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Bayesian Reliability Assessment with Spatially Variable Measurements: The Spatial Averaging Approach

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ABSTRACT: A large portion of the hydraulic structures in the German waterways network have been in service for over 70 years. Verification of these structures according to current standards is not always possible, as they were built according to the standards valid at the time of construction. The massive shape of large hydraulic structures can lead to spatially variable properties, which should be explicitly addressed in a reliability assessment through a random field modeling. Spatially distributed measurements of the concrete properties are available for many structures. These measurements can be used to update the random field model of the concrete properties through Bayesian analysis. In this contribution, we develop and investigate a simplified approach to model non-homogeneous spatial variability in Bayesian posterior random field models that is consistent with the common semi-probabilistic assessment format. The proposed method reduces the random field to a small number of random variables that correspond to local averages of the random field and, hence, generalizes the spatial averaging method originally developed for approximating homogeneous random fields. This should facilitate its application in engineering practice. We apply the proposed methodology to a numerical example and compare the results to those obtained with the random field approach.

1. INTRODUCTION

An important task in infrastructure management is the reliability assessment of existing structures that are part of the network, especially when these structures are approaching their intended service life. This is the case for a large portion of the hydraulic structures in the German waterways network, a substantial number of which have been in service for more than 70 years (Westendarp et al., 2015). Due to the characteristics of hydraulic structures, in particular their massive size and specific load conditions in combination with the long service life, the German Federal Waterways Engineering and Research Institute has developed a guideline for the structural verification of existing hydraulic structures (Federal Waterways Engineering and Research Institute, 2016). It allows the use of probabilistic methods for modeling the material and geometrical properties as well as load conditions.

Modeling the uncertainties of a system as ran-

dom variables may not suffice in the case where an uncertain property is spatially variable, as is the case for material properties in large hydraulic structures. To correctly account for this spatial variability, uncertainties have to be modeled as random fields instead of random variables (Vanmarcke, 2010).

The structural verification of existing structures differs from the verification of newly built structures as often data is available from inspection and repair actions that have taken place throughout the service life. A powerful tool to include these data into an existing (prior) probabilistic model is Bayesian analysis (Gelman et al., 2013).

By combining random field modeling and Bayesian analysis to incorporate available data, it is possible to define an accurate probabilistic model of the material properties for reliability analysis of a structure. However, this requires to perform all computational steps with a full random field model, which is computationally expensive and not easy to couple with most commercial finite element software. The complexity of the random field model can be reduced by employing a discretization technique which expresses the random field by a finite number of random variables. A variety of such methods exist, including point discretization methods, averaging discretization methods and series expansion methods (Sudret and Der Kiureghian, 2000). In this contribution we use the spatial averaging method (SA) (Vanmarcke, 2010) for the discretization of random fields conditional on spatially variable measurements. First, we employ direct measurements of the parameter to update the random field model. The reduction to a set of random variables by SA is then performed on the (non-homogeneous) posterior random field. This generalizes the application of the SA method, which was originally developed for the discretization of homogeneous random fields. The proposed approach accounts for the spatial correlation of the measurement locations in the Bayesian analysis, while the SA discretization allows direct application of commercial finite element software in the reliability analysis process.

The outline of the paper is as follows: First, a brief introduction to spatial variability and the modeling of random fields with the SA method is given. This is followed by a short discussion of Bayesian analysis and the conjugate prior concept that is used to update the random field parameters. The combination of Bayesian analysis and SA is then applied to a numerical example to investigate the effects and efficiency of the proposed approach.

2. Modeling spatial variability

The traditional approach of modeling uncertainties in material properties with random variables is sufficient for cases where the spatial variability of the modeled quantity is negligible or not of importance to the performance of the investigated system. In large hydraulic structures, some material properties may vary significantly in space. Modeling these properties with random variables would assume perfect correlation of all spatial locations and hence underestimate the uncertainty of the system. By modeling these uncertainties with random fields instead of random variables, this spatial variability can be quantified and accounted for in the reliability analysis.

2.1. Random fields

A random field is defined as an indexed collection of random variables $X(\mathbf{t})$, $\mathbf{t} \in \Omega$, where Ω describes the one-, two- or three-dimensional domain on which the random field is defined (Vanmarcke, 2010). The index $\mathbf{t}$ denotes the spatial location within Ω. The random field follows the probability density function (PDF) $f(x(\mathbf{t}))$ at each point $\mathbf{t} \in \Omega$. The correlation of different spatial locations is described by the autocorrelation function $\rho(\mathbf{t}_1, \mathbf{t}_2)$. The auto-covariance function is then expressed as

$$\mathrm{Cov}[X(\mathbf{t}_1), X(\mathbf{t}_2)] = \sigma_X(\mathbf{t}_1) \cdot \sigma_X(\mathbf{t}_2) \cdot \rho(\mathbf{t}_1, \mathbf{t}_2), \tag{1}$$

where $\sigma_X(\mathbf{t})$ is the standard deviation function of the random field. If $\mu_X(t)$ is constant in space, i.e. $\mu_X(t) = \mu_X \ \forall \ t \in \Omega$ and $\mathrm{Cov}[X(\mathbf{t}_1), X(\mathbf{t}_2)]$ is a function of the spatial distance of $X(\mathbf{t}_1)$ and $X(\mathbf{t}_2)$ only, $X(\mathbf{t})$ is said to be (weakly) homogeneous.

In this contribution we model spatial variability with Gaussian random fields, which means that

$f(x(\mathbf{t}))$ is a Gaussian density for any $\mathbf{t} \in \Omega$. That is, the random field is fully described by the spatial function for the mean $\mu_X(\mathbf{t})$ and the autocovariance function $\mathrm{Cov}[X(\mathbf{t}_1), X(\mathbf{t}_2)]$. However, the methodology could be extended to a class of non-Gaussian random fields that can be expressed by a nonlinear transformation of an underlying Gaussian random field - the so-called translation fields (Grigoriu, 2009).

2.2. Spatial average method

The exact representation of a random field at any point $\mathbf{t} \in \Omega$ typically requires an infinite number of random variables. With discretization methods, a random field is approximated in terms of a finite set of random variables. We choose the SA method to approximate a random field (Vanmarcke, 2010) because it lends itself to easy implementation in practice. It represents the random field as a set of local integrals that correspond to the average of the random field over small portions of the domain Ω. For the sake of lucidity, the following derivations restrict to the one-dimensional case but the concept applies to the multidimensional case as well.

The spatial average X_{T_i} of the random field $X(t)$ over the interval $[t_1, t_2]$ is defined as the following integral (Vanmarcke, 2010):

$$X_{T_i} = \frac{1}{T_i} \int_{t_1}^{t_2} X(t)\,\mathrm{d}t, \tag{2}$$

where T_i is the length of the integration domain, i.e. $T_i = t_2 - t_1$. Figure 1 shows a single realization of a homogeneous Gaussian random field and the spatial average in the interval T_i determined by application of Equation 2. It can be seen that the realization of $X(t)$ is approximated by a constant value in the interval $[t_1, t_2]$.

Accordingly, the mean of the spatial average can be determined:

$$\mu_{X_{T_i}} = \frac{1}{T_i} \int_{t_1}^{t_2} \mu_X(t)\,\mathrm{d}t \tag{3}$$

If $\mu_X(t)$ is constant in space (e.g. in a homogeneous random field), this integration has no effect on the mean. Then $\mu_{X_{T_i}} = \mu_X \;\forall\, t_1, t_2 \in \Omega$.

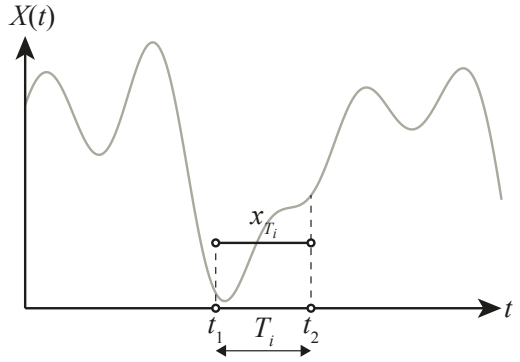

Figure 1: Approximation of the random field $X(t)$ via the local spatial average X_{T_i} in the interval $[t_1, t_2]$.

In contrast, the variance of the random field is always affected by the averaging process. For a homogeneous random field it can be calculated by multiplying the variance of the random field by a reduction factor found by an integration over the autocorrelation function, which is a function of the spatial distance only (Vanmarcke, 2010). This function is often termed variance reduction function. In the non-homogeneous case, however, the correlation of two points may depend not only on their spatial distance and the variance of the random field is not constant in space. Therefore, the variance of the averaging integral has to be calculated by the following double integral over the autocovariance function:

$$\mathrm{Var}[X_T] = \frac{1}{T^2} \int_{t_1}^{t_2} \int_{t_1}^{t_2} \mathrm{Cov}[X(a), X(b)]\,\mathrm{d}a\mathrm{d}b \tag{4}$$

In order to obtain a realization of the random field in terms of local spatial averaging elements, it is necessary to correctly account for the covariances and hence the correlation of these local averages. Consider two intervals, T_1 and T_2, for which the covariance is to be calculated as shown in Figure 2. The covariance of the two spatial averages $\mathrm{Cov}[X_{T_2}, X_{T_1}]$ can then be calculated from the local averaging variances as follows (Vanmarcke, 2010):

$$\begin{aligned} &\mathrm{Cov}[X_{T_2}, X_{T_1}] = \\ &= \frac{1}{2T_1T_2} \left(\Delta(L_1) - \Delta(L_2) + \Delta(L_3) - \Delta(L_4)\right), \end{aligned} \tag{5}$$

where

$$\Delta(L_i) = L_i^2 \cdot \mathrm{Var}[X_{L_i}],\ i = 1, \ldots 4 \tag{6}$$

L_1, L_2, L_3 and L_4 are the distances between the endpoints of T_1 and T_2 in Figure 2. The local averaging variances of Equation (6) are calculated by means of Equation (4). It is noted that Equation (5) is also valid for overlapping, containing or coinciding intervals T_1 and T_2.

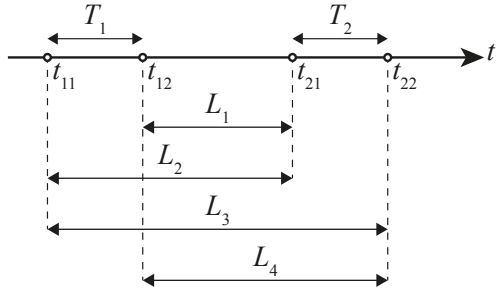

Figure 2: Intervals for the calculation of the covariance of the local averaging intervals T_1 and T_2.

Based on Equations (3) to (6), all necessary parameters of the local averaging random variables used to represent the Gaussian random field can be calculated. An approximation of the realization of the random field of Figure 1 with the spatial average method with five elements is illustrated in Figure 3. The random field is approximated with a set of five random variables, that follow a multivariate Gaussian distribution with mean vector and covariance matrix calculated following Equations (3) to (6).

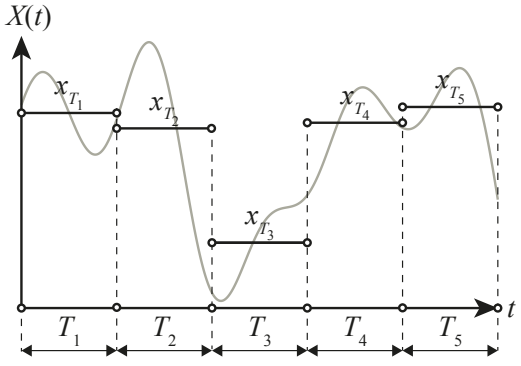

Figure 3: Approximation of the random field $X(t)$ with SA with $n_{SA} = 5$.

The number of averaging elements determines the accuracy of the approximation. To quantify the approximation error, i.e. the difference between the true random field and the approximation, different error measures can be employed, e.g. the error variance, the bias or the mean square error (e.g. Sudret and Der Kiureghian (2000); Betz et al. (2014)). As these quantities are usually defined point-wise, global error measures can be defined by integration over Ω or by taking the supremum norm of the point-wise error (Sudret and Der Kiureghian, 2000). The required number of elements for the SA method can be determined by setting a target global discretization error.

3. Bayesian updating of random fields

The incorporation of available information in the analysis can be done via Bayesian analysis, which allows to update the probabilistic model based on the information at hand.

3.1. Bayesian analysis

Bayesian analysis with available information M is based on the application of Bayes' rule (Gelman et al., 2013):

$$f''_{\mathbf{X}}(\mathbf{x}) \propto L(\mathbf{x}) \cdot f'_{\mathbf{X}}(\mathbf{x}), \tag{7}$$

where $f'_{\mathbf{X}}(\mathbf{x})$ is the prior joint PDF of some random vector $\mathbf{X}$, $f''_{\mathbf{X}}(\mathbf{x})$ is the posterior joint PDF of the random vector $\mathbf{X}$ after the update and $L(\mathbf{x})$ denotes the likelihood function, which is proportional to the probability of the information M conditional on $\mathbf{x}$. M can be available as measurements of a continuous quantity Q. The measurements are often subject to a measurement error. In the case of an additive measurement error, the measurement outcome q_m is expressed as

$$q_m = q + \varepsilon_m. \tag{8}$$

The mathematical expression for the likelihood function $L(\mathbf{x})$ is then obtained in terms of the PDF of the measurement error:

$$L(\mathbf{x}) = f_{\varepsilon_m}(q_m - q) \tag{9}$$

Under the relatively mild assumption of independence between multiple measurement errors, the joint likelihood function of a set of n_m measurements is defined as the product of the marginal likelihoods:

$$L(\mathbf{x}) = \prod_{i=1}^{n_m} f_{\varepsilon_m}(q_{m,i} - q_i) \tag{10}$$

3.2. Conjugate prior for Gaussian random fields

Equation (7) often has to be evaluated numerically, as a closed-form solution for the posterior distribution is typically not available. However, it is possible to choose the prior PDF and the likelihood function such that the posterior PDF can be obtained in functional form. In these cases, the prior PDF is termed a conjugate prior for the likelihood function (Gelman et al., 2013).

This concept can be employed for updating the parameters of a Gaussian random field. To this end, we model the additive measurement error with a normal distribution with zero mean and standard deviation σ_{ε_m}. In such case, the conjugate prior for the likelihood function is the Gaussian distribution. The posterior random field will also be Gaussian with mean and auto-covariance functions as follows (Papaioannou and Straub, 2017):

$$\mu_X''(t) = \mu_X' + \Sigma_X(t) \cdot \Sigma_{m,\varepsilon}^{-1} \cdot \delta_m \tag{11}$$

$$\Sigma_X''(t_1,t_2) = \Sigma_X'(t_1,t_2) - \Sigma_X(t_1) \cdot \Sigma_{m,\varepsilon}^{-1} \cdot \Sigma_X(t_2)^{\mathsf{T}} \tag{12}$$

μ_X' and $\Sigma_X'(t_1,t_2)$ are the prior mean and prior covariance function of the Gaussian random field. $\Sigma_X(t)$ is a $1 \times n_m$ row vector function with element i equal to $\Sigma_X'(t,t_i)$, where $t_i,\ i = 1,\ldots,n_m$ denotes the measurement locations. $\Sigma_{m,\varepsilon}$ is defined as $\Sigma_m + \Sigma_\varepsilon$, where Σ_m is a $n_m \times n_m$ matrix with element (i,j) equal to $\Sigma_X'(t_i,t_j)$ and Σ_ε is a $n_m \times n_m$ diagonal matrix with the error variance $\sigma_{\varepsilon_m}^2$ on the diagonal. The off-diagonal terms of Σ_ε are 0 due to the assumption of statistical independence between the measurements. δ_m is a $n_m \times 1$ column vector containing the difference of the measured data from the prior mean with element i equal to $(x_m(t_i) - \mu_X')$.

The resulting non-homogeneous random field can then be approximated with any discretization method, e.g. with the spatial average method presented in Section 2.2.

4. NUMERICAL EXAMPLE

This section applies the SA method to represent the spatial variability of a material property and assesses the accuracy of the obtained approximation prior and posterior to the inclusion of spatially variable measurements. We apply the method to the structural example of a one-dimensional pinned-fixed beam subject to a uniformly distributed vertical load illustrated in Figure 4. The length of the beam and the applied load are chosen deterministically as $L = 10\,\mathrm{m}$ and $q = 1.2\,\mathrm{kN\,m^{-1}}$.

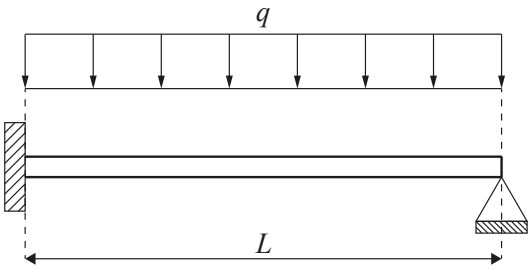

Figure 4: Pinned-fixed beam under uniformly distributed vertical load.

The prior flexibility of the beam is defined by a homogeneous Gaussian random field with mean value $\mu_F' = 1 \times 10^{-4}\,\mathrm{N^{-1}mm^{-2}}$ and standard deviation $\sigma_F' = 2 \times 10^{-5}\,\mathrm{N^{-1}mm^{-2}}$. The spatial correlation is modeled with the exponential autocorrelation function with a correlation length of $l_C = 5\,\mathrm{m}$:

$$\rho(t_1,t_2) = \exp\left(-\frac{|t_2 - t_1|}{l_C}\right) \tag{13}$$

The system response is evaluated with the linear finite element method based on the Euler-Bernoulli beam theory with a finite element size of $l_{FE} = 0.01\,\mathrm{m}$.

The accuracy of SA is measured by means of the statistics of the output of the numerical model. To this end we compare the bending moment and the displacement of the beam computed with SA to a reference solution obtained with the midpoint method with a very fine discretization. We employ the supremum of the point-wise root mean square error normalized by the reference mean value as error measure:

$$\varepsilon_Q = \frac{\sup_{t\in\Omega} \sqrt{\mathrm{E}\left[\left(Q(t) - Q_T(t)\right)^2\right]}}{\mathrm{E}[Q(t)]}, \tag{14}$$

where Q refers to the quantity of interest, i.e. the bending moment or the displacement. We calculate the error of Equation (14) based on Monte Carlo Simulation, i.e. for each realization of the random

field, $Q_T(t)$ is obtained by averaging the realization of the reference solution $Q(t)$. It is noted that the error in Equation (14) lumps both the error in the representation of the mean as well as the error in the variance of the random field.

4.1. Parameter update with measurements

In this study, it is assumed that two direct measurements of the beam flexibility are available:

$$\begin{bmatrix} \mathbf{t}_m & \mathbf{F}_m \end{bmatrix} = \begin{bmatrix} 3\,\mathrm{m} & 0.8\times10^{-4}\,\mathrm{N^{-1}mm^{-2}} \\ 7\,\mathrm{m} & 1.2\times10^{-4}\,\mathrm{N^{-1}mm^{-2}} \end{bmatrix} \quad (15)$$

This data can be used to update the random field parameters by means of Equations (11) and (12). The additive measurement error is modeled with the normal distribution with zero mean and standard deviation $\sigma_{\varepsilon_m} = 0.5\times10^{-5}\,\mathrm{N^{-1}mm^{-2}}$. The resulting posterior parameters compared to the prior parameters are illustrated in Figures 5 and 6. Obviously, the posterior random field is not homogeneous anymore, the influence of the measurements is clearly visible.

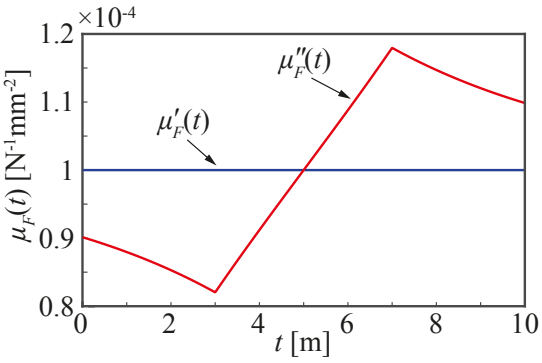

Figure 5: Prior (blue) and posterior (red) point-wise mean of the flexibility of the beam.

4.2. Representation with spatial averages

The number of averaging elements determines the accuracy and the computational effort for the representation with SA as one random variable is required for the realization of each element. Figures 7 and 8 show the parameters of an approximation with five elements. While the prior mean is represented exactly with SA, the approximation of the prior standard deviation underestimates the point-wise standard deviation of the random field. For

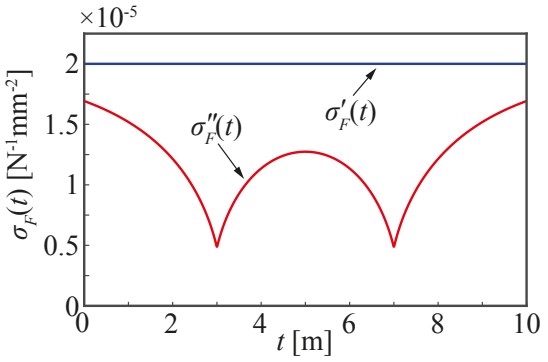

Figure 6: Prior (blue) and posterior (red) point-wise standard deviation of the flexibility of the beam.

the posterior parameters, the approximation of both parameters differs from the true point-wise value.

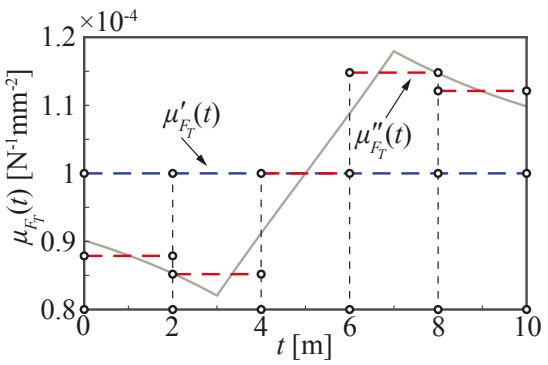

Figure 7: Prior (blue, dashed) and posterior (red, dashed) point-wise mean with $n_{SA} = 5$.

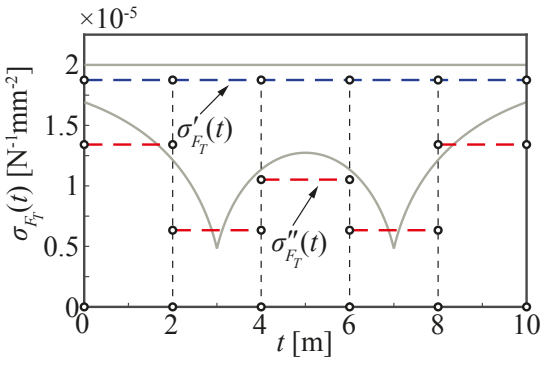

Figure 8: Prior (blue, dashed) and posterior (red, dashed) point-wise standard deviation with $n_{SA} = 5$.

4.3. Evaluation of the finite element model

To determine the accuracy of SA in terms of the response of the finite element model, a parameter

study on n_{SA} is conducted. The error in the system response according to Equation (14) is evaluated with Monte Carlo Simulation based on 1×10^5 repeated simulations. Figures 9 and 10 show the error in the bending moment and the displacement respectively as function of n_{SA}.

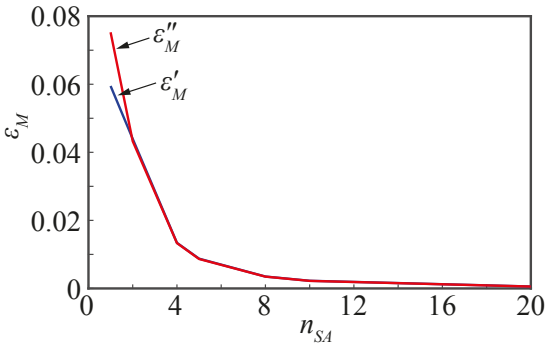

Figure 9: Prior (blue) and posterior (red) error in the bending moment.

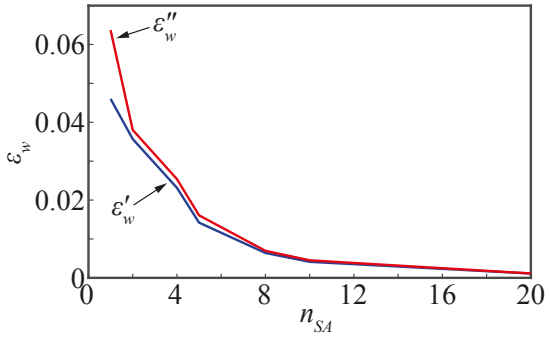

Figure 10: Prior (blue) and posterior (red) error in the displacement.

The error in the representation of the bending moment is very similar for the prior and the posterior random field, albeit slightly higher for the posterior random field if only one averaging element is used. With $n_{SA} \geq 5$ both prior and posterior random field approximate the bending moment with an error of less than 1 %. It is noted that the location of the maximum moment error is at the fixed end of the beam independent of the random field parameters and n_{SA}.

The error in the representation of the displacement of the beam is also higher for the posterior random field than it is for the prior random field if n_{SA} is small and it decreases rapidly for increasing n_{SA}. However, it requires more elements to achieve an error of less than 1 %, namely $n_{SA} \geq 8$ in this example. Contrary to the error in the moment, the location of the maximum displacement error depends on the random field parameters and n_{SA}.

4.4. *Reliability analysis*

The application of a discretization method to approximate random fields in the reliability analysis of a structural system requires good approximation in the tails of the distribution for the system response. To illustrate this behavior, a reliability analysis is conducted for the simply supported beam with the following limit state function:

$$g\left(X(t)\right) = M_{\text{crit}} - M\left(X(t)\right), \tag{16}$$

where M_{crit} is chosen 15 % higher than the bending moment at the fixed end of a beam with a constant flexibility, i.e. $M_{\text{crit}} = 17.25\,\text{kN m}$.

The reliability analysis is performed with Monte Carlo Simulation with 1×10^5 independent simulation runs. The results are illustrated in Figure 11. It appears that a small n_{SA} gives non-conservative estimates of the system behavior as the failure probability is underestimated significantly for the prior and posterior parameters. With increasing n_{SA} the estimate for P_F converges towards the reference solution. It is noted that the prior estimate of P_F is better than the posterior estimate for the same number of averaging elements. This performance gap decreases with increasing n_{SA}. For $n_{SA} \geq 10$ the estimates are sufficiently exact for the prior and posterior random field parameters.

5. Concluding remarks

The reliability assessment of massive existing structures requires a probabilistic model that can account for spatially variable parameters. In general, such models are not easy to couple with commercial finite element software. This study investigates the suitability of the spatial average method to reduce the complexity of the spatially variable model. Bayesian analysis with a conjugate prior approach is used to include available data in the analysis.

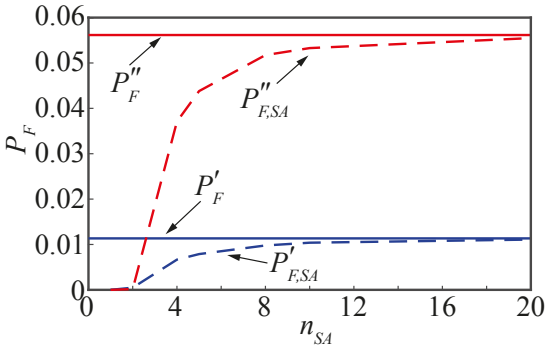

Figure 11: Convergence of the prior (blue, dashed) and posterior (red, dashed) estimate of P_f.

The numerical example of a pinned-fixed beam under uniformly distributed load shows that by using spatial averages, the random field can be expressed by a relatively small number of random variables with sufficient accuracy. The error in the point-wise bending moment is in the order of 1 % if five or more averaging elements are used. For the point-wise displacement of the beam a minimum of eight averaging elements is required to achieve the same level of accuracy. Remarkably, the required number of elements is not significantly influenced by the Bayesian analysis although the posterior random field, unlike the prior, is not homogeneous.

To investigate the accuracy of the spatial average method in the distribution tails of the system response, a reliability analysis was conducted. The failure criterion was defined by the exceedance of a critical bending moment. The failure probability was estimated for variable number of averaging elements and for the prior and posterior random field. The results indicate that for an accurate estimate of the failure probability, the required number of averaging elements is larger than the one required for estimating the mean and variance of the system response. For this example, the estimate of the posterior failure probability is slightly less accurate than that of the prior failure probability with the same number of averaging elements.

In conclusion, the spatial average method has the potential to be applied in engineering practice to describe material properties in the reliability analysis of existing structures. The reduction to a set of random variables, where each of them represents a specific part of the structure, facilitates the combination of such a probabilistic model with commercial finite element software. The accuracy must be studied for additional examples before general recommendations can be made.

ACKNOWLEDGEMENTS

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