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이학박사 학위논문

Classifications of regular ternary triangular forms

(정규 삼변수 삼각형식의 분류)

2019년 8월

서울대학교 대학원

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Classifications of regular ternary triangular forms

A dissertation
submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
to the faculty of the Graduate School of
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by

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Abstract

Let $T_x = \frac{x(x+1)}{2}$. For positive integers $a_1, a_2, \ldots, a_k$, a polynomial of the form $a_1T_{x_1} + a_2T_{x_2} + \cdots + a_kT_{x_k}$ is called a triangular form.

In this thesis, we study various properties of representations of integers by ternary and quaternary triangular forms. A triangular form is called regular if it represents all positive integers that are locally represented. We classify the regular ternary triangular forms. We also prove several conjectures of Sun regarding the number of representations of integers by ternary and quaternary triangular forms.

Key words: representation of ternary quadratic forms, regular forms, triangular numbers, Watson transformation
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Contents

Chapter 1

Introduction

Lagrange's celebrated four squares theorem says that every positive integer is a sum of four squares, that is, the quaternary quadratic form $x^2+y^2+z^2+t^2$ represents all positive integers. A positive definite integral quadratic form is called *universal* if it represents all positive integers. In 1916, Ramanujan gave in [29] the list of 55 diagonal quaternary universal quadratic forms. Later, Dickson confirmed 54 forms among them are actually universal, whereas the quaternary form $x^2 + 2y^2 + 5z^2 + 5w^2$ in the Ramanujan's list turns out to be non-universal for it does not represent 15. In 2000, Bhargava [5] gave short proof of the Conway-Schneeberger's, so called, "15-theorem", and proved that there are exactly 204 (classic integral) positive definite integral quaternary universal quadratic forms. Bhargava and Hanke [6] also proved the "290-theorem" and derived that there are exactly 6436 (non-classic integral) positive definite universal quaternary forms. A positive definite integral quadratic form is called *regular* if it globally represents every integer which is locally represented. Dickson [12] who initiated the study of regular quadratic forms first coined the term regular. Jones and Pall [19] gave the list of all 102 primitive diagonal regular ternary quadratic forms. Watson proved that there are only finitely many equivalence classes of primitive positive definite ternary regular forms in his thesis [32]. Jagy, Kaplansky and Schiemann [17] succeeded Watson's study and provide the list of 913 candidates for such forms. All but 22 of them are already proved at the time. Recently, the regularities of 8 forms among 22 candidates were proved by Oh [25]. A conditional proof for the remaining 14 candidates under the Generalized Riemann Hypothesis was given by Lemke Oliver [24].

Now we look into *the representations of triangular forms*. An integer of the form $T_x = \frac{x(x+1)}{2}$ for some integer x is called *a triangular number*. For positive integers $a_1, a_2, \ldots, a_k$, we call a polynomial of the form

$$\Delta(a_1, a_2, \ldots, a_k)(x_1, x_2, \ldots, x_k) := a_1\frac{x_1(x_1+1)}{2}+a_2\frac{x_2(x_2+1)}{2}+\cdots+a_k\frac{x_k(x_k+1)}{2}$$

a k-ary triangular form. We define the universality and regularity of a triangular form similarly to the case of a quadratic form. Gauss' Eureka Theorem says that every positive integer is a sum of at most three triangular numbers. This is equivalent to say that the triangular form $\Delta(1, 1, 1)$ is universal. In 1862, Liouville classified all ternary universal triangular forms and in fact, they are the following seven forms:

$$\Delta(1,1,1),\quad \Delta(1,1,2),\quad \Delta(1,1,4),\quad \Delta(1,1,5),$$
$$\Delta(1,2,2),\quad \Delta(1,2,3),\quad \Delta(1,2,4).$$

In 2013, Chan and Oh [10] showed that there are only finitely many ternary regular triangular forms. In 2015, Chan and Ricci [11] proved the finiteness of ternary regular triangular forms in a more general setting. They actually proved that for any given positive integer c, there are only finitely many inequivalent positive ternary regular primitive complete quadratic polynomials with conductor c. From this follows the finiteness of regular ternary m-gonal forms. Note that an integer of the form $\frac{(m-2)x^2-(m-4)x}{2}$ for some integer x is called an m-gonal number, and a ternary m-gonal form is defined similarly.

Now we move on to the subject of the number of representations by a quadratic form or a triangular form. Given a positive definite integral quadratic form, the problem of determining the representation number of integers by the form is quite old and is still complicate, in general. If the class number of a quadratic form is one, then a closed formula for the number of representations may be obtained from the Minkowski-Siegel formula by calculating the local densities. The theory of theta function identities or theory of modular forms gives some information about the number of representations of certain quadratic forms. A closed formula for the number of representations of integers by a quadratic form is only known for some specific forms. As we will see later, the number of representations of an integer by a triangular form can be transformed to the number of representations of the corresponding integer by the corresponding diagonal quadratic forms with certain congruence condition. For positive integers $a_1, a_2, \ldots, a_k$ and n, we define

$$r(n, \langle a_1, \ldots, a_k \rangle) = \left| \left\{ (x_1, \ldots, x_k) \in \mathbb{Z}^k : a_1 x_1^2 + \cdots + a_k x_k^2 = n \right\} \right|,$$

and

$$t(n, \langle a_1, \ldots, a_k \rangle) = \left| \left\{ (x_1, \ldots, x_k) \in \mathbb{Z}^k : a_1 T_{x_1} + \cdots + a_k T_{x_k} = n \right\} \right|.$$

Legendre proved that

$$t(n, \langle 1, 1, 1, 1 \rangle) = 16\sigma(2n+1),$$

where σ is the sum of divisors function. In 2003, Williams [34] showed

$$t(n, \langle 1, 1, 2, 2 \rangle) = 4 \sum_{d \mid 4n+3} \left(d - (-1)^{d - \frac{1}{2}} \right).$$

Finding a closed formula for the number of representations of integers by general triangular forms seems to be challenging and in fact, little is known. There are some results on the relation between the number of representations of a triangular form and the number of representations of some quadratic forms. For example, Adiga, Cooper and Han [1] showed, for $5 \leq a+b+c+d \leq 7$,

$$C(a,b,c,d)t(n, \langle a,b,c,d \rangle) = r(8n + a + b + c + d, \langle a,b,c,d \rangle),$$

where $C(a,b,c,d)$ is an explicit constant depending only on a, b, c and d. In 2008, Baruah, Cooper and Hirschhorn [2] proved that if $a+b+c+d = 8$, then

$$C(a,b,c,d)t(n, \langle a,b,c,d \rangle) = r(8n+8, \langle a,b,c,d \rangle) - r(2n+2, \langle a,b,c,d \rangle).$$

In 2016, Sun [30] verified several relations between the number of representations of quaternary triangular forms and the number of representations of corresponding quadratic forms. In that paper, he also proposed 23 intriguing conjectures on the relations between $t(n, \langle a, b, c, d\rangle)$ and $r(n', \langle a, b, c, d\rangle)$, where n' is a positive integer determined by a, b, c, d and n. Yao [37] proved 11 conjectures among them by using (p, k)-parametrization of theta functions. Sun [31] himself proved 2 conjectures by using elementary method. Xia and Zhong [35] proved 3 conjectures by using theta function identities. Sun [31] discovered further relations on ternary and quaternary cases and posed some conjectures some of which are on ternary triangular forms.

In the first part of the thesis, we prove that there are exactly 49 regular ternary triangular forms. In the previous papers [10] and [11], the authors use Burgess' estimation of character sums (for this, see [7] and [14]) to prove the finiteness of regular ternary triangular forms. It seems to be quite difficult to find an explicit upper bound of the discriminant of regular ternary triangular forms by using Burgess' estimation. So, we use purely arithmetic method to find such an explicit upper bound, and finally, we classify all regular ternary triangular forms. We also prove five conjectures given in [31] on the number of representations of ternary triangular forms. In fact, Xia, Zhang and Baruah, Kaur proved these conjectures independently. However, our method is quite different from their methods. Furthermore, we prove quite a generalized version of the original conjectures. Finally, we prove Conjecture 2.5 given in [30]. Most results were done by joint work with B.-K. Oh.

In Chapter 2, we present some basic notations and terminologies that will be used throughout the thesis. The explicit definition of a triangular form and its regularity will be given in the chapter. Some properties of the Watson transformation on triangular forms will be presented in this chapter, which will play a key role in classification.

In Chapter 3, we classify all regular ternary triangular forms. First of all, we show that in some cases, Watson transformations preserve the regularity. Then we find all stable regular ternary triangular forms which are, by definition, at the floor level with respect to Watson transformations. To find stable regular ternary triangular forms, we obtain a nice upper bound of the number of primes at which the corresponding ternary quadratic forms are anisotropic and then proceed to find an upper bound of the discriminant of regular ternary triangular forms. After determining all candidates for stable regular forms, we prove the regularities of those candidates. Next, we show there is no missing prime greater than 7 (for the definition of a missing prime, see Chapter 3). Then, we trace back Watson transformations for primes 3,5, and 7 to enumerate all candidates of regular ternary triangular forms. Finally, we verify the regularities of all candidates. The regularities of most candidates are easily proved by using some elementary method. Only five of the candidates need to be treated delicately and the proof of regularity of each triangular form involves some techniques parallel to the ones appeared in [25] and some properties of the graph defined on ternary quadratic forms(cf. [4],[18]). Most results of this chapter are part of [22].

In Chapter 4, we discover some general relations between the number of representations of a ternary triangular forms and the number of representations of the corresponding quadratic forms. We prove some relation between $r(n, f)$ and $r(4n, f)$ for three binary quadratic forms $f = \langle 3, 5\rangle, \langle 1, 7\rangle$, and $\langle 1, 15\rangle$. Using this relation, we generalize some conjectures. To prove the conjectures, we first deform the representation of corresponding quadratic form with congruence condition into the representation of subform. Then, the main method of the proofs is to use certain rational isometries between quadratic forms. Most results of this chapter are part of [21].

In Chapter 5, we prove Conjecture 2.5 given in [30]. Most results of this chapter are part of [3].

Chapter 2

Preliminaries

In this chapter, we introduce some definitions, notations and well-known results which we frequently use throughout the thesis.

2.1 Triangular numbers and triangular forms

A nonnegative integer of the form

$$T_x = \frac{x(x+1)}{2} \quad (x \in \mathbb{N})$$

is called a triangular number. For example, $0, 1, 3, 6, 10, 15, \cdots$ are triangular numbers. Since $T_x = T_{-x-1}$, T_x is a triangular number for any integer x. For positive integers $a_1, a_2, \dots, a_k$, we call a polynomial of the form

$$\Delta(a_1, a_2, \dots, a_k) := a_1T_{x_1} + a_2T_{x_2} + \cdots + a_kT_{x_k}$$

a *k-ary triangular form*. A triangular form $\Delta(a_1, a_2, \dots, a_k)$ is called *primitive* if $\gcd(a_1, a_2, \dots, a_k) = 1$. Unless stated otherwise, we always assume that

every triangular form is primitive.

For an integer n and a k-ary triangular form $\Delta(a_1, a_2, \ldots, a_k)$, we say that n *is represented by* $\Delta(a_1, a_2, \ldots, a_k)$ if the Diophantine equation

$$a_1T_{x_1} + a_2T_{x_2} + \cdots + a_kT_{x_k} = n$$

has an integral solution. We also define

$$T(n, \langle a_1, \ldots, a_k\rangle) = \left\{(z_1, \ldots, z_k) \in \mathbb{Z}^k : a_1T_{z_1} + \cdots + a_kT_{z_k} = n\right\}$$

and $t(n, \langle a_1, a_2, \ldots, a_k\rangle)$ to be the cardinality of the above set. Note that $t(n, \langle a_1, a_2, \ldots, a_k\rangle)$ is always finite since we are assuming $a_i > 0$ for every i.

A triangular form $\Delta(a_1, a_2, \ldots, a_k)$ is called *universal* if it represents every positive integer, that is,

$$a_1T_{x_1} + a_2T_{x_2} + \cdots + a_kT_{x_k} = n \text{ is soluble in } \mathbb{Z}$$

for any positive integer n. A triangular form $\Delta(a_1, a_2, \ldots, a_k)$ is called *regular* if it globally represents every integer which is locally represented. In other words, $\Delta(a_1, a_2, \ldots, a_k)$ is regular if the following implication holds for any positive integer n; if $a_1T_{x_1} + a_2T_{x_2} + \cdots + a_kT_{x_k} = n$ is soluble in $\mathbb{Z}_p$ for any prime p, then $a_1T_{x_1} + a_2T_{x_2} + \cdots + a_kT_{x_k} = n$ is soluble in $\mathbb{Z}$.

The following lemma appear in [10] says that we may ignore the prime 2 when we consider the regularity of triangular forms.

Lemma 2.1.1. *Any primitive triangular form is universal over $\mathbb{Z}_2$.*

Note that $\Delta(a_1, a_2, \cdots, a_k)$ represents n if and only if the equation

$$a_1(2x_1 + 1)^2 + a_2(2x_2 + 1)^2 + \cdots + a_k(2x_k + 1)^2 = 8n + a_1 + a_2 + \cdots + a_k$$

is soluble in $\mathbb{Z}$. This equivalence shows how the representation of a triangular form is transformed into the representation of a diagonal quadratic form with congruence conditions. Now we can reformulate the regularity in a practical way. A ternary triangular form $\Delta(a, b, c)$ is regular if the following implication holds for any positive integer n; if $ax^2 + by^2 + cz^2 = 8n + a + b + c$ is soluble in $\mathbb{Z}_p$ for any odd prime p, then $ax^2 + by^2 + cz^2 = 8n + a + b + c$ for some odd integers x, y and z.

2.2 Quadratic spaces and lattices

Let $\mathbb{Q}$ be the field of rational numbers. For a prime number p, let $\mathbb{Q}_p$ denote the field of p-adic numbers and $\mathbb{Q}_\infty$ denote the field of real numbers $\mathbb{R}$. We always assume that $F = \mathbb{Q}$ or $F = \mathbb{Q}_p$. Let V be a finite dimensional vector space over F. Let

$$B : V \times V \to F$$

be a symmetric bilinear form on V, which means that

$$B(x, y) = (y, x) \quad \text{and} \quad B(cx + y, z) = cB(x, z) + B(y, z)$$

for any $x, y, z \in V$ and $c \in F$. We call (V, B) be a *quadratic space* over F. We define the quadratic map $Q : V \to F$ associated with B by

$$Q(x) = B(x, x)$$

for $x \in V$. Let V be a quadratic space over F with symmetric bilinear form B and $\mathfrak{B} = \{x_1, x_2, \cdots, x_k\}(\mathrm{dim}V = k)$ be a basis for V, where $\mathrm{dim}V = k$. The matrix

$$(B(x_i, x_j))_{1\le i,j\le k}$$

is called the *matrix presentation* of V and we write

$$V \simeq (B(x_i, x_j)) \quad \text{in} \quad \mathfrak{B}.$$

We define the *discriminant* dV of V by the determinant of the matrix of V. In other words,

$$dV = \det(B(x_i, x_j)) \in (F^\times/((F^\times)^2) \cup \{0\},$$

where $F^\times$ is the multiplicative group of non-zero elements of the field F. Note that dV is independent of the choice of basis for V. We call V a *regular* quadratic space if $dV \neq 0$. In this thesis, the term quadratic space always refer to a regular quadratic space and thus we omit the adjective *regular*. Let (V', B') be another quadratic space over F and let $\sigma : V \to V'$ be a linear transformation. We call σ a *representation* if

$$B(x, y) = B'(\sigma x, \sigma y)$$

for all $x, y \in V$. If further σ is a linear isomorphism, then we say that σ is an *isometry.* The set of all isometries from V onto V itself is denoted by $O(V)$.

Let $\mathbb{Z}$ be the ring of rational integers and $\mathbb{Z}_p$ be the ring of p-adic integers where p is a prime number. Let $R = \mathbb{Z}$ or $R = \mathbb{Z}_p$ for a prime p and let F be the field of fractions of R. An *R-lattice* L on V is a finitely generated free R-module which spans V over F. Note that L has induced symmetric bilinear map B and quadratic map Q from V. Let L be an R-lattice on a quadratic space V over F. For an R-basis $\mathfrak{C} = \{y_1, y_2, \ldots, y_k\}$ of L, we call

$$M_L := (B(y_i, y_j))_{1 \le i,j \le k}$$

the *matrix presentation* of L in $\mathfrak{C}$. If M_L is diagonal, then we simply write

$$L \simeq \langle Q(x_1), Q(x_2), \ldots, Q(x_k) \rangle.$$

We define the *discriminant* dL of L by the determinant of the $k \times k$ matrix $(B(y_i, y_j))$. The *scale* $\mathfrak{s}L$ of L is defined by the ideal in R generated by the set

$$\{B(x, y) \mid x, y \in L\}$$

and the *norm* $\mathfrak{n}L$ of L is the ideal generated by the set

$$\{Q(x) \mid x \in L\}.$$

We say L is *integral* if $\mathfrak{s}L \subset R$ and *primitive* if $\mathfrak{s}L = R$. We call L *isotropic* if there is a non-zero vector $x \in L$ such that $Q(x) = 0$, *anisotropic* otherwise. The corresponding quadratic form of L is defined by

$$f_L = f_L(x_1, x_2, \ldots, x_k) = \sum_{1 \le i,j \le k} B(x_i, x_j) x_i x_j.$$

We say that the quadratic form is *primitive, integral,* $\cdots$, *etc,* if the corresponding lattice is. Let L and K be R-lattices on quadratic spaces V and W, respectively. A *representation* of L by K is a space representation $\sigma : V \to W$ such that $\sigma(L) \subset K$. If $\sigma(L) = K$, then we say that L *is isometric to* K and write

$$L \simeq K.$$

Let L be a $\mathbb{Z}$-lattice on a quadratic space V over $\mathbb{Q}$ and $\mathrm{rank}(L) = k$. We say L is *positive definite* if the corresponding matrix M_L is. For a prime p, we define

$$L_p = \mathbb{Z}_p \otimes L.$$

Note that L_p is a $\mathbb{Z}_p$-lattice on the quadratic space $V_p = \mathbb{Q}_p \otimes V$. We say that L is *anisotropic(isotropic) at* p if L_p is anisotropic(isotropic, respectively). In this thesis, we always assume that

every $\mathbb{Z}$-lattice is positive definite and integral,

unless stated otherwise. Note that a $\mathbb{Z}$-lattice is called *unary, binary, ternary,* $\cdots$, *k-ary,* $\cdots$, if the rank of the lattice is $1, 2, 3, \cdots, k, \cdots$.

Let V and W be quadratic spaces over $\mathbb{Q}$ and let L and K be $\mathbb{Z}$-lattice on V and W, respectively. The set of all representations of L by K is denoted by $R(L, K)$. For the case of $L = K$, we let $O(L) = R(L, L)$ and $o(L) = |O(L)|$. If $\mathrm{rank}(L) = 1$, then we abuse the notation and make the following definition. For a quadratic form $f(x_1, x_2, \dots, x_k)$ over $\mathbb{Z}$ and an integer n, we define

$$R(n, f) = \big\{(z_1, z_2, \dots, z_k) \in \mathbb{Z}^k : f(z_1, z_2, \dots, z_k) = n\big\}$$

and define $r(n, f)$ to be the cardinality of the above set. Clearly, $R(n, f) = R(\langle n\rangle, K_f)$, where $\langle n\rangle$ is a unary lattice and K_f is a $\mathbb{Z}$-lattice corresponding to f. Note that $r(n, f)$ is always finite since the quadratic form f is positive definite. We use the notation

$$\langle a_1, a_2, \dots, a_k\rangle$$

for diagonal quadratic form $a_1x_1^2 + a_2x_2^2 + \cdots + a_kx_k^2$ also. Now for a vector $\mathbf{d} = (d_1, \dots, d_k) \in (\mathbb{Z}/2\mathbb{Z})^k$, we define

$$R_{\mathbf{d}}(n, f) = \{(x_1, x_2, \dots, x_k) \in R(n, f) : x_i \equiv d_i \pmod 2 \text{ for } i = 1, 2, \dots, k\}.$$

The cardinality of the above set will be denoted by $r_{\mathbf{d}}(n, f)$. For the diagonal binary quadratic forms, we also define

$$\widetilde{R}_{(1,1)}(N, \langle a, b\rangle) = \{(x, y) \in R_{(1,1)}(N, \langle a, b\rangle) : x \not\equiv y \pmod 4\}.$$

Note that if we define the cardinality of $\widetilde{R}_{(1,1)}(N, \langle a, b\rangle)$ by $\widetilde{r}_{(1,1)}(N, \langle a, b\rangle)$, then we have

$$r_{(1,1)}(N, \langle a, b\rangle) = 2 \cdot \widetilde{r}_{(1,1)}(N, \langle a, b\rangle).$$

Lemma 2.2.1. *Let m be a positive integer.*

(i) If $m \equiv 1 \pmod 4$, then we have

$$2r_{(1,0)}(m, \langle 1, 3\rangle) = r_{(1,1)}(4m, \langle 1, 3\rangle).$$

(ii) If $m \equiv 3 \pmod 4$, then we have

$$2r_{(0,1)}(m, \langle 1, 3\rangle) = r_{(1,1)}(4m, \langle 1, 3\rangle).$$

(iii) If $m \equiv 4 \pmod 8$, then we have

$$2r_{(0,0)}(m, \langle 1, 3\rangle) = r_{(1,1)}(m, \langle 1, 3\rangle).$$

Proof. (i) If we define a map

$$\psi_1 : R_{(1,0)}(m, x^2+3y^2) \to \widetilde{R}_{(1,1)}(4m, x^2+3y^2) \text{ by } \psi_1(x, y) = (x+3y, -x+y)$$

then one may easily check that it is bijective.
(ii) We define a map

$$\psi_2 : R_{(0,1)}(m, x^2+3y^2) \to \widetilde{R}_{(1,1)}(4m, x^2+3y^2) \text{ by } \psi_2(x, y) = (x+3y, -x+y).$$

Then one may show that it is bijective.
(iii) One may easily show that if we define a map

$$\psi_3 : R_{(0,0)}(m, x^2+3y^2) \to \widetilde{R}_{(1,1)}(m, x^2+3y^2) \text{ by } \psi_3(x, y) = \left(\frac{x+3y}{2}, \frac{-x+y}{2}\right),$$

then it is bijective. $\square$

Now for a triangular form $\Delta(a_1, a_2, \ldots, a_k)$, the corresponding quadratic form is defined by

$$\langle a_1, a_2, \ldots, a_k \rangle = a_1 x_1^2 + a_2 x_2^2 + \cdots + a_k x_k^2$$

Using these notations, the number of representations of an integer n by a triangular form can be rewritten as the number of representations of n by the corresponding quadratic form with a congruence condition;

$$t(n, \langle a_1, a_2, \ldots, a_k \rangle) = r_{(1,1,\ldots,1)}(8n + a_1 + a_2 + \cdots + a_k, \langle a_1, a_2, \ldots, a_k \rangle).$$

Any unexplained notations and terminologies can be found in [23] or [28].

2.3 Watson transformations

Let L be a $\mathbb{Z}$-lattice and m be a positive integer. We define the Watson transformation of L modulo m by

$$\Lambda_m(L) = \{x \in L : Q(x + z) \equiv Q(z) \pmod{m} \text{ for any } z \in L\}.$$

We denote by $\lambda_m(L)$ the primitive $\mathbb{Z}$-lattice obtained from $\Lambda_m(L)$ by scaling $\mathbb{Q} \otimes_{\mathbb{Z}} L$ by a suitable rational number. Throughout this section, we further assume that

every $\mathbb{Z}$-lattice is primitive and diagonal

for convenience.

Let $\Delta(a, b, c)$ be a ternary triangular form and let p be an odd prime. We define

$$\lambda_p(\Delta(a, b, c)) = \Delta(a', b', c'),$$

where $\langle a', b', c' \rangle \simeq \lambda_p(\langle a, b, c \rangle)$. Let p be an odd prime. Let $L = \langle a, p^m b, p^n c \rangle$

be a ternary $\mathbb{Z}$-lattice, where $(abc, p) = 1$ and $0 \le m \le n$. Then we have

$$\lambda_p(L) \simeq \begin{cases} \langle a, b, c \rangle & \text{if } m = n = 0, \\ \langle pa, b, p^{n-1}c \rangle & \text{if } 1 = m \le n, \\ \langle a, p^{m-2}b, p^{n-2}c \rangle & \text{if } 1 < m \le n. \end{cases}$$

Chapter 3

Regular ternary triangular forms

In this chapter, we classify all regular ternary triangular forms. We adopt some notations which will be used throughout the chapter. For an integer n and a diagonal quadratic form $\langle a_1, a_2, \ldots, a_k \rangle$, we write

$$n \xrightarrow{2} \langle a_1, a_2, \ldots, a_k \rangle$$

if there is a vector $(x_1, x_2, \ldots, x_k) \in \mathbb{Z}^n$ with $(x_1 x_2 \cdots x_k, 2) = 1$ such that $a_1 x_1^2 + a_2 x_2^2 + \cdots + a_k x_k^2 = n$. We also use the notation

$$n \overset{2}{\nrightarrow} \langle a_1, a_2, \ldots, a_k \rangle$$

if there is no such vector. Note that

$$n \xrightarrow{2} \langle a_1, a_2, \ldots, a_k \rangle \quad \text{if and only if} \quad r_{(1,1,\ldots,1)}(n, \langle a_1, a_2, \ldots, a_k \rangle) > 0.$$

3.1 The descending trick

The following lemma is just a reformulation of [10, Lemma 3.3].

Lemma 3.1.1. *Let p be an odd prime and a, b, c be positive integers which are not divisible by p. If the ternary triangular form $\Delta(a, p^r b, p^s c)$ with $1 \le r \le s$*

is regular, then $\lambda_p(\Delta(a, p^r b, p^s c))$ *is also regular.*

Though the proof of the next lemma is quite similar to the proof of Lemma 3.1.1, we provide the proof for completeness.

Lemma 3.1.2. *Let* p *be an odd prime. Let* a, b, c *and* s *be positive integers such that* $(p, abc) = 1$ *and* $\left(\frac{-ab}{p}\right) = -1$. *If the ternary triangular form* $\Delta(a, b, p^s c)$ *is regular, then* $\lambda_p(\Delta(a, b, p^s c))$ *is also regular.*

Proof. It is enough to show that $\Delta(p^2 a, p^2 b, p^s c)$ is regular. Let n be a positive integer such that the equation

$$p^2 a T_x + p^2 b T_y + p^s c T_z = n \tag{3.1.1}$$

is soluble over $\mathbb{Z}_p$ for any prime p. Then

$$8n + p^2 a + p^2 b + p^s c \longrightarrow \mathrm{gen}(\langle p^2 a, p^2 b, p^s c\rangle).$$

Thus

$$8\left(n + \frac{p^2 - 1}{8} a + \frac{p^2 - 1}{8} b\right) + a + b + p^s c \longrightarrow \mathrm{gen}(\langle a, b, p^s c\rangle).$$

Since $\Delta(a, b, p^s c)$ is regular, there is a vector $(x, y, z) \in \mathbb{Z}^3$ with $xyz \equiv 1 \pmod 2$ such that $ax^2 + by^2 + p^s cz^2 = 8n + p^2 a + p^2 b + p^s c$. Since n is divided by p, we have $ax^2 + by^2 \equiv 0 \pmod p$. From the assumption $\left(\dfrac{-ab}{p}\right) = -1$, we have $x \equiv y \equiv 0 \pmod p$. So

$$p^2 a \left(\frac{x}{p}\right)^2 + p^2 b \left(\frac{y}{p}\right)^2 + p^s c z^2 = 8n + p^2 a + p^2 b + p^s c$$

with $\frac{x}{p} \cdot \frac{y}{p} \cdot z \equiv 1 \pmod 2$. Thus Equation (3.1.1) is soluble in $\mathbb{Z}$. This completes the proof. $\square$

For an odd prime p and a ternary $\mathbb{Z}$-lattice L, we say that L is *p-stable* if

$$\langle 1, -1\rangle \longrightarrow L_p \quad \text{or} \quad L_p \simeq \langle 1, -\Delta_p\rangle \perp \langle p\epsilon_p\rangle$$

for some $\epsilon_p \in \mathbb{Z}_p^\times$. Furthermore, we say that L is *stable* if L is p-stable for every odd prime p. A ternary triangular form is called *p-stable* (*stable*) if the corresponding quadratic form is p-stable (stable, respectively). Let $\Delta(a, b, c)$ be a regular ternary triangular form. By Lemma 3.1.1 and Lemma 3.1.2, we may take λ_q-transformations to $\Delta(a, b, c)$ several times for odd primes dividing the discriminant and obtain a stable regular ternary triangular form $\Delta(a', b', c')$. In general, the corresponding quadratic form $\langle a', b', c'\rangle$ has smaller discriminant and simpler local structure than $\langle a, b, c\rangle$.

3.2 Stable regular ternary triangular forms

In this section, we prove that there are exactly 17 stable regular ternary triangular forms. Throughout this section, r_k denotes the k-th odd prime so that $\{r_1 = 3 < r_2 = 5 < r_3 = 7 < \cdots\}$ is the set of all odd primes. Let $\Delta(a, b, c)$ be a stable regular ternary triangular form. We always assume that $0 < a \le b \le c$.

Lemma 3.2.1. *For an integer s greater than 1, let $p_1 < p_2 < \cdots < p_s$ be odd primes. Let u be an integer with $(u, p_1p_2\cdots p_s) = 1$ and let v be an arbitrary integer. Then there is an integer n with $0 \le n < (s+2)2^{s-1}$ such that $(un + v, p_1p_2\cdots p_s) = 1$.*

Proof. See [20, Lemma 3]. □

Though Lemma 3.2.1 gives, in general, a nice upper bound of the longitude of arithmetic progression satisfying the assumption, there is a shaper bound in some restricted situation.

Lemma 3.2.2. *Under the same notations given in Lemma 3.2.1, if $s < p_1$, then there is an integer n with $0 \le n \le s$ such that $(un + v, p_1p_2\cdots p_s) = 1$.*

Proof. Trivial. □

Lemma 3.2.3. *Let $p \ge 5$ be a prime and let d be a positive integer with $(d, p) = 1$. Let $L = \langle a, b, c\rangle$ be a p-stable $\mathbb{Z}$-lattice that is anisotropic over $\mathbb{Z}_p$. Then there is an integer g such that* *(i)* $0 < g < p^2$;

(ii) $dg + a + b \nrightarrow \langle a, b\rangle$ *over* $\mathbb{Z}_p$;

(iii) $dg + a + b + c \longrightarrow \langle a, b, c\rangle$ *over* $\mathbb{Z}_p$;

(iv) $\max\{\mathrm{ord}_p(dg + a + b), \mathrm{ord}_p(dg + a + b + c)\} \le 1$.

Proof. Since L is p-stable and is anisotropic over $\mathbb{Z}_p$ by assumption, we have

$$\langle a, b, c\rangle \simeq \langle 1, -\Delta_p\rangle \perp \langle p\epsilon_p\rangle \text{ over } \mathbb{Z}_p,$$

for some $\epsilon_p \in \mathbb{Z}_p^\times$. First, we assume that p divides c. Since $\langle a, b\rangle \simeq \langle 1, -\Delta_p\rangle$, it does not represent $\gamma \in \mathbb{Z}_p$ satisfying $\mathrm{ord}_p\gamma \equiv 1 \pmod 2$. Since $p \ge 5$, there exists a positive integer g_1 with $g_1 < p^2$ such that

$$dg_1 + a + b \equiv 3c \pmod{p^2}.$$

Then one may easily check that g_1 satisfies all conditions given above. Now, assume that p divides ab. Without loss of generality, we may assume that p divides b. Since $p \ge 5$, there exists an integer a' with $(p, a') = 1$ such that aa' is not a square modulo p and $a' \not\equiv -c \pmod p$. We take a positive integer g_2 with $g_2 < p$ such that $dg_2 + a + b \equiv a' \pmod p$. One may easily show that g_2 satisfies all conditions given above, which completes the proof. $\square$

Let T be the set of odd primes p such that the diagonal ternary quadratic form $\langle a, b, c\rangle$ is anisotropic over $\mathbb{Z}_p$. Since such primes are only finitely many, we let

$$\begin{aligned} T &= \{p : p \ge 3,\ \langle a, b, c\rangle \text{ is anisotropic over } \mathbb{Z}_p\} \\ &= \{p_1 < p_2 < \cdots < p_t\}. \end{aligned}$$

Let

$$T' = T - \{3\} = \{q_1 < q_2 < \cdots < q_{t'}\}.$$

Note that $t' = t$ if $3 \notin T$, and $t' = t - 1$ otherwise.

Lemma 3.2.4. *Under the assumptions given above, we have* $t' \le 17$.

Proof. Note that $\langle a,b,c\rangle$ represents every integer of the form $24n+a+b+c$ over $\mathbb{Z}_3$. Let g be a positive integer satisfying Lemma 3.2.3 in the case when $p=q_1$ and $d=24$.

By Lemma 3.2.1, there is an integer h with $0\le h<(t'+1)2^{t'-2}$ such that $(24q_1^2h+24g+a+b+c, q_2\cdots q_{t'})=1$. If we let $k=q_1^2h+g$, then one may easily show that

$$24k+a+b\nrightarrow\langle a,b\rangle \tag{3.2.1}$$

and

$$24k+a+b+c\longrightarrow \mathrm{gen}(\langle a,b,c\rangle).$$

Since $\Delta(a,b,c)$ is regular, there is a vector $(x,y,z)\in\mathbb{Z}^3$ with $xyz\equiv 1 \pmod 2$ such that $ax^2+by^2+cz^2=24k+a+b+c$. From Equation (3.2.1), we have $z^2\ge 9$. So $a+b+9c\le 24k+a+b+c$ and we have $c\le 3k$. Now

$$q_1q_2\cdots q_{t'}\le abc\le c^3\le(3k)^3\le(3q_1^2(t'+1)2^{t'-2})^3.$$

Assume to the contrary that $t'\ge 18$. Then one may easily show that

$$r_8r_9\cdots r_{t'+1}>(3(t'+1)2^{t'-2})^3.$$

Since $q_i\ge r_{i+1}$ for any i, we have

$$(q_1\cdots q_6)q_7q_8\cdots q_{t'}>q_1^6\cdot r_8r_9\cdots r_{t'+1}>(3q_1^2(t'+1)2^{t'-2})^3,$$

which is a contradiction. Therefore we have $t'\le 17$. This completes the proof. □

If we are able to use Lemma 3.2.2 instead of Lemma 3.2.1, then we may have more effective upper bound of t' than the previous lemma.

Lemma 3.2.5. *Under the same notations given above, if* $0<t'-j<q_{j+1}$ *for some* j *such that* $1\le j\le t'-1$, *then we have*

$$q_1q_2\cdots q_{t'}<a(3q_1^2q_2\cdots q_j(t'-j+1))^2\le(3q_1^2q_2\cdots q_j(t'-j+1))^3.$$

Proof. Note that $\langle a,b,c\rangle$ represents every integer of the form $24n+a+b+c$ over $\mathbb{Z}_3$. Let g be a positive integer satisfying Lemma 3.2.3 in the case when

$p = q_1$ and $d = 24$. Let

$$g_j = \begin{cases} g & \text{if } j = 1, \\ g + \epsilon_1 q_1^2 & \text{if } j = 2, \\ g + \epsilon_1 q_1^2 + \epsilon_2 q_1^2 q_2 + \cdots + \epsilon_{j-1} q_1^2 q_2 q_3 \cdots q_{j-1} & \text{if } j \geq 3, \end{cases}$$

where for each i, ϵ_i is suitably chosen in $\{0, 1\}$ so that

$$24g_j + a + b + c \not\equiv 0 \pmod{q_2 \cdots q_j}$$

for any $j \geq 2$. Note that $g_1 = g < q_1^2$ and $g_j < q_1^2 q_2 \cdots q_j$ for any $j \geq 2$. Since $0 < t' - j < q_{j+1}$ by assumption, we apply Lemma 3.2.2 with odd primes $q_{j+1} < q_{j+2} < \cdots < q_{t'}$, $u = 24q_1^2 q_2 \cdots q_j$ and $v = 24g_j + a + b + c$ so that we may conclude that there is an integer s with $0 \leq s \leq t' - j$ such that

$$(24q_1^2 q_2 \cdots q_j s + 24g_j + a + b + c, q_{j+1} q_{j+2} \cdots q_{t'}) = 1.$$

Therefore, by a similar reasoning to Lemma 3.2.4, we have $c \leq 3q_1^2 q_2 \cdots q_j (t' - j + 1)$. The lemma follows directly from this. □

Lemma 3.2.6. *Under the assumptions given above, we have $t \leq 10$.*

Proof. By Lemma 3.2.4, we may assume that $t' \leq 17$. First, assume that $q_1 \geq 13$. Since $t' - 1 < 17 \leq q_2$, we may apply Lemma 3.2.5 so that

$$q_1 q_2 \cdots q_{t'} < (3q_1^2 t')^3.$$

From this, one may easily show that $t' \leq 8$.

Now, assume that $q_1 = 11$. Since $t' - 2 < 17 \leq q_3$, we may apply Lemma 3.2.5 so that we may conclude that

$$q_1 q_2 \cdots q_{t'} < \left(3q_1^2 q_2 (t' - 1)\right)^3.$$

Suppose that $t' \geq 11$. Since $r_8 = 23, r_9 = 29, r_{10} = 31, \ldots$, one may directly show that

$$11 \cdot r_8 r_9 \cdots r_{t'+3} > \left(3 \cdot 11^2 \cdot (t' - 1)\right)^3.$$

Since $q_i \geq r_{i+3}$ for any i, we have

$$q_1q_2\cdots q_{t'} > 11q_2^3r_8r_9\cdots r_{t'+3} > \left(3\cdot 11^2\cdot q_2\cdot (t'-1)\right)^3,$$

which is a contradiction. Therefore we have $t' \leq 10$. Now, since $t'-1 < 13 \leq q_2$, we deduce, similarly to the above, that

$$q_1q_2\cdots q_{t'} < (3q_1^2t')^3,$$

and thus $t' \leq 7$.

Assume that $q_1 = 7$. Since $t'-3 < 17 \leq q_4$ in this case, one may deduce that

$$q_1q_2\cdots q_{t'} < \left(3q_1^2q_2q_3(t'-2)\right)^3,$$

and thus we have $t' \leq 12$. Now, since $t'-2 < 13 \leq q_3$, we may have

$$q_1q_2\cdots q_{t'} < \left(3q_1^2q_2(t'-1)\right)^3,$$

and hence $t' \leq 9$. Since $t'-1 < 11 \leq q_2$,

$$q_1q_2\cdots q_{t'} < \left(3q_1^2t'\right)^3.$$

Therefore, we have $t' \leq 7$.

Finally, assume that $q_1 = 5$. Since $t'-4 < 17 \leq q_5$, we have

$$q_1q_2\cdots q_{t'} < \left(3q_1^2q_2q_3q_4(t'-3)\right)^3 \quad \text{and thus } t' \leq 14.$$

Now, since $t'-3 < 13 \leq q_4$, we have

$$q_1q_2\cdots q_{t'} < \left(3q_1^2q_2q_3(t'-2)\right)^3 \quad \text{and } t' \leq 12.$$

Then, since $t'-2 < 11 \leq q_3$, we have

$$q_1q_2\cdots q_{t'} < \left(3q_1^2q_2(t'-1)\right)^3, \quad \text{and finally we have } t' \leq 9.$$

The lemma follows directly from this. □

Recall that we are assuming that $\Delta(a,b,c)$ is stable. Hence for any odd prime p,

$$\langle 1,-1\rangle \longrightarrow \langle a,b,c\rangle \ \text{ over } \mathbb{Z}_p \quad \text{or} \quad \langle a,b,c\rangle \simeq \langle 1,-\Delta_p\rangle \perp \langle p\epsilon_p\rangle \ \text{ over } \mathbb{Z}_p,$$

for some $\epsilon_p \in \mathbb{Z}_p^\times$. In the former case, every element in $\mathbb{Z}_p$ is represented by $\langle a,b,c\rangle$ over $\mathbb{Z}_p$. In the latter case,

$$\begin{aligned}&\{\gamma \in \mathbb{Z}_p : \gamma \nrightarrow \langle a,b,c\rangle \ \text{ over } \ \mathbb{Z}_p\}\\ &= \left\{p^{2w-1}\delta_p : w \in \mathbb{N},\ \delta_p \in \mathbb{Z}_p^\times,\ \delta_p\epsilon_p \notin \left(\mathbb{Z}_p^\times\right)^2\right\}.\end{aligned}$$

Recall that r_j is the j-th odd prime. Let u be a positive integer not divisible by r_j and let v be an integer. Let $\eta_{r_j} \in \{1, \Delta_{r_j}\}$. For a positive integer i, we define

$$\begin{aligned}&\Psi_{u,v}(i,j;\eta_{r_j})\\ &= \left|\left\{un+v : 1 \le n \le i,\ un+v \nrightarrow \langle 1,-\Delta_{r_j}\rangle \perp \langle \eta_{r_j}\cdot r_j\rangle \ \text{ over } \ \mathbb{Z}_{r_j}\right\}\right|.\end{aligned}$$

We also define

$$\Psi_{u,v}(i,j) = \max\{\Psi_{u,v}(i,j;1), \Psi_{u,v}(i,j;\Delta_{r_j})\}.$$

Let $i = b_{e-1}b_{e-2}\ldots b_{0(r_j)}$ be the base-r_j representation of i, that is,

$$i = b_{e-1}r_j^{e-1} + b_{e-2}r_j^{e-2} + \cdots + b_0$$

with $0 \le b_\nu < r_j$ for $\nu = 1,2,\ldots,e-1$ and $b_{e-1} > 0$. We define

$$\epsilon_{i,j}(k) = \begin{cases} 0 & \text{if } i \equiv 0 \pmod{r_j^{2k-1}},\\ 1 & \text{if } i \not\equiv 0 \pmod{r_j^{2k-1}}.\end{cases}$$

We also define

$$\psi_{i,j}(k) = \begin{cases} \min\left(b_{2k-1} + \epsilon_{i,j}(k), \frac{r_j-1}{2}\right) & \text{if } k < \left[\frac{e+1}{2}\right], \\ \min\left(b_{2\delta-1} + \epsilon_{i,j}(\delta), \frac{r_j+1}{2}\right) & \text{if } e = 2\delta \text{ and } k = \delta, \\ 1 & \text{if } e = 2\delta - 1 \text{ and } k = \delta. \end{cases}$$

Lemma 3.2.7. *Under the notations and assumptions given above, we have*

$$\Psi_{u,v}(i,j) \le \sum_{k=1}^{\delta} \frac{r_j - 1}{2} \left[\frac{i}{r_j^{2k}}\right] + \psi_{i,j}(k).$$

Proof. Since both cases can be done in a similar manner, we only provide the proof of the case when $e = 2\delta$ for some positive integer δ. Without loss of generality, we may assume that $u = 1$. We have to show that the number of integers of the form $r_j^{2k-1}\eta_{r_j}$ $(r_j^{2k-1}\eta'_{r_j})$ in the set $\{1+v, 2+v, \dots, i+v\}$ is less than or equal to the right hand side, where η_{r_j} (η'_{r_j}) is a square (nonsquare, respectively) in $\mathbb{Z}_{r_j}^{\times}$.

For any integer k such that $1 \le k \le \delta$, let

$$i = r_j^{2k-1}(r_j\alpha_k + b_{2k-1}) + \beta_k, \quad (0 \le \beta_k \le r_j^{2k-1} - 1).$$

Let $r_j^{2k-1}(x+1)$ be the smallest integer greater than v that is divisible by r_j^{2k-1}. Then any integer in the set $\{r_j^{2k-1}(x+s) : 1 \le s \le r_j\alpha_k + b_{2k-1}\}$ is less than or equal to $i + v$. Note that there is at most one more integer other than these integers that is divisible by r_j^{2k-1}, and that is less than or equal to $i + v$. Note that such an integer exists only when $\epsilon_{i,j}(k) \ne 0$ (or $\beta_k \ne 0$). Furthermore, if such an integer exists, then it must be $r_j^{2k-1}(x + r_j\alpha_k + b_{2k-1} + 1)$. Note that there are exactly $\frac{r_j-1}{2}$ quadratic residues and $\frac{r_j-1}{2}$ quadratic non-residues in the consecutive r_j integers. Therefore there are exactly $\frac{r_j-1}{2}\alpha_k$ quadratic residues and $\frac{r_j-1}{2}\alpha_k$ quadratic non-residues in

$$\{r_j^{2k-1}(x+s) : 1 \le s \le r_j\alpha_k\}.$$

Note that $\alpha_k = \left[\frac{i}{r_j^{2k}}\right]$ for any $1 \le k \le \delta$. The remaining multiples of r_j^{2k-1}

are contained in

$$\{r_j^{2k-1}(x+r_j\alpha_k+1), r_j^{2k-1}(x+r_j\alpha_k+2), \dots, r_j^{2k-1}(x+r_j\alpha_k+b_{2k-1}+\epsilon_{i,j}(k))\}.$$

Among them, there are at most $\psi_{i,j}(k)$ quadratic residues, and at most $\psi_{i,j}(k)$ quadratic non-residues. Note that there is at most one multiple of $r_j^{2\delta+1}$ in $\{1+v, 2+v, \dots, i+v\}$ which is, if exists, contained in the set

$$\{r_j^{2\delta-1}(x+1), r_j^{2\delta-1}(x+2), \dots, r_j^{2\delta-1}(x+b_{2\delta-1}+\epsilon_{i,j}(\delta))\}.$$

Note that there are at most $\psi_{i,j}(\delta)$ quadratic residues or a multiple of r_j, and at most $\psi_{i,j}(\delta)$ quadratic non-residues or a multiple of r_j in the set $\{x+1, x+2, \dots, x+b_{2\delta-1}+\epsilon_{i,j}(\delta)\}$. The lemma follows from this. □

For the sake of brevity, we let

$$a_{ij} = \sum_{k=1}^{\delta} \frac{r_j-1}{2}\left[\frac{i}{r_j^{2k}}\right] + \psi_{i,j}(k)$$

for positive integers i and j.

Remark 3.2.8. *One may easily show that $a_{ij} \le \left\lceil \frac{i}{r_j} \right\rceil$ for any positive integers i and j, where $\lceil \cdot \rceil$ is the ceiling function. It is a little bit complicate to compute an upper bound of $\Psi_{u,v}(i,j)$ by using Lemma 3.2.7. Instead of that, one may easily show that*

$$\Psi_{u,v}(i,j) \le \frac{r_j+1}{2}\left\lceil \frac{i}{{r_j}^2} \right\rceil.$$

Recall that T is the set of all odd primes at which $\langle a,b,c\rangle$ is anisotropic, and $|T| = t \le 10$ by Lemma 3.2.6.

Lemma 3.2.9. *Let i be a positive integer. For any integer $s > t$, we define $b_{ij}(s) = max\left(a_{ij}, \left\lceil \frac{i}{r_s} \right\rceil\right)$ for $j = 1, 2, \dots, s-1$. Then we have*

$$|\{1 \le n \le i : 8n+a+b+c \xrightarrow{2} \langle a,b,c\rangle\}| \ge i - b_{i,1}(s) - b_{i,2}(s) - \cdots - b_{i,s-1}(s).$$

Proof. Let s be any integer greater than t and let $J = \{j \in \mathbb{N} : r_j \in T\}$. We also let $J_1 = \{j \in J : j \le s-1\}$, $J_2 = J - J_1$, and $J_3 = \{1, 2, \ldots, s-1\} - J_1$. Note that $|J_2| \le |J_3|$ and for any $j \in J_3$, $\left\lceil \frac{i}{r_s} \right\rceil \le b_{ij}(s)$ by assumption. From Remark 3.2.8, for any $j \in J_2$, we have $a_{ij} \le \left\lceil \frac{i}{r_j} \right\rceil \le \left\lceil \frac{i}{r_s} \right\rceil$. Thus we have

$$\begin{aligned}
\sum_{j\in J} a_{i,j} = \sum_{j_1\in J_1} a_{i,j_1} + \sum_{j_2\in J_2} a_{i,j_2} \quad &\le \sum_{j_1\in J_1} a_{i,j_1} + |J_2| \cdot \left\lceil \frac{i}{r_s} \right\rceil \\
&\le \sum_{j_1\in J_1} b_{i,j_1}(s) + \sum_{j_3\in J_3} b_{i,j_3}(s) \le \sum_{j=1}^{s-1} b_{i,j}(s).
\end{aligned}$$

Since $\Delta(a, b, c)$ is stable regular, we have

$$\begin{aligned}
&|\{1 \le n \le i : 8n + a + b + c \xrightarrow{2} \langle a, b, c\rangle\}| \\
&\qquad = |\{1 \le n \le i : 8n + a + b + c \longrightarrow \mathrm{gen}(\langle a, b, c\rangle)\}| \\
&\qquad \ge i - \sum_{j\in J} a_{i,j} \ge i - \sum_{j=1}^{s-1} b_{i,j}(s).
\end{aligned}$$

This completes the proof. □

Remark 3.2.10. *In the remaining of this section, we need the exact values of a_{ij}'s for some integers i and j. We provide some of these values in Table 3.1 below.*

Lemma 3.2.11. *Under the assumptions given above, we have $t \le 7$.*

Proof. By Lemma 3.2.9 with $i = 25$ and $s = 11$, one may easily show that $8n_1 + a + b + c \xrightarrow{2} \langle a, b, c\rangle$ for some $1 \le n_1 \le 25$. From our assumption of $a \le b \le c$, we have $9a + b + c \le 8n_1 + a + b + c$, and thus we have $a \le 25$. To prove the lemma, we will use Lemma 3.2.5 repeatedly.

First, assume that $q_1 \ge 7$. Since $t' - 1 < 11 \le q_2$, we may apply Lemma 3.2.5 so that

$$q_1 q_2 \cdots q_{t'} < 25 \left(3q_1^2 t'\right)^2.$$

Table 3.1: Some values of a_{ij}

$i \backslash j$	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1	1	1	1
3	1	1	1	1	1	1	1	1	1	1	1
4	2	1	1	1	1	1	1	1	1	1	1
5	2	1	1	1	1	1	1	1	1	1	1
6	2	2	1	1	1	1	1	1	1	1	1
7	2	2	1	1	1	1	1	1	1	1	1
9	2	2	2	1	1	1	1	1	1	1	1
19	4	3	3	2	2	2	1	1	1	1	1
20	4	3	3	2	2	2	2	1	1	1	1
25	4	3	4	3	2	2	2	2	1	1	1
26	4	4	4	3	2	2	2	2	1	1	1
29	6	4	4	3	3	2	2	2	1	1	1
32	6	5	4	3	3	2	2	2	2	2	1
35	6	5	4	4	3	3	2	2	2	2	1
41	7	5	4	4	4	3	3	2	2	2	2
47	8	5	4	5	4	3	3	3	2	2	2
49	8	5	4	5	4	3	3	3	2	2	2
83	13	9	7	6	7	5	5	4	3	3	3
314	41	29	22	16	13	11	10	12	11	11	9

This is possible only when $t' \leq 6$. Now, assume that $q_1 = 5$. Since $t' - 2 < 11 \leq q_3$, one may deduce that

$$q_1 q_2 \cdots q_{t'} < 25 \left(3q_1^2 q_2 (t'-1)\right)^2$$

and thus $t' \leq 7$. Finally, since $t' - 1 < 7 \leq q_2$, we have

$$q_1 q_2 \cdots q_{t'} < 25(3q_1^2 t')^2$$

and thus $t' \leq 6$. This completes the proof. $\square$

Lemma 3.2.12. *For any stable regular ternary triangular form $\Delta(a,b,c)$ with $0 < a \leq b \leq c$, we have $a = 1$ or 2.*

Proof. For any positive integer n, we define $s_n = 8n + a + b + c$. Since

$$\begin{aligned}&\{s_n : s_n < 25a + b + c,\ s_n \xrightarrow{2} \langle a,b,c\rangle\}\\ &\subset \{9a+b+c,\ a+9b+c,\ a+b+9c,\ 9a+9b+c,\ 9a+b+9c,\ a+9b+9c\},\end{aligned}$$

we have

$$|\{1 \leq n \leq 3a-1 : s_n \xrightarrow{2} \langle a,b,c\rangle\}| \leq 6.$$

On the other hand, by Lemma 3.2.9 with $i = 32$ and $s = 8$, one may check that

$$|\{1 \leq n \leq 32 : s_n \xrightarrow{2} \langle a,b,c\rangle\}| \geq 7.$$

By comparing these two inequalities, we have $a \leq 10$.

Now, we will show that if $3 \leq a \leq 10$, then c is bounded. For each positive odd integer k, we let

$$U_k(a,b,c) = \left\{1 \leq n < \frac{k^2-1}{8}a : s_n \xrightarrow{2} \langle a,b,c\rangle\right\},$$
$$V_k(a,b,c) = \left\{1 \leq n < \frac{k^2-1}{8}a : s_n - c \xrightarrow{2} \langle a,b\rangle\right\},$$

and we also let $u_k = |U_k|$ and $v_k = |V_k|$. Note that V_k does not depend on c. For each integer a with $3 \leq a \leq 10$, we will choose an integer k suitably so that $v_k < u_k$. Note that if this inequality holds, then $a+b+9c \leq 8(\frac{k^2-1}{8}a - 1) + a + b + c$ and therefore, we have

$$c \leq \frac{k^2-1}{8}a - 1.$$

In fact, we choose

$$(a,k) = (10,5), (9,5), (8,7), (7,7), (6,7), (5,9), (4,13) \text{ and } (3,29).$$

Now, by using Lemma 3.2.9 with $i = \dfrac{k^2-1}{8}a - 1$ and $s = 8$, one may easily compute the lower bound of u_k:

(a,k)	(10,5)	(9,5)	(8,7)	(7,7)	(6,7)	(5,9)	(4,13)	(3,29)
u_k	≥ 5	≥ 5	≥ 15	≥ 11	≥ 8	≥ 17	≥ 31	≥ 164

To compute an upper bound of v_k, note that

$$V_k = \{\alpha^2 a + \beta^2 b : a + b < \alpha^2 a + \beta^2 b < k^2 a + b,\ \alpha\beta \equiv 1 \pmod 2\}.$$

Hence one may easily show that

$$v_5 \leq 3,\ v_7 \leq 7,\ v_9 \leq 14,\ v_{13} \leq 30 \quad \text{and} \quad v_{29} \leq 161.$$

By comparing the lower bound for u_k and the upper bound for v_k, we have an upper bound of c for each $a = 3, 4, \cdots, 10$, as follows:

a	10	9	8	7	6	5	4	3
c	≤ 29	≤ 26	≤ 47	≤ 41	≤ 35	≤ 49	≤ 83	≤ 314

Now, by using MAPLE program, one may check that there is no stable regular ternary triangular form $\Delta(a,b,c)$ for $3 \leq a \leq 10$. Therefore, we have $a \leq 2$. □

Lemma 3.2.13. *Under the assumptions given above, we have $t \leq 5$.*

Proof. By the proof of Lemma 3.2.11, we have $t' \leq 6$. First, assume that $a = 2$. By Lemma 3.2.9 with $i = 29$ and $s = 8$, one may easily show, by using Table 1, that

$$|\{1 \leq n \leq 29 : s_n \xrightarrow{2} \langle 2, b, c\rangle\}| \geq 5.$$

On the other hand,

$$\begin{aligned}&|\{1 \le n \le 29 : 8n + 2 + b + c = 2\alpha^2 + b + c \text{ for some odd integer } \alpha\}| \\ &\qquad = |\{\alpha \ge 3 : 2\alpha^2 + b + c \le 8 \cdot 29 + 2 + b + c,\ \alpha \equiv 1 \pmod 2\}| = 4.\end{aligned}$$

Thus we have $2 + 9b + c \le 8 \cdot 29 + 2 + b + c$ and $b \le 29$. Let g be a positive integer satisfying Lemma 3.2.3 in the case when $p = q_1$ and $d = 24$. Note that

$$24q_1^2 n + 24g + 2 + b + c \longrightarrow \langle 2, b, c\rangle \quad \text{over} \quad \mathbb{Z}_3$$

for any integer n. For any positive integer r, define

$$h(r) = 24q_1^2(r-1) + 24g + 2 + b + c.$$

Clearly $h(r)$ is represented by $\langle 2, b, c\rangle$ over $\mathbb{Z}_q$ for any $q \in \{2, 3, q_1\}$. Note that

$$t' - 1 \le 5, \quad b_{7,2}(6) = 2 \quad \text{and} \quad b_{7,j}(6) = 1 \ \text{ for any } \ j \ge 3,$$

where $b_{ij}(s)$ is an integer defined in Lemma 3.2.9. From this, similarly with the proof of Lemma 3.2.9, one may easily show that there exists a positive integer r with $1 \le r \le 7$ such that $h(r)$ is represented by $\langle 2, b, c\rangle$ over $\mathbb{Z}_{q_i}$ for any $i = 2, 3, \ldots, t'$. Therefore, we have

$$h(r) = 24q_1^2(r-1) + 24g + 2 + b + c \longrightarrow \text{gen}(\langle 2, b, c\rangle).$$

Furthermore, since $\Delta(2, b, c)$ is regular, we have

$$h(r) = 24q_1^2(r-1) + 24g + 2 + b + c \xrightarrow{\ 2\ } \langle 2, b, c\rangle.$$

From our choices of g and r, we have $h(r) - c \nrightarrow \langle 2, b\rangle$. Thus, $2 + b + 9c \le h(r)$, which implies that $c \le 21q_1^2$. Therefore we have

$$q_1 q_2 \cdots q_{t'} \le abc \le 58c \le 1218q_1^2.$$

This implies that $t' \le 4$.

Now, assume that $a = 1$. By Lemma 3.2.9 with $i = 35$ and $s = 8$, one may easily show that

$$|\{1 \le n \le 35 : s_n \xrightarrow{2} \langle 1, b, c\rangle\}| \ge 8.$$

On the other hand,

$$\begin{aligned}&|\{1 \le n \le 35 : 8n + 1 + b + c = \alpha^2 + b + c \ \text{ for some odd integer } \ \alpha\}| \\ &\qquad = |\{\alpha \ge 3 : \alpha^2 + b + c \le 8 \cdot 35 + 1 + b + c,\ \alpha \equiv 1 \pmod 2\}| = 7.\end{aligned}$$

Thus we have $1 + 9b + c \le 8 \cdot 35 + 1 + b + c$ and $b \le 35$. Similarly to the case when $a = 2$, one may deduce that $c \le 21q_1^2$. Therefore, we have

$$q_1 q_2 \cdots q_{t'} \le abc \le 35c \le 735q_1^2,$$

which implies that $t' \le 4$. This completes the proof. □

Lemma 3.2.14. *For any stable regular ternary triangular form* $\Delta(a, b, c)$ *with* $0 < a \le b \le c$*, we have* $a + b \le 21$.

Proof. Note that $a = 1$ or 2 by Lemma 3.2.12. First, assume that $a = 2$. By Lemma 3.2.9 with $i = 19$ and $s = 6$, one may easily show that

$$|\{1 \le n \le 19 : 8n + 2 + b + c \xrightarrow{2} \langle 2, b, c\rangle\}| \ge 5.$$

On the other hand,

$$\begin{aligned}&|\{1 \le n \le 19 : 8n + 2 + b + c = 2\alpha^2 + b + c \ \text{ for some odd integer } \ \alpha\}| \\ &\qquad = |\{\alpha \ge 3 : 2\alpha^2 + b + c \le 8 \cdot 19 + 2 + b + c,\ \alpha \equiv 1 \pmod 2\}| = 3.\end{aligned}$$

Thus we have $2 + 9b + c \le 8 \cdot 19 + 2 + b + c$, and $b \le 19$. Now, assume that $a = 1$. By Lemma 3.2.9 with $i = 20$ and $s = 6$, one may check that

$$|\{1 \le n \le 20 : 8n + 1 + b + c \xrightarrow{2} \langle 1, b, c\rangle\}| \ge 6.$$

On the other hand,

$$\begin{aligned}&|\{1 \le n \le 20 : 8n+1+b+c = \alpha^2+b+c \ \text{ for some odd integer } \ \alpha\}| \\ &\qquad = |\{\alpha \ge 3 : \alpha^2+b+c \le 8\cdot 20+1+b+c,\ \alpha \equiv 1 \pmod 2\}| = 5.\end{aligned}$$

Thus we have $1+9b+c \le 8\cdot 20+1+b+c$, and $b \le 20$. □

Now, we are ready to classify all stable regular ternary triangular forms. The following lemma which is a direct consequence of Lemma 2.2.1(iii) is very useful to prove the regularity.

Lemma 3.2.15. *Let m be a positive integer congruent to 4 modulo 8. Then*

$$r_{(1,1)}(m, \langle 1,3\rangle) = \frac{2}{3} r(m, \langle 1,3\rangle).$$

Theorem 3.2.16. *There are exactly* 17 *stable regular ternary triangular forms.*

$$\begin{array}{llll}
\Delta_1 = \Delta(1,1,1), & \Delta_2 = \Delta(1,1,2), & \Delta_3 = \Delta(1,1,3), & \Delta_4 = \Delta(1,1,4), \\
\Delta_5 = \Delta(1,2,2), & \Delta_6 = \Delta(1,1,5), & \Delta_7 = \Delta(1,1,6), & \Delta_8 = \Delta(1,2,3), \\
\Delta_9 = \Delta(1,2,4), & \Delta_{10} = \Delta(1,2,5), & \Delta_{11} = \Delta(1,1,12), & \Delta_{12} = \Delta(1,3,4), \\
\Delta_{13} = \Delta(2,2,3), & \Delta_{14} = \Delta(1,2,10), & \Delta_{15} = \Delta(1,1,21), & \Delta_{16} = \Delta(1,4,6), \\
\Delta_{17} = \Delta(1,3,10). & & &
\end{array}$$

Proof. By Lemmas 3.2.12, 3.2.13 and 3.2.14, we have

$$t \le 5, \quad 1 \le a \le 2, \quad \text{and} \quad a+b \le 21.$$

First, we want to find an upper bound for c for each possible pair (a,b). Since all the other cases can be done in a similar manner, we only consider 3 representative cases here.

(i) $(a,b) = (2,2)$. Let $E_1 = \{4\cdot 3, 4\cdot 7, 4\cdot 11, 4\cdot 19, 4\cdot 23, 4\cdot 31\}$. Suppose that $c \ge 16$. For any $e_1 \in E_1$, e_1 is not represented by $\langle 2,2\rangle$. Furthermore, since $e_1 + c < 4+9c$ by assumption, $e_1 + c \overset{2}{\nrightarrow} \langle 2,2,c\rangle$. Since $\Delta(2,2,c)$ is stable regular, there is an odd prime divisor p of e_1+c such that $\langle 2,2,c\rangle$ is anisotropic over $\mathbb{Z}_p$. Therefore, p divides c and also divides e_1. Furthermore, since $|E_1| = 6$, there are at least six such odd primes. This is a contradiction to the fact that $t \le 5$. Thus, we have $c \le 15$ if $(a, b) = (2, 2)$.

(ii) $(a, b) = (2, 3)$. Let $E_2 = \{69, 117, 141, 213, 285, 333\}$. Suppose that $c \ge 42$. Since we are assuming that $\Delta(2, 3, c)$ is 3-stable, c is not divisible by 3. Any element of E_2 is of the from $8n+2+3$ for some positive integer n, and the elements of E_2 share no odd prime divisors other than 3. Let $e_2 \in E_2$. From the assumption that $c \ge 42$, one may easily check that $e_2+c \overset{2}{\nrightarrow} \langle 2, 3, c\rangle$. Since $\Delta(2, 3, c)$ is stable regular, there is an odd prime p dividing $e_2 + c$ and $\langle 2, 3, c\rangle$ is anisotropic over $\mathbb{Z}_p$. Hence p is greater than 3 and divides e_2. Thus there are at least six such odd primes. This is a contradiction, and we have $c \le 41$.

(iii) $(a, b) = (2, 6)$. Since $\Delta(2, 6, c)$ is 3-stable, c is not a multiple of 3. Note that $48 + c = 8 \cdot 5 + 2 + 6 + c \overset{2}{\nrightarrow} \langle 2, 6, c\rangle$. Thus there is an odd prime $p > 3$ dividing $48 + c$ and $\langle 2, 6, c\rangle$ is anisotropic over $\mathbb{Z}_p$. Therefore, 48 is divisible by p, which is a contradiction. Therefore, the pair $(a, b) = (2, 6)$ is impossible.

All the other cases can be done in a similar manner to one of the above three cases, and one may obtain an upper bound for c in each case. After that, with the help of MAPLE program, one may show that there are 17 candidates of stable regular ternary triangular forms given above.

For each $i = 1, 2, \cdots, 17$, we write $\Delta_i = \Delta(a_i, b_i, c_i)$ and $L_i = \langle a_i, b_i, c_i\rangle$. For any $i \in U = \{1, 2, 4, 5, 6, 8, 9\}$, it is well known that Δ_i is universal (see [13, p.23]). Hence we may assume that $i \notin U$. Let n_i be any positive integer such that

$$\widetilde{n_i} := 8n_i + a_i + b_i + c_i \longrightarrow \mathrm{gen}(L_i).$$

Note that L_i has class number 1 for any $1 \le i \le 17$ and thus $\widetilde{n_i} \longrightarrow L_i$.

For $i \in \{11, 13, 14, 15, 16\}$, one may easily check that

$$R(\widetilde{n_i}, L_i) = R_{(1,1,1)}(\widetilde{n_i}, L_i),$$

that is, if $a_i x^2 + b_i y^2 + c_i z^2 = \widetilde{n_i}$, then $xyz \equiv 1 \pmod 2$. Assume that $i \in \{7, 10\}$. Since the class number of L_i is 1 and it primitively represents $\widetilde{n_i}$ over $\mathbb{Z}_2$, there is a vector $(x, y, z) \in R(\widetilde{n_i}, L_i)$ with $(x, y, z, 2) = 1$. One may easily check that $(x, y, z, 2) = 1$ implies $xyz \equiv 1 \pmod 2$ in this case. If $i = 12$, then one may easily show that

$$r(\widetilde{n_i}, L_i) = r_{(0,0,0)}(\widetilde{n_i}, L_i) + r_{(0,0,1)}(\widetilde{n_i}, L_i) + r_{(1,1,1)}(\widetilde{n_i}, L_i).$$

Similarly to the previous case, the existence of a vector $(x, y, z) \in R(\widetilde{n_i}, L_i)$ with $(x, y, z, 2) = 1$ implies that

$$r_{(0,0,1)}(\widetilde{n_i}, L_i) + r_{(1,1,1)}(\widetilde{n_i}, L_i) > 0.$$

By Lemma 3.2.15,

$$\begin{aligned} r_{(1,1,1)}(8n_i + 8, x^2 + 3y^2 + 4z^2) &= \sum_{z:\text{odd}} r_{(1,1)}(8n_i + 8 - 4z^2, x^2 + 3y^2) \\ &= \sum_{z:\text{odd}} \frac{2}{3} r(8n_i + 8 - 4z^2, x^2 + 3y^2) \\ &= \frac{2}{3} r_{(0,0,1)}(\widetilde{n_i}, L_i) + \frac{2}{3} r_{(1,1,1)}(\widetilde{n_i}, L_i). \end{aligned}$$

Therefore we have $r_{(1,1,1)}(\widetilde{n_i}, x^2 + 3y^2 + 4z^2) = 2r_{(0,0,1)}(\widetilde{n_i}, x^2 + 3y^2 + 4z^2) > 0$. If $i = 3$, then one may easily check that

$$r(\widetilde{n_i}, L_i) = 2r_{(1,0,0)}(\widetilde{n_i}, L_i) + r_{(1,1,1)}(\widetilde{n_i}, L_i).$$

By Lemma 2.2.1(iii), we have

$$\begin{aligned} r_{(1,1,1)}(8n_i + 5, x^2 + y^2 + 3z^2) &= \sum_{x:\text{odd}} r_{(1,1)}(8n_i + 5 - x^2, y^2 + 3z^2) \\ &= \sum_{x:\text{odd}} 2r_{(0,0)}(8n_i + 5 - x^2, y^2 + 3z^2) \\ &= 2r_{(1,0,0)}(8n_i + 5, x^2 + y^2 + 3z^2). \end{aligned}$$

Thus we have $r_{(1,1,1)}(\widetilde{n_i}, x^2 + y^2 + 3z^2) = \frac{1}{2} r(\widetilde{n_i}, x^2 + y^2 + 3z^2) > 0$. Finally, assume that $i = 17$. Note that if $x^2 + 3y^2 + 10z^2 = 8n + 14$, then $x \equiv y \pmod 2$

and $z \equiv 1 \pmod 2$. By Lemma 3.2.15 again, we have

$$\begin{aligned} r_{(1,1,1)}(8n_i+14, x^2+3y^2+10z^2) &= \sum_{z\in\mathbb{Z}} r_{(1,1)}(8n_i+14-10z^2, x^2+3y^2) \\ &= \sum_{z\in\mathbb{Z}} \frac{2}{3} r(8n_i+14-10z^2, x^2+3y^2) \\ &= \frac{2}{3} r(8n_i+14, x^2+3y^2+10z^2). \end{aligned}$$

This completes the proof. □

3.3 Classifications of regular ternary triangular forms

In this section, we prove that there are exactly 49 regular ternary triangular forms. Let $\Delta(a',b',c')$ be a regular ternary triangular form and let $\Delta(a,b,c)$ be the stable regular ternary triangular form obtained from it by taking λ-transformations, if necessary, repeatedly. Here, we are not assuming that $a \le b \le c$. It might happen that there is an odd prime l dividing $a'b'c'$ such that $(abc,l)=1$. We call such a prime l a *missing prime.* Note that $\lambda_p \circ \lambda_q = \lambda_q \circ \lambda_p$ for any odd primes p and q. Thus if l is a missing prime, then one of the followings holds:

(i) $\Delta(a,l^2b,l^2c)$ is regular.

(ii) $\Delta(a,b,l^2c)$ is regular and $\left(\dfrac{-ab}{l}\right) = -1$.

Lemma 3.3.1. *There is no missing prime l greater than* 7*.*

Proof. Let l be a missing prime. Then there is a stable regular ternary triangular form $\Delta(a,b,c)$ such that $(abc,l)=1$, and (i) or (ii) given above holds.

Assume that the case (i) holds, that is, $\Delta(a,l^2b,l^2c)$ is regular. We let

$$s_n = 8n + a + l^2b + l^2c \quad \text{for} \quad n = 1,2,3,\cdots.$$

First, we prove that $l \le 131$. Assume to the contrary that $l \ge 137$. One may easily check that if

$$\alpha^2 a + \beta^2 l^2 b + \gamma^2 l^2 c \le 8l + a + l^2 b + l^2 c$$

with odd integers α, β and γ, then $\beta^2 = \gamma^2 = 1$. Thus we have

$$\left|\left\{1 \le n \le l : s_n \xrightarrow{2} \langle a, l^2 b, l^2 c\rangle\right\}\right| \le \left[\sqrt{\frac{2l}{a} + \frac{1}{4}} - \frac{1}{2}\right] \le \left[\sqrt{2l + \frac{1}{4}}\right].$$

On the other hand, by Theorem 3.2.16, the set of odd primes at which $\langle a, b, c\rangle$ is anisotropic is

$$\emptyset,\ \{3\},\ \{5\},\ \{7\},\ \{3,5\}\ \text{ or }\ \{3,7\}.$$

From Remark 3.2.8, we have

$$|\{1 \le n \le l : s_n \nrightarrow \langle a, l^2 b, l^2 c\rangle \text{ over } \mathbb{Z}_p\}| \le \begin{cases} 2\left\lceil \dfrac{l}{9} \right\rceil & \text{if } p = 3, \\ 3\left\lceil \dfrac{l}{25} \right\rceil & \text{if } p = 5, \\ 4\left\lceil \dfrac{l}{49} \right\rceil & \text{if } p = 7, \\ \dfrac{l+1}{2} & \text{if } p = l. \end{cases}$$

From the assumption that $l \ge 137$, we have $\dfrac{3}{25}l + 3 \ge \dfrac{4}{49}l + 4$. Since

$$l - \left(\frac{2}{9}l + 2 + \frac{3}{25}l + 3 + \frac{l+1}{2}\right) = \frac{71}{450}l - \frac{11}{2},$$

we must have

$$\left|\left\{1 \le n \le l : s_n \xrightarrow{2} \langle a, l^2 b, l^2 c\rangle\right\}\right| \ge \left\lceil \frac{71}{450}l - \frac{11}{2} \right\rceil.$$

However, one may directly show that if $l \geq 137$, then $\left\lceil \frac{71}{450}l - \frac{11}{2} \right\rceil >$ $\left[\sqrt{2l+\frac{1}{4}}\right]$. This is a contradiction and hence we have $l \leq 131$. Now, by a direct calculation with the help of MAPLE, one may check that for any prime $11 \leq q \leq 131$ and any stable regular ternary triangular form $\Delta(a,b,c)$, all of the triangular forms $\Delta(a,q^2b,q^2c)$ are not regular.

Now, assume that $\Delta(a,b,l^2c)$ $(a \leq b)$ is regular with $\left(\frac{-ab}{l}\right) = -1$. By Theorem 3.2.16, (a,b) is one of the following pairs:

$$\begin{array}{lllllllll} (1,1), & (1,2), & (1,3), & (1,4), & (2,2), & (1,5), & (1,6), & (2,3), & (2,4), \\ (1,10), & (2,5), & (1,12), & (3,4), & (2,10), & (1,21), & (4,6), & (3,10). & \end{array}$$

First, suppose that $l \geq 29$. Since all the other cases can be done in a similar manner, we only consider the cases when $(a,b) = (1,1)$ or $(1,5)$. Assume that $(a,b) = (1,1)$. Since

$$418 + l^2c = 8 \cdot 52 + 1 + 1 + l^2c \longrightarrow \mathrm{gen}(\langle 1,1,l^2c\rangle),$$

and $\Delta(a,b,l^2c)$ is regular, there is a vector $(x,y,z) \in z^3$ with $xyz \equiv 1 \pmod 2$ such that $x^2 + y^2 + l^2cz^2 = 418 + l^2c$. From the assumption that $l \geq 29$, we have $z^2 = 1$. This is a contradiction, for 418 is not a sum of two integer squares. Next, assume that $(a,b) = (1,5)$. Note that

$$110 + l^2c = 8 \cdot 13 + 1 + 5 + l^2c \longrightarrow \mathrm{gen}(\langle 1,5,l^2c\rangle).$$

Since we are assuming that $\Delta(1,5,l^2c)$ is regular, there is a vector $(x_1,y_1,z_1) \in \mathbb{Z}^3$ with $x_1y_1z_1 \equiv 1 \pmod 2$ such that $x_1^2 + 5y_1^2 + l^2cz_1^2 = 110 + l^2c$. Since $l \geq 29$, we have $z_1^2 = 1$. This is a contradiction, for 110 is not represented by $\langle 1,5\rangle$. Therefore, we have $l \leq 23$. Now, by a direct calculation with the help of MAPLE, one may check that for any prime $11 \leq l \leq 23$ and any stable regular ternary triangular form $\Delta(a,b,c)$, all of the forms $\Delta(a,b,l^2c)$ are not regular. This completes the proof. $\square$

Remark 3.3.2. *From Theorem 3.2.16 and Lemma 3.3.1, one may easily deduce that any prime divisor of the discriminant of a regular ternary triangu-*

lar form is less than or equal to 7.

Let $\Delta(a', b', c')$ be a regular ternary triangular form. Then there are nonnegative integers e_3, e_5 and e_7 such that

$$\lambda_3^{e_3}(\lambda_5^{e_5}(\lambda_7^{e_7}(\Delta(a', b', c')))) = \Delta(a, b, c),$$

is stable regular. Hence, to find all regular ternary triangular forms, it suffices to find all regular ternary triangular forms in the inverse image of the λ_p-transformation of each regular triangular form for each $p \in \{3, 5, 7\}$. Note that any triangular form in the inverse image $\lambda_p^{-1}(\Delta(a, p^r b, p^s c))$, for $abc \not\equiv 0 \pmod{p}$ and $0 \le r \le s$, is given in Table 3.2.

Table 3.2: Inverse image of λ_p-transformations

Cases	Triangular forms in $\lambda_p^{-1}(\Delta(a, p^r b, p^s c))$
$r = s = 0$	$\Delta(p^2 a, b, c), \Delta(a, p^2 b, c), \Delta(a, b, p^2 c),$ $\Delta(p^2 a, p^2 b, c), \Delta(p^2 a, b, p^2 c), \Delta(a, p^2 b, p^2 c)$
$r = 0, s = 1$	$\Delta(pa, pb, c), \Delta(a, p^2 b, p^3 c), \Delta(p^2 a, b, p^3 c), \Delta(a, b, p^3 c)$
$r = 0, s \ge 2$	$\Delta(a, p^2 b, p^{s+2} c), \Delta(p^2 a, b, p^{s+2} c), \Delta(a, b, p^{s+2} c)$
$r = s = 1$	$\Delta(pa, b, p^2 c), \Delta(pa, p^2 b, c), \Delta(pa, b, c), \Delta(a, p^3 b, p^3 c)$
$r = 1, s \ge 2$	$\Delta(pa, b, p^{s+1} c), \Delta(a, p^3 b, p^{s+2} c)$
$r \ge 2$	$\Delta(a, p^{r+2} b, p^{s+2})$

First, we find all regular triangular forms in the inverse images of stable regular ternary triangular forms via λ_p-transformation for each $p \in \{3, 5, 7\}$, and then we repeat this process again until any inverse image does not contain a regular triangular form. As a sample, ternary triangular forms lying over $\Delta(1, 1, 1)$ are given in Figure 3.1. In that figure, if the triangular form is not regular, then the smallest positive integer which is represented locally, but not globally by the triangular form is given in parentheses.

Finally, one may have a list of 49 candidates for the regular ternary triangular forms including 17 stable regular forms, which is given in Table 3.4. The regularities of 32 forms except 17 stable regular forms will be proved

$$
\begin{tikzcd}
\begin{array}{c}\Delta(1,1,81)(19)\\ \Delta(1,9,81)(19)\end{array} \arrow[d, "\lambda_3"] & & \Delta(1,81,81)(19) \arrow[d, "\lambda_3"] & & & & \\
\Delta(1,1,9) \arrow[drrr, "\lambda_3"'] & & \Delta(1,9,9) \arrow[dr, "\lambda_3"'] & & \begin{array}{c}\Delta(1,1,25)(5)\\ \Delta(1,25,25)(5)\end{array} \arrow[dl, "\lambda_5"] & & \begin{array}{c}\Delta(1,1,49)(8)\\ \Delta(1,49,49)(8)\end{array} \arrow[dlll, "\lambda_7"] \\
& & & \Delta(1,1,1) & & &
\end{tikzcd}
$$

Figure 3.1: Triangular forms lying over $\Delta(1,1,1)$ via λ-transformations

here. Before doing that, we need some lemmas.

Let p be an odd prime and let k be a positive integer relatively prime to p. Assume that p is represented by the binary quadratic form $x^2 + ky^2$. In 1928, B. W. Jones proved in his unpublished thesis that if the Diophantine equation $x^2 + ky^2 = N(N > 0)$ has an integral solution, then it also has an integral solution x, y with $(x, y, p) = 1$. The following lemma follows immediately from this.

Lemma 3.3.3. *Let N be a positive integer. If $x^2 + 2y^2 = N$ for some $(x, y) \in \mathbb{Z}^2$, then there is a vector $(\tilde{x}, \tilde{y}) \in \mathbb{Z}^2$ such that*

$$\tilde{x} \not\equiv \tilde{y} \pmod 3,\ \tilde{x} \equiv x \pmod 4,\ \tilde{y} \equiv y \pmod 2 \quad \textit{and} \quad \tilde{x}^2 + 2\tilde{y}^2 = N.$$

We also need the following lemma which appeared in the middle of the proof of [25, Theorem 3.1].

Lemma 3.3.4. *Let $S \in M_3(\mathbb{Z})$ be a positive-definite symmetric matrix and let $T \in M_3(\mathbb{Q})$ such that ${}^tTST = S$. Let $(u, v, w) \in \mathbb{Z}^3$ and define*

$$\begin{pmatrix} u_n \\ v_n \\ w_n \end{pmatrix} = T^n \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \quad n = 1, 2, 3, \cdots.$$

Assume that

(i) T *has an infinite order.*

(ii) $(u_n, v_n, w_n) \in \mathbb{Z}^3$ *for any* n.

Then $(u, v, w) \in \ker(T - \det(T)I)$ *and* $\dim_{\mathbb{R}} \ker((T - \det(T)I)) = 1$.

In the following 5 consecutive propositions, we prove the regularities of 5 candidates, all of whose corresponding quadratic forms are not regular(see [19]).

Proposition 3.3.5. *The ternary triangular form* $\Delta(1, 4, 9)$ *is regular.*

Proof. Let $L = \langle 1, 4, 9 \rangle$ be a ternary quadratic form and let $\ell = 8n + 14$ be an integer such that $\ell \longrightarrow \text{gen}(L)$. One may easily check that $R(\ell, L) = R_{(1,1,1)}(\ell, L)$. Thus it suffices to show that $\ell \longrightarrow L$. Since

$$\text{gen}(L) = \{L, K = \langle 1, 1, 36 \rangle\},$$

we may assume that $\ell \longrightarrow K$.

First, assume that $\ell \equiv 0, 1 \pmod 3$. Since $\ell \longrightarrow K$, there is a vector $(x, y, z) \in \mathbb{Z}^3$ such that $x^2 + y^2 + 36z^2 = \ell$. We have $x \equiv 0 \pmod 3$ or $y \equiv 0 \pmod 3$ and thus $\ell \longrightarrow \langle 1, 9, 36 \rangle \longrightarrow L$.

Now, assume that $\ell \equiv 2 \pmod 3$. We assert that there is a vector $(x_1, y_1, z_1) \in R(\ell, K)$ such that $x_1 \not\equiv \pm y_1 \pmod 9$ or $z_1 \not\equiv 0 \pmod 3$. Assume to the contrary that there is no such vector. Then, we may assume that there is a vector $(u, v, w) \in R(\ell, K)$ such that $u \equiv v \pmod 9$ and $w \equiv 0 \pmod 3$. Let

$$T = \frac{1}{9}\begin{pmatrix} 3 & 6 & 36 \\ 6 & 3 & -36 \\ -1 & 1 & -3 \end{pmatrix}.$$

Note that

$$M_K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 36 \end{pmatrix} \quad \text{and} \quad {}^tTM_KT = M_K.$$

If we let

$$\begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} = T \begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

then one may check that $(u_1, v_1, w_1) \in \mathbb{Z}^3$ and thus $(u_1, v_1, w_1) \in R(\ell, K)$. Thus $u_1 \equiv \pm v_1 \pmod 9$ and $w_1 \equiv 0 \pmod 3$ by assumption. Since

$$u_1 - v_1 = \frac{-u+v}{3} + 8w \equiv 0 \pmod 3,$$

we have $u_1 \equiv v_1 \pmod 9$. From this, one may easily check that T satisfies all conditions given in Lemma 3.3.4 with $S = M_K$, and thus we have $(u, v, w) \in \ker(T - I)$. Since $\ker(T - I) = \langle(1, 1, 0)\rangle$, we have $(u, v, w) = k(1, 1, 0)$ for some integer k and $u^2 + v^2 + 36w^2 = 2k^2$. This is a contradiction to the fact that $\ell \equiv 6 \pmod 8$, and we may conclude that there is a vector $(x_2, y_2, z_2) \in R(\ell, K)$ such that

$$x_2 \not\equiv \pm y_2 \pmod 9 \quad \text{or} \quad z_2 \not\equiv 0 \pmod 3.$$

By changing signs of x_2, y_2, z_2 and by interchanging the role of x_2 and y_2, if necessary, we may assume that there is a vector $(x_3, y_3, z_3) \in R(\ell, K)$ such that $2x_3 + y_3 + 12z_3 \equiv 0 \pmod 9$. If we let

$$(x_4, y_4, z_4) = \left(\frac{x_3 + 2y_3 - 12z_3}{3}, \frac{x_3 - y_3 - 3z_3}{3}, \frac{2x_3 + y_3 + 12z_3}{9}\right),$$

then one may easily show that $(x_4, y_4, z_4) \in R(\ell, L)$. This completes the proof. $\square$

Proposition 3.3.6. *The ternary triangular form $\Delta(1, 3, 27)$ is regular.*

Proof. Let $L = \langle 1, 3, 27\rangle$ be a ternary quadratic form and let $\ell = 8n + 31$ be an integer such that $\ell \longrightarrow \text{gen}(L)$. Note that

$$\text{gen}(L) = \left\{ L, K = \langle 3\rangle \perp \begin{pmatrix} 4 & 1 \\ 1 & 7 \end{pmatrix} \right\}.$$

By [25, Theorem 2.3] one may show that any integer congruent to 7 modulo 8 that is represented by K is also represented by L. Therefore, ℓ is represented by L. Note that if $x^2+3y^2+27z^2=\ell$, then

$$(x^2,3y^2,27z^2)\equiv(1,3,3),(0,4,3),(4,0,3),(0,3,4)\ \text{ or }\ (4,3,0)\pmod 8.$$

Therefore, if there is a vector $(x,y,z)\in R(\ell,L)$ with $x\equiv y\pmod 2$, then we are done by Lemma 3.2.15. Thus we may assume that for any $(x,y,z)\in R(\ell,L)$,

$$y\equiv 1\pmod 2,\ x\equiv z\equiv 0\pmod 2\ \text{ and }\ x\not\equiv z\pmod 4.$$

Suppose that $xy\not\equiv 0\pmod 3$ for any $(x,y,z)\in R(\ell,L)$. Let $(u,v,w)\in R(\ell,L)$ with $u\equiv v\pmod 3$. For a rational isometry

$$T=\frac{1}{12}\begin{pmatrix}-3&18&-27\\6&0&-18\\1&2&9\end{pmatrix},$$

of M_L, we apply Lemma 3.3.4. Then we have $(u,v,w)\in\ker(T+I)$. Since $\ker(T+I)=\langle(2,-1,0)\rangle$, we have $(u,v,w)=k(2,-1,0)$ for some integer k. One may easily check that $|k|>1$ and $(k,6)=1$. Hence there is a prime $q\geq 5$ such that $k=qs$ and $s\in\mathbb{Z}$. Then

$$\ell=u^2+3v^2+27w^2=7q^2s^2. \tag{3.3.1}$$

On the other hand,

$$r_{(1,1,1)}(\ell,L)=\frac{2}{3}r\left(\ell,(y-2x)^2+3y^2+27z^2\right)=\frac{2}{3}r\left(\ell,\langle 27\rangle\perp\begin{pmatrix}4&2\\2&4\end{pmatrix}\right).$$

If we let $M_1 = \langle 27\rangle \perp \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$, then

$$\begin{aligned} \operatorname{gen}(M_1) &= \left\{ M_1, M_2 = \begin{pmatrix} 7 & 1 & 1 \\ 1 & 7 & 1 \\ 1 & 1 & 7 \end{pmatrix}, M_3 = \langle 3\rangle \perp \begin{pmatrix} 4 & 2 \\ 2 & 28 \end{pmatrix} \right\}, \\ \operatorname{spn}(M_1) &= \{M_1, M_2\}. \end{aligned}$$

Note that $7 \longrightarrow M_2$. By [4, Proposition 1], we have $7q^2 \longrightarrow M_1$ and thus $\ell = 7q^2s^2 \longrightarrow M_1$. Thus $r_{(1,1,1)}(\ell, L) > 0$ and we are done with this case.

Now, suppose that there is a vector $(x_1, y_1, z_1) \in R(\ell, L)$ such that $x_1y_1 \equiv 0 \pmod 3$. We define

$$(x_2, y_2, z_2) = \begin{cases} \left(\dfrac{x_1 + 9z_1}{2}, y_1, \dfrac{-x_1 + 3z_1}{6} \right) & \text{if } x_1 \equiv 0 \pmod 3, \\ \left(\dfrac{x_1 + 9z_1}{2}, \dfrac{-x_1 + 3z_1}{2}, \dfrac{y_1}{3} \right) & \text{otherwise.} \end{cases}$$

Then, one may easily check that $(x_2, y_2, z_2) \in R_{(1,1,1)}(\ell, L)$. □

Proposition 3.3.7. *The ternary triangular form* $\Delta(1, 6, 27)$ *is regular.*

Proof. Let $L = \langle 1, 6, 27\rangle$ be a ternary quadratic form and let $\ell = 8n + 34$ be an integer such that $\ell \longrightarrow \operatorname{gen}(L)$. Note that

$$\operatorname{gen}(L) = \left\{ L, K = \langle 6\rangle \perp \begin{pmatrix} 4 & 1 \\ 1 & 7 \end{pmatrix} \right\}.$$

Since $\lambda_2(L) \simeq \langle 3\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 14 \end{pmatrix} \simeq \lambda_2(K)$, we have

$$r(\ell, L) = r\left(\frac{\ell}{2}, \langle 3\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 14 \end{pmatrix} \right) = r(\ell, M)$$

and thus $\ell \longrightarrow L$. If $(x, y, z) \in R(\ell, L)$, then

$$(x^2, 6y^2, 27z^2) \equiv (0, 6, 4), (4, 6, 0) \text{ or } (1, 6, 3) \pmod 8.$$

Thus we may assume that for any $(x, y, z) \in R(\ell, L)$,

$$y \equiv 1 \pmod 2, \quad x \equiv z \equiv 0 \pmod 2, \quad \text{and} \quad x \not\equiv z \pmod 4.$$

First, suppose that there is a vector $(x_1, y_1, z_1) \in R(\ell, L)$ with $x_1 \equiv 0 \pmod 3$. If we let

$$(x_2, y_2, z_2) = \left(\frac{x_1 + 9z_1}{2}, y_1, \frac{-x_1 + 3z_1}{6}\right),$$

then one may easily check that $(x_2, y_2, z_2) \in R_{(1,1,1)}(\ell, L)$. Hence we may further assume that for any $(x, y, z) \in R(\ell, L)$, $x \not\equiv 0 \pmod 3$.

Now, suppose that there is a vector $(x_3, y_3, z_3) \in R(\ell, L)$ with $y_3 \equiv 0 \pmod 3$. Let $y_3 = 3y_3'$. Then we have $x_3^2 + 27(2y_3'^2 + z_3^2) = \ell$. Since $y_3' \equiv 1 \pmod 2$, we have $2y_3'^2 + z_3^2 \neq 0$. By Lemma 3.3.3, there is a vector $(x_4, y_4, z_4) \in \mathbb{Z}^3$ with $y_4 \not\equiv z_4 \pmod 3$ such that $x_4^2 + 27(2y_4^2 + z_4^2) = \ell$. Thus $(x_4, 3y_4, z_4) \in R(\ell, L)$ such that $y_4 \not\equiv 0 \pmod 3$ or $z_4 \not\equiv 0 \pmod 3$. By changing signs of x_4, y_4, z_4, if necessary, we may assume that $x_4 \equiv y_4 + z_4 \pmod 3$. If we let

$$(x_5, y_5, z_5) = \left(\frac{x_4 + 12y_4 + 3z_4}{2}, \frac{-3y_4 + 6z_4}{3}, \frac{-3x_4 + 12y_4 + 3z_4}{18}\right),$$

then one may easily check that $(x_5, y_5, z_5) \in R_{(1,1,1)}(\ell, L)$. Therefore, we further assume that for any $(x, y, z) \in R(\ell, L)$, $xy \not\equiv 0 \pmod 3$.

Suppose that there is a vector $(x_6, y_6, z_6) \in R(\ell, L)$ such that $y_6 \not\equiv \pm 4x_6 \pmod 9$ or $z_6 \not\equiv 0 \pmod 3$. Then one may check that by changing signs of x_6, y_6, z_6, if necessary, we may assume that

$$x_6 + y_6 - 3z_6 \equiv 0 \pmod 9 \text{ or } x_6 - 4y_6 - 3z_6 \equiv 0 \pmod 9.$$

If $x_6 + y_6 - 3z_6 \equiv 0 \pmod 9$, then we define

$$(x_7, y_7, z_7) = \left(\frac{x_6 + 9z_6}{2}, \frac{-x_6 - y_6 + 3z_6}{3}, \frac{-x_6 + 8y_6 + 3z_6}{18}\right).$$

If $x_6 - 4y_6 - 3z_6 \equiv 0 \pmod 9$, then we define

$$(x_7, y_7, z_7) = \left(\frac{x_6 + 4y_6 + 3z_6}{2}, \frac{-x_6 + y_6 + 3z_6}{3}, \frac{x_6 - 4y_6 + 15z_6}{18} \right).$$

Then one may easily check that $(x_7, y_7, z_7) \in R_{(1,1,1)}(\ell, L)$ in each case. Now, we further assume that for any $(x, y, z) \in R(\ell, L)$,

$$y \equiv \pm 4x \pmod 9 \quad \text{and} \quad z \equiv 0 \pmod 3). \tag{3.3.2}$$

Suppose that there is a vector $(x_8, y_8, z_8) \in R(\ell, L)$ such that $z_8 \not\equiv 0 \pmod 9$. By changing signs of y_8 and z_8, if necessary, we may assume that $y_8 \equiv 4x_8 \pmod 9$ and $\dfrac{x_8 - y_8}{3} + z_8 \not\equiv \pm 4x_8 \pmod 9$. If we let

$$(x_9, y_9, z_9) = \left(2y_8 + 3z_8, \frac{x_8 - y_8 + 3z_8}{3}, \frac{-x_8 - 2y_8 + 6z_8}{9} \right),$$

then $(x_9, y_9, z_9) \in R(\ell, L)$ and $y_9 \not\equiv \pm 4x_9 \pmod 9$. This contradicts to our assumption (3.3.2). Therefore, we further assume that for any $(x, y, z) \in R(\ell, L)$,

$$y \equiv \pm 4x \pmod 9 \quad \text{and} \quad z \equiv 0 \pmod 9).$$

Take a vector $(u, v, w) \in R(\ell, L)$ with $u \equiv v \pmod 3$ so that $u + 2v + 6w \equiv 0 \pmod 9$. If we let

$$T = \frac{1}{9} \begin{pmatrix} 0 & 18 & -27 \\ 3 & -3 & -9 \\ 1 & 2 & 6 \end{pmatrix},$$

then one may easily check that

$$M_L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 27 \end{pmatrix} \quad \text{and} \quad {}^tTM_LT = M_L.$$

If we let
$$\begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} = T \begin{pmatrix} u \\ v \\ w \end{pmatrix},$$
then clearly, $(u_1, v_1, w_1) \in \mathbb{Z}^3$, and thus $(u_1, v_1, w_1) \in R(\ell, L)$. Note that $u_1 - v_1 \equiv 0 \pmod 3$. From this, one may show that T satisfies all conditions given in Lemma 3.3.4 with $S = M_L$, and thus we have $(u, v, w) \in \ker(T + I)$. Since $\ker(T + I) = \langle (2, -1, 0) \rangle$, we have $(u, v, w) = k(2, -1, 0)$ for some integer k with $|k| > 1$ and $(k, 6) = 1$. Thus there is a prime divisor $q \geq 5$ of k. Now $\ell = 10q^2s^2$ for some odd integer s. Note that
$$r_{(1,1,1)}(\ell, L) = 2r\left(\ell, (z - 4x)^2 + 6y^2 + 27z^2\right) = 2r\left(\ell, \langle 6 \rangle \perp \begin{pmatrix} 16 & 4 \\ 4 & 28 \end{pmatrix}\right).$$

Let $M_1 = \langle 6 \rangle \perp \begin{pmatrix} 16 & 4 \\ 4 & 28 \end{pmatrix}$. Then
$$\text{gen}(M_1) = \text{spn}(M_1) = \{M_1, M_2 = \langle 4, 6, 108 \rangle\}.$$

Note that $10 \longrightarrow M_2$. By [4, Proposition 1], we have $r(10q^2s^2, M_1) > 0$, and this completes the proof. $\square$

Proposition 3.3.8. *The ternary triangular form* $\Delta(1, 9, 18)$ *is regular.*

Proof. Let $L = \langle 1, 9, 18 \rangle$ be a ternary quadratic form and let $\ell = 8n + 28$ be an integer such that $\ell \longrightarrow \text{gen}(L)$. Note that
$$\text{gen}(L) = \left\{ L, K = \begin{pmatrix} 4 & 1 & -1 \\ 1 & 7 & 2 \\ -1 & 2 & 7 \end{pmatrix} \right\}.$$

Since $\lambda_2(L) \simeq \langle 9 \rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix} \simeq \lambda_2(K)$, we have $\ell \longrightarrow L$. Let $(x, y, z) \in R(\ell, L)$. We may assume that $x \equiv y \equiv z \equiv 0 \pmod 2$. Then $x \not\equiv y \pmod 4$.

First, assume that $x \not\equiv 0 \pmod 3$ and $y^2 + 2z^2 > 0$. Then by Lemma 3.3.3, there is a vector $(y_1, z_1) \in \mathbb{Z}^2$ with $y_1 \not\equiv z_1 \pmod 3$, $y_1 \equiv y \pmod 4)$

and $z_1 \equiv z \pmod 2$ such that $y_1^2 + 2z_1^2 = y^2 + 2z^2$. So $x^2 + 9y_1^2 + 18z_1^2 = \ell$. By replacing x by $-x$, if necessary, we may assume $x + y_1 - z_1 \equiv 0 \pmod 3$. If we let

$$(x_2, y_2, z_2) = \left(\frac{3x + 9y_1 + 18z_1}{6}, \frac{-x + 5y_1 - 2z_1}{6}, \frac{-x - y_1 + 4z_1}{6}\right),$$

then one may easily check that $(x_2, y_2, z_2) \in R_{(1,1,1)}(\ell, L)$.

Now, assume that $x \not\equiv 0 \pmod 3$ and $y = z = 0$. Note that

$$r_{(1,1,1)}(\ell, L) = 2r\left(\ell, (v - 4u)^2 + 9v^2 + 18w^2\right) = 2r\left(\frac{\ell}{2}, \langle 9\rangle \perp \begin{pmatrix} 5 & 2 \\ 2 & 8 \end{pmatrix}\right).$$

If we let $M_1 = \langle 9\rangle \perp \begin{pmatrix} 5 & 2 \\ 2 & 8 \end{pmatrix}$, then

$$\mathrm{gen}(M_1) = \mathrm{spn}(M_1) = \left\{M_1, M_2 = \langle 36\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}\right\}.$$

Then by [4, Proposition 1], $2p^2 \longrightarrow M_1$ for any prime $p \geq 5$. Note that

$$\frac{\ell}{2} = 2\left(\frac{x}{2}\right)^2, \quad \left(\frac{x}{2}, 6\right) = 1 \quad \text{and} \quad \frac{x}{2} > 1.$$

So there is a prime divisor q of $\frac{x}{2}$ with $q \geq 5$. Thus we have $r\left(\frac{\ell}{2}, M_1\right) > 0$.

Finally, assume that $x \equiv 0 \pmod 3$. If we let

$$(x_3, y_3, z_3) = \left(\frac{3x + 9y + 18z}{6}, \frac{-x - 3y + 6z}{6}, \frac{-x + 3y}{6}\right),$$

then one may easily check that $(x_3, y_3, z_3) \in R_{(1,1,1)}(\ell, L)$. □

Proposition 3.3.9. *The ternary triangular form $\Delta(1, 1, 18)$ is regular.*

Proof. Let $L = \langle 1, 1, 18\rangle$ be a ternary quadratic form and let $\ell = 8n + 20$ be an integer such that $\ell \longrightarrow \text{gen}(L)$. Note that

$$\text{gen}(L) = \left\{ L, K = \langle 2 \rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix} \right\}.$$

Since $\lambda_2(L) \simeq \langle 1,1,9 \rangle \simeq \lambda_2(K)$, we have $\ell \longrightarrow L$. Let $(x,y,z) \in R(\ell, L)$.

First, assume that $\ell \equiv 0 \pmod 3$. Then $x \equiv y \equiv 0 \pmod 3$ and thus $\ell \equiv 0 \pmod 9$. So

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 + 2z^2 = \frac{\ell}{9}.$$

Note that $\frac{\ell}{9} \geq 4$ and $\frac{\ell}{9} \equiv 4 \pmod 8$. Since the triangular form $\Delta(1,1,2)$ is universal, there is a vector $(x_1, y_1, z_1) \in R_{(1,1,1)}\left(\frac{\ell}{9}, \langle 1,1,2 \rangle\right)$ and thus $(3x_1, 3y_1, z_1) \in R_{(1,1,1)}(\ell, L)$.

Now, assume $\ell \equiv 1 \pmod 3$. Note that $xy \equiv 0 \pmod 3$. Without loss of generality, we may assume that $y \equiv 0 \pmod 3$. Then

$$\ell = x^2 + 9\left(\frac{y}{3}\right)^2 + 18z^2.$$

Note that $\ell \geq 28$, $\ell \equiv 4 \pmod 8$. Since $\Delta(1,9,18)$ is regular by Proposition 3.3.8, there is a vector $(x_2, y_2, z_2) \in R_{(1,1,1)}(\ell, \langle 1,9,18 \rangle)$ and thus $(x_2, 3y_2, z_2) \in R_{(1,1,1)}(\ell, \langle 1,1,18 \rangle)$.

Finally, assume that $\ell \equiv 2 \pmod 3$. Since $x^2 + y^2 + 18z^2 \equiv 4 \pmod 8$, we may assume that $x \equiv 0 \pmod 4$, $y \equiv 2 \pmod 4$ and $z \equiv 0 \pmod 2$. Since $xy \not\equiv 0 \pmod 3$, we may further assume that $x \equiv y \pmod 3$. If we let

$$(x_3, y_3, z_3) = \left(\frac{x+y}{2} + 3z, -\frac{x+y}{2} + 3z, \frac{-x+y}{6}\right),$$

then one may easily check that $(x_3, y_3, z_3) \in R_{(1,1,1)}(\ell, L)$. $\square$

Theorem 3.3.10. *There are exactly* 49 *regular ternary triangular forms, which are listed in Table 4.*

Proof. For $1 \leq i \leq 49$, we write $\Delta_i = \Delta(a_i, b_i, c_i)$. Let $L_i = \langle a_i, b_i, c_i \rangle$ be a ternary quadratic form and let $\ell_i(n) = 8n + a_i + b_i + c_i$ be any integer such

Table 3.3: Regular ternary triangular forms

$\Delta_1 = \Delta(1,1,1)$,	$\Delta_2 = \Delta(1,1,2)$,	$\Delta_3 = \Delta(1,1,3)$,
$\Delta_4 = \Delta(1,1,4)$,	$\Delta_5 = \Delta(1,1,5)$,	$\Delta_6 = \Delta(1,1,6)$,
$\Delta_7 = \Delta(1,2,2)$,	$\Delta_8 = \Delta(1,2,3)$,	$\Delta_9 = \Delta(1,2,4)$,
$\Delta_{10} = \Delta(1,1,9)$,	$\Delta_{11} = \Delta(1,3,3)$,	$\Delta_{12} = \Delta(1,2,5)$,
$\Delta_{13} = \Delta(1,1,12)$,	$\Delta_{14} = \Delta(1,3,4)$,	$\Delta_{15} = \Delta(2,2,3)$,
$\Delta_{16} = \Delta(1,1,18)$,	$\Delta_{17} = \Delta(1,3,6)$,	$\Delta_{18} = \Delta(2,3,3)$,
$\Delta_{19} = \Delta(1,2,10)$,	$\Delta_{20} = \Delta(1,1,21)$,	$\Delta_{21} = \Delta(1,4,6)$,
$\Delta_{22} = \Delta(1,5,5)$,	$\Delta_{23} = \Delta(1,3,9)$,	$\Delta_{24} = \Delta(1,3,10)$,
$\Delta_{25} = \Delta(1,3,12)$,	$\Delta_{26} = \Delta(1,4,9)$,	$\Delta_{27} = \Delta(1,6,6)$,
$\Delta_{28} = \Delta(3,3,4)$,	$\Delta_{29} = \Delta(1,5,10)$,	$\Delta_{30} = \Delta(1,3,18)$,
$\Delta_{31} = \Delta(1,6,9)$,	$\Delta_{32} = \Delta(2,3,9)$,	$\Delta_{33} = \Delta(3,3,7)$,
$\Delta_{34} = \Delta(2,3,12)$,	$\Delta_{35} = \Delta(1,3,27)$,	$\Delta_{36} = \Delta(1,9,9)$,
$\Delta_{37} = \Delta(1,3,30)$,	$\Delta_{38} = \Delta(2,5,10)$,	$\Delta_{39} = \Delta(1,9,12)$,
$\Delta_{40} = \Delta(2,3,18)$,	$\Delta_{41} = \Delta(1,5,25)$,	$\Delta_{42} = \Delta(3,7,7)$,
$\Delta_{43} = \Delta(2,5,15)$,	$\Delta_{44} = \Delta(1,6,27)$,	$\Delta_{45} = \Delta(1,9,18)$,
$\Delta_{46} = \Delta(1,9,21)$,	$\Delta_{47} = \Delta(1,21,21)$,	$\Delta_{48} = \Delta(5,6,15)$,
$\Delta_{49} = \Delta(3,7,63)$.		

that $\ell_i(n) \longrightarrow \text{gen}(L_i)$. In Theorem 3.2.16 and Propositions 3.3.5∼3.3.9, we have already proved the regularity of each Δ_i when

$$i \in \{k : 1 \le k \le 9, 12 \le k \le 16, \text{ or } k = 19, 20, 21, 24, 26, 35, 44, 45\}.$$

Hence we may assume that i is not contained in the above set. Note that for any integer i which is not contained in $\{16, 26, 35, 44, 45\}$, which we alreay considered in Propositions 3.3.5∼3.3.9, the corresponding quadratic form L_i has class number 1 and thus $\ell_i(n) \longrightarrow L_i$. If $i \in \{10, 36, 39, 40, 41, 49\}$, then one may easily show that $R(\ell_i(n), L_i) = R_{(1,1,1)}(\ell_i(n), L_i)$. Hence $\ell_i(n) \xrightarrow{2} L_i$ in this case.

Now, we consider the case when $i = 30$. Note that if $x^2 + 3y^2 + 18z^2 = 8n + 22$, then we have $z \equiv 1 \pmod 2$ and $x \equiv y \pmod 2$. By Lemma 3.2.15, we have

$$\begin{aligned} r_{(1,1,1)}(8n+22, \langle 1,3,18\rangle) &= \sum_{z\in\mathbb{Z}} r_{(1,1)}(8n+22-18z^2, \langle 1,3\rangle) \\ &= \frac{2}{3} r(8n+22, \langle 1,3,18\rangle). \end{aligned}$$

Since the proof of the case when $i = 48$ is quite similar to this, we omit the proof.

Assume that $i = 31$. Since the quadratic form $\langle 1,6,9\rangle$ has class number 1 and it primitively represents $8n + 16$ over $\mathbb{Z}_2$, there is a vector

$$(x, y, z) \in R(8n + 16, \langle 1,6,9\rangle), \quad (x, y, z, 2) = 1.$$

Since $x^2 + 6y^2 + 9z^2 \equiv 0 \pmod 8$, we have $xyz \equiv 1 \pmod 2$.

For the remaining i, that is,

$$i \in \{11, 17, 18, 22, 23, 25, 27, 28, 29, 32, 33, 34, 37, 38, 42, 43, 46, 47\},$$

one may check that $\Delta(a_i, b_i, c_i)$ can be obtained from a ternary triangular form whose regularity is already proved by taking λ_p-transformations several times for some $p \in \{3, 5, 7\}$. Furthermore, one may easily check that the regularity is preserved during taking the λ_p-transformation. This completes the proof. □

Chapter 4

The number of representations of ternary triangular forms

4.1 The number of representations of ternary triangular forms

Let a, b and c be positive integers such that $(a, b, c) = 1$. Throughout this section, we assume, without loss of generality, that a is odd. We show that the number $t(n, \langle a, b, c\rangle)$ is equal to the number of representations of a subform of the ternary diagonal quadratic form $ax^2 + by^2 + cz^2$, if $a + b + c$ is not divisible by 8, or a difference of the numbers of representations of two ternary quadratic forms otherwise.

Let $f(x, y, z) = ax^2 + by^2 + cz^2$ be a ternary diagonal quadratic form. Recall that

$$t(n, \langle a, b, c\rangle) = |\{(x, y, z) \in \mathbb{Z}^3 : f(x, y, z) = 8n+a+b+c,\ xyz \equiv 1 \pmod 2\}|.$$

Lemma 4.1.1. *Assume that $a + b + c$ is odd. For any positive integer n, we have*

$$t(n, \langle a, b, c\rangle) = r(8n + a + b + c, f(x, x - 2y, x - 2z)).$$

In particular, if $a \equiv b \equiv c \pmod 4$*, then we have*

$$t(n, \langle a, b, c\rangle) = r(8n + a + b + c, f(x, y, z)).$$

Proof. Let $g(x, y, z) = f(x, x-2y, x-2z)$. Define a map $\phi : T(n, \langle a, b, c\rangle) \to R(n, g)$ by $\phi(x, y, z) = (x, \frac{x-y}{2}, \frac{x-z}{2})$. Then one may easily show that it is a bijective map.

Now, assume that $a \equiv b \equiv c \pmod 4$. If $ax^2 + by^2 + cz^2 = 8n + a + b + c$ for some integers x, y and z, then one may easily show that x, y and z are all odd. The lemma follows directly from this. □

Lemma 4.1.2. *Assume that* $S = a + b + c$*, both* a *and* b *are odd and* c *is even. Then, for any positive integer* n*, we have*

$$t(n, \langle a, b, c\rangle) = \begin{cases} r(8n + S, f(x, y, z)) & \text{if } S \equiv 2 \pmod 4 \text{ and } c \equiv 4 \pmod 8, \\ r(8n + S, f(x, y, y-2z)) & \text{if } S \equiv 2 \pmod 4 \text{ and } c \not\equiv 4 \pmod 8, \\ 2r(8n + S, f(x, x-4y, z)) & \text{if } S \equiv 4 \pmod 8 \text{ and } c \equiv 2 \pmod 4, \\ 2r(8n + S, f(x, x-4y, x-2z)) & \text{if } S \equiv 4 \pmod 8 \text{ and } c \equiv 0 \pmod 4, \end{cases}$$

and if $S \equiv 0 \pmod 8$*, then*

$$t(n, \langle a, b, c\rangle) = r(8n + S, f(x, x-2y, x-2z)) - r\left(2n + \frac{S}{4}, f(x, y, z)\right).$$

Proof. Since the proof is quite similar to each other, we only provide the proof of the fourth case, that is, the case when $S \equiv 4 \pmod 8$ and $c \equiv 0 \pmod 4$. Let $g(x, y, z) = f(x, x-4y, x-2z)$. We define a map

$$\psi : \{(x, y, z) \in R_{(1,1,1)}(8n + S, f),\ x \equiv y \pmod 4\} \to R(8n + S, g) \ \text{ by } \ \psi(x, y, z) = \left(x, \frac{x-y}{4}, \frac{x-z}{2}\right).$$

From the assumption, it is well defined. Conversely, assume that $g(x, y, z) =$

$8n+S$ for some $(x,y,z)\in\mathbb{Z}^3$. Since

$$\begin{aligned} f(x,x-4y,x-2z) &= ax^2+b(x-4y)^2+c(x-2z)^2 \\ &\equiv ax^2+bx^2+cx^2 \equiv Sx^2 \equiv S \pmod 8 \end{aligned}$$

and $S\equiv 4 \pmod 8$, the integer x is odd. Therefore, the map $(x,y,z)\to(x,x-4y,x-2z)$ is an inverse map of ψ. The lemma follows from this and the fact that

$$t(n,\langle a,b,c\rangle) = 2\left|\left\{(x,y,z)\in R_{(1,1,1)}(8n+S,f) : x\equiv y \pmod 4\right\}\right|.$$

This completes the proof. $\square$

4.2 Triangular forms and diagonal quadratic forms

In this section, we generalize some conjectures given by Sun in [31] on the relations between $t(n,\langle a,b,c\rangle)$ and the numbers of representations of integers by some ternary quadratic forms, and prove these generalized conjectures.

Lemma 4.2.1. *Let a,b $(a<b)$ be positive odd integers such that* $\gcd(a,b)=1$ *and* $a+b\equiv 0 \pmod 8$. *Then*

$$r_{(1,1)}(m,\langle a,b\rangle) = r_{(1,1)}(4m,\langle a,b\rangle) \tag{4.2.1}$$

for any integer m divisible by 8 if and only if $(a,b)\in\{(3,5),(1,7),(1,15)\}$.

Proof. Assume that Equation (4.2.1) holds for any integer m divisible by 8. Let $a+b=2^uk$ for some integer $u\geq 3$ and an odd integer k. Note that $1\leq a<2^{u-1}k$.

First, we assume $u\geq 5$. Since

$$a\cdot 1^2+(2^uk-a)\cdot 1^2 = 4\cdot 2^{u-2}k \quad \text{and} \quad 2^{u-2}k\equiv 0 \pmod 8,$$

there exist odd integers x and y satisfying $ax^2+(2^uk-a)y^2=2^{u-2}k$, which is a contradiction.

Next, assume that $u = 4$. Since

$$a \cdot 7^2 + (16k - a) \cdot 1^2 = 4(4k + 12a) \quad \text{and} \quad 4k + 12a \equiv 0 \pmod 8,$$

there exist two odd integers x_1, y_1 such that $ax_1^2 + (16k - a)y_1^2 = 4k + 12a$. Thus, $4k+12a \geq 16k$ and hence $k \leq a$. Now, since $a \cdot 1^2 + (16k-a) \cdot 1^2 = 16k$, there are two positive odd integers x_2, y_2 with $ax_2^2 + (16k - a)y_2^2 = 64k$. Since $16k - a > 8k$ by assumption, we have $y_2^2 = 1$. Furthermore, since $ax_2^2 = a + 48k \leq 49a$, $(x_2, a) = (3, 6k), (5, 2k)$ or $(7, k)$. Since a is odd, we have $(a, b) = (1, 15)$ in this case.

Finally, we assume that $u = 3$. Since $a \cdot 1^2 + (8k - a) \cdot 1^2 = 8k$, there are positive odd integers x_3, y_3 such that $ax_3^2 + (8k - a)y_3^2 = 32k$. Hence we have

$$y_3^2 = 1 \quad \text{and} \quad ax_3^2 = a + 24k. \tag{4.2.2}$$

Note that if $x_3 = 3$, then $(a, b) = (3, 5)$ and if $x_3 = 5$, then $(a, b) = (1, 7)$. Assume that $x_3 \geq 7$, that is, $2a \leq k$. Since $a \cdot 3^2 + (8k - a) \cdot 1^2 = 8k + 8a$, there are two odd integers x_4, y_4 such that $ax_4^2 + (8k - a)y_4^2 = 32k + 32a$. If $y_4^2 \geq 9$, then $a + 72k - 9a \leq 32k + 32a$, which is a contradiction to the assumption that $2a \leq k$. Hence we have

$$y_4^2 = 1 \quad \text{and} \quad ax_4^2 = 33a + 24k. \tag{4.2.3}$$

Now, by Equations (4.2.2) and (4.2.3), we have $x_4^2 - x_3^2 = 32$. Therefore, $x_3^2 = 49$, $x_4^2 = 81$, and $k = 2a$. which is a contradiction to the assumption that k is odd.

To prove the converse, we define three maps

$$\chi_1 : \widetilde{R}_{(1,1)}(m, 3x^2 + 5y^2) \to \widetilde{R}_{(1,1)}(4m, 3x^2 + 5y^2)$$
$$\text{by} \quad \chi_1(x, y) = \left(\frac{x - 5y}{2}, \frac{3x + y}{2} \right),$$

$$\chi_2 : \widetilde{R}_{(1,1)}(m, x^2 + 7y^2) \to \widetilde{R}_{(1,1)}(4m, x^2 + 7y^2)$$
$$\text{by} \quad \chi_2(x, y) = \left(\frac{3x - 7y}{2}, \frac{x + 3y}{2} \right),$$

and

$$\chi_3 : \widetilde{R}_{(1,1)}(m, x^2+15y^2) \to \widetilde{R}_{(1,1)}(4m, x^2+15y^2) \quad \text{by} \quad \chi_3(x,y) = \left(\frac{x+15y}{2}, \frac{-x+y}{2}\right).$$

One may easily show that the above three maps are all bijective. $\square$

Theorem 4.2.2. *Let a, b, c be positive integers such that $(a,b,c) \neq (1,1,1)$ and $\gcd(a,b,c) = 1$. Assume that two of three fractions $\frac{b}{a}, \frac{c}{b}, \frac{c}{a}$ are contained in $\left\{1, \frac{5}{3}, 7, 15\right\}$. Then, for any positive integer n, we have*

$$2t(n, \langle a,b,c\rangle) = r(4(8n+a+b+c), \langle a,b,c\rangle) - r(8n+a+b+c, \langle a,b,c\rangle).$$

Proof. Note that all of a, b and c are odd. Furthermore, from the assumption, one may easily show that

$$-a \equiv b \equiv c \pmod 8, \quad a \equiv -b \equiv c \pmod 8 \quad \text{or} \quad a \equiv b \equiv -c \pmod 8.$$

By switching the roles of a, b and c if necessary, we may assume $a \equiv b \equiv -c \pmod 8$. Then we have

$$\left(\frac{a}{(a,c)}, \frac{c}{(a,c)}\right), \left(\frac{b}{(b,c)}, \frac{c}{(b,c)}\right) \in \{(3,5), (5,3), (1,7), (7,1), (1,15), (15,1)\}.$$

Let

$$f = f(x,y,z) = ax^2 + by^2 + cz^2 \quad \text{and} \quad N = 8n + a + b + c.$$

One may easily show that if $f(x,y,z) = 4N$, then

$$\left(ax^2, by^2, cz^2\right) \equiv \begin{array}{l} (0,0,4), (0,4,0), (a,4,c), (4,0,0), (4,b,c), \\ \text{or} \ \ (4,4,4) \pmod 8. \end{array}$$

Let

$$A = \{(x,y,z) \in R(4N, f) : y \equiv 2 \pmod 4, xz \equiv 1 \pmod 2\},$$
$$B = \{(x,y,z) \in R(4N, f) : x \equiv 2 \pmod 4, yz \equiv 1 \pmod 2\}.$$

Note that

$$r(4N,f)-r(N,f)=|A|+|B|.$$

Thus it is sufficient to show $t(N,\langle a,b,c\rangle)=|A|$ and $t(N,\langle a,b,c\rangle)=|B|$. To show the first equality, we apply Lemma 4.2.1 to show that

$$r_{(1,1,1)}(N,f)=\sum_{y\in\mathbb{Z}} r_{(1,1)}(N-by^2,ax^2+cz^2) \tag{4.2.4}$$

$$=\sum_{y\in\mathbb{Z}} r_{(1,1)}(4(N-by^2),ax^2+cz^2)=|A|. \tag{4.2.5}$$

The proof of $t(N,\langle a,b,c\rangle)=|B|$ is quite similar to this. This completes the proof. □

Remark 4.2.3. *All triples (a,b,c) satisfying the assumption of Theorem 4.2.2 are listed in Table 4.1 below. The triples marked with asterisks are exactly those that are listed in Conjecture 6.1 of [31].*

$(1,1,7)^*$, $(1,1,15)^*$, $(3,3,5)$, $(1,7,7)^*$, $(3,5,5)$, $(1,7,15)^*$, $(1,9,15)^*$
$(1,15,15)^*$, $(3,5,21)$, $(1,7,49)$, $(1,15,25)^*$, $(3,5,35)$, $(3,5,45)$
$(1,7,105)$, $(3,5,75)$, $(1,15,105)$, $(3,21,35)$, $(1,15,225)$, $(9,15,25)$
$(5,21,35)$, $(7,15,105)$

Table 4.1:

Theorem 4.2.4. *Let a,b be relatively prime positive odd integers such that one of four fractions $\frac{b}{a},\frac{a}{b},\frac{3a}{b},\frac{b}{3a}$ is contained in $\{\frac{5}{3},7,15\}$. Then, for any positive integer n, we have*

$$2t(n,\langle a,3a,b\rangle)=3r(8n+4a+b,\langle a,3a,b\rangle)-r(4(8n+4a+b),\langle a,3a,b\rangle).$$

Proof. Since all the other cases can be treated in a similar manner, we only consider the case when $\frac{b}{3a}=\frac{5}{3}$, that is, $(a,3a,b)=(1,3,5)$. One may easily show that if $x^2+3y^2+5z^2=4(8n+9)$, then

$$\left(x^2,3y^2,5z^2\right)\equiv(0,0,4),(1,3,0),(4,0,0),(4,3,5),\quad\text{or}\quad(4,4,4)\pmod 8.$$

Let

$$f = f(x,y,z) = x^2 + 3y^2 + 5z^2 \quad \text{and} \quad N = 8n + 9.$$

From the above observation, we have

$$\begin{aligned}
3r(N,f) - r(4N,f) &= 3r_{(0,0,0)}(4N,f) - r(4N,f)\\
&= 2r_{(0,0,0)}(4N,f) - r_{(1,1,0)}(4N,f) - r_{(0,1,1)}(4N,f).
\end{aligned}$$

Therefore, it suffices to show that

$$2r_{(1,1,1)}(N,f) = 2r_{(0,0,0)}(4N,f) - r_{(1,1,0)}(4N,f) - r_{(0,1,1)}(4N,f).$$

Since $r_{(0,0,0)}(4N,f) = r(N,f)$ and

$$r(N,f) = r_{(1,1,1)}(N,f) + r_{(1,0,0)}(N,f) + r_{(0,0,1)}(N,f),$$

it is enough to show that

$$r_{(1,0,0)}(N,f) = \frac{1}{2}r_{(1,1,0)}(4N,f) \quad \text{and} \quad r_{(0,0,1)}(N,f) = \frac{1}{2}r_{(0,1,1)}(4N,f).$$

To prove the first assertion, we apply (i) of Lemma 2.2.1 to show that

$$\begin{aligned}
r_{(1,0,0)}(N,f) &= \sum_{z\in\mathbb{Z}} r_{(1,0)}(N-5z^2, x^2+3y^2)\\
&= \frac{1}{2}\sum_{z\in\mathbb{Z}} r_{(1,1)}(4(N-5z^2), x^2+3y^2) = \frac{1}{2}r_{(1,1,0)}(4N,f).
\end{aligned}$$

For the second assertion, we apply (iii) of Lemma 2.2.1 and Lemma 4.2.1 to show that

$$\begin{aligned}
r_{(0,0,1)}(N,f) &= \sum_{z\in\mathbb{Z}} r_{(0,0)}(N-5z^2, x^2+3y^2)\\
&= \frac{1}{2}\sum_{z\in\mathbb{Z}} r_{(1,1)}(N-5z^2, x^2+3y^2)\\
&= \frac{1}{2} r_{(1,1,1)}(N, x^2+3y^2+5z^2)\\
&= \frac{1}{2}\sum_{x\in\mathbb{Z}} r_{(1,1)}(N-x^2, 3y^2+5z^2)\\
&= \frac{1}{2}\sum_{x\in\mathbb{Z}} r_{(1,1)}(4(N-x^2), 3y^2+5z^2)\\
&= \frac{1}{2} r_{(0,1,1)}(4N, f).
\end{aligned}$$

This completes the proof. $\square$

Remark 4.2.5. *All triples $(a, 3a, b)$ satisfying the assumption of the Theorem 4.2.4 are listed in Table 4.2 below. Those triples marked with asterisks are exactly those that are listed in Conjecture 6.2 of [31].*

$(1,3,5)^*$, $(1,3,7)^*$, $(1,3,15)^*$, $(1,3,21)^*$, $(1,5,15)^*$, $(1,3,45)$
$(3,5,9)^*$, $(1,7,21)^*$, $(3,5,15)^*$, $(3,7,21)^*$, $(1,15,45)$, $(5,9,15)$

Table 4.2:

Theorem 4.2.6. *Let $(a,b,c) \in \{(1,2,15),(1,15,18),(1,15,30)\}$. For any positive even integer n, we have*

$$2t(n,\langle a,b,c\rangle) = r(4(8n+a+b+c),\langle a,b,c\rangle) - r(8n+a+b+c,\langle a,b,c\rangle). \quad (4.2.6)$$

Proof. First, assume that $(a,b,c) = (1,2,15)$. Let

$$f = f(x,y,z) = x^2+2y^2+15z^2 \quad \text{and} \quad N = 8n+18.$$

One may easily show that if $f(x, y, z) = 4N$, then

$$\left(x^2, 2y^2, 15z^2\right) \equiv (0,0,0), (1,0,7), \quad \text{or} \quad (4,0,4) \pmod 8.$$

Hence the right-hand side of Equation (4.2.6) is

$$r(4N, f) - r(N, f) = r_{(1,0,1)}(4N, f).$$

Note that

$$\begin{aligned}
r_{(1,1,1)}(N, f) &= \sum_{y \in \mathbb{Z}} r_{(1,1)}((N - 2y^2), x^2 + 15z^2) \\
&= \sum_{y \in \mathbb{Z}} r_{(1,1)}(4(N - 2y^2), x^2 + 15z^2) \\
&= r_{(1,1,1)}(4N, x^2 + 8y^2 + 15z^2) \\
&= |\{(x, y, z) \in R(4N, f) : xz \equiv 1 \pmod 2,\ y \equiv 2 \pmod 4\}|
\end{aligned}$$

by Lemma 4.2.1. Since

$$|\{(x, y, z) \in R(4N, f) : xz \equiv 1 \pmod 2, y \equiv 0 \pmod 4\}| = r(4N, \langle 1, 32, 15\rangle),$$

it suffices to show that

$$r_{(1,1,1)}(N, f) = r(4N, \langle 1, 32, 15\rangle). \tag{4.2.7}$$

It is well known that

$$\text{gen}(f_1 = 4x^2 + 4y^2 + 8z^2 + 2xy) = \{f_1, f_2, f_3\},$$

where $f_2 = 4x^2+6y^2+6z^2+4yz+2xz+2xy$, $f_3 = 2x^2+6y^2+12z^2+6yz+2xz$, and

$$\text{gen}(g_1 = 4x^2 + 8y^2 + 18z^2 + 8yz + 4xz) = \{g_1, g_2 = 2x^2 + 10y^2 + 24z^2\}.$$

Note that

$$r_{(1,1,1)}(N,f) = r(N, x^2 + 2(x-2y)^2 + 15(x-2z)^2) = r(N, g_1).$$

On the other hand, the right-hand side of Equation (4.2.7) is

$$\begin{aligned}
&r\left(4N, x^2 + 15y^2 + 32z^2\right) = r\left(4N, (3x+y)^2 + 15(x+y)^2 + 32z^2\right) \\
&= r\left(2N, 12x^2 + 8y^2 + 16z^2 + 18xy\right) \\
&= r\left(2N, 48x^2 + 8y^2 + 16z^2 + 36xy\right) + r\left(2N, 12x^2 + 32y^2 + 16z^2 + 36xy\right) \\
&= 2r\left(N, f_1\right).
\end{aligned}$$

Therefore, it suffices to show that for any positive even integer $n = 2m$,

$$2r(16m+18, f_1) = r(16m+18, g_1). \tag{4.2.8}$$

By the Minkowski-Siegel formula, we have

$$\begin{aligned}
&r(16m+18, f_1) + 2r(16m+18, f_2) + r(16m+18, f_3) \\
&= r(16m+18, g_1) + r(16m+18, g_2).
\end{aligned}$$

If $f_1(x,y,z) = 16m+18$, then one may easily check that $x + 3y - 4z \equiv 0 \pmod 8$, and if $f_2(x,y,z) = 16m+18$, then $x - 6y + 2z \equiv 0 \pmod 8$. If we define a map

$$\begin{aligned}
\phi_1 :\ &\{(x,y,z) \in R(16m+18, f_1) : x + 3y - 4z \equiv 0 \pmod{16}\} \\
&\to \{(x,y,z) \in R(16m+18, f_2) : x - 6y + 2z \equiv 0 \pmod{16}\}
\end{aligned}$$

by $\phi_1(x,y,z) = \left(\dfrac{12x+4y+16z}{16}, \dfrac{-11x-y+12z}{16}, \dfrac{x-13y-4z}{16}\right)$, then it is bijective. Furthermore, the map

$$\begin{aligned}
\phi_2 :\ &\{(x,y,z) \in R(16m+18, f_1) : x + 3y - 4z \equiv 8 \pmod{16}\} \\
&\to \{(x,y,z) \in R(16m+18, f_2) : x - 6y + 2z \equiv 8 \pmod{16}\}
\end{aligned}$$

defined by $\phi_2(x,y,z) = \left(\frac{4x+12y-16z}{16}, \frac{-13x+y+4z}{16}, \frac{-x-11y-12z}{16}\right)$ is also bijective. Therefore, we have

$$r(16m+18, f_1) = r(16m+18, f_2). \tag{4.2.9}$$

Note that the above equation does not hold, in general, if n is odd. If we define two maps

$$\phi_3 : R(16m+18, \langle 8,10,24\rangle) \to R(16m+18, f_1)$$
$$\text{by} \quad \phi_3(x,y,z) = (y+2z, y-2z, x)$$

and

$$\phi_4 : R(16m+18, \langle 2,24,40\rangle) \to R(16m+18, f_3)$$
$$\text{by} \quad \phi_4(x,y,z) = (x+z, 2y+z, -2z),$$

then one may easily check that both of them are bijective. Hence we have

$$\begin{aligned} r(16m+18, g_2) &= r\left(16m+18, \langle 8,10,24\rangle\right) + r\left(16m+18, \langle 2,24,40\rangle\right) \\ &= r(16m+18, f_1) + r(16m+18, f_3), \end{aligned}$$

for any non negative integer m. Therefore, from the Minkowski-Siegel formula given above, we have $2r(16m+18, f_2) = r(16m+18, g_1)$ for any non-negative integer m. Equation (4.2.8) follows directly from this and Equation (4.2.9).

For the other two cases, one may easily show Equation (4.2.6) by replacing N, f_i, g_i and ϕ_i with the following data:

(1) $(a,b,c) = (1,15,18)$**.** In this case, we let $N = 8n+34$ and

$$\begin{aligned} f_1 &= 4x^2+4y^2+72z^2+2xy, \\ f_2 &= 4x^2+16y^2+22z^2+14yz-2xz+4xy, \\ f_3 &= 6x^2+16y^2+16z^2-8yz+6xz+6xy, \end{aligned}$$

and

$$g_1 = 4x^2 + 34y^2 + 34z^2 + 8yz + 4xz + 4xy, \quad g_2 = 10x^2 + 18y^2 + 24z^2.$$

Define

$$\begin{aligned} \phi_1 : \ &\{(x,y,z) \in R(16m+34, f_1) : 3x + y + 4z \equiv 0 \pmod{16}\} \\ &\quad \to \{(x,y,z) \in R(16m+34, f_2) : 3x - y + 2z \equiv 0 \pmod{16}\} \end{aligned}$$

by

$$\phi_1(x,y,z) = \left(\frac{x - 5y - 68z}{16}, \frac{-5x - 7y + 20z}{16}, \frac{-4x + 4y - 16z}{16} \right),$$

$$\begin{aligned} \phi_2 : \ &\{(x,y,z) \in R(16m+34, f_1) : 3x + y + 4z \equiv 8 \pmod{16}\} \\ &\quad \to \{(x,y,z) \in R(16m+34, f_2) : 3x - y + 2z \equiv 8 \pmod{16}\} \end{aligned}$$

by

$$\phi_2(x,y,z) = \left(\frac{9x - 5y - 52z}{16}, \frac{3x + 9y + 4z}{16}, \frac{4x - 4y + 16z}{16} \right),$$

and

$$\begin{aligned} \phi_3 : \ &R(16m+34, 10x^2 + 24y^2 + 72z^2) \\ &\quad \to R(16m+34, f_1) \ \text{ by } \ \phi_3(x,y,z) = (x - 2y, x + 2y, z), \end{aligned}$$

$$\begin{aligned} \phi_4 : \ &R(16m+34, 18x^2 + 24y^2 + 40z^2) \\ &\quad \to R(16m+34, f_3) \ \text{ by } \ \phi_4(x,y,z) = (x + 2y, -x + z, -x - z). \end{aligned}$$

(2) $(a,b,c) = (1,15,30)$**.** In this case, we let $N = 8n + 46$ and

$$\begin{aligned} f_1 &= 4x^2 + 4y^2 + 120z^2 + 2xy, \\ f_2 &= 4x^2 + 16y^2 + 34z^2 + 14yz - 2xz + 4xy, \\ f_3 &= 10x^2 + 16y^2 + 16z^2 + 8yz + 10xz + 10xy, \end{aligned}$$

and

$$g_1 = 4x^2 + 46y^2 + 46z^2 + 32yz + 4xz + 4xy, \quad g_2 = 6x^2 + 30y^2 + 40z^2.$$

Define

$$\begin{aligned}\phi_1 :\ & \{(x,y,z) \in R(16m+46, f_1) : 3x - y - 4z \equiv 0 \pmod{16}\} \\ & \to \{(x,y,z) \in R(16m+46, f_2) : 3x - y + 2z \equiv 8 \pmod{16}\}\end{aligned}$$

by

$$\phi_1(x,y,z) = \left(\frac{7x-13y-4z}{16}, \frac{-3x+y-44z}{16}, \frac{-4x-4y+16z}{16}\right),$$

$$\begin{aligned}\phi_2 :\ & \{(x,y,z) \in R(16m+46, f_1) : 3x - y - 4z \equiv 8 \pmod{16}\} \\ & \to \{(x,y,z) \in R(16m+46, f_2) : 3x - y + 2z \equiv 0 \pmod{16}\}\end{aligned}$$

by

$$\phi_2(x,y,z) = \left(\frac{9x-11y+20z}{16}, \frac{3x+7y+28z}{16}, \frac{-4x-4y+16z}{16}\right),$$

and

$$\begin{aligned}\phi_3 :\ & R(16m+46, 6x^2+40y^2+120z^2) \\ & \to R(16m+46, f_1) \ \text{ by } \ \phi_3(x,y,z) = (x+2y, -x+2y, z),\end{aligned}$$

$$\begin{aligned}\phi_4 :\ & R(16m+46, 24x^2+30y^2+40z^2) \\ & \to R(16m+46, f_3) \ \text{ by } \ \phi_4(x,y,z) = (-y-2z, x+y, -x+y).\end{aligned}$$

This completes the proof. $\square$

Theorem 4.2.7. *For any positive integer n such that $n \not\equiv 1 \pmod 3$, we have*

$$2t(n, \langle 1,1,27\rangle) = r(4(8n+29), x^2+y^2+27z^2) - r(8n+29, x^2+y^2+27z^2). \tag{4.2.10}$$

Proof. Let $N = 8n+29$ and

$$\begin{aligned} f &= f(x,y,z) = x^2+y^2+27z^2, \\ g &= g(x,y,z) = 8x^2+20y^2+29z^2+4yz+8xz+8xy, \\ h &= h(x,y,z) = 2x^2+5y^2+27z^2+2xy. \end{aligned}$$

For any positive integer $m \not\equiv 1 \pmod 3$, we let

$$\delta_m = \begin{cases} 1 & \text{if } \ m \equiv 0 \pmod 3, \\ 2 & \text{if } \ m \equiv 2 \pmod 3. \end{cases}$$

Note that

$$r(m, f) = \delta_m \left| \{(x, y, z) \in R(m, f) : x \equiv y \pmod 3\} \right| . \tag{4.2.11}$$

Since

$$r(4N, f) = \delta_N \cdot r(4N, x^2 + (x - 3y)^2 + 27z^2) = \delta_N \cdot r(4N, h)$$

and

$$\begin{aligned} &\left|\{(x, y, z) \in R(4N, f) : y \equiv 0 \pmod 2\}\right| \\ &= \delta_N \cdot r(4N, x^2 + 4(x - 3y)^2 + 27z^2) = \delta_N \cdot r(4N, 8x^2 + 5y^2 + 27z^2 + 4xy) \\ &= \delta_N \left|\{(x, y, z) \in R(4N, h) : x \equiv 0 \pmod 2\}\right| , \end{aligned}$$

we have

$$|\,\{(x, y, z) \in R(4N, f) : y \text{ is odd}\}\,| = \delta_N |\,\{(x, y, z) \in R(4N, h) : x \text{ is odd}\}\,|. \tag{4.2.12}$$

One may easily show that if $(x, y, z) \in R(4N, f)$, then

$$\left(x^2, y^2, 27z^2\right) \equiv (0, 0, 4), (0, 1, 3), (0, 4, 0), (1, 0, 3), (4, 0, 0), (4, 4, 4) \pmod 8.$$

From this and Equation (4.2.12), the right hand side of Equation (4.2.10) becomes

$$R(4N, f) - R(N, f) = 2\delta_N \left|\{(x, y, z) \in R(4N, h) : x \equiv 1 \pmod 2\}\right| .$$

On the other hand, by Equation (4.2.11),

$$\begin{aligned}
t(n,\langle 1,1,27\rangle) &= r_{(1,1,1)}(N,f)\\
&= \delta_N \left|\left\{(x,y,z)\in R(N,f):\ \begin{array}{c} x\equiv y \pmod 3, \\ x\equiv y\equiv z \pmod 2 \end{array}\right\}\right|\\
&= \delta_N\cdot r(N, x^2+(x-6y)^2+27(x-2z)^2) = \delta_N\cdot r(N,g).
\end{aligned}$$

Therefore, it is enough to show that

$$r(N,g) = \left|\{(x,y,z)\in R(4N,h):\ x\equiv 1 \pmod 2\}\right|.$$

Now, we let

$$\begin{aligned}
A &= \{(x,y,z)\in R(N,g) : x\equiv 0 \pmod 2\}\,,\\
B &= \{(x,y,z)\in R(4N,h):\ x\equiv 1 \pmod 2, x+z\equiv 0 \pmod 8\}\,.
\end{aligned}$$

Note that $x+z\equiv 8 \pmod{16}$ if $(x,y,z)\in B$. Define a map $\phi : A\to B$ by

$$\phi(x,y,z) = (x-7z,\ -x-4y+z,\ -x-z)\,.$$

Then, one may easily show that ϕ is a bijection. Since $g(x+z,y,-z) = g(x,y,z)$ and z_0 is odd for any $(x_0,y_0,z_0)\in R(N,g)$, we have

$$\begin{aligned}
&\left|\{(x,y,z)\in R(N,g) : x\equiv 0 \pmod 2\}\right|\\
&= \left|\{(x,y,z)\in R(N,g) : x\equiv 1 \pmod 2\}\right|
\end{aligned}$$

and thus

$$r(N,g) = 2\left|\{(x,y,z)\in R(N,g) : x\equiv 0 \pmod 2\}\right|.$$

Now, we are ready to prove the assertion. Note that if $(x,y,z)\in R(4N,h)$

and $x \equiv 1 \pmod 2$, then $z \equiv \pm x \pmod 8$. Therefore, we have

$$\begin{aligned}
&|\{(x,y,z) \in R(4N,h) : \ x \equiv 1 \pmod 2\}| \\
&= 2\,|\{(x,y,z) \in R(4N,h) : \ x \equiv 1 \pmod 2, \ x+z \equiv 0 \pmod 8\}| \\
&= 2|B| = 2|A| = r(N,g).
\end{aligned}$$

This completes the proof. $\square$

Since the ternary quadratic form $x^2+y^2+6z^2$ has class number 1, the following Conjecture 6.7 in [31] follows directly from Theorem 3.2.16.

Theorem 4.2.8. *For a positive integer n, the Diophantine equation*

$$\mathcal{T}_{(1,1,6)}(x,y,z) := \frac{x(x+1)}{2} + \frac{y(y+1)}{2} + 6\frac{z(z+1)}{2} = n$$

has an integer solution if and only if $n \not\equiv 2 \cdot 3^{2r-1} - 1 \pmod{3^{2r}}$ for any positive integer r.

Chapter 5

The number of representations of quaternary triangular forms

In this chapter, we prove Conjecture 2.5 in [30]. To prove this, we need a proposition which relates the representation numbers between two quaternary quadratic forms in the genus of $\langle 1, 2, 4, 17\rangle$.

Proposition 5.0.1. *For any positive integer* $n \equiv 3, 5 \pmod 8$*, we have*

$$r(n, x^2+2y^2+4z^2+17w^2) = r(n, 2x^2+3y^2+4z^2+8w^2+2xy+2yz+2yw).$$

Proof. Let

$$\begin{aligned} f &= f(x, y, z, w) = x^2 + 2y^2 + 4z^2 + 17w^2, \\ g &= g(x, y, z, w) = 2x^2 + 3y^2 + 4z^2 + 8w^2 + 2xy + 2yz + 2yw. \end{aligned}$$

First, we consider the case when n is a positive integer congruent to 3 modulo 8. Note that if $(x, y, z, w) \in R(n, f)$, then $x \not\equiv w \pmod 2$. Furthermore, one may easily show that if $(x, y, z, w) \in R(n, f)$ and $x \equiv 1 \pmod 2$, then $2x+2y-2z-3w \equiv 0 \pmod 4$. Since $f(-x, y, z, w) = f(x, y, z, w)$, the map $\eta_1 : R(n, f) \to R(n, f)$ defined by

$$\eta_1(x, y, z, w) = (-x, y, z, w),$$

is a well-defined bijective map. Hence we have

$$\begin{aligned}
&|\{(x,y,z,w)\in R(n,f) : x\equiv 1 \pmod 2,\ 2x+2y-2z-3w\equiv 0 \pmod 8\}|\\
&=\left|\left\{(x,y,z,w)\in R(n,f) : \begin{array}{c} x\equiv 1 \pmod 2, \\ 2x+2y-2z-3w\equiv 4 \pmod 8 \end{array}\right\}\right|,
\end{aligned}$$

which implies that

$$\begin{aligned}
&|\{(x,y,z,w)\in R(n,f) : x\equiv 1 \pmod 2\}|\\
&=2\left|\left\{(x,y,z,w)\in R(n,f) : \begin{array}{c} x\equiv 1 \pmod 2, \\ 2x+2y-2z-3w\equiv 0 \pmod 8 \end{array}\right\}\right|.
\end{aligned}$$

Note that if $(x,y,z,w)\in R(n,f)$ and $x\equiv 0 \pmod 2$, then $x+6y-6z+6w\equiv 0 \pmod 4$. Since $f(x,y,z,-w)=f(x,y,z,w)$, the map $\eta_2 : R(n,f)\to R(n,f)$ defined by

$$\eta_2(x,y,z,w)=(x,y,z,-w),$$

is a well-defined bijective map. Hence we have

$$\begin{aligned}
&|\{(x,y,z,w)\in R(n,f) : x\equiv 0 \pmod 2,\ x+6y-6z+6w\equiv 0 \pmod 8\}|\\
&=\left|\left\{(x,y,z,w)\in R(n,f) : \begin{array}{c} x\equiv 0 \pmod 2, \\ x+6y-6z+6w\equiv 4 \pmod 8 \end{array}\right\}\right|,
\end{aligned}$$

which implies that

$$\begin{aligned}
&|\{(x,y,z,w)\in R(n,f) : x\equiv 0 \pmod 2\}|\\
&=2\left|\left\{(x,y,z,w)\in R(n,f) : \begin{array}{c} x\equiv 0 \pmod 2, \\ x+6y-6z+6w\equiv 0 \pmod 8 \end{array}\right\}\right|.
\end{aligned}$$

Now, if we define

$$F_1 = \left\{ (x,y,z,w) \in R(n,f) : \begin{array}{c} x \equiv 1 \pmod 2, \\ 2x + 2y - 2z - 3w \equiv 0 \pmod 8 \end{array} \right\},$$
$$F_2 = \left\{ (x,y,z,w) \in R(n,f) : \begin{array}{c} x \equiv 0 \pmod 2, \\ x + 6y - 6z + 6w \equiv 0 \pmod{16} \end{array} \right\},$$
$$F_3 = \left\{ (x,y,z,w) \in R(n,f) : \begin{array}{c} x \equiv 0 \pmod 2, \\ x + 6y - 6z + 6w \equiv 8 \pmod{16} \end{array} \right\},$$

then we have

$$R(n,f) = 2(|F_1| + |F_2| + |F_3|).$$

Now, we analyze the set $R(n,g)$. First, we note that $y \equiv 1 \pmod 2$ for any $(x,y,z,w) \in R(n,g)$. Since $g(x+y,-y,-z,-w) = g(x,y,z,w)$, the map $\eta_3 : R(n,g) \to R(n,g)$ defined by

$$\eta_3(x,y,z,w) = (x+y,-y,-z,-w)$$

is a well-defined bijective map. Therefore, we have

$$R(n,g) = 2\left|\{(x,y,z,w) \in R(n,g) : x \equiv 0 \pmod 2\}\right|.$$

One may easily check that for $(x,y,z,w) \in R(n,g)$, if $x \equiv 0 \pmod 2$, then $x - z + w \equiv 0 \pmod 4$. Furthermore, if $x - z + w \equiv 4 \pmod 8$, then $7x - 4y + 9z - w \equiv 0 \pmod 8$. Thus if we define

$$G_1 = \{(x,y,z,w) \in R(n,g) : x \equiv 0 \pmod 2,\ x - z + w \equiv 0 \pmod 8\},$$
$$G_2 = \left\{ (x,y,z,w) \in R(n,g) : \begin{array}{r} x \equiv 0 \pmod 2,\ x - z + w \equiv 4 \pmod 8, \\ 7x - 4y + 9z - w \equiv 8 \pmod{16} \end{array} \right\},$$

$$G_3 = \left\{ (x,y,z,w) \in R(n,g) : \begin{array}{r} x \equiv 0 \pmod 2,\ x - z + w \equiv 4 \pmod 8, \\ 7x - 4y + 9z - w \equiv 0 \pmod{16} \end{array} \right\},$$

then the set $\{(x,y,z,w) \in R(n,g) : x \equiv 0 \pmod 2\}$ is a disjoint union of

G_1, G_2 and G_3. Hence we have

$$R(n,g) = 2(|G_1| + |G_2| + |G_3|).$$

Now, for $i = 1, 2, 3$, we define maps $\phi_i : G_i \to F_i$ by

$$\phi_1(x,y,z,w) = \frac{1}{8}\begin{pmatrix} 4 & 8 & -4 & -12 \\ -2 & -8 & -6 & -10 \\ -3 & 0 & -5 & 5 \\ -2 & 0 & 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix},$$
$$\phi_2(x,y,z,w) = \frac{1}{16}\begin{pmatrix} 2 & 24 & -2 & 18 \\ -10 & -8 & -6 & 22 \\ -3 & -4 & -13 & -11 \\ -4 & 0 & 4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix},$$
$$\phi_3(x,y,z,w) = \frac{1}{16}\begin{pmatrix} 6 & -8 & -6 & -42 \\ 2 & -8 & 14 & 2 \\ 7 & 12 & 9 & -1 \\ 4 & 0 & -4 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}.$$

Then one can easily check that all of them are well-defined bijective maps. Therefore, we have

$$R(n,f) = 2(|F_1| + |F_2| + |F_3|) = 2(|G_1| + |G_2| + |G_3|) = R(n,g).$$

Next, we consider the case when n is a positive integer congruent to 5 modulo 8. Note that if $(x,y,z,w) \in R(n,f)$ and $x \equiv 1 \pmod 2$, then $2x + 2y - 2z + 5w \equiv 0 \pmod 4$. Since $f(-x,y,z,w) = f(x,y,z,w)$, we have

$$\begin{aligned} &|\{(x,y,z,w) \in R(n,f) : x \equiv 1 \pmod 2,\ 2x + 2y - 2z + 5w \equiv 0 \pmod 8\}| \\ &= \left|\left\{(x,y,z,w) \in R(n,f) : \begin{array}{c} x \equiv 1 \pmod 2, \\ 2x + 2y - 2z + 5w \equiv 4 \pmod 8 \end{array}\right\}\right|, \end{aligned}$$

which implies that

$$\begin{aligned}&|\{(x,y,z,w)\in R(n,f) : x\equiv 1 \pmod 2\}|\\ &=2\left|\left\{(x,y,z,w)\in R(n,f) : \begin{array}{c} x\equiv 1 \pmod 2, \\ 2x+2y-2z+5w\equiv 0 \pmod 8\end{array}\right\}\right|.\end{aligned}$$

For $(x,y,z,w)\in R(n,f)$, if $x\equiv 0 \pmod 2$, then we have $w\equiv 1 \pmod 2$ and $x+6y-6z+6w\equiv 0 \pmod 4$. Since $f(x,y,z,-w)=f(x,y,z,w)$,

$$\begin{aligned}&|\{(x,y,z,w)\in R(n,f) : x\equiv 0 \pmod 2,\ x+6y-6z+6w\equiv 0 \pmod 8\}|\\ &=|\{(x,y,z,w)\in R(n,f) : x\equiv 0 \pmod 2,\ x+6y-6z+6w\equiv 4 \pmod 8\}|,\end{aligned}$$

which implies that

$$\begin{aligned}&|\{(x,y,z,w)\in R(n,f) : x\equiv 0 \pmod 2\}|\\ &=2\left|\left\{(x,y,z,w)\in R(n,f) : \begin{array}{c} x\equiv 0 \pmod 2, \\ x+6y-6z+6w\equiv 0 \pmod 8\end{array}\right\}\right|.\end{aligned}$$

Thus if we define

$$\begin{aligned}X_1&=\left\{(x,y,z,w)\in R(n,f) : \begin{array}{c} x\equiv 1 \pmod 2, \\ 2x+2y-2z+5w\equiv 0 \pmod 8\end{array}\right\},\\ X_2&=\left\{(x,y,z,w)\in R(n,f) : \begin{array}{c} x\equiv 0 \pmod 2, \\ x+6y-6z+6w\equiv 0 \pmod{16}\end{array}\right\},\\ X_3&=\left\{(x,y,z,w)\in R(n,f) : \begin{array}{c} x\equiv 0 \pmod 2, \\ x+6y-6z+6w\equiv 8 \pmod{16}\end{array}\right\},\end{aligned}$$

then we have

$$R(n,f)=2(|X_1|+|X_2|+|X_3|).$$

Now, we analyze the set $R(n,g)$. One may check the followings;

(i) if $(x,y,z,w)\in R(n,g)$ and $x\equiv 0 \pmod 2$, then $x+y+z-w\equiv 0 \pmod 4$;

(ii) if $(x,y,z,w)\in R(n,g)$ and $x\equiv 1 \pmod 2$, then $x-z+w\equiv 0 \pmod 4$.

Since $g(x+y,-y,-z,-w)=g(x,y,z,w)$, we have

$$\begin{aligned}&|\{(x,y,z,w)\in R(n,g): x\equiv 0 \pmod 2,\ x+y+z-w\equiv 0 \pmod 8\}|\\&=|\{(x,y,z,w)\in R(n,g): x\equiv 1 \pmod 2,\ x-z+w\equiv 0 \pmod 8\}|\end{aligned}$$

and

$$\begin{aligned}&|\{(x,y,z,w)\in R(n,g): x\equiv 0 \pmod 2,\ x+y+z-w\equiv 4 \pmod 8\}|\\&=|\{(x,y,z,w)\in R(n,g): x\equiv 1 \pmod 2,\ x-z+w\equiv 4 \pmod 8\}|.\end{aligned}$$

Therefore, we have

$$\begin{aligned}R(n,g)=&2\,|\{(x,y,z,w)\in R(n,g): x\equiv 0 \pmod 2\}|\\=&2\left|\left\{(x,y,z,w)\in R(n,g): \begin{array}{c}x\equiv 0 \pmod 2,\\ x+y+z-w\equiv 0 \pmod 8\end{array}\right\}\right|\\&+2\left|\left\{(x,y,z,w)\in R(n,g): \begin{array}{c}x\equiv 0 \pmod 2,\\ x+y+z-w\equiv 4 \pmod 8\end{array}\right\}\right|\\=&2\left|\left\{(x,y,z,w)\in R(n,g): \begin{array}{c}x\equiv 0 \pmod 2,\\ x+y+z-w\equiv 0 \pmod 8\end{array}\right\}\right|\\&+2\left|\left\{(x,y,z,w)\in R(n,g): \begin{array}{c}x\equiv 1 \pmod 2,\\ x-z+w\equiv 4 \pmod 8\end{array}\right\}\right|.\end{aligned}$$

One may easily show that for $(x,y,z,w)\in R(n,g)$, if $x\equiv 1 \pmod 2$ and $x-z+w\equiv 4 \pmod 8$, then $7x-4y+9z-w\equiv 0 \pmod 8$. Thus if we define

$$\begin{aligned}Y_1&=\{(x,y,z,w)\in R(n,g): x\equiv 0 \pmod 2,\ x+y+z-w\equiv 0 \pmod 8\},\\Y_2&=\left\{(x,y,z,w)\in R(n,g): \begin{array}{r}x\equiv 1 \pmod 2,\ x-z+w\equiv 4 \pmod 8,\\ 7x-4y+9z-w\equiv 8 \pmod{16}\end{array}\right\},\\Y_3&=\left\{(x,y,z,w)\in R(n,g): \begin{array}{r}x\equiv 1 \pmod 2,\ x-z+w\equiv 4 \pmod 8,\\ 7x-4y+9z-w\equiv 0 \pmod{16}\end{array}\right\},\end{aligned}$$

then we have

$$R(n,g)=2(|Y_1|+|Y_2|+|Y_3|).$$

For $i = 1, 2, 3$, if we define maps $\psi_i : Y_i \to X_i$ by

$$\psi_1(x,y,z,w) = \frac{1}{8}\begin{pmatrix} 4 & -4 & 4 & 12 \\ -2 & 6 & 6 & 10 \\ -3 & -3 & 5 & -5 \\ -2 & -2 & -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix},$$

$$\psi_2(x,y,z,w) = \frac{1}{16}\begin{pmatrix} 2 & 24 & -2 & 18 \\ -10 & -8 & -6 & 22 \\ -3 & -4 & -13 & -11 \\ -4 & 0 & 4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix},$$

$$\psi_3(x,y,z,w) = \frac{1}{16}\begin{pmatrix} 6 & -8 & -6 & -42 \\ 2 & -8 & 14 & 2 \\ 7 & 12 & 9 & -1 \\ 4 & 0 & -4 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix},$$

then one may check that they are all bijective. Therefore, we have

$$R(n,f) = 2(|X_1| + |X_2| + |X_3|) = 2(|Y_1| + |Y_2| + |Y_3|) = R(n,g),$$

which completes the proof. $\square$

Theorem 5.0.2. *For any positive integer* $n \equiv 0, 2 \pmod 8$, *we have*

$$t(n, \langle 1,2,4,17\rangle) = 4r(n+3, \langle 1,2,4,17\rangle).$$

Proof. Let

$$\begin{aligned} f &= f(x,y,z,w) = x^2 + 2y^2 + 4z^2 + 17w^2, \\ g &= g(x,y,z,w) = 2x^2 + 3y^2 + 4z^2 + 8w^2 + 2xy + 2yz + 2yw, \\ h_1 &= h_1(x,y,z,w) = 2x^2 + 4y^2 + 4z^2 + 6w^2 + 2xw + 2yw + 4zw, \\ h_2 &= h_2(x,y,z,w) = x^2 + 2y^2 + 2z^2 + 9w^2 + 2zw. \end{aligned}$$

First, note that

$$\begin{aligned}
&|\{(x,y,z,w) \in R(2n+6,h_1) : w \equiv 0 \pmod 2\}| \\
&= |\{(x,y,z,w) \in \mathbb{Z}^4 : h_1(x,y,z,2w) = 2n+6\}| \\
&= |\{(x,y,z,w) \in \mathbb{Z}^4 : 2 \cdot h_2(x+w,z+w,y,w) = 2n+6\}| = r(n+3,h_2).
\end{aligned}$$

Note that for $(x,y,z,w) \in R(2n+6,h_1)$, if $w \equiv 1 \pmod 2$, then $y \equiv 0 \pmod 2$. Hence we have

$$\begin{aligned}
&|\{(x,y,z,w) \in R(2n+6,h_1) : x \equiv y \equiv 0 \pmod 2,\ w \equiv 1 \pmod 2\}| \\
&= |\{(x,y,z,w) \in \mathbb{Z}^4 : h_1(2x,2y,z,w) = 2n+6\}| \\
&= |\{(x,y,z,w) \in \mathbb{Z}^4 : 2 \cdot g(z,w,x,y) = 2n+6\}| = r(n+3,g).
\end{aligned}$$

Finally, since $h_1(w-2x,2y,z,w) = 2 \cdot g(z,w,x-w,y)$, we have,

$$\begin{aligned}
&|\{(x,y,z,w) \in R(2n+6,h_1) : x \equiv w \equiv 1 \pmod 2,\ y \equiv 0 \pmod 2\}| \\
&= |\{(x,y,z,w) \in \mathbb{Z}^4 : h_1(w-2x,2y,z,w) = 2n+6\}| = r(n+3,g).
\end{aligned}$$

Therefore, we have

$$r(2n+6,h_1) = r(n+3,h_2) + 2r(n+3,g), \tag{5.0.1}$$

for any nonnegative even integer n.

By Proposition 5.0.1 and Equation (5.0.1), we have

$$2r(n+3,f) = r(2n+6,h_1) - r(n+3,h_2) \quad \text{for any } n \equiv 0,2 \pmod 8. \tag{5.0.2}$$

Now, note that if $8x^2+y^2+2z^2+9w^2-4xw \equiv 0 \pmod 4$, then $y \equiv z \equiv w \pmod 2$. Since $h_1(y,x,z,-w) = 4x^2+2y^2+4z^2+6w^2-2xw-2yw-4zw$, we have

$$\begin{aligned}
&|\{(x,y,z,w) \in R(8n+24,f) : x \equiv w \pmod 4\}| \\
&= r(8n+24,(w-4x)^2+2y^2+4z^2+17w^2) \\
&= r(4n+12,8x^2+y^2+2z^2+9w^2-4xw) \\
&= r(4n+12,8x^2+(w-2y)^2+2(w-2z)^2+9w^2-4xw) \\
&= r(2n+6,4x^2+2y^2+4z^2+6w^2-2xw-2yw-4zw) \\
&= r(2n+6,h_1).
\end{aligned}$$

Note that if $x^2+8y^2+16z^2+17w^2 = 8n+24$, then $x \equiv w \equiv 0 \pmod 2$. Since $2 \cdot h_2(y,z,x,-w) = (w-2x)^2+2y^2+4z^2+17w^2$,

$$\begin{aligned}
&|\{(x,y,z,w) \in \mathbb{Z}^4 : x^2+8y^2+16z^2+17w^2 = 8n+24\}| \\
&= |\{(x,y,z,w) \in \mathbb{Z}^4 : 4x^2+8y^2+16z^2+68w^2 = 8n+24\}| \\
&= r(2n+6,x^2+2y^2+4z^2+17w^2) \\
&= r(2n+6,(w-2x)^2+2y^2+4z^2+17w^2) \\
&= r(n+3,h_2).
\end{aligned}$$

From these equalities and Equation (5.0.2), we have

$$\begin{aligned}
2r(n+3,f) = &|\{(x,y,z,w) \in R(8n+24,f) : x \equiv w \pmod 4\}| \\
&- \left|\{(x,y,z,w) \in \mathbb{Z}^4 : x^2+8y^2+16z^2+17w^2 = 8n+24\}\right|
\end{aligned} \tag{5.0.3}$$

for any $n \equiv 0,2 \pmod 8$. Note that if $x^2+2y^2+4z^2+17w^2 = 8n+24$, then

$$\begin{aligned}
(x^2,2y^2,4z^2,17w^2) \equiv\ &(0,0,0,0),(4,0,0,4),(4,0,4,0),(0,0,4,4) \\
&\text{or} \quad (1,2,4,1) \pmod 8.
\end{aligned}$$

From this and Equation (5.0.3), we may easily deduce that

$$\begin{aligned}
\frac{1}{2}t(n,\langle 1,2,4,17\rangle) =& \left|\left\{(x,y,z,w)\in R(8n+24,f) : \begin{array}{l} x\equiv w \pmod 4, \\ y\equiv z\equiv 1 \pmod 2 \end{array}\right\}\right| \\
=& |\{(x,y,z,w)\in R(8n+24,f) : x\equiv w \pmod 4\}| \\
&- |\{(x,y,z,w)\in R(8n+24,f) : y\equiv z\equiv 0 \pmod 2\}| \\
=& |\{(x,y,z,w)\in R(8n+24,f) : x\equiv w \pmod 4\}| \\
&- r(8n+24, x^2+8y^2+16z^2+17w^2) \\
=& 2r(n+3,f).
\end{aligned}$$

This completes the proof. □

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국문초록

다항식 T_x 를 $\dfrac{x(x+1)}{2}$ 라 정의하자. 자연수 $a_1, a_2, \ldots, a_k$ 에 대해 $a_1T_{x_1} + a_2T_{x_2} + \cdots + a_kT_{x_k}$ 형태의 다항식을 삼각형식이라 부른다.

이 논문에서는 삼변수 삼각형식과 사변수 삼각형식의 표현에 관한 다양한 성질에 관해 연구한다. 어떠한 삼각형식이 국소적으로 표현하는 모든 자연수를 대역적으로 표현하는 경우 이를 정규 삼각형식이라 부른다. 이 논문에서는 정규 삼변수 삼각형식을 분류한다. 또한 삼변수 혹은 사변수 삼각형식에 관한 Sun의 다양한 추측들을 증명한다.

주요어휘: 삼변수 이차형식의 표현, 정규 형식, 삼각수, 왓슨 변환
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