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Stochastic Fragility Analysis of MDOF Systems using Gaussian Mixture-Based Equivalent Linearization Method (GM-ELM)

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ABSTRACT: This research improves the recently developed Gaussian mixture based equivalent linearization method (GM-ELM) to promote its applications to the seismic fragility analysis. GM-ELM is a stochastic linearization method, which defines the equivalent linear system by a set of linear oscillators. This linear system is identified from Gaussian mixture distribution-based decomposition of the instantaneous response probability distribution function (PDF). The method has merits in terms of the applicability to general multi-degree of freedom (MDOF) systems. However, the method may suffer from limitations of stationary assumption and excitation-intensity-dependency of ELS. As a further development to facilitate earthquake engineering applications, we extended GM-ELM to nonstationary responses by proposing to use the temporal average of response PDF. Furthermore, the concept of universal ELS which is invariant to the scaling of excitation is established by utilizing intensity-augmented PDF. Numerical experiments demonstrate viability of the proposed approach on general MDOF systems and nonstationary excitations.

1. INTRODUCTION

Seismic reliability assessment of a structural system plays an important role in design and maintenance decisions. However, the random nature of earthquake shakings and the complex behavior of nonlinear structures may make evaluation of the stochastic response a challenging task. The fragility analysis is often used to derive the probability of failure for a range of ground motion intensities.

In the common practice of the fragility analysis, the randomness is addressed by repeated nonlinear dynamic simulations using a set of earthquake motion recordings. The simulated responses, e.g. peak responses, are fitted to a probability density function (PDF) to evaluate the conditional probability of structural failures given seismic intensity. Therefore, the results could be strongly influenced by the selection of ground motions. In most of the design practice, the reference response spectrum for the specific site is first evaluated, and then earthquake records selected from a database are artificially scaled to be compatible with the spectrum. However, the appropriateness of different selection and scaling algorithms still remain elusive (Der Kiureghian and Fujimura 2009).

Another branch of earthquake engineering method avoids these difficulties by mathematically modeling the site-specific ground motions in terms of its frequency contents, e.g. as a power spectral density (PSD) function. Thereby, earthquake ground motion could be generated as many times as desired. Moreover, the propagation of randomness to the structural responses can be quantified by means of stochastic parameters and the random vibration analyses. Especially when dealing with a nonlinear system, adoption of these random vibration analysis methods could save a significant amount of computational cost, compared to brute force Monte Carlo simulations.

Equivalent linearization method (ELM) is one of this nonlinear random vibration analysis approaches that gains the efficiency by substituting the complicated behavior of the nonlinear system by that of an equivalent linear system (ELS). Most widely used ELS is the linear system that minimizes the mean square error, or the error in the variance of the structural responses. This approach is appealing for the computational efficiency since it does not require the repeated dynamic simulations. However, it does not conserve several stochastic properties, due to the intrinsic restriction. For example, the stationary response of an equivalent linear system subjected to the stationary Gauss excitation strictly follows Gaussian distribution, but the response of the original nonlinear system is generally non-Gaussian. Thus, the estimation by conventional ELS could be erroneous especially at the tail-range of the response PDF.

Several approaches has been established to overcome this limitation, and among those, a reliability-theory based linearization approach, termed tail-equivalent linearization method (TELM) (Der Kiureghian and Fujimura 2009) has gained recognition recently. TELM overcomes the prescribed limitation by identifying a series of ELS that each fits for a specific threshold value of the interest. TELM has proven its accuracy and efficiency in diverse application examples. This approach was also applied to assess seismic fragility analysis, utilizing its desirable features such that it could be applied for multi-degree of freedom (MDOF) structures and nonstationary ground motions. Moreover, the ELS identified by TELM is proven to be scale-invariant. However, TELM might require relatively high computational cost, due to repeated first-order reliability method (FORM) analysis and nonlinear dynamic analysis, and that the resulting ELS is threshold dependent.

Inspired by above linearization methods, Gaussian Mixture based ELM (GM-ELM) was recently developed by Wang and Song (2017). Unlike conventional ELM, GM-ELM conserves the non-Gaussianity of the response by defining a set of multiple linear oscillators, which collectively resembles the nonlinear responses. This decomposition or the *linearization* process is done by means of the Gaussian mixture fitting. GM-ELM has shown the superior accuracy and efficiency for predicting the failure probabilities of nonlinear system, and has an attractive feature that it is applicable to generic nonlinear MDOF structures. However it has yet been used for seismic fragility analysis, which mainly attributes to the two practical restrictions: (**i**) GM-ELM requires the assumption of stationary response, and (**ii**) ELS identified by GM-ELM is highly dependent on the scaling of the input excitation. These are main obstacles to earthquake applications of GM-ELM, and make GM-ELM-based seismic fragility analysis inefficient. In this research, we overcome these challenges by two improved versions of GM-ELM, which are either jointly or individually applicable. Throughout the paper each improvement will be referred to as "temporal-average" and "intensity-augmented" GM-ELM, respectively. The proposed methods are demonstrated by the numerical example of a 6-story building showing a hysteretic behavior, located at Gyeongju, Korea and corresponding design code-conforming nonstationary ground motion model.

2. BRIEF REVIEW OF GM-ELM

2.1. Concept of GM-ELM

The Gaussian Mixture based Equivalent Linearization Method (GM-ELM) approximates the complicated behavior of a nonlinear system by that of an equivalent linear system. To assure the equivalent stochastic properties, GM-ELM equalizes the nonlinear response probability density function (PDF) to that of an equivalent linear system. Particularly, a general probability distribution of a *stationary* nonlinear response (e.g. the displacement at specific location of the system) is approximated by a set of Gaussian distributions by means of the Gaussian Mixture distribution model. Since the stationary response of a linear system subjected to the Gauss process excitation also follows a Gaussian process, each component of GM model can be regarded as the response of one of the imaginary linear oscillators. Therefore, the linear system of GM-ELM consists of a multiple linear oscillators that have different locations and structural parameters.

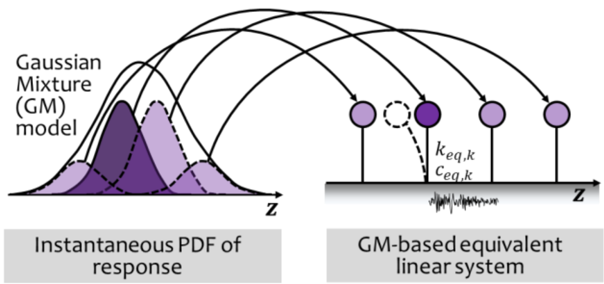

Figure 1: Identification of equivalent linear system

2.2. Linearization of GM-ELM

Consider the response of nonlinear structure at time t subjected to the stationary excitation, for which the PDF of the instantaneous responses are time-invariant, i.e. $f_{\boldsymbol{z}}(\boldsymbol{z};t)=f_{\boldsymbol{z}}(\boldsymbol{z})$. Then, a GM-based approximation of the PDF is

$$f_{\boldsymbol{z}}(\boldsymbol{z}) \simeq \sum_{k=1}^{K} \alpha_k f_N(\boldsymbol{z};\boldsymbol{\mu}_k,\boldsymbol{\Sigma}_\mathrm{k}) \qquad (1)$$

where K is the total number of involving Gauss components. The parameter α_k is the relative contribution of each component, and $\boldsymbol{\mu}_k$ and $\boldsymbol{\Sigma}_\mathrm{k}$ are respectively its mean and covariance. The vector of responses, denoted by $\boldsymbol{z}$, could be determined from the response of interest. For example, if one is interested in the passage event of the displacement at specific location of a nonlinear system, denoted by z, one can either use the PDF of $\boldsymbol{z}=z$ or the joint PDF of $\boldsymbol{z}=\{z,\dot{z}\}$. The analysis with the former information is named as the univariate GM-ELM (Wang and Song, 2017), and that with the later information is bivariate GM-ELM (Yi *et al.*, 2018).

The GM parameters $\{\alpha_k,\boldsymbol{\mu}_k,\boldsymbol{\Sigma}_k\}$ each gives the information on the corresponding equivalent linear oscillator. While α_k and $\boldsymbol{\mu}_k$ respectively represents the relative occurrence rate and underlying displacement/velocity shift of the k-th oscillator, $\boldsymbol{\Sigma}_k$ is used to infer its structural parameters. The explicit relationship between a linear single DOF system parameters and its response variances are (Lutes and Sarkani, 2004)

$$\sigma_z^2 = 2\int_0^\infty |H(\omega)|^2 S_g(\omega)d\omega \qquad (2a)$$

$$\sigma_{\dot{z}}^2 = 2\int_0^\infty \omega^2 |H(\omega)|^2 S_g(\omega)d\omega \qquad (2b)$$

where $H(\omega)$ is the frequency response function (FRF) of the single degree-of freedom (SDOF) system. For example, if the excessive displacement due to the acceleration excitation is of concern, the following FRF is used

$$H_k(\omega) = \frac{m_{eq,k}}{m_{eq,k} + i\omega c_{eq,k} - \omega^2 k_{eq,k}} \qquad (3)$$

in which $m_{k,eq}$, $c_{k,eq}$ and $k_{k,eq}$ are respectively the mass, damping and stiffness of k-th equivalent linear oscillator. Eq. (2a) is used to optimize the equivalent stiffness $k_{k,eq}$, and Eq. (2b) can be use to optimize equivalent damping $c_{k,eq}$ when $\sigma_{\dot{z}}^2$ is available. Meanwhile, the redundant parameters can be pre-fixed, for example, lumped mass at the response of interest could be used as equivalent mass in bivariate GM-ELM.

2.3. Analysis using Equivalent Linear System

The collective response of ELS can approximate the nonlinear response. First, the mean up-crossing rate of the response can be derived as the weighted sum of the crossing rates of each linear oscillators (Wang and Song, 2017), i.e.

$$\nu^+(a) = \sum_{k=1}^{K} \alpha_k \nu_k^+(a) \qquad (4)$$

where a denotes the response threshold, and

$$\nu_k^+(a) = \frac{1}{2\pi}\sqrt{\frac{\lambda_{2,k}}{\lambda_{0,k}}}\exp\left[-\frac{0.5(z-\mu_{z,k})^2}{\lambda_{0,k}}\right] \quad (5)$$

in which $\lambda_{j,k}$ is j-th order spectral moment (Lutes and Sarkani 2004) of the k-th linear oscillator. For bivariate GM-ELM (Yi *et al*., 2018), it is shown that the shifted velocity could also be accounted for by modifying the contribution factor α_k into

$$\hat{\alpha}_k = F_\nu(\delta_{\dot{z},k}) \cdot \alpha_k \quad (6)$$

where the $F_\nu(\delta_{\dot{z},k})$ is the modification factor proposed as

$$F_\nu(\delta_{\dot{z},k}) = \sqrt{2\pi}\,[\varphi(\delta_{\dot{z},k}) + \delta_{\dot{z},k} - \delta_{\dot{z},k}\Phi(-\delta_{\dot{z},k})] \quad (7)$$

in which $\delta_{\dot{z},k} = \mu_{\dot{z},k}/\sigma_{\dot{z},k}$, and the parameters $\mu_{\dot{z},k}$ and $\sigma_{\dot{z},k}$ are respectively the mean and standard deviation of the derivative response of the k-th Gauss component.

By Poisson assumption of the crossing events, the first passage probability during the excitation duration T_d is also derived as

$$P_f(a) = 1 - A\exp\left(-\sum_{k=1}^{K}\alpha_k\nu_k^+(a)\,T_d\right) \quad (8)$$

where A is the probability of the safe start, i.e. $A = \Pr(z < a)$.

2.4. *Drawbacks of GM-ELM for nonstationary seismic fragility analysis*

The existing GM-ELMs have two limitations in terms of the application to seismic fragility analysis: **a**) GM-ELM cannot accommodate the nonstationary response, and **b**) the properties of the oscillators in the ELS are highly dependent on the intensity of the excitations. The first one becomes restriction when dealing with the realistic nonstationary earthquake models. On the other hand, the second issue can make GM-ELM inefficient for fragility analysis which requires the repeated evaluation of failure probabilities for a range of intensity levels.

3. IMPROVEMENT 1: TEMPORAL-AVERAGE GM-ELM

3.1. *Concept of temporal-average ELS*

To embrace the nonstationary responses, we first consider the *temporal-average* PDF of the response. Since the nonstationary response has a time-variant PDF, $f_{\mathbf{z}}(\mathbf{z};t)$, its temporal average is represented as

$$\hat{f}_{\mathbf{z}}(\mathbf{z}) = \frac{1}{T_d}\int_{T_i}^{T_e} f_{\mathbf{z}}(\mathbf{z};t)\,dt \quad (9)$$

where T_i and T_e respectively denote the starting and ending time points of the strong motion duration $T_d = T_e - T_i$. By using $\hat{f}_{\mathbf{z}}(\mathbf{z})$ instead of $f_{\mathbf{z}}(\mathbf{z})$ in Eq. (1), one could identify the temporal-average ELS. This ELS enables us to utilize the same GM-ELM analysis procedures (Figure 2). Unlike ELS identified by the existing GM-ELM, which only incorporates stationary responses, temporal-average ELS has the following properties:

- The *nonstationary* nonlinear response can be approximated by a set of *stationary* linear responses. The corresponding stationary linear responses are acquired from the equivalent stationary excitation, which is defined as the temporal average of

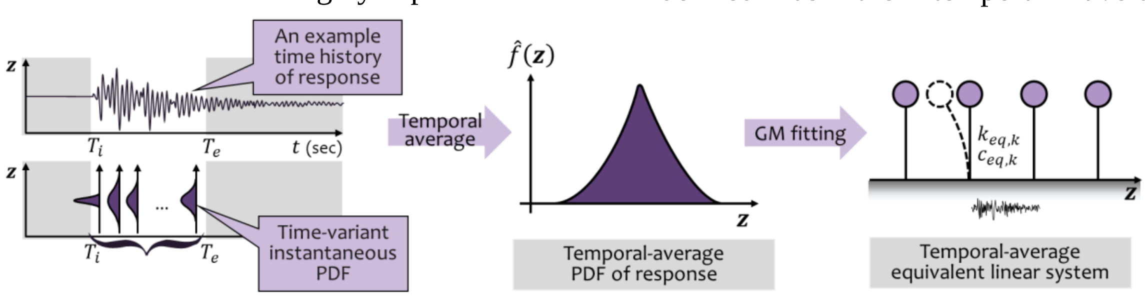

Figure 2: Concept of temporal-average GM-ELM

nonstationary PSD. If the nonstationary excitation is represented by an evolutionary PSD $S_g(\omega, t)$ (Lutes and Sarkani, 2004), its temporal average can be calculated as:

$$\hat{S}_g(\omega) = \frac{1}{T_d}\int_{T_i}^{T_e} S_g(\omega, t)dt \tag{10}$$

- While the response statistics such as the instantaneous failure probabilities or the mean up-crossing rate are also time-variant in nonstationary process, the proposed temporal-average GM-ELM analysis produces the time-averaged values of those. Specifically, the mean up-crossing rate obtained using Eq.(5) will correspond to the time averaged value of the crossing rate, i.e.

$$\hat{v}_k(z) = \frac{1}{T_d}\int_{T_i}^{T_e} \nu_k(z; t)dt \tag{11}$$

  This is a temporal average of the crossing rate formulation of Rice (1945). Under the nonhomogeneous Poisson assumptions of the crossing events, the first passage probability is then estimated by

$$P_f(a) = 1 - \exp\left(-\sum_{k=1}^{K} \alpha_k \hat{v}_k^{+}(z; a)\, T_d\right) \tag{12}$$

- Note that, for stationary inputs and responses, time-average GM-ELM analysis is identical to the existing GM-ELM.

3.2. Strong motion duration of temporal-average GM-ELM

We next discuss how the strong motion duration of the nonstationary response is identified for the temporal-average GM-ELM. Since the critical response of a structure is expected to occur while the external energy is being supplied, we select the criteria defined in terms of the cumulative energy of input ground motions. Particularly, Arias intensity is adopted which is evaluated by the integration of the squared excitation history:

$$I_A(t) = \frac{\pi}{2g}\int_0^t f(t)^2 dt \tag{13}$$

The average intensity is derived in terms of the evolutionary PSD model in Eq. (10) using (Zembaty 1988)

$$E[I_A(t)] = \frac{\pi}{g}\int_0^t \int_0^\infty S_g(\omega, \tau)\, d\omega\, d\tau \tag{14}$$

The cumulated amount of energy by time *t* can be normalized by that of the overall duration T, i.e.

$$r_s(t) = \frac{E[I_A(t)]}{E[I_A(T)]} \tag{15}$$

The starting and ending time of the strong motion duration are defined as the time points when the normalized value reaches 5% and 99% respectively, in order to cover the time interval which critical response is likely to occur.

4. IMPROVEMENT 2: INTENSITY-AUGMENTED GM-ELM

4.1. Universal Equivalent Linear System

To identify an ELS that is invariant to the scaling of excitation, we propose to incorporate the information on the intensity by means of the *intensity-augmented* PDF. Consider an auxiliary variable *I* which represents the intensity scale of an earthquake event. By assigning arbitrary distribution, e.g. uniform distribution, to the variable, we define the intensity-augmented PDF as

$$f_{\mathbf{z},I}(\mathbf{z}, I) = f_{\mathbf{z}|I}(\mathbf{z}|I)f_I(I) \tag{16}$$

which could be approximated using Gaussian mixture distribution model with a higher dimension. By imposing independent condition between *I* and $\boldsymbol{z}$, we obtain

$$\begin{aligned} f_{\mathbf{z},I}(\mathbf{z}, I) &\simeq \sum_{k=1}^{K} \tilde{\alpha}_k f_N\left(z, I \middle| \tilde{\boldsymbol{\mu}}_k, \tilde{\boldsymbol{\Sigma}}_k\right) \\ &= \sum_{k=1}^{K} \tilde{\alpha}_k f_N\left(I \middle| \tilde{\mu}_{I,k}, \tilde{\sigma}^2{}_{I,k}\right) f_N\left(z \middle| \tilde{\boldsymbol{\mu}}_{z,k}, \tilde{\boldsymbol{\Sigma}}_{z,k}\right) \end{aligned} \tag{17}$$

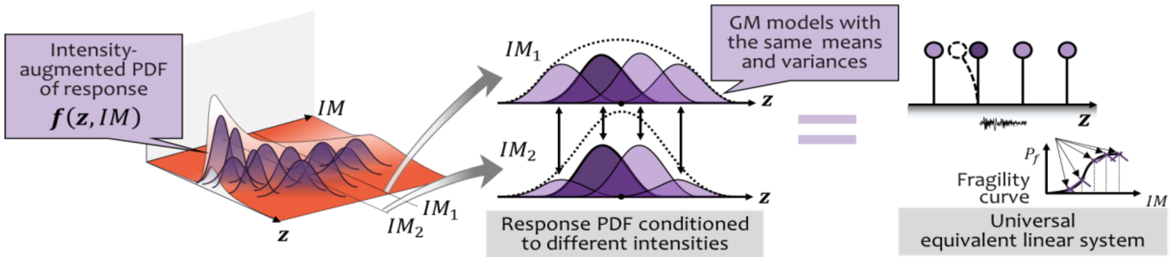

Figure 3: Concept of intensity-augmented GM-ELM

Then, the response PDF conditional to I is

$$f_{\mathbf{z},I}(\mathbf{z}|I) = \frac{f_{\mathbf{z},I}(\mathbf{z}, I)}{f_I(I)} \quad (18)$$

By substituting Eq. (17) and its marginalized distribution with respect to I into Eq. (18), we obtain

$$f_{\mathbf{z},I}(\mathbf{z}|I) = \sum_{k=1}^{K} \alpha_k^*(I) f_N\left(z\middle|\tilde{\boldsymbol{\mu}}_{z,k}, \tilde{\boldsymbol{\Sigma}}_{z,k}\right) \quad (19)$$

where

$$\alpha_k^*(I) = \frac{\tilde{\alpha}_k f_k\left(I\middle|\tilde{\mu}_{I,k}, \tilde{\sigma}^2{}_{I,k}\right)}{\sum_{n=1}^{K} \tilde{\alpha}_n f_n\left(I\middle|\tilde{\mu}_{I,n}, \tilde{\sigma}^2{}_{I,n}\right)} \quad (20)$$

quantifies the contribution of the k-th component given the intensity scale I. The conditional distribution is reduced back to the Gauss mixture approximation with the dimension of $\mathbf{z}$. Note that, in Eq. (19), the parameter related to the augmented dimension only affects the contribution factor $\alpha_k^*(I)$, and the parameters of each Gauss distribution component do not change along the intensity as depicted in Figure 3. The ELS identified from intensity-augmented PDF is termed as the *universal* ELS, owing to the following properties:

- The structural properties and base locations of the universal equivalent linear oscillators are invariant to the scaling of excitations. Therefore this ELS needs to be identified only once for fragility analysis.
- The relative rate of occurrence or contribution between the oscillators changes for different intensities. It can be re-evaluated by simple calculation of Eq. (20), once the mixture parameters are determined.

5. PROCEDURE FOR GM-ELM-BASED FRAGILITY ANALYSIS

By incorporating both temporal-average and intensity-augmented PDF, the fitted GM model can be obtained as

$$\hat{f}_{\mathbf{z},I}(z, I) = \int_{T_i}^{T_e} f(z, I; t)\, dt \quad (21)$$

Figure 4 illustrates the GM-ELM fragility analysis procedures.

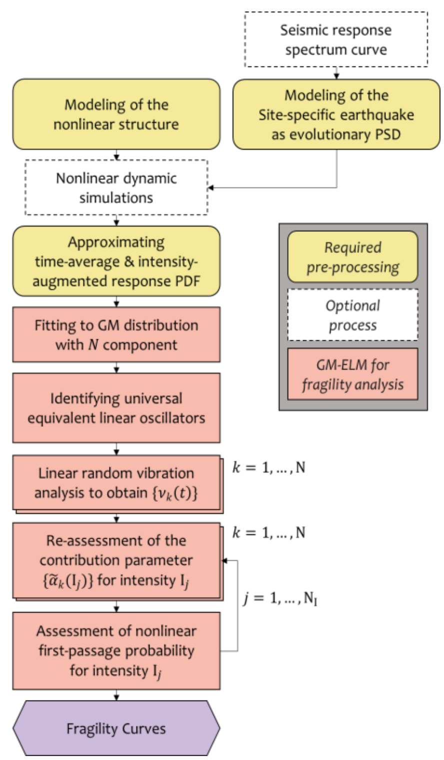

Figure 4: Procedures of GM-ELM fragility

In general, it is difficult to know the exact shape of the temporal-average and intensity-augmented PDF. Therefore, the most practical choice is to approximate it by discretized result of nonlinear dynamic simulations. In particular, to obtain samples of the intensity-augmented PDF, one could first generate an intensity measure of the ground motion, and perform nonlinear dynamic simulations, by generating artificial ground motion time history that fits to the intensity. An enough number of samples can be obtained by repeating the procedure.

6. NUMERICAL EXAMPLES

The proposed approaches are applied to the fragility analysis of a 6 story building structure located at Gyeongju, Korea. The nonlinearity is described by the Bouc-Wen hysteresis model (Wen 1976). The first and second initial natural periods are 0.576 sec and 0.238 sec, respectively. The initial damping ratio is 0.05 and pre- and post-yield stiffness ratio is 0.1. The yield displacement is $u_y = 0.025$ m and the three (floor) displacement limit states $\{1.0u_y,\ 1.5u_y,\ 2.0u_y\}$ are considered. Figure 5 shows the structure the typical hysteresis behavior at the 1st and 6th floor.

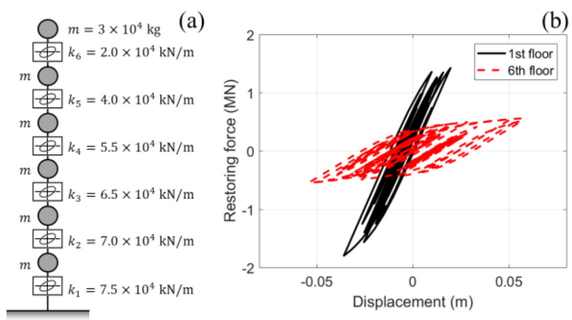

Figure 5: (a) Shear building model, and (b) hysteretic behavior at 1st and 6th floor

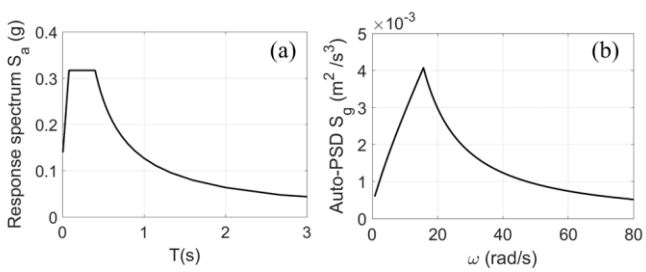

Figure 6: (a) Design response spectrum, and (b) corresponding power spectrum density function

Design code-conforming ground motion is considered (Alibrandi and Mosalam 2018), whose nonstationarity is expressed by a uniformly modulated PSD, i.e., stationary base process multiplied by time-modulating function. The response spectrum for Gyeongju is evaluated from the Building Design Code of Korea (KBC 2016) and shown in Figure 6(a). The response spectrum is first converted to the following base stationary PSD function in Figure 6(b) using the iterative formulation of Cacciola *et al.* (2004):

$$S_g(\omega_i) = \begin{cases} 0 & 0 \le \omega_i \le \omega_o \\ \dfrac{4\zeta_o}{\omega_i \pi - 4\zeta_o \omega_{i-1}} \left(\dfrac{S_a^{\,2}(\omega_i, \zeta_o)}{\eta_x^2} - \Delta\omega \sum_{j=1}^{i-1} S_g(\omega_j) \right) & \omega_o < \omega_i \end{cases} \tag{22}$$

where ω_i is the i-th value of the discretized frequency domain, and $\zeta_o = 0.05$ is the damping ratio of given response spectrum, $\omega_o = 0.36$ rad/s and η_x is the peak factor evaluated by the closed-form equations by Der Kiureghian (1980). For the time modulating function, the model proposed by Hsu and Bernard (1978) is adopted, i.e.

$$q(t) = \varepsilon_{HB} t \cdot \exp(-\mu_{HB} t) \tag{23}$$

where $\varepsilon_{HB} = 1/t_{max}$, $\mu_{HB} = e/t_{max}$, and $t_{max} = 5$ sec.

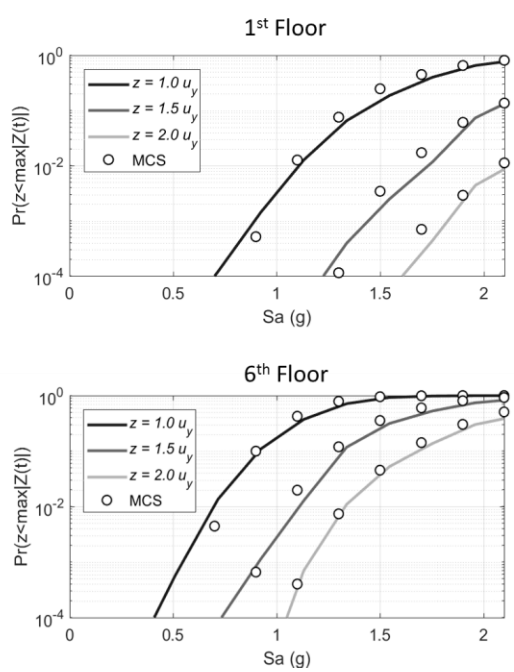

Figure 7: Fragility curves for 1st and 6th floor

Figure 7 illustrates the fragility analysis results in semi-log scale. For the GM-ELM analysis, total 700 rounds of dynamic analysis results are used to approximate tri-variate temporal-average and intensity-augmented PDF, which is fitted by GM model with 196 components. By comparing it with the Monte Carlo simulation results with a sufficient number of simulations (with total 1×10^5 dynamic analysis), one could verify that the proposed approach accurately captures the fragility of the MDOF structure.

7. CONCLUSIONS

This paper proposes two further developments of GM-ELM that aims to increase applicability to seismic fragility analysis. First, the development of temporal-average GM-ELM allows us to embrace the nonstationary response, by substituting temporal-average PDF in the place of instantaneous PDF. Second, in the development of the intensity-augmented GM-ELM, we newly introduced the concept of universal ELS which is invariant to the scaling of excitations, and proposed corresponding linearization process. Two methods are jointly applied to the numerical example of a nonlinear MDOF system under the code-conforming nonstationary excitations. The encouraging results of the numerical example demonstrated the viability of the GM-ELM in the earthquake engineering applications.

As future research topics, one could develop more efficient methods to identify temporal-average and intensity-augmented PDF, in place of the current brute force sampling approach. Also current practice of GM-ELM adopts Poisson assumption for the first passage events. It is desirable to have more sophisticated estimation equations to increase the accuracy, especially when the narrow-band responses are of interest.

8. ACKNOWLEDGEMENT

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