이학박사 학위논문

On intersection forms of 4-manifolds with boundary

(경계를 가진 4차원 다양체의 교차형식에 대하여)

2018년 8월

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위 원 장	Otto Van Koert	(인)
부위원장	박 종 일	(인)
위 원	조 철 현	(인)
위 원	윤 기 헌	(인)
위 원	김 상 현	(인)

On intersection forms of 4-manifolds with boundary

A dissertation
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Dong Heon Choe

Dissertation Director : Professor Jongil Park

Department of Mathematical Sciences
Seoul National University

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Abstract

On intersection forms of 4-manifolds with boundary

Dong Heon Choe

Department of Mathematical Sciences
The Graduate School
Seoul National University

In the early 1980s, Freedman showed that every unimodular bilinear form is realized as the intersection form of a simply connected closed topological 4-manifold. On the other hand, Donaldson proved that the intersection form of a smooth definite closed 4-manifold is diagonalizable over the integer. The intersection form of a 4-manifold is the integral symmetric bilinear form on the second homology group of the 4-manifold. In this thesis, we study the intersection forms of the smooth definite 4-manifolds with a rational homology 3-sphere boundary.

The first part of this thesis provides the finiteness result of the intersection forms of the smooth definite 4-manifolds bounded by a fixed rational homology 3-sphere up to stable equivalence. We also show that the finiteness result holds for the family of spherical 3-manifolds. In the second part, we focus on the spherical 3-manifolds which admit a rational homology ball filling. Hence we provide a complete classification of spherical 3-manifolds which bound a smooth rational homology 4-ball. We also determine the orders of spherical 3-manifolds in the rational homology cobordism group.

Key words: intersection forms, slice-ribbon conjecture, spherical 3-manifolds, Heegaard Floer correction terms.
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Contents

Chapter 1

Introduction

In 4-dimensional topology, there are two celebrated results in the early 1980s by M. Freedman and S. Donaldson. Commonly, these results involve the topological invariant called the intersection form of 4-manifolds. It is an integral symmetric bilinear form on the second homology group modulo torsion given by the algebraic intersection number between two surfaces in the 4-manifold.

In 1981, Freedman proved the 4-dimensional Poincaré conjecture [12], i.e., he showed that a homotopy 4-sphere is homeomorphic to the standard 4-sphere. More generally, the homeomorphism type of the simply connected 4-manifold is unique if the intersection form is even. For an odd form, there are two homeomorphism types and at least one of the manifolds does not admit any smooth structures. He also proved that every unimodular form is realized by the intersection forms of simply connected *topological* 4-manifolds.

In contrast, Donaldson proved the diagonalization theorem[6] which states that the only diagonal forms are realized by the intersection forms of definite *smooth* 4-manifolds. It is known that there are more than 10^7 equivalence classes of the even definite unimodular form with rank 32. Thus, there is no general classification theorem for unimodular definite forms.

As in [13], Frøyshov defined the rational valued invariant $\gamma(Y, \mathfrak{s})$ for a rational homology 3-sphere $(Y, \mathfrak{s})$ with a spin^c structure $\mathfrak{s}$ by analyzing the solution space of Seiberg-Witten equations. This invariant puts a constraint on the intersection form of a smooth definite 4-manifold bounded by Y.

He gave an alternate proof of Donaldson diagonalization theorem by using Frøyshov's invariant combining Elkies's theorem [9].

Ozsváth and Szabó introduced a Frøyshov type invariant called the correction term invariant derived from Heegaard Floer theory. This invariant shares many properties of Frøyshov's invariant. Moreover, it is a spin^c homology cobordism invariant. There are algorithms for computing the invariants for a large family of 3-manifolds (e.g., lens spaces, some families of plumbed 3-manifolds, Dehn surgery manifolds).

1.1 Intersection forms of definite 4-manifolds with boundary

In this thesis, we study the intersection forms of smooth definite 4-manifolds bounded by a given 3-manifold. We say that a lattice Λ is *smoothly* (resp. *topologically*) bounded by a 3-manifold Y if Λ is realized by the intersection form of a *smooth* (resp. *topological*) 4-manifold with the boundary Y. We can check that Y bounds $\Lambda \oplus \langle -1 \rangle$ if Y bounds Λ by the connected sum with $\overline{\mathbb{CP}}^2$. We call the procedure *stabilization*.

Definition 1.1.1. Two negative definite forms Λ_1 and Λ_2 are stably equivalent if $\Lambda_1 \oplus \langle -1 \rangle^n \cong \Lambda_2 \oplus \langle -1 \rangle^m$ for some non-negative integers n and m.

Let $\mathcal{I}(Y)$(resp. $\mathcal{I}^{TOP}(Y)$) be the set of all negative definite lattices that can be smoothly (resp. topologically) bounded by Y, up to stable-equivalence. Freedman and Donaldson's results can be interpreted as follows.

$$\mathcal{I}^{TOP}(S^3) = \{[\Lambda] | \Lambda \text{ is any unimodular negative definite lattice}\}$$

$$\mathcal{I}(S^3) = \{[\langle -1 \rangle]\}$$

We can check that $|\mathcal{I}^{TOP}(Y)|$ is also infinite for a rational homology 3-sphere Y. See theorem 2.2.1.

Our main result is the finiteness of $\mathcal{I}(Y)$ for 3-manifolds with some conditions. Although it has been known that $\mathcal{I}(Y)$ is finite only for 3-manifolds with small correction term, we conjecture that it is finite for any rational homology 3-sphere Y.

Theorem 1.1.2. *Let Y_1 and Y_2 be rational homology 3-spheres. Suppose that there is a negative definite smooth cobordism from Y_1 to Y_2 and $|\mathcal{I}(Y_2)| < \infty$. Then $|\mathcal{I}(Y_1)| < \infty$.*

Since we know that $|\mathcal{I}(S^3)| < \infty$ based on Donaldson's diagonalization theorem, we have the following corollary.

Corollary 1.1.3. *Let Y be a rational homology 3-sphere. If Y bounds a positive (resp. negative) definite smooth 4-manifold, then there are only finitely many negative (resp. positive) definite lattices, up to stable-equivalence, which can be realized as the intersection form of a smooth 4-manifold bounded by Y.*

In other words, if $\mathcal{I}(-Y) \neq \emptyset$, then $|\mathcal{I}(Y)| < \infty$.

Proof. Let W be a positive definite smooth 4-manifold with the boundary Y. Then we construct a negative definite cobordism from Y to S^3 by reversing the orientation of the punctured W. $\square$

In [3], Boyer showed that there are the only finitely many homeomorphism types of the simply connected 4-manifolds which have a given intersection form and boundary Y. From this, we obtain the following corollary.

Corollary 1.1.4. *For a rational homology 3-sphere Y, if Y bounds a positive (resp. negative) definite smooth 4-manifold, then there are the finitely many homeomorphism types of simply connected negative (resp. positive) definite smooth 4-manifolds bounded by Y, up to stabilization.*

To prove our main theorem, we consider a set of lattices $\mathcal{L}$ defined purely algebraically in terms of the invariants of a given 3-manifold Y. This set $\mathcal{L}$ contains negative definite lattices, up to stable-equivalence, which satisfy the conditions for a definite lattice to be smoothly bounded by Y, induced from the correction term invariants and fundamental obstructions from algebraic topology. See Chapter 2 for details of these conditions, where we obtain Theorem 1.1.2 by showing the finiteness of $\mathcal{L}$.

Theorem 1.1.5. *Let Γ_1 and Γ_2 be fixed negative definite lattices, and $C > 0$ and $D \in \mathbb{Z}$ be constants. Define $\mathcal{L}(\Gamma_1, \Gamma_2; C, D)$ to be the set of negative definite lattices Λ, up to stable-equivalence, satisfying the following conditions:*

- *$det(\Lambda) = D$,*
- *$\delta(\Lambda) \leq C$, and*
- *$\Gamma_1 \oplus \Lambda$ is embedded into $\Gamma_2 \oplus \langle -1 \rangle^N$, $N = rk(\Gamma_1) + rk(\Lambda) - rk(\Gamma_2)$.*

Then $\mathcal{L}(\Gamma_1, \Gamma_2; C, D)$ is finite.

Our proof of Theorem 1.1.5 is highly inspired by the work of Owens and Strle in [36], where they studied non-unimodular lattices in terms of the lengths of characteristic covectors. The key idea of our proof is to improve one of their inequalities on the length of a characteristic covector, to give an upper bound on the rank of the lattices in $\mathcal{L}(\Gamma_1, \Gamma_2; C, D)$.

It is interesting to ask which 3-manifolds satisfy the condition in Corollary 1.1.3, i.e., which 3-manifolds bound a positive definite smooth 4-manifold. Since many of 3-manifolds, including all Seifert fibered rational homology 3-spheres, bound a definite smooth 4-manifold up to the sign (see Proposition 4.2.2), it is more reasonable to find the families of 3-manifolds bounding the definite smooth 4-manifolds of both signs.

It is well known that any lens space satisfies such property. Note that lens spaces can be obtained by the double covering of S^3 branched along 2-bridge knots. If we generalize lens spaces to this direction, the double branched covers of S^3 along quasi-alternating links are known to bound the definite smooth 4-manifolds of both signs (with trivial first homology) [40, Proof of Lemma 3.6]. Also, notice that lens spaces can be obtained by Dehn-surgery along the unknot. In [37], Owens and Strle classified 3-manifolds obtained by Dehn-surgery on torus knots which can bound definite smooth 4-manifolds of both signs. We consider this question for another class of 3-manifolds, Seifert fibered rational homology 3-spheres, which also contains lens spaces. In particular, we determine spherical 3-manifolds which bound definite smooth 4-manifolds of both signs, and finally, show that any spherical 3-manifold Y has the property that $|\mathcal{I}(Y)| < \infty$.

Theorem 1.1.6. *Let Y be a spherical 3-manifold. Then, there are finitely many stable classes of negative definite lattices which can be realized as the intersection form of a smooth 4-manifold bounded by Y, i.e.* $|\mathcal{I}(Y)| < \infty$.

1.2 Spherical 3-manifolds bounding rational homology 4-balls

Another interesting question is which 3-manifolds bound rational homology 4-balls. The question is related to the study of the knot concordance group. It is well known that the double branched cover $\Sigma(K)$ of S^3 along a knot K bounds a rational homology ball if the K is slice [4]. So, one can show that a knot K is not slice by proving that $\Sigma(K)$ cannot bound any rational homology 4-ball. For example, the celebrated slice-ribbon conjecture posted by Fox [11] has been established for all 2-bridge knots [26], a certain family of pretzel knots [15],[23] and some families of Montesinos knots [22],[24].

By using Donaldson diagonalization theorem, Lisca classified all lens spaces which bound rational homology balls. Moreover, he completely determined which connected sums of lens spaces bound rational homology balls. Aceto and Golla studied this question for the 3-manifolds given by Dehn surgery on S^3 along a knot.

In this thesis, we consider this question for spherical 3-manifolds. The spherical 3-manifolds are the closed orientable 3-manifolds which admit a complete metric with the constant curvature $+1$. A spherical 3-manifold has a form S^3/G where G is a finite subgroup of $SO(4)$ generated by rotations. Notice that all 3-manifolds with a finite fundamental group are spherical 3-manifolds by Perelman's elliptization theorem. Spherical 3-manifolds are divided into 5-classes with respect to their fundamental groups: **C**(cyclic), **D**(dihedral), **T**(tetrahedral), **O**(octahedral) and **I**(icosahedral) type. The cyclic type manifolds are the lens spaces, of which Lisca gave a complete classification. Let $\mathcal{R}$ be the set of rational numbers greater than 1 defined in [26], so that the lens space $L(p,q)$ with $p > q > 0$ bounds a rational homology ball if and only if $p/q \in \mathcal{R}$. Our main result is the following.

Theorem 1.2.1. *A spherical 3-manifold Y bounds a rational homology 4-ball if and only if Y is homeomorphic to one of the following manifolds:*

- $L(p,q)$ *such that* $\frac{p}{q} \in \mathcal{R}$
- $D(p,q)$ *such that* $\frac{p-q}{q'} \in \mathcal{R}$
- T_3, T_{27} *and* I_{49},

where p and q are relatively prime integers such that $p > q > 0$ and $p - q > q' > 0$ is the reduction of q modulo $p - q$.

The spherical 3-manifolds are obtained by the double branched cover of some Montesinos links. Let $\mathcal{S}$ be the set of Montesinos knots which give spherical double branched covers.

Corollary 1.2.2. *The slice-ribbon conjecture is true for knots in $\mathcal{S}$.*

We say that two 3-manifolds Y_1 and Y_2 are *rational homology cobordant* if there is a 4-dimensional cobordism W from Y_1 to Y_2 such that the embeddings $Y_i \longrightarrow W$ $i = 1, 2$ induce isomorphisms between homology groups with rational coefficients. The set of rational homology 3-spheres up to the cobordism relation forms an abelian group called *the rational homology cobordism group*, denoted by $\Theta^3_{\mathbb{Q}}$. Theorem 1.2.1 classifies spherical 3-manifolds which represent the identity element in $\Theta^3_{\mathbb{Q}}$. It is natural to ask what the orders of spherical manifolds are. The orders of all lens spaces are known by Lisca in [27]. We determine the order of non-cyclic spherical manifolds.

Theorem 1.2.3. *The order of a spherical manifold of type* **D**, *$D(p,q)$ in $\Theta^3_{\mathbb{Q}}$ is the same as that of the lens space $L(p-q, q)$. The order of a spherical manifold Y of type* **T**, **O** *or* **I** *is given as:*

- 1 *if and only if* $\pm Y \cong T_3, T_{27}$ *or* I_{49},
- 2 *if and only if* $\pm Y \cong T_{15}$,
- ∞ *otherwise.*

As a corollary, we compute the orders of elements of $\mathcal{S}$ in the knot concordance group.

Corollary 1.2.4. *For a Montesinos knot $K \in \mathcal{S}$, the order of K in the knot concordance group is equal to that of $\Sigma(K)$ in $\Theta^3_{\mathbb{Q}}$.*

Organization

In the next chapter, we collect some background materials of integral lattices, intersection forms and Seifert 3-manifolds. In Chapter 3, we introduce Heegaard Floer homology and the correction term invariant that will be used to prove the main theorem. In Chapter 4, we investigate the finiteness of the equivalence classes of the definite intersection forms bounded by a given rational homology 3-sphere. In Chapter 5, we classify the spherical 3-manifolds which bound a rational homology 4-ball. We also determine the order of elements in $\Theta^3_{\mathbb{Q}}$ which are represented by spherical 3-manifolds.

Chapter 2

Preliminaries

In this chapter, we introduce the intersection forms of 4-manifolds and linking pairings of 3-manifolds. Algebraically, these are bilinear forms over a finitely generated abelian group. We also briefly explain the Seifert 3-manifolds and Dehn surgery manifolds.

2.1 Integral lattices

A lattice of rank n is a free abelian group of rank n with a nondegenerate symmetric bilinear form over the group. An integral lattice denotes the lattice with an integer valued bilinear form. Let $\Lambda = (\mathbb{Z}^n, Q)$ be an integral lattice where

$$Q : \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \mathbb{Z}$$

is a non-degenerate bilinear form. By tensoring $\mathbb{R}$ to Q, the bilinear form Q is diagonalizable and we denote the number of positive and negative diagonal entries by b^+ and b^- respectively. The signature $\sigma(\Lambda)$ is defined by $\sigma(\Lambda) := b^+ - b^-$. If $|\sigma(\Lambda)| = \mathrm{rk}(\Lambda)$, then we say that Λ is definite lattice. By fixing a basis $\{v_1, \ldots, v_n\}$ for Λ, we can represent Λ by an $n \times n$ matrix, $[Q(v_i, v_j)]$. The *determinant* of a lattice Λ denoted by $\det(\Lambda)$ is the determinant of a matrix representation of Λ. In particular, if the determinant of a lattice is ± 1, or equivalently a corresponding matrix is invertible over $\mathbb{Z}$, we say that the lattice is *unimodular*.

For simplicity, we denote $Q(v, w)$ by $v \cdot w$. We say that an integral lattice Λ is even if $v \cdot v$ is even for any $v \in \mathbb{Z}^n$. Otherwise, Λ is odd. The dual lattice $\Lambda^* := \mathrm{Hom}(\Lambda, \mathbb{Z})$ is defined as follows.

$$\mathrm{Hom}(\Lambda, \mathbb{Z}) = \{v \in \Lambda \otimes \mathbb{R} | v \cdot w \in \mathbb{Z} \text{ for any } w \in \Lambda\}$$

We say that a lattice Λ_1 is embedded into a lattice Λ_2 if there is a monomorphism preserving bilinear forms. Note that there is the natural lattice embedding Λ into Λ^* by definition of the dual lattice. $\xi \in \Lambda^*$ is called a characteristic covector if $\xi \cdot v \equiv v \cdot v \pmod 2$ for any $v \in \Lambda$. Let $\mathrm{Char}(\Lambda)$ be the set of characteristic covectors of Λ. Observe that $\xi - \xi' \in 2\Lambda^*$ for any $\xi, \xi' \in \mathrm{Char}(\Lambda)$. This implies that there is an affine structure on $\mathrm{Char}(\Lambda)$: $\mathrm{Char}(\Lambda) = \xi_0 + 2\Lambda^*$ for a characteristic covector $\xi_0 \in \mathrm{Char}(\Lambda)$.

There is a classification theorem for indefinite unimodular lattices.

Theorem 2.1.1 (J. Serre). The equivalence class of an indefinite unimodular lattice is determined by its rank, signature and type.

Lemma 2.1.2 (Van der Blij). For any unimodular lattice Λ and characteristic covector ξ of Λ, we must have

$$\sigma(\Lambda) \equiv \xi \cdot \xi \pmod 8$$

We denote the rank 1 unimodular lattice by $\langle \pm 1 \rangle$ and $\langle \pm 1 \rangle^n$ for its n-direct sum. For an odd unimodular lattice with rank n and signature σ, it is represented by $(\frac{n+\sigma}{2})\langle +1 \rangle \oplus (\frac{n-\sigma}{2})\langle -1 \rangle$ for a suitable choice of basis. For an even unimodular form, there are two building blocks denoted by E_8 and H.

$$E_8 = \begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{bmatrix}, H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

In particular, if a unimodular lattice is even, then the signature is divided by 8 by lemma 2.1.2. Then the even lattice with rank n and signature σ is represented by

$$\frac{\sigma}{8}E_8 \oplus \frac{n-|\sigma|}{2}H.$$

However, there is no classification theorem for definite lattices. The following numerical invariant of definite lattice is defined by using the characteristic covectors.

$$\delta(\Lambda) \coloneqq \max_{\xi \in \mathrm{Char}(\Lambda)} (\mathrm{rk}(\Lambda) - |\xi \cdot \xi|)$$

Note that $\delta(\langle +1 \rangle^n) = 0$ and $\delta(\lambda) = \mathrm{rk}(\Lambda)$ for an even lattice Λ. Elkies proved the characterization theorem of the standard definite lattice by using this invariant.

Theorem 2.1.1 ([9])**.** *Let Λ be a positive definite unimodular lattice. Then $\delta(\Lambda) \geq 0$. Moreover, $\delta(\Lambda) = 0$ if and only if $\Lambda \cong \langle +1 \rangle^n$ for some n.*

Later we will see that there is an alternate proof of Donaldson diagonalization theorem by using the result above with the correction term invariant. Owens and Strle obtained an analogous result for nonunimodular definite lattices.

Theorem 2.1.2 ([36])**.** *For a positive definite lattice Λ with rank n and $d = \det \Lambda$,*

- *$\delta(\Lambda) \geq 1 - 1/d$ if d is odd*
- *$\delta(\Lambda) \geq 1$ if d is even*

Moreover, the inequalities become equalities if and only if $\Lambda \cong \langle +1 \rangle^{n-1} \oplus \langle d \rangle$.

2.2 Intersection forms of 4-manifolds

Let X be a compact oriented 4-manifold. The intersection form is defined as follows:

$$Q_X : H_2(X, \mathbb{Z})/\mathrm{Tor} \times H_2(X, \mathbb{Z})/\mathrm{Tor} \longrightarrow \mathbb{Z}$$

defined by $Q_X(a,b) = \mathrm{PD}(a) \cup \mathrm{PD}(b)[X, \partial X]$ where PD is the poincaré duality map and $[X, \partial X]$ is the relative fundamental class compatible with the given orientation of X. Note that Q_X is a symmetric bilinear form. Moreover, if ∂X is a rational homology 3-sphere, Q_X is nondegenerate. In this case, $\Lambda_X = (H_2(X,\mathbb{Z})/\mathrm{Tor}, Q_X)$ forms a lattice. We call a closed 3-manifold a rational homology 3-sphere if the homology groups of the manifold are isomorphic to those of the standard 3-sphere with rational coefficients. Hence $(H_2(X,\mathbb{Z})/\mathrm{Tor}, Q_X)$ forms an integral lattice. In particular, if ∂X is an integral homology sphere, then $(H_2(X,\mathbb{Z})/\mathrm{Tor}, Q_X)$ is an unimodular lattice.

Example 2.2.1. The simplest example is the standard 4-sphere S^4. Since there is no nontrivial class in $H_2(S^4)$, Λ_{S^4} is trivial. The second example is the complex projective plane $\mathbb{CP}^2$. Note that two distinct projective lines in $\mathbb{CP}^2$ intersect positively at one point. This implies $[\mathbb{CP}^1] \cdot [\mathbb{CP}^1] = 1$ for $[\mathbb{CP}^1] \in H_2(\mathbb{CP}^2)$, which means that $Q_{\mathbb{CP}^2} = \langle +1 \rangle$ from $b_2(\mathbb{CP}^2) = 1$. The third example is $S^2 \times S^2$. There are two homology classes $S_1 = [S^2 \times p]$, $S_2 = [p \times S^2] \in H_2(S^2 \times S^2)$ for a point p in S^2. Then we can check $S_1 \cdot S_1 = 0$, $S_2 \cdot S_2 = 0$ and $S_1 \cdot S_2 = 1$. Hence we obtain $Q_{S^2 \times S^2} \cong H$ from $b_2(S^2 \times S^2) = 2$.

The remarkable theorem of Freedman says that all unimodular lattices are realized by the intersection forms of the simply connected topological 4-manifolds.

Theorem 2.2.2 ([12])*.* For any integral unimodular lattice Λ, there is a closed simply connected topological 4-manifold X with $\Lambda_X \cong \Lambda$. Moreover, the number of homeomorphism types is given as follows.

- If Λ is even, there is exactly one such manifold.
- If Λ is odd, there are two such manifolds.

Now we introduce the linking pairing on 3-manifolds. The linking pairing is related to the intersection form of 4-manifolds bounded by the 3-manifold. For a rational homology 3-sphere Y, the *linking pairing* of Y is the map

$$\lambda_Y : H^2(Y;\mathbb{Z}) \times H^2(Y;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

induced by the Poincaré duality of Y. Suppose that Y bounds a lattice $\Lambda = (\mathbb{Z}^n, Q)$ and X is a 4-manifold realizing Λ. From a long exact sequence of the pair (X, Y), we have

$$|H^2(Y;\mathbb{Z})| = |\det(\Lambda)|t^2 \tag{2.2.1}$$

for some integer t, and it induces the following chain map

$$H^2(X,Y;\mathbb{Z})/Tors \xrightarrow{Q} H^2(X;\mathbb{Z})/Tors \to H^2(Y;\mathbb{Z})/\mathcal{T},$$

where $\mathcal{T}$ is the image of the torsion subgroup of $H^2(X;\mathbb{Z})$. This gives a necessary condition for a lattice to be bounded by Y. Freedman's classification theorem can be generalized for 4-manifolds with boundary. See [3]. We can obtain an infinity result in the topological category.

Theorem 2.2.1. *Let Y be a rational homology 3-sphere. Then*

$$|\mathcal{I}^{TOP}(Y)| = \infty.$$

Proof. For the linking pairing of a 3-manifold Y, it is known by Edmonds [8, Section 6] that there is a definite form presenting it. This form is realized by a topological 4-manifold W by Boyer's result. Now, we have the infinitely many definite lattices bounded by Y, up to stabilization, by the connected summing W which is a closed topological 4-manifold with a non-standard definite intersection form. □

Proposition 2.2.2. *If a lattice Λ is bounded by a rational homology 3-sphere Y, then there exists an integer t such that $|H^2(Y;\mathbb{Z})| = |\det(\Lambda)|t^2$, a subgroup $\mathcal{T}$ of $H^2(Y)$ of order t, and a monomorphism*

$$\psi : \mathbb{Z}^n/Q(\mathbb{Z}^n) \to H^2(Y)/\mathcal{T}$$

such that ψ is equivariant between the linking form of $\mathbb{Z}^n/Q(\mathbb{Z}^n)$ and the induced linking pairing on the image of ψ.

Indeed, for our purpose, we only need a much weaker obstruction to the lattices bounded by a rational homology 3-sphere.

Proposition 2.2.3. *If a lattice Λ is bounded by a rational homology 3-sphere Y, then* $\det(\Lambda)$ *divides* $|H^2(Y;\mathbb{Z})|$.

2.3 Spherical manifolds as Seifert manifolds and Dehn surgery manifolds

2.3.1 Seifert fibered rational homology 3-spheres

Spherical 3-manifolds are included in a broad class of 3-manifolds called Seifert fibered rational homology 3-spheres, shorten to Seifert manifolds in this thesis. We briefly recall some notions and properties of Seifert manifolds. See [34] and [32] for some detailed expositions.

A Seifert manifold can be represented by the surgery diagram depicted in Figure 2.1. The collection of integers in the diagram,

$$(b; (\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_r, \beta_r)),$$

of which $b \in \mathbb{Z}$, and $\alpha_i > 0$ and β_i are coprime integers, is called the *Seifert invariant*. Let $Y(b; (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r))$ denote the Seifert manifold corresponding to the invariant. It is clear from the surgery diagram that permutations among (α_i, β_i)'s in an invariant result in the same 3-manifold. By the blow-down procedure and Rolfsen's twist respectively, we have the following homeomorphisms,

$$Y(b; (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r), (1, \pm 1)) \cong Y(b \mp 1; (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r)),$$

and

$$Y(b; (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r)) \cong Y(b - n; (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r - n\alpha_r)).$$

See [14] for more details. Hence any Seifert invariant can be normalized so that $b \in \mathbb{Z}$ and $\alpha_i > \beta_i > 0$ are coprime integers to represent the same manifold, called the *normalized Seifert invariant*. From now on, all Seifert invariants are assumed to be normalized unless stated otherwise. Any Seifert manifolds with $r \leq 2$ are homeomorphic to lens spaces, and the (orientation preserving) homeomorphism classes of lens spaces are classified as follows:

$$L(p, q) \cong L(p', q') \text{ if and only if } p = p' \text{ and } q \equiv (q')^{\pm 1} \text{ modulo } p.$$

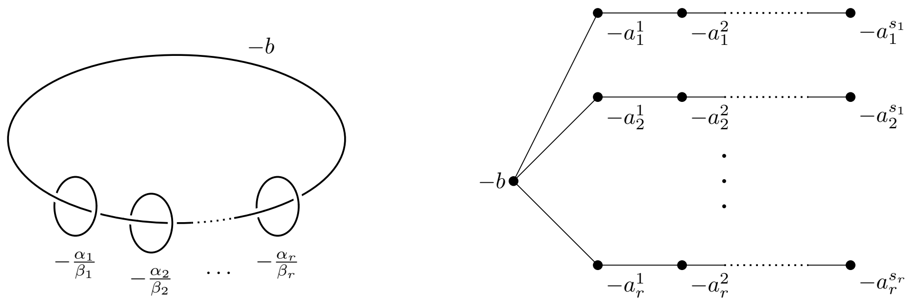

Figure 2.1: A surgery diagram of the Seifert manifold with the invariant $(b; (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r))$ and its associated plumbing graph.

In this thesis, $L(p, q)$ denote the lens space obtained by (p/q)-surgery along the unknot. By the homeomorphism classification of Seifert manifolds (See [34, Chapter 5]), the homeomorphism types of Seifert manifolds with $r \geq 3$ are classified by their normalized invariants up to the permutations among the pairs (α_i, β_i).

Let Y be a Seifert manifold with the invariant $(b; (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r))$. We define

$$e(Y) := b - \sum_{i=1}^{r} \frac{\beta_i}{\alpha_i},$$

called the *Euler number* of the Seifert manifold. Note that the Euler number is independent from the choices of the Seifert invariants of Y. It is easy to check that Y is a rational homology 3-sphere if and only if $e(Y) \neq 0$. From the linking matrix of the surgery diagram of Y, the order of $H_1(Y, \mathbb{Z})$ is given as follows.

$$|H_1(Y, \mathbb{Z})| = \alpha_1 \alpha_2 \ldots \alpha_k \left| b - \sum_{i=1}^{k} \frac{\beta_i}{\alpha_i} \right|.$$

By changing the orientation of Y if necessary, we can assume that $e(Y) > 0$. Note that $e(-Y) = -e(Y)$. Then there is a natural negative definite 4-manifold with the boundary Y constructed as follows. Extend each pair

(α_i, β_i) in the invariant by the following Hirzebruch-Jung continued fraction:

$$\frac{\alpha_i}{\beta_i} = a_i^1 - \cfrac{1}{a_i^2 - \cfrac{1}{\cdots - \cfrac{1}{a_i^{s_i}}}} =: [a_i^1, a_i^2, \ldots, a_i^{s_i}],$$

where $a_i^j \geq 2$. The *canonical (negative) definite 4-manifold* of Y is the plumbed 4-manifold corresponding to the weighted graph in Figure 2.1, i.e., each vertex with weight n represents the disk bundle over the 2-sphere with the Euler number n and each edge connecting two vertices represents the plumbing of the associated two disk bundles. See [14, Example 4.6.2] for the plumbing construction. The assumption that $e(Y)$ is greater than 0 implies that the intersection form of the 4-manifold is negative definite. See lemma 4.2.1.

2.3.2 Spherical 3-manifolds as Seifert manifolds

We recall that all spherical manifolds are known to be Seifert manifolds with the following normalized Seifert invariants [43]:

- Type **C**: $(b; (\alpha_1, \beta_1))$,
- Type **D**: $(b; (2,1), (2,1), (\alpha_3, \beta_3))$,
- Type **T**: $(b; (2,1), (3,\beta_2), (3,\beta_3))$,
- Type **O**: $(b; (2,1), (3,\beta_2), (4,\beta_3))$,
- Type **I**: $(b; (2,1), (3,\beta_2), (5,\beta_3))$,

where $\alpha_i > \beta_i > 0$ are coprime integers. Up to the orientations of the manifolds, we can further assume that b is greater than or equal to 2 for **D**, **T**, **O** and **I** type manifolds. In terms of the choices of $\{\beta_i\}$, we organize the notations of all spherical manifolds, up to the orientations, as follows:

- Type **C**: $L(p,q)$, $p > q > 0, (p,q) = 1$,

- Type **D**: $D(p,q)$, $p>q>0, (p,q)=1$,
- Type **T**: $T_{6(b-2)+k}, b \geq 2, k=1,3,5$,
- Type **O**: $O_{12(b-2)+k}, b \geq 2, k=1,5,7,11$,
- Type **I**: $I_{30(b-2)+k}, b \geq 2, k=1,7,11,13,17,19,23,29$.

See Table 5.4.5 for the identification of these notations with the corresponding Seifert invariants. Notice that the subscript in the notation for **T**, **O** or **I** type manifold is equal to the order of $H_1(-;\mathbb{Z})$ of the manifold divided by 3, 2 and 1 respectively.

2.3.3 Spherical 3-manifolds as Dehn surgery manifolds

Let $S^3_r(K)$ denote the 3-manifold obtained by r-framed Dehn surgery of S^3 along a knot K, and $T_{p,q}$ denote the (p,q)-torus knot. Recall that any Dehn surgery manifolds along torus knots are Seifert manifolds.

Lemma 2.3.1 ([30, Proposition 3.1], [37, Lemma 4.4])**.** *Let p and q be relative prime integers. Then for any rational r,*

$$S^3_r(T_{p,q}) \cong -Y(2;(p,q^*),(q,p^*),(m,n))$$

where $0<p^*<q$ (resp. $0<q^*<p$) is the multiplicative inverse of p (resp. q) modulo q (resp. p), and $\frac{m}{n}=\frac{pq-r}{pq-r-1}$.

In particular, since a Seifert invariant of any spherical 3-manifold of the type **T**, **O** or **I** can be written as

$$\pm Y(2;(2,1),(3,2),(*,*))$$

by changing the orientations if necessary, it can be obtained by Dehn surgery along the right-handed trefoil knot T by Lemma 2.3.1.

Example 2.3.2. The manifold $O_{12(b-2)+11}$ is homeomorphic to the manifold gotten by $(\frac{2(12b-13)}{4b-5})$-surgery along T.

$$\begin{aligned}
Y(b;(2,1),(3,1),(4,1)) &\cong -Y(-b;(2,-1),(3,-1),(4,-1)) \\
&\cong -Y(-b+3;(2,1),(3,2),(4,3)) \\
&\cong -Y(-b+3+(b-1);(2,1),(3,2),(4,3+4(b-1))) \\
&\cong S^3_{\frac{2(12b-13)}{4b-5}}(T).
\end{aligned}$$

Chapter 3

Heegaard Floer homology and correction term invariants

In this chapter, we overview Heegaard Floer homology developed by Ozsváth and Szabó. From the absolute grading on the homology groups, the correction term invariant can be induced. This invariant puts some constraints on the intersection form of definite 4-manifolds bounded by a given 3-manifold.

3.1 Heegaard Floer homology

For an oriented closed 3-manifold Y equipped with spin^c-structure $\mathfrak{t}$, one can associate a $\mathbb{Z}$-filtered chain complex $(CF^\infty(Y,\mathfrak{t}),\partial)$. If $\mathfrak{t}$ is a torsion spin^c-structure, the chain complex has a relative $\mathbb{Z}$-grading and a finitely and freely generated $\mathbb{Z}[U,U^{-1}]$-module structure. Note that the filtration is given by the negative power of U and the U-action decreases the homological grading by 2. Denote the filtration level by i, then the subcomplex can be defined as follows.

$$CF^-(Y,\mathfrak{t}) := CF^\infty(Y,\mathfrak{t})\{i<0\}$$

There are various versions of Heegaard Floer chain complexes

$$CF^+(Y,\mathfrak{t}) := CF^\infty(Y,\mathfrak{t})/CF^-(Y,\mathfrak{t}),\ \widehat{CF}(Y,\mathfrak{t}) := CF^-(Y,\mathfrak{t})/U\cdot CF^-(Y,\mathfrak{t})$$

These chain complexes form short exact sequences as follows.

$$0 \longrightarrow CF^-(Y,\mathfrak{t}) \longrightarrow CF^\infty(Y,\mathfrak{t}) \longrightarrow CF^+(Y,\mathfrak{t}) \longrightarrow 0$$

$$0 \longrightarrow \widehat{CF}(Y,\mathfrak{t}) \longrightarrow CF^+(Y,\mathfrak{t}) \xrightarrow{\cdot U} CF^+(Y,\mathfrak{t}) \longrightarrow 0$$

We obtain long exact sequences of homologies from the short exact sequences.

$$\cdots \longrightarrow HF^-(Y,\mathfrak{t}) \xrightarrow{\iota} HF^\infty(Y,\mathfrak{t}) \xrightarrow{\pi} HF^+(Y,\mathfrak{t}) \longrightarrow \cdots$$

$$\cdots \longrightarrow \widehat{HF}(Y,\mathfrak{t}) \xrightarrow{\widehat{\iota}} HF^+(Y,\mathfrak{t}) \xrightarrow{\cdot U} HF^+(Y,\mathfrak{t}) \longrightarrow \cdots$$

From the map π, we define the reduced part of the group $HF^+_{red}(Y,\mathfrak{s}) := \operatorname{coker}(\pi)$. Also one can check that

$$HF^+_{red}(Y,\mathfrak{s}) := \operatorname{coker}(\pi) \cong \ker(\iota) =: HF^-_{red}(Y,\mathfrak{s}).$$

3.2 Absolute grading and correction terms

A $(n+1)$-dimensional cobordism from Y_1 to Y_2 is an oriented $(n+1)$-dimensional manifold X with $\partial X \cong Y_2 \coprod -Y_1$. A 4-dimensinal cobordism W with a spin^c-structure $\mathfrak{s}$ induces maps between the boundary 3-manifolds with spin^c structures given by the restriction of $\mathfrak{s}$. More precisely, the cobordism $(W,\mathfrak{s})$ from $(Y_1,\mathfrak{t}_1)$ to $(Y_2,\mathfrak{t}_2)$ induces the following commutative diagram.

$$\begin{tikzcd}
\cdots \arrow[r] & HF^-(Y_1,\mathfrak{t}_1) \arrow[r, "\iota"] \arrow[d, "F^-_{W,\mathfrak{s}}"'] & HF^\infty(Y_1,\mathfrak{t}_1) \arrow[r, "\pi"] \arrow[d, "F^\infty_{W,\mathfrak{s}}"'] & HF^+(Y_1,\mathfrak{t}_1) \arrow[r] \arrow[d, "F^+_{W,\mathfrak{s}}"'] & \cdots \\
\cdots \arrow[r] & HF^-(Y_2,\mathfrak{t}_2) \arrow[r, "\iota"] & HF^\infty(Y_2,\mathfrak{t}_2) \arrow[r, "\pi"] & HF^+(Y_2,\mathfrak{t}_2) \arrow[r] & \cdots
\end{tikzcd}$$

Ozsváth and Szabó showed that each of the maps $F^{\circ}_{W,\mathfrak{s}}$ is an invariant of the spin^c cobordism $(W, \mathfrak{s})$ up to sign [41]. Moreover the cobordism map induces a lift called the absolute grading of the relative $\mathbb{Z}$-grading on $HF^{\circ}(Y, \mathfrak{t})$ for a torsion spin^c structure $\mathfrak{t}$. The absolute grading $\widetilde{\mathrm{gr}}$ is characterized by the following properties.

- In diagram 3.2 above, the row maps preserve the absolute grading.
- The generator of $\widehat{HF}(S^3)$ has the absolute grading 0.
- Let $(W, \mathfrak{s})$ be a spin^c cobordism from $(Y_1, \mathfrak{t}_1)$ to $(Y_2, \mathfrak{t}_2)$, and $\xi \in HF^{\infty}(Y_1, \mathfrak{t}_1)$, then

$$\widetilde{\mathrm{gr}}(F_{W,\mathfrak{s}}(\xi)) - \widetilde{\mathrm{gr}}(\xi) = \frac{c_1(\mathfrak{s})^2 - 2\chi(W) - 3\sigma(W)}{4}$$

From the absolute grading, Ozsváth and Szabó defined the numerical invariant of rational homology spheres with a spin^c structure.

Definition 3.2.1 ([38])**.** The *correction term* $d(Y, \mathfrak{t})$ is the minimal grading of the non-torsion elements in the image of $\pi : HF^{\infty}(Y, \mathfrak{t}) \longrightarrow HF^{+}(Y, \mathfrak{t})$.

Let Y be a rational homology 3-sphere and $\mathfrak{t}$ be a spin^c structure over Y. The correction term is an analogous invariant to Frøyshov's in Seiberg-Witten theory [13]. The correction terms are rational homology spin^c cobordism invariants and satisfy

$$d(-Y, \mathfrak{t}) = -d(Y, \mathfrak{t})$$

and

$$d(Y_1 \# Y_2, \mathfrak{t}_1 \# \mathfrak{t}_2) = d(Y_1, \mathfrak{t}_1) + d(Y_2, \mathfrak{t}_2),$$

It also puts a constraint on the intersection forms of the negative definite spin^c 4-manifolds with boundary.

Theorem 3.2.1 ([38, Theorem 9.6])**.** *If X is a negative definite smooth 4-manifold bounded by Y, then for each spin^c structure $\mathfrak{s}$ over X*

$$c_1(\mathfrak{s})^2 + n \leq 4d(Y, \mathfrak{s}|_Y),$$

where $c_1(-)$ denotes the first Chern class, n is the rank of $H_2(X;\mathbb{Z})$, and $\mathfrak{s}|_Y$ is the restriction of $\mathfrak{s}$ over Y.

Since any characteristic covector in $H^2(X,\mathbb{Z})/Tors$ is identified with $c_1(\mathfrak{s})$ for some spin^c structure $\mathfrak{s}$ on X, we obtain the following inequality.

Proposition 3.2.2. If a negative definite lattice Λ is smoothly bounded by a rational homology sphere Y, then

$$\delta(\Lambda) \leq 4d(Y)$$

where $d(Y) := \max_{\mathfrak{t}\in Spin^c(Y)} d(Y,\mathfrak{t})$.

From $d(S^3) = 0$ and the characterization theorem of Elkies 2.1.1, we can obtain Donaldson's diagonalization theorem.

Theorem 3.2.3 ([7]). Suppose that X is a smooth closed 4-manifold. If the intersection form of X is negative definite, then it is isomorphic to the diagonal lattice $(\mathbb{Z}^n, \langle -1\rangle^n)$.

3.3 Correction terms of knot surgery manifolds

Although it is a hard problem in general to compute the correction terms of rational homology 3-spheres, it can become tractable if the manifold can be obtained by Dehn-surgery along a knot. Let K be a knot in S^3, and $S^3_{p/q}(K)$ denote the 3-manifold obtained by the (p/q)-framed Dehn surgery along K. Note that there is a natural enumeration $i \in \mathbb{Z}/p\mathbb{Z}$ of spin^c structures over $S^3_{p/q}(K)$ [38, Section 4.1]. If K is the unknot U, the correction terms of $S^3_{p/q}(U)$ or lens space $L(p,q)$ can be computed by the following recursive formula [38]:

$$d(-L(p,q),i) = \left(\frac{pq-(2i+1-p-q)^2}{4pq}\right) - d(-L(q,r),j) \quad \text{and} \quad d(S^3) = 0, \tag{3.3.1}$$

where r and j are the reduction modulo q of p and i, respectively. According to Jabuka, Robins and Wang [19], the correction terms for lens spaces can be also obtained by a closed formula in terms of the Dedekind-Rademacher sums as follows.

$$d(L(p,q),i) = 2s(q,p;i) + s(q,p) - \frac{1}{2p} \tag{3.3.2}$$

The Dedekind-Rademacher sum $s(q,p;i)$ is defined as

$$s(q,p;i) = \sum_{k=0}^{|p|-1} \bar{B}_1\left(\frac{kq+i}{p}\right) \cdot \bar{B}_1\left(\frac{k}{p}\right)$$

and

$$s(q,p) = s(q,p;0) - \frac{1}{4},$$

where $\bar{B}_1(x) = x - \lfloor x \rfloor - \frac{1}{2}$ for the usual floor function $x \mapsto \lfloor x \rfloor$.

For a general knot K, it is known that the correction terms of $S^3_{p/q}(K)$ can be computed in terms of the correction terms of lens spaces and a sequence of non-negative integers $\{V_s(K)\}_{s=0}^{\infty}$, which are the invariants of K derived from the knot Floer chain complex of K, by the following formula due to Ni and Wu.

Theorem 3.3.1 ([33, Proposition 1.6])**.** *Suppose $p, q > 0$, and fix $0 \leq i \leq p-1$. Then*

$$d(S^3_{p/q}(K),i) = d(L(p,q),i) - 2\max\{V_{\lfloor \frac{i}{q} \rfloor}, V_{\lfloor \frac{p+q-1-i}{q} \rfloor}\}. \tag{3.3.3}$$

If K is a torus knot (more generally, an L-space knot), $V_s(K)$ can be gotten by the Alexander polynomial of K [40]. For instance, the (either left or right handed) trefoil knot T has

$$V_s(T) = \begin{cases} 1, & s = 0 \\ 0, & s > 0. \end{cases}$$

Thus we can compute the correction terms of all spherical manifolds of the types **T**, **O** and **I**, in principle, by the discussion in Section 2.3.3 and the formula above.

There is another advantage of having Dehn surgery descriptions of our 3-

manifolds. Applying the correction term obstruction usually takes more work since we need to determine in advance which spin^c structures over a rational homology 3-sphere can be extended to any hypothetical rational homology 4-ball which it bounds. See [35, 18, 16] for some resolutions of this issue. However, if the 3-manifold is obtained by Dehn surgery along a knot, then we can easily identify such spin^c structures. Suppose $Y \cong S^3_{p/q}(K)$. If Y bounds a rational homology ball W, then $p = m^2$ for some $m > 0$ and there are exactly m spin^c structures over Y that extend to W. It is known that there are exactly m spin^c structures on $S^3_{m^2/q}(K)$ that admit integer-valued correction terms, according to Aceto and Golla [1, Lemma 4.9]. Therefore, in order to argue that Y does not bound a rational homology ball, we only need to find a nonzero integer correction term of a spin^c structure over Y.

Proposition 3.3.2. *Let Y be a manifold obtained by (m^2/q)-framed Dehn surgery along a knot in S^3. If Y admits a non-vanishing integral correction term for some $\mathfrak{s} \in Spin^c(Y)$, then Y does not bound a rational homology 4-ball.*

We call those spin^c structures on $S^3_{m^2/q}(K)$ admitting integer-valued correction terms the *extendable spin^c structures*. In particular, in terms of the natural identification

$$\mathrm{Spin}^c(S^3_{m^2/q}(K)) \cong \mathbb{Z}/m^2\mathbb{Z},$$

the set of extendable spin^c structures is

$$\{[i_0 + m \cdot k] \in \mathbb{Z}/m^2\mathbb{Z} \mid k = 0, \ldots, m\},$$

where

$$i_0 = \begin{cases} \frac{q-1}{2} & \text{for odd } m \text{ and odd } q, \\ \frac{m+q-1}{2} & \text{for odd } m \text{ and even } q, \\ \frac{q-1}{2} \text{ or } \frac{m+q-1}{2} & \text{for even } m. \end{cases}$$

See [1, Lemma 4.7]. We can say that if m is even, the correction term of either $[\frac{q-1}{2}]$ or $[\frac{m+q-1}{2}]$ has an integer value. In particular, if m is odd, then i_0 is the spin^c structure induced from the unique spin structure on $S^3_{m^2/q}(K)$.

Chapter 4

Definite intersection forms of 4-manifolds with boundary

4.1 Finiteness of definite lattices bounded by a rational homology 3-sphere

The purpose of this section is to prove Theorem 1.1.5 and consequently Theorem 1.1.2. First, recall the following well-known fact: see [29, p. 18] for example.

Lemma 4.1.1. *There are finitely many isomorphism classes of definite lattices which have a given rank and determinant.*

We first show a special case of Theorem 1.1.5 in which Γ_1 and Γ_2 are the trivial empty lattices.

Proposition 4.1.2. *Let $C > 0$ and $D \in \mathbb{Z}$ be constants. There are finitely many negative definite lattices Λ, up to stable-equivalence, which satisfy the following conditions:*

- $\det \Lambda = D$,
- $\delta(\Lambda) \leq C$, *and*
- Λ *is embedded into* $\langle -1 \rangle^{rk(\Lambda)}$ *with prime index.*

Proof. Let Λ be a negative definite lattice of rank n that satisfies the conditions above. Without loss of generality, we may assume that there is no vector of square -1 in Λ. By Lemma 4.1.1 the theorem follows if we find an upper bound for the rank of Λ, only depending on D and C. Let $\{e, e_1, \ldots, e_{n-1}\}$ be the standard basis of the standard negative definite lattice $\langle -1 \rangle^n$, i.e. $e^2 = -1$, $e \cdot e_i = 0$, $e_i \cdot e_j = -\delta_{ij}$ for $i, j = 1, \ldots, n-1$.

Let p be the index of the embedding ι of Λ into $\langle -1 \rangle^n$. If $p = 1$, then the embedding is an isomorphism and so Λ should be the empty lattice, up to stable-equivalence. Now suppose p is an odd prime. Then the cokernel of ι is a cyclic group of order p, and it is generated by $e + \Lambda$ since $e \notin \Lambda$. Observe that $-e_i \in s_i e + \Lambda$ for some odd integer $s_i \in [-p+1, p-1]$. Consider a set of elements of Λ,

$$\mathcal{B} \coloneqq \{pe, e_1 + s_1 e, \ldots, e_{n-1} + s_{n-1} e\}.$$

Since the determinant of the coordinates matrix corresponding to the above set equals to p, it is in fact a basis for Λ. The matrix representation of Λ with respect to the basis is given as

$$Q = -\begin{pmatrix} p^2 & ps_1 & ps_2 & \cdots & ps_{n-1} \\ ps_1 & 1+s_1^2 & s_1 s_2 & \cdots & s_1 s_{n-1} \\ ps_2 & s_1 s_2 & 1+s_2^2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & s_{n-2}s_{n-1} \\ ps_{n-1} & s_1 s_{n-1} & \cdots & s_{n-2}s_{n-1} & 1+s_{n-1}^2 \end{pmatrix}.$$

We also compute the inverse of Q as follows

$$Q^{-1} = \left(\begin{array}{c|cccc} -\frac{1+\sum_{i=1}^{n-1} s_i^2}{p^2} & \frac{s_1}{p} & \frac{s_2}{p} & \cdots & \frac{s_{n-1}}{p} \\ \hline \frac{s_1}{p} & & & & \\ \frac{s_2}{p} & & & & \\ \vdots & & -I_{(n-1)\times(n-1)} & & \\ \frac{s_{n-1}}{p} & & & & \end{array}\right).$$

Note that Q^{-1} represents the dual lattice of Λ with respect to the dual basis of $\mathcal{B}$. Hence a characteristic covector ξ of Λ can be written as a vector

$$\xi = (k, k_1, \ldots, k_{n-1}),$$

where k is an odd integer and k_i's are even integers, in terms of the dual basis of Q since p^2 is odd and $1 + s_i^2$ is even for each i. From the matrix Q^{-1} we compute

$$|\xi \cdot \xi| = \frac{1}{p^2}(k^2 + \sum_{i=1}^{n-1}(ks_i - pk_i)^2).$$

Applying Lemma 4.1.3 below,

$$\min\{|\xi \cdot \xi| : \xi \text{ characteristic covector of } \Lambda\} \leq \frac{n+2}{3}.$$

Therefore, by theorem 3.2.1,

$$\delta(\Lambda) = \frac{n - \min_{\xi \in Char(\Lambda)} |\xi \cdot \xi|}{4} \leq C,$$

and we conclude that

$$n \leq 6C + 1.$$

In the case of $p = 2$, the lattice Λ admits a basis

$$\{2e, e_1 + e, \ldots, e_{n-1} + e\},$$

and hence the zero vector is characteristic. Therefore,

$$\delta(\Lambda) = \frac{n}{4} \leq C.$$

By Lemma 4.1.1, there are only finitely many negative definite lattices satisfying the given conditions. □

Now, we prove the following algebraic lemma used in the proof above.

Lemma 4.1.3. *For an odd prime p and odd integers $s_1, s_2, \ldots, s_{n-1}$ in $[-p+1, p-1]$, there exist an odd integer k and even integers $k_1, k_2, \ldots, k_{n-1}$ such* *that*

$$k^2 + \sum_{i=1}^{n-1}(ks_i - pk_i)^2 < \frac{n+2}{3}p^2.$$

Proof. For an odd prime p, consider the set $K \coloneqq \{-p+2, -p+4, \dots, p-2\}$ of odd integers in the interval $[-p+1, p-1]$. Note, for each k and s_i in K, there is a unique even integer k_i such that $ks_i - pk_i \in K$. Denote this k_i by $k_i(k, s_i)$. Since $ks_i \equiv k's_i \pmod{2p}$ implies $k \equiv k' \pmod{2p}$ for k and k' in K, we obtain $\{ks_i - p \cdot k_i(k, s_i) | k \in K\} = K$ for each $s_i \in K$. Therefore,

$$\begin{aligned}\sum_{k \in K}\sum_{i=1}^{n-1}(ks_i - p \cdot k_i(k, s_i))^2 &= (n-1) \cdot 2(1^2 + 3^2 + \cdots + (p-2)^2) \\ &= \frac{n-1}{3}p(p-1)(p-2).\end{aligned}$$

Since $|K| = p - 1$, there exists $k \in K$ such that

$$\sum_{i=1}^{n-1}(ks_i - p \cdot k_i(k, s_i))^2 \leq \frac{n-1}{3}p(p-2).$$

Since $|k| < p$, we obtain the desired inequality. $\square$

Now, Theorem 1.1.5 is proved by applying a similar argument of Proposition 4.1.2. First, observe the following.

Lemma 4.1.4. *Let Γ_1 and Γ_2 be negative definite lattices. Then the set of stable classes of lattices,*

$$\mathcal{C}(\Gamma_1, \Gamma_2) \coloneqq \{(Im(\iota))^\perp \mid \iota\colon \Gamma_1 \hookrightarrow \Gamma_2 \oplus \langle -1 \rangle^N, \textit{ an embedding for some } N \in \mathbb{N}\}/\sim,$$

is finite.

Proof. First, we claim that the set

$$\{(\mathrm{Im}(\iota))^\perp \mid \iota\colon \Gamma_1 \hookrightarrow \Gamma_2 \oplus \langle -1 \rangle^N, \text{ an embedding}\}/\sim$$

is stabilized for some large enough N. Let $\{v_1, \dots, v_{\mathrm{rk}(\Gamma_1)}\}$ be a basis for Γ_1.

By considering the representations of v_i in terms of a basis for $\Gamma_2 \oplus \langle -1\rangle^N$, if

$$N > \sum_{i=1}^{\mathrm{rk}(\Gamma)} |v_i \cdot v_i|,$$

then there is a vector $e \in \langle -1\rangle^N$ such that $e \cdot e = -1$ and $e \cdot v_i = 0$ for all $i = 1, \ldots, \mathrm{rk}(\Gamma_1)$. Hence in order to prove the finiteness of $\mathcal{C}(\Gamma_1, \Gamma_2)$, it is enough to consider some fixed large N.

Let $\{w_j\}_{j=1}^m$ be a basis for $\Gamma_2 \oplus \langle -1\rangle^N$. Note that an embedding of Γ_1 into $\Gamma_2 \oplus \langle -1\rangle^N$ can be presented by the system of equations

$$v_i = \sum_{j=1}^{m} a_{i,j} w_j,$$

where $a_{i,j}$ are integers. Since Γ_1 and $\Gamma_2 \oplus \langle -1\rangle^N$ are definite, the possible choices of $a_{i,j}$ are finite for each i, j, and hence the number of possible embedding maps is also finite. $\square$

Proof of Theorem 1.1.5. Fix negative definite lattices Γ_1 and Γ_2, and constants $C > 0$ and $D \in \mathbb{Z}$. Let Λ be a negative definite lattice which satisfies the conditions in the theorem. Without loss of generality, we may assume that there is no square -1 vector in Λ. By Lemma 4.1.1, the theorem follows if we show that the rank of Λ is bounded by some constant only depending on Γ_1, Γ_2, C and D.

From the third condition of Λ, there is an embedding

$$\iota|_{\Gamma_1} \colon \Gamma_1 \hookrightarrow \Gamma_2 \oplus \langle -1\rangle^N,$$

where $N = \mathrm{rk}(\Lambda) + \mathrm{rk}(\Gamma_1) - \mathrm{rk}(\Gamma_2)$. Let $(\mathrm{Im}(\iota|_{\Gamma_1}))^\perp \cong \langle -1\rangle^n \oplus E$, where E is a lattice without square -1 vectors and $n = \mathrm{rk}(\Lambda) - \mathrm{rk}(E)$. Note that $[E]$ is one of the elements in the finite set $\mathcal{C}(\Gamma_1, \Gamma_2)$ in Lemma 4.1.4.

Now we need to find a bound of the rank of Λ embedded in $\langle -1\rangle^n \oplus E$. This will be obtained by the similar argument in the proof of Theorem 4.1.2. The main difference is that we have an extra summand E.

Let ι' be an embedding Λ into $\langle -1\rangle^n \oplus E$, and p be the index of ι'. If

$n = 0$, i.e. the rank of Λ is same as the rank of E, then we have a bound of the rank of Λ by Lemma 4.1.4. Similarly, if $p = 1$, then $\Lambda \cong \langle -1 \rangle^n \oplus E$ and we have the same rank bound of Λ.

Now suppose $n \neq 0$ and that p is an odd prime. Let

$$\{e, e_1, \ldots, e_{n-1}, f_1, \ldots, f_r\}$$

be a basis for $\langle -1 \rangle^n \oplus E$ so that $e^2 = -1$, $e \cdot e_i = 0$, $e_i \cdot e_j = -\delta_{ij}$ and $e \cdot f_j = e_i \cdot f_j = 0$ for any i, j and E is generated by $\{f_1, \ldots, f_r\}$. By the same argument in Proposition 4.1.2, we can choose a basis for Λ ,

$$\{pe, e_1 + s_1 e, \ldots, e_{n-1} + s_{n-1} e, f_1 + t_1 e, \ldots, f_r + t_r e\} \tag{4.1.1}$$

where s_i's are odd integers in $[-p+1, p-1]$ and t_j's are integers in $[-p+1, p-1]$. Now with respect to the dual coordinates for this basis, write a characteristic covector ξ as

$$\xi = (k, k_1, \ldots, k_{n-1}, l_1, \ldots, l_r)$$

where k is odd, k_i's are even and $l_j \equiv (f_j + t_j e) \cdot (f_j + t_j e) \pmod 2$ for each i, j. To find the matrices of Λ and Λ^{-1}, introduce an $(n+r) \times (n+r)$ matrix M and a $r \times r$ matrix A as

$$M_{ij} \coloneqq \begin{cases} p & \text{if } \ i = 1, j = 1 \\ 1 & \text{if } \ i = j, 2 \leq j \leq n + r \\ s_{j-1} & \text{if } \ i = 1, 2 \leq j \leq n \\ t_{j-n} & \text{if } \ i = 1, n+1 \leq j \leq n + r \\ 0 & \text{otherwise,} \end{cases}$$

and

$$A_{ij} \coloneqq f_i \cdot f_j.$$

Note that M represents the embedding of Λ into $\langle -1 \rangle^n \oplus E$ and A represents

E. By the basis in (4.1.1), Λ and the dual of Λ are represented as follows:

$$Q_\Lambda = M^t \begin{pmatrix} -I_{n\times n} & 0 \\ 0 & A \end{pmatrix} M$$

and

$$\begin{aligned} Q_\Lambda^{-1} &= M^{-1} \begin{pmatrix} -I_{n\times n} & 0 \\ 0 & A^{-1} \end{pmatrix} (M^t)^{-1} \\ &= -M^{-1}(M^t)^{-1} + M^{-1} \begin{pmatrix} 0 & 0 \\ 0 & A^{-1} + I_{r\times r} \end{pmatrix} (M^t)^{-1}. \end{aligned} \tag{4.1.2}$$

We obtain that

$$|\xi \cdot \xi| \leq \frac{1}{p^2}(k^2 + \sum_{i=1}^{n-1}(ks_i - pk_i)^2) + |S(k, l_1, \ldots, l_r, t_1, \ldots, t_r)|,$$

for some function S. We emphasize that the function S is independent to n, since it is obtained from the last term of Equation (4.1.2). Then by Lemma 4.1.3,

$$\min\{|\xi \cdot \xi| : \xi \text{ characteristic covector of } L\} \leq \frac{n}{3} + |S|$$

Therefore, we obtain

$$n \leq \frac{3}{2}(4C + |S|)$$

from $\delta(\Lambda) \leq C$. The case that $p = 2$ is easier to find a similar bound for n by applying the same argument.

For an arbitrary index p, we use an idea in [36] to have a sequence of embeddings

$$\Lambda = E_0 \hookrightarrow E_1 \hookrightarrow E_2 \hookrightarrow \cdots \hookrightarrow E_s = \langle -1 \rangle^n \oplus E$$

such that each embedding $E_i \hookrightarrow E_{i+1}$ has a prime index. The length of this steps is also bounded by some constant related to D. Moreover, $\delta(E_i) \leq \delta(\Lambda) \leq C$ for any i since $E_i^* \hookrightarrow \Lambda^*$ and so $Char(E_i) \subset Char(\Lambda)$. Thus we complete the proof of theorem by an induction along each prime index embedding. $\square$

Proof of Theorem 1.1.2. Let W be a negative definite, smooth cobordism from Y_1 and Y_2. If X is a negative definite 4-manifold with the boundary Y_1, then $X \cup_{Y_1} W$ is a negative definite 4-manifold bounded by Y_2. Moreover, the intersection form of X embeds into the intersection form of $X \cup_{Y_1} W$. By the necessary conditions for a negative definite lattice to be bounded by a rational homology sphere discussed in Section 2, $\mathcal{I}(Y_1)$ is a subset of the union of

$$\mathcal{L}(Q_W, \Gamma; \max_{\mathfrak{t} \in \mathrm{Spin}^c(Y_1)} d(Y_1, \mathfrak{t}), D)$$

over all integers D dividing $|H_1(Y_1; \mathbb{Z})|$ and $[\Gamma] \in \mathcal{I}(Y_2)$. Then Theorem 1.1.2 follows from Theorem 1.1.5. □

4.2 Definite lattices bounded by Seifert fibered 3-manifolds

In this section, we discuss which Seifert fibered 3-manifolds satisfy the condition in Corollary 1.1.3. In particular, we completely classify spherical 3-manifolds Y such that both $\mathcal{I}(Y)$ and $\mathcal{I}(-Y)$ are nonempty, and show that $|\mathcal{I}(Y)| < \infty$ for any spherical 3-manifold Y.

4.2.1 Seifert fibered spaces

Seifert fibered 3-manifolds are a large class of 3-manifolds that contains 6 geometries among Thurston's 8 geometries of 3-manifolds. We remind that These 3-manifolds have a definite bounding from the plumbing construction. See Section 2.3.1.

Lemma 4.2.1. *The intersection form of the corresponding plumbed 4-manifold of a normal form $Y(e_0; (a_1, b_1), \ldots, (a_k, b_k))$ is negative definite if and only if*

$$e(Y) = e_0 - \frac{b_1}{a_1} - \cdots - \frac{b_k}{a_k} > 0.$$

Proof. This is directly obtained by diagonalizing the corresponding intersection form of the plumbed 4-manifold after extending over $\mathbb{Q}$. □

We refer $e(Y)$ by the *Euler number* of a Seifert form $Y(e_0; (a_1, b_1), \ldots, (a_k, b_k))$. Notice that the Euler number is in fact an invariant for the Seifert fibered 3-manifold Y, and $e(-Y) = -e(Y)$.

Proposition 4.2.2. *Any Seifert fibered rational homology 3-sphere can bound positive or negative definite smooth 4-manifolds.*

Now we introduce a condition for a Seifert fibered rational homology 3-sphere to bound definite 4-manifolds of both signs.

Proposition 4.2.3. *Let Y be a Seifert fibered rational homology 3-sphere of the normal form*

$$(e_0; (a_1, b_2), \ldots, (a_k, b_k)).$$

If $-e_0 + k \leq 0$, then Y bounds both positive and negative definite smooth 4-manifolds, i.e. both $\mathcal{I}(Y)$ and $\mathcal{I}(-Y)$ are not empty.

Proof. Let Y be a Seifert 3-manifold with the normal form $(e_0; (a_1, b_2), \ldots, (a_k, b_k))$ such that $-e_0 + k \leq 0$. By the previous proposition, Y bounds a negative definite 4-manifold. To find a positive definite bounding of Y, consider the plumbed 4-manifold X corresponding to $-Y \cong Y(-e_0 + k; (a_1, a_1 - b_1), \ldots, (a_k, a_k - b_k))$. Note that $b_2^+(X) = 1$. By blowing up $(e_0 - k)$ points on the sphere in X corresponding to the central vertex, we get a sphere with self intersection 0 in $X\#(e_0 + k)\overline{\mathbb{CP}^2}$. By doing a surgery on this sphere, we obtain a desired negative definite 4-manifold. More precisely, we remove the interior of the tubular neighborhood of the sphere, $S^2 \times D^2 \subset X\#(e_0 - k)\overline{\mathbb{CP}^2}$, and glue $D^3 \times S^1$ along the boundary, and it reduces $b_2^+(X\#(e_0 - k)\overline{\mathbb{CP}^2})$ by 1. □

Remark. This proposition can be alternatively proved by the fact that these Seifert fibered spaces can be obtained by the branched double covers of S^3 along alternating Montesinos links. See [28, Section 4].

Note that the condition in Proposition 4.2.3 is not a necessary condition. For example, the Brieskorn manifold,

$$\Sigma(2, 3, 6n + 1) \cong M(-1, (2, 1), (3, 1), (6n + 1, 1))$$

bounds both negative and positive definite 4-manifolds since $e(M) < 0$ and it can be obtained by $(+1)$-surgery of S^3 on the n-twist knot.

On the other hand, the inequality $-e_0 + k \leq 0$ is sharp since the Brieskorn manifold $\Sigma(2,3,5) \cong Y(2,(2,1),(3,2),(5,4))$, of which $-e_0 + k = 1$, cannot bound any positive definite 4-manifold by the constraint from Donaldson's diagonalization theorem.

4.2.2 Spherical 3-manifolds

We claim that most of the spherical 3-manifolds can bound smooth definite 4-manifolds of both signs, except the following cases:

$$T_1 = Y(2;(2,1),(3,2),(3,2)),$$

$$O_1 = Y(2;(2,1),(3,2),(4,3)),$$
$$I_1 = Y(2;(2,1),(3,2),(5,4)),$$

and

$$I_7 = Y(2;(2,1),(3,2),(5,3)).$$

Remark that we follow the notations of Bhupal and Ono in [2] for this class of 3-manifolds.

Proposition 4.2.4. *The manifolds T_1, O_1, I_1 and I_7 cannot bound a positive definite smooth 4-manifolds.*

Proof. In [25, Lemma 3.3], Lecuona and Lisca showed that if $1 < \frac{a_1}{b_1} \leq \frac{a_2}{b_2} \leq \frac{a_3}{b_3}$ and $1 < \frac{b_2}{a_2} + \frac{b_3}{a_3}$, then the intersection lattice of the plumbing associated to $Y(2;(a_1,b_1),(a_2,b_2),(a_3,b_3))$ cannot be embedded into a negative definite standard lattice.

Observe that the manifolds, T_1, O_1, I_1 and I_7 satisfy the conditions of the lemma. Hence these manifolds cannot bound any positive definite 4-manifolds by the standard argument using Donaldson's theorem. □

Proposition 4.2.5. *Any spherical 3-manifolds except T_1, O_1, I_1 and I_7 can bound both positive and negative definite smooth 4-manifolds.*

Proof. Since we orient a spherical 3-manifold to bound a natural negative definite 4-manifold, it is enough to construct a positive definite one with the given boundary Y. Type-**C** manifolds are lens spaces, and it is well known that they bound positive definite 4-manifolds either.

Let (n, q) be a pair of integers such that $1 < q < n$ and $\gcd(n, q) = 1$, and $\frac{n}{q} = [b, b_1, \ldots, b_{r-1}, b_r]$. We denote by $D_{n,q}$, the dihedral manifold

$$Y(b; (2, 1), (2, 1), (q, bq - n)).$$

The canonical plumbed 4-manifold of $D_{n,q}$ is given in Figure 4.1. If $b > 2$, there is a positive definite bounding by Proposition 4.2.3. In the case $b = 2$, we can check that $-D_{n,q} \cong Y(1; (2, 1), (2, 1), (q, n - q))$, and the corresponding plumbed 4-manifold X satisfies $b_2^+(X) = 1$. As seen in Figure 4.2, we get a 2-sphere with self-intersection 0 after blowing down twice from X, and the sphere intersects algebraically twice with the sphere of self-intersection $-c + 3$. Hence the sphere with self-intersection 0 is homologically essential, and we obtain a desired negative definite 4-manifold by a surgery along the sphere.

In the other cases (tetrahedral, octahedral and icosahedral cases), we can apply a similar argument except for the 4-cases. $\square$

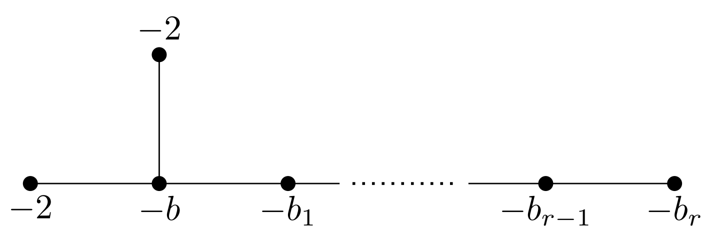

Figure 4.1: The plumbing graph of the canonical plumbed 4-manifold of $D_{n,q}$, where $\frac{n}{q} = [b, b_1, \ldots, b_{r-1}, b_r]$.

By Proposition 4.2.5 and Theorem 1.1.2, we know that the most of the spherical manifolds have finitely many stable classes of definite lattices to bound them. Finally, we show that the exceptional cases of spherical 3-manifolds also satisfy such finiteness property.

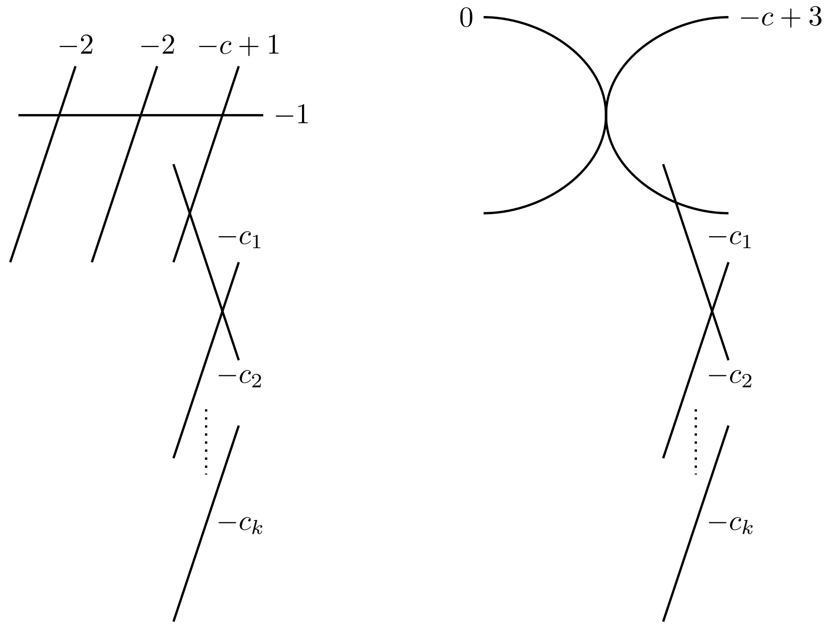

Figure 4.2: The canonical plumbed 4-manifold of $-D_{n,q}$ and the configuration after blow-down twice, where $\frac{n}{n-q} = [c, c_1, \ldots, c_k]$.

Proof of Theorem 1.1.6. Note that $I_1 \cong \Sigma$, and in this case we have $|\mathcal{I}(\Sigma)| \leq 15$ from the lattice theoretic result in [10] as mentioned in the introduction. We utilize this for the other cases.

Observe that the canonical plumbed 4-manifold X_{T_1} of T_1 can be embedded in the canonical plumbed 4-manifold X_Σ of Σ. See Figure 4.3. Let W be a 4-manifold constructed by removing X_{T_1} from X_Σ. Then W is a negative definite cobordism from T_1 to Σ. The finiteness of $\mathcal{I}(T_1)$ follows from Theorem 1.1.2.

Since the canonical plumbed 4-manifold O_1 is also embedded in X_Σ, we have $|\mathcal{I}(O_1)| < \infty$. For the manifold I_7, we blow up on the sphere in X_Σ to contain the canonical plumbed 4-manifold of I_7. Then the same argument works to show that $|\mathcal{I}(I_7)| < \infty$. $\square$

As we mentioned in the introduction, it is known that the 3-manifolds obtained by the double branched cover of quasi-alternating links in S^3 bound

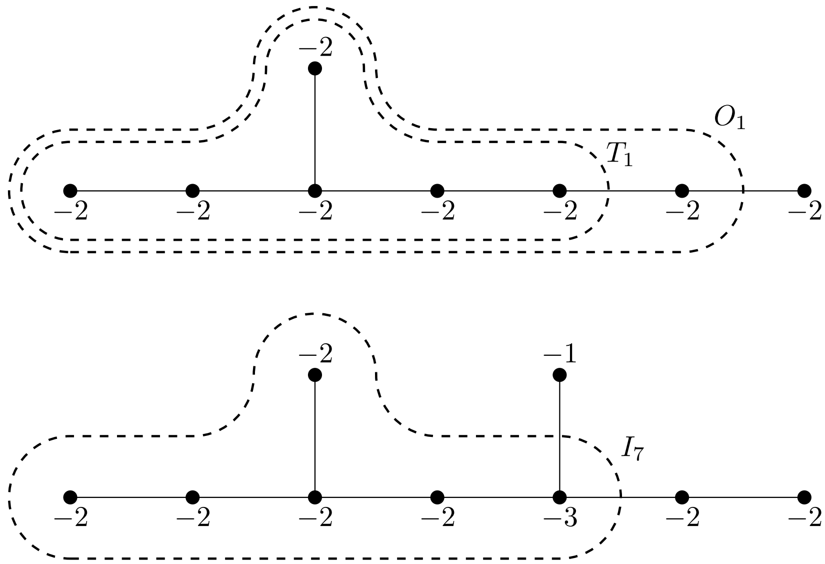

Figure 4.3: The embedding of the plumbed 4-manifold corresponding to the manifolds T_1 and O_1, and I_7 into $-E_8$-manifold and $-E_8\#\overline{\mathbb{CP}^2}$, respectively.

both positive and negative definite 4-manifolds with trivial H_1. Indeed, there is some family of Seifert fibered 3-manifolds that are not obtained by the double branched cover on a quasi-alternating link but can be shown to bound definite 4-manifolds of both signs by our result. For example, the dihedral manifolds $D_{n,n-1}$ are such 3-manifolds.

Proposition 4.2.6. *The dihedral manifold $D_{n,n-1}$ cannot bound a positive definite smooth 4-manifold with trivial H_1, and consequently cannot be obtained by the double branched cover of S^3 along a quasi-alternating link in S^3.*

Proof. Let Y be $D_{n,n-1}$ manifold, and X be the canonical plumbed 4-manifold of Y. If W is a positive definite 4-manifold bounded by Y, then, as usual, $Q_X \oplus -Q_W$ embeds into $\langle -1\rangle^{\mathrm{rk}(Q_X)+\mathrm{rk}(Q_W)}$. First observe that an embedding ι of Q_X to a standard definite lattice is unique, up to the automorphism of the standard definite lattice, as depicted in Figure 4.4, in terms of the standard basis $\{e_1, e_2, \ldots, e_{n+1}\}$. For this unique embedding, we have $(\mathrm{Im}\iota)^{\perp} \cong \langle -1\rangle^{\mathrm{rk}(Q_W)}$.

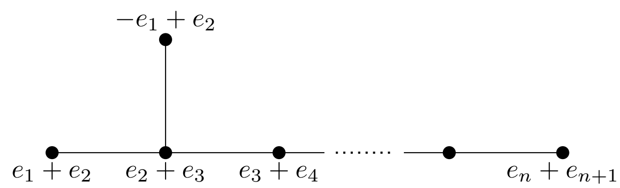

Figure 4.4: The canonical plumbed 4-manifold of $D_{n,n-1}$.

Since $H_1(X;\mathbb{Z})$ is trivial, $-Q_W$ is isomorphic to $(\mathrm{Im}\iota)^\perp \cong \langle -1\rangle^{\mathrm{rk}(Q_W)}$. Suppose $H_1(W)$ is trivial. Then $H^2(W,\partial W)$ and $H^2(W)$ are torsion free, and $H^2(Y)$ have to be trivial from the following long exact sequence:

$$\cdots \to H^2(W,\partial W) \xrightarrow[Q_W]{\cong} H^2(W) \to H^2(Y) \to H^3(W,\partial W) = 0$$

However, we know that $H^2(Y)$ is non-trivial. $\square$

Remark. Recently, all quasi-alternating Montesinos links are completely classified due to Issa in [17]. Since any Seifert fibered rational homology 3-sphere is the double branched cover of S^3 along a Montesinos link, the above proposition might be followed from his result.

Chapter 5

Spherical 3-manifolds bounding rational homology 4-balls

5.1 Obstructions for 3-manifolds admitting rational ball fillings

In this section, we recall our two main obstructions to 3-manifolds bounding rational homology balls, one from Donaldson's diagonalization theorem 3.2.3 and the other from Heegaard Floer correction terms. We also discuss how one can compute the correction terms of spherical manifolds.

5.1.1 Donaldson obstruction

One of the main obstructions we shall apply to spherical manifolds bounding rational homology balls or having finite order in $\Theta^3_{\mathbb{Q}}$ is given from Donaldson's diagonalization theorem.

Let Y be a rational homology 3-sphere, and suppose Y bounds a negative definite 4-manifold X and a rational homology 4-ball W. Then by summing X and W along Y, one constructs a smooth negative definite closed 4-manifold $X \cup_Y -W$. Then Donaldson's diagonalization theorem [6, 7] implies that the intersection form of $X \cup_Y -W$ is diagonalizable over $\mathbb{Z}$, in other words, the form is isometric to the standard negative definite lattice $(\mathbb{Z}^{b_2(X)}, \langle -1 \rangle^{b_2(X)})$. In particular, the intersection form of X, Q_X, embeds into $(\mathbb{Z}^{b_2(X)}, \langle -1 \rangle^{b_2(X)})$, which is induced by the map $H_2(X;\mathbb{Z}) \to H_2(X \cup_Y -W;\mathbb{Z})$, namely there exists a map ρ from $(H_2(X;\mathbb{Z})/\mathrm{Tors}, Q_X)$ to $(\mathbb{Z}^{b_2(X)}, \langle -1 \rangle^{b_2(X)})$ such that $\rho(v_1) \cdot \rho(v_2) = v_1 \cdot v_2$ for any v_1 and v_2 in $H_2(X;\mathbb{Z})/\mathrm{Tors}$. We summarize this condition as follows.

Theorem 5.1.1 (Donaldson obstruction)**.** *Let Y be a rational homology 3-sphere. Suppose Y bounds a negative definite smooth 4-manifold X. If Y bounds a rational homology ball, then the intersection form of X embeds into the standard negative definite lattice of the same rank.*

Type-C manifolds (lens spaces)

The **C**-type spherical manifolds are lens spaces. As mentioned in the introduction, lens spaces bounding rational homology 4-balls are completely classified by Lisca in [26], mainly using the Donaldson obstruction. We briefly recall his strategy and results, which are also applied to some other types of spherical 3-manifolds in Section 5.4.2.

Let $p > q > 0$ be relative prime integers, and $L(p, q)$ be the lens space obtained by (p/q)-framed Dehn surgery along the unknot. Given the Hirzebruch-Jung continued fraction, $p/q = [a_1, a_2, \ldots, a_r]$, let $\Gamma_{p,q}$ be the linear graph with r vertices of weights $-a_1, \ldots, -a_r$ consecutively. Note that $-L(p, q)$ bounds a negative definite plumbed 4-manifolds corresponding to the graph $\Gamma_{p,q}$. Define a numerical value $\mathcal{I}$ associated to $\Gamma_{p,q}$ as

$$\mathcal{I}(\Gamma) := \sum_{i=1}^{r} (|a_i| - 3).$$

One of the main ingredients in [26] and [27] is the complete classification of the intersection lattices associated to incident forms of linear graphs provided that $\mathcal{I}(-) < 0$ and their direct sums that can be embedded into the standard diagonal lattice of the same rank. For the lens spaces whose corresponding definite form is possibly embedded into the standard diagonal one with the same rank, Lisca showed that there are in fact rational homology balls bounded by the lens spaces.

Theorem 5.1.2 ([26]). *Suppose $p > q > 0$ are relative primes and $\mathcal{I}(\Gamma_{p,q}) < 0$. Then $L(p,q)$ bounds a rational ball if and only if $\Gamma_{p,q}$ is embedded into the standard definite lattice of the same rank.*

Since lens spaces have a symmetry, $-L(p,q) \cong L(p,p-q)$, and $\mathcal{I}(\Gamma_{p,q}) + \mathcal{I}(\Gamma_{p,p-q}) = -2$, either $\Gamma_{p,q}$ or $\Gamma_{p,p-q}$ have negative $\mathcal{I}$ value. Thus the above gives the complete classification for lens spaces admitting rational homology ball fillings.

5.1.2 Heegaard Floer correction terms

The correction terms satisfy the following property for rational homology 3-spheres that bound rational homology balls.

Theorem 5.1.3 ([38, Proposition 9.9]). *If Y is a rational homology 3-sphere that bounds a rational homology four-ball W, then*

$$d(Y, \mathfrak{t}) = 0$$

for any spin^c structure $\mathfrak{t}$ that extends to W.

In order to obtain a more effective obstruction, one might need to employ some algebro-topological aspects in this setting. Let X be a closed oriented 3-manifold or a compact oriented 4-manifold. Recall that the set $\mathrm{spin}^c(X)$ of spin^c structures of X is affine isomorphic to $H^2(X;\mathbb{Z})$. Namely, by fixing a spin^c structure $\mathfrak{s}$ in $\mathrm{spin}^c(X)$, we get an isomorphism $H^2(X;\mathbb{Z}) \cong \mathrm{spin}^c(X)$, and we denote the image of $\alpha \in H^2(X;\mathbb{Z})$ in the isomorphism by $\mathfrak{s}+\alpha$. For a 4-manifold W with the boundary 3-manifold Y, after fixing a spin^c structure $\mathfrak{s}$ on W, we have the following commutative diagram:

$$\begin{array}{ccc} H^2(W;\mathbb{Z}) & \xrightarrow[\mathfrak{s}+]{\cong} & \mathrm{spin}^c(W) \\ \downarrow & & \downarrow \\ H^2(Y;\mathbb{Z}) & \xrightarrow[\mathfrak{s}|_Y+]{\cong} & \mathrm{spin}^c(Y), \end{array}$$

where the vertical maps are induced by the natural restriction map.

Let Y be a rational homology 3-sphere, and λ be the linking form of Y. We say a subgroup $\mathcal{M}$ in $H^2(Y;\mathbb{Z})$ is called a *metabolizer* of Y if $\mathcal{M} = \mathcal{M}^\perp$ with respect to λ. In particular, the order of $\mathcal{M}$ is the square root of that of $H^2(Y;\mathbb{Z})$. Suppose W is a rational homology 4-ball bounded by Y. Then by the properties of Y and W and the long exact sequence of the pair (W, Y), one can show that the image of the map $H^2(W;\mathbb{Z}) \to H^2(Y;\mathbb{Z})$ is a metabolizer in $H^2(Y;\mathbb{Z})$ [4]. The discussion so far provides the following more detailed obstruction to a rational homology 3-sphere bounding a rational homology ball.

Correction term obstruction. Let Y be a rational homology 3-sphere. If Y bounds a rational homology 4-ball, then there exists a spin^c structure $\mathfrak{s}_0$ on Y and a metabolizer $\mathcal{M}$ in $H^2(Y;\mathbb{Z})$ such that

$$d(Y, \mathfrak{s}_0 + \alpha) = 0$$

for any $\alpha \in \mathcal{M}$.

5.2 Spherical manifolds of type D and T

In this section, we recall Lecuona's results in [22, 24], which allows us to determine the order of manifolds of type **D** and of the form $T_{6(b-2)+3}$. We also classify **T**-type manifolds admitting rational homology ball fillings.

5.2.1 Seifert manifolds with complementary legs

We say two pairs of integers (α_1, β_1) and (α_2, β_2) in a Seifert invariant are *complementary legs* if $\beta_1/\alpha_1 + \beta_2/\alpha_2 = 1$. In [22, 24], Lecuona studied the set of 3-legged Seifert manifolds with complementary legs. We recall Lecuona's results.

Proposition 5.2.1 ([24, Proposition 3.1], [22, Section 3.1]). *Let Y be a Seifert manifold with an invariant,*

$$(b; (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3)).$$

Suppose (α_1, β_1) *and* (α_2, β_2) *are complementary legs, namely* $\beta_1/\alpha_1 + \beta_2/\alpha_2 = 1$*. Then* Y *is rational homology cobordant to the manifold with the invariant* $(b-1; (\alpha_3, \beta_3))$*.*

One can find a more general statement for Seifert manifolds with more than 3-legs and a complementary legs in [1, Lemma 6.2]. Lecuona further showed that if such Seifert manifold bounds a rational homology ball, then the corresponding Montesinos links admit a ribbon surface. More precisely,

Proposition 5.2.2 ([24, Proposition 3.4])**.** *Let* Γ *be a 3-legged star shaped graph with two complementary legs. The Montesinos link* $ML_\Gamma \subset S^3$ *associated to* Γ *is the boundary of a ribbon surface* F *with* $\chi(F) = 1$ *if and only if the Seifert space* Y_Γ *associated to* Γ *is the boundary of a rational homology ball.*

5.2.2 Type-D (Prism manifolds)

Recall that a type-**D** manifold admits the Seifert invariant

$$(b_0; (2,1), (2,1), (\alpha_3, \beta_3)),$$

such that $b_0 \geq 2$ and $\alpha_3 > \beta_3 > 0$ are coprime integers, up to the orientations. The manifolds are usually enumerated by two coprime integers $p > q > 0$ likewise lens spaces. Let $\frac{p}{q} = [b_0, b_1, \ldots, b_r]$, and $D(p,q)$ denote the manifold homeomorphic to the boundary of the 4-manifold corresponding to the left plumbing graph in Figure 5.1. Since $(2,1)$ and $(2,1)$ are complementary pairs, we have the following direct corollary of Proposition 5.2.1.

Proposition 5.2.3. *Let* $p > q > 0$ *be coprime integers. The dihedral manifold* $D(p,q)$ *is rational homology cobordant to the lens space* $-L(p-q, q)$*.*

In particular, the manifold $D(p,q)$ has the same order in $\Theta^3_{\mathbb{Q}}$ as that of the lens space $L(p, p-q)$. Notice that if $b_0 = 2$, then $\frac{p-q}{q} < 1$. Since $L(p-q, q) \cong L(p-q, q')$ for $q' \equiv q$ modulo $p-q$, we have that $D(p,q)$ bounds a rational homology ball if and only if $\frac{p-q}{q'} \in \mathcal{R}$, where $0 < q' < p-q$ is the reduction of q modulo $p-q$, as the statement in Theorem 1.2.1.

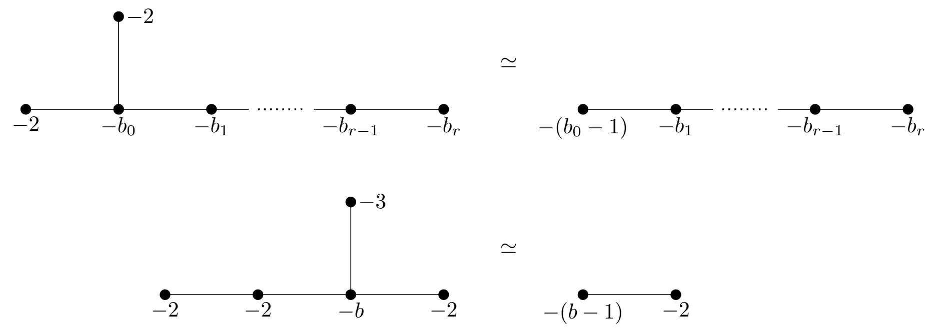

Figure 5.1: The canonical plumbing graphs of $D(p, q)$ and $T_{6(b-2)+3}$, and linear graphs associated to manifolds that are rational homology cobordant to them respectively.

5.2.3 Type-T

The **T**-type manifolds of the form $T_{6(b-2)+3}$ admitting the Seifert invariant

$$(b; (2, 1), (3, 1), (3, 2)),$$

also possesses complementary legs, $(3, 1)$ and $(3, 2)$. Thus the manifold $T_{6(b-2)+3}$ is rational homology cobordant to the lens space $-L(2b-3, 2)$ by Proposition 5.2.1. Then by Lisca's result on the order of lens spaces in $\Theta^3_{\mathbb{Q}}$ [27, Corollary 1.3], the order of $T_{6(b-2)+3}$ is given as

- 1 if $b = 2, 6$,
- 2 if $b = 4$, and
- ∞ otherwise.

In fact, this together with a topological condition for rational homology spheres bounding rational homology balls gives a complete answer for type-**T** manifolds admitting rational ball filling. Let Y be a rational homology 3-sphere that bounds a rational homology 4-ball W. From the long exact sequence of the pair (W, Y), it is easy to see that the order of $H_1(Y, \mathbb{Z})$ is m^2 for some non-negative integer m. In fact, m is the order of the image of the restriction map $H^2(Y;\mathbb{Z}) \to H^2(W;\mathbb{Z})$. See[4, Lemma 3] or the discussion in Section 5.1.2.

Now, let Y be a **T**-type spherical manifold, which has the normalized Seifert form

$$(b; (2,1), (3,\beta_2), (3,\beta_3)).$$

Observe that the order of $H_1(Y;\mathbb{Z})$ is equal to

$$3\,|6b - 3 - 2(\beta_2 + \beta_3)|\,.$$

If Y bounds a rational homology 4-ball, then $|H_1(Y;\mathbb{Z})|$ is a perfect square. It forces Y to have $\{\beta_2, \beta_3\} = \{2, 3\}$, namely Y is of the form $T_{6(b-2)+3}$. Therefore, any manifolds of the form $T_{6(b-2)+1}$ or $T_{6(b-2)+5}$ cannot bound a rational homology 4-ball.

Proposition 5.2.4. *A spherical manifold Y of type* **T** *bounds a smooth rational homology 4-ball if and only if Y or $-Y$ is homeomorphic to T_3 or T_{27}.*

5.3 Spherical manifolds of type O and I

In this section, we classify the spherical 3-manifolds of type **O** and **I** bounding rational homology 4-balls. In fact, this can be answered as a corollary of the results in Section 5.4, where we will determine the order of those spherical manifolds in $\Theta^3_{\mathbb{Q}}$. Nonetheless we present this section because the argument becomes much easier if one just want to determine whether a spherical manifold bounds a rational homology ball or not, instead of the exact order of the manifold. This is because the condition that the order of H_1 of a rational homology 3-sphere bounding a rational homology ball is a square number, reduces the cases we need to examine. Whereas this condition is not applicable to the connected sums of a manifold since any even number of connected sum admits a square order of H_1.

5.3.1 Type-O

The order of H_1 of a **O**-type manifold, which has a Seifert invariant

$$(b; (2,1), (3,\beta_2), (4,\beta_3)),$$

where $b \geq 2$, and $\beta_2 \in \{1,2\}$ and $\beta_3 \in \{1,3\}$, equals

$$2\left|12b - 6 - 4\beta_2 - 3\beta_3\right|.$$

Note that it cannot be a square number since β_3 is coprime to 4. Therefore, a **O**-type manifold cannot bound any rational homology 4-ball.

5.3.2 Type-I

Let Y be a **I**-type manifold, which has a Seifert invariant

$$(b; (2,1), (3,\beta_2), (5,\beta_3)),$$

where $b \geq 2$, $\beta_2 \in \{1,2\}$ and $\beta_3 \in \{1,2,3,4\}$. Note that the order of $H_1(Y;\mathbb{Z})$ equals

$$\left|30(b-2) + 45 - 10\beta_2 - 6\beta_3\right|.$$

As considering the quadratic residues modulo 30, the only cases to make $|H_1(Y,\mathbb{Z})|$ a square number are $(\beta_2,\beta_3) = (2,4)$ or $(\beta_2,\beta_3) = (2,1)$. Hence if a **I**-type manifold bounds a rational homology ball, then it should be one of the following families:

$$I_{30(b-2)+1} = Y(b; (2,1), (3,2), (5,4))$$

or

$$I_{30(b-2)+19} = Y(b; (2,1), (3,2), (5,1)).$$

Manifolds of the form $I_{30(b-2)+1}$

We can show that the manifold $I_{30(b-2)+1}$ cannot bound any rational homology ball by using the Donaldson obstruction. Note that the manifold

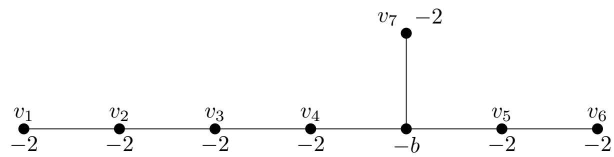

Figure 5.2: The canonical plumbing graph of the manifold $I_{30(b-2)+1}$.

$I_{30(b-2)+1}$ bounds the plumbed definite 4-manifold X in Figure 5.2. Let us label the vertices with weight -2 on the plumbing diagram by v_1 to v_7 as depicted in Figure 5.2. We let v_i simultaneously denotes a generator of $H_2(X;\mathbb{Z})$ represented by the sphere corresponding to the vertex. Suppose there is an embedding ρ of Q_X into the standard negative definite lattice $(\mathbb{Z}^n, \langle -1\rangle^n)$ with the standard basis $\{e_1, \ldots, e_n\}$; namely $e_i \cdot e_i = -1$ for each i and $e_i \cdot e_j = 0$ for $i \neq j$. Since $v_1 \cdot v_1 = -2$, we have $\rho(v_1) = \pm e_i \pm e_j$ for some i and j. After re-indexing and re-scaling by ± 1, we may assume that $\rho(v_1) = e_1 - e_2$ without loss of generality. By a similar argument, the image of v_2 has the form $\rho(v_2) = e_2 - e_3$ since $v_1 \cdot v_2 = 1$. For the image of v_3 on the embedding, there could be two choices: $\rho(v_3) = e_3 - e_4$ or $-e_2 - e_1$. However, assuming the latter implies that $v_1 \cdot v_4 \equiv v_3 \cdot v_4 \equiv 1 \pmod 2$, which gives a contradiction. Hence we admit the former case. By performing this procedure consequently, the embedding is expressed as $\rho(v_1) = e_1 - e_2$, $\rho(v_2) = e_2 - e_3$, $\rho(v_3) = e_3 - e_4$, $\rho(v_4) = e_4 - e_5$, $\rho(v_5) = e_6 - e_7$, $\rho(v_6) = e_7 - e_8$ and $\rho(v_7) = e_9 - e_{10}$, up to the automorphisms of $(\mathbb{Z}^n, \langle -1\rangle^n)$. Therefore, the rank of the image of ρ is at least 10, and $I_{30(b-2)+1}$ cannot bound any rational homology ball by the Donaldson obstruction.

Manifolds of the form $I_{30(b-2)+19}$

Suppose $I_{30(b-2)+19}$ bounds a rational homology ball. Then

$$|H_1(I_{30(b-2)+19};\mathbb{Z})| = 30(b-2) + 19$$

must be a square number m^2. In order to satisfy this property, m^2 should be equal to $(30k+13)^2$ or $(30k+23)^2$ for some $k \in \mathbb{Z}$ by considering square numbers in $\mathbb{Z}/30\mathbb{Z}$. Let us further assume this.

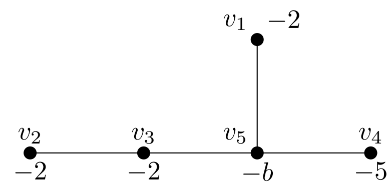

Figure 5.3: The canonical plumbing graph of the manifold $I_{30(b-2)+19}$.

For the manifold $I_{30(b-2)+19}$, we cannot apply the same obstruction that used for $I_{30(b-2)+1}$ since the intersection form of the canonical plumbed manifold of $I_{30(b-2)+19}$ can be possibly embedded into the standard definite diagonal lattice of the same rank. We first observe this. Let $v_1, \dots, v_5$ denote the basis vectors of the intersection form Q_X of the canonical plumbed 4-manifold X corresponding to $I_{30(b-2)+19}$ as depicted in Figure 5.3, and let $\{e_1, \dots, e_5\}$ be the standard basis of the diagonal lattice $(\mathbb{Z}^5, \langle -1\rangle^5)$. If $m^2 = (30k+13)^2$ (i.e. $b = 30k^2 + 26k + 7$), we have an embedding of Q_X into $\langle -1\rangle^5$ so that

$$\begin{aligned}
&\rho(v_1) = e_1 - e_2,\\
&\rho(v_2) = e_3 - e_4,\\
&\rho(v_3) = e_4 - e_5,\\
&\rho(v_4) = e_1 + e_2 - (e_3 + e_4 + e_5), \text{ and}\\
&\rho(v_5) = e_1 + 2e_2 + e_3 + e_4 + 2e_5 - (k+1)\{3(e_1+e_2) + 2(e_3+e_4+e_5)\}.
\end{aligned}$$

If $m^2 = (30k+23)^2$ (i.e. $b = 30k^2 + 46k + 19$), then we get an embedding by assigning v_5 instead as

$$\rho(v_5) = e_2 - e_3 - e_4 - (k+1)\{3(e_1+e_2) - 2(e_3+e_4+e_5)\}.$$

Instead of the Donaldson obstruction, we make use of the condition from Heegaard Floer correction terms. Observe that $I_{30(b-2)+19}$ can be obtained by $(-\frac{30(b-2)+19}{5b-6})$-framed Dehn-surgery along the left-handed trefoil knot. We first consider the correction terms for the associated lens spaces.

Lemma 5.3.1. *Let $k \in \mathbb{Z}$ and $m^2 = (30k+13)^2$ and $q = 150k^2 + 130k + 29$,* *or* $m^2 = (30k+23)^2$ *and* $q = 150k^2 + 230k + 89$*, and* $i_0 = \frac{q-1}{2}$*. Then*

$$d(L(m^2, q), i_0 + 2|m|) = 6$$

for $k \neq 0, -1$*.*

Proof. This is obtained by a direct computation using the reciprocal formula (3.3.1) for the correction terms of lens spaces. We give tables of triple (p, q, i)'s in the computation. The condition $k \neq 0, -1$ ensures that $p > q > 0$ and $p > i \geq 0$ in each steps.

Steps	p	q	i
1	$(30k+13)^2$	$150k^2+130k+29$	$75k^2+65k+14+2m$
2	$150k^2+130k+29$	$150k^2+130k+24$	$75k^2+65k+14+2m$
3	$150k^2+130k+24$	5	$75k^2+65k+14+2m$
4	5	4	0

Steps	p	q	i
1	$(30k+23)^2$	$150k^2+230k+89$	$75k^2+115k+44+2m$
2	$150k^2+230k+89$	$150k^2+230k+84$	$75k^2+115k+44+2m$
3	$150k^2+230k+84$	5	$75k^2+115k+44+2m$
4	5	4	0

□

Let T be the right-handed trefoil knot. Then by the formula (3.3.3) we have

$$\begin{aligned} d(I_{30(b-2)+19}, i_0 + 2|m|) &= d(S^3_{-\frac{30(b-2)+19}{5b-6}}(-T), i_0 + 2|m|) \\ &= -d(S^3_{\frac{30(b-2)+19}{5b-6}}(T), i_0 + 2|m|) \\ &= -(d(L(m^2, q), i_0 + 2|m|) - 2\max\{V_{\lfloor \frac{i}{q} \rfloor}(T), V_{\lfloor \frac{p+q-1-i}{q} \rfloor}(T)\}). \end{aligned}$$

Therefore, $I_{30(b-2)+19}$ cannot bound a rational homology ball for any $b \neq 3, 7, 11, 19$ by Lemma 5.3.1 and Proposition 3.3.2. Similarly we can compute that

$$d(L(23^2, 89), i_0 + 2 \cdot 23) = 4,$$

and hence I_{529} ($b = 19$) cannot bound a rational ball neither. The manifolds I_{289} ($b = 11$) and I_{169} ($b = 7$) also admit nonvanishing integral correction terms,

$$d(I_{289}, i_0 + 2 \cdot 17) = 2 \quad \text{and} \quad d(I_{169}, i_0 + 6 \cdot 13) = 2,$$

by the straightforward computation using the formula (3.3.3) and the correction term formula of [19] for lens spaces.

Proposition 5.3.2. *The spherical manifold of the form $I_{30(b-2)+19}$, $b \geq 2$, does not bound a rational homology ball if $b \neq 3$.*

5.3.3 The manifold I_{49} bounding a rational homology ball

The manifold I_{49} is the only remaining case of spherical manifolds that we have not determined if it bounds a rational homology ball. Note that all correction terms of extendable spin^c structures on I_{49} vanish, i.e.

$$d(I_{49}, i_0 + 7s) = 0,$$

for $i_0 = 4$ and $s = 0, \ldots, 6$. We shall show that the manifold I_{49} in fact bounds a rational homology ball by proving a stronger statement that this manifold is the double cover of S^3 branched along a *ribbon* knot. Recall that any Seifert fibered rational homology sphere is the double cover of S^3 along a Montesinos link. The corresponding link to a Seifert manifold can be found from the plumbing graph associated to the manifold as follows. To each vertex with weight n in the graph, we associate D^1 bundle over S^1 with n half twists, and to each adjacent vertices, we perform plumbings of the corresponding pair of bundles. Then the boundary of the 2-manifold constructed in the 3-sphere is the link to produce the Seifert manifold by the double branched cover. We denote the Montesinos link corresponding to the

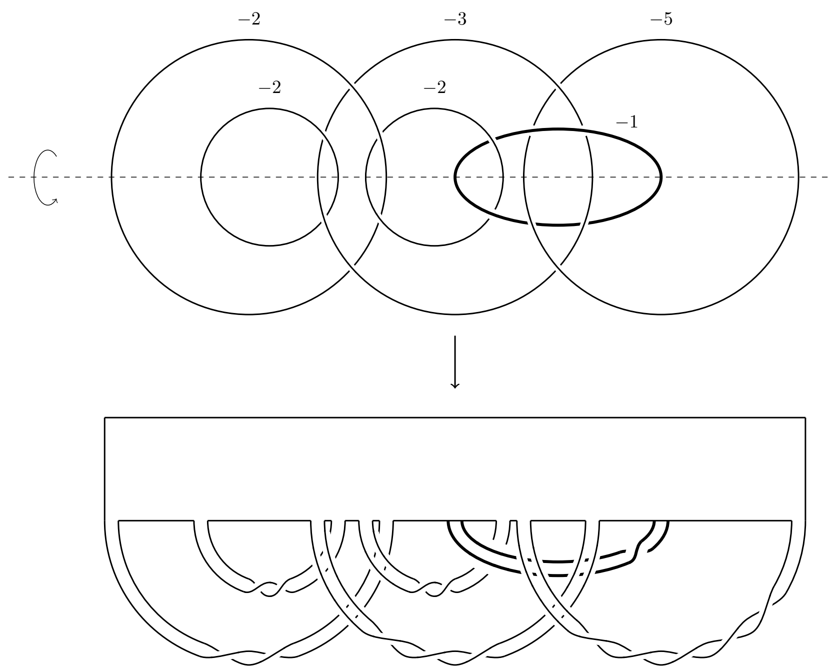

Figure 5.4: A surgery diagram of I_{49} as a strongly invertible link and its canonical branch set, the Montesinos knot $M(3;(2,1),(3,2),(5,1))$. The thick (-1)-framed unknot and its corresponding band induces a ribbon move to the two component unlink.

Seifert invariant $(b;(\alpha_1,\beta_1),\ldots,(\alpha_r,\beta_r))$ by $M(b;(\alpha_1,\beta_1),\ldots,(\alpha_r,\beta_r))$. For instance, the Montesinos knot associated with the manifold I_{49} is depicted in the below of Figure 5.4.

Proposition 5.3.3. *The Montesinos knot $M(3;(2,1),(3,2),(5,1))$ is a ribbon knot.*

Proof. It is a well-known fact that if there is a band that induces a ribbon move from a knot to the two component unlink, then the knot is ribbon. A band attached to the Montesinos knot $M(3;(2,1),(5,1),(3,2))$ is given in Figure 5.4 and one can check that this can be isotoped to the two component unlink by performing the ribbon move along the band.

We found the band by using a technique of Lecuona in [22, Section 3]. Lecuona considered the class of Montesinos knots whose branched cover has the plumbing graph with $\mathcal{I}(-) \leq -2$ and gave an algorithm to find a band if it is a ribbon knot. Although the graph Γ associated to our Montesinos knot $M(3;(2,1),(5,1),(3,2))$ has $\mathcal{I}(\Gamma) = -1$, Lecuona's technique was still useful to find a band attached to it. As considering an embedding of the intersection lattice of I_{49} to the standard diagonal one of the same rank, we find a (-1)-framed unknot added to the surgery diagram of I_{49}, as a *strongly invertible link* depicted in Figure 5.4. One can easily check that a consecutive blow-downs from the framed link diagram with the (-1)-framed unknot results to $S^1 \times S^2$. In this case, the band corresponding to the branch set of the (-1)-framed unknot, induces a ribbon move to the two component unlink. □

The discussion in the last two sections gave the proof of Theorem 1.2.1, responding which spherical 3-manifolds bound rational homology balls.

5.4 Orders of spherical 3-manifolds in $\Theta^3_{\mathbb{Q}}$

In this section, we discuss more generally the order of spherical 3-manifolds in $\Theta^3_{\mathbb{Q}}$. The orders of lens spaces were completely determined by Lisca in [27]. In Section 5.2, we observed that any **D**-type manifolds and manifolds of the form $T_{6(b-2)+3}$ are rational homology cobordant to lens spaces, and the order of them are same as that of the corresponding lens spaces. Now we determine the order of all other types of spherical manifolds, using both Donaldson and correction term obstructions again.

5.4.1 Finite order obstruction from Heegaard Floer correction terms

Heegaard Floer correction term for a certain spin^c structure can be used to obstruct some rational homology 3-spheres to have finite order in $\Theta^3_{\mathbb{Q}}$. Note that if a rational homology 3-sphere Y has odd $|H_1(Y;\mathbb{Z})|$, then it admits a unique spin structure on Y since $H_1(Y;\mathbb{Z}/2\mathbb{Z}) = 0$.

Proposition 5.4.1. *Let Y be a rational homology 3-sphere of which $|H_1(Y;\mathbb{Z})|$ is odd, and let $\mathfrak{s}_0$ be the spin^c structure induced from the unique spin structure on Y. If Y has finite order in $\Theta^3_{\mathbb{Q}}$, then*

$$d(Y,\mathfrak{s}_0)=0.$$

Proof. Let Y be a rational homology 3-sphere with odd $|H_1(Y;\mathbb{Z})|$. We first claim that the spin^c structure $\mathfrak{s}_0$ induced from the unique spin structure on Y extends to any rational homology ball bounded by Y. Recall that if a spin^c structure on Y extends to W, then also does its conjugation. Since the set of extendable spin^c structures on Y to W is also of odd order (the square root of $|H_1(Y;\mathbb{Z})|$), it must contain $\mathfrak{s}_0$, which is the unique spin^c structure preserved by the conjugation, i.e. the one from the spin structure. Remark that the same argument appears in the proof of [45, Proposition 4.2], although it is stated only for rational homology 3-spheres obtained from plumbed graphs.

Notice that $\#^n\mathfrak{s}_0$ is the spin^c structure induced from the unique spin structure on $\#^nY$. Hence if $\#^nY$ bounds a rational homology ball for some n, in other words, Y has finite order in $\Theta^3_{\mathbb{Q}}$, then

$$d(\#^nY,\#^n\mathfrak{s}_0)=n\cdot d(Y,\mathfrak{s}_0)=0$$

by the property of correction terms under the connected sum and Theorem 5.1.2. □

Remark. We remark that the parity condition in the proposition is necessary. For instance, it is well known that $L(p^2,p-1)$ bounds a rational homology ball, but one can check that the correction terms for spin structures on $L(p^2,p-1)$ are non-vanishing when p is even.

Now we apply the above to spherical 3-manifolds Y with odd $|H_1(Y;\mathbb{Z})|$. In [45] Stipsicz showed that the correction term for spin spherical 3-manifold, more generally for any spin link of rational surface singularities can be identified with the $\overline{\mu}$-invariant of Neumann [31] and Siebenmann [44] as

$$-4d(Y,\mathfrak{s}_0)=\overline{\mu}(Y,\mathfrak{s}_0).$$

In particular, $\overline{\mu}(Y,\mathfrak{s}_0)$ can be computed from the plumbing graph of the canonical definite 4-manifold X of Y [31] (See also Section 2 of [45]). Let Σ denote the *Wu surface* of $\mathfrak{s}_0$, namely the surface that represents the Poincaré dual of $c_1(\mathfrak{s}_0)$ and has coordinates 0 or 1 in the basis of $H_2(X;\mathbb{Z})$ represented by embedded 2-spheres in the plumbing diagram of X. Then

$$\overline{\mu}(Y,\mathfrak{s}_0) = \sigma(X) - [\Sigma]^2,$$

where $\sigma(X)$ is the signature of the intersection form of X.

Corollary 5.4.2. *The following spherical manifolds admit non-vanishing correction terms for the spin structures, and hence have infinite order in* $\Theta^3_{\mathbb{Q}}$.

- $T_{6(b-2)+k}$ *with any* b *and* $k = 1, 5$.
- $I_{30(b-2)+k}$ *with any* b *and* $k = 1, 7, 11, 13, 29$.
- $I_{30(b-2)+17}$ *with odd* b.
- $I_{30(b-2)+19}$ *and* $I_{30(b-2)+23}$ *with even* b.

Proof. For those manifolds, we compute the $\overline{\mu}$-invariants (and hence d-invariants) for the spin structures by the algorithm in [45, Section 2] using the canonical plumbing graphs. For instance, the plumbing graph associated to $T_{6(b-2)+1}$ is depicted in Figure 5.5. The Wu surface Σ corresponding to the spin structure on $T_{6(b-2)+1}$ is the sphere corresponding to the -2 vertex on the short leg if b is odd, and the empty surface if b is even. Hence,

$$-4d(T_{6(b-2)+1},\mathfrak{s}_0) = \overline{\mu}(T_{6(b-2)+1},\mathfrak{s}_0) = \sigma(X) - [\Sigma]^2 = \begin{cases} -4 & \text{for odd } b \\ -6 & \text{for even } b. \end{cases}$$

By applying the algorithm to other types of manifolds listed, one can check that the correction terms for the spin structures on the manifolds do not vanish. Therefore, those have infinite order in $\Theta^3_{\mathbb{Q}}$ by Proposition 5.4.1. □

On the other hand, the manifolds of type $I_{30(b-2)+17}$ with even b, and $I_{30(b-2)+19}$ and $I_{30(b-2)+23}$ with odd b have vanishing correction terms for spin structures, and any **O** type manifolds admit even order first homology group. Hence we cannot apply Proposition 5.4.1 to them.

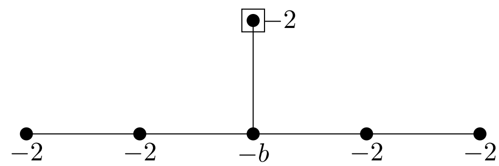

Figure 5.5: The canonical plumbed graph of $T_{6(b-2)+1}$, and the boxed vertex represents the Wu surface corresponding to the spin structure on it for even b.

5.4.2 Finite order obstruction from Donaldson's theorem

The Donaldson obstruction can be also used to give information for the order of rational homology 3-manifolds. If a rational homology 3-sphere Y bounds a negative definite 4-manifold X and have order n in $\Theta^3_{\mathbb{Q}}$ then the direct sum of n copies of the intersection form of X is embedded into the standard diagonal lattice of the rank $n \cdot b_2(X)$. This condition was sufficient enough to obstruct sums of lens spaces to have finite order by Lisca [27]. For the non-cyclic spherical manifolds, we consider two kinds of definite fillings of them. One is the canonical definite plumbed 4-manifold we have used in Section 5.3.2 and the other is the 4-manifold induced by the Dehn surgery descriptions of them, which we will introduce first.

Donaldson obstruction using Dehn surgery descriptions of spherical manifolds

Let $p > q > 0$ be the relative prime integers and K be a knot in S^3. Let $\frac{p}{q}$ admit the Hirzebruch-Jung continued fraction $[a_1, a_2, \ldots, a_r]$, with $a_i \geq 2$, and let $\Gamma_{p,q}$ be the linear graph with weights $a_1, \ldots, a_r$ consecutively. By Rolfsen's twist $S^3_{p/q}(K)$ bounds a definite 4-manifold X of which intersection form is isomorphic to $Q_{\Gamma_{p,q}}$, the incidence form of $\Gamma_{p,q}$; see Figure 5.6 for a framed link diagram of X. As we introduced in Section 5.1.1, Lisca studied the embedding of sums of intersection lattices $Q_{\Gamma_{p,q}}$ into the standard diagonal lattice of the same rank, provided that $\mathcal{I}(\Gamma_{p,q}) < 0$. Thus the result of Lisca can be also applied to $S^3_{p/q}(K)$ such that $\mathcal{I}(\Gamma_{p,q}) < 0$.

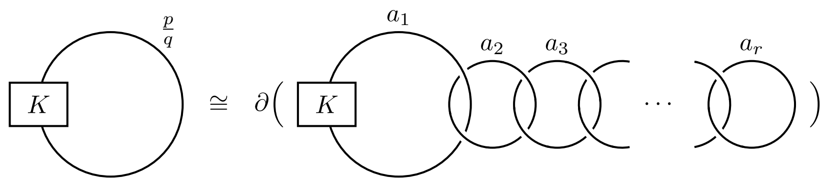

Figure 5.6: A definite 4-manifold bounded by a knot surgery manifold.

Proposition 5.4.3. *Let $p > q > 0$ be relative prime integers such that $\mathcal{I}(\Gamma_{p,q}) < 0$. Then for any knot K in S^3, the order of $S^3_{p/q}(K)$ in $\Theta^3_{\mathbb{Q}}$ is greater than or equals to that of $L(p,q)$.*

Proof. In Lisca's works [26, 27], a lower bound for the order of a lens spaces $L(p,q)$ in $\Theta^3_{\mathbb{Q}}$ is obtained by the Donaldson obstruction using the definite lattice $\oplus^n Q_{\Gamma_{p,q}}$ bounded by the lens spaces. Therefore, the lower bound is also applicable to the 3-manifolds that bound a 4-manifold with the same intersection lattice $Q_{\Gamma_{p,q}}$. Moreover, for the lens spaces, the bound was turned out to be the exact order in the Lisca's work. □

For example, consider manifolds of the form $O_{12(b-2)+11}$. By Lemma 2.3.1, we have the following surgery description of it (see Example 2.3.2):

$$O_{12(b-2)+11} \cong S^3_{\frac{2(12b-13)}{4b-5}}(T).$$

The surgery coefficients have the following continued fraction:

$$\frac{2(12b-13)}{4b-5} = \begin{cases} [8,2,2] & \text{if } b = 2 \\ [7,\underbrace{2,\ldots,2}_{b-3},3,2,2] & \text{if } b \geq 3 \end{cases}$$

Notice that if $b > 5$, then the corresponding linear graph has $\mathcal{I}(\Gamma_{2(12b-13),4b-5}) < 0$.

Similarly, we have

$$I_{30(b-2)+17} \cong S^3_{\frac{30b-43}{5b-8}}(T) \quad \text{and} \quad I_{30(b-2)+23} \cong S^3_{\frac{30b-37}{5b-7}}(T),$$

and the continued fraction of the surgery coefficients are given as

$$\frac{30b-43}{5b-8} = \begin{cases} [9,2] & \text{if } b=2 \\ [7,\underbrace{2,\ldots,2}_{b-3},4,2] & \text{if } b \geq 3 \end{cases}$$

and

$$\frac{30b-37}{5b-7} = \begin{cases} [8,3] & \text{if } b=2. \\ [7,\underbrace{2,\ldots,2}_{b-3},3,3] & \text{if } b \geq 3. \end{cases}$$

Hence if $b > 7$ in both cases, each linear graph corresponding to the surgery coefficients have $\mathcal{I}(-) < 0$.

Corollary 5.4.4. *The following spherical manifolds have infinite order in* $\Theta^3_{\mathbb{Q}}$*.*

- $O_{12(b-2)+11}$ *for any* $b > 5$
- $I_{30(b-2)+17}$ *for any* $b > 7$
- $I_{30(b-2)+23}$ *for any* $b > 7$

Proof. This directly follows from the result of [26, 27] and Proposition 5.4.3. More precisely one check that the incidence forms of linear plumbing graphs corresponding to the surgery slopes of the manifolds listed above are not included in the Lisca's set of linear lattices or sums of them that can be embedded into the standard diagonal one with the same rank. □

Remark that the remaining spherical manifolds that we have not determined the order, admit the Dehn-surgery descriptions (by Lemma 2.3.1) whose corresponding linear graphs have positive $\mathcal{I}$ values.

Donaldson obstruction using canonical plumbed 4-manifolds

For the manifolds of type $O_{12(b-2)+1}$, $O_{12(b-2)+5}$, and $O_{12(b-2)+7}$, we make use of the canonical plumbed 4-manifolds of them to apply the Donaldson obstruction and show they have infinite order in $\Theta^3_{\mathbb{Q}}$.

Proposition 5.4.5. *Let X be the canonical negative definite 4-manifold with the boundary $O_{12(b-2)+k}$ for $b \geq 2$ and $k = 1, 5$ or 7. Then the direct sum of n-copies of Q_X does not embed into the standard diagonal lattice of rank $n \cdot rk(Q_X)$ for any $n \geq 1$.*

Proof. We first consider the manifold of the form $O_{12(b-2)+1}$. Let Q_X be the intersection form of the canonical negative definite 4-manifold X with the boundary $O_{12(b-2)+1}$, and let Q_X^n denote the direct sum of n copies of Q_X. Suppose that there is an embedding ρ of Q_X^n into the standard lattice $\langle -1 \rangle^r$ for some $r > 0$. Label the basis vectors of the i-th copy of Q_X^n by $v_1^i, \ldots, v_7^i$, corresponding them to the spheres in the plumbing diagram of X as seen in the top of Figure 5.7. Let $\{e_1, \cdots, e_r\}$ be the standard basis of $\langle -1 \rangle^r$. Then after re-indexing and re-scaling by ± 1, we may assume $\rho(v_1^1) = e_1 - e_2$, $\rho(v_2^1) = e_3 - e_4$, $\rho(v_3^1) = e_4 - e_5$, $\rho(v_4^1) = e_5 - e_6$, $\rho(v_6^1) = e_7 - e_8$ and $\rho(v_7^1) = e_8 - e_9$ for the embedding of the first copy of Q_X as usual. Then there are essentially two choices of the image of v_1^2, namely $\rho(v_1^2) = \pm(e_1 + e_2)$ or $e_{10} - e_{11}$. However notice that the former case cannot happen since $1 = v_1^1 \cdot v_5^1 \equiv v_1^2 \cdot v_5^1 \pmod 2$ assuming this. By using this argument inductively, we may assume that

$$\begin{aligned}
\rho(v_1^i) &= e_{9(i-1)+1} - e_{9(i-1)+2}, \\
\rho(v_2^i) &= e_{9(i-1)+3} - e_{9(i-1)+4}, \\
\rho(v_3^i) &= e_{9(i-1)+4} - e_{9(i-1)+5}, \\
\rho(v_4^i) &= e_{9(i-1)+5} - e_{9(i-1)+6} \\
\rho(v_6^i) &= e_{9(i-1)+7} - e_{9(i-1)+8}, \\
\rho(v_7^i) &= e_{9(i-1)+8} - e_{9(i-1)+9}.
\end{aligned}$$

Then it follows that the rank of the image of ρ is at least $9n$. Hence Q_X^n cannot be embedded into the standard diagonal lattice of the same rank.

Now, let X be the canonical definite 4-manifold of $O_{12(b-2)+5}$. Label by $v_1^i, \ldots, v_6^i$, basis vectors of each i-th copy of Q_X^n, as shown in Figure 5.7. As usual, an embedding ρ of Q_X^n into the standard diagonal lattice $\langle -1 \rangle^r$ should have the form,

$$\begin{aligned}
\rho(v_1^i) &= e_{6(i-1)+1} - e_{6(i-1)+2},\\
\rho(v_2^i) &= e_{6(i-1)+3} - e_{6(i-1)+4},\\
\rho(v_3^i) &= e_{6(i-1)+4} - e_{6(i-1)+5}\\
\rho(v_4^i) &= e_{6(i-1)+5} - e_{6(i-1)+6}
\end{aligned}$$

for $i = 1, \dots, n$, without loss of generality. Notice that there appeared $6n$ basis vectors in the image of $v_1^i \dots v_4^i$. Thus we only need to show that the image of v_6^1 contains a basis vector of other than $\{e_1, \dots, e_{6n}\}$. In fact, one can easily see that $\rho(v_6^1)$ should be one of the following forms:

$$e_{6n+1} \pm (e_{6(j-1)+1} + e_{6(j-1)+2}) \quad \text{or} \quad e_{6n+1} + e_{6n+2} + e_{6n+3}$$

for some $j \in \{1, \dots, n\}$. Hence the rank of the image of ρ is strictly greater than $6n$.

A similar argument will be applied to $O_{12(b-2)+7}$. We label by $v_1^i, \cdots, v_5^i$, $i = 1, \cdots, n$, the basis vectors of Q_X^n for the plumbed manifold X of $O_{12(b-2)+7}$ as in Figure 5.7. By the same argument of the previous cases, an embedding ρ of Q_X^n into $\langle -1 \rangle^r$ should have the form:

$$\begin{aligned}
\rho(v_1^i) &= e_{5(i-1)+1} - e_{5(i-1)+2}\\
\rho(v_2^i) &= e_{5(i-1)+3} - e_{5(i-1)+4}\\
\rho(v_3^i) &= e_{5(i-1)+4} - e_{5(i-1)+5}
\end{aligned}$$

for $i = 1, \cdots, n$. Then one can check that the followings are the only choices for $\rho(v_5^1)$:

$$\begin{gathered}
e_{5n+1} + e_{5n+2} + e_{5n+3} + e_{5n+4},\\
e_{5n+1} + e_{5n+2} \pm (e_{5(j-1)+1} + e_{5(j-1)+2}),\\
e_{5n+1} \pm (e_{5(j-1)+3} + e_{5(j-1)+4} + e_{5(j-1)+5}), \text{ or}\\
\pm(e_{5(j-1)+1} + e_{5(j-1)+2}) \pm (e_{5(k-1)+1} + e_{5(k-1)+2})
\end{gathered}$$

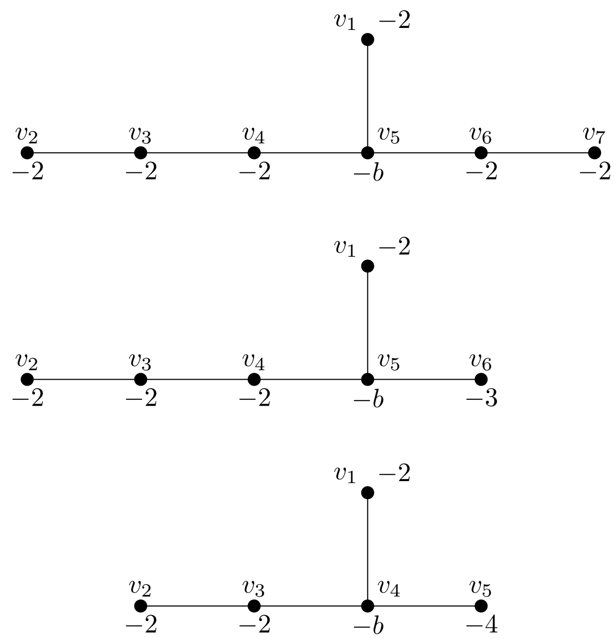

Figure 5.7: The plumbing graphs for the canonical definite 4-manifolds bounded by $O_{12(b-2)+1}$, $O_{12(b-2)+5}$ and $O_{12(b-2)+7}$, respectively.

for some $j, k \in \{1, \cdots, n\}$ such that $j \neq k$. If we admit one of the first 3 cases, then the rank of the image of ρ is greater than $5n$. If we assume the last case, then $v_5^1 \cdot w \equiv (v_1^j + v_1^k) \cdot w$ modulo 2 for any w in Q_X^n. Since $(v_1^j + v_1^k) \cdot v_4^j = (v_1^j + v_1^k) \cdot v_4^k = 1$, we have $v_5^1 \cdot v_4^j \neq 0$ and $v_5^1 \cdot v_4^k \neq 0$, which induces a contradiction. $\square$

5.4.3 Greene-Jabuka technique

The spherical 3-manifold of the form $I_{30(b-2)+19}$ with odd b is the manifold to which the previous finite order obstructions cannot be applied. The correction terms for spin structures on them vanish, and the intersection lattice of the canonical plumbed manifold of them can be possibly embedded into the standard diagonal one of the same rank as we observed in Section 5.3.2. However, a technique of Greene and Jabuka in [15] is turned out to be useful for showing $I_{30(b-2)+19}$ have infinite order for $b > 5$.

We briefly sketch their idea. To prove the slice-ribbon conjecture for a certain family of 3-strands pretzel knots, they combined the information from the Donaldson's theorem and Heegaard Floer correction terms. Let Y be a rational homology 3-sphere bounding a negative definite plumbed 4-manifold X with at most two bad vertices. If Y bounds a rational homology ball W, then the inclusion map $X \hookrightarrow X \cup_Y W$ induces lattices embedding $A\colon (H_2(X;\mathbb{Z}), Q_X) \to (H_2(X \cup_Y W;\mathbb{Z})/Tor, \langle -1 \rangle^{b_2(X)})$ by Donaldson's diagonalization theorem. They observed that the set of the spin^c-structures on Y extending to W can be described by the map A. Moreover, combined with a formula of Ozsváth and Szabó [39] for the correction terms of the boundary 3-manifold of plumbed 4-manifold, they obtained an upper bound for the number of extendable spin^c structures to W admitting vanishing correction terms. We summarize their result as follows.

Proposition 5.4.6 ([15])**.** *Let Y be a rational homology 3-sphere with $|H_1(Y;\mathbb{Z})|$ odd, and X be a plumbed 4-manifold with the boundary Y associated to a negative definite forest of trees with at most two bad vertices. Suppose Y bounds a rational homology 4-ball W. Denote by A a matrix representation of the map*

$$H_2(X;\mathbb{Z}) \to H_2(X \cup_Y W;\mathbb{Z})/Tors$$

in terms of the standard basis of the image. Then the number of spin^c structures that admit vanishing correction terms is less than or equal to the order of the quotient set

$$\{(\pm 1, \ldots, \pm 1) \in \mathbb{Z}^n\}/ \sim,$$

where $v_1 \sim v_2$ if $v_1 - v_2$ is a linear combination of column vectors of A over $\mathbb{Z}$.

We now apply the above to the manifold $I_{30(b-2)+19}$. Let Q be the intersection form of the canonical negative definite 4-manifold bounded by $I_{30(b-2)+19}$ for a fixed b. Let Q^n denote the direct sum of n-copies of Q. We also denote basis vectors of the i-th summand of Q^n by $v_1^i, v_2^i, \ldots, v_5^i$ following the assignment in Figure 5.3. Note that $\mathrm{rk}(Q^n)$=$5n$.

Lemma 5.4.7. *Let ρ be an embedding of Q^n into the standard negative definite lattice of the rank $5n$. Then ρ has the following form, up to the automorphisms of the standard lattice:*

$$\rho(v_1^i) = e_{1+5(i-1)} - e_{2+5(i-1)}$$
$$\rho(v_2^i) = e_{3+5(i-1)} - e_{4+5(i-1)}$$
$$\rho(v_3^i) = e_{4+5(i-1)} - e_{5+5(i-1)}$$
$$\rho(v_4^i) = \pm(e_{1+5(\sigma_1(i)-1)} + e_{2+5(\sigma_1(i)-1)}) \pm (e_{3+5(\sigma_2(i)-1)} + e_{4+5(\sigma_2(i)-1)} + e_{5+5(\sigma_2(i)-1)})$$

for $i = 1, \ldots, n$, where σ_1 and σ_2 are some permutations in $\{1, \ldots, n\}$, and $\{e_1, \ldots e_{5n}\}$ is the standard basis of $(\mathbb{Z}^{5n}, \langle -1\rangle^{5n})$.

Proof. Suppose that there is an embedding of Q^n into the standard lattice of the rank $5n$. Then by the same argument of Proposition 5.4.5, the embedding ρ have to have the following form:

$$\rho(v_1^i) = e_{1+5(i-1)} - e_{2+5(i-1)}$$
$$\rho(v_2^i) = e_{3+5(i-1)} - e_{4+5(i-1)}$$
$$\rho(v_3^i) = e_{4+5(i-1)} - e_{5+5(i-1)}$$

for $i = 1, \ldots, n$, without loss of generality. Note that all $5n$ standard basis vectors are already appeared in the image of v_1^i, v_2^i and v_3^i. Since $v_4^i \cdot v_4^i = -5$ and $v_4^i \cdot v_k^j = 0$ for $k = 1, 2, 3$ and $j = 1, \ldots, n$, one can observe that the image of v_4^i has the form,

$$\rho(v_4^i) = \pm(e_{1+5(\sigma_1(i)-1)} + e_{2+5(\sigma_1(i)-1)}) \pm (e_{3+5(\sigma_2(i)-1)} + e_{4+5(\sigma_2(i)-1)} + e_{5+5(\sigma_2(i)-1)}).$$

□

Corollary 5.4.8. *For $b > 5$, $I_{30(b-2)+19}$ has infinite order in $\Theta^3_{\mathbb{Q}}$.*

Proof. If $I_{30(b-2)+19}$ has order n in $\Theta^3_{\mathbb{Q}}$, then there is a lattice embedding $\rho\colon Q^n \hookrightarrow \langle -1\rangle^{5n}$ by the Donaldson obstruction. Let A be the matrix that represents the embedding ρ in terms of the basis $\{v_1^i, \cdots, v_5^i\}_{i=1}^n$ and $\{e_1, e_2, \cdots, e_{5n}\}$ of Q^n and $\langle -1\rangle^{5n}$ respectively as in the proof of Lemma 5.4.7. Then by the lemma and re-indexing and re-scaling of the basis $\{v_1^i, \cdots, v_5^i\}_{i=1}^n$, one may assume that the $5n \times 5n$-matrix A has the following form:

$$A = \begin{bmatrix} B & 0 & 0 & 0 & * \\ 0 & B & 0 & 0 & * \\ 0 & 0 & \ddots & 0 & * \\ 0 & 0 & 0 & B & * \end{bmatrix},$$

where B is a block matrix of the form either

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & -1 \end{bmatrix}.$$

Note that the last n columns of A correspond to the image of $\{v_5^1, \cdots, v_5^n\}$. Now we apply Proposition 5.4.6. Consider the set $V = \{(\pm 1, \cdots, \pm 1) \in \mathbb{Z}^{5n}\}$ and define a relation $\sim$ on V such that $v \sim w$ if and only if $v - w$ is a linear combination over $\mathbb{Z}$ of the first $4n$ column vectors of A. Define $W := \{(\pm 1, \cdots, \pm 1) \in \mathbb{Z}^5\}$ and a relation $\sim$ on W such that $v \sim w$ if and only if $v - w$ is a linear combination over $\mathbb{Z}$ of the column vectors of B. It is easy to check that $|V/{\sim}| = |W/{\sim}|^n$ because the initial $4n$ columns of A has a block matrix form.

To compute $|W/{\sim}|$, we follow the Greene and Jabuka's argument in [15]. Define a function $l : \mathbb{Z}^5 \to \mathbb{Z}$ as $l(v) = 3(x_1 + x_2) \pm 2(x_3 + x_4 + x_5)$ where $v = (x_1, x_2, x_3, x_4, x_5)$ is a element in $\mathbb{Z}^5$. Observe that $\ker(l)$ is exactly same as the subspace generated by the column vectors of B ($\pm$ sign of l follows from the choices of B). One can check

$$\mathrm{Im}(l|_{V_1}) = \{-12, -8, -6, -4, -2, 0, 2, 4, 6, 8, 12\}.$$

Hence $|W/{\sim}| = |\mathrm{Im}(l|_W)| = 11$ and so $|V/{\sim}| = 11^n$. Proposition 5.4.6 implies that 11^n is an upper bound for the number of extendable spin^c structures with vanishing correction terms of $\#^n I_{30(b-2)+19}$. By the correction term obstruction, if $I_{30(b-2)+19}$ has the order n in $\Theta^3_{\mathbb{Q}}$, there are at least $\sqrt{(30(b-2)+19)^n}$ spin^c structures on $\#^n I_{30(b-2)+19}$ with vanishing correction terms. Therefore, we obtain that $30(b-2)+19 \le 11^2$ and $b \le 5$. $\square$

5.4.4 Correction terms revisited

In this section, we finally discuss the following 11 manifolds whose order has not been determined so far.

$$\begin{cases} O_{12(b-2)+11} & \text{with} \quad b = 2,3,4,5, \\ I_{30(b-2)+17} & \text{with} \quad b = 2,4,6, \\ I_{30(b-2)+19} & \text{with} \quad b = 5, \\ I_{30(b-2)+23} & \text{with} \quad b = 3,5,7. \end{cases}$$

We claim that these manifolds also have infinite order in $\Theta^3_{\mathbb{Q}}$. The main ingredients of our proof are the correction term obstruction together with the study of metabolizer subgroups in products of finite abelian groups by Kim and Livingston in [21]. We also take advantage of that it is possible to compute all the correction terms explicitly since we only have finitely many manifolds at hand. We first develop the following finite order obstruction.

Proposition 5.4.9. *Let Y be a rational homology 3-sphere with $|H_1(Y;\mathbb{Z})|$ square free. Then if Y has finite order in $\Theta^3_{\mathbb{Q}}$, then*

$$\sum_{\mathfrak{s}\in Spin^c(Y)} d(Y,\mathfrak{s}) = 0.$$

Proof. Let Y be a rational homology 3-sphere with square free $m = |H_1(Y;\mathbb{Z})|$. Suppose $\#^{2k}Y$ bounds a rational homology 4-ball. Then there exists a spin^c structure $\mathfrak{t}$ on $\#^{2k}Y$ and a metabolizer $\mathcal{M}$ in $H^2(\#^{2k}Y;\mathbb{Z}) \cong \oplus_{i=1}^{2k} H^2(Y_i;\mathbb{Z}) \cong (\mathbb{Z}_m)^{2k}$ such that

$$d(\#^{2k}Y, \mathfrak{t}+\alpha) = 0$$

for each $\alpha \in \mathcal{M}$. Note that since m is square free, the projection map from the metabolizer $\mathcal{M}$ to each summand $H^2(Y_i;\mathbb{Z}) \cong \mathbb{Z}_m$ is a surjective homomorphism by [21, Corollary 3]. Let α_i denote the restriction of $\alpha \in$

$\mathcal{M}$ on each $H^2(Y_i;\mathbb{Z})$. Then by summing all the correction terms for spin^c structures extending to the rational ball, we have

$$0 = \sum_{\alpha\in\mathcal{M}} d(\#^{2k}Y, \mathfrak{t}+\alpha) = \sum_{\alpha\in\mathcal{M}}\sum_{i=1}^{2k} d(Y_i, \mathfrak{t}|_{Y_i}+\alpha_i) = \sum_{i=1}^{2k}\sum_{\alpha\in\mathcal{M}} d(Y_i, \mathfrak{t}|_{Y_i}+\alpha_i)$$
$$= \sum_{i=1}^{2k} m^{k-1} \sum_{\mathfrak{s}\in Spin^c(Y_i)} d(Y_i,\mathfrak{s}) = 2km^{k-1}\sum_{\mathfrak{s}\in Spin^c(Y_i)} d(Y_i,\mathfrak{s}).$$

□

Corollary 5.4.10. *The following manifolds have infinite order in $\Theta^3_{\mathbb{Q}}$:*

- $O_{12(b-2)+11}$ *for* $b = 2,3,4,5,$
- $I_{30(b-2)+17}$ *for* $b = 2,6,$
- $I_{30(b-2)+19}$ *for* $b = 5,$
- $I_{30(b-2)+23}$ *for* $b = 3,5,7.$

Proof. Observe that each manifold listed above admits square free order of H_1. Then by a direct computation of correction terms using the formula (3.3.3) of Ni and Wu, we find each manifold admits nonzero sum of correction terms over all spin^c structures on it. Hence they have infinite order in $\Theta^3_{\mathbb{Q}}$ by Proposition 5.4.9. □

Now, we are left with only one manifold, I_{77}. Notice that the sum of all correction terms on I_{77} is zero. The order of I_{77} is answered by the following finite order obstruction.

Proposition 5.4.11. *Let Y be a rational homology 3-sphere with $|H_1(Y;\mathbb{Z})| = \mathbb{Z}_p \oplus \mathbb{Z}_q$, where p is a positive prime, $(p,q)=1$, and pq is odd. Let $\mathfrak{s}_0$ be the unique spin^c structure whose first Chern class is trivial. Consider the subset in $Spin^c(Y)$ defined as*

$$A = \{\mathfrak{s}_0 + n\cdot q \mid n = 0,\dots,p-1\}.$$

Then, if Y has finite order in $\Theta^3_{\mathbb{Q}}$, then

$$\max_{\mathfrak{s}\in A}\{d(Y,\mathfrak{s})\} + \min_{\mathfrak{s}\in A}\{d(Y,\mathfrak{s})\} = 0.$$

Proof. Let Y be a rational homology sphere of finite order in $\Theta^3_{\mathbb{Q}}$, satisfying the hypothesis in the proposition. With out loss of generality, we may assume that there is $k > 0$ such that $\#^{4k}Y$ bounds a rational homology ball. Let $\mathcal{M}$ be a metabolizer of the linking form λ of $\#^{4k}Y$ in $\oplus^{4k}H^2(Y;\mathbb{Z}) \cong (\mathbb{Z}_p)^{4k} \oplus (\mathbb{Z}_q)^{4k}$ of which corresponding correction terms vanish. By the classification of linking forms over finite abelian groups, the linking form λ can be also decomposed into the direct sum of linking forms on each summand of $(\mathbb{Z}_p)^{4k} \oplus (\mathbb{Z}_q)^{4k}$. Then, by a result of Kim and Livingston [21, Theorem 4], $\mathcal{M}$ contains an element of the form

$$\gamma = (1,\dots,1,a_{2k+1},\dots,a_{4k}) \oplus \beta \in (\mathbb{Z}_p)^{4k} \oplus (\mathbb{Z}_q)^{4k}$$

for some $a_i \in \mathbb{Z}_p$ and some $\beta \in (\mathbb{Z}_q)^{4k}$. For any $m \in \{0,\dots,p-1\}$,

$$\begin{aligned}0 &= d(\#^{4k}Y, \#^{4k}\mathfrak{s}_0 + mq\cdot(\gamma\oplus\beta)) \\ &= d(\#^{4k}Y, \#^{4k}\mathfrak{s}_0 + mq\cdot\gamma\oplus 0) \\ &= 2k\cdot d(Y,\mathfrak{s}_0 + mq) + \sum_{i=2k+1}^{4k} d(Y,\mathfrak{s}_0 + mqa_i).\end{aligned}$$

Note that the first equation follows from Theorem 5.1.2, and in the third equation we use $d(Y,\mathfrak{s}_0) = 0$ from Proposition 5.4.1. Now, choose m so that $d(Y,\mathfrak{s}_0 + mq)$ is the maximum in A. Then in order for the above equation to hold, the minimum value in $\{d(Y,\mathfrak{s}_0 + nq) | n = 0,\dots p\}$ must be the negative of the maximum value. □

Corollary 5.4.12. *The manifold of type I_{77} has infinite order in $\Theta^3_{\mathbb{Q}}$.*

Proof. By taking $p = 7$ and $q = 11$, the manifold I_{77} satisfies the hypothesis of the above proposition. The correction terms of $\{\mathfrak{s}_0 + 11n | n = 0,\dots,6\}$

are given as:

$$\left\{0, -\frac{2}{7}, \frac{6}{7}, -\frac{4}{7}, -\frac{4}{7}, \frac{6}{7}, -\frac{2}{7}\right\}$$

by the computation using that $I_{77} \cong S_{\frac{77}{12}}(T_{2,3})$, and Ni and Wu's formula (3.3.3). Since $\frac{6}{7} - \frac{4}{7} \neq 0$, I_{77} has infinite order in $\Theta^3_{\mathbb{Q}}$ by Proposition 5.4.11. $\square$

Remark. We notice that the generalizations of our finite order obstructions, Proposition 5.4.9 and 5.4.11, can be found in [16].

5.4.5 Concordance of Montesinos knots admitting the spherical branched cover

We now prove by-products of our results, Corollary 1.2.2 and 1.2.4, about the concordance of the family of Montesinos knots admitting spherical branched double covers. Let $\mathcal{S}$ be the set of such Montesinos knots. The corollaries can be considered as generalizations of the results of Lisca, [26, Corollary 1.3] and [27, Corollary 1.3], for 2-bridge knots (of which branched covers are lens spaces).

Proof of Corollary 1.2.2. By the classifications of Seifert manifolds [43] and Montesinos links [46], the Montesinos links which admit spherical manifolds as the branched cover are exactly those corresponding to the natural Montesinos branch sets of spherical manifolds. Then, by the following clear implication,

$$K \text{ is ribbon} \Rightarrow K \text{ is slice} \Rightarrow \Sigma(K) \text{ bounds a rational ball},$$

the corollary is proved if we show that all Montesinos knots K in $\mathcal{S}$ such that $\Sigma(K)$ bounds a rational ball are ribbon. This follows from Proposition 5.2.3 for the branch sets of **D**-type manifolds of order 1, T_3, and T_{27}, and from Proposition 5.3.3 for the branch set of I_{49}. $\square$

Proof Corollary 1.2.4. Recall that the concordance order of a knot K is bounded below by the cobordism order of $\Sigma(K)$. In the proof of Corollary 1.2.2 above, we show that for any knot K in $\mathcal{S}$, $\Sigma(K)$ bounds a rational homology ball if and only if K is slice (of concordance order 1). Thus we only need to show that the branch sets of order 2 spherical manifolds have the same order in the concordance group.

For the lens spaces, this is a result of Lisca [27, Corollary 1.3]. Since the order of H_1 of **D**-type manifolds are even, they cannot be obtained by the double cover along a knot. Hence it remains to show that the Montesinos knot $M(4;(2,1),(3,2),(3,1))$, the canonical branch set of T_{15}, has concordance order two. In fact, this Montesinos knot is isotopic to the knot 9_{24} in terms of the notation of the Rolfsen's table [42]. See [20, Appendix F.2] for the fact that $9_{24} = M(4;(2,1),(3,2),(3,1))$. According to KnotInfo of Cha and Livingston [5], the knot 9_{24} has concordance order two. $\square$

As a final remark, we note that our work did not employ any particular facts of spherical geometry, but make use of the properties of Seifert manifolds and knot surgery manifolds. Thus it is natural to expect to generalize our results to larger families of manifolds, and we address the following question.

Question. Which Dehn surgeries on the trefoil knot, or more generally 3-legged Seifert manifolds bound rational homology balls?

Remark. We remark that Aceto and Golla classified $S^3_{p/q}(T)$ bounding rational balls provided that $p \equiv 1$ modulo q [1, Corollary 4.14].

Type	Spherical manifolds	Order in $\Theta^3_{\mathbb{Q}}$	Proved in
C	$L(p,q) \cong Y(-1;(q,p-q))$	1 if $\frac{p}{q} \in \mathcal{R}$	Lisca [27]
		2 if $\frac{p}{q} \in (\mathcal{S} \setminus \mathcal{R}) \cup \mathcal{F}_2$	
		∞ if $\frac{p}{q} \notin \mathcal{S} \cup \mathcal{R} \cup \mathcal{F}_2$	
D	$D(p,q) \cong Y(1;(2,1),(2,1),(q,q-p))$	same as $L(p-q,q)$	Section 5.2.2
T	$T_{6(b-2)+1} \cong Y(b;(2,1),(3,2),(3,2))$	∞ for any b	Section 5.4.1
	$T_{6(b-2)+3} \cong Y(b;(2,1),(3,1),(3,2))$	1 if $b=2,6$	Section 5.2.3
		2 if $b=4$	
		∞ otherwise	
	$T_{6(b-2)+5} \cong Y(b;(2,1),(3,1),(3,1))$	∞ for any b	Section 5.4.1
O	$O_{12(b-2)+1} \cong Y(b;(2,1),(3,2),(4,3))$	∞ for any b	Section 5.4.2
	$O_{12(b-2)+5} \cong Y(b;(2,1),(3,1),(4,3))$	∞ for any b	Section 5.4.2
	$O_{12(b-2)+7} \cong Y(b;(2,1),(3,2),(4,1))$	∞ for any b	Section 5.4.2
	$O_{12(b-2)+11} \cong Y(b;(2,1),(3,1),(4,1))$	∞ for any $b>5$	Section 5.4.2
		∞ for $b=2,3,4,5$	Section 5.4.4
I	$I_{30(b-2)+1} \cong Y(b;(2,1),(3,2),(5,4))$	∞ for any b	Section 5.4.1
	$I_{30(b-2)+7} \cong Y(b;(2,1),(3,2),(5,3))$	∞ for any b	Section 5.4.1
	$I_{30(b-2)+11} \cong Y(b;(2,1),(3,1),(5,4))$	∞ for any b	Section 5.4.1
	$I_{30(b-2)+13} \cong Y(b;(2,1),(3,2),(5,2))$	∞ for any b	Section 5.4.1
	$I_{30(b-2)+17} \cong Y(b;(2,1),(3,1),(5,3))$	∞ for any odd b	Section 5.4.1
		∞ for even $b>7$	Section 5.4.2
		∞ for $b=2,4,6$	Section 5.4.4
	$I_{30(b-2)+19} \cong Y(b;(2,1),(3,2),(5,1))$	∞ for any even b	Section 5.4.1
		∞ for any odd $b>5$	Section 5.4.3
		1 if $b=3$	Proposition 5.3.3
		∞ if $b=5$	Section 5.4.4
	$I_{30(b-2)+23} \cong Y(b;(2,1),(3,1),(5,2))$	∞ for even b	Section 5.4.1
		∞ for odd $b>7$	Section 5.4.2
		∞ for $b=3,5,7$	Section 5.4.4
	$I_{30(b-2)+29} \cong Y(b;(2,1),(3,1),(5,1))$	∞ for any b	Section 5.4.1

Table 5.1: The order of spherical manifolds in $\Theta^3_{\mathbb{Q}}$, where $p > q > 0$ are relative prime integers and $b \geq 2$.

Bibliography

[1] P. Aceto and M. Golla. Dehn surgeries and rational homology balls. *Algebr. Geom. Topol.*, 17(1):487–527, 2017.

[2] M. Bhupal and K. Ono. Symplectic fillings of links of quotient surface singularities. *Nagoya Math. J.*, 207:1–45, 2012.

[3] S. Boyer. Simply-connected 4-manifolds with a given boundary. *Trans. Amer. Math. Soc.*, 298(1):331–357, 1986.

[4] A. J. Casson and C. McA. Gordon. Cobordism of classical knots. In *À la recherche de la topologie perdue*, volume 62 of *Progr. Math.*, pages 181–199. Birkhäuser Boston, Boston, MA, 1986. With an appendix by P. M. Gilmer.

[5] J. C. Cha and C. Livingston. Knotinfo: Table of Knot Invariants. available at `http://www.indiana.edu/~knotinfo`, February 13, 2018.

[6] S. K. Donaldson. An application of gauge theory to four-dimensional topology. *J. Differential Geom.*, 18(2):279–315, 1983.

[7] S. K. Donaldson. The orientation of Yang-Mills moduli spaces and 4-manifold topology. *J. Differential Geom.*, 26(3):397–428, 1987.

[8] A. L. Edmonds. Homology lens spaces in topological 4-manifolds. *Illinois J. Math.*, 49(3):827–837 (electronic), 2005.

[9] N. D. Elkies. A characterization of the $\mathbf{Z}^n$ lattice. *Math. Res. Lett.*, 2(3):321–326, 1995.

[10] N. D. Elkies. Lattices and codes with long shadows. *Math. Res. Lett.*, 2(5):643–651, 1995.

[11] R. H. Fox. Some problems in knot theory. In *Topology of 3-manifolds and related topics (Proc. The Univ. of Georgia Institute, 1961)*, pages 168–176. Prentice-Hall, Englewood Cliffs, N.J., 1962.

[12] M. H. Freedman. The topology of four-dimensional manifolds. *J. Differential Geom.*, 17(3):357–453, 1982.

[13] K. A. Frøyshov. The Seiberg-Witten equations and four-manifolds with boundary. *Math. Res. Lett.*, 3(3):373–390, 1996.

[14] R. E. Gompf and A. I. Stipsicz. *4-manifolds and Kirby calculus*, volume 20 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1999.

[15] J. Greene and S. Jabuka. The slice-ribbon conjecture for 3-stranded pretzel knots. *Amer. J. Math.*, 133(3):555–580, 2011.

[16] J. E. Grigsby, D. Ruberman, and S. Strle. Knot concordance and Heegaard Floer homology invariants in branched covers. *Geom. Topol.*, 12(4):2249–2275, 2008.

[17] A. Issa. The classification of quasi-alternating montesinos links, 2017. arXiv:1701.08425.

[18] S. Jabuka and S. Naik. Order in the concordance group and Heegaard Floer homology. *Geom. Topol.*, 11:979–994, 2007.

[19] S. Jabuka, S. Robins, and X. Wang. Heegaard Floer correction terms and Dedekind-Rademacher sums. *Int. Math. Res. Not. IMRN*, (1):170–183, 2013.

[20] A. Kawauchi. *A survey of knot theory.* Birkhäuser Verlag, Basel, 1996. Translated and revised from the 1990 Japanese original by the author.

[21] S. Kim and C. Livingston. Nonsplittability of the rational homology cobordism group of 3-manifolds. *Pacific J. Math.*, 271(1):183–211, 2014.

[22] A. G. Lecuona. On the slice-ribbon conjecture for Montesinos knots. *Trans. Amer. Math. Soc.*, 364(1):233–285, 2012.

[23] A. G. Lecuona. On the slice-ribbon conjecture for pretzel knots. *Algebr. Geom. Topol.*, 15(4):2133–2173, 2015.

[24] A. G. Lecuona. Complementary legs and rational balls. *to appear in Michigan Math. J.* , 2017. available at `https://arxiv.org/abs/1701.01980`.

[25] A. G. Lecuona and P. Lisca. Stein fillable Seifert fibered 3-manifolds. *Algebr. Geom. Topol.*, 11(2):625–642, 2011.

[26] P. Lisca. Lens spaces, rational balls and the ribbon conjecture. *Geom. Topol.*, 11:429–472, 2007.

[27] P. Lisca. Sums of lens spaces bounding rational balls. *Algebr. Geom. Topol.*, 7:2141–2164, 2007.

[28] C. Manolescu and B. Owens. A concordance invariant from the Floer homology of double branched covers. *Int. Math. Res. Not. IMRN*, (20):Art. ID rnm077, 21, 2007.

[29] J. Milnor and D. Husemoller. *Symmetric bilinear forms*. Springer-Verlag, New York-Heidelberg, 1973. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 73.

[30] L. Moser. Elementary surgery along a torus knot. *Pacific J. Math.*, 38:737–745, 1971.

[31] W. D. Neumann. An invariant of plumbed homology spheres. In *Topology Symposium, Siegen 1979 (Proc. Sympos., Univ. Siegen, Siegen, 1979)*, volume 788 of *Lecture Notes in Math.*, pages 125–144. Springer, Berlin, 1980.

[32] W. D. Neumann and F. Raymond. Seifert manifolds, plumbing, μ-invariant and orientation reversing maps. In *Algebraic and geometric topology (Proc. Sympos., Univ. California, Santa Barbara, Calif., 1977)*,

volume 664 of *Lecture Notes in Math.*, pages 163–196. Springer, Berlin, 1978.

[33] Y. Ni and Z. Wu. Cosmetic surgeries on knots in S^3. *J. Reine Angew. Math.*, 706:1–17, 2015.

[34] P. Orlik. *Seifert manifolds.* Lecture Notes in Mathematics, Vol. 291. Springer-Verlag, Berlin-New York, 1972.

[35] B. Owens and S. Strle. Rational homology spheres and the four-ball genus of knots. *Adv. Math.*, 200(1):196–216, 2006.

[36] B. Owens and S. Strle. A characterization of the $\mathbb{Z}^n \oplus \mathbb{Z}(\delta)$ lattice and definite nonunimodular intersection forms. *Amer. J. Math.*, 134(4):891–913, 2012.

[37] B. Owens and S. Strle. Dehn surgeries and negative-definite four-manifolds. *Selecta Math. (N.S.)*, 18(4):839–854, 2012.

[38] P. Ozsváth and Z. Szabó. Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary. *Adv. Math.*, 173(2):179–261, 2003.

[39] P. Ozsváth and Z. Szabó. On the Floer homology of plumbed three-manifolds. *Geom. Topol.*, 7:185–224 (electronic), 2003.

[40] P. Ozsváth and Z. Szabó. On the Heegaard Floer homology of branched double-covers. *Adv. Math.*, 194(1):1–33, 2005.

[41] Peter Ozsváth and Zoltán Szabó. Holomorphic triangles and invariants for smooth four-manifolds. *Adv. Math.*, 202(2):326–400, 2006.

[42] D. Rolfsen. *Knots and links.* Publish or Perish, Inc., Berkeley, Calif., 1976. Mathematics Lecture Series, No. 7.

[43] H. Seifert. Topologie Dreidimensionaler Gefaserter Räume. *Acta Math.*, 60(1):147–238, 1933.

[44] L. Siebenmann. On vanishing of the Rohlin invariant and nonfinitely amphicheiral homology 3-spheres. In *Topology Symposium, Siegen 1979 (Proc. Sympos., Univ. Siegen, Siegen, 1979)*, volume 788 of *Lecture Notes in Math.*, pages 172–222. Springer, Berlin, 1980.

[45] A. I. Stipsicz. On the $\overline{\mu}$-invariant of rational surface singularities. *Proc. Amer. Math. Soc.*, 136(11):3815–3823, 2008.

[46] H. Zieschang. Classification of Montesinos knots. In *Topology (Leningrad, 1982)*, volume 1060 of *Lecture Notes in Math.*, pages 378–389. Springer, Berlin, 1984.

국문초록

행렬식이 ± 1인 임의의 정수이중선형형식이 단순연결된 닫힌 4차원 다양체의 교차형식이 될 수 있다는 것이 M. Freedman에 의해 알려져있다. 반면 닫힌 매끈한 4차원 다양체의 교차형식이 정부호이면 그 교차형식은 대각화 가능하다는 것이 S. Donaldson에 의해 알려져있다. 이 논문에서는 유리적 호몰로지 3차원 구를 경계로 갖는 매끈한 정부호 4차원 다양체의 교차형식에 대해 연구한다. 첫 번째로 이 논문에서는 고정된 유리적 호몰로지 3차원 구에 대해 그것을 경계로 갖는 매끈한 정부호 4차원 다양체의 교차형식의 유한성에 대해 논의한다. 또한 3차원 구면 다양체에 대하여 위의 유한성이 만족됨을 보인다. 두 번째로 어떠한 3차원 구면다양체가 유리적 호몰로지 4차원 공의 경계로 주어지는지를 중점적으로 살펴본다. 그리하여 어떠한 3차원 구면다양체가 유리적 호몰로지 4차원 공의 경계로 주어지는지 분류한다. 또한 유리적 호몰로지 보충경계 군 안에서 3차원 구면다양체의 위수를 결정한다.

주요어휘**:** 교차형식, 슬라이스-리본 가설, 3차원 구면 다양체, Heegaard Floer 수정 항.
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