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A Multivariate Gamma Process for Dependent Degradation Modelling and Life Prediction

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ABSTRACT: Remaining service life of infrastructure assets is governed by functionality and structural integrity, both deteriorate with time. In this paper, we propose a multivariate gamma process model to model the stochastically dependent deterioration phenomena that collectively define the asset life. The temporal uncertainty is characterized by nonstationary gamma processes with independent increments while dependence among degradation processes is characterized with a correlation matrix in the copula space. Parameter estimation is done by the maximum likelihood method. For the lifetime prediction, a component experiencing multiple degradation phenomena is said to fail based on a number of scenarios. So the remaining lifetime distribution will be based on the current state of the component as well as failure thresholds of all phenomena. The proposed methodology is illustrated with a case study of a highway pavement experiencing multiple degradation such as rutting, cracking, and surface smoothness.

1. INTRODUCTION

Degradation modeling is a core component of infrastructure asset management. In some previous studies, the focus has been on modeling different degradation phenomena individually with univariate stochastic processes. However, degradation phenomena observed in a physical system are often correlated and modeled together (Rodriguez-Picon, 2017). The observation may be due to the degradation phenomena being in close proximity to one another and having some shared underlying causes. Hence, an assumption of independence may underestimate lifetime prediction of such structures or components.

The main contribution of this paper will be to demonstrate the use of the multivariate stochastic process model for competing degradation and lifetime prediction. The paper is arranged as follows. Section 2 deals with the literature review. The model and methodology are presented in section 3. Section 4 discusses a case study of multiple degradation in a flexible pavement. Section 5 concludes the paper.

2. LITERATURE REVIEW

Many previous research focused on modeling degradation phenomena as independent stochastic processes. However, in reality, many structures or components experience multiple degradation phenomena which are dependent on one another. There have been previous research on stochastic modeling of multiple degradation. In the early days of bivariate degradation modeling, Whitmore et al. (1998) proposed a two-dimensional Wiener process to model degradation. Their model comprises two processes - the component, which is directly observable, is the marker while the other component, which is unobservable, determines the failure time. Both components are correlated and have a bivariate Gaussian distribution. More recently, Shemehsavar (2014) proposed a monotonically increasing bivariate gamma model with latent component and marker. In a similar vein, the latent process cannot be observed and determines the failure time while the second (i.e. the marker) can be observed. Both processes have

Kibble's bivariate gamma distribution with the same positive shape parameter and a scale parameter of 1. Liu et al. (2014) proposed a model for multiple degradation processes with marginal inverse Gaussian process. In their model, copulas were used to characterize dependence among degradation processes.

Wang et al. (2015) proposed a bivariate nonstationary gamma degradation process. Their model assumed that a product state could be described by two dependent performance characteristics whose degradation mechanisms both follow nonstationary gamma processes. Also, a copula function was used to characterize the dependence structure. An earlier paper by Pan and Balakrishnan (2011) proposed a bivariate stationary gamma degradation model for reliability analysis of products with two dependent performance characteristics. Caballé et al. (2015) and Castro et al. (2015) modeled multiple degradation growths and sudden shocks in a system using gamma processes with initiation times following a nonhomogeneous Poisson process. Both competing degradation growths and sudden shocks were treated as dependent but the degradation processes were assumed to be independent of one another.

A major benefit derivable from degradation modeling of a structure is being able to estimate reliability and predict the lifetime of the structure. A structure, component or system is considered to have failed when the cumulative degradation in it reaches a predetermined failure threshold ζ. This means that failure does not have to be catastrophic. The failure is characterized by a lifetime distribution which basically is a probability density function defined over a range of time. Its cumulative distribution function (CDF) $F(t)$ is the probability that the component fails before or at time t. The CDF is defined as $F(t) = P(T \leq t) = P(X(t) \geq \zeta)$. For more information on lifetime distribution, see Van Noortwijk (2007) and Yu et al. (2008).

Gamma process has been used to model degradation, predict reliability, and compute lifetime and remaining lifetime distribution of components (Yuan, 2007). Also, Wei and Xu (2014) presented a method to estimate remaining useful life of components using a gamma process. In their paper, Monte Carlo simulation was used to obtain lifetime distribution. Nystad et al. (2012) proposed a nonstationary gamma process to model a degradation phenomenon with gamma-distributed failure threshold. The remaining useful life was estimated by taking the integral of a function, while taking into account degradation state of the component.

3. MULTIVARIATE GAMMA PROCESS

The purpose of this chapter is to present a model suitable for modeling competing degradation phenomenon.

3.1. Definition

We formally define the process below. An n-dimensional multivariate gamma process $X(t) = \{X_1(t), \dots, X_n(t)\}$ with $t \geq 0$ satisfies the following conditions:

1. $X_j(0) = 0$ almost surely for all $j = 1, \dots, n$.
2. For any time $t \geq 0$, $X_j(t)$ is a nonstationary gamma process with increments that follow a gamma distribution with shape $\alpha_j(t)$ and scale β_j, i.e., $\Delta X_j(t) \sim Ga\big(\alpha_j(t), \beta_j\big)$.
3. For any times $0 \leq t_1 < t_2$, the increments $X_j(t_2) - X_j(t_1)$ follow a multivariate gamma distribution that is defined as Eq. (1) below with α_j and β_j for $j = 1, \dots, n,$ and the correlation coefficient is defined as correlation between two stochastic processes.

$$g_n(\boldsymbol{x}) = \frac{1}{(2\pi)^{n/2}|\mathbf{R}|^{1/2}} \exp\left(-\frac{1}{2}\boldsymbol{z}^T\mathbf{R}^{-1}\boldsymbol{z}\right) \prod_{j=1}^{n} \frac{g(x_j;\alpha_j,\beta_j)}{\phi(z_j)} \quad (1)$$

where $\phi(z_j)$ denotes the probability density function (PDF) of a standard normal distribution; $\mathbf{R}$ is an n by n correlation matrix; $z_j = \Phi^{-1}(u_j)$; $u_j = G(x_j;\alpha_j,\beta_j)$, and $\Phi^{-1}(u)$ denotes the inverse of the standard normal cumulative distribution function (CDF) at probability u.

The gamma distribution mentioned in the second condition is a two-parameter continuous probability distribution whose PDF and CDF are

$$g(x;\alpha,\,\beta)=\frac{x^{\alpha-1}\,e^{-\frac{x}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)} \tag{2}$$

$$G(x;\alpha,\beta)=\frac{\Gamma(\alpha,x/\beta)}{\Gamma(\alpha)} \tag{3}$$

for $x\geq 0$, where $\alpha>0$ and $\beta>0$ are the shape and scale parameters, respectively, and $\Gamma(p,q)=\int_0^q x^{p-1}e^{-x}\mathrm{d}u$ is called the lower incomplete gamma function, and $\Gamma(p)=\Gamma(p,\infty)$ the complete gamma function. The cumulative distribution function (CDF) is expressed as a ratio of two gamma functions.

The shape parameter is assumed to follow a power law $\alpha_j(t)=a_j t^{c_j}$ for $j=1,\dots,n$ and $a_j,c_j>0$. This implies $\alpha_j(0)=0$. When $0<c<1$, the rate of increase of the shape parameter decreases with time. On the other hand, when $c>1$, the rate of increase of the shape parameter increases with time. In both scenarios, the stochastic process is nonstationary. However, the stochastic process is said to be stationary when the shape parameter is linear with time i.e. $c=1$.

In the context of degradation modeling, we consider $X_j(t)$ to be a degradation process that represents the cumulative amount of deterioration observed in a component.

3.2. Simulation

Sample paths of a multivariate gamma process can be simulated by the procedure described in this section. The procedure involves generating multivariate Gaussian variates and then transforming them to multivariate random variates with gamma-distributed marginals via copula. Suppose we are interested in simulating a multivariate gamma process whose dimension is four over a specified planning horizon. Random variates from the multivariate normal distribution $\boldsymbol{z}$ of dimension $n=4$, with zero mean and a positive definite correlation matrix $\boldsymbol{R}$, are generated. This is followed by a double transformation of the zero-mean multivariate Gaussian variates to multivariate gamma variates. Basically, the transformation involves calculating the standard normal CDF u_j at each value of $\boldsymbol{z}$ and setting $x_j=F_j^{-1}(u_j)$ where $j=1,\dots,n$ and F_j^{-1} is the inverse univariate gamma cumulative distribution function with shape and scale parameters $\alpha_j\Delta t$ and β_j respectively. For illustration, see Figure 1.

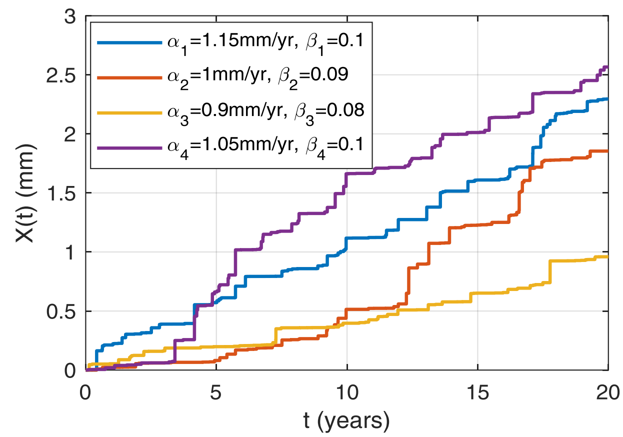

Figure 1: Simulated degradation paths for a multivariate gamma process ($c_j=1$ for $j=1,\dots,4$; $\{\rho_{12}=0.7,\rho_{13}=0.5,\rho_{14}=0.7,\rho_{23}=0.4,\rho_{24}=0.6,\rho_{34}=0.5\}$).

3.3. Parameter Estimation

Suppose there exists datasets from m inspection outages of a component experiencing n number of competing degradation phenomena. It is also assumed that all degradation phenomena have common inspection times $t_0,t_1,\dots,t_m$, where t_0 is the time the component was put into service. Considering the initial state of the component x_{0j}, where $j=1,\dots,n$, there will be m increments for each degradation phenomenon. An increment is defined as $\Delta x_{ij}=x_{ij}-x_{i-1,j}$; $1\leq i\leq m$ for a fixed j. In other words, the degradation data are $X_1(t)=[x_{01}\;x_{11}\dots x_{m1}],\dots,X_n(t)=[x_{0n}\;x_{1n}\dots x_{mn}]$ while the increments are $\Delta X_1(t)=[\Delta x_{11}\;\Delta x_{21}\dots\Delta x_{m1}],\dots,\Delta X_n(t)=[\Delta x_{1n}\;\Delta x_{2n}\dots\Delta x_{mn}]$. For any two consecutive inspection outages, the joint PDF of the multivariate gamma distribution (Eq.(4)) is expressed as

$$g_n(\boldsymbol{\Delta x})=\frac{1}{(2\pi)^{n/2}|\mathbf{R}|^{1/2}}\exp\left(-\frac{1}{2}\mathbf{z}^T\mathbf{R}^{-1}\mathbf{z}\right)\prod_{j=1}^{n}\frac{g(\Delta x_{ij};a,\beta_j,c_j)}{\phi(z_{ij})} \tag{4}$$

Consequently, the likelihood function for the joint distribution function is the product of independent multivariate gamma densities of the increments

$$L\left(\boldsymbol{a}_j, \boldsymbol{\beta}_j, \boldsymbol{c}_j \middle| \Delta x_{11}, \dots, \Delta x_{mn}\right) = \prod_{i=1}^{m} g_n\left(\boldsymbol{\Delta x}_{ij}, \boldsymbol{a}_j, \boldsymbol{\beta}_j, \boldsymbol{c}_j\right) \quad (5)$$

The maximum likelihood estimates of $\boldsymbol{a}$,$\boldsymbol{\beta}$ and $\boldsymbol{c}$ are obtained by numerically maximizing the likelihood function. This is equivalent to computing the first partial derivatives of the likelihood function with respect to each of the parameters of the multivariate gamma process.

It is always mathematically convenient to take the logarithm of the likelihood function during parameter estimation. The parameter estimation was done in MATLAB using *fmincon*. To ensure that the correlation matrix $\boldsymbol{R}$ remained positive definite at every iteration during the parameter estimation, the Cholesky decomposition of $\boldsymbol{R}$ was used in the likelihood function. After the solution converged, the correlation matrix was reassembled i.e. $\boldsymbol{R} = \boldsymbol{L}\boldsymbol{L}^T$, where $\boldsymbol{L}$ is a lower triangular matrix.

4. CASE STUDY

Flexible pavements experience multiple degradation over time as a result of normal wear and tear. Other contributing factors to pavement material breakdown are construction failure and prolonged exposure to atmospheric substances such as rain and sunlight. Examples of common degradation phenomena in pavement include cracking and rutting.

4.1. Multiple Degradation Modeling in Highway Pavement

In this case study, three measures of pavement degradation are considered. These are rutting, International Roughness Index (IRI) and Distress Management Index (DMI). Rutting is a permanent deformation along the wheel path on the road surface and increases over time. A newly constructed road, for instance, has a zero rut depth. The IRI is a dimensionless measure of road roughness. It increases over time until there is an intervention in terms of maintenance. Ideally, a newly-built road is expected to have a zero IRI, but this is hardly the case. DMI refers to the sum of all distresses and is a measure of overall service damage for the road section. Its value, however, decreases with time.

For the case study, the assumptions are:

1. The road section is subjected to multiple degradation processes $X_j(t)$, where $j = 1, 2, 3$ and these processes are assumed to be dependent. Each $\{X_j(t), t \geq 0\}$ is a non-stationary gamma process with shape and scale parameters $\alpha_j(t)$ and β_j respectively.
2. Contrary to the first condition in the definition of the multivariate gamma process, degradation phenomena do not necessarily start from zero, so $X_j(t) = x_{0j} \pm Ga(\alpha_j(t), \beta_j)$ where x_0 is the initial measure of the degradation.

Table 1 presents the degradation data for a section of a flexible pavement road. The table shows the measurements of DMI, rut depths and IRI covering a 7-year period with measurements taken on a yearly basis. To incorporate the DMI values in the increasing gamma process, the absolute values of the changes are used in the parameter estimation. Measurement error in the observed data is not accounted for in the model.

Table 1: Degradation data for a road section in Ontario.

Year	*DMI*	*IRI*	*Rut depth*
2005	*9.49*	*1.12*	*3.49*
2006	*9.03*	*1.21*	*4.56*
2007	*8.73*	*1.29*	*4.85*
2008	*8.54*	*1.35*	*5.44*
2009	*7.83*	*1.44*	*5.76*
2010	*7.51*	*1.54*	*5.99*
2011	*7.02*	*1.68*	*6.61*

The procedure described in section 2.3 was used to estimate the parameters of the multivariate gamma process model. The objective function was found to have several local minima. Therefore, the *fmincon* solver was run repeatedly in an attempt to find a global minimum. The estimated parameters from the solution that has the lowest objective function value are shown in Table 2. Table 3 shows the correlation coefficients between the stochastic processes.

Table 2: Estimated shape and scale parameters of the multivariate gamma process.

Parameter	*DMI*	*IRI*	*Rut depth*
$\hat{\alpha}$	*5.89*	*15.1*	*12.8*
$\hat{c}$	*1.01*	*1.10*	*0.65*
$\hat{\beta}$	*0.07*	*0.01*	*0.08*

Table 3: Estimated correlation coefficients of the multivariate gamma process.

Parameter	*DMI*	*IRI*	*Rut depth*
DMI	*1*	*0.59*	*0.08*
IRI	*sym.*	*1*	*0.32*
Rut depth			*1*

Table 2 reveals that the power term, $\hat{c}$ of the shape parameters is less than 1 for rut depth. This confirms the initial assumption of nonstationarity i.e. the mean rates of the degradation phenomena are not linear with time. However, the mean rate of increase of the DMI and IRI are close to 1. The correlation coefficients shown in Table 3 shows positive correlations among DMI, IRI and rut depth.

To study the effect of modeling the stochastic processes as dependent as against individual monovariate stochastic processes, the parameters of individual nonstationary gamma processes were estimated by numerically maximizing the likelihood function in Eq.(6).

$$L(a \ ,\beta \ ,c\,|\Delta x_{1j},\dots,\Delta x_{mj}) = \prod_{i=1}^{m} \frac{\Delta x_{ij}^{(a(t_i^c-t_{i-1}^c)-1)}\exp(-\Delta x_{ij}/\beta)}{\beta^{\left(a\left(t_i^c-t_{i-1}^c\right)\right)}\Gamma\left(a\left(t_i^c-t_{i-1}^c\right)\right)} \tag{6}$$

The parameters in Table 2 are compared with corresponding parameters of individual nonstationary gamma processes shown in Table 4. Both tables reveal that shape parameters for the multivariate gamma process model are greater than the shape parameters from corresponding individual gamma process models. Meanwhile, the tables suggest that the scale parameters and the power term in both multivariate and individual gamma process models are comparable.

Table 4: Estimated parameters of nonstationary gamma processes.

Parameter	*DMI*	*IRI*	*Rut depth*
$\hat{\alpha}$	*5.87*	*14.7*	*13.2*
$\hat{c}$	*1.02*	*1.12*	*0.64*
$\hat{\beta}$	*0.07*	*0.01*	*0.08*

4.2. Remaining Lifetime Prediction

Generally, failure is said to occur in engineering when degradation exceeds the threshold specified in the code(s). In pavement engineering, however, the definition of failure is dependent on what really matters to the planner. For example, a pavement experiencing multiple degradation phenomena may be said to fail when any individual degradation process X_j(t) reaches its critical threshold ζ_j . In other words, each degradation process determines the failure of the component. Mathematically, the probability of failure is defined in Eq. (7) as

$$F(t) = P(T \le t) = 1 - P(X_1(t) < \zeta_1, \dots, X_n(t) < \zeta_n) \tag{7}$$

The other extreme is when failure is defined as when degradation phenomena all reach their respective thresholds (Eq. (8)).

$$F(t) = P(T \le t) = P(X_1(t) \ge \zeta_1, \dots, X_n(t) \ge \zeta_n) \tag{8}$$

Alternatively, a pavement subjected to multiple degradation processes may be said to have failed when a process reaches its failure threshold, two specific processes both reach their thresholds, any two processes both reach their thresholds or any combination thereof.

As the state of each degradation process can be observed, the probability density function takes into account this information. Suppose the degradation processes are last observed at surviving time s , then at future time t the probability of a degradation increment of $\zeta_j -$

$X_j(s)$ is an updated PDF $f_{X_j(t)-X_j(s)}$. To estimate the remaining lifetime distribution, growth of each process over time has to be predicted based on the updated PDF. So, Eq. (9) shows future degradation process as

$$X_j(t) = X_j(s) + \Delta X_j(t-s) \tag{9}$$

where $\Delta X_j(t-s)$ is the addition of all future increments up to time t. Monte Carlo simulation is used to generate one million sample paths and failure probability evaluated by dividing the number of times $X_j(t)$ exceeds ζ_j by the total number of simulation runs. The simulation uses the parameters shown in Table 2 and Table 3 in the previous section together with failure thresholds in Table 5.

Table 5: Failure thresholds for degradation phenomena.

Phenomenon	ζ
DMI	*6*
IRI	*2.17*
Rut depth	*9.5*

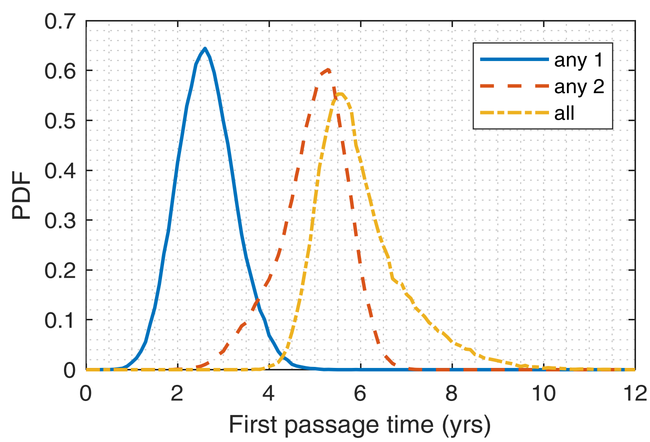

Figure 2: Remaining lifetime distribution for 3 scenarios.

Figure 2 shows the remaining lifetime distribution based on current state of the pavement section. Three scenarios are considered namely when failure is defined as any degradation phenomenon reaching its failure threshold, any 2 phenomena both reaching their thresholds and all three reaching their thresholds. Figure 2 reveals that as the definition of failure is relaxed, the distribution of remaining lifetime shifts to the right as expected. Furthermore, the mean of the distribution for each scenario estimated numerically is shown in Table 6. The means are estimated to be 2.63, 4.99 and 6.02 years, respectively.

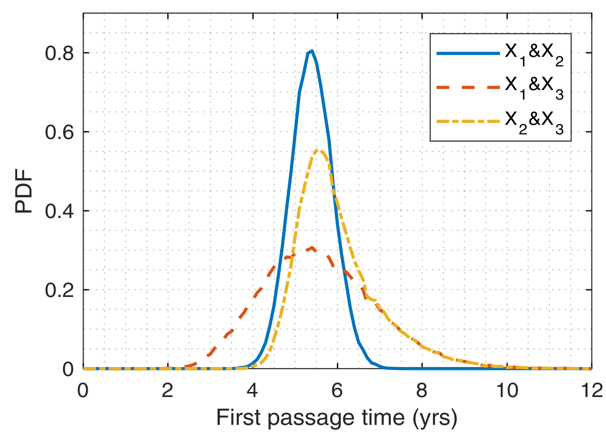

Figure 3: Remaining lifetime distribution for specific pairs of degradation phenomena.

Figure 3 shows another three scenarios as well. These are when failure is defined as when DMI (X_1) and IRI (X_2) both reach their failure thresholds, DMI (X_1) and rut depth (X_3) both reach their thresholds and IRI (X_2) and rut depth (X_3) both reach their thresholds. Numerical estimation of the expectations of these distributions are presented in Table 6.

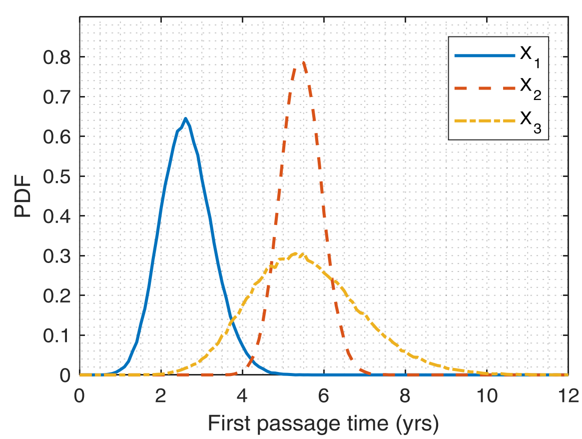

Figure 4: Remaining lifetime distribution when only one phenomenon matters.

In Figure 4, it is assumed that failure occurs when a specific stochastic process exceeds its failure threshold. The resulting three scenarios are presented in the figure while corresponding means of the distributions are shown in Table 6.

Table 6: Remaining lifetime means based on different failure criteria.

Scenario	*Lifetime mean (years)*
any 1	*2.63*
any 2	*4.99*
All	*6.02*
$X_1 \& X_2$	*5.44*
$X_1 \& X_3$	*5.58*
$X_2 \& X_3$	*6.02*
X_1	*2.64*
X_2	*5.44*
X_3	*5.57*

5. CONCLUSIONS

This paper presents a multivariate nonstationary gamma process model suitable for modeling multiple degradation phenomena in civil infrastructure such as a highway pavement section. The estimated parameters of the model were compared with parameters of independent stochastic processes. In addition, the parameters were used to generate realizations of future degradation paths which are subsequently used to evaluate the remaining lifetime distribution based on a number of failure scenarios.

The results from the multivariate gamma process modeling serve as an input for a condition-based inspection and maintenance optimization. Work on this is ongoing and will be presented in a future publication.

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