Vector Autoregressive-based Structural Identification Method by Means of Bayesian Inference

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ABSTRACT: Modal identification involves the determination of natural frequencies, damping ratios, and mode shapes of a mechanical system using measured vibration data. The vector autoregressive (VAR) method and its variants are popular techniques capable of quickly extracting the modal properties, whose parameters are entries in the system matrices and are estimated by linear regression. However, those methods originally provide only the best estimates of modal parameters. Given the identified parameters are often used as a basis for structural control and health monitoring, it is important to know the statistics of those estimates. Probability logic with Bayesian updating provides a rigorous framework to obtain VAR model coefficients, quantify their uncertainty and moreover, calculate the statistics of modal parameters derived from the VAR model. In this study, an approach based on the VAR and Bayesian inference is investigated to obtain the most probable value and statistical features of modal frequencies of a steel plate girder bridge.

1. INTRODUCTION

A variety of methods have been developed to perform modal identification on operational structure, in which the structure is usually considered as a linear-elastic structure subjected to white noise excitation. Among them, the vector autoregressive modeling (VAR), stochastic subspace identification (SSI) and its variants are popular techniques capable of quickly extracting the modal properties based on a time-invariant linear model whose parameters are entries in the system matrices and are estimated by linear regression based on least-square. They originally provide only the best estimates of modal parameters. Given the identified modal parameters are often used as a basis for structural

control and health monitoring, it is important to know the accuracy of those estimates.

Besides VAR and SSI methods, modal identification methods based on Bayesian logic has recently attracted considerable attention. However, the implementation of Bayesian methods usually comes with demanding prerequisite or high computational cost: a time-domain formulation commonly gives rise to computational problems due to the large number of parameters involved; on the other hand, though methods such as the Bayesian Operational Modal Analysis works on the frequency domain and managed to obtain the most probable modal properties and assess their posterior uncertainties, it usually assumes that structural modes are well
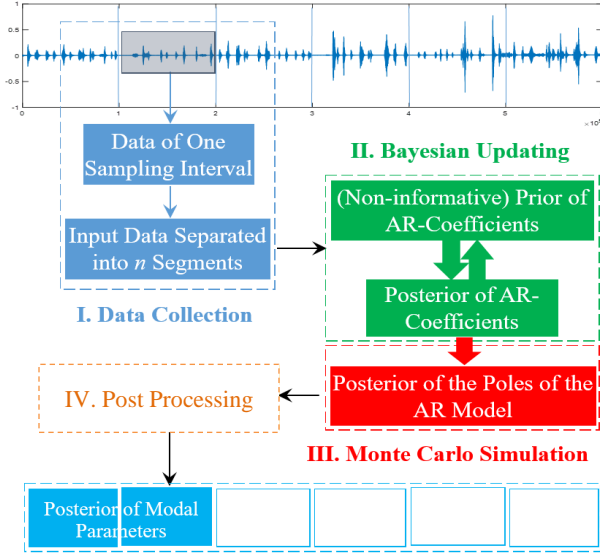
Figure 1: Flow of VAR-Bayesian Identification.

separated, and the corresponding frequency bands are supposed to be known in advance.

To explore a method of better practicability and applicability, an approach based on the VAR and Bayesian inference is proposed, and the Monte Carlo method is applied to approach the statistical features of identified modal frequencies. The proposed method is verified via a case study based on the data of a steel-plate girder bridge. To address the problem of computational cost in the Monte Carlo simulation, an investigation based on singular value decomposition is also discussed.

2. METHODOLOGY

The concept of the modal parameter identification by means of vector autoregressive-based Bayesian identification is summarized in Figure 1. In operational modal analysis, the structure is usually considered as a time-invariant linear system subjected to white noise (denoted by $\boldsymbol{e}_k \in R^{m \times 1}$ in Eq. (1)), and its response can be modelled by the VAR of sufficient model order P (Goi and Kim 2017a):

$$\boldsymbol{y}_k = \sum_{i=1}^{P} A_i \boldsymbol{y}_{k-i} + \boldsymbol{e}_k \qquad (1)$$

where $\boldsymbol{y}_k \in R^{m \times 1}$ is a column vector of the discrete time series of acceleration data from m measurements and $A_i \in R^{m \times m}$ is the i-th AR coefficient matrix. Through Bayesian inference,

posterior distribution of the model parameters can be obtained through Bayesian updating:

$$p(\boldsymbol{\theta}|D) = p(D|\boldsymbol{\theta})p(\boldsymbol{\theta})p(D)^{-1} \qquad (2)$$

where regressive parameters $\boldsymbol{\theta} = \{A_1, A_2, \ldots, A_P, \beta_1, \ldots, \beta_m\}$, in which β_i is the precision parameter of the regression, and D is observed data. Here, with a time window of length lw, n times of Bayesian updating are conducted within a sampling interval, a posterior of the regressive parameters is obtained and the most probable values of the posterior is considered as the estimates identified from the interval. Also, the derived distribution of natural frequencies $(f_1, f_2, \ldots, f_n)$ of the structure can be obtained through Monte Carlo simulation, therefore:

$$p(\boldsymbol{\theta}|D) \longrightarrow p(f_1, f_2, \ldots, f_n|D) \qquad (3)$$

2.1. Bayesian Inference

Focusing on j-th row in Eq. (1), the following regressive model is obtained:

$$y^{\{j\}}{}_k = \sum_{i=1}^{P} \boldsymbol{a}_i{}^{\{j\}} \boldsymbol{y}_{k-i} + e^{\{j\}}{}_k \qquad (4)$$

Therein, $y^{\{j\}}{}_k$ and $e^{\{j\}}{}_k$ represent j-th element of $\boldsymbol{y}_k$ and $\boldsymbol{e}_k$, and $\boldsymbol{a}_i{}^{\{j\}} \in R^{1 \times m}$ represents j-th row of A_i. Assuming elements of $\boldsymbol{e}_k$ are statistically independent each other and following Gaussian distribution with expectation 0, then $y^{\{j\}}{}_k$ also follows Gaussian distribution with expectation $\sum_{i=1}^{P} \boldsymbol{a}_i{}^{\{j\}} \boldsymbol{y}_{k-i}$. Letting $t_k = y_k{}^{\{j\}}$, $\boldsymbol{w} = [\boldsymbol{a}_i{}^{\{j\}}, \ldots, \boldsymbol{a}_P{}^{\{j\}}]^T \in R^{mP \times 1}$ and $\boldsymbol{\phi}_k = [\boldsymbol{y}_{k-1}{}^T, \ldots, \boldsymbol{y}_{k-P}{}^T]^T \in R^{mP \times 1}$ for simplicity, probability distribution function (PDF) of t is:

$$p(t_k|\boldsymbol{\phi}_k, \boldsymbol{w}, \beta) = N(t_k|\boldsymbol{w}^T \boldsymbol{\phi}_k, \beta^{-1}) \qquad (5)$$

where, $N(x|\mu, \sigma^2)$ is PDF of x following Gaussian distribution with expectation μ and variance σ^2, and β represents the precision parameter of the regression, which is the inverse of the variance of the noise term $e_k{}^{\{j\}}$. Assuming n samples of t_k and $\boldsymbol{\phi}_k$ are observed, and letting $\boldsymbol{t} = [t_1, \ldots, t_n] \in R^{n \times 1}$ and $\Phi = [\boldsymbol{\phi}_1 \ldots \boldsymbol{\phi}_n]^T \in R^{n \times mP}$, then the likelihood function for the parameters $\boldsymbol{w}$ and β is as follows (Goi and Kim 2017b):

$$p(t|\Phi, w, \beta) = \prod_{k=1}^{n} N\left(t_k \middle| w^T \phi_k, \beta^{-1}\right) \quad (6)$$

With the observed t, the updated posterior of the model, defined by the parametric vector and consequently the parametric matrix composed of VAR coefficients, can be obtained through Bayes' Theorem as:

$$p(w, \beta|t) = p(t|w, \beta)p(w, \beta)p(t)^{-1} \quad (7)$$

where $p(w, \beta)$ is the prior for w and β, $p(t|w, \beta)$ as a function of w is the likelihood, and $p(t)$ is the normalizing constant given observation t, of which the evaluation requires an intractable integration over the model parameter space. To address the problem, the prior PDF is formulated as a conjugate prior:

$$p(w, \beta) = N\left(w \middle| m_0, \beta^{-1}L_0^{-1}\right)Gam(\beta|a_0, b_0) \quad (8)$$

Here, $N(x|\mu, \Sigma)$ is the joint PDF of a vector x following the multivariate Gaussian distribution with expectation μ and covariance matrix Σ, which in this case, m_0 and $\beta^{-1}L_0^{-1}$ respectively. And a_0 and b_0 are hyperparameters that govern the distribution of β through Gamma distribution characterized by *shape* a_0 and *rate* b_0. Therefore the posterior can be obtained by:

$$\begin{aligned} &p(w, \beta|t) \\ &= N\left(w \middle| m_N, \beta^{-1}L_N^{-1}\right)Gam(\beta|a_N, b_N) \end{aligned}$$

$$(9)$$

$$L_N = L_0 + \phi^T\phi \quad (10)$$

$$m_N = L_N^{-1}(L_0 m_0 + \phi^T t) \quad (11)$$

$$a_N = a_0 + \frac{n}{2} \quad (12)$$

$$\begin{aligned} b_N = b_0 + \frac{1}{2}(\|t - \phi m_N\|^2 + (m_0 - m_N)^T \\ L_0(m_0 - m_N)) \end{aligned} \quad (13)$$

In practice, for a structure without prior information, it is usually recommended to utilize non-informative prior. Then, the posterior PDF of VAR coefficients can be updated iteratively with subsequent observations.

2.2. Modal Parameters and Monte Carlo Simulation

Using the *z*-transform, Eq. (1) is transformed into *z*-domain as:

$$Y(z) = H(z)E(z) \quad (14)$$

$$H(z) = \left(I_m - \sum_{i=1}^{P} z^{-i}A_i\right)^{-1} \quad (15)$$

where $Y(z)$ and $E(z)$ are *z*-transforms of y_k and e_k, respectively, I_m denotes the identity matrix of *m*-order. Matrix $H(z)$ in Eq. (14) and (15) is the transfer function of the linear system shown in Eq. (1). The conjugated pairs of the poles of $H(z)$ are related to the modal characteristics of the structure as shown in Eq. (16):

$$\lambda_i\lambda_i^* = \exp((-\xi_i \pm j\left(1 - \xi_i^2\right)^{\frac{1}{2}})\omega_i\Delta t) \quad (16)$$

where ω_i and ξ_i are the natural angular frequency and damping ratio of the *i*-th mode, respectively, Δt is the sampling time of t and j represents an imaginary unit. These poles are obtained by solving the eigenvalue problem with respect to *z*:

$$|I_l z - S| = 0 \quad (17)$$

$$S = \begin{bmatrix} A_1 & A_2 & \dots & A_{P-1} & A_P \\ I_m & 0 & \dots & 0 & 0 \\ 0 & I_m & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I_m & 0 \end{bmatrix} \quad (18)$$

where $l = mP$, $|\cdot|$ denotes matrix determinant, and the updated coefficient matrix A_i ($i = 1, \dots, P$) can be obtained through Bayesian inference (the entries in A_i are random variables following the corresponding posterior distribution).

However, the derived distribution of modal parameters, which concerns eigenvalue spectrum of sparse random matrices, is analytically intractable. To address the problem, a Monte Carlo simulation is conducted as the following:
(i). Sample the transformed system matrix *S*, in which $A_i \sim N(m_N, \beta^{-1}L_N)Gam(\beta|a_N, b_N)$;
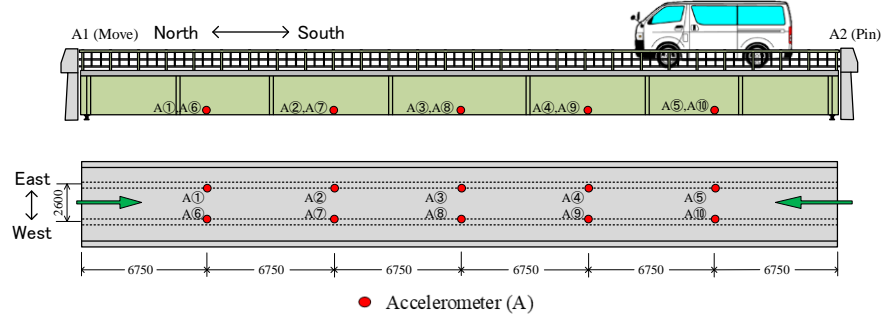(ii). With the sampled matrix *S*, obtain the corresponding modal parameters by Eq. (16)-(18);

Figure 2: Observation Bridge and Sensor Layout.

(iii). Obtain statistical features of the identified modal parameters through the samples obtained in last step, such as a simulated variance:

$$\sigma^2{}_{sim} = \frac{1}{n-1}\sum_{i=1}^{n}(f_{sam} - \mu)^2 \qquad (19)$$

where "*sim*" denotes simulated value, and "*sam*" denotes sampled value, and μ is the mean of the sampled modal frequencies. Thus, the modeling uncertainty can be exhibited by the statistical features of the sampled distribution. For example, a simulated variance $\sigma^2{}_{sim}$ calculated from the distribution approached by the Monte Carlo simulation may serve as a good reference in structural assessment.

3. CASE STUDY

3.1. Observation Bridge
The observation bridge is a steel plate-girder bridge with a span of 40.5 m long and a width of 4.5m. A vibration series of 113 hours was analyzed in this study. The monitoring system includes 10 accelerometers installed separately on each side of the bridge as shown in Figure 2. The accelerations were sampled at 200 Hz.

3.2. System Identification
Based on the procedure presented in Figure 1, 113 sets of the modal parameters are obtained hourly. Here, the time window *lw* is 15 minutes (therefore $n = 4$), and the model order of the VAR model *P* was set as 20 for the balance of necessary model complexity and identification efficiency. Due to low excitation (especially at nights of low traffic), it was difficult to identify all the modes of the
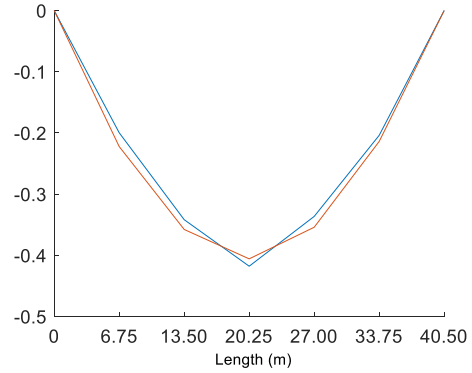
1st *Bending Mode*, $f = 3.0266, \xi = 0.0170$
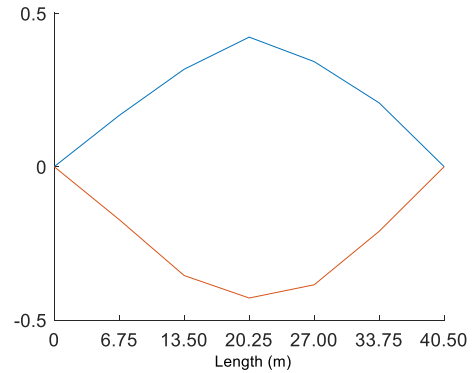
1st *Torsional Mode*, $f = 4.8919, \xi = 0.0172$
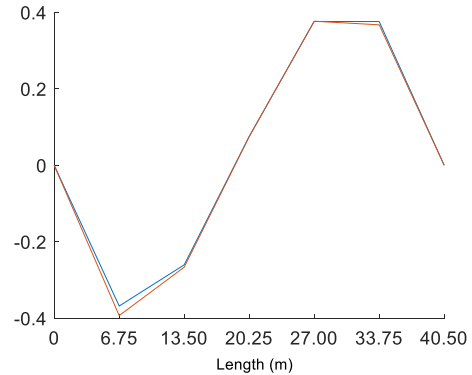
2nd *Bending Mode*, $f = 9.3084, \xi = 0.0066$

Figure 3: Identified Result (First Interval).

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Table 1: Discrepancy between SSI and VAR-Bayesian.

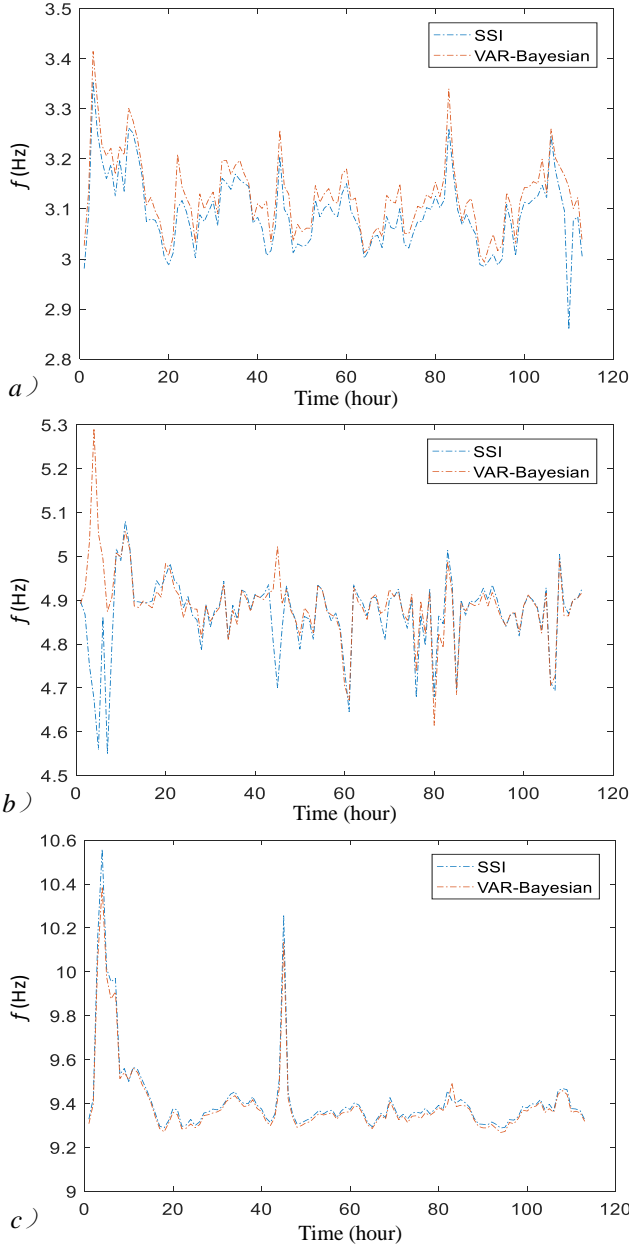|	$f_{SSI} - f_{VAR-Bayesian}$		
	Min.	*Avg. of Abs.*	*Max.*
1st *bending*	-0.2843	0.0347	-0.0049
1st *torsional*	-0.6173	0.0348	0.0649
2nd *bending*	-0.0811	0.0192	0.1710

Figure 4: Results of Two Methods: a) 1st Bending Mode; b) 1st Torsional Mode; c) 2nd Bending Mode.

bridge at every time interval. This study focuses on the frequencies of the first three modes (first bending, first torsional and second bending), of which typical outcomes are shown in Figure 3. In Figure 4, the comparison between the results of the vector autoregressive-based Bayesian identification and the SSI (the maximum order of the SSI is set as 50), which is a widely considered reliable method, can serve as an intuitive representative of the performance of the method.

It can be found that the results of both methods showed considerable agreement to each other in the most intervals, especially for the bending modes; meanwhile, the result of the torsional mode showed less agreement. Those observation can also be verified by the summary shown in Table 1.

For all the modes, the natural frequencies of both methods varied throughout the monitoring with a period of approximately 24 hours, which is very likely caused by the environmental variation (mainly temperature). Meanwhile, some of the irregular variation might be due to the operational variations, such as the traffic volume.

3.3. Uncertainty Analysis

The posterior distribution in the context of Bayesian inference of the modal parameters can be approached through the Monte-Carlo simulation in sub-section 2.2, and a simulated variance of the frequencies can be found. In this case, 10,000 instances are sampled from the posterior distribution of the VAR coefficient matrices to obtain the posterior distribution statistics of the modal frequencies, the distributions for the frequencies are shown in Figure 5, along with a summary in Table 2.

From the diagram, it can be concluded that the frequencies of the torsional mode sampled from the posterior distribution varies more than the bending modes. This result indicates that the bending modes can be identified with better assurance. It is possibly because the observation bridge is a single lane bridge, and the bending modes are more easily excited than the torsional mode.

More intuitively, the convergence of the simulated variance is shown in the Figure 6. Here, the sampling of an interval of busy traffic (so that the first six modes can be identified) is shown to make the comparison more straightforward. Generally, the first three modes come to converge (though with little fluctuation) just after around 2000 times of sampling. On the other hand, there is no clear sign of convergence for the other modes, especially the second torsional mode that fluctuate intensively to the end of the sampling.
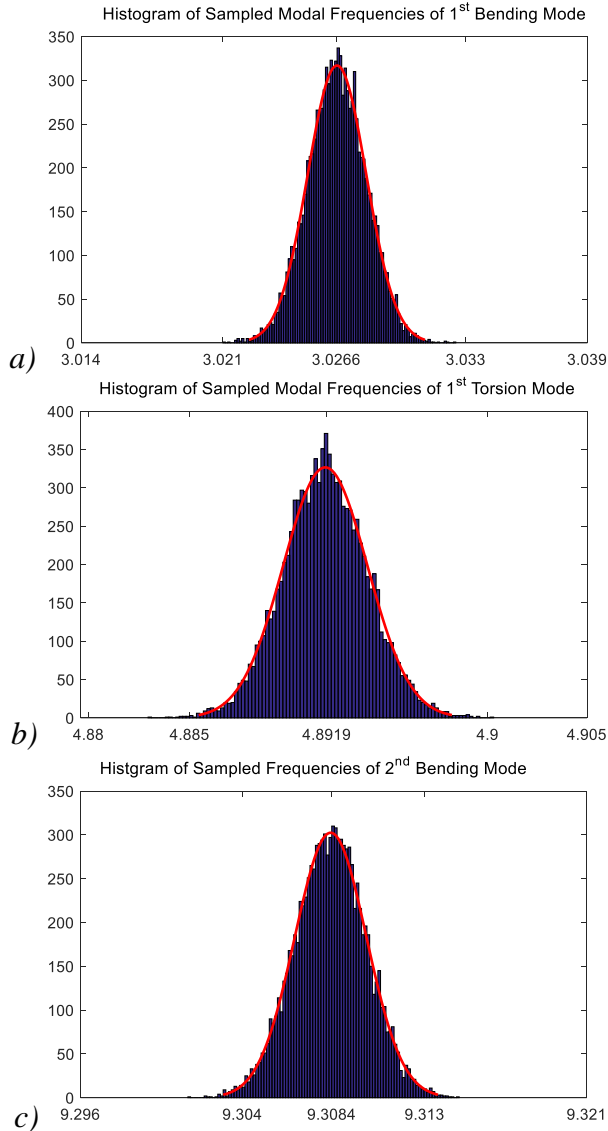
Table 2: Statistics of Identified Frequencies.

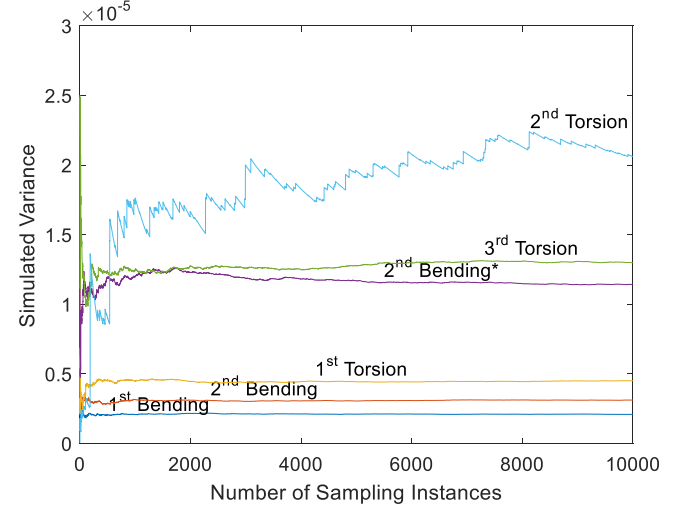|	*1ˢᵗ bending*	*1ˢᵗ torsional*	*2ⁿᵈ bending*
μ	3.03	4.90	9.31
σ_{sim}	2.1×10^{-6}	4.4×10^{-6}	3.1×10^{-6}
σ_{sim}/μ	6.9×10^{-7}	9.0×10^{-7}	3.3×10^{-7}

Figure 6: Convergence of Simulated Variance.

A noticeable issue is that, in the process of uncertainty quantification through Monte Carlo simulation, the sampling of large matrix can be very time-consuming: in this case, it would take more than 25 hours to process the vibration data. To address that problem, a further investigation is presented in next section.

Figure 5: Gaussian Fitted Histogram of Sampled Frequencies of First Three Modes: a) 1ˢᵗ Bending Mode; b) 1ˢᵗ Torsional Mode; c) 2ⁿᵈ Bending Mode.

4. FURTHER INVESTIGATION

One route to deal with the problem of computational cost would be a feature extraction procedure based on SVD. Through the SVD, the principle components of the posterior distribution are extracted.

In Eq. (8)-(9), the covariance matrix L contains the information of observed time series of the bridge. The SVD of it is shown as following:

$$L = U \Lambda U^T = [U_1 U_2] \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} \quad (20)$$

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where $\Lambda \in R^{mP \times mP}$ is the diagonal matrix consisting of the singular values and $U \in R^{mP \times mP}$ is the orthogonal matrix consisting of the singular vector. $\Lambda_1 \in R^{q \times q}$ and $U_1 \in R^{mP \times q}$ represent the q largest singular values and the corresponding singular vectors; Λ_2 and U_2 represent the other singular values and singular vectors. Let $\widetilde{w}$ denote the orthogonal transformation of the parameter vector w, $\widetilde{w} = U^T w$; $\widetilde{w}_1$ and $\widetilde{w}_2$ denote components of $\widetilde{w}$, therefore $\widetilde{w}_1 = U_1{}^T w$ and $\widetilde{w}_2 = U_2{}^T w$. Then Eq. (9) leads to the posterior distribution of $\widetilde{w}_1$, $\widetilde{w}_2$ and β as follows:

$$\begin{aligned} &p(\widetilde{w}, \beta | t) \\ &= N\big(\widetilde{w}\big|\widetilde{m},\, \beta^{-1}\Lambda^{-1}\big)Gam(\beta|a_N, b_N) \\ &= N\big(\widetilde{w}_1\big|\widetilde{m}_1,\, \beta^{-1}\Lambda_1{}^{-1}\big)N\big(\widetilde{w}_2\big|\widetilde{m}_2,\, \beta^{-1}\Lambda_2{}^{-1}\big) \\ &\quad Gam(\beta|a_N, b_N) \end{aligned}$$

$$(21)$$

Therein, $\widetilde{m} = U^T m_N$, $\widetilde{m}_1 = U_1{}^T m_N$ and $\widetilde{m}_2 = U_2{}^T m_N$. In the context of Bayesian inference, the observed structure response t includes the uncertainty (measurement error, operational effect, etc.) consequently expressed through the posterior distribution in Eq. (21). In Eq. (21), the parameter $\widetilde{w}_1$ would have less variances compared to the $\widetilde{w}_2$. Therefore, the $\widetilde{w}_1$ represents the parametric subspace derived from the observation that is more inferable. The parametric subspace for $\widetilde{w}_1$ gives a reduced form of the transformed system matrix S:

$$\tilde{S}_1 = U_1{}^T S U_1 \qquad (22)$$

With the reduced and transformed system matrix $\tilde{S}_1$, the modal parameters and the associated statistical features are expected to be efficiently obtained: the size of transformed system matrix S is reduced from $mP \times mP$ to $q \times q$. In this case, when setting $q = 14$, it needs only 12% of the original computational time. A comparison of the identified modal frequency before and after the model reduction is shown in Figure 7. Also, after the reduction, similar patterns of posterior distribution of modal

frequencies and associated statistics can be found in Table 3 and Figure 8.

Table 3: Statistics of Identified Frequencies.

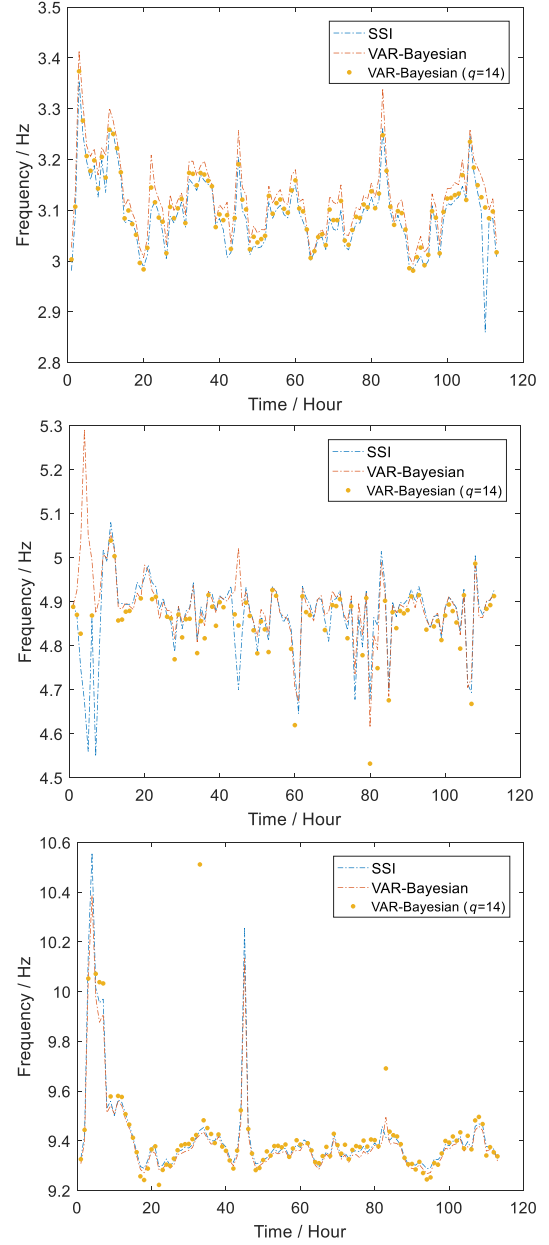|	1st bending	1st torsional	2nd bending
μ	3.00	4.89	9.33
σ_{sim}	6.3×10^{-8}	4.1×10^{-7}	3.8×10^{-7}
σ_{sim}/μ	2.1×10^{-8}	8.4×10^{-8}	4.0×10^{-8}

Figure 7: Comparison of Three Methods: a) 1st Bending Mode; b) 1st Torsional Mode; c) 2nd Bending Mode.
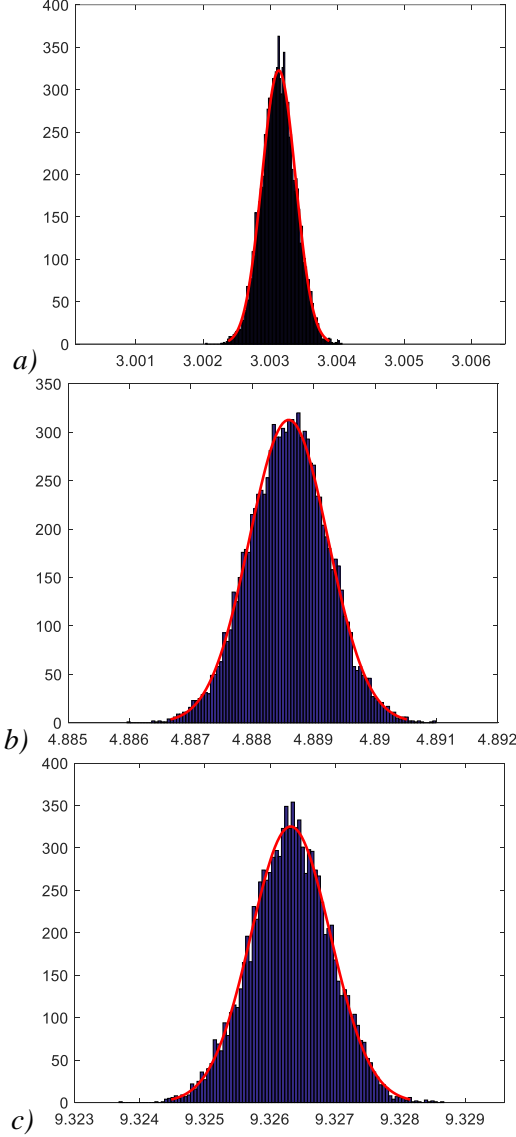
Figure 8: Gaussian Fitted Histogram of Sampled Frequencies Based on Reduced Matrix: a) 1st Bending Mode; b) 1st Torsional Mode; c) 2nd Bending Mode.

Table 4: Consequence of Parametric Matrix Reduction.

	Identification Rate		Average MAC	
	Original	*Reduced*	*Original*	*Reduced*
1st *bending*	100%	100%	0.9999	0.9998
1st *torsional*	100%	69%	0.9883	0.9444
2nd *bending*	100%	95%	0.9917	0.9866

However, the improvement of efficiency comes with a cost on the identification. In the Table 4, a successful identification indicates that it presents structural mode identified with promising value of Modal Assurance Criterion (MAC, here the critical value is set as 0.95). It shows that the identification may not always be successful, even for the most excited modes of the bridge. The main reason is that the reduction of the parametric matrix leads to the distortion of identified mode shape, which is revealed as the decrease of MAC value. How to improve the algorithm efficiency without losing important information is a problem remained to be solved.

5. CONCLUSIONS

In this study, an autoregressive-based structural identification method by means of Bayesian Inference was performed to obtain the modal frequency. Although some discrepancy and instability are remained, its result showed high agreement with those obtained from the existed method.

However, to assess the posterior uncertainty, the method based on Monte-Carlo simulation would be time-consuming. Though an attempt based on SVD and principle components analysis (PCA) was conducted, the improvement in efficiency actually comes with a sacrifice of identification quality. How to improve the algorithm efficiency without losing important information is a problem remained to be solved.

6. REFERENCES

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