5.1]). Moreover, we can easily show that the restriction of the normal bundle of the exceptional divisor $\mathsf{H}_2(\mathrm{Gr}(2, \mathcal{U}))$ onto the exceptional divisor of $\widetilde{\mathsf{S}}(\mathsf{Y})$ is $\mathcal{O}(-1)$ (cf. [18, Proposition 3.6]). So, we can apply Fujiki-Nakano criterion([76, Main Theorem]), that we conclude that the space obtained by blow-down is smooth. Thus we can conclude that the Hilbert scheme $\mathsf{H}_2(\mathsf{Y})$ is smooth if we can show that $\mathsf{H}_2(\mathsf{Y})$ is reduced and irreducible. Moreover, we can directly check that $H_2(Y)$ is irreducible from the diagram (4.7). Also, we can show that $H_2(Y)$ is reduced in the same manner as we used in the proof of Lemma 4.3.2. Hence we complete the proof. $\square$

4.3.3 Duality between $T^{3,1}(Y^m)$ and $T^{2,2}(Y^m)$ for $m = 4, 5, 6$

It is well-known that $T(G)$ is equal to $\mathrm{OG}(3,6) = \mathbb{P}^3 \sqcup \mathbb{P}^3$-bundle over $\mathrm{Gr}(4,5)$ as we already mentioned in Proposition 4.2.2 and [16, Lemma 3.9], and each $\mathbb{P}^3$ are set of $\sigma_{2,2}$-planes and $\sigma_{3,1}$-planes with an envelope information. Furthermore, we can check that $T(Y^4) = \mathbb{P}^1 \sqcup \mathbb{P}^1$ where each $\mathbb{P}^1$ are set of $\sigma_{2,2}$-planes and $\sigma_{3,1}$-planes with an envelope information, and $T(Y^5)$ is $\mathbb{P}^1 \sqcup \mathbb{P}^1$-bundle over $\mathrm{Gr}(4,5)$ where each $\mathbb{P}^1$ are set of $\sigma_{2,2}$-planes and $\sigma_{3,1}$-planes, by direct local chart computations.

So, it is natural to think about there exist some kind of duality between $\sigma_{2,2}$-planes and $\sigma_{3,1}$ planes. In fact, there is a representation-theoretic duality between $\sigma_{2,2}$ and $\sigma_{3,1}$-planes in $\mathrm{Gr}(2,4) \subset \mathbb{P}^5$ from S. Hosono and H. Takagi's paper.

Proposition 4.3.10. [46, (4.5)] Let W be a 4-dimensional vector space. Then planes in $\mathrm{Gr}(2,4) \subset \mathbb{P}^5$ are elements of $\mathrm{Gr}(3, \wedge^2 W) = \mathrm{Gr}(3,6) \subset \mathbb{P}(\wedge^3(\wedge^2 W)) = \mathbb{P}(S^2W \otimes \det(W) \oplus S^2W^* \otimes \det(W)^{\otimes 2})$.

Then the set of $\sigma_{2,2}$-type planes is identified with $\mathbb{P}(W^*)$ and embeds to $\mathbb{P}(S^2W^*)$ as a Veronese embedding, and the set of $\sigma_{3,1}$-type planes is identified with $\mathbb{P}(W)$ and embeds to $\mathbb{P}(S^2W)$ as a Veronese embedding.

The above proposition express the duality between $T^{3,1}(G)$ and $T^{2,2}(G)$. We can express it more simply. Over a rank 4 subspace $V_4 \in \mathrm{Gr}(4,5)$ of $\mathbb{C}^5$, the fiber $T^{3,1}(G) = F(1,4,5)$ is identified by the pairs (V_1, V_4), where V_1 is a 1-dimensional subspace of V_4, so that the fiber is isomorphic to $\mathbb{P}(V_4)$. On the other hand, the fiber $T^{2,2}(G) = F(3,4,5)$ is identified by the pairs

(V_3, V_4), where V_3 is a 3-dimensional subspace of V_4, hence the fiber is isomorphic to $\mathbb{P}(V_4)^*$. Therefore, there is a duality between a projective space and its dual projective space.

In this section, we explain dualities in $T(Y^5)$ and $T(Y^4)$. In fact, all dualities in this section arise in a similar manner as above, i.e. a duality between a projective space and its dual projective space. We fix a basis $\{e_0, e_1, e_2, e_3, e_4\}$ of $\mathbb{C}^5$.

Proposition 4.3.11 (Duality of $T^{3,1}(Y^5)$ and $T^{2,2}(Y^5)$)**.** $T(Y^5)$ is a $\mathbb{P}^1 \sqcup \mathbb{P}^1$-bundle over $\mathrm{Gr}(3,4)$ linearly embedded in $\mathrm{Gr}(4,5)$, where the linear embedding is given by the 1-1 correspondence between 3-dimensional subspaces in $\mathbb{C}^5/\langle e_4\rangle$ and 4-dimensional subspaces in $\mathbb{C}^5$ containing $\langle e_4\rangle$. Consider the rank 4 skew-symmetric 2 form $\Omega := p_{12} - p_{03}$ on $\mathbb{C}^5$, where p_{ij} are Plücker coordinates on $\wedge^2\mathbb{C}^5$. Then, for a 4-dimensional subspace $V_4 \in \mathrm{Gr}(3,4) \subset \mathrm{Gr}(4,5)$ of $\mathbb{C}^5$ in the sublocus, the restriction $\Omega|_{V_4}$ becomes a rank 2 singular 2-form $\Omega|_{V_4}$ on V_4. Then the fiber of $T^{3,1}(Y^5) \subset F(1,4,5)$ over V_4 canonically identified with $\mathbb{P}(\ker\Omega|_{V_4}) \cong \mathbb{P}^1 \subset \mathbb{P}(\mathbb{C}^5)$ and the fiber of $T^{2,2}(Y^5) \subset F(3,4,5)$ over V_4 canonically identified with $\mathbb{P}((\mathbb{C}^5/\ker\Omega|_{V_4})^*) \cong \mathbb{P}^1 \subset \mathbb{P}((V_4)^*)$.

Proof. Consider an arbitrary 4-dimensional vector space $V_4 \in\subset \mathrm{Gr}(4,5)$. We can observe that rank $\Omega|_{V_4} \geq 2$ Since we have $\operatorname{rank}\Omega = 4$ and $\operatorname{rank}\Omega \leq \operatorname{rank}\Omega|_{V_4} + 2$. If $\operatorname{rank}\Omega|_{V_4} = 4$, then there cannot exist a vector $v \in \mathbb{C}^5$ such that v is orthogonal to V_4 with respect to the 2-form Ω. Hence there does not exist any $\sigma_{3,1}$-plane contained in the fiber of $T(Y^5)$ on V_4. Moreover, there cannot exist a 3-dimensional subspace $V_3 \subset V_4$ of V_4 such that $\Omega|_{V_3} = 0$ since we have $\operatorname{rank}\Omega|_{V_4} \leq \operatorname{rank}\Omega|_{V_3} + 2$. Therefore, there is no $\sigma_{3,1}$-plane in the fiber of $T(Y^5)$ over V_4. In summary, the fiber of $T(Y^5)$ over V_4 is empty whenever $\operatorname{rank}\Omega|_{V_4} = 4$.

Next, consider the case when $\operatorname{rank}\Omega|_{V_4} = 2$. Assume that $V \cap \ker\Omega = V \cap \langle e_4\rangle = \langle 0\rangle$. Then, since $\Omega = p_{12} - p_{03}$ descent to the rank 4 skew-symmetric 2-form $\overline{\Omega}$ on quotient space $V/\langle e_4\rangle$. Since $V_4 \cap \langle 0\rangle = 0$, we can easily observe that the natural isomorphism $\phi : V_4 \xrightarrow{\cong} V/\langle e_4\rangle$ preserves skew-symmetric two forms, i.e. $\phi^*\overline{\Omega} = \Omega|_{V_4}$. Therefore $\operatorname{rank}\Omega|_{V_4} = 4$, which is a contradiction. Thus we have $\langle e_4\rangle \subset \mathbb{C}^5$. Conversely if $\langle e_4\rangle \subset \mathbb{C}^5 = \ker\Omega$, then we have $\operatorname{rank}\Omega = 2$. Therefore $\operatorname{rank}\Omega|_{V_4} = 2$ if and only if $V_4 \in \mathrm{Gr}(3,4) \subset \mathrm{Gr}(4,5)$, where $\mathrm{Gr}(3,4) \subset \mathrm{Gr}(4,5)$ is a linear embedding given by the 1-1 correspondence between 3-dimensional subspaces in $\mathbb{C}^5/\langle e_4\rangle$ and 4-dimensional subspaces in $\mathbb{C}^5$ containing e_4.

Moreover, the fiber of $T^{3,1}(Y^5) \subset F(1,4,5)$ over $V_4 \in \mathrm{Gr}(3,4) \subset \mathrm{Gr}(4,5)$ is represented by pairs (p, V_4) such that $\Omega(p, V_4) = 0$. Therefore, the fiber is canonically identified with $\mathbb{P}(\ker\Omega) \cong \mathbb{P}^1 \subset \mathbb{P}(\mathbb{C}^5)$.

The fiber of $T^{2,2}(Y^5) \subset F(3,4,5)$ over V_4 is represented by pairs (V_3, V_4) such that $V_3 \subset V_4$, $\Omega|_{V_3} = 0$. Assume that $V_3 \cap \ker\Omega|_{V_4} = 1$. Then there is a natural isomorphism $\phi : V_3/(V_3 \cap \ker\Omega|_{V_4}) \xrightarrow{\cong} V_4/\ker\Omega|_{V_4}$. Then, when we denote by $\overline{\Omega}$ the induced 2-form on $V_4/\ker\Omega|_{V_4}$, and $\overline{\Omega'}$ be the induced 2-form on $V_3/(V_3 \cap \ker\Omega|_{V_4})$, we can observe that $\phi^*\overline{\Omega} = \overline{\Omega'}$. But we have $\operatorname{rank}\overline{\Omega'} = 0$ since $\operatorname{rank}\Omega|_{V_3} = 0$ and $\operatorname{rank}\overline{\Omega} = 2$ since $\operatorname{rank}\Omega|_{V_4} = 2$, which leads to the contradiction. Therefore, we have $\ker\Omega|_{V_4} \subset V_3$. Conversely, if $\ker\Omega|_{V_4} \subset V_3$, then it is clear that $\operatorname{rank}\Omega|_{V_3} = 0$. Therefore, the fiber is canonically identifed with $\mathbb{P}((V/\ker\Omega|_{V_4})^*) \cong \mathbb{P}^1 \subset \mathbb{P}((\mathbb{C}^5)^*)$.

□

Proposition 4.3.12 (Duality in $T^{3,1}(Y^4)$ and $T^{2,2}(Y^4)$)**.** $T(Y^4)$ is a double cover over $\mathbb{P}^1 \cong \mathrm{Gr}(1,2) \subset \mathrm{Gr}(4,5)$, with 2 connected components, where $\mathrm{Gr}(1,2) \subset \mathrm{Gr}(4,5)$ is a linear embedding given by 1-1 correspondence between 1-dimensional subspaces in $\mathbb{C}^5/\langle e_0, e_1, e_4\rangle$ and 4-dimensional subspaces in $\mathbb{C}^5$ containing $\langle e_0, e_1, e_4\rangle$. Let $\Omega_1 := p_{12} - p_{03}$, and $\Omega_2 := p_{13} - p_{24}$ be the skew-symmetric 2-forms on $\mathbb{C}^5$.

Then, the fiber of $T(Y^4)$ over $V_4 \in \mathrm{Gr}(1,2)$ is a 2 point set, one point is the fiber of $T^{3,1}(Y^4) \subset F(1,4,5)$ over V_4 defined by a pair $(\ker\Omega_1|_{V_4} \cap \ker\Omega_2|_{V_4}, V_4)$, and the other point is a fiber of $T^{2,2}(Y^4) \subset F(3,4,5)$ over V_4 defined by a pair $(\ker\Omega_1|_{V_4} + \ker\Omega_2|_{V_4}, V_4)$.

Proof. From the proof of the previous proposition, we can obtain that rank Ω_1 and rank $\Omega_2 \geq 2$, and the fiber of $T(Y^4)$ over V_4 is empty if rank $\Omega_1|_{V_4}$ or rank $\Omega_2|_{V_4}$ is 4. Therefore, it enough to consider the case that rank $\Omega_1|_{V_4} =$ rank $\Omega_2|_{V_4} = 2$.

Assume that $\ker\Omega_1|_{V_4} = \ker\Omega_2|_{V_4}$. Since $\langle e_4\rangle \subset \ker\Omega_1|_{V_4}$ and $\langle e_0\rangle \subset \ker\Omega_2|_{V_4}$, we have $\ker\Omega_1|_{V_4} = \ker\Omega_2|_{V_4} = \langle e_0, e_4\rangle$. Then, for an element $ae_1 + be_2 + ce_3 \in V_4$, we have $c = b = 0$ from the relation $\Omega_1|_{V_4} = \Omega_2|_{V_4} = 0$ which contradicts to the fact that V_4 is a 4-dimensional vector space. Therefore $\ker\Omega_1|_{V_4}$ and $\ker\Omega_2|_{V_4}$ cannot be equal.

Next, consider the case when $\ker\Omega_1|_{V_4} \cap \ker\Omega_2|_{V_4} = \langle v\rangle$, i.e. 1-dimensional vector space generated by $v \in \mathbb{C}^5$. If we write $v = a_0e_0 + \cdots + a_4e_4$, then from the condition that $\Omega_1(v, e_0) = \Omega_2(v, e_4) = 0$, we have $b_2 = b_3 = 0$. Therefore we conclude that $\langle e_0, e_1, e_4\rangle \subset V_4$. Conversely, if $\langle e_0, e_1, e_4\rangle \subset V_4$, then we can observe that $\ker\Omega_1|_{V_4} \subset \langle e_0, e_1, e_4\rangle$, $\ker\Omega_2|_{V_4} \subset \langle e_0, e_1, e_4\rangle$ in the same manner. Therefore we have $\ker\Omega_1|_{V_4} \cap \ker\Omega_2|_{V_4}$ is a 1-dimensional vector space. Hence, the locus where $\ker\Omega_1|_{V_4} \cap \ker\Omega_2|_{V_4}$ is 1-dimensional is the image of the linear embedding $\mathrm{Gr}(1,2) \subset \mathrm{Gr}(4,5)$, given by the 1-1 correspondence between 1-dimensional subspaces in $\mathbb{C}^5/\langle e_0, e_1, e_4\rangle$ and 4-dimensional subspaces in $\mathbb{C}^5$ containing $\langle e_0, e_1, e_4\rangle$.

Furthermore, when we consider a 4-dimensional subspace $V_4 \in \mathrm{Gr}(1,2) \subset \mathrm{Gr}(4,5)$ of $\mathbb{C}^5$, the fiber $T^{3,1}(Y^4) \subset F(1,4,5)$ over V_4 is represented by a pair $(\ker\Omega_1|_{V_4} \cap \ker\Omega_2|_{V_4}, V_4)$, and the fiber $T^{2,2}(Y^4) \subset F(3,4,5)$ over V_4 is represented by a pair $(\ker\Omega_1|_{V_4} + \ker\Omega_2|_{V_4}, V_4)$.

It is obvious that the fiber of $T(Y^4)$ is empty over the 4-dimensional subspace V_4 of $\mathbb{C}^5$ where $\ker\Omega_1|_{V_4} \cap \ker\Omega_2|_{V_4} = \langle 0\rangle$. □

We conclude this subsection with the following result about the Fano variety of planes in the hyperplane section of the Grassmannian $\mathrm{Gr}(2,2n) \cap H$. We can show this in a similar manner we proved Proposition 4.3.11. This result will be used when we discuss the birational geometry of $H_d(\mathrm{Gr}(2,2n) \cap H)$ in Chapter 6 using the result of Chung, Hong, and Kiem [18].

Proposition 4.3.13 (Fano variety of planes in $\mathrm{Gr}(2,2n)\cap H$). Fano variety of planes $F_2(\mathrm{Gr}(2,2n)\cap H)$ is smooth.

Proof. Similar to the $\mathrm{Gr}(2,5)$ case, $\sigma_{2,2}$-planes in $\mathrm{Gr}(2,2n)$ are parametrized by the Flag variety $F(3,4,2n)$ and $\sigma_{3,1}$-planes in $\mathrm{Gr}(2,2n)$ are parametrized by the Flag variety $F(1,4,2n)$. Then $F_2(\mathrm{Gr}(2,2n)\cap H))$ is a the sublocus of $F(3,4,2n)\sqcup F(1,4,2n)$, we denote it by $T_H^{2,2}\sqcup T_H^{3,1}$. We want to determine a sublocus $Z_H\subset \mathrm{Gr}(4,2n)$ where $T_H^{2,2}\sqcup T_H^{3,1}$ supported on.

Then, when we fix a $(2n-1)$-vector space $V_{2n-1}\subset\mathbb{C}^{2n}$, then by the proof of Proposition 4.3.11, we can easily observe that for any 4-dimensional vector space $V_4\subset V_{2n-1}$, $V_4\in Z_H$ if and only if Z_H contains the kernel of the skew-symmetric form $\Omega_H|_{V_{2n-1}}$. Since H is a general hyperplane section, H has rank $2n$, so $\Omega_H|_{V_{2n-1}}$ has rank $2n-2$ and the kernel $\ker\Omega_{H|_{V_{2n-1}}}$ is 1-dimensional. So when we consider a Grassmannian $\mathrm{Gr}(2n-1,2n)$ and a rank $(2n-1)$-tautological bundle $\mathcal{U}$, we can have the following fiber diagram :

$$\begin{tikzcd}
\mathrm{Gr}(3,\mathcal{U}/\ker\Omega_{H|_{V_{2n-1}}}) \arrow[r, hook, "\iota"', "\text{linear}"'] \arrow[d, two heads, "q"] \arrow[dr, phantom, "\square"] & \mathrm{Gr}(4,\mathcal{U}) = F(4,2n-1,2n) \arrow[d, "p"] \\
Z_H \arrow[r, hook] & \mathrm{Gr}(4,\mathbb{C}^{2n})
\end{tikzcd}$$

where the upper horizontal arrow is a linear embedding, hence its image is smooth in $\mathrm{Gr}(4,\mathcal{U})$. Since tautological bundle $\mathcal{U}$ has local trivialization, p is a fibration. Therefore Z_H is also smooth. Furthermore, by the proof of Proposition 4.3.11, we can observe that $T_H^{2,2}\sqcup T_H^{3,1}$ is a $\mathbb{P}^1\sqcup\mathbb{P}^1$-bundle over Z_H, hence it is smooth. Moreover, in a similar manner to the proof of Proposition 4.3.4, $T_H^{3,1}$ is isomorphic to the Fano variety of $\sigma_{3,1}$-type planes $F^{3,1}(\mathrm{Gr}(2,2n))\cap H$ and $T_H^{2,2}$ is a locally trivial $\mathbb{P}^1$-fibration over the Fano variety of $\sigma_{2,2}$-type planes $F^{2,2}(\mathrm{Gr}(2,2n)\cap H)$. Therefore $F^{3,1}(\mathrm{Gr}(2,2n))\cap H$ and $F^{2,2}(\mathrm{Gr}(2,2n)\cap H)$ are both smooth. $\square$

4.3.4 Scheme-theoretic intersection of $S(Y)$ and $T(G)$

In this subsection, we compute the scheme-theoretic intersection of $S(Y)$ and $T(G)$, i.e. $I_{T(Y),S(G)} = I_{S(Y),S(G)} + I_{T(G),S(G)}$. By presenting the defining equation of $T(Y)$, the smoothness of $T(Y)$ is proved. By Proposition 4.2.2[16, Lemma 3.9], we know that $T(G)$ is an $\mathrm{OG}(3,6) \cong \mathbb{P}^3 \sqcup \mathbb{P}^3$-bundle over $\mathrm{Gr}(4,5)$, $\sigma_{3,1}$-planes and $\sigma_{2,2}$-planes corresponds to each disjoint $\mathbb{P}^3$. Denote them by $T(G)_{2,2}$ and $T(G)_{3,1}$. Since they are disjoint, we can consider them independently, i.e. it is enough to show that $I_{T(Y)_{2,2},S(G)} = I_{S(Y),S(G)} + I_{T(G)_{2,2},S(G)}$, $I_{T(Y)_{3,1},S(G)} = I_{S(Y),S(G)} + I_{T(G)_{3,1},S(G)}$ where $T(Y)_{2,2} := T(G)_{2,2} \cap S(Y)$,
$T(Y)_{3,1} := T(G)_{3,1} \cap S(Y)$.

We first state the Cauchy-Binet formula here, which is useful for further calculations :

Proposition 4.3.14 (Cauchy-Binet)**.** [53, Example 2.15] Let A be a $n \times m$ matrix and B be a $m \times n$ matrix where $n \leq m$. Then we have the following formula for the determinant of the matrix AB :

$$\det(AB) = \sum_{S \in \binom{[m]}{n}} \det A_{[n],S} \cdot \det B_{S,[n]}$$

where $[m] = 1, 2, ..., m$ is a set and $\binom{[m]}{n}$ is a set of n combinations of elements in $[m]$.

For $n = 2, m = 3$ case, we can check the following corollary by direct calculation :

Corollary 4.3.15. [8, Example 4.9] Let A be a 2×3 matrix and B be a 3×2 matrix. Let $[A]_0, [A]_1$ be a row vector of A and $[B]^0, [B]^1$ be a column vector of B. Then we have :

$$\det AB = ([A]_0 \times [A]_1) \cdot ([B]^0 \times [B]^1)$$

where $'\times'$ is a cross product defined in $\mathbb{C}^3$.

We check $\mathrm{I}_{\mathsf{T}(\mathsf{Y})_{2,2},\mathsf{S}(\mathsf{G})} = \mathrm{I}_{\mathsf{S}(\mathsf{Y}),\mathsf{S}(\mathsf{G})} + \mathrm{I}_{\mathsf{T}(\mathsf{G})_{2,2},\mathsf{S}(\mathsf{G})}$ for affine local charts. Consider a chart for $\mathsf{S}(\mathsf{G})$. Since $\mathsf{S}(\mathsf{G})$ is $\mathrm{Gr}(3,6)$-bundle over $\mathrm{Gr}(4,5)$, we should consider chart for $\Lambda \in \mathrm{Gr}(4,5)$ and $\mathsf{F} \in \mathrm{Gr}(3,6) = \mathrm{Gr}(3, \wedge^2\Lambda)$. There are 5 standard charts for $\Lambda \in \mathrm{Gr}(4,5)$:

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 & a \\ 0 & 1 & 0 & 0 & b \\ 0 & 0 & 1 & 0 & c \\ 0 & 0 & 0 & 1 & d \end{pmatrix}, \Lambda = \begin{pmatrix} 1 & 0 & 0 & a & 0 \\ 0 & 1 & 0 & b & 0 \\ 0 & 0 & 1 & c & 0 \\ 0 & 0 & 0 & d & 1 \end{pmatrix}, \Lambda = \begin{pmatrix} 1 & 0 & a & 0 & 0 \\ 0 & 1 & b & 0 & 0 \\ 0 & 0 & c & 1 & 0 \\ 0 & 0 & d & 0 & 1 \end{pmatrix},$$

$$\Lambda = \begin{pmatrix} 1 & a & 0 & 0 & 0 \\ 0 & b & 1 & 0 & 0 \\ 0 & c & 0 & 1 & 0 \\ 0 & d & 0 & 0 & 1 \end{pmatrix} \text{ and } \Lambda = \begin{pmatrix} a & 1 & 0 & 0 & 0 \\ b & 0 & 1 & 0 & 0 \\ c & 0 & 0 & 1 & 0 \\ d & 0 & 0 & 0 & 1 \end{pmatrix}.$$

But in the first chart :

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 & a \\ 0 & 1 & 0 & 0 & b \\ 0 & 0 & 1 & 0 & c \\ 0 & 0 & 0 & 1 & d \end{pmatrix}$$

the equation of $\mathsf{Y}^5 : \mathsf{p}_{12} - \mathsf{p}_{03}$ has no solution. Furthermore, Since the symmetry interchanging the index $1,2$ and $0,3$ does not change the equation $\mathsf{p}_{12} - \mathsf{p}_{03}$, it is enough to consider the following two chart of $\mathrm{Gr}(4,5)$:

$$\Lambda = \begin{pmatrix} 1 & 0 & a & 0 & 0 \\ 0 & 1 & b & 0 & 0 \\ 0 & 0 & c & 1 & 0 \\ 0 & 0 & d & 0 & 1 \end{pmatrix} \text{ and } \Lambda = \begin{pmatrix} 1 & 0 & 0 & a & 0 \\ 0 & 1 & 0 & b & 0 \\ 0 & 0 & 1 & c & 0 \\ 0 & 0 & 0 & d & 1 \end{pmatrix}.$$

Let us start with the first chart :

$$\Lambda = \begin{pmatrix} 1 & 0 & a & 0 & 0 \\ 0 & 1 & b & 0 & 0 \\ 0 & 0 & c & 1 & 0 \\ 0 & 0 & d & 0 & 1 \end{pmatrix}.$$

Let $q_{01}, ..., q_{23}$ be a coordinate of a fiber of $\wedge^2 \mathcal{U}$ over this chart, where $\mathcal{U}$ is a tautological rank 4 bundle over Gr$(4,5)$. Then we have $p_{12} - p_{03} = -aq_{01} - q_{02} + cq_{12} + dq_{13}$. By Proposition 4.3.11, $T^{2,2}(Y)$ is a fibration over Gr$(3,4)$ linearly embedded in Gr$(4,5)$, whose images are $\Lambda \in$ Gr$(4,5)$ such that $e_4 \in \Lambda$. Therefore, we have equation $d = 0$ in $I_{T(Y)_{2,2}}$.

Next, $\sigma_{2,2}$-plane corresponds to $\mathbb{P}^2$-plane in $\mathbb{P}\Lambda \cong \mathbb{P}^3 \subset \mathbb{P}^4$ must one be of the following form(i.e. it correspond to the row space of the matrix $R \cdot \Lambda$) :

$$R = \begin{pmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & \gamma \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & \beta & 0 \\ 0 & 0 & \gamma & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & \alpha & 0 & 0 \\ 0 & \beta & 1 & 0 \\ 0 & \gamma & 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} \alpha & 1 & 0 & 0 \\ \beta & 0 & 1 & 0 \\ \gamma & 0 & 0 & 1 \end{pmatrix}.$$

Therefore, the intersection of $S(Y)$ and $T(G)$ arises only in the following three charts for fibers $F \in$ Gr$(3, \wedge^2 \Lambda)$:

$$F = \begin{array}{c} \begin{matrix} 01 & 02 & 03 & 12 & 13 & 23 \end{matrix} \\ \begin{pmatrix} 1 & 0 & e & 0 & f & g \\ 0 & 1 & h & 0 & i & j \\ 0 & 0 & k & 1 & l & m \end{pmatrix} \end{array}, F = \begin{array}{c} \begin{matrix} 01 & 02 & 03 & 12 & 13 & 23 \end{matrix} \\ \begin{pmatrix} 1 & e & 0 & f & 0 & g \\ 0 & h & 1 & i & 0 & j \\ 0 & k & 0 & l & 1 & m \end{pmatrix} \end{array},$$

$$F = \begin{array}{c} \begin{matrix} 01 & 02 & 03 & 12 & 13 & 23 \end{matrix} \\ \begin{pmatrix} e & 1 & 0 & f & g & 0 \\ h & 0 & 1 & i & j & 0 \\ k & 0 & 0 & l & m & 1 \end{pmatrix} \end{array} \text{and } F = \begin{array}{c} \begin{matrix} 01 & 02 & 03 & 12 & 13 & 23 \end{matrix} \\ \begin{pmatrix} e & f & g & 1 & 0 & 0 \\ h & i & j & 0 & 1 & 0 \\ k & l & m & 0 & 0 & 1 \end{pmatrix} \end{array}$$

where the upper indices are indices of Plücker coordinates. Let us start with the first chart :

$$F = \begin{pmatrix} 1 & 0 & e & 0 & f & g \\ 0 & 1 & f & 0 & i & j \\ 0 & 0 & g & 1 & l & m \end{pmatrix},$$

In this case, we can easily observe that a $\sigma_{2,2}$-plane contained in this chart must correspond to the row space of a matrix of the form :

$$R\Lambda = \begin{pmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & \gamma \end{pmatrix} \cdot \Lambda.$$

For a matrix M, we let M_i^j be a matrix obtained from M by deleting i-th row and j-th column. From the equation $p_{12} - p_{03} = 0$, and since $d = 0$ for $\sigma_{2,2}$-planes in $T(Y)_{2,2}$, using corollary 4.3.15, we can observe that the equation for $T(Y)_{2,2}$ in this chart is $([R^4]_i \times [R^4]_j) \cdot ([\Lambda_4^5]^1 \times [\Lambda_4^5]^2 - [\Lambda_4^5]^0 \times [\Lambda_4^5]^3) = ([R^4]_i \times [R^4]_j) \cdot (c, 1, -a) = 0$ for all $0 \le i < j \le 2$. Then, since $([R^4]_0 \times [R^4]_2) = (0, -1, 0)$, we have no solution. Therefore, the intersection of $T(G)$ and $S(Y)$ does not happens in this chart.

Next, we consider the second chart :

$$F = \begin{pmatrix} 1 & e & 0 & f & 0 & g \\ 0 & f & 1 & i & 0 & j \\ 0 & g & 0 & l & 1 & m \end{pmatrix}$$

then we can easily observe that a $\sigma_{2,2}$-plane contained in this chart must correspond to the row space of a matrix of the form :

$$R\Lambda = \begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & \beta & 0 \\ 0 & 0 & \gamma & 1 \end{pmatrix} \cdot \Lambda.$$

In the same manner we can show that $([R^4]_i \times [R^4]_j) \cdot ([\Lambda_4^5]^1 \times [\Lambda_4^5]^2 - [\Lambda_4^5]^0 \times [\Lambda_4^5]^3) = ([R^4]_i \times [R^4]_j) \cdot (c, 1, -a) = 0$ for all $0 \leq i < j \leq 2$ is the equation for $T(Y)_{2,2}$ in this chart under the condition $d = 0$. By direct calculations, we have $\gamma = 0, \alpha c + \beta + a = 0$.

We observe that this $\sigma_{2,2}$-plane which correspond to the row space of the matrix $R\Lambda$ correspond to the following matrix form in the chart of F :

$$\begin{pmatrix} 1 & \beta & 0 & -\alpha & 0 & 0 \\ 0 & \gamma & 1 & 0 & 0 & \alpha \\ 0 & 0 & 0 & \gamma & 1 & \beta \end{pmatrix}$$

In summary, we obtain the full description of the equation of $T(Y)_{2,2}$ in the chart :

$$I_{T(Y)_{2,2}} = \langle g, i, k, f + j, e - m, h, l, d, -fc + e + a \rangle.$$

On the other hand, from the equation $-aq_{01} - q_{02} + cq_{12} + dq_{13}$, we have :

$$I_{S(Y)} = \langle -a - e + cf, -h + ci, -k + cl + d \rangle.$$

And clearly the equation for $T(G)_{2,2}$ is given by :

$$I_{T(G)_{2,2}} = \langle g, i, k, f + j, e - m, h - l \rangle.$$

Therefore, we can check the following clean intersection by direct calculation :

$$I_{T(G)_{2,2}} + I_{S(Y)} = I_{T(Y)_{2,2}}.$$

Next, we consider the third chart :

$$F = \begin{pmatrix} e & 1 & 0 & f & g & 0 \\ h & 0 & 1 & i & j & 0 \\ k & 0 & 0 & l & m & 1 \end{pmatrix}.$$

Then we can easily observe that a $\sigma_{2,2}$-plane contained in this chart must correspond to the row space of a matrix of the form :

$$R\Lambda = \begin{pmatrix} 1 & \alpha & 0 & 0 \\ 0 & \beta & 1 & 0 \\ 0 & \gamma & 0 & 1 \end{pmatrix} \cdot \Lambda.$$

Then in the same manner, we can calculate $I_{T(Y)_{2,2}}$, $I_{S(Y)}$ and $I_{T(G)_{2,2}}$ by direct calculation :

$$I_{T(Y)_{2,2}} = \langle g, i, k, f-j, e-m, h, l, d, fc-1-ea \rangle$$
$$I_{S(Y)} = \langle -ae+cf+dg-1, -ah+ci+dj, -ak+dl+dm \rangle$$
$$I_{T(G)_{2,2}} = \langle g, i, k, f-j, e-m, h+l \rangle$$

Therefore we can check the clean intersection $I_{T(Y)_{2,2}} = I_{S(Y)} + I_{T(G)_{2,2}}$ by direct calculation.

At last, we consider the fourth chart :

$$F = \begin{pmatrix} e & f & g & 1 & 0 & 0 \\ h & i & j & 0 & 1 & 0 \\ k & l & m & 0 & 0 & 1 \end{pmatrix}.$$

Then we can easily observe that a $\sigma_{2,2}$-plane contained in this chart must correspond to the row space of a matrix of the form :

$$R\Lambda = \begin{pmatrix} \alpha & 1 & 0 & 0 \\ \beta & 0 & 1 & 0 \\ \gamma & 0 & 0 & 1 \end{pmatrix} \cdot \Lambda.$$

Then in the same manner, we can calculate $I_{T(Y)_{2,2}}$, $I_{S(Y)}$ and $I_{T(G)_{2,2}}$ by direct calculation :

$$I_{T(Y)_{2,2}} = \langle g, i, k, f-j, e+m, h, l, d, c-f-ae\rangle$$
$$I_{S(Y)} = \langle -ae+c-f, -ah+d-i, -ak-l\rangle$$
$$I_{T(G)_{2,2}} = \langle g, i, k, f-j, e+m, h-l\rangle$$

Therefore we can check the clean intersection $I_{T(Y)_{2,2}} = I_{S(Y)} + I_{T(G)_{2,2}}$ by direct calculation.

In summary, we checked the clean intersection $I_{S(Y)} + I_{T(G)_{2,2}}$ for the chart

$$\Lambda = \begin{pmatrix} 1 & 0 & a & 0 & 0 \\ 0 & 1 & b & 0 & 0 \\ 0 & 0 & c & 1 & 0 \\ 0 & 0 & d & 0 & 1 \end{pmatrix} \in \mathrm{Gr}(4,5).$$

and all charts for $F \in \mathrm{Gr}(3, \wedge^2\Lambda)$.

We can also check the clean intersection for the second chart :

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & a & 0 \\ 0 & 1 & 0 & b & 0 \\ 0 & 0 & 1 & c & 0 \\ 0 & 0 & 0 & d & 1 \end{pmatrix}.$$

But the computation proceeds exactly in the same manner as the case of first chart so we do not write it down here.

Next, we can also check clean intersection at $T(Y)_{3,1}$. We should check $I_{T(Y)_{3,1},S(G)} = I_{S(Y),S(G)} + I_{T(G)_{3,1},S(G)}$.

We first consider an open chart for $S(G)$. Same as in the case of $T(Y)_{2,2}$, it is enough to consider 2 chart for Λ :

$$\Lambda = \begin{pmatrix} 1 & 0 & a & 0 & 0 \\ 0 & 1 & b & 0 & 0 \\ 0 & 0 & c & 1 & 0 \\ 0 & 0 & d & 0 & 1 \end{pmatrix} \text{ and } \Lambda = \begin{pmatrix} 1 & 0 & 0 & a & 0 \\ 0 & 1 & 0 & b & 0 \\ 0 & 0 & 1 & c & 0 \\ 0 & 0 & 0 & d & 1 \end{pmatrix}.$$

Let us start with the first chart :

$$\Lambda = \begin{pmatrix} 1 & 0 & a & 0 & 0 \\ 0 & 1 & b & 0 & 0 \\ 0 & 0 & c & 1 & 0 \\ 0 & 0 & d & 0 & 1 \end{pmatrix}.$$

Let $q_{01}, ..., q_{23}$ be a coordinate of a fiber of $\wedge^2\mathcal{U}$ over this chart. Then we have $p_{12} - p_{03} = -aq_{01} - q_{02} + cq_{12} + dq_{13}$.

Next, by Proposition 4.3.11, $T^{3,1}(Y)$ is a fibration over $\mathrm{Gr}(3,4)$ linearly embedded in $\mathrm{Gr}(4,5)$, whose images are $\Lambda \in \mathrm{Gr}(4,5)$ such that $e_4 \in \Lambda$. Therefore, we have equation $d = 0$ in $\mathrm{I}_{T(Y)_{3,1}}$. Furthermore, by Proposition 4.3.11, a pair $(x, \Lambda) \in T^{3,1}(G)$ over Λ contained in $T^{3,1}(Y)$ if and only if the vertex x must be contained in the projectivized kernel of the 2-form $(-ap_{01} + cp_{12} + dp_{13} - p_{02})$, which is equal to $\mathbb{P}^1 = \mathbb{P}\langle(c, 1, -a, 0), (0, 0, 0, 1)\rangle$.

Therefore, we should consider two types of the vertex x :

$$x = (c, 1, -a, s) \text{ and } x = (sc, s, -sa, 1)$$

where $s \in k$.

Let us start with the first vertex type :

$$x = (c, 1, -a, s).$$

Then, the corresponding $\sigma_{3,1}$-plane is spanned by $(c, 1, -a, s) \wedge (1, 0, 0, 0)$,

$(c, 1, -a, s) \wedge (0, 0, 1, 0), (c, 1, -a, s) \wedge (0, 0, 0, 1)$. So we can rewrite it by a following 3×6-matrix :

$$\begin{pmatrix} 1 & -a & s & 0 & 0 & 0 \\ 0 & c & 0 & 1 & 0 & -s \\ 0 & 0 & c & 0 & 1 & -a \end{pmatrix}.$$

Thus, intersection of $S(Y)$ and $T(G)_{3,1}$ only occurs in the following chart of F :

$$F = \begin{pmatrix} 1 & e & f & 0 & 0 & g \\ 0 & h & i & 1 & 0 & j \\ 0 & k & l & 0 & 1 & m \end{pmatrix}.$$

Therefore, we have $I_{T(Y)_{3,1}} = \langle f + j, e - m, e + a, h - l, c - h, g, i, k, d \rangle$.

On the other hand, $\sigma_{3,1}$-plane contained in this chart of F is defined by the vertex of the form :

$$x = (\alpha, 1, \beta, \gamma)$$

which correspond to the following 3×6-matrix :

$$\begin{pmatrix} 1 & \beta & \gamma & 0 & 0 & 0 \\ 0 & \alpha & 0 & 1 & 0 & -\gamma \\ 0 & 0 & \alpha & 0 & 1 & \beta \end{pmatrix}.$$

Thus, we have $I_{T(G)_{3,1}} = \langle f + j, e - m, h - l, g, i, k \rangle$.

Furthermore, from the equation $-aq_{01} - q_{02} + cq_{12} + dq_{13}$, we obtain the equation for $S(Y)$, i.e. $I_{S(Y)} = \langle -a - e, c - h, d - k \rangle$. Finally, we can check the clean intersection $I_{T(Y)_{3,1}} = I_{T(G)_{3,1}} + I_{S(Y)}$ by direct calculation.

Next, we consider the second vertex type :

$$x = (sc, s, -sa, 1)$$

Then, the corresponding $\sigma_{3,1}$-plane is spanned by $(sc, s, -sa, 1) \wedge (1, 0, 0, 0)$,

$(sc, s, -sa, 1) \wedge (0, 1, 0, 0), (sc, s, -sa, 1) \wedge (0, 0, 1, 0)$. So we can rewrite it by a following 3×6-matrix :

$$\begin{pmatrix} s & -sa & 1 & 0 & 0 & 0 \\ -sc & 0 & 0 & -sa & 1 & 0 \\ 0 & -sc & 0 & -s & 0 & 1 \end{pmatrix}.$$

Thus, intersection of $S(Y)$ and $T(G)_{3,1}$ only occurs in the following chart of F :

$$F = \begin{pmatrix} e & f & 1 & g & 0 & 0 \\ h & i & 0 & j & 1 & 0 \\ k & l & 0 & m & 0 & 1 \end{pmatrix}.$$

Therefore, we have $I_{T(Y)_{3,1}} = \langle f - j, h - l, e + m, g, i, k, f + ea, l - cm, d \rangle$.

On the other hand, $\sigma_{3,1}$-plane contained in this chart of F is defined by the vertex of the form :

$$x = (\alpha, \beta, \gamma, 1)$$

which correspond to the following 3×6-matrix :

$$\begin{pmatrix} \beta & \gamma & 1 & 0 & 0 & 0 \\ -\alpha & 0 & 0 & \gamma & 1 & 0 \\ 0 & -\alpha & 0 & -\beta & 0 & 1 \end{pmatrix}.$$

Thus, we have $I_{T(G)_{3,1}} = \langle f - j, h - l, e + m, g, i, k \rangle$. Furthermore, from the equation $-aq_{01} - q_{02} + cq_{12} + dq_{13}$, we obtain the equation for $S(Y)$, i.e. $I_{S(Y)} = \langle -ae - f + eg, -ah - i + cj + d, -ak - l + cm \rangle$. Finally, we can check the clean intersection $I_{T(Y)_{3,1}} = I_{T(G)_{3,1}} + I_{S(Y)}$ by direct calculation.

In summary, we checked the clean intersection $I_{S(Y)} + I_{T(G)_{3,1}} = I_{T(Y)_{3,1}}$ for the chart

$$\Lambda = \begin{pmatrix} 1 & 0 & a & 0 & 0 \\ 0 & 1 & b & 0 & 0 \\ 0 & 0 & c & 1 & 0 \\ 0 & 0 & d & 0 & 1 \end{pmatrix}.$$

We can also check the clean intersection for the second chart :

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & a & 0 \\ 0 & 1 & 0 & b & 0 \\ 0 & 0 & 1 & c & 0 \\ 0 & 0 & 0 & d & 1 \end{pmatrix}.$$

But the it proceeds exactly in the same manner as the case of first chart so we do not write it down here.

In summary, we checked the clean intersection of $S(Y)$ and $T(G)$ in $S(G)$ by direct calculation.

Proposition 4.3.16. For Fano 5-fold $Y := Y^5 = \mathrm{Gr}(2,5) \cap H$, $S(Y)$ and $T(G)$ cleanly intersect in $S(G)$, i.e. we have :

$$I_{T(Y)} = I_{T(G)} + I_{S(Y)}.$$

4.4 Fano 4-fold Y^4

In this section, we denote by $Y = Y^4$ the smooth Fano 4-fold defined by the intersection of the image of the Grassmannian $\mathrm{Gr}(2,5)$ under the Plücker embedding into $\mathbb{P}(\wedge^2\mathbb{C}^5) = \mathbb{P}^9$ with two general hyperplanes H_1, H_2. We denote p_{ij} the Plücker coordinates. For explicit computations, we may assume that :

$$H_1 = \{p_{12} - p_{03} = 0\}, \quad H_2 = \{p_{13} - p_{24} = 0\}.$$

We use the same strategy as the case of Fano 5-fold Y^5 to show the smoothness of Hilbert compactification $H_2(Y^4)$. So we should first study Fano variety of projecctive planes in the Fano 4-fold $Y = Y^4$.

The results on projective planes and lines in the Fano 4-fold Y are due to Todd [97]. We also introduce elementary proofs for the convenience of the reader. On the other hand, our result on the Hilbert scheme of conics seems to be new.

4.4.1 Fano varieties of lines and planes in Y^4

First, we introduce the results for the Fano variety of projective lines and planes in the Fano 4-fold Y. The results introduced in this section are due to Todd [97].

Lemma 4.4.1. [97] There is a unique $\sigma_{2,2}$-plane in the Fano 4-fold Y. In other words, there is a unique projective plane $\Pi \subset \mathbb{P}^4$ such that every line $\ell \subset \Pi$, which are considered as elements of $\mathrm{Gr}(2,5)$, is contained in Y.

Proof. Consider the following affine open chart

$$\begin{pmatrix} 1 & 0 & a_2 & a_3 & a_4 \\ 0 & 1 & b_2 & b_3 & b_4 \end{pmatrix}$$

of the Grassmannian $\mathrm{Gr}(2,5)$. In this chart we have $p_{12} - p_{03} = -a_2 - b_3$ and $p_{13} - p_{24} = -a_3 - a_2 b_4 + a_4 b_2$. Therefore, finding the plane Π is equivalent to finding pair of linearly independent linear equations in variables $x_0, \cdots, x_4$ such that both $(1, 0, a_2, a_3, a_4)$ and $(0, 1, b_2, b_3, b_4)$ satisfy the two equations. By direct calculation, we can check that the unique pair of linear equations satisfying the above condition is $(x_2 = 0, x_3 = 0)$. By doing same chart calculations for other affine open charts, we conclude that $\{x_2 = x_3 = 0\} \subset \mathbb{P}^4$ determines the unique $\sigma_{2,2}$-plane Π. $\square$

Remark 4.4.2. The plane $\Pi \subset \mathbb{P}^4$ in Lemma 4.4.1 plays a crucial role in the structure of the Fano 4-fold Y^4 ([87, Section 3], [27, Section 3] and [33]).

We recall that a $\sigma_{3,1}$-plane is a set of projective lines in a 3-dimensional linear space $\mathbb{P}^3 \subset \mathbb{P}^4$ which pass through a fixed point p. We call p the *vertex* of the $\sigma_{3,1}$-plane.

Lemma 4.4.3. [97] There is a 1-dimensional family of $\sigma_{3,1}$-planes in the Fano 4-fold Y whose vertices lies on a smooth conic C_0 in the plane $\Pi \subset \mathbb{P}^4$. Other $\sigma_{3,1}$-planes in Y does not exist.

Proof. Consider a $\sigma_{3,1}$-plane with a vertex $(1, a_1, a_2, a_3, a_4)$. Then a point in the plane is represented by the following matrix :

$$\begin{pmatrix} 1 & a_1 & a_2 & a_3 & a_4 \\ 0 & b_1 & b_2 & b_3 & b_4 \end{pmatrix}.$$

We have $p_{12} - p_{03} = a_1b_2 - a_2b_1 - b_3$ and $p_{13} - p_{24} = a_1b_3 - a_3b_1 - a_2b_4 + a_4b_2$, and these two equations are linear in (b_1, b_2, b_3, b_4). These two linear equations in (b_1, b_2, b_3, b_4) are linearly dependent if and only if

$$\operatorname{rank} \begin{pmatrix} a_2 & -a_1 & 1 & 0 \\ -a_3 & a_4 & a_1 & -a_2 \end{pmatrix} = 1.$$

This condition hold if and only if $a_2 = a_3 = 0$ and $a_1^2 + a_4 = 0$. The first equation implies that vertices of $\sigma_{3,1}$-planes in Y contained in the plane $\Pi = \{x_2 = x_3 = 0\} \subset \mathbb{P}^4$ in Lemma 4.4.1 and the second equation says that the vertices of $\sigma_{3,1}$-planes in Y lies on the smooth conic $C_0 := \{x_1^2 + x^4x_0 = 0\}$ in Π. Through the similar computations for all other local charts, we complete the proof. □

Corollary 4.4.4. The Fano variety of projective planes in the Fano 4-fold Y is isomorphic to the smooth conic $C_0 \sqcup \{\Pi\}$.

Proposition 4.4.5. [97] Let $H_1(Y) = F_1(Y)$ be the Hilbert scheme(or the Fano variety) of lines in the Fano 4-fold $Y = Y^4$. Then the Hilbert scheme

$H_1(Y)$ is isomorphic to the blow-up space of $\mathbb{P}^4$ at the smooth conic $C_0 \subset \Pi$ which is defined in Lemma 4.4.3.

Proof. We recall that an arbitrary line $\mathbb{L}$ in the Grassmannian $G = \mathrm{Gr}(2,5)$ is a set of lines in $\mathbb{P}^4$ contained in a projective plane $P \subset \mathbb{P}^4$ which pass through a fixed point $p \in P$. The point p is said to be the vertex of the line $\mathbb{L}$. Assigning each line $\mathbb{L}$ to its vertex p gives the following morphism :

$$\psi : H_1(Y) \subset H_1(G) = \mathrm{Gr}(1,3,5) \longrightarrow \mathrm{Gr}(1,5) = \mathbb{P}^4.$$

By the proof of Lemma 4.4.3, for a point $p \notin C_0$, the Schubert variety $\sigma_{3,1}(p)$ with Y along a line. If $p \in C_0$, the Schubert variety $\sigma_{3,1}(p)$ intersects with Y along the $\sigma_{3,1}$-plane in Y. Thus we conclude that $\psi^{-1}(p)$ is a single point for a point $p \notin C_0$ and $\psi^{-1}(p)$ is a projective plane $\mathbb{P}^2$ for a point $p \in C_0$.

By local chart computation similar as in the proof of Proposition 4.3.1, we can to show that the map ψ is the blow-up map along the smooth conic C_0. Also, using the same argument as in Lemma 4.3.2, we can check that the Hilbert scheme(or the Fano variety) $H_1(Y)$ is reduced. So we complete the proof. $\square$

Proposition 4.4.6. We denote $C_0^\vee \subset H_1(Y)$ the dual conic which is the set of projective tangent lines of C_0 in the plane $\Pi \subset \mathbb{P}^4$.

Let $\mathbb{L} \in H_1(Y)$ be an arbitrary projective line in the Fano 4-fold Y. Then the normal bundle $N_{\mathbb{L}/Y}$ of $\mathbb{L}$ in Y is isomorphic to $\mathcal{O}_{\mathbb{L}}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{L}}(1)$ if $\mathbb{L} \notin C_0^\vee$ and $N_{\mathbb{L}/Y}$ is isomorphic to $\mathcal{O}_{\mathbb{L}}(-1) \oplus \mathcal{O}_{\mathbb{L}}(1)^{\oplus 2}$ if $\mathbb{L} \in C_0^\vee$.

Proof. Consider a line $\mathbb{L}$ the dual projective space $\Pi^\vee \subset Y$ with a vertex $p = (1, a_1, a_2, a_3, a_4)$. Then the point in the Schubert variety $\sigma_{3,1}(p)$ is represented by the following matrix :

$$\begin{pmatrix} 1 & a_1 & a_2 & a_3 & a_4 \\ 0 & x_1 & x_2 & x_3 & x_4 \end{pmatrix}.$$

Since $x_2 = x_3 = 0$ is the equation for Π, a_2, a_3, b_2, b_3 are coordinates of the fiber of the normal bundle $N_{\Pi^\vee/G}|_{\mathbb{L}}$. Since a_2, a_3 has homogeneous degree 0 and x_2, x_3 has homogeneous degree 1, we have $N_{\Pi^\vee/G}|_{\mathbb{L}} \cong \mathcal{O}_{\mathbb{L}}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{L}}(1)^{\oplus 2}$. Moreover, the two equations $p_{12} - p_{03}$ and $p_{13} - p_{24}$ give us homomorphisms

$$\begin{aligned} &\mathcal{O}_{\mathbb{L}}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{L}}(1)^{\oplus 2} \to \mathcal{O}(1)^{\oplus 2}, \\ &\quad (a_2, a_3, x_2, x_3) \mapsto (a_1 b_2 - a_2 x_1 - x_3, a_1 x_3 - a_3 x_1 - a_2 x_4 + a_4 x_2). \end{aligned}$$

If $a_4 + a_1^2 \neq 0$, then we can observe that the kernel of this homomorphism is $\mathcal{O}_{\mathbb{L}}^{\oplus 2}$. If $a_4 + a_1^2 = 0$, then the kernel is $\mathcal{O}_{\mathbb{L}}(-1) \oplus \mathcal{O}_{\mathbb{L}}(1)$. The equation $a_4 + a_1^2 = 0$ is exactly same to the equation for the smooth conic C_0. Because the normal bundle $N_{\mathbb{L}/\Pi^\vee}$ is isomorphic to $\mathcal{O}_{\mathbb{L}}(1)$, the normal bundle sequence $0 \to N_{\mathbb{L}/\Pi^\vee} \to N_{\mathbb{L}/Y} \to N_{\Pi^\vee/Y}|_{\mathbb{L}} \to 0$ splits, i.e. $N_{\mathbb{L}/Y} \cong \mathcal{O}_{\mathbb{L}}(1) \oplus N_{\Pi^\vee/Y}|_{\mathbb{L}}$. We can do same computations for other open charts. So we obtain the proof. $\square$

4.4.2 Hilbert scheme of conics $H_2(Y^4)$ in Y^4

In this section, we construct birational morphisms which connects $H_2(Y^4)$ and its projective models in a similar manner as in the diagram (4.1). The next theorem is an analogue of Theorem 4.3.9 in the case of Fano 4-fold Y^4.

Theorem 4.4.7. We denote $H_2(Y)$ the Hilbert scheme of conics in the Fano 4-fold $Y = Y^4$. Let $\mathcal{U}$ be the tautological sub-bundle on the Grassmannian Gr$(4, 5)$. We define

$$\mathcal{K} := \ker\{\wedge^2 \mathcal{U} \hookrightarrow \wedge^2 \mathcal{O}^{\oplus 5} \to \mathcal{O}^{\oplus 2}\}$$

as the kernel of the composition of the above sequence where the second arrow in the above diagram is induced from the equations $p_{12} - p_{03}$ and $p_{13} - p_{24}$. Then $H_2(Y)$ is a blow-down of $\widetilde{S}(Y)$, which is a blow-up of the relative Grassmannian bundle $\mathsf{S}(\mathsf{Y}) := \mathrm{Gr}(3, \mathcal{K})$:

$$\begin{tikzcd} & \widetilde{\mathsf{S}}(\mathsf{Y}) \arrow[dl, "\Xi"'] \arrow[dr, "\Phi"] & \\ \mathsf{S}(\mathsf{Y}) & & \mathsf{H}_2(\mathsf{Y}). \end{tikzcd} \tag{4.8}$$

where Ξ is the blow-up along $\mathsf{T}(\mathsf{Y}) := \mathsf{T}(\mathsf{G}) \cap \mathsf{S}(\mathsf{Y})$ and Φ is the blow-up along the locus consists of conics contained in $\sigma_{2,2}$-type planes. Furthermore, $\mathsf{H}_2(\mathsf{Y})$ is a smooth, irreducible variety with dimension 7.

Proof. We can fill in the proof in a similar manner as in Proposition 4.3.9 so we omit here. □

4.4.3 Scheme-theoretic intersection of $\mathsf{S}(\mathsf{Y})$ and $\mathsf{T}(\mathsf{G})$

We show scheme-theoretic intersection of $\mathsf{S}(\mathsf{Y})$ and $\mathsf{T}(\mathsf{G})$ here, so we can show clean intersection by [68, Lemma 5.1], so we have another proof of the clean intersection in the case of Fano 4-fold Y^4.

First we consider charts for $\mathsf{S}(\mathsf{G})$. Since $\mathsf{S}(\mathsf{G})$ is $\mathrm{Gr}(3, 6)$-bundle over $\mathrm{Gr}(4, 5)$, we should consider chart for $\Lambda \in \mathrm{Gr}(4, 5)$ and $\mathsf{F} \in \mathrm{Gr}(3, 6) = \mathrm{Gr}(3, \wedge^2\Lambda)$.

There are 5 standard charts for $\Lambda \in \mathrm{Gr}(4, 5)$:

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 & a \\ 0 & 1 & 0 & 0 & b \\ 0 & 0 & 1 & 0 & c \\ 0 & 0 & 0 & 1 & d \end{pmatrix}, \Lambda = \begin{pmatrix} 1 & 0 & 0 & a & 0 \\ 0 & 1 & 0 & b & 0 \\ 0 & 0 & 1 & c & 0 \\ 0 & 0 & 0 & d & 1 \end{pmatrix}, \Lambda = \begin{pmatrix} 1 & 0 & a & 0 & 0 \\ 0 & 1 & b & 0 & 0 \\ 0 & 0 & c & 1 & 0 \\ 0 & 0 & d & 0 & 1 \end{pmatrix},$$

$$\Lambda = \begin{pmatrix} 1 & a & 0 & 0 & 0 \\ 0 & b & 1 & 0 & 0 \\ 0 & c & 0 & 1 & 0 \\ 0 & d & 0 & 0 & 1 \end{pmatrix} \text{ and } \Lambda = \begin{pmatrix} a & 1 & 0 & 0 & 0 \\ b & 0 & 1 & 0 & 0 \\ c & 0 & 0 & 1 & 0 \\ d & 0 & 0 & 0 & 1 \end{pmatrix}.$$

By Proposition 4.3.12 we know that $T(Y)$ is the double cover over the linear embedding $\mathbb{P}^1 \cong \mathrm{Gr}(1, \mathbb{C}^4/\langle e_0, e_1, e_4\rangle) \subset \mathrm{Gr}(4,5)$. Therefore, The intersection between $T(G)$ and $S(Y)$ only occurs in the following two charts :

$$\Lambda = \begin{pmatrix} 1 & 0 & a & 0 & 0 \\ 0 & 1 & b & 0 & 0 \\ 0 & 0 & c & 1 & 0 \\ 0 & 0 & d & 0 & 1 \end{pmatrix} \text{ and } \Lambda = \begin{pmatrix} 1 & 0 & 0 & a & 0 \\ 0 & 1 & 0 & b & 0 \\ 0 & 0 & 1 & c & 0 \\ 0 & 0 & 0 & d & 1 \end{pmatrix}.$$

and $a = b = d = 0$ contained in the equations of $T(Y)$ in both cases, i.e. $a, b, d \in I_{T(Y)}$. Since $T(Y) = T(Y)_{2,2} \coprod T(Y)_{3,1}$, we can consider each part independently. We consider the clean intersection at $T(Y)_{3,1}$ first.

First, consider the first chart :

$$\Lambda = \begin{pmatrix} 1 & 0 & a & 0 & 0 \\ 0 & 1 & b & 0 & 0 \\ 0 & 0 & c & 1 & 0 \\ 0 & 0 & d & 0 & 1 \end{pmatrix}.$$

Let $q_{01}, \ldots, q_{23}$ be a coordinate of a fiber of $\wedge^2 \mathcal{U}$ over this chart. Then we have $p_{12} - p_{03} = -aq_{01} - q_{02} + cq_{12} + dq_{13}$ and $p_{13} - p_{24} = -aq_{03} + q_{12} - bq_{13} - cq_{23}$.

Each $\sigma_{3,1}$-plane in $T(Y)_{3,1}$ which correspond to the vertex $x \in \mathbb{P}\Lambda \subset \mathbb{C}^5$ such that $(-q_{02} + cq_{12})(x, y) = 0, (q_{12} - cq_{23})(x, y) = 0$(here, we consider q_{ij} as a skew-symmetric two form) for all $y \in \Lambda$, because we have $a = b = d = 0$ in $T(Y)_{3,1}$. Then, by direct calculation, we can check that the sigma $\sigma_{3,1}$-plane correspond to the vertex x contained in $T(Y)_{3,1}$ if and only if it satisfies the equations :

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 0 & -ay_1 & -y_2 & 0 \\ ay_0 & 0 & cy_2 & dy_3 \\ y_0 & -cy_1 & 0 & 0 \\ 0 & -dy_1 & 0 & 0 \end{pmatrix} = 0$$

$$\text{and } \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -ay_3 \\ 0 & 0 & y_2 & -by_3 \\ 0 & -y_1 & 0 & -cy_3 \\ ay_0 & by_1 & cy_2 & 0 \end{pmatrix} = 0.$$

for all $y = (y_0, y_1, y_2, y_3) \in \Lambda$. Thus, we conclude that $x = [-c^2 : -c : 0 : 1] \in \mathbb{P}\Lambda$. Then, the corresponding $\sigma_{3,1}$-plane is spanned by $(-c^2, -c, 0, 1) \wedge (1, 0, 0, 0), (-c^2, -c, 0, 1) \wedge (0, 1, 0, 0), (-c^2, -c, 0, 1) \wedge (0, 0, 1, 0)$. So we can rewrite it by a following 3×6-matrix :

$$\begin{pmatrix} -c & 0 & 1 & 0 & 0 & 0 \\ c^2 & 0 & 0 & 0 & 1 & 0 \\ 0 & c^2 & 0 & c & 0 & 1 \end{pmatrix}.$$

Thus, intersection of $S(Y)$ and $T(G)_{3,1}$ only occurs in the following chart of F :

$$F = \begin{pmatrix} e & f & 1 & g & 0 & 0 \\ h & i & 0 & j & 1 & 0 \\ k & l & 0 & m & 0 & 1 \end{pmatrix}.$$

In this chart, we can compute the ideal of $T(Y)_{3,1}$:

$$T(Y)_{3,1} = \langle a, b, d, g, i, k, f, j, e + m, h - l, h - c^2, e + c \rangle$$

On the other hand, $\sigma_{3,1}$-plane contained in this chart of F is defined by the vertex of the form :

$$x = (\alpha, \beta, \gamma, 1)$$

which correspond to the following 3×6-matrix :

$$\begin{pmatrix} \beta & \gamma & 1 & 0 & 0 & 0 \\ -\alpha & 0 & 0 & \gamma & 1 & 0 \\ 0 & -\alpha & 0 & -\beta & 0 & 1 \end{pmatrix}.$$

Thus, we have $I_{T(G)_{3,1}} = \langle f - j, e + m, h - l, g, i, k \rangle$.

On the other hand, from the equations $-aq_{01} - q_{02} + cq_{12} + dq_{13}$ and $-aq_{03} + q_{12} - bq_{13} - cq_{23}$, we obtain ideal for $S(Y)$:

$$I_{S(Y)} = \langle -ae - f + cg, -ah - i + cj + d, -ak - l + cm, -a + g, j - b, m - c \rangle$$

Thus, we can check the clean intersection $I_{T(Y)_{3,1}} = I_{S(Y)} + I_{T(G)_{3,1}}$ in the first chart of Λ by direct calculation.

Next, consider the second chart :

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & a & 0 \\ 0 & 1 & 0 & b & 0 \\ 0 & 0 & 1 & c & 0 \\ 0 & 0 & 0 & d & 1 \end{pmatrix}.$$

Let $q_{01}, ..., q_{23}$ be a coordinate of a fiber of $\wedge^2 \mathcal{U}$ over this chart. Then we have $p_{12} - p_{03} = -bq_{01} - cq_{02} - dq_{03} + q_{12}$ and $p_{13} - p_{24} = -aq_{01} + cq_{12} + dq_{13} - q_{23}$.

Then, in the same manner as in the first chart case, we can show that $\sigma_{3,1}$-plane in $T(Y)_{3,1}$ correspond to the vertex $x = [1 : c : 0 : -c^2] \in \mathbb{P}\Lambda$. The corresponding $\sigma_{3,1}$-plane is spanned by $(-c^2, -c, 0, 1) \wedge (0, 1, 0, 0), (-c^2, -c, 0, 1) \wedge (0, 0, 1, 0), (-c^2, -c, 0, 1) \wedge (0, 0, 0, 1)$. So we can rewrite it by a following

3×6-matrix :

$$\begin{pmatrix} 1 & 0 & 0 & 0 & c^2 & 0 \\ 0 & 1 & 0 & c & 0 & c^2 \\ 0 & 0 & 1 & 0 & c & 0 \end{pmatrix}.$$

Thus, the intersection of $S(Y)$ and $T(G)_{3,1}$ only occurs in the following chart of $F \in Gr(3, \wedge^2\Lambda)$:

$$F = \begin{pmatrix} 1 & 0 & 0 & e & f & g \\ 0 & 1 & 0 & h & i & j \\ 0 & 0 & 1 & k & l & m \end{pmatrix}.$$

In this chart, we can compute the ideal of $T(Y)_{3,1}$:

$$T(Y)_{3,1} = \langle a, b, d, g, i, k, e, m, h - l, f - j, h - c, f - c^2 \rangle$$

On the other hand, $\sigma_{3,1}$-plane contained in this chart of F is defined by the vertex of the form :

$$x = (1, \alpha, \beta, \gamma)$$

which correspond to the following 3×6-matrix :

$$\begin{pmatrix} 1 & 0 & 0 & -\beta & -\gamma & 0 \\ 0 & 1 & 0 & \alpha & 0 & -\gamma \\ 0 & 0 & 1 & 0 & \alpha & \beta \end{pmatrix}.$$

Thus, we have $I_{T(G)_{3,1}} = \langle g, i, k, f - j, h - l, e + m \rangle$.

On the other hand, from the equations $-bq_{01} - cq_{02} - dq_{03} + q_{12}$ and $-aq_{01} + cq_{12} + dq_{13} - q_{23}$, we obtain ideal for $S(Y)$:

$$I_{S(Y)} = \langle -b + e, -c + h, -d + k, -a + ce + df - g, ch + di - j, ck + dl - m \rangle$$

So, we can check the clean intersection $I_{T(Y)_{3,1}} = I_{S(Y)} + I_{T(G)_{3,1}}$ in the second chart of Λ by direct calculation. In summary, we checked clean intersection at $T(Y)_{2,2}$

Next, we check clean intersection at $T(Y)_{2,2}$. Let us start with the first chart for Λ :

$$\Lambda = \begin{pmatrix} 1 & 0 & a & 0 & 0 \\ 0 & 1 & b & 0 & 0 \\ 0 & 0 & c & 1 & 0 \\ 0 & 0 & d & 0 & 1 \end{pmatrix}$$

Next, $\sigma_{2,2}$-plane corresponds to $\mathbb{P}^2$-plane in $\mathbb{P}\Lambda \cong \mathbb{P}^3 \subset \mathbb{P}^4$ must one be of the following form(i.e. it correspond to the row space of the matrix $R \cdot \Lambda$) :

$$R = \begin{pmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & \gamma \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & \beta & 0 \\ 0 & 0 & \gamma & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & \alpha & 0 & 0 \\ 0 & \beta & 1 & 0 \\ 0 & \gamma & 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} \alpha & 1 & 0 & 0 \\ \beta & 0 & 1 & 0 \\ \gamma & 0 & 0 & 1 \end{pmatrix}.$$

Therefore, intersection of $S(Y)$ and $T(G)_{2,2}$ arises only in the following four charts of F:

$$F = \begin{array}{c} \begin{array}{cccccc} 01 & 02 & 03 & 12 & 13 & 23 \end{array} \\ \begin{pmatrix} 1 & 0 & e & 0 & f & g \\ 0 & 1 & h & 0 & i & j \\ 0 & 0 & k & 1 & l & m \end{pmatrix} \end{array}, F = \begin{array}{c} \begin{array}{cccccc} 01 & 02 & 03 & 12 & 13 & 23 \end{array} \\ \begin{pmatrix} 1 & e & 0 & f & 0 & g \\ 0 & h & 1 & i & 0 & j \\ 0 & k & 0 & l & 1 & m \end{pmatrix} \end{array},$$

$$F = \begin{array}{c} \begin{array}{cccccc} 01 & 02 & 03 & 12 & 13 & 23 \end{array} \\ \begin{pmatrix} e & 1 & 0 & f & g & 0 \\ h & 0 & 1 & i & j & 0 \\ k & 0 & 0 & l & m & 1 \end{pmatrix} \end{array} \text{and } F = \begin{array}{c} \begin{array}{cccccc} 01 & 02 & 03 & 12 & 13 & 23 \end{array} \\ \begin{pmatrix} e & f & g & 1 & 0 & 0 \\ h & i & j & 0 & 1 & 0 \\ k & l & m & 0 & 0 & 1 \end{pmatrix} \end{array}$$

where the upper indices are indices of Plücker coordinates. Let us start with the first chart :

$$F = \begin{pmatrix} 1 & 0 & e & 0 & f & g \\ 0 & 1 & f & 0 & i & j \\ 0 & 0 & g & 1 & l & m \end{pmatrix},$$

In this case, we can easily observe that $\sigma_{2,2}$-plane contained in the intersection of $T(Y)_{2,2}$ and this chart must correspond to the row space of the matrix :

$$R\Lambda = \begin{pmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & \gamma \end{pmatrix} \cdot \Lambda.$$

We observe that this $\sigma_{2,2}$-plane which correspond to the row space of the matrix $R\Lambda$ correspond to the following matrix form in the chart of F :

$$\begin{pmatrix} 1 & 0 & \beta & 0 & -\alpha & 0 \\ 0 & 1 & \gamma & 0 & 0 & -\alpha \\ 0 & 0 & 0 & 1 & \gamma & -\beta \end{pmatrix}$$

But, in this case, the equations $-aq_{01} - q_{02} + cq_{12} + dq_{13}$ and $-aq_{03} + q_{12} - bq_{13} - cq_{23}$ does not have solutions since we have $a = b = d = 0$ on $T(Y)_{2,2}$. Therefore, we can show that intersection of $S(Y)$ and $T(G)_{2,2}$ does not happens in the chart for F :

$$F = \begin{pmatrix} 1 & 0 & e & 0 & f & g \\ 0 & 1 & h & 0 & i & j \\ 0 & 0 & k & l & l & m \end{pmatrix}.$$

In the similar manner, we can also show that no intersection of $S(Y)$ and $T(G)_{2,2}$ does not happens in the chart for F :

$$F = \begin{pmatrix} e & f & g & 1 & 0 & 0 \\ h & i & j & 0 & 1 & 0 \\ k & l & m & 0 & 0 & 1 \end{pmatrix}.$$

Therefore, it is enough to consider only two chart for F. Let us start with the following chart for F :

$$F = \begin{pmatrix} 1 & e & 0 & f & 0 & g \\ 0 & h & 1 & i & 0 & j \\ 0 & k & 0 & l & 1 & m \end{pmatrix}.$$

In this case, we can easily observe that a $\sigma_{2,2}$-plane contained in this chart must correspond to the row space of a matrix of the form :

$$R\Lambda = \begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & \beta & 0 \\ 0 & 0 & \gamma & 1 \end{pmatrix} \cdot \Lambda.$$

Then, by the equation $-aq_{01} - q_{02} + cq_{12} + dq_{13}$ and $-aq_{03} + q_{12} - bq_{13} - cq_{23}$, we can observe that this $\sigma_{2,2}$-plane contained in $T(Y)_{2,2}$ if and only if it satisfies the following matrix equations :

$$\begin{aligned} &- a[R]^0 \times [R]^1 + c[R]^1 \times [R]^2 + d[R]^1 \times [R]^3 - [R]^0 \times [R]^2 = 0 \text{ and} \\ &[R]^1 \times [R]^2 - a[R]^0 \times [R]^3 - b[R]^1 \times [R]^3 - c[R]^2 \times [R]^3 = 0 \end{aligned}$$

Since we already have $a = b = d = 0$ satisfied in $T(Y)_{2,2}$, by Proposition 4.3.12, the above equations reduce to :

$$\begin{aligned} &c[R]^1 \times [R]^2 - [R]^0 \times [R]^2 = 0 \text{ and} \\ &[R]^1 \times [R]^2 - c[R]^2 \times [R]^3 = 0 \end{aligned}$$

Therefore, we have :

$$\begin{aligned} c(-\gamma, 0, \alpha) - (0, 0, 1) = 0 \\ (-\gamma, 0, \alpha) - c(1, 0, 0) = 0 \end{aligned}$$

Thus, there is no solution for these equations. So intersection of $T(G)_{2,2}$ and

$S(Y)$ does not occur in this chart of F.

So, in summary, we checked the clean intersection $I_{T(Y)_{2,2}} = I_{S(Y)} + I_{T(G)_{2,2}}$ in the first chart of Λ and all chart of F. We can also check the clean intersection in the second chart for Λ :

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & a & 0 \\ 0 & 1 & 0 & b & 0 \\ 0 & 0 & 1 & c & 0 \\ 0 & 0 & 0 & d & 1 \end{pmatrix}.$$

in the same manner, as we used in the case of the first chart of Λ. But since all process is parallel, we do not write it down here. In summary, we obtain the following result.

Proposition 4.4.8. For Fano 4-fold $Y := Y^4 = \mathrm{Gr}(2,5) \cap H_1 \cap H_2$, $S(Y)$ and $T(G)$ cleanly intersect in $S(G)$, i.e.

$$I_{T(Y)} = I_{T(G)} + I_{S(Y)}.$$

4.5 Fano threefold Y^3

We can also apply arguments in previous sections on the case of Fano threefold Y^3. Applying similar methods as in the previous sections, we reprove well-known results on the moduli space of projective lines and conics. For concrete local chart computations, we let

$$H_1 = \{p_{12} - p_{03} = 0\}, \quad H_2 = \{p_{13} - p_{24} = 0\}, \quad H_3 = \{p_{14} - p_{02} = 0\}.$$

Proposition 4.5.1. [30, Lemma 3.3] The Hilbert scheme of lines $F_1(Y)$ in the Fano 3-fold Y is isomorphic to $\mathbb{P}^2$.

Proof. Consider the projection $F_1(Y) \subset \mathrm{Gr}(1,3,5) \to \mathrm{Gr}(1,5)$. Then this map assigns each line to its vertex. Then for a vertex $p = [a_0 : a_1 : a_2 : a_3 :$

$a_4]$, a line in the Gr$(2,5)$ whose vertex is p lies in the Fano 3-fold Y if and only if

$$\mathrm{rank}\begin{pmatrix} a_3 & 0 & a_2 \\ a_2 & a_3 & a_4 \\ -a_1 & -a_4 & -a_0 \\ -a_0 & -a_1 & 0 \\ 0 & a_2 & -a_1 \end{pmatrix} \leq 2.$$

The rank condition determines the ideal of the image of the projection map Im$(F_1(Y))$ in Gr$(1,5) = \mathbb{P}^4$, which is is given by

$$\langle a_1a_2a_3+a_0a_3^2-a_2^2a_4+a_3a_4^2,\, a_2^3-a_1a_3^2-a_2a_3a_4,\, a_1a_2^2+a_0a_2a_3-a_1a_3a_4,\, a_0a_2^2+a_1^2a_3,$$
$$a_1^2a_2+a_0a_1a_3+a_0a_2a_4,\, a_0a_1a_2+a_0^2a_3+a_1^2a_4+a_0a_4^2,\, a_1^3-a_0^2a_2+a_0a_1a_4\rangle.$$

It is well-known that the zero set of the above ideal is isomorphic to $\mathbb{P}^2$, which is a projection of the Veronese surface ([84, Theorem 1.1]). We note that Im$(F_1(Y)) \cong \mathbb{P}^2$ contains the smooth conic C_0 appeared in Proposition 4.4.5. Thus $F_1(Y) = \mathrm{Bl}_{C_0}\mathrm{Im}(F_1(Y)) \cong \mathbb{P}^2$. □

The following theorem is an analogue of Theorem 4.3.9 in the case of Fano threefold Y^3.

Theorem 4.5.2. [30, Lemma 3.3] The Hilbert scheme $H_2(Y)$ of conics in the Fano 3-fold Y is isomorphic to the Grassmannian Gr$(4,5) \cong \mathbb{P}^4$.

Proof. In a similar manner as in the proof of Lemma 4.4.1, we can easily check that there is no plain contained in Y. Moreover, we define

$$\mathcal{K} := \ker\{\wedge^2\mathcal{U} \subset \wedge^2\mathcal{O}^{\oplus 5} \to \mathcal{O}^{\oplus 3}\}$$

as the kernel of the above composition map where $\mathcal{U}$ is the tautological bundle on Gr$(4,5)$ and the second arrow is induced from the three linear equations $(p_{12}-p_{03}=0, p_{13}-p_{24}=0, p_{14}-p_{02}=0)$ of the Fano 3-fold Y. Then we can easily show that the rank of $\mathcal{K}$ is 3 by direct calculation so that $\mathcal{K}$ is a vector bundle. Hence we obtain isomorphisms $\mathsf{H}_2(\mathsf{Y})_{\mathrm{red}} \cong \mathsf{S}(\mathsf{Y}) \cong \mathrm{Gr}(4, 5)$. In a same manner as in the proof of Lemma 4.3.2, we can also prove $\mathsf{H}_2(\mathsf{Y})_{\mathrm{red}} = \mathsf{H}_2(\mathsf{Y})$. □

Chapter 5

Compactifications of the moduli spaces of degree 3 smooth rational curves in $\mathcal{N}$

The results presented in this chapter are based on the results obtained joint with Chung in [20].

We studied that there are two irreducible component $R_3(0)$ and $R_3(1)$ of $R_3(\mathcal{N})$ in Chapter 3, Proposition 3.1.3. In this chapter, we study their Kontsevich compactification. For the definition of the stable map space and Kontsevich compactification, see Chapter 4, Section 4.1.

But since $R_3(0)$ is a fiber bundle over $\mathrm{Pic}^0(X)$, whose fiber over a line bundle $L \in \mathrm{Pic}^0(X)$ is an open subscheme of the degree 3 map space $\mathrm{Hom}_3(\mathbb{P}^1, \mathbb{P}\mathrm{Ext}^1(L, L^{-1}(-x)))$, Kontsevich compactification of this space is already well-known by Kiem-Moon [56]. So we concentrate on the Kontsevich compactification of the component $R_3(1)$ here. Let $\mathbf{M}_0(\mathcal{N}, d)$ be the stable map space of genus zero, degree d stable maps with no marked points. Here, the degree of the map is defined via the very ample divisor Θ on $\mathcal{N}$. We denote $\overline{R_3(1)} \subset \mathbf{M}_0(\mathcal{N}, 3)$ by $\Lambda_1 := \overline{R_3(1)}$.

5.1 Notations

In this chapter, let us fix some notations as follows.

- X: a smooth projective curve with genus $g \geq 4$ over $\mathbb{C}$.

- x: a fixed point of X.

- $\mathcal{N}$: Moduli space of rank 2 stable vector bundles on the smooth projective curve X with a fixed determinant line bundle $\mathcal{O}_X(-x)$.

- $V_L^d := \mathrm{Ext}^1(L, L^{-1}(-x))$ where d is a dimension of the vector space, $(V_L^d)^s$ is the sublocus of $\mathrm{Ext}^1(L, L^{-1}(-x))$ parametrizing extensions which have stable rank 2 vector bundles in their middle terms. Therefore, by Riemann-Roch formula, $\mathrm{Ext}^1(L, L^{-1}(-x)) = V_L^g$ if $L \in \mathrm{Pic}^0(X)$ and $\mathrm{Ext}^1(L, L^{-1}(-x)) = V_L^{g+2}$ if $L \in \mathrm{Pic}^1(X)$.

- $\mathbb{P}_L^{g-1} := \mathbb{P}V_L^g$ for a line bundle $L \in \mathrm{Pic}^0(X)$ and $(\mathbb{P}_L^{g+1})^s := \mathbb{P}(V_L^{g+2})^s$, where $(V_L^{g+2})^s$ is the sublocus of $\mathrm{Ext}^1(L, L^{-1}(-x))$ parametrizing extensions which have stable rank 2 vector bundles in their middle terms.

  We sometimes abbreviate $\mathbb{P}_L^{g-1}$ by $\mathbb{P}^{g-1}$ if there is no confusion on the choice of the line bundle L. Also, we sometimes abbreviate $\mathbb{P}_L^{g+1}$ by (respectively, $(\mathbb{P}_L^{g+1})^s$) by $\mathbb{P}^{g+1}$ (respectively, $(\mathbb{P}^{g+1})^s$) if there is confusion on the choice of the line bundle $L \in \mathrm{Pic}^1(X)$. Moreover, also sometimes abbreviate stable locus of $\mathbb{P}V_L^{g+2} := (\mathbb{P}V_L^{g+2})^s$ by $(\mathbb{P}^{g+1})^s$, when there is no confusion on the choice of the line bundle $L \in \mathrm{Pic}^1(X)$.

5.2 Review of the resolution of unstable locus $(\mathbb{P}_L^{g+1})^{us}$

To understand the compactification Λ_1 of the component $R_3(1)$, whose elements are lines, i.e. degree one map $f : \mathbb{P}^1 \to (\mathbb{P}_L^{g+1})^s$, we should understand what happens that these lines get close to the unstable locus, $(\mathbb{P}_L^{g+1})^{us}$, because boundary elements of Λ_1 arise in this kind of limits. So what we should do first is to determine the unstable locus $(\mathbb{P}_L^{g+1})^{us}$. In fact, Castravet already studied about this unstable locus in [13, Section 2.2], [12, Section 2.1]. The following result directly follows from the simple observation of the proof of [13, Lemma 2.1].

Proposition 5.2.1. [13, Proof of Lemma 2.1] For a degree 1 line bundle $L \in \mathrm{Pic}^1(X)$, the unstable locus $\mathbb{P}(\mathrm{Ext}^1(L, L^{-1}(-x)))^{us} = (\mathbb{P}_L^{g+1})^{us}$ is isomorphic to the image of the following morphism, induced by the complete linear system :

$$i = |L^2(x) \otimes K_X| : X \hookrightarrow \mathbb{P}_L^{g+1}$$

where K_X is a canonical line bundle of the curve X.

We note that $L^2(x) \otimes K_X$ is very ample, therefore i is a closed embedding, so we can identify the unstable locus with the smooth projective curve X. In the upcoming contents, we will reinterpret the unstable locus using elementary modification we introduced in Chapter 3, Definition 3.1.1, which will give us some geometric intuition about the rational map $\Psi_L : \mathbb{P}_L^{g+1} \to \mathcal{N}$.

5.2.1 Some remarks about the rational map $\Psi_L : \mathbb{P}_L^{g+1} \dashrightarrow \mathcal{N}$

Let us recall the definition of the elementary modification in Chapter 3. Recall the sequence 3.1, which gives the elementary modification

$$0 \longrightarrow E^{\nu_p} \longrightarrow E \xrightarrow{\nu_p} \mathbb{C}_p \longrightarrow 0.$$

Then let us assume that $E = \xi \oplus \xi'$ decomposes to line bundles ξ and ξ' on the curve X. If $[\nu_p] \in \mathbb{C}^* = \mathbb{P}^1 \setminus \{[1:0],[0:1]\}$, we can easily check the elementary modifications E^{ν_p} are isomorphic to each other. Thus, we can introduce the following definition.

Definition 5.2.1. We define the rank 2 vector bundle on the curve X as follows :

$$(\xi \oplus \xi')^p := (\xi \oplus \xi')^{\nu_p}$$

for any $\nu_p \in \mathbb{C}^* = \mathbb{P}^1 \setminus \{[1 : 0], [0 : 1]\}$. This vector bundle is well-defined since elementary modifications $(\xi \oplus \xi')^{\nu_p}$ are all isomorphic to each other for different choice of $\nu_p \in \mathbb{C}^* = \mathbb{P}^1 \setminus \{[1 : 0], [0 : 1]\}$.

Furthermore, when $L \in \mathrm{Pic}^1(X)$, we can easily observe that there is a short exact sequence :

$$0 \to L^{-1}(-x) \to (L \oplus L^{-1}(p - x))^p \to L \to 0$$

and $(L \oplus L^{-1}(p - x))^p$ is a non-split vector bundle.

From the above definition, it is natural to consider a morphism from the curve X to a $\mathbb{P}\mathrm{Ext}^1(L, L^{-1}(-x))$. But it is unclear what this morphism exactly is. The following lemma gives an answer to this question.

Lemma 5.2.2. (cf. [96, (3.4)] and [4, Section 3]) Consider

$$f : X \to \mathbb{P}^{g+1}_L = \mathbb{P}\mathrm{Ext}^1(L, L^{-1}(-x)), \quad p \mapsto (L \oplus L^{-1}(p - x))^p$$

the map defined as the elementary modification. Then the map f coincide with the map induced from the following complete linear system :

$$i = |L^2(x) \otimes K_X| : X \hookrightarrow \mathbb{P}^{g+1}_L$$

where K_X is the canonical line bundle of X.

Proof. We first note that it was shown in [96, (3.4)] that the map i coincide with the map $g : X \to \mathbb{P}H^1(\Lambda^{-1}) = M_0$ $(\Lambda = L^2(x))$ where M_0 is the moduli space parametrizing pairs of stable bundles on the curve X and their sections. Here, the map g is given by $g : X = \mathbb{P}W \to \mathbb{P}H^1(L^{-2}(-x))$ where W is a line bundle on the curve X and $g(p) = \mathbb{P}H^0(L^{-2}(-x)|_p) \in \mathbb{P}H^1(L^{-2}(-x))$ (See the last paragraph in [96, 329p]). In fact, we get this map by taking the projectivization of the map μ in the short exact sequence in the following

$$0 \to \mathrm{Ext}^1(L|_p, L^{-1}(-x)) \xrightarrow{\mu} \mathrm{Ext}^1(L, L^{-1}(-x)) \xrightarrow{\gamma} \mathrm{Ext}^1(L(-p), L^{-1}(-x)) \to 0. \tag{5.1}$$

Here, we get (5.1) by applying the functor $\mathrm{Hom}(-, L^{-1}(-x))$ to the following exact sequence :

$$0 \to L(-p) \to L \to L|_p \to 0.$$

Since we have $\mathrm{Ext}^1(L|_p, L^{-1}(-x)) = \mathbb{C}$, it is enough to check $\gamma(f(p)) = L(-p) \oplus L^{-1}(-x)$ to prove $g(p) = f(p)$. Therefore, what we have to show is the following :

$$\gamma(f(p)) = (L \oplus (L^{-1}(p-x))^p \oplus_L L(-p) \cong L(-p) \oplus L^{-1}(-x). \tag{5.2}$$

We can easily observe that the left hand side fit to the following pull-back diagram :

$$\begin{tikzcd}
0 \arrow[r] & L^{-1}(-x) \arrow[r] \arrow[d, equal] & (L \oplus L^{-1}(p-x))^p \oplus_L L(-p) \arrow[r] \arrow[d] & L(-p) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & L^{-1}(-x) \arrow[r] & (L \oplus L^{-1}(p-x))^p \arrow[r] & L \arrow[r] & 0.
\end{tikzcd}$$

Using the above pull-back diagram, we can show the isomorphism (5.2) as follows. Since isomorphism is a local property, it is enough to show locally.

Consider any open set $U \subset X$, then we have :

$$\begin{aligned}
&((L \oplus L^{-1}(p-x))^p \oplus_L L(-p))(U) \\
&= \{((s_1, s_2), s_3) | (s_1, s_2) \in ((L \oplus L^{-1}(p-x))^p(U), s_3 \in L(-p)(U), s_1 = s_3\} \\
&\cong \{(s_1, s_2) \in ((L \oplus L^{-1}(p-x))^p(U) | s_1 \in L(-p)\} \\
&\cong \{(s_1, s_2) \in L(U) \oplus L^{-1}(p-x)(U)) | as_1(p) + bs_2(p) = 0, s_1(p) = 0\} \\
&\cong \{(s_1, s_2) \in L(U) \oplus L^{-1}(p-x))(U) | s_1(p) = s_2(p) = 0\} \\
&\cong \{(s_1, s_2) \in (L(-p) \oplus L^{-1}(-x))(U)\}.
\end{aligned}$$

Here, we can observe that the third isomorphism obtained directly follows from the definition of the elementary modification, and the fourth isomorphism follows since we should choose $\nu_p = [a : b] \in \mathbb{C}^*$ such that $ab \neq 0$. $\square$

Remark 5.2.3. $\deg(f(X)) = 2g + 1$.

Proposition 5.2.4. ([96, 4]) For a line bundle $L \in \mathrm{Pic}^1(X)$, consider

$$\Psi_L : \mathbb{P}_L^{g+1} \dashrightarrow \mathcal{N}$$

the rational map induced from the middle term of the universal extension sequence of the projectivized extension group $\mathbb{P}\mathrm{Ext}^1(L, L^{-1}(-x))$. Then we have following properties for the map Ψ_L :

(1) The base locus of the rational map Ψ_L is identified with the curve X (Lemma 5.2.2). By taking the blow-up of $\mathbb{P}_L^{g+1}$ along the base locus X, we obtain a regular morphism

$$\widetilde{\Psi}_L : \mathrm{Bl}_X \mathbb{P}_L^{g+1} (:= \widetilde{\mathbf{P}}_L) \longrightarrow \mathcal{N}.$$

which is an extension of Ψ_L.

(2) The fiber of the exceptional divisor E over a point $p \in X$ of the blow-up morphism π is isomorphic to $\mathbb{P}^{g-1}$, exactly coincides with the degree

0 extension type space $\mathbb{P}^{g-1}_{L(-p)}$. Therefore, each fiber of the exceptional divisor linearly embedded into $\mathcal{N}$ via the map $\widetilde{\Psi}_L$.

(3) If the map Ψ_L is injective and $H^0(L^2(x)) = 0$ for some $L \in \mathrm{Pic}^1(X)$, then the morphism $\widetilde{\Psi}_L$ is a closed embedding for the line bundle L.

$$\begin{tikzcd} \widetilde{\mathbf{P}}_L \arrow[d, "\pi"'] \arrow[dr, "\widetilde{\Psi}_L"] & \\ \mathbb{P}^{g+1}_L \arrow[r, dashed, "\Psi_L"] & \mathcal{N}. \end{tikzcd}$$

Proof. We follow the notations in [96]. So if we let $\Lambda = L^2(x)$ we have the short exact sequence $0 \to \mathcal{O}_X \to E \to L^2(x) \to 0$. Furthermore, we have $\widetilde{\mathbf{P}}_L \cong M_1$ where the space M_1 parametrizes pairs (s, E) such that the bundle E is stable and $s \subset H^0(E)$ is a section of E. ([96]). Part (1) follows from Lemma 5.2.2 and [96, (2.1)]. Part (2) obtained from part (2) of [4, Theorem 1]. In part (3), injectiveness of the map $\widetilde{\Psi}_L$ follows from [96, (3.20)] since we have $H^0(E) = \mathbb{C}$. We note that the map $\widetilde{\Psi}_L$ is in fact equals to the forgetful map $(s, E) \mapsto E$ where $s \subset H^0(E)$. Therefore, the induced tangential map $T\widetilde{\Psi}_{L*} : T_{[(s,E)]}\widetilde{\mathbf{P}}_L \to T_{[E]}\mathcal{N}$ identified with the last morphism in the following exact sequence ([96, (2.1)]) :

$$0 \to \mathrm{Ext}^0(E, E) \to H^0(E) \to T_{[(s,E)]}\widetilde{\mathbf{P}}_L \xrightarrow{T\widetilde{\Psi}_{L*}} T_{[E]}\mathcal{N}.$$

Since we have $\mathrm{Ext}^0(E, E) = \mathbb{C}$ and $H^0(E) = \mathbb{C}$, we deduce that the extended map $\widetilde{\Psi}_L$ is an embedding. □

We note that the conditions in the item (3) of the above proposition are satisfied for the the line bundle L which satisfies the property defined in the following. (cf. Lemma 5.2.11).

Definition 5.2.2. We call a line bundle L *non-trisecant* if $H^0(L^2(x)) = 0$, otherwise, we call L *trisecant*.

Furthermore, we can interpret non-tresecant condition in a geometric way using the following corollary.

Corollary 5.2.5. ([45, Lemma 5.1]) Let $L \in \mathrm{Pic}^1(X)$ be a line bundle. Then we have the followings :

(a) For the curve $X \subset \mathbb{P}_L^{g+1}$ (embedded in $\mathbb{P}_L^{g+1}$ by the linear system $|L^2(x) \otimes K_X|$), there exist a line in $\mathbb{P}_L^{g+1}$ which is trisecant to X if and only if :

$$L^2(x) \cong \mathcal{O}_X(p+q+r)(\text{ equivalently, } \mathrm{H}^0(L^2(x)) \neq 0) \tag{5.3}$$

for some points p, q, r on the curve X. Then there exist a trisecant line ℓ which intersect with the curve X on the points p, q, r. If $p = q$, then ℓ is tangent to X at the point p. Also, if $p = q = r$, then ℓ is tangent to X at the point p and it intersect with X on p with multiplicity 3.

(b) Let us assume that the curve X is not trigonal and also not hyperelliptic. Then the points p, q, r on the curve X which satisfies the equation (5.3) are uniquely defined. Therefore there is unique trisecant line passing through the points p, q, r.

Proof. Part (a) : Assume that there is a trisecant line ℓ and let $\ell \cap X = \{p, q, r\}$(intersection points p, q, r need not to be distinct). So we can write $\ell = \overline{pqr}$. Since the line ℓ is trisecant, in the similar manner as in the Proposition 5.2.7, we can observe that the line ℓ is identified with the projectivized kernel of the morphism δ_1 in the following diagram :

$$\begin{aligned}\mathrm{Ext}^0(L, L^{-1}(-x)) &\to \mathrm{Ext}^0(L(-p-q-r), L^{-1}(-x)) \to \mathrm{Ext}^1(L|_{p+q+r}, L^{-1}(-x)) \\ &\to \mathrm{Ext}^1(L, L^{-1}(-x)) \xrightarrow{\delta_1} \mathrm{Ext}^1(L(-p-q-r), L^{-1}(-x)) \to 0.\end{aligned}$$

Since the first term of the diagram is zero, the dimension of the kernel of δ_1 is 2(since $\overline{pqr}$ is the line ℓ), and the dimension of $\mathrm{Ext}^1(L|_{p+q+r}, L^{-1}(-x))$ is 3, we deduce that the dimension of $\mathrm{Ext}^0(L(-p-q-r), L^{-1}(-x)) \cong \mathrm{H}^0(L^{-2}(-x) \otimes$

$\mathcal{O}_X(p+q+r))$ is equal to 1. Thus we obtain $L^{-2}(-x)\otimes\mathcal{O}_X(p+q+r)\cong\mathcal{O}_X$. Therefore we obtain that $L^2(x)\cong\mathcal{O}_X(p+q+r)$.

Conversely, consider a line bundle L which satisfies the equation (5.3). Then, we take the functor $\mathrm{Hom}(L,-)$ in the following short exact sequence :

$$0\to L^{-1}(-x)\to L^{-1}(p+q+r-x)\to L^{-1}(p+q+r-x)|_{p+q+r}\to 0,$$

so we have the following long exact sequence.

$$\begin{aligned}&\mathrm{Ext}^0(L,L^{-1}(p+q+r-x))\to\mathrm{Ext}^0(L,L^{-1}(p+q+r-x)|_{p+q+r})\to\mathrm{Ext}^1(L,L^{-1}(-x))\\&\xrightarrow{\delta_2}\mathrm{Ext}^1(L,L^{-1}(p+q+r-x))\to\mathrm{Ext}^1(L,L^{-1}(p+q+r-x)|_{p+q+r})=0.\end{aligned}\tag{5.4}$$

Since the final term of the above sequence is clearly zero, we obtain $\dim(\mathrm{Ext}^1(L,L^{-1}(-x)))=g+2$ and $\dim\mathrm{Ext}^1(L,L^{-1}(p+q+r-x))=\dim\mathrm{Ext}^1(L,L)$ $=g$ by (5.3). Therefore we have $\dim\ker\delta_2=2$. Furthermore, since the map δ_2 is in fact equal to the composition of the map δ_1 and a natural isomorphism $\mathrm{Ext}^1(L(-p-q-r),L^{-1}(-x))\cong\mathrm{Ext}^1(L,L^{-1}(p+q+r-x))$, we conclude that $\ker\delta_2=\ker\delta_1$. Thus, the vector space $\ker\delta_2$ is the affine cone $\widehat{\ell}$ of the sub-linear space $\ell=\overline{pqr}\subset\mathbb{P}^{g+1}$, which is turned out to be a line. By definition, we have $X\cap\ell=p+q+r$.

Part (b) : Assume that there exist three points s,t,u(such that $s+t+u\neq p+q+r$) on the curve X which satisfies $L^2(x)\cong\mathcal{O}_X(s+t+u)$. Then we have $\mathcal{O}_X(p+q+r-s-t-u)\cong\mathcal{O}_X$. But this says that the curve X is hyperelliptic or trigonal. □

Remark 5.2.6. We note that the non-trisecant condition $H^0(L^2(x))=0$ is general for degree 1 line bundles $L\in\mathrm{Pic}^1(X)$ by Riemann-Roch theorem, since our curve X satisfies $g(X)\geq 4$. In my joint work paper [20], we assumed that $g(X)\geq 3$, but if $g(X)=3$, then for degree 1 line bundle L,

$H^0(L^2(x)) \neq 0$. So If $g(X) = 3$, non-trisecant condition is not general condition. So we changed the genus condition to be $g(X) \geq 4$. This part was advised by Atanas Iliev.

5.2.2 Geometry of lines in $\mathbb{P}^{g+1}$ meeting X

In this section, we make a more precise description of the line $\overline{pq} := \langle f(p), f(q)\rangle$ which pass through the points p, q in the curve X embedded in $\mathbb{P}^{g+1} = \mathbb{P}V_L^{g+2}$ for a line bundle $L \in \mathrm{Pic}^1(X)$ via the map f. Since this kind of lines appear as a component of a boundary curve of Λ_1, this precise description helps us to understand the structure of the boundary of Λ_1.

If two points p and q coincides, then $\overline{pp}$ denotes the projective line tangent to X at the point $f(p)$ in the projective space $\mathbb{P}_L^{g+1}$. For a point $t \in X$ on the curve X, we have the image $f(t)$ (see Lemma 5.2.2) which fits into the exact sequences as follows :

$$0 \to L^{-1}(-x) \to f(t) = (L \oplus L^{-1}(t-x))^t \to L \to 0.$$

Proposition 5.2.7. Let $M := L \oplus L^{-1}(p+q-x)$. Then the punctured line $\overline{pq} \setminus \{p, q\}$ is parametrized by rank 2 vector bundles obtained by *double* elementary modifications, $(M^{\nu_p})^{\nu_q} (= (M^{\nu_q})^{\nu_p})$ which fit into the following short exact sequence:

$$0 \longrightarrow (M^{\nu_p})^{\nu_q} \longrightarrow M \stackrel{(\nu_p \oplus \nu_q)}{\longrightarrow} \mathbb{C}_p \oplus \mathbb{C}_q \longrightarrow 0.$$

Here $\nu_p \in \mathbb{C}^* \subset \mathbb{P}(H^0(M|_p)^\vee) = \mathbb{P}^1$ and $\nu_q \in \mathbb{C}^* \subset \mathbb{P}(H^0(M|_q)^\vee) = \mathbb{P}^1$.

Proof. By diagram chasing, we can check that the vector bundle obtained by the double elementary modification exactly coincide with the kernel of the morphism $\nu_p \oplus \nu_q$.

First we describe the $\overline{pq} \subset \mathbb{P}_L^{g+1}$ in an algebraic way. Applying the functor $\mathrm{Hom}(-, L^{-1}(x))$ to the exact sequence $0 \to L(-p-q) \to L \to L|_{p+q} \to 0$, we have the following long exact sequence :

$$\begin{aligned} 0 = \mathrm{Ext}^0(L(-p-q), L^{-1}(-x)) &\to \mathrm{Ext}^1(L|_{p+q}, L^{-1}(-x)) \xrightarrow{i} \mathrm{Ext}^1(L, L^{-1}(-x)) \\ &\xrightarrow{j} \mathrm{Ext}^1(L(-p-q), L^{-1}(-x)) \overset{\varphi}{\cong} \mathrm{Ext}^1(L, L^{-1}(p+q-x)) \to 0 \end{aligned}$$

where φ is the natural isomorphism tensoring the line bundle $\mathcal{O}(p+q)$ and the first identity holds because of degree reasons.

So we claim that the image of $\mathbb{P}\mathrm{Ext}^1(L|_{p+q}, L^{-1}(-x))$ in $\mathbb{P}\mathrm{Ext}^1(L, L^{-1}(-x)) = \mathbb{P}_L^{g+1}$ is equal to the line $\overline{pq}$. To prove this, we first show that the line $\mathbb{P}\mathrm{Ext}^1(L|_{p+q}, L^{-1}(-x)) \subset \mathbb{P}^{g+1}$ is parametrized by bundles comes from double elementary modifications. Let $E \in \mathrm{Ext}^1(L|_{p+q}, L^{-1}(-x))$. Then the image $i(E)$ of the bundle E fits into the exact sequence in the following

$$0 \to L^{-1}(-x) \to i(E) \to L \to 0.$$

We have $\varphi(j(i(E))) = L^{-1}(p+q-x) \oplus L$, which means that we can construct the push-out diagram as follows

$$\begin{tikzcd} 0 \arrow[r] & L^{-1}(-x) \arrow[r] \arrow[d, hook] & i(E) \arrow[r] \arrow[d, "a"] & L \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & L^{-1}(p+q-x) \arrow[r] & L^{-1}(p+q-x) \oplus L \arrow[r] & L \arrow[r] & 0. \end{tikzcd} \tag{5.5}$$

Then we obtain the following exact sequence using some diagram chasing

$$0 \to i(E) \xrightarrow{a} L^{-1}(p+q-x) \oplus L \xrightarrow{\nu_p \oplus \nu_q} \mathbb{C}_{p+q} \to 0 \tag{5.6}$$

where $p_1 \circ a$ is a surjection and the map $p_1 : L \oplus L^{-1}(p+q-x) \to L$ is the natural projection to the first summand. Furthermore, it is trivial that $p_1 \circ a$ is a surjection if and only if $\nu_p, \nu_q \neq [1:0]$.

On the contrary, consider a rank 2 vector bundle E fits into the short exact sequence (5.6) where the map $p_1 \circ a$ is surjective. Then we can observe that the vector bundle E fits into the commutative diagram (5.5) using the snake lemma.

As a result, extensions arises as a form of (5.6) where $\nu_p \neq [1:0]$ and $\nu_q \neq [1:0]$ are elements of $\mathrm{Ext}^1(L|_{p+q}, L^{-1}(-x))$. Next, we can check that the vector bundle correspond to $f(p)$ (resp. $f(q)$) which is obtained by elementary modification fits into the diagram of (5.6) where $\nu_p \in \mathbb{C}^*$ and $\nu_q = [0:1]$ (resp. $\nu_q \in \mathbb{C}^*$ and $\nu_p = [0:1]$). Hence, when $p \neq q$, then we have $f(p) \neq f(q)$ and $f(p)$ and $f(q)$ lie on the linear space $\mathbb{P}\mathrm{Ext}^1(L|_{p+q}, L^{-1}(-x)) \subset \mathbb{P}^{g+1}_L$ which coincide with the line $\overline{pq}$ by definition.

We can observe that the extension group $\mathrm{Ext}^1(L|_{2p}, L^{-1}(-x))$ is equal to limit of the family of extension groups $\mathrm{Ext}^1(L|_{p+q}, L^{-1}(-x))$ when p approaches to q. So we obtain the same conclusion for the case of $p = q$. □

Next, we specify which vector bundles are contained in the following intersections of the projectivized extension groups :

1. $\mathbb{P}V^g_\zeta \cap \mathbb{P}V^g_\eta$ for $\zeta, \eta \in \mathrm{Pic}^0(X)$.
2. $\mathbb{P}(V^{g+2}_\zeta)^s \cap \mathbb{P}V^g_\eta$ for $\zeta \in \mathrm{Pic}^1(X)$ and $\eta \in \mathrm{Pic}^0(X)$.
3. $\mathbb{P}(V^{g+2}_\zeta)^s \cap \mathbb{P}(V^{g+2}_\eta)^s$ for $\zeta, \eta \in \mathrm{Pic}^1(X)$.

It should be noted that these intersections arise in the moduli space of vector bundle $\mathcal{N}$. Case (1) is already covered in [77, 6.19]. $\mathbb{P}V^g_\zeta$ and $\mathbb{P}V^g_\eta$ cleanly intersect at a point or their intersection locus is empty. Thus we concentrate on the case (2) and (3).

Proposition 5.2.8. For line bundles $\zeta \in \mathrm{Pic}^1(X)$ and $\eta \in \mathrm{Pic}^0(X)$, vector bundles in $\mathbb{P}(V^{g+2}_\zeta)^s \cap \mathbb{P}V^g_\eta \subset \mathcal{N}$ arise in one of the following forms :

i) If $\zeta \otimes \eta \cong \mathcal{O}(p+q-x)$, for some points $p, q \in X$, then the image of $\mathbb{P}(V^g_\zeta)^s \cap \mathbb{P}V^g_\eta$ and the image of $\overline{pq} \setminus \{p, q\}$ are exactly the same in $\mathcal{N}$.

ii) Otherwise, we have $\mathbb{P}(V^{g+2}_\zeta)^s \cap \mathbb{P}V^g_\eta = \varnothing$

Proof. Consider a bundle E in the intersection $\mathbb{P}(V_\zeta^{g+2})^s \cap \mathbb{P}V_\eta^g$. Then we have the following diagram :

$$\begin{array}{ccccccccc} 0 & \to & \zeta^{-1}(-x) & \xrightarrow{a} & E & \xrightarrow{b} & \zeta & \to & 0 \\ & & & & \downarrow = & & & & \\ 0 & \to & \eta^{-1}(-x) & \xrightarrow{c} & E & \xrightarrow{d} & \eta & \to & 0. \end{array}$$

If $d \circ a$=0, then we can see that d factors through $E \xrightarrow{b} \zeta$. But in this case, since deg$(\zeta) = 1 >$ deg$(\eta) = 0$ we have $d = 0$ and it leads to the contradiction. Hence the map $d \circ a$ is injective. Furthermore, since the degree of the map η is 0 and the degree of $\zeta^{-1}(-x)$ is -2, we obtain Coker$(d \circ a)) = \mathbb{C}_{p+q}$ for $p, q \in X$. Hence we have $\zeta^{-1}(-x) \cong \eta(-p-q)$. Next, consider the diagram in the following

$$\begin{tikzcd} & & 0 & 0 & \\ 0 \arrow[r] & \zeta^{-1}(-x) \arrow[r] & \eta \arrow[u] \arrow[r, "r"] & \mathbb{C}_{p+q} \arrow[u] \arrow[r] & 0 \\ 0 \arrow[r] & \zeta^{-1}(-x) \arrow[u, equal] \arrow[r, "a"] & E \arrow[u, "d"] \arrow[r, "b"] & \zeta \arrow[u, "s"] \arrow[r] & 0 \\ & & \eta^{-1}(-x) \arrow[u, "c"] \arrow[r, equal] & \eta^{-1}(-x) \arrow[u, "b \circ c"] & \\ & & 0 \arrow[u] & 0 \arrow[u] & . \end{tikzcd} \tag{5.7}$$

Then, one can observe that Coker$(d \circ a)$ is isomorphic to Coker$(b \circ c)$, and this observation leads to the following diagram :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \zeta^{-1}(-x) & \xrightarrow{a} & E & \xrightarrow{b} & \zeta & \longrightarrow & 0 \\ & & & & \downarrow = & & & & \\ 0 & \longrightarrow & \zeta(-p-q) & \xrightarrow{c} & E & \xrightarrow{d} & \zeta^{-1}(p+q-x) & \longrightarrow & 0. \end{array}$$

From the above diagram, we can construct the morphism in the following

$$b \oplus d : E \to \zeta \oplus \zeta^{-1}(p+q-x).$$

Then, we also construct the following commutative diagram :

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C}_{p+q} \arrow[r, "\Delta"] & \mathbb{C}_{p+q} \oplus \mathbb{C}_{p+q} \arrow[r, "g"] & \mathbb{C}_{p+q} \arrow[r] & 0 \\
0 \arrow[r] & E \arrow[r, "b\oplus d"] \arrow[u, "s\circ b"] & \zeta \oplus \zeta^{-1}(p+q-x) \arrow[r] \arrow[u, "s\oplus r"] & C_3 \arrow[r] \arrow[u, "h"] & 0
\end{tikzcd} \tag{5.8}$$

where the map g is defined to be $g(z, w) := z - w$. Because the map $g \circ (s \oplus r)$ is surjective, the map h is also surjective. We can observe that the degree of the coherent sheaf C_3 is 2 and supported at $\{p, q\}$, so we deduce that h is an isomorphism.

As a result, the vector bundle E fit into the following short exact sequence :

$$0 \longrightarrow E \xrightarrow{b\oplus d} \zeta \oplus \zeta^{-1}(p+q-x) \xrightarrow{\nu_p \oplus \nu_q} \mathbb{C}_{p+q} \longrightarrow 0, \tag{5.9}$$

where $\nu_t \in \mathbb{C}^*$ for some $t \in \{p, q\}$. We note that the the class of ν_p and ν_q should not be $[1:0] \in \mathbb{P}^1$ or $[0:1] \in \mathbb{P}^1$ since the morphism b and d are both surjective. Conversely, if a vector bundle E satisfy the conditions mentioned above, then it is easy to show that E is contained in $\mathbb{P}(V_\zeta^{g+2})^s \cap \mathbb{P}V_\eta^g$ when $\eta \cong \zeta^{-1}(p+q-x)$. Hence, we have the conclusion from Proposition 5.2.7. $\square$

Proposition 5.2.9. Let $M := L \oplus L^{-1}(p+q+r-x)$. Then the scraped linear space $\overline{pqr} \setminus \{\overline{pq} \cup \overline{qr} \cup \overline{pr}\}$ is parametrized by rank 2 vector bundles obtained by *triple* elementary modifications, $((M^{\nu_p})^{\nu_q})^{\nu_r} (= ((M^{\nu_p})^{\nu_r})^{\nu_q} = \cdots = ((M^{\nu_r})^{\nu_q})^{\nu_p})$ which fit into a short exact sequence of the following form

$$0 \longrightarrow ((M^{\nu_p})^{\nu_q})^{\nu_r} \longrightarrow M \xrightarrow{(\nu_p \oplus \nu_q \oplus \nu_r)} \mathbb{C}_p \oplus \mathbb{C}_q \oplus \mathbb{C}_r \longrightarrow 0.$$

Here, $\nu_p \in \mathbb{C}^* \subset \mathbb{P}(H^0(M|_p)^\vee) = \mathbb{P}^1$, $\nu_q \in \mathbb{C}^* \subset \mathbb{P}(H^0(M|_q)^\vee) = \mathbb{P}^1$ and

$\nu_r \in \mathbb{C}^* \subset \mathbb{P}(H^0(M|_r)^\vee) = \mathbb{P}^1$.

Proof. First, we can check that vector bundles obtained by the triple elementary modifications exactly correspond to vector bundles which are kernels of the morphisms $\nu_p \oplus \nu_q \oplus \nu_r$ by diagram chasing.

First, we describe the $\overline{pqr} \subset \mathbb{P}_L^{g+1}$ in an algebraic way. By applying the functor $\mathrm{Hom}(-, L^{-1}(x))$ to the exact sequence $0 \to L(-p-q-r) \to L \to L|_{p+q+r} \to 0$, we have the following long exact sequence :

$$\mathrm{Ext}^0(L(-p-q-r), L^{-1}(-x)) \to \mathrm{Ext}^1(L|_{p+q+r}, L^{-1}(-x)) \xrightarrow{i} \mathrm{Ext}^1(L, L^{-1}(-x)) \tag{5.10}$$

$$\xrightarrow{j} \mathrm{Ext}^1(L(-p-q-r), L^{-1}(-x)) \overset{\varphi}{\cong} \mathrm{Ext}^1(L, L^{-1}(p+q+r-x)) \to 0 \tag{5.11}$$

where φ is induced by twisting the line bundle $\mathcal{O}(p+q+r)$ and we can check the first identity using degree reasons. So we claim the following

$$\mathbb{P}\mathrm{Ext}^1(L|_{p+q+r}, L^{-1}(-x)) = \overline{pqr}.$$

To prove this claim, we first show that the line $\mathbb{P}\mathrm{Ext}^1(L|_{p+q+r}, L^{-1}(-x)) \subset \mathbb{P}^{g+1}$ is represented by bundles comes from triple elementary modifications. Let $E \in \mathrm{Ext}^1(L|_{p+q+r}, L^{-1}(-x))$. Then the image $i(E)$ of the bundle E fits into the following exact sequence:

$$0 \to L^{-1}(-x) \to i(E) \to L \to 0.$$

Because $\varphi(j(i(E))) = L^{-1}(p+q+r-x) \oplus L$, we can consider the push-out diagram as follows

$$\begin{tikzcd}
0 \arrow[r] & L^{-1}(-x) \arrow[r] \arrow[d, hook] & i(E) \arrow[r] \arrow[d, "a"] & L \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L^{-1}(p+q+r-x) \arrow[r] & L^{-1}(p+q+r-x) \oplus L \arrow[r] & L \arrow[r] & 0.
\end{tikzcd} \tag{5.12}$$

By diagram chasing, one can check that there is an exact sequence in the following

$$0 \to i(E) \xrightarrow{a} L^{-1}(p+q+r-x) \oplus L \xrightarrow{\nu_p \oplus \nu_q \oplus \nu_r} \mathbb{C}_{p+q+r} \to 0 \tag{5.13}$$

where the composition $p_1 \circ a$ is surjective for the projection $p_1 : L \oplus L^{-1}(p+q+r-x) \to L$ into the first factor. Furthermore, it is obvious that the composition $p_1 \circ a$ is a surjection if and only if $\nu_p \neq [1:0]$, $\nu_q \neq [1:0]$ and $\nu_r \neq [1:0]$.

On the contrary, we assume that the rank 2 vector bundle E fits into the short exact sequence (5.13) such that the map $p_1 \circ a$ is a surjection. Then we can easily show that the vector bundle E fits into the push-out diagram (5.12) using the snake lemma.

As a result, extensions appeared as a form of (5.13) where $\nu_p \neq [1:0]$, $\nu_q \neq [1:0]$ and $\nu_r \neq [1:0]$ are the elements of $\mathrm{Ext}^1(L|_{p+q+r}, L^{-1}(-x))$. Next, we can easily check that bundles correspond to $\overline{pq}$ (resp. $\overline{qr}$, $\overline{pr}$) obtained by elementary modification fits into the diagram of (5.13) where $\nu_r = [0:1]$ and $\nu_p, \nu_q \in \mathbb{P}^1 \setminus [1:0]$ (resp. $\nu_p = [0:1]$ and $\nu_q, \nu_r \in \mathbb{P}^1 \setminus [1:0]$, $\nu_q = [0:1]$ and $\nu_p, \nu_r \in \mathbb{P}^1 \setminus [1:0]$). Hence, when p, q, r are all distinct, three points $f(p), f(q), f(r)$ are also distinct and they lie on the linear space $\mathbb{P}\mathrm{Ext}^1(L|_{p+q+r}, L^{-1}(-x))$. If $f(p), f(q), f(r)$ are colinear, then the line bundle L is trisecant by Corollary 5.2.5. Therefore, we have $\dim \mathrm{Ext}^0(L(-p-q-r), L^{-1}(-x)) = 1$ in the sequence (5.10). Therefore by dimension counting, we conclude that $\mathbb{P}(\mathrm{Ext}^1(L|_{p+q+r}, L^{-1}(-x))/\mathrm{Ext}^0(L(-p-q-r), L^{-1}(-x)))$ has dimension 1, i.e. a projective line. If $f(p), f(q), f(r)$ is not colinear, than $\dim\mathrm{Ext}^0(L(-p-q-r), L^{-1}(-x)) = 0$ by 5.2.5 and therefore we conclude that $\mathbb{P}\mathrm{Ext}^1(L|_{p+q}, L^{-1}(-x))$ is a projective plane, and therefore $f(p), f(q), f(r)$ spans the projective plane. So, in both cases, $f(p), f(q), f(r)$ spans the linear space $\mathbb{P}\mathrm{Ext}^1(L|_{p+q}, L^{-1}(-x))$ so we can write it by $\overline{pqr}$. In summary, The bundles obtained by triple elementary modifications parametrizes the scraped linear space $\overline{pqr} \setminus \{\overline{pq} \cup \overline{qr} \cup \overline{pr}\}$.

Since the extension groups $\mathrm{Ext}^1(L|_{2p+q}, L^{-1}(-x))$, $\mathrm{Ext}^1(L|_{3p}, L^{-1}(-x))$ are equal to the limits of the extension groups $\mathrm{Ext}^1(L|_{p+q+r}, L^{-1}(-x))$ by taking $p \to q$, and $q \to r$, we obtain the same conclusion for the case of $p = q$. Here, $\overline{ppq}$ is the linear space spanned by the projective tangent line of the curve X at p and the point r, and $\overline{ppp}$ is the osculating plane of the curve X at the point p. $\square$

Proposition 5.2.10. Let $\zeta \in \mathrm{Pic}^1(X)$ and $\eta \in \mathrm{Pic}^1(X)$, $[\zeta] \neq [\eta]$. Then bundles in the intersection $\mathbb{P}(V_\zeta^{g+2})^s \cap \mathbb{P}(V_\eta^{g+2})^s \subset \mathcal{N}$ arise as one of the following types :

i) If $\zeta \otimes \eta \cong \mathcal{O}(p+q+r-x)$, for some points $p, q, r \in X$, then the image of $\mathbb{P}(V_\zeta^{g+2})^s \cap \mathbb{P}(V_\eta^{g+2})^s$ and the image of $\overline{pqr} \setminus \{\overline{pq} \cup \overline{qr} \cup \overline{pr}\}$ are exactly the same in $\mathcal{N}$. Here, $\overline{pqr}$ is the linear space in $\mathbb{P}(V_\zeta^{g+2})$ spanned by points p, q, r.

ii) Otherwise, we have $\mathbb{P}(V_\zeta^{g+2})^s \cap \mathbb{P}(V_\eta^{g+2})^s = \varnothing$

Proof. The proof proceeds in the similar manner as 5.2.8. Consider a bundle E in the intersection $\mathbb{P}(V_\zeta^{g+2})^s \cap \mathbb{P}(V_\eta^{g+2})^s$. Then we have the following diagram :

$$\begin{array}{ccccccccc} 0 & \to & \zeta^{-1}(-x) & \xrightarrow{a} & E & \xrightarrow{c} & \zeta & \to & 0 \\ & & & & \downarrow{=} & & & & \\ 0 & \to & \eta^{-1}(-x) & \xrightarrow{b} & E & \xrightarrow{d} & \eta & \to & 0. \end{array}$$

If $d \circ a$=0, then we can see that d factors through $E \xrightarrow{c} \zeta$. But in this case, since $\deg(\zeta) = 1 = \deg(\eta) = 1$, we have $\zeta \cong \eta$ and it leads to the contradiction. Hence the map $d \circ a$ is injective. By degree reason, we have $\mathrm{Coker}(d \circ a) = \mathbb{C}_{p+q+r}$ for points $p, q, r \in X$. Hence we have $\zeta^{-1}(-x) \cong \eta(-p-q-r)$. In the same manner, we can show that $c \circ b$ is also injective. Next, consider the following commutative diagram :

$$\begin{tikzcd}
 & & 0 & 0 & \\
0 \arrow[r] & \zeta^{-1}(-x) \arrow[r] & \eta \arrow[r, "r"] \arrow[u] & \mathbb{C}_{p+q+r} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \zeta^{-1}(-x) \arrow[r, "a"] \arrow[u, equal] & E \arrow[r, "c"] \arrow[u, "d"] & \zeta \arrow[r] \arrow[u, "s"'] & 0 \\
 & & \eta^{-1}(-x) \arrow[r, equal] \arrow[u, "b"] & \eta^{-1}(-x) \arrow[u, "c \circ b"'] & \\
 & & 0 \arrow[u] & 0. \arrow[u] &
\end{tikzcd} \tag{5.14}$$

From the above diagram, we can observe that $\mathrm{Coker}(d \circ a)$ is isomorphic to $\mathrm{Coker}(c \circ b)$, and this fact leads to the following diagram :

$$\begin{tikzcd}
0 \arrow[r] & \zeta^{-1}(-x) \arrow[r, "a"] & E \arrow[r, "c"] \arrow[d, "="] & \zeta \arrow[r] & 0 \\
0 \arrow[r] & \zeta(-p-q-r) \arrow[r, "b"] & E \arrow[r, "d"] & \zeta^{-1}(p+q+r-x) \arrow[r] & 0.
\end{tikzcd}$$

Thus, we can construct the following map

$$b \oplus d : E \to \zeta \oplus \zeta^{-1}(p+q+r-x).$$

Then, we can also construct the following commutative diagram :

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C}_{p+q+r} \arrow[r, "\Delta"] & \mathbb{C}_{p+q+r} \oplus \mathbb{C}_{p+q+r} \arrow[r, "g"] & \mathbb{C}_{p+q+r} \arrow[r] & 0 \\ 0 \arrow[r] & E \arrow[r, "c\oplus d"] \arrow[u, "s\circ c"] & \zeta \oplus \zeta^{-1}(p+q+r-x) \arrow[r] \arrow[u, "s\oplus r"] & C_3 \arrow[r] \arrow[u, "h"] & 0 \end{tikzcd} \tag{5.15}$$

where the map g is defined to be $g(z, w) := z - w$. Because the map $g\circ(s\oplus r)$ is a surjection, the map h is also a surjection. We have the degree of the coherent sheaf C_3 is 3 and supported at $\{p, q, r\}$, so we deduce that h is an isomorphism.

As a result, the vector bundle E fit into the following short exact sequence :

$$0 \longrightarrow E \xrightarrow{c\oplus d} \zeta \oplus \zeta^{-1}(p+q+r-x) \xrightarrow{\nu_p\oplus\nu_q\oplus\nu_r} \mathbb{C}_{p+q+r} \longrightarrow 0, \tag{5.16}$$

where $\nu_t \in \mathbb{C}^*$ for each $t \in \{p, q, r\}$. We note that the classes of ν_p, ν_q and ν_r should not be in $\{[1:0], [0:1]\} \subset \mathbb{P}^1$ since the morphism c and d are both surjective. Conversely, if a vector bundle E satisfy the conditions mentioned above, then it is easy to show that E is contained in $\mathbb{P}(V_\zeta^{g+2})^s \cap \mathbb{P}(V_\eta^{g+2})^s$ when $\eta \cong \zeta^{-1}(p+q+r-x)$. Hence, we obtain the proof from Proposition 5.2.9. $\square$

For classifying stable maps in the space $\widetilde{\mathbf{P}}_L$, we will use the following result in Corollary 5.2.12, which computes the degree of the map given by the composition $\mathbb{P}^1 \xrightarrow{f} (\mathbb{P}_L^{g+1})^s \xrightarrow{\Psi_L} \mathcal{N}$ from some geometric information.

Lemma 5.2.11. For elements $\alpha \neq \beta \in (\mathbb{P}V_L^{g+2})^s$, we have $\Psi_L(\alpha) = \Psi_L(\beta)$ if and only if the points $\alpha, \beta \in \mathbb{P}V_L^{g+2} = \mathbb{P}_L^{g+1}$ lie in a trisecant line of the curve $X \subset \mathbb{P}_L^{g+1}$.

Proof. Let us assume that $\Psi_L(\alpha) = \Psi_L(\beta)$, then we have :

$$\begin{tikzcd}
\alpha : [\ 0 \arrow[r] & L^{-1}(-x) \arrow[r, "a"] & E \arrow[r, "c"] \arrow[d, equal] & L \arrow[r] & 0\ ] \\
\beta : [\ 0 \arrow[r] & L^{-1}(-x) \arrow[r, "b"] & E \arrow[r, "d"] & L \arrow[r] & 0\ ].
\end{tikzcd}$$

If the composition $d \circ a = 0$, then the map a should factors through the map $L^{-1}(-x) \xrightarrow{b} E$. Therefore the map a should be a scalar multiplication on $L^{-1}(-x)$, so we have $a = \lambda b$ for some $\lambda \in \mathbb{C}^*$. Again since $d \circ a = 0$, d factored by the quotient map d and the descent map from L to L, which should be an isomorphism. Thus d is a scalar multiplication on L, so we obtain $d = \lambda' c$ for some $\lambda' \in \mathbb{C}^*$. This means the extension classes α and β are equal, which is a contradiction. Therefore the composition $d \circ a$ is not zero. Thus the map $d \circ a$ is an injection, which says that the coherent sheaf Coker$(d \circ a)$ is equal to the skyscraper sheaf $\mathbb{C}_{p+q+r}$ for three points $p, q, r \in X$. Then we observe that $L^{-1}(-x)$ is isomorphic to $L(-p-q-r)$, which means that $L^2(x) \cong \mathcal{O}_X(p+q+r)$. In a similar manner, we can also check that $c \circ b \neq 0$. Then, there exist a trisecant line ℓ of X in $\mathbb{P}_L^{g+1}$ which intersects with the curve X on the points p, q, r by Corollary 5.2.5. Next, consider the map $c \oplus d : E \to L \oplus L$. Using the same argument used to construct the diagram (5.15), we obtain Coker$(c \oplus d) = \mathbb{C}_{p+q+r}$. Then, look at the commutative diagram in the following

$$\begin{tikzcd}
0 \arrow[r] & L^{-1}(-x) \arrow[r, "d \circ a"] \arrow[d, hook, "a"] & L \arrow[r] \arrow[d, hook, "0 \oplus id"] & \mathbb{C}_{p+q+r} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & E \arrow[r, "c \oplus d"] & L \oplus L \arrow[r] & \mathbb{C}_{p+q+r} \arrow[r] & 0.
\end{tikzcd}$$

Using the snake lemma, we obtain the commutative diagram in the following

$$\begin{tikzcd}
0 \arrow[r] & L^{-1}(-x) \arrow[r, "a"] \arrow[d, hook, "d\circ a"] & E \arrow[r, "c"] \arrow[d, hook, "c\oplus d"] & L \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L \arrow[r] & L \oplus L \arrow[r] & L \arrow[r] & 0.
\end{tikzcd}$$

Since $L \cong L^{-1}(p+q+r-x)$, we can check that α and β are elements of the projectivized kernel of the map δ_2 in (5.4). Thus by Corollary 5.2.5, we have α and β are contained in the trisecant line ℓ of the curve $X \subset \mathbb{P}^{g+1}_L$.

Next, we check the necessary condition. To show this, it is enough to check that a trisecant line ℓ of the curve X contracts to a point when we take its image in the moduli space $\mathcal{N}$. Consider any trisecant line ℓ of X such that ℓ intersect with X at the points p, q, r. Then we have $\ell = \overline{pq}$. There is a long exact sequence

$$\begin{aligned}
&0 \to \mathrm{Ext}^0(L^{-1}(p+q-x), L) \to \mathrm{Ext}^0(L^{-1}(p+q-x), L|_{p+q}) \to \\
&\mathrm{Ext}^1(L^{-1}(p+q-x), L(-p-q)) \xrightarrow{\delta_3} \mathrm{Ext}^1(L^{-1}(p+q-x), L) \\
&\to \mathrm{Ext}^1(L^{-1}(p+q-x), L|_{p+q}) = 0,
\end{aligned} \tag{5.17}$$

which is obtained by applying the functor $\mathrm{Hom}(L^{-1}(p+q-x), -)$ to the exact sequence $0 \to L(-p-q) \to L \to L|_{p+q} \to 0$.

We can consider equivalence classes of elements of $\ker\delta_3$ as a subset of $\mathbb{P}^{g-1}_{L^{-1}(p+q-x)} = \mathbb{P}V^g_{L^{-1}(p+q-x)}$, which are represented by the following short exact sequence :

$$[0 \to L(-p-q) \to E \to L^{-1}(p+q-x) \to 0].$$

In a similar manner as in the proof of Proposition 5.2.7, we can show that the vector bundle E fits into the following short exact sequence :

$$0 \longrightarrow E \xrightarrow{b\oplus d} L \oplus L^{-1}(p+q-x) \xrightarrow{\nu_p \oplus \nu_q} \mathbb{C}_{p+q} \longrightarrow 0 \tag{5.18}$$

where the map d is surjective. Hence by Proposition 5.2.7, the extension classes represented by the above short exact sequence corresponds to points in the punctured line $\overline{pq} \setminus \{p, q\}$. On the contrary, we can show that a rank 2 vector bundle E which fits into the short exact sequence (5.18) where the map d is surjective corresponds to the equivalence classes of the kernel of the map δ_3 by some diagram chasing. Hence we conclude that the image of $\overline{pq} \setminus \{p, q\}$ in $\mathcal{N}$ is a single point if and only if $\dim \ker\delta_3 \leq 1$.

Thus, by the equation (5.17), we have :

$$\ker\delta_3 \cong \mathrm{Ext}^0(L^{-1}(p+q-x), L|_{p+q})/\mathrm{Ext}^0(L^{-1}(p+q-x), L) \cong \mathbb{C}^2/H^0(L^2(x)(-p-q)).$$

However, since $L^2(x)(-p-q) \cong \mathcal{O}_X(r)$, the claim holds. □

Corollary 5.2.12. If the projective line $\ell \subset \mathbb{P}^{g+1}_L$ intersects the curve $X \subset \mathbb{P}^{g+1}_L$ with multiplicity m. Then $\iota : \ell \setminus (\ell \cap X) \to \mathcal{N}$ is a degree $3 - m$ map for $m = 0, 1, 2, 3$.

Proof. $m = 0$: This case is trivial because the degree of the map Ψ_L is 3.

$m = 1$: It is clear that $\deg \iota \in \{0, 1, 2\}$. If $\deg \iota = 0$, then the image of $\ell \setminus (\ell \cap X)$ by the map ι is a single point in the moduli space $\mathcal{N}$. Hence by the Lemma 5.2.11, ℓ is a line trisecant to the curve X, which is a contradiction. If deg $\iota = 1$, then by [17], ι should factors through the space $\mathbb{P}^{g-1}_M = \mathbb{P}V^g_M$ for a line bundle $M \in \mathrm{Pic}^0(X)$. Therefore, by Corollary 5.2.8, we obtain that the line ℓ intersect X two times, which is a contradiction. Hence we conclude that deg $\iota = 2$.

$m = 2$: We may assume that ℓ intersect with X at p, q and we can write $\ell = \overline{pq}$. Because the line ℓ is not trisecant, we have $H^0(L^2(x)(-p-q)) = 0$ by Proposition 5.2.5. Hence by the proof of Lemma 5.2.11, we can observe that the map $\iota : \ell \to \mathcal{N}$ factors through the space $\mathbb{P}^{g-1}_{L^{-1}(p+q-x)}$. Thus we have deg $\iota = 1$ since we already know that $\Psi_{L^{-1}(p+q-x)}$ is a linear embedding.

$m = 3$: We assume that ℓ intersect with X on p, q, r. By Lemma 5.2.11, the image of ℓ by the map ι is a single point in $\mathcal{N}$. Hence the degree of the map ι is 0. $\square$

Remark 5.2.13. Recall the case of $m = 1$ in Corollary 5.2.12. Since the degree of the map ι is 2, the closure $\bar{\ell} := \overline{\iota(\ell \setminus (\ell \cap X))}$ is a smooth conic in the moduli space $\mathcal{N}$. By [54, Proposition 3.6], $\bar{\ell}$ becomes a *Hecke* curve or a smooth conic in $\mathbb{P}^{g-1}_M$ for a line bundle $M \in \mathrm{Pic}^0(X)$. In the latter case, the line ℓ intersects with X at a point r, and $\ell \setminus r \subset (\mathbb{P}V_L^{g+2})^s \cap \mathbb{P}V_M^g$ for a line bundle $M \in \mathrm{Pic}^0(X)$, which contradicts to the part i) of Proposition 5.2.8. Therefore the line $\bar{\ell}$ is a Hecke conic of the moduli space $\mathcal{N}$.

5.3 Stable maps in the moduli space $\mathcal{N}$

5.3.1 Conjectural picture

In Chapter 3, Proposition 3.1.3, we reviewed about the classification of irreducible components of $R_3(\mathcal{N})$ studied by Castravet in [13, 54]. In this section, we study the compactification Λ_1 of the component $R_3(1)$ of $R_3(\mathcal{N})$ as we announced at the beginning of the chapter. By 5.2.4, we know that the rational map $\Psi_L : \mathbb{P}^{g+1}_L \dashrightarrow \mathcal{N}$ extends to the regular map $\widetilde{\Psi_L} : \widetilde{\mathbf{P}}_L \to \mathcal{N}$, which is an embedding when L is a non-trisecant. Since we can find a limit of a family of lines $\mathbb{P}^1 \to (\mathbb{P}^{g+1}_L)^s$ which getting close to the unstable locus in $\widetilde{\mathbf{P}}_L = \mathrm{Bl}_X \mathbb{P}^{g+1}_L$.

Next, consider a relativization of the space $\widetilde{\mathbf{P}}_L$. Consider a universal line bundle $\mathcal{L}$ on $\mathrm{Pic}^1(X) \times X$. Let p_1, p_2 are projections from $\mathrm{Pic}^1(X) \times X$ to $\mathrm{Pic}^1(X)$ and X. We define projective bundle $\mathbb{P}\mathrm{Ext}^1(\mathcal{L}, \mathcal{L}^{-1}(-(\{x\}\times \mathrm{Pic}^1(X)))) := (p_1)_*(\mathcal{L}^2(\{x\}\times \mathrm{Pic}^1(X))$. Then in a similar manner, we can show that the unstable locus of $\mathbb{P}\mathrm{Ext}^1(\mathcal{L}, \mathcal{L}^{-1}(-(\{x\} \times \mathrm{Pic}^1(X))))$ is isomorphic to $X \times \mathrm{Pic}^1(X)$ embedded in $\mathbb{P}\mathrm{Ext}^1(\mathcal{L}, \mathcal{L}^{-1}(-(\{x\} \times \mathrm{Pic}^1(X))))$ via the complete linear system $|\mathcal{L}^2 \otimes (p_2)^* K_X|$.

Then, similar to the Proposition 5.2.4, we conjecture that there is an extended morphism :

$$\mathrm{Bl}_{X\times \mathrm{Pic}^1(X)}\mathbb{P}\mathrm{Ext}^1(\mathcal{L},\mathcal{L}^{-1}(-(x\times \mathrm{Pic}^1(X)))) := \widetilde{\mathbf{P}} \xrightarrow{\widetilde{\Psi}} \mathcal{N}. \tag{5.19}$$

In summary, we have a conjectural diagram :

$$\begin{tikzcd} \widetilde{\mathbf{P}}_L \arrow[r] \arrow[d] & \mathrm{Bl}_{X\times \mathrm{Pic}^1(X)}\mathbb{P}\mathrm{Ext}^1(\mathcal{L},\mathcal{L}^{-1}(-(x\times \mathrm{Pic}^1(X)))) := \widetilde{\mathbf{P}} \arrow[r, "\widetilde{\Psi}"] \arrow[d, "q"] & \mathcal{N} \\ \{L\} \arrow[r, hook] & \mathrm{Pic}^1(X) & \end{tikzcd}$$

Thus we have the following morphisms of stable maps :

$$\mathbf{M}_0(\widetilde{\mathbf{P}}_L,\beta) \xrightarrow{i} \mathbf{M}_0(\widetilde{\mathbf{P}},\beta) \xrightarrow{j} \mathbf{M}(\mathcal{N},3)$$

where β is the homology class which is an l.c.i pull back of homology class of line blow-up morphism $\pi : \widetilde{\mathbf{P}}_L \to \mathbb{P}^{g+1}_L$.

Our first goal is to figure out which types of nodal curves are contained in the boundary of Λ_1. Since coarse moduli spaces of the stable map spaces are projective, j is proper. Therefore the image of j contains the component Λ_1 since the image of j contains lines in $\mathbb{P}^{g+1}_L \setminus X$ for arbitrary $L \in \mathrm{Pic}^1(X)$ and the image of j is closed.

Therefore, it is enough to study which types of nodal curves are contained in $\mathbf{M}_0(\widetilde{\mathbf{P}},\beta)$. We also conjecture that for the projection $q : \mathbf{M}_0(\widetilde{\mathbf{P}},\beta) \to \mathrm{Pic}^1(X)$, its fiber over a line bundle $L \in \mathrm{Pic}^1(X)$ is equal to $\mathbf{M}_0(\widetilde{\mathbf{P}}_L,\beta)$. So it is enough to study which types of nodal curves are contained in the boundary of $\mathbf{M}_0(\widetilde{\mathbf{P}}_L,\beta)$ for each $L \in \mathrm{Pic}^1(X)$, under this conjectural picture. Therefore the study of stable map spaces $\mathbf{M}_0(\widetilde{\mathbf{P}}_L,\beta)$ and the study of the irreducible component Λ_1 is closely related.

Furthermore, for any smooth rational map $f : \mathbb{P}^1 \xrightarrow{\deg 1} \mathbb{P}(\mathrm{Ext}^1(L,L^{-1}(-x)))^s \xrightarrow{\Psi_L} \mathcal{N} \in \mathsf{R}_3(1)$, we assign a line bundle $L \in \mathrm{Pic}^1(X)$. Then by Proposition

5.2.10, any line or conic, or twisted cubic cannot be contained in the intersection of two different $\mathbb{P}^{g+1}_L$(since intersections only arises on stable part), so we can observe that this line bundle L is unique for each rational map f. Therefore, we can conjecture that there is a morphism $R_3(1) \to \mathrm{Pic}^1(X)$. Moreover, by observing nodal curves in $\mathbf{M}_0(\widetilde{\mathbf{P}}_L, \beta)$, where the homology class $\beta = \pi^*[\text{line}] \in H_2(\widetilde{\mathbf{P}}_L)$ is the l.c.i pull-back of the homology class of a projective line in $\mathbb{P}^{g+1}_L$, we can guess further that there may be a morphism :

$$p : \Lambda_1 \to \mathrm{Pic}^1(X). \tag{5.20}$$

On the other hand, for a non-trisecant line bundle $L \in \mathrm{Pic}^1(X)$, we recall that the extended morphism $\widetilde{\Psi}_L : \widetilde{\mathbf{P}}_L \to \mathcal{N}$ is a closed embedding by Proposition 5.2.4. therefore the induced morphism of stable maps $\mathbf{M}_0(\widetilde{\mathbf{P}}_L, \beta) \to \mathbf{M}_0(\mathcal{N}, 3)$ is a closed embedding. We also conjecture that the conjecture morphism p compatible with the morphisms j and q.

Then we can expect that the fiber of the morphism p over the non-trisecant line bundle L, $p^{-1}(L)$ is isomorphic to the irreducible component of $\mathbf{M}_0(\widetilde{\mathbf{P}}_L, \beta)$, which is a closure of the locus of lines in $(\mathbb{P}^{g+1}_L)^s = \widetilde{\mathbf{P}}_L \setminus E$, where E is the exceptional divisor of $\widetilde{\mathbf{P}}_L$.

Therefore, based on this conjectural picture, we focus on the study of the stable map space $\mathbf{M}_0(\widetilde{\mathbf{P}}_L, \beta)$ in this thesis, for a non-trisecant degree 1 line bundle L. Furthermore, if we let $U \subset \mathrm{Pic}^1(X)$ be the open subset of non-trisecant degree 1 line bundles, then we expect that $\Lambda_1 \times_{\mathrm{Pic}^1(X)} U$ is isomorphic to an irreducible component of $\mathbf{M}_0(\widetilde{\mathbf{P}}, \beta) \times_{\mathrm{Pic}^1(X)} U$ which is expected to has a fiber bundle structure over U with fiber isomorphic to $\mathbf{M}_0(\widetilde{\mathbf{P}}_L, \beta)$. From now on, we fix L to be a degree 1 non-trisecant line bundle.

5.3.2 Stable maps in the blow-up space $\widetilde{\mathbf{P}}_{\mathrm{L}}$

In this subsection, we work on the following moduli space

$$\mathbf{M}_0(\widetilde{\mathbf{P}}_{\mathrm{L}}, \beta) \ (\subset \mathbf{M}_0(\mathcal{N}, 3))$$

of genus zero stable maps of degree 3, which is embedded in $\mathbf{M}_0(\mathcal{N}, 3)$. We start from the topological classification of genus zero stable maps in $\widetilde{\mathbf{P}}_{\mathrm{L}}$ with homology class β.

Lemma 5.3.1. Stable maps correspond to the closed points in the stable map space $\mathbf{M}_0(\widetilde{\mathbf{P}}, \beta)$ are classified by one of the following types. Recall that $\pi : \widetilde{\mathbf{P}}_{\mathrm{L}} \to \mathbb{P}^{g+1}_{\mathrm{L}}$ is the blow-up morphism.

1. Projective lines in $\mathbb{P}^{g+1}_{\mathrm{L}} \backslash \mathrm{X}$. Stable maps of this type form $2g$-dimensional open sublocus in $\mathbf{M}_0(\widetilde{\mathbf{P}}_{\mathrm{L}}, \beta)$.

2. Union of the strict transformation of a projective line in $\mathbb{P}^{g+1}_{\mathrm{L}}$ that intersects X on a point p and a projective line in the exceptional fiber of the point p, $\pi^{-1}(p) = \mathbb{P}^{g-1}_{\mathrm{L}(-p)}$. Stable maps of this type form $(2g-1)$-dimensional locally closed sublocus in $\mathbf{M}_0(\widetilde{\mathbf{P}}_{\mathrm{L}}, \beta)$.

3. Union of the strict transformation of a line in $\mathbb{P}^{g+1}_{\mathrm{L}}$ that intersects X on two distinct points p, q, a line in in the exceptional fiber $\pi^{-1}(p) = \mathbb{P}^{g-1}_{\mathrm{L}(-p)}$, and a projective line in another exceptional fiber $\pi^{-1}(q) = \mathbb{P}^{g-1}_{\mathrm{L}(-p)}$. Stable maps of this type form $(2g-2)$-dimensional locally closed sublocus in $\mathbf{M}_0(\widetilde{\mathbf{P}}_{\mathrm{L}}, \beta)$.

4. Union of the strict transformation of a projective line in $\mathbb{P}^{g+1}_{\mathrm{L}}$ that intersects X on two distinct points p, q and a stable map of degree two in the exceptional fiber $\pi^{-1}(p) = \mathbb{P}^{g-1}_{\mathrm{L}(-p)}$. Stable maps of this type form $2g$-dimensional locally closed sublocus in $\mathbf{M}_0(\widetilde{\mathbf{P}}_{\mathrm{L}}, \beta)$.

5. Union of the strict transformation of a projective line in $\mathbb{P}^{g+1}_L$ which is tangent to the curve X on a point p and a stable map of degree two in the exceptional fiber $\pi^{-1}(p) = \mathbb{P}^{g-1}_{L(-p)}$. Stable maps of this type form $(2g-1)$-dimensional closed sublocus in $\mathbf{M}_0(\widetilde{\mathbf{P}}_L, \beta)$.

Proof. We already know that $\mathrm{H}_2(\widetilde{\mathbf{P}}_L) \cong \mathbb{Z} \oplus \mathbb{Z}$ such that $(1,0)$ correspond to the homology class of the l.c.i.(locally complete intersection morphism) pull-back π^*[line] and $(0,1)$ correspond to the homology class of a projective line in the exceptional fiber $\pi^{-1}(p)$. Hence the homology class of the strict transform $\widetilde{\ell}$ of a line $\ell \subset \mathbb{P}^{g+1}$ that intersects the curve X with multiplicity m is $(1,-m)$. Then we can classify stable maps in the blow-up space $\widetilde{\mathbf{P}}_L$ using the equivalent conditions of the non-trisecant property of the curve X appeared in Corollary 5.2.5. The dimension counting is not difficult. For instance, we calculate the dimension of the sublocus of type (4) stable maps. We can observe that the locus of type (4) stable maps is a fibration over the base space $X \times X \setminus \Delta$. Let F be the fiber space of the fibration. Then F parametrizes stable maps of degree two in the projective space $\mathbb{P}^{g-1}$ which pass through a fixed point. Then the space F is irreducible by [59] and [43, Chapter III, Corollary 9.6]. Thus, the dimension of locus of type (4) of stable maps is equal to $2 + \dim Z = 2 + (2g-2) = 2g$. □

Next, we can consider the stable map space $\mathbf{M}_0(\widetilde{\mathbf{P}}_L, \beta)$ locally as a zero locus of a regular section of a vector bundle on a smooth space by the proof of [55, Corollary 4.6]. Therefore, we can observe that all irreducible components of the space $\mathbf{M}_0(\widetilde{\mathbf{P}}_L, \beta)$ have the dimension greater or equal than $\int_{\beta=\pi^*[\mathrm{line}]} c_1(T_{\widetilde{\mathbf{P}}_L}) + \dim\widetilde{\mathbf{P}}_L - 3 = 2g$. Now we introduce the main result of this Chapter.

Theorem 5.3.2. The stable map space $\mathbf{M}_0(\widetilde{\mathbf{P}}_L, \beta)$ has two irreducible components $\overline{B}_1$ and $\overline{B}_2$ such that:

1. B_1 parametrizes projective lines in $\mathbb{P}^{g+1}_L \setminus X$. Moreover, the union of subloci of types (1)-(3) and (5) stable maps is equal to the closure $\overline{B}_1$.

2. B_2 parametrizes the union of a smooth conic in the exceptional divisor of $\widetilde{\mathbf{P}}_L$ and strict transformation of a projective line ℓ intersect on a point for a projective line ℓ which intersects the curve X with multiplicity 2 (Thus ℓ can be a tangent line of X). Moreover, the union of subloci of stable maps of types (4) and (5) is equal to the closure $\overline{B}_2$.

In particular, the intersection $\overline{B}_1 \cap \overline{B}_2$ is equal to the sublocus of the type (5) stable maps of Lemma 5.3.1.

We note that component $\overline{B}_1$ is expected to be equal to $p^{-1}(L)$ where p is the conjectural morphism (5.20). For the proof of this theorem, we start by the computing the obstruction spaces of the type (4) stable maps.

Lemma 5.3.3. Consider a projective line $\ell \subset \widetilde{\mathbf{P}}_L$ where $\pi(\ell)$ intersects with X at two distinct point p, q. Then we have the following formula for the the normal bundle $N_{\ell/\widetilde{\mathbf{P}}_L}$ of the projective line ℓ in $\widetilde{\mathbf{P}}_L$

$$N_{\ell/\widetilde{\mathbf{P}}_L} \cong \mathcal{O}_\ell(-1)^{\oplus(g-2)} \oplus \mathcal{O}_\ell(-1) \oplus \mathcal{O}_\ell(1) \text{ or } \mathcal{O}_\ell(-1)^{\oplus(g-2)} \oplus \mathcal{O}_\ell^{\oplus 2}.$$

Proof. Consider a line ℓ_0 in $\mathbb{P}^{g+1}$ which cleanly intersecting the curve X at two distinct points p, q. We denote ℓ be the proper transform of the projective line ℓ_0 for the blow-up morphism $\pi : \widetilde{\mathbf{P}}_L = \mathrm{Bl}_X\mathbb{P}^{g+1} \to \mathbb{P}^{g+1}$. From the proof of [58, Lemma 1], we observe that the normal bundle $N_{\ell/\widetilde{\mathbf{P}}_L}$ fits into the short sequence in the following

$$0 \to \pi^* N_{\ell_0/\mathbb{P}^{g+1}} \otimes \mathcal{O}(-E)|_\ell \to N_{\ell/\widetilde{\mathbf{P}}_L} \to \mathbb{C}_p \oplus \mathbb{C}_q \to 0.$$

where the map $N_{\ell/\widetilde{\mathbf{P}}_L} \to \mathbb{C}_p \oplus \mathbb{C}_q$ is locally constructed by the following(cf. [34, Appendix B.6.10] way.

Consider $T_1, ..., T_{g+1}$ a local coordinate of $\mathbb{P}^{g+1}$ around the point p so such that locally we have $I_{\ell_0/\mathbb{P}^{g+1}} = \langle T_1, T_3, ..., T_{g+1}\rangle$, $I_{X/\mathbb{P}^{g+1}} = \langle T_2, T_3, ..., T_{g+1}\rangle$. Thus, we have a local coordinate $t_1, t_2, x_3, ..., x_{g+1}$ of $\widetilde{\mathbf{P}}_L$ around the point $\tilde{p}$ which is the lift of p in ℓ such that $\pi \circ T_1 = t_1, \pi \circ T_2 = t_2, \pi \circ T_i =$

$t_2 \circ x_i$ for $3 \leq i \leq g+1$. Hence we obtain $\pi^* I_{\ell_0/\mathbb{P}^{g+1}} = \pi^*\langle t_1, t_3, ..., t_{g+1}\rangle = \langle t_1, t_2x_3, ..., t_2x_{g+1}\rangle$. Therefore, locally we conclude that $\langle t_2\rangle$ is the defining ideal of the exceptional divisor E of $\widetilde{\mathbf{P}}_L$. Thus we observe that there is the following exact sequence :

$$0 \to I_{\ell/\widetilde{\mathbf{P}}_L} \cdot I_{E/\widetilde{\mathbf{P}}_L} \to \pi^* I_{\ell_0/\mathbb{P}^{g+1}} \to \pi^* I_{\ell_0/\mathbb{P}^{g+1}} / I_{\ell/\widetilde{\mathbf{P}}_L} \cdot I_{E/\widetilde{\mathbf{P}}_L} \to 0.$$

By taking pull-back of above sequence on the projective line ℓ, we have the following short exact sequence

$$0 \to I_{\ell/\widetilde{\mathbf{P}}_L}/I^2_{\ell/\widetilde{\mathbf{P}}_L} \otimes \mathcal{O}_{\widetilde{\mathbf{P}}_L}(-E) \to \pi^*(I_{\ell_0/\mathbb{P}^{g+1}}/I^2_{\ell_0/\mathbb{P}^{g+1}}) \xrightarrow{\tilde{\partial}_p} \mathbb{C}_p \to 0$$

where the map $\tilde{\partial}_p$ is given by the differentiation of the tangent vector $\frac{\partial}{\partial t_1}$ in the tangent space $T_p\widetilde{\mathbf{P}}_L$. By taking dual of this sequence, we obtain a map $N_{\ell/\widetilde{\mathbf{P}}_L} \to \mathbb{C}_p$. In a similar manner, we can also define a map $N_{\ell/\widetilde{\mathbf{P}}_L} \to \mathbb{C}_q$.

Since we have $N_{\ell_0/\mathbb{P}^{g+1}} = \mathcal{O}_{\ell_0}(1)^{\oplus g}$ and $\mathcal{O}(-E)|_\ell = \mathcal{O}_\ell(-2)$, we complete the proof. $\square$

Similar to the cases of other Fano varieties, normal bundle of the projective lines in the blow-up space can be classified in a geometric method as follows.

Corollary 5.3.4. If two projective tangent lines T_pX and T_qX are coplanar (respectively, skew lines), then $N_{\ell/\widetilde{\mathbf{P}}_L} \cong \mathcal{O}_\ell(-1)^{\oplus(g-1)} \oplus \mathcal{O}_\ell(1)$ (respectively, $N_{\ell/\widetilde{\mathbf{P}}_L} \cong \mathcal{O}_\ell(-1)^{\oplus(g-2)} \oplus \mathcal{O}_\ell^{\oplus 2}$).

Proof. We easily obtain the conclusions by computations using local coordinates in a similar manner as in the proof of Lemma 5.3.3. $\square$

Next, consider a smooth conic Q contained in the exceptional divisor E. Then the conic Q should be contained in some exceptional fiber $\mathbb{P}^{g-1}$ of the projective bundle $E = \mathbb{P}(N_{X/\mathbb{P}^{g+1}}) \to X$, we can observe that :

$$N_{Q/E} \cong N_{Q/\mathbb{P}^{g-1}} \oplus N_{\mathbb{P}^{g-1}/E}|_Q \cong (\mathcal{O}_Q(2) \oplus \mathcal{O}_Q(1)^{\oplus(g-3)}) \oplus \mathcal{O}_Q$$

because $H^1(\mathcal{O}_Q(i)) = 0$ for $i = 1, 2$. Hence we have the following normal bundle sequence :

$$0 \to N_{Q/E} \to N_{Q/\widetilde{\mathbf{P}}_L} \to N_{E/\widetilde{\mathbf{P}}_L}|_Q \cong \mathcal{O}_Q(-1) \to 0, \tag{5.21}$$

which implies the following isomorphism

$$N_{Q/\widetilde{\mathbf{P}}_L} \cong (\mathcal{O}_Q(2) \oplus \mathcal{O}_Q(1)^{\oplus(g-3)}) \oplus \mathcal{O}_Q \oplus \mathcal{O}_Q(-1). \tag{5.22}$$

Proposition 5.3.5. Let $[C] \in B_2$ be a stable map of the form $C = \ell \cup Q$, which is the union of a projective line ℓ in $\widetilde{\mathbf{P}}_L$ and a smooth conic Q in the exceptional divisor, cleanly intersecting on a point z. Then we have $H^1(N_{C/\widetilde{\mathbf{P}}_L}) = 0$.

Proof. By construction, C is a nodal curve. Hence, the conormal sheaf of C in the blow-up space $N^\vee_{C/\widetilde{\mathbf{P}}_L} := I_{C/\widetilde{\mathbf{P}}_L}/I^2_{C/\widetilde{\mathbf{P}}_L}$ is locally free. Then, from two exact sequences in the following

- $0 \to N^\vee_{C/\widetilde{\mathbf{P}}_L} \to \Omega_{\widetilde{\mathbf{P}}_L}|_C \to \Omega_C \to 0$ and
- $0 \to \mathcal{O}_\ell(-1) \to \mathcal{O}_C \to \mathcal{O}_Q \to 0,$

we obtain the following commutative diagram :

$$\begin{tikzcd}
\mathrm{Ext}^1(\Omega_C, \mathcal{O}_C) \arrow[r] \arrow[d, two heads] & \mathrm{Ext}^1(\Omega_{\widetilde{\mathbf{P}}_L}, \mathcal{O}_C) \arrow[r] \arrow[d, "\cong"] & \mathrm{Ext}^1(N^\vee_{C/\widetilde{\mathbf{P}}_L}, \mathcal{O}_C) \arrow[r] \arrow[d] & 0 \\
\mathrm{Ext}^1(\Omega_C, \mathcal{O}_Q) \arrow[r] & \mathrm{Ext}^1(\Omega_{\widetilde{\mathbf{P}}_L}, \mathcal{O}_Q) \arrow[r] & \mathrm{Ext}^1(N^\vee_{C/\widetilde{\mathbf{P}}_L}, \mathcal{O}_Q) \arrow[r] & 0.
\end{tikzcd}$$

Since the curve $C = \ell \cup Q$ has the unique nodal point z, we have $\mathrm{Ext}^1(\Omega_C, \mathcal{O}_C) \cong$

$\mathbb{C}$. Moreover, $\mathrm{Ext}^2(\Omega_C, \mathcal{O}_\ell(-1)) = 0$ implies the surjectiveness of the first vertical map. By Lemma 5.3.6, we check the second vertical map $\mathrm{H}^1(T_{\widetilde{\mathbf{P}}_L}|_C) \cong \mathrm{Ext}^1(\Omega_{\widetilde{\mathbf{P}}_L}, \mathcal{O}_C) \to \mathrm{Ext}^1(\Omega_{\widetilde{\mathbf{P}}_L}, \mathcal{O}_Q) \cong \mathrm{H}^1(T_{\widetilde{\mathbf{P}}_L}|_Q) = \mathbb{C}$ is an isomorphism. Therefore the claim is true whenever $\mathrm{H}^1(N_{C/\widetilde{\mathbf{P}}_L}|_Q) = 0$. Next, consider the following structure sequence :

$$0 \to N^\vee_{C/\widetilde{\mathbf{P}}_L}|_Q \to N^\vee_{Q/\widetilde{\mathbf{P}}_L} \xrightarrow{\partial_z} \mathbb{C}_z \to 0$$

where the map ∂_z is given by the differentiation of the tangent vector $T_z\ell$. We can show this by the following local computation. We can choose local coordinates $x_1, ..., x_{g+1}$ of $\widetilde{\mathbf{P}}_L$ around the point z where locally we have $I_{Q/\widetilde{\mathbf{P}}_L} = \langle x_2, x_3, ..., x_{g+1}\rangle, I_{\ell/\widetilde{\mathbf{P}}_L} = \langle x_1, x_3, ..., x_{g+1}\rangle$. Then, we obtain $I_{C/\widetilde{\mathbf{P}}_L} = \langle x_1x_2, x_3, ..., x_{g+1}\rangle$. Hence, we have the following short exact sequence :

$$0 \to I_{C/\widetilde{\mathbf{P}}_L} \to I_{Q/\widetilde{\mathbf{P}}_L} \to I_{Q/\widetilde{\mathbf{P}}_L}/I_{C/\widetilde{\mathbf{P}}_L} \to 0$$

By taking pull-back to the smooth conic Q, we obtain the sequence :

$$0 \to I_{C/\widetilde{\mathbf{P}}_L}/I_{C/\widetilde{\mathbf{P}}_L}|_Q \to I_{Q/\widetilde{\mathbf{P}}_L}/I^2_{Q/\widetilde{\mathbf{P}}_L} \to \mathbb{C}_z \to 0$$

Here, we can observe the map $I_{Q/\widetilde{\mathbf{P}}_L}/I^2_{Q/\widetilde{\mathbf{P}}_L} \to \mathbb{C}_z$ is given by the differentiation of the tangent vector $T_p\ell$ since it kills the local coordinates $x_1, x_3, ..., x_{g+1}$. Then we can show that the composition map $\mathcal{O}_Q(1) \cong N^\vee_{E/\widetilde{\mathbf{P}}_L}|_Q \subset N^\vee_{Q/\widetilde{\mathbf{P}}_L} \xrightarrow{r} \mathbb{C}_p$(see (5.21).) is not zero since the projective line ℓ transversally intersects the exceptional divisor E. Therefore we easily show that $N^\vee_{C/\widetilde{\mathbf{P}}_L}|_Q \cong \mathcal{O}_Q(s) \oplus N^\vee_{Q/E}$ for some point $s \in Q$. Because $\mathrm{H}^1(\mathcal{O}_Q(-s)) = \mathrm{H}^1(N_{Q/E}) = 0$, we completes the proof. $\square$

Lemma 5.3.6. (cf. [54, Lemma 6.4]) Consider the following long exact sequence :

$$\mathrm{H}^0(T\widetilde{\mathbf{P}}_L|_\ell) \oplus \mathrm{H}^0(T\widetilde{\mathbf{P}}_L|_Q) \xrightarrow{\alpha} \mathrm{H}^0(T_p\widetilde{\mathbf{P}}_L) \to \mathrm{H}^1(T\widetilde{\mathbf{P}}_L|_C) \to \mathrm{H}^1(T\widetilde{\mathbf{P}}_L|_\ell) \oplus \mathrm{H}^1(T\widetilde{\mathbf{P}}_L|_Q) \to 0.$$

which comes from the structure sequence $0 \to \mathcal{O}_C \to \mathcal{O}_\ell \oplus \mathcal{O}_Q \to \mathbb{C}_p \to 0$. Then the first map α is surjective and therefore we have $\mathrm{H}^1(T\widetilde{\mathbf{P}}_L|_C) \cong \mathrm{H}^1(T\widetilde{\mathbf{P}}_L|_Q) \cong \mathbb{C}$.

Proof. Since the line ℓ and the exceptional divisor E transversally intersect on the point p, we have $T_p\widetilde{\mathbf{P}}_L \cong T_p\ell \oplus T_pE$. From $\mathrm{H}^0(T\widetilde{\mathbf{P}}_L|_\ell) = \mathrm{H}^0(T\ell) \oplus \mathrm{H}^0(N_{\ell/\widetilde{\mathbf{P}}_L})$, we observe that the map :

$$\mathrm{H}^0(T\widetilde{\mathbf{P}}_L|_\ell) \twoheadrightarrow \mathrm{H}^0(T_p\ell) \tag{5.23}$$

is surjective because of the positive degree part $\mathrm{H}^0(T\ell) = \mathrm{H}^0(\mathcal{O}_\ell(2))$.

On the other hand, we obtain $\mathrm{H}^0(T\widetilde{\mathbf{P}}_L|_Q) = \mathrm{H}^0(T\mathbb{P}^{g-1}|_Q) \oplus \mathrm{H}^0(N_{\mathbb{P}^{g-1}/\widetilde{\mathbf{P}}_L}|_Q)$ from $Q \subset \mathbb{P}^{g-1} \subset E$. Then we can easily check that the projection to the first summand

$$\mathrm{H}^0(T\mathbb{P}^{g-1}|_Q) \twoheadrightarrow \mathrm{H}^0(T_p\mathbb{P}^{g-1})$$

is a surjection. Moreover, with some calculation, we can easily show that $N_{\mathbb{P}^{g-1}/\widetilde{\mathbf{P}}_L}|_Q \cong \mathcal{O}_Q \oplus \mathcal{O}_Q(-1)$ where $N_{\mathbb{P}^{g-1}/E}|_Q = \mathcal{O}_Q$ and hence the positive degree part $\mathrm{H}^0(N_{\mathbb{P}^{g-1}/\widetilde{\mathbf{P}}_L}|_Q) = \mathrm{H}^0(N_{\mathbb{P}^{g-1}/E}|_Q)$ maps to $\mathrm{H}^0(N_{\mathbb{P}^{g-1}/E,p}) = \mathbb{C}$. Therefore we check the following map

$$\begin{aligned} \mathrm{H}^0(T\widetilde{\mathbf{P}}_L|_Q) = \mathrm{H}^0(T\mathbb{P}^{g-1}|_Q) \oplus \mathrm{H}^0(N_{\mathbb{P}^{g-1}/\widetilde{\mathbf{P}}_L}|_Q) \\ \twoheadrightarrow \mathrm{H}^0(T_p\mathbb{P}^{g-1}) \oplus \mathrm{H}^0(N_{\mathbb{P}^{g-1}/E,p}) = \mathrm{H}^0(T_pE) \end{aligned} \tag{5.24}$$

is surjective. Moreover, By (5.23) and (5.24), we check the map α is surjective. The last isomorphisms obtained from the equation (5.22) and the Lemma 5.3.3. $\square$

Finally, we are ready to prove Theorem 5.3.2, our main theorem.

Proof of Theorem 5.3.2. We can easily observe that the locus B_1 is isomorphic to an open subset in the Grassmannian $\mathrm{Gr}(2, g+2)$ that parametrizes projective lines in $\mathbb{P}^{g+1}$ which do not intersect the curve X. So we have $\overline{B}_1$ is irreducible. Moreover, we already know that B_2 is irreducible by the proof of Lemma 5.3.1. By Lemma 5.3.1, $\overline{B}_1$ and $\overline{B}_2$ both have the expected dimension $2g$. Furthermore, there does not exist other irreducible component whose dimension is greater or equal than $2g$ ([55, Proof of corollary 4.6]). So we conclude that $\overline{B}_1$ and $\overline{B}_2$ are all irreducible components of the stable map space $\mathbf{M}_0(\widetilde{\mathbf{P}}_L, \beta)$. Since the loci of type (1), (2) and (3) stable maps are not included in $\overline{B}_2$, they must be contained in the component $\overline{B}_1$. Moreover, the loci of type (4) and (5) stable maps should be in component $\overline{B}_2$ by definition.

Since every irreducible component in the stable map space $\mathbf{M}_0(\widetilde{\mathbf{P}}_L, \beta)$ has expected dimension $2g$, we conclude that $\mathbf{M}_0(\widetilde{\mathbf{P}}_L, \beta)$ is locally a complete intersection through the proof of [55, Corollary 4.6]. When the point p approaches to the point q, a type (3) stable map degenerates to a union of the strict transformation of a projective line in $\mathbb{P}^{g+1}$ that is tangent to the curve X and a singular conic in the exceptional fiber $\pi^{-1}(p) \cong \mathbb{P}^{g-1}$, which is a type (4) stable map. Therefore we have $\overline{B}_1 \cap \overline{B}_2 \neq \varnothing$. Furthermore, since there exists only two irreducible components $\overline{B}_1$ and $\overline{B}_2$, we can show that their intersection $\overline{B}_1 \cap \overline{B}_2$ is pure dimensional with dimension $2g-1$ by using Hartshorne's connectedness theorem([41, Theorem 3.4]).

Through the proof of Proposition 5.3.5, we can check that a type (4) stable map (4) which has smooth conic component has no obstruction. Thus it corresponds to a smooth closed point in the moduli space. Therefore it is not possible to be an element of the intersection $\overline{B}_1 \cap \overline{B}_2$. Also, the sublocus consists of type (4) stable maps which have singular conic component is $(2g-2)$-dimensional, and the sublocus consists of type (5) stable maps is $(2g-1)$-dimensional and clearly irreducible. In summary, we obtain the conclusion that the intersection $\overline{B}_1 \cap \overline{B}_2$ consists of type (5) stable maps. $\square$

Chapter 6

Further questions and research

6.1 Hilbert space of conics for hyperplane sections of Grassmannians $\mathrm{Gr}(2, n)$ for general n

By the result of Piontkowski and Van de Ven, Chapter 2, Proposition 2.6.7, we know that the automorphism group of the hyperplane section $\mathrm{Gr}(2, 2n) \cap H$ is $\mathrm{Sp}(2n, \mathbb{C})/\mathbb{Z}_2$ and it acts on $\mathrm{Gr}(2, 2n) \cap H$ homogeneously. Using this group action, we want to use the result of Chung, Hong and Kiem [18]. But we cannot sure $\mathrm{Gr}(2, 2n) \cap H$ is a homogeneous variety. Instead, we can check that $\mathrm{Gr}(2, 2n) \cap H$ satisfies the condition $(1) - (4)$ in [18, Lemma 2.1], which is necessary to use the machinery in the paper. We state the conditions as follows :

Lemma 6.1.1. [18, Lemma 2.1] Let X be a projective homogeneous variety, fix a projective embedding $\phi : X \to \mathbb{P}^k$, and define $\mathcal{O}_X(1) := \phi^*\mathcal{O}_{\mathbb{P}^k}(1)$. Then we have the following.

1. $H^1(\mathbb{P}^1, f^*T_X) = 0$ for every morphism $\mathbb{P}^1 \to X$

2. Let us define the moduli space of lines with one marked point $M_{0,1}(X,1)$ to be :

$$M_{0,1}(X,1) := \{(f : \mathbb{P}^1 \to X, p \in \mathbb{P}^1) \mid \deg f^*\mathcal{O}_X(1) = 1\}.$$

   Then the evaluation morphism $ev : M_{0,1}(X,1) \to X$ at the marked point p is smooth.

3. The Fano variety of planes $F_2(X)$ in X is smooth.

4. The defining ideal I_X of $X \subset \mathbb{P}^k$ is generated by quadric polynomials.

Since $\mathrm{Gr}(2,2n) \cap H$ is a homogeneous variety. Therefore it automatically satisfies the condition (1) and (2) of [18, Lemma 1.4]. We checked that $\mathrm{Gr}(2,2n) \cap H$ satisfies the condition (3), the smoothness of the Fano variety of planes, in Chapter 4, Proposition 4.3.13. Moreover, the condition (4), that the defining ideal of the variety X in the projective embedding $X \subset \mathbb{P}^k$ is generated by quadratic equations, is automatically satisfied since $\mathrm{Gr}(2,2n) \cap H$ is a hyperplane section of the Grassmannian. The defining ideal of the Grassmannian in the Plücker embedding is generated by quadratic equations [93, Chapter I, Section 4, Example 1]. Therefore, we can use [18, Theorem 3.7, Theorem 4.11 and Theorem 4.16], to study the birational geometry of Hilbert scheme of conics $H_2(\mathrm{Gr}(2,2n) \cap H)$ and Hilbert scheme of twisted cubics $H_3(\mathrm{Gr}(2,2n) \cap H)$ on the hyperplane section $\mathrm{Gr}(2,2n) \cap H$.

For other cases, $\mathrm{Gr}(2,2n) \cap H_1 \cap H_2$, $\mathrm{Gr}(2,2n+1) \cap H$, $\mathrm{Gr}(2,2n+1) \cap H_1 \cap H_2$, there are lots of geometric structure including automorphism groups and their orbits classified by Piontkowski and Van de Ven, which we introduced in Chapter 2. In the case of $\mathrm{Gr}(2,2n+1) \cap H$, $\mathrm{Gr}(2,2n+1) \cap H_1 \cap H_2$, geometry comes from center point and center curves looks interesting. So we expect that these geometric structures may help us to study Hilbert scheme of conics on these spaces.

On the other hand, for the case of $\mathrm{Gr}(2,6) \cap H$ and $\mathrm{Gr}(2,6) \cap H_1 \cap H_2$, we can directly use our key blow-up blow-down diagram in Chapter 5, (4.7) again :

$$\begin{tikzcd} & \widetilde{S}(Y) \arrow[dl, "\Phi"'] \arrow[dr, "\Xi"] & \\ S(Y) & & H_2(Y), \end{tikzcd}$$

since we can compute blow-up and blow-down locus a by direct local chart computation in $\mathrm{Gr}(2,6)\cap H$ and $\mathrm{Gr}(2,6)\cap H_1\cap H_2$ in the same manner. We expect that we can show the smoothness of Hilbert schemes $H_2(\mathrm{Gr}(2,6)\cap H)$ and $H_2(\mathrm{Gr}(2,6)\cap H_1\cap H_2)$.

6.2 Conjectural picture in Chapter 5

The conjectural picture in Chapter 5 is not verified yet. But we are quite sure about the existence of the conjectural morphisms in Chapter 5, (5.19)

$$\mathrm{Bl}_{X\times \mathrm{Pic}^1(X)}\mathbb{P}\mathrm{Ext}^1(\mathcal{L},\mathcal{L}^{-1}(-(x\times \mathrm{Pic}^1(X)))) := \widetilde{\mathbf{P}} \xrightarrow{\widetilde{\Psi}} \mathcal{N},$$

and (5.20) :

$$p : \Lambda_1 \to \mathrm{Pic}^1(X).$$

But we still have no idea how to construct it explicitly.

Moreover, for a projection

$$q : \mathbf{M}_0(\widetilde{\mathbf{P}}, \beta) \to \mathrm{Pic}^1(X),$$

it is not clear that the fiber of q over $L \in \mathrm{Pic}^1(L)$, $q^{-1}(L)$ is isomorphic to $\mathbf{M}_0(\widetilde{\mathbf{P}}_L, \beta)$ since there can be a limit of a family of lines in $(\mathbb{P}^{g+1}_L)^s$, varying the line bundle $L \in \mathrm{Pic}^1(X)$.

Then, consider a component $\overline{B}_{rel,1}$ of $\mathbf{M}_0(\widetilde{\mathbf{P}}_L, \beta)$ the closure of the locus of lines in $(\mathbb{P}^{g+1}_L)^s$ where L runs over all elements in $\mathrm{Pic}^1(X)$. Let $U \subset \mathrm{Pic}^1(X)$, which is a locus of non-trisecant line bundles. Then we cannot be sure that elements of $\overline{B}_{rel,1} \times_{\mathrm{Pic}^1(X)} U$ consists of stable maps of types $(1), (2), (3), (5)$ in Chapter 5, Lemma 5.3.1.

So completing this conjectural picture is a task we should do afterwards.

6.3 Classify all topological types of stable maps in Λ_1

Although we classified all topological types of stable maps in $M_0(\widetilde{\mathbf{P}}_L, \beta)$ in Chapter 5, it is still unclear these are all topological types of stable maps in Λ_1. One reason is that the conjectural picture is not completed yet, and the other reason is that we only studied non-trisecant line bundle cases.

Let us assume that we succeed to complete the conjectural picture in Chapter 5, Section 5.3.1, then we conclude that elements of $\Lambda_1 \times_{\mathrm{Pic}^1(X)} U$, which is the surjective image of $\overline{B}_{rel,1} \times_{\mathrm{Pic}^1(X)} U$ consists of stable maps of types $(1), (2), (3), (5)$ in Chapter 5, Lemma 5.3.1. Then we should study topological types of stable maps in $\Lambda_1 \times_{\mathrm{Pic}^1(X)} \mathrm{Pic}^1(X) \setminus U$, which is covered by the image of $\mathbf{M}_0(\widetilde{\mathbf{P}}_L, \beta)$ where a line bundle L runs over $L \in \mathrm{Pic}^1(X) \setminus U$, which is the locus of trisecant line bundles. Therefore we should study topological types of stable maps in the moduli space $\mathbf{M}_0(\widetilde{\mathbf{P}}_L, \beta)$ for a trisecant line bundle L to complete the classification of stable maps in the component $\Lambda_1 \subset \mathbf{M}_0(\mathcal{N}, 3)$.

6.4 Hilbert compactifications for $R_3(\mathcal{N})$

In this thesis, we only considered the Kontsevich compactification of a moduli of smooth rational curves $R_3(\mathcal{N})$ in $\mathcal{N}$. In [54], the authors considered Kontsevich compactification $\mathbf{M}_0(\mathcal{N}, 2)$ and Hilbert compactification $H_2(\mathcal{N})$ of the degree 2 smooth rational curves $R_2(\mathcal{N})$ and related them by blow-ups and contractions. So we want to consider the Hilbert compactification of the component $R_3(1)$ either, and find a birational relation with the Kontsevich compactification Λ_1.

6.5 Generalization to Moduli space of vector bundles with even determinants

Although we only considered Moduli space $\mathcal{N}$ of rank 2 stable vector bundles with odd determinant on the smooth projective curve $\mathcal{X}$ over $\mathbb{C}$ with genus $g \geq 4$, by the results in Chapter 2, Subsection 2.6.1, it is also meaningful to consider the even determinant case, let us denote this moduli space by $\mathcal{N}_e$. Drezet-Narasimhan [29] showed that it has a Picard group isomorphic to $\mathbb{Z}$, generated by generalized Theta divisor Θ. Furthermore, Brivio-Verra [6] showed that Θ is very ample if $g(X) \geq 3$ and X is not hyperelliptic. So we can define a degree of a rational curve via the projective embedding given by the very ample divisor Θ. So it is reasonable to consider a Hilbert scheme of lines, smooth conics and twisted cubics in this space $\mathcal{N}_e$. Unfortunately, this space is singular on the locus of strictly semi-stable bundles, we cannot consider stable map space on $\mathcal{N}_e$, but the existence of the singular locus may lead to an interesting phenomenon, and it will be interesting to observe relations with these Hilbert schemes with the Hilbert schemes in $\mathcal{N}$, the moduli space bundles with odd determinant.

6.6 Generalization to the moduli space of symplectic vector bundles

Furthermore, there are various examples of moduli space of vector bundles with additional structures, for examples, moduli space of symplectic vector bundles. We state the definition.

Definition 6.6.1 (Moduli space of symplectic vector bundles). A rank $2n$ symplectic vector bundle E on a variety V is defined by the following data :

(i) A local trivialization on an open cover $\{U_i\}_{i\in\mathcal{I}}$ of V, $E|_{U_i} = U_i \times k^n$.

(ii) A transition morphism $\varphi_{ij} : U_i \cap U_j \times k^{2n} \to U_i \cap U_j \times k^{2n}$ on each intersection $U_i \cap U_j$ such that its restriction to each fiber k^n is an element of $\mathrm{Sp}_n(k)$. Furthermore, transition morphisms satisfies the cocycle condition $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$.

We denote a moduli of semi-stable rank $2n$ symplectic vector bundles on a curve X as $\overline{M}_X(\mathrm{Sp}_n(k))$.

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Moduli space $\overline{M}_X(\mathrm{Sp}_n(k))$ is deeply studied by the thesis of Hitching [44]. By the result of Ramanathan [89], [90, Theorem 5.9], moduli space of semi-stable principal G-bundle on a smooth projective connected curve over $\mathbb{C}$ with genus ≥ 2, for a reductive group G is normal projective, Cohen-Macaulay scheme. Then we can check that $\overline{M}_X(\mathrm{Sp}_n(k))$ is isomorphic to the moduli space semi-stable principal $\mathrm{Sp}_n(k)$-bundle. Then by the result of [64], Picard groups of moduli spaces of semi-stable principal G-bundles, for a simply connected algebraic group G is isomorphic to an infinite cyclic group $\mathbb{Z}$.

For the case of $G = \mathrm{Sp}_n(k)$, by [64], $\overline{M}_X(\mathrm{Sp}_n(k))$ is generated by determinant bundle L on $\overline{M}_X(\mathrm{Sp}_n(k))$. But it is not certain to determine the integer m, that L^m becomes a very ample line bundle. So what we should to first is to fix a polarization of the projective variety $\overline{M}_X(\mathrm{Sp}_n(k))$, then we can study Hilbert scheme of rational curves of various degrees in this moduli space.

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국문초록

이 논문에서 우리는 두 가지 부류의 파노 대수다양체들을 다룬다. 하나는 복소체 위의 매끄러운 사영 곡선 위의 determinant가 고정된 2차원 벡터 다발들의 모듈라이 공간이고, 다른 한 가지 부류는 그라스마니안 다양체 $\mathrm{Gr}(2,5)$의 초평면들의 교집합으로 나타내어지는 파노 대수다양체들이다. 먼저 우리는 이러한 파노 대수다양체들 위의 매끄러운 유리곡선들의 모듈라이 공간을 공부한다. 특히 그라스마니안 다양체 $\mathrm{Gr}(2,5)$의 초평면들의 교집합에 대해서는, 차수가 3 이하인 경우 매끄러운 유리곡선들의 모듈라이 공간이 유리 다양체가 됨을 보인다.

다음으로 우리는 이러한 매끄러운 유리곡선들의 모듈라이 공간의 다양한 긴밀화를 생각한다. 매끄러운 사영 곡선 위의 determinant가 고정된 2차원 벡터 다발들의 모듈라이 공간에 대해서는, 매끄러운 유리곡선들의 모듈라이 공간을 스테이블 맵 공간안에서 긴밀화를 하여 그 긴밀화된 공간의 경계에 어떠한 원소들이 포함될 것인지를 공부하였다. 그라스마니안 다양체 $\mathrm{Gr}(2,5)$의 초평면들의 교집합에 대해서는, 매끄러운 유리곡선들의 모듈라이 공간을 힐버트 스킴 안에서 긴밀화를 하여 긴밀화된 공간의 쌍유리 기하학을 공부하고, 이를 통해 긴밀화된 공간이 매끄러운 공간이 됨을 보인다.

주요어휘: 모듈라이 공간, 유리곡선, 파노 대수다양체, 힐버트 스킴, 스테이블 맵 공간, 그라스마니안, 벡터 다발들의 모듈라이
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감사의 글

먼저 지도교수님이신 김영훈 선생님께 감사를 드립니다. 긴 대학원 생활 동안 학문적으로 지도해 주시고 제가 나아갈 방향을 제시해 주신 것은 물론 한 인간으로써의 자세에 대해서도 많은 조언과 지도를 아끼지 않아 주신 것에 감사드립니다. 잠시나마 제자로써 선생님의 번뜩이는 수학적 업적들을 옆에서 지켜볼 수 있었던 것은 제 인생의 큰 행운이었고 행복한 순간이었다고 생각합니다. 그런 빛나는 기억들이 제가 앞으로 흔들리거나 나태해질 때 저를 붙잡아 줄 것이라고 믿습니다. 다시 한 번 진심으로 감사를 드립니다.

또한 바쁘신 중에도 저의 논문심사를 흔쾌히 맡아 주신 Atanas Iliev 선생님, 현동훈 선생님, 최진원 선생님, 정기룡 선생님께도 감사드립니다. Atanas Iliev 선생님께서는 저의 부족한 영어실력으로 인해 의사소통이 원활하지 않은데도 제가 찾아가면 시간을 아끼지 않고 항상 많은 것을 알려주신 것에 감사드립니다. 정기룡 선생님께서는 제가 대학원 생활동안 쓴 두 편의 논문의 공동저자로써 논문을 함께 쓰는 동안 정말 많은 도움을 주셨고, 부족했던 저로써는 그 과정에서 정말 세세한 부분까지 많은 것을 배울 수 있었기에 감사드립니다. 또 학부부터 대학원까지 저에게 많은 관심을 보여 주시고 많은 조언과 격려와 조언을 해 주신 홍재현 선생님, 김범식 선생님, 김창호 선생님, 오병권 선생님, 최인송 선생님, 허석문 선생님께 감사드립니다. 홍재현 선생님께서는 정기룡 교수님과 저와 함께 3명이 함께 쓴 논문의 공동저자로써 논문을 함께 쓰는 동안 많은 도움을 주셨고, 그 과정에서 많은 것을 배울 수 있었기에 감사드립니다.

그리고 많은 학회에서 만나 연구에 대한 질문을 받아 주시고 많은 조언을 해 주신 오정석 박사님과, 역시 많은 학회에서 만나 서로 교류하며 함께 논문을 썼던 신재선에게도 감사의 뜻을 전합니다.

대학원에 처음 들어갔을 때부터 좋은 세미나를 열어 주시고 공부에 대해서 그리고 대학원 생활에 대해서 많은 조언을 해 주시고, 부족한 저의 대학원 생활에 많은 정신적인 의지가 되어 주신 이경석 선배님, 나주한 선배님, 노현호 선배님, 김윤환 선배님께 감사의 말씀을 전합니다. 다음으로 학부생 때부터 즐거운

시간을 함께하고 수리과학부 학부생 세미나를 진행하며 많은 추억을 공유했고 계속해서 많은 교류를 나누었던 유필상 선배님, 김동관 선배님, 박인성 선배님과 좌동욱 형에게 감사의 뜻을 전합니다. 또 이 졸업논문의 영어 문법 교정에 많은 도움을 주었고, 평소에도 영어 문법 교정과 관련해서 흔쾌히 많은 도움을 준 김현문에게도 감사의 뜻을 전합니다.

중학교 시절 공부하고 생각하는 일의 즐거움을 가르쳐 주시고 소중한 추억을 함께한 은사이신 안신근 선생님과, 초등학교 시절 과학자로의 꿈을 키울 수 있었던 소중한 장을 제공해 준 한국과학우주청소년단에도 감사의 뜻을 전합니다.

그밖에도 여기에 쓰지 못한 11년 동안의 학부와 대학원 생활을 하며 즐거운 일들을 함께하고 힘들 때 곁에 있어 준 수많은 선후배들과 친애하는 친구들이 많지만 일일이 거명하자면 너무 길어지기에, 기회가 되어 만날 때마다 감사의 뜻을 계속해서 전하고자 합니다. 분명한 점은 쉽지만은 않았던 저의 생활을 제가 무사히 마칠 수 있었던 것은, 결코 저 혼자만의 힘이 아니라 제가 힘들 때 곁에 있어주고 격려와 조언을 아끼지 않아 주었던 소중한 많은 이들의 도움이 있었기 때문에 가능했다는 점입니다. 이를 항상 기억하고 저 주변의 누군가의 어려운 일에 도움을 주는 사람이 되고자 다짐해 봅니다.

그리고 또한 학부시절부터 대학원 시절까지 제가 11년 반 동안 몸담았던 서울대학교에 감사함을 표하고, 또 이제 정든 터전을 떠나는 사람으로써의 아쉬움을 느낍니다. 서울대학교의 훌륭한 시설의 혜택을 받고, 서울대학교 수리과학부의 장학금을 통해 많은 경제적인 도움을 받았고, 수리과학부의 행정 직원분들께서는 저에게 친절하게 대해 주셨고 많은 도움을 주셨습니다. 다시 한 번 서울대학교와 서울대학교 수리과학부에 감사드립니다.

마지막으로는 제가 학문에 최대한 집중할 수 있도록 모든 지원을 아끼지 않아 주시고, 항상 저를 사랑해 주시고 믿어 주신 부모님께 이루 말할 수 없는 감사와 사랑의 마음을 전하고 싶습니다. 부모님께서 제가 가는 이 길에 무한한 신뢰와 지지를 보내 주시지 않으셨다면, 저는 결코 지금처럼 학문에 집중할 수가 없었을 것입니다. 11년간 묵묵히 뒷바라지 해 주신 부모님께 이 논문을 바치며, 앞으로도 훌륭한 연구자, 훌륭한 사람이 될 수 있도록 힘껏 살아갈 것을 다짐해 봅니다.