Pathways for Uncertainty Quantification through Stochastic Damage Constitutive Models of Concrete

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ABSTRACT: The constitutive model of concrete is of paramount significance for the design of concrete structures and the corresponding reliability assessment. In the present paper, the uniaxial damage model of concrete based on Chinese design code is introduced. It is noticed that there are seven crucial parameters in this model, while five of them are of physical significance and generally should be regarded as random variables. Therefore, the major task of the present paper is to study the effects, variations and randomness of these five parameters. Starting with the fuzzy analysis method (FAM), a brief uncertainty quantification scheme is described. This method is straightforward and easy to implement. Nevertheless, the prior knowledge (i.e., the engineering experience of designers or published literature) is required in FAM. Alternatively, the probability density evolution method (PDEM) is utilized with less needs of prior knowledge, while the type of marginal distribution of parameters is still required or assumed. Thus the epistemic uncertainty may be, more or less, brought in when applying these two methods. To improve this situation, i.e., to reduce the involvement of prior knowledge, a probabilistic learning method (PLM) is applied, in which the prior knowledge is reduced as it is nearly of data-driven background. The research results indicate that these three different methods of uncertainty quantification provide some basic and common conclusions, showing that all of them can capture the main characters of the experimental data. In addition, they individually offer various aspects of information due to different perspectives of these three methods. Therefore, these three methods might derive a series of powerful tools for uncertainty quantification in structural engineering, and be of future interest for opening new perspectives.

1. INTRODUCTION

In civil engineering, there are some inevitable uncertainties according to the observations and experiments. For instance, the compressive strength and modulus of elasticity of concrete are found to be random variables (Ang & Tang 2007), or random fields (Chen et al 2018). Thus, the deterministic damage models cannot well capture the randomness that exists in the properties of concrete materials. Therefore, by introducing the micro-fracture strain as a 1-D random field, the microscopic stochastic fracture model (MSFM, Li & Ren 2009) was proposed. Although MSFM is well behaved to quantify the randomness as well as the nonlinearity of concrete compared with experimental data, it is, however, inconvenient for complex engineering applications. In current Chinese design code (MHURD 2010), the deterministic macro-scale uniaxial damage model for concrete is adopted. This model, according to further study of Li et al (2017), can be regarded as a mean value of the constitutive relationship of concrete with respect to (w.r.t.) MSFM in Li & Ren (2009).

Inspired by the researches of Li et al (2017), the parameters in the deterministic Chinese design code can be regarded directly as random variables, and thus the task of uncertainty quantification (UQ) is in need. In fact, there are seven crucial parameters in the Chinese design code model (CDCM), including the compressive strength f_c and tensile strength f_t of concrete, respectively, w.r.t. the ultimate compressive strain ε_c and tensile strain ε_t, the initial modulus of concrete E_c and two shape parameters (α_c and α_t) to describe the curves in descending phase. In the present paper, the first five parameters of physical significance are then taken as random variables, which is studied based on experimental data. The mostly tough task of this case, propagation of uncertainty in UQ, is handled with three different methods in various aspects, namely the fuzzy analysis method (FAM, Möller & Beer 2004), probability density evolution method (PDEM, Li & Chen 2006) as well as the probabilistic learning method (PLM, Soize & Ghanem 2016). The above three UQ methods are aimed at studying the aleatory along with epistemic uncertainties embedded in CDCM, which indicate their individual properties in different perspectives as powerful UQ tools.

2. CHINESE DESIGN CODE MODEL OF CONCRETE MATERIALS

In Chinese design code (MHURD 2010), the uniaxial constitutive model of concrete for compression and tension is given by

$$\sigma^{\pm} = (1 - d^{\pm}) E_c \varepsilon^{\pm} \tag{1}$$

where $\sigma^{\pm}$ is the stress and $\varepsilon^{\pm}$ is the elastic strain of concrete, $d^{\pm}$ is the damage variable and E_c is the initial modulus of elasticity. The superscript "$\pm$" denotes cases for tension and compression, respectively. For the compressive curve, we have

$$d^{-} = \begin{cases} 1 - \dfrac{\rho_c n}{n - 1 + x^n}, & x \leqslant 1 \\ 1 - \dfrac{\rho_c}{\alpha_c (x-1)^2 + x}, & x > 1 \end{cases} \tag{2}$$

in which $x = \varepsilon^{-}/\varepsilon_c$, $\rho_c = f_c/(E_c \varepsilon_c)$ and $n = E_c \varepsilon_c/(E_c \varepsilon_c - f_c)$. While for the tensile curve, there is

$$d^{+} = \begin{cases} 1 - \rho_t (1.2 - 0.2x^5), & x \leqslant 1 \\ 1 - \dfrac{\rho_t}{\alpha_t (x-1)^{1.7} + x}, & x > 1 \end{cases} \tag{3}$$

where $x = \varepsilon^{+}/\varepsilon_t$ with $\rho_t = f_t/(E_c \varepsilon_t)$. Noticing that in the CDCM, there are totally seven important parameters that are perhaps mutually dependent. On the basis of researches by Guo et al (1982) and Guo & Zhang (1988), the relationship between ε_c and f_c can be given by

$$\varepsilon_c = \left(700 + 172\sqrt{f_c}\right) \times 10^{-6}, \tag{4}$$

while ε_t can be a function of f_t as

$$\varepsilon_t = f_t^{0.54} \times 65 \times 10^{-6}. \tag{5}$$

Also, it is found that f_t is related to f_c by

$$f_t = 0.251 f_c^{2/3}. \tag{6}$$

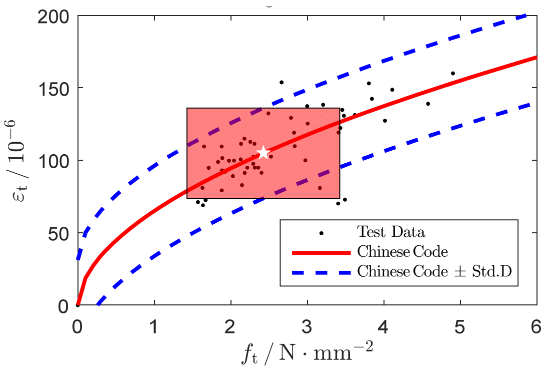
Finally, E_c is of regression by f_c with (MHURD 2010)

$$E_c = 10^5 \Big/ \left(2.2 + \frac{34.7}{f_c}\right). \tag{7}$$

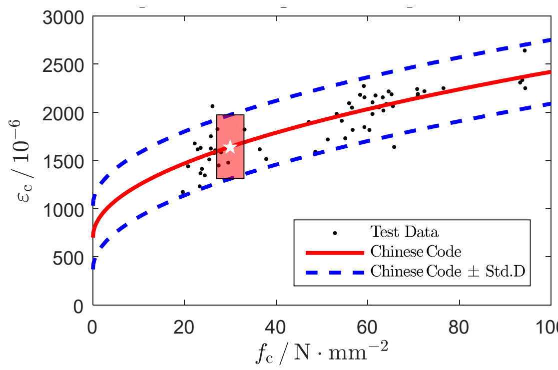
The above Eqs.(4) to (7) are utilized for CDCM in Eqs.(1) to (3). Moreover, two shape parameters are specified, by fitting, as (MHURD 2010)

$$\begin{aligned} \alpha_t &= 0.312 f_t^2, \\ \alpha_c &= 0.157 f_c^{0.785} - 0.905. \end{aligned} \tag{8}$$

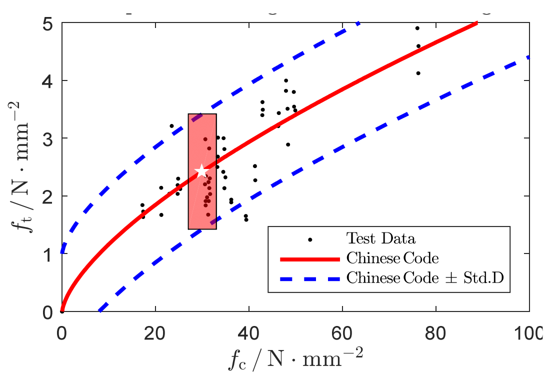
Noticing that Eqs. (4) to (7) are based on experimental data summarized in Figure 1 (red solid lines), which are actually empirical formulas.



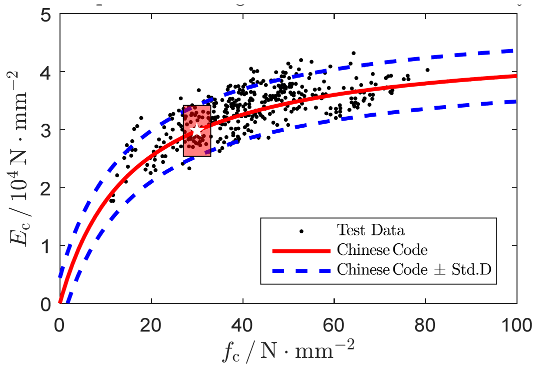
(a) Tensile strength vs. Tensile strain



(b) Compressive strength vs. Compressive strain



(c) Compressive strength vs. Tensile strength



(d) Compressive strength vs. Modulus of elasticity

Figure 1: Empirical formulas in CDCM.

It is obvious that, according to these formulae, for each design value of concrete strength, e.g., if we have $f_c = 30\,\text{MPa}$ (the star mark in Figure 1), the constitutive relationship is only deterministic. Nevertheless, as mentioned before, randomness or uncertainty is objectively existent in the mechanical properties of concrete materials and cannot be negligible. In Li et al (2017), the CDCM is applied for identifying the parameters in the MSFM; similarly but more straightforwardly, the parameters in CDCM graphed in Figure 1 are taken as uncertain variables in our consideration. It is noted that the observed data are nearly all located in the range bounded by the two blue dashed lines ($\pm$ standard derivation value, Std.D); or simultaneously, data are scattered in the region of 10% uncertainty (rectangular region). Therefore, the uncertain parameters in CDCM can be characterized as:

1) fuzzy variables

$$\begin{aligned} \tilde{f}_c &= \langle 0.9 f_c, f_c, 1.1 f_c \rangle, \\ \tilde{f}_t &= \langle f_t - \sigma_{f_t}, f_t, f_t + \sigma_{f_t} \rangle, \\ \tilde{\varepsilon}_c &= \langle \varepsilon_c - \sigma_{\varepsilon_c}, \varepsilon_c, \varepsilon_c + \sigma_{\varepsilon_c} \rangle, \\ \tilde{\varepsilon}_t &= \langle \varepsilon_t - \sigma_{\varepsilon_t}, \varepsilon_t, \varepsilon_t + \sigma_{\varepsilon_t} \rangle, \\ \tilde{E}_c &= \langle E_c - \sigma_{E_c}, E_c, E_c + \sigma_{E_c} \rangle \end{aligned} \tag{9}$$

where $\langle\cdot\rangle$ denotes the membership function of triangular form, and the middle values represents the corresponding design variables. This is related to the FAM (Möller & Beer 2004). Besides, these uncertain parameters can also be quantified as:

2) random variables

$$\begin{aligned}
&\boldsymbol{f}_c \sim N(f_c,(0.05f_c)^2),\\
&\boldsymbol{f}_t \sim N(f_t,(\sigma_{f_t}/2)^2),\\
&\boldsymbol{\varepsilon}_c \sim N(\varepsilon_c,(\sigma_{\varepsilon_c}/2)^2),\\
&\boldsymbol{\varepsilon}_t \sim N(\varepsilon_t,(\sigma_{\varepsilon_t}/2)^2),\\
&\boldsymbol{E}_c \sim N(E_c,(\sigma_{E_c}/2)^2)
\end{aligned} \tag{10}$$

where $N(\cdot,\cdot)$ denotes the normal distribution. Noticing that all Std.D values can be estimated by the experimental data shown in Figure 1. The propagation of uncertainty embedded by random variables can be performed with PDEM easily (Li & Chen 2006).

Moreover, due to the limitation of original data in the range of design region (the rectangular area of engineering consideration) in Figure 1, one can firstly do some "learning" work based on the whole observed data by the so-called probabilistic learning method (PLM, Soize & Ghanem 2016). Compared with FAM as well as PDEM, it is no need to assume the uncertain parameters as fuzzy or random variables anymore. The basic idea for PLM is to find the manifold structure of original data and then generate more suitable new data for analysis, which is the so-called data-driven mode. Clearly, these three methods stand for different viewpoints in UQ framework, and will give various results of quantification due to their own unique properties.

3. THREE UNCERTAINTY QUANTIFICATION METHODS

The basic theories of the three UQ methods mentioned above are summarized hereinafter.

3.1. Fuzzy analysis method

In the fuzzy analysis method, each uncertain parameter is considered as an interval variable w.r.t. a membership value. In this manner, the membership value is used as a parameter to control the extent of uncertainty quantified by the interval variables (Beer et al. 2013). Based on the extension principle and α-level optimization strategy (Möller et al 2000, Möller & Beer 2004), CDCM with assumptions in Eq.(9) can be quantified by the flowchart graphed in Figure 2. In general, it is enough that the number of α-levels is set to be 5 to 20, depending on the features of the analysis.

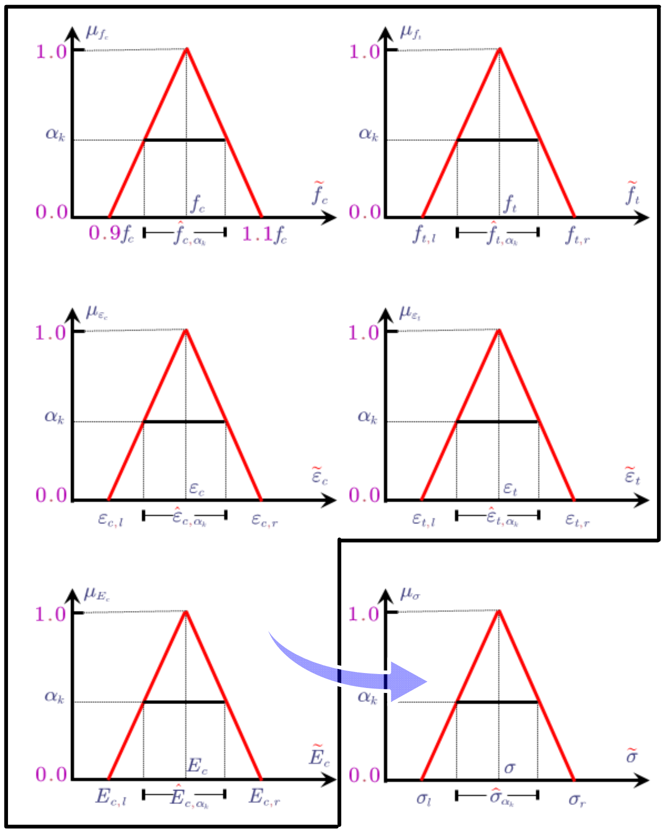


Figure 2: α-level optimization.

3.2. Probability density evolution method

Denote the random source as $\boldsymbol{\Theta}$, and the physical evolution system as $X = G(\boldsymbol{\Theta}, t)$. On the basis of principle of preservation of probability (Li & Chen 2008), we have

$$\frac{\partial p_{X\Theta}(x,\boldsymbol{\theta},t)}{\partial t}+\dot{G}\frac{\partial p_{X\Theta}(x,\boldsymbol{\theta},t)}{\partial x}=0 \quad (11)$$

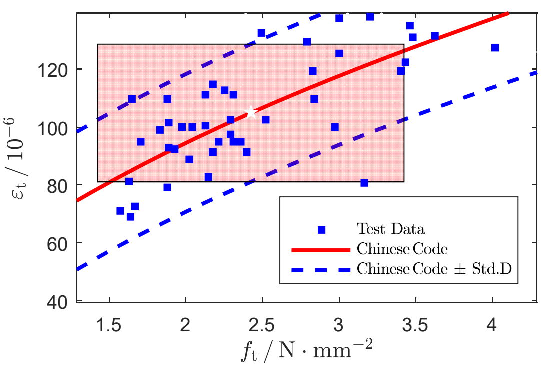
where $\dot{G}$ denotes the generalized velocity of the physical system. By numerically partitioning of the probability space of $\boldsymbol{\Theta}$, i.e., via GF-discrepancy strategy (Chen et al 2016), the probability density function (PDF) of response of system $p_X(x,t)$ can be simply obtained as

$$p_X(x,t)=\int_{\Omega_\Theta} p_{X\Theta}(x,\boldsymbol{\theta},t)\mathrm{d}\boldsymbol{\theta} \doteq \sum_{q=1}^{n_{\rm sel}} p_{X\Theta}(x,\boldsymbol{\theta}_q,t) \quad (12)$$

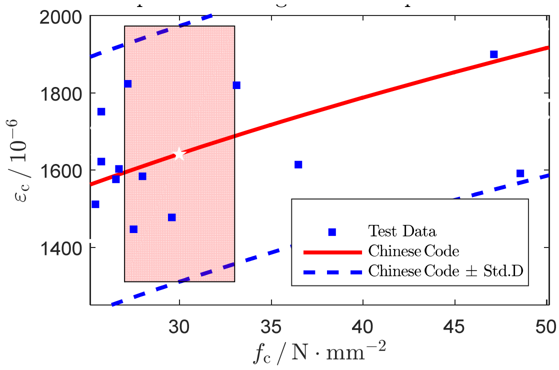
where $\{\boldsymbol{\theta}_q\}_{q=1}^{n_{\rm sel}}$ stands for the representative point set, $n_{\rm sel}$ is the number of representative points, which is dependent in the present case. This value is chosen to be 300 by experience.

3.3. Probabilistic learning method

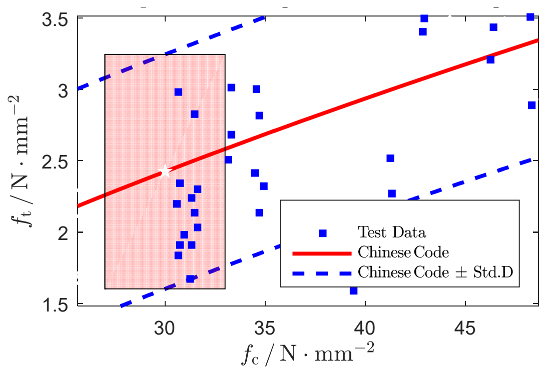
In the scope of design region graphed in Figure 3, data for analysis is sparse indeed. Hence, epistemic uncertainty may be brought in since no adequate data are available for supporting assumptions in above two methods. Noticed that the data distributed in all (Figure 1) are believed to follow the same physical principles (material properties), which can be utilized for "learning".



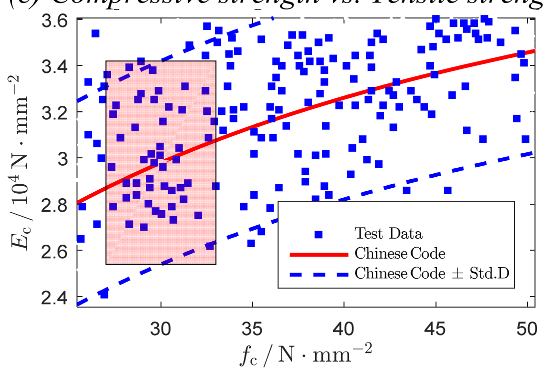
(a) Tensile strength vs. Tensile strain



(b) Compressive strength vs. Compressive strain



(c) Compressive strength vs. Tensile strength



(d) Compressive strength vs. Modulus of elasticity

Figure 3: Available data in the region of design.

Thus the PLM can be applied here to enrich the data in the design region, namely, "learning" the manifold structure of all available data and then generate some "new" data for UQ analysis. The basic theoretical framework for PLM can be summarized into two parts as followings (Soize & Ghanem 2016):

1) Monte Carlo Markov Chain (MCMC):

In general, for a stochastic dissipative Hamiltonian dynamical system

$$\left[\ddot{\mathrm{U}}(r)\right] + \frac{1}{2} f_0 \left[\dot{\mathrm{U}}(r)\right] = \sqrt{f_0}\,[\mathrm{W}(r)] - \nabla_{[\mathrm{U}]}\Phi([\mathrm{U}(r)]) \tag{13}$$

where $[\mathrm{U}(r)]$, $\dot{\mathrm{U}}(r)$ and $\ddot{\mathrm{U}}(r)$ are generalized displacement, velocity and acceleration; $[\mathrm{W}(r)]$ is the Wiener process; $\Phi(\cdot)$ is the potential function; f_0 is the damping term. If $[\mathrm{U}(0)]$ is distributed w.r.t. $q(u)$ and assume we have

$$\Phi(u) = -\ln\{q(u)\}, \tag{14}$$

then when $r \to \infty$ there is

$$\lim_{r\to\infty} [\mathrm{U}(r)] = [\mathrm{U}(0)]. \tag{15}$$

Eq.(15) holds in probability distribution $q(u)$.

2) Diffusion Maps:

Denote two arbitrary points in Euclidean space as ξ and ζ, and the distance is measured by

$$k_\epsilon(\xi,\zeta) = \exp\left(-\frac{1}{4\epsilon}\|\xi-\zeta\|^2\right). \tag{16}$$

For all points in the original data set $[\eta]_{s\times n}$, using Eq.(16) to structure the adjacent matrix as

$$[K]_{ij} = k_\epsilon(\eta^i,\eta^j) \text{ for } i,j=1,\cdots,n \tag{17}$$

By rebuilding the diffusion-maps basis, the affine transformation is established as

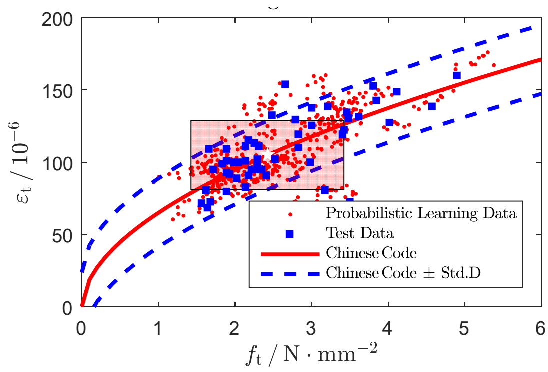
$$[\eta]_{s\times n} \overset{\text{To Manifold Space}}{\longmapsto} [z]_{s\times m} \qquad [z]_{s\times m} \overset{\text{To Euclidean Space}}{\longmapsto} [\eta]_{s\times n} \tag{18}$$

The quantities in Eq.(18) satisfy

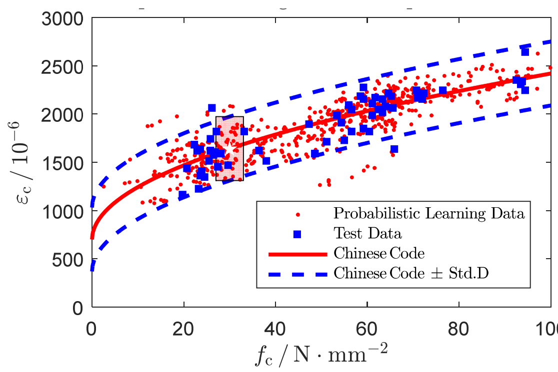
$$[z] = [\eta][a] \quad \text{s.t. } [a] = [g]([g]^{\mathrm{T}}[g])^{-1}$$
$$[\eta] = [z][g]^{\mathrm{T}} \text{ s.t. } g^i = \lambda_i \varphi_i \quad i=1,\cdots,m \tag{19}$$

in which $\boldsymbol{\lambda}$ and $\boldsymbol{\varphi}$ are eigenvalues and eigenvectors w.r.t. $[\mathbb{P}_s]\boldsymbol{\varphi} = \boldsymbol{\lambda}\boldsymbol{\varphi}$, in which $[\mathbb{P}_s]$ is calculated by $[K]$, see Soize & Ghanem (2016) for details.

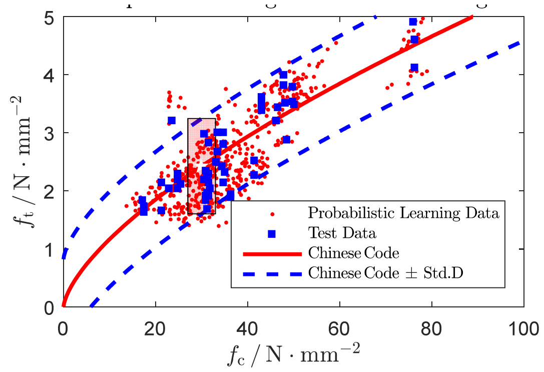
The PLM is then applied to "generate" more data that all follow the same original probability distribution and the manifold structure, as is shown in Figure 4. Then these data can be analyzed statistically, which is totally data-driven.



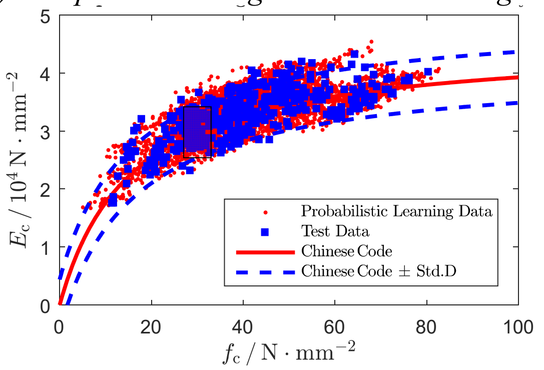
(a) Tensile strength vs. Tensile strain



(b) Compressive strength vs. Compressive strain



(c) Compressive strength vs. Tensile strength



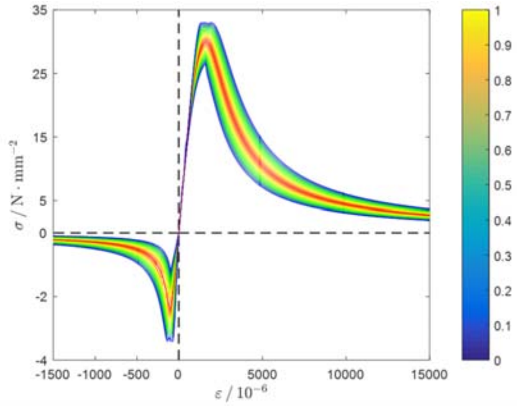
(d) Compressive strength vs. Modulus of elasticity

Figure 4: Generated data by PLM.

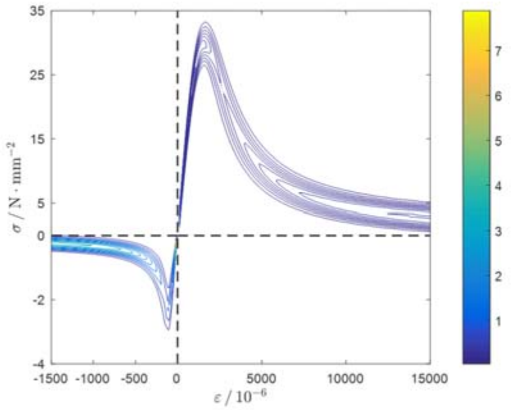
4. RESULTS OF UQ AND DISCUSSIONS

The UQ results by the three different methods are illustrated in Figure 5 (curves of tension are amplified 5 times compared with compression parts for clarity). It can be seen that all of them can capture the basic features embedded in the CDCM, i.e., the ascending curve is convex. Nonetheless, different aspects are:

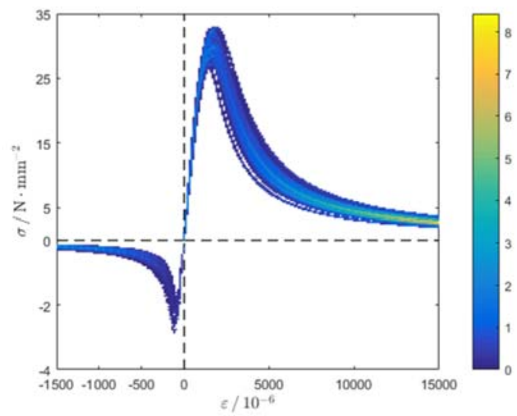
1) For FAM, it is indicated that the ascending curves of compression and tension, are not obviously effected by uncertain parameters in CDCM. Besides, when the membership value is high (1~0.7), the descending curves are narrowly distributed, indicating a relatively low uncertain effect. It should be noted that the curves in Figure 5a are not w.r.t. any real stress-strain curve (but a set of many stress-strain curves), and that is why there is a platform at the peak points.



(a) Concrete uniaxial constitutive model via FAM



(b) Concrete uniaxial constitutive model via PDEM



(c) Concrete uniaxial constitutive model via PLM

Figure 5: Results of UQ for CDCM via (a) FAM, (b) PDEM and (c) PLM.

2) The results in Figure 5b via PDEM provide some other useful information of UQ. It turns out that the value of probability density contour curves, for the compression part, is lower than that of tension, which demonstrates that the compression part is more likely to be affected. Nevertheless, the compressive behavior of concrete materials is certainly of importance; therefore, more UQ works are needed to be carried out in future.
3) Based on the generated "new" data via PLM, the histogram results are displayed in Figure 5c. It is somewhat similar to PDEM but reveals some new features. Firstly, it can be seen that the tails of curves in compression and tension parts are much thinner; meanwhile, the curves evolve asymmetrically, especially for compression part. This may be reasonable as the original data are distributed slightly above average.

Moreover, the enriched data by PLM can be reused to modify the uncertainty characterizations in FAM and PDEM, i.e., the membership function can be asymmetric, or the probability distribution can be skewed. In summary, all of these three UQ methods can, more or less, help engineers to form a better understanding of engineering issues that involving either aleatory or epistemic uncertainties, or both of them.

5. ACKNOWLEDGEMENTS

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