Quantification of Aleatory Uncertainty in Modal Updating Problems using a New Hierarchical Bayesian Framework

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ABSTRACT: Identification of structural damage requires reliable assessments of damage-sensitive quantities, including natural frequencies, mode shapes, and damping ratios. Lack of knowledge about the correct value of these parameters introduces a particular sort of uncertainty often referred to as epistemic uncertainty. This class of uncertainty is reducible in a sense that it can be decreased by enhancing the modeling accuracy and collecting new information. On the contrary, such damage-sensitive parameters might also have intrinsic randomness arising from unknown phenomena and effects, which gives rise to an irreducible category of uncertainty often referred to as aleatory uncertainty. The present Bayesian modal updating methodologies can produce reasonable quantification of the epistemic uncertainties, while they often fail to account for the aleatory uncertainties. In this paper, a new multilevel (hierarchical) probabilistic modeling framework is proposed to bridge this significant gap in uncertainty quantification and propagation of structural dynamics inverse problems. Since multilevel model calibration schemes establish a complicated model structure associated with additional parameters and variables, their computational costs are often considerable, if not prohibitive. To reduce the computational costs, the modal updating procedure is simplified using a second-order Taylor expansion approximation. This approximation is combined with a Markov chain Monte-Carlo (MCMC) sampling method to compute marginal posterior distributions of quantities of interest. The proposed framework is illustrated using one simple experimental example. As a result, it is demonstrated that the proposed framework surpasses the present Bayesian modal updating methods as it accounts for both the aleatory and epistemic uncertainties.

1. INTRODUCTION

Bayesian operational modal analysis (BOMA) is originally developed by Katafygiotis and Yuen (2001 and 2003). Despite great novelty in the original formulations proposed to identify modal parameters, they are computationally demanding. Among different variants of the BOMA, fast Fourier transform (FFT) methodologies have gained greater publicity and attention. Au (2011) developed a new FFT method to dramatically reduce the computational costs involved with this approach, while theoretical, computational, and practical issues involved with applying this method to ambient vibration tests are discussed in Au et al. (2013). Yan and Katafygiotis (2015b) have continued this line of research to develop a method to separate the identification of mode frequencies and damping ratios from the estimation of mode shapes.

In general, the main advantage of using the BOMA over deterministic methods is its capability to estimate the involved uncertainties. The uncertainty considered by the present BOMA methods is solely due to the lack of knowledge about the modal parameters. This category of uncertainty is often referred to as epistemic uncertainty, which can be reduced when additional observations are obtained (Kiureghian and Ditlevsen 2009). However, there is a prospect that the unknown parameters can also be subjected to inherent randomness when different data sets are used for the Bayesian inference. This category of uncertainty is irreducible, often originates from modeling errors, and cannot be accounted for by the present Bayesian methods. In this paper, a new multilevel probabilistic framework is proposed to bridge this significant gap in modal identification problems. Subsequently, a basic experimental example is used to demonstrate the proposed framework.

2. PROPOSED BAYESIAN FRAMEWORK

2.1. Vibration data

Let $D=\left\{\hat{\mathbf{y}}_j \in \mathbb{R}^{n\times 1}, j=0,...,N-1\right\}$ denote a data set comprising discrete-time response of a dynamical system measured at n degrees-of-freedom (DOF), where the index j corresponds to the discrete-time $t_j = j\Delta t$ and Δt is the sampling interval. The scaled FFT of the response can be computed as (Yuen and Katafygiotis 2003):

$$\hat{\mathbf{F}}_k = \sqrt{\frac{\Delta t}{N}} \sum_{j=0}^{N-1} \hat{\mathbf{y}}_j \, \mathrm{e}^{-2\pi ijk/N} \tag{1}$$

where $\hat{\mathbf{F}}_k \in \mathbb{R}^{n\times 1}$ is the Fourier transform of the response at the frequency $f_k = k/(N\Delta t)$.

2.2. Frequency-domain model

To construct a parametric model in frequency-domain, the Fourier transform of the theoretical response can be used. When the resonance bands are well-separated, the modal inference can be performed over each individual band. Considering the system to be linear having clear resonance peaks, the Scaled FFT of the response over a frequency band containing only one dynamical mode can be expressed as (Au 2017):

$$\mathbf{F}_k = \boldsymbol{\varphi}_i h_{ik} p_{ik} \tag{2}$$

where $\boldsymbol{\varphi}_i \in \mathbb{R}^{n\times 1}$ is the i^{th} mode shape, p_{ik} is the scaled FFT of the i^{th} modal force corresponding to f_k, and h_{ik} is the transfer function at f_k given by (Au 2017):

$$h_{ik} = \frac{(2\pi i f_k)^{-q}}{1-\beta_{ik}^2 - 2\xi_i \beta_{ik} i} \tag{3}$$

Here, q takes on 0, 1, and 2 for acceleration, velocity, and displacement response measurements, respectively; β_{ik} is the frequency ratio, f_k / f_i; f_i and ξ_i are the modal frequency and damping ratio corresponding to the i^{th} dynamical mode. Therefore, the free parameters can be collected into $\phi = \{f_i, \xi_i, \boldsymbol{\varphi}_i\}$, which should be calibrated based on the vibration data described in the next section.

2.3. Modal identification

Over a particular frequency band, which contains only the resonance peak corresponding to the i^{th} dynamical mode, the scaled FFT of the response can be predicted as:

$$\hat{\mathbf{F}}_k = \mathbf{F}_k + \boldsymbol{\varepsilon}_k \tag{4}$$

where $\boldsymbol{\varepsilon}_k \in \mathbb{R}^{n\times 1}$ is prediction errors assumed to have constant power spectral density (PSD) over the frequency band of interest. Considering the prediction errors to be statistically independent and identically distributed (*i.i.d.*) and describing them by using a Gaussian distribution leads to:

$$\boldsymbol{\varepsilon}_k \underset{i.i.d.}{\sim} N(0, S_e \mathbf{I}_n) \tag{5}$$

Here, S_e is prediction error variance assumed to be constant over the entire frequency band of interest and across all DOF. Given this assumption, the FFT response can be described by a complex Gaussian distribution expressed as (Au 2017):

$$p\left(\hat{\mathbf{F}}_k \mid \boldsymbol{\theta}\right) = \frac{\pi^{-n}}{\left|\mathbf{E}_k\right|^{-1/2}} \exp\left(-\hat{\mathbf{F}}_k^* \ \mathbf{E}_k(\boldsymbol{\theta})^{-1} \ \hat{\mathbf{F}}_k\right) \quad (6)$$

and

$$\mathbf{E}_k(\boldsymbol{\theta}) = E\left[\mathbf{F}_k\mathbf{F}_k^* \mid \boldsymbol{\theta}\right] = SD_k\boldsymbol{\varphi}_i\boldsymbol{\varphi}_i^T + S_e\mathbf{I}_n \quad (7)$$

where $\mathbf{E}_k(\boldsymbol{\theta})$ is the theoretical PSD, S is the PSD of the modal force p_{ik}, and $\boldsymbol{\theta}$ denotes the parameters of this probabilistic model given by:

$$\boldsymbol{\theta} = \{ f_i, \xi_i, \boldsymbol{\varphi}_i, S, S_e \} \quad (8)$$

Considering $\hat{\mathbf{F}}_k$'s to be statistically independent over the entire frequency band allows constructing the likelihood function as follows:

$$p\left(\left\langle \hat{\mathbf{F}}_k \right\rangle \mid \boldsymbol{\theta}\right) = \prod_k p\left(\hat{\mathbf{F}}_k \mid \boldsymbol{\theta}\right) \quad (9)$$

By using the Bayes' rule the posterior distribution of the parameters can be computed readily. Given a uniform prior distribution for the modal parameters, the negative log-likelihood function should be minimized to yield the MAP estimations:

$$\begin{aligned} L(\boldsymbol{\theta}) &= nN_f \ln \pi + \sum_k \ln\left|\mathbf{E}_k(\boldsymbol{\theta})\right| \\ &+ \sum_k \hat{\mathbf{F}}_k^* \ \mathbf{E}_k(\boldsymbol{\theta})^{-1} \ \hat{\mathbf{F}}_k \end{aligned} \quad (10)$$

where N_f is the number of data points over the frequency band of interest. This optimization problem yields the most probable values (MPV) of the parameters. An asymptotic approximation can simplify the posterior distribution giving (Papadimitriou et al. 1997):

$$p(\boldsymbol{\theta} \mid D) \approx N\left(\boldsymbol{\theta} \mid \hat{\boldsymbol{\theta}} \ , \ \hat{\boldsymbol{\Sigma}}_{\boldsymbol{\theta}} \right) \quad (11)$$

and

$$\hat{\boldsymbol{\Sigma}}_{\boldsymbol{\theta}} = \left[\nabla_{\boldsymbol{\theta}}^T \nabla_{\boldsymbol{\theta}} L(\boldsymbol{\theta})\right]_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}}^{-1} \quad (12)$$

In the latest equations, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\Sigma}}_{\boldsymbol{\theta}}$ denotes the MPV and covariance matrix of $\boldsymbol{\theta}$, respectively, where $\hat{\boldsymbol{\Sigma}}_{\boldsymbol{\theta}}$ is approximated as the inverse of the Hessian matrix of $L(\boldsymbol{\theta})$ evaluated at $\hat{\boldsymbol{\theta}}$. The Hessian can be computed both analytically and numerically as addressed in Au (2017).

2.4. Hierarchical Bayesian Approach

The posterior distribution computed earlier only accounts for the epistemic sources of uncertainty. To account for the aleatory uncertainty involved with the modal parameters, multiple sets of vibration data should be combined under a hierarchical probabilistic modeling.

Let $\mathbf{D} = \{D_r, r = 1,\dots,N_D\}$ be the full data set comprising N_D independent sets of vibration data. Corresponding to each data set, the foregoing Bayesian inference can be applied, and N_D independent realizations for the modal parameters are thus obtained. Based on Eq. (11), one can write:

$$p(\boldsymbol{\theta}_r \mid D_r) = N\left(\boldsymbol{\theta}_r \mid \hat{\boldsymbol{\theta}}_r \ , \ \hat{\boldsymbol{\Sigma}}_{\boldsymbol{\theta},r} \right) \quad (13)$$

where $\boldsymbol{\theta}_r$ denotes the modal parameters inferred from the data set D_r. To account for the aleatory uncertainties, we assume that the dataset-specific parameters, $\boldsymbol{\theta}_r$'s, follow a Gaussian distribution with the unknown mean $\boldsymbol{\mu}_{\boldsymbol{\theta}}$ and covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{\theta}}$. These parameters are often referred to as hyper-parameters, and this probabilistic model is called the hierarchical modeling technique (Nagel and Sudret 2016). Given these assumptions, Sedehi et al. (2019) have recently shown that the marginal distribution of the hyper-parameters updated based on multiple data sets can be computed from:

$$p\left(\boldsymbol{\mu}_{\boldsymbol{\theta}}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}} \mid \mathbf{D}\right) \propto p\left(\boldsymbol{\mu}_{\boldsymbol{\theta}}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}\right) \times \prod_{r=1}^{N_D} N\left(\boldsymbol{\mu}_{\boldsymbol{\theta}} \mid \hat{\boldsymbol{\theta}}_r , \boldsymbol{\Sigma}_{\boldsymbol{\theta}} + \hat{\boldsymbol{\Sigma}}_{\boldsymbol{\theta},r} \right) \tag{14}$$

Moreover, the posterior predictive distribution of the modal parameters can be computed using a Markov chain Monte-carlo (MCMC) sampling method giving (Sedehi et al. 2019):

$$p\left(\boldsymbol{\theta}^{new} \mid \mathbf{D}\right) = \frac{1}{N_s} \sum_{m=1}^{N_s} N\left(\boldsymbol{\theta}^{new} \mid \boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(m)}\right) \tag{15}$$

where $\boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)}$ and $\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(m)}$ denote samples of the hyper-parameters drawn from $p\left(\boldsymbol{\mu}_{\boldsymbol{\theta}}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}} \mid \mathbf{D}\right)$ given by Eq. (14). The latest equation provides a new formulation to combine multiple data sets accounting for the inherent randomness of the modal parameters observed over different data sets. Sedehi et al. (2018) have proven that the second-moment statistics of $p\left(\boldsymbol{\theta}^{new} \mid \mathbf{D}\right)$ can easily be computed from:

$$E\left(\boldsymbol{\theta}^{new}\right) = \frac{1}{N_s} \sum_{m=1}^{N_s} \boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)} \tag{16}$$

$$\begin{aligned} \mathrm{CoV}\left(\boldsymbol{\theta}^{new}\right) &= \frac{1}{N_s} \sum_{m=1}^{N_s} \left(\boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)} \boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)T} + \boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(m)}\right) \\ &- \left(\frac{1}{N_s} \sum_{m=1}^{N_s} \boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)}\right) \left(\frac{1}{N_s} \sum_{m=1}^{N_s} \boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)}\right)^T \end{aligned} \tag{17}$$

where $E\left(\boldsymbol{\theta}^{new}\right)$ and $\mathrm{CoV}\left(\boldsymbol{\theta}^{new}\right)$ denote the expected value and covariance matrix of $\boldsymbol{\theta}^{new}$, respectively. In the next section, the proposed hierarchical Bayesian approach is demonstrated using an experimental example.

3. EXPERIMENTAL EXAMPLE

Figure 1 shows a three-story shear building prototype structure tested on a Shaking table at the Hong Kong University of Science and Technology (HKUST). The acceleration responses of the three stories were measured when the prototype was subjected to $N_D = 20$ independent Gaussian White noise (GWN) input excitations. Each set of time-history response measurements is 120s long, sampled at 0.005s time intervals.

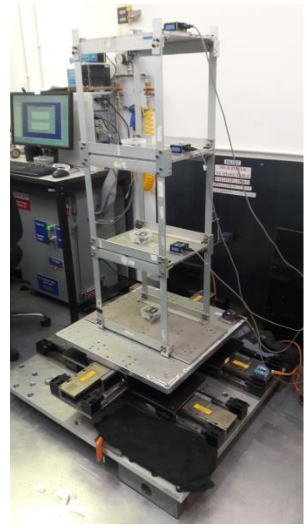


Figure 1: Structure prototype tested subjected to white Gaussian noise base excitations.

The FFT of discrete-time acceleration responses corresponding to a particular data set is shown in Figure 2. The three well-separated peaks appearing on this plot correspond to the three dynamical modes of the structure. Thus, the presented BOMA approach can simply be applied to compute the posterior distribution of dynamical properties from each individual data set. The proposed hierarchical Bayesian approach should next be applied to combine multiple data sets. For the sake of simplicity, we neglect the correlation between the modal parameters and assign only one pair of hyper-parameters to each quantity of interest, the mean and standard deviation. By using Eq. (14), we computed the marginal posterior distribution of the hyper-parameters. Figure 3 shows the marginal posterior distribution of the modal frequencies. As can be seen, the MPV of the mean and standard deviation of the three mode frequencies are estimated as (4.21Hz, 0.022Hz), (13.04Hz, 0.045Hz), and (18.82Hz, 0.018Hz). Figure 4 shows the marginal posterior distribution of the modal damping ratios. The MPV of the mean and standard deviations of the damping ratios are obtained as (0.038, 0.01), (0.013, 0.002), and (0.0036, 0.0075). The validity of this results can be confirmed when compared with the past studies (Zhouquan 2013).

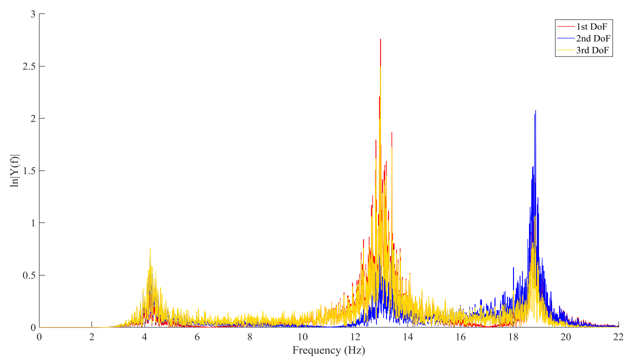


Figure 2: Frequency response function of measured output accelerations.

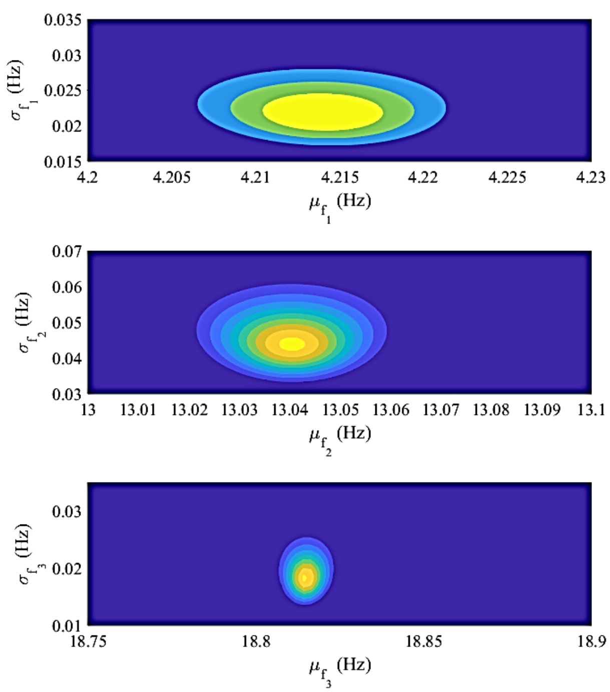


Figure 3: Posterior distributions of the hyper-distributions chosen to represent mode frequencies.

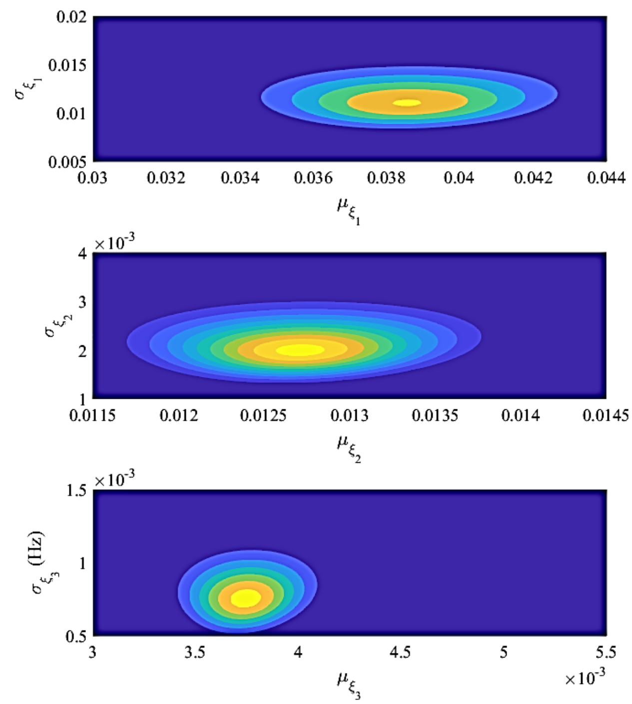


Figure 4: Posterior distributions of the hyper-distributions chosen to represent damping ratios.

Transitional MCMC method (Ching and Chen 2007) is next used to draw the samples from the marginal posterior distribution of the hyper-parameters. Eqs. (16) and (17) are used to compute the mean and standard deviation of the modal parameters according to multiple data sets. Figures 5 and 6 compare the data set specific posterior distributions, shown by the blue error bars, with the mean and standard deviations computed by the hierarchical method indicated by the shaded areas plotted along with the red line. As shown, the uncertainty bounds computed by the hierarchical Bayesian approach are in good agreement with the dataset-specific uncertainties shown by the blue error bars. In other words, the uncertainty computed by using one single data set is fairly consistent with the uncertainty estimated by using one single data set. Thus, the hierarchical Bayesian approach is not only a tool to combine different data sets reliably, but also it can be used test the robustness of estimated uncertainties obtained from the classical BOMA approaches.

The example demonstrated herein was rather simple, the modeling assumptions were flawless, and the full acceleration response measurements were fed into the algorithm. However, these results confirm the robustness of the present BOMA methods to quantify the involved uncertainty with reasonable accuracy when there are not considerable modeling errors. Nevertheless, the prospect that this nice accuracy cannot be maintained in the presence of drastic modeling errors still remains open.

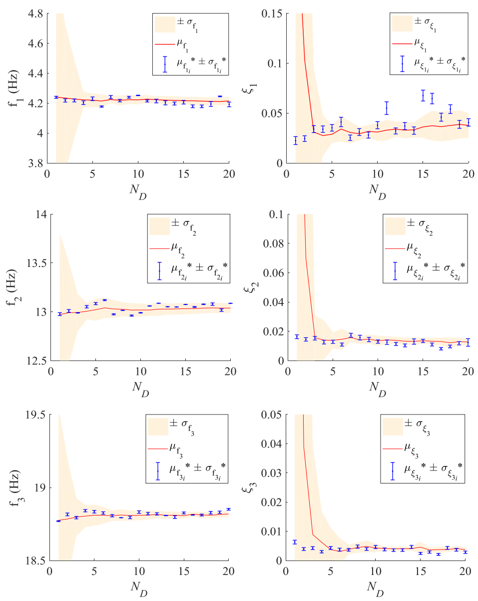
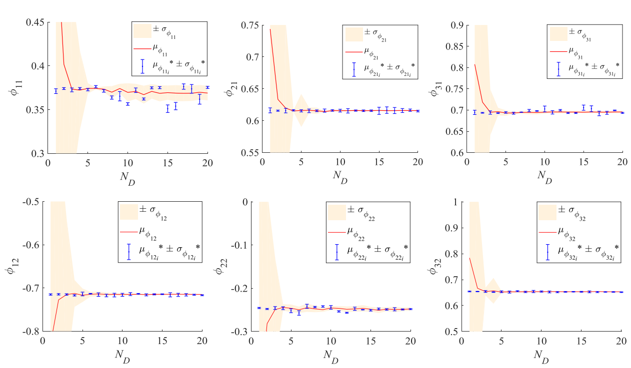


Figure 5: Posterior distribution of mode frequencies and damping ratios (The shaded area and the red lines show the results of hierarchical Bayesian approach and the blue error bars show posterior distributions estimated using a particular data set)



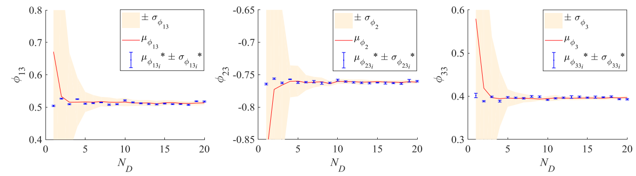


Figure 6: Posterior distribution of mode shapes (The shaded area and the red lines show the results of hierarchical Bayesian approach and the blue error bars show posterior distributions estimated using a particular data set)

4. CONCLUSIONS

A hierarchical Bayesian formulation is presented to account for both aleatory and epistemic uncertainties involved with the operational modal analysis problems. An experimental example is adopted to demonstrate the efficacy of the present BOMA methods, when the models are sufficiently accurate. Although the hierarchical method is used herein to combine different data sets, it is powerful to test and verify the robustness of estimations obtained from the classical BOMA methods.

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