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Multiple Equilibria in Morris and Shin Model (2002)

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This note reconsiders the model of Morris and Shin (2002). We show that when the random variable of interest is generated by two independent pieces of uncertainty, there are multiple equilibria. Moreover, the social welfare is increasing in the precision of the public information.

I. Introduction

In their seminal paper, Morris and Shin (2002) study the interaction of public information and private information in a setting with features of a beauty contest type. In their model, agents use both the public and the private information to get close not only to the target but also to the average action of other agents. Public information helps them to achieve both objectives whereas private information may lead them away from the average action of others. Thus agents tend to downplay their private information, putting too much weight on public information. As a result, social welfare may decrease when the public information gets more accurate.

In this paper, we extend their analysis to a case where the underlying random variable of interest is generated by two independent components, and each agent can choose which of the

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two components to investigate. We retain most of the original notations.

II. Analysis

1. The Setup

Let $\tilde{\theta}$ be the underlying random variable of interest. We consider the case where the variable $\tilde{\theta}$ is a sum of two independent components:

$$\tilde{\theta} = \tilde{u} + \tilde{w}$$

We assume that the two random variables $\tilde{u}$ and $\tilde{w}$ are independently (written as $\tilde{u} \perp \tilde{w}$), distributed normal random variables: $\tilde{u} \sim N(\overline{u}, 1/h_u)$ and $\tilde{w} \sim N(\overline{w}, 1/h_w)$ where $h_u = \sigma_u^{-2}$ and $h_w = \sigma_w^{-2}$ are precision of $\tilde{u}$ and $\tilde{w}$, respectively. All random variables in this paper are normally distributed.

Morris and Shin use improper prior of $\tilde{\theta} \sim \text{Unif}(-\infty, +\infty)$, which simplifies calculations a great deal. In their model the assumption of the improper prior does not cause any trouble because they are interested only in the conditional distributions of random variables. We cannot utilize the convenience of the improper prior since we need to compute unconditional means as well.

The public information $\tilde{y}$, which will be known to everyone in the economy at no cost, is a signal about the first component $\tilde{u}$. So, we posit

$$\tilde{y} = \tilde{u} + \tilde{\epsilon}_y$$

Here, $\tilde{\epsilon}_y \sim N(0, 1/h_y)$ is independent of random variables $(\tilde{u}, \tilde{w})$. We will denote the error term associated with a random variable $\tilde{s}$ by $\tilde{\epsilon}_y$ and its precision by h_s, so that $\tilde{\epsilon}_s \sim N(0, 1/h_s)$. All error terms are assumed to be independent of each other.

We consider public information as something like announcing quarterly unemployment data. Public information of this sort may provide agents with some idea where the economy is headed. Yet, it will not be directly applicable when agents are interested in using the information to trade in the stock market, for example. To be able to take a full advantage of the information, they will typically need to acquire more information about the stock market reaction to the public information. That information about the stock market is represented by the orthogonal random variable $\tilde{w}$.

Each agent in the economy can acquire private information about $\tilde{u}$ or $\tilde{w}$. We assume that each agent can acquire only one piece of information, probably because of the cost of getting additional information. Specifically, agent i can acquire either signal $\tilde{x}_i$ or signal $\tilde{z}_i$, which are defined by

$$\tilde{x}_i = \tilde{u} + \tilde{\epsilon}_{xi}$$

$$\tilde{z}_i = \tilde{w} + \tilde{\epsilon}_{zi}$$

Here, $\tilde{\epsilon}_{x_i} \perp \tilde{\epsilon}_{x_j}$, $\tilde{\epsilon}_{z_i} \perp \tilde{\epsilon}_{z_j}$ for all $i \neq j$, and $\tilde{\epsilon}_{x_i} \perp \tilde{\epsilon}_{z_j}$, for all i and j with $i \neq j$. Thus signal $\tilde{x}$ provides information about $\tilde{u}$, and signal $\tilde{z}$ provides information about $\tilde{w}$. If agents acquire private information $\tilde{x}$, they are getting more information about the same component that public information deals with. If they acquire information $\tilde{z}$, they are getting information about a component that public information does not provide any clue to. We assume $\tilde{\epsilon}_{si} \sim N(0, 1/h_s)$ for all i where s=x, z. Note that precision of each signal does not depend on the identity of agent who acquires it.

As in Morris and Shin, agent i chooses his action a_i to maximize preferences given by:

$$u_i(a_i) = -(1-r)\left(a_i - \tilde{\theta}\right)^2 - r\left[\left(a_i - \overline{a}\right)^2 - \int_0^1 \left(a_j - \overline{a}\right)^2 dj\right] \qquad \text{(P)}$$

They showed that the optimal solution is given by:

$$a_i = (1-r)\sum_{k=0}^{\infty} r^k E_i\left(\overline{E}^k(\theta)\right)$$

We can use the same formula here as well.

2. Benchmark Case

If there is only public information without any private information, each agent's forecast of $\tilde{u}$ will be based on the prior distribution of $\tilde{u}$ and the public information $\tilde{y}$:

$$E_i(\tilde{u}) = \frac{h_u \bar{u} + h_y y}{h_u + h_y}$$

Define

$$v \equiv \frac{h_u \bar{u} + h_y y}{h_u + h_y}, \quad \text{and} \quad h \equiv h_u + h_y$$

The variable h is the combined precision of the prior of $\tilde{u}$ and the public information $\tilde{y}$.

III. All Agents Acquiring Private Information of the Same Type

We start with the case where all agents acquire the same kind of private information. If all agents acquire information x, then their forecast of $\tilde{\theta}$ is given by:

$$E_i(\tilde{\theta}) = E(\tilde{\theta} \mid y, x_i) = \frac{h_u \bar{u} + h_y y + h_x x_i}{h_u + h_y + h_x} + \bar{w} = \frac{hv + h_x x_i}{h + h_x} + \bar{w}$$

It is straightforward to see that when all agents acquire information x, the optimal action of an agent is given by the following formula.

$$a_i = \frac{hv + (1-r) h_x x_i}{h + (1-r) h_x} + \bar{w}$$

Since

$$E(a_i-\overline{a})^2=\int_0^1 E(a_j-\overline{a})^2\,dj=E(\int_0^1 (a_j-\overline{a})^2\,dj),$$

the term in the bracket in the expression (P) of preferences get cancelled out. Thus, we have:

$$U_i(x)=Eu_i(a_i)=-(1-r)\left[\frac{h+(1-r)^2 h_x}{\left(h+(1-r)h_x\right)^2}+\frac{1}{h_w}\right]$$

Similarly, if all agents acquire private information only about z, each agent gets an expected utility of

$$U_i(z)=-(1-r)\left[\frac{1}{h}+\frac{h_w+(1-r)^2 h_z}{\left(h_w+(1-r)h_z\right)^2}\right]$$

Comparing the two, we arrive at:

$$U_i(z)>U_i(x) \text{ if and only if } \frac{h_z\left[(1+r)h_w+(1-r)h_z\right]}{h_w\left(h_w+(1-r)h_z\right)^2}>\frac{h_x\left[(1+r)h+(1-r)h_x\right]}{h\left(h+(1-r)h_x\right)^2}$$

It is straightforward to show that for fixed values of h_w, h_x and h_z, there is only one positive root $\hat{h}$ to the following equation:

$$f(h)=\frac{h_z[(1+r)h_w+(1-r)h_z]}{h_w(h_w+(1-r)h_z)^2}-\frac{h_x[(1+r)h+(1-r)h_x]}{h(h+(1-r)h_x)^2}=0$$

Thus, $U_i(x)>U_i(z)$ holds if and only if $h<\hat{h}$. It is also easy to show that $h_w>\hat{h}$ if and only if $h_z>h_x$. Thus agents will switch to information z only when the public information is accurate enough relative to the prior information on w. In other words, if public information alone is very accurate, then acquiring more information on u is not necessary, so agents switch to acquiring information on w.

Note that the fact $U_i(z) > U_i(x)$ does not guarantee that acquiring information z rather than x is an equilibrium because computations of $U_i(z)$ and $U_i(x)$ are based on the assumption that *all* agents acquire the same information. To tell whether all agents will acquire information z or x, we need to check whether each agent has incentives to deviate and get a different kind of information when all the other agents acquire the same kind of information.

1. Deviation

Suppose that it is known that all the other agents in the economy are acquiring information x. If an agent deviates and acquires information z, how much will he get? Since all other agents are using information x, the average action $\bar{a}$ is given by

$$\bar{a} = \frac{hv + (1-r)h_x u}{h + (1-r)h_x} + \bar{w}$$

Note that the action of the deviant agent does not enter the picture because he is so small. Therefore, the deviant agent's action is given by the formula $a_d = (1-r)E_d(\theta) + rE_d(\bar{a})$, or

$$a_d = (1-r)\left[v + \frac{h_w \bar{w} + h_z z_d}{h_w + h_z}\right] + r(v + \bar{w})$$

Recall the definition of v. Thus, the deviant agent gets:

$$E(a_d - \bar{a})^2 = (1-r)^2 \left[\frac{h_x^2}{h\left(h + (1-r)h_x\right)^2} + \frac{h_z}{h_w(h_w + h_z)}\right]$$

For other agents, we have:

$$E\left(a_j - \bar{a}\right)^2 = \frac{(1-r)^2 h_x}{\left(h + (1-r)h_x\right)^2}$$

Therefore,

$$E\left(a_d-\bar{a}\right)^2-\int_0^1\left(a_j-\bar{a}\right)^2 dj=(1-r)^2\left[\frac{h_x(h_x-h)}{h\left(h+(1-r)h_x\right)^2}+\frac{h_z}{h_w(h_w+h_z)}\right]$$

From this, we can compute the deviating agent's expected utility as follows:

$$U_d(z)=-(1-r)\left[\frac{1}{h}+\frac{\left(h_w+rh_z\right)^2+(1-r)^2 h_z h_w}{h_w\left(h_w+h_z\right)^2}\right]-r\left[E\left(a_d-\bar{a}\right)^2-\int_0^1\left(a_j-\bar{a}\right)^2 dj\right]$$

Compute $U_j(x)$ and we get the following:

$$U_j(x)=-(1-r)\left[\frac{h+(1-r)^2 h_x}{\left(h+(1-r)h_x\right)^2}+\frac{1}{h_w}\right]-r\left[\frac{(1-r)^2 h_x}{\left(h+(1-r)h_x\right)^2}-\int_0^1\left(a_j-\bar{a}\right)^2 dj\right]$$

Thus by deviating, the agent gets, relative to non-deviation:

$$U_d(z)-U_j(x)=(1-r)^2\left[\frac{h_z}{h_w(h_w+h_z)}-\frac{h_x\left(h+h_x\right)}{h\left(h+(1-r)h_x\right)^2}\right]$$

From this we get:

$$U(x)>U_d(z) \quad \text{if and only if} \quad \frac{h_x\left(h+h_x\right)}{h\left(h+(1-r)h_x\right)^2}>\frac{h_z}{h_w(h_w+h_z)}$$

Thus deviation does not pay only when h is low enough to satisfy the following

$$\frac{h_x\left(h+h_x\right)}{h\left(h+(1-r)h_x\right)^2}>\frac{h_z}{h_w(h_w+h_z)} \qquad (D_x)$$

As long as this inequality holds true, agents acquiring information x is an equilibrium.

Similarly, we can show that all agents acquiring information z is an equilibrium if and only if the following inequality holds:

$$\frac{h_z\left(h_w+h_z\right)}{h_w\left(h_w+(1-r)h_z\right)^2} > \frac{h_x}{h(h+h_x)} \qquad (D_z)$$

2. Acquisition of Different Private Information by Agents of Positive Measure

Now, let's consider whether there is a sort of mixed strategy equilibrium. Suppose that agents of measure $m \in (0,1)$ acquires information x and the remaining agents of measure $(1-m)$ acquires information z.

Then, since the two groups of agents have different expectations, we have:

$$\bar{E}(\theta) = m\left[\frac{hv+h_x u}{h+h_x}+\bar{w}\right]+(1-m)\left[v+\frac{h_w\bar{w}+h_z w}{h_w+h_z}\right]$$

Thus those who acquire information x see this as:

$$E_x\left(\bar{E}(\theta)\right) = m\left[\frac{hv}{h+h_x}+\frac{h_x}{h+h_x}\left(\frac{hq+h_x x_j}{h+h_x}\right)+\bar{w}\right]+(1-m)(v+\bar{w})$$

And those who acquire information z see this as:

$$E_z\left(\bar{E}(\theta)\right) = m(v+\bar{w})+(1-m)\left[v+\frac{h_w\bar{w}}{h_w+h_z}+\frac{h_z}{h_w+h_z}\left(\frac{h_w\bar{w}+h_z z_i}{h_w+h_z}\right)\right]$$

Using the formula

$$a_i = (1-r)\sum_{k=0}^{\infty} r^k E_i(\bar{E}^k(\theta))$$

we can compute the optimal action of each agent. For those who acquires information x:

$$a_{x_i} = \frac{\left[h+r(1-m)h_x\right]v+(1-r)h_x x_i}{h+(1-rm)h_x}+\bar{w}$$

For those who acquires information z:

$$a_{z_j} = v + \frac{\left[h_w + rmh_z\right]\overline{w} + (1-r)h_z z_j}{h_w + [1 - r(1-m)]h_z}$$

The average action is given by

$$\overline{a} = \frac{\left(h + (1-m)h_x\right)v + m(1-r)h_x u}{h + (1-rm)h_x} + \frac{\left(h_w + mh_z\right)\overline{w} + (1-m)(1-r)h_z w}{h_w + \left[1 - r(1-m)\right]h_z}$$

From this, we can compute the expected utility of each group of agents. Below the subscript *m* refers to the fact that a *m* fraction of agents are acquiring information *x*.

$$\begin{aligned} U_m(z) = &-(1-r)\left[\frac{\left(h_w + rmh_z\right)^2 + (1-r)^2 h_w h_z}{h_w\left(h_w + \left[1 - r(1-m)\right]h_z\right)^2} + \frac{1}{h}\right] \\ &-r(1-r)^2\left[\frac{m^2 h_z^2 + h_w h_z}{h_w(h_w + \left[1 - r(1-m)\right]h_z)^2} + \frac{m^2 h_x^2}{h\left(h + (1-rm)h_x\right)^2}\right] \\ &+ r\int_0^1 \left(a_j - \overline{a}\right)^2 dj \end{aligned}$$

and

$$\begin{aligned} U_m(x) = &-(1-r)\left[\frac{\left(h + r(1-m)h_x\right)^2 + (1-r)^2 hh_x}{h\left(h + (1-rm)h_x\right)^2} + \frac{1}{h_w}\right] \\ &-r(1-r)^2\left[\frac{hh_x + (1-m)^2 h_x^2}{h\left(h + (1-rm)h_x\right)^2} + \frac{(1-m)^2 h_z^2}{h_w\left(h_w + [1 - r(1-m)]h_z\right)^2}\right] + r\int_0^1 \left(a_j - \overline{a}\right)^2 dj \end{aligned}$$

The difference between the two is given by

$$U_m(z) - U_m(x) = (1-r)^2\left[\frac{h_z(h_w + h_z)}{h_w\left(h_w + [1 - r(1-m)]h_z\right)^2} - \frac{h_x(h + h_x)}{h\left(h + (1-rm)h_x\right)^2}\right]$$

For this to be an equilibrium, there must be an $m\in(0,1)$ that equalizes the two expected utilities: $U_m(z)-U_m(x)=0$. Since the expression in the bracket is strictly decreasing in m, the following two conditions are necessary and sufficient for the existence of the mixed strategy equilibrium:

$$U_0(z)-U_0(x)>0 \Leftrightarrow \frac{h_z(h_w+h_z)}{h_w\left(h_w+(1-r)h_z\right)^2}>\frac{h_x}{h(h+h_x)}$$

$$U_1(z)-U_1(x)<0 \Leftrightarrow \frac{h_x(h+h_x)}{h\left(h+(1-r)h_x\right)^2}>\frac{h_z}{h_w(h_w+h_z)}$$

Note that the first condition is identical to the condition (D_z) that all agents acquiring information z is an equilibrium and the second condition is identical to condition (D_x) that all agents acquiring information x is an equilibrium. Therefore, the mixed equilibrium exists only when conditions (D_x) and (D_z) both are satisfied.

Proposition: There are two critical values of h, called h^* and h^{**} such that

(a) If $h<h^*$, all agents acquiring information x is the unique equilibrium.

(b) If $h>h^{**}$, then all agents acquiring information z is the unique equilibrium.

(c) If $h^*<h<h^{**}$, there are three types of equilibrium. All agents acquiring either information x or z only is an equilibrium. All agents playing a mixed strategy of getting information x with a probability $m\in(0,1)$ is also an equilibrium.

(Sketch of Proof) Consider the following two equations

$$f_x(h)=h_wh_x(h_w+h_z)(h+h_x)-h_zh(h+(1-r)h_x)^2=0$$

$$f_z(h)=h_zh(h_w+h_z)(h+h_x)-h_wh_x(h+(1-r)h_z)^2=0$$

It is easy to see $f_x(-(1-r)h_x)>0$, $f_x'(-(1-r)h_x)>0$ and $f_x(0)>0$. Thus, equation $f_x(h)=0$ has only one positive solution. Call it h^{**}. Also, since $f_z(h)$ is quadratic with a positive coefficient on h^2, and $f_z(0)<0$, there is only one positive solution to equation $f_z(h)=0$. Call it h^*. One can also show $h^* < h^{**}$. Note that all agents acquiring information x is an equilibrium if and only if $h < h^{**}$, and all agents acquiring information w is an equilibrium if and only if $h > h^*$. Since $h^* < \mathrm{h}^{**}$. Acquiring either information is an equilibrium if $h \in [h^*, h^{**}]$. The mixed strategy constitutes an equilibrium only when both are equilibrium. Thus the conclusion follows.

3. Social Welfare

We now compute the social welfare associated with each equilibrium. One can show:

$$W(x) = -(1-r)\left[\frac{h+(1-r)^2 h_x}{\left(h+(1-r)h_x\right)^2} + \frac{1}{h_w}\right]$$

$$W(z) = -(1-r)\left[\frac{1}{h} + \frac{h_w+(1-r)^2 h_z}{\left(h_w+(1-r)h_z\right)^2}\right]$$

and

$$W_m = -(1-r)\left[\frac{\left(h_w+rmh_z\right)^2+(1-r)^2 h_w h_z}{h_w\left(h_w+\left[1-r(1-m)\right]h_z\right)^2} + \frac{1}{h}\right]$$

It is obvious that the social welfare is increasing in the precision of the public information represented by the variable h.

IV. Conclusion

This note shows that when the uncertainty is generated by two dimensional random variables, the conclusions of Morris and Shin (2002) do not follow. While this may be indicative of the robustness of their results, how the mixed strategy equilibrium is actually obtained in an economy with many agents. More work needs to be done in this regard.

References

Morris, S. and Hyun Song Shin (2002), "Social value of public information," *The American Economic Review*, 92(5), 1521-1534.