Reliability-based Bayesian Updating using Machine Learning

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ABSTRACT: Bayesian updating is a powerful tool for model calibration and uncertainty quantification when new observations are available. By reformulating Bayesian updating into a structural reliability problem and introducing an auxiliary random variable, the state-of-the-art *BUS* algorithm has showcased large potential to achieve higher accuracy and efficiency compared with the conventional Markov-Chain-Monte-Carlo approach. However, *BUS* faces a number of limitations. The transformed reliability problem often investigates a very rare event problem especially when the number of measurements increases. Moreover, conventional reliability analysis techniques are not efficient and in some cases not capable of accurately estimating the probability of rare events. To overcome the aforementioned limitations, we propose integrating *BUS* algorithm with adaptive Kriging-based reliability analysis method. This approach improves the accuracy of the Bayesian updating and requires considerably smaller number of evaluations of the time-consuming likelihood function, compared to *BUS*.

1. INTRODUCTION

The advancement of monitoring and measuring techniques in structures and infrastructure systems facilitates the uncertainty reduction in decision-making. Up-to-date information on responses and features including, for example, system capacities, structural deformations, system dynamic features and geometric deteriorations assist in comprehensively perceiving the probabilistic information of all system variables with randomness. The main purpose of Bayesian updating is to assess the posterior probabilistic information from the often empirically defined prior statistical assumptions using observations.

In the past, the posterior distribution has commonly been estimated through the implementation of Markov chain Monte Carlo (MCMC) simulation (Beck James L. and Au Siu-Kui 2002). In the MCMC-based Bayesian updating approach, points are generated by the proposal function, whose mean value changes corresponding to the last accepted point, while the samples generated by the proposal function are compared with a standard uniformly distributed random value to determine if those points are accepted or rejected. It has been shown that the probability density of the accumulated accepted points converges to the posterior probability density. However, failing to converge to a stationary state corresponding to Markov chain is the major limitation of the MCMC-based Bayesian updating method (Giovanis et al. 2017; Straub Daniel and Papaioannou Iason 2015; Betz et al. 2018). To address the aforementioned limitation, the transitional Markov chain Monte Carlo simulation (TMCMC) has been proposed by Ching et al. (Ching Jianye and Chen Yi-Chu 2007), which attempts to sequentially sample a series of distributions that gradually approaches to posterior distribution. Although, the TMCMC-based Bayesian updating method improves the performance by avoiding the burn-in phenomenon in the conventional approach, the gained efficiency is not significant when the dimension of the parameter space increases (Giovanis et al. 2017; Betz Wolfgang, Papaioannou Iason, and Straub Daniel 2016). Considering those issues, an innovative methodology, called *BUS* (Bayesian Updating with Structural reliability methods), is proposed by Straub et al. (Straub Daniel and Papaioannou Iason 2015). The primary idea behind BUS is to reformulate the Bayesian updating problem into structural reliability problem. By introducing an auxiliary standard uniform random variable, P, the Bayesian updating problem with simple rejection sampling strategy targets selecting points that satisfy the limit state equation: $P \leq cL(\boldsymbol{X})$, where c is a constant ensuring a good acceptance ratio. Avoiding the process for ensuring the stationarity of Markov Chain in MCMC, BUS applies the subset simulation technique (Au and Beck 2001) to focus on the points with corresponding accepted domain regardless of the dimension of random variables. Accordingly, this approach adaptively prescribes and constrains the failure domain until the estimated intermediate failure threshold is smaller than zero and draws the samples in the last subset with MCMC.

However, the process of estimating the posterior distribution through BUS is quite computationally expensive especially when the likelihood functions (e.g. Finite element model) becomes very complex (Betz et al. 2018). It is the reason that the method for solving the structural reliability problem in BUS is through Subset simulation, which needs a large number of evaluations to the performance function. Second, due to the fact that the failure probability estimated from the limit state function in BUS is typically very rare as the number of measurements increases, other non-simulation-based techniques such as First and Second Order Reliability Methods (FORM & SORM) become inefficient or even invalid. As pointed out in (Giovanis et al. 2017), this computationally expensive problem can be tackled through the application of surrogate model-based structural reliability analysis methods. These methods can use surrogate models such as Polynomial Chaos Expansion (Blatman and Sudret 2010) or Kriging (Echard, Gayton, and Lemaire 2011; Fauriat and Gayton 2014). Among all these methods, adaptive Kriging-based reliability analysis methods have been proven to be highly accurate and efficient in solving structural reliability problems (Gaspar, Teixeira, and Soares 2014; Wang and Shafieezadeh 2019, 2018), and therefore, have gained a lot of attention in recent years (Wang and Shafieezadeh 2019).

In this article, an algorithm named $BUAK$ (Bayesian Updating using Adaptive Kriging), is proposed to combine the BUS algorithm with advanced Kriging-based reliability analysis methods. Several examples are investigated to showcase the computational efficiency and accuracy of $BUAK$. Specifically, number of evaluations to the likelihood function has been significantly reduced, while the posterior probabilistic distributions are estimated very accurately. Eventually, with the well-trained Kriging surrogate model, samples with posterior distribution can be generated unlimitedly, which does not rely on the computationally expensive likelihood function.

Bayesian updating with structural reliability methods and BUS algorithm are briefly introduced in Section 2. Afterwards, the elements of Kriging-based reliability analysis methods are presented in Section 3. The proposed method, $BUAK$, is subsequently presented in Section 4. An example is implemented in section 5 to showcase the performance of $BUAK$. Conclusions are drawn in Section 6.

2. BAYESIAN UPDATING

Bayesian updating facilitates inferring the status of a system's variables by assuming a reasonable prior probability distribution for those variables, denoted as $f(\boldsymbol{x})$, and then estimating the posterior probability distribution, denoted as $f'(\boldsymbol{x})$, according to existing observations. $f'(\boldsymbol{x})$ can be estimated by the Bayes' theorem, which can be represented as:

$$f'(\boldsymbol{x}) = \frac{L(\boldsymbol{x})f(\boldsymbol{x})}{\int_{\Omega} L(\boldsymbol{x})f(\boldsymbol{x})d\boldsymbol{x}} \tag{1}$$

where Ω is the probabilistic domain of random variable $\boldsymbol{x}$ and $L(\boldsymbol{x})$ is the so-called likelihood function, which is proportional to the conditional probability of observations, and can be read as:

$$L(\boldsymbol{x}) \propto \Pr(Z|\boldsymbol{X}=\boldsymbol{x}) \tag{2}$$

In estimating $f'(\boldsymbol{x})$ through MCMC, the denominator $\int_{\Omega} L(\boldsymbol{x})f(\boldsymbol{x})d\boldsymbol{x}$ in Eq. (1) can be ignored since it is only a normalizing constant ensuring that $f'(\boldsymbol{x})$ integrates to one (Giovanis et al. 2017). Typically, the likelihood function $L(\boldsymbol{x})$ is composed of three parts: observations Z, responses from the model $s(\boldsymbol{x})$ and error ε that represents the deviation of $s(\boldsymbol{x})$ from Z. Because of the measuring error and modeling errors, observations Z can not really reflect $s(\boldsymbol{x})$. Therefore, the relationship below always holds:

$$\varepsilon = Z - s(\boldsymbol{x}) \tag{3}$$

Generally $L(\boldsymbol{x})$ can be estimated through the probability density function (pdf) of the error ε. This relation can be read as:

$$L(\boldsymbol{x}) = \rho_{\varepsilon}(\varepsilon) = \rho_{\varepsilon}\big(Z - s(\boldsymbol{x})\big) \tag{4}$$

where $\rho_{\varepsilon}(\cdot)$ denotes the pdf of ε. Although the type of pdf of $L(\boldsymbol{x})$ is commonly assumed to be a multivariate Gaussian distribution with zero mean, it can be any other unbiased distribution. Moreover, when m mutually independent observations are available, the likelihood function in Eq. (4) can be reinterpreted as:

$$L(\boldsymbol{x}) = \prod_{i=1}^{m} L_i(\boldsymbol{x}) = \prod_{i=1}^{m} \rho_{\varepsilon_i}\big(Z_i - s_i(\boldsymbol{x})\big) \tag{5}$$

In this article, the likelihood function is denoted as $L(\boldsymbol{x})$ for both independent and dependent observations.

2.1. Simple Rejection Sampling (SRS)

The idea of transforming the Bayesian updating problem into a structural reliability problem according to the simple rejection algorithm was initially proposed by Straub and Papaioannou (Straub Daniel and Papaioannou Iason 2015). It is known that the goal of Bayesian Updating is to estimate the posterior distribution $f'(\boldsymbol{x})$, which is proportional to the product of the likelihood function $L(\boldsymbol{x})$ and prior distribution $f(\boldsymbol{x})$:

$$f'(\boldsymbol{x}) \propto L(\boldsymbol{x})f(\boldsymbol{x}) \tag{6}$$

Achieving stability in the Markov Chain via the conventional MCMC approach is not computationally efficient. Thus, a simple rejection sampling algorithm can be applied here. First, the accepted domain Ω_{acc} can be defined corresponding to the augmented outcome space $[\boldsymbol{x}, p]$ with an auxiliary random variable P, which can be expressed as:

$$\Omega_{acc} = [p \le cL(\boldsymbol{x})] = [h(\boldsymbol{x}, p) \le 0] \tag{7}$$

where $h(\boldsymbol{x}, p) = p - cL(\boldsymbol{x})$ and c is a constant satisfying $cL(\boldsymbol{x}) \le 1$ for all the outcomes for $\boldsymbol{X}$. Moreover, it is suggested that c can be defined as:

$$c = \frac{1}{max\big(L(\boldsymbol{x})\big)} \tag{8}$$

Based on this scenario, a simple rejection sampling algorithm is available according to (Smith and Gelfand 1992), which is shown in Algorithm 1.

2.2. Bayesian Updating with Structural Reliability Methods (BUS)

Due to the inherent disadvantages of simple rejection sampling-based Bayesian updating, the MCMC method was proposed. However, to ensure a stable Markov chain, the MCMC-based Bayesian updating needs to investigate a large number of evaluations to the likelihood function. Thus, the MCMC-based Bayesian updating needs to investigate even a larger number of evaluations to the likelihood function. On the other hand, though the acceptance rate of simple rejection sampling-based Bayesian updating approach is low, it is very straightforward to implement and can guarantee an exact and uncorrelated posterior distributed samples.

Algorithm 1. Simple Rejection Sampling	
1.	$i = 1$
2.	Generate a sample $\boldsymbol{x}^i$ from $f(\boldsymbol{x})$
3.	Generate a sample p^i from the standard uniform distribution [0.1]
4.	If $[\boldsymbol{x}^i, p^i] \in \Omega_{acc}$
	(a). Accept $\boldsymbol{x}^i$
	(b). $i = i + 1$
5.	Stop if $i = N_s$, else go to step 2

By maintaining these advantages in simple rejection-based approach, Straub and Papaioannou (Straub Daniel and Papaioannou Iason 2015) proposed Bayesian Updating with Structural reliability methods (*BUS*) that strategically integrates the simple rejection sampling with structural reliability analysis techniques.

In *BUS* algorithm, the Bayesian updating problem is handled in a way of solving a structural reliability analysis problem. The equivalent limit state function in *BUS* approach can be read as:

$$h(\boldsymbol{x}, p) = p - cL(\boldsymbol{x}) \tag{9}$$

note that the task of Bayesian updating is different from that in reliability analysis. In the process of reliability analysis, the target is to estimate the probability of failure, while drawing the samples in the accepted (failure) domain is the main purpose of *BUS*. Concerning this point, many existing reliability analysis methods such as First & Second Order Reliability Methods (FORM & SORM), Importance Sampling (IS) and Subset Simulation (SS) can be adjusted to be applicable in association with *BUS*. For instance, the combination of subset simulation and *BUS* has shown great efficiency in drawing samples from posterior distributions. Details of *BUS* with subset simulation can be found in (Straub Daniel and Papaioannou Iason 2015).

3. ADAPTIVE KRIGING-BASED RELIABILITY ANALYSIS

Kriging-based reliability analysis methods are known for their capabilities in substituting the limit state function and reducing the number of evaluations to the performance function (Echard, Gayton, and Lemaire 2011). The main idea of using adaptive Kriging-based reliability analysis through *BUS* is to train a surrogate model $\hat{h}(\boldsymbol{x}, p)$ to substitute the computationally demanding limit state function $h(\boldsymbol{x}, p)$ in Eq. (9). Then, SRS can be conducted directly on the computationally efficient surrogate model. In this section, the elements of Kriging model and Kriging-based reliability analysis are briefly reviewed.

The Kriging surrogate model, also known as the Gaussian Process Regression, has been widely used in computer-based experiment design ("UQLab Kriging (Gaussian Process Modelling) Manual" n.d.). In this model, the estimated responses are mean values and variances following a normal distribution ("UQLab Kriging (Gaussian Process Modelling) Manual" n.d.). In this model, the responses $\hat{h}(\boldsymbol{X})$ ($\boldsymbol{X}$ represents $[\boldsymbol{X}, P]$ in this section) are defined as

$$\begin{aligned}\hat{h}(\boldsymbol{X}) &= F(\boldsymbol{\beta}, \boldsymbol{x}) + \psi(\boldsymbol{x}) \\ &= \boldsymbol{\beta}^T \boldsymbol{f}(\boldsymbol{x}) + \psi(\boldsymbol{x})\end{aligned} \tag{10}$$

where $\boldsymbol{X}$ is the vector of random variables, $F(\boldsymbol{\beta}, \boldsymbol{x})$ are the regression elements, and $\psi(\boldsymbol{x})$ is the Gaussian process. In $F(\boldsymbol{\beta}, \boldsymbol{x})$, $\boldsymbol{f}(\boldsymbol{x})$ is the Kriging basis and β is the corresponding coefficient. There are multiple formulations of $\boldsymbol{\beta}^T f(\boldsymbol{x})$ including ordinary (β_0), linear ($\beta_0 +$

$\sum_{i=1}^{N}\beta_i\boldsymbol{x}_i$), or quadratic ($\beta_0$+$\sum_{i=1}^{N}\beta_i\boldsymbol{x}_i$+$\sum_{i=1}^{N}\sum_{j=i}^{N}\beta_{ij}\boldsymbol{x}_i\boldsymbol{x}_j$), where N is the number of dimensions of $\boldsymbol{x}$. In this article, the ordinary Kriging model is used. The Gaussian process $\psi(\boldsymbol{x})$ has a zero mean and a covariance matrix that can be represented as:

$$COV\left(\psi(\boldsymbol{x}_i),\psi(\boldsymbol{x}_j)\right) = \sigma^2 R(\boldsymbol{x}_i,\boldsymbol{x}_j;\boldsymbol{\theta}) \quad (11)$$

where σ^2 is the process variance or the generalized mean square error (MSE) from the regression, $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ are two observations, and $R(\boldsymbol{x}_i,\boldsymbol{x}_j;\boldsymbol{\theta})$ is known as the kernel function representing the correlation between observations $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ parametrized by $\boldsymbol{\theta}$. The correlation functions implemented in Kriging can include, among others, linear, exponential, Gaussian, and Matérn functions. The Gaussian kernel function is used in this paper, which has the following form:

$$R(\boldsymbol{x}_i,\boldsymbol{x}_j;\boldsymbol{\theta}) = \prod_{k=1}^{N}\exp\left(-\theta^k\left(x_i^k - x_j^k\right)^2\right) \quad (12)$$

where x_i^k is the k_{th} dimension of $\boldsymbol{x}_i$ and $\boldsymbol{\theta}$ is estimated via the Maximum Likelihood Estimation (MLE) method ("UQLab Kriging (Gaussian Process Modelling) Manual" n.d.). To maintain consistency, θ^k is searched in (0,10) according to the optimization algorithm in the MATLAB toolbox DACE (Lophaven, Nielsen, and Søndergaard 2002b), (Lophaven, Nielsen, and Søndergaard 2002a). Here, MLE can be represented as:

$$\boldsymbol{\theta} = \underset{\boldsymbol{\theta}^*}{\operatorname{argmin}}\left(\left|\boldsymbol{R}(\boldsymbol{x}_i,\boldsymbol{x}_j;\boldsymbol{\theta})\right|^{\frac{1}{m}}\sigma^2\right) \quad (13)$$

where m is the number of training points. Accordingly, the regression coefficient $\boldsymbol{\beta}$, and the predicted mean and variance can be determined as follows ("UQLab Kriging (Gaussian Process Modelling) Manual" n.d.):

$$\boldsymbol{\beta} = (\boldsymbol{F}^T\boldsymbol{R}^{-1}\boldsymbol{F})^{-1}\boldsymbol{F}^T\boldsymbol{R}^{-1}\boldsymbol{Y}$$

$$\mu_{\hat{h}}(\boldsymbol{x}) = \boldsymbol{f}^T(\boldsymbol{x})\boldsymbol{\beta} + \boldsymbol{r}^T(\boldsymbol{x})\boldsymbol{R}^{-1}(\boldsymbol{y} - \boldsymbol{F}\boldsymbol{\beta})$$

$$\sigma_{\hat{h}}^2(\boldsymbol{x}) = \sigma^2\begin{pmatrix} 1 - \boldsymbol{r}^T(\boldsymbol{x})\boldsymbol{R}^{-1}\boldsymbol{r}(\boldsymbol{x}) \\ +\left(\boldsymbol{F}^T\boldsymbol{R}^{-1}\boldsymbol{r}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x})\right)^T \\ (\boldsymbol{F}^T\boldsymbol{R}^{-1}\boldsymbol{F})^{-1}(\boldsymbol{F}^T\boldsymbol{R}^{-1}\boldsymbol{r}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x})) \end{pmatrix} \quad (14)$$

where $\boldsymbol{F}$ is the matrix of the basis function $\boldsymbol{f}(\boldsymbol{x})$ evaluated at the training points, i.e., $F_{ij} = f_j(\boldsymbol{x}_i)$, $i = 1, 2, \dots, m$; $j = 1,2,\dots,p$, $\boldsymbol{r}(\boldsymbol{x})$ is the correlation between known training points $\boldsymbol{x}_i$ and an untried point $\boldsymbol{x}$: $r_i = R(\boldsymbol{x},\boldsymbol{x}_i,\boldsymbol{\theta})$, $i = 1,2 \dots m$, and $\boldsymbol{R}$ is the autocorrelation matrix for known training points: $R_{ij} = R(\boldsymbol{x}_i,\boldsymbol{x}_j,\boldsymbol{\theta})$, $i = 1,2,\dots,m; j = 1,2,\dots,m$. Therefore, the estimated Kriging mean $\mu_{\hat{h}}(\boldsymbol{x})$ and variance $\sigma_{\hat{h}}^2(\boldsymbol{x})$ can be presented as:

$$\hat{h}(\boldsymbol{u}) \sim N\left(\mu_{\hat{h}}(\boldsymbol{x}),\sigma_{\hat{h}}^2(\boldsymbol{x})\right) \quad (15)$$

It is obvious that the responses from the Kriging model $\hat{h}(\boldsymbol{x})$ are not deterministic but probabilistic in the form of a normal distribution with mean $\mu_{\hat{h}}(\boldsymbol{x})$ and variance $\sigma_{\hat{h}}^2(\boldsymbol{x})$.

Details of the implementation of Kriging-based reliability analysis can be found in (Echard, Gayton, and Lemaire 2011; Wang and Shafieezadeh 2019, 2018). Note that learning functions have a crucial role in adaptive Kriging-based reliability analysis methods. As the name implies, the 'learning' refers to the process of iterative selection of points for Kriging refinement based on the stochastic information for each design point. A popular learning function is U that is concerned with uncertainties in sign ($\pm$) estimation of $\hat{h}(\boldsymbol{x})$=0. This learning function is used in this paper. In this regard, U takes the probabilistic distribution of estimated responses into consideration, and quantifies the probability of making a wrong sign estimation in $\hat{h}(\boldsymbol{x})$. The formulation of U is:

$$U(\boldsymbol{x}) = \frac{|\mu_{\hat{h}}(\boldsymbol{x})|}{\sigma_{\hat{h}}(\boldsymbol{x})} \quad (16)$$

The general principle of adaptive Kriging-based reliability analysis method is to start with a small number of candidate design samples to estimate $\hat{P}_f$ and then adaptively refine the limit state.

4. NUMERICAL STUDIES

In this section, an examples is investigated to demonstrate the performance of the proposed method BUAK. This example has also been studied in (Betz et al. 2018). A one-dimensional random variable is used here which follows a standard normal prior distribution, and is denoted as $\varphi(x)$. The likelihood of this problem also follows a normal distribution with mean $\mu_l = 3$ and standard deviation $\sigma_l = 0.3$. Thus the maximum value of this likelihood is $L_{max} = \frac{1}{\sigma_l\sqrt{2\pi}} = 1.33$, which means $c = \frac{1}{max(L_{max})} = 0.752$. Then the limit state function according to Eq. (9) can be represented as:

$$h(x,p) = p - c\phi(x|\mu_l,\sigma_l) \quad (17)$$

where p is an auxiliary random variable following uniform distribution and $\phi(\boldsymbol{x}|\mu_l,\sigma_l)$ denotes the probability density function (PDF) of normal distribution parameterized by μ_l and σ_l.

To reduce the nonlinearity of Eq. (17), a logarithmic formulation of the limit state function can be represented as (Betz et al. 2018),

$$g(x,p) = ln(p) - ln(c) - ln\big(\phi(x|\mu_l,\sigma_l)\big) \quad (18)$$

The performance of this method is evaluated in terms of the number of calls to the likelihood function, N_{call}. Since the acceptance ratio, P_{acc}, of this example is 4.63×10^{-3}, the total number of calls to the performance function through *BUS* in association with subset simulation is more than 1500 according to (Straub Daniel and Papaioannou Iason 2015). However, the total number of calls to the likelihood function N_{call} is equal to 20 through the proposed Kriging-based Bayesian updating method in this example. Fig. 1 showcases the limit state of Eq. (17) or (18) estimated through BUAK with $N_{call} = 15$ and $N_{call} = 20$. According to Fig 2., the limit state $h(\boldsymbol{x},p) = 0$ is gradually refined as the number of training points x_{tr} increases. The accepted samples and inaccurately classified samples are illustrated in Fig. 2. There are totally 17 points out of 10^4(as denoted in the magenta cross dots in Fig. 2(b)), whose labels are estimated wrongly (i.e., points that should be accepted but are rejected, and points that should be rejected but are accepted).

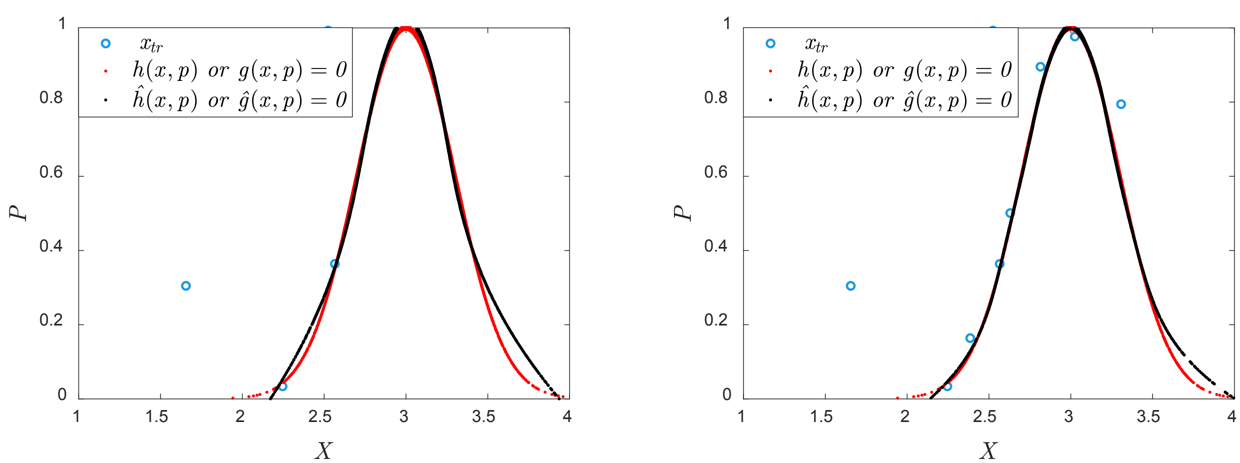
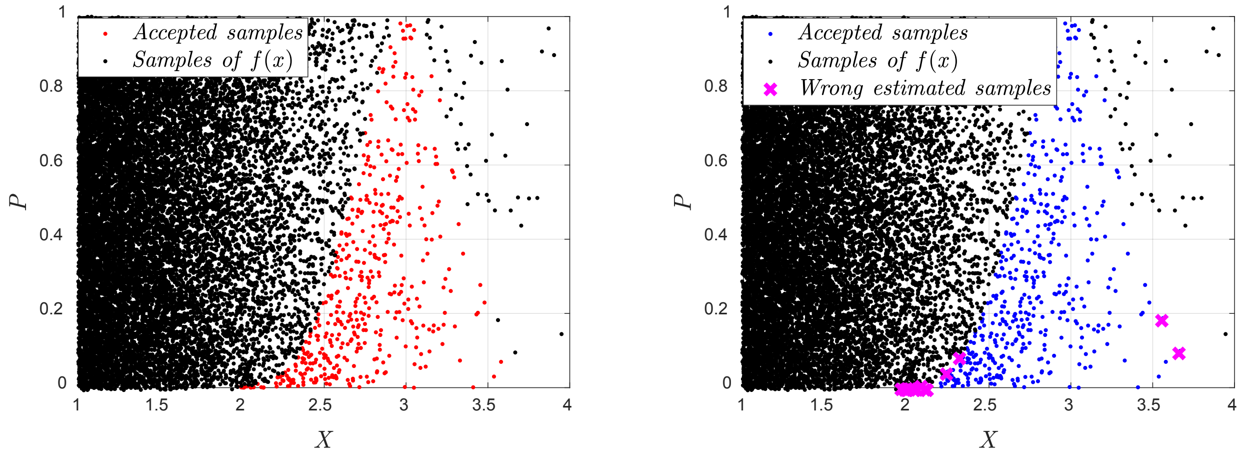


Fig. 1. The limit state of $h(\boldsymbol{x},p) = 0$ and $\hat{h}(\boldsymbol{x},p) = 0$ with (a) $N_{call} = 15$, (b) $N_{call} = 20$



(a) (b)

Fig. 2. Illustration of accepted samples through (a) the true limit state $h(\boldsymbol{x}, p)$ (b) the estimated limit state $\hat{h}(\boldsymbol{x}, p)$

5. CONCLUSIONS

A novel framework, termed *BUAK*, that combines Bayesian Updating with structural reliability methods (*BUS*) and Adaptive Kriging surrogate model is proposed in this article to improve the performance of Bayesian updating. The main idea behind *BUAK* is to integrate the *BUS* algorithm with the advanced adaptive Kriging-based reliability analysis methods, which have shown great capability in reducing the number of evaluations to the performance function and simultaneously training very accurate surrogate models. By training multiple parallel Kriging surrogates for these decomposed limit state functions, the *BUS* algorithm can be well implemented based on the advanced adaptive Kriging-based reliability analysis. A numerical example is investigated to analyze the performance of the proposed method. Results indicate that *BUAK* offers great computational efficiency in accurately estimating the posterior distribution of correction factors.

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