이학박사학위논문

Uniqueness problems of diffusion operators on Euclidean space and on abstract Wiener space

유클리드 공간과 추상적인 위너 공간 위에서의 확산 작용소들의 유일성에 관한 문제들

2018 년 8 월

서울대학교 대학원

수리과학부

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Abstract

Uniqueness problems of diffusion operators on Euclidean space and on abstract Wiener space

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The central question discussed in this thesis is whether a given diffusion operators, i.e., a second order linear elliptic differential operator without zeroth order term, which is a priori only defined on test functions over some (finite or infinite dimensional) state space, uniquely determines a strongly continuous semigroup on a corresponding weighted L^p space.

On the first part of the thesis, we are mainly focus on equivalence of different definitions of capacities, and removability of singularities. More precisely, let L be either a fractional powers of Laplacian of order less than one whose domain is smooth compactly supported functions on $\mathbb{R}^d \setminus \Sigma$ of a given compact set $\Sigma \subset \mathbb{R}^d$ of zero Lebesgue measure or integral powers of Ornstein-Uhlenbeck operator defined on suitable algebras of functions vanishing in a neighborhood of a given closed set Σ of zero Gaussian measure in abstract Wiener space. Depending on the size of Σ, the operator under consideration, may or may not be L^p unique. We give descriptions for the critical size of Σ in terms of capacities and Hausdorff measures. In addition, we collect some known results for certain multi-parameter stochastic processes.

On the second part of this thesis, we are mainly focus on Neumann problems on $L^p(U,\mu)$, where $U \subset \mathbb{R}^d$ is an open set. More precisely, let L be a nonsymmetric operator of type $Lu = \sum a_{ij}\partial_i\partial_j u + \sum b_i\partial_i u$, whose domain is $C^2_{0,Neu}(\overline{U})$. We give some results about Markov uniqueness, L^p-uniqueness, relation of L^1-uniqueness and conservativeness, uniqueness of invariant measures, elliptic regularity, etc under certain assumption on μ and on the coefficients of L.

Keywords: generalized Dirichlet forms, non-symmetric Dirichlet forms, conservativeness, diffusion processes, Neumann problem, abstract Wiener space, capacity, Ornstein-Uhlenbeck operator, Markov uniqueness, L^p-uniqueness, essential self-adjointness, elliptic regularity, invariant measure, Hausdorff measure.

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Contents

Chapter 1 General Introduction

This thesis is based on [48, 49, 64, 65]. For this chapter, we mainly follow the book [29]. Let U be an open subset in $\mathbb{R}^d$, and let $\mathcal{A}$ be space of test functions on U, e.g., $\mathcal{A} = C_0^\infty(U)$. Suppose we are given a second order differential operator

$$Lf = \sum_{i,j=1}^{d} a_{ij}\partial_i\partial_j f + \sum_{i=1}^{d} b_i\partial_i f, \quad f \in \mathcal{A},$$

with measurable coefficients $a_{ij},\ b_i : U \to \mathbb{R}$, $1 \le i, j, \le d$, such that the matrix $A = (a_{ij})_{i,j}$ is positive definite for all $x \in U$. We call such operators, as well as operators of a similar type on more general (in particular infinite dimensional) state spaces E, diffusion operators, cf. [29, Appendix B] for a general definition.

Besides their theoretical importance in analysis and probability, singular finite dimensional diffusion operators occur in many applications, including in particular stochastic mechanics. Moreover, considering singular finite dimensional diffusion operators can be viewed as a pre-study for the more difficult infinite dimensional case. The interplay between finite and infinite dimensional analysis is very powerful. We will show some finite dimensional uniqueness problem, and try to lift some dimension independent finite dimensional results to infinite dimensions.

Our main focus are uniqueness problems for diffusion operators on L^p spaces. Let $\mathcal{A}$ be a space of test functions over the corresponding state space U, e.g., $\mathcal{A} = C_0^\infty(U)$, if U is an open subset of $\mathbb{R}^d$.

Let μ be a σ-finite measure on U. The measure μ we choose is an invariant measure for $(L, \mathcal{A})$. Let $1 \le p < \infty$. For the operators we are interested in, it is known that there exists a C_0 semigroup $(T_t)_{t\ge0}$ on $L^p(U, \mu)$ such that its generator extends the operator

$(L, \mathcal{A})$. We also know that $(T_t)_{t \geq 0}$ is sub-Markovian, i.e., $0 \geq T_t f \geq 1$ μ-a.e., whenever $0 \leq f \leq 1$ μ-a.e. If $(T_t)_{t \geq 0}$ is the only C_0 semigroup on $L^p(U, \mu)$ such that its generator extends $(L, \mathcal{A})$, we call $(L, \mathcal{A})$ L^p unique. If $(T_t)_{t \geq 0}$ is the only sub-Markovian C_0 semigroup on $L^2(U, \mu)$ such that its generator extends $(L, \mathcal{A})$, we call $(L, \mathcal{A})$ Markov unique. Clearly, L^2 uniqueness implies Markov uniqueness. If the closure of symmetric operator $(L, \mathcal{A})$ on $L^2(U, \mu)$ is self-adjoint, we say $(L, \mathcal{A})$ is essentially self-adjoint. For a non-positive symmetric operator $(L, \mathcal{A})$ on $L^2(U, \mu)$, i.e., $(f, Lf)_{L^2(U,\mu)} \leq 0$ for all $f \in \mathcal{A}$ and $(L, \mathcal{A})$ is symmetric, it is known that essential self-adjointnesss is equivalent to L^2 uniqueness.

This general uniqueness problem is related to several specific questions arising in different areas, e.g., uniqueness of martingale problems, existence of operator cores consisting of "nice" functions, essential self-adjointness and uniqueness problems in mathematical physics, as well as uniqueness problems for Dirichlet forms.

Part I

Equivalence of capacities and removability of singularities

Chapter 2 Probabilistic characterizations of essential self-adjointness and removability of singularities

2.1 Introduction

In this chapter we would like to point out an interesting connection between some traditional and well-studied notions in analysis and an interesting, but perhaps slightly less known area in probability theory. More precisely, we outline the relation between uniqueness questions for self-adjoint extensions of the Laplacian and its powers on the one hand and hitting probabilities for certain two-parameter stochastic processes on the other. Although both, the analytic part and the probabilistic part of the results stated below are well-established, it seems that the existing literature did never merge these two different aspects.

Recall that if a symmetric operator in a Hilbert space, considered together with a given dense initial domain, has a unique self-adjoint extension, then it is called *essentially self-adjoint*. The question of essential self-adjointness has strong physical relevance, because the evolution of a quantum system is described in terms of a unitary group, the generator of a unitary group is necessarily self-adjoint, and different self-adjoint operators determine different unitary groups, i.e. different physical dynamics. See for instance [88, Section X.1]. Self-adjointness, and therefore also essential self-adjointness, are notions originating from quantum mechanics.

A related notion of uniqueness comes up in probability theory, more precisely, in the theory of Markov semigroups. Recall that any non-positive definite self-adjoint operator L on a Hilbert space H is uniquely associated with a non-negative definite closed and densely defined symmetric bilinear form Q on H by $Q(u,v) = -\langle u, Lv\rangle_H$, [88, Section VIII.6], where $\langle\cdot,\cdot\rangle_H$ denotes the scalar product in H and u and v are arbitrary elements of the domain of Q and the domain of L, respectively. Now assume that H is an L^2-space of real-valued (classes of) functions. Then, if for any u from the domain of Q also $|u|$ is in the domain of Q and we have $Q(|u|,|u|) \leq Q(u,u)$, the form Q is said to satisfy the *Markov property*. In this case it is called a *Dirichlet form*, and L is the infinitesimal generator of a uniquely determined strongly continuous semigroup of symmetric Markov operators on H, sometimes also called a *Markov generator*, [17, 23, 35]. We say that a non-positive definite symmetric operator in an L^2-space H, together with a given dense initial domain, is *Markov unique*, if it has a unique self-adjoint extension in H that generates a Markov semigroup. Different Markov generators determine different Markov semigroups and (disregarding for a moment important issues of construction and regularity) this means that they define different Markov processes. So the notion of Markov uniqueness belongs to probability theory. It has strong relevance in the context of classical mechanics and statistical physics.

For a non-positive definite densely defined symmetric operator on an L^2-space essential self-adjointness implies Markov uniqueness, but the converse implication is false, see Examples 1 and 2 below or [99]. Even if an operator is Markov unique, it may still have other self-adjoint extensions that do not generate Markov semigroups. It is certainly fair to say that a priori the notion of essential self-adjointness is a not a probabilistic notion. However, and this is what we would like to point out here, in certain situations essential self-adjointness can still be characterized in terms of classical probability.

We consider specific exterior boundary value problems in $\mathbb{R}^d$. It is well-known that the Laplacian Δ, endowed with the initial domain $C_0^\infty(\mathbb{R}^d)$ of smooth compactly supported functions on $\mathbb{R}^d$, has a unique self-adjoint extension in $L^2(\mathbb{R}^d)$. This unique self-adjoint extension is given by $(\Delta, H^2(\mathbb{R}^d))$, where given $\alpha > 0$, the symbol $H^\alpha(\mathbb{R}^d)$ denotes the Bessel potential space of order α, see Section 2.2 below. Similarly, the fractional Laplacians $-(-\Delta)^{\alpha/2}$ of order $\alpha > 0$, endowed with the domain $C_0^\infty(\mathbb{R}^d)$, have unique self-adjoint extensions, respectively, namely $(-(-\Delta)^{\alpha/2}, H^\alpha(\mathbb{R}^d))$. In the present note we focus on the cases $0 < \alpha \leq 2$.

Given a compact set $\Sigma \subset \mathbb{R}^d$ of zero d-dimensional Lebesgue measure, we denote its complement by $N := \mathbb{R}^d \setminus \Sigma$. For any $0 < \alpha \leq 2$ the operator $(-(-\Delta)^{\alpha/2}, C_0^\infty(N))$ is non-positive definite and symmetric on $L^2(N) = L^2(\mathbb{R}^d)$. We are interested in conditions on the size of Σ so that $(-(-\Delta)^{\alpha/2}, C_0^\infty(N))$ is essentially self-adjoint. Of course one possible self-adjoint extension is the global operator $(-(-\Delta)^{\alpha/2}, H^\alpha(\mathbb{R}^d))$, which 'ignores' Σ. If Σ is 'sufficiently small', it will not be seen, and there is no other self-adjoint extension. If Σ is 'too big', it will registered as a boundary, leading to a self-adjoint extension different from the global one.

As mentioned, the analytic background of this problem is classical and can for instance be found in the textbooks [3, 35, 77]. See in particular [77, Sections 13.3 and 13.4]. For integer powers of the Laplacian on $\mathbb{R}^d$ a description of the critical size of Σ in terms of capacities and Hausdorff measures had been given in [5, Section 10], and to our knowledge this was the first reference that gave such a characterization of essential self-adjointness. For fractional powers a characterization of essential self-adjointness for the case $\Sigma = \{0\}$ follows from [34, Theorem 1.1]. For more general compact sets Σ such descriptions do not seem to exist in written form. A probabilistic description for the critical size of Σ, which we could not find anywhere in the existing literature, can be given in terms of suitable two-parameter processes as for instance studied in [52, 59, 61, 62]. In essence, these descriptions are straightforward applications of Kakutani type theorems for multiparameter processes, see for instance [59, Chapter 11, Theorems 3.1.1 and 4.1.1]. In fact, using processes with more than two parameters one could even extend this type of results to fractional Laplacians of arbitrary order. A philosophically related idea, namely a connection between Riesz capacities and the hitting behaviour of certain one-parameter Gaussian processes (that are not Markov processes except in the Brownian case) had already been studied in [57]. Taking into consideration also processes with a more general state space, another idea is to test the size of small sets with one-parameter processes taking values in the space of finite measures over $\mathbb{R}^d$, see for instance [26, 82, 83]. Interestingly, they exhibit exactly the hitting behaviour needed to characterize the essential self-adjointness of the Laplacian, [83, Theorem III.5.2].

We would like to announce related forthcoming results for Laplacians on complete Riemannian manifolds, [50]. An analytic description of essential self-adjointness for the Laplacian via capacities reads as in the Euclidean case, instead of traditional arguments for Euclidean spaces based on convolutions, [3], our proof uses the regularity theory for the Laplacian on manifolds, [40], and basic estimates on the gradients of resolvent densities, [6, Section 4.2]. To proceed to a geometric description we use asymptotics of the resolvent densities, they are basically the same as those for Green functions, see for instance [6, Section 4.2], [39, Section 4.2] or [70, Section 4.2]. For a probabilistic description we restrict ourselves, at least for the time being, to the case of Lie groups. In this case we can still work with relatively simple two-parameter processes and use the potential developed in [52, 53] to connect them to capacities and essential self-adjointness. In the case of general complete Riemannian manifolds one first has to raise the quite non-trivial question what could be suitable two-parameter processes taking values in manifolds. It might even turn out that it is more natural to use measure-valued processes.

A subsequent idea to be addressed in the near future concerns details of the relationship between stochastic processes and specific boundary value problems. For many interesting cases it is well understood how boundary value problems (such as Dirichlet, Neumann or mixed), encoded in the choice of domain for the associated Dirichlet form, determine the behaviour of associated one-parameter Markov processes. It would be interesting to see whether, and if yes, in what sense, the behaviour of related two-parameter processes can reflect given boundary value problems for the Laplacian, encoded in the choice of its domain as a self-adjoint operator.

In the next section we collect some preliminaries. In Section 2.3 we discuss analytic characterizations of Markov uniqueness and essential self-adjointness for fractional Laplacians. In Section 2.4 we provide geometric descriptions, and in Section 2.5 we give probabilistic characterizations in terms of hitting probabilities for two-parameter processes.

2.2 Bessel potential spaces, capacities and kernels

We provide some preliminaries on function spaces, fractional Laplacians, related capacities and kernels. Our exposition mainly follows [3, Chapters 1-3]. Given $\alpha > 0$ we define the *Bessel potential space of order* α by

$$H^\alpha(\mathbb{R}^d) = \left\{u \in L^2(\mathbb{R}^d) : (1 + |\xi|^2)^{\alpha/2}\hat{u} \in L^2(\mathbb{R}^d)\right\},$$

where $u \mapsto \hat{u}$ denotes the Fourier transform of u. Together with the norm

$$\|u\|_{H^\alpha(\mathbb{R}^d)} = \left\|(1 + |\xi|^2)^{\alpha/2}\hat{u}\right\|_{L^2(\mathbb{R}^d)}$$

it becomes a Hilbert space. See for instance [3, 77, 100, 101]. Using the fact that

$$-\Delta f = (|\xi|^2 \hat{f})^\vee$$

for any $f \in \mathcal{S}(\mathbb{R}^d)$, where $\mathcal{S}(\mathbb{R}^d)$ denotes the space of Schwartz functions on $\mathbb{R}^d$ and $u \mapsto \check{u}$ the inverse Fourier transform, we can easily see that $(\Delta, C_0^\infty(\mathbb{R}^d))$ is essentially self-adjoint on $L^2(\mathbb{R}^d)$ with the unique self-adjoint extension $(\Delta, H^2(\mathbb{R}^d))$, see for instance [24, Theorem 3.5.3]. For $\alpha > 0$ we can define the *fractional Laplacians* $-(-\Delta)^{\alpha/2}$ *of order* $\alpha/2$ in terms of Fourier transforms by

$$(-\Delta)^{\alpha/2} f = (|\xi|^\alpha \hat{f})^\vee.$$

Again it is not difficult to show that $(-\Delta)^{\alpha/2}$, endowed with the domain $C_0^\infty(\mathbb{R}^d)$, has a unique self-adjoint extension, namely $((-\Delta)^{\alpha/2}, H^\alpha(\mathbb{R}^d))$. One can proceed similarly as in [24, Theorem 3.5.3], see also [24, Theorem 1.2.7 and Lemma 1.3.1].

Given $\alpha > 0$, we write

$$\gamma_\alpha := ((1 + |\xi|^2)^{-\alpha/2})^\vee \tag{2.1}$$

to denote the *Bessel kernel of order* α and $\mathcal{G}_\alpha f := \gamma_\alpha * f$ to denote the *Bessel potential operator* $\mathcal{G}_\alpha$ *of order* α, which defines a bijection from $\mathcal{S}(\mathbb{R}^d)$ into itself and also a bounded linear operator $\mathcal{G}_\alpha : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$. In both interpretations we have $\mathcal{G}_\alpha = (I - \Delta)^{-\alpha/2}$. The image $\mathcal{G}_\alpha f$ of a measurable function $f : \mathbb{R}^d \to [0, +\infty]$ is a lower semicontinuous nonnegative function on $\mathbb{R}^d$, see [3, Proposition 2.3.2]. This implies that for any $f \in L^2_+(\mathbb{R}^d)$, where the latter symbol denotes the cone of nonnegative elements in $L^2(\mathbb{R}^d)$, its image $\mathcal{G}_\alpha f$ is a $[0, +\infty]$-valued function on $\mathbb{R}^d$, i.e. defined for any $x \in \mathbb{R}^d$. We can therefore define the $\alpha, 2$-*capacity* $\mathrm{Cap}_{\alpha,2}(E)$ of a set $E \subset \mathbb{R}^d$ by

$$\mathrm{Cap}_{\alpha,2}(E) = \inf \left\{ \|f\|^2_{L^2(\mathbb{R}^d)} : f \in L^2_+(\mathbb{R}^d) \text{ and } \mathcal{G}_\alpha f(x) \geq 1 \text{ for all } x \in E \right\},$$

with the convention that $\mathrm{Cap}_{\alpha,2}(E) = +\infty$ if no such f exists, see [3, Definition 2.3.3].

There is another, 'more algebraic' definition of a $\alpha, 2$-capacity. For a compact set $K \subset \mathbb{R}^d$, define

$$\mathrm{Cap}'_{\alpha,2}(K) = \inf\Big\{\|\varphi\|^2_{H^\alpha(\mathbb{R}^d)} : \ \varphi \in C_0^\infty(\mathbb{R}^d) \text{ such that } \varphi(x) = 1 \\ \text{for all } x \text{ from a neighborhood of } K\Big\}. \quad (2.2)$$

Exhausting open sets by compact ones and approximating arbitrary sets from outside by open ones, this definition can be extended in a consistent manner to arbitrary subsets of $\mathbb{R}^d$. Now it is known that there exist constants $c_1, c_2 > 0$ such that for any compact set $K \subset \mathbb{R}^d$, we have

$$c_1 \ \mathrm{Cap}_{\alpha,2}(K) \leq \mathrm{Cap}'_{\alpha,2}(K) \leq c_2 \ \mathrm{Cap}_{\alpha,2}(K), \quad (2.3)$$

see [76, Theorem 3.3] for integer α and [3, Section 2.7 and Corollary 3.3.4] or [4, Theorem A] for general α. We would like to remark that (2.3) is based on certain truncation results for potentials. For $0 < \alpha \leq 1$ the spaces $H^\alpha(\mathbb{R}^d)$ are domains of Dirichlet forms so that truncation properties are immediate from the Markov property. However, for $\alpha > 1$ one needs to invest additional arguments, see for instance [3, Sections 3.3, 3.5 and 3.7].

As before, let $\alpha > 0$. We say that a Radon measure μ on $\mathbb{R}^d$ has *finite α-energy* if

$$\int_{\mathbb{R}^d} |v| d\mu \leq c \, \|v\|_{H^\alpha(\mathbb{R}^d)} \qquad \text{for all } v \in C_0^\infty(\mathbb{R}).$$

For a measure μ having finite α-energy we can find a function $U^\alpha\mu \in H^\alpha(\mathbb{R}^d)$ such that

$$\langle U^\alpha \mu, v\rangle_{H^\alpha(\mathbb{R}^d)} = \int_{\mathbb{R}^d} v \, d\mu \quad \text{for all } v \in \mathcal{S}(\mathbb{R}^d). \quad (2.4)$$

Using Fourier transforms this seen to be equivalent to requiring

$$\Big\langle (1 + |\xi|^2)^\alpha \widehat{U^\alpha \mu}, \hat{v}\Big\rangle_{L^2(\mathbb{R}^d)} = \hat{\mu}(\widehat{v(-\cdot)}) \quad \text{for all } v \in \mathcal{S}(\mathbb{R}^d),$$

what implies that $\widehat{U^\alpha \mu} = (1+|\xi|^2)^{-\alpha}\hat{\mu}$ in the sense of Schwartz distributions, and finally,

$$U^\alpha \mu = \gamma_{2\alpha} * \mu.$$

Note that by (2.1) we have

$$\gamma_{2\alpha} = \gamma_\alpha * \gamma_\alpha. \tag{2.5}$$

We can define the α-energy of μ as

$$E_\alpha(\mu) := \int_{\mathbb{R}^d} U^\alpha \mu \, d\mu,$$

and by (2.4) this can be seen to equal $\|U^\alpha \mu\|^2_{H^\alpha(\mathbb{R}^d)}$. There is a dual definition of the $\alpha, 2$-capacity: For a compact set $K \subset \mathbb{R}^d$ we have

$$\operatorname{Cap}_{\alpha,2}(K) = \sup\left\{\frac{\mu(K)^2}{E_\alpha(\mu)} : \ \mu \text{ is a Radon measure on } K\right\} \tag{2.6}$$

with the interpretation $\frac{1}{\infty} := 0$, see [3, Theorem 2.2.7].

We finally collect some well-known asymptotics of the Bessel kernels. For $0 < \alpha < d$ we have

$$\gamma_\alpha \sim c_{d,\alpha}|x|^{\alpha-d} \qquad \text{as } |x| \to 0 \tag{2.7}$$

with a positive constant $c_{d,\alpha}$ depending only on d and α, and for the limit case $\alpha = d$,

$$\gamma_d(x) \sim c_d(-\log|x|) \qquad \text{as } |x| \to 0 \tag{2.8}$$

with a positive constant c_d depending on only on d. Moreover, it is known that

$$\gamma_\alpha \text{ is continuous away from 0 and } \gamma_\alpha(x) = O(e^{-c|x|}) \text{ as } |x| \to \infty. \tag{2.9}$$

By (2.1) we have $\hat{\gamma}_\alpha(\xi) \leq |\xi|^{-\alpha}$ for all sufficiently large $\xi \in \mathbb{R}^d$. In the case $d < \alpha$ we therefore see that the Bessel kernel γ_α is an element of $L^1(\mathbb{R}^d)$ and equals

$$\gamma_\alpha(x) = \int_{\mathbb{R}^d} \frac{e^{i\langle x,\xi\rangle}}{(1+|\xi|^2)^{\alpha/2}} \, d\xi, \quad x \in \mathbb{R}^d. \tag{2.10}$$

See [3, Sections 1.2.4 and 1.2.5].

2.3 Markov uniqueness, essential self-adjointness and capacities

Recall that $\Sigma \subset \mathbb{R}^d$ is a given compact set of zero Lebesgue measure and $N := \mathbb{R}^d \setminus \Sigma$. We first state a well-known known result on Markov uniqueness. Using the definition (2.2) of capacities together with traditional approximation arguments, which we will formulate below for the question of essential self-adjointness, one can obtain the following.

Theorem 1 *Let $0 < \alpha \le 2$. The fractional Laplacian $((-\Delta)^{\alpha/2}, C_0^\infty(N))$ is Markov-unique if and only if $\mathrm{Cap}_{\alpha/2,2}(\Sigma) = 0$.*

A classical guiding example for the case $\alpha = 2$ is the following, which will be complemented for the cases $0 < \alpha < 2$ in Section 2.4.

Example 1 *Consider the case that $\Sigma = \{0\}$. Then $(\Delta, C_0^\infty(N))$ is Markov unique if and only if $d \ge 2$. See [99, p.114].*

We turn to essential self-adjointness. The following theorem provides a characterization in term of the $\alpha, 2$-capacity of Σ.

Theorem 2 *Let $0 < \alpha \le 2$. The fractional Laplacian $((-\Delta)^{\alpha/2}, C_0^\infty(N))$ is essentially self-adjoint if and only if $\mathrm{Cap}_{\alpha,2}(\Sigma) = 0$.*

For the case $\alpha = 2$ Theorem 2 is partially implied by [5, Theorems 10.3 and 10.5], which also imply corresponding results for powers of the Laplacian of higher integer order. In [50] we provide a version of Theorem 2 for the Laplacian ($\alpha = 2$) on complete Riemannian manifolds, generalizing earlier results given in [74, Theorem 3] and [20, Theoreme 1].

The following is a well-known guiding example for $\alpha = 2$, for the case $0 < \alpha < 2$ see Section 2.4.

Example 2 *Consider the case that $\Sigma = \{0\}$. Then $(\Delta, C_0^\infty(N))$ is essentially self-adjoint if and only if $d \geq 4$. See [99, p.114] and [88, Theorem X.11, p.161].*

We formulate a proof of Theorem 2. Theorem 1 can be obtained by similar arguments.

Proof Suppose that $\mathrm{Cap}_{\alpha,2}(\Sigma) = 0$. Let $(\mathcal{L}^{(\alpha)}, \mathrm{dom}\,\mathcal{L}^{(\alpha)})$ denote the closure in $L^2(\mathbb{R}^d)$ of $-(-\Delta)^{\alpha/2}$ with initial domain $C_0^\infty(N)$. Since clearly $\mathrm{dom}\,\mathcal{L}^{(\alpha)} \subset H^\alpha(\mathbb{R}^d)$, it suffices to show the converse inclusion. Given $u \in H^\alpha(\mathbb{R}^d)$, let $(u_n)_n \subset C_0^\infty(\mathbb{R}^d)$ be a sequence approximating u in $H^\alpha(\mathbb{R}^d)$. By (2.2) there is a sequence $(v_k)_k \subset C_0^\infty(N)$ such that $v_k \to 0$ in $H^\alpha(\mathbb{R}^d)$ and for each k, v_k equals one on a neighborhood of Σ. Set $w_{nk} := (1 - v_k)u_n$ to obtain functions $w_{nk} \in C_0^\infty(N)$. Let n be fixed. It is easy to see that $u_n - w_{nk} = u_n v_k \to 0$ in $L^2(\mathbb{R}^d)$ as $k \to \infty$. Because the graph norm of $(-\Delta)^{\alpha/2}$ provides an equivalent norm in $H^\alpha(\mathbb{R}^d)$, it now suffices to note that

$$(-\Delta)^{\alpha/2}(u_n - w_{nk}) = (-\Delta)^{\alpha/2}(u_n v_k) \to 0 \quad \text{in } L^2(\mathbb{R}^d) \text{ as } k \to \infty. \tag{2.11}$$

For any $f, g \in C_0^\infty(\mathbb{R}^d)$ we can use the identity

$$-(-\Delta)^{\alpha/2}(fg) = 2\Gamma^{(\alpha)}(f,g) - f(-\Delta)^{\alpha/2}g - g(-\Delta)^{\alpha/2}f \tag{2.12}$$

to define the *carré du champ* $\Gamma^{(\alpha)}(f,g)$ of f and g associated with $-(-\Delta)^{\alpha/2}$, see for instance [7, Section 1.4.2]. We have

$$\left\| f(-\Delta)^{\alpha/2}g \right\|_{L^2(\mathbb{R}^d)} \leq \|f\|_{L^\infty(\mathbb{R}^d)} \|g\|_{H^\alpha(\mathbb{R}^d)}$$

for the second summand on the right hand side, and

$$\left\| g(-\Delta)^{\alpha/2}f \right\|_{L^2(\mathbb{R}^d)} \leq \left\| (-\Delta)^{\alpha/2}f \right\|_{L^\infty(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)}$$

for the third. For the first summand on the right hand side of (2.12) we can use Cauchy-Schwarz, $|\Gamma^{(\alpha)}(f,g)| \le \Gamma^{(\alpha)}(f,f)^{1/2}\Gamma^{(\alpha)}(g,g)^{1/2}$, and since $(-(-\Delta)^{\alpha/2}, C_0^\infty(\mathbb{R}^d))$ also extends to a Feller generator on $\mathbb{R}^d$ (see for instance [89]), we have $\Gamma^{(\alpha)}(f,f) \in L^\infty(\mathbb{R}^d)$, so that

$$\left\|\Gamma^{(\alpha)}(f,g)\right\|_{L^2(\mathbb{R}^d)} \le \left\|\Gamma^{(\alpha)}(f,f)\right\|_{L^\infty(\mathbb{R}^d)} \|g\|_{H^\alpha(\mathbb{R}^d)}.$$

Here we have used that $\left\|\Gamma^{(\alpha)}(g,g)\right\|^2_{L^1(\mathbb{R}^d)}$ is nothing but the energy $\left\langle(-\Delta)^{\alpha/4}g, (-\Delta)^{\alpha/4}g\right\rangle_{L^2(\mathbb{R}^d)}$ of g, clearly dominated by the square of the $H^\alpha(\mathbb{R}^d)$-norm of g. Considering (2.12) with u_n and v_k in place of f and g and applying the preceding estimates, we see (2.11). As a consequence, we see that $H^\alpha(\mathbb{R}^d) \subset \operatorname{dom}\mathcal{L}^{(\alpha)}$.

Conversely, suppose that $((-\Delta)^{\alpha/2}, C_0^\infty(N))$ is essentially self-adjoint in $L^2(\mathbb{R}^d)$. Then its unique self-adjoint extension must be $((-\Delta)^{\alpha/2}, H^\alpha(\mathbb{R}^d))$. Let $u \in C_0^\infty(\mathbb{R}^d)$ be a function that equals one on a neighborhood of Σ. Since $C_0^\infty(\mathbb{R}^d) \subset H^\alpha(\mathbb{R}^d)$ and by hypothesis $C_0^\infty(N)$ must be dense in $H^\alpha(\mathbb{R}^d)$, we can find a sequence $(u_n)_n$ approximating u in $H^\alpha(\mathbb{R}^d)$. The functions $e_n := u - u_n$ then are in $C_0^\infty(\mathbb{R}^d)$, equal one on a neighborhood of Σ, and converge to zero in $H^\alpha(\mathbb{R}^d)$, so that $\mathrm{Cap}_{\alpha,2}(\Sigma) \le \lim_n \|e_n\|^2_{H^\alpha(\mathbb{R}^d)} = 0$.

□

Finally, we would like to mention known removability results for Δ. One says that a compact set $K \subset \mathbb{R}^d$ is *removable* (or a *removable singularity*) for Δ in L^2 if any solution u of $\Delta u = 0$ in $U \setminus K$ for some bounded open neighborhood U of K such that $u \in L^2(U \setminus K)$, can be extended to a function $\widetilde{u} \in L^2(U)$ satisfying $\Delta\widetilde{u} = 0$ in U. See [3, Definition 2.7.3]. By Corollary [3, 3.3.4] (see also [77, Section 13.4] and [5, Proposition 10.2]) a compact set $K \subset \mathbb{R}^d$ is removable for Δ in L^2 if and only if $\mathrm{Cap}_{2,2}(K) = 0$.

Removability results for fractional Laplacians are for instance discussed in [56].

2.4 Riesz capacities and Hausdorff measures

In this section we consider some geometric descriptions for the critical size of Σ. For the case of Markov uniqueness they have been discussed in many places. For the case of essential self-adjointness of integer powers of the Laplacian they were already stated in [5].

We first give a quick review of Riesz energies and capacities. Given $s > 0$ and a Radon measure μ on $\mathbb{R}^d$, let

$$I_s\mu = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |x-y|^{-s}\mu(dy)\mu(dx)$$

denote the *Riesz energy of order s* of μ. The *Riesz energy of order zero* of a Radon measure μ on $\mathbb{R}^d$ we define to be

$$I_0\mu = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (-\ln|x-y|)_+ \,\mu(dy)\mu(dx).$$

For a Borel set $E \subset \mathbb{R}^d$ we can the define the *Riesz capacity of order $s \geq 0$* of E by

$$\mathrm{Cap}_s(E) = \left[\inf\left\{I_s(\mu) : \mu \text{ Borel probability measure on } E\right\}\right]^{-1}$$

with the agreement that $\frac{1}{\infty} := 0$. See for instance [59, Appendix C].

Now suppose $0 < 2\alpha \leq d$ and that $K \subset \mathbb{R}^d$ is compact. Then

$$\mathrm{Cap}_{\alpha,2}(K) > 0 \quad \text{if and only if} \quad \mathrm{Cap}_{d-2\alpha}(K) > 0. \tag{2.13}$$

To see this note that if there exists a Borel probability measure μ on K with $I_{d-2\alpha}(\mu) < +\infty$, then by (2.9) and (2.7) respectively (2.8) we have $E_\alpha(\mu) < +\infty$, and by (2.6) therefore $\mathrm{Cap}_{\alpha,2}(K) > 0$. Conversely, if the $\alpha, 2$-capacity of K is positive, we can find a nonzero Radon measure μ on K with $E_\alpha(\mu) < +\infty$, so that again by (2.9) and (2.7) respectively (2.8) the Borel probability measure $\frac{\mu}{\mu(K)}$ has finite Riesz energy of order $d - 2\alpha$.

Consider the Dirac measure δ_0 with total mass one at the origin, it is the only possible probability measure on the compact set $\{0\}$. If $2\alpha \le d$ then obviously $I_{d-2\alpha}(\delta_0) = +\infty$, so that by (2.13) we have $\mathrm{Cap}_{\alpha,2}(\{0\}) = 0$. On the other hand, for $d < 2\alpha$ identity (2.10) implies that $U^\alpha\delta_0(x) = \gamma_{2\alpha} * \delta_0(x) = \gamma_{2\alpha}(x)$, $x \in \mathbb{R}^d$, so that $E_\alpha(\delta_0) = \gamma_{2\alpha}(0) < +\infty$ and therefore $\mathrm{Cap}_{\alpha,2}(\{0\}) > 0$. Similar arguments are valid with α in place of 2α. This produces fractional versions of Examples 1 and 2.

Example 3 *Consider the case that* $0 < \alpha < 2$ *and* $\Sigma = \{0\}$. *Then* $((-\Delta)^{\alpha/2}, C_0^\infty(N))$ *is always Markov unique for* $d \ge 2$. *For* $d = 1$ *it is Markov unique if* $0 < \alpha \le 1$ *but not if* $1 < \alpha < 2$. *See also [16, Section II.5, p.63]. So a necessary and sufficient condition for Markov uniqueness is* $d \ge \alpha$.

Example 4 *Consider the case that* $0 < \alpha < 2$ *and* $\Sigma = \{0\}$. *Then* $((-\Delta)^{\alpha/2}, C_0^\infty(N))$ *is always essentially self-adjoint for* $d \ge 4$. *For* $d \le 3$ *it is essentially self-adjoint if* $0 < 2\alpha \le d$ *but not if* $d < 2\alpha < 4$. *Therefore a necessary and sufficient condition for essential self-adjointness is* $d \ge 2\alpha$.

As before let $\Sigma \subset \mathbb{R}^d$ be compact and of zero Lebesgue measure and write $N := \mathbb{R}^d \setminus \Sigma$. Using theorems of Frostman-Taylor type, [59, Appendix C, Theorems 2.2.1 and 2.3.1], see also [33, 58, 75, 79], we can give another description of the critical size of Σ, now in terms of its Hausdorff measure and dimension. Given $s \ge 0$, the symbol $\mathcal{H}^s$ denotes the s-dimensional Hausdorff measure on $\mathbb{R}^d$, [33, 58, 75, 79]. By $\dim_H$ we denote the Hausdorff dimension. Again we begin with a result on Markov uniqueness.

Corollary 1 *Let* $0 < \alpha \le 2$ *and suppose* $\alpha \le d$.

(i) If $\mathcal{H}^{d-\alpha}(\Sigma) < +\infty$ *then* $((-\Delta)^{\alpha/2}, C_0^\infty(N))$ *is Markov unique. This is true in particular if* $\alpha < d$ *and* $\dim_H \Sigma < d - \alpha$.

(ii) If $((-\Delta)^{\alpha/2}, C_0^\infty(N))$ *is Markov unique then* $\dim_H \Sigma \le d - \alpha$.

For the essential self-adjointness we have the following result, it partially generalizes [5, Theorem 10.3, Corollary 10.4 and Theorem 10.5]

Corollary 2 *Let* $0 < \alpha \le 2$ *and suppose* $2\alpha \le d$.

(i) If $\mathcal{H}^{d-2\alpha}(\Sigma) < +\infty$ *then* $((-\Delta)^{\alpha/2}, C_0^\infty(N))$ *is essentially self-adjoint. This is true in particular if* $2\alpha < d$ *and* $\dim_H \Sigma < d - 2\alpha$.

(ii) If $((-\Delta)^{\alpha/2}, C_0^\infty(N))$ *is essentially self-adjoint then* $\dim_H \Sigma \le d - 2\alpha$.

We provide some arguments for Corollary 2, it follows from Theorem 2. In a similar manner one can deduce Corollary 1 from Theorem 1. *Proof* If $2\alpha < d$ and $\mathcal{H}^{d-2\alpha}(\Sigma) < +\infty$ in Corollary 2 (i), then by Frostman-Taylor, [59, Appendix C, Theorem 2.3.1], we have $\mathrm{Cap}_{d-2\alpha}(\Sigma) = 0$, and by (2.13) therefore also $\mathrm{Cap}_{\alpha,2}(\Sigma) = 0$. If $2\alpha = d$ and $\mathcal{H}^0(\Sigma) < +\infty$, then Σ must be a finite set of points, note that $\mathcal{H}^0$ is the counting measure. Since capacities are subadditive, we have $\mathrm{Cap}_{d/2,2}(\Sigma) = 0$ once we know a single point has zero $d/2, 2$-capacity. However, the only probability measure a single point $p \in \mathbb{R}^d$ can carry is a Dirac point mass measure δ_p with total mass one, and clearly $I_0(\delta_p) = +\infty$, so that $\mathrm{Cap}_0(\{p\}) = 0$. By (2.13) this implies that $\mathrm{Cap}_{d/2,2}(\{p\}) = 0$, as desired. Conversely, if we have $2\alpha < d$ and $\mathrm{Cap}_{\alpha,2}(\Sigma) = 0$ Corollary 2 (ii), then by (2.13) $\mathrm{Cap}_{d-2\alpha}(\Sigma) = 0$, and Frostman-Taylor implies that for any $\varepsilon > 0$, $\mathcal{H}^{d-2\alpha+\varepsilon}(\Sigma) = 0$, showing $\dim_H \Sigma \le d - 2\alpha$. If $2\alpha = d$ and $\mathrm{Cap}_{d/2,2}(\Sigma) = 0$, then by (2.13) we have $\mathrm{Cap}_0(\Sigma) = 0$. It is not difficult to see that this implies $\mathrm{Cap}_\varepsilon(\Sigma) = 0$ for all $\varepsilon > 0$, and therefore $\dim_H \Sigma = 0$.

$\square$

2.5 Additive processes and a probabilistic characterization

In this section we provide probabilistic characterizations of Markov uniqueness and essential self-adjointness. We use the notation $\mathbb{R}_+ = [0, +\infty)$.

We are aiming only at results on hitting probabilities, so there is ambiguity what sort of stochastic process to use. Potential theory suggests to use Markov processes, and due to the group structure of $\mathbb{R}^d$ a particularly simple choice is to use certain Lévy processes, [16, 89]. Recall that a *Lévy process on* $\mathbb{R}^d$ is a stochastic process $(X_t)_{t\in\mathbb{R}_+}$, modelled on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking values in $\mathbb{R}^d$ that has independent and stationary increments, is stochastically continuous, is $\mathbb{P}$-a.s. right-continuous with left limits ('càdlàg') and such that $\mathbb{P}(X_0 = 0) = 1$. See for instance [89, Chapter I, Section 1, Definition 1.6].

Let $(B_t)_{t\in\mathbb{R}_+}$ denote a *Brownian motion* on $\mathbb{R}^d$ (starting at the origin), modelled on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, that is a Lévy process on $\mathbb{R}^d$ with $\mathbb{P}$-a.s. continuous paths and such that for any $t > 0$ and any Borel set $A \subset \mathbb{R}^d$,

$$\mathbb{P}(B_t \in A) = \int_A p(t,x)dx,$$

where

$$p(t,x) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|x|^2}{2t}\right), \quad t > 0, \ x \in \mathbb{R}^d.$$

Alternatively, in terms of characteristic functions, a Brownian motion is a Lévy process on $\mathbb{R}^d$ satisfying

$$\mathbb{E}\left[\exp\left\{i\langle \xi, B_t\rangle\right\}\right] = \exp\left\{-2^{-1}t|\xi|^2\right\} \quad t \geq 0, \ \xi \in \mathbb{R}^d.$$

More generally, given $0 < \alpha \leq 2$ let $(X_t^{(\alpha)})_{t\in\mathbb{R}_+}$ denote an *isotropic α-stable Lévy process* on $\mathbb{R}^d$, modelled on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, that is a Lévy process on

$\mathbb{R}^d$ satisfying

$$\mathbb{E}\left[\exp\left\{i\left\langle\xi, X_t^{(\alpha)}\right\rangle\right\}\right] = \exp\left\{-2^{-\alpha/2}t|\xi|^\alpha\right\} \quad t \geq 0,\ \xi \in \mathbb{R}^d.$$

Obviously for $\alpha = 2$ the process $(X_t^{(2)})_{t\in\mathbb{R}_+}$ is equal in law to a Brownian motion $(B_t)_{t\in\mathbb{R}_+}$. For $0 < \alpha < 2$ an isotropic α-stable Lévy process can be obtained from a Brownian motion by subordination, see [89, Chapter 6, in particular Example 30.6]. For general existence results for Lévy processes see [16, Section I.1, Theorem 1] or [89, Corollary 11.6].

To prepare the discussion of related two-parameter processes below, we collect some properties. Let $0 < \alpha \leq 2$. By

$$T_t^{(\alpha)} f(x) = \mathbb{E}[f(X_t^{(\alpha)} + x)], \quad t \geq 0,\ x \in \mathbb{R}^d,$$

we can define a strongly continuous contraction semigroup $(T_t^{(\alpha)})_{t>0}$ of Markov operators on $L^2(\mathbb{R}^d)$ (and on the space $C_\infty(\mathbb{R}^d)$ of continuous functions vanishing at infinity), they are symmetric in $L^2(\mathbb{R}^d)$. Its infinitesimal generator (in both spaces) is $-2^{-\alpha/2}(-\Delta)^{\alpha/2}$. The associated 1-resolvent operators $R_1^{(\alpha)} = (I + 2^{-\alpha/2}(-\Delta)^{\alpha/2})^{-1}$ satisfy

$$R_1^{(\alpha)} f = \int_0^\infty e^{-t} T_t^{(\alpha)} f \, dt,$$

they are bounded linear operators on $L^2(\mathbb{R}^d)$ (and on $C_\infty(\mathbb{R}^d)$). The operators $R_1^{(\alpha)}$ admit radially symmetric densities $u^{(\alpha)}$, that is

$$R_1^{(\alpha)} f(x) = \int_{\mathbb{R}^d} f(y) u_1^{(\alpha)}(x-y) dy.$$

For $0 < \alpha < d$ we have

$$c_1|x|^{\alpha-d} \leq u_1^{(\alpha)}(x) \leq c_2|x|^{\alpha-d} \tag{2.14}$$

whenever $|x|$ is sufficiently small, where c_1 and c_2 are two positive constants. See for instance [59, Section 10, Lemma 3.1.1 and 3.4.1].

Versions of Kakutani's theorem, [59, Section 10, Theorems 3.1.1 and 3.4.1], now allow to use Brownian motions (in case $\alpha = 2$) or isotropic α-stable Lévy processes (in case $0 < \alpha < 2$) to characterize Markov uniqueness. As before, $\Sigma \subset \mathbb{R}^d$ is a compact set of zero Lebesgue measure and $N := \mathbb{R}^d \setminus \Sigma$.

Corollary 3 *Let $0 < \alpha \leq 2$ and assume $d \geq \alpha$. The operator $((-\Delta)^{\alpha/2}, C_0^\infty(N))$ is Markov unique if and only if for any $x \notin \Sigma$ we have*

$$\mathbb{P}(\exists\ t \in \mathbb{R}_+ \text{ such that } X_t^{(\alpha)} + x \in \Sigma) = 0.$$

The main aim of the present note is to point out a similar characterization for essential self-adjointness. Because their definition and structure is particularly simple, we will use two-parameter additive stable processes to describe the critical size of Σ. Let $0 < \alpha \leq 2$. Given two independent isotropic α-stable Lévy processes $(X^{(\alpha)})_{t \in \mathbb{R}_+}$ and $(\widetilde{X}^{(\alpha)})_{t \in \mathbb{R}_+}$ on $\mathbb{R}^d$ we consider the process $(\mathcal{X}_{\mathbf{t}}^{(\alpha)})_{\mathbf{t} \in \mathbb{R}_+^2}$ defined by

$$\mathcal{X}_{\mathbf{t}}^{(\alpha)} = X_{t_1}^{(\alpha)} + \widetilde{X}_{t_2}^{(\alpha)}, \quad \mathbf{t} = (t_1, t_2) \in \mathbb{R}_+^2. \tag{2.15}$$

It is called the *two-parameter additive stable process if index α*, see [59, Section 11.4.1]. In the case $\alpha = 2$ it is called the *two-parameter additive Brownian motion*, we also denote it by $(\mathcal{B}_{\mathbf{t}})_{\mathbf{t} \in \mathbb{R}_+}$, where

$$\mathcal{B}_{\mathbf{t}} = B_{t_1} + \widetilde{B}_{t_2}, \quad \mathbf{t} = (t_1, t_2) \in \mathbb{R}_+^2,$$

with two independent Brownian motions $(B_t)_{t \in \mathbb{R}_+}$ and $(\widetilde{B}_t)_{t \in \mathbb{R}_+}$ on $\mathbb{R}^d$. Additive stable processes or, more generally, additive Lévy processes have been studied intensely in [59, 61, 62] and follow up articles.

It seems plausible that, as two processes are added, these two-parameter processes move 'more actively' than their one-parameter versions, so they should be able to hit smaller sets with positive probability. This is indeed the case and can be used for our purpose. The next satement is a simple application of known Kakutani-type theorems for two-parameter processes, [59, Section 11, Theorem 4.1.1].

Corollary 4 *Let $0 < \alpha \leq 2$ and assume $d \geq 2\alpha$. The operator $((-\Delta)^{\alpha/2}, C_0^\infty(N))$ is essentially self-adjoint if and only if for any $x \notin \Sigma$ we have*

$$\mathbb{P}(\exists\ \mathbf{t} \in \mathbb{R}_+^2 \textit{ such that } \mathcal{X}_{\mathbf{t}}^{(\alpha)} + x \in \Sigma) = 0.$$

Applying Corollary 4 with $\alpha = 2$ and $d \geq 4$ we can conclude that a compact set $K \subset \mathbb{R}^d$ is removable for Δ in L^2 if and only if it is not hit by the additive Brownian motion with positive probability.

We collect some notions and facts related to additive stable processes and then briefly comment on the case $d = 2\alpha$ in Corollary 4 which is the only case not covered by [59, Section 11, Proposition 4.1.1 and Theorem 4.1.1].

One can define a two-parameter family $(\mathcal{T}_{\mathbf{t}}^{(\alpha)})_{\mathbf{t}>0}$ of bounded linear operators $\mathcal{T}_{\mathbf{t}}^{(\alpha)}$ on $L^2(\mathbb{R}^d)$ (or $C_\infty(\mathbb{R}^d)$) by

$$\mathcal{T}_{\mathbf{t}}^{(\alpha)} := T_{t_1}^{(\alpha)} T_{t_2}^{(\alpha)}, \quad \mathbf{t} = (t_1, t_2) > 0.$$

Here we write $(t_1, t_2) > (s_1, s_2)$ if $t_1 > s_1$ and $t_2 > s_2$. They satisfy the semigroup property $\mathcal{T}_{\mathbf{t}}^{(\alpha)} \mathcal{T}_{\mathbf{s}}^{(\alpha)} = \mathcal{T}_{\mathbf{s}+\mathbf{t}}^{(\alpha)}$ for all $\mathbf{s}, \mathbf{t} > 0$ and also the strong limit relation

$$\lim_{\mathbf{t}\to 0} \left\| \mathcal{T}_{\mathbf{t}}^{(\alpha)} f - f \right\|_{\sup} = 0, \quad f \in C_\infty(\mathbb{R}^d),$$

and, using the density of $C_\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$, also for the $L^2(\mathbb{R}^d)$-norm and $f \in L^2(\mathbb{R}^d)$. By the independence of the summands in (2.15) it is not difficult to see that

$$\mathcal{T}_{\mathbf{t}}^{(\alpha)} \mathbf{1}_A(x) = \mathbb{P}(\mathcal{X}_{\mathbf{t}}^{(\alpha)} + x \in A)$$

for all Borel sets $A \subset \mathbb{R}^d$ and starting points $x \in \mathbb{R}^d$. See for instance [52] or [59, Sections 11.1 and 11.2]. Mimicking the one-parameter case, on can introduce associated **1**-resolvent operators $\mathcal{R}_{\mathbf{1}}^{(\alpha)}$ by

$$\mathcal{R}_{\mathbf{1}}^{(\alpha)} f(x) = \int_{\mathbb{R}_+^2} e^{-(s_1+s_2)} \mathcal{T}_{\mathbf{s}}^{(\alpha)} f(x) d\mathbf{s}.$$

Here, in accordance with the notation used above, we write $\mathbf{1} = (1,1)$. Obviously

$$\mathcal{R}_{\mathbf{1}}^{(\alpha)} = R_1^{(\alpha)} R_1^{(\alpha)},$$

and consequently the $\mathcal{R}_{\mathbf{1}}^{(\alpha)}$ are bounded and linear operators on $L^2(\mathbb{R}^d)$ (and $C_\infty(\mathbb{R}^d)$) and admit the densities

$$u_{\mathbf{1}}^{(\alpha)} = u_1^{(\alpha)} * u_1^{(\alpha)}, \tag{2.16}$$

that is

$$\mathcal{R}_{\mathbf{1}}^{(\alpha)} f(x) = \int_{\mathbb{R}^d} f(y) u_{\mathbf{1}}^{(\alpha)}(x-y) dy.$$

We provide the arguments for the special case $2\alpha = d$ in Corollary 4. By Giraud's lemma, [6, Chapter 4, Proposition 4.12], together with (2.14) and (2.16), the densities $u_{\mathbf{1}}^{(d/2)}$ are continuous away from the origin and satisfy

$$u_{\mathbf{1}}^{(d/2)}(x) \le c_3 \left(-\log|x|\right)$$

for sufficiently small x, where c_3 is a positive constant. We also have

$$u_{\mathbf{1}}^{(d/2)}(x) \ge c_4 \left(-\log|x|\right)$$

for sufficiently small x with a positive constant c_4: Suppose $x \in \mathbb{R}^d$ and $|x| < 1$. We have

$$u_{\mathbf{1}}^{(d/2)}(x) \ge \int_{\{|x-y|\le|y|\}} |x-y|^{-d/2}|y|^{-d/2} dy \ge \int_{\{|x-y|\le|y|\}} |y|^{-d} dy.$$

Let $P_{(x)}$ denote the hyperplane orthogonal to the straight line connecting x and the origin 0 and containing the point $\frac{1}{2}x$. (For $d = 1$ it just equals the one-point set containing $\frac{1}{2}x$.) Then, if $H_{(x)}$ denotes the closed half space having boundary $P_{(x)}$ and containing x, any $y \in H_{(x)}$ satisfies $|x-y| \leq |y|$. Writing λ^d for the d-dimensional Lebesgue measure and κ_d for the volume of the d-dimensional unit ball, we have, given $|x|/2 \leq r \leq 1$,

$$\lambda^d(B(0,r) \cap H_{\{x\}}) = \kappa_{d-1} \int_{|x|/2}^{r} (r^2 - h^2)^{(d-1)/2} dh = \kappa_{d-1} r^d \int_{|x|/(2r)}^{1} (1-\eta^2)^{(d-1)/2} d\eta$$

for the volume of the spherical cap $B(0,r) \cap H_{\{x\}}$. For $|x| \leq r \leq 1$ this is bounded below by $c(d)\, r^d$ with a constant $c(d) > 0$ depending on d only. Writing $m_{(x)}(r) \coloneqq \lambda^d(B(0,r) \cap H_{\{x\}})$ we therefore have

$$u_{\mathbf{1}}^{(d/2)}(x) \geq \int_{B(0,1)\cap H_{(x)} \setminus B(0,|x|)} |u|^{-d} dy = \int_{|x|}^{1} r^{-d} dm_x(r) \geq c(d) d \int_{|x|}^{1} r^{-1} dr = c_4(-\log|x|),$$

as desired. Now an application of [59, Section 11.3, Theorem 3.1.1] yields Corollary 4 for $2\alpha = d$.

Remark 1

(i) Alternatively, one can use the $\mathbb{R}^d$-valued two-parameter Brownian sheet to characterize the essential self-adjointness of $(\Delta, C_0^\infty(N))$. A real-valued Gaussian process indexed by $\mathbb{R}_+^2$ is called a two-parameter Brownian sheet *if it has mean zero and covariance function $C(\mathbf{s}, \mathbf{t}) = (s_1 \wedge t_1)(s_2 \wedge t_2)$, $\mathbf{s}, \mathbf{t} \in \mathbb{R}_+^2$. An* $\mathbb{R}^d$-valued two-parameter Brownian sheet *is a process $(\mathbb{B}_{\mathbf{t}})_{\mathbf{t}\in\mathbb{R}_+^2}$, where*

$$\mathbb{B}_{\mathbf{t}} = (\mathbb{B}_{\mathbf{t}}^1, ..., \mathbb{B}_{\mathbf{t}}^d),$$

and the components $(\mathbb{B}_{\mathbf{t}}^i)_{\mathbf{t}\in\mathbb{R}_+^2}$, $i = 1, ..., d$, are independent two-parameter Brownian sheets. See for instance [59] or [60]. Using the arguments of [53] one can *conclude that* $(\Delta, C_0^\infty(N))$ *is essentially self-adjoint if and only if* Σ *is polar for the two-parameter Brownian sheet, more precisely, if and only if*

$$\int_{\mathbb{R}^d} \mathbb{P}(\mathbb{B}_{\mathbf{t}} + x \in \Sigma) dt = 0.$$

(ii) As mentioned, yet another stochastic process that can be used to characterize the essential self-adjointness of $(\Delta, C_0^\infty(N))$ *is the super-Brownian motion. It is a one-parameter process but its state space is a space of measures, and its construction is probabilistically more involved. See for instance [26, 82, 83] and in particular, [83, Theorem III.5.2].*

Chapter 3 Capacities, removable sets and L^p-uniqueness on Wiener spaces

3.1 Introduction

The present chapter deals with capacities associated with Ornstein-Uhlenbeck operators on abstract Wiener spaces (B, μ, H), [13, 17, 42, 55, 67, 72, 73, 92, 98], and applications to L^p-uniqueness problems for Ornstein-Uhlenbeck operators and their integer powers, endowed with algebras of functions vanishing in a neighborhood of a small closed set.

Our original motivation comes from L^p-uniqueness problems for operators L endowed with a suitable algebra $\mathcal{A}$ of functions, the special case $p = 2$ is the problem of essential self-adjointness. For the 'globally defined' operator L on the entire space L^p-uniqueness is well understood, see for instance [29] and the references cited there. If the globally defined operator is L^p-unique one can ask whether the removal of a small set (or, in other words, the introduction of a small boundary) destroys this uniqueness or not. A loss of uniqueness means that extensions to generators of C_0-semigroups, [81], with different boundary conditions exist. The answer to this question depends on the size of the removed set. The most classical example may be the essential self-adjointness problem for the Laplacian Δ on $\mathbb{R}^n$, endowed with the algebra $C_c^\infty(\mathbb{R}^n \setminus \{0\})$ of smooth compactly supported functions on $\mathbb{R}^n$ with the origin $\{0\}$ removed. It is well known that this operator is essential self-adjoint in $L^2(\mathbb{R}^n)$ if and only if $n \geq 4$, [99, p.114] and [84, Theorem X.11, p.161]. Generalizations of this example to manifolds have been provided in [20] and [74], more general examples on

Euclidean spaces can be found in [5] and [48], further generalizations to manifolds and metric measure spaces will be discussed in [50]. For the Laplacian on $\mathbb{R}^n$ one main observation is that, if a compact set Σ of zero measure is removed from $\mathbb{R}^n$, the essential self-adjointness of $(\Delta, C_c^\infty(\mathbb{R}^n \setminus \Sigma))$ in $L^2(\mathbb{R}^n)$ implies that $\dim_H \Sigma \leq n-4$, where $\dim_H$ denotes the Hausdorff dimension. See [5, Theorems 10.3 and 10.5] or [48, Theorem 2]. This necessary 'codimension four' condition can be rephrased by saying that we must have $\mathcal{H}^{n-d}(\Sigma) = 0$ for all $d < 4$, where $\mathcal{H}^{n-d}$ denotes the Hausdorff measure of dimension $n-d$.

Having in mind coefficient regularity or boundary value problems for operators in infinite dimensional spaces, see e.g. [15, 21, 22, 46, 47], one may wonder whether a similar 'codimension four' condition can be observed in infinite dimensional situations. For the case of Ornstein-Uhlenbeck operators on abstract Wiener spaces an affirmative answer to this question follows from the present results in the special case $p = 2$.

The basic tools to describe the critical size of a removed set $\Sigma \subset B$ are capacities associated with the Sobolev spaces $W^{r,p}(B,\mu)$ for the H-derivative respectively the Ornstein-Uhlenbeck semigroup, [13, 17, 42, 55, 67, 72, 73, 92, 98]. Such capacities can be introduced following usual concepts of potential theory, [17, 31, 73, 91, 92, 94, 95, 98], see Definition 3.1 below, and they are known to be connected to Gaussian Hausdorff measures, [32]. Uniqueness problems connect easier to another, slightly different definition of capacities, where the functions taken into account in the definition are recruited from the initial algebra $\mathcal{A}$ and, roughly speaking, are required to be equal to one on the set in question, see Definitions 3.2 and 3.3. This type of definition connects them to an algebraic ideal property which is helpful to investigate extensions of operators initially defined on ideals of $\mathcal{A}$. For Euclidean Sobolev spaces these two types of capacities are known to be equivalent, see for instance [3, Section 2.7]. The proofs of these equivalences go back to Mazja, Khavin, Adams, Hedberg, Polking and others, [2, 3, 4, 76, 77, 78], and rely on bounds in Sobolev norms for certain nonlinear composition operators acting on the cone of nonnegative Sobolev functions, see e.g. [2, Theorem 3], or the cone of potentials of nonnegative functions, see e.g. [2, Theorem 2] or [3, Theorem 3.3.3]. Apart from the first order case $r = 1$ this is nontrivial, because in finite dimensions Sobolev spaces are not stable under such compositions, see for instance [3, Theorem 3.3.2]. Apart from the case $p = 2$, where one can also use an integration by parts argument, [2, Theorem 3], the desired bounds are shown using suitable Gagliardo-Nirenberg inequalities, [4, 76], or suitable multiplicative estimates of Riesz or Bessel potential operators involving Hardy-Littelwood maximal functions and the L^p-boundedness of the latter, [3, Theorem 1.1.1, Proposition 3.1.8] The constants in these estimates are dimension dependent.

Sobolev spaces $W^{r,p}(B,\mu)$ over abstract Wiener spaces (B,μ,H) are stable under compositions with bounded smooth functions, [13, Remark 5.2.1 (i)], but one still needs to establish quantitative bounds. We establish Sobolev norm bounds for nonlinear composition operators acting on potentials of nonnegative functions, Lemma 3.1. To obtain it, we use the L^p-boundedness of the maximal function in the sense of Rota and Stein for the Ornstein-Uhlenbeck semigroup, [92, Theorem 3.3], this provides a similar multiplicative estimate as in the finite dimensional case, see Lemma 3.3. From the Sobolev norm estimate for compositions we can then deduce the desired equivalence of capacities, Theorem 3.1, where $\mathcal{A}$ is chosen to be the set of smooth cylindrical functions or the space of Watanabe test functions. Applications of this equivalence provide L^p-uniqueness results for the Ornstein-Uhlenbeck operator and, under a sufficient condition that ensures they generate C_0-semigroups, also for its integer powers, see Theorem 3.2. In particular, if $\Sigma \subset B$ is a given closed set of zero Gaussian measure, then the Ornstein-Uhlenbeck operator, endowed with the algebra of cylindrical functions vanishing in a neighborhood of Σ (or the algebra of Watanabe test functions vanishing q.s. on a neighborhood of Σ) is L^p-unique if and only if the $(2,p)$-capacity of Σ is zero, see Theorem 3.2. Combined with results from [32] on Gaussian Hausdorff measures, we then observe that the L^p-uniqueness of this Ornstein-Uhlenbeck operator 'after the removal of Σ' implies that the Gaussian Hausdorff measure $\varrho_d(\Sigma)$ of codimension d of Σ must be zero for all $d < 2p$, see Corollary 3.1. In particular, if the operator is essentially self-adjoint on $L^2(B,\mu)$, then $\varrho_d(\Sigma)$ must be zero for all $d < 4$, what is an analog of the necessary 'codimension four' condition knwon from the Euclidean case.

In the next section we recall standard items from the analysis on abstract Wiener spaces. In Section 3.3 we define Sobolev capacities and prove their equivalence, based on the norm bound on nonlinear compositions, which is proved in Section 3.4. Section 3.5 contains the mentioned L^p-uniqueness results. The connection to Gaussian Hausdorff measures is briefly discussed in Section 3.6, followed by some remarks on related Kakutani theorems for multiparameter processes in Section 3.7.

3.2 Preliminaries

Following the presentation in [92], we provide some basic definitions and facts.

Let (B,μ,H) be an *abstract Wiener space*. That is, B is a real separable Banach space, H is a real separable Hilbert space which is embedded densely and continuously on B, and μ is a Gaussian measure on B with

$$\int_B \exp\{\sqrt{-1}\langle\varphi,y\rangle\}\mu(dy) = \exp\{-\frac{1}{2}|\varphi|^2_{H^*}\}, \quad \varphi \in B^*,$$

see for instance [92, Definition 1.2]. Here we identify H^* with H as usual, so that

$B^* \subset H \subset B$. Since every $\varphi \in B^*$ is $N(0, \|\varphi\|_H^2)$-distributed, it is an element of $L^2(B;\mu)$ and the map $\varphi \mapsto \langle \varphi, \cdot \rangle$ is an isometry from B^*, equipped with the scalar product $\langle \cdot, \cdot \rangle_H$, into $L^2(B,\mu)$. It extends uniquely to an isometry

$$h \mapsto \hat{h} \tag{3.1}$$

from H into $L^2(B,\mu)$. A function $f : B \to \mathbb{R}$ is said to be *H-differentiable at $x \in B$* if there exists some $h^* \in H^*$ such that

$$\frac{d}{dt} f(x + th)|_{t=0} = \langle h, h^* \rangle$$

for all $h \in H$. If f is H-differentiable at x then h^* is uniquely determined, denoted by $Df(x)$ and refereed to as the *H-derivative of f at x*. See [92, Definition 2.6]. For a function f that is H-differentiable at $x \in B$ and an element h of H we can define the *directional derivative $\partial_h f(x)$ of f at x* by

$$\partial_h f(x) := \langle Df(x), h \rangle_H .$$

A function $f : B \to \mathbb{R}$ is said to be *k-times H-differentiable at $x \in B$* if there exists a continuous k-linear mapping $\Phi_x : H^k \to \mathbb{R}$ such that

$$\frac{\partial^k}{\partial t_1 \cdots \partial t_k} f(x + t_1 h_1 + \cdots + t_k h_k)|_{t_1 = \cdots = t_k = 0} = \Phi_x(h_1, \ldots h_k)$$

for all $h_1, \ldots, h_k \in H$. If so, Φ_x is unique and denoted by $D^k f(x)$. A function $f : B \to \mathbb{R}$ is called a *(smooth) cylindrical function* if there exist an integer $n \geq 1$, linear functionals $l_1, ..., l_n \in B^*$ and a function $F \in C_b^\infty(\mathbb{R}^n)$ such that

$$f = F(l_1, ..., l_n). \tag{3.2}$$

The space of all such cylindrical functions on B we denote by $\mathcal{F}C_b^\infty$. Clearly $\mathcal{F}C_b^\infty$ is an algebra under pointwise multiplication and stable under the composition with functions $T \in C_b^\infty(\mathbb{R})$.

A cylindrical function $f \in \mathcal{F}C_b^\infty$ as in (3.2) is infinitely many times H-differentiable at any $x \in B$, and for any $k \geq 1$ we have

$$D^k f(x) = \sum_{j_1, \dots j_k = 1}^{\infty} \partial_{j_1} \cdots \partial_{j_k} F(\langle x, l_1 \rangle, \dots, \langle x, l_n \rangle)\, l_{j_1} \otimes \cdots \otimes l_{j_k},$$

where ∂_j denotes the j-th partial differentiation in the Euclidean sense. The space $\mathcal{F}C_b^\infty$ is dense in $L^p(B, \mu)$ for any $1 \leq p < +\infty$, see e.g. [12, Lemma 2.1].

We write $\mathcal{H}_0 := \mathbb{R}$, $\mathcal{H}_1 := H$ and generalizing this, denote by $\mathcal{H}_k$ the space of k-linear maps $A : H^k \to \mathbb{R}$ such that

$$\|A\|_{\mathcal{H}_k}^2 := \sum_{j_1, \dots, j_k = 1}^{\infty} (A(e_{j_1}, \dots, e_{j_k}))^2 < +\infty, \tag{3.3}$$

where $(e_i)_{i=1}^\infty$ is an orthonormal basis in H. The value of this norm does not depend on the choice of this basis. See [14, p.3]. Clearly every such k-linear map A can also be seen as a linear map $A : H^{\otimes k} \to \mathbb{R}$, where $H^{\otimes k}$ denotes the k-fold tensor product of H, with this interpretation we have $A(e_{j_1} \otimes \dots \otimes e_{j_k}) = A(e_{j_1}, \dots, e_{j_k})$ and by (3.3) the operator A is a Hilbert-Schmidt operator. For later use we record the following fact.

PROPOSITION **3.1** *For any $A \in \mathcal{H}_k$ we have*

$$\|A\|_{\mathcal{H}_k} \leq 2k^k \sup \{ |A(h_1, \dots, h_k)| : \ h_1, \cdots, h_k \text{ are members of an orthonormal system in } H\text{, not necessarily distinct} \}.$$

Proof By Parseval's identity and Cauchy-Schwarz in $H^{\otimes k}$ we have

$$\|A\|_{\mathcal{H}_k} = \sup \left\{ |Ay| : \ y \in H^{\otimes k} \text{ and } \|y\|_{H^{\otimes k}} = 1 \right\}.$$

Choose an element $y = y_1 \otimes \dots \otimes y_k \in H^{\otimes k}$ such that $\|y\|_{H^{\otimes k}} = 1$ and $\|A\|_{\mathcal{H}^k} \leq 2|Ay|$. Without loss of generality we may assume that $\|y_j\|_H = 1$, $1 \leq j \leq k$. Choosing an orthonormal basis $(b_i)_{i=1}^n$ in the subspace span $\{y_1, ..., y_k\}$ of H we observe $n \le k$ and $y_j = \sum_{i=1}^n b_i \lambda_{ij}$ with some $|\lambda_{ij}| \le 1$. Since this implies

$$|Ay| \le \sum_{i_1,\cdots,i_k \in \{1,\cdots,n\}} |A(b_{i_1}, \cdots, b_{i_k})|,$$

we obtain the desired result.

□

We recall the definition of Sobolev spaces on B. For any $1 \le p < +\infty$ and $k \ge 0$ let $L^p(B, \mu, \mathcal{H}_k)$ denote the L^p-space of functions from B into $\mathcal{H}_k$. For any $1 \le p < +\infty$ and integer $r \ge 0$ set

$$\|f\|_{W^{r,p}(B,\mu)} := \sum_{k=0}^{r} \left\|D^k f\right\|_{L^p(B,\mu,\mathcal{H}_k)}, \tag{3.4}$$

$f \in \mathcal{F}C_b^\infty$. The *Sobolev class* $W^{r,p}(B, \mu)$ is defined as the completion of $\mathcal{F}C_b^\infty$ in this norm, see [13, Section 5.2] or [14, Section 8.1]. In particular, $W^{0,p}(B, \mu) = L^p(B, \mu)$. For $f \in W^{r,p}(B, \mu)$ the derivatives $D^k f$, $k \le r$, are well defined as elements of $L^p(B, \mu)$, see [13, Section 5.2]. By definition the spaces $W^{r,p}(B, \mu)$ are Banach spaces, Hilbert if $p = 2$. The space W^∞ of *Watanabe test functions* is defined as

$$W^\infty := \bigcap_{r \ge 1,\, 1 \le p < +\infty} W^{r,p}(B, \mu).$$

We have $\mathcal{F}C_b^\infty \subset W^\infty$, in particular, W^∞ is a dense subset of every $L^p(B, \mu)$ and $W^{r,p}(B, \mu)$.

In contrast to Sobolev spaces over finite dimensional spaces, [3, Theorem 3.3.2], also the Sobolev classes $W^{r,p}(B, \mu)$, $r \ge 2$, are known to be stable under compositions $u \mapsto T(u) = T \circ u$ with functions $T \in C_b^\infty(\mathbb{R})$, as follows from the evaluation of an integration by parts identity together with the chain rule, applied to cylindrical functions. See [13, Remark 5.2.1 (i)] or [14, Proposition 8.7.5]. In particular, the space

W^∞ is stable under compositions with functions from $C_b^\infty(\mathbb{R})$. Also, it is an algebra with respect to pointwise multiplication, [73, Corollary 5.8].

Given a bounded (or nonnegative) Borel function $f : B \to \mathbb{R}$ and $t > 0$ set

$$P_t f(x) := \int_B f(e^{-t}x + \sqrt{1 - e^{-2t}}y)\mu(dy), \quad x \in B. \tag{3.5}$$

The function $P_t f$ is again bounded (resp. nonnegative) Borel on B and the operators P_t form a semigroup, i.e. that for any $s, t > 0$ we have $P_{t+s} = P_t P_s$. The semigroup $(P_t)_{t>0}$ is called the *Ornstein-Uhlenbeck semigroup on B*. For any $1 \leq p \leq +\infty$ it extends to a contraction semigroup $(P_t^{(p)})_{t>0}$ on $L^p(B, \mu)$, [92, Proposition 2.4], strongly continuous for $1 \leq p < +\infty$. The semigroup $(P_t^{(2)})_{t>0}$ is a sub-Markovian symmetric semigroup on $L^2(B, \mu)$ in the sense of [17, Definition I.2.4.1]. The infinitesimal generators $(\mathcal{L}^{(p)}, \mathcal{D}(\mathcal{L}^{(p)}))$ of $(P_t^{(p)})_{t>0}$ is called the *Ornstein-Uhlenbeck operator on* $L^p(B, \mu)$, [92, Section 2.1.4]. We will always write P_t and $\mathcal{L}$ instead of $P_t^{(p)}$ and $\mathcal{L}^{(p)}$, the meaning will be clear from the context. Given $r > 0$ and a bounded (or nonnegative) Borel function $f : B \to \mathbb{R}$, set

$$V_r f := \frac{1}{\Gamma(r/2)} \int_0^\infty t^{r/2-1} e^{-t} P_t f dt, \tag{3.6}$$

where Γ denotes the Euler Gamma function. The function $V_r f$ is again bounded (resp. nonnegative) Borel, and for any $1 \leq p < \infty$ the operators V_r form a strongly continuous contraction semigroup $(V_r)_{r>0}$ on $L^p(B, \mu)$, see [13, Corollary 5.3.3] or [92, Proposition 4.7], symmetric for $p = 2$. In any of these spaces the operators V_r are the powers $(I - \mathcal{L})^{-r/2}$ of order $r/2$ of the respective 1-resolvent operators $(I - \mathcal{L})^{-1}$. Meyer's equivalence, [14, Theorem 8.5.2], [92, Theorem 4.4], states that for any integer $r \geq 1$ and any $1 < p < +\infty$ and any $u \in W^{r,p}(B, \mu)$ we have

$$c_1 \|u\|_{W^{r,p}(B,\mu)} \leq \left\|(I - \mathcal{L})^{r/2} u\right\|_{L^p(B,\mu)} \leq c_2 \|u\|_{W^{r,p}(B,\mu)} \tag{3.7}$$

with constants $c_1 > 0$ and $c_2 > 0$ depending only on r and p. By the continuity of the V_r and the density of cylindrical functions we observe $W^{r,p}(B,\mu) = V_r(L^p(B,\mu))$. The operator V_r acts as an isometry from $W^{s,p}(B,\mu)$ onto $W^{s+r,p}(B,\mu)$, [17, Chapter II, Theorem 7.3.1]. For later use we record the following well known fact.

PROPOSITION **3.2** *For any $r > 0$ we have $V_r(\mathcal{F}C_b^\infty) \subset \mathcal{F}C_b^\infty$ and $V_r(W^\infty) \subset W^\infty$.*

Proof From the preceding lines it is immediate that $V_r(W^\infty) \subset W^\infty$. To see the remaining statement suppose $f \in \mathcal{F}C_b^\infty$ with $f = F(l_1, ..., l_n)$, $l_i \in B^*$, $F \in C_b^\infty(\mathbb{R}^n)$, and by applying Gram-Schmidt we may assume $\{l_1, ..., l_n\}$ is an orthonormal system in H. The Ornstein-Uhlenbeck semigroup $(T_t^{(n)})_{t>0}$ on $L^2(\mathbb{R}^n)$, defined by

$$T_t^{(n)} F(\xi) = \int_{\mathbb{R}^n} F(e^{-t}\xi + \sqrt{1-e^{-2t}}\eta)(2\pi)^{-n/2}\, e^{-|\eta|^2/2} d\eta,$$

preserves smoothness, i.e. $T_t^{(n)} F \in C_b^\infty(\mathbb{R}^n)$ for any $F \in C_b^\infty(\mathbb{R}^n)$.

Given $x \in B$ and writing $\xi = (\langle x, l_1\rangle_H, ..., \langle x, l_n\rangle_H)$, we have

$$\begin{aligned}
&P_t f(x) \\
&= \int_B F(\langle e^{-t}x + \sqrt{1-e^{-2t}}y, l_1\rangle_H, ..., \langle e^{-t}x + \sqrt{1-e^{-2t}}y, l_n\rangle_H)\mu(dy) \\
&= \int_B F(e^{-t}\xi + \sqrt{1-e^{-2t}}(\langle y, l_1\rangle_H, ..., \langle y, l_n\rangle_H))\mu(dy) \\
&= \int_{\mathbb{R}^n} F(e^{-t}\xi + \sqrt{1-e^{-2t}}\eta)(2\pi)^{-n/2}\, e^{-|\eta|^2/2} d\eta \\
&= F_t^{(n)}(\langle x, l_1\rangle_H, ..., \langle x, l_n\rangle_H),
\end{aligned}$$

where $F_t^{(n)} = T_t^{(n)} F$. Consequently $P_t f \in \mathcal{F}C_b^\infty$, and using (3.6) and dominated convergence it follows that $V_r f \in \mathcal{F}C_b^\infty$.

□

Although different in nature both $\mathcal{F}C_b^\infty$ and W^∞ can serve as natural replacements in infinite dimensions for algebras of smooth differentiable functions in Euclidean spaces or on manifolds.

3.3 Capacities and their equivalence

We define two types of capacities related to $W^{r,p}(B,\mu)$-spaces and verify their equivalence.

The following definition is standard, see for instance [31, 91].

DEFINITION **3.1** *Let $1 \leq p < +\infty$ and let $r > 0$ be an integer. For open $U \subset B$, let*

$$\mathrm{Cap}_{r,p}(U) := \inf\{\|f\|_{L^p}^p \mid f \in L^p(B,\mu),\ V_r f \geq 1\ \mu\text{-}a.e.\ on\ U\}$$

and for arbitrary $A \subset B$,

$$\mathrm{Cap}_{r,p}(A) := \inf\{\mathrm{Cap}_{r,p}(U) |\ A \subset U,\ U\ open\}.$$

We give two further definitions of (r,p)-capacities. The first one is based on cylindrical functions and resembles [3, Definition 2.7.1] and [77, Chapter 13].

DEFINITION **3.2** *Let $1 \leq p < +\infty$ and let $r > 0$ be an integer. For an open set $U \subset B$ define*

$$\mathrm{cap}_{r,p}^{(\mathcal{F}C_b^\infty)}(U) := \inf\left\{\|u\|_{W^{r,p}(B,\mu)}^p \mid u \in \mathcal{F}C_b^\infty,\ u = 1\ on\ U\right\},$$

and for an arbitrary set $A \subset B$,

$$\mathrm{cap}_{r,p}^{(\mathcal{F}C_b^\infty)}(A) := \inf\{\mathrm{cap}_{r,p}^{(\mathcal{F}C_b^\infty)}(U) |\ A \subset U,\ U\ open\}.$$

The capacities $\mathrm{cap}_{r,p}^{(\mathcal{F}C_b^\infty)}$ have useful 'algebraic' properties which we will use in Section 3.5.

One can give a similar definition based on the space W^∞. To do so, we recall some potential theoretic notions. If a property holds outside a set $E \subset B$ with $\mathrm{Cap}_{r,p}(E) = 0$ then we say it holds (r,p)*-quasi everywhere (q.e.)*. We follow [73, Chapter IV, Section 1.2] and call a set $E \subset B$ *slim* if $\mathrm{Cap}_{r,p}(E) = 0$ for all $1 < p < +\infty$ and all integer $r > 0$, and if a property holds outside a slim set, we say it holds *quasi surely (q.s.)*. A function $u : B \to \mathbb{R}$ is said to be (r,p)-*quasi continuous* if for any $\varepsilon > 0$ we can find an open set $U_\varepsilon \subset B$ such that $\mathrm{Cap}_{r,p}(U) < \varepsilon$ and the restriction $u|_{U_\varepsilon^c}$ of u to U_ε^c is continuous. Every function $u \in W^{r,p}(B,\mu)$ admits a (r,p)-quasi-continuous version $\widetilde{u}$, unique in the sense that two different quasi continuous versions can differ only on a set of zero (r,p)-capacity. Since continuous functions are dense in $W^{r,p}(B,\mu)$ this follows by standard arguments, see for instance [17, Chapter I, Section 8.2]. Now one can follow [73, Chapter IV, Section 2.4] to see that for any $u \in W^\infty$ there exists a function $\widetilde{u} : B \to \mathbb{R}$ such that $u = \widetilde{u}$ μ-a.e. and for all r and p the function $\widetilde{u}$ is (r,p)-quasi continuous. It is referred to as the *quasi-sure redefinition* of u and it is unique in the sense that the difference of two quasi-sure redefinitions of u is zero (r,p)-quasi everywhere for all r and p, [73].

DEFINITION **3.3** *Let $1 \le p < +\infty$ and let $r > 0$ be and integer. For an open set $U \subset B$ define*

$$\mathrm{cap}_{r,p}^{(W^\infty)}(U) := \inf\left\{ \|u\|^p_{W^{r,p}(B,\mu)} \mid u \in W^\infty,\ \widetilde{u} = 1 \ \ on\ U\ \ q.s. \right\},$$

where $\widetilde{u}$ denotes the quasi-sure redefinition of u with respect to the capacities from Definition 3.1, and for an arbitrary set $A \subset B$,

$$\mathrm{cap}_{r,p}^{(W^\infty)}(A) := \inf\{\mathrm{cap}_{r,p}^{(W^\infty)}(U) \mid A \subset U,\, U\ \ open\}.$$

This definition may seem a bit odd because it refers to Definition 3.1. However, for some applications capacities based on the algebra W^∞ may be more suitable that those based on cylcindrical functions.

The following equivalence can be observed.

THEOREM **3.1** *Let $1 < p < +\infty$ and let $r > 0$ be an integer. Then there are positive constants c_3 and c_4 depending only on p and r such that for any set $A \subset B$ we have*

$$c_3 \ \mathrm{cap}_{r,p}^{(\mathcal{F}C_b^\infty)}(A) \le \mathrm{Cap}_{r,p}(A) \le c_4 \ \mathrm{cap}_{r,p}^{(\mathcal{F}C_b^\infty)}(A) \tag{3.8}$$

and

$$c_3 \ \mathrm{cap}_{r,p}^{(W^\infty)}(A) \le \mathrm{Cap}_{r,p}(A) \le c_4 \ \mathrm{cap}_{r,p}^{(W^\infty)}(A). \tag{3.9}$$

Theorem 3.1 is an analogue of corresponding results in finite dimensions, [4, Theorem A], [76, Theorem 3.3], see also [3, Section 2.7 and Corollary 3.3.4] or [77, Sections 13.3 and 13.4].

One ingredient of our proof of Theorem 3.1 is a bound in $W^{r,p}(B,\mu)$-norm for compositions with suitable smooth truncation functions. For the spaces $W^{1,p}(B,\mu)$ such a bound is clear from the chain rule for D respectively from general Dirichlet form theory, see [17]. Norm estimates in $W^{r,p}(B,\mu)$ for compositions $T\circ u$ of elements $u \in W^{r,p}(B,\mu)$ with suitable smooth functions $T : \mathbb{R} \to \mathbb{R}$ can be obtained via the chain rule. For instance, in the special case $r = 2$ the chain and product rules and the definition of the generator $\mathcal{L}$ imply

$$\mathcal{L}T(u) = T'(u)\mathcal{L}u + T''(u)\left\langle Du, Du\right\rangle_H , \quad u \in W^{2,p}.$$

By (3.7) it would now suffice to show a suitable bound for $\mathcal{L}T(u)$ in L^p, and the summand more difficult to handle is the one involving the first derivatives Du. In the finite dimensional Euclidean case an L^p-estimate for it follows immediately from a simple integration by parts argument, [2, Theorem 3], or by a use of a suitable Gagliardo-Nirenberg inequality, [4, 76]. Integration by parts for Gaussian measures comes with an additional 'boundary' term involving the direction $h \in H$ of differentiation that spoiles the original trick, and the classical proof of the Gagliardo-Nirenberg inequality involves dimension dependent constants. A simple alternative approach, suitable for any integer $r > 0$, is to prove truncation results for potentials in a similar way as in [3, Theorem 3.3.3], so that a quick evaluation of the first order term above follows from estimates in terms of the maximal function, [3, Proposition 3.1.8]. This method can be made dimension independent if the Hardy-Littlewood maximal function is replaced by the maximal function in terms of the semigroup operators (3.5) in the sense of Rota and Stein, [92, Theorem 3.3], [97, Chapter III, Section 3], see Lemma 3.3 below. We obtain the following variant of a Theorem due to Mazja and Adams, [2, Theorems 2 and 3], [3, Theorem 3.3.3], now for Sobolev spaces $W^{r,p}(B,\mu)$ over abstract Wiener spaces. A proof will be given in Section 3.4 below.

LEMMA **3.1** *Assume $1 < p < +\infty$ and let $r > 0$ be an integer. Let $T \in C^\infty(\mathbb{R}^+)$ and suppose that T satisfies*

$$\sup_{t>0} |t^{i-1} T^{(i)}(t)| \leq L < \infty, \quad i = 0, 1, 2, ... \tag{3.10}$$

Then for every nonnegative $f \in L^p(B,\mu)$ the function $T \circ V_r f$ is an element of $W^{r,p}(B,\mu)$, and there is a constant $c_T > 0$ depending only on p, r and L such that for every nonnegative $f \in L^p(B,\mu)$ we have

$$\|T \circ V_r f\|_{W^{r,p}(B,\mu)} \leq c_T \|f\|_{L^p}. \tag{3.11}$$

Another useful tool in our proof of Theorem 3.1 is the following 'intermediate' description of $\mathrm{Cap}_{r,p}$. By $\mathcal{F}C_{b,+}^\infty$ we denote the cone of nonnegative elements of $\mathcal{F}C_b^\infty$.

LEMMA **3.2** *Let $1 \leq p < +\infty$ and let $r > 0$ be an integer. For any open set $U \subset B$ we have*

$$\mathrm{Cap}_{r,p}(U) = \inf \left\{ \|f\|_{L^p}^p \mid f \in \mathcal{F}C_{b,+}^\infty,\ V_r f \geq 1 \text{ on } U \right\}. \tag{3.12}$$

Due to Proposition 3.2 the right hand side in (3.12) makes sense. The lemma can be proved using standard techniques, we partially follow [72, III. Proposition 3.5].

Proof For $U \subset B$ open let the right hand side of (3.12) be denoted by $\mathrm{Cap}'_{r,p}(U)$. Then clearly

$$\mathrm{Cap}'_{r,p}(\{|V_r f| > R\}) \leq R^{-p} \, \|f\|^p_{L^p} \tag{3.13}$$

for all $f \in \mathcal{F}C_b^\infty$ and $R > 0$.

Now let $U \subset B$ open be fixed. The value of $\mathrm{Cap}_{r,p}(U)$ does not change if in its definition we require that $V_r f \geq 1 + \delta$ μ-a.e. on U with an arbitrarily small number $\delta > 0$. It does also not change if in addition we consider only nonnegative $f \in L^p$ in the definition: For any $f \in L^p$ the positivity and linearity of V_r imply that $(V_r f)^+ \leq V_r(f^+)$. Consequently, if $f \in L^p$ is such that $V_r f \geq 1 + \delta$ μ-a.e. on U, then also $V_r(f^+) \geq 1 + \delta$ μ-a.e. on U, and clearly $\|f^+\|_{L^p} \leq \|f\|_{L^p}$.

Given $\varepsilon > 0$ choose a nonnegative function $f \in L^p(B, \mu)$ such that $u := V_r f \geq 1 + \delta$ μ-a.e. on U with some $\delta > 0$ and

$$\|f\|^p_{L^p} \leq \mathrm{Cap}_{r,p}(U) + \frac{\varepsilon}{3}.$$

Approximating f by bounded nonnegative functions in $L^p(B, \mu)$, taking their cylindrical approximations, which are nonnegative as well, and smoothing by convolution in finite dimensional spaces, we can approximate f in $L^p(B, \mu)$ by a sequence of nonnegative functions $(f_n)_{n=1}^\infty \subset \mathcal{F}C_{b,+}^\infty$, see for instance [73, Chapter II, Theorem 5.1] or [67, Theorem 7.4.5]. Clearly the functions $u_n := V_r f_n$ satisfy $\lim_n u_n = u$ in $W^{r,p}(B, \mu)$.

By (3.13) and the convergence in $W^{r,p}(B, \mu)$ we can now choose a subsequence $(u_{n_i})_{i=1}^\infty$ such that

$$\mathrm{Cap}'_{r,p}(\{|u_{n_{i+1}} - u_{n_i}| > 2^{-i}\}) \leq 2^{-i} \quad \text{and} \quad \|u_{n_{i+1}} - u_{n_i}\|_{L^p} \leq 2^{-2i}$$

for all $i = 1, 2, \ldots$ For any $k = 1, 2, \ldots$ let now

$$A_k := \bigcup_{i \geq k} \{|u_{n_{i+1}} - u_{n_i}| > 2^{-i}\}, \quad k = 1, 2, \ldots$$

Then for each k the sequence $(u_{n_i})_{i=1}^{\infty}$ is Cauchy in supremum norm on A_k^c. On the other hand,

$$\mu(\{|u_{n_{i+1}} - u_{n_i}| > 2^{-i}\}) \leq 2^{-ip},$$

so that $\mu(A_k) \leq \sum_{i=k}^{\infty} 2^{-ip}$, what implies

$$\mu\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_{k \to \infty} \mu(A_k) = 0.$$

Consequently, setting $\overline{u}(x) := \lim_{n \to \infty} u_n(x)$ for all $x \in \bigcup_{k=1}^{\infty} A_k^c$ and $\overline{u}(x) = 0$ for all other x, we obtain a μ-version $\overline{u}$ of u.

Now choose l such that $\mathrm{Cap}'_{r,p}(A_l) < \frac{\varepsilon}{3}$ and then j large enough so that

$$\left\|f_{n_j} - f\right\|_{L^p}^p < \frac{\varepsilon}{3} \quad \text{and} \quad \sup_{x \in A_l^c} |u_{n_j}(x) - \overline{u}(x)| < \delta/2.$$

Then $u_{n_j} \geq 1$ μ-a.e. on some neighborhood V of $U \cap A_l^c$. The topological support of μ is B, see for instance [13, Theorem 3.6.1, Definition 3.6.2 and the remark following it]. Since u_{n_j} is continuous by Proposition 3.2 we therefore have $u_{n_j} \geq 1$ everywhere on V. Now, since $\mathrm{Cap}'_{r,p}$ is clearly subadditive and monotone,

$$\begin{aligned}
\mathrm{Cap}_{r,p}(U) \leq \mathrm{Cap}'_{r,p}(U) \leq \mathrm{Cap}'_{r,p}(V) + \mathrm{Cap}'_{r,p}(A_l) &\leq \left\|f_{n_j}\right\|_{L^p}^p + \frac{\varepsilon}{3} \\
&\leq \|f\|_{L^p}^p + \frac{2\varepsilon}{3} \\
&\leq \mathrm{Cap}_{r,p}(U) + \varepsilon.
\end{aligned}$$

□

Using Lemmas 3.1 and 3.2 we can now verify Theorem 3.1.

Proof We show (3.8). It suffices to consider open sets U. Since $\mathcal{F}C_b^\infty \subset W^{r,p}(B,\mu)$, we have

$$\mathrm{Cap}_{r,p}(U) \leq c_2^p \, \mathrm{cap}_{r,p}^{(\mathcal{F}C_b^\infty)}(U)$$

with c_2 as in (3.7), so that it suffices to show

$$\mathrm{cap}_{r,p}^{(\mathcal{F}C_b^\infty)}(U) \leq c \, \mathrm{Cap}_{r,p}(U)$$

with a suitable constant $c > 0$ depending only on r and p.

Let $T \in C^\infty(\mathbb{R})$ be a function such that $0 \leq T \leq 1$, $T(t) = 0$ for $0 \leq t \leq 1/2$ and $T(t) = 1$ for $t \geq 1$, and let c_T be as in Lemma 3.1. Given $\epsilon > 0$, let $f \in \mathcal{F}C_{b,+}^\infty$ be such that $u := V_r f \geq 1$ on U and

$$\|f\|_{L^p}^p \leq \mathrm{Cap}_{r,p}(U) + \frac{\varepsilon}{c_T^p},$$

due to Lemma 3.2 such f can be found. Clearly $T \circ u \in \mathcal{F}C_b^\infty$ and $T \circ u = 1$ on U. Therefore, using Lemma 3.1, we have

$$\mathrm{cap}_{r,p}^{(\mathcal{F}C_b^\infty)}(U) \leq \|T \circ u\|_{W^{r,p}(B,\mu)}^p \leq c_T^p \, \|f\|_{L^p}^p \leq c_T^p \, \mathrm{Cap}_{r,p}(U) + \varepsilon,$$

and we arrive at (3.8) with $c_3 := 1/c_T^p$ and $c_4 := c_2^p$. Since $\mathcal{F}C_b^\infty \subset W^\infty \subset W^{r,p}(B,\mu)$, (3.9) is an easy consequence.

□

3.4 Smooth truncations

To verify Lemma 3.1 we begin with the following generalization of [13, formula (5.4.4) in Proposition 5.4.8].

PROPOSITION **3.3** *Assume $p > 1$ and $f \in L^p(B,\mu)$. Then for any $t > 0$ and μ-a.e. $x \in B$ the mapping $h \mapsto P_t f(x+h)$ from H to B is infinitely Fréchet differentiable, and given $h_1, ..., h_k \in H$ we have*

$$\partial_{h_1} \cdots \partial_{h_k} P_t f(x) = \left(\frac{e^{-t}}{\sqrt{1-e^{-2t}}} \right)^k \int_B f(e^{-t}x + \sqrt{1-e^{-2t}}y) Q(\hat{h}_1(y), ..., \hat{h}_k(y)) \mu(dy),$$

where the functions $\hat{h}_i$ are as in (3.1) and $Q : \mathbb{R}^n \to \mathbb{R}$, $n \le k$, is a polynomial of degree k whose coefficients are constants or products of scalar products $\langle h_i, h_j \rangle_H$. If the $h_1, ..., h_k$ are elements of an orthonormal system $(g_i)_{i=1}^k$ in H, not necessarily distinct, then each coefficient of Q depends only on the multiplicity according to which the respective element of $(g_i)_{i=1}^k$ occurs in $\{h_1, ..., h_k\}$.

Proof The infinite differentiability was shown in [13, Proposition 5.4.8] as a consequence of the Cameron-Martin formula. By the same arguments we can see that

$$\partial_{h_1} \cdots \partial_{h_k} P_t f(x) = \int_B f(e^{-t}x + \sqrt{1-e^{-2t}}y) \times \\ \times \frac{\partial^k}{\partial \lambda_1 \cdots \partial \lambda_k} \varrho(t, \lambda_1 h_1 + ... + \lambda_k h_k, y)|_{\lambda_1 = ... = \lambda_k = 0}\, \mu(dy),$$

where

$$\varrho(t,h,y) = \exp\left\{ \frac{e^{-t}}{\sqrt{1-e^{-2t}}} \hat{h}(y) - \frac{e^{-2t}}{2(1-e^{-2t})} \|h\|_H^2 \right\}.$$

A straightforward calculation shows that

$$\frac{\partial^k}{\partial \lambda_1 \cdots \partial \lambda_k} \varrho(t, \lambda_1 h_1 + ... + \lambda_k h_k, y)|_{\lambda_1 = ... = \lambda_k = 0} = \left(\frac{e^{-t}}{\sqrt{1-e^{-2t}}} \right)^k Q(\hat{h}_1(y), ..., \hat{h}_k(y))$$

with a polynomial Q as stated.

□

The next inequality is a counterpart to [3, Proposition 3.1.8]. It provides a pointwise multiplicative estimate for derivatives of potentials in terms of powers of the potential and a suitable maximal function.

LEMMA **3.3** *Let* $1 < q < +\infty$*, let* $r > 0$ *be an integer and let* $k < r$*. Then for any nonnegative Borel function* f *on* B *and all* $x \in B$ *we have*

$$\|D^k V_r f(x)\|_{\mathcal{H}_k} \leq c(k,q,r)\ (V_r f(x))^{1-\frac{k}{r}} \left(\sup_{t>0} P_t(f^q)(x) \right)^{\frac{k}{rq}}. \tag{3.14}$$

Note that lemma 3.3 is interesting only for $r \geq 2$.

Proof Suppose $h_1, ..., h_k \in H$ are members of an orthonormal system in H, not necessarily distinct. Then for any $\delta > 0$ we have, by dominated convergence,

$$\begin{aligned}
&D^k V_r f(x)(h_1, ..., h_k) \\
&\qquad = \partial_{h_1} \cdots \partial_{h_k} V_r f(x) \\
&\qquad = \int_0^\delta \int_B e^{-t} t^{r/2-1} \left(\frac{e^{-t}}{\sqrt{1-e^{-2t}}} \right)^k f(e^{-t}x + \sqrt{1-e^{-2t}}y) \times \\
&\qquad\qquad \times Q(\hat{h}_1(y), ..., \hat{h}_k(y))\, \mu(dy) dt \\
&\qquad + \int_\delta^\infty \int_B e^{-t} t^{r/2-1} \left(\frac{e^{-t}}{\sqrt{1-e^{-2t}}} \right)^k f(e^{-t}x + \sqrt{1-e^{-2t}}y) \times \\
&\qquad\qquad \times Q(\hat{h}_1(y), ..., \hat{h}_k(y))\, \mu(dy) dt \\
&\qquad := I_1(\delta) + I_2(\delta)
\end{aligned}$$

with a polynomial Q of degree k as in Proposition 3.3. Now let $\beta > 1$ be a real number such that

$$\frac{r}{2k} \leq \beta < \frac{r}{k}. \tag{3.15}$$

Hölder's inequality yields

$$|I_1(\delta)| \le \left(\int_0^\delta \int_B e^{-t} t^{r/2-1} \left(\frac{e^{-t}}{\sqrt{1-e^{-2t}}} \right)^{\beta k} f(e^{-t}x + \sqrt{1-e^{-2t}}y) \times \right.$$

$$\left. \times |Q(\hat{h}_1(y), ..., \hat{h}_k(y))|^\beta \mu(dy)dt \right)^{1/\beta} \times \tag{3.16}$$

$$\left(\int_0^\delta \int_B e^{-t} t^{r/2-1} f(e^{-t}x + \sqrt{1-e^{-2t}}y)\, \mu(dy)dt \right)^{1/\beta'}.$$

Using the elementary inequality $e^{-t}t \le 1 - e^{-2t}$ for $t \ge 0$ and (3.15),

$$e^{-t} t^{r/2-1} \left(\frac{e^{-t}}{\sqrt{1-e^{-2t}}} \right)^{\beta k} \le (1-e^{-2t})^{r/2-k\beta/2-1}\, e^{-2t},$$

so that another application of Hölder's inequality, now with q, shows that the first factor on the right hand side of (3.16) is bounded by

$$\left(\int_0^\delta \int_B f(e^{-t}x + \sqrt{1-e^{-2t}}y)^q \mu(dy) (1-e^{-2t})^{r/2-k\beta/2-1} e^{-2t} dt \right)^{1/(\beta q)} \times$$

$$\times \left(\int_B |Q(\hat{h}_1(y), ..., \hat{h}_k(y))|^{\beta q'} \mu(dy) \int_0^\delta (1-e^{-2t})^{r/2-k\beta/2-1} e^{-2t} dt \right)^{1/(\beta q')}.$$

According to Proposition 3.3 the coefficients of the polynomial Q are bounded for fixed k, and since its degree does not exceed k, it involves only finitely many distinct products of powers of the functions $\hat{h}_i$. Together with the fact that each $\hat{h}_i$ is $N(0,1)$-distributed, this implies that there is a constant $c_1(k,q,\beta) > 0$, depending on k but not on the particular choice of the elements $h_1, ..., h_k$, such that

$$\left(\int_B |Q(\hat{h}_1(y), ..., \hat{h}_k(y))|^{\beta q'} \mu(dy) \right)^{1/(\beta q')} < c_1(k,q,\beta).$$

Taking into account (3.15), we therefore obtain

$$|I_1(\delta)| \le c_1(k,q,\beta) \left(\frac{r}{2} - \frac{\beta k}{2} \right)^{-1/\beta} (1-e^{-2\delta})^{r/(2\beta)-k/2} \times$$

$$\times (V_r f(x))^{1/\beta'} \left(\sup_{t>0} P_t(f^q)(x) \right)^{\frac{1}{\beta q}}. \tag{3.17}$$

To estimate $I_2(\delta)$ let

$$\frac{r}{k} < \gamma. \tag{3.18}$$

In a similar fashion we can then obtain the estimate

$$|I_2(\delta)| \le c_2(k,q,\gamma)\left(\frac{r}{2}-\frac{\gamma k}{2}\right)^{-1/\gamma}(1-e^{-2\delta})^{r/(2\gamma)-k/2}\times$$
$$\times (V_r f(x))^{1/\gamma'}\left(\sup_{t>0} P_t(f^q)(x)\right)^{\frac{1}{\gamma q}}, \tag{3.19}$$

where $c_2(k,q,\gamma)>0$ is a constant depending on n but not on the particular choice of $h_1,...,h_k$.

We finally choose suitable $\delta>0$. The function

$$\delta \mapsto (1-e^{-2\delta}), \quad \delta>0,$$

can attain any value in $(0,1)$. Since Jensen's inequality implies

$$(V_r f(x))^q \le \sup_{t>0}(P_t(f)(x))^q \le \sup_{t>0} P_t(f^q)(x), \tag{3.20}$$

we have $\sup_{t>0}(P_t(f^q)(x))^{1/q} \ge V_r f(x)$ and can choose $\delta>0$ such that

$$(1-e^{-2\delta}) = \frac{V_r f(x)^{2/r}}{2\sup_{t>0}(P_t(f^q)(x))^{2/(qr)}}, \tag{3.21}$$

note that the denominator cannot be zero unless f is zero μ-a.e. Combining with (3.17) and (3.19) we obtain

$$|D^k V_r f(x)(h_1,...,h_k)|$$
$$\le \left\{c_1'(k,q,\beta)\left(\frac{r}{2}-\frac{\beta k}{2}\right)^{-1/\beta} + c_2'(k,q,\gamma)\left(\frac{r}{2}-\frac{\gamma k}{2}\right)^{-1/\gamma}\right\}\times$$
$$\times (V_r f(x))^{1-k/r}\left(\sup_{t>0} P_t(f^q)(x)\right)^{k/(qr)}$$

for some constants $c_1'(k,q,\beta)$, $c_2'(k,q,\gamma)$. For any given r there exist only finitely many numbers $k < r$ and for any such k numbers β and γ as in (3.15) and (3.18) can be fixed. Using Proposition 3.1 we can therefore find a constant $c(k,q,r)$ depending only on k, q and r such that (3.14) holds.

□

We prove Lemma 3.1, basically following the method of proof used for [3, Theorem 3.3.3].

Proof If $r = 1$ then T has a bounded first derivative, and the desired bound is immediate from the definition of the norm $\|\cdot\|_{W^{1,p}}$, the chain rule for the gradient D and Meyer's equivalence, [92, Theorem 4.4]. In the following we therefore assume $r \geq 2$.

We verify that for any $k \leq r$ the inequality

$$\left\|D^k(T \circ V_r f)\right\|_{L^p(B,\mu,\mathcal{H}^k)} \leq c(k,L,p,r)\ \|f\|_{L^p(B,\mu)} \tag{3.22}$$

holds with a constant $c(k,L,p,r) > 0$ depending only on k, L, p and r. If so, then summing up yields

$$\|T \circ V_r f\|_{W^{r,p}(B,\mu)} = \sum_{k=0}^{r} \left\|D^k(T \circ V_r f)\right\|_{L^p(B,\mu,\mathcal{H}^r)} \leq c_T\ \|f\|_{L^p(B,\mu)}$$

with a constant $c_T > 0$ depending on L, p and r, as desired.

To see (3.22) suppose $k \leq r$ and that $h_1, ..., h_k$ are members of an orthonormal system $(g_i)_{i=1}^k$, not necessarily distinct. To simplify notation, we use multiindices with respect to this orthonormal system: Given a multiindex $\alpha = (\alpha_1, ..., \alpha_k)$ we write $D^\alpha := \partial_{g_1}^{\alpha_1} \cdots \partial_{g_k}^{\alpha_k}$, where for $\beta = 0,1,2,...$, a function $u : B \to \mathbb{R}$ and an element $g \in H$ we define $\partial_g^\beta u$ as the image of u under the application β differentiations in direction g,

$$\partial_g^\beta u(x) := \partial_g \cdots \partial_g u(x) = D^\beta u(x)(g, ..., g).$$

Now let α be a multiindex such that $D^\alpha = \partial_{h_1}\cdots\partial_{h_k}$. Then clearly $|\alpha| = k$. Moreover, we have

$$D^\alpha(T\circ V_rf)(x)$$
$$= \sum_{j=1}^k T^{(j)}\circ V_rf(x)\sum C_{\alpha^1,\dots,\alpha^j}D^{\alpha^1}V_rf(x)\cdots D^{\alpha^j}V_rf(x)$$

by the chain rule, where the interior sum is over all j-tuples $(\alpha^1,\dots,\alpha^j)$ of multiindices α^i such that $|\alpha^i|\geq 1$ for all i and $\alpha^1+\alpha^2+\dots+\alpha^j=\alpha$. The interior sum has $\binom{k-1}{j-1}$ summands. The $C_{\alpha^1,\dots,\alpha^j}$ are real valued coefficients, and since there are only finitely many different $C_{\alpha^1,\dots,\alpha^j}$, there exists a constant $C(k)>0$ which for all multiindices α with $|\alpha|=k$ dominates these constants, $C_{\alpha^1,\dots,\alpha^j}\leq C(k)$. In particular, $C(k)$ does not depend on the particular choice of the elements $h_1,\dots,h_k$. More explicit computations can for instance be obtained using [45].

The hypothesis (3.10) on T implies

$$|D^\alpha(T\circ V_rf)(x)|\leq c(k)L\sum_{j=1}^k(V_rf(x))^{1-j}\sum|D^{\alpha^1}V_rf(x)\cdots D^{\alpha^j}V_rf(x)|$$

with a constant $c(k)>0$ depending only on k and with L being as in (3.10). Since $\sum_{i=1}^j(1-|\alpha^i|/k) = j-|\alpha|/k = j-1$ and

$$|D^{\alpha^i}V_rf(x)|\leq\|D^{|\alpha^i|}V_rf(x)\|_{\mathcal{H}_{|\alpha^i|}},$$

Lemma 3.1 implies that

$$\sum_{j=2}^k(V_rf(x))^{1-j}\sum|D^{\alpha^1}V_rf(x)\cdots D^{\alpha^j}V_rf(x)|$$
$$\leq\sum_{j=2}^k(V_rf(x))^{1-j}\sum\|D^{|\alpha^1|}V_rf(x)\|_{\mathcal{H}_{|\alpha^1|}}\cdots\|D^{|\alpha^j|}V_rf(x)\|_{\mathcal{H}_{|\alpha^j|}}$$
$$\leq c(k,q,r)\sum_{j=2}^k\binom{k-1}{j-1}(\sup_{t>0}P_t(f^q)(x))^{1/q},$$

where $1 < q < +\infty$ is arbitrary and $c(k,q,r) > 0$ is a constant depending only on k, q and r. For the case $j = 1$ we have

$$|D^\alpha V_r f(x)| \le \|D^k V_r f(x)\|_{\mathcal{H}_k}.$$

Taking the supremum over all $h_1, ..., h_k \in H$ as above we obtain

$$\left\|D^k T \circ V_r f(x)\right\|_{\mathcal{H}^k} \le c(k,L,q,r)\left[(\sup_{t>0} P_t(f^q)(x))^{1/q} + \|D^k V_r f(x)\|_{\mathcal{H}_k}\right]$$

with a constant $c(k,L,q,r) > 0$ by Proposition 3.1.

Fixing $1 < q < p$ and using the boundedness of the semigroup maximal function, [92, Theorem 3.3], we see that there is a constant $c(p,q) > 0$ depending only on p and q such that

$$\left\|(\sup_{t>0} P_t(f^q))^{1/q}\right\|_{L^p(B,\mu)} \le c(p,q)\ \|f\|_{L^p(B,\mu)}.$$

On the other hand, by (3.7), we have

$$\left\|D^k V_r f\right\|_{L^p(B,\mu,\mathcal{H}_k)} \le \frac{1}{c_1}\ \|f\|_{L^p(B,\mu)}.$$

Combining, we arrive at (3.22).

□

3.5 L^p-uniqueness

We discuss related uniqueness problems for the Ornstein Uhlenbeck operator $\mathcal{L}$ and its integer powers.

Recall first that a densely defined operator $(L,\mathcal{A})$ on $L^p(B,\mu)$, $1 \le p < +\infty$ is said to be L^p*-unique* if there is only one C_0-semigroup on $L^p(B,\mu)$ whose generator extends $(L,\mathcal{A})$, see e.g. [29, Chapter I b), Definition 1.3]. If $(L,\mathcal{A})$ has an extension generating a C_0-semigroup on $L^p(B,\mu)$ then $(L,\mathcal{A})$ is L^p-unique if and only if the closure of $(L, \mathcal{A})$ generates a C_0-semigroup on $L^p(B,\mu)$, see [29, Chapter I, Theorem 1.2 of Appendix A].

From (3.7) it follows that for any $m = 1, 2, ...$ and $1 < p < +\infty$ we have $\mathcal{D}((-\mathcal{L})^m) = W^{2m,p}(B,\mu)$. The density of $\mathcal{F}C_b^\infty$ and W^∞ in the spaces $W^{2m,p}(B,\mu)$ and the completeness of the latter imply that $((-\mathcal{L})^m, W^{2m,p}(B,\mu))$ is the closure in $L^p(B,\mu)$ of $((-\mathcal{L})^m, \mathcal{F}C_b^\infty)$ and also of $((-\mathcal{L})^m, W^\infty)$.

Since obviously $(P_t)_{t>0}$ is a C_0-semigroup, $(\mathcal{L}, \mathcal{F}C_b^\infty)$ and $(\mathcal{L}, W^\infty)$ are L^p-unique in all $L^p(B,\mu)$, $1 \le p < +\infty$. To discuss the its powers $-(-\mathcal{L})^m$ for $m \ge 2$ we quote well known facts to provide a sufficient condition for them to generate C_0-semigroups. Since $(P_t)_{t>0}$ is a symmetric Markov semigroup on $L^2(B,\mu)$, for any $1 < p < +\infty$ the operator $\mathcal{L} = \mathcal{L}^{(p)}$ generates a bounded holomorphic semigroups on $L^p(B,\mu)$ with angle θ satisfying $\frac{\pi}{2} - \theta \le \frac{\pi}{2}|\frac{2}{p} - 1|$, see for instance [23, Theorem 1.4.2]. On the other hand [27, Theorem 4] tells that if L is the generator of a bounded holomorphic semigroup with angle θ satisfying $\frac{\pi}{2} - \theta < \frac{\pi}{2m}$, then also $-(-L)^m$ generates a bounded holomorphic semigroup. Combining, we can conclude that $-(-\mathcal{L})^m$ generates a bounded holomorphic semigroup on $L^p(B,\mu)$ and therefore in particular a (bounded) C_0-semigroup if

$$|\frac{2}{p} - 1| < \frac{1}{m} \ . \tag{3.23}$$

[28, Theorem 8] shows that (up to a discussion of limit cases) this is a sharp condition for $-(-\mathcal{L})^m$ to generate a bounded C_0-semigroup. For $1 < p < +\infty$ this also recovers the L^p-uniqueness in the case $m = 1$. For $p = 2$ condition (3.23) is always satisfied. Alternatively we can conclude the generation of C_0-semigroups on $L^2(B,\mu)$ directly from the spectral theorem.

For later use we fix the following fact.

PROPOSITION **3.4** *Let $1 < p < +\infty$ and let $m > 0$ be an integer satisfying (3.23).*

Then the operators $(-(-\mathcal{L})^m, \mathcal{F}C_b^\infty)$ and $(-(-\mathcal{L})^m, W^\infty)$ are L^p-unique in $L^p(B,\mu)$. In particular, they are essentially self-adjoint in $L^2(B,\mu)$ for all $m > 0$.

The last statement is true because a semi-bounded symmetric operator (L, A) on $L^2(B,\mu)$ is L^2-unique if and only if it is essential self-adjoint, see [29, Chapter I c), Corollary 1.2].

Here we are interested in L^p-uniqueness after the removal of a small closed set $\Sigma \subset B$ of zero measure. This is similar to our discussion in [48] and, in a sense, similar to a removable singularities problem, see for instance [76] or [77] or [3, Section 2.7].

Let $\Sigma \subset B$ be a closed set of zero Gaussian measure and $N := B \setminus \Sigma$. We define

$$\mathcal{F}C_b^\infty(N) := \{f \in \mathcal{F}C_b^\infty \mid f = 0 \text{ on an open neighborhood of } \Sigma\}$$

and

$$W^\infty(N) := \{f \in W^\infty \mid \widetilde{f} = 0 \text{ q.s. on an open neighborhood of } \Sigma\}.$$

The L^p-uniqueness of $-(-\mathcal{L})^m$, restricted to $\mathcal{F}C_b^\infty(N)$ and $W^\infty(N)$, respectively, now depends on the size of the set Σ. If it is small enough not to cause additional boundary effects then from the point of view of operator extensions it is removable.

THEOREM **3.2** *Let $1 < p < +\infty$, let $m > 0$ be an integer and assume that $\Sigma \subset B$ is a closed set of zero measure μ. Write $N := B \setminus \Sigma$.*

(i) If $\mathrm{Cap}_{2m,p}(\Sigma) = 0$ then the closure of $(-(-\mathcal{L})^m, \mathcal{F}C_b^\infty(N))$ in $L^p(B,\mu)$ is

$$(-(-\mathcal{L})^m, W^{2m,p}(B,\mu)).$$

If in addition m satisfies (3.23) then $(-(-\mathcal{L})^m, \mathcal{F}C_b^\infty(N))$ is L^p-unique.

(ii) If $(-(-\mathcal{L})^m, \mathcal{F}C_b^\infty(N))$ is L^p-unique, then $\mathrm{Cap}_{2m,p}(\Sigma) = 0$.

The same statements are true with $W^\infty(N)$ in place of $\mathcal{F}C_b^\infty(N)$.

Proof To see (i) suppose that $\mathrm{Cap}_{2m,p}(\Sigma) = 0$. Let $((-\mathcal{L})^m, \mathcal{D}((-\mathcal{L})^m))$ denote the closure of $((-\mathcal{L})^m, \mathcal{F}C_b^\infty(N))$ in $L^p(B,\mu)$. Since $\mathcal{F}C_b^\infty(N) \subset \mathcal{F}C_b^\infty$ we trivially have

$$\mathcal{D}((-\mathcal{L})^m) \subset W^{2m,p}(B,\mu),$$

and it remains to show the converse inclusion.

Given $u \in W^{2m,p}(B,\mu)$, let $(u_j)_{j=1}^\infty \subset \mathcal{F}C_b^\infty$ be a sequence approximating u in $W^{2m,p}(B,\mu)$. By Theorem 3.1 there is a sequence $(v_l)_{l=1}^\infty \subset \mathcal{F}C_b^\infty$ such that $\lim_{l\to\infty} v_l = 0$ in $W^{2m,p}(B,\mu)$ and for each l the function v_l equals one on an open neighborhood of Σ. Set $w_{jl} := (1-v_l)u_j$ to obtain functions $w_{jl} \in \mathcal{F}C_b^\infty(N)$. Now let j be fixed. For any $1 \le k \le 2m$ let $h_1, ..., h_k$ be members of an orthonormal system $(g_i)_{i=1}^k$, not necessarily distinct. As in the proof of Lemma 3.1 we use multiindex notation with respect to this orthonormal system. Let α be such that $D^\alpha = \partial_{h_1}\cdots\partial_{h_k}$. Then, by the general Leibniz rule,

$$D^\alpha(u_j - w_{jl})(x) = D^\alpha(u_j v_l)(x) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} D^\beta u_j(x) D^{\alpha-\beta} v_l(x),$$

where for two multiindices α and β we write $\beta \le \alpha$ if $\beta_i \le \alpha_i$ for all $i = 1, ..., k$. For any such β we clearly have

$$|D^\beta u_j(x)| \le \left\| D^{|\beta|} u_j(x) \right\|_{\mathcal{H}_{|\beta|}} \quad \text{and} \quad |D^{\alpha-\beta} v_l(x)| \le \left\| D^{|\alpha-\beta|} v_l(x) \right\|_{\mathcal{H}_{|\alpha-\beta|}},$$

and taking the supremum over all $h_1, ..., h_k$ as above,

$$\left\| D^k(u_j - w_{jl})(x) \right\|_{\mathcal{H}_k} \le c(k) \max_{n\le k} \|D^n u_j(x)\|_{\mathcal{H}_n} \max_{n\le k} \|D^n v_l(x)\|_{\mathcal{H}_n}$$

with a constant $c(k) > 0$ depending only on k. Taking into account that

$$\sup_{x\in B} \|D^n u_j(x)\|_{\mathcal{H}_n} < +\infty$$

for any $n \geq 1$ and summing up, we see that

$$\begin{aligned}\lim_l \sum_{k=1}^{2m} \left\| D^k(u_j - w_l) \right\|_{L^p(B,\mu,\mathcal{H}_k)} \\ &\leq c(m) \max_{n \leq 2m} \sup_{x \in B} \left\| D^n u_j(x) \right\|_{\mathcal{H}_n} \lim_l \|v_l\|_{W^{2m,p}} \\ &= 0,\end{aligned}$$

here $c(m) > 0$ is a constant depending on m only. Since u_j is bounded, we also have $\lim_l (u_j - w_{jl}) = \lim_l u_j v_l = 0$ in $L^p(B,\mu)$ so that

$$\lim_l w_{jl} = u_j \quad \text{in } W^{2m,p}(B,\mu),$$

what implies $u \in \mathcal{D}((-\mathcal{L})^m)$ and therefore

$$W^{2m,p}(B,\mu) \subset \mathcal{D}((-\mathcal{L})^m).$$

To see (ii) suppose that $(-(-\mathcal{L})^m, \mathcal{F}C_b^\infty(N))$ is L^p-unique in $L^p(B,\mu)$. Then its unique extension must be $(-(-\mathcal{L})^m, W^{2m,p}(B,\mu))$. Let $u \in \mathcal{F}C_b^\infty$ be a function that equals one on a neighborhood of Σ. Since $\mathcal{F}C_b^\infty \subset W^{2m,p}(B,\mu)$ and by hypothesis $\mathcal{F}C_b^\infty(N)$ is dense in $W^{2m,p}(B,\mu)$, we can find a sequence $(u_l)_l \subset \mathcal{F}C_b^\infty(N)$ approximating u in $W^{2m,p}(B,\mu)$. The functions $e_l := u - u_l$ then are in $\mathcal{F}C_b^\infty$, each equals one on an open neighborhood of Σ, and they converge to zero in $W^{2m,p}(B,\mu)$, so that by Theorem 3.1 we have

$$\mathrm{Cap}_{2m,p}(\Sigma) \leq c_2 \lim_l \|e_l\|_{W^{2m,p}} = 0.$$

The proof for W^∞ is similar.

□

3.6 Comments on Gaussian Hausdorff measures

For finite dimensional Euclidean spaces the link between Sobolev type capacities and Hausdorff measures is well known and the critical size of a set Σ in order to have (r,p)-capacity zero or not is, roughly speaking, determined by its Hausdorff codimension, see e.g. [3, Chapter 5]. For Wiener spaces one can at least provide a partial result of this type.

Hausdorff measures on Wiener spaces of integer codimension had been introduced in [32, Section 1]. We briefly sketch their method but allow non-integer codimensions, this is an effortless generalization and immediate from their arguments.

Given an m-dimensional Euclidean space F and a real number $0 \le d \le m$ the spherical Hausdorff measure $\mathcal{S}^d$ of dimension d can be defined as follows: For any $\varepsilon > 0$ set

$$\mathcal{S}^d_\varepsilon(A) := \inf\left\{\sum_{i=1}^\infty r_i^d : \{B_i\}_{i=1}^\infty \text{ is a collection of balls}\right.$$
$$\left.\text{of radius } r_i < \varepsilon/2 \text{ such that } A \subset \bigcup_{i=1}^\infty B_i\right\},$$

and finally, $\mathcal{S}^d(A) := \sup_{\varepsilon>0} \mathcal{S}^d_\varepsilon(A)$, $A \subset F$. A priori $\mathcal{S}^d$ is an outer measure, but its σ-algebra of measurable sets contains all Borel sets. For any $0 \le d \le m$ and we define

$$\theta_d^F(A) := (2\pi)^{-m/2} \int_A \exp\left(-\frac{|y|_F^2}{2}\right) \mathcal{S}^{m-d}(dy),$$

for Borel sets $A \subset F$, [32, 1. Definition], by approximation from outside it extends to an outer measure on F, defined in particular for any analytic set. Recall that a set $A \subset F$ is called *analytic* if it is a continuous image of a Polish space.

We return to the abstract Wiener space (B, μ, H). Let $d \ge 0$ be a real number and let F be a subspace of H of finite dimension $m \ge d$. Let p^F denote the orthogonal projection from H onto F, it extends to a linear projection p^F from B onto F which is (r,p)-quasi continuous for all r and p, [31, 11. Théorème]. We write $\widetilde{F}$ for the kernel of p^F. The spaces B and $F \times \widetilde{F}$ are isomorphic under the map $p^F \times (I - p^F)$. If $A \subset B$ is analytical and for any $x \in \widetilde{F}$ the section with respect to the above product is denotes by $A_x \subset F$, then for any $a \in \mathbb{R}$ the set $\{x \in \widetilde{F} : \theta_d^F(A_x) > a\}$ is analytic up to a slim set, as shown in [32, 4. Lemma]. We follow [32, 5. Definition] and set $\mu^F(B) := \mu((I - p^F)^{-1}(B))$ for any analytic subset B of F. Then by [32, 4. Lemma] we can define

$$\varrho_d^F(A) := \int_B \theta_d^F(A_x)\mu(dx)$$

for any analytic subset A of B. As in [32, 8. Definition] we define the *Gaussian Hausdorff measure* ϱ_d *of codimension* $d \geq 0$ by

$$\varrho_d(A) := \sup\left\{\varrho_d^F(A) : F \subset H \quad \text{and } d \leq \dim F < +\infty\right\}$$

for any analytic set $A \subset B$. Restricted to the Borel σ-algebra it is a Borel measure. The next result follows in the same way as [32, 9. Theorem] from [31, 32. Théorème] and [78], see also [3, Theorem 5.1.13].

THEOREM **3.3** *If a Borel set* $A \subset B$ *satisfies* $\mathrm{Cap}_{r,p}(A) = 0$*, then* $\varrho_d(A) = 0$ *for all* $d < rp$*.*

Combined with Theorem 3.2 this yields a necessary codimension condition which is similar as in the case of Laplacians on Euclidean spaces, [5, 48].

COROLLARY **3.1** *Assume* $1 < p < +\infty$*. Let* $\Sigma \subset B$ *be a closed set of zero measure and* $N := B \setminus \Sigma$*.*

If $(-(-\mathcal{L})^m, \mathcal{F}C_b^\infty(N))$ *is* L^p*-unique, then*

$$\varrho_d(\Sigma) = 0 \quad \textit{for all } d < 2mp.$$

In particular, if $(\mathcal{L}, \mathcal{F}C_b^\infty(N))$ *is essentially self-adjoint, then*

$$\varrho_d(\Sigma) = 0 \quad \text{for all } d < 4.$$

The same is true with $W^\infty(N)$ *in place of* $\mathcal{F}C_b^\infty(N)$.

3.7 Comments on stochastic processes

We finally like to briefly point out connections to known Kakutani type theorems for related multiparameter Ornstein-Uhlenbeck processes. The connection between Gaussian capacities, [31], and the hitting behaviour of multiparameter processes, [51, 53, 54], has for instance been investigated in [8, 94, 95]. We briefly sketch the construction and main result of [95], later generalized in [8].

Let $\Theta^{(0)} := B$ and for integer $k \geq 1$, $\Theta^{(k+1)}(B) := C(\mathbb{R}_+, \Theta^{(k)}(B))$. The space $\Theta^k(B)$ can be identified with $C(\mathbb{R}_+^k, B)$. Moreover, set $\mu^{(0)} := \mu$, $T_t^{(0)} := P_t$, $t > 0$, and let $Z^{(1)}$ be the Ornstein-Uhlenbeck process taking values in $\Theta^{(0)}(B) = B$ with semigroup $T_t^{(0)}$ and initial law $\mu^{(0)}$. Let $\mu^{(1)}$ denote the law of the process $Z^{(1)}$, clearly a centered Gaussian measure on $\Theta^{(1)}(B)$. Next, let $(T_t^{(1)})_{t<0}$ be the Ornstein-Uhlenbeck semigroup on $\Theta^{(1)}(B)$ defined by

$$T_t^{(1)} f(x) = \int_{\Theta^{(1)}(B)} f(e^{-t}x + \sqrt{1 - e^{-2t}}y)\mu^{(1)}(dy), \quad x \in \Theta^{(1)}(B),$$

for any bounded Borel function f on $\Theta^{(1)}(B)$, and let $Z^{(2)}$ be the Ornstein-Uhlenbeck process taking values in $\Theta^{(1)}(B)$ with semigroup $(\Theta^{(1)})_{t>0}$ and initial law $\mu^{(1)}$. Iterating this construction yields, for any integer $r \geq 1$, an Ornstein-Uhlenbeck process $Z^{(r)}$ taking values in $\Theta^{(r-1)}(B)$. This process may also be viewed as an r-parameter process $Z^{(r)} = (Z_{\mathbf{t}}^{(r)})_{\mathbf{t}\in\mathbb{R}_+^r}$ taking values in B. Now [95, §6, Théorème 1] tells that a Borel set $A \subset B$ has zero $(r,2)$-capacity $\mathrm{Cap}_{r,2}(A) = 0$ if and only if the event

$$\left\{\text{there exists some } \mathbf{t} \in \mathbb{R}_+^r \text{ such that } Z_{\mathbf{t}}^{(r)} \in A\right\}$$

has probability zero. See also [8, 13. Corollary].

Combined with Theorem 3.2 this result gives a preliminary characterization of L^2-uniqueness (that is, essential self-adjointness) in terms of the hitting behaviour of the $2m$-parameter Ornstein-Uhlenbeck process $(X_{\mathbf{t}}^{(m)})_{\mathbf{t}\in\mathbb{R}_+^{2m}}$.

COROLLARY **3.2** *Let $m > 0$ be an integer. Let $\Sigma \subset B$ be a closed set of zero measure and $N := B \setminus \Sigma$. The operators $(-(-\mathcal{L})^m, \mathcal{F}C_b^\infty(N))$ and $(-(-\mathcal{L})^m, W^\infty(N))$ are L^2-unique (resp. essentially self-adjont) if and only if $Z^{(2m)}$ does not hit Σ with positive probability.*

A more causal connection between uniqueness problems for operators and classical probability should involve certain branching diffusions rather than multiparameter processes, but even for finite dimensional Euclidean spaces the problem is not fully settled and remains a future project.

Part II

Markov uniqueness, L^p uniqueness and elliptic regularity on reflected Dirichlet space

Chapter 4 Markov uniqueness and L^2-uniqueness on reflected Dirichlet space

4.1 Introduction

Let U be an open set in $\mathbb{R}^d$ whose boundary is smooth enough in a sense that will be precised later and let $\mu = \varphi^2 dx$, $\varphi \in H^{1,2}_{loc}(U)$, $\varphi > 0$ dx-a.e. Let $A = (a_{ij})_{i,j=1}^d$ be a locally uniformly strictly elliptic matrix consisting of measurable functions and $B := (B_1, \dots, B_d)$ be a weakly μ-divergence free vector field (for the precise conditions, see later sections). Let S be a non-symmetric linear operator on $L^2(U, \mu)$ with the domain $D(S) \subset C_0^2(\overline{U})$ being densely defined in $L^2(U, \mu)$. Let

$$Su = \frac{1}{2}\sum_{i,j}(a_{ij}\partial_i\partial_j u + \partial_i a_{ij}\partial_j u) + \sum_{i=1}^d b_i \partial_i u \quad f \in D(S).$$

Consider the non-symmetric bilinear form $(\mathcal{E}, D(S))$ defined by

$$\mathcal{E}(f,g) = \frac{1}{2}\int_U \langle A\nabla f, \nabla g\rangle d\mu - \int_U \langle B, \nabla f\rangle g d\mu, \quad f, g \in D(S).$$

Assume this form can be extended to a Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ with generator $(L, D(L))$ extending $(S, D(S))$. The Markov uniqueness problem consists of finding conditions which ensure that a diffusion operator has a unique sub-Markovian extension, i.e., an extension which is the generator of a Dirichlet form. That is, we want to find a condition that guarantees that $(L, D(L))$ is the sole sub-Markovian extension of $(S, D(S))$. To show Markov uniqueness, we have to know that there exists a maximum extension, and then show $(S, D(S))$ is dense in this maximum extension in some sense(see Section 4.3 for more detail). When $U = \mathbb{R}^d$ and $D(S) = C_0^\infty(\mathbb{R}^d)$, it is the so-called Dirichlet problem and well-studied by many other authors (see [19, 29, 35, 46, 69, 85] and the reference therein).

However, we cannot guarantee Markov uniqueness on the domain even in the simplest case when $\varphi = 1$ and U is an open ball so that $(S, D(S)) = (\frac{1}{2}\Delta, C_0^\infty(U))$, since we know that there exists at least two different extensions $(\frac{1}{2}\Delta, H_{Neu}^{2,2}(U))$ and $(\frac{1}{2}\Delta, H^{2,2}(U) \cap H_0^{1,2}(U))$. Hence we need some different test functions to guarantee Markov uniqueness. One of possible test functions is $D(L) \cap C_0^2(\overline{U})$ and we will see in Section 3 that it corresponds to the so-called Neumann problem.

Markov uniqueness problems are related to a number of other uniqueness problems. The L^2 uniqueness problem consists of finding conditions which ensure that a diffusion operator has a unique extension which generates a C_0 semigroup. In particular, it is easy to see that L^2 uniqueness implies Markov uniqueness. In the symmetric case, it is known that L^2 uniqueness is equivalent to essential self-adjointness. Essential self-adjointness has been well-studied by many authors (see [9, 29, 46, 49, 66, 71] and the references therein).

In this paper, we will present generalizations (in some sense) of previous results on uniqueness problems on domains of Euclidean space in the case of Neumann problem.

In Section 4.2, we present our general setting, some preliminary results and notations that will be used throughout this paper.

In Section 4.3, we will see Markov uniqueness results in simple cases, when $A = I$, $B = 0$, and U is of class C^2 or certain type of d-dimensional polytope. We will use a reflection technique and collect some known tools from partial differential equations to show our main result. Moreover, we will show some density result for more general A. However, we cannot see that this density result implies Markov uniqueness, since we don't know which is the maximum extension.

In Section 4.4, we use similar tool from Section 4.3 to show L^2 uniqueness, for more general A and B, but we assume more restrictions on the density φ.
In Section 4.5, we merge our result with a known result from [85]. There, a certain condition on A near the boundary is assumed to guarantee Markov uniqueness even in Dirichlet problem. So, we call this merged problem as Robin boundary problem.

4.2 Functional analytic framework, preliminary results and notations

In general, we shall denote by $\|\cdot\|_B$ the norm of a Banach space (or vector space) B. In the special case of $\mathbb{R}^d$, $d \geq 1$, $|\cdot|$ will denote the corresponding Euclidean norm and $\langle \cdot, \cdot \rangle$ the Euclidean inner product. For $x \in \mathbb{R}^d$, let x_i denote the i-th coordinate of x, $1 \leq i \leq d$, and $B_r(x) := \{y \in \mathbb{R}^d : \|y - x\| < r\}, \overline{B}_r(x) := \{y \in \mathbb{R}^d : \|y - x\| \leq r\}$, $r > 0$. In general $\overline{A}$ shall denote the closure of A in the corresponding topological space. By $\mathbb{R}_+$ we denote the set of all positive (≥ 0) real numbers.
Let $U \subset \mathbb{R}^d$ be a possibly unbounded open set whose boundary is locally the graph of a Lipschitz function. Let σ denote the surface measure on the boundary ∂U of U and η be the inward normal vector on ∂U. It is known that η is σ-almost everywhere (σ-a.e.) uniquely defined.
For arbitrary open $\Omega \subset \mathbb{R}^d$, let $L^p(\Omega, \mu)$, $p \in [1, \infty]$, denote the usual L^p-spaces with respect to the measure μ and we omit μ if it is the Lebesgue measure. By $L^p_{loc}(U, \mu)$ we denote all measurable functions $f : U \to \mathbb{R}$ with $f \in L^p(V, \mu)$ for any bounded and open set $V \subset U$. We denote by $(\cdot, \cdot)_H$ the inner product in Hilbert space H.
Let A, B be sets. For a function $f : A \to \mathbb{R}$ and $B \subset A$, denote the restriction of f to B by $f_{|B}$. We denote the closure (in a topological space that will be mentioned) of B by $\overline{B}$. For a set $A \subset \mathbb{R}^d$, a function $f : A \to \mathbb{R}$, let $\mathrm{supp}_A f$ denote the essential support

of f in A with respect to the Lebesgue measure. For convenience, we write supp$f$ or support of f instead of supp$f$. Let $n \in \mathbb{N} \cup \{\infty\}$. Denote by $C_0^n(\mathbb{R}^d)$ the set of n-times continuously differentiable functions on $\mathbb{R}^d$ with compact support, and $C_0^n(\overline{\Omega})$ denote the restriction of $C_0^n(\mathbb{R}^d)$ to $\overline{\Omega}$, where $\Omega \subset \mathbb{R}^d$ is an open set.

If $W \subset L^p(U,\mu)$ is an arbitrary subspace, let W_0 denote the space of all elements $u \in W$ such that supp $u \cap U$ is a bounded set in U, by $W_b = W \cap L^\infty(U,\mu)$ the space of all (μ-)essentially bounded elements in W. Finally, let $W_{0,b} = W_0 \cap W_b$.

Let $C_{loc}^{0,1}(\overline{U})$ be the set of functions whose restriction to a compact set K is a Lipschitz continuous function, where $K \subset \overline{U}$ is arbitrary.

For $n = 1,2$ and arbitrary open set $\Omega \subset \mathbb{R}^d$, let $H^{n,p}(\Omega)$ be the classical Sobolev space of order n in $L^p(\Omega)$, i.e. the space of all measurable functions that are together with their weak derivatives up to order n again in $L^p(\Omega)$, $p \in [1,\infty]$. For a weakly differentiable function u, let $\partial_i u$ denote its i-th partial weak derivative, and let $\nabla u := (\partial_1 u, \ldots, \partial_d u)$, $1 \le i \le d$. Let

$$H_{loc}^{1,2}(\Omega) := \{u : u \cdot \chi \in H^{1,2}(\Omega) \text{ for all } \chi \in C_0^\infty(\overline{\Omega})\}.$$

Fix $\varphi \in H_{loc}^{1,2}(U)$ such that $\varphi > 0$ dx-a.e. and let

$$d\mu = \varphi^2 dx.$$

Note that $H_{loc}^{1,2}(\Omega) = H^{1,2}(\Omega)$, if Ω is bounded.

A family $(T_t)_{t \ge 0}$ of linear operators on $L^2(U,\mu)$ with $D(T_t) = L^2(U,\mu)$ for all $t > 0$ is called a sub-Markovian strongly continuous contraction semigroup or shortly sub-Markovian C_0 semigroup of contractions, if

(i) $\lim_{t\to 0} T_t u = u$ for all $u \in L^2(U,\mu)$ (strong continuity),

(ii) T_t is a contraction on $L^2(U,\mu)$ for all $t > 0$,

(iii) $T_t T_s = T_{t+s}$ for all $t, s > 0$ (semigroup property),

(iv) $0 \le u \le 1$ μ-a.e. implies $0 \le T_t u \le 1$ μ-a.e. for all $u \in L^2(U,\mu)$.

Consider

$$\mathcal{E}(f,g) := \frac{1}{2}\int_U \langle \nabla f, \nabla g \rangle d\mu, \quad f,g \in C_0^\infty(\overline{U}). \tag{4.1}$$

The regularity properties of φ imply that $(\mathcal{E}, C_0^\infty(\overline{U}))$ is closable in $L^2(U,\mu)$ and the closure $(\mathcal{E}, D(\mathcal{E}))$ is actually a Dirichlet form on $L^2(U,\mu)$. Note however, that the closability of $(\mathcal{E}, C_0^\infty(\overline{U}))$ also follows from the existence of a "maximum" Dirichlet form $(\mathcal{E}^+, H^{1,2}(U,\mu))$ as defined in (4.3) below. We denote its generator (resp. its C_0-semigroup of sub-Markovian contractions) by $(L, D(L))$ (resp. $(T_t)_{t\ge 0}$).

Assume a densely defined symmetric linear operator $(S, D(S))$ in $L^2(U,\mu)$ is given such that $-S$ is non-negative definite, i.e. $(-Su, u)_{L^2(U,\mu)} \ge 0$, $u \in D(S)$. Define (for more details, we refer to [35, Chapter 3.3])

$$\begin{aligned} \mathcal{A}_M(S, D(S)) \;:=\; & \{A \,|\, A \text{ with domain } D(A) \text{ is self-adjoint extension of } (S, D(S)), \\ & -A \text{ is non-negative definite and the semigroup on } L^2(U,\mu) \\ & \text{generated by } A \text{ is sub-Markovian}\}. \end{aligned}$$

If there is only one element in $\mathcal{A}_M(S, D(S))$, we call $(S, D(S))$ **Markov unique**. For $A \in \mathcal{A}_M(S, D(S))$, let $(\mathcal{E}_A, D(\mathcal{E}_A))$ denote the Dirichlet form with generator $(A, D(A))$. If $A_1, A_2 \in \mathcal{A}_M(S, D(S))$, we can define a partial order in the following sense

$$A_1 \le A_2 \Leftrightarrow D(\mathcal{E}_{A_1}) \subset D(\mathcal{E}_{A_2}),\ \mathcal{E}_{A_1}(f,g) \ge \mathcal{E}_{A_2}(f,g)\ \ \forall f,g \in D(\mathcal{E}_{A_1}).$$

For $k \in \{2,3,\dots\} \cup \{\infty\}$ define

$$C^k_{0,Neu}(\overline{U}) = \{f \in C^k_0(\overline{U}) : \langle \nabla f,\ \eta\rangle = 0\ \varphi^2 d\sigma\text{-a.e. on } \partial U\}.$$

Note that $\varphi^2 d\sigma$ is well-defined, because φ^2 has a trace on ∂U (see e.g. [30]) that we denote for simplicity again by φ^2.

LEMMA **4.1** *Let* $k \in \{2,3,\dots\} \cup \{\infty\}$. *Then*

$$C^k_0(\overline{U}) \cap D(L) = C^k_{0,Neu}(\overline{U}).$$

Proof Assume $g \in C^\infty_0(U)$ and $f \in C^k_0(\overline{U}) \cap D(L)$. Using integration by parts, we can see that

$$\mathcal{E}(f,g) = -\frac{1}{2}\int_U (\Delta f + 2\nabla f \cdot \frac{\nabla\varphi}{\varphi}) g d\mu.$$

By denseness of $C^\infty_0(U)$ in $L^2(U,\mu)$, we can see that

$$Lf = \frac{1}{2}\Delta f + \nabla f \cdot \frac{\nabla\varphi}{\varphi} \quad \mu\text{-a.e.}$$

Now take $g \in C^\infty_0(\overline{U})$ and get

$$\mathcal{E}(f,g) = -\frac{1}{2}\int_U (\Delta f + 2\nabla f \cdot \frac{\nabla\varphi}{\varphi}) g d\mu + \frac{1}{2}\int_{\partial U} g\langle \nabla f, \eta\rangle \varphi^2 d\sigma.$$

Hence $\int_{\partial U} g\langle \nabla f, \eta\rangle\varphi^2 d\sigma = 0$ for $g \in C^\infty_0(\overline{U})$. Since the support of f is bounded, choose a compact K such that $\partial U \cap \operatorname{supp} f \subset K \subset \partial U$. Then

$$\int_K g\langle \nabla f, \eta\rangle \varphi^2 d\sigma = 0 \quad \text{for } g \in C^\infty_0(\overline{U}). \tag{4.2}$$

In particular, (4.2) is valid for any polynomial g, since polynomials on a compact set can be extended to a function in $C^\infty_0(\overline{U})$. By the Stone-Weierstrass Theorem, we can see that (4.2) is also valid for continuous functions on K. Therfore

$$C^k_0(\overline{U}) \cap D(L) \subset \{f \in C^k_0(\overline{U}) : \langle \nabla f, \eta\rangle = 0\ \varphi^2 d\sigma\text{-a.e. on } \partial U\}.$$

The converse inclusion is clear.

$\square$

Let for some $k \in \{2, 3, \dots\} \cup \{\infty\}$,

$$D(S) := C_0^k(\overline{U}) \cap D(L) \text{ and } Sf := Lf = \frac{1}{2}\Delta f + \nabla f \cdot \frac{\nabla \varphi}{\varphi} \text{ for } f \in D(S).$$

Then by Lemma 4.1

$$(S, D(S)) = (L, C_0^k(\overline{U}) \cap D(L)) = (L, C_{0,Neu}^k(\overline{U})).$$

DEFINITION **4.1** *For open $\Omega \subset \mathbb{R}^d$, $\psi \in H_{loc}^{1,2}(\Omega)$ and $\psi > 0$ dx-a.e. on Ω, the* ***weighted Sobolev space*** *$H^{1,2}(\Omega, \psi^2 dx)$ is the set of elements $f \in L^2(\Omega, \psi^2 dx)$ such that there exists $(f_1, \dots, f_d) \in (L^2(\Omega, \psi^2 dx))^d$ satisfying*

$$(f_i, g)_{L^2(\Omega,\psi^2 dx)} = -\left(f, \partial_i g + 2g\frac{\partial_i \psi}{\psi}\right)_{L^2(\Omega,\psi^2 dx)} \text{ for any } g \in C_0^\infty(\Omega).$$

By [99, Lemma 6], we have $\partial_i f = f_i$, $1 \le i \le d$ if $f \in H^{1,2}(\Omega, \psi^2 dx) \cap H^{1,2}(\Omega, dx)$. For notational convenience, we also write $\partial_i f = f_i$ for any $f \in H^{1,2}(\Omega, \psi^2 dx)$.

By [99, Theorem on page 114], the symmetric bilinear form on $L^2(U, \mu)$ defined by

$$\mathcal{E}^+(f, g) := \frac{1}{2}\int_U \langle \nabla f, \nabla g \rangle d\mu, \quad f, g \in H^{1,2}(U, \mu) \tag{4.3}$$

is the Dirichlet form of the maximum element of $\mathcal{A}_M(L, C_0^\infty(U))$ with repect to the above partial order. In particular, $(\mathcal{E}^+, H^{1,2}(U, \mu))$ is closed. Thus, since $C_0^\infty(\overline{U}) \subset H^{1,2}(U, \mu)$, we can see again that $(\mathcal{E}, C_0^\infty(\overline{U}))$ as defined in (4.1) is closable in $L^2(U, \mu)$.

LEMMA **4.2** *Suppose $C_{0,Neu}^k(\overline{U})$ is dense in $H^{1,2}(U, \mu)$. Then $(L, C_{0,Neu}^k(\overline{U}))$ is Markov unique.*

Proof Since $(L, D(L)) \in \mathcal{A}_M(L, C_0^\infty(U)) \cap \mathcal{A}_M(L, C^k_{0,Neu}(\overline{U}))$, the maximal element of $\mathcal{A}_M(L, C_0^\infty(U))$, i.e. the generator of $(\mathcal{E}^+, H^{1,2}(U,\mu))$, is equal to the maximal element of $\mathcal{A}_M(L, C^k_{0,Neu}(\overline{U}))$.

Using [35, Lemma 3.3.1] the minimal element of $\mathcal{A}_M(L, C^k_{0,Neu}(\overline{U}))$ is the generator of the closure of $(\mathcal{E}, C^k_{0,Neu}(\overline{U}))$ in $L^2(U,\mu)$ which coincides with $(\mathcal{E}^+, H^{1,2}(U,\mu))$, if $C^k_{0,Neu}(\overline{U})$ is dense in $H^{1,2}(U,\mu)$. Therefore, in this case, the minimal element of $\mathcal{A}_M(L, C^k_{0,Neu}(\overline{U}))$ coincides with the maximal element of $\mathcal{A}_M(L, C^k_{0,Neu}(\overline{U}))$ and Markov uniqueness holds.

□

Before we start the proof, we need the following Lemma which gives an explicit description of the weighted Sobolev space.

LEMMA **4.3** *Let* Ω *and* ψ *as in Definition* 4.1. *To distinguish d-dimensional Lebesgue measure and* $(d-1)$*-dimensional Lebesgue measure, we let* λ^k *be the k-dimensional Lebesgue measure on this lemma,* $k = d,\ d-1$. *Let*

$$\Omega_1 = \{(x_2, \dots, x_d) \in \mathbb{R}^{d-1} |\ \textit{there exists } x_1 \textit{ such that } (x_1, x_2, \dots, x_d) \in \Omega\}.$$

Let $\tilde{\rho}^{(1)}$ *be a* λ^d*-version of* $\rho := \psi^2$ *which is absolutely continuous on the* x_1*-axis for* λ^{d-1}*-a.e.* $(x_2, \dots, x_d)$ *in* Ω_1. *Define the following space*

$$D(\bar{\mathcal{E}})_1 := \left\{ \begin{array}{l} u \in L^2(\Omega, \rho dx) :\ \textit{there exists a function } \tilde{u}^{(1)} \textit{ such that} \\ i)\ \tilde{u}^{(1)} = u, \rho dx\textit{-a.e.} \\ ii)\ \textit{for } \lambda^{d-1}\textit{-a.e. } (x_2, \dots, x_d) \in \Omega_1,\ \tilde{u}^{(1)}(x_1, x_2, \dots, x_d) \\ \quad \textit{is absolutely continuous in } x_1 \textit{ on } \{x_1 \in \mathbb{R} | \tilde{\rho}^{(1)}(x_1, x_2, \dots, x_d) > 0\} \\ \quad \textit{and } \partial \tilde{u}^{(1)} / \partial x_1 (\textit{classical partial derivative}) \in L^2(\Omega, \rho dx) \end{array} \right\}$$

Then $D(\bar{\mathcal{E}})_1$ is independent of the choice of the version $\tilde{\rho}^{(1)}$ and $\partial\tilde{u}^{(1)}/\partial x_1$ is defined μ-a.e. Define $D(\bar{\mathcal{E}})_i$, $i = 2, \dots, d$ analogously and let $D(\bar{\mathcal{E}}) = \cap_i D(\bar{\mathcal{E}})_i$. Then we have $D(\bar{\mathcal{E}}) = H^{1,2}(\Omega, \rho dx)$ and $\partial\tilde{u}^{(i)}/\partial x_i = \partial_i u$.

Proof See [99, Lemma 6].

□

We get the following corollary of Lemma 4.3.

COROLLARY **4.1** *For $\chi \in C_0^\infty(\overline{U})$ and $f \in H^{1,2}(U,\mu)$, $\chi f \in H^{1,2}(U,\mu)$ and $\partial_i(\chi f) = \partial_i \chi f + \chi \partial_i f$.*

4.3 Main result on Markov Uniqueness

First, we obtain Markov Uniqueness in the special case when U is a cube.

THEOREM **4.1** *Let $U = (0,1)^d$ be the d-dimensional unit cube. Then $C_{0,Neu}^\infty(\overline{U})$ is dense in $H^{1,2}(U,\mu)$. Moreover, if $f \in H^{1,2}(U,\mu)_0$, then for any $\varepsilon > 0$, we can choose $\{f_n\}_{n\geq 1} \subset C_{0,Neu}^\infty(\overline{U})$ such that f_n converges to f in $H^{1,2}(U,\mu)$ and $supp f_n \subset \{x | dist(x, supp f) < \varepsilon\}$.*

Proof Since $(\mathcal{E}^+, H^{1,2}(U,\mu))$ is a Dirichlet form, $H^{1,2}(U,\mu)_b$ is dense in $H^{1,2}(U,\mu)$. Let $\tilde{f}$ denote a fixed μ-version of $f \in H^{1,2}(U,\mu)_b$. Extend $\tilde{f}$ to $3U := (-1,2)^d$ by reflection, i.e.,

$$\tilde{f}(x_1, x_2, \dots, x_d) = f(\phi(x_1), \phi(x_2), \dots, \phi(x_d)),$$

where $\phi : [-1,2] \to [0,1]$ is defined by

$$\phi(x) = \begin{cases} x & \text{if } x \in [0,1], \\ 2 - x & \text{if } x \in [1,2], \\ -x & \text{if } x \in [-1,0]. \end{cases}$$

Consider a fixed μ-version $\tilde{\varphi}$ of φ, extend $\tilde{\varphi}$ to $3U$ analogously, and let $d\tilde{\mu} = \tilde{\varphi}^2 dx$. By Lemma 4.3, we can see that $\tilde{f} \in H^{1,2}(3U, \tilde{\mu})_b$. Choose $g \in C_0^\infty(\mathbb{R}^d)$ with

$$g(x) = \begin{cases} 1 & if \quad x \in [-\frac{1}{4}, \frac{5}{4}]^d, \\ 0 & if \quad x \notin [-\frac{1}{2}, \frac{3}{2}]^d. \end{cases}$$

Let $\hat{f} = \tilde{f}g$, $\hat{\varphi}$ be an extension of $\tilde{\varphi}$ to $\mathbb{R}^d$ with compact support, and let $d\hat{\mu} = \hat{\varphi}^2 dx$. Then we have $\hat{f} \in H^{1,2}(\mathbb{R}^d, \hat{\mu})_{0,b}$. Define

$$f_\varepsilon(x) := \int_{3U} \eta_\varepsilon(y)\hat{f}(x-y)dy,$$

where η_ε is a standard mollifier, and $\varepsilon < \frac{1}{4}$. We want to show $f_\varepsilon \in C^\infty_{0,Neu}(\overline{U})$. Clearly, $f_\varepsilon \in C_0^\infty(\overline{U})$. Note that $g \in C^\infty_{0,Neu}(\overline{U})$, if and only if $\partial_i g|_{x_i=0} = \partial_i g|_{x_i=1} = 0$ and $g \in C_0^\infty(\overline{U})$. Define $d\widehat{y_i} := dy_i dy_1 \dots dy_{i-1} dy_{i+1} \dots dy_d$

$$\partial_i f_\varepsilon(x)|_{x_i=0} = \int_{-1}^2 \cdots \int_{-1}^2 \int_{-1}^2 \partial_i \eta_\varepsilon(x_1 - y_1, \dots, -y_i, \dots, x_d - y_d)\hat{f}(y_1, y_2, \dots, y_d) d\widehat{y_i}$$

and

$$\begin{aligned}
\int_{-1}^2 \partial_i\eta_\varepsilon(\dots)\hat{f}(\dots)dy_i &= \int_{-\varepsilon}^{\varepsilon} \partial_i\eta_\varepsilon(\dots)\hat{f}(\dots)dy_i \\
&= \int_0^{\varepsilon} \partial_i\eta_\varepsilon(\dots)\hat{f}(\dots)dy_i + \int_{-\varepsilon}^0 \partial_i\eta_\varepsilon(\dots)\hat{f}(\dots)dy_i \\
&= \int_0^{\varepsilon} \partial_i\eta_\varepsilon(x_1 - y_1, \dots, -y_i, \dots, x_d - y_n) f(y_1, \dots, y_n) dy_i \\
&- \int_{\varepsilon}^0 \partial_i\eta_\varepsilon(x_1 - y_1, \dots, y_i, \dots, x_d - y_n)\hat{f}(y_1, \dots, -y_i, \dots, y_n) dy_i \\
&= 0.
\end{aligned}$$

Silmilarily, $\partial_i f_\varepsilon(x)|_{x_i=1} = 0$, and we get the desired result. Now, the proof of [19, Theorem 2.7] shows that f_ε converges to $\hat{f}$ in $H^{1,2}(\mathbb{R}^d, \hat{\mu})$. In particular, f_ε converges to f in $H^{1,2}(U, \mu)$.

□

Next, we want to prove Markov Uniqueness for more general domains.

DEFINITION **4.2** *Define* ***partial d-polytope*** *by* $\{x | x = \sum_i^d \lambda_i \boldsymbol{v}_i,$ *either* $\lambda_i \in \mathbb{R}$ *or* $\lambda_i \in \mathbb{R}_+$ *for each* $1 \leq i \leq d$, $\boldsymbol{v}_i$*'s are linearly independent vectors with norm one*$\}$.

DEFINITION **4.3** *A partial d-polytope* V *is called* ***tessellationable*** *if we can cover* $\mathbb{R}^d$ *by copying some* V *and gluing some points only if they are copied from same point. (In particular, if* $d = 2$*, this is possible, if and only if angle at the origin* 0 *is* $\frac{\pi}{n}$ *for some natural number* n*.)*

THEOREM **4.2** *Theorem 4.1 also holds for tesselationable* U*.*

Proof Since $(\mathcal{E}^+, H^{1,2}(U,\mu))$ is a Dirichlet form, $H^{1,2}(U,\mu)_b$ is dense in $H^{1,2}(U,\mu)$. Choose $f \in H^{1,2}(U,\mu)_b$ and let $g_l \in C_0^\infty(\mathbb{R}^d)$ be such that it is 1 on $B_k(0)$, $0 \leq g_l \leq 1$ and $|\nabla g_l| \leq 1$, for $k \in \mathbb{N}$. By Lebesgue's dominated convergence theorem and Corollary 4.1, $f_l := f \cdot g_l|_U$ converges to f in $H^{1,2}(U,\mu)$. Hence $H^{1,2}(U,\mu)_{0,b}$ is dense in $H^{1,2}(U,\mu)$.

Now, the remaining parts of the proof is similar to Theorem 4.1. More precisely, we can use the reflection method on the boundary to extend a given function $f \in H^{1,2}(U,\mu)_{0,b}$ and density φ, and then use mollification method to get approximation by functions in $C^\infty_{0,Neu}(\overline{U})$.

□

DEFINITION **4.4** *Let* $k \in \{2, 3, \dots\} \cup \{\infty\}$ *and* U *be an open subset of* $\mathbb{R}^d$*. We say that* U *has a* $\mathbf{C^k}$ ***locally tesselationable boundary****, if* $\forall x \in \partial U$ *there is a tesselationable* V_x*, some open neighborhood* U_x *of* x*,* $\delta_x > 0$*, and there exists* $\psi_x : \overline{U}_x \to \overline{B}_{\delta_x}(0)$*, such that* *(i)* ψ_x *is a* C^k *diffeomorphism,*

(ii) $\psi_x(x) = 0$,

(iii) $B_{\delta_x}(0) \cap V_x = \psi_x(U \cap U_x)$,

(iv) $B_{\delta_x}(0) \cap \partial V_x = \psi_x(\partial U \cap U_x)$.

For $k \in \{0, 1, \dots\} \cup \{\infty\}$, *let* $C^{k,1}(\overline{U}_x)$ *denote the set of all* k *times continuously differentiable functions* f *(resp. the set of continuous functions* f *if* $k = 0$*) such that* $\partial_1^{\alpha_1} \dots \partial_d^{\alpha_d} f$, $\sum_{i=1}^d \alpha_i = k$ *and* $\alpha_i \in \mathbb{N} \cup \{0\}$, *is Lipschitz continuous. If* ψ_x, ψ_x^{-1} *above can be chosen to be of class* $C^{k,1}$, *we say that* U *has a* $C^{k,1}$ *locally tesselationable boundary.*

Since a half-space in $\mathbb{R}^d$ is tesselationable, the following definition is a special case of Definition 4.4.

DEFINITION **4.5** *Let* $k \in \{2, 3, \dots\} \cup \{\infty\}$. *A possibly unbounded open set* U *in* $\mathbb{R}^d$ *is said to have a* C^k ***boundary****, if for all* $x \in \partial U$, *there exists* $\delta_x > 0$, *an open neighborhood* U_x *of* x, *and* $\psi_x : \overline{U}_x \to \overline{B}_{\delta_x(0)}$ *such that*

(i) ψ_x *is a* C^k *diffeomorphism,*

(ii) $\psi_x(x) = 0$,

(iii) $B_{\delta_x}(0) \cap \{x_d > 0\} = \psi_x(U \cap U_x)$,

(iv) $B_{\delta_x}(0) \cap \{x_d = 0\} = \psi_x(\partial U \cap U_x)$.

For $k \in \{0, 1, \dots\} \cup \{\infty\}$, *if* ψ_x, ψ_x^{-1} *can be chosen to be of class* $C^{k,1}$, *we say* U *has a* $C^{k,1}$ *boundary.*

LEMMA **4.4** *Let Ω_1, Ω_2 be bounded open subsets of $\mathbb{R}^d$. Let $F = (F_1, \dots, F_d) : \overline{\Omega}_1 \to \overline{\Omega}_2$ be a one-to-one mapping which is Lipschitz continuous together with its inverse $F^{-1} : F(\overline{\Omega}_1) \to \Omega_1$. Let $\psi \in H^{1,2}(\Omega_2)$, $\psi > 0$ dx-a.e. Then $\psi(F) \in H^{1,2}(\Omega_1)$ and for all $g \in H^{1,2}(\Omega_2, \psi^2 dx)$, $g(F) \in H^{1,2}(\Omega_1, \psi(F)^2 dx)$. Moreover, $\partial_i g(F) = \Sigma_{k=1}^d (\partial_k g \circ F)\partial_i F_k$ a.e. on Ω_1.*

Proof By [36, Lemma 1.3.3.1], we have $\psi(F) \in H^{1,2}(\Omega_1)$. Then we can show $C^\infty(\Omega_2) \cap H^{1,2}(\Omega_2, \psi^2 dx)$ is dense in $H^{1,2}(\Omega_2)$ analogously to [35, Lemma 3.3.3]
(see [19, Theorem 2.7] also). Since F is Lipschitz continuous, it maps Lebesgue zero sets to Lebesgue zero sets. Assume $g \in C^\infty(\Omega_2) \cap H^{1,2}(\Omega_2, \psi^2 dx)$ first. The last assertion for this g follows easily from elementary calculation. Then $g(F) \in H^{1,2}(\Omega_1, \psi(F)^2 dx)$ by change of variable (see [87, Theorem 7.26] for example).
Now, assume $g \in H^{1,2}(U_2, \psi^2 dx)$ and choose $g_l \in C^\infty(\Omega_2) \cap H^{1,2}(\Omega_2, \psi^2 dx)$ converging to g. Then we can easily see that $g_l(F)$ (resp. $\partial_i g_l(F)$) converges to $g(F)$ (resp. $\Sigma_{k=1}^d (\partial_k g \circ F)\partial_i F_k$) in $L^2(\Omega_1, \psi(F)^2 dx)$ by change of variable. Now we can see that $g(F)$ satisfies the equation in Definition 4.1 with $\partial_i g(F) = \Sigma_{k=1}^d (\partial_k g \circ F)\partial_i F_k$ and the result follows.

□

We are ready to prove Markov uniqueness result.

THEOREM **4.3** *Assume that U is an open set in $\mathbb{R}^d$ with C^k locally tesselationable boundary for some $k \in \{2, 3, \dots\} \cup \{\infty\}$. Assume further that for all $x \in \partial U$, the diffeomorphism ψ_x of Definition 4.4 maps the normal vector at $y \in \partial U \cap U_x$ to the normal vector at $\psi_x(y) \in \partial V_x \cap B_{\delta_x}(0)$ for $\varphi^2\, d\sigma$-a.e. $y \in \partial U \cap U_x$. Then $(L, C^k_{0,Neu}(\overline{U}))$ is Markov unique.*

Proof Choose $f \in H^{1,2}(U,\mu)$ and let $g_l \in C_0^\infty(\mathbb{R}^d)$ be such that it is 1 on $B_l(0)$, $0 \le g_l \le 1$ and $|\nabla g_l| \le 1$, for $l \in \mathbb{N}$. By Lebesgue's dominated convergence theorem and Corollary 4.1, $f_l := f \cdot g_l|_U$ converges to f in $H^{1,2}(U,\mu)$ and have bounded supports. Hence $H^{1,2}(U,\mu)_0$ is dense in $H^{1,2}(U,\mu)$.

Choose $f \in H^{1,2}(U,\mu)_0$. For $x \in K := \partial U \cap \mathrm{supp} f$ let U_x, ψ_x be defined as in Definition 4.4. Since $(U_x)_{x \in K}$ is an open cover of the compact set K, we can find a finite subcover $(U_{y^i})_{i=1,\dots,m}$. Then the two compact sets $\mathrm{supp} f \setminus \cup_{i=1}^m U_{y^i}$ and $\mathrm{supp} f \cap \partial U$ are disjoint and have hence positive distance to each other. Therefore, we can choose a bounded open set O such that $\overline{O} \subset U$ and $\{O, U_{y^1}, \dots, U_{y^m}\}$ is an open cover of $\mathrm{supp} f$. Choose a partition of unity $(\zeta_i)_{i=0,\dots,m} \subset C_0^\infty(\mathbb{R}^d)$ subordinate to $\{O, U_{y^1}, \dots, U_{y^m}\}$, i.e. $\mathrm{supp}\zeta_0 \subset O$, $\mathrm{supp}\zeta_i \subset U_{y^i}$, $1 \le i \le m$ and $\sum_{i=0}^m \zeta_i = 1$ on $\mathrm{supp} f$. Now, let $\bar{\varphi}_i := \varphi(\psi_{y^i}^{-1})$, $i = 1, \dots, m$. Then $\bar{\varphi}_i \in H^{1,2}(\{x_d > 0\} \cap B_{\delta_{y^i}}(0))$. Let $f_i := \zeta_i f$, $i = 0, \dots, m$. Then $f_0 \in H^{1,2}(O \cap U, \mu)$, $f_i \in H^{1,2}(U_{y^i} \cap U, \mu)$, $i = 1, \dots, m$, and $f_i(\psi_{y^i}^{-1}) \in H^{1,2}(\{x_d > 0\} \cap B_{\delta_{y^i}(0)}, \bar{\varphi}^2 dx)$ by Lemma 4.4. By Remark 4.2 we can choose $g_l^i \in C_{0,Neu}^\infty(\{x_d \ge 0\} \cap \overline{B}_{\delta_{y^i}}(0))$ which converge to $f_i(\psi_{y^i}^{-1})$ for $1 \le i \le m$. Note that $g_l^i(\psi_{y^i}) \in C_{0,Neu}^k(\overline{U})$ since ψ_{y^i} preserves normal vectors by assumption. Moreover, $g_l^i(\psi_{y^i}) \to f_i$ in $H^{1,2}(U,\mu)$, $1 \le i \le m$, by change of variable. By [19, Theorem 2.7] we can find a sequence $(g_l)_{l \ge 1} \subset C_0^\infty(O) \subset C_{0,Neu}^k(\overline{U})$ which converges to f_0 in $H^{1,2}(U,\mu)$. Thus

$$g_l + \sum_{i=1}^m g_l^i(\psi_{y^i}) \to f_0 + \sum_{i=1}^m f_i = f$$

in $H^{1,2}(U,\mu)$. Therefore $C_{0,Neu}^k(\overline{U})$ is dense in $H^{1,2}(U,\mu)$.

□

Using the technique of Theorem 4.3, we can show the following two corollaries.

COROLLARY **4.2** *Let $U \subset \mathbb{R}^d$ be an open subset with a $C^{0,1}$ boundary. Then $f \in$*

$H^{1,2}(U,\mu)_0$ can be extended to $\hat{f} \in H^{1,2}(\mathbb{R}^d, \hat{\varphi}^2 dx)_0$, where $\hat{\varphi} \in H^{1,2}(\mathbb{R}^d)_0$ is an extension of $\varphi_{|suppf}$ to $\mathbb{R}^d$.

Proof Choose $f \in H^{1,2}(U,\mu)_0$. Take a partition of unity (ζ_0, O), $(\zeta_i, U_{y_i})_{i=1,\ldots,m}$ as in Theorem 4.3. For $i = 1,\ldots,m$, let $f_i = \zeta_i f$ and extend $\varphi(\psi_{y^i}^{-1})$, $f_i(\psi_{y^i}^{-1})$ by reflection to the whole ball $B_{\delta_{y^i}}(0)$ such that the support of $f_i(\psi_{y^i}^{-1})$ is a compact set contained in the ball $B_{\delta_{y_i}}(0)$ as in Theorem 4.1 and call it as $\tilde{\varphi}_i$, $\tilde{f}_i$. Choose a bounded open Ω containing the closure of $O \cup_{i=1}^m U_m$ such that $\partial\Omega$ is locally the graph of a Lipchitz function and a cut-off function $\chi \in C_0^\infty(\Omega)$ such that $\chi = 1$ on $O \cup_{i=1}^m U_m$. Then we can extend $\zeta_0\varphi + \sum_{i=1}^m \zeta_i \tilde{\varphi}(\psi_{y^i})$ to a function $\tilde{\varphi} \in H^{1,2}(O \cup_{i=1}^m U_m)$. Finally $\hat{\varphi} := \chi\tilde{\varphi}$ and $\hat{f} := \zeta_0 f + \sum_{i=1}^m f_i(\psi_{y^i})$ are the desired functions by Lemma 4.4.

□

REMARK **4.1** *Assume $U \subset \mathbb{R}^d$ be an open subset with a $C^{0,1}$ boundary. Then $C_0^\infty(\overline{U})$ is dense in $H^{1,2}(U,\mu)$.*

Proof By the proof of Theorem 4.3, we can see that $H^{1,2}(U,\mu)_0$ is dense in $H^{1,2}(U,\mu)$. Now choose arbitrary $f \in H^{1,2}(U,\mu)_0$ and extend it to $\hat{f} \in H^{1,2}(\mathbb{R}^d, \hat{\varphi}^2 dx)_0$ by Corollary 4.2. Now we get the desired result by [19, Theorem 2.7].

□

The condition in Theorem 4.3 that "ψ maps a normal vector on $\partial U \cap U_x$ to a normal vector on $\partial V \cap B_{\delta_x}(0)$ dx-a.e." is a quite strong assumption. For the rest of this section we will consider some specific cases where this is possible. At first, we will deal with C^k-boundaries, $k \geq 2$.

LEMMA **4.5** *Let $\mathbb{R}^d_+$ be the open half-space with $x_d > 0$ and $F = (F_1, F_2, \ldots, F_d) : \partial\mathbb{R}^d_+ \to \mathbb{R}^d$ be of class C^k, $F_i = F_i(x_1, x_2, \ldots, x_{d-1})$ for $1 \le i \le d$, and $F_d \neq 0$ on some neighborhood of zero. Then there exist open neighborhoods V_1, V_2 of zero in $\mathbb{R}^d$, a C^{k+1}-diffeomorphism $\phi = (\phi_1, ..., \phi_d)$ from V_1 to V_2 satisfying $(i) - (iv)$ of Definition 4.5 with $x = 0$, such that*

$$\begin{pmatrix} \frac{\partial\phi_1}{\partial x_1} \cdots \frac{\partial\phi_1}{\partial x_d} \\ \vdots \ddots \vdots \\ \frac{\partial\phi_d}{\partial x_1} \cdots \frac{\partial\phi_d}{\partial x_d} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ F_d \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_{d-1} \\ F_d \end{pmatrix} \text{ on } V_1 \cap \partial\mathbb{R}^d_+ \tag{4.4}$$

Proof Using [105, Theorem I], we can extend F_i to a function $\tilde{F}_i = \tilde{F}_i(x_1, x_2, \ldots, x_d)$ on $\mathbb{R}^d$, $1 \le i \le d$ such that it is C^k on $\partial\mathbb{R}^d_+$ and C^∞ outside. Let $\phi_i(x_1, x_2, \ldots, x_d) = \int_0^{x_d} \frac{\tilde{F}_i(x_1, \ldots, x_{d-1}, y_d)}{\tilde{F}_d(x_1, \ldots, x_{d-1}, y_d)} dy_d + x_i$ for $1 \le i \le d-1$ and $\phi_d(x_1, x_2, \ldots, x_d) = x_d$ on some neighborhood of zero, where $\tilde{F}_d$ is nonzero there. One can check $\det D\phi(0) \neq 0$. Thus, by the Inverse-function theorem, we can find some open neighborhood V_1, V_2 of 0 in $\mathbb{R}^d$ and $\phi := (\phi_1, \ldots, \phi_d)$ is C^{k+1}-diffeomorphism there. It is easy to see that ϕ satisfies $(i) - (iv)$ of Definition 4.5 with $x = 0$, and satisfies (4.4).

□

COROLLARY **4.3** *Let $k \in \{2, 3, 4, ...\} \cup \{\infty\}$ and U be an open set in $\mathbb{R}^d$ with C^k boundary. Then $(L, C^k_{0,Neu}(\overline{U}))$ is Markov unique.*

Proof The normal vector is locally defined on $\partial U \cap U_x$ by $\frac{\nabla[(\psi_x)_d]}{|\nabla[(\psi_x)_d]|}$, where ψ_x is any function as in Definition 4.5. Let $f \in C^\infty(\overline{B}_{\delta_x(0)} \cap \{x_d \ge 0\})$ whose normal derivative is zero on $B_{\delta_x(0)} \cap \{x_d = 0\}$. Then $f(\psi)$ has a zero normal derivative on

$\partial U \cap U_x$, if and only if $(\nabla f)(\psi) \begin{pmatrix} \frac{\partial \psi_1}{\partial x_1} \cdots \frac{\partial \psi_1}{\partial x_d} \\ \vdots \ddots \vdots \\ \frac{\partial \psi_d}{\partial x_1} \cdots \frac{\partial \psi_d}{\partial x_d} \end{pmatrix} \begin{pmatrix} \frac{\partial \psi_d}{\partial x_1} \\ \vdots \\ \frac{\partial \psi_d}{\partial x_d} \end{pmatrix} = 0$. Since d^{th} component of $\begin{pmatrix} \frac{\partial \psi_1}{\partial x_1} \cdots \frac{\partial \psi_1}{\partial x_d} \\ \vdots \ddots \vdots \\ \frac{\partial \psi_d}{\partial x_1} \cdots \frac{\partial \psi_d}{\partial x_d} \end{pmatrix} \begin{pmatrix} \frac{\partial \psi_d}{\partial x_1} \\ \vdots \\ \frac{\partial \psi_d}{\partial x_d} \end{pmatrix}$ is positive and using Lemma 4.5 with $F_i = (\sum_{k=1}^d \partial_k \psi_i \partial_k \psi_d) \circ \psi^{-1}$, we can assume $\begin{pmatrix} \frac{\partial \psi_1}{\partial x_1} \cdots \frac{\partial \psi_1}{\partial x_d} \\ \vdots \ddots \vdots \\ \frac{\partial \psi_d}{\partial x_1} \cdots \frac{\partial \psi_d}{\partial x_d} \end{pmatrix} \begin{pmatrix} \frac{\partial \psi_d}{\partial x_1} \\ \vdots \\ \frac{\partial \psi_d}{\partial x_d} \end{pmatrix}$ is a normal vector on $\partial U \cap U_x$.
Now, the remaining part of the proof follows from Theorem 4.3.

□

Now we will deal with C^2 locally tesselationable boundary case in $\mathbb{R}^2$.

LEMMA **4.6** *Let f be a real-valued C^m function defined on the hyperspace $A := \{x \in \mathbb{R}^d | x_d = 0\}$, where m is a nonnegative integer and $I \subset \{1, 2, \ldots, d-1\}$. We assume that $f \equiv 0$ on some of the $d-2$ dimensional spaces $A_i = \{x \in A | x_i = 0\}$, $i \in I$. For $x \in \mathbb{R}^d$, let Px be the orthogonal projection of x onto A. Then we can extend f to a function g which is defined on the whole space such that g is C^∞ on A^c. Moreover, $g(x) = 0$, if $Px \in A_i$.*

Proof The proof is attained from classical result, [105, Theoreom I]. However, we have to modify this proof a little bit to attain additional condition that g is constantly 0 if $Px \in A_i$. From now on, we will use the notation from [105] for this proof with $E = \mathbb{R}^d$. Clearly, $f(x)$ is of class C^m in A in terms of $f_k(x)$, $\sigma_k \le m$, with $f_k(x) = D_k f(x)$ if $k_d = 0$, $f_l(x) = 0$ if $k_d \ne 0$. When we divide E into n-cubes of side 1 ([105, page 67, 8.]), we will take a special cubes, which are of the form $I_1 \times I_2 \times \cdots \times I_d$, where

$I_j = [N_j, N_j + 1]$, where N_j's are integers.

If $Px \in A_i$ and $y \in A_i$, we check $\psi(x; y) = \sum_{\sigma_l \le m} \dfrac{f_l(y)}{l!}(y-x)^l = 0$ by using the fact $f_l = 0$ for $l_i = 0$, $x_i = y_i = 0$ for $l_i \neq 0$. Since $\phi_\eta(x) \neq 0$ iff $x \in I_\eta - B_\eta$, if $Px \in A_i$ and $x^\eta \in A \backslash A_i$, then $\phi_\eta(x) = \pi_\eta(x) = 0$. Hence, $g(x) = \sum_\eta \phi_\eta(x)\psi(x; x^\eta) = 0$ if $Px \in A_i$.

□

LEMMA **4.7** *Let $V \subset \mathbb{R}^2$ be the defined by $V := \{x \in \mathbb{R}^2 | x = \lambda_1 \boldsymbol{v}_1 + \lambda_2 \boldsymbol{v}_2,\ \lambda_i \ge 0,\ i = 1, 2\}$, where $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly independent unit vectors. Assume there exists $V_0 \subset \mathbb{R}^d$ which is a neighborhood of 0 satisfying following properties.*

For $i = 1, 2$, $F^i : C_i (:= \mathbb{R}\boldsymbol{v}_i) \to \mathbb{R}^2$ is of class C^k with $\langle \eta_i, F^i \rangle \neq 0$ on $C_i \cap V_0$, where η_i is a inward normal vector which is orthogonal to $\boldsymbol{v}_i$. Assume further that $\langle \boldsymbol{v}_i, F^i \rangle = 0$ at 0. Then there exists an open set U_0 and a C^{k+1}-diffeomorphism ϕ as in Definition 4.4 such that U_0 is an open neighborhood of 0 and ϕ is a diffeomorphism of U_0 into V and $\nabla\phi$ maps F^i into normal vector (possibly multiplied by scalar function) on $C_i \cap U_0, i = 1, 2$.

Proof Using Lemma 4.6, we can extend F^1 to a C^k function $\tilde{F^1} = \tilde{F}^1(x_1, x_2, \dots, x_d)$ on $\mathbb{R}^d$ such that it is infinitely differentiable outside C_1, $\langle \tilde{F}^1, \mathbf{v}_1 \rangle = 0$ on $C_2 \cap V_0$. By shrinking to a subset if necessary, we can assume $\langle \eta_i, \tilde{F^i} \rangle \neq 0$ is nonzero on V_0. Note that every $x \in \mathbb{R}^d$ can be uniquely written in the form $x = \sum_{k=1}^2 \lambda_{k,x} \mathbf{v}_k$, we will write $[x, \mathbf{v}_k] := \lambda_{k,x}$ for notational convenience. Let

$$[\phi^1, \mathbf{v}_1](x) =$$

$$\cos(\eta_1, \mathbf{v}_2) \int_0^{[x,\mathbf{v}_2]} \frac{[\tilde{F}^1, \mathbf{v}_1]([x,\mathbf{v}_1]\mathbf{v}_1 + [y,\mathbf{v}_2]\mathbf{v}_2)}{\langle \eta_1, \tilde{F^1} \rangle([x,\mathbf{v}_1]\mathbf{v}_1 + [y,\mathbf{v}_2]\mathbf{v}_2)\cos(\eta_2, \mathbf{v}_1) - [\tilde{F}^1, \mathbf{v}_1]([x,\mathbf{v}_1]\mathbf{v}_1 + [y,\mathbf{v}_2]\mathbf{v}_2)\cos(\eta_2, \eta_1)} d[y, \mathbf{v}_2] + [x, \mathbf{v}_1],$$

and

$$[\phi^1, \mathbf{v}_2] = \int_0^{[x,\mathbf{v}_2]} \frac{\langle \eta_1, \tilde{F}^1\rangle([x,\mathbf{v}_1]\mathbf{v}_1+[y,\mathbf{v}_2]\mathbf{v}_2)}{\langle \eta_1, \tilde{F}^1\rangle([x,\mathbf{v}_1]\mathbf{v}_1+[y,\mathbf{v}_2]\mathbf{v}_2)\cos(\eta_2,\mathbf{v}_1)-[\tilde{F}^1,\mathbf{v}_1]([x,\mathbf{v}_1]\mathbf{v}_1+[y,\mathbf{v}_2]\mathbf{v}_2)\cos(\eta_2,\eta_1)} d[y,\mathbf{v}_2]$$

on V_0. We can see that $\nabla\phi^1$ preserves normal vectors (possibly multiplied by scalar function) on $C_2 \cap V_0$, and also preserves the regularity of F^2. We can also see that $\nabla\phi^1$ maps η_1 (possibly multiplied by scalar function) into F^1 on $C_1 \cap V_0$, and is of class C^{k+1}. Since $\nabla\phi$ is invertible at 0, we can apply inverse function theorem. We can define ϕ^2 similarily and using composition with ϕ^1, we get the desired result.

□

Let $U \subset \mathbb{R}^d$ be an open set with C^k locally tesselationable boundary. We say that $x \in \partial U$ is a **singular point**, if for any open neighborhood U_x of x in ∂U, U_x is not the graph of a C^k function. Note that there are two natural tangent vectors at a singular point x, and we define angle at x be the angle of two natural tangent vectors.

COROLLARY **4.4** *Assume that* U *is an open set in* $\mathbb{R}^2$ *with* C^2 *locally tesselationable boundary. Assume* $\nabla\psi$ *defined in Definition 4.4 preserves angle at singular points. Then* $(S, D(S))$ *is Markov unique,* $k \geq 2$*. Moreover, if boundary of* U *is of class* C^k*, we get* $(L, C^k_{0,Neu}(\overline{U}))$ *is Markov unique,* $k \geq 2$*.*

Proof For a singular point $x \in \partial U$, choose a ψ_x as in Definition 4.4. Let $E_{i,x}$ be the set which is mapped into C_i by ψ_x, where C_i is as in Lemma 4.7. We can naturally define normal vector $\eta_{i,x}$ on $E_{i,x}$, and let $F^i := \nabla\psi_x \ \eta_{i,x}$(we can multiply cut-off function so that we can assume F^i is of C^{k-1} on C_i). If $\nabla\psi$ preserves the angle between two vectors in some plane, it also preserves all angles between two lines who lie on that plane. Hence $\langle \mathbf{v}_i, F^i\rangle = 0$ at 0 is nothing but $\nabla\psi_x$ preserves the angle between normal vectors on $E_{1,x}$ and $E_{2,x}$ which is same with angle at x. Now remaining parts of the proof is analogous to Theorem 4.3.

□

REMARK **4.2** *On the previous Corollary, we infer that if $\nabla\psi_x$ preserves normal vectors near x, it also preserves the angle at x. Hence previous Corollary is maximal result. From this, we can imagine the following conjecture.*

(**Conjecture 1**) Let V be tesselationable, and let I be the set of indices i, where $\lambda \in \mathbb{R}_+$ (note ∂V consists of $d-1$ dimensional spaces $C_i := \{x \in \partial V | x = \sum_{j\neq i}^{d} \lambda_j \mathbf{v}_j,\ \lambda_j \in \mathbb{R}$ if $j \notin I, \lambda_j \in \mathbb{R}_+$ otherwise, for each $1 \leq j \leq d, j \neq i\}$, $i \in I$.) Assume there exists $V_0 \subset \mathbb{R}^d$ which is a neighborhood of 0 satisfying following properties.
Assume for each $i \in I$, $F^i = (F_1^i, F_2^i, \ldots, F_d^i) : C_i \to \mathbb{R}^d$ is of class C^k(it means, F^i is restriction of C^k function defined on the hyperspace containing C_i.) with $\langle \eta_i, F^i \rangle \neq 0$ on $C_i \cap V_0$, where η_i is a normal vector which is orthogonal to $\mathbf{v}_j$ for all $j \neq i$. Assume further that for each point $x \in \partial C_i \cap \partial C_j \cap V_0$, for some $j \neq i \in I$, $\langle \mathbf{v}_j, F^i \rangle = 0$. Then there exists an open set U_0 and a C^{k+1}-diffeomorphism ϕ as in Definition 4.4 such that U_0 is an open neighborhood of 0 and ϕ is a diffeomorphism of U_0 into V and $\nabla\phi$ maps F^i into normal vector(possibly multiplied by scalar function) on $C_i \cap U_0$.
If this conjecture is true, we can show the following conjecture.
(**Conjecture 2**) Assume that U is an open set in $\mathbb{R}^d$ with C^k locally tesselationable boundary. For a singular point $x \in \partial U$, choose a ψ_x as in Definition 4.4. Let $E_{i,x}$ be the set which is mapped into C_i by ψ_x, where C_i is as in Conjecture 1. If ψ_x preserves the angle between normal vectors on $E_{i,x}$ and $E_{j,x}$, $i,j \in I$, then $(L, C^k_{0,Neu}(\overline{U}))$ is Markov unique.

REMARK **4.3** *Let $k \geq 2$ be an integer. Assume that U is an open set in $\mathbb{R}^d$ with*

C^k locally tesselationable boundary. Assume $A = (a_{ij})_{1\le i,j\le d}$ is a matrix of real-valued measurable functions on U. For each $x \in \partial U$, let ψ_x be as in Definition 4.4, $E_{i,x}$ be the set which is mapped into C_i by ψ_x, where C_i is as in Conjecture 1, and let n_i be the normal vector on U_x corresponding to η_i. We assume that for each $x \in \partial U$, $\nabla \psi_x A^T n_i = \eta_i$ on U_x, $i \in I$ for some appropriate (ψ_x, U_x). Then we can use the reflection method of Theorem 4.3 to cover oblique Neumann boundary conditions, that is, the directional derivative with respect to $A^T\eta$ is 0 on ∂U.
In particular,

$$C^k_{0,ObNeu}(\overline{U}) := \{f \in C^k_0(\overline{U}) \,|\, \langle A^T\eta, \nabla f\rangle = 0 \ \varphi^2 d\sigma\text{-a.e. on } \partial U\}.$$

is dense in $H^{1,2}(U,\mu)_0$.

Note that under the assumption of following proposition, the condition of Remark 4.3 is satisfied.

PROPOSITION **4.1** *Let $k \ge 2$ be an integer, and assume $A = (a_{ij})_{1\le i,j\le d}$ is a matrix of real-valued measurable functions on $\overline{U}$, where U has a C^k boundary and*

$$a_{ij} \in C^{k-1}(\partial U) := \{f : f(\psi_x^{-1}) \in C^{k-1}(B_{\delta_x}(0) \cap \{x_d = 0\}) \text{ for each } x \in \partial U\},$$

with ψ_x defined as in Definition 4.5. Assume further that

$$0 < \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \qquad \forall \xi = (\xi_1,\dots,\xi_d) \in \mathbb{R}^d,\ x \in \partial U. \tag{4.5}$$

Assume also that Then

$$C^k_{0,ObNeu}(\overline{U}) := \{f \in C^k_0(\overline{U}) \,|\, \langle A^T\eta, \nabla f\rangle = 0 \ \varphi^2 d\sigma\text{-a.e. on } \partial U\}$$

is dense in $H^{1,2}(U,\mu)_0$.

Proof Almost same with Corollary 4.3. Just note that $\langle A^T\eta, \nabla f\rangle = 0$ on $\partial U \cap U_x$ iff
$Df \begin{pmatrix} \frac{\partial\psi_1}{\partial x_1} \cdots \frac{\partial\psi_1}{\partial x_d} \\ \vdots \ddots \vdots \\ \frac{\partial\psi_d}{\partial x_1} \cdots \frac{\partial\psi_d}{\partial x_d} \end{pmatrix} A^T \begin{pmatrix} \frac{\partial\psi_d}{\partial x_1} \\ \vdots \\ \frac{\partial\psi_d}{\partial x_d} \end{pmatrix} = 0$, where U_x is as in Theorem 4.3.

Note also that d^{th} component of $\begin{pmatrix} \frac{\partial\psi_1}{\partial x_1} \cdots \frac{\partial\psi_1}{\partial x_d} \\ \vdots \ddots \vdots \\ \frac{\partial\psi_d}{\partial x_1} \cdots \frac{\partial\psi_d}{\partial x_d} \end{pmatrix} A^T \begin{pmatrix} \frac{\partial\psi_d}{\partial x_1} \\ \vdots \\ \frac{\partial\psi_d}{\partial x_d} \end{pmatrix}$ is always positive by (4.5).

□

4.4 L^2-uniqueness

Let U and A satisfy the condition in Remark 4.3, and $d \geq 2$, and let

$$\bar{A} := (\bar{a}_{ij})_{i,j=1}^d, \quad \bar{a}_{ij} := \frac{a_{ij} + a_{ji}}{2}, \qquad \check{A} := (\check{a}_{ij})_{i,j=1}^d, \quad \check{a}_{ij} := \frac{a_{ij} - a_{ji}}{2}.$$

We assume further that $a_{ij} \in C_{loc}^{0,1}(\overline{U})$ and for any bounded $V \subset U$ there exist constants $K, L, \lambda_V \in (0,\infty)$ with

$$\lambda_V^{-1}|\xi|^2 \leq \sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \leq \lambda_V|\xi|^2 \qquad \forall \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d,\ \mu\text{-a.e. } x \in V \tag{4.6}$$

and for $f,\ g \in C_0^\infty(\overline{U})$,

$$\int_U \langle \check{A}\nabla f, \nabla g\rangle d\mu \leq K\mathcal{E}_1(f,f)^{1/2}\mathcal{E}_1(g,g)^{1/2} \tag{4.7}$$

In particular, (4.7) is satisfied if

$$\max_{1\leq i,j\leq d} \|\check{a}_{ij}\|_{L^\infty(V,\mu)} \leq L\lambda_V^{-1}. \tag{4.8}$$

We also assume that $B = (b_1, \dots, b_d)$, with

$$b_i \in L_{loc}^\infty(U,\mu) \tag{4.9}$$

satisfy

$$\int_U \langle B, \nabla u\rangle \, d\mu = 0 \quad \forall u \in C_0^\infty(\overline{U}). \tag{4.10}$$

Note that (4.10) implies

$$2\int_U \langle B, \nabla u\rangle u \, d\mu = \int_U \langle B, \nabla u^2\rangle \, d\mu = 0 \quad \forall u \in H^{1,2}(U)_0. \tag{4.11}$$

On this section, we define a closable form.

$$\mathcal{E}(f,g) := \frac{1}{2}\int_U \langle A\nabla f, \nabla g\rangle d\mu - \int_U \langle B, \nabla f\rangle g \, d\mu, \quad f, g \in C_0^\infty(\overline{U}).$$

Note that closablity of $(\mathcal{E}, C_0^\infty(\overline{U}))$ is equivalent with closability of its symmetric part $(\tilde{\mathcal{E}}, C_0^\infty(\overline{U}))$. Moreover, closability of the latter form is equivalent with closability of $(\tilde{\mathcal{E}}, \{f \in C_0^2(\overline{U}) \,|\, \langle \bar{A}^T\eta, \nabla f\rangle = 0 \ \varphi^2 d\sigma\text{-a.e. on } \partial U\})$ by previous section. But we know that the last form is closable by [72, Proposition 3.3].

We also assume that $(\mathcal{E}, C_0^\infty(\overline{U}))$ can be uniquely extended to a Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$. Let $(L, D(L))$ be a generator of the Dirichlet form.

LEMMA **4.8** $D(\mathcal{E})_0 = H^{1,2}(U,\mu)_0$.

Proof Let $f \in D(\mathcal{E})_0$ and $(f_n)_{n\geq 1} \in C_0^\infty(\overline{U})$ which converges to f in $D(\mathcal{E})$. Take $\chi \in C_0^\infty(\overline{U})$ which is 1 on the support of f. By direct calculation using (4.6), we can see that $\sup_{n\geq 1}\big(\frac{1}{2}\int_U \langle \nabla(\chi f_n), \nabla(\chi f_n)\rangle d\mu\big) < \infty$. By [72, I. Lemma 2.12], we have $f \in H^{1,2}(U,\mu)_0$ and $D(\mathcal{E})_0 \subset H^{1,2}(U,\mu)_0$. The converse inclusion also holds by similar reason.

□

Now we will see partition of unity and their corollaries.

For the following lemma, we used the technique as in [86, Theorem 10.8].

LEMMA **4.9** *(Partition of Unity with Neumann boundary condition) Suppose K is a compact subset of $\overline{U}$, and $\{U_\alpha\} \subset \mathbb{R}^d$ is an open cover of K. For the sake of simplicity, we further assume that U_α is either a domain of diffeomorphsim in Definition 4.4 or contained in U. Then there exist functions $\zeta_1, \dots, \zeta_n \in C_0^2(\overline{U})$ such that*

$$
\begin{aligned}
&(a) \quad 0 \le \zeta_i \le 1 \text{ for } 1 \le i \le n, \\
&(b) \quad supp\zeta_i \subset U_\alpha \text{ for some } \alpha, \\
&(c) \quad \text{Sum of } \zeta_i(x) \text{ is } 1 \text{ for every } x \in K, \text{ and} \\
&(d) \quad \langle A^T\eta, \nabla\zeta\rangle = 0 \ \varphi^2 d\sigma\text{-a.e. on } \partial U.
\end{aligned}
$$

Proof Associate with each $x \in K$ an index $\alpha(x)$ so that $x \in U_{\alpha(x)}$. Then there are open balls $B_k(x)$, $1 \le k \le 3$, centered at x, with

$$\overline{B}_1(x) \subset B_2(x) \subset \overline{B}_2(x) \subset B_3(x) \subset \overline{B}_3(x) \subset U_{\alpha(x)}. \tag{4.12}$$

Since K is compact, there are points $x_1, \dots, x_d$ in K such that K is contained in the union of $B_1(x_i)$.

We define a measurable function $\zeta_i^0 : \overline{U} \to \mathbb{R}$ such that $\zeta_i^0(x) = 1$ on $B_2(x_i) \cap \overline{U}$ and 0 otherwise. Using reflection method as in Theorem 4.1 and Corollary 4.3, and take ε small enough when we do mollification with standard mollifier η_ε so that we can find ζ_i^1 satisfying $\zeta_i^1(x) = 1$ on $B_1(x_i) \cap \overline{U}$, $\zeta_i^1(x) = 0$ on $\overline{U} \backslash B_3(x_i)$, $0 \le \zeta_i^0(x) \le 1$, and satisfies condition (d). This is possible by (4.12). Now, remainder of proof follows from usual argument (see [86, Theorem 10.8]).

□

COROLLARY **4.5** *For any bounded $V \subset U$, there exists $\chi \in C^2_{0,Neu}(\overline{U})$ such that χ is 1 on V, and $0 \le \chi \le 1$ on U. Note that if ∂U is of class C^k locally tesselationable, $k \ge 2$ is an integer, we can take $\chi \in C^k_{0,Neu}(\overline{U})$.*

LEMMA **4.10** *If $h : U \to \mathbb{R}$ be a measurable function such that $\chi h \in H^{1,2}(U,\mu)_0$ for all $\chi \in C^2_{0,Neu}(\overline{U})$, then $\psi h \in H^{1,2}(U,\mu)_0$ for all $\psi \in C_0^\infty(\overline{U})$.*

Proof For arbitrary $\psi \in C_0^\infty(\overline{\Omega})$, choose $\chi \in C^2_{0,Neu}(\overline{\Omega})$ which is 1 on the support of ψ. This is possible by Corollary 4.5. Then $\psi h = \chi\psi h = \underbrace{\psi \underbrace{\chi h}_{\in H^{1,2}(U,\mu)_0}}_{\in H^{1,2}(U,\mu)_0 \text{ by } Corollary 4.1.}$.

□

Before we prove L^2 uniqueness, we will see some regularity result for certain type of weak solutions.

LEMMA **4.11** *Assume $\varphi(x) = 1$ for all $x \in U$. We assume $U \subset \mathbb{R}^d$ be an open set with $C^{1,1}$-boundary. Then $D(L) \subset H^{2,2}_{loc}(U)$, where*

$$H^{2,2}_{loc}(U) := \{f | f\chi \in H^{2,2}(U) \text{ for all } \chi \in C^2_{0,ObNeu}(\overline{U})\}.$$

Moreover, we can see

$$D(L) = H^{1,2}(U) \cap H^{2,2}_{loc,ObNeu}(U), where$$

$$H^{2,2}_{loc,ObNeu}(U) := \{f \in H^{2,2}_{loc}(U) | \langle A^T\eta, \nabla f\rangle = 0 \ \varphi^2 d\sigma\text{-}a.e. \ on \ \partial U\},$$

and

$$Lu = \frac{1}{2}\sum_{i,j}(a_{ij}\partial_i\partial_j u + \partial_i a_{ij}\partial_j u) + \sum_{i=1}^d b_i\partial_i u.$$

Proof Choose $f \in D(L)$. For all $g \in H^{1,2}(U)_0$, we have

$$\mathcal{E}(f,g) = \frac{1}{2}\int \langle A\nabla f, \nabla g\rangle d\mu - \int_U \langle B, \nabla f\rangle g \, d\mu = \int -Lf g d\mu.$$

Note that first equality holds by Lemma 4.8, and Sobolev inequality.

Therefore, if $\chi \in C^2_{0,ObNeu}(\overline{U})$, $\chi f \in H^{1,2}(U)_0$ by Lemma 4.8 and

$$\int \langle A\nabla(\chi f), \nabla g\rangle d\mu = \int f' g d\mu \text{ for all } g \in H^{1,2}(U)_0$$

, where $f' = -2\chi Lf - \sum_{i,j}\{a_{ij}(\partial_i\chi)(\partial_j f) + \partial_i(a_{ij}(\partial_j\chi)f)\} + 2\sum_{i=1}^{d} b^i\partial_i\chi f$.
The case $U = \mathbb{R}^d$ and $U = \mathbb{R}^d_+$ is entirely analogous of those of [18, Proof of Theorem 9.25 A,B]. Using the compactness of $\partial U \cap \mathrm{supp}\chi$, choose finitely many points $x_i \in \partial U$ and neighborhoods U_{x_i} which is an open cover of $\partial U \cap \mathrm{supp} f$ and let $\psi_i := \psi_{x_i}$, where ψ_{x_i} and U_{x_i} are defined in Definition 4.5, $1 \le i \le m$. Choose y_0 and a neighborhood U_{y_0} such that $\overline{U}_{y_0} \subset U$ and $\{U_{x_i}\}$ is an open cover of $\overline{U} \cap \mathrm{supp} f$, $0 \le i \le m$. Choose a partition of unity $\zeta_i \in C_{0,ObNeu}(\overline{U})$ such that $\mathrm{supp}\zeta_i \subset U_{x_i}$ and $\sum_i \zeta_i = 1$ on $\overline{U} \cap \mathrm{supp} f$ by Lemma 4.9. We write $u = \sum_{i=0}^{m} \zeta_i$. Now remaining parts of the proof is analogous to [18, Proof of Theorem 9.25 C].

The last statement can be derived similarly as in Lemma 4.1.

□

COROLLARY **4.6** *Assume $\varphi(x) = 1$ for all $x \in U$. We assume $U \subset \mathbb{R}^d$ be an open bounded set with $C^{1,1}$-boundary. Then there exists a weak solution $u \in H^{2,2}(U)$ of*

$$\begin{cases} (1-L)u = f \ \text{ in } U \\ \langle u, A^T\eta\rangle = 0 \ \text{ on } \partial U \end{cases}$$

for all $f \in L^2(U)$.

Proof Existence of solution is derived from [72, I.Exercise 2.7] and regularity comes from previous lemma.

□

LEMMA **4.12** *Assume $\varphi(x) = 1$ for all $x \in U$. We assume $U \subset \mathbb{R}^d$ be a bounded convex set. We further assume that $A^T\eta = \bar{A}^T\eta$ $d\sigma$-a.e. on ∂U. Then there exists a* *weak solution* $u \in H^{2,2}(U)$ *of*

$$\begin{cases} (1-L)u = f \ \textit{in } U \\ \langle u, A^T\eta\rangle = 0 \ \textit{on } \partial U \end{cases}$$

for all $f \in L^2(U)$.

Proof We follow the proof of [36, Theorem 3.2.1.3] with some changes. We choose a sequence $(\Omega_m)_{m\in\mathbb{N}}$ of bounded convex open subsets of $\mathbb{R}^d$ with C^2 boundaries Γ_m such that $U \subset \Omega_m$ and the distance of ∂U and Γ_m tends to zero as $m \to \infty$. We consider the solution u_m of the Neumann problem in Ω_m, i.e.

$$\begin{cases} (1-L)u_m = \tilde{f} \text{ in } \Omega_m \\ \langle u_m, A^T\eta\rangle = 0 \text{ on } \Gamma_m \end{cases}$$

(it has a solution in $H^{2,2}(U)$ by Corollary 4.6).
Now we need the following Lemma which corresponds to [36, Theorem 3.1.3.3].

LEMMA **4.13** *Assume* $\varphi(x) = 1$ *for all* $x \in U$. *Let* U *be a convex, bounded open subset of* $\mathbb{R}^d$ *with a* C^2 *boundary, and* $\alpha > 0$ *be a real number. Assume* $A^T\eta = \bar{A}^T\eta$ $d\sigma$*-a.e. on* ∂U. *Then there exists a constant* $C = C(\alpha, A, d)$ *such that*

$$\|u\|_{2,2,U} \leq C(\alpha, A, d)\|(\alpha - L)u\|_2$$

for all $u \in H^{2,2}(U)$ *such that* $\langle A^T\eta, \nabla f\rangle = 0$ $d\sigma$*-a.e. on* ∂U.

Proof (of Lemma 4.13)
Apply [36, Theorem 3.1.1.1] to $\mathbf{v} = \bar{\mathbf{A}}\nabla\mathbf{u}$. Then, we have

$$\int_U |L^{\bar{A}}u|^2 dx - \sum_{i,j=1}^{d}\int_U \frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i}dx \geq 0,$$

where v_i is i'th component of $\mathbf{v}$ and $L^{\bar{A}}u := \sum_{i,j=1}^d \partial_i(\bar{a}_{ij}\partial_j u)$.

Observe that

$$
\begin{aligned}
\frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i} &= (\sum_{k=1}^d \partial_j(\bar{a}_{ik}\partial_k u))(\sum_{l=1}^d \partial_i(\bar{a}_{jl}\partial_l u)) \\
&= \underbrace{(\sum_{k=1}^d \partial_j\bar{a}_{ik}\partial_k u)(\sum_{l=1}^d \partial_i\bar{a}_{jl}\partial_l u)}_{=:I} + \underbrace{(\sum_{k=1}^d \partial_j\bar{a}_{ik}\partial_k u)(\sum_{l=1}^d \bar{a}_{jl}\partial_i\partial_l u)}_{=:II} \\
&+ \underbrace{(\sum_{k=1}^d \bar{a}_{ik}\partial_j\partial_k u)(\sum_{l=1}^d \partial_i\bar{a}_{jl}\partial_l u)}_{=:III} + \underbrace{(\sum_{k=1}^d \bar{a}_{ik}\partial_j\partial_k u)(\sum_{l=1}^d \bar{a}_{jl}\partial_i\partial_l u)}_{=:IV}
\end{aligned}
$$

$\sum_{i,j=1}^d (IV) \geq \lambda_U^{-2}\sum_{i,j=1}^d |\partial_i\partial_j u|^2$ by [36, Lemma 3.1.3.4].(Note that the term (I) is missing on the proof in [36, Theorem 3.1.3.3]). For the below, $C = C(\alpha, A, d)$ could be different from line by line.

By estimation as in [36, Theorem 3.1.3.3], we get $\sum_{i,j=1}^d \int_U |\partial_i\partial_j u|^2 dx \leq C(\int_U |L^{\bar{A}}u|^2 dx + \sum_{i=1}^d \int_U |\partial_i u|^2 dx)$.

Using the fact that $\int_U (\alpha - L)u u dx = \frac{1}{2}\sum_{i,j=1}^d \int_U a_{ij}\partial_i u \partial_j u + \alpha \int_U |u|^2 dx$ by (4.11), we can see $\|u\|_2$ and $\|\nabla u\|_2$ are bounded by $C\|(\alpha - L)u\|_2$. Since

$$
\begin{aligned}
\int_U |L^{\bar{A}}u|^2 dx &= \int_U |(2\alpha - 2L)u + \sum_{i,j=1}^d \partial_i(\breve{a}_{ij}\partial_j u) - 2\alpha u + 2\langle B, \nabla u\rangle|^2 dx \\
&\leq C(\int_U (|(\alpha - L)u|^2 + |\underbrace{\sum_{i,j=1}^d \partial_i(\breve{a}_{ij}\partial_j u)}_{=\sum_{i,j=1}^d \partial_i\breve{a}_{ij}\partial_j u}|^2 + |2\alpha u|^2 + |2\langle B, \nabla u\rangle|^2) dx) \\
&\leq C(\|(\alpha - L)u\|_2^2 + \|\nabla u\|_2^2 + \|u\|_2^2),
\end{aligned}
$$

we get the desired result.

$\square$

Now, we return to the proof of Lemma 4.12.

From previous lemma, we can see that there exists a constant C such that

$$\|u_m\|_{H^{2,2}(\Omega_m)} \leq C.$$

Again, argument as in [36, Theorem 3.2.1.3] shows u_m converges to a weak solution u weakly in $H^{2,2}(U)$ up to a subsequence.

□

COROLLARY **4.7** *Let U be a C^2 locally tesselationable set. Assume $\varphi(x) = 1$ for all $x \in U$. We further assume that $A^T\eta = \bar{A}^T\eta$ $d\sigma$-a.e. on ∂U. Then $D(L) \subset H^{2,2}_{loc}(U)$. Moreover, we can see*

$$D(L) = H^{1,2}(U) \cap H^{2,2}_{loc,ObNeu}(U),$$

and

$$Lu = \frac{1}{2}\sum_{i,j}(a_{ij}\partial_i\partial_j u + \partial_i a_{ij}\partial_j u) + \sum_{i=1}^{d} b_i\partial_i u.$$

Proof Choose $f \in D(L)$. Similar to Lemma 4.11, we have for $\chi \in C^2_{0,ObNeu}(\overline{U})$,

$$\frac{1}{2}\int \langle A\nabla(\chi f), \nabla g\rangle dx + \int \chi f g dx = \int f' g dx \text{ for all } g \in H^{1,2}(U)_0, \tag{4.13}$$

where $f' = \chi(1-L)f - \frac{1}{2}\sum_{i,j}\{a_{ij}(\partial_i\chi)(\partial_j f) + \partial_i(a_{ij}(\partial_j\chi)f)\} + \sum_{i=1}^{d} b^i\partial_i\chi f$.
Take $x \in U$ and choose (ψ_x, U_x) as in Definition 4.4. Take χ whose support is in U_x in (4.13). Note that if V is a tesselationable set, it must be convex (and unbounded). Note also that intersection of unbounded convex set with a ball is a bounded convex set. Using the argument as in Lemma 4.11 and by Lemma 4.6, we have a weak solution $u \in H^{2,2}(U \cap U_x)$ satisfying

$$\frac{1}{2}\int \langle A\nabla u, \nabla g\rangle dx + \int u g dx = \int f' g dx \text{ for all } g \in H^{1,2}(U \cap U_x).$$

Since χf is also a weak solution, $\chi f = u \in H^{2,2}(U \cap U_x)$ by uniqueness. Since interior regularity is well-known by classical result, we get the desired result.

□

Now, we will show you some definitions and well-known facts about L^p uniqueness.

If there is only one C_0 semigroup on $L^p(U,\mu)$ whose generator extends given densely defined unbounded operator (A,D), then the operator (A,D) is said to be L^p-unique. If (A,D) is semi-bounded symmetric operator, it is known that L^2-uniqueness is equivalent to essential self-adjointness(see [29, cor 1.2]).

Let B be a Banach space, and $f \in B$. An element $l \in B'$ that satisfies $\|l\|_{B'} = \|f\|_B$, and $l(f) = \|f\|_B^2$ is called a normalized tangent functional to f.

A densely defined operator (S,D) on a Banach space B is said to be dissipative, if for each $f \in D$ there exists a normalized tangent functional l with $l(Lf) \le 0$.

LEMMA **4.14** *A densely defined operator (S,D) on $L^p(U,\mu)$ is L^p-unique, if and only (S,D) is dissipative and $(\alpha - S)D$ is dense in $L^p(U,\mu)$ for some $\alpha > 0$, $p \ge 1$.*

Proof

"$\Rightarrow$": Assume (S,D) is L^p-unique. Let $(\overline{S},\overline{D})$ be the unique extension of (S,D) who generates C_0-semigroup. Note that $(\overline{S},\overline{D})$ is closed by [85, Proposition on page 237]. By [85, Theorem X.48], we can see that $(\overline{S},\overline{D})$ is dissipative, hence its restriction (S,D) is also dissipative. By [81, A-II,Theorem1.33], (S,D) must be dense in $(\overline{S},\overline{D})$ with respect to the graph norm. Let (U_α) be the corresponding resolvent. For any $f \in L^p(U,\mu)$, $U_\alpha f \in \overline{D}$. Thus we can choose $(f_n)_{n\ge1} \subset D$ converging to $U_\alpha f$ with respect to the graph-norm. Then $(\alpha - S)f_n$ converges to $(\alpha - S)U_\alpha f = f$.

"$\Leftarrow$": Assume (S, D) is dissipative and $(\alpha - S)D$ is dense in $L^p(U, \mu)$. We first note that every dissipative operator is closable and that its closure is again dissipative. Denote hence by $(\overline{S}, \overline{D})$ the closure of (S, D) on $L^p(U, \mu)$ which is again dissipative. Let l be a normalized tangent functional to $u \in \overline{D}$, which exists since $(\overline{S}, \overline{D})$ is dissipative. Then

$$\begin{aligned}
\alpha \|u\|^2_{L^p(U,\mu)} &\leq \alpha l(u) - l(\overline{S}u) \\
&= l((\alpha - \overline{S})u) \\
&\leq \|u\|_{L^p(U,\mu)} \|(\alpha - \overline{S})u\|_{L^p(U,\mu)}.
\end{aligned}$$

Thus the range of $(\alpha - \overline{S})$ is closed and we get that $(\overline{S}, \overline{D})$ generates a C_0-semigroup by [85, Theorem X.48]. Let $(\overline{J}, \tilde{D})$ be another generator that extends (S, D). Then it also extends $(\overline{S}, \overline{D})$. But since both $(\alpha - \overline{S})$ and $(\alpha - \overline{J})$ are invertible, they must be the same.

□

Now, we are ready to prove our L^2 uniqueness results.

PROPOSITION **4.2** *Assume (4.8),* $\dfrac{\nabla\varphi}{\varphi} \in L^\gamma_{loc}(U, \mu)$*, where* $\gamma > 2$ *if* $d = 2$*,* $\gamma = d$ *if* $d \geq 3$*, and* a_{ij}*,* b_i *are in* $L^\infty(U)$*,* $1 \leq i, j \leq d$*. Then* $(L, D(L)_0)$ *is* L^2*-unique.*

Proof Here, we used the idea from [96] and [9].

$(L, D(L))$ is dissipative by [85, Theorem X.48], and hence its restriction $(L, D(L)_0)$ is also dissipative. Using Lemma 4.14, it is sufficient to show if $h \in L^2(U, \mu)$ is such that $\int (1 - L)uh d\mu = 0$ for all $u \in D(L)_0$ it follows that $h = 0$. Let $\chi \in C^2_{0,ObNeu}(\overline{U})$. If $u \in D(L)$, we can easily see that $\chi u \in D(L)_0$ and $L(\chi u) = \chi Lu + \langle \bar{A}\nabla\chi, \nabla u\rangle + uL\chi$.

Hence

$$\begin{aligned}\int (1-L)u(\chi h)d\mu = & \int (1-L)(u\chi)hd\mu + \int \langle \bar{A}\nabla\chi, \nabla u\rangle hd\mu + \int uL\chi hd\mu \\ = & \int \langle \bar{A}\nabla\chi, \nabla u\rangle hd\mu + \int uL\chi hd\mu.\end{aligned}$$

Let $p^* := \dfrac{dp}{d-p}$ so that $\dfrac{1}{p^*} + \dfrac{1}{d} = \dfrac{1}{p}$. Take p such that $\dfrac{1}{p^*} + \dfrac{1}{\gamma} = \dfrac{1}{2}$. Since $L\chi = \frac{1}{2}\sum_{i,j=1}^d \partial_i(a_{ij}\partial_j\chi) + \sum_{i,j=1}^d a_{ij}\dfrac{\partial_j\varphi}{\varphi}\partial_i\chi + \sum_{i=1}^d b_i\partial_i\chi$, we have $\int uL\chi hd\mu \le C(\|u\|_2\|h\|_2 + \|u\|_{p^*}\|\dfrac{\nabla\varphi}{\varphi}\|_\gamma\|h\|_2) \le C(\|u\|_2\|h\|_2+\|u\|_{H^{1,2}(U,\mu)}\|\dfrac{\nabla\varphi}{\varphi}\|_\gamma\|h\|_2)$ by Sobolev inequality, where C is a constant which does not depend on u. We obtain $u \to \int(1-L)u(\chi h)d\mu$, $u \in D(L)$, is continuous with respect to the norm $\tilde{\mathcal{E}}_1^{1/2}$. Since $D(L)$ is dense in $D(\mathcal{E})$, there exists $v \in D(\mathcal{E})$ such that $\mathcal{E}_1(u,v) = \int(1-L)u(\chi h)d\mu$ for $u \in D(L)$ by [72, I.Exercise 2.7]. Hence, $\int(1-L)u(v-\chi h)d\mu$ for all $u \in D(L)$, and $\chi h = v \in H^{1,2}(U,\mu)_0$ and that

$$\mathcal{E}_1(u,\chi h) = \int \langle A\nabla u, \nabla\chi\rangle h\, d\mu + \int (L^0\chi)uh\, d\mu.$$

Given arbitrary $u \in C^2_{0,ObNeu}(\overline{U})$, we can choose $\chi \in C^2_{0,ObNeu}(\overline{U})$ which is 1 on the support of u Corollary 4.5. Hence, the previous equality implies

$$\mathcal{E}_1(u,h) = 0 \text{ for all } u \in C^2_{0,ObNeu}(\overline{U}).$$

Now, let $u \in H^{1,2}(U,\mu)_0$, and choose a sequence $u_n \in C^2_{0,ObNeu}(\overline{U})$, $n \ge 1$, such that $\lim_{n\to\infty} u_n = u$ in $H^{1,2}(U,\mu)$ by Remark 4.1. Take $\psi \in C^2_{0,ObNeu}(\overline{U})$ such that ψ is 1 on support of u by Corollary 4.5. Using Corollary 4.1, we can see that $\lim_{n\to\infty}\psi u_n = \psi u = u$ in $H^{1,2}(U,\mu)$. Hence

$$\frac{1}{2}\int \langle A\nabla u, \nabla h\rangle d\mu - \int \langle B, \nabla u\rangle hd\mu + \int uhd\mu = \mathcal{E}_1(u,h) = 0 \text{ for all } u \in H^{1,2}(U,\mu)_0.$$

By Lemma 4.10, we can see $\chi h \in H^{1,2}(U,\mu)_0$ for all $\chi \in C_0^\infty(\overline{U})$.

Now, for $\chi \in C_0^\infty(\overline{U})$,

$$
\begin{aligned}
&\frac{1}{2}\int \langle A\nabla(\chi h), \nabla(\chi h)\rangle d\mu + \int (\chi h)^2 d\mu \\
&= \frac{1}{2}\int \langle A\nabla(\chi^2 h), \nabla h\rangle d\mu - \frac{1}{2}\int (\chi h)\langle A\nabla\chi, \nabla h\rangle d\mu + \frac{1}{2}\int h\langle A\nabla(\chi h), \nabla\chi\rangle d\mu + \int (\chi h)^2 d\mu \\
&= \int \langle B, \nabla(\chi^2 h)\rangle h d\mu - \frac{1}{2}\int (\chi h)\langle A\nabla\chi, \nabla h\rangle d\mu + \frac{1}{2}\int h\langle A\nabla(\chi h), \nabla\chi\rangle d\mu \\
&= -\int \langle B, \nabla h\rangle \chi^2 h d\mu - \frac{1}{2}\int (\chi h)\langle A\nabla\chi, \nabla h\rangle d\mu + \frac{1}{2}\int h\langle A\nabla(\chi h), \nabla\chi\rangle d\mu \\
&= -\frac{1}{2}\int \langle B, \nabla(h^2)\rangle \chi^2 d\mu - \int (\chi h)\langle \check{A}\nabla\chi, \nabla h\rangle d\mu + \frac{1}{2}\int h^2\langle A\nabla\chi, \nabla\chi\rangle d\mu \\
&= \underbrace{\frac{1}{2}\int \langle B, \nabla(\chi^2)\rangle h^2 d\mu}_{=:I} - \underbrace{\int h\langle \check{A}\nabla\chi, \nabla(\chi h)\rangle d\mu}_{=:II} + \underbrace{\frac{1}{2}\int h^2\langle A\nabla\chi, \nabla\chi\rangle d\mu.}_{=:III}
\end{aligned}
$$

Now, choose a sequence $\chi_k \in C_0^\infty(\overline{U})$ such that it is 1 on $B_k(0)\cap U$, $0 \le \chi_k \le 1$ and $|\nabla\chi_k| \le \frac{1}{k}$, $k \in \mathbb{N}$ and put it in above equation instead of χ.

Then, for some constant C which could be different on each inequality but independent of k, we get

$$
|I| \le C\sum_{i=1}^{d} \|b_i\|_{L^\infty(U,\mu)} \underbrace{|\nabla(\chi_k^2)|}_{\le \frac{2}{k}} \|h\|^2_{L^2(U,\mu)},
$$

$$
\begin{aligned}
|II| &\le C\int_{\mathrm{supp}\chi_k} |h|\lambda^{-1}_{\mathrm{supp}\chi_k} \underbrace{|\nabla\chi_k|}_{\frac{1}{k}} |\nabla(\chi_k h)| d\mu \\
&\le \frac{C}{k}\|h\|_{L^2(U,\mu)}\Big(\int_{\mathrm{supp}\chi_k} \lambda^{-1}_{\mathrm{supp}\chi_k}|\nabla(\chi_k h)|^2 d\mu\Big)^{1/2} \quad (\text{since } \lambda^{-1}_{\mathrm{supp}\chi_k} \le 1) \\
&\le \frac{C}{k}\left(\mathcal{E}(\chi_k h, \chi_k h)\right)^{1/2},
\end{aligned}
$$

$$
|III| \le C\sum_{i,j=1}^{d} \|a_{ij}\|_{L^\infty(U,\mu)} \underbrace{|\nabla\chi_k|^2}_{\le \frac{1}{k^2}} \|h\|^2_{L^2(U,\mu)}.
$$

Hence

$$
\mathcal{E}_1(\chi_k h, \chi_k h) \le \frac{C}{k}\left(\mathcal{E}_1(\chi_k h, \chi_k h)\right)^{1/2} + \frac{C}{k}. \tag{4.14}
$$

By letting $k \to \infty$, we get $\mathcal{E}_1^0(\chi_k h, \chi_k h) \to 0$. In particular, $\|h\|^2_{L^2(U,\mu)} = 0$, hence $h = 0$ and we get the desired result.

$\square$

THEOREM **4.4** *Assume $\varphi(x)^2 > 0$ for all $x \in U$ and is in $C^{0,1}_{loc}(\overline{U})$. We also assume that (4.8), and $a_{ij},\ b_i \in L^\infty(U)$, $1 \le i,j \le d$. We assume further that $A^T\eta = \bar{A}^T\eta\ d\sigma$-a.e. on ∂U if U contains singular points. Then $(L, C^2_{0,ObNeu}(\overline{U}))$ is L^2-unique.*

Proof By [30, Theorem 1.2 of Appendix A], it suffices to prove $C^2_{0,ObNeu}(\overline{U})$ is dense in $D(L)$ with respect to the graph norm. However, we know that $D(L)_0$ is dense in $D(L)$ with respect to graph norm by Proposition 4.2. Hence it suffices to prove $C^2_{0,ObNeu}(\overline{U})$ is dense in $D(L)_0$. Note that $H^{2,2}(U,\mu)$ convergence implies convergence with respect to the graph norm. Proof is similar with Theorem 4.3. We just give a sketch of proof here. Choose $f \in D(L)_0$. Let $A' = (a'_{ij}),\ B' = (b'_1, \dots, b'_d)$, where $a'_{ij} = \varphi^2 a_{ij}$ and $b'_i = \varphi^2 b_i$, $1 \le i,j \le d$. Then A', B' satisfies the same condition as A and B locally. Choose diffeomorphism $(U_{x_i}, \psi_i)_{1\le i\le m}$ as in Theorem 4.3 which is an open cover of supp$f$. Use partition of unity(Lemma 4.9) $(\zeta_i)_{i=0}^m$ such that supp$\zeta_i \subset U_{x_i}$ with K =supp$f$. We can use Lemma 4.11 (resp. Corollorary 4.7) to get $(f\zeta_i)(\psi_i^{-1}) =: u_i \in H^{2,2}_{Neu}(B_i) := \{f \in H^{2,2}(B_i) | \langle \eta, \nabla f\rangle = 0\ \varphi^2 d\sigma\text{-a.e. on } \partial B_i\}$, where $B_i := \psi(U_{x_i} \cap U)$. For simplicity, we assume $B_i = \mathbb{R}^d_+$. Define $\tilde{u}$ by an extension of u to $\mathbb{R}^d$ by reflection. We want to show $\tilde{u} \in H^{2,2}(\mathbb{R}^d)$. We just give a reason why $\tilde{u} \in H^{2,2}(\mathbb{R}^d)$ below. The rest of proof is same with Theorem 4.3.

Let $f \in C_0^\infty(\mathbb{R}^d)$, then

$$\begin{aligned}
\int_{\mathbb{R}^d} \tilde{u}(x)\partial_d\partial_d f(x)dx &= \int_{\mathbb{R}^d_+} u(x)\partial_d\partial_d f(x)dx + \int_{\mathbb{R}^d_-} u(x_1,\dots,x_{d-1},-x_d)\partial_d\partial_d f(x)dx \\
&= \int_{\mathbb{R}^d_+} u(x)\partial_d\partial_d f(x)dx + \int_{\mathbb{R}^d_+} u(x)(\partial_d\partial_d f)(x_1,\dots,x_{d-1},-x_d)dx \\
&= -\int_{\mathbb{R}^d_+} \partial_d u(x)(\partial_d f(x) - (\partial_d f)(x_1,\dots,x_{d-1},-x_d))dx + (*) \\
&= \int_{\mathbb{R}^d_+} \partial_d\partial_d u(x)(f(x) + f(x_1,\dots,x_{d-1},-x_d))dx + (**) \\
&= \int_{\mathbb{R}^d} (\partial_d\partial_d u(x)|_{\mathbb{R}^d_+} + (\partial_d\partial_d u)(x_1,\dots,x_{d-1},-x_d)|_{\mathbb{R}^d_-})f(x)dx \\
&= \int_{\mathbb{R}^d} \partial_d\partial_d \tilde{u}(x)f(x)dx
\end{aligned}$$

$(*)$ on the 4'th equality is the boundary term derived from integration by parts and it disappears, since $\partial_d(f(x_1,\dots,x_{d-1},-x_d))|_{x_d=0} = -\partial_d f(x)|_{x_d=0}$.
$(**)$ on the 5'th equality is the boundary term derived from integration by parts and it disappears by Neumann boundary condition of u. Similarly, the following holds for all $1 \le i, j \le d$.

$$\int_{\mathbb{R}^d} \tilde{u}(x)\partial_i\partial_j f(x)dx = \int_{\mathbb{R}^d} \partial_i\partial_j \tilde{u}(x)f(x)dx.$$

□

REMARK **4.4** *By the reflection method as above, any* $u \in H^{2,p}_{ObNeu}(U)_0$ *with* $(\alpha - L)u = f$, $f \in L^p(U)$, *can be extended to* $\tilde{u} \in H^{2,p}(\mathbb{R}^d)$ *such that* $(\alpha - L)u = \tilde{f}$ *in* $\mathbb{R}^d$, *where* $\tilde{f}$ *is an extension of* f *such that* $\|\tilde{f}\|_{L^p(\mathbb{R}^d)} \le C\|f\|_{L^p(U)}$ *for some constant* C *independent of* u *and* f.

REMARK **4.5** *If* $\dfrac{\nabla\varphi}{\varphi} \in L^\gamma_{loc}(U)$ *for some* $\gamma > d$, φ *is locally uniformly positive and is locally bounded, i.e.,* $essinf_V \varphi > 0$ *and* $\|\varphi\|_{L^\infty(V,\mu)} < \infty$ *on each bounded set* $V \subset U$.

Proof Assume U is a half space $\mathbb{R}^d_+ := \{(x_1, \dots, x_d) | x_d > 0\}$ first. Extend φ to $\mathbb{R}^d$ by reflection, i.e., $\varphi(x_1, \dots, x_d) := \varphi(x_1, \dots, -x_d)$ for $\{x \in \mathbb{R}^d | x_d < 0\}$. Now the result follows from [9, Corollary 8].

Now U is as in the beginning of this section. Choose bounded $V \subset U$. Now the result follows from usual argument using partition of unity as in Lemma 4.11.

□

REMARK **4.6** *With the condition in [68, Chapter 3, Theorem 3.1 page 135], we can get* L^2*-uniqueness of* $(L, C^2_{Neu}(\overline{U}))$*. In particular, if* U *is bounded* $C^{2,\alpha}$*-domain,* $a_{ij} \in C^{1,\alpha}(U)$*,* $b_i \in C^{0,\alpha}(U)$ *and* $\dfrac{\nabla\varphi}{\varphi} \in C^{0,\alpha}(U)$ *for some* $0 < \alpha < 1$*, we can easily get* L^2*-uniqueness. Previous Theorem 4.4 clearly extends this classical method.*

Proof Assume $\int_U (1 - L)uh d\mu = 0$ for all $u \in C^2_{0,ObNeu}(\overline{U})$ for some $h \in L^2(U)$. By [68, Chapter 3, Theorem 3.1 page 135], for any $f \in C_0^\infty(U)$, there exists $u_f \in C^2_{0,ObNeu}(\overline{U})$ such that $(1 - L)u_f = f$. Hence, we are done.

□

4.5 Markov uniqueness of Robin boundary condition

Let $U \subset \mathbb{R}^d$ with 2 disjoint boundaries Γ_1 and Γ_2, $dist(\Gamma_1, \Gamma_2) := \inf_{x \in \Gamma_1,\ y \in \Gamma 2} \|x - y\| > 0$.

Let $a_{ij} = a_{ji} \in H^{1,\infty}(U) \cap C^1(\Gamma_1)$, which is strictly positive for all $x \in U \cup \Gamma_1$. For $\chi \in C_0^\infty(\bar{U})$, χa_{ij} can be extended to $H^{1,\infty}(\mathbb{R}^d)$. Note that $H^{1,\infty}(U)$ consists of locally Lipschitz continuous functions (See [44, Theorem 4.1]). Hence, every point $x \in U \cup \Gamma_1$ has a neighborhood such that a_{ij} is Lipschitz continuous there. Let

$$C^2_{ObNeu}(U \cup \Gamma_1) := \{f \in C_0^2(\mathbb{R}^d \backslash \Gamma_2)_{|U \cup \Gamma_1} | \langle A^T \eta, \nabla f \rangle = 0 \ \varphi^2 d\sigma\text{-a.e. on } \Gamma_1\} \quad (4.15)$$

Define

$$\mathcal{E}(f,g) = \int_U \langle A\nabla f, \nabla g\rangle dx, \tag{4.16}$$

where the $\partial_i f$ denote the distributional derivatives and the domain $D(\mathcal{E})$ of the form is defined to be the space of all $f \in L^2(U)$ for which the integral is finite. By [85, Theorem 1.1 I], it is a Dirichlet form. We can also see that $C^2_{ObNeu}(U \cup \Gamma_1)$ is contained in the generator $(L, D(L))$ of the form.

THEOREM **4.5** *Suppose $x \in \Gamma_1$ has a neighborhood which C^2-deffeomorphic to tesselationable set in the sense of Definition 4.4. Then $\big(L, C^2_{ObNeu}(U \cup \Gamma_1)\big)$ is Markov unique if and only if $cap_U(\Gamma_2) = 0$, where cap_U is defined as in [85, page 3 (4)](we use cap_U instead of cap_Ω).*

Proof By [85, Theorem 1.1 III], $(\mathcal{E}, D(\mathcal{E}))$ is a maximal extension in the sense of Section 3 of this paper. Therefore, it is enough to show that $C^2_{ObNeu}(U \cup \Gamma_1)$ is dense in $D(\mathcal{E})$. Choose $f \in D(\mathcal{E})$. Take g_0 be a measurable function which is 1 near Γ_1 and 0 near Γ_2. By convolution with usual mollifier, we can get C^∞ function g_1 which is 1 near Γ_1 and 0 near Γ_2. Note g_1 and ∇g_1 are both globally bounded. Let $h_1 = fg_1,\ h_2 = 1-f$. Note $h_1,\ h_2 \in D(\mathcal{E})$. Now h_1 can be approximated as in Theorem 4.3 and h_2 can be approximated as in [85, Proposition 4.1].

□

Chapter 5 L^1-uniqueness and conservativeness on reflected Dirichlet space

5.1 Introduction

Let U be an open set in $\mathbb{R}^d$ whose boundary is smooth enough in certain sense and let $\mu = \varphi^2 dx$, $\varphi \in H^{1,2}_{loc}(U)$, $\varphi > 0$ dx-a.e. Let $A = (a_{ij})_{i,j=1}^d$ (possibly non-symmetric) be a locally strictly elliptic matrix with elements in $H^{1,2}_{loc}(U)$ and $\beta := (\beta_1, \cdots, \beta_d)$ be a divergence free vector field (for the precise conditions, see later sections). Let L be a non-symmetric linear operator on $L^2(U,\mu)$ with the domain $D(L) \subset C_0^2(\overline{U})$ being dense in $L^2(U,\mu)$. Let

$$Lu = \frac{1}{2}\sum_{i,j}(a_{ij}\partial_i\partial_j u + \partial_i a_{ij}\partial_j u) + \sum_{i=1}^d b_i \partial_i u \quad f \in D(L).$$

The uniqueness problem for diffusion operators are studied by many articles and book, for example, [10, 29, 46, 49, 66, 71] (For background material, motivation and survey, see [29] for detail). In this chapter, in particular, we are interested in constructing extension of L which generates C_0-semigroup, finding regularity of a_{ij} and b_i which guarantee equivalence of conservativeness and L^1 uniqueness, and certain type of elliptic regularity and L^2 uniqueness result. Main importance of our chapter is, since we deal with Neumann problem, we can also investigate boundary behaviour of some problems. From previous result, they usually assume that $U = \mathbb{R}^d$ because it is very difficult to control boundary effect.

In Section 2, we use the idea from [96] to construct extension $\overline{L}$ of L whose generator extends L. Moreover, we also show that L^1 uniqueness of $(L, C^2_{0,Neu}(\overline{U}))$ is equivalent to conservativeness in certain sense, which will give us analytic criteria to show conservativeness. We also contain some concrete example when this holds. Note that if $U = \mathbb{R}^d$, our result extends the previous result [96].

In Section 3, we use the idea from [10] to show elliptic regularity and some applications including L^2 uniqueness result and invariant measure result. On this section, we only assume that a_{ij} is locally Hölder continuous rather than $H^{1,p}_{loc}(U)$ which extends previous result [9, 10].

In Section 4, we will see when distorted Brownian motion will satisfy the condition of Section 2.

Although many statements in this chapter resemble statements in [96] and [10], we will state them here again to make this thesis self-contained.

5.2 Functional analytic framework and notations

In general, we shall denote by $\|\cdot\|_B$ the norm of a Banach space (or vector space) B. We denotye the topological dual space of a Banach space B by B'. In the special case of $\mathbb{R}^d$, $d \geq 1$, $|\cdot|$ will denote the corresponding Euclidean norm and $\langle\cdot,\cdot\rangle$ the Euclidean inner product.

Let $U \subset \mathbb{R}^d$ be a possibly unbounded open set with Lipschitz boundary, where the definition of Lipschitz boundary is given in the Appendix. Let σ denote the surface measure on the boundary ∂U of U and η be the inward normal vector on ∂U.

For any $V \subset U$, V open, let $L^p(V,\mu)$, $p \in [1,\infty]$, denote the usual L^p-spaces with respect to the measure μ and we omit μ if it is the Lebesgue measure. We denote by $(\cdot,\cdot)_H$ the inner product in Hilbert space H. By $L^p_{loc}(U,\mu)$ we denote all measurable functions $f : U \to \mathbb{R}$ with $f \in L^p(V,\mu)$ for any bounded and open set $V \subset U$. Let $n \in \mathbb{N} \cup \{0,\infty\}$. Denote by $C^n_0(\mathbb{R}^d)$ the set of n-times continuously differentiable functions on $\mathbb{R}^d$ with compact support. For a set $A \subset \mathbb{R}^d$, a function $f : A \to \mathbb{R}$, let $\mathrm{supp}_A f$ denote the essential support of f in A with respect to the Lebesgue measure. For convenience, we write $\mathrm{supp} f$ or support of f instead of $\mathrm{supp}_{\overline{U}} f$. For a compact K in $\mathbb{R}^d$, let $C^{n,s}(K)$ be the usual Hölder space of order (n, s), which consists of n-times continuously differentiable functions with Hölder exponent s, $n \in \mathbb{N} \cup \{0\}$, $0 < s < 1$. For open $\Omega \subset \mathbb{R}^d$, a continuous function $f : \Omega \to \mathbb{R}$ is called locally Hölder continuous if its restriction to K for all compact $K \subset \overline{\Omega}$ is Hölder continuous. Let A, B be sets. For a function $f : A \to \mathbb{R}$ and $B \subset A$, denote the restriction of f to B by $f_{|B}$. We denote the closure (in a topological space that will be mentioned) of B by $\overline{BKR}$. For $V \subset U$, V open, let

$$\begin{aligned} &C_0^n\big(V \cup (\partial U \cap \overline{V})\big) \\ &:= \{f_{|\overline{V}} \,|\, f \in C_0^n(\mathbb{R}^d), \mathrm{supp}_{\mathbb{R}^d} f \subset V_0 \text{ for some } V_0 \subset \mathbb{R}^d \text{ open with } V_0 \cap U = V\}. \end{aligned}$$

Note that for $V = U$, we get $C_0^n(\overline{U}) = \{f_{|\overline{U}} \,|\, f \in C_0^n(\mathbb{R}^d)\}$. For $n = 1, 2$ and arbitrary open set $V \subset U$, let $H^{n,p}(V)$, $p \in [1, \infty]$, be the classical Sobolev space of order n in $L^p(V)$, i.e. the space of all measurable functions that are together with their weak derivatives up to order n again in $L^p(V)$. For a weakly differentiable function u, let $\partial_i u$ denotes the directional derivative with respect to the direction e_i which is 1 on the i'th coordinate and 0 on the other coordinates, and let $\nabla u := (\partial_1 u, \cdots, \partial_d u)$, $1 \le i \le d$. Let

$$H^{1,p}_{loc}(U) := \{u \,|\, u \cdot \chi \in H^{1,p}(U) \text{ for all } \chi \in C_0^\infty(\overline{U})\}, \ p \in [1, \infty].$$

If $W \subset L^p(U, \mu)$ is an arbitrary subspace, let W_0 denote the space of all elements $u \in W$ such that supp u is a bounded set in U, and by $W_b = W \cap L^\infty(U, \mu)$ the space of all (μ-)essentially bounded elements in W. Finally, let $W_{0,b} = W_0 \cap W_b$.

Fix $\varphi \in H^{1,2}_{loc}(U)$ such that $\varphi > 0$ dx-a.e. and let $d\mu = \varphi^2 dx$. For $1 \le i, j \le d$ let

$a_{ij} : \overline{U} \longrightarrow \mathbb{R}$ be measurable functions. Let further $A = (a_{ij})_{1 \le i,j \le d}$ and for $1 \le i,j \le d$

$$\bar{a}_{ij} := \frac{a_{ij} + a_{ji}}{2}, \qquad \check{a}_{ij} := \frac{a_{ij} - a_{ji}}{2}.$$

We assume that A is locally strictly elliptic, i.e., for any bounded $V \subset U$ there exist constants $M, \lambda_V \in (0, \infty)$ with

$$\lambda_V^{-1} |\xi|^2 \le \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \le \lambda_V |\xi|^2 \qquad \forall \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d,\ \mu\text{-a.e. } x \in V \tag{5.1}$$

and (**actually, we only need that the Dirichlet form defined in Lemma 4.5 below satisfies the weak sector condition**)

$$\max_{1 \le i,j \le d} \|\check{a}_{ij}\|_{L^\infty(V,\mu)} \le M \lambda_V^{-1} \tag{5.2}$$

and

$$a_{ij} \in H^{1,2}_{loc}(U,\mu) := \{u \,|\, u \cdot \chi \in H^{1,2}(U,\mu) \text{ for all } \chi \in C_0^\infty(\overline{U})\}, \quad 1 \le i,j \le d.$$

For an arbitrary open set $V \subset U$, $H^{1,2}(V,\mu)$ be the closure of $C_0^2\big(V \cup (\partial U \cap \overline{V})\big)$ in $L^2(V,\mu)$ with respect to the norm $(\int_V u^2 d\mu + \int_V |\nabla u|^2 d\mu)^{1/2}$. The closure exists (see, e.g. Lemma 4.5 below) and moreover $H^{1,2}(V_1,\mu) \subset H^{1,2}(V_2,\mu)$ whenever $V_1 \subset V_2$. For $V \subset U$ open, let

$$C^2_{0,Neu}\big(V \cup (\partial U \cap \overline{V})\big) := \{f \in C_0^2\big(V \cup (\partial U \cap \overline{V})\big) \,|\, \langle A^T\eta, \nabla f\rangle = 0 \ \varphi^2 d\sigma\text{-a.e. on } \partial U \cap \overline{V}\}.$$

where $A^T = (a^T_{ij}) = (a_{ji})$ is the transposed matrix of A. **Attention: Here we assume that $A^T\eta$ is uniquely defined $\varphi^2 d\sigma$-a.e. on ∂U.** Note that for $V = U$, we get

$$C^2_{0,\text{Neu}}(\overline{U}) := \{f \in C_0^2(\overline{U}) \,|\, \langle A^T\eta, \nabla f\rangle = 0 \ \varphi^2 d\sigma\text{-a.e. on } \partial U\}.$$

Let $B = (b_i)_i \in L^2_{loc}(U; \mathbb{R}^d, \mu)$, i.e. $\int_V |B|^2 d\mu < +\infty$ for any bounded and open $V \subset U$. Suppose

$$\int_U L^A u + \langle B, \nabla u\rangle \, d\mu = 0 \quad \forall u \in C^2_{0,\text{Neu}}(\overline{U}), \tag{5.3}$$

where $L^A u := \frac{1}{2}\sum_{i,j=1}^d a_{ij}\partial_i\partial_j u$. Let

$$\beta^{A,\varphi^2} = (\beta_1^{A,\varphi^2}, \cdots, \beta_d^{A,\varphi^2}), \quad \beta_i^{A,\varphi^2} = \sum_{j=1}^d \left(\frac{\partial_j a_{ij}^T}{2} + a_{ij}^T \frac{\partial_j \varphi}{\varphi} \right), \quad 1 \le i \le d.$$

Let $\beta := (\beta_1, \cdots, \beta_d) = B - \beta^{A,\varphi^2}$.

Then, using integration by parts and noting that the boundary terms disappear, we obtain from (5.3) that

$$\int_U \langle \beta, \nabla u \rangle \, d\mu = 0 \quad \forall u \in C^2_{0,\mathrm{Neu}}(\overline{U}). \tag{5.4}$$

LEMMA **5.1** *We have that*

$$\mathcal{E}^0(u,v) := \frac{1}{2}\int_U \langle A\nabla u, \nabla v \rangle d\mu, \quad u, v \in C_0^2(\overline{U}), \tag{5.5}$$

is closable in $L^2(U,\mu)$ *and its closure* $(\mathcal{E}^0, D(\mathcal{E}^0))$ *is a sectorial Dirichlet form.*

Proof Assume $(f_n)_{n\ge1} \subset C_0^2(\overline{U}) \longrightarrow 0$ in $L^2(U,\mu)$ and $\mathcal{E}^0(f_n - f_m, f_n - f_m) \longrightarrow 0$ as $n, m \to \infty$. Then $(f_n)_{n\ge1} \subset C_0^2(\overline{V}) := \{f_{|\overline{V}} \,|\, f \in C_0^2(\mathbb{R}^d)\} \longrightarrow 0$ in $L^2(V, u)$ and $\frac{1}{2}\int_V \langle A\nabla(f_n - f_m), \nabla(f_n - f_m)\rangle d\mu \longrightarrow 0$ for any bounded and open set $V \subset U$ as $n, m \to \infty$. By [103, Lemma 1.1], we obtain that $\langle A\nabla f_n, \nabla f_n\rangle \to 0$ in $L^1(V, m)$. From this it easily follows that there exists a subsequence such that $\lim_{k\to\infty}\langle A\nabla f_{n_k}, \nabla f_{n_k}\rangle \longrightarrow 0$ μ-a.e. on U. Therefore, by Fatou

$$\mathcal{E}^0(f_n, f_n) \le \liminf_{k\to\infty} \mathcal{E}^0(f_n - f_{n_k}, f_n - f_{n_k}),$$

which can be made arbitrarily small for big n. Hence $(\mathcal{E}^0, C_0^2(\overline{U}))$ is closable in $L^2(U,\mu)$. Finally, the conditions (5.2) on $\breve{a}_{ij}$ and (5.1) imply that the closure satisfies the strong sector condition and it is a Dirichlet form.

□

Consider the Dirichlet form $(\mathcal{E}^0, D(\mathcal{E}^0))$ and let $(L^0, D(L^0))$, $(G^0_\alpha)_{\alpha>0}$, $(T^0_t)_{t>0}$ be the corresponding generator, resolvent, and semigroup respectively. We denote by $(\widehat{\mathcal{E}}^0, D(\widehat{\mathcal{E}}^0))$ the co-form of $(\mathcal{E}^0, D(\mathcal{E}^0))$, i.e. the form defined by $\widehat{\mathcal{E}}^0(u,v) := \mathcal{E}^0(v,u)$. Note that $D(\mathcal{E}^0) = D(\widehat{\mathcal{E}}^0)$. Objects corresponding to the co-form $(\widehat{\mathcal{E}}^0, D(\widehat{\mathcal{E}}^0))$ are all noted with a hat. For instance, the corresponding co-resolvent, and co-semigroup are denoted by $(\widehat{G}^0_\alpha)_{\alpha>0}$, and $(\widehat{T}^0_t)_{t>0}$ respectively. Since for any $t > 0$, $\widehat{T}^0_t$ is sub-Markovian, $T^0_t|_{L^2(U,\mu)\cap L^1(U,\mu)}$ can be uniquely extended to a sub-Markovian contraction $\overline{T}^0_t$ on $L^1(U,\mu)$.

We further assume

(C) equality (5.4) extends to any $u \in C^2_0(\overline{U})$.

REMARK **5.1** *If U has a C^2 boundary, and $a_{ij} \in C^1(\partial U)$, $1 \le i,j \le d$, then* **(C)** *holds. (Definition of $C^1(\partial U)$ is given in Appendix.)*

Proof See Proposition 4.1.

□

(C) implies that (5.4) extends to any $u \in H^{1,2}(U,\mu)_0$, in particular,

$$\int_U \langle \beta, \nabla u\rangle v\, d\mu = -\int_U \langle \beta, \nabla v\rangle u\, d\mu \quad \forall u,\ v \in H^{1,2}(U,\mu)_{0,b}. \tag{5.6}$$

It is easy to see that $C^2_{0,\mathrm{Neu}}(\overline{U}) \subset D(L^0)$. Therefore, $Lu := L^0 u + \langle \beta, \nabla u\rangle, u \in D(L^0)_{0,b}$ is an extension of $L^A u + \langle B, \nabla u\rangle, u \in C^2_{0,Neu}(\overline{U})$.

For any bounded and open set $V \subset U$, denote by $(L^{0,V}, D(L^{0,V}))$ the generator of $\mathcal{E}^0(u,v)$; $u, v \in H^{1,2}(V,\mu)$, by $(G^{0,V}_\alpha)_{\alpha>0}$ the associated resolvent, by $(T^{0,V}_t)_{t>0}$ the associated sub-Markovian C_0-semigroup of contractions and by $(\overline{T}^{0,V}_t)_{t>0}$ its unique extension to $L^1(V,\mu)$. Let further $(\overline{L}^{0,V}, D(\overline{L}^{0,V}))$ be the generator and $(\overline{G}^{0,V}_\alpha)_{\alpha>0}$ be the resolvent of $(\overline{T}_t^{0,V})_{t>0}$. We denote here the dual objects on $L^1(V,\mu)$which are obtained through the co-form $(\widehat{\mathcal{E}}^0, D(\widehat{\mathcal{E}}^0))$ by $(\overline{L}^{0,V,'}, D(\overline{L}^{0,V,'}))$, $(\overline{G}_\alpha^{0,V,'})_{\alpha>0}$, $(\overline{T}_t^{0,V,'})_{t>0}$.

DEFINITION **5.1** *Let (S,D) be a densely defined linear operator on $L^p(U,\mu)$. If there is only one C^0-semigroup on $L^p(U,\mu)$ whose generator extends (S,D), then (S,D) is said to be L^p-unique. If (S,D) is a semi-bounded symmetric operator, it is known that L^2-uniqueness is equivalent to essential self-adjointness (see [29, 1 c) Corollary 1.2 and Lemma 1.4]).*

DEFINITION **5.2** *Let B be a Banach space, $f \in B$. An element $l \in B'$ that satisfies $\|l\|_{B'} = \|f\|_B$, and $l(f) = \|f\|_B^2$ is called a normalized tangent functional to f.*

DEFINITION **5.3** *A densely defined operator (S,D) on a Banach space B is said to be dissipative, if for each $f \in D$ there exists a normalized tangent functional l with $l(Lf) \le 0$.*

PROPOSITION **5.1** *Let $V \subset U$ be open and bounded. Then:*

(a) The operator $L^V = L^{0,V}u + \langle \beta, \nabla u\rangle, u \in D(L^{0,V})_b$, is dissipative, hence in particular closable, on $L^1(V,\mu)$. The closure $(\overline{L}^V, D(\overline{L}^V))$ generates a sub-Markovian C_0-semigroup of contractions $(\overline{T}_t^V)_{t\ge0}$.

(b) $D(\overline{L}^V)_b \subset H^{1,2}(V,\mu)$ and

$$\mathcal{E}^0(u,v) - \int \langle \beta, \nabla u\rangle v d\mu = -\int \overline{L}^V u\, v\, d\mu; u \in D(\overline{L}^V)_b, v \in H^{1,2}(V,\mu)_b.$$

In particular,

$$\mathcal{E}^0(u,u) = -\int \overline{L}^V u\, u\, d\mu; u \in D(\overline{L}^V)_b.$$

Before proving Proposition 5.1, we need some lemmas.

LEMMA **5.2** *Let* $(S, D(S))$ *be the generator of a strongly continuous contraction semigroup on a Hilbert space* $(H, (\cdot,\cdot))$. *Suppose there exists a constant* $K > 0$ *such that*

$$|((1-S)u, v)| \leq K((1-S)u, u)^{1/2}((1-S)v, v)^{1/2} \text{ for all } u, v \in D(S).$$

Then $T_t f \in D(S)$ *for all* $t > 0$, *and* $f \in H$.

Proof See [72, I.Theorem 2.20, I.Corollary 2.21].

□

LEMMA **5.3** *Let* $f \in H^{1,2}(V, \mu)$. *Suppose there exists* $g \in L^1(V, \mu)$ *such that*

$$\mathcal{E}^0(f, h) = \int g\, h\, d\mu \quad \forall h \in H^{1,2}(V, \mu) \cap L^\infty(V, \mu).$$

Then $f \in D(\overline{L}^{0,V})$ *and* $\overline{L}^{0,V} f = -g$.

Proof (we adapt here the proof of [17, I. Lemma 4.2.2.1] to the non-symmetric case). For any $v \in L^\infty(V, \mu)$, we have

$$\mathcal{E}^0(f, \widehat{G}_1^{0,V} v) = \int_V g\, \widehat{G}_1^{0,V} v\, d\mu = \lim_{n\to\infty} \int_V ((g \wedge n) \vee (-n))\, \widehat{G}_1^{0,V} v\, d\mu = \int_V (\overline{G}_1^{0,V} g)\, v\, d\mu.$$

On the other hand

$$\mathcal{E}^0(f, \widehat{G}_1^{0,V} v) = \int_V f(v - \widehat{G}_1^{0,V} v) d\mu = \int_V (f - \overline{G}_1^{0,V} f) v\, d\mu.$$

Hence $f = \overline{G}_1^{0,V}(g + f)$ and we get the desired result.

□

LEMMA **5.4** *Let* $(\mathcal{B}, D(\mathcal{B}))$ *be a coercive closed form on a Hilbert space* $(H, (\cdot,\cdot))$ *and let* C *be a non-empty closed linear subspace of* $D(\mathcal{B})$. *Let* J *be a continuous linear functional on* $D(\mathcal{B})$ *and* $\alpha > 0$. *Then there exists a unique* $v \in C$ *such that*

$$\mathcal{B}(v, w) + \alpha(v, w) = J(w) \text{ for all } w \in C.$$

Proof See [72, I.Exercise 2.7].

□

Proof (of Proposition 5.1) Because of Lemma 4.14, 5.2, 5.3, 5.4, we can proceed as in [96, Proposition 1.1], although we are in the non-symmetric case.

□

REMARK **5.2** *Since $-\beta$ satisfies the same assumption as β, the closure $(\overline{L}^{V,'}, D(\overline{L}^{V,'}))$ of $\hat{L}^{0,V}u - \langle \beta, \nabla u\rangle, u \in D(\hat{L}^{0,V})_{0,b}$ on $L^1(V,\mu)$ generates a sub-Markovian C_0-semigroup of contractions. If $(L^{V,'}, D(L^{V,'}))$ is the part of $(\overline{L}^{V,'}, D(\overline{L}^{V,'}))$ on $L^2(V,\mu)$ and $(L^V, D(L^V))$ is the part of $(\overline{L}^V, D(\overline{L}^V))$ on $L^2(V,\mu)$, then $(L^{V,'}, D(L^{V,'}))$ is the adjoint operator of $(L^V, D(L^V))$. For more details, see [96, Remark 1.3].*

LEMMA **5.5** *To distinguish d-dimensional Lebesgue measure and $(d-1)$-dimensional Lebesgue measure, we let λ^k be the k-dimensional Lebesgue measure on this lemma, $k = d,\ d-1$. Let $U_1 = \{(x_2, \cdots, x_d) \in \mathbb{R}^{d-1} |$ there exists x_1 such that $(x_1, x_2, \cdots, x_d) \in U\}$. Let $\tilde{\rho}^{(1)}$ be a λ^d-version of $\rho := \varphi^2$ which is absolutely continuous on the x_1-axis for λ^{d-1}-a.e. $(x_2, \cdots, x_d)$ in U_1. Then define the following space*

$$D(\bar{\mathcal{E}})_1 := \left\{ u \in L^2(U,\mu);\ \begin{array}{l} \textit{there exists a function } \tilde{u}^{(1)} \textit{ such that} \\ i)\ \tilde{u}^{(1)} = u, \mu - a.e. \\ ii)\ \textit{for } \lambda^{d-1} - a.e.\ (x_2, \cdots, x_d) \in U_1,\ \tilde{u}^{(1)}(x_1, x_2, \cdots, x_d) \\ \quad \textit{is absolutely continuous in } x_1 \textit{ on } \{x_1 \in \mathbb{R} | \tilde{\rho}^{(1)}(x_1, x_2, \cdots, x_d) > 0\} \\ \quad \textit{and } \partial \tilde{u}^{(1)}/\partial x_1 \in L^2(U,\mu) \end{array} \right\}$$

Then $D(\bar{\mathcal{E}})_1$ is independent of the choice of the version $\tilde{\rho}^{(1)}$ and $\partial \tilde{u}^{(1)}/\partial x_1$ is defined μ-a.e. Define $D(\bar{\mathcal{E}})_i$, $i = 2, \cdots, d$ analogously and let $D(\bar{\mathcal{E}}) = \cap_i D(\bar{\mathcal{E}})_i$. Then we have $D(\bar{\mathcal{E}}) = H^{1,2}(U,\mu)$ and $\partial \tilde{u}^{(1)}/\partial x_i = \partial_i u$.

Proof See [99, Lemma 6].

□

We get the following corollary by above Lemma.

COROLLARY **5.1** *For $\chi \in C_0^\infty(\overline{U})$ and $f \in H^{1,2}(U,\mu)$, $\chi f \in H^{1,2}(U,\mu)$ and $\partial_i(\chi f) = \partial_i \chi f + \chi \partial_i f$.*

THEOREM **5.1** *There exists a closed extension $(\overline{L}, D(\overline{L}))$ of*

$$Lu := L^0 u + \langle \beta, \nabla u \rangle, u \in D(L^0)_{0,b},$$

on $L^1(U,\mu)$ satisfying the following properties:

(a) *$(\overline{L}, D(\overline{L}))$ generates a sub-Markovian C_0-semigroup of contractions $(\overline{T}_t)_{t\geq 0}$.*

(b) *Let $(U_n)_{n\in\mathbb{N}}$ be any sequence of bounded open sets in $\mathbb{R}^d$, with $\overline{U}_n \subset U_{n+1}$ for any n, and $\cup_{n\geq 1} U_n = \mathbb{R}^d$. Set $V_n := U_n \cap U$, $n \geq 1$. Then $\lim_{n\to\infty} \overline{G}_\alpha^{V_n} f = (\alpha - \overline{L})^{-1} f$ in $L^1(U,\mu)$ for all $f \in L^1(U,\mu)$ and $\alpha > 0$.*

(c) *$D(\overline{L})_b \subset D(\mathcal{E}^0)$ and*

$$\mathcal{E}^0(u,v) - \int \langle \beta, \nabla u \rangle v \, d\mu = -\int \overline{L} u \, v \, d\mu; u \in D(\overline{L})_b, v \in H^{1,2}(U,\mu)_{0,b}.$$

Moreover,

$$\mathcal{E}^0(u,u) \leq -\int \overline{L} u \, u \, d\mu; u \in D(\overline{L})_b.$$

Before proving Theorem 5.1, we need the following lemma.

LEMMA **5.6** *Let Ω_1, Ω_2 be bounded open subsets of U with $\Omega_1 \subset \Omega_2$. Let $u \in L^1(U,\mu), u \geq 0$, and $\alpha > 0$. Then $\overline{G}_\alpha^{\Omega_1} u \leq \overline{G}_\alpha^{\Omega_2} u$.*

Proof The proof is basically the one of [96, Lemma 1.6], but we have to change the reason why $w_\alpha^+ \in H^{1,2}(\Omega_1, \mu)$.
Let $v \in C_0^2\big(\Omega_1 \cup (\partial U \cap \overline{\Omega}_1)\big)$ and let $\chi \in C_0^2\big(\Omega_1 \cup (\partial U \cap \overline{\Omega}_1)\big)$ be positive with $\chi \equiv 1$ on $\mathrm{supp}_{\Omega_1}(v)$. Then $(v - \overline{G}_\alpha^{\Omega_2} u)^+ = (\chi(v - \overline{G}_\alpha^{\Omega_2} u))^+$. Assume $(u_n)_{n\in\mathbb{N}} \subset C_0^2\big(\Omega_2 \cup (\partial U \cap \overline{\Omega}_2)\big)$ converges to $\overline{G}_\alpha^{\Omega_2} u$ in $H^{1,2}(\Omega_2, \mu)$. Since $(\chi u_n)_{n\in\mathbb{N}} \subset C_0^2\big(\Omega_1 \cup (\partial U \cap \overline{\Omega}_1)\big)$ converges to $\chi \overline{G}_\alpha^{\Omega_2} u$ in $H^{1,2}(\Omega_1, \mu)$ by Corollay 5.1, it follows $(v - \overline{G}_\alpha^{\Omega_2} u)^+ \in H^{1,2}(\Omega_1, \mu)$. Finally, choose $(v_n)_{n\in\mathbb{N}} \subset C_0^2\big(\Omega_1 \cup (\partial U \cap \overline{\Omega}_1)\big)$, such that $v_n \to \overline{G}_\alpha^{\Omega_1} u$ strongly in $H^{1,2}(\Omega_1, \mu)$. Then, $\mathcal{E}^0\left((v_n - \overline{G}_\alpha^{\Omega_2} u)^+, (v_n - \overline{G}_\alpha^{\Omega_2} u\right) \le \mathcal{E}^0\left(v_n - \overline{G}_\alpha^{\Omega_2} u, v_n - \overline{G}_\alpha^{\Omega_2} u)^+\right)$ which is bounded independent of n. Since $(v_n - \overline{G}_\alpha^{\Omega_2} u)^+ \to w_\alpha^+$ in $L^2(\Omega_1, \mu)$, we get that $(v_n - \overline{G}_\alpha^{\Omega_2} u)^+ \to w_\alpha^+$ weakly in $H^{1,2}(\Omega_1, \mu)$. Therefore, $w_\alpha^+ \in H^{1,2}(\Omega_1, \mu)$ by [72, I. Lemma 2.12].

□

Proof (of Theorem 5.1) Using Lemma 5.6, the proof is similar to [96, Theorem 1.5]. In particular, we note that $\mathcal{E}_\alpha^0(\cdot, v)$ for some fixed $v \in D(\mathcal{E}^0)$ is a linear functional on $D(\mathcal{E}^0)$ by sector condition.

□

REMARK **5.3** *(a) By Lemma 5.6, $(\overline{L}, D(\overline{L}))$ satisfying Theorem 5.1(a),(b), is uniquely determined.*

(b) Analogously to Theorem 5.1 (cf. Remark 5.2), we can construct a closed extension $(\overline{L}', D(\overline{L}'))$ of $\widehat{L}^0 u - \langle \beta, \nabla u \rangle, u \in D(\widehat{L}^0)_{0,b}$ on $L^1(U, \mu)$ and which satisfies the analogous properties to Theorem 5.1(a)-(c), with β replaced by $-\beta$ and $\mathcal{E}^0$ replaced by $\widehat{\mathcal{E}}^0$. We can easily see that

$$\int \overline{G}_\alpha u v d\mu = \int u \overline{G}'_\alpha v d\mu \text{ for all } u, v \in L^1(U, \mu)_b,$$

where $\overline{G}_\alpha = (\alpha - \overline{L})^{-1}$ and $\overline{G}'_\alpha = (\alpha - \overline{L}')^{-1}$.

(c) $(\overline{L}, D(\overline{L}))$ *is dissipative by Theorem 5.1(a) and [88, Theorem X.48], and hence its restriction* $(L, D(L^0)_{0,b})$ *is also dissipative.*

(d) $D(\overline{L})_b$ *(and then of course also* $D(\overline{L}')_b$*) is an algebra:*

Proof *The proof is basically same as in [96, Remark 1.7]. But we have to make some modification due to the non-symmetry. We shall precise the details below. Let* $u \in D(\overline{L})_b$*. It is enough to show that* $u^2 \in D(\overline{L})_b$*. To this end it suffices to prove that if* $g := 2u\overline{L}u + \langle \bar{A}\nabla u, \nabla u\rangle$ *then*

$$\int \overline{L}' v u^2 d\mu = \int g v d\mu \text{ for all } v = \overline{G}_1' h, h \in L^1(U,\mu)_b, \tag{5.7}$$

since then $\int \overline{G}_1(u^2 - g) h d\mu = \int (u^2 - g)\overline{G}_1' h d\mu = \int u^2(\overline{G}_1' h - \overline{L}'\overline{G}_1' h) d\mu = \int u^2 h d\mu$ *for all* $h \in L^1(U,\mu)_b$*, where* $\bar{A} = \{\bar{a}_{ij}\}$*. Consequently,* $u^2 = \overline{G}_1(u^2 - g) \in D(\overline{L})_b$*. For the proof of (5.7) fix* $v = \overline{G}_1' h, h \in L^1(U,\mu)_b$*, and suppose first that* $u = \overline{G}_1 f$ *for some* $f \in L^1(U,\mu)_b$*. Let* $u_n := \overline{G}_1^{V_n} f$ *and* $v_n = \overline{G}^{V_n,'} h$*, where* $(V_n)_{n\geq 1}$ *is as in Theorem 5.1(b). Then by Proposition 5.1 and Theorem 5.1,*

$$\begin{aligned}
\int \overline{L}^{V_n,'} v_n u u_n d\mu &= -\mathcal{E}^0(u u_n, v_n) - \int \langle \beta, \nabla v_n\rangle u u_n d\mu \\
&= -\mathcal{E}^0(u, v_n u_n) - \frac{1}{2}\int \langle A\nabla u_n, \nabla v_n\rangle u d\mu + \frac{1}{2}\int \langle A\nabla u, \nabla u_n\rangle v_n d\mu \\
&+ \int \langle \beta, \nabla u\rangle v_n u_n d\mu - \int \langle \beta, \nabla u_n\rangle u v_n d\mu \\
&= \int \overline{L} u v_n u_n d\mu + \int \overline{L}^{V_n} u_n v_n u d\mu + \frac{1}{2}\int \langle A\nabla u_n, \nabla(v_n u)\rangle d\mu \\
&- \frac{1}{2}\int \langle A\nabla u_n, \nabla v_n\rangle u d\mu + \frac{1}{2}\int \langle A\nabla u, \nabla u_n\rangle v_n d\mu \\
&= \int (\overline{L}u) v_n u_n d\mu + \int (\overline{L}^{V_n} u_n) v_n u d\mu + \int \langle \bar{A}\nabla u, \nabla u_n\rangle v_n d\mu.
\end{aligned}$$

Since u_n *converges to* u *weakly in* $D(\mathcal{E}^0)$*,* $\int \langle \bar{A}\nabla u, \nabla u_n\rangle v_n \, d\mu \to \int \langle \bar{A}\nabla u, \nabla u\rangle v \, d\mu$ *by the sector condition.*

(Indeed, $\lim_{n\to\infty}\int\langle\bar{A}\nabla u,\nabla u_n\rangle v\,d\mu=\lim_{n\to\infty}\int\langle\bar{A}\nabla u_n,\nabla u\rangle v_n\,d\mu$, *since* $v_n\to v$ μ*-a.e.,* $|v-v_n|$ *is bounded uniformly in* n *and* $|\langle\bar{A}\nabla u,\nabla u_n\rangle|$ *is bounded in* $L^1(U,\mu)$ *uniformly in* n*, because it converges to* $|\langle\bar{A}\nabla u,\nabla u\rangle|$ *in* $L^1(U,\mu)$*. So we can apply Lebesgue's theorem). Now, the remaining part of the proof is the same as [96, Remark 1.7], if we change* A *to* $\bar{A}$ *in the definition of* g_α *there.*

□

LEMMA **5.7** *Let* $u\in H^{1,2}(U,\mu)$ *and* $u=constant$ μ*-a.e. on a Borel measurable subset* B *of* u*. Then* $1_B|\nabla u|^2 d\mu=0$*, i.e.*

$$|\nabla u|=0\quad \mu\text{-a.e. on } B.$$

Proof See [104, Lemma 3.8 (iii)].

□

COROLLARY **5.2** *For* $u\in H^{1,2}(U,\mu)$*, support of* ∇u *is contained in support of* u*.*

PROPOSITION **5.2** *Assume* U *has a* C^2 *boundary and* $a_{ij}\in C^1(\partial U)$*. The following statements are equivalent:*

(a) *There exists* $(\chi_n)_{n\ge1}\subset H^{1,2}_{loc}(U,\mu)$ *and* $\alpha>0$ *such that* $(\chi_n-1)^-\in H^{1,2}(U,\mu)_{0,b}$, $n\ge1$, $\lim_{n\to\infty}\chi_n=0$ μ*-a.e. and*

$$\mathcal{E}^0_\alpha(v,\chi_n)+\int\langle\beta,\nabla\chi_n\rangle v d\mu\ge0 \text{ for all } v\in H^{1,2}(U,\mu)_{0,b}, v\ge0$$

for any $n\ge1$.

(b) $(L,D(L^0)_{0,b})$ *is* L^1*-unique.*

(c) μ *is* $(\overline{T}_t)$*-invariant.*

Proof (a)⇒(b): By Remark 5.3(c) and Lemma 4.14, it is enough to show that if $h \in L^\infty(U,\mu)$ is such that $\int_U (\alpha - L)uhd\mu = 0$ for all $u \in D(L^0)_{0,b}$ and some $\alpha > 0$, then $h = 0$. The rest of the proof is almost the same as [96, Proposition 1.9], but we have to change $C_0^\infty(\mathbb{R}^d)$ to $C^2_{0,Neu}(\overline{U})$, A to $\frac{1}{2}\bar{A}$, $\mathcal{E}^0$ to $\hat{\mathcal{E}}^0$ and $H_0^{1,2}(\mathbb{R}^d,\mu)_0$ to $H^{1,2}(U,\mu)_0$. Then similarly to [96, (1.22)], we obtain for $u \in D(L^0)_b$, $\chi \in C^2_{0,Neu}(\overline{U})$ that $\chi u \in D(L^0)_{0,b}$ and $\chi h \in D(\mathcal{E}^0)$ and that

$$\mathcal{E}^0_\alpha(u,\chi h) = \int \langle \beta, \nabla(\chi u)\rangle h\, d\mu + \int \langle A\nabla u, \nabla \chi\rangle h\, d\mu + \int (L^0\chi) uh\, d\mu.$$

Given arbitrary $u \in C^2_{0,Neu}(\overline{U})$, we can choose $\chi \in C^2_{0,Neu}(\overline{U})$ which is 1 on the support of u (see Lemma 4.5). Hence, the previous equality implies

$$\mathcal{E}^0_\alpha(u,h) - \int \langle \beta, \nabla u\rangle hd\mu = 0 \text{ for all } u \in C^2_{0,Neu}(\overline{U}).$$

Now, let $u \in H^{1,2}(U,\mu)_0$, and choose a sequence $u_n \in C^2_{0,Neu}(\overline{U})$, $n \geq 1$, such that $\lim_{n\to\infty} u_n = u$ in $H^{1,2}(U,\mu)$ (see Proposition 4.1). Take $\psi \in C^2_{0,Neu}(\overline{U})$ such that ψ is 1 on support of u by Lemma 4.5. Using Corollay 5.1, we can see that $\lim_{n\to\infty} \psi u_n = \psi u = u$ in $H^{1,2}(U,\mu)$. Hence,

$$\mathcal{E}^0_\alpha(u,h) - \int \langle \beta, \nabla u\rangle hd\mu = 0 \text{ for all } u \in H^{1,2}(U,\mu)_0. \tag{5.8}$$

Let $v_n := \|h\|_{L^\infty(U,\mu)}\chi_n - h$. Then $v_n^- \leq (\|h\|_{L^\infty(U,\mu)}\chi_n - \|h\|_{L^\infty(U,\mu)})^-$ μ-a.e. In particular, v_n^- is essentially bounded and has compact support. Choose a nonnegative $\psi \in C_0^2(\overline{U})$ such that $\psi = 1$ on the support of v_n^-. $\psi(\|h\|_{L^\infty(U,\mu)} - h) \in H^{1,2}(U,\mu)_{0,b}$ by Lemma 4.10 and $\psi(\|h\|_{L^\infty(U,\mu)}\chi_n - \|h\|_{L^\infty(U,\mu)})^- \in H^{1,2}(U,\mu)_{0,b}$ by Corollay 5.1. Hence

$$v_n^- = (\|h\|_{L^\infty(U,\mu)}\chi_n - h)^- = (\underbrace{\psi\|h\|_{L^\infty(U,\mu)}\chi_n}_{\in H^{1,2}(U,\mu)_0 \text{ by Corollay 5.1}} - \underbrace{\psi h}_{\in H^{1,2}(U,\mu)_0 \text{ by Lemma 4.10}})^- \in H^{1,2}(U,\mu)_{0,b}\,.$$

Moreover,

$$0 \le \mathcal{E}^0_\alpha(v_n^-, v_n) - \int \langle \beta, \nabla v_n^- \rangle v_n d\mu \le -\alpha \int (v_n^-)^2 d\mu.$$

First inequality holds by using the fact that $-\int \langle \beta, \nabla v_n^- \rangle \chi_n d\mu \underset{(I)}{=} \int \langle \beta, \nabla \chi_n \rangle v_n^- d\mu$, the assumption on χ_n, and (5.8).

(I) holds by (5.6), Corollary 5.2 and replacing χ_n by $-M \vee \psi\chi_n \wedge M$, where M is large enough constant such that $\chi_n = -M \vee \psi\chi_n \wedge M$ on the support of v_n^-.

Second inequality holds by using the fact that $\int \langle \beta, \nabla v_n^- \rangle v_n d\mu \underset{(II)}{=} \int \langle \beta, \nabla v_n^- \rangle v_n^- d\mu \underset{(III)}{=} 0$, $\mathcal{E}^0(v_n^-, v_n) \underset{(IV)}{\le} 0$.

(II) holds by Corollary 5.2.

(III) holds by (5.6).

(IV) holds by Lemma 5.7.

Thus $v_n^- = 0$, i.e., $h \le \|h\|_{L^\infty(U,\mu)}\chi_n$. Similarily, $-h \le \|h\|_{L^\infty(U,\mu)}\chi_n$, hence $|h| \le \|h\|_{L^\infty(U,\mu)}\chi_n$. Since $\lim_{n\to\infty} \chi_n = 0$ μ-a.e. it follows that $h = 0$ μ-a.e.

(b)⇒(c): Same as [96, Proposition 1.9](ii)⇒(iii).

(c)⇒(a): Let (V_n) be as in Theorem 5.1. By Remark 5.2, the closure of $\hat{L}^{0,V_n}u - \langle \beta, \nabla u \rangle$, $u \in D(L^{0,V_n})_b$, on $L^1(V_n, \mu)$ generates a sub-Markovian C_0-semigroup. Let $\chi_n := 1 - \overline{G}_1^{V_n,\prime} 1_{V_n}$, $n \ge 1$. By the dual version of Proposition 5.1 (see Remark 5.2), we have $\overline{G}_1^{V_n,\prime} 1_{V_n} \in H^{1,2}(V_n, \mu) \subset H^{1,2}(U, \mu)$. Hence $\chi_n \in H^{1,2}_{loc}(U, \mu)$ and $(\chi_n - 1)^- \in H^{1,2}(U, \mu)_{0,b}$.

Fix $n \ge 1$ and let $w_\gamma := \gamma \overline{G}'_{\gamma+1} \overline{G}_1^{V_n,\prime} 1_{V_n}, \gamma > 0$. Since $w_\gamma \ge \gamma \overline{G}_{\gamma+1}^{V_n,\prime} \overline{G}_1^{V_n,\prime} 1_{V_n}$ and

$$\gamma \overline{G}_{\gamma+1}^{V_n,\prime} \overline{G}_1^{V_n,\prime} 1_{V_n} = \overline{G}_1^{V_n,\prime} 1_{V_n} - \overline{G}_{\beta+1}^{V_n,\prime} 1_{V_n} \ge \overline{G}_1^{V_n,\prime} 1_{V_n} - \frac{1}{\gamma+1}$$

by the resolvent equation it follows that

$$w_\gamma \geq \overline{G}_1^{V_n,\prime} 1_{V_n} - \frac{1}{\gamma+1} \quad \text{for all } \gamma > 0. \tag{5.9}$$

Now, we have (using in particular the dual version of Theorem 5.1, see Remark 5.3(b) for the equality)

$$\begin{aligned}
\mathcal{E}_1^0(w_\gamma, w_\gamma) &\leq \gamma(\overline{G}_1^{V_n,\prime} 1_{V_n} - w_\gamma, w_\gamma)_{L^2(U,\mu)} \\
&\leq \gamma(\overline{G}_1^{V_n,\prime} 1_{V_n} - w_\gamma, \overline{G}_1^{V_n,\prime} 1_{V_n})_{L^2(U,\mu)} \\
&= \mathcal{E}_1^0(\overline{G}_1^{V_n,\prime} 1_{V_n}, w_\gamma) + \int \langle \beta, \nabla w_\gamma \rangle \overline{G}_1^{V_n,\prime} 1_{V_n} d\mu \\
&\leq K \mathcal{E}_1^0(w_\gamma, w_\gamma)^{\frac{1}{2}} \left(\mathcal{E}_1^0(\overline{G}_1^{V_n,\prime} 1_{V_n}, \overline{G}_1^{V_n,\prime} 1_{V_n})^{\frac{1}{2}} + \sqrt{\lambda_{V_n}} \| |\beta| 1_{V_n} \|_{L^2(U,\mu)} \right)
\end{aligned}$$

where K is a weak-sector constant of $(\mathcal{E}^0, D(\mathcal{E}^0))$.

By the above, we see that $\lim_{\gamma\to\infty} w_\gamma = \overline{G}_1^{V_n,\prime} 1_{V_n}$ weakly in $D(\mathcal{E}^0)$. Let $J(f) := \mathcal{E}_1^0(u, f)$. Then by the weak sector condition, J is a continuous linear form on $D(\mathcal{E}^0)$. Hence, $J \in D(\mathcal{E}^0)'$ and $J(w_\gamma) \to J(\overline{G}_1^{V_n,\prime} 1_{V_n})$. For $u \in H^{1,2}(U,\mu)_{0,b}, u \geq 0$,

$$\begin{aligned}
\mathcal{E}_1^0(u, \chi_n) + \int \langle \beta, \nabla \chi_n \rangle u \, d\mu &= \lim_{\beta\to\infty} \left(\int u \, d\mu - \mathcal{E}_1^0(u, w_\gamma) - \int \langle \beta, \nabla w_\gamma \rangle u \, d\mu \right) \\
&= \lim_{\beta\to\infty} \left(\int u \, d\mu - \gamma \int (\overline{G}_1^{V_n,\prime} 1_{V_n} - w_\gamma) u \, d\mu \right) \geq 0
\end{aligned}$$

by (5.9). Since χ_n is decreasing by Lemma 5.6, $\chi_\infty := \lim_{n\to\infty} \chi_n$ exists μ-a.e. If $g \in L^1(U,\mu)_b$, then

$$\begin{aligned}
\int g \chi_\infty d\mu &= \lim_{n\to\infty} \int g \chi_n \, d\mu = \lim_{n\to\infty} \left(\int g \, d\mu - \int g \overline{G}_1^{V_n,\prime} 1_{V_n} \, d\mu \right) \\
&= \lim_{n\to\infty} \left(\int g \, d\mu - \int \overline{G}_1^{V_n} g 1_{V_n} \, d\mu \right) \\
&= \int g \, d\mu - \int \overline{G}_1 g \, d\mu = 0
\end{aligned}$$

since μ is $(\overline{T}_t)$-invariant. Hence $\int g\chi_\infty\, d\mu = 0$ for all $g \in L^1(U,\mu)_b$ and we get the desired result.

$\square$

REMARK **5.4** *The proof of (c)⇒(a) in Proposition 5.2 shows that if μ is $(\overline{T}_t)$-invariant then there exists for all $\alpha > 0$ a sequence $(\chi_n)_{n\ge1} \subset H^{1,2}_{loc}(U,\mu)$ such that $(\chi_n - 1)^- \in H^{1,2}(U,\mu)_{0,b}$, $n \ge 1$, $\lim_{n\to\infty} \chi_n = 0$ μ-a.e. and*

$$\mathcal{E}^0_\alpha(v,\chi_n) + \int \langle \beta, \nabla\chi_n\rangle v d\mu \ge 0 \textit{ for all } v \in H^{1,2}(U,\mu)_{0,b}, v \ge 0$$

for any $n \ge 1$.
Indeed, it suffices to take $\chi_n := 1 - \overline{G}_1^{V_n,'} 1_{V_n}$, $n \ge 1$.

THEOREM **5.2** *Let $d \ge 2$. Assume that for all compact K in $\overline{U}$, there exist $L_K \ge 0$ and $s_K \in (0,1)$ such that*

$$|\bar{a}_{ij}(x) - \bar{a}_{ij}(y)| \le L_K|x-y|^{s_K} \textit{ for all } x,y \in K.$$

Assume further that there exist bounded and open sets $U_n \subset \mathbb{R}^d$, $n \ge 1$ such that $\cup_n U_n = \mathbb{R}^d$ and $V_n := U_n \cap U$ is a bounded $C^{2,s_{\overline{V}_n}}$-domain and $a_{ij} \in C^{1,s_{\overline{V_n}}}(\partial U \cap \overline{V_n})$. Let $h \in L^\infty(U,\mu)$ be such that $\int (1-L)uhd\mu = 0$ for all $u \in C^2_{0,Neu}(\overline{U})$. Then $h \in H^{1,2}_{loc}(U,\mu)$ and $\mathcal{E}^0_1(u,h) - \int \langle\beta,\nabla u\rangle hd\mu = 0$ for all $u \in H^{1,2}(U,\mu)_0$.

Before proving Theorem 5.2, we need following lemmas.

LEMMA **5.8** *(Hopf's Lemma)*
Let $\Omega \subset \mathbb{R}^d$ be a bounded C^2-domain and $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$. Let $B = (b_{ij})$ be a non necessarily symmetric matrix of continuous functions $b_{ij} : \overline{\Omega} \to \mathbb{R}$, $1 \le i,j \le d$, which is uniformly strictly elliptic, i.e., there exists a constant $\theta > 0$

$$\sum_{i,j=1}^d b_{ij}(x)\xi_i\xi_j \ge \theta|\xi|^2 \qquad \forall \xi = (\xi_1,\dots,\xi_d) \in \mathbb{R}^d,\ x \in \Omega.$$

Let $L_\alpha u := (\alpha - \sum_{ij} b_{ij}\partial_i\partial_j)u$*. Assume that for some* $\alpha \geq 0$*,* $L_\alpha u \leq 0$ *in* Ω *and there exists* $x_0 \in \partial\Omega$ *such that* $u(x_0) \geq 0$ *and* $u(x_0) > u(x)$ *for all* $x \in \Omega$*. Then*

$$\langle B^T \eta, \nabla u\rangle(x_0) < 0.$$

Proof We follow the proof as in [30, LEMMA (ii) on page 347].
Since Ω is a C^2 domain, it satisfies the interior ball condition at x_0, i.e., there exists an open ball $B \subset \Omega$ such that $x_0 \in \partial B$. We may assume $B = B_r(0)$, where $B_r(0)$ is the open ball of radius r centered at 0 for some $r > 0$. Define

$$v(x) := e^{-\lambda|x|^2} - e^{-\lambda r^2}, \quad x \in B_r(0),$$

for $\lambda > 0$ as selected below. Let $\delta_{ij} := 1$ if $i = j$ and 0 otherwise, $1 \leq i, j \leq d$. Then using the uniform strict ellipticity condition, we compute

$$\begin{aligned} L_\alpha v &= (\alpha - \sum_{ij} b_{ij}\partial_i\partial_j)v \\ &= e^{-\lambda|x|^2}\sum_{ij} b_{ij}(-4\lambda^2 x_i x_j + 2\lambda\delta_{ij}) + \alpha(e^{-\lambda|x|^2} - e^{-\lambda r^2}) \\ &\leq e^{-\lambda|x|^2}(-4\theta\lambda^2|x|^2 + 2\lambda trace(B) + \alpha) \end{aligned}$$

Consider next the open annular region $R := B_r(0) - \overline{B_{r/2}(0)}$. We have (since all b_{ij} are continuous)

$$L_\alpha v \leq e^{-\lambda|x|^2}(-\theta\lambda^2 r^2 + 2\lambda trace(B) + \alpha) \leq 0 \tag{5.10}$$

in R, provided $\lambda > 0$ is fixed large enough.
Since $u(x_0) > u(x)$ for all $x \in \Omega$, there exists a constant $\varepsilon > 0$ so small that

$$u(x_0) \geq u(x) + \varepsilon v(x), \quad x \in \partial B_{r/2}(0). \tag{5.11}$$

In addition note

$$u(x_0) \geq u(x) + \varepsilon v(x), \quad x \in \partial B_r(0). \tag{5.12}$$

since $v \equiv 0$ on $\partial B_r(0)$.

From (5.10) we see

$$L_\alpha(u + \varepsilon v - u(x_0)) \le -\alpha u(x_0) \le 0 \text{ in } R,$$

and from (5.11), (5.12) we observe

$$u + \varepsilon v - u(x_0) \le 0 \text{ on } \partial R.$$

In view of weak maximum principle, [30, p346, Theorem 2], $u + \varepsilon v - u(x_0) \le 0$ in R. But $u(x_0) - \varepsilon v(x_0) - u(x_0) = 0$, and so

$$\langle B^T \eta, \nabla u\rangle(x_0) + \varepsilon \langle B^T \eta, \nabla v\rangle(x_0) \le 0$$

by uniform strict ellipticity.

Consequently,

$$\langle B^T \eta, \nabla u\rangle(x_0) \le -\varepsilon \langle B^T \eta, \nabla v\rangle(x_0) = -\frac{\varepsilon}{r}\langle -B^T x_0, \nabla v(x_0)\rangle = \frac{-2\lambda\varepsilon}{r}\langle B^T x_0, x_0\rangle e^{-\lambda r^2} < 0$$

as desired.

□

LEMMA **5.9** *Assume $B = (b_{ij})$ satisfies the same assumptions as in Lemma 5.8 and let L_α be defined as in Lemma 5.8. Let $\Omega \subset \mathbb{R}^d$ be a bounded connected C^2-domain and $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$. Assume that for some $\alpha > 0$, $L_\alpha u \le 0$ in Ω and $\langle B^T \eta, \nabla u\rangle \ge 0$ on $\partial\Omega$. Then $u \le 0$ in Ω. Moreover, $u < 0$ in Ω if u is not a constant.*

Proof If u is constant, it is easy to see that the conclusion holds. Assume u is not constant. Assume to the contrary that there exists $y \in \Omega$ such that $u(y) \ge 0$.

By the strong maximum principle [30, Theorem 4 on page 350] u attains its nonnegative maximum at some point x_0 of the boundary and $u(x_0) > u(x)$ for any $x \in \Omega$. But this leads to a contradiction by Lemma 5.8, since $\langle B^T \eta, \nabla u\rangle \ge 0$.

□

REMARK **5.5** *In Lemma 5.9, we can get $u \le 0$ in Ω even if Ω is not connected, since we can apply this result on each connected component.*

LEMMA **5.10** *Assume $B = (b_{ij})$ satisfies the same assumptions as in Lemma 5.8 and let L_α be defined as in Lemma 5.8. Let $\alpha > 0$ and $\Omega \subset \mathbb{R}^d$ be a bounded open C^2-domain. Suppose $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ satisfies $\langle B^T \eta, \nabla u \rangle = 0$ on $\partial\Omega$. Then in Ω*

$$\alpha |u| \le sup_\Omega |L_\alpha u|.$$

Proof It is enough to show $\alpha u \le sup_\Omega (L_\alpha u)$, since $-u$ satisfies the same assumptions than u. Then $\pm \alpha u \le sup_\Omega |L_\alpha u|$ and the result follows.
Define $M := sup_\Omega (L_\alpha u)$ and let $w = u - \alpha^{-1} M$. Then $L_\alpha w = L_\alpha u - M \le 0$ in Ω. By Lemma 5.9 and Remark 5.5, we get $w \le 0$ in Ω, which implies the assertion.

□

LEMMA **5.11** *Assume $d \ge 2$ here. Assume $B = (b_{ij})$ satisfies the same assumptions as in Lemma 5.8 and let L_α be defined as in Lemma 5.8. Moreover, assume $\bar{b}_{ij} := \dfrac{b_{ij} + b_{ji}}{2} \in C^{0,s}(\overline{\Omega})$ and $b_{ij} \in C^{1,s}(\partial\Omega)$. Let $\Omega \subset \mathbb{R}^d$ be a bounded $C^{2,s}$-domain and let $f \in C^s(\overline{\Omega})$, $0 < s < 1$, $\alpha > 0$. Then*

$$\begin{cases} L_\alpha u = f & \text{in } \Omega, \\ \langle B^T \eta, \nabla u \rangle = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.13}$$

has a unique solution $u := R_\alpha f \in C^{2,s}(\overline{\Omega})$. Moreover:

(i) R_α is positivity preserving, i.e.

$$f \in C^\alpha(\overline{\Omega}),\ f(x) \ge 0,\ \forall x \in \Omega \implies R_\alpha f(x) \ge 0,\ \forall x \in \Omega.$$

(ii) $\sup_\Omega |\alpha R_\alpha f| \le \sup_\Omega |f|$.

Proof The first statement follows from [68, Chapter 3, Theorem 3.1 page 135](i) and (ii) follow from Remark 5.5 and Lemma 5.10, respectively.

□

Proof (of Theorem 5.2) We follow the proof of [96, Theorem 2.1], but since the changes are subtle, we shall explain the details of the whole proof.

Let $\chi \in C^2_{0,Neu}(\overline{U})$ and $r > 0$ be such that $\mathrm{supp}(\chi) \subset B_r(0)$. Choose $n \in \mathbb{N}$ such that $\overline{B_{2r}(0)} \subset U_n$. Choose $\psi \in C_0^\infty(U_n)$ which is 1 on $\overline{B_r(0)}$, 0 on $U_n - B_{2r}(0)$, and $0 \le \psi \le 1$. Let $\tilde{A} = \{\tilde{a}_{ij}\}$ be such that $\tilde{A} := \psi A + (1-\psi)I$, where I is the identity matrix. Then $\tilde{a}_{ij}(x) = a_{ij}(x)$ for all $x \in B_r(0) \cap U$, there exists some constants $L > 0$, $0 < s < 1$ such that $|\hat{a}_{ij}(x) - \hat{a}_{ij}(y)| \le L|x-y|^s$ for all $x, y \in U$, and $\hat{a}_{ij} = \dfrac{\tilde{a}_{ij} + \tilde{a}_{ji}}{2}$. Moreover, $\tilde{a}_{ij} \in C^{1,s}\overline{V_n}(\partial V_n)$ and satisfies the uniform ellipticity condition. Let L_α be defined as in Lemma 5.8 with $b_{ij} := \frac{\tilde{a}_{ij}}{2}, 1 \le i,j \le d$, satisfying the conditions of Theorem 5.2. Then by Lemma 5.11, for any $f \in C_0^\infty(V_n)$ there exists a unique $R_\alpha^n f \in C^{2,\alpha}(\overline{V_n})$ with $L_\alpha R_\alpha^n f = f$ in V_n, satisfying Lemma 5.11(i) and (ii). By a standard procedure (see e.g. proof of [96, Theorem 2.1]) αR_α^n can be extended to a sub-Markovian operator αV_α^n on the bounded Borel measurable functions $\mathcal{B}_b(V_n)$ on V_n such that $V_\alpha^n f = R_\alpha^n f$ for any function $f \in C_\infty(V_n)$, i.e. any continuous function f on V_n that vanishes at infinity. Choose $(f_k)_{k\ge1} \subset C_0^\infty(V_n)$ such that $f_k \to h1_{V_n}$ μ-a.e. and $\|f_k\|_{L^\infty(V_n,\mu)} \le \|h\|_{L^\infty(V_n,\mu)}$. Then $\lim_{k\to\infty} \alpha V_\alpha^n f_k = \alpha V_\alpha^n(h1_{V_n})$ pointwise on V_n by Lebesgue's theorem, hence $\lim_{k\to\infty} \chi\alpha V_\alpha^n f_k = \chi\alpha V_\alpha^n(h1_{V_n})$ on $L^2(V_n,\mu)$ again by Lebesgue's theorem. Next, one shows that $(\chi\alpha V_\alpha^n f_k)_{k\ge1}$ is $\mathcal{E}^0$-bounded, which further implies that $\chi\alpha V_\alpha^n(h1_{V_n}) \in D(\mathcal{E}^0)$ and that $(\chi\alpha V_\alpha^n f_k)_{k\ge1}$ converges weakly to $\chi\alpha V_\alpha^n(h1_{V_n})$ in $D(\mathcal{E}^0)$ by [72, I. Lemma 2.12]. Then, we derive the analogous equalities and inequalities to (2.2) and (2.3) in the proof of [96, Theorem 2.1]. By this we get that $(\chi\alpha V_\alpha^n(h1_{V_n}))_{\alpha>0}$ is bounded in $D(\mathcal{E}^0)$. Then again as in the proof of [96, Theorem 2.1] we get that the limit of some weakly convergent subsequence $(\chi\alpha_k V_{\alpha_k}^n(h1_{V_n}))_{k\geq 1}$ with $\alpha_k \to \infty$ equals $\chi h 1_{V_n} = \chi h$ μ-a.e, since every closed ball in Hilbert space is weakly sequentially compact. In particular, $\chi h \in H^{1,2}(U,\mu)_{0,b}$. Moreover, Lemma 4.10 implies $h \in H^{1,2}_{loc}(U,\mu)$.

Let $u \in H^{1,2}(U,\mu)_0$. Then by Proposition 5.2, there exists a sequence $v_n \in C^2_{0,Neu}(\overline{U})$, $n \geq 1$, such that $\lim_{n\to\infty} v_n = u$ in $H^{1,2}(U,\mu)$. Let $\psi \in C^2_{0,Neu}(\overline{U})$ be such that ψ is 1 on support of u (see Lemma 4.5). Using Corollay 5.1, we get $\lim_{n\to\infty}\psi v_n = \psi u = u$ in $H^{1,2}(U,\mu)$. Let $u_n := \psi v_n$ and $\chi \in C_0^\infty(\overline{U})$ be such that χ is 1 on support of ψ.

Then

$$
\begin{aligned}
\mathcal{E}_1^0(u,h) &- \int \langle \beta, \nabla u\rangle h d\mu = \lim_{n\to\infty}(\mathcal{E}_1^0(u_n,h) - \int \langle \beta, \nabla u_n\rangle h d\mu) \\
&= \lim_{n\to\infty}\int (1-L)u_n \chi h d\mu = 0.
\end{aligned}
$$

□

REMARK **5.6** *Assume the boundary of U is of class $C^{l-1,1}$ and $a_{ij} \in C^{l-2,1}(\overline{U}) = \{f_{|\overline{U}} | f \in C^{l-2,1}(\mathbb{R}^d)\}$, where $l = 3 + \max\{k \in \mathbb{Z} | k \leq \frac{d}{2}\}$. Let $h \in L^\infty(U,\mu)$ be such that $\int (1-L)uhd\mu = 0$ for all $u \in C^2_{0,Neu}(\overline{U})$. Then $h \in H^{1,2}_{loc}(U,\mu)$ and $\mathcal{E}_1^0(u,h) - \int \langle\beta,\nabla u\rangle hd\mu = 0$ for all $u \in H^{1,2}(U,\mu)_0$.*

Proof Let $(\mathcal{A}, D(\mathcal{A}))$ be the closure of (5.5) (see Lemma 5.1) with $\mu = dx$, and $(W_\alpha)_{\alpha>0}$ (resp. $L^{\mathcal{A}}$) be corresponding resolvent (resp. generator). By Lemma 4.8, we have $D(\mathcal{A} = H^{1,2}(U)_0$ and if $f \in L^2(U)$ and $g \in H^{1,2}(U)_0$, then

$$
\mathcal{A}_\alpha(W_\alpha f, g) = \frac{1}{2}\int \langle A\nabla W_\alpha f, \nabla g\rangle dx + \alpha \int W_\alpha f g dx = \int f g dx. \tag{5.14}
$$

If $\chi \in C^l_{0,Neu}(\overline{U})$ and $f \in C_0^\infty(U)$, then $\chi W_\alpha f \in D(\mathcal{A})_0$. Then, using the product rule, (5.14), and the Gauss-Green Theorem, we get

$$\frac{1}{2}\int \langle A\nabla(\chi W_\alpha f), \nabla g\rangle dx + \alpha \int \chi W_\alpha f g dx = \int f' g dx \text{ for all } g \in H^{1,2}(U)_0, \quad (5.15)$$

where $f' = \chi f - \frac{1}{2}\sum_{i,j}\{a_{ij}(\partial_i \chi)(\partial_j W_\alpha f) + \partial_i(a_{ij}(\partial_j \chi)W_\alpha f)\}$. By a classical regularity result for weak solutions of the Neumann problem(cf. [18, Remark 24 written under Theorem 9.26]), $\chi W_\alpha f \in H^{l,2}(U)$ (This theorem also holds for unbounded U by exactly analogous proof using the compactness of supp$\chi \cap \overline{U}$. The only part using the boundedness of U is when we apply partition of unity on $\overline{U}$, but it's OK if supp$\chi \cap \overline{U}$ is compact. See also Lemma 4.11. Note that we can assume boundary of U is of class $C^{1-1,1}$ instead of C^l by [37, Lemma 1.3.3.1].), and hence of class $C_0^2(\overline{U})$ by [30, 5.6.3. Theorem 6] (This theorem also holds for unbounded U by exactly analogous proof using the compactness of supp$\chi \cap \overline{U}$ similar to above.) Now, (5.15) implies $\chi W_\alpha f \in C^2_{0,Neu}(\overline{U})$.

Now, choose $V \subset U$ such that V is open and bounded, supp$(\chi) \subset V$ and choose $(f_n)_{n\geq 1} \subset C_0^\infty(V)$ such that $f_n \to h1_V$ μ-a.e. and $\|f_n\|_{L^\infty(V,\mu)} \leq \|h\|_{L^\infty(V,\mu)}$. Since μ is equivalent to dx, we get $\lim_{n\to\infty} f_n = h1_V$ in $L^2(V)$ by Lebesgue's theorem and $\lim_{n\to\infty} \alpha W_\alpha f_n = \alpha W_\alpha(h1_V)$ in $L^2(V)$. Passing to a subsequence, we can assume $\lim_{n\to\infty} \alpha W_\alpha f_n = \alpha W_\alpha(h1_V)$ μ-a.e. Now the remaining parts of the proof is similar to Theorem 5.2, but we explain the details. Let $B' := (b'_1, \cdots, b'_d)$, where $b'_i = \sum_{j=1}^d \frac{\partial_j a^T_{ij}}{2}$.

Then

$$\begin{aligned}
\mathcal{E}^0(\chi\alpha W_\alpha f_n, \chi\alpha W_\alpha f_n) &= -\int L^0(\chi\alpha W_\alpha f_n)\chi\alpha W_\alpha f_n d\mu \\
&= -\int \chi L^{\mathcal{A}}\chi(\alpha W_\alpha f_n)^2 d\mu - \int \langle \bar{A}\nabla\chi, \nabla(\chi\alpha W_\alpha f_n)\rangle \alpha W_\alpha f_n d\mu \\
&\quad + \int \langle \bar{A}\nabla\chi, \nabla\chi\rangle(\alpha W_\alpha f_n)^2 d\mu - \alpha\int (\alpha W_\alpha f_n - f_n)\chi^2\alpha W_\alpha f_n d\mu \\
&\quad - \int \langle \beta^{A,\varphi^2} - B', \nabla(\chi\alpha W_\alpha f_n)\rangle \chi\alpha W_\alpha f_n d\mu.
\end{aligned}$$

Hence $(\chi\alpha W_\alpha f_n)_{n\geq 1}$ is $\mathcal{E}^0$-bounded, which further implies that $\chi\alpha W_\alpha(h1_V) \in D(\mathcal{E}^0)$ and that $(\chi\alpha W_\alpha f_k)_{k\geq 1}$ converges weakly to $\chi\alpha W_\alpha(h1_V)$ in $D(\mathcal{E}^0)$. Then, we also derive the analogous inequality to [96, (2.3)], which is as follows.

$$\begin{aligned}
\mathcal{E}^0(\chi\alpha W_\alpha h, \chi\alpha W_\alpha h) &\leq \liminf_{k\to\infty} \mathcal{E}^0(\chi\alpha W_\alpha f_k, \chi\alpha W_\alpha f_k) \\
\leq -\int \chi L^{\mathcal{A}}\chi(\alpha W_\alpha(h1_V))^2 d\mu &- \int \langle \bar{A}\nabla\chi, \nabla(\chi\alpha W_\alpha(h1_V))\rangle \alpha W_\alpha(h1_V) d\mu \\
+ \int \langle \bar{A}\nabla\chi, \nabla\chi\rangle(\alpha W_\alpha(h1_V))^2 d\mu &- \int \chi^2(\alpha W_\alpha(h1_V))h d\mu \\
+ \int \langle B - B', \nabla(\chi\alpha W_\alpha(h1_V))\rangle \chi h d\mu &+ \int \langle B - B', \nabla\chi\rangle\chi(\alpha W_\alpha(h1_V))h d\mu \\
+2\int \langle \bar{A}\nabla\chi, \nabla(\chi\alpha W_\alpha(h1_V))\rangle h d\mu &- 2\int \langle \bar{A}\nabla\chi, \nabla\chi\rangle(\alpha W_\alpha(h1_V))h d\mu \\
+ \int L^{\mathcal{A}}(\chi^2)(\alpha W_\alpha(h1_V))h d\mu &- \int \langle \beta^{A,\varphi^2} - B', \nabla(\chi\alpha W_\alpha(h1_V))\rangle \chi\alpha W_\alpha(h1_V) d\mu
\end{aligned}$$

By this we get that $(\chi\alpha W_\alpha(h1_V))_{\alpha>0}$ is bounded in $D(\mathcal{E}^0)$. Then again as in the proof of [96, Theorem 2.1] we get that the limit of some weakly convergent subsequence $(\chi\alpha_k W_{\alpha_k}(h1_V))_{k\geq 1}$ with $\alpha_k \to \infty$ equals $\chi h1_V = \chi h$ μ-a.e. In particular, $\chi h \in H^{1,2}(U,\mu)_{0,b}$.

Then we get

$$\mathcal{E}_1^0(u,h) - \int \langle \beta, \nabla u\rangle h d\mu = 0 \quad h \in H^{1,2}(U,\mu)_0$$

as in Theorem 5.2.

□

COROLLARY **5.3** *Assume all the conditions as in Remark 5.6. Fix* $\alpha > 0$*. For all* $f \in C_0^\infty(\overline{U})$*, there exists* $u \in C^2_{0,Neu}(\overline{U})$ *such that* $(\alpha - L)u = f$ *on* $suppf$*. For arbitrary open set* $\Omega \subset \mathbb{R}^d$ *containing* $suppf$*, we can further assume that* $suppu$ *is contained in* Ω*.*

Proof f' defined in Remark 5.6 coincides with f on supp$f$.

□

COROLLARY **5.4** *Under the assumptions of Theorem 5.2 (resp. Remark 5.6),* $(L, C^2_{0,Neu}(\overline{U}))$ *is* L^1*-unique if and only if* μ *is* $(\overline{T}_t)$*-invariant.*

Proof Almost same with [96, Corollary 2.2]. Just change $\mathbb{R}^d$ to U, $C_0^\infty(\mathbb{R}^d)$ to $C^2_{0,Neu}(\overline{U})$, $H_0^{1,2}(\mathbb{R}^d, \mu)_0$ to $H^{1,2}(U, \mu)_0$, [96, Proposition 1.9] to Proposition 5.2, [96, Theorem 2.1] to Theorem 5.2 (resp. Remark 5.6).

□

Since $(\overline{T}_t)$-invariance of μ is equivalent to conservativeness of $(\overline{T}'_t)$, we get the following:

COROLLARY **5.5** *Under the assumptions of Theorem 5.2 (resp. Remark 5.6),* $(L, C^2_{0,Neu}(\overline{U}))$ *is* L^1*-unique if and only if* $(\overline{T}'_t)$ *is conservative, where* $(\overline{T}'_t)$ *is the semigroup of* $(\overline{L}', D(\overline{L}'))$*.*

PROPOSITION **5.3** *Let* U *have a* C^2 *boundary and* $a_{ij} \in C^1(\partial U)$*. Each of the following conditions (a) and (b) imply that* μ *is* $(\overline{T}_t)$*-invariant.*

(a) $\bar{a}_{ij}$*,* $b_i - \beta_i^{A,\varphi^2} \in L^1(U, \mu)$*,* $1 \le i, j \le d$*.*

(b) There exists $u \in C^2(\overline{U})$ *with* $\langle \nabla u, A\eta \rangle \le 0$ $\varphi^2 d\sigma$*-a.e. on* ∂U *and* $\alpha > 0$ *such that*

$\lim_{|x|\to\infty} u(x) = +\infty$ *(If* U *is unbounded) and* $L^A u + \langle \beta^{A^T,\varphi^2} - \beta, \nabla u\rangle \le \alpha u$.

(c) $-\langle A(x)x, x\rangle/(|x|^2+1) + \frac{1}{2}trace(A(x)) + \langle(\beta^{A^T,\varphi^2} - \beta)(x), x\rangle \le M(|x|^2\ln(|x|^2+1)+1)$ *for some* $M \ge 0$. *We further assume one of the following :*

(i) $A = I$ *and* U *is star-shaped centered at* 0, *i.e., for all* x *in* U *the line segment from* 0 *to* x *is contained in* U.

(ii) U *is a ball centered at* 0.

Proof (a) By Proposition 5.2, it is enough to show that $(L, D(L^0)_{0,b})$ is L^1-unique. By Remark 5.3(c) and Lemma 4.14, it hence suffices to show that if $h \in L^\infty(U,\mu)$ is such that

$$\int_U (\alpha - L)uh\,d\mu = 0 \text{ for all } u \in D(L^0)_{0,b}, \tag{5.16}$$

for some $\alpha > 0$, then $h = 0$. From the proof of Proposition 5.2 (a)⇒(b), we know that (5.16) implies $\chi h \in H^{1,2}(U,\mu)$ for $\chi \in C^2_{0,Neu}(\overline{U})$ and

$$\mathcal{E}^0_\alpha(u,h) - \int \langle \beta, \nabla u\rangle h\,d\mu = 0 \text{ for all } u \in H^{1,2}(U,\mu)_0. \tag{5.17}$$

Moreover, Lemma 4.10 implies $h \in H^{1,2}_{loc}(U,\mu)$. Now, for $\chi \in C_0^\infty(\overline{U})$ (for intermediate steps see proof of Proposition 4.2.)

$$\frac{1}{2}\int_U \langle A\nabla(\chi h), \nabla(\chi h)\rangle d\mu + \alpha\int_U (\chi h)^2 d\mu$$
$$= \underbrace{\frac{1}{2}\int \langle \beta, \nabla(\chi^2)\rangle h^2 d\mu}_{=:I} - \underbrace{\int h\langle \check{A}\nabla\chi, \nabla(\chi h)\rangle d\mu}_{=:II} + \underbrace{\frac{1}{2}\int h^2\langle A\nabla\chi, \nabla\chi\rangle d\mu}_{=:III}.$$

Now, choose a sequence $\chi_k \in C_0^\infty(\overline{U})$ such that it is 1 on $B_k(0)\cap U$, $0 \le \chi_k \le 1$ and $|\nabla\chi_k| \le \frac{1}{k}$, $k \in \mathbb{N}$. Replacing χ by χ_k in the latter equation, we get for constant C (which could be different from inequality to inequality but is always independent of

k), we get using in particular (5.2)

$$|I| \leq \frac{1}{2}\underbrace{|\nabla(\chi_k^2)|}_{\leq \frac{2}{k}}\|h\|^2_{L^\infty(U,\mu)}\sum_{i=1}^d \int |b_i - \beta_i^{A,\varphi^2}|d\mu,$$

$$\begin{aligned}
|II| &\leq M\int |h|\lambda^{-1}_{\operatorname{supp}\chi_k}|\nabla\chi_k\|\nabla(\chi_k h)|d\mu \\
&\leq C\|h\|_{L^\infty(U,\mu)}\underbrace{|\nabla\chi_k|}_{\frac{1}{k}}\int_{\operatorname{supp}\chi_k}\lambda^{-1/2}_{\operatorname{supp}\chi_k}\lambda^{-1/2}_{\operatorname{supp}\chi_k}|\nabla(\chi_k h)|d\mu \\
&\leq \frac{C}{k}(\int_{\operatorname{supp}\chi_k}\lambda^{-1}_{\operatorname{supp}\chi_k}d\mu)^{1/2}(\int_{\operatorname{supp}\chi_k}\lambda^{-1}_{\operatorname{supp}\chi_k}|\nabla(\chi_k h)|^2 d\mu)^{1/2} \\
&\leq \frac{C}{k}\|\bar{a}_{11}\|^{1/2}_{L^1(U,\mu)}\left(\mathcal{E}^0(\chi_k h,\chi_k h)\right)^{1/2},
\end{aligned}$$

$$|III| \leq C\underbrace{|\nabla\chi_k|^2}_{\leq \frac{1}{k^2}}\|h\|^2_{L^\infty(U,\mu)}\sum_{i,j=1}^d \int |\bar{a}_{ij}|d\mu.$$

Hence we get

$$\mathcal{E}^0_\alpha(\chi_k h,\chi_k h) \leq \frac{C}{k}\left(\mathcal{E}^0_\alpha(\chi_k h,\chi_k h)\right)^{1/2} + \frac{C}{k}.$$

Hence $\mathcal{E}^0_\alpha(\chi_k h,\chi_k h) \to 0$ as $k \to \infty$. In particular, $\alpha\|h\|^2_{L^2(U,\mu)} = 0$, hence $h = 0$ and we get the desired result.

(b) Let $\chi_n := \frac{u}{n}$. Then $\lim_{n\to\infty}\chi_n = 0$ and $\chi_n \in H^{1,2}_{loc}(U,\mu)$, $(\chi_n - 1)^-$ is bounded and has compact support. Thus $(\chi_n - 1)^- \in H^{1,2}(U,\mu)_{0,b}$.

For all $v \in H^{1,2}(U,\mu)_{0,b}, v \geq 0$,

$$\begin{aligned}
\mathcal{E}^0_\alpha(v,\chi_n) + \int \langle\beta,\nabla\chi_n\rangle v d\mu &= \frac{1}{2}\int_U\langle A\nabla v,\nabla\chi_n\rangle d\mu + \alpha\int_U v\chi_n d\mu + \int\langle\beta,\nabla\chi_n\rangle v d\mu \\
&= -\int_U(L^A\chi_n + \langle\beta^{A^T,\varphi^2} - \beta,\nabla\chi_n\rangle)v d\mu + \alpha\int_U v\chi_n d\mu \\
&\quad -\frac{1}{2}\int_{\partial U}\langle\nabla\chi_n, A\eta\rangle v\varphi^2 d\sigma \geq 0
\end{aligned}$$

Now Proposition 5.2 implies the desired result.

(c) By taking $u(x) = \ln(|x|^2 + 1) + r$ for sufficiently large r, we can apply (b).

$\square$

REMARK **5.7** *Let U have a C^2 boundary and $a_{ij} \in C^1(\partial U)$.*
(a) We can replace the assumption (5.2) on Proposition 5.3-(a) by weak sector condition, i.e., there exists a constant $K > 0$ such that

$$\int_U \langle \check{A}\nabla f, \nabla g\rangle d\mu \le K\mathcal{E}_1(f,f)^{1/2}\mathcal{E}_1(g,g)^{1/2}, \quad f,\ g \in C_0^\infty(\overline{U})$$

if $\bar{a}_{ij}$, $b_i - \beta_i^{A,\varphi^2}$, $\partial_i \check{a}_{ij} \in L^1(U,\mu)$, $\check{a}_{ij} \in L^p(U,\mu)$, $\dfrac{\nabla\varphi}{\varphi} \in L^{p'}(U,\mu)$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, $1 \le p, p' \le \infty$, $1 \le i,j \le d$, and $\check{A}^T\eta = 0$ $\varphi^2 d\sigma$-a.e. on ∂U.
(b) Suppose μ is finite. Then with the same proof as in [96, Remark 1.11.(i)], we can see that μ is $(\overline{T}_t)$-invariant if and only if μ is $(\overline{T}'_t)$-invariant. Then we can replace $\beta^{A^T,\varphi^2} - \beta$ in Proposition 5.3 (b) by $\beta^{A,\varphi^2} + \beta = B$, and $A\eta$ by $A^T\eta$ and the implication still holds true.
(c) Suppose that there exists a bounded, nonnegative and nonzero function $u \in C^2(\overline{U})$ with $\langle \nabla u, A\eta\rangle \ge 0$ $d\sigma$-a.e. on ∂U and $\alpha > 0$ such that $L^A u + \langle \beta^{A^T,\varphi^2} - \beta, \nabla u\rangle \ge \alpha u$. Then μ is not $(\overline{T}_t)$-invariant.

Proof (a) As in the proof of Proposition 5.3-(a), it is enough to show that if $h \in L^\infty(U,\mu)$ is such that (5.16) holds for some $\alpha > 0$, then $h = 0$. We also know that $h \in H^{1,2}_{loc}(U,\mu)$ and (5.17) holds. Now, for $\chi \in C_0^\infty(\overline{U})$ (for intermediate steps see proof of Proposition 4.2.)

$$
\begin{aligned}
&\frac{1}{2}\int_U \langle A\nabla(\chi h), \nabla(\chi h)\rangle d\mu + \alpha\int_U (\chi h)^2 d\mu \\
&= \frac{1}{2}\int_U \langle \beta, \nabla(\chi^2)\rangle h^2 d\mu - \int_U (\chi h)\langle \breve{A}\nabla\chi, \nabla h\rangle d\mu + \frac{1}{2}\int_U h^2\langle A\nabla\chi, \nabla\chi\rangle d\mu \\
&= \frac{1}{2}\int_U \langle \beta, \nabla(\chi^2)\rangle h^2 d\mu - \frac{1}{4}\int_U \langle \breve{A}\nabla\chi^2, \nabla h^2\rangle d\mu + \frac{1}{2}\int_U h^2\langle A\nabla\chi, \nabla\chi\rangle d\mu \\
&= \underbrace{\frac{1}{2}\int_U \langle \beta, \nabla(\chi^2)\rangle h^2 d\mu}_{=:I} + \underbrace{\frac{1}{4}\int_U \sum_{i,j}\partial_i \breve{a}_{ij}\partial_j(\chi^2)h^2 d\mu}_{=:II} \\
&\quad + \underbrace{\frac{1}{4}\int_U \sum_{i,j}\breve{a}_{ij}\partial_j(\chi^2)\frac{2\partial_i\varphi}{\varphi}h^2 d\mu}_{=:III} + \underbrace{\frac{1}{2}\int_U h^2\langle A\nabla\chi, \nabla\chi\rangle d\mu}_{=:IV}.
\end{aligned}
$$

Now, choose a sequence $\chi_k \in C_0^\infty(\overline{U})$ such that it is 1 on $B_k(0)\cap U$, $0 \le \chi_k \le 1$ and $|\nabla\chi_k| \le \frac{1}{k}$, $k \in \mathbb{N}$ and put it in above equation instead of χ.

Then, for some constant C which could be different on each inequality but independent of k, we get

$$|I| \le C\underbrace{|\nabla(\chi_k^2)|}_{\le\frac{2}{k}}\|h\|^2_{L^\infty(U,\mu)}\sum_{i=1}^d \int |b_i - \beta_i^{A,\varphi^2}|d\mu,$$

$$|II| \le C\underbrace{|\nabla(\chi_k^2)|}_{\le\frac{2}{k}}\|h\|^2_{L^\infty(U,\mu)}\sum_{i,j=1}^d \int |\partial_i \breve{a}_{ij}|d\mu,$$

$$|III| \le C\underbrace{|\nabla(\chi_k^2)|}_{\le\frac{2}{k}}\|h\|^2_{L^\infty(U,\mu)}\Big(\sum_{i,j=1}^d \|\breve{a}_{ij}\|_{L^p(U,\mu)}\Big)\|\nabla\varphi\|_{L^{p'}(U,\mu)},$$

$$|IV| \le C\underbrace{|\nabla\chi_k|^2}_{\le\frac{1}{k^2}}\|h\|^2_{L^\infty(U,\mu)}\sum_{i,j=1}^d \int |\tilde{a}_{ij}|d\mu.$$

By letting $k \to \infty$, we can conclude as in the proof of Proposition 5.3(a).

(c) It is almost same with [96, Remark 1.11.(ii)], but we repeat the proof here.

We may suppose that $u \le 1$. If μ would be $(\overline{T}_t)$-invariant it would follow that there exist $\chi_n \in H^{1,2}_{loc}(U,\mu)$, $n \ge 1$, such that $(\chi_n - 1)^- \in H^{1,2}(U,\mu)_{0,b}$, $\lim_{n\to\infty} \chi_n = 0$ μ-a.e. and $\mathcal{E}^0_\alpha(v,\chi_n) + \int \langle \beta, \nabla \chi_n \rangle v d\mu \ge 0$ for all $v \in H^{1,2}(U,\mu)_{0,b}$, $v \ge 0$ (by Remark 5.4). Let $v_n := (\chi_n - u)$. Then $v_n^- \le (\chi_n - 1)^-$ μ-a.e. In particular, v_n^- is essentially bounded and has compact support. Choose a nonnegative $\psi \in C_0^2(\overline{U})$ such that $\psi = 1$ on the support of v_n^-. Note that $v_n^- = (\chi_n - u)^- = \big(\underbrace{\psi(\chi_n - u)}_{\in H^{1,2}(U,\mu)_0}\big)^- \in H^{1,2}(U,\mu)_{0,b}$ by Corollay 5.1.

$$0 \le \mathcal{E}^0_\alpha(v_n^-, v_n) - \int \langle \beta, \nabla v_n^- \rangle v_n d\mu \le -\alpha \int (v_n^-)^2 d\mu.$$

First inequality holds by using the fact that $-\int \langle \beta, \nabla v_n^- \rangle \chi_n d\mu \underset{(I)}{=} \int \langle \beta, \nabla \chi_n \rangle v_n^- d\mu$, and the assumptions on χ_n and u.

(I) holds by (5.6), Corollary 5.2 and replacing χ_n by $-M \vee \psi\chi_n \wedge M$, where M is large enough constant such that $\chi_n = -M \vee \psi\chi_n \wedge M$ on the support of v_n^-.

Second inequality holds by using the fact that $\int \langle \beta, \nabla v_n^- \rangle v_n d\mu \underset{(II)}{=} -\int \langle \beta, \nabla v_n^- \rangle v_n^- d\mu \underset{(III)}{=} 0$, $\mathcal{E}^0(v_n^-, v_n) \underset{(IV)}{\le} 0$.

(II) holds by Corollary 5.2.

(III) holds by (5.6).

(IV) holds by Lemma 5.7.

Thus $v_n^- = 0$, i.e., $u \le \chi_n$. Since $\lim_{n\to\infty} \chi_n = 0$ μ-a.e. and $u \ge 0$ it follows that $u = 0$ which is a contradiction to our assumption $u \ne 0$.

□

EXAMPLE **5.1** *Let* $\mu := e^{-x^2}dx$, $B(x) = -2x - 6e^{x^2}$, $Lu := u'' + Bu'$, $u \in C^2_{0,Neu}(\overline{U})$, *where* $U = (a, \infty)$, $-\infty \le a < +\infty$, $(\overline{L}, D(\overline{L}))$ *be the maximal extension having properties (a)-(c) in Theorem 5.1 and* $(\overline{T}_t)_{t\ge 0}$ *be the associated semigroup. Let* $h(x) := \int_{-\infty}^x e^{-t^2} dt$, $x \in U$. *Then* h *satisfies the assumption of Remark 5.7 (c), hence* μ *is not*

$(\overline{T}_t)$*-invariant.*

5.3 Elliptic regularity and L^2-uniqueness

In this section for $\Omega \subset \mathbb{R}^d$, Ω open, $L^p(\Omega)$ be a usual L^p space on Ω and $H^{1,p}(\Omega)$ be a classical sobolev space of order 1 in $L^p(\Omega)$. Assume $d \geq 2$ throughout this section. In this section, we follow the proof of [10, Chapter 2. Elliptic case.] . Assume U and A be as in Remark 5.6(resp. Theorem 5.2). On this section, because we need one of two different conditions, we will use (resp.) for the condition of Theorem 5.2. For $0 < p < \infty$, let p' denote the number satisfying $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

LEMMA **5.12** *Let ν be a locally finite (not necessarily non-negative) Borel measure on U such that for any compact $K \subset U$ some $C_K > 0$,*

$$\int_{U\cap K} \sum_{i,j} a_{ij}\partial_i\partial_j f d\nu \leq C_K \left(\sup_{U\cap K} |f| + \sup_{U\cap K} |\nabla f| \right) \tag{5.18}$$

for all nonnegative $f \in C^2_{0,Neu}(\overline{U})$. Then ν is absolutely continuous with respect to Lebesgue measure with $\dfrac{d\nu}{dx} \in L^r_{loc}(U)$ for every $r \in [1, d')$.

Proof The proof is analogous to [10, Theorem 2.1 (ii)], but we will repeat again here, because there are some changes.

Let D_0 be a ball in $\mathbb{R}^d$ and let $\xi \in C^2_{0,Neu}(\overline{U})$ be such that $0 \leq \xi \leq 1$, $\xi = 1$ on $D_0 \cap \overline{U}$ and the suppξ belongs to $D \cap \overline{U}$, where D is a ball in $\mathbb{R}^d$(resp. we take $D = U_n$ satisfying supp$\xi \subset U_n \cap \overline{U}$ instead), this is possible by Corollary 4.5. Let us consider the measure $\nu' = \xi\nu$. By substituting $\xi\psi$ in place of f in (5.18), for every nonnegative $\psi \in C^2_{0,Neu}(\overline{U})$, we obtain

$$\left| \int_{D\cap U} \sum_{i,j} a_{ij}\partial_i\partial_j \psi d\nu' \right| \leq C_1 \left(\sup_{D\cap U} |\psi| + \sup_{D\cap U} |\nabla\psi| \right) \tag{5.19}$$

where C_1 is a constant independent of ψ as in the proof of [10, Theorem 2.1 (ii)]. Note that above inequality still remains true for every $\psi \in C^2(D \cap U)$ with $\langle A^T \eta, \psi \rangle = 0$ on $\partial(D \cap U)$ again as in the proof of [10, Theorem 2.1 (ii)]. Now fix λ big enough to satisfy the inequality on [36, Theorem 2.3.3.6](resp. arbitrary $\lambda > 0$). Now let $r > d$. By Corollary 5.3(resp. the proof of Theorem 5.2), given $g \in C_0^\infty(D \cap U)$, there exists a function $u \in C^2(\overline{D \cap U})$ such that

$$\lambda u - \sum_{i,j} a_{ij} \partial_i \partial_j u = g$$

on $D \cap U$ and $\langle A^T \eta, u \rangle = 0$ on ∂U(resp. $\langle A^T \eta, u \rangle = 0$ on $\partial(D \cap U)$). Now [36, Theorem 2.3.3.6](resp. [37, Lemma 9.17]) implies, there exists a constant C_2 independant of g such that

$$\|u\|_{H^{2,r}(D \cap U)} \leq C_2 \|g\|_{L^r(D \cap U)}.$$

By Morrey's theorem,

$$\sup_{D \cap U} |\nabla u| + \sup_{D \cap U} |u| \leq C_3 \|g\|_{L^r(D \cap U)}$$

for some constant $C_3 > 0$.

Together with (5.19) yields

$$\int_{D \cap U} g d\nu' \leq C_4 \|g\|_{L^r(D \cap U)}, \quad \forall g \in C_0^\infty(D \cap U) \tag{5.20}$$

for some $C_4 > 0$. Hence we get the desired result.

□

For a Banach space B and $n \in \mathbb{N}$, let $B^n := \underbrace{B \oplus \cdots \oplus B}_{n}$ with norm $\|f\|_{B^n} := \sum_{i=1}^b \|f_i\|_B$, where $f = (f_1, \cdots, f_n) \in B^n$.

LEMMA **5.13** *Let Ω be an open set. For $\nu \in L^1_{loc}(\Omega)$, define $L_\nu : (C_0^\infty(\Omega))^{d+1} \to \mathbb{R}$ by*

$$L_\nu(v) = \int \nu(v_0 - \sum_{i=1}^d \partial_i v_i) dx,$$

where $v = (v_0, \cdots, v_d) \in (C_0^\infty(\Omega))^{d+1}$. *If* $|L_\nu(v)| \le C\|v\|_{(L^{p'}(\Omega))^{d+1}}$ *for some constant* $C > 0$, *then* $\nu \in H^{1,p}(\Omega)$.

Proof Take $v = (v_0, 0, \cdots, 0)$. Then $|\int \nu v_0 dx| \le C\|v_0\|_{L^{p'}(\Omega)}$. Hence $\nu \in L^p(\Omega)$. Now, take $v = (0, v_1, 0, \cdots, 0)$. Then $|\int \nu \partial_1 v_1 dx| \le C\|v_1\|_{L^{p'}(\Omega)}$. Therefore, there exsits $\nu_1 \in L^p(\Omega)$ such that $\int \nu \partial_1 v_1 dx = \int \nu_1 v_1 dx$. Hence ν is weakly differentialble with respect to the first coordinate, and $\partial_1 \nu = \nu_1 \in L^p(\Omega)$. Similarily, we can get the desired result.

□

LEMMA **5.14** *Let* $r > 1$, $q \in [r', \infty)$. *Fix arbitrary* $x \in \overline{U}$. *Then there exists a bounded open neighborhood* $D \subset \mathbb{R}^d$ *of* x *with the following property. Let* $\nu \in L^r_{loc}(U)$ *such that for any* $f \in C_0^2(\overline{D \cap U})$ *with* $\langle A^T \eta, f \rangle = 0$ *on* $\partial(D \cap U)$, *we have*

$$\left| \int_{D \cap U} \sum_{i,j} a_{ij} \partial_i \partial_j f \nu dx \right| \le C\|f\|_{H^{1,q}(D \cap U)} \tag{5.21}$$

with C *independent of* f.
Then $\nu \in H^{1,q'}(D \cap U)$.

Proof Take arbitrary D which is a neighborhood of x(resp. take $D = U_n$ such that $x \in U_n$). By Corollary 5.3(resp. the proof of Theorem 5.2), given $g = (g_0, g_1, \cdots, g_d)$, $(g_i)_{0 \le i \le d} \in C_0^\infty(D \cap U)$, there exists a function $u \in C_0^2(\overline{D \cap U})$ such that

$$\lambda u - \sum_{i,j} a_{ij} \partial_i \partial_j u = g_0 + \sum_i \partial_i g_i \tag{5.22}$$

on $D \cap U$ and $\langle A^T \eta, u \rangle = 0$ on $\partial(D \cap U)$ for any λ.
By putting into (5.21), we get

$$\left| \int_{D \cap U} (g_0 + \sum_i \partial_i g_i - \lambda u) \nu dx \right| \le C\|u\|_{H^{1,q}(D \cap U)}. \tag{5.23}$$

Hence

$$\left|\int_{D\cap U}(g_0+\sum_i \partial_i g_i)\nu dx\right| \le C_0\|u\|_{H^{1,q}(D\cap U)} \tag{5.24}$$

for some constant $C_0 >$. By Lemma 5.13, it remains to prove $\|u\|_{H^{1,q}(D\cap U)} \le C_1\|g\|_{(L^q(\Omega))^{d+1}}$ for some constant $C_1 > 0$.

By Remark 4.4, u, g_i can be extended to a function $\tilde{u} \in H^{2,2}(\mathbb{R}^d)$, $\tilde{g}_i \in L^p(\mathbb{R}^d)$ such that $\|\tilde{g}_i\|_{L^p(\mathbb{R}^d)} \le C_2\|g_i\|_{L^p(D\cap U)}$ for some constant C_2 independent of u, g_i. Now the result follow from [63, 4.4 Theorem 2].

□

LEMMA **5.15** *Let $p > d$, $r \in (p', \infty)$, $\nu \in L^r_{loc}(U)$, and let $\gamma \in L^p_{loc}(U)$ or $\gamma \in L^p_{loc}(U, \nu dx)$. Assume that, for every $f \in C^2_{0,Neu}(U)$, we have*

$$\left|\int_U a_{ij}\partial_i\partial_j f\nu dx\right| \le \int_U (|f|+|\nabla f|)|\gamma\nu|dx. \tag{5.25}$$

Then, $\nu \in H^{1,p}_{loc}(U)$.

Proof Fix arbitrary $x \in \overline{U}$ and choose D as in Lemma 5.14. Let $q = pr/(pr-p-r) > r'$. By analogous argument of [10, Theorem 2.8], we can see that $|\gamma\nu| \in L^{q'}_{loc}(U)$. Again, analogous argument of [10, Theorem 2.8] shows

$$\left|\int_{D\cap U} a_{ij}\partial_i\partial_j f(\zeta\nu)dx\right| \le C\|\nabla f\|_{L^q(D\cap U)} \tag{5.26}$$

for any $f \in C^2_{0,Neu}(U)$, and $\zeta \in C^2_{0,Neu}(U)$ such that supp$\zeta \subset D$ for some constant C independent of f. Since we can take any $x \in \overline{U}$, $\nu \in H^{1,q'}_{loc}(U)$. Now, exactly analogous proof of [10, Theorem 2.8] can be applied.

□

COROLLARY **5.6** *Let $p > d$, and let $b_i \in L^p_{loc}(U)$. Let ν be a locally finite Borel measure satisfying*

$$\int_U L^A u + \langle B, \nabla u\rangle \, d\nu = 0 \quad \forall u \in C^2_{0,Neu}(\overline{U}). \tag{5.27}$$

Then ν is absolutely continuous with respect to Lebesgue measure with $\frac{d\nu}{dx} \in H^{1,p}_{loc}(U)$.

Proof It easily follows from Lemma 5.12 and Lemma 5.15.

□

By Riesz-Thorin Interpolation Theorem, $(\overline{T}_t)$ determines uniquely a semigroup of contractions (T_t) on $L^2(U,\mu)$ whose generator extends $(L, C^2_{0,Neu}(\overline{U}))$.
Note that the following theorem extends [9, Theorem 7].

THEOREM **5.3** *Let $p > d$, $b_i \in L^p_{loc}(U,\mu)$, $\beta_i \in L^\infty(U)$, $\bar{a}_{ij} \in L^\infty(U)$ and φ is locally uniformly positive and is locally bounded, i.e., $essinf_V \varphi > 0$ and $\|\varphi\|_{L^\infty(V,\mu)} < \infty$ on each bounded set $V \subset U$, where $\gamma > d$, then we can get L^2-uniqueness of $(L, C^2_{0,Neu}(\overline{U}))$.*

Proof Assume $h \in L^2(U,\mu)$ such that

$$\int_U (1-L)uh d\mu = 0 \text{ for all } u \in C^2_{0,Neu}(\overline{U}).$$

By Corollary 5.6, we have $h \in H^{1,p}_{loc}(U)$. Therefore, for $\chi \in C_0^\infty(U)$,

$$\begin{aligned} &\frac{1}{2}\int_U \langle A\nabla(\chi h), \nabla(\chi h)\rangle d\mu + \int_U (\chi h)^2 d\mu \\ &= \frac{1}{2}\int \langle \beta, \nabla(\chi^2)\rangle h^2 d\mu - \int h\langle \check{A}\nabla\chi, \nabla(\chi h)\rangle d\mu + \frac{1}{2}\int h^2\langle A\nabla\chi, \nabla\chi\rangle d\mu. \end{aligned}$$

Now the remaining parts of the proof is exactly analogous with Proposition 4.2.

□

REMARK **5.8** *Assume ∂U is of class $C^{0,1}$. If $\dfrac{\nabla\varphi}{\varphi} \in L^{\gamma}_{loc}(U)$ for some $\gamma > d$, φ is locally uniformly positive and is locally bounded.*

Proof Proof is exactly analogous of that of Remark 4.5 using [37, Lemma 1.3.3.1].

□

Now we will show an application of our result.

PROPOSITION **5.4** *Assume further that $A = I$ and U is a star-shaped domain centered at 0. Let $b_i \in L^p_{loc}(U)$ for some $p > d$. Suppose that there exists $M \geq 0$ such that*

$$\langle B(x), x\rangle \leq M(|x|^2 \ln(|x|^2 + 1) + 1) \quad \textit{for all } x \in U. \tag{5.28}$$

Then there exists at most one probability measure μ satisfying

$$b_i \in L^1_{loc}(U, \mu) \textit{ and } \int \Delta u + \langle B, \nabla u\rangle d\mu = 0 \quad \textit{for all } u \in C^2_{0,Neu}(\overline{U}). \tag{5.29}$$

Proof Let μ_1, μ_2 be two probability measures satisfying (5.29) and let $\mu = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$. Clearly, μ satisfies (5.29) again. By Corollary 5.6, μ is absolutely continuous with respect to Lebesgue measure and for the density ρ we have that $\rho \in H^{1,p}_{loc}(U)$. By [102, Corollary 5.3], ρ admits a positive continuous modification, thus $\varphi := \sqrt{\rho} \in H^{1,2}_{loc}(U)$ and $b_i \in L^2_{loc}(U, \mu)$.
Now, we follow the proof [96, Proposition 2.8] to show $(L, C^2_{0,Neu}(\overline{U}))$ is L^1-unique by Remark 5.6(resp. Theorem 5.2).
Again, analogous proof of [96, Proposition 2.8] shows $h := \frac{d\mu_1}{d\mu} \in D(\mathcal{E}^0)$ and $\mathcal{E}^0(h, h) = 0$, and hence $\mu_1 = \mu_2$.

□

5.4 Examples

(This is the condition for Distorted Brownian motion to satisfy the assumption of this chapter.)

All the conditions are as in the Introduction and $d \geq 2$(The case $d = 1$ can be similarly, but easier. Here we omit the proof). Assume additionally that $A = I + \check{A}, a_{ij} \in C^1(\overline{U}), U : C^2$ domain, $\varphi \in C^1(\partial U)$.
Additionally, we also assume that the one of Condition Theorem 5.2 (resp. Remark 5.6) is satisfied.

$$\begin{aligned}
&\int_{\partial U} \langle \check{A}\nabla f, \eta\rangle \varphi^2 d\sigma = 0 \quad \text{ for all } f \in C^2_{0,Neu}(\overline{U}) \\
\iff &\int_{\partial U} \langle \nabla f, \check{A}^T \eta\rangle \varphi^2 d\sigma = 0 \quad \text{ for all } f \in C^2_{0,Neu}(\overline{U}) \\
\iff &\int_{\partial U} \langle \nabla f, \check{A}^T \eta\rangle \varphi^2 d\sigma = 0 \quad \text{ for all } f \in C^2(\partial U)
\end{aligned}$$

The last equivalence holds because, if $f \in C^2_{0,Neu}(\overline{U})$, the restriction of f to ∂U is clearly $C^2(\partial U)$. Conversely, let $f \in C^2(\partial U)$. Let (Ω_x, ϕ_x) be usual C^2 diffeomorphisms as in Definition 4.1 whose domains are neighborhoods Ω_x of $x \in \partial U$. As in previous chapter, we can assume ϕ_i maps $A^T\eta$ to $(0, 0, \cdots, 0, 1)$. Since $supp f$ is compact, we can assume $supp f \subset \cup_{i=1}^n \Omega_i$ for some $\Omega_i = \Omega_{x_i}$, $x_i \in \partial U$. Let η_i be partition of unity subordinated to Ω_i such that $\sum_i \eta_i = 1$ on $supp f$. Then $(\eta_i f)\circ\phi^{-1}$ is a function on $\mathbb{R}^d \cap x_d = 0$. We can easily extend $(\eta_i f)\circ\phi^{-1}$ to $C^2(\overline{\mathbb{R}}^d_+)$ whose support is contained in image of ϕ_i(For example, we can define a function $F \in C^2(\overline{\mathbb{R}}^d_+)$ such that $F(x_1, \cdots, x_{d-1}, x_d) = (\eta_i f)\circ\phi^{-1}(x_1, \cdots, x_{d-1})$ and we can multiply some function $C_0^\infty(\overline{\mathbb{R}}^d_+)$ whose support(in $\overline{\mathbb{R}}^d_+$) is contained in image of ϕ_i and is 1 on the support(in $\partial\overline{\mathbb{R}}^d_+$) of $\phi^{-1}(x_1, \cdots, x_{d-1})$).
By returning back this extended functions to U, we get the desired result.
To get further result from this, we need integration by parts on the Riemannian

manifold. Note that ∂U is a $d-1$ dimensional Riemannian manifold. Since $\langle \check{A}^T\eta, \eta\rangle = 0$, $\check{A}^T\eta$ is a tangent vector field on the manifold ∂U.

$$\begin{aligned}
&\int_{\partial U} \langle \nabla f, \check{A}^T\eta\rangle \varphi^2 d\sigma = 0 \quad \text{ for all } f \in C^2(\partial U) \\
\iff &\int_{\partial U} g(grad\ f, \check{A}^T\eta\varphi^2) d\sigma = 0 \quad \text{ for all } f \in C^2(\partial U) \\
\iff &-\int_{\partial U} f div(\check{A}^T\eta\varphi^2) d\sigma = 0 \quad \text{ for all } f \in C^2(\partial U)
\end{aligned}$$

Therefore, $div(\check{A}^T\eta\varphi^2) = 0$ $d\sigma$-a.e.

Here, $g(grad\ f, \check{A}^T\eta\varphi^2) = df(\check{A}^T\eta\varphi^2)$, and divergence is in the sense of Riemannian manifold structure, i.e., $divX = \frac{1}{\sqrt{det(g)}}\partial_i(\sqrt{det(g)}X^i)$.

Note also that although ∂U may not be an oriented compact manifold, we can do integration by parts, since we only need to integrate locally.

5.5 Appendix

DEFINITION **5.4** *An open set U is called to have Lipschitz boundary if it is locally a graph of a Lipschitz function. Alternatively, we also say that U is a Lipschitz domain.*

DEFINITION **5.5** *An open set U is called to have C^n boundary if for all $x \in \partial U$, there exists $\delta_x > 0$, there exists U_x an open neighborhood of x, there exists $\psi_x : B_{\delta_x}(0) \to U_x$ such that*

$$\begin{aligned}
&(i) \quad \psi_x,\ \psi_x^{-1} \text{ are of class } C^n \\
&(ii) \quad \psi_x(0) = x \\
&(iii) \quad \psi_x(B_{\delta_x}(0) \cap \{x_1 < 0\}) = \psi_x(B_{\delta_x}(0)) \cap U \\
&(iv) \quad \psi_x(B_{\delta_x}(0) \cap \{x_1 = 0\}) = \psi_x(B_{\delta_x}(0)) \cap \partial U
\end{aligned}$$

, where $n \in \mathbb{N} \cup \{0\}$. Moreover, if ψ_x, ψ_x^{-1} are of class $C^{n,s}$, we say U has $C^{n,s}$-boundary, $0 < s < 1$. Alternatively, we also say that U is a $C^{n,s}$-domain.

DEFINITION **5.6** *Let U be a $C^{n,s}$-domain.*

$$C^{n',s'}(\partial U) := \{f | f(\psi_x) \in C^{n',s'}(\overline{B_{\delta_x}(0) \cap \{x_d = 0\}}) \text{ for each } x \in \partial U\}$$

, where ψ_x is defined in Definition 5.5, $n' \in \mathbb{N} \cup \{0\}$, $0 < s' < 1$, $n' + s' \leq n + s$.

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국문초록

이 학위 논문에서 중심적으로 논의된 질문은 주어진 확산 작용소, 즉, 어떠한 (유한 또는 무한 차원의) 상태공간 위의 시험함수들에서만 연역적으로 정의된 0차항의값이 없는 2차 선형 타원 미분 작용소가 강하게 연속적인 준군(semigroup)을 대응되는 가중된 L^p 공간 위에서 유일하게 결정하는지 아닌지이다.

이 학위 논문의 첫번째 부분에서, 우리는 여러가지 서로다른 수용성의 정의의 동등성과 특이점의 제거가능성에 관해 주로 초점을 맞춘다. 좀 더 정확히 말하면, L을 르벡 측도가 0인 콤팩트 집합 $\Sigma \subset \mathbb{R}^d$에 대하여 정의역이 $\mathbb{R}^d \setminus \Sigma$ 위에서 매끄럽고(smooth) 콤팩트한 받침을 갖는(compactly supported) 함수들인 라플라스의 1 미만의 분수승 이거나 추상적인 위너공간 안에서 주어진 가우스 측도가 0인 집합 Σ의 주변에서 사라지는 적절한 함수들의 algebra에서 정의된 온슈타인-울렌벡 작용소의 정수승이라고 하자. Σ의 크기에 따라서, 고려되는 작용소들이 L^p 유일할지도 그렇지 않을지도 모른다. 우리는 Σ의 critical한 크기를 수용성과 하우스돌프 측도를 이용하여 서술한다. 게다가, 우리는 특정한 여러 매개변수 확률과정들에 대한 알려진 결과들을 모은다.

이 학위 논문의 두번째 부분에서, $U \subset \mathbb{R}^d$가 열린 집합일 때 우리는 $L^p(U, \mu)$ 위에서의 노이만 문제에 대해 주로 초점을 맞출 것이다. 좀 더 정확히 말하면, L을 정의역이 $C^2_{0,Neu}(\overline{U})$이고 모양이 $Lu = \sum a_{ij}\partial_i\partial_j u + \sum b_i \partial_i u$인 비대칭적인 작용소라고 하자. 우리는 μ와 L의 계수들의 특정한 가정하에 마르코브 유일성, L^p-유일성, L^1-유일성과 보존성의 관계, 불변 측도의 유일성, 타원형 미분성, 기타등등에 관한 결과를 줄 것이다.

주요어: 일반화된 디리클레 형식, 비대칭 디리클레 형식, 보존성, 확산과정, 노이만 문제, 추상적인 위너 공간, 수용성, 온슈타인-울렌벡 작용소, 마르코프 유일성, L^p-유일성, 본

질적인 자가 수반성, 타원형 미분성, 불변 측도, 하우스돌프 측도.

학번: 2013-20225

Acknowledgement

제일먼저 저의 지도교수님이셨던 이우영 교수님, 현재 저의 지도교수님이신 Gerald Trutnau 교수님, 그리고 독일에서 현지 지도교수님이셨던 Michael Hinz 박사님께 감사하단 말씀을 전하고 싶습니다. 처음 대학원에 들어오고 아무것도 모르던 저를 학생으로 받아주시고 공부에 관해 다양한 방식으로 끊임없이 지원해주시고 믿어주셨던 이우영 교수님, 전공을 변경한 저를 받아주시고 여러 학회들을 데려가주시고 논문을 쓰는데 정말 다양한 방식으로 조언해주시고 도와주셨던 Gerald Trutnau 지도교수님, 해외에 장기간 방문하는게 처음인 저를 인간적으로도 학문적으로도 정말 크게 도움을 주시고 좋은 논문을 쓸 수 있게 도와주셨던 Michael Hinz 박사님, 모든 분들이 전부 감사할 말이 너무나도 많아서 차마 이 좁은 종이에 다 쓸 수 없을만큼 감사드리고 싶습니다.

저에게 IRTG2235라는 좋은 독일방문 프로그램에 참가할 수 있게 해주시고 논문자격 시험심사도 도와주시고 좋은 말씀을 아낌없이 해주신 김판기 교수님, 논문심사를 도와주신 강남규 교수님, 박형빈 교수님, 서인석 교수님께도 감사드립니다.

아무것도 모르는 저에게 여러 조언들을 아낌없이 해주신 김민중, 신지용 선배님, 같이 IRTG2235에 참가해 공부했던 여러 친구들, 다른 공부를 하고있지만 응원해주던 여러 친구들에게도 감사의 마음을 전합니다.

마지막으로 제가 힘들어 지쳐 우울해 할 때도 늘 응원해주고 챙겨주시던 부모님께 정말 감사하단 말씀을 드리고 싶습니다.