A unified framework for two-level reliability-based design optimization using metamodels

Maliki Moustapha
Post-doctoral researcher, Chair of Risk, Safety and Uncertainty Quantification, ETH Zurich, Switzerland

Bruno Sudret
Professor, Chair of Risk, Safety and Uncertainty Quantification, ETH Zurich, Switzerland

ABSTRACT: Reliability-based design optimization (RBDO) is an active field of research that combines reliability analysis and optimization techniques. The early RBDO techniques were limited in applications because of the approximation techniques (FORM) used in the reliability part. More sophisticated techniques that rely on simulations were not of practical interest until the recent introduction of surrogate modeling to the field of structural reliability. Many approaches that couple surrogate modeling, simulation-based reliability analysis and optimization for the solution of RBDO problems have been recently developed. This paper proposes a global and unified framework for surrogate-assisted RBDO that allows for the solution of various types of RBDO formulations. This framework is built using three distinct and independent blocks which are coupled *non-intrusively*. To enhance the overall efficiency, the surrogate models are built adaptively in a single augmented space prior to starting optimization. The proposed framework is illustrated using different techniques in each of the three blocks.

1. INTRODUCTION

Reliability-based design optimization (RBDO) is one of the most widely used approaches for the design of structures under uncertainties. The problem basically consists of a combination of optimization and reliability analysis. A substantial amount of research has been devoted to developing methods within the RBDO framework that allows for an efficient solution, i.e. with the smallest number of calls to the model used in the computation of various designs failure probabilities. Early formulations relied either on analytical approximate solutions or on the reformulation of the RBDO problem. These methods have been reviewed in Chateauneuf and Aoues (2008); Valdebenito and Schuëller (2010) and classified into *two-level*, *mono-level* and *decoupled* approaches. Benchmark studies (*e.g.* Aoues and Chateauneuf (2010)) have shown the limitation of these approaches which, for the most part, boils down to the use of approximation techniques such as the first-order reliability method (FORM). A direct consequence is the development of simulation-based methods which however comes with a large computational burden. More recently, another line of research has been drawn on the use of surrogate models, *i.e.* cheap and easy-to-evaluate proxies of the usually expensive original models. These have facilitated the development of more accurate and robust solution schemes in RBDO problems. However, the way in which such metamodels are introduced and the chosen formulations make it difficult to solve non-intrusively all types of RBDO problems. In this paper, we propose a unified and global framework for the solution of RBDO problems. The interest of this approach can be declined in the following points. First, it allows to solve all types of RBDO formulations regardless of the probabilistic input model, *i.e.* a) whether considering design variables only or both design and environmental variables (see below for a detailed definition) and b) whether the design parameters are deterministic or also affected by uncertainties. Second, the framework is made of three independent blocks (surrogate modeling, structural reliability, and optimization) which are coupled non-intrusively. In the first part of the paper, the RBDO problem is formulated followed by a brief review and classification of surrogate-assisted methods. The proposed framework is then presented together with some numerical considerations for a proper implementation. Finally, applications are made on two analytical benchmark examples using three different realizations of the framework.

2. FORMULATION OF THE RBDO PROBLEM

Various formulations have been proposed over time for the solution of RBDO problems. The most popular one consists in minimizing a deterministic cost $\mathfrak{c}$ under probabilistic constraints (Dubourg et al., 2011):

$$\boldsymbol{d}^* = \arg\min_{\boldsymbol{d}\in\mathbb{D}} \mathfrak{c}(\boldsymbol{d}) \quad \text{subject to:}$$
$$\begin{cases} \mathfrak{f}_j(\boldsymbol{d}) \leq 0, & \{j = 1, \dots, n_s\}, \\ \mathbb{P}\left(\mathfrak{g}_k(\boldsymbol{X}(\boldsymbol{d}), \boldsymbol{Z}) \leq 0\right) \leq \bar{P}_{f_k}, & \{k = 1, \dots, n_h\}, \end{cases} \tag{1}$$

where $\boldsymbol{d} \in \mathbb{D} \subset \mathbb{R}^{M_d}$ denotes the design variables, $\mathfrak{f}_j$ are a set of soft constraints (*e.g.* feasible bounds) and $\mathfrak{g}_k$ are the limit-state functions for which negative value indicates failure of the system. The latter depend on random variables which are here classified into two groups, namely the random *design variables* $\boldsymbol{X}(\boldsymbol{d}) \sim f_{\boldsymbol{X}|\boldsymbol{d}}$ which account for any variability that may be encountered in the decision parameters and the *environmental variables* $\boldsymbol{Z} \sim f_{\boldsymbol{Z}}$, which describe the random parameters that influence the limit-state function without being considered as decision parameters for the optimization problem. The specificity of RBDO lies in the fact that the constraints are assessed in terms of failure probability $\mathbb{P}\left(\mathfrak{g}_k(\boldsymbol{X}(\boldsymbol{d}), \boldsymbol{Z}) \leq 0\right)$ whose value is expected to be lower than a predefined threshold $\bar{P}_{f_k}$.

Equivalent formulations such as those based on the reliability index can be used. In this paper, we will also consider another formulation where the probabilistic constraint in Eq. (1) is replaced by the following quantile constraints (Moustapha et al., 2016):

$$Q_{\alpha_k}\left(\boldsymbol{d}; \mathfrak{g}_k(\boldsymbol{X}(\boldsymbol{d}), \boldsymbol{Z})\right) \leq 0, \quad \{k = 1, \dots, n_h\}, \tag{2}$$

where $\alpha_k = \bar{P}_{f_k}$ and

$$\begin{aligned} &Q_{\alpha_k}\left(\boldsymbol{d}; \mathfrak{g}_k(\boldsymbol{X}(\boldsymbol{d}), \boldsymbol{Z})\right) = \\ &\inf\left\{q \in \mathbb{R} : \mathbb{P}\left(\mathfrak{g}_k(\boldsymbol{X}(\boldsymbol{d}), \boldsymbol{Z}) \leq q\right) \geq \alpha_k\right\}. \end{aligned} \tag{3}$$

Various methods have been introduced in the literature to solve Eq. (1). A thorough review can be found in Chateauneuf and Aoues (2008); Valdebenito and Schuëller (2010). Most of the existing contributions rely on a reformulation of the problem and/or on approximation techniques for the reliability analysis. Chateauneuf and Aoues (2008) classified these approaches into three categories, namely two-level, mono-level and decoupled approaches. The latter two attempt to tackle an equivalent but easier-to-solve problem through the introduction of optimality conditions or upon separating the optimization problem from the reliability analysis. The double-loop approach however solves directly the RBDO problem through a nested loop which consists of an outer optimization loop and an inner reliability analysis. Despite being easy to implement, such methods have shown to be flawed on many levels. This is mainly related to the use of approximation techniques for the reliability analysis, more specifically the *first-order reliability method* (FORM). FORM is indeed known for leading to spurious results when the limit-state function is highly non-linear or when there exists more than one possible most probable failure point. In the benchmark performed in Aoues and Chateauneuf (2010), it is shown that most of these methods require an unaffordably large number of model evaluations to converge and lack of robustness as the solution strongly depends on the optimization starting point and on the degree of non-linearity of the limit-state surface.

Alternative approaches which rely on direct simulation rather than approximation techniques for the reliability analysis have been proposed in the literature in the last few years. To cope with the high number of model evaluations that Monte Carlo simulation-based methods require, surrogate modeling is often introduced in the framework. This has been done following different schemes as illustrated in Figure 1. The two major distinct groups consist in either approximating the relationship between the design and the corresponding failure probability or approximating directly the limit-state surface. The latter, which is more efficient, can further be declined according to how the metamodel is built. In this paper, we focus on the most general approach which consists in building one *unique* and *global* metamodel that will be used for all reliability analyses throughout the optimization process.

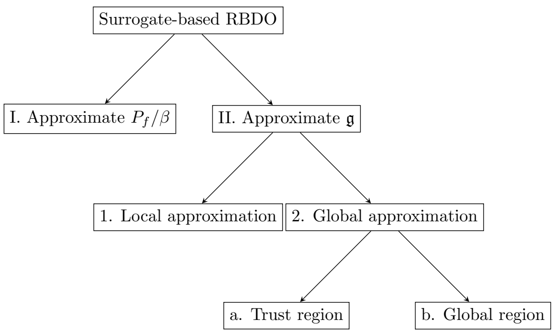


Figure 1: Classification of surrogate-assisted RBDO approaches.

3. PROPOSED FRAMEWORK

As introduced in the previous section, the use of surrogate models has been a focus of the recent RBDO contributions. Most of the proposed methods are however problem-dependent or require specialized developments. For instance, methods based on score functions to compute the sensitivity of the reliability index or failure probability require using Monte Carlo simulation to compute the latter. Moreover, the way the surrogate model is built often depends on the probabilistic input model, *i.e.* whether there are environmental variables or not or whether the design variables are deterministic or considered as means of random variables.

In this work, we propose a generalized *non-intrusive* framework for surrogate-assisted RBDO. The framework is made of three modular and independent blocks, namely surrogate modeling, reliability analysis and optimization. Each of them communicates with the others in a black-box fashion as illustrated in Figure 2. This means that the method chosen in each block can be modified without affecting the remaining ones. This idea is illustrated in the examples section where three "realizations" of the frameworks corresponding to different methods in each block is used.

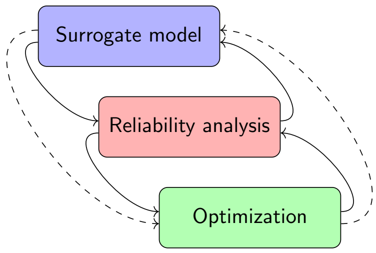


Figure 2: Proposed global and non-intrusive RBDO framework.

3.1. Implementation of the framework

3.1.1. Surrogate modeling

The basic idea of surrogate modeling is to build a cheap approximation of an expensive-to-evaluate function. This is achieved by training a metamodel over a limited set of data known as the *experimental design*. In the proposed framework, a unique and global surrogate model is built to approximate the limit-state function. The interest in such an approach is that the same metamodel can be used for any reliability analysis regardless of the design choice. The space in which the metamodel is built should therefore be large enough and account for both the design and environmental variables. This is achieved by using the so-called *augmented space*. In practice, this consists of a tensor product of uni-dimensional confidence regions defined on each variable. For the design variable, such a confidence region can be derived by simply extending the bounds of the design space so as to account for possible sample points that correspond to extreme design variables. Mathematically, this reads (Moustapha et al., 2016):

$$x_i^- = F^{-1}_{X_i|d_i^-}(\alpha_{d_i}), \qquad x_i^+ = F^{-1}_{X_i|d_i^+}(1-\alpha_{d_i}), \quad (4)$$

where $F^{-1}_{X_i|d_i^-}$ and $F^{-1}_{X_i|d_i^+}$ are respectively the inverse cumulative distribution functions (CDF) associated to the lower and upper design bounds and α_{d_i} denotes the confidence level associated to the design d_i. When the design variable is deterministic, the bounds of the augmented and design spaces simply coincide. The environmental variables do not evolve during optimization and a simple approach would consist in simply setting the corresponding augmented space to the tensor product of the marginal environmental variables space. However, we may also construct a marginal confidence region similar to that of the design space as follows:

$$z_i^- = F_{Z_i}^{-1}(\alpha_{z_i}/2), \qquad z_i^+ = F_{Z_i}^{-1}(1-\alpha_{z_i}/2), \quad (5)$$

where $F_{Z_i}^{-1}$ is the inverse CDF associated to the environmental variable Z_i. Eventually, the augmented space reads:

$$\mathbb{W} = \mathbb{X} \times \mathbb{Z}. \quad (6)$$

where $\mathbb{X} = \prod_{i=1}^{M_d} \left[x_i^-, x_i^+\right]$ and $\mathbb{Z} = \prod_{i=1}^{M_z} \left[z_i^-, z_i^+\right]$.

Making such a global surrogate model accurate enough for all reliability analyses throughout the optimization process would require an extremely large number of training points. To overcome the induced hurdle, one may rely on active learning schemes that allow one to efficiently build metamodels by directing the computational resources to points that contribute to the accurate estimation of the limit-state surface.

3.1.2. Reliability analysis

In RBDO, one or more reliability analyses are carried out at each iteration of the optimization process in order to estimate failure probability corresponding to the current design. As proposed in this framework, any reliability technique can be used. As explained earlier approximation-based approaches may lack of accuracy while the introduction of surrogate modeling allows for the use of direct simulation-based approaches. Examples of state-of-the-art simulation approaches include crude Monte Carlo simulation (MCS) for small probabilities (*e.g.* $< 10^{-2}$), importance sampling or subset simulation. These or any more advanced method can be seamlessly plugged into the proposed framework in a black-box fashion.

3.2. Optimization

In the proposed framework, any general-purpose optimization algorithm can be used. In general, optimization algorithms are split into two groups, namely local and global optimizers. Local optimization algorithms rely on local information about the cost and constraint functions to iteratively search for better designs. Except for the double-loop approaches, the approximation-based RBDO methods often rely on built-in optimization algorithms. In double loop approaches, very often sensitivities of the reliability index are used to compute the gradients. Even though such an approach may be more efficient, it violates the black-box constraint of the proposed framework. For local search, we rely on the computation of gradients using the finite-difference method. Here again the cost is leveraged by the fact that we are using a surrogate model that can be evaluated at practically no cost. Numerical precautions need to be taken for this scheme to work. More specifically, the constraint, *i.e.* the estimator of the failure probability or reliability index, is stochastic in nature when Monte Carlo simulation is used. For the finite-difference scheme to make sense, it is necessary, on the one hand, to control the variance of the failure probability estimates and on the other hand, to use concepts such as the so-called *common random numbers* (Spall, 2003). Essentially the latter means using the same stream of random numbers to compute failure probabilities for different design points. Even when using such approaches, the reliability technique itself may lead to slight variation within infinitesimally close designs. In such case, one should resort to global optimization algorithms which do not require any sensitivity information.

4. APPLICATIONS

4.1. Problem set up

To illustrate this framework we consider three different realizations of the proposed framework as summarized in Table 1. In this table, SQP and CMA-ES stand respectively for sequential quadratic programming and covariance matrix adaptation - evolution scheme (Arnold and Hansen, 2012). They are respectively local and global optimization algorithms.

4.1.1. Framework #1

In the first framework, Kriging is used as surrogate model. In short, Kriging is a metamodeling tech-

Table 1: Three different realizations of the proposed framework to be used in the application examples.

Framework	Metamodel	Reliability	Optimization
Case #1	Kriging	Subset simulation	CMA-ES
Case #2	PCE	MCS	SQP
Case #3	Kriging	Quantile MCS	Intrusive CMA-ES
Reference	Original model	MCS	Hybrid CMA-ES

nique which considers the model to approximate as a realization of a Gaussian process, which reads:

$$\mathscr{M}(\boldsymbol{w}) = \sum_{j=1}^{p} \beta_j f_j(\boldsymbol{w}) + Z(\boldsymbol{w}), \tag{7}$$

where β_j and f_j are respectively a set of P weights and basis functions and Z is a zero-mean stationary Gaussian process with auto-covariance $\mathrm{Cov}[Z(\boldsymbol{w}), Z(\boldsymbol{w}')] = \sigma^2 R(\boldsymbol{w}, \boldsymbol{w}', \boldsymbol{\theta})$. The covariance function is defined here by the constant variance σ^2 and the auto-correlation R with parameters $\boldsymbol{\theta}$.

Given an experimental design, the Kriging predictor is considered to be the mean of a Gaussian random variable $\mathscr{N}\left(\mu_{\widehat{G}_k}, \sigma^2_{\widehat{G}_k}\right)$, where:

$$\begin{aligned} \mu_{\widehat{G}_k}(\boldsymbol{w}) &= \boldsymbol{f}^T(\boldsymbol{w})\widehat{\boldsymbol{\beta}} + \boldsymbol{r}^T(\boldsymbol{w})\boldsymbol{R}^{-1}\left(\boldsymbol{y} - \boldsymbol{F}^T\widehat{\boldsymbol{\beta}}\right), \\ \sigma^2_{\widehat{G}_k}(\boldsymbol{w}) &= \sigma^2(1 - \boldsymbol{r}^T(\boldsymbol{w})\boldsymbol{R}^{-1}\boldsymbol{r}(\boldsymbol{w}) \\ &\quad + \boldsymbol{u}^T(\boldsymbol{w})\left(\boldsymbol{F}^T\boldsymbol{R}^{-1}\boldsymbol{F}\right)^{-1}\boldsymbol{u}(\boldsymbol{w})). \end{aligned} \tag{8}$$

In these equations, $\widehat{\boldsymbol{\beta}} = \left(\boldsymbol{F}^T\boldsymbol{R}^{-1}\boldsymbol{F}\right)^{-1}\boldsymbol{F}^T\boldsymbol{R}^{-1}\boldsymbol{y}$ is the generalized least-square estimate of the weight coefficients, $\boldsymbol{r}(\boldsymbol{w})$ is a vector of cross-correlations between the point $\boldsymbol{w}$ and each point of the training set, $\boldsymbol{F} = \left[f_j\left(\boldsymbol{w}^{(i)}\right)\right]_{1\leq i\leq N, 1\leq j\leq P}$ and $\boldsymbol{u}(\boldsymbol{w}) = \boldsymbol{F}^T\boldsymbol{R}^{-1}\boldsymbol{r}(\boldsymbol{w}) - \boldsymbol{f}(\boldsymbol{w})$.

The Kriging variance $\sigma^2_{\widehat{G}_k}(\boldsymbol{w})$ represents a measure of epistemic uncertainty of the prediction at the current point $\boldsymbol{w}$ and is often relied upon when building learning functions. In this paper, we consider the expected feasibility function which reads:

$$\begin{aligned} EF(\boldsymbol{s}) = \mu_{\widehat{G}_k}(\boldsymbol{s})&\left[2\Phi\left(\frac{\mu_{\widehat{G}_k}(\boldsymbol{s})}{\sigma_{\widehat{G}_k}(\boldsymbol{s})}\right) - \Phi\left(-\delta^+(\boldsymbol{s})\right)\right. \\ &\left. - \Phi\left(-\delta^-(\boldsymbol{s})\right)\right] - \sigma_{\widehat{G}_k}(\boldsymbol{s})\left[2\varphi\left(\frac{\mu_{\widehat{G}_k}(\boldsymbol{s})}{\sigma_{\widehat{G}_k}(\boldsymbol{s})}\right)\right. \\ &\left. - \varphi\left(-\delta^+(\boldsymbol{s})\right) - \varphi\left(-\delta^-(\boldsymbol{s})\right)\right] \\ &+ 2\sigma_{\widehat{G}_k}(\boldsymbol{s})\left[\Phi\left(-\delta^-(\boldsymbol{s})\right) - \Phi\left(-\delta^+(\boldsymbol{s})\right)\right], \end{aligned} \tag{9}$$

where $\delta^-(\boldsymbol{s}) = \frac{\mu_{\widehat{G}_k}(\boldsymbol{s}) - 2\sigma_{\widehat{G}_k}(\boldsymbol{s})}{\sigma_{\widehat{G}_k}(\boldsymbol{s})}$ and $\delta^+(\boldsymbol{s}) = \frac{\mu_{\widehat{G}_k}(\boldsymbol{s}) + 2\sigma_{\widehat{G}_k}(\boldsymbol{s})}{\sigma_{\widehat{G}_k}(\boldsymbol{s})}$.

In practice, a candidate set for enrichment is sampled uniformly in the augmented space: $\mathscr{S} = \{\boldsymbol{s} \in \mathbb{W}, i = 1, \dots, N_C\}$. The enrichment is then made iteratively by adding to the current experimental design, the one that maximizes Eq. (9):

$$\boldsymbol{s}_{\text{next}} = \arg\max_{\boldsymbol{s}\in\mathscr{S}} EF(\boldsymbol{s}). \tag{10}$$

The enrichment is stopped when the metamodel is deemed accurate enough. In this work this is controlled by the stability of the surrogate model in approximating a classifier based on the limit-state surface.

4.1.2. Framework #2

In this framework, polynomial chaos expansions (PCE) are used as surrogate model. Let us consider a random vector $\boldsymbol{W} \sim f_{\boldsymbol{W}}$ whose components are assumed independent and further assume that the random output, $Y = \mathfrak{g}_k(\boldsymbol{W})$, has a finite variance. The latter can then be cast as the following polynomial chaos expansion (Xiu and Karniadakis, 2002):

$$Y = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^M} y_{\boldsymbol{\alpha}}\Psi_{\boldsymbol{\alpha}}(\boldsymbol{W}), \tag{11}$$

where $y_{\boldsymbol{\alpha}}$ is a set of coefficients to calibrate and $\Psi_{\boldsymbol{\alpha}}$ is a collection of multivariate orthonormal polynomials that are given with respect to the inputs random distributions. In the present setting, the inputs are all sampled uniformly in the augmented space, which leads to using *Legendre* polynomials.

In practice, this expansion is truncated into a finite and limited set of polynomials. The coefficients are then calibrated using various approaches among which regression techniques that are based on a sampled experimental design. *Least-angle regression* is considered here (Blatman and Sudret, 2011). Such an algorithm allows one to enforce sparsity in the PCE representation.

Similarly to the previous case, the PCE model is built adaptively off-line prior to starting the optimization. For the applications in this paper, we consider a distance-based learning function that combines exploration and exploitation of the augmented space. More specifically, the next best point is chosen as:

$$\boldsymbol{s}^{\text{next}} = \arg\max_{\boldsymbol{s}\in\mathscr{S}'} \min_{i=1,\dots,N} \|\boldsymbol{s} - \boldsymbol{w}^{(i)}\|, \tag{12}$$

where $\mathscr{S}' = \{\boldsymbol{s} \in \mathscr{S} : |\widehat{\mathfrak{g}}(\boldsymbol{s})| \leq q\}$ with q being an α-quantile of $|\widehat{\mathfrak{g}}(\boldsymbol{s})|$. In other words, the set of candidates is first reduced to the closest points to the currently estimated limit-state surface. By setting $\alpha = 0.01$, the candidate set is reduced to the 1% closest points to the limit-state surface. Among these points, the next best point is chosen as the one that is furthest away from all existing training points. The process is repeated until convergence is achieved.

4.1.3. *Framework* #3

This framework is a slight adaptation of the proposed framework which can be qualified gray-box. By adding a simple coupling between the optimization algorithm and the surrogate model, the efficiency of the RBDO solution can be greatly enhanced. In fact, the previous cases focus on accurately defining the limit-state surface in the entire augmented space. However, this is not necessary as only a subset of the limit-state surface will be of interest for optimization, *i.e.* where the cost function is improving with respect to the initial design. To integrate such an information, a two-stage enrichment scheme was proposed in Moustapha et al. (2016). The first stage of the enrichment proceeds as above but stops prematurely before convergence. This only serves the purpose of grossly identifying the contours of the limit-state surface. The second stage of enrichment is coupled to optimization. At each iteration of the optimization process, the accuracy of the estimated failure probability, or in this case quantile, is assessed. Enrichment is made locally whenever this estimate is not deemed accurate enough, otherwise the optimization algorithm, henceforth called *intrusive* CMA-ES, proceeds to the next iteration.

In practice, upper and lower bounds of the quantiles estimates (resp. $q_\alpha^-(\boldsymbol{d})$ and $q_\alpha^+(\boldsymbol{d})$) are introduced using the models defined respectively by $\mu_{\widehat{G}_k}(\boldsymbol{w}) - 1.96\,\sigma_{\widehat{G}_k}(\boldsymbol{w})$ and $\mu_{\widehat{G}_k}(\boldsymbol{w}) + 1.96\,\sigma_{\widehat{G}_k}(\boldsymbol{w})$. Even though those values may not actually bound the true quantile, they can be used as a measure of the surrogate model accuracy in estimating the true quantile. Hence, the following criterion is computed at each iteration:

$$\varepsilon_q(\boldsymbol{d}) = \frac{|q_\alpha^-(\boldsymbol{d}) - q_\alpha^+(\boldsymbol{d})|}{1 + |q_\alpha(\boldsymbol{d})|}. \tag{13}$$

Enrichment is made if ε_q is larger than a predefined threshold, herein set to 0.1. This is achieved by considering the same learning function defined in Eq. (9) where the set of candidates for enrichment is taken locally as the samples used to compute the quantile.

4.2. *Examples*

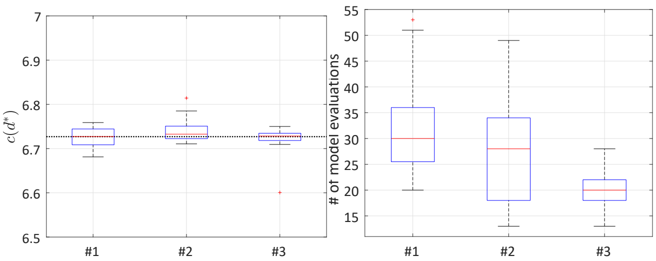
4.2.1. *Three non-linear constraints*

The first example is a two-dimensional problem with three non-linear limit-state functions widely used in the literature for RBDO methods benchmarks. In a deterministic context, it reads:

$$\boldsymbol{d}^* = \arg\min_{\boldsymbol{d}\in[0,10]^2} d_1 + d_2 \quad \text{s.t.:}$$

$$\begin{cases} \mathfrak{g}_1(\boldsymbol{d}) = 1 - \dfrac{d_1^2 d_2}{20} \geq 0 \\ \mathfrak{g}_2(\boldsymbol{d}) = 1 - \dfrac{(d_1+d_2-5)^2}{30} - \dfrac{(d_1-d_2-12)^2}{120} \geq 0 \\ \mathfrak{g}_3(\boldsymbol{d}) = \dfrac{80}{1-\left(d_1^2+8d_2+5\right)} \geq 0 \end{cases}. \tag{14}$$

The equivalent RBDO problem as in Eq. (1) is obtained by considering the design variables random with the following distribution $X_i \sim \mathcal{N}\left(d_i, 0.3^2\right)$ and setting the target failure probability (resp. reliability index) to $\bar{P}_f = 1.35 \cdot 10^{-3}$ (resp. $\bar{\beta} = 3$). The solution is carried out starting with an initial experimental design consisting of 10 points sampled using Latin Hypercube sampling (LHS). The analysis is repeated 20 times to account for the statistical variability in the results. Figure 3 shows the resulting optimal costs and number of model evaluations using box-plots. The corresponding median results are gathered in Table 2, together with some literature benchmark results. The three cases lead



(a) Optimal cost (b) # of model evaluations

Figure 3: Results for the highly non-linear limit-sate functions problems

to accurate results within a reasonable number of calls to the limit-state functions. In particular, their efficiency is larger than most of the approximation-based methods. From these figures, it can be observed that case #3 is more robust and efficient than cases #1 and #2. This is due to the fact that in case #3 enrichment is mostly local and follows the optimization process, therefore the estimated quantiles are extremely accurate in the final solutions. In contrary, cases #1 and #2 try to approximate the limit-state surface accurately in the entire augmented space with a convergence criterion that may not be well calibrated.

Table 2: Results comparison for the three non-linear constraints problem.

Method	d_1^*	d_2^*	$\mathfrak{c}(\boldsymbol{d}^*)$	$\mathfrak{g}$-calls
Reference	3.45	3.30	6.75	$\approx 10^6$
Case #1	3.45	3.28	6.73	28
Case #2	3.45	3.28	6.73	30
Case #3	3.44	3.29	6.73	19
PMA[a]	3.43	3.29	6.72	1,551
SORA[a]	3.44	3.29	6.73	151
Single loop[a]	3.43	3.29	6.72	19
RDS[a]	3.44	3.28	6.72	27
Meta-RBDO[a]	3.46	3.27	6.74	20

[a] Results gathered from Dubourg (2011).

4.2.2. *Bracket structure*

This example deals with the bracket structure illustrated in Figure 4. The structure consists of two connected members hinged to a wall to which is applied a vertical load. The RBDO problem consists in minimizing the weight of the bracket while ensuring that the constraints are satisfied, namely the maximum bending stress σ_b is below the yield stress σ_y and the compression force in the oblique bar F_{AB} is lower than the critical Euler Force F_b.

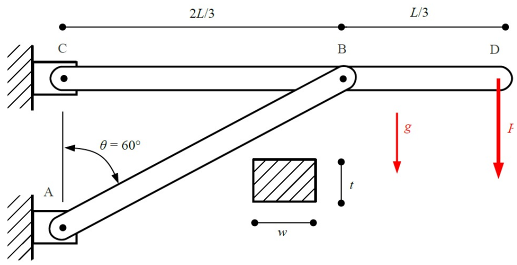


Figure 4: A sketch of the bracket structure (as illustrated in Dubourg (2011))

The RBDO problem therefore reads:

$$\boldsymbol{d}^* = \arg\min_{\boldsymbol{d}\in\mathbb{D}} \mathfrak{c}(\boldsymbol{d}) \quad \text{subject to:} \\ \begin{cases} \mathbb{P}\left(\mathfrak{g}_1\left(\boldsymbol{X}(\boldsymbol{d}),\boldsymbol{Z}\right)\leq 0\right)\leq \bar{P}_{f_1} \\ \mathbb{P}\left(\mathfrak{g}_2\left(\boldsymbol{X}(\boldsymbol{d}),\boldsymbol{Z}\right)\leq 0\right)\leq \bar{P}_{f_2}, \end{cases} \tag{15}$$

where $\bar{P}_{f_1} = \bar{P}_{f_2} = 0.00227$, $\mathfrak{c}(\boldsymbol{d}) = \rho t L\left(4\sqrt{3}/9 w_{AB} + w_{CD}\right)$ and the limit-state functions are given by:

$$\mathfrak{g}_1\left(\boldsymbol{X}(\boldsymbol{d}),\boldsymbol{Z}\right) = \sigma_y - \frac{6M_B}{w_{CD}t^2}, \\ \mathfrak{g}_2\left(\boldsymbol{X}(\boldsymbol{d}),\boldsymbol{Z}\right) = \frac{\pi^2 EI}{L_{AB}^2} - \frac{1}{cos\theta}\left(\frac{3P}{2} + \frac{3\rho g w_{CD} t L}{4}\right) \tag{16}$$

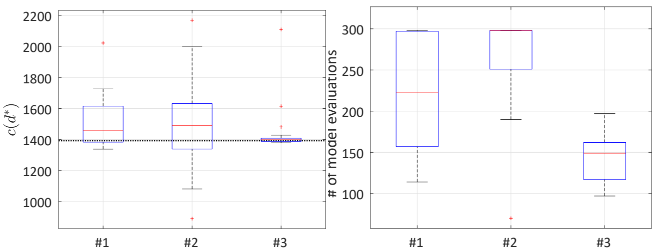
with $M_B = PL/3 + \rho g w_{CD} t L^2/18$, $I = t w_{AB}^3/12$ and $L_{AB} = 2L/3 sin\theta$. The eight random variables are described in Table 3.

As in the previous case, the solution is repeated 20 times starting with an initial experimental design of 20 drawn using LHS. Figure 5 shows the resulting optimal costs and number of limit-state evaluations. Again the metamodel-based solutions are better in efficiency than the approximation-based ones. Cases #1 and #2 require a relatively large amount of model evaluations to converge and somehow lack of robustness. This can be explained by

Table 3: Parameters of the variables defining the probabilistic model for the bracket structure problem: $\boldsymbol{d} = \{w_{AB}, w_{CD}, t\}^T$ *are the design variables and* $\boldsymbol{z} = \{P, E, \sigma_y, \rho, L\}^T$ *are the environmental variables.*

Parameter	Distribution	Mean	COV (δ%)
Width of AB (w_{AB} in m)	Normal	w_{AB}	0.05
Width of CD (w_{CD} in m)	Normal	w_{CD}	0.05
Thickness (t in m)	Normal	t	0.05
Applied load (P in kN)	Gumbel	100	0.15
Young's modulus (E in GPa)	Gumbel	200	0.08
Yield stress (σ_y in MPa)	Lognormal	225	0.08
Unit mass (ρ in kg/m^3)	Weibull	7860	0.10
Length (L in m)	Normal	5	0.05

the size of the augmented space and the complexity of the limit-state functions. In contrary, the case #3 due to the two-stage enrichment scheme is extremely efficient and robust.



(a) Optimal cost (b) # of model evaluations

Figure 5: Results for the highly non-linear limit-sate functions problems

5. CONCLUSIONS

This paper presents a global framework for the solution of RBDO problems. This framework is made of three independent and non-intrusive blocks, namely surrogate modeling, reliability analysis and optimization. Such a framework solution unifies various schemes introduced in the literature that solely focus on solving a specific type of RBDO problems. Further, the proposed framework is non-intrusive in the sense that one can plug any method in any of three blocks independently from the others. Two examples were used to showcase the application of this framework considering three different cases. The efficiency of the resulting scheme was shown based on comparison with literature benchmark results. The accuracy and robustness of the solution strongly relies on the surrogate model that is built adaptively in the augmented space. These results show that a key point is the set-up of an efficient enrichment scheme and the calibration of the corresponding convergence criterion.

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