Bayesian multi-parameter estimation using the mechanical equivalent of logical inference

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ABSTRACT: In this work we illustrate how the mathematics of rational thinking is formally equivalent to that of structural mechanics. Concepts from the wold of logic, such as accuracy, uncertainty, Maximum a Posteriori (MAP) and rationality correspond, in the world of mechanics, to stiffness, flexibility, equilibrium and conservativeness. For instance, a linear Gaussian N-parameter estimation problem can be solved through a N-dof linear elastic system, as the analogy goes along these lines: the parameters' covariance matrix is the system's flexibility matrix; the Fisher's information is the stiffness matrix; the negative log-distribution of the parameters is the elastic potential energy of the system; the Maximum a Posteriori (MAP) is the state of static equilibrium. In principle, based on this analogy, we could reproduce any logical inference problem with a finite element model, and make a judgment by finding its equilibrium state. We will show application of this analogy to a number of civil engineering inference problems, including Bayesian estimation, Bayesian networks and Kalman filter.

1. INTRODUCTION

Structural engineers usually have a solid background in mechanics, yet not always a good relationship with probability theory. In most cases, this is not that critical because code-based design is practically probability-free, with serious probabilistic analysis typically being confined to the most recondite annexes of the codes (EN 1990:2002). It is different for those engineers who grapple with structural health monitoring (SHM), an activity where the objective is to estimate the state of a structure from an uncertain batch of observations provided by different kind of sensors, such as strain gauge (Zonta, et al., 2003), or fiber optic sensor (Inaudi & Glisic, 2006). A consistent framework for making inferences from uncertain information is Bayesian probability theory (Sohn & Law, 2000). Yet structural engineers are often unenthusiastic about Bayesian formal logic, finding its application complicated and burdensome, and they prefer to make inference by using heuristics. In this contribution, we wish to help structural engineers reconcile with probabilistic logic (Jaynes, 2003) by suggesting a quantitative method for logical inference based on a formal analogy between mechanics and Bayesian probability. To start, we will limit the analogy to the case of linear Gaussian single-parameter estimation, which corresponds in the mechanical counterpart to mere linear elastic single-degree-of-freedom analysis: a cakewalk for structural engineers. In section 3, we apply this formal analogy to a classical inference problem: the estimation of the deformation of a cable belonging to a cable-stayed bridge, characterized by two independent parameters. We will carry out the simple problem of linear regression by solving the equivalent mechanical system of springs.

2. FORMULATION OF THE ANALOGY FOR A SINGLE PARAMETER

In this section, we refer to the problem of logical inference of a single parameter based on uncertain information (Cappello, et al., 2015). The goal is to estimate a parameter θ based on a set of uncertain information y_i. Further assumptions are that all the uncertain quantities have Gaussian distribution, and that the relationship between information and parameter is linear. When the problem is linear and Gaussian, in principle we can solve any logical inference problem using the following two fundamental rules.

First inference rule or inverse-variance weighting rule (Ku, 1966). Given a set of n observations y_i of variance σ_i^2, the inverse of the variance σ_θ^2 of the parameter is the sum of the inverse-variances of the observations, and the expected value of the parameter μ_θ is the inverse-variance weighted sum of the observations:

$$\frac{1}{\sigma_\theta^2} = \sum_{i=1}^{n} \frac{1}{\sigma_i^2}, \quad \mu_\theta = \frac{\sum_{i=1}^{n} \frac{y_i}{\sigma_i^2}}{\sum_{i=1}^{n} \frac{1}{\sigma_i^2}}. \tag{1a,b}$$

Second inference rule or linear propagation of uncertainties (Kirkup & Frenkel, 2006). The indirect measurement $y = x_1 + \ldots + x_m$, being the sum of m different arguments x_j of variance σ_j^2, the variance of the observations is the sum of the variance of the arguments and the mean value of the indirect observation is the sum of the arguments:

$$\sigma_y^2 = \sum_{j=1}^{m} \sigma_j^2, \quad \mu_y = \sum_{j=1}^{m} x_j. \tag{2a,b}$$

Before proceeding it is also convenient, primarily to lighten notation, to introduce the quantity

$$w = \sigma^{-2} = \frac{1}{\sigma^2}. \tag{3}$$

The quantity w is compatible with the official definition of *accuracy* (ISO5725-6:1994) and the word itself intuitively connects to the practical meaning of w: the higher the *accuracy* w of an observation is, the more *accurate* our knowledge about the parameter becomes. Therefore, in the rest of the paper we will refer to the inverse-variance w simply as *accuracy*. Based on that, we can reword and reformulate the two basic inference rules.

First inference rule. Given a set of n observations y_i with accuracy w_i, the accuracy w_θ of the parameter estimation is the sum of the accuracy of the observations, and the mean value of the parameter μ_θ is the sum of the observations weighted with their accuracy:

$$w_\theta = \sum_{i=1}^{n} w_i, \quad \mu_\theta = \frac{\sum_{i=1}^{n} y_i w_i}{w_\theta}. \tag{4a,b}$$

Second inference rule. The indirect measurement $y = x_1 + \ldots + x_m$ being the sum of m different arguments x_j of accuracy w_j, the inverse-accuracy of the observation is the sum of the inverse-accuracy of the arguments and the mean value of the indirect observation is the sum of the arguments:

$$\frac{1}{w_y} = \sum_{j=1}^{m} \frac{1}{w_j}, \quad \mu_y = \sum_{j=1}^{m} x_j. \tag{5a,b}$$

At this point, it's not difficult for a structural engineer to spot in (4a) the same form of the expression that provides the stiffness of a set of springs in parallel; and similarly, (5a) reminds of the stiffness expression of a set of springs in series. This allows to set an analogy between the world of logic and the world of mechanics. Particularly, the analogy statements (Cappello, et al., 2015) are summarized in Table 1, while Figure 1 shows the mechanical representation of simple linear Gaussian inference problems.

Table 1: Analogy between inference and mechanical models.

Symbol	Logical meaning	Mechanical meaning
w, σ^{-2}	accuracy, inverse-variance	stiffness
σ^2	variance	Flexibility
y	observation	pre-stretch
μ	expected value	equilibrium displacement

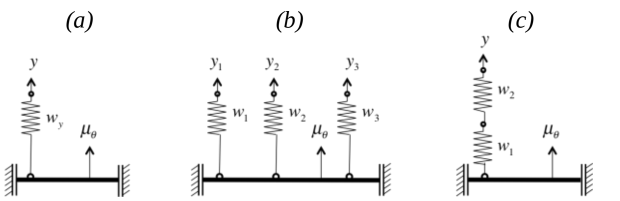

Figure 1: Mechanical analogy of simple linear Gaussian inference problems: parameter estimation based on one observation (a), three uncorrelated observations (b), one observation affected by two uncorrelated sources of uncertainty (c).

3. EXTENSION OF THE ANALOGY TO N PARAMETERS

Now, we analyse a generic inference problem with N unknown parameters to estimate, represented by the vector $\boldsymbol{\theta}=(\theta_1,...,\theta_N)^{\mathrm{T}}$. We imagine that each parameter is characterized by a prior mean value μ_{θ_i} and a prior standard deviation s_{q_i}; the latter is linked by the equation $w_{\theta_i}=\sigma_{\theta_i}^{-2}$ to the i$^{\mathrm{th}}$ accuracy, which in our mechanical analogy represents the stiffness of the spring associated to each single parameter. The multivariate Gaussian distribution (Bishop, 2006), linked to the N-dimensional vector **θ**, takes the form:

$$\mathrm{N}(\boldsymbol{\mu},\Sigma;\boldsymbol{\theta})=\frac{1}{(2\pi)^{\frac{N}{2}}}\frac{1}{|\Sigma|^{\frac{1}{2}}}\mathrm{e}^{\left\{-\frac{1}{2}(\boldsymbol{\theta}-\boldsymbol{\mu})^{T}\Sigma^{-1}(\boldsymbol{\theta}-\boldsymbol{\mu})\right\}}, \qquad (6)$$

where $\boldsymbol{\mu}$ is the N-dimensional mean vector, containing the N values μ_{θ_i} associated to each parameter, Σ is the NxN covariance matrix, and $|\Sigma|$ denotes the determinant of Σ. We can notice that the exponent is characterized by a quadratic form that corresponds to the potential energy $\mathrm{E_p}(\boldsymbol{\theta})$ of a mechanical system with N degrees of freedom, related to the inference problem in question. It takes the following form:

$$\mathrm{E_p}(\boldsymbol{\theta})=-\mathrm{In}(\mathrm{N}(\boldsymbol{\mu},\Sigma;\boldsymbol{\theta}))=\frac{1}{2}(\boldsymbol{\mu}-\boldsymbol{\theta})^{\mathrm{T}}\Sigma^{-1}(\boldsymbol{\mu}-\boldsymbol{\theta}). \quad (7)$$

Here, we name the inverse of the covariance matrix $\Lambda=\Sigma^{-1}$; this is also known as *accuracy matrix* (Bishop, 2006). Its diagonal terms represent the posterior stiffness $w_{\theta_i|\mathbf{y}}$ of each single parameter q_i. Now, to obtain the N diagonal elements to Λ we must get the second derivative of $\mathrm{E_p}(\boldsymbol{\theta})$ with respect to each of the parameters θ_{i}; the elements out of diagonal are instead obtained by calculating the mixed derivatives of each parameter with respect to all other parameters. To obtain the covariance matrix we simply make the inverse of Λ. The diagonal elements of S represent the posterior variance $s^2_{q_i|\mathbf{y}}$ of each single parameter θ_{i}. The posterior mean values $\mu_{\theta_i|\mathbf{y}}$ of each parameter q_i correspond to those values that minimize the potential energy of our mechanical system. Therefore, to discover them, we have to resolve an algebraic system with N variables in which there are the partial derivatives of $\mathrm{E_p}(\boldsymbol{\theta})$, each with respect to each parameter θ_{i}, set equal to zero.

4. A CASE STUDY: ELONGATION OF A CABLE BELONGING TO ADIGE BRIDGE

Structural monitoring has been recognized as a powerful information tool, especially about bridges management (Pozzi, et al., 2010), and requests a deep knowledge of Bayesian rules. For this reason, we apply our method to the Adige Bridge (Cappello, et al., 2015), a two-span cable-stayed bridge located ten kilometres north of the city of Trento, Italy (Figure 2). The composite deck is made from 4 "I"-section steel girders and a 25 cm cast-on-site concrete slab. The deck is also supported by 12 stay cables, 6 on each side, which have a diameter of 116 mm and 128 mm. Their operational design load varies from 5,000 kN to 8,000 kN. The cables are anchored to the bridge tower, consisting of four pylons and located in the middle of the bridge. When the construction was completed, the Italian Autonomous Province of Trento, which owns and manages the bridge, decided to install a monitoring system to continuously record force and elongation of the stay cables. Elongations are recorded by 1 m long gauge sensors, placed on each of the 12 cables. These fiber-optical sensors (FOS) (Glisic, et al., 2007) are based on fiber Bragg gratings (FBG) and they also record local temperature for thermal compensation.

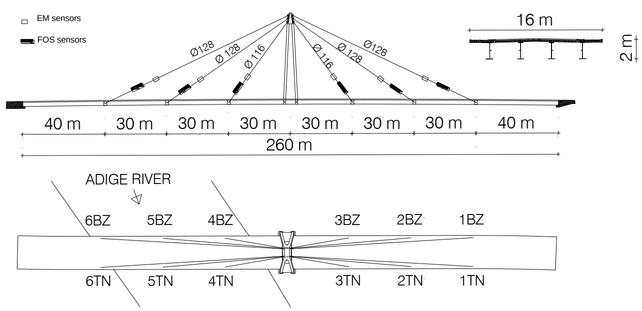

Figure 2: Longitudinal section of the bridge and sensor layout (upper); plan view of the bridge (lower).

4.1. Two parameters to estimate

As an example, we use data acquired from October 12, 2011, to November 25, 2012, for cable 1TN, purified of the effect of temperature. We consider only one sample a day between 4 AM and 6 AM, as we assume a constant temperature in this period. We have discarded those days in which no samples were found. Figure 3 shows the data acquired, expressed in terms of difference of deformation and time:

$$\Delta y = y_i - y_1, \quad \Delta t = t_i - t_1. \tag{8a,b}$$

During the analysis, 411 deformation measurements were recorded with an uncertainty for each measurement equal to $w_y = 0.0016\ \mu\varepsilon^{-2}$, i.e. $\sigma_y = 25\,\mu\varepsilon$; this is clearly a classical problem of linear regression. We have to estimate the two parameters that best characterize the straight line fitting our time-dependent data set. The function is:

$$y = y_0 + \varphi \cdot t, \tag{9}$$

where y_0 is the intercept and φ the slope of the straight line fitting our dataset. As we said before, the goal is to estimate the vector of the parameters $\boldsymbol{\theta} = (y_0, \varphi)^T$ that characterizes the parametric model resulting in the observations $\mathbf{y} = (y_1, y_2, ..., y_N)^T$, linearly dependent on the time t, as shown in Figure 4. We can represent the problem as a bar with two degrees of freedom: vertical translation and rotation. According to the parametric model defined in (9), we consider the slope of the bar linked to the parameter φ, its length to the time t and its distance from the ground floor to the parameter y_0. Based on our experience, we assign to the two parameters φ and t two prior Gaussian distributions that give us the initial information about the state of the bar. We connect the left-hand end of the rigid bar to a vertical linear elastic spring with flexibility equal to the standard deviation of the prior distribution associated to the parameter y_0 and pre-stretch equal to its mean value. We connect the same end to a torsion spring with flexibility and imposed rotation equal respectively to the standard deviation and the mean value of the prior distribution associated to the parameter φ (Figure 4). Finally, we introduce the measurements as a system of linear springs, each with flexibility and pre-stretch equal respectively to the standard deviation and value associated to a single measurement. Each spring is placed at a distance from the torsion spring equal to the corresponding interval of time t_i. The elastic potential of the mechanical system of Figure 4 becomes:

$$\begin{aligned} \mathrm{E}_p(y_0,\varphi) &= \frac{1}{2} w_y (y_0 - \mu_{y_0})^2 + \frac{1}{2} w_\varphi (\varphi - \mu_\varphi)^2 + \\ &+ \frac{1}{2} w_y \sum_{i=1}^{N} [(y_0 + \varphi t_i) - y_i]^2, \end{aligned} \tag{10}$$

where $\Delta y_i = y_0 + \varphi \cdot t_i - y_i$ represents the elongation suffered by the N springs linked to the observations, due to a generic translation y_0 and a generic rotation φ imposed on the system. The accuracy matrix is simply the Hessian matrix of (10):

$$\mathbf{\Lambda} = \begin{bmatrix} \dfrac{\delta^2 \mathrm{E_p}(y_0,\varphi)}{\delta y_0{}^2} & \dfrac{\delta^2 \mathrm{E_p}(y_0,\varphi)}{\delta y_0 \delta\varphi} \\ \dfrac{\delta^2 \mathrm{E_p}(y_0,\varphi)}{\delta\varphi\delta y_0} & \dfrac{\delta^2 \mathrm{E_p}(y_0,\varphi)}{\delta\varphi^2} \end{bmatrix}. \qquad (11)$$

The inverse of the matrix (11) represents the covariance matrix Σ: the first term of its diagonal is the posterior variance associated to the parameter y_0 while the second term on the same diagonal is the posterior variance associated to the parameter φ. To identify instead the values $\mu_{y_0|\mathbf{y}}$ and $\mu_{\varphi|\mathbf{y}}$ that represent the posterior mean values associated respectively to the parameters y_o and φ, we must solve the system formed by the first derivative of (10) with respect to the parameter y_0 and the parameter φ, set equal to zero.

$$\begin{cases} \dfrac{\partial \mathrm{E_p}(\boldsymbol{\theta})}{\partial y_0} = w_{y_0}(y_0 - \mu_{y_0}) + w_y \sum_{i=1}^{N} [(y_0 + \varphi t_i) - y_i] = 0 \\ \dfrac{\partial \mathrm{E_p}(\boldsymbol{\theta})}{\partial \varphi} = w_\varphi(\varphi - \mu_\varphi) + w_y \sum_{i=1}^{N} t_i [(y_0 + \varphi t_i) - y_i] = 0 \end{cases} . \qquad (12)$$

The solutions of the system (12) give us the values of $\mu_{y_0|\mathbf{y}}$ and $\mu_{\varphi|\mathbf{y}}$, that represent the posterior mean values associated respectively to the parameters y_0 and φ and that minimize the potential $\mathrm{E_p}(y_0,\varphi)$ of our mechanical system. Now we can substitute the numerical values into the equations formulated above, and we obtain the final outcomes reported in Table 2, compared with the prior values of the parameters. Figure 3 reports the two straight lines interpolating our dataset. We obtain the same results as applying the flexibility method to the same mechanical system (Cappello, et al., 2015), although, with the potential energy, we considerably reduce the computational algebra cost.

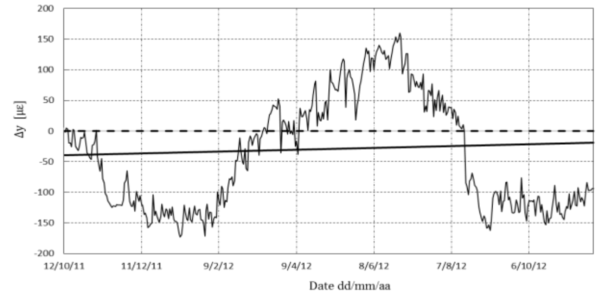

Figure 3: Relative strain of cable 1TN and interpolating lines.

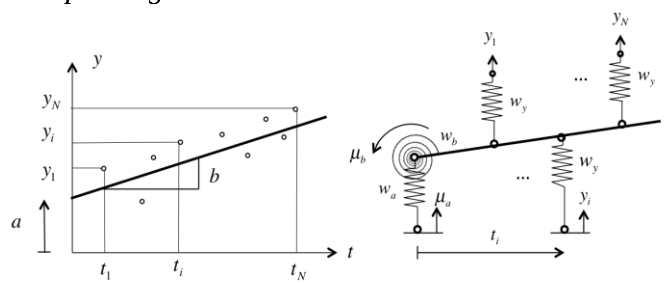

Figure 4: Linear regression problem in the world of Mechanics.

Table 2: Prior and posterior values of the parameters to estimate.

Prior distributions			
Parameter y_0		Parameter φ	
w_{y_0} $[\mu\varepsilon^{-2}]$	0.0025	w_φ $[\mu\varepsilon^{-2} day^2]$	1
σ_{y_0} $[\mu\varepsilon]$	20.00	σ_φ $[\mu\varepsilon\ day^{-1}]$	1.0000
μ_{y_0} $[\mu\varepsilon]$	0.00	μ_φ $[\mu\varepsilon\ day^{-1}]$	0.0000
Posterior distributions			
Parameter y_0		Parameter φ	
w_{y_0} $[\mu\varepsilon^{-2}]$	0.6601	w_φ $[\mu\varepsilon^{-2} day^2]$	36893
σ_{y_0} $[\mu\varepsilon]$	2.44	σ_φ $[\mu\varepsilon\ day^{-1}]$	0.0103
μ_{y_0} $[\mu\varepsilon]$	-49.07	μ_φ $[\mu\varepsilon\ day^{-1}]$	0.0473

4.2. Three parameters to estimate

We now extend the case of Adige Bridge, presented in the previous Section, by introducing the effect of temperature $\Delta\hat{T}$. Thus, we must estimate an additional parameter α and the model that fits our time dependent dataset becomes the following:

$$\Delta\hat{y} = y_0 + \alpha \cdot \Delta\hat{T} + \varphi \cdot \Delta\hat{t}\ . \qquad (13)$$

In Figure 5, we can note the N translation springs linked to the different measurements with stiffness $w_{LH} = \sigma_{LH}{}^{-2} = 0.0016\,\mu\varepsilon^{-2}$ and the springs linked to the prior distribution: a translation spring associated to the parameter y_0, a rotational spring associated to α and a rotational spring associated to φ, whose numerical values are the same as the case in the previous Section. To determine the posterior standard deviation of the three parameters to estimate (y_0, a, j) we have to express the potential energy $E_p(y_0, a, j)$ of the mechanical system represented in Figure 5, as a function of the three unknown parameter. We can now obtain the accuracy matrix Λ simply by calculating the Hessian Matrix associated to $E_p(y_0, a, j)$, and the covariance matrix from the inverse of Λ. To discover the values $m_{y_0|\mathbf{y}}$, $m_{a|\mathbf{y}}$ and $m_{j|\mathbf{y}}$, which represent the posterior mean values associated respectively to the parameters y_0, α and j, we must solve the system formed by the first derivative of the potential energy with respect to the three parameters, set equal to zero. Figure 6 shows the graphical representation of the two surfaces fitting our data set. Finally, Table reports the numerical values obtained from the posterior distribution of the parameters.

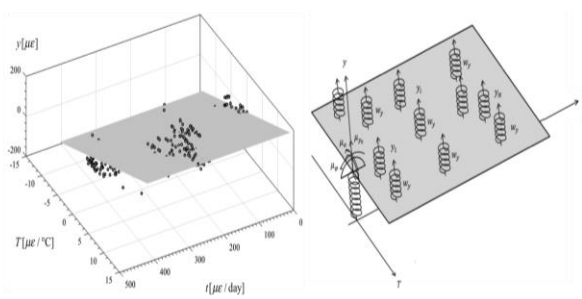

Figure 5: Three parameters estimation problem in the world of Mechanics.

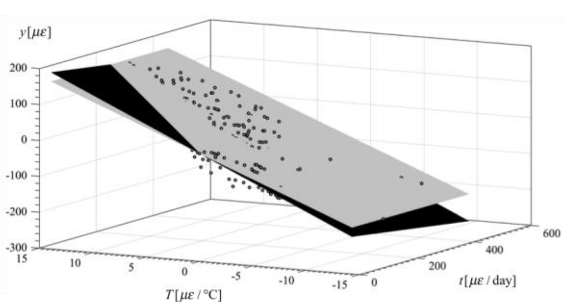

Figure 6: Two fitting surfaces related to prior parameters (grey) and posterior parameters (black).

Table 3: Posterior values of the three parameters to estimate.

Posterior distributions			
Parameter y_0		Parameter φ	
w_{y_0} $[\mu\varepsilon^{-2}]$	0.6601	w_φ $[\mu\varepsilon^{-2} day^2]$	36893
σ_{y_0} $[\mu\varepsilon]$	2.54	σ_φ $[\mu\varepsilon\ day^{-1}]$	0.0106
μ_{y_0} $[\mu\varepsilon]$	0.48	μ_φ $[\mu\varepsilon\ day^{-1}]$	-0.1209
Parameter α			
w_α $[\mu\varepsilon^{-2}\ °C^2]$	27.88		
σ_α $[\mu\varepsilon\ °C^{-1}]$	0.20		
μ_α $[\mu\varepsilon\ °C^{-1}]$	13.80		

5. NON GAUSSIAN SINGLE PARAMETER ESTIMATION

How does change the theory of the mechanical equivalent if we decide to involve non-Gaussian variable? As is logical, we will obtain non-linear springs, whose constitutive laws vary depending on the probability distributions that characterize them. To extend the mechanical analogy to distribution other than the Gaussian results very simple thanks to the three basic rules explained below. We denote with $f(q;a,b)$ a generic probability distribution, where θ is the unknown parameter to estimate, a and b the parameters that characterize the probability distribution in exam. The potential energy, the elastic force and the stiffness linked to the spring representing the generic distribution f are the following (14a,b,c):

$$E_p(\theta) = -\mathrm{In}(f),\ F_e(\theta) = \frac{\partial E_p(\theta)}{\partial \theta},\quad k(\theta) = \frac{\partial^2 E_p(\theta)}{\partial^2 \theta}. \tag{14a,b,c}$$

In the following sections we will report some examples, regarding the main probability distributions used in the world of logic, and we will try to define for each the mechanical features of the spring that represent them. We remember that, in case of Gaussian distribution, we solve any inference problem through mechanical systems composed by elastic linear springs, with a constant stiffness and a quadratic potential.

5.1. Lognormal distribution

The lognormal distribution (Forbes, 2011) is applicable to random variables that are constrained by zero but have a few very large values. The resulting distribution is asymmetrical and positively skewed. In particular, in engineering field, the lognormal distribution is often used to describe the fatigue behavior of many mechanical components and the mechanical resistance of structural materials, as the steel. The application of a logarithmic transformation to the data can allow the data to be approximated by the symmetrical normal distribution, although the absence of negative values may limit the validity of this procedure. In other words, it is the probability distribution of a random variable θ whose logarithm $\mathrm{In}(\theta)$ follows a normal distribution, and it takes the following form:

$$\mathrm{l}(\lambda,\varepsilon;\theta) = \frac{1}{\theta\varepsilon\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2\varepsilon^2}\left(\mathrm{In}(\theta)-\lambda\right)^2} \quad \text{with } 0<\theta<+\infty, \tag{15}$$

where λ is the mean of $\mathrm{In}(q)$ and ε the standard deviation of $\mathrm{In}(q)$, which are both dimensionless. But how can we model a spring representing $\mathrm{l}(l,e;q)$? The answer is simple: we must use the three aforementioned expressions (14a,b,c), to spot the trend of the potential, of the elastic force and of the stiffness of the spring linked to the lognormal distribution (16-18).

$$\begin{aligned}\mathrm{E}_p(\theta) &= -\mathrm{In}(\mathrm{l}(\lambda,\varepsilon;\theta)) = \\ &= \frac{1}{2\varepsilon^2}\left(\mathrm{In}(\theta)-\lambda\right)^2 + \mathrm{In}(\theta) + a\,,\end{aligned} \tag{16}$$

where a is an additive constant that we can neglect.

$$\mathrm{F}_e(\theta) = \frac{\partial \mathrm{E}_p(\theta)}{\partial\theta} = \frac{1}{\theta\varepsilon^2}\left(\mathrm{In}(\theta) - \lambda + \varepsilon^2\right), \tag{17}$$

$$k(\theta) = \frac{\partial^2 \mathrm{E}_p(\theta)}{\partial^2\theta} = \frac{1}{\theta^2\varepsilon^2}\left(1 - \mathrm{In}(\theta) + \lambda - \varepsilon^2\right). \tag{18}$$

Figure 7 shows that the constitutive low of the spring is absolutely non-linear. We note that the potential energy, Figure 7a, has a minimum in correspondence to the mode of the probability distribution $e^{\lambda-\varepsilon^2}=1$ and not in correspondence to the mean. This time the potential energy is not symmetric with respect to its minimum value. When the displacements become remarkable, the elastic force becomes constant, tending to a value little greater than zero; consequently, the stiffness, i.e. the first derivative of $\mathrm{F}_e(q)$, tends to zero.

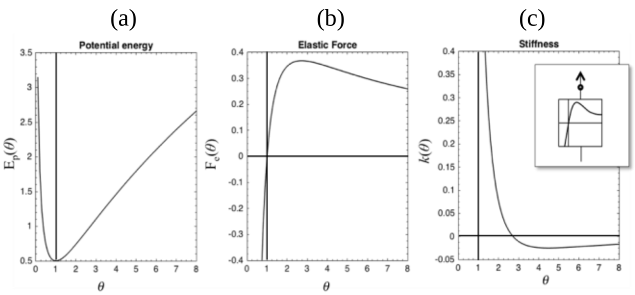

Figure 7: Three mechanical properties of a lognormal distribution $\mathrm{l}(1,1;\theta)$

5.2. Cauchy distribution

The Cauchy distribution (Forbes, 2011) is of mathematical interest due to the absence of defined moments. Its probability density function takes the following form:

$$\mathrm{c}(a,b;\theta) = \left\{\pi b\left[1+\left(\frac{\theta-a}{b}\right)^2\right]\right\}^{-1} \quad \text{with } -\infty<\theta<+\infty, \tag{19}$$

where a and b are the parameters that characterize the distribution. The Cauchy distribution is unimodal and symmetric, with much heavier tails than the normal. The probability density function is symmetric about a, with upper and lower quartiles, $a \pm b$. The potential, the elastic force and the stiffness function linked to the Cauchy distribution $\mathrm{c}(a,b;\theta)$ are:

$$\mathrm{E_p}(\theta) = \mathrm{In}\left(1+\left(\frac{\theta-a}{b}\right)^2\right) + c, \tag{20}$$

$$\mathrm{F_e}(\theta) = \frac{\partial \mathrm{E_p}(\theta)}{\partial\theta} = \frac{2(\theta-a)}{b^2+(\theta-a)^2}, \tag{21}$$

$$k(\theta) = \frac{\partial^2 \mathrm{E_p}(\theta)}{\partial\theta^2} = \frac{2[(\theta-a)^2+b^2]-4(\theta-a)^2}{\left(b^2+(\theta-a)^2\right)^2}. \tag{22}$$

Figure 8 shows these mechanical properties linked to a Cauchy distribution c(2,10;q) with mode value a=2. Also, this time the potential energy has a minimum in correspondence to its mode a=2, but unlike the non-linear previous examples, here the potential energy is symmetric respect to the mode. The elastic force, in correspondence to the mode, has an inflection point and changes its curvature. We observe that if the non-linear probability distribution is symmetric respect its mode, the equivalent potential energy results symmetric respect to the same value, where it yields also its minimum value.

(a) (b) (c)

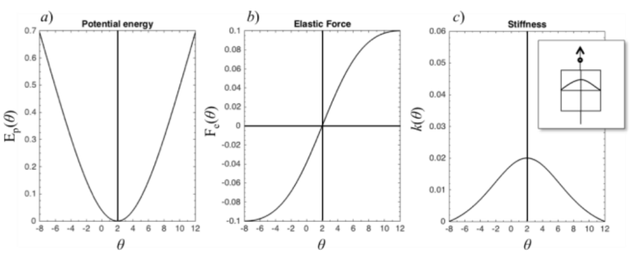

Figure 8: Three mechanical properties of a Cauchy distribution c(2,10;θ)

6. CONCLUSIONS

We have stated an analogy between the world of logic and the world of mechanics, allowing us to solve, using the methods of classical structural engineering, any complex inference parameter estimation problem, in which the values of the parameters have to be estimated based on multiple Gaussian-distributed uncertain observations. By simply expressing the potential energy of the mechanical system associated to our inference scheme, we are able, with a few trivial algebraic steps, to determine the posterior mean values and standard deviations of the parameters to estimate. With the aid of real-life structural health monitoring cases, we have showed how our approach allows structural engineers to solve simply general problems of linear regression. Although the examples shown in this paper are incidentally all structural engineering cases, the scope of application of the method is evidently the most general, and we seek to demonstrate in the future its applicability to inference problem arising from various disciplinary fields, including cognitive science, economics and law.

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