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A New Approach for Capturing the Probability Density Function of the Maximum Value of a Markov Process

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ABSTRACT: In the present paper a method to determine the probability distribution of the maximum value process (MVP) of a Markov process is proposed. In this method, an augmented vector process of a physical process and its MVP is constructed. The joint probability density function is then calculated by the path integral solution (PIS), and further the probability density function of the MVP can be obtained as the marginal probability density. A numerical example is shown to validate the proposed method.

Key words: Maximum value process (MVP), Dynamics reliability, Path integral, Markov process

1. INTRODUCTION

Stochastic dynamical systems or stochastic processes are studied and applied widely in various science and engineering fields. The probability distribution of the extreme value of a stochastic process, e.g., the response of stochastic dynamical systems, is of important concern, in particular in dynamical reliability of structures in civil/marine engineering (Li & Chen, 2009; Naess & Moan, 2013; Melchers & Beck, 2018). As a matter of fact, more than half a century ago, the probability distribution of the extreme value of a series of events or discrete time series was well studied (Weibull, 1951; Gumbel 1958; Ang & Tang, 1984). However, for the extreme value distributions (EVD) of stochastic processes in continuous time domain, much less understanding has been achieved. Only limited results for some special stochastic processes were available (Newland, 1993; Finkenstädt & Rootzén, 2004; Naess & Moan, 2013). The methods that can capture the probability distribution of extreme value of generic stochastic processes were unavailable till the end of last century. The probability density evolution method (PDEM) (Li & Chen, 2009; Chen & Zhang, 2013) provides a potential tool. Based on the generalized probability density evolution equation (GDEE) (Li & Chen, 2008), the EVD can be evaluated through constructing a virtual stochastic process (Chen & Li, 2007). The structural global dynamic reliability can then be evaluated (Li et al, 2007).

An alternative approach is to apply an absorbing boundary condition at a certain threshold and calculate the remain probability density (Redner, 2001). However, again only for some special processes the analytical solutions of the remain probability density are available, e.g., the Brown motion process (Dudley, 1989; Kou & Zhong, 2016), the process with purely time dependent drift and diffusion (Molini et al, 2011), and so on. For general processes, even Markov processes, how to capture its remain probability density analytically at a certain threshold is still a challenging problem.

The study on maximum value process (MVP) of the Markov process is another perspective on this issue. In the present paper, an augmented vector Markov process is constructed by combining the MVP with the underlying Markov process. The path integral solution (PIS) is then adopted to yield the probability distribution of this augmented process. A numerical example is illustrated to demonstrate the effectiveness of the proposed method.

2. MAXIMUM VALUE PROCESS AND PATH INTEGRAL SOLUTION OF PROBABILITY DENSITY FUNCTION

For a continuous stochastic process $X(t)$, the corresponding MVP can be defined as

$$Z(t) = \max_{0 \leqslant \tau \leqslant t} \{X(\tau)\} \tag{1}$$

Clearly, $Z(t)$ is a stochastic process, and is non-decreasing monotonous.

Such a time-dependent maximum value is of great engineering significance. For instance, the probability without any excursion of the system, i.e., the reliability, is

$$R(t) = \Pr\{X(\tau) < b,\ 0 \leqslant \tau \leqslant t\} \tag{2}$$

where $\Pr\{\cdot\}$ denotes the probability of the bracketed event. Obviously, the reliability is equivalent to (Chen & Li 2007)

$$R(t) = \int_{-\infty}^{b} p_Z(z,t)\mathrm{d}z \tag{3}$$

where $p_Z(z,t)$ is the PDF of the MVP $Z(t)$ defined in Eq. (1).

Unfortunately, as discussed in the preceding section, for a general stochastic process, it is difficult to obtain the analytical solution of the probability density function (PDF) of MVP, except several very special cases (Dudley, 1989; Redner, 2001; Molini et al, 2011; Kou & Zhong, 2016). In the present paper, a new method will be developed for the determination of the PDF of MVP of a Markov process.

To make it clear, for convenience, consider a Markov process $X(t)$ governed by the following state equation, namely, the Itô SDE

$$\mathrm{d}X(t) = f[X(t),t]\mathrm{d}t + g[X(t),t]\mathrm{d}W(t) \tag{4}$$

where f and g are two functions satisfying appropriate regulation conditions; $W(t)$ is a Wiener process with $E[\mathrm{d}W(t)] = 0$ and $E\{[\mathrm{d}W(t)]^2\} = D\mathrm{d}t$; D is the intensity. It is known that the stochastic process $X(t)$ governed by the Itô SDE [Eq. (4)] is a Markov process (Gardiner, 1985).

The MVP $Z(t)$ itself is not a Markov process. However, the augmented vector process $(Z(t),X(t))^{\mathrm{T}}$ is a Markov vector process.

Though the analytical solution for the augmented Markov vector process $(Z(t),X(t))^{\mathrm{T}}$ is still unavailable for general problems, numerical methods are feasible and will be elaborated by PIS.

According to the Chapman-Kolmogorov equation

$$\begin{aligned} &p_{ZX}(z_3,x_3,t_3|z_1,x_1,t_1) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[p_{ZX}(z_3,x_3,t_3|z_2,x_2,t_2) \\ &\quad \cdot p_{ZX}(z_2,x_2,t_2|z_1,x_1,t_1)]\mathrm{d}z_2\mathrm{d}x_2 \end{aligned} \tag{5}$$

in which $t_1 < t_2 < t_3$ are three arbitrary different time instants, $p_{ZX}(z_2,x_2,t_2|z_1,x_1,t_1)$ is the transition probability density (TPD) from the state (z_1,x_1) at time t_1 to the state (z_2,x_2) at time t_2. Similar meaning apply to the other symbols. According to Eq. (5), the recursive format of PIS to obtain the joint PDF of $(Z(t),X(t))^{\mathrm{T}}$ can be written as

$$\begin{aligned} &p_{ZX}(z,x,t+\tau) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[p_{ZX}(z',x',t) \\ &\quad \cdot p_{ZX}(z,x,t+\tau|z',x',t)]\mathrm{d}z'\mathrm{d}x' \end{aligned} \tag{6}$$

where $p_{ZX}(z,x,t)$ is the joint PDF of $(Z(t),X(t))^{\mathrm{T}}$; $\tau>0$ is an arbitrary time increment; $p_{ZX}(z,x,t+\tau|z',x',t)$ is the TPD given by

$$p_{ZX}(z,x,t+\tau|z',x',t) = \frac{u(z'-x)\delta(z-z')+u(x-z')\delta(z-x)}{g(x',t)\sqrt{2\pi\tau}} \cdot e^{-\frac{[x-x'-f(x',t)\tau]^2}{2g^2(x',t)\tau}} \quad (7)$$

where $u(\cdot)$ is the Heaviside step function; $\delta(\cdot)$ is Dirac's δ function.

Once the joint PDF $p_{ZX}(z,x,t)$ is obtained, the PDF of MVP can be further obtained by

$$p_Z(z,t) = \int_{-\infty}^{\infty} p_{ZX}(z,x,t)\mathrm{d}x \quad (8)$$

Simultaneously, the state PDF can also be obtained as a marginal PDF

$$p_X(x,t) = \int_{-\infty}^{\infty} p_{ZX}(z,x,t)\mathrm{d}z \quad (9)$$

3. NUMERICAL EXAMPLE

In this section, a numerical example is studied to illustrate the approach for determining the PDF of the MVP of Markov processes.

Consider a nonlinear, additive white noise excited, one-dimensional diffusion process $X(t)$ with the following Itô SDE (Er, 2000)

$$\mathrm{d}X(t) = \frac{1}{2}[\alpha X(t) - \beta X^3(t)]\mathrm{d}t + \mathrm{d}W(t) \quad (10)$$

where $\alpha,\beta>0$ are coefficients; $W(t)$ is a Brownian motion process with intensity D. As $t\to\infty$, the stationary PDF of $X(t)$ has the expression (Er, 2000)

$$p_X(x) = Ce^{\frac{1}{2D}\left(\alpha x^2 - \frac{\beta}{2}x^4\right)} \quad (11)$$

where C is the normalization constant given by

$$C = \frac{1}{\int_{-\infty}^{\infty} e^{\frac{1}{2D}\left(\alpha x^2 - \frac{\beta}{2}x^4\right)}\mathrm{d}x} \quad (12)$$

In a small time increment τ, the TPD of $X(t)$ can be written as

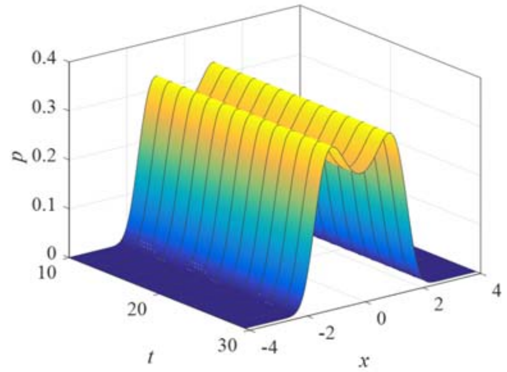
$$p_X(x,t+\tau|x',t) = \frac{1}{\sqrt{2\pi D\tau}}e^{-\frac{1}{2D\tau}\left[x-\left(1+\frac{\alpha\tau}{2}\right)x'+\frac{\beta\tau}{2}x'^3\right]^2} \quad (13)$$

The MVP $Z(t)$ of $X(t)$ is defined by Eq. (1). Then, the TPD of the augmented vector process $(Z(t),X(t))^{\mathrm{T}}$ can be obtained as

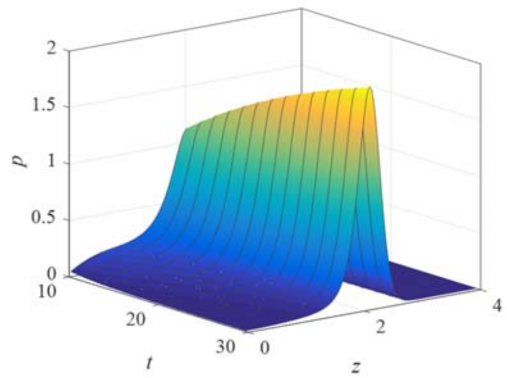
$$p_{ZX}(z,x,t+\tau|z',x',t) = \frac{u(z'-x)\delta(z-z')+u(x-z')\delta(z-x)}{\sqrt{2\pi D\tau}} \cdot e^{-\frac{1}{2D\tau}\left[x-\left(1+\frac{\alpha\tau}{2}\right)x'+\frac{\beta\tau}{2}x'^3\right]^2} \quad (14)$$

Therefore, the PDF $p_{ZX}(z,x,t)$ of $(Z(t),X(t))^{\mathrm{T}}$ can be obtained by the proposed method. Then, the PDF $p_Z(z,t)$ of $Z(t)$ can be obtained by a marginal integral.

In this case, the initial value is taken as $x_0=0$; the intensity of white noise is $D=1$; the coefficients are $\alpha=\beta=1$; the time step is $\Delta t=0.02$; the solving domain is $z\in[0,5]$ and $x\in[-5,5]$; and the grid size is $\Delta z=\Delta x=0.05$. The PDF surfaces of $Z(t)$ and $X(t)$ against time are shown, respectively, in Figure 1.

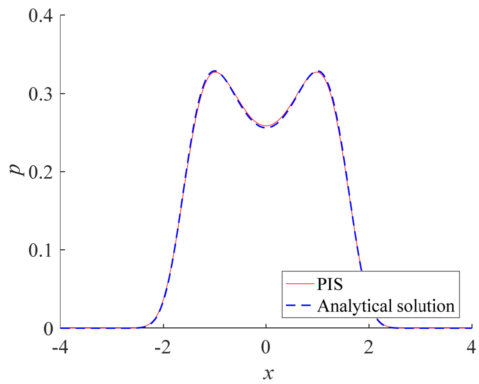


(a) PDF surface of the underlying process $X(t)$;

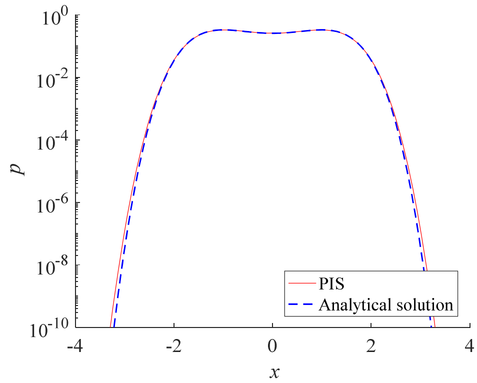


(b) PDF surface of the MVP $Z(t)$.
Figure 1: The PDF surfaces and their contours of $Z(t)$ and $X(t)$ of a scalar diffusion process.

At the time $t=30$, the comparison between the numerical result and the stationary analytical solution of PDF of $X(t)$ is shown in Figure 2. Again perfect agreement is observed between the analytical and numerical results.

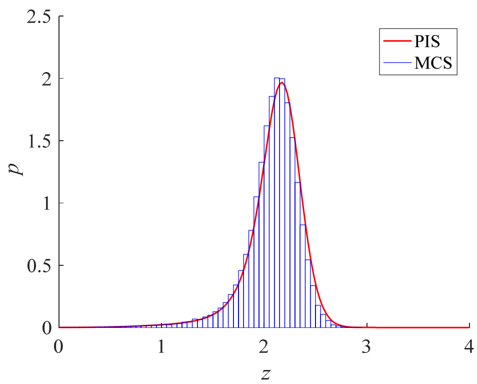


(a) In linear coordinates;

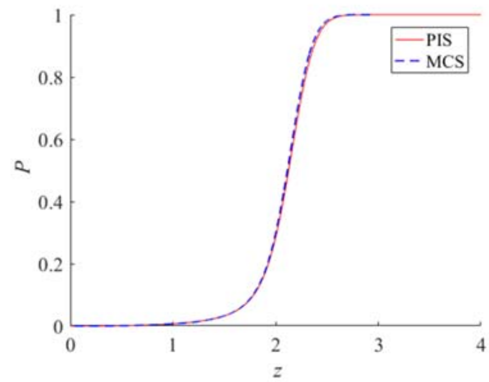


(b) In logarithmic coordinates.
Figure 2: The comparison between the numerical and stationary analytical solution of PDF of a scalar diffusion process $X(t)$ at $t=30$.

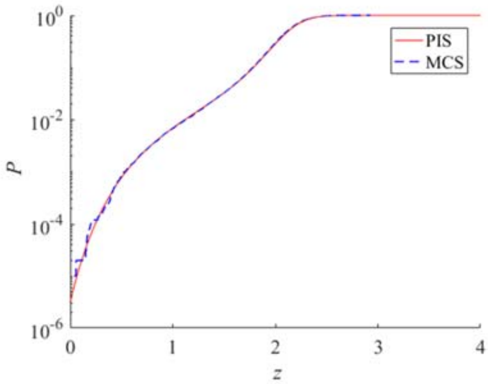
To compare with the numerical solution of PDF of $Z(t)$ by the proposed method, MCS is carried out. 10^5 samples of $Z(t)$ are conducted in the MCS. The comparison between the PDF and cumulative distribution function (CDF) of $Z(t)$ at the time $t=30$ by the proposed method and MCS, respectively, are shown in Figure 3. Again, fairly good agreement between the results by the two methods is observed. Obvious skewness is also observed from Figure 3(a). It is also noted that the accuracy of the tail of CDF is at least in the order of magnitude of 10^{-5}.



(a) PDF;



(b) CDF (in linear coordinates);



(c) CDF (in logarithmic coordinates).

Figure 3: The comparison between PIS and MCS of PDF and CDF of $Z(t)$ of a scalar diffusion process at $t = 30$.

4. CONCLUDING REMARKS

In this paper, the theoretical and numerical investigations on the maximum value process (MVP) of Markov process are carried out. The MVP of a Markov process, combined with the underlying process, constitutes an augmented vector Markov process. The numerical solution of its PDF can then be obtained by the PIS. The proposed method is verified by a numerical example, where the numerical solution obtained by PIS agrees well with the analytical solution and MCS. It is shown that the results obtained by PIS are of high accuracy even in the tail. The basic idea can be extended to more complex systems.

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