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Time-variant Reliability Analysis Based on AK-SYS

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ABSTRACT:

The objective of this paper is to investigate system reliability methods for time-variant reliability analysis. The main challenge in time-variant reliability analysis is dealing with low probabilities of failure while computationally expensive performance functions are involved. Time-variant problems are often simplified by discretizing time intervals. This enables one to transform the initial problem to a system problem. System reliability methods can therefore be used to address time-variant reliability problems. AK-SYS is an efficient system reliability method which can be applied when costly-to-evaluate performance functions are considered. Application of this method for time-variant reliability problems is introduced in this paper that is supported with two academic examples from literature.

1. INTRODUCTION

Time-variant reliability analysis is necessary for many of engineering systems. It provides an indicator that shows how proper a system performs its duties over a given period of time under specified conditions. This indicator might be fruitful to analyze the life-cycle cost, to decide about the maintenance and inspection allocation, and also to measure the durability of a system Hawchar et al. (2017). Time adds an extra dimension to the classical reliability problem and makes it more difficult to solve. Achieving a reasonable trade-off between efficiency and accuracy is still a challenge in this field. This is even more difficult for problems with low probabilities of failure while computationally expensive functions are involved. Therefore, the main goal of this study is to introduce an efficient methodology to address time-variant reliability problems that has a good level of accuracy as well. Generally, methods for time-variant reliability analysis can be divided into two categories: first passage based methods, and extreme value based methods.

Methods in the first category estimate the failure probability based on out-crossing events in which a time-dependent performance function crosses the threshold from the safe to the failure zone. Averaging the out-crossing rate during the lifetime of a structure is used to calculate the probability of failure. Rice formula (Rice, 1944), PHI2 method (Andrieu-Renaud et al., 2004), and PHI2+ (Sudret, 2008) are among the popular methods in this category.

Methods in the second category are based on estimating the extreme response of a time-variant performance function with respect to time. Failure happens when the extreme value violates a given threshold. Therefore, failure probabilities can be calculated by knowing the distribution of the extreme response. Computing the probability distribution of the extreme response is a challenge especially when computationally expensive performance functions are involved. Hence, surrogate models are used to approximate the extreme response. In the literature, some Kriging based methods such as NERS (Wang and Wang, 2012), Mixed-EGO (Hu and Du, 2015), and SILK (Hu and Mahadevan, 2016) can be found in addition to some methods using Polynomial Chaos Expansion like t-PCE by Hawchar et al. (2017).

Dealing with time parameter is one of the sources of complexity in time-variant reliability analysis. Discretizing the time interval into a finite number of time nodes is one way to overcome this difficulty. It converts the problem into the reliability analysis of a serially connected system. A performance function is defined for each time node that represents a component in the system. This brings the idea of employing efficient system reliability methods for time-variant reliability analysis. Therefore, the goal here is to apply the efficient Kriging based system reliability method, namely AK-SYS (Fauriat and Gayton, 2014), for time-variant problems. The aim of AK-SYS is to solve system reliability problems involving costly-to-evaluate performance functions.

The remaining of this study is organized as follows. Time-variant reliability analysis is reviewed in Section 2. Section 3 is related to the proposed methodology. The main principles and steps of the methodology are introduced in this section. In Section 4 two examples from literature are selected to show the efficiency and accuracy of the new approach. In the end, a short conclusion is provided in Section 5.

2. TIME-VARIANT RELIABILITY ANALYSIS

Generally speaking, there are two types of reliability problems: time-invariant (static) and time-variant (dynamic) problems depending on the performance function of a system. Time-variant reliability assessment is much more complicated than time-invariant problems due to the added complexity to the problem by considering time as an input parameter. In most of reliability problems, time-variant analysis cannot be circumvented due to the temporal nature of material properties, loadings, and geometric parameters.

The general type of performance function ($G(\mathbf{X}, \mathbf{Y}(t), t)$) in a time-variant reliability problem involves input random variables $\mathbf{X}$, random processes $\mathbf{Y}(t)$ and time t. However, this performance function can be converted to an explicit performance function ($G(\mathbf{X}, t)$) by employing some methods such as Karhunen-Loeve (KL) transformation (Loeve, 1977).

In time-variant reliability analysis two types of failure probability can be computed: the cumulative and instantaneous failure probabilities. The former measures the probability of having at least one failure over a given period of time $[t_0, t_l]$, see Equation 1, while the latter measures the probability of failure at a given time instant t, see Equation 2. It should be noted that in both cases $G(\mathbf{X}, t) = 0$ denotes the limit state function that separates the safe domain ($G(\mathbf{X}, t) > 0$) from the failure domain ($G(\mathbf{X}, t) \leq 0$).

$$P_{f,c}(t_0,t_l) = \text{Prob}(\exists \tau \in [t_0,t_l], G(\mathbf{X},\tau) \leq 0) \quad (1)$$

$$P_{f,i}(t) = \text{Prob}(G(\mathbf{X},t) \leq 0) \quad (2)$$

Instantaneous probability of failure can be calculated by employing time-invariant reliability methods such as sampling techniques: Monte Carlo Simulation (MCS), importance sampling, directional sampling, etc., most probable failure point-based techniques: FORM and SORM (Melchers, 1999), and meta-model-based techniques (AK-MCS (Echard et al., 2011), EGRA (Ranjan et al., 2008), etc. Hence, the goal of time-variant reliability methods is to approximate the cumulative failure probability.

System reliability methods can be utilized to address time-variant reliability problems. As it was mentioned before, when the lifetime of a system is discretized into several time nodes, time-variant reliability problem is equivalent to a series system problem. Each time node represents a component of the system. Therefore, it looks reasonable to get advantage of system reliability methods to solve time-variant problems. Among recently developed methods for system reliability, AK-SYS has a promising efficiency. Consequently, application of this method for time-variant reliability analysis will be explained in next section.

3. PROPOSED METHODOLOGY AND PRINCIPLES

In this section a new approach for time-variant reliability analysis is proposed that is based on the system reliability method AK-SYS. Hence, the correspondence of time-variant and system reliability is explained first, then system reliability and AK-SYS method are reviewed. Afterwards, the novel approach is introduced.

3.1. *From time-variant to system reliability*

In fact, the similarity between the time-variant and series system reliability analysis gives us the motivation to propose a new approach for time-variant problems that involve time consuming performance functions.

As said before, one basic method to handle the complexity in time-variant problems is to discretize the system's lifetime into N_t time nodes. An instantaneous performance function $G_n(\mathbf{X})$ is considered for each node t_n, $n = 0,...,N_t$ and the failure event can be defined for each time node as formulated in Equation 3. The probability of the union of all failure events approximates the cumulative failure probability in Equation 4. It is worth mentioning again that the discretization strategy is very important to be able to properly detect the failure events. Therefore, the time interval between consecutive time nodes Δt needs to be determined carefully. An illustration of time discretization and failure events is provided in Figure 1.

$$E_n = \{\mathbf{x} : \; G_n(\mathbf{x}) < 0)\} \quad (3)$$

$$P_{f,c}(t_0,t_l) \approx \text{Prob}\{\cup_{n=0}^{N_t} E_n\} = \text{Prob}\{\cup_{n=0}^{N_t} (G_n(\mathbf{X}) < 0)\} \quad (4)$$

Comparison between Equation 4 and Equation 6 (see Section 3.2) shows the equivalence between time-variant and series systems reliability analyses. Hence, efficient system reliability methods like AK-SYS can be modified for time-variant reliability problems.

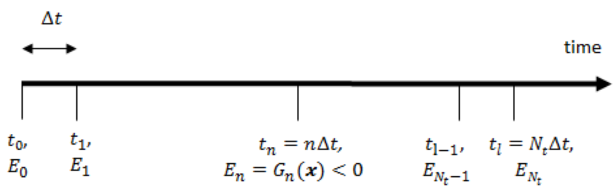

Figure 1: A representation of time discretization

3.2. *System reliability*

Unlike reliability analysis of components, performing system reliability analysis is a hard task since multiple failure modes might be involved. For a given system, failure modes need to be identified very carefully. Our concern in this study is reliability of serially connected systems. However, parallel systems and series systems can be easily converted to each other using De Morgan's law, see Equation 5).

$$\overline{\cup_{j=1}^{p} E_j} = \cap_{j=1}^{p} \overline{E_j} \quad (5)$$

where E_j is the jth failure event while there are p failure events in the system and j can change from 1 to p.

In series systems with p components, a performance function $G_j(.),\ j = 1,...,p$ is defined for each component. The jth failure event can be formulated as $E_j = \{G_j(\mathbf{X}) \leq 0\}$ and failure probability for a series system can be calculated as follow:

$$P_f = \text{Prob}(\cup_{j=1}^{p} E_j) = \text{Prob}(\cup_{j=1}^{p} G_j(\mathbf{X}) \leq 0) \quad (6)$$

The system fails when one component fails. Therefore, one way to calculate the failure probability is to find the composite limit state as it is formulated in Equation 7. A composite limit state searches for the minimum performance function in the system (Singh et al., 2010). It converts the system problem into a component problem and conventional methods for component reliability analysis can be used. However, this might lead to a very complex limit state that will be difficult to evaluate and often makes available methods inefficient and/or inaccurate.

$$\{\mathbf{x} : \cup_{j=1}^{p} G_j(\mathbf{x}) \leq 0\} = \{\mathbf{x} : G_{comp}(\mathbf{x}) \equiv \min_j G_j(\mathbf{x}) \leq 0\} \quad (7)$$

Employing surrogate models associated with an appropriate learning process can be helpful when costly-to-evaluate functions are involved. Using a Kriging meta-model for each performance function in the system and a specific learning process has led to the AK-SYS method that is very efficient (Fauriat and Gayton, 2014). This method is briefly explained in the following part.

3.3. *A review on AK-SYS*

As it was mentioned before, AK-SYS is a recently developed methodology for system reliability analysis which aims at reducing the computational burden. Actually, this method is a modification of AK-MCS (Echard et al., 2011) towards system reliability. In AK-MCS the original performance function G is replaced with a Kriging meta-model $\hat{G}$ which is calibrated from an adaptive Design of Experiments (DOE). To improve the accuracy of Kriging meta-model with respect to the objective of reliability assessment, the learning function U (Equation 8) is used to find the best next training point to enrich the DOE.

$$U(\mathbf{x}) = \frac{|\hat{G}(\mathbf{x})|}{\sigma_{\hat{G}}(\mathbf{x})} \quad (8)$$

where $\sigma^2_{\hat{G}}(\mathbf{x})$ is the variance of estimation at point $\mathbf{x}$.

Among all the sample points inside the initially generated population from a classical MCS of size N, the point $\mathbf{x}$ that minimizes the learning function is considered as the best training point. The learning process stops when the minimum value of U is greater than 2 for all sample points of the population. This indicates that the Kriging limit state $\hat{G} = 0$ sufficiently matches the true limit state $G = 0$.

Employing the learning function U for a system with p components, a composite criterion learning function has been introduced in AK-SYS that is defined in Equation 9.

$$U_s(\mathbf{x}^{(i)}) = \frac{|\hat{G}_s(\mathbf{x}^{(i)})|}{\sigma_{\hat{G}_s(\mathbf{x}^{(i)})}} \quad (9)$$

where for each point $\mathbf{x}^{(i)},\ i = 1,...,N$ the minimum performance function $\hat{G}_s$ among $\hat{G}_j(\mathbf{X}),\ j = 1,...,p$ is found first. The learning process is then performed on this function to find the best training point. The point $\mathbf{x}^{(i)}$ that minimizes U_s is the best training point. The Kriging meta-model $\hat{G}_s$ is subsequently updated. The learning process stops when for all sample points $\min U_s \geq 2$. This learning function tries to find the best training point by searching through the most vulnerable components that have significant contribution to the system's failure. This makes the third approach very efficient. This approach will be used for time-variant reliability analysis.

3.4. *Extension of AK-SYS to time-variant reliability*

The aim of introducing this new approach is to exploit the efficiency of AK-SYS towards time-variant reliability analysis. As in AK-MCS, one advantage of AK-SYS is to select new enrichments from a Monte Carlo population. Therefore, no continues optimization is required. Then, by using a composite criterion learning function it searches only among those components that have considerable contribution to the system's failure.

The algorithm of the new approach is illustrated in Figure 2. The algorithm starts by discretizing the lifetime into N_t time nodes, generating the initial Monte Carlo population of size N_{MCS} from the joint distribution of $\mathbf{X}$, and preparing the DOE of size N_{DOE}. Different sampling approaches such as random sampling, Latin hypercube sampling, and Hammersley sampling can be used to prepare an appropriate DOE (Hu and Du, 2015). The choice of Δt (the time interval between nodes) is very important since it should be chosen in a way that between two consecutive instants there is at most one passage of the performance function from the safe to the failure zone. This ensures the detection of failure.

Initial Kriging meta-models ($\hat{G}_n(\mathbf{X})$, $n = 0,...,N_t$) are prepared for all instantaneous performance functions ($G_n(\mathbf{X})$, $n = 0,...,N_t$). This is done by using the primary DOE. Kriging meta-models need to be trained to become accurate enough to be able to properly evaluate the failure probability. The composite criterion learning function from AK-SYS is utilized for such purpose. The learning procedure continues until the minimum value of learning function over the Monte Carlo population is greater than 2. Otherwise, the point that minimizes U_s will be used to enrich the DOE. Adding each point to the DOE needs an evaluation of the original performance function G.

The cumulative probability of failure over a time period $[t_0,t_l]$ can be evaluated in the same way as MCS. The difference here is that the original performance functions $G_n(\mathbf{X})$ are replaced by the Kriging meta-models that are prepared in the previous step. By generating N_{MCS} realizations of the time-variant performance function, the failure probability is the ratio of failed realizations to the total number of realizations N_{MCS} as in Equation 10.

$$\hat{P}_{f,c}(t_0,t_l) = \frac{\hat{N}_{fail}(t_0,t_l)}{N_{MCS}} \tag{10}$$

where $\hat{N}_{fail}$ is the number of failed realizations calculated by Kriging meta-models.

The final stopping criterion for the algorithm is defined on the coefficient of variation of $\hat{P}_{f,c}$ to check if the size of Monte Carlo population is large enough to reach a sufficient accuracy on the Kriging prediction of probability of failure. The coefficient of variation can be calculated using Equation 11.

$$C.O.V_{\hat{P}_{f,c}} = \sqrt{\frac{1-\hat{P}_{f,c}(t_0,t_l)}{N_{MCS}\times\hat{P}_{f,c}(t_0,t_l)}} \tag{11}$$

4. CASE STUDIES

The application of the proposed methodology is investigated on two numerical examples from literature in this section. The results of the proposed methodology are compared with MCS. To evaluate the efficiency of the approach, two indicators are calculated:

1. The relative percentage error between the cumulative failure probabilities obtained by AK-SYS based approach and MCS. This error can be calculated by Equation 12.

2. The number of miss-classified realizations of the performance function using Equation 13. Well-classified realizations are those that have the same sign (positive or negative) at each time node.

It should be noted that for comparison purpose, the same time discretization strategy is used for both methods.

$$Error(\%) = \frac{|P_{MCS}-\hat{P}_{f,c}|}{P_{MCS}}\times 100 \tag{12}$$

$$N_{misclass} = \sum_{n=1}^{N_t}\sum_{i=1}^{N_{MCS}} I(\mathbf{x}^{(i)},t_n) \tag{13}$$

where

$$I(\mathbf{x}^{(i)},t_n) = \begin{cases} 1 & if \quad \hat{G}(\mathbf{x}^{(i)},t_n)\times G(\mathbf{x}^{(i)},t_n) < 0 \\ 0 & \text{Otherwise} \end{cases} \tag{14}$$

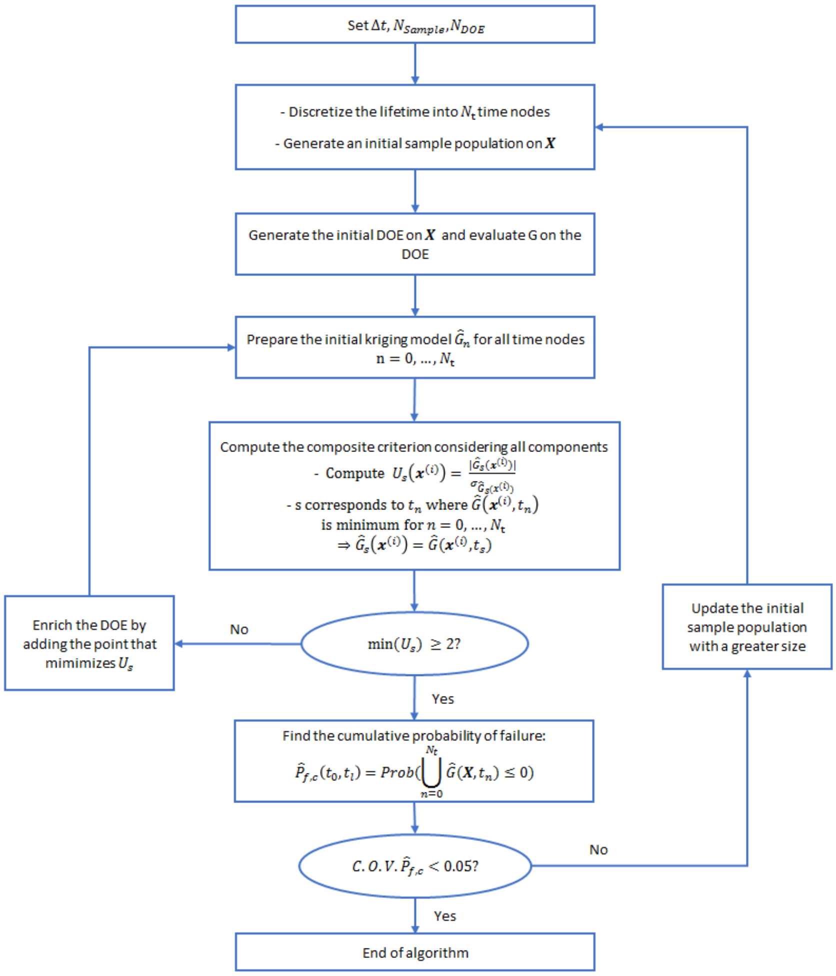

Figure 2: General algorithm for proposed methodology

4.1. Example 1: A nonlinear model

The first numerical example is a nonlinear performance function which is an explicit relationship with respect to time. This example is taken from (Hu and Du, 2015) with small modifications. This performance function is formulated in Equation 15. It includes one random variable X where $X \sim N(10,1)$.

$$G(X,t) = 0.014 - \frac{1}{X^2+4}\sin(2.5X)\cos(t+0.4)^2 \tag{15}$$

the cumulative probability of failure over $[1,2.5]$ is determined by:

$$P_{f,c}(1,2.5) = \text{Prob}(\exists \tau \in [1,2.5],\ G(X,\tau) \leq 0) \tag{16}$$

A Monte Carlo population of size 5×10^5 is generated. Latin hypercube sampling is used to prepare the DOE of size 10. Different discretization scenarios are investigated (see Table 1). A first obvious comment is that the probability of failure increases when the number of time nodes increases. It can be seen that for all discretization schemes, the number of miss-classified realizations is zero and therefore the probability of failure is exactly the same as MCS. It should be noted that this perfect level of accuracy is achieved only by a few number of calls (around 20) to the original performance function that makes the method very efficient as well. Also, it should be mentioned that the *COV* for all cases is less than 0.05.

To emphasize the importance of time discretization, the simulation has been performed for 100 time nodes as well. The cumulative failure probability for this case is 0.011590 using MCS or AK-SYS-based method. If we use this outcome as a reference value for the probability of failure, one can say that by increasing the number of time nodes for a given time interval, the approximated failure probability converges to the reference value. However, this requires extra computational time. Hence, an appropriate discretization strategy is necessary to have a sufficient accuracy and to avoid extra computational costs. It should be noted that finding the best discretization strategy is out of scope of this paper that remains a perspective.

Table 1: Results for example 1

Time Nodes	P_MCS	P_AK-SYS-based	N_calls	Misclass	Error(%)
5	0.002048	0.002048	19	0	0
10	0.002046	0.002046	20	0	0
20	0.010068	0.010068	23	0	0
50	0.011588	0.011588	21	0	0

4.2. Example 2: A general model

The second numerical example includes a stochastic process (Hu and Du, 2015). A corroded beam subjected to a stochastic loading is considered. The performance function for this example is defined in Equation 17 where $\mathbf{X} = [\sigma_u, a_0, b_0]$, $Y(t) = F(t)$, $L = 5m$, $k = 5 \times 10^{-5}$, and $\rho_{st} = 7.85 \times 10^4 N$.

$$G(\mathbf{X}, Y(t), t) = -(F(t)L/4 + \rho_{st} a_0 b_0 L^2/8) + (a_0 - 2kt)(b_0 - 2kt)^2 \sigma_u/4 \quad (17)$$

$F(t)$ is the involved stochastic loading. A spectral representation method is used to decompose this process, see Equation 18. Table 2 provides the distributions and their parameters for the random variables $\mathbf{x} = [\sigma_u, a_0, b_0]$ and ξ_i, $i = 1, ..., 7$. They are assumed to be independent. The matrices for coefficients a_{ij}, b_{ij}, and c_{ij} can be found in (Hu and Du, 2015).

$$F(t) = 6500 + \sum_{i=1}^{7} \xi_i \Big(\sum_{j=1}^{7} (a_{ij} sin(b_{ij} t + c_{ij})) \Big) \quad (18)$$

The cumulative probability of failure for 35 years lifetime for such corroded beam is defined as:

$$P_{f,c}(0, 35) = \text{Prob}(\exists \tau \in [0, 35],\ G(\mathbf{x}, Y(t), \tau) \leq 0) \quad (19)$$

Table 2: Distribution of random variables for example 2

variable	Mean	Standard deviation	Distribution
σ_u(Pa)	2.4×10^8	2×10^7	Normal
a_0(m)	0.2	0.01	Normal
b_0(m)	0.04	4×10^{-3}	Normal
ξ_1	0	100	Normal
ξ_2	0	50	Normal
ξ_3	0	98	Normal
ξ_4	0	121	Normal
ξ_5	0	227	Normal
ξ_6	0	98	Normal
ξ_7	0	121	Normal

To evaluate the cumulative failure probability for this case, a Monte Carlo population of size 5×10^4 is generated. An initial DOE of size 50 is generated randomly. Several time discretization scenarios are investigated for this example too. As it can be seen from Table 3, in this numerical case with a higher dimension ($D = 10$), a few realizations are mis-classified. However, the estimated probability failure remains accurate.

By comparing the results of the first example with dimension one and the second example with dimension 10, one can conclude that more calls to the true performance function is required in the second example to reach the stopping criterion. However, the novel method still performs well with a significant reduction of calls to the original performance function.

Table 3: Results for example 2

Time Nodes	P_MCS	P_AK-SYS-based	N_calls	Misclass	Error(%)
5	0.04416	0.04418	73	7	0.045
10	0.044065	0.044025	79	12	0.091
20	0.044163	0.044063	77	17	0.226
50	0.044203	0.044243	77	14	0.090

5. Conclusion

This paper investigates the application of system reliability methods like AK-SYS for time-variant reliability analysis. This utilization is reasonable according to the similarity between system and time-variant reliability problems. In fact, the discretization of lifetime of a system leads to a serially connected reliability problem. An extension of AK-SYS for time-variant reliability analysis has been proposed.

In the new approach, an instantaneous performance function can be defined for each time node (i.e after discretization). These functions are subsequently replaced by Kriging meta-models. The enriching process in AK-SYS can help to find the best training points. Besides, it searches only through the vulnerable instantaneous functions that have a significant contribution to the system's failure. Therefore, the new approach owes its efficiency to AK-SYS.

Two illustrative examples show the efficiency and accuracy of the proposed methodology. It can be seen that for problems with low dimensionality the results are exactly the same as MCS, but by increasing the dimension, the error slightly increases. However, the results are still very accurate. It should be noted that, this level of accuracy has been reached by only few calls to the original performance functions comparing to MCS.

Some perspectives for this study would consist in comparing the method with the state of the art methods, studying the time discretization strategy, and finally our objective is to apply this method proposing a relevant approach for fatigue reliability analysis.

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