M.S. THESIS

Gaussian states and channels with mathematical rigor

가우시안 양자상태와 양자채널의 엄밀한 정의

BY

GWAK, SEUNG-HYUN

FEBURARY 2020

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Abstract

Since the establishment of quantum mechanics, quantum information theory have been one of the main interests in mathematical physics. Especially, a bosonic gaussian state is one of the main objects in quantum information, since it is relatively easy to physically realize, so that we can conduct experiments, for example, with photons. There are many monographs and papers addressing (bosonic) gaussian states and gaussian quantum channels, but unfortunately, they usually show insufficient mathematical rigor. In this thesis we fill up omitted contents which are related to functional analysis and operator algebra.

We will mainly follow the approach of Alessio Serafini's monograph [Ser17] but clarify some omitted contents such as domain issues, self-adjointness of Hamiltonian operator and well-defineness of gaussian states, etc, which will allow us to analyze bosonic gaussian states and gaussian quantum channels without any ambiguity.

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Chapter 1

INTRODUCTION

The main objective of this thesis is to define (bosonic) gaussian states and analyze gaussian quantum channels more rigorously in terms of functional analysis and operator theory mainly following the approach of Alessio Serafini's monograph [Ser17].

In Chapter 2, we will define self-adjoint operators on infinite dimensional Hilbert spaces and study conditions ensuring self-adjointness. In Chapter 3, we introduce important unbounded operators, such as the position operator, the momentum operator and the numbering operator based on the creation and annihilation operators. Then, we will carefully check self-adjointness of these operators. In Chapter 4, we define 2nd order Hamiltonian operators $\hat{H}$ and we will verify the self-adjointness of $\hat{H}$ in full generality.

Secondly, we study bosonic gaussian states and gaussian quantum channels which send gaussian states to gaussian states. In Chapter 5, gaussian states are defined and statistical moments ($\bar{r}$: mean, σ : covariance matrix) of gaussian states are introduced, which can completely determine gaussian states. In Chapter 6, we define gaussian unitary transforms and introduce gaussian quantum channels. Furthermore, the action of gaussian channels on gaussian states will be precisely shown in this chapter.

Throughout this thesis, all $\mathcal{H}$ are infinite dimensional separable Hilbert spaces. We will freely use notations from Alessio Serafini's monograph [Ser17].

Chapter 2

PRELIMINARY

From Chapter 2 to Chapter 4, we will mainly follow [Hal13].

2.1 Distribution

Definition 2.1.1 (Distribution)**.** A sequence $\{\phi_m\}_{m\geq 1} \subseteq C_c^\infty(\mathbb{R})$ is said to converge to $\phi \in C_c^\infty(\mathbb{R})$ if (1) there exists a single compact set K containing the support of all the ϕ_m's and (2) ϕ_m converges uniformly to ϕ, (3) each derivative of ϕ_m converges uniformly to the corresponding derivative of ϕ. A *distribution* on $\mathbb{R}$ is a linear map $T : C_c^\infty(\mathbb{R}) \to \mathbb{C}$, having the following continuity property: if ϕ_m converges to ϕ in the sense of definition above, $T(\phi_m)$ converges to $T(\phi)$.

If T is a distribution, and f is a locally integrable function, the expression "T is given by f" means that

$$T(\phi) = \int_{\mathbb{R}} \phi(x) f(x) dx$$

for all $\phi \in C_c^\infty(\mathbb{R})$.

Definition 2.1.2 (Derivative of distribution)**.** If T is a distribution given by f, define the distribution $\frac{dT}{dx}$ by

$$\frac{dT}{dx}(\phi) = -T\left(\frac{d\phi}{dx}\right), \qquad \phi \in C_c^\infty(\mathbb{R}).$$

Remark 2.1.3. Since $\frac{dT}{dx}(\phi_m) = -T\left(\frac{d\phi_m}{dx}\right)$ and $\frac{d\phi_m}{dx} \to \frac{d\phi}{dx}$,

$$\frac{dT}{dx}(\phi_m) \to \frac{dT}{dx}(\phi)$$

Furthermore, suppose that T is given by a continuously differentiable function $f(x)$. Then, for all differentiable ϕ,

$$\begin{aligned}\frac{dT}{dx}(\phi) &= \int_{\mathbb{R}} \phi \frac{d}{dx} f(x) dx \\ &= [f(x)\phi(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x) \frac{d}{dx}\phi(x) dx \\ &= \int_{-\infty}^{\infty} f(x)\left(-\frac{d}{dx}\phi(x)\right) dx \\ &= -T\left(\frac{d\phi}{dx}\right).\end{aligned}$$

So, the definition above is well defined.

Proposition 2.1.4. Suppose that ϕ and ψ are in $L^2(\mathbb{R})$, the $\frac{d\psi}{dx} = \phi$ holds in the distribution sense if and only if,

$$-\left\langle \frac{d\chi}{dx}, \psi \right\rangle = \langle \chi, \phi \rangle$$

for all $\chi \in C_c^\infty(\mathbb{R})$.

Proof. $(\Rightarrow)$

$$\begin{aligned}-\int_{\mathbb{R}} \overline{\frac{d\chi(x)}{dx}} \psi(x) dx = -T\left(\frac{d\overline{\chi}}{dx}\right) &= \frac{dT}{dx}(\overline{\chi}) \\ &= \int_{\mathbb{R}} \overline{\chi(x)} \frac{d}{dx}\psi(x) dx \\ &= \langle \chi, \phi \rangle, \qquad \forall \chi \in C_c^\infty(\mathbb{R}).\end{aligned}$$

($\Leftarrow$) for all $\chi \in C_c^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} \overline{\chi(x)} \left(\frac{d}{dx}\psi(x) - \phi(x) \right) dx = 0$$

Therefore, $\frac{d}{dx}\psi(x) = \phi(x)$ almost everywhere. $\square$

2.2 Self-adjoint operator

Definition 2.2.1 (Unbounded operator)**.** An unboudned operator A on $\mathcal{H}$ is a linear map from a dense subspace $Dom(A) \subset \mathcal{H}$ into $\mathcal{H}$.

Definition 2.2.2 (Adjoint operator)**.** For an unbounded operator A on $\mathcal{H}$, the adjoint A^* of A is defined as follows.

$$Dom(A^*) = \{\phi \in \mathcal{H} : \exists! \chi \in \mathcal{H}, such\ that\ \langle \phi, A\psi \rangle = \langle \chi, \psi \rangle, \forall \psi \in Dom(A)\}$$

and $A^*\phi = \chi$, for $\phi \in Dom(A^*)$.

It means that a vector $\phi \in \mathcal{H}$ belongs to the $Dom(A^*)$ if the linear functional

$$\langle \phi, A\cdot \rangle ,$$

defined on $Dom(A)$, is bounded. For $\phi \in Dom(A^*)$, let $A^*\phi$ be the unique vector χ such that

$$\langle \chi, \psi \rangle = \langle \phi, A\psi \rangle$$

for all $\psi \in Dom(A)$.

Definition 2.2.3 (Self-adjoint operator)**.** An unbounded operator A on $\mathcal{H}$ is *symmetric* if $\langle \phi, A\psi \rangle = \langle A\phi, \psi \rangle$ for all $\phi, \psi \in Dom(A)$. The operator A is *self-adjoint* if $Dom(A^*) = Dom(A)$ and $A^*\phi = A\phi$ for all $\Phi \in Dom(A)$.

Definition 2.2.4 (Extension of an operator)**.** An unbounded operator A is extension of unbounded operator B if $Dom(B) \subset Dom(A)$ and $A = B$ on $Dom(B)$.

Proposition 2.2.5. An unbounded operator A is symmetric if and only if A^* is extension of A.

Proof. If A is symmetric then for all $\phi \in Dom(A)$,

$$|\langle \phi, A\psi \rangle| = |\langle A\phi, \psi \rangle| \leq \|A\phi\| \|\psi\| < \infty,$$

showing that $\phi \in Dom(A^*)$ so that $Dom(A) \subset Dom(A^*)$.

If A^* is a extension of A, then for each $\phi \in Dom(A) \subset Dom(A^*)$,

$$\langle \phi, A\psi \rangle = \langle A^*\phi, \psi \rangle = \langle A\phi, \psi \rangle ,$$

for all $\psi \in Dom(A)$ which shows that A is symmetric. □

2.3 Conditions for self-adjointness

Definition 2.3.1 (Closed, closable operator)**.** An unbounded operator A on $\mathcal{H}$ is said to be *closed* if the graph of A is a closed subset of $\mathcal{H} \times \mathcal{H}$. An unbounded operator A on $\mathcal{H}$ is said to be *closable* if the closure in $\mathcal{H} \times \mathcal{H}$ of the graph of A is the graph of a function.

If A is closable, then the closure A^{cl} of A is the operator with graph equal to the closure of the graph of A.

Remark 2.3.2. An operator A is closed if and only if the following condition holds: Suppose a sequence ψ_n belongs to $Dom(A)$ and suppose that there exist vectors ψ and ϕ in $\mathcal{H}$ with $\psi_n \to \psi$ and $A\psi_n \to \phi$. Then ψ belongs to $Dom(A)$ and $A\psi = \phi$.

Definition 2.3.3 (Essentially self-adjoint)**.** An unbounded operator A on $\mathcal{H}$ is said to be *essentially self-adjoint* if A is symmetric and closable and A^{cl} is self-adjoint.

Lemma 2.3.4. If A is an unbounded operator on $\mathcal{H}$, then the graph of the operator A^* is closed in $\mathcal{H} \times \mathcal{H}$. So that a symmetric operator is always closable.

Proof. Suppowe ψ_n is a sequence in the domain of A^* that converges to some $\psi \in \mathcal{H}$. Suppose also that $A^*\phi_n$ converges to some $\phi \in \mathcal{H}$. Then $\langle \psi_n, A\cdot \rangle = \langle A^*\psi_n, \cdot \rangle$ and for any $\chi \in Dom(A)$, we have

$$\langle \psi, A\chi \rangle = \lim_{n\to\infty} \langle \psi_n, A\chi \rangle = \lim_{n\to\infty} \langle A^*\psi_n, \chi \rangle = \langle \phi, \chi \rangle .$$

This shows that ψ belongs to the domain of A^* and that $A^*\psi = \phi$, establishing that the graph of A^* is closed. If A is symmetric, A^* is an extension of A. Since, as we have just proved, A^* is closed, A has a closed extension and is therefore closable. □

Lemma 2.3.5. If A is a closable operator on $\mathcal{H}$, then $A^{cl^*} = A^*$.

Proof. Suppose that for some $\psi \in \mathcal{H}$ there exists a ϕ such that $\left\langle \psi, A^{cl}\chi \right\rangle = \langle \phi, \chi \rangle$ for all $\chi \in Dom(A^{cl})$. Since A^{cl} is an extension of A, it follows that $\langle \psi, A\chi \rangle = \langle \phi, \chi \rangle$ for all $\chi \in Dom(A)$. This shows that $Dom(A^*) \supset Dom((A^{cl})^*)$.

In the other direction, suppose that for some $\psi \in \mathcal{H}$ there exists a ϕ such that $\langle \psi, A\chi \rangle = \langle \phi, \chi \rangle$ for all $\chi \in Dom(A)$. Suppose now $\xi \in Dom(A^{cl})$ with $A^{cl}\xi = \eta$. Then there exists a sequence $\chi_n \in Dom(A)$ with $\chi_n \to \xi$ and $A\chi_n \to \eta$, and we have

$$\langle \psi, A\chi_n \rangle = \langle \phi, \chi_n \rangle$$

for all n. Letting n tend to infinity, we obtain $\langle \psi, \eta \rangle = \langle \phi, \xi \rangle$ so that $\left\langle \psi, A^{cl}\xi \right\rangle = \langle \phi, \eta \rangle$. This shows that $\psi \in Dom((A^{cl})^*)$ and $A^{cl}\psi = \phi$. Thus, $Dom(A^*) \subset Dom((A^{cl})^*)$.

□

Lemma 2.3.6. If A is symmetric on $Dom(A)$, then for $\psi \in Dom(A)$,

$$\langle (A - iI)\psi, (A - iI)\psi \rangle \geq \langle \psi, \psi \rangle .$$

Proof.

$$\langle A\psi, A\psi \rangle - i \langle \psi, A\psi \rangle + i \langle A\psi, \psi \rangle - i \langle \psi, -i\psi \rangle = \langle A\psi, A\psi \rangle + \langle \psi, \psi \rangle \geq \langle \psi, \psi \rangle .$$

□

Lemma 2.3.7. Let A be a closed operator and λ be an element of $\mathbb{C}$. Suppose that there exists $\varepsilon > 0$ such that

$$\|(A - \lambda I)\psi\| \geq \varepsilon \|\psi\|$$

for all $\psi \in Dom(A)$. Then the range of $A - \lambda I$ is a closed subspace of $\mathcal{H}$.

Proof. Assume that ϕ_n is a sequence in the range of $A - \lambda I$ converging to some ϕ. Then $\phi_n = (A - \lambda I)\psi_n$, for some sequence $\psi_n \in Dom(A)$. Then with $\psi = \psi_n - \psi_m$ shows that

$$\|\psi_n - \psi_m\| \leq \frac{1}{\varepsilon} \|\phi_n - \phi_m\|.$$

So ϕ_n is Cauchy sequence and thus convergent to some vector ψ. Since $\phi_n \to \phi$ and $(A - \lambda I)\psi_n = \phi_n \to \phi$, we have that

$$A\psi_n = \lambda\psi_n + \phi_n \to \lambda\psi + \phi.$$

Thus by the definition of a closed operator, $\psi \in Dom(A)$ and $A\psi = \lambda\psi + \phi$. This means that $(A - \lambda I)\psi = \phi$ and so the $Range(A - \lambda I)$ is closed. □

Lemma 2.3.8. If A is an unbounded operator on $\mathcal{H}$, then

$$(Range(A))^{\perp} = ker(A^*).$$

Proof. First assume that $\psi \in (Range(A))^{\perp}$. Then for all $\phi \in Dom(A)$ we have

$$\langle \psi, A\phi \rangle = 0.$$

Thus, from the definition of the adjoint, we conclude that $\phi \in Dom(A^*)$ and $A^*\phi = 0$. Meanwhile, suppose that $\psi \in Dom(A^*)$ and that $A^*\psi = 0$. The only way this can happen is if the linear functional $\langle \psi, A\cdot \rangle$ is zero on $Dom(A)$, which means that ψ is orthogonal to the image of A. □

Theorem 2.3.9 (Conditions for self-adjointness)**.** If A is a symmetric operator on $L^2(\mathbb{R})$ and $Range(A - iI)$ and $Range(A + iI)$ are dense subspace of $L^2(\mathbb{R})$ then A is essentially self-adjoint.

Proof. If A is symmetric, then A is closable, and $A^{cl} \subset A^*$. By Lemma 2.3.5.,
$A^{cl} \subset A^* = (A^{cl})^*$. So $(A^{cl})^*$ is extension of A^{cl}, therefore A^{cl} is symmetric.
By Lemma 2.3.6.,

$$\|(A^{cl} - iI)\psi\|^2 \geq \|\psi\|^2 \quad \psi \in Dom(A^{cl})$$

and showing that $A^{cl} - iI$ is injective.
Since the range of $A - iI$ is dense, $Range(A^{cl} - iI)$ is also dense. By Lemma 2.3.7., $Range(A^{cl} - iI)$ is closed, hence $Range(A^{cl} - iI) = \mathcal{H}$. In the same way, $Range(A^{cl} + iI) = \mathcal{H}$.

Now, $(A^{cl} - iI)^* = (A^{cl})^* + iI$,which is an extension of $A^{cl} + iI$. Suppose $(A^{cl})^* + iI$ is a proper extension of $A^{cl} + iI$. Then since $A^{cl} + iI$ already maps onto $\mathcal{H}$, $(A^{cl})^* + iI$ cannot be injective. Thus, the operator

$$(A^{cl})^* + iI = A^* + iI = (A - iI)^*$$

must have a non-trivial kernel. Then by Lemma 2.3.8., $Range(A - iI)$ is not dense, contradicting our assumptions. We conclude, therefore, that $(A^{cl})^* + iI = A^{cl} + iI$, so that $(A^{cl})^* = A^{cl}$. □

Corollary 2.3.10. If A is a symmetric operator on $L^2(\mathbb{R})$, and $A^* \pm iI$ are injective on $Dom(A^*)$ then, A is essentially self-adjoint.

2.4 Spectral theorem

Definition 2.4.1 (Spectrum)**.** Suppose A is an unbounded operator on $\mathcal{H}$. A number $\lambda \in \mathbb{C}$ belongs to the *resolvent set* of A if there exists a bounded operator B with the following properties:

(1) for all $\psi \in \mathcal{H}$, $B\psi$ belongs to $Dom(A)$ and $(A - \lambda I)B\psi = \psi$, and

(2) for all $\psi \in Dom(A)$ we have $B(A - \lambda I)\psi = \psi$.

If no such bounded operator B exists, then λ belongs to the *spectrum* of A denoted as $\sigma(A)$.

Definition 2.4.2 (Projection valued measure)**.** Let X be a set and Ω a σ-algebra in X. A map $\mu : \Omega \to \mathcal{B}(\mathcal{H})$ is called a *projection-valued measure* if the following properties are satisfied.

1. For each $E \in \Omega$, $\mu(E)$ is an orthogonal projection.
2. $\mu(\varnothing) = 0$ and $\mu(X) = I$.
3. If $E_1, E_2, E_3, \ldots$ in Ω are disjoint, then for all $v \in \mathcal{H}$, we have

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) v = \sum_{j=1}^{\infty} \mu(E_j)v,$$

where the convergence of the sum is the norm topology on $\mathcal{H}$.

4. For all $E_1, E_2 \in \Omega$, we have $\mu(E_1 \cap E_2) = \mu(E_1)\mu(E_2)$.

Remark 2.4.3. Given that μ is projection-valued measure defined as above, we can construct a non-negative, real-valued measure μ_ϕ by

$$\mu_\phi(E) = \langle \phi, \mu(E)\phi \rangle$$

for each measurable set E.

Theorem 2.4.4 (Spectral theorem)**.** Suppose A is a self-adjoint operator on $\mathcal{H}$. Then there is a unique projection-valued measure μ on $\sigma(A)$ with values in $\mathcal{B}(\mathcal{H})$ such that

$$\int_{\sigma(A)} \lambda d\mu(\lambda) = A.$$

Furthermore,

$$\langle \psi, A\psi \rangle = \left\langle \psi, \int_{\sigma(A)} \lambda d\mu(\lambda)\psi \right\rangle = \int_{\sigma(A)} \lambda d\mu_\psi(\lambda).$$

Definition 2.4.5 (Functional calculus)**.** For any measurable function f on $\sigma(A)$, define a (possibly unbounded) operator, denoted $f(A)$, by

$$f(A) = \int_{\sigma(A)} f(\lambda) d\mu(\lambda).$$

2.5 Stone's theorem

Definition 2.5.1 (One-parameter unitary group)**.** A *one-parameter unitary group* on $\mathcal{H}$ is a family $U(t)$, $t \in \mathbb{R}$, of unitary operators with the property that $U(0) = I$ and that $U(s+t) = U(s)U(t)$ for all $s, t \in \mathbb{R}$. A one-parameter unitary group is said to be strongly continuous if

$$\lim_{s \to t} \|U(t)\psi - U(s)\psi\| = 0$$

for all $\psi \in \mathcal{H}$ and all $t \in \mathbb{R}$.

And if $U(\cdot)$ is a strongly continuous one-parameter unitary group, the "infinitesimal generator" of $U(\cdot)$ is the operator A given by

$$A\psi = \lim_{t \to 0} \frac{1}{i} \frac{U(t)\psi - \psi}{t},$$

with $Dom(A)$ consisting of the set of $\psi \in \mathcal{H}$ for which the limit above exists in the norm topology on $\mathcal{H}$.

Theorem 2.5.2. Suppose A is a self-adjoint operator on $\mathcal{H}$ and let $U(\cdot)$ be defined by

$$U(t) = \mathrm{e}^{itA} \quad t \in \mathbb{R}$$

where the operator e^{itA} is defined by functional calculus for A. Then the following hold.

1. $U(t)$ is unitary operator.
2. For all $\psi \in Dom(A)$, we have

$$iA\psi = \lim_{t \to 0} \frac{U(t)\psi - \psi}{t},$$

where the limit is in the norm topology on $\mathcal{H}$.

Proof. 1. Since $\sigma(A) \subset \mathbb{R}$, the function $f(\lambda) := \mathrm{e}^{it\lambda}$ is bounded on $\sigma(A)$ and satisfies $f(\lambda)\overline{f(\lambda)} = 1$ for all $\lambda \in \sigma(A)$. Thus, the operator $f(A)$ is bounded and satisfies

$$f(A)f(A)^* = f(A)^*f(A) = I,$$

which shows that $f(A) = \mathrm{e}^{itA}$ is unitary.

2. $A = \int_{\mathbb{R}} \lambda d\mu(\lambda)$, and take $\psi \in Dom(A)$. Then

$$\left\| \frac{U(t)\psi - \psi}{t} - iA\psi \right\|^2 = \int_{\mathbb{R}} \left| \frac{\mathrm{e}^{it\lambda} - 1}{t} - i\lambda \right|^2 d\mu_\psi(\lambda).$$

Since

$$\left|\frac{(\mathrm{e}^{it\lambda}-1)}{t}\right| = \left|\frac{\mathrm{e}^{it\lambda/2}(\mathrm{e}^{it\lambda/2}-\mathrm{e}^{-it\lambda/2})}{t}\right| = \left|\frac{2i\sin(t\lambda/2)}{t}\right| \leq \lambda,$$

we have $\left|(\frac{\mathrm{e}^{it\lambda}-1}{t}) - i\lambda\right| \leq 2\lambda$. Meanwhile, since ψ is in the domain of the operator $A = \int_{\mathbb{R}} \lambda d\mu(\lambda)$, we have $\int_{\mathbb{R}} \lambda^2 d\mu_\psi(\lambda) < \infty$. Thus, by dominated convergence theorem with $4\lambda^2$ as our dominating function,

$$\left|\frac{\mathrm{e}^{it\lambda}-1}{t} - i\lambda\right|^2 \leq 4\lambda^2,$$

$$\lim_{t\to 0}\left\|\frac{U(t)\psi-\psi}{t} - iA\psi\right\|^2 = \lim_{t\to 0}\int_{\mathbb{R}}\left|\frac{\mathrm{e}^{it\lambda}-1}{t} - i\lambda\right|^2 d\mu_\psi(\lambda) = 0.$$

□

Theorem 2.5.3. If A is self-adjoint operator, $\psi \in Dom(A)$ and $U(t)$ is defined above, then for all $t \in \mathbb{R}$, the vector $U(t)\psi$ belongs to $Dom(A)$ and

$$\lim_{h\to 0}\frac{U(t+h)\psi - U(t)\psi}{h} = iU(t)A\psi = iAU(t)\psi.$$

Proof. We compute that

$$\frac{U(t+h)\psi - U(t)\psi}{h} = U(t)\frac{[U(h)\psi - \psi]}{h}.$$

Since $\psi \in Dom(A)$,

$$\lim_{h\to 0}\frac{U(t+h)\psi - U(t)\psi}{h} = \lim_{h\to 0}U(t)\frac{[U(h)\psi - \psi]}{h} = iU(t)A\psi.l$$

On the other hand,

$$\frac{U(t+h)\psi - U(t)\psi}{h} = \frac{U(h)(U(t)\psi) - (U(t)\psi)}{h}.$$

Thus,

$$\lim_{h\to 0}\frac{U(t+h)\psi - U(t)\psi}{h} = \lim_{h\to 0}\frac{U(h)(U(t)\psi) - (U(t)\psi)}{h} = iA(U(t)\psi) = iU(t)A\psi.$$

This shows $U(t)\psi \in Dom(A)$. □

Theorem 2.5.4 (Stone's theorem)**.** Suppose $U(\cdot)$ is a strongly continuous one-parameter unitary group on $\mathcal{H}$. Then the infinitesimal generator A of $U(\cdot)$ is densely defined and self-adjoint, and $U(t) = \mathrm{e}^{itA}$ for all $t \in \mathbb{R}$.

Chapter 3

SOME SELF-ADJOINT OPERATORS

3.1 Position and momentum operators

Theorem 3.1.1 (Position operator $\hat{x}$)**.** Let $\hat{x}$ be an unbounded operator called *position* operator, with domain

$$Dom(\hat{x}) = \{\psi \in L^2(\mathbb{R}) : \hat{x}\psi(x) \in L^2(\mathbb{R})\}$$

and given by

$$\hat{x}\psi(x) = x\psi(x).$$

Then $Dom(\hat{x})$ is dense in $L^2(\mathbb{R})$ and $\hat{x}$ is self-adjoint on this domain.

Proof. Let us define subset E_m of $\mathbb{R}$ by

$$\mathrm{E}_m = \{x \in L^2(\mathbb{R}) : |x| < m\}$$

so that $\bigcup_m \mathrm{E}_m = \mathbb{R}$. Then for any $\psi \in L^2(\mathbb{R})$, the function ψ_{E_m} defined by

$$\psi_{\mathrm{E}_m}(x) = \begin{cases} \psi(x), & x \in \mathrm{E}_m \\ 0 & x \in \mathbb{R} - \mathrm{E}_m \end{cases}$$

belongs to $Dom(\hat{x})$. On the other hand, using dominated convergence theorem, $\|\psi_{\mathrm{E}_m} - \psi\|_2 \to 0$ as $m \to \infty$, so that $Dom(\hat{x})$ is dense in $L^2(\mathbb{R})$.

Since

$$\langle \hat{x}\phi, \psi \rangle = \int_{\mathbb{R}} \overline{\hat{x}\phi(x)}\psi(x)dx = \int_{\mathbb{R}} x\overline{\phi(x)}\psi(x)dx = \int_{\mathbb{R}} \overline{\phi(x)}x\psi(x)dx = \langle \phi, \hat{x}\psi \rangle ,$$

$\hat{x}$ is symmetric on $Dom(\hat{x})$.

Thus $\hat{x}^*$ is an extension of $\hat{x}$, so that $Dom(\hat{x}^*) \supset Dom(\hat{x})$.

Meanwhile, suppose $\phi \in Dom(\hat{x}^*)$, meaning that

$$\psi \mapsto \langle \phi, \hat{x}\psi \rangle , \qquad \psi \in Dom(\hat{x})$$

is bounded linear functional, and there exists a unique $\chi \in L^2(\mathbb{R})$ such that

$$\langle \phi, \hat{x}\psi \rangle = \int_{\mathbb{R}} \overline{\phi(x)}\hat{x}\psi(x)dx = \int_{\mathbb{R}} \overline{\chi(x)}\psi(x)dx = \langle \chi, \psi \rangle$$

so that

$$\int_{\mathbb{R}} \left[\overline{\phi(x)}\hat{x} - \overline{\chi(x)}\right] \psi(x)dx = 0$$

for all $\psi \in Dom(\hat{x})$.

Taking $\psi = (\phi(x)x - \chi(x))_{\mathrm{E}_m} \in Dom(\hat{x})$,

$$\begin{aligned}\int_{\mathbb{R}} \left[\overline{\phi(x)}\hat{x} - \overline{\chi(x)}\right] (\phi(x)x - \chi(x))_{\mathrm{E}_m} dx &= \int_{\mathrm{E}_m} \left[\overline{\phi(x)}x - \overline{\chi(x)}\right] [\phi(x)x - \chi(x)]\, dx \\ &= \int_{\mathrm{E}_m} \overline{[\phi(x)x - \chi(x)]}\, [\phi(x)x - \chi(x)]\, dx \\ &= 0\end{aligned}$$

So $\phi(x)x - \chi(x) = 0$ alomst everywhere on $\mathbb{R}$. Thus

$$\hat{x}\phi(x) = x\phi(x) = \chi(x) \in L^2(\mathbb{R})$$

So $\phi \in Dom(\hat{x}) \Rightarrow Dom(\hat{x}^*) \subset Dom(\hat{x})$

$\therefore Dom(\hat{x}^*) = Dom(\hat{x})$ and $\hat{x}$ is self-adjoint. $\square$

Theorem 3.1.2 (Momentum operator $\hat{p}$)**.** Let us define a domain $Dom(\hat{p}) \subset L^2(\mathbb{R})$ as follows.

$$Dom(\hat{p}) = \left\{ \psi \in L^2(\mathbb{R}) : p\widehat{\psi}(p) \in L^2(\mathbb{R}) \right\}$$

where $\widehat{\psi}$ is the Fourier transform of ψ and $\hat{p}$ is given by

$$\hat{p}\psi = \mathcal{F}^{-1}(p\widehat{\psi}(p)).$$

Then, the *momentum* operator $\hat{p}$ is self-adjoint on $Dom(\hat{p})$.

Proof. Since $\hat{x}$ is self-adjoint,

$$\langle \hat{x}\mathcal{F}(\psi), \mathcal{F}(\phi) \rangle = \langle \mathcal{F}(\psi), \hat{x}\mathcal{F}(\phi) \rangle$$

for all $\mathcal{F}(\psi), \mathcal{F}(\phi) \in Dom(\hat{x})$. Then,

$$\left\langle \mathcal{F}^{-1}\hat{x}\mathcal{F}(\psi), \phi \right\rangle = \left\langle \psi, \mathcal{F}^{-1}\hat{x}\mathcal{F}(\phi) \right\rangle$$

for all $\psi, \phi \in \mathcal{F}^{-1}(Dom(\hat{x}))$ so that $\hat{p} = \mathcal{F}^{-1}\hat{x}\mathcal{F}$ is self adjoint on $\mathcal{F}^{-1}(Dom(\hat{x}))$.

From now on, what to show is $Dom(\hat{p}) = \mathcal{F}^{-1}(Dom(\hat{x}))$. For all $\phi \in \mathcal{F}^{-1}(Dom(\hat{x}))$, $p\widehat{\phi}(p)$ belongs to $L^2(\mathbb{R})$ so that $\mathcal{F}^{-1}(Dom(\hat{x})) \subset Dom(\hat{p})$. On the other hand, For all $\psi \in Dom(\hat{p})$, $p\mathcal{F}(\psi)$ belongs to $L^2(\mathbb{R})$ so that $\hat{x}\mathcal{F}(\psi) \in L^2(\mathbb{R})$. Then, $\mathcal{F}(\psi)$ belongs to $Dom(\hat{x})$ and $\psi \in \mathcal{F}^{-1}(Dom(\hat{x}))$. This means $Dom(\hat{p}) \subset \mathcal{F}^{-1}(Dom(\hat{x}))$.
$\therefore Dom(\hat{p}) = \mathcal{F}^{-1}(Dom(\hat{x}))$ □

Lemma 3.1.3. Suppose $\psi \in L^2(\mathbb{R})$ has the property that $\frac{d\psi}{dx}$(computed in the distribution sense) is equal to an L^2 function ϕ, then, $ip\widehat{\psi}(p) \in L^2(\mathbb{R})$.

Conversely, Suppose $\psi \in L^2(\mathbb{R})$ has property that $p\widehat{\psi}(p)$ belongs to $L^2(\mathbb{R})$, then $\frac{d\psi}{dx}$(computed in distribution sense) is equal to
$\mathcal{F}^{-1}(ip\mathcal{F}(\psi)) = \mathcal{F}^{-1}(ip\widehat{\psi}(p)) \in L^2(\mathbb{R})$, where $\mathcal{F}$ is Fourier transform,
and each $\widehat{\psi}$, $\widehat{\phi}$ is the Fourier transform of ψ, ϕ.

Proof. ($\Rightarrow$) Suppose $\psi \in L^2(\mathbb{R})$ has the property that $\frac{d\psi}{dx}$(computed in the distribution sense) is equal to an L^2 function ϕ, then, by the unitarity of the Fourier transform (Parseval theorem)

$$\langle \chi, \phi \rangle = -\left\langle \frac{d\chi}{dx}, \psi \right\rangle = -\left\langle \mathcal{F}\left(\frac{d\chi}{dx}\right), \mathcal{F}(\psi) \right\rangle = -\left\langle ip\mathcal{F}(\chi), \mathcal{F}(\psi) \right\rangle$$

for all $\chi \in C_c^\infty(\mathbb{R})$. Thus,

$$\langle \mathcal{F}(\chi), \mathcal{F}(\phi) \rangle = -\left\langle ip\mathcal{F}(\chi), \mathcal{F}(\psi) \right\rangle, \quad \chi \in C_c^\infty(\mathbb{R})$$

so that

$$\int_{\mathbb{R}} \overline{\widehat{\chi}(p)}\widehat{\phi}(p)dp = -\int_{\mathbb{R}} \overline{ip\widehat{\chi}(p)}\widehat{\psi}(p)dp = \int_{\mathbb{R}} \overline{\widehat{\chi}(p)}ip\widehat{\psi}(p)dp.$$

for all $\chi \in C_c^\infty(\mathbb{R})$.

So, $\widehat{\phi}(p) = ip\widehat{\psi}(p)$ for almost every $p \in \mathbb{R}$. Since $\widehat{\phi}(p)$ belongs to $L^2(\mathbb{R})$, $ip\widehat{\psi}(p)$ belongs to $L^2(\mathbb{R})$.

($\Leftarrow$) Since $p\mathcal{F}(\psi) \in L^2(\mathbb{R})$ for all $\chi \in C_c^\infty(\mathbb{R})$,

$$\begin{aligned}
\langle \mathcal{F}(\chi), ip\mathcal{F}(\psi) \rangle &= -\left\langle ip\mathcal{F}(\chi), \mathcal{F}(\psi) \right\rangle \\
&= -\left\langle \mathcal{F}\left(\frac{d\chi}{dx}\right), \mathcal{F}(\psi) \right\rangle \\
&= -\left\langle \frac{d\chi}{dx}, \psi \right\rangle \\
&= \left\langle \chi, \frac{d\psi}{dx} \right\rangle.
\end{aligned}$$

Therefore,

$$\left\langle \chi, \mathcal{F}^{-1}(ip\mathcal{F}(\psi)) \right\rangle = \langle \mathcal{F}(\chi), ip\mathcal{F}(\psi) \rangle = \left\langle \chi, \frac{d\psi}{dx} \right\rangle$$

for all $\chi \in C_c^\infty(\mathbb{R})$.

□

Corollary 3.1.4. Define a domain $Dom(\hat{p}) \subset L^2(\mathbb{R})$ as follows.

$$Dom(\hat{p}) = \left\{ \psi \in L^2(\mathbb{R}) : -i\frac{d\psi}{dx} \in L^2(\mathbb{R}) \right\}$$

where $\frac{d\psi}{dx}$ is computed in distributional sense. And $\hat{p}$ is given by

$$\hat{p}\psi = -i\frac{d\psi}{dx}.$$

Then, the $momentum$ operator $\hat{p}$ is self-adjoint on $Dom(\hat{p})$.

3.2 Creation and annihilation operators

Definition 3.2.1 (Schwartz space)**.** The *Schwartz space* is the subspace of $L^2(\mathbb{R}^n)$ given by

$$\mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : \|f\|_{\alpha,\beta} < \infty \quad \forall \alpha, \beta \in \mathbb{N}^n\}$$

where α, β are multi-indices, $C^\infty(\mathbb{R}^n)$ is the set of smooth functions from $\mathbb{R}^n$ to $\mathbb{C}$ and

$$\|f\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} \left| x^\alpha D^\beta f(x) \right|$$

with $Df = \frac{df}{dx}$.

Theorem 3.2.2. The position and momentum operator $\hat{x}$ and $\hat{p}$ do not commute, but satisfy the relation

$$[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = iI,$$

on $\mathcal{S}(\mathbb{R})$.

Proof. For all $\psi \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned}
\hat{p}\hat{x}\psi &= -i\frac{d}{dx}(x\psi(x)) \\
&= -i\psi(x) - ix\frac{d\psi}{dx} \\
&= -i\psi(x) + \hat{x}\hat{p}\psi.
\end{aligned}$$

$\square$

Theorem 3.2.3. We define $\hat{a} := \frac{\hat{x}+i\hat{p}}{\sqrt{2}}$ and $\hat{a}^{\dagger} := \frac{\hat{x}-i\hat{p}}{\sqrt{2}}$ with $Dom(\hat{a}) = Dom(\hat{a}^{\dagger}) = \mathcal{S}(\mathbb{R})$.

Then for all $\phi, \psi \in \mathcal{S}(\mathbb{R})$,

$$\left\langle \phi, \hat{a}^{\dagger}\psi \right\rangle = \langle \hat{a}\phi, \psi \rangle$$

and $\hat{a}^{\dagger}\hat{a}$ is symmetric on $\mathcal{S}(\mathbb{R})$.

Proof.

$$\begin{aligned}\int_{\mathbb{R}} \overline{\phi(x)} \left(x\psi(x) - \frac{d}{dx}\psi(x) \right) dx &= \int_{\mathbb{R}} \overline{\phi(x)} x\psi(x)dx - \int_{\mathbb{R}} \overline{\phi(x)} \frac{d}{dx}\psi(x)dx \\ &= \int_{\mathbb{R}} x\overline{\phi(x)}\psi(x)dx + \int_{\mathbb{R}} \frac{d}{dx}\overline{\phi(x)}\psi(x)dx \\ &= \int_{\mathbb{R}} \left(x\overline{\phi(x)} + \frac{d}{dx}\overline{\phi(x)} \right) \psi(x)dx\end{aligned}$$

So $\left\langle \phi, \hat{a}^{\dagger}\psi \right\rangle = \langle \hat{a}\phi, \psi \rangle$.

Since $\hat{a}\phi$ and $\hat{a}\psi$ belong to $\mathcal{S}(\mathbb{R})$,

$$\left\langle \phi, \hat{a}^{\dagger}\hat{a}\psi \right\rangle = \langle \hat{a}\phi, \hat{a}\psi \rangle = \left\langle \hat{a}^{\dagger}\hat{a}\phi, \psi \right\rangle$$

for all $\psi, \phi \in \mathcal{S}(\mathbb{R})$, so that $\hat{a}^{\dagger}\hat{a}$ is symmetric. $\square$

Lemma 3.2.4. Suppose A is a symmetric operator on $Dom(A)$ and λ is eigenvalue for A with eigenstate $\phi \in Dom(A)$, then $\lambda \in \mathbb{R}$.

Proof.

$$\lambda \langle \phi, \phi \rangle = \langle \phi, A\phi \rangle = \langle A\phi, \phi \rangle = \overline{\lambda} \langle \phi, \phi \rangle .$$

So, for all $\phi \neq 0$, $\lambda = \overline{\lambda}$. $\square$

Theorem 3.2.5. Suppose there exists $\phi \in \mathcal{S}(\mathbb{R})$ such that $\hat{a}^{\dagger}\hat{a}\phi = \lambda\phi$ where $\lambda \in \mathbb{R}$. Then,

$$\hat{a}^{\dagger}\hat{a}(\hat{a}\phi) = (\lambda - 1)\hat{a}\phi$$

$$\hat{a}^{\dagger}\hat{a}(\hat{a}^{\dagger}\phi) = (\lambda + 1)\hat{a}^{\dagger}\phi$$

so that $\hat{a}\phi, \hat{a}^\dagger\phi \in \mathcal{S}(\mathbb{R})$ are eigenstates of $\hat{a}^\dagger\hat{a}$.

Proof. By commutative relation, $\hat{a}^\dagger\hat{a} = \hat{a}\hat{a}^\dagger - I$ on $\mathcal{S}(\mathbb{R})$,

$$\begin{aligned}\hat{a}^\dagger\hat{a}(\hat{a}\phi) &= (\hat{a}\hat{a}^\dagger - I)(\hat{a}\phi) \\ &= \hat{a}(\hat{a}^\dagger\hat{a}\phi) - \hat{a}\phi \\ &= \hat{a}(\lambda - 1)\phi \\ &= (\lambda - 1)\hat{a}\phi.\end{aligned}$$

In the same way,

$$\hat{a}^\dagger\hat{a}(\hat{a}^\dagger\phi) = (\lambda + 1)\hat{a}^\dagger\phi.$$

□

In fact, $\phi_0 = \pi^{-1/4}\mathrm{e}^{-x^2/2} \in \mathcal{S}(\mathbb{R})$ is an eigenstate of $\hat{a}^\dagger\hat{a}$. So we can generate eigenstates with real eigenvalues by applying $\hat{a}^\dagger$ and $\hat{a}$. If $\phi \in \mathcal{S}(\mathbb{R})$ is an eigenstate for $\hat{a}^\dagger\hat{a}$ with eigenvalue $\lambda \in \mathbb{R}$, then

$$\lambda \langle \phi, \phi \rangle = \left\langle \phi, \hat{a}^\dagger\hat{a}\phi \right\rangle = \langle \hat{a}\phi, \hat{a}\phi \rangle \geq 0$$

and

$$\hat{a}^\dagger\hat{a}(\hat{a}^n\phi) = (\lambda - n)\hat{a}^n\phi, \quad \hat{a}^n\phi \in \mathcal{S}(\mathbb{R}).$$

If for all $m \in \mathbb{N}_0$, $\lambda - m$ is non-zero, then there exists $n \in \mathbb{N}_0$, such that $\lambda - n < 0$. It's contradict. So for each eigenstate ϕ, there exists some $N \in \mathbb{N}_0$ such that $\hat{a}^N\phi \neq 0$ but $\hat{a}^{N+1}\phi = 0$.

We define such $\hat{a}^N\phi =: \phi_0 \in \mathcal{S}(\mathbb{R})$ as an eigenstate of $\hat{a}^\dagger\hat{a}$ such that $\hat{a}\phi_0 = 0$. By solving

$$\frac{1}{\sqrt{2}}\left(x\phi_0(x) + \frac{d}{dx}\phi_0(x)\right) = 0,$$

we get $\phi_0 = \pi^{-1/4}\mathrm{e}^{-x^2/2} \in \mathcal{S}(\mathbb{R})$.

Theorem 3.2.6. If ϕ_0 is a unit with the property that $\hat{a}\phi_0 = 0$, then the states ϕ_n are defined as

$$\phi_n := (\hat{a}^\dagger)^n \phi_0, \quad n \in \mathbb{N}_0,$$

and satisfy the following relations for all $n, m \in \mathbb{N}_0$:

1. $\hat{a}^\dagger \phi_n = \phi_{n+1}$
2. $\hat{a}^\dagger \hat{a} \phi_n = n\phi_n$
3. $\langle \phi_n, \phi_m \rangle = m! \delta_{n,m}$
4. $\hat{a}\phi_{n+1} = (n+1)\phi_n$.

By those relations $\|\phi_n\| = \sqrt{n!}$ and if we make all ϕ_n normalized then,

$$\phi_n = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} \phi_0$$

and satisfy following relations

$$\hat{a}^\dagger \phi_n = \sqrt{n+1}\phi_{n+1}$$
$$\hat{a}\phi_n = \sqrt{n}\phi_{n-1}$$

Proof. 1. obvious.

2.

$$\begin{aligned}
\hat{a}^\dagger \hat{a} \phi_n &= \hat{a}^\dagger \hat{a} (\hat{a}^\dagger)^n \phi_0 \\
&= \hat{a}^\dagger \hat{a} \hat{a}^\dagger (\hat{a}^\dagger)^{n-1} \phi_0 \\
&= \hat{a}^\dagger (\hat{a}^\dagger \hat{a} + 1)(\hat{a}^\dagger)^{n-1} \phi_0 \\
&= (\hat{a}^\dagger)^2 \hat{a} (\hat{a}^\dagger)^{n-1} \phi_0 + (\hat{a}^\dagger)^n \phi_0 \\
&= \cdots \\
&= (\hat{a}^\dagger)^{n+1} \hat{a} \phi_0 + n(\hat{a}^\dagger)^n \phi_0 \\
&= 0 + n\phi_n
\end{aligned}$$

3.

$$
\begin{aligned}
\langle \phi_n, \phi_m \rangle &= \left\langle \phi_{n-1}, \hat{a}\hat{a}^\dagger \phi_{m-1} \right\rangle \\
&= \left\langle \phi_{n-1}, \{\hat{a}^\dagger \hat{a} + I\} \phi_{m-1} \right\rangle \\
&= \left\langle \phi_{n-1}, \hat{a}^\dagger \hat{a} \phi_{m-1} \right\rangle + \langle \phi_{n-1}, \phi_{m-1} \rangle \\
&= (m-1) \langle \phi_{n-1}, \phi_{m-1} \rangle + \langle \phi_{n-1}, \phi_{m-1} \rangle \\
&= m \langle \phi_{n-1}, \phi_{m-1} \rangle .
\end{aligned}
$$

So if $n \neq m$ especially $n > m$, then,

$$\langle \phi_n, \phi_m \rangle = m! \langle \phi_{n-m}, \phi_0 \rangle = m! \left\langle \phi_0, (\hat{a})^{n-m} \phi_0 \right\rangle = 0.$$

If $n = m$ then

$$\langle \phi_m, \phi_m \rangle = m! \langle \phi_0, \phi_0 \rangle = m!$$

4. Since $\langle \phi_n, \phi_n \rangle = n!$ so, $\|\phi_n\| = \sqrt{n!}$.

By normalizing ϕ_n,

$$\phi_n = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} \phi_0.$$

Then,

$$\hat{a}^\dagger \phi_n = \frac{(\hat{a}^\dagger)^{n+1}}{\sqrt{n!}} \phi_0 = \sqrt{n+1} \frac{(\hat{a}^\dagger)^{n+1}}{\sqrt{(n+1)!}} \phi_0 = \sqrt{n+1} \phi_{n+1},$$

and

$$\hat{a}\phi_n = \hat{a} \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} \phi_0 = \frac{\hat{a}\hat{a}^\dagger}{\sqrt{n!}} \sqrt{(n-1)!} \phi_{n-1} = \frac{\hat{a}\hat{a}^\dagger}{\sqrt{n}} \phi_{n-1} = \frac{(\hat{a}^\dagger \hat{a} + 1)}{\sqrt{n}} \phi_{n-1} = \sqrt{n} \phi_{n-1}.$$

□

3.3 Orthonormal basis of $L^2(\mathbb{R})$

Theorem 3.3.1. The operator $\hat{a} = \frac{\hat{x}+i\hat{p}}{\sqrt{2}}$, and there is $\phi_0 = \pi^{-1/4}\mathrm{e}^{-x^2/2}$ such that $\hat{a}\phi_0 = 0$. Then, we can redefine ϕ_n as follow.

$$\phi_n := H_n\phi_0$$

where H_n is a polynomial of degree n, given inductively by formulas,

$$\begin{aligned} H_0(x) &= 1 \\ H_{n+1}(x) &= \frac{1}{\sqrt{2(n+1)}}\left(2xH_n(x) - \frac{dH_n(x)}{dx}\right). \end{aligned}$$

Proof. When $n = 0$, $\phi_0 = H_0\phi_0$. Assuming $\phi_n = H_n\phi_0$ for some n, we compute ϕ_{n+1} as

$$\begin{aligned} \phi_{n+1} = \frac{\hat{a}^\dagger}{\sqrt{n+1}}\phi_n &= \frac{1}{\sqrt{2(n+1)}}\left(xH_n(x)\pi^{-1/4}\mathrm{e}^{-x^2/2} - \frac{d}{dx}\left[H_n(x)\pi^{-1/4}\mathrm{e}^{-x^2/2}\right]\right) \\ &= \frac{1}{\sqrt{2(n+1)}}\left(2xH_n(x) - \frac{dH_n(x)}{dx}\right)\pi^{-1/4}\mathrm{e}^{-x^2/2} \end{aligned}$$

□

Lemma 3.3.2. For all $\alpha \in \mathbb{C}$, the partial sums of the series

$$\sum_{n=0}^{\infty}\frac{\alpha^n x^n}{n!}\mathrm{e}^{-x^2/2}$$

converge in L^2 to the function $\mathrm{e}^{\alpha x}\mathrm{e}^{-x^2/2}$.

Proof.

$$\left|\sum_{n=N+1}^{\infty}\frac{\alpha^n x^n}{n!}\mathrm{e}^{-x^2/2}\right|^2 \leq \left(\sum_{n=0}^{\infty}\frac{|\alpha x|^n}{n!}\mathrm{e}^{-x^2/2}\right)^2 = \mathrm{e}^{2|\alpha||x|}\mathrm{e}^{-x^2} \in L^1(\mathbb{R}).$$

Since

$$\lim_{N\to\infty}\left|\sum_{n=N+1}^{\infty}\frac{\alpha^n x^n}{n!}\mathrm{e}^{-x^2/2}\right|^2 = 0,$$

by dominated convergence theorem,

$$\lim_{N\to\infty}\int_{\mathbb{R}}\left|\sum_{n=N+1}^{\infty}\frac{\alpha^n x^n}{n!}\mathrm{e}^{-x^2/2}\right|^2 dx = 0.$$

Therefore,

$$\lim_{N\to\infty}\left\|\mathrm{e}^{\alpha x}\mathrm{e}^{-x^2/2}-\sum_{n=0}^{N}\frac{\alpha^n x^n}{n!}\mathrm{e}^{-x^2/2}\right\|_2^2 = \lim_{N\to\infty}\int\left|\sum_{n=N+1}^{\infty}\frac{\alpha^n x^n}{n!}\mathrm{e}^{-x^2/2}\right|^2 dx = 0$$

and $\sum_{n=0}^{N}\frac{\alpha^n x^n}{n!}\mathrm{e}^{-x^2/2}$ converges in L^2 to $\mathrm{e}^{\alpha x}\mathrm{e}^{-x^2/2}$. $\square$

Theorem 3.3.3. The functions

$$\phi_n(x) = H_n(x)\phi_0(x) = H_n(x)\pi^{-1/4}\mathrm{e}^{-x^2/2}$$

form an orthonormal basis for Hilbert space $L^2(\mathbb{R})$.

Proof. For all normalized ϕ_n, $\langle\phi_n,\phi_m\rangle = \delta_{n,m}$. So, $\{\phi_n\}_{n\in\mathbb{N}_0}$ form an orthonormal set. It remain to show that $\{\phi_n\}_{n\in\mathbb{N}_0}$ form an basis for $L^2(\mathbb{R})$.

Let V denote the space of finite linear combinations of ϕ_n's. Since H_n is a polynomial of degree n, $\mathrm{e}^{ikx}\mathrm{e}^{-x^2/2}$ belongs to the L^2- closure of V for all $k\in\mathbb{R}$. Thus, if $\phi\in L^2(\mathbb{R})$ is orthogonal to every element of $\overline{V}$, we have

$$\int_{\mathbb{R}}\mathrm{e}^{ikx}\mathrm{e}^{-x^2/2}\phi(x)dx = 0$$

for all $k\in\mathbb{R}$.

Now, since $\mathrm{e}^{-x^2/2}$ belongs to $L^\infty(\mathbb{R})\cap L^2(\mathbb{R})$ and $\phi\in L^2(\mathbb{R})$, their product belongs to $L^2(\mathbb{R})\cap L^1(\mathbb{R})$ by *Cauchy-Schwarz* inequality

$$\int_{\mathbb{R}}|fg|^2dx \le \int_{\mathbb{R}}|f|^2dx\int_{\mathbb{R}}|g|^2dx < \infty,$$

and *Hölder* inequality

$$\|fg\|_1 \leq \|f\|_2 \|g\|_2 < \infty.$$

Thus, $\int_{\mathbb{R}} \mathrm{e}^{ikx} \mathrm{e}^{-x^2/2} \phi(x) dx$ is well defined as Fourier transform of $\mathrm{e}^{-x^2/2}\phi(x)$, and this is identically zero by hypothesis.

Thus $\mathrm{e}^{-x^2/2}\phi(x)$ must be the zero element of $L^2(\mathbb{R})$ by *Plancherel* theorem ($\|f\|_2 = \|\hat{f}\|_2$). So $\phi(x) = 0$ almost everywhere. This shows that $V^{\perp} = \{0\}$, meaning that V is dense in $L^2(\mathbb{R})$. So $\{\phi_n\}_{n\in\mathbb{N}_0}$ is maximal orthonormal set, which is orthonormal basis. □

3.4 Self-adjointness of operator $\hat{a}^{\dagger}\hat{a}$

Now let us check whether $\hat{a}^{\dagger}\hat{a}$ is essentially self-adjoint operator.

Theorem 3.4.1. The symmetric operator $\hat{a}^{\dagger}\hat{a}$ is essentially self-adjoint operator.

Proof. Let $Dom(\hat{a}^{\dagger}\hat{a})$ is a set of all finite linear combination of $\{\phi_n\}$. For all $\psi = \sum_m^{fin} c_m \phi_m \in Dom(\hat{a}^{\dagger}\hat{a})$,

$$(\hat{a}^{\dagger}\hat{a} - iI)\psi = \sum_m^{fin} c_m(m - i)\phi_m.$$

Since $m - i \neq 0$ and each $(m - i)\phi_m$ are linearly independant, $Range(\hat{a}^{\dagger}\hat{a} - iI)$ is also dense in $L^2(\mathbb{R})$. In the same way, $Range(\hat{a}^{\dagger}\hat{a} + iI)$ is also dense in $L^2(\mathbb{R})$. So $\hat{a}^{\dagger}\hat{a}$ is essentially self-adjoint. □

Remark 3.4.2. By changing to useful physics notation,

$$|m\rangle := \phi_m, \qquad m \in \mathbb{N}_0.$$

Then we can define a projection operator as $|m\rangle\langle m|$ acting like

$$|m\rangle\langle m|\,|\psi\rangle = \langle \phi_m, \psi\rangle\,|m\rangle\,,$$

so that $\sum_{m=0}^{\infty} |m\rangle\langle m| = I$.

Theorem 3.4.3. The unbounded self-adjoint operator $\hat{a}^\dagger\hat{a}$ can be written by

$$\hat{a}^\dagger\hat{a} = \sum_{m\in\mathbb{N}_0} m\,|m\rangle\langle m|\,.$$

Proof. Suppose that $Dom(\hat{a}^\dagger\hat{a})$ is finite linear combination of orthonormal basis and $\lambda \notin \mathbb{N}_0$. Then for any eigenvalue λ_j with eigenstate ϕ_j, $\lambda_j - \lambda \neq 0$.
So, if we set $b_j := \frac{1}{\lambda_j - \lambda}$, then

$$(\hat{a}^\dagger\hat{a} - \lambda I)b_j a_j \phi_j = a_j \phi_j.$$

Since $\|\frac{1}{\lambda_j-\lambda}\| < \infty$, there exists bounded operator B satisfying

$$(\hat{a}^\dagger\hat{a} - \lambda I)B\psi = \psi$$

where $\psi \in L^2(\mathbb{R})$, so that $B\psi \in Dom(\hat{a}^\dagger\hat{a})$.
In the same way, for $\psi \in Dom(\hat{a}^\dagger\hat{a})$, there exists bounded operator B satisfying

$$B(\hat{a}^\dagger\hat{a} - \lambda I)\psi = \psi.$$

So the $\lambda \notin \sigma(\hat{a}^\dagger\hat{a})$ and $\sigma(\hat{a}^\dagger\hat{a}) = \mathbb{N}_0$.

Let $E \subset \sigma(\hat{a}^\dagger\hat{a})$, then by spectral theorem, there exists projection valued measure μ such that

$$\mu(E) := \sum_{m\in E} |m\rangle\langle m|\,,$$

and

$$\hat{a}^\dagger\hat{a} = \int_{\sigma(\hat{a}^\dagger\hat{a})} \lambda d\mu(\lambda) = \int_{\mathbb{N}_0} \lambda d\mu(\lambda) = \sum_{m\in\mathbb{N}_0} m\,|m\rangle\langle m|\,.$$

□

Chapter 4

HAMILTONIAN OPERATOR

In this chapter, the main theorem(Theorem 4.2.2.) will be proved by using methods introduced from [BD07].

4.1 Hamiltonian operator

Remark 4.1.1. Now we extend single-mode system to n-mode system, i.e. we set our Hilbert space as $L^2(\mathbb{R}^n)$. Since $L^2(\mathbb{R}^n) = \bigotimes^n L^2(\mathbb{R})$, orthonormal basis of $L^2(\mathbb{R}^n)$ is

$$\{\phi_M := \phi_{m_1}(x_1)\phi_{m_2}(x_2)\cdots\phi_{m_n}(x_n)\}_{M:=(m_1,m_2,\cdots,m_n)\in\mathbb{N}_0^n} \in \mathcal{S}(\mathbb{R}^n)$$

where each ϕ_{m_k} is orthonormal basis of $L^2(\mathbb{R})$.

We define $\hat{x}_j$ and $\hat{p}_k$ on $L^2(\mathbb{R}^n)$ with $Dom(\hat{x}_j) = Dom(\hat{p}_k) = \mathcal{S}(\mathbb{R}^n)$ as follows,

$$\hat{x}_j\phi(\mathbf{x}) = x_j\phi(\mathbf{x})$$
$$\hat{p}_k\phi(\mathbf{x}) = -i\frac{\partial}{\partial x_k}\phi(\mathbf{x})$$

where $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ and $\phi(\mathbf{x}) \in \mathcal{S}(\mathbb{R}^n)$.

Then $\hat{x}_j$ and $\hat{p}_k$ are known to be essentially self-adjoint.

Definition 4.1.2 (Canonical commutation relations)**.**

$$[\hat{x}_j, \hat{p}_k] = i\delta_{jk},$$

on $\mathcal{S}(\mathbb{R}^n)$, where $j, k = \{1, 2 \ldots, n\}$.

Theorem 4.1.3. By defining the vector of canonical operators $\hat{r} = (\hat{x}_1, \hat{p}_1 \ldots \hat{x}_n, \hat{p}_n)^{\mathsf{T}}$, we have

$$\left[\hat{r}, \hat{r}^{\mathsf{T}}\right] = i\Omega,$$

on $\mathcal{S}(\mathbb{R}^n)$, where

$$\Omega = \bigoplus_{j=1}^{n} \Omega_1, \quad with \quad \Omega_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Definition 4.1.4 (Hamiltonian operator)**.** The *Hamiltonain* operator $\hat{H}$ is defined by

$$\hat{H} := \frac{1}{2}\hat{r}^{\mathsf{T}} H \hat{r} + \hat{r}^{\mathsf{T}} r,$$

where r is $2n$-dimensional real vector and H is $2n \times 2n$ positive-definite, symmetric, real Matrix.

4.2 Essentially self-adjointness of Hamiltonian

Let $\{|m\rangle\}_{m \in \mathbb{N}_0}$ is basis of $L^2(\mathbb{R})$. Then we can similarily set a basis of $L^2(\mathbb{R}^n)$ by

$$\{|m_1\rangle |m_2\rangle |m_3\rangle \cdots |m_n\rangle\}_{(m_1, m_2, \cdots, m_n) \in \mathbb{N}_0^n}$$

Definition 4.2.1. We define orthogonal projection P_N on $L^2(\mathbb{R}^n)$ as follows

$$P_N := \sum_{\substack{(m_1, \cdots, m_n) \in \mathbb{N}_0^n \\ \sum_{j=1}^n m_j = N}} \left(\bigotimes_{i=1}^{n} P_{m_i} \right)$$

where P_m is orthogonal projection onto $V_m = \{c\,|m\rangle : c \in \mathbb{C}\}$.
It is well defined, since $(P_N)^2 = P_N$. Furthermore, for any $\Psi \in L^2(\mathbb{R}^n)$ and $N, M \in \mathbb{N}_0$ with $N \neq M$, we can see that $\langle P_N\Psi, P_M\Psi\rangle = 0$. So, P_N is orthogonal projection.

Now we define a Hamiltonian operator $\hat{H}$ on $L^2(\mathbb{R}^n)$
with $Dom(\hat{H}) = span(\{|m_1\rangle\,|m_2\rangle\,|m_3\rangle \cdots |m_n\rangle\}_{(m_1,m_2,\cdots,m_n)\in\mathbb{N}_0^n})$ as follows.

For n-mode, H is a $2n \times 2n$ real symmetric matrix. Let define 2×2 matrix

$$H_{j,k} := \begin{pmatrix} a_{2j-1,2k-1} & a_{2j-1,2k} \\ a_{2j,2k-1} & a_{2j,2k} \end{pmatrix} \qquad j, k \in \mathbb{N} \leq n,$$

Then,

$$\begin{aligned}\frac{1}{2}\hat{r}^{\mathsf{T}}H\hat{r} + \hat{r}^{\mathsf{T}}r = \frac{1}{4}\sum_{j,k}\Big\{&(a_{2j-1,2k-1} + a_{2j,2k} + a_{2j,2k-1}i - a_{2j-1,2k}i)\hat{a}_j^{\dagger}\hat{a}_k \\ &+ (a_{2j-1,2k-1} + a_{2j,2k} - a_{2j,2k-1}i + a_{2j-1,2k}i)\hat{a}_j\hat{a}_k^{\dagger}\Big\} \\ + \frac{1}{\sqrt{2}}\sum_{j}&\Big\{(b_{2j-1} - b_{2j}i)\hat{a}_j + (b_{2j-1} + b_{2j}i)\hat{a}_j^{\dagger}\Big\} \\ + \frac{1}{4}\sum_{j,k}&\{(a_{2j-1,2k-1} - a_{2j,2k} - a_{2j,2k-1}i - a_{2j-1,2k}i)\hat{a}_j\hat{a}_k\} \\ + \frac{1}{4}\sum_{j,k}&\Big\{(a_{2j-1,2k-1} - a_{2j,2k} + a_{2j,2k-1}i + a_{2j-1,2k}i)\hat{a}_j^{\dagger}\hat{a}_k^{\dagger}\Big\}\end{aligned}$$

By applying simple notation,

$$\hat{r}^{\mathsf{T}}H\hat{r} = d\Gamma + \hat{A} + \hat{A}^{\dagger}$$

where

$$
\begin{aligned}
d\Gamma = \frac{1}{4}\sum_{j,k} &\Big\{(a_{2j-1,2k-1} + a_{2j,2k} + a_{2j,2k-1}i - a_{2j-1,2k}i)\hat{a}_j^\dagger \hat{a}_k \\
&+ (a_{2j-1,2k-1} + a_{2j,2k} - a_{2j,2k-1}i + a_{2j-1,2k}i)\hat{a}_j \hat{a}_k^\dagger\Big\} \\
&+ \frac{1}{\sqrt{2}}\sum_{j}\Big\{(b_{2j-1} - b_{2j}i)\hat{a}_j + (b_{2j-1} + b_{2j}i)\hat{a}_j^\dagger\Big\} \\
\hat{A} = \frac{1}{4}\sum_{j,k} &\{(a_{2j-1,2k-1} - a_{2j,2k} - a_{2j,2k-1}i - a_{2j-1,2k}i)\hat{a}_j \hat{a}_k\} \\
\hat{A}^\dagger = \frac{1}{4}\sum_{j,k} &\Big\{(a_{2j-1,2k-1} - a_{2j,2k} + a_{2j,2k-1}i + a_{2j-1,2k}i)\hat{a}_j^\dagger \hat{a}_k^\dagger\Big\}
\end{aligned}
$$

Theorem 4.2.2. The operator $\hat{H} = d\Gamma + \hat{A} + \hat{A}^\dagger$ defined as above is essentially self-adjoint.

Proof. This theorem is proved by using methods in [BD07]. First set dense domain $Dom(\hat{H}) = span(\{|m_1\rangle\,|m_2\rangle\,|m_3\rangle\cdots|m_n\rangle\}_{(m_1,m_2,\cdots,m_n)\in\mathbb{N}_0^n})$, then $\hat{H}$ is symmetric on $Dom(\hat{H})$. Now what to show is $ker(\hat{H}^* - i) = 0$.

For all $\Psi \in L^2(\mathbb{R}^n)$,

$$
P_N\Psi \in Dom(\hat{H}).
$$

Let $\Phi \in ker(\hat{H}^* - i)$. Then for all N,

$$
-i\|P_N\Phi\|^2 = -i\langle P_N\Phi, \Phi\rangle = \left\langle P_N\Phi, \hat{H}^*\Phi\right\rangle = \left\langle \hat{H}P_N\Phi, \Phi\right\rangle
$$

and $i\|P_N\Phi\|^2 = \left\langle \Phi, \hat{H}P_N\Phi\right\rangle$ in the same way.

Therefore

$$\begin{aligned}
-2i\|P_N\Phi\|^2 &= [\langle d\Gamma P_N\Phi, \Phi\rangle - \langle \Phi, d\Gamma P_N\Phi\rangle \\
&\qquad + \left\langle (\hat{A}+\hat{A}^\dagger)P_N\Phi, \Phi\right\rangle - \left\langle \Phi, (\hat{A}+\hat{A}^\dagger)P_N\Phi\right\rangle\Big] \\
&= \left\langle (\hat{A}+\hat{A}^\dagger)P_N\Phi, \Phi\right\rangle - \left\langle \Phi, (\hat{A}+\hat{A}^\dagger)P_N\Phi\right\rangle \\
&= \left\langle \hat{A}P_N\Phi, \Phi\right\rangle + \left\langle \hat{A}^\dagger P_N\Phi, \Phi\right\rangle - \left\langle \Phi, \hat{A}P_N\Phi\right\rangle - \left\langle \Phi, \hat{A}^\dagger P_N\Phi\right\rangle \\
&= \left\langle \hat{A}P_N\Phi, P_{N-2}\Phi\right\rangle + \left\langle \hat{A}^\dagger P_N\Phi, P_{N+2}\Phi\right\rangle \\
&\qquad - \left\langle \hat{A}^\dagger P_{N-2}\Phi, P_N\Phi\right\rangle - \left\langle \hat{A}P_{N+2}\Phi, P_N\Phi\right\rangle.
\end{aligned}$$

Then, we have

$$\begin{aligned}
-2i\sum_{N=0}^{M}\|P_N\Phi\|^2 &= \left\langle \hat{A}^\dagger P_M\Phi, P_{M+2}\Phi\right\rangle + \left\langle \hat{A}^\dagger P_{M-1}\Phi, P_{M+1}\Phi\right\rangle \\
&\qquad - \left\langle \hat{A}P_{M+2}\Phi, P_M\Phi\right\rangle - \left\langle \hat{A}P_{M+1}\Phi, P_{M-1}\Phi\right\rangle.
\end{aligned}$$

Since

$$\begin{aligned}
&\|\hat{A}P_M\Phi\| \\
&\quad = \|\frac{1}{4}\sum_{j,k}\{(a_{2j-1,2k-1} - a_{2j,2k} - a_{2j,2k-1}i - a_{2j-1,2k}i)\hat{a}_j\hat{a}_k\}\, P_M\Phi\| \\
&\quad \le \frac{1}{4}\sum_{j,k}(\|a_{2j-1,2k-1} - a_{2j,2k} - a_{2j,2k-1}i - a_{2j-1,2k}i\|\|\hat{a}_j\hat{a}_k P_M\Phi\|) \\
&\quad \le \frac{n^2}{4}\max_{j,k}\{|a_{2j-1,2k-1} - a_{2j,2k} - a_{2j,2k-1}i - a_{2j-1,2k}i|\}M\|P_M\Phi\|,
\end{aligned}$$

and

$$\begin{aligned}
&\|\hat{A}^\dagger P_M\Phi\| \\
&\quad = \|\frac{1}{4}\sum_{j,k}\left\{(a_{2j-1,2k-1} - a_{2j,2k} + a_{2j,2k-1}i + a_{2j-1,2k}i)\hat{a}_j^\dagger\hat{a}_k^\dagger\right\} P_M\Phi\| \\
&\quad \le \frac{1}{4}\sum_{j,k}\left(\|a_{2j-1,2k-1} - a_{2j,2k} + a_{2j,2k-1}i + a_{2j-1,2k}i\|\|\hat{a}_j^\dagger\hat{a}_k^\dagger P_M\Phi\|\right) \\
&\quad \le \frac{n^2}{4}\max_{j,k}\{|a_{2j-1,2k-1} - a_{2j,2k} + a_{2j,2k-1}i + a_{2j-1,2k}i|\}(M+2)\|P_M\Phi\|,
\end{aligned}$$

if we set $\max_{j,k}\{|a_{2j-1,2k-1} - a_{2j,2k} \pm a_{2j,2k-1}i \pm a_{2j-1,2k}i|\} = h$, we can get this inequality

$$\begin{aligned}
&2\sum_{N=0}^{M}\|P_N\Phi\|^2 \\
&\le \|\hat{A}^\dagger P_M\Phi\|\|P_{M+2}\Phi\| + \|\hat{A}^\dagger P_{M-1}\Phi\|\|P_{M+1}\Phi\| \\
&\qquad\qquad + \|\hat{A}P_{M+2}\Phi\|\|P_M\Phi\| + \|\hat{A}P_{M+1}\Phi\|\|P_{M_1}\Phi\| \\
&\le \frac{n^2}{4}h(M+2)\|P_M\Phi\|\|P_{M+2}\Phi\| + \frac{n^2}{4}h(M+1)\|P_{M-1}\Phi\|\|P_{M+1}\Phi\| \\
&\quad + \frac{n^2}{4}h(M+2)\|P_{M+2}\Phi\|\|P_M\Phi\| + \frac{n^2}{4}h(M+1)\|P_{M+1}\Phi\|\|P_{M-1}\Phi\| \\
&\le \frac{n^2}{4}h(M+2)\left[2\|P_M\Phi\|\|P_{M+2}\Phi\| + 2\|P_{M-1}\Phi\|\|P_{M+1}\Phi\|\right] \\
&\le \frac{n^2}{4}h(M+2)\left[\|P_{M-1}\Phi\|^2 + \|P_M\Phi\|^2 + \|P_{M+1}\Phi\|^2 + \|P_{M+2}\Phi\|^2\right].
\end{aligned}$$

Suppose $\Phi \neq 0$, then there exists M_0 and $C \in \mathbb{R}^+$ such that

$$\sum_{N=0}^{M_0}\|P_N\Phi\|^2 = C > 0,$$

and $\sum_{N=0}^{M}\|P_M\Phi\|^2 \geq C$ for all $M > M_0$. So $\forall M \geq M_0$,

$$\frac{8C}{n^2h(M+2)} \le \sum_{j=M-1}^{M+2}\|P_j\Phi\|^2,$$

and we have

$$\sum_{M=M_0}^{\infty}\frac{8C}{n^2h(M+2)} \le \sum_{M=M_0}^{\infty}\sum_{j=M-1}^{M+2}\|P_j\Phi\|^2. \tag{4.1}$$

However, left side of (4.1) is diverge to ∞ but right side of (4.1) converge. Since this is a contradiction, Φ must be zero. $\square$

4.3 Weyl operator and displacement operation

Theorem 4.3.1. Fix $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$ and let $\mathbf{X} = (\hat{x}_1, \ldots, \hat{x}_n)$, $\mathbf{P} = (\hat{p}_1, \ldots, \hat{p}_n)$. The operator given by

$$(\mathbf{a} \cdot \mathbf{X} + \mathbf{b} \cdot \mathbf{P})\psi(\mathbf{x}) = (\mathbf{a} \cdot \mathbf{x})\psi(\mathbf{x}) - i \sum_{j=1}^{n} b_j \frac{\partial \psi}{\partial x_j}.$$

Then $\mathbf{a} \cdot \mathbf{X} + \mathbf{b} \cdot \mathbf{P}$ is essentially self-adjoint.

Proof. Assume for two-mode such that $\mathbf{b} = \alpha \mathbf{e}_1$, $\alpha \neq 0$ and $\mathbf{a} = \beta \mathbf{e}_1 + \gamma \mathbf{e}_2$, so that

$$(A\psi)(\mathbf{x}) = (\beta x_1 + \gamma x_2)\psi(\mathbf{x}) - i\alpha \frac{\partial \psi}{\partial x_1}.$$

Since the adjoint A^* of A has same formula as A, with $Dom(A^*)$ consisting of those elements $\psi \in L^2(\mathbb{R}^2)$ for which $A\psi$, computed in distributional sense, belongs to $L^2(\mathbb{R}^2)$.

After rewriting the equation $A^*\psi = i\psi$ as

$$\frac{\partial \psi}{\partial x_1} = -\frac{i}{\alpha}(\beta x_1 + \gamma x_2)\psi(\mathbf{x}) - \frac{1}{\alpha}\psi(\mathbf{x}),$$

then we can find the general distributional solution as

$$\psi(\mathbf{x}) = c(x_2)\text{exp}\left\{ -\frac{i\beta}{2\alpha}x_1^2 - \frac{i\gamma}{\alpha}x_1 x_2 - \frac{1}{\alpha}x_1 \right\}.$$

Since the exponential factor is never square integrable as a function of x_1 with x_2 fixed, therefore $c(x_2)$ is zero for almost every value of x_2. So there is no nonzero solution of $A^*\psi = \pm i\psi$ □

Definition 4.3.2. If $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ are unbounded self-adjoint operators on $\mathcal{H}$, the A's and B's satisfy the *exponentiated commutation relations* if the following relations hold for all $1 \leq j, k \leq n$ and $s, t \in \mathbb{R}$,

$$\begin{aligned} \mathrm{e}^{isA_j}\mathrm{e}^{itA_k} &= \mathrm{e}^{itA_k}\mathrm{e}^{isA_j} \\ \mathrm{e}^{isB_j}\mathrm{e}^{itB_k} &= \mathrm{e}^{itB_k}\mathrm{e}^{isB_j} \\ \mathrm{e}^{isA_j}\mathrm{e}^{itB_k} &= \mathrm{e}^{-ist\delta_{jk}}\mathrm{e}^{itB_k}\mathrm{e}^{isA_j} \end{aligned}$$

Lemma 4.3.3. Let $\mathcal{H} = L^2(\mathbb{R}^n)$ and let $U_{\mathbf{e}_j}(t)$ be the translation operator given by

$$(U_{\mathbf{e}_j}(t)\psi)(\mathbf{x}) = \psi(\mathbf{x} + t\mathbf{e}_j),$$

where $\mathbf{e}_j = (0, \ldots, 1_j, \ldots, 0) \in \mathbb{R}^n$. Then $U_{\mathbf{e}_j}$ is a strongly continuous one-parameter unitary group which infinitesimal generator is $\hat{p}_j$ such that for each $\psi \in \mathcal{S}(\mathbb{R}^n)$,

$$\hat{p}_j\psi = -i\frac{\partial\psi}{\partial x_j}.$$

Furthermore, $\hat{p}_j$ is essentially self-adjoint on $\mathcal{S}(\mathbb{R}^n)$.

Theorem 4.3.4. the position operator $\hat{x}_j$ and momentum operator $\hat{p}_k$ satisfy the exponentiated commutation relations, where $j, k \in \mathbb{N}$.

Proof. Since $\hat{x}_j$ is just multiplication by x_j, $\mathrm{e}^{is\hat{x}_j}$ is just multiplication by e^{isx_j}. Meanwhile, the exponentiated momentum operators satisfy

$$(\mathrm{e}^{it\hat{p}_k}\psi)(\mathbf{x}) = \psi(\mathbf{x} + t\mathbf{e}_j).$$

Furthermore, $\mathrm{e}^{is\hat{x}_j}$ commutes with $\mathrm{e}^{it\hat{x}_k}$ and $\mathrm{e}^{is\hat{p}_j}$ commutes with $\mathrm{e}^{it\hat{p}_k}$. So,

$$\begin{aligned}(\mathrm{e}^{it\hat{p}_k}\mathrm{e}^{is\hat{x}_j}\psi)(\mathbf{x}) &= \mathrm{e}^{is(\mathbf{x}+t\mathbf{e}_k)_j}\psi(\mathbf{x} + t\mathbf{e}_k) \\ &= \mathrm{e}^{ist\delta_{jk}}(\mathrm{e}^{is\hat{x}_j}\mathrm{e}^{it\hat{p}_k}\psi)(\mathbf{x}).\end{aligned}$$

□

Theorem 4.3.5. For $a, b \in \mathbb{R}$, if we define

$$U_{a,b}(t) := \mathrm{e}^{-\frac{i}{2}t^2ab}\mathrm{e}^{ita\hat{x}}\mathrm{e}^{itb\hat{p}},$$

then $U_{a,b}(t)$ is a unitary group on $L^2(\mathbb{R}^n)$, and the infinitesimal generator of $U_{a,b}(t)$ is $i(a\hat{x} + b\hat{p})$.

Proof. Let $\alpha(t) = \mathrm{e}^{ita\hat{x}}\mathrm{e}^{itb\hat{p}}\mathrm{e}^{-\frac{i}{2}t^2ab}$, then following equivalences are vaild in $Dom(\alpha(t)) = span(\{|m\rangle\}_{m\in\mathbb{N}_0})$.

$$\begin{aligned}\frac{d\alpha}{dt} &= \mathrm{e}^{ita\hat{x}}ia\hat{x}\mathrm{e}^{itb\hat{p}}\mathrm{e}^{-\frac{i}{2}t^2ab} + \mathrm{e}^{ta\hat{x}}\mathrm{e}^{itb\hat{p}}ib\hat{p}\mathrm{e}^{-\frac{i}{2}t^2ab} \\ &\quad + \mathrm{e}^{ita\hat{x}}\mathrm{e}^{itb\hat{p}}\mathrm{e}^{-\frac{i}{2}t^2ab}(-tabi).\end{aligned}$$

To simplify $d\alpha/dt$, we need an intermediate result. By the product rule

$$\frac{d}{dt}\mathrm{e}^{-itb\hat{p}}a\hat{x}\mathrm{e}^{itb\hat{p}} = \mathrm{e}^{-itb\hat{p}}abi\mathrm{e}^{itb\hat{p}} = abi$$

Noting that $\mathrm{e}^{-itb\hat{p}}a\hat{x}\mathrm{e}^{itb\hat{p}} = a\hat{x}$ when $t = 0$, we may get

$$\mathrm{e}^{-itb\hat{p}}a\hat{x}\mathrm{e}^{itb\hat{p}} = a\hat{x} + tabi.$$

Using this, we obtain

$$\mathrm{e}^{ita\hat{x}}a\hat{x}\mathrm{e}^{itb\hat{p}} = \mathrm{e}^{ita\hat{x}}\mathrm{e}^{itb\hat{p}}(\mathrm{e}^{-itb\hat{p}}a\hat{x}\mathrm{e}^{itb\hat{p}}) = \mathrm{e}^{ita\hat{x}}\mathrm{e}^{itb\hat{p}}(a\hat{x} + tabi).$$

So,

$$\begin{aligned}\frac{d\alpha}{dt} &= \alpha(t)(ia\hat{x} + tabi + ib\hat{p} - tabi)\\ &= \alpha(t)i(a\hat{x} + b\hat{p}).\end{aligned}$$

Now, the unique solution to the differential equation $d\alpha/dt = \alpha(t)i(a\hat{x} + b\hat{p})$ is $\alpha(t) = \alpha(0)\mathrm{e}^{it(a\hat{x}+b\hat{p})}$ so we can get

$$\mathrm{e}^{ita\hat{x}}\mathrm{e}^{itb\hat{p}}\mathrm{e}^{-\frac{i}{2}t^2ab} = \mathrm{e}^{it(a\hat{x}+b\hat{p})} \qquad \text{on } Dom(\alpha(t)).$$

Since both sides are all unitary, $\mathrm{e}^{it(a\hat{x}+b\hat{p})}|_{Dom(\alpha(t))} \subseteq \mathrm{e}^{ita\hat{x}}\mathrm{e}^{itb\hat{p}}\mathrm{e}^{-\frac{i}{2}t^2ab}$.
Then by taking closure, we prove the theorem. ☐

Definition 4.3.6 (Weyl operator)**.** Let $\hat{r}$ is the vector of canonical operators $\hat{r} = (\hat{x}_1, \hat{p}_1 \ldots \hat{x}_n, \hat{p}_n)^\mathsf{T}$ and r is $2n$-dimensions real vector. The *Weyl* operator on $L^2(\mathbb{R}^n)$

$$\hat{D}_r := \mathrm{e}^{ir^\mathsf{T}\Omega\hat{r}}.$$

Remark 4.3.7. Since $r^\mathsf{T}\Omega\hat{r}$ which the closure has been taken is self-adjoint, $\hat{D}_r$ is unitary operator by Stone's theorem and its adjoint is $\hat{D}_r^\dagger = \hat{D}_{-r}$.

Theorem 4.3.8 (Composition of Weyl operators)**.**

$$\hat{D}_{r_1+r_2} = \hat{D}_{r_1}\hat{D}_{r_2}\mathrm{e}^{ir_1^\top \Omega r_2/2}$$

Proof. Since for all $j,k \in \mathbb{N}$, $\hat{x}_j, \hat{p}_k$ satisfy the exponentiated commutation ralations,

$$\begin{aligned}
\mathrm{e}^{ir_1^\top \Omega \hat{r}}\mathrm{e}^{ir_2^\top \Omega \hat{r}} &= \mathrm{e}^{i[\sum(r_1^\top\Omega)_{2j-1}\hat{x}_j+\sum(r_1^\top\Omega)_{2j}\hat{p}_j]}\mathrm{e}^{i[\sum(r_2^\top\Omega)_{2k-1}\hat{x}_k+\sum(r_2^\top\Omega)_{2k}\hat{p}_k]}\\
&= \mathrm{e}^{i(\mathbf{a}\cdot\mathbf{X}+\mathbf{b}\cdot\mathbf{P})}\mathrm{e}^{i(\mathbf{a}'\cdot\mathbf{X}+\mathbf{b}'\cdot\mathbf{P})}\\
&= \mathrm{e}^{i(\mathbf{a}\cdot\mathbf{b})/2}\mathrm{e}^{ia_1\hat{x}_1}\ldots\mathrm{e}^{ia_n\hat{x}_n}\mathrm{e}^{ib_1\hat{p}_1}\ldots\mathrm{e}^{ib_n\hat{p}_n}\\
&\qquad\times \mathrm{e}^{i(\mathbf{a}'\cdot\mathbf{b}')/2}\mathrm{e}^{ia'_1\hat{x}_1}\ldots\mathrm{e}^{ia'_n\hat{x}_n}\mathrm{e}^{ib'_1\hat{p}_1}\ldots\mathrm{e}^{ib'_n\hat{p}_n}\\
&= \mathrm{e}^{i(\mathbf{a}\cdot\mathbf{b}+\mathbf{a}'\cdot\mathbf{b}')/2}\mathrm{e}^{i\mathbf{b}\cdot\mathbf{a}'}\mathrm{e}^{ia_1\hat{x}_1}\ldots\mathrm{e}^{ia_n\hat{x}_n}\mathrm{e}^{ia'_1\hat{x}_1}\ldots\mathrm{e}^{ia'_n\hat{x}_n}\\
&\qquad\times \mathrm{e}^{ib_1\hat{p}_1}\ldots\mathrm{e}^{ib_n\hat{p}_n}\mathrm{e}^{ib'_1\hat{p}_1}\ldots\mathrm{e}^{ib'_n\hat{p}_n}\\
&= \mathrm{e}^{i(\mathbf{a}\cdot\mathbf{b})/2+i\mathbf{b}\cdot\mathbf{a}'-i(\mathbf{a}+\mathbf{a}')(\mathbf{b}+\mathbf{b}')/2}\mathrm{e}^{i[(\mathbf{a}+\mathbf{a}')\cdot\mathbf{X}+(\mathbf{b}+\mathbf{b}')\cdot\mathbf{P}]}\\
&= \mathrm{e}^{-i(\mathbf{a}\cdot\mathbf{b}'-\mathbf{b}\cdot\mathbf{a}')/2}\mathrm{e}^{i[(\mathbf{a}+\mathbf{a}')\cdot\mathbf{X}+(\mathbf{b}+\mathbf{b}')\cdot\mathbf{P}]}\\
&= \mathrm{e}^{-ir_1^\top\Omega r_2/2}\mathrm{e}^{ir_1^\top\Omega\hat{r}+ir_2^\top\Omega\hat{r}} = \mathrm{e}^{-ir_1^\top\Omega r_2/2}\hat{D}_{r_1+r_2},
\end{aligned}$$

where
$\mathbf{a} = [(r_1^\top\Omega)_1, (r_1^\top\Omega)_3, \ldots, (r_1^\top\Omega)_{2n-1}]$, $\mathbf{b} = [(r_1^\top\Omega)_2, (r_1^\top\Omega)_4, \ldots, (r_1^\top\Omega)_{2n}]$
$\mathbf{a}' = [(r_2^\top\Omega)_1, (r_2^\top\Omega)_3, \ldots, (r_2^\top\Omega)_{2n-1}]$, $\mathbf{b}' = [(r_2^\top\Omega)_2, (r_2^\top\Omega)_4, \ldots, (r_2^\top\Omega)_{2n}]$
$\mathbf{X} = (\hat{x}_1, \ldots, \hat{x}_n)$, and $\mathbf{P} = (\hat{p}_1, \ldots, \hat{p}_n)$.

□

Theorem 4.3.9 (Displacement operators)**.** Let $\hat{D}_{\bar{r}} = \mathrm{e}^{i\bar{r}^\top\Omega\hat{r}}$, then

$$\hat{D}_{\bar{r}}^\dagger \hat{r}\hat{D}_{\bar{r}} = \hat{r} - \bar{r}$$

where it is understood that the same Weyl operator acts on all entries of the vector $\hat{r}$.

Proof. It is easily proved by showing that for $t \in \mathbb{R}$,

$$\mathrm{e}^{-i\bar{r}^\top\Omega\hat{r}}\mathrm{e}^{it\hat{r}_j}\mathrm{e}^{i\bar{r}^\top\Omega\hat{r}} = \mathrm{e}^{-it\bar{r}_j}\mathrm{e}^{it\hat{r}_j}\mathrm{e}^{-i\bar{r}^\top\Omega\hat{r}}\mathrm{e}^{i\bar{r}^\top\Omega\hat{r}} = \mathrm{e}^{-it\bar{r}_j}\mathrm{e}^{it\hat{r}_j} = \mathrm{e}^{it(\hat{r}-\bar{r})_j}$$

$$\iff \hat{D}_{\bar{r}}^\dagger\hat{r}_j\hat{D}_{\bar{r}} = (\hat{r}-\bar{r})_j.$$

□

Theorem 4.3.10 (Re-defining Hamiltonian)**.** Let $\bar{r} = -H^{-1}r$ and $\hat{H}' = \frac{1}{2}(\hat{r} - \bar{r})^{\mathsf{T}} H(\hat{r} - \bar{r}) = \hat{H} - \gamma I$, where $\gamma = \bar{r}^{\mathsf{T}} r \in \mathbb{R}$. Then $\hat{H}'$ is equivalent to

$$\hat{H}' := \frac{1}{2}(\hat{r} - \bar{r})^{\mathsf{T}} H(\hat{r} - \bar{r}) = \frac{1}{2}\hat{D}_{-\bar{r}}\hat{r}^{\mathsf{T}} H\hat{r}\hat{D}_{\bar{r}}.$$

Proof. Since

$$\hat{D}_{-\bar{r}} H\hat{r}\hat{D}_{\bar{r}} = H(\hat{r} - \bar{r}),$$

we can get following

$$\begin{aligned}\hat{H}' &= \frac{1}{2}(\hat{r} - \bar{r})^{\mathsf{T}} H(\hat{r} - \bar{r}) \\ &= \frac{1}{2}\hat{D}_{-\bar{r}}\hat{r}^{\mathsf{T}}\hat{D}_{\bar{r}}\hat{D}_{-\bar{r}} H\hat{r}\hat{D}_{\bar{r}} \\ &= \frac{1}{2}\hat{D}_{-\bar{r}}\hat{r}^{\mathsf{T}} H\hat{r}\hat{D}_{\bar{r}}\end{aligned}$$

□

4.4 Symplectic transformation

Theorem 4.4.1. Given that $\hat{H}$ is second-order n-bosonic mode Hamiltonian, and $\hat{r}(t) := \mathrm{e}^{i\hat{H}t}\hat{r}\mathrm{e}^{-i\hat{H}t}$ where $\hat{r} = (\hat{x}_1, \hat{p}_1 \ldots \hat{x}_n, \hat{p}_n)^{\mathsf{T}}$.
If $\hat{H}$ is purly quadratic Hamiltonian, then

$$\mathrm{e}^{i\hat{H}t}\hat{r}\mathrm{e}^{-i\hat{H}t} = \mathrm{e}^{\Omega Ht}\hat{r}$$

Proof. On $Dom(\hat{H}) = span(\{|m_1\rangle\,|m_2\rangle\,|m_3\rangle \cdots |m_n\rangle\}_{(m_1,m_2,\cdots,m_n)\in\mathbb{N}_0^n})$,

$$
\begin{aligned}
i[\hat{H}, \hat{r}_j] &= \frac{i}{2}\sum_{k,l}[\hat{r}_k H_{kl}\hat{r}_l, \hat{r}_j] = \frac{i}{2}\sum_{k,l} H_{kl}[\hat{r}_k\hat{r}_l, \hat{r}_j] \\
&= \frac{i}{2}\sum_{k,l} H_{kl}(\hat{r}_k\hat{r}_l\hat{r}_j - \hat{r}_j\hat{r}_k\hat{r}_l) \\
&= \frac{i}{2}\sum_{k,l} H_{kl}(\hat{r}_k\hat{r}_l\hat{r}_j - \hat{r}_k\hat{r}_j\hat{r}_l + \hat{r}_k\hat{r}_j\hat{r}_l - \hat{r}_j\hat{r}_k\hat{r}_l) \\
&= \frac{i}{2}\sum_{k,l} H_{kl}(\hat{r}_k[\hat{r}_l, \hat{r}_j] + [\hat{r}_k, \hat{r}_j]\hat{r}_l) \\
&= \frac{i}{2}\sum_{k,l} H_{kl}(\hat{r}_k i\Omega_{lj} + i\Omega_{kj}\hat{r}_l) \\
&= -\sum_{k,l} H_{kl}\Omega_{kj}\hat{r}_l = \sum_{k,l}\Omega_{jk}H_{kl}\hat{r}_l
\end{aligned}
$$

So we have $i\left[\hat{H}, \hat{r}\right] = \Omega H\hat{r}$ on $Dom(\hat{H})$.

Since, for all $f \in Dom(\hat{H})$ and $m \in \mathbb{N}_0$, there exists $C_f \in \mathbb{C}$, such that $\|\hat{H}^m f\| \le |C_f|^m m!$. This means the function which $z \mapsto \left\langle e^{iz\hat{H}}f, g\right\rangle$ is analytic for $z \in \mathbb{R}$ and $f, g \in Dom(\hat{H})$.

For $f, g \in Dom(\hat{H}) \otimes \mathbb{C}^{2n}$, If we define

$$
\begin{aligned}
F(z) &:= \left\langle e^{iz\hat{H}}\hat{r}f, g\right\rangle \\
G(z) &:= \left\langle e^{iz\hat{H}}f, \hat{r}^{\mathsf{T}} e^{t(\Omega H)^{\mathsf{T}}} g\right\rangle,
\end{aligned}
$$

where $z \in \mathbb{R}$, then $F(z), G(z)$ are analytic.
Since $i\left[\hat{H}, \hat{r}\right] = \Omega H\hat{r}$ on $Dom(\hat{H})$, we get $F^{(m)}(0) = G^{(m)}(0)$ for all $m \ge 0$, so that

$$
\left\langle e^{it\hat{H}}\hat{r}f, g\right\rangle = \left\langle e^{it\hat{H}}f, \hat{r}^{\mathsf{T}} e^{t(\Omega H)^{\mathsf{T}}} g\right\rangle, \qquad \forall f, g \in Dom(\hat{H}) \otimes \mathbb{C}^{2n}.
$$

Furthermore, we can verfy $e^{it\hat{H}} f \in Dom(\hat{r}^\intercal e^{t(\Omega H)^\intercal})$, so

$$\begin{aligned}
&\left\langle e^{it\hat{H}}\hat{r}f, g\right\rangle = \left\langle e^{it\hat{H}}f, \hat{r}^\intercal e^{t(\Omega H)^\intercal} g\right\rangle \\
\iff &\left\langle e^{it\hat{H}}\hat{r}f, g\right\rangle = \left\langle e^{t(\Omega H)}\hat{r}e^{it\hat{H}} f, g\right\rangle \\
\iff &e^{it\hat{H}}\hat{r}f = e^{t(\Omega H)}\hat{r}e^{it\hat{H}} f \\
\iff &\hat{r}f = e^{-it\hat{H}}e^{t(\Omega H)}\hat{r}e^{it\hat{H}} f, \qquad \forall f, g \in Dom(\hat{H}) \otimes \mathbb{C}^{2n}.
\end{aligned}$$

By the way, $\hat{r}$ and $e^{-it\hat{H}}e^{t(\Omega H)}\hat{r}e^{it\hat{H}}$ are self-adjoint operator with $\hat{r}|_{Dom(\hat{H})\otimes\mathbb{C}^{2n}} \subseteq e^{-it\hat{H}}e^{t(\Omega H)}\hat{r}e^{it\hat{H}}$. Finally, by taking closure, we can say

$$\hat{r} = e^{-it\hat{H}}e^{t(\Omega H)}\hat{r}e^{it\hat{H}}.$$

[BD07] is a reference of this proof. □

Definition 4.4.2 (Symplectic group)**.** The *symplectic* group $Sp_{2n,\mathbb{R}}$ is subgroup of General linear group, such that
$\forall S \in Sp_{2n,\mathbb{R}}$

$$S\Omega S^\intercal = \Omega$$

where Ω is $2n \times 2n$ nonsingular skew-symmetric matrix.

Theorem 4.4.3. For $t \in \mathbb{R}$,

$$e^{\Omega H t} \in Sp_{2n,\mathbb{R}}$$

where H is $2n \times 2n$ symmetric real Hamiltonian matrix.

Proof. If H is symmetric real matrix,

$$\begin{aligned}
\frac{d}{dt}\left(e^{\Omega H t}\Omega e^{-H^\intercal \Omega t}\right) &= e^{\Omega H t}\Omega H \Omega e^{-H^\intercal \Omega t} + e^{\Omega H t}\Omega e^{-H^\intercal \Omega t}(-H^\intercal \Omega) \\
&= e^{\Omega H t}(\Omega H \Omega - \Omega H^\intercal \Omega)e^{-H^\intercal \Omega t} = 0.
\end{aligned}$$

So, $e^{\Omega H t}\Omega(e^{\Omega H t})^\intercal = \Omega$ and $e^{\Omega H t} \in Sp_{2n,\mathbb{R}}$.
Conversely, all elements of Lie algebra $\mathfrak{sp}_{2n,\mathbb{R}}$ is ΩH where H is $2n \times 2n$ symmetric matrix. □

Theorem 4.4.4 (Action of quadratic Hamiltonian on canonical operations)**.** Let $\hat{S}_H = e^{\frac{i}{2}\hat{r}^\mathsf{T} H \hat{r}}$. Since $\hat{r}^\mathsf{T} H \hat{r}$ which the clousure has been taken is self-adjoint, $\hat{S}_H$ is unitary, and satisfy following

$$\hat{S}_H \hat{r} \hat{S}_H^\dagger = S_H \hat{r},$$

where $S_H = e^{\Omega H} \in Sp_{2n,\mathbb{R}}$.

4.5 Decomposition of Hamiltonian

Theorem 4.5.1 (Williamson decomposition)**.** [Wil36] Given a $2n \times 2n$ (strictly)positive real matrix M, there exists a symplectic transformation $S \in Sp_{2n,\mathbb{R}}$, such that

$$SMS^\mathsf{T} = D$$

with $D = diag(d_1, d_1 \cdots \cdots d_n, d_n), \quad d_j \in \mathbb{R}^+, \forall j \in [1, 2 \cdots n]$.

Theorem 4.5.2 (Decomposition of Hamiltonian)**.** Given that
$\hat{H}' = \frac{1}{2}(\hat{r} - \bar{r})^\mathsf{T} H (\hat{r} - \bar{r}) = \hat{H} - \gamma I$, where $\bar{r} = -H^{-1} r$ and $\gamma = \bar{r}^\mathsf{T} r \in \mathbb{R}$, then

$$\hat{H}' = \hat{D}_{-\bar{r}} \hat{S} \left(\sum_{j=0}^{n} \hat{H}_{w_j} \right) \hat{S}^\dagger \hat{D}_{\bar{r}}$$

where $\hat{H}_{w_j} = \frac{w_j}{2}(\hat{x}_j^2 + \hat{p}_j^2)$, $w_j \in \mathbb{R}^+$ for all j and $r \in \mathbb{R}^{2n}$.
$\hat{S}\hat{r}\hat{S}^\dagger$ is symplectic transform of $\hat{r}$ and there exists $S^H \in Sp_{2n,\mathbb{R}}$ such that $\hat{S}\hat{r}\hat{S}^\dagger = S^H \hat{r}$.

Proof. The $2n \times 2n$ Hamiltonian matrix H is strictly positive, real, symmetric matrix. Then, $\exists S \in Sp_{2n,\mathbb{R}}$, such that

$$SHS^\mathsf{T} = D = \bigoplus_{j=1}^{n} w_j I_2.$$

So,

$$H = S^{-1}\left(\bigoplus_{j=1}^{n} w_j I_2\right)(S^{\mathsf{T}})^{-1} = S^{H^{\mathsf{T}}}\left(\bigoplus_{j=1}^{n} w_j I_2\right)S^{H}$$

where $S^H = (S^{\mathsf{T}})^{-1} \in Sp_{2n,\mathbb{R}}$.
Then, $\frac{1}{2}\hat{r}^{\mathsf{T}} H \hat{r} = \frac{1}{2}\hat{r}^{\mathsf{T}} S^{H^{\mathsf{T}}}\left(\bigoplus_{j=1}^{n} w_j I_2\right) S^H \hat{r}$.
For $S^H \in Sp_{2n,\mathbb{R}}$, there are symmetric matrix A and B, so that S^H is decomposed as $S^H = \mathrm{e}^{\Omega A}\mathrm{e}^{\Omega B}$. Then, there exist $\hat{S}_A, \hat{S}_B$, such that

$$S^H \hat{r} = \mathrm{e}^{\Omega A}\mathrm{e}^{\Omega B}\hat{r} = \hat{S}_B \hat{S}_A \hat{r} \hat{S}_A^{\dagger} \hat{S}_B^{\dagger}.$$

If we set $\hat{S}_B\hat{S}_A = \hat{S}$ then,

$$\begin{aligned}
\hat{H}' &= \frac{1}{2}(\hat{r}-\bar{r})^{\mathsf{T}} H (\hat{r}-\bar{r}) \\
&= \frac{1}{2}\hat{D}_{-\bar{r}}\hat{S}\hat{r}^{\mathsf{T}}\left(\bigoplus_{j=1}^{n} w_j I_2\right)\hat{r}\hat{S}^{\dagger}\hat{D}_{\bar{r}} \\
&= \hat{D}_{-\bar{r}}\hat{S}\left(\sum_{j=1}^{n}\frac{w_j}{2}(\hat{x}_j^2+\hat{p}_j^2)\right)\hat{S}^{\dagger}\hat{D}_{\bar{r}} \\
&= \hat{D}_{-\bar{r}}\hat{S}\left(\sum_{j=1}^{n}\hat{H}_{w_j}\right)\hat{S}^{\dagger}\hat{D}_{\bar{r}}
\end{aligned}$$

□

Theorem 4.5.3. By using orthonormal basis $\{|m\rangle\}_{m\in\mathbb{N}_0}$,
$\hat{H}_w = \sum_{m=0}^{\infty} w(m+\frac{1}{2})\,|m\rangle\langle m|$. And for $w, \beta \in \mathbb{R}^+$, $\mathrm{e}^{-\beta \hat{H}_w}$ is a bounded operator belongs to trace-class.

Proof.

$$\hat{H}_w = \frac{w}{2}(\hat{x}^2+\hat{p}^2) = w(\hat{a}^{\dagger}\hat{a}+\frac{1}{2}I)$$

Since $\hat{a}^{\dagger}\hat{a} = \sum_{m=0}^{\infty} m\,|m\rangle\langle m|$,

$$\hat{H}_w = w\left(\sum_{m=0}^{\infty} m\,|m\rangle\langle m| + \sum_{m=0}^{\infty}\frac{1}{2}\,|m\rangle\langle m|\right) = \sum_{m=0}^{\infty} w\left(m+\frac{1}{2}\right)|m\rangle\langle m|.$$

Then, by functional calculus,

$$\mathrm{e}^{-\beta \hat{H}_w} = \sum_{m=0}^{\infty} \mathrm{e}^{-\beta w\left(m+\frac{1}{2}\right)} |m\rangle \langle m| \,.$$

Now let us check whether $\mathrm{e}^{-\beta \hat{H}_w}$ is trace-class. Since $\{|n\rangle\}$ is orthonormal basis,

$$\begin{aligned}\mathrm{Tr}(\mathrm{e}^{-\beta \hat{H}_w}) &= \mathrm{Tr}\left(\sum_{m=0}^{\infty} \mathrm{e}^{-\beta w\left(m+\frac{1}{2}\right)} |m\rangle \langle m|\right) \\ &= \sum_{i=0}^{\infty} \mathrm{e}^{-\beta w(n+1/2)} < \infty\end{aligned}$$

□

Chapter 5

GAUSSIAN STATE

From Chapter 5 to Chapter 7, we will mainly follow [Ser17].

5.1 Gaussian state

Theorem 5.1.1. Suppose that U is unitary operator, and $f(A)$ is functional calculus of unbounded self-adjoint operator A. Then,

$$f(UAU^\dagger) = Uf(A)U^\dagger.$$

Proof. Let μ is projection valued measure, so that $A = \int_{\sigma(A)} \lambda d\mu(\lambda)$, and $f(A) = \int_{\mathbb{R}} f(\lambda) d\mu(\lambda)$.
Since for all E in σ-algebra of $\mathbb{R}$, $\mu(E)$ orthogonal projection, $\mu'(E) = U\mu(E)U^\dagger$ is also orthogonal projection, so that

$$UAU^\dagger = \int_{\sigma(A)} \lambda d\mu'(\lambda).$$

Then,

$$Uf(A)U^\dagger = U\int_{\mathbb{R}} f(\lambda)d\mu(\lambda)U^\dagger = \int_{\mathbb{R}} f(\lambda)d\mu'(\lambda) = f(UAU^\dagger)$$

□

Definition 5.1.2 (Gaussian state)**.** The *guassian* state ϱ_G is density operator defined as

$$\varrho_G := \frac{\mathrm{e}^{-\beta\hat{H}}}{\mathrm{Tr}\left[\mathrm{e}^{-\beta\hat{H}}\right]} \quad \beta \in \mathbb{R}^+,$$

so that $\mathrm{Tr}[\varrho_G] = 1$. It is well defined. Because, For $\beta > 0$, $\mathrm{e}^{-\beta\hat{H}}$ belongs to trace-class.

Since

$$\hat{H}' = \hat{D}_{-\bar{r}}\hat{S}\left(\sum_{j=0}^{n}\hat{H}_{w_j}\right)\hat{S}^\dagger\hat{D}_{\bar{r}},$$

we can obtain

$$\varrho_G = \frac{\mathrm{e}^{-\beta\hat{H}}}{\mathrm{Tr}\left[\mathrm{e}^{-\beta\hat{H}}\right]} = \frac{\mathrm{e}^{-\beta\hat{H}'}}{\mathrm{Tr}\left[\mathrm{e}^{-\beta\hat{H}'}\right]} = \hat{D}_{-\bar{r}}\hat{S}\frac{\left(\bigotimes_{j=1}^{n}\mathrm{e}^{-\beta\hat{H}_{w_j}}\right)}{\prod_{j=1}^{n}\mathrm{Tr}\left[\mathrm{e}^{-\beta\hat{H}_{w_j}}\right]}\hat{S}^\dagger\hat{D}_{\bar{r}},$$

and represention of gaussian states by basis $\{\phi_m\}_{m\in\mathbb{N}_0}$ is

$$\begin{aligned}\varrho_G &= \left(\prod_{j=1}^{n}(1-\mathrm{e}^{-\beta\omega_j})\right)\hat{D}_{\bar{r}}^\dagger\hat{S}\left(\bigotimes_{j=1}^{n}\left(\sum_{m=0}^{\infty}\mathrm{e}^{-\beta\omega_j m}|m\rangle_{jj}\langle m|\right)\right)\hat{S}^\dagger\hat{D}_{\bar{r}} \\ &= \left(\prod_{j=1}^{n}(1-\mathrm{e}^{-\xi_j})\right)\hat{D}_{\bar{r}}^\dagger\hat{S}\left(\bigotimes_{j=1}^{n}\left(\sum_{m=0}^{\infty}\mathrm{e}^{-\xi_j m}|m\rangle_{jj}\langle m|\right)\right)\hat{S}^\dagger\hat{D}_{\bar{r}}\end{aligned}$$

where $\xi_j \in \mathbb{R}^+$

Definition 5.1.3 (Pure gaussian state)**.** The all pure gaussian states are described by,

$$\begin{aligned}\lim_{\beta\to\infty}\frac{\mathrm{e}^{-\beta\hat{H}}}{\mathrm{Tr}\left[\mathrm{e}^{-\beta\hat{H}}\right]} &= \lim_{\beta\to\infty}\left(\prod_{j=1}^{n}(1-\mathrm{e}^{-\beta\omega_j})\right)\hat{D}_{\bar{r}}^\dagger\hat{S}\left(\bigotimes_{j=1}^{n}\left(\sum_{m=0}^{\infty}\mathrm{e}^{-\beta\omega_j m}|m\rangle_{jj}\langle m|\right)\right)\hat{S}^\dagger\hat{D}_{\bar{r}} \\ &= \hat{D}_{\bar{r}}^\dagger\hat{S}\left(\bigotimes_{j=1}^{n}\left(|0\rangle_{jj}\langle 0|\right)\right)\hat{S}^\dagger\hat{D}_{\bar{r}} = \hat{D}_{\bar{r}}^\dagger\hat{S}\left(|0\rangle\langle 0|\right)\hat{S}^\dagger\hat{D}_{\bar{r}}\end{aligned}$$

Lemma 5.1.4. $\mathrm{Tr}\left(\sum_{m=0}^{\infty}\mathrm{e}^{-\beta\omega m}|m\rangle\langle m|\hat{x}\right) = 0.$

Proof.

$$\begin{aligned}
\mathrm{Tr}\left(\sum_{m=0}^{\infty}\mathrm{e}^{-\beta\omega m}|m\rangle\langle m|\hat{x}\right) &= \sum_{n=0}^{\infty}\left\langle \phi_n, \sum_{m=0}^{\infty}\mathrm{e}^{-\beta\omega m}|m\rangle\langle m|\hat{x}\phi_n\right\rangle \\
&= \sum_{n=0}^{\infty}\left\langle \phi_n, \sum_{m=0}^{\infty}\mathrm{e}^{-\beta\omega m}|m\rangle\langle m|\left(\frac{\hat{a}+\hat{a}^\dagger}{\sqrt{2}}\right)\phi_n\right\rangle \\
&= \frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}\left\langle \phi_n, \sum_{m=0}^{\infty}\mathrm{e}^{-\beta\omega m}|m\rangle\langle m|\hat{a}\phi_n\right\rangle \\
&\quad + \frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}\left\langle \phi_n, \sum_{m=0}^{\infty}\mathrm{e}^{-\beta\omega m}|m\rangle\langle m|\hat{a}^\dagger\phi_n\right\rangle = 0
\end{aligned}$$

So does $\mathrm{Tr}\left(\sum_{m=0}^{\infty}\mathrm{e}^{-\beta\omega m}|m\rangle\langle m|\hat{p}\right) = 0.$ □

5.2 Moments of gaussian states

Theorem 5.2.1 (The first moments of gaussian states)**.** The 1st moment of gaussian states $\bar{r}_m$ is defined as

$$\bar{r}_m := \mathrm{Tr}[\varrho_G\hat{r}]$$

then, $\bar{r}_m = \bar{r}$.

Proof.

$$\begin{aligned}
\bar{r}_m = \mathrm{Tr}[\varrho_G\hat{r}] &= \prod_{j=1}^{n}(1-\mathrm{e}^{-\xi_j})\mathrm{Tr}\left(\left(\bigotimes_{j=1}^{n}\left(\sum_{m=0}^{\infty}\mathrm{e}^{-\xi_j m}|m\rangle_{jj}\langle m|\right)\right)\hat{S}\hat{D}_{\bar{r}}\hat{r}\hat{D}_{\bar{r}}^\dagger\hat{S}^\dagger\right) \\
&= \prod_{j=1}^{n}(1-\mathrm{e}^{-\xi_j})\mathrm{Tr}\left(\bigotimes_{j=1}^{n}\left(\sum_{m=0}^{\infty}\mathrm{e}^{-\xi_j m}|m\rangle_{jj}\langle m|\right)(S\hat{r}+\bar{r})\right)
\end{aligned}$$

Since each $S\hat{r}_j$ are linear combination of $\{\hat{a}_j\}, \{\hat{a}_k\}$,
$\mathrm{Tr}\left(\bigotimes_{j=1}^{n}\left(\sum_{m=0}^{\infty} \mathrm{e}^{-\xi_j m}|m\rangle_{jj}\langle m|\right) S\hat{r}\right) = 0$ So,

$$\bar{r}_m = \prod_{j=1}^{n}(1-\mathrm{e}^{-\xi_j})\mathrm{Tr}\left(\bigotimes_{j=1}^{n}\left(\sum_{m=0}^{\infty} \mathrm{e}^{-\xi_j m}|m\rangle_{jj}\langle m|\right)\bar{r}\right) = \bar{r}$$

□

Definition 5.2.2 (Covariance matrix of gaussian states)**.** The covariance matrix(2nd moment) of gaussian states σ is defined as

$$\sigma := \mathrm{Tr}[\{(\hat{r}-\bar{r}), (\hat{r}-\bar{r})^{\mathsf{T}}\}\varrho_G],$$

and

$$\begin{aligned}
\sigma &= \mathrm{Tr}[\{(\hat{r}-\bar{r}), (\hat{r}-\bar{r})^{\mathsf{T}}\}\varrho_G] \\
&= \prod_{j=1}^{n}(1-\mathrm{e}^{-\xi_j})\mathrm{Tr}\left(\left(\bigotimes_{j=1}^{n}\left(\sum_{m=0}^{\infty} \mathrm{e}^{-\xi_j m}|m\rangle_{jj}\langle m|\right)\right)\hat{S}\hat{D}_{\bar{r}}\{(\hat{r}-\bar{r}), (\hat{r}-\bar{r})^{\mathsf{T}}\}\hat{D}_{\bar{r}}^{\dagger}\hat{S}^{\dagger}\right) \\
&= \prod_{j=1}^{n}(1-\mathrm{e}^{-\xi_j})\mathrm{Tr}\left(\left(\bigotimes_{j=1}^{n}\left(\sum_{m=0}^{\infty} \mathrm{e}^{-\xi_j m}|m\rangle_{jj}\langle m|\right)\right)\hat{S}\{\hat{r}, \bar{r}^{\mathsf{T}}\}\hat{S}^{\dagger}\right) \\
&= S\prod_{j=1}^{n}(1-\mathrm{e}^{-\xi_j})\mathrm{Tr}\left(\left(\bigotimes_{j=1}^{n}\left(\sum_{m=0}^{\infty} \mathrm{e}^{-\xi_j m}|m\rangle_{jj}\langle m|\right)\right)\{\hat{r}, \bar{r}^{\mathsf{T}}\}\right)S^{\mathsf{T}}.
\end{aligned}$$

Theorem 5.2.3.

$$\sigma = S\left(\bigoplus_{j=1}^{n} v_j I_2\right)S^{\mathsf{T}}$$

where $v_j = \frac{1+\mathrm{e}^{-\xi_j}}{1-\mathrm{e}^{-\xi_j}} \geq 1$ and $S \in Sp_{2n,\mathbb{R}}$.

Proof. For single mode,

$$\begin{aligned}
\{(\hat{x}_1, \hat{p}_1), (\hat{x}_1, \hat{p}_1)^{\mathsf{T}}\} &= \begin{pmatrix} 2\hat{x}_1^2 & \hat{x}_1\hat{p}_1 + \hat{p}_1\hat{x}_1 \\ \hat{x}_1\hat{p}_1 + \hat{p}_1\hat{x}_1 & 2\hat{p}_1^2 \end{pmatrix} \\
&= \begin{pmatrix} 2\hat{a}_1^{\dagger}\hat{a}_1 + 1 + \hat{a}_1^2 + (\hat{a}_1^{\dagger})^2 & i((\hat{a}_1^{\dagger})^2 - \hat{a}_1^2) \\ i((\hat{a}_1^{\dagger})^2 - \hat{a}_1^2) & 2\hat{a}_1^{\dagger}\hat{a}_1 + 1 - \hat{a}_1^2 - (\hat{a}_1^{\dagger})^2 \end{pmatrix},
\end{aligned}$$

$$\mathrm{Tr}\left(\sum_{m=0}^{\infty} \mathrm{e}^{-\xi_1 m}|m\rangle_{11}\langle m|\left(\{(\hat{x}_1,\hat{p}_1),(\hat{x}_1,\hat{p}_1)^{\mathsf{T}}\}\right)\right) = \sum_{m=0}^{\infty} \mathrm{e}^{-\xi_1 m}(2m+1)I_2 = \frac{1+\mathrm{e}^{-\xi_1}}{(1-\mathrm{e}^{-\xi_1})^2} I_2.$$

Then,

$$\begin{aligned}(1-\mathrm{e}^{-\xi_1})\mathrm{Tr}\left(\sum_{m=0}^{\infty} \mathrm{e}^{-\xi_1 m}|m\rangle_{11}\langle m|\left(\{(\hat{x}_1,\hat{p}_1),(\hat{x}_1,\hat{p}_1)^{\mathsf{T}}\}\right)\right) &= (1-\mathrm{e}^{-\xi_1})\left(\frac{1+\mathrm{e}^{-\xi_1}}{(1-\mathrm{e}^{-\xi_1})^2} I_2\right) \\ &= \frac{1+\mathrm{e}^{-\xi_1}}{1-\mathrm{e}^{-\xi_1}} I_2.\end{aligned}$$

So we have

$$\prod_{j=1}^{n}(1-\mathrm{e}^{-\xi_j})\mathrm{Tr}\left(\left(\bigotimes_{j=1}^{n}\left(\sum_{m=0}^{\infty} \mathrm{e}^{-\xi_j m}|m\rangle_{jj}\langle m|\right)\right)\{\hat{r},\bar{r}^{\mathsf{T}}\}\right) = \bigoplus_{j=1}^{n} v_j I_2$$

where $v_j = \frac{1+\mathrm{e}^{-\xi_j}}{1-\mathrm{e}^{-\xi_j}} \geq 1$

Finally,

$$\begin{aligned}\sigma &= \prod_{j=1}^{n}(1-\mathrm{e}^{-\xi_j})S\mathrm{Tr}\left(\left(\bigotimes_{j=1}^{n}\left(\sum_{m=0}^{\infty} \mathrm{e}^{-\xi_j m}|m\rangle_{jj}\langle m|\right)\right)\{\hat{r},\bar{r}^{\mathsf{T}}\}\right)S^{\mathsf{T}} \\ &= S\left(\bigoplus_{j=1}^{n} v_j I_2\right)S^{\mathsf{T}}.\end{aligned}$$

□

5.3 Coherent states

Definition 5.3.1 (Coherent state)**.** Let $\alpha = (x+ip)/\sqrt{2}$, $r = (x,p)^{\mathsf{T}}$, $\hat{r} = (\hat{x},\hat{p})^{\mathsf{T}}$, where $x,p \in \mathbb{R}$. The *coherent* state $|\alpha\rangle$ is the eigenstate of the operator $\hat{a} = (\hat{x}+\hat{p})/\sqrt{2}$ with eigenvalue $\alpha \in \mathbb{C}$. An operator $\hat{D}_\alpha$ is defined as

$$\hat{D}_\alpha := \hat{D}_{-r} = \mathrm{e}^{-ir^{\mathsf{T}}\Omega\hat{r}} = \mathrm{e}^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}},$$

Remark 5.3.2. The operator $\hat{D}_\alpha$ acts on $\hat{a}$ as follows

$$\hat{D}_\alpha^\dagger \hat{a} \hat{D}_\alpha = \hat{D}_r \left(\frac{\hat{x} + i\hat{p}}{\sqrt{2}} \right) \hat{D}_{-r} = \hat{D}_r \left(\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}} \right) \hat{r} \hat{D}_{-r}$$
$$= \left(\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}} \right) (\hat{r} + r) = \left(\frac{\hat{x} + i\hat{p}}{\sqrt{2}} + \frac{x + ip}{\sqrt{2}} \right) = \hat{a} + \alpha.$$

Proposition 5.3.3.

$$\hat{a} \hat{D}_\alpha |0\rangle = \alpha \hat{D}_\alpha |0\rangle$$

Proof.

$$\hat{a} \hat{D}_\alpha |0\rangle = \hat{D}_\alpha \hat{D}_\alpha^\dagger \hat{a} \hat{D}_\alpha |0\rangle = \hat{D}_\alpha (\hat{a} + \alpha) |0\rangle$$
$$= \hat{D}_\alpha \hat{a} |0\rangle + \alpha \hat{D}_\alpha |0\rangle = \alpha \hat{D}_\alpha |0\rangle$$

so that $|\alpha\rangle := \hat{D}_\alpha |0\rangle$ is eigen state of $\hat{a}$ for every $\alpha \in \mathbb{C}$ □

Theorem 5.3.4. Let $|\alpha\rangle$ be a coherent state such that $\hat{a} |\alpha\rangle = \alpha |\alpha\rangle$, then

$$|\alpha\rangle = \mathrm{e}^{-\frac{|\alpha|^2}{2}} \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} |m\rangle .$$

Proof. Since $\{|m\rangle\}_{m \in \mathbb{N}_0}$ are basis of $L^2(\mathbb{R})$, $|\alpha\rangle = \sum_{m=0}^{\infty} c_m |m\rangle$ for $c_m \in \mathbb{C}$. Then

$$\hat{a} |\alpha\rangle = \sum_{m=0}^{\infty} c_m \sqrt{m} |m-1\rangle = \sum_{m=0}^{\infty} c_{m+1} \sqrt{m+1} |m\rangle = \alpha \sum_{m=0}^{\infty} c_m |m\rangle$$

then $c_{m+1}\sqrt{m+1} = \alpha c_m$, so that $c_m = c_0 \alpha^m / \sqrt{m!}$.
So we have $|\alpha\rangle = \sum_{m=0}^{\infty} \frac{c_0 \alpha^m}{\sqrt{m!}} |m\rangle$. Since $\sum_{m=0}^{\infty} \frac{c_0^2 \alpha^{2m}}{m!} = 1$, $c_0^2 \mathrm{e}^{|\alpha|^2} = 1$,
$\therefore |\alpha\rangle = \mathrm{e}^{-\frac{|\alpha|^2}{2}} \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} |m\rangle$. □

Remark 5.3.5. By composition of Weyl operator,

1. $\hat{D}_\alpha \hat{D}_\beta = \hat{D}_{\alpha+\beta} \mathrm{e}^{(\alpha\beta^* - \alpha^*\beta)/2}$
2. $\langle \beta | \alpha \rangle = \langle 0 | \hat{D}_{-\beta} \hat{D}_\alpha |0\rangle = \mathrm{e}^{-\frac{1}{2}|\alpha-\beta|^2} \mathrm{e}^{\frac{1}{2}(\alpha\beta^* - \alpha^*\beta)}$.

Theorem 5.3.6 (Coherent states resolution of the identity)**.** The $\{|\alpha\rangle\}_{\alpha\in\mathbb{C}}$ are coherent states then,

$$\frac{1}{\pi}\int_{\mathbb{C}} |\alpha\rangle\langle\alpha|\, \mathrm{d}^2\alpha = \hat{I}$$

where $\int_{\mathbb{C}} \cdot \,\mathrm{d}^2\alpha := \int_{\mathbb{R}^2} \cdot\, d(Re[\alpha])d(Im[\alpha])$.

Proof. We give this resloution of the identity in the weak operator topology by this

$$\langle v|\,\frac{1}{\pi}\int_{\mathbb{C}} |\alpha\rangle\langle\alpha|\,\mathrm{d}^2\alpha\,|w\rangle := \lim_{R\to\infty}\frac{1}{\pi}\int_{|\alpha|\le R}\langle v|\alpha\rangle\langle\alpha|w\rangle\,\mathrm{d}^2\alpha. \qquad |v\rangle, |w\rangle \in L^2(\mathbb{R})$$

Then what to show is $\lim_{R\to\infty}\frac{1}{\pi}\int_{|\alpha|\le R}\langle v|\alpha\rangle\langle\alpha|w\rangle\,\mathrm{d}^2\alpha = \langle v|w\rangle$.

Since $\mathrm{e}^{-|\alpha|^2}\sum_{m,n}\frac{\alpha^m(\alpha^*)^n}{\sqrt{m!}\sqrt{n!}}v_m w_n$ is integrable on $|\alpha| \le R < \infty$,

$$\begin{aligned}\lim_{R\to\infty}\frac{1}{\pi}\int_{|\alpha|\le R}\langle v|\alpha\rangle\langle\alpha|w\rangle\,\mathrm{d}^2\alpha &= \lim_{R\to\infty}\frac{1}{\pi}\int_{|\alpha|\le R}\mathrm{e}^{-|\alpha|^2}\sum_{m,n}\frac{\alpha^m(\alpha^*)^n}{\sqrt{m!}\sqrt{n!}}v_m w_n\,\mathrm{d}^2\alpha \\ &= \lim_{R\to\infty}\sum_{m,n}v_m w_n\int_{|\alpha|\le R}\frac{\mathrm{e}^{-|\alpha|^2}}{\pi}\frac{\alpha^m(\alpha^*)^n}{\sqrt{m!}\sqrt{n!}}\,\mathrm{d}^2\alpha\end{aligned}$$

where $|v\rangle = \sum_m v_m|m\rangle$ and $|w\rangle = \sum_n w_n|n\rangle$.
If $n \neq m$,

$$\lim_{R\to\infty}\sum_{m,n}v_m w_n\int_0^R\int_0^{2\pi}\frac{\mathrm{e}^{-\rho^2}}{\pi}\frac{\rho^m(\rho^*)^n}{\sqrt{m!}\sqrt{n!}}\mathrm{e}^{i(m-n)\theta}\rho d\theta d\rho = 0.$$

So,

$$\begin{aligned}&\lim_{R\to\infty}\sum_{m,n}v_m w_n\int_{|\alpha|\le R}\frac{\mathrm{e}^{-|\alpha|^2}}{\pi}\frac{\alpha^m(\alpha^*)^n}{\sqrt{m!}\sqrt{n!}}\,\mathrm{d}^2\alpha \\ &\quad= \lim_{R\to\infty}\sum_m v_m w_m\int_0^R\int_0^{2\pi}\frac{\mathrm{e}^{-\rho^2}}{\pi}\frac{\rho^{2m+1}}{m!}d\theta d\rho \\ &\quad= \sum_m v_m w_m = \langle v|w\rangle\end{aligned}$$

□

5.4 Characteristic function of gaussian states

Lemma 5.4.1. For all $f, g \in L^2(\mathbb{R})$, we have

$$\frac{1}{2\pi}\int_{\mathbb{C}} |\langle f|\,\hat{D}_{-\alpha}\,|g\rangle|^2 \,\mathrm{d}^2\alpha = \|f\|^2\|g\|^2.$$

so that $\mathrm{Tr}\left[\hat{D}_\alpha\,|f\rangle\,\langle g|\right]$ is L^2 function, and

$$\frac{1}{2\pi}\int_{\mathbb{C}} \mathrm{Tr}\left[\hat{D}_\alpha\,|f\rangle\,\langle g|\right]\hat{D}_{-\alpha}\,\mathrm{d}^2\alpha$$

is well defined as an weak-convergent bounded operator on $L^2(\mathbb{R})$.

Proof. Let $\alpha = \frac{r_1+ir_2}{\sqrt{2}}$ where $(r_1, r_2) \in \mathbb{R}^2$.

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\mathbb{C}} |\langle f|\,\hat{D}_{-\alpha}\,|g\rangle|^2 \,\mathrm{d}^2\alpha \\
&= \frac{1}{2\pi}\int_{\mathbb{C}} \langle f|\,\hat{D}_{-\alpha}\,|g\rangle\,\langle g|\,\hat{D}_{\alpha}\,|f\rangle\,\mathrm{d}^2\alpha \\
&= \frac{1}{2\pi}\int_{\mathbb{C}} \mathrm{e}^{ir_1r_2}\left[\int_{\mathbb{R}} \mathrm{e}^{-ir_2x_1} g(x_1+r_1)\overline{f(x_1)}dx_1\right]\left[\int_{\mathbb{R}} \mathrm{e}^{ir_2x_2} f(x_2-r_1)\overline{g(x_2)}dx_2\right]\mathrm{d}^2\alpha \\
&= \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{e}^{ir_1r_2}\left[\widehat{g(r_2+r_1)\overline{f(r_2)}}\right]\left[\widehat{f(-r_2-r_1)\overline{g(-r_2)}}\right]dr_2dr_1 \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \left[g(x+r_1)\overline{f(x)}\right]\left[f(r_1+x-r_1)\overline{g(r_1+x)}\right]dxdr_1 \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} g(x+r_1)\overline{f(x)}f(x)\overline{g(x+r_1)}dxdr_1 \\
&= \|g\|^2\|f\|^2
\end{aligned}$$

□

Theorem 5.4.2 (Fourier-Weyl relation)**.** Given a bounded operator $\hat{O}$ in trace-class on the Hilbert space of one (bosonic)mode, one has

$$\hat{O} = \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{Tr}\left[\hat{D}_\alpha\hat{O}\right]\hat{D}_{-\alpha}\,\mathrm{d}^2\alpha.$$

For n-bosonic mode,

$$\hat{O} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} \mathrm{Tr}\left[\hat{D}_{-r}\hat{O}\right]\hat{D}_r\,\mathrm{d}\,r,$$

Proof. First, we intend to show for rank 1 operator.

$$|\alpha\rangle\langle\beta| = \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{Tr}\left[\hat{D}_\gamma |\alpha\rangle\langle\beta|\right]\hat{D}_{-\gamma}\,\mathrm{d}^2\gamma$$

Since, $\hat{D}_\alpha|0\rangle = |\alpha\rangle$,

$$\hat{D}_\alpha|0\rangle\langle 0|\hat{D}_\beta^\dagger = \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{Tr}\left[\hat{D}_\gamma |\alpha\rangle\langle\beta|\right]\hat{D}_{-\gamma}\,\mathrm{d}^2\gamma.$$

So,

$$\begin{aligned}
|0\rangle\langle 0| &= \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{Tr}\left[|\alpha\rangle\langle\beta|\hat{D}_\gamma\right]\hat{D}_{-\alpha}\hat{D}_{-\gamma}\hat{D}_\beta\,\mathrm{d}^2\gamma \\
&= \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{Tr}\left[|\alpha\rangle\langle 0|\hat{D}_{-\beta}\hat{D}_\gamma\right]\hat{D}_{-\alpha}\hat{D}_{-\gamma}\hat{D}_\beta\,\mathrm{d}^2\gamma \\
&= \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{Tr}\left[|\alpha\rangle\langle\beta-\gamma|\right]\mathrm{e}^{\frac{1}{2}(\gamma\beta^*-\gamma^*\beta)}\hat{D}_{-\alpha}\hat{D}_{-\gamma}\hat{D}_\beta\,\mathrm{d}^2\gamma \\
&= \frac{1}{\pi}\int_{\mathbb{C}} \langle\beta-\gamma \mid \alpha\rangle\,\mathrm{e}^{\frac{1}{2}(\gamma\beta^*-\gamma^*\beta)}\hat{D}_{-\alpha}\hat{D}_{-\gamma}\hat{D}_\beta\,\mathrm{d}^2\gamma \\
&= \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{e}^{-\frac{1}{2}|\beta-\alpha-\gamma|^2}\hat{D}_{\beta-\alpha-\gamma}\,\mathrm{d}^2\gamma \\
&= \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{e}^{-\frac{1}{2}|\gamma|^2}\hat{D}_\gamma\,\mathrm{d}^2\gamma
\end{aligned}$$

From now on, In aspect of weak topology, we will show that

$$\langle v|0\rangle\langle 0|w\rangle = \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{e}^{-\frac{1}{2}|\gamma|^2}\langle v|\hat{D}_\gamma|w\rangle\,\mathrm{d}^2\gamma.$$

where $|v\rangle, |w\rangle \in L^2(\mathbb{R})$. Let

$$|v\rangle = \sum_{l=0}^{\infty} v_l|l\rangle$$
$$|w\rangle = \sum_{m=0}^{\infty} w_m|m\rangle$$

where $|l\rangle, |m\rangle$ are orthonormal basis of $L^2(\mathbb{R})$.

Since $|\langle v|\hat{D}_\gamma|w\rangle| \le \|v\|\|w\| < \infty$,
$\mathrm{e}^{-\frac{1}{2}|\gamma|^2}\langle v|\hat{D}_\gamma|w\rangle$ is integrable, so

$$\frac{1}{\pi}\int_{\mathbb{C}} \mathrm{e}^{-\frac{1}{2}|\gamma|^2}\langle v|\hat{D}_\gamma|w\rangle\,\mathrm{d}^2\gamma = \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{e}^{-\frac{1}{2}|\gamma|^2}\sum_{l,m} v_l^* w_m\langle l|\hat{D}_\gamma|m\rangle\,\mathrm{d}^2\gamma,$$

and

$$\begin{aligned}
\langle l|\, \hat{D}_\gamma \,|m\rangle &= \langle l|\, \frac{(\hat{a}^* - \gamma^*)^m}{\sqrt{m!}} \,|\gamma\rangle \\
&= \langle l|\, \frac{(\hat{a}^* - \gamma^*)^m}{\sqrt{m!}} \sum_{n=0}^{\infty} \mathrm{e}^{-\frac{1}{2}|\gamma|^2} \frac{\gamma^n}{\sqrt{n!}} \,|n\rangle \\
&= \mathrm{e}^{-\frac{1}{2}|\gamma|^2} \sum_{n=0}^{\infty} \langle l|\, \frac{(\hat{a}^* - \gamma^*)^m \gamma^n}{\sqrt{m!}\sqrt{m!}} \,|n\rangle \\
&= \mathrm{e}^{-\frac{1}{2}|\gamma|^2} \sum_{n=0}^{\infty} \sum_{j=0}^{m} (-1)^j \binom{m}{j} (\gamma^*)^j \gamma^n \,\langle l|\, \frac{(\hat{a}^*)^{(m-j)}}{\sqrt{m!}\sqrt{n!}} \,|n\rangle \,.
\end{aligned}$$

Now we know that

$$\begin{aligned}
&\mathrm{e}^{-\frac{1}{2}|\gamma|^2} \sum_{l,m} v_l^* w_m \,\langle l|\, \hat{D}_\gamma \,|m\rangle \\
&\qquad = \mathrm{e}^{-|\gamma|^2} \sum_{l,m} \sum_{n=0}^{\infty} \sum_{j=0}^{m} (-1)^j \binom{m}{j} v_l^* w_m (\gamma^*)^j \gamma^n \,\langle l|\, \frac{(\hat{a}^*)^{(m-j)}}{\sqrt{m!}\sqrt{n!}} \,|n\rangle \,.
\end{aligned}$$

So we have

$$\begin{aligned}
&\frac{1}{\pi} \int_{\mathbb{C}} \mathrm{e}^{-\frac{1}{2}|\gamma|^2} \sum_{l,m} v_l^* w_m \,\langle l|\, \hat{D}_\gamma \,|m\rangle \,\mathrm{d}^2\gamma \\
&= \frac{1}{\pi} \int_{\mathbb{C}} \mathrm{e}^{-|\gamma|^2} \sum_{l,m} \sum_{n=0}^{\infty} \sum_{j=0}^{m} (-1)^j \binom{m}{j} v_l^* w_m (\gamma^*)^j \gamma^n \,\langle l|\, \frac{(\hat{a}^*)^{(m-j)}}{\sqrt{m!}\sqrt{n!}} \,|n\rangle \,\mathrm{d}^2\gamma \\
&= \sum_{l,m} \sum_{j=0}^{m} \sum_{n=0}^{\infty} \left[\frac{1}{\pi} \int_{\mathbb{C}} \mathrm{e}^{-|\gamma|^2} (\gamma^*)^j \gamma^n \,\mathrm{d}^2\gamma \right] (-1)^j \binom{m}{j} v_l^* w_m \,\langle l|\, \frac{(\hat{a}^*)^{(m-j)}}{\sqrt{m!}\sqrt{n!}} \,|n\rangle \\
&= \sum_{l,m} \sum_{j=0}^{m} \sum_{n=0}^{\infty} n! \delta_{jn} (-1)^j \binom{m}{j} v_l^* w_m \,\langle l|\, \frac{(\hat{a}^*)^{(m-j)}}{\sqrt{m!}\sqrt{n!}} \,|n\rangle \\
&= \sum_{l,m} \sum_{j=0}^{m} (-1)^j \binom{m}{j} v_l^* w_m \,\langle l|\, \frac{j!(\hat{a}^*)^{(m-j)}}{\sqrt{m!}\sqrt{j!}} \,|j\rangle \\
&= \sum_{l,m} v_l^* w_m \sum_{j=0}^{m} (-1)^j \binom{m}{j} \langle l|m\rangle = \sum_{l,m} v_l^* w_m \delta_{m0} \delta_{lm} = v_0^* w_0.
\end{aligned}$$

We varify that

$$|\alpha\rangle\langle\beta| = \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{Tr}\left[\hat{D}_\gamma |\alpha\rangle\langle\beta|\right] \hat{D}_{-\gamma}\, \mathrm{d}^2\gamma.$$

Since $span\{|\alpha\rangle : \alpha \in \mathbb{C}\}$ is dense in $L^2(\mathbb{R})$, for every $n, m \in \mathbb{N}$,

$$|f_n\rangle\langle g_m| = \left|\sum_{i=0}^{n} c_i |\alpha_i\rangle\right\rangle\left\langle\sum_{j=0}^{m} d_j \langle\beta_j|\right| = \frac{1}{\pi}\int_{\mathbb{C}} \langle g_m| \hat{D}_\gamma |f_n\rangle \hat{D}_{-\gamma}\, \mathrm{d}^2\gamma$$

where $f_n := \sum_{i=0}^{n} c_i |\alpha_i\rangle$ and $g_m := \sum_{j=0}^{m} d_j |\beta_j\rangle$ belong to $L^2(\mathbb{R})$ with $c_i, d_j \in \mathbb{C}$.

By Lemma 5.4.1., for every $f, g \in L^2(\mathbb{R})$,

$$|f\rangle\langle g| = \lim_{n,m\to\infty} \left|\sum_{i=0}^{n} c_i |\alpha_i\rangle\right\rangle\left\langle\sum_{j=0}^{m} d_j \langle\beta_j|\right| = \frac{1}{\pi}\int_{\mathbb{C}} \langle g| \hat{D}_\gamma |f\rangle \hat{D}_{-\gamma}\, \mathrm{d}^2\gamma.$$

Then now, let's check about arbitrary trace-class operator $\hat{O}$.

$\hat{O}$ can be written as

$$\hat{O} = \sum_{i=0}^{\infty} \lambda_i |e_i\rangle\langle h_i|$$

for some orthonormal basis $\{e_i\}, \{h_i\}$ for $L^2(\mathbb{R})$ and a sequence of $\{\lambda_i\}$ with $\sum_i |\lambda_i| < \infty$.

In the same way, by Lemma 5.4.1.,

$$\frac{1}{\pi}\int_{\mathbb{C}} \left|\sum_{i=0}^{\infty} \lambda_i \langle h_i| \hat{D}_\gamma |e_i\rangle\right| \langle f| \hat{D}_{-\gamma} |g\rangle\, \mathrm{d}^2\gamma \leq \sum_{i=0}^{\infty} |\lambda_i| \|f\|_2 \|g\|_2 < \infty.$$

So, for every $\hat{O}$ in trace-class, the integral

$$\frac{1}{\pi}\int_{\mathbb{C}} \mathrm{Tr}\left[\hat{D}_\gamma \hat{O}\right] \hat{D}_{-\gamma}\, \mathrm{d}^2\gamma$$

is well defined as an weak -convergent bounded operator on $L^2(\mathbb{R})$.

Finally, for any $|f\rangle, |g\rangle \in L^2(\mathbb{R})$

and $\hat{O}_n := \sum_{i=0}^{n} \lambda_i |e_i\rangle\langle h_i|$ for some orthonormal basis $\{e_i\}, \{h_i\}$ for $L^2(\mathbb{R})$ with a sequence of $\{\lambda_i\}$ with $\sum_i |\lambda_i| < \infty$,

$$\begin{aligned}
&\left|\left\langle f \middle| \hat{O} - \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{Tr}\left[\hat{D}_\gamma \hat{O}\right] \hat{D}_{-\gamma}\, \mathrm{d}^2\gamma \middle| g \right\rangle\right| \\
&\quad = \left|\left\langle f \middle| \hat{O} - \hat{O}_n + \hat{O}_n - \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{Tr}\left[\hat{D}_\gamma \hat{O}\right] \hat{D}_{-\gamma}\, \mathrm{d}^2\gamma \middle| g \right\rangle\right| \\
&\quad \leq |\langle f| \hat{O} - \hat{O}_n |g\rangle| + \left|\frac{1}{\pi}\int_{\mathbb{C}} \mathrm{Tr}\left[\hat{D}_\gamma (\hat{O}_n - \hat{O})\right] \langle f| \hat{D}_{-\gamma} |g\rangle\, \mathrm{d}^2\gamma\right| \\
&\quad \leq |\langle f| \hat{O} - \hat{O}_n |g\rangle| + \sum_{i=n+1}^{\infty} |\lambda_i| \|f\|_2 \|g\|_2
\end{aligned}$$

So

$$\left|\langle f| \hat{O} - \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{Tr}\left[\hat{D}_\gamma \hat{O}\right] \hat{D}_{-\gamma}\, \mathrm{d}^2\gamma\, |g\rangle\right| \to 0, \qquad n \to \infty.$$

Therefore, we show that

$$\hat{O} = \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{Tr}\left[\hat{D}_\gamma \hat{O}\right] \hat{D}_{-\gamma}\, \mathrm{d}^2\gamma.$$

We can extend this to n-mode case by using tensor product as follow

$$\begin{aligned}
\bigotimes_{j=1}^{n} |\alpha_j\rangle\langle\beta_j| &= \bigotimes_{j=1}^{n} \left(\frac{1}{\pi}\int_{\mathbb{C}} \mathrm{Tr}\left[\hat{D}_{\gamma_j} |\alpha_j\rangle\langle\beta_j|\right] \hat{D}_{-\gamma_j}\, \mathrm{d}^2\gamma_j\right) \\
&= \frac{1}{\pi^n}\int_{\mathbb{C}^n} \mathrm{Tr}\left[\hat{D}_{-r} \bigotimes_{j=1}^{n} |\alpha_j\rangle\langle\beta_j|\right] \hat{D}_r\, \mathrm{d}^2\gamma_1 \cdots \mathrm{d}^2\gamma_n \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} \mathrm{Tr}\left[\hat{D}_{-r} \bigotimes_{j=1}^{n} |\alpha_j\rangle\langle\beta_j|\right] \hat{D}_r dr.
\end{aligned}$$

□

Definition 5.4.3 (Characteristic function)**.** Given a state ϱ which belongs to trace-class, define characteristic function χ_ϱ

$$\chi_\varrho(\alpha) := \mathrm{Tr}(\hat{D}_\alpha \varrho).$$

For n-bosonic mode,

$$\chi_\varrho(r) := \mathrm{Tr}(\hat{D}_{-r} \varrho).$$

It is well defined since ϱ belongs to trace-class. Since $\hat{D}_\alpha$ is unitary so that be bounded, $\varrho\hat{D}_\alpha$ belongs to trace-class.

Proposition 5.4.4. Given a density state ϱ belongs to trace-class, and characteristic function χ_ϱ defined as before,

1. χ_ϱ is bounded so continuous.
2. $\chi_\varrho(0) = 1$
3. $\chi_\varrho(-r) = \overline{\chi(r)}$ if and only if $\varrho^\dagger = \varrho$
4. $\chi_\varrho(r) \in L^2(\mathbb{R}^{2n})$

Remark 5.4.5. By the Fourier-Weyl relation, every state ϱ in trace-class can be represented by

$$\varrho = \frac{1}{\pi}\int_{\mathbb{C}} \chi(\alpha)\hat{D}_{-\alpha}\,\mathrm{d}^2\,\alpha$$

for single mode,

$$\varrho = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} \chi(r)\hat{D}_r\,\mathrm{d}\,r$$

for n-mode.

Let's determine the characteristic function χ_G of general gaussian state.

Theorem 5.4.6 (Characteristic function of Gaussian states)**.**

$$\chi_G(r) = \mathrm{e}^{-\frac{1}{4}r^\mathsf{T}\Omega^\mathsf{T}\sigma\Omega r}\mathrm{e}^{ir^\mathsf{T}\Omega^\mathsf{T}\bar{r}}$$

Proof.

$$\begin{aligned}
\chi_G(r) &= \prod_{j=1}^{n}(1-\mathrm{e}^{-\xi_j})\mathrm{Tr}\left(\left(\bigotimes_{j=1}^{n}\left(\sum_{m=0}^{\infty}\mathrm{e}^{-\xi_j m}|m\rangle_{jj}\langle m|\right)\right)\hat{S}\hat{D}_{\bar{r}}\hat{D}_{-r}\hat{D}_{\bar{r}}^\dagger\hat{S}^\dagger\right)\\
&= \prod_{j=1}^{n}(1-\mathrm{e}^{-\xi_j})\mathrm{Tr}\left(\left(\bigotimes_{j=1}^{n}\left(\sum_{m=0}^{\infty}\mathrm{e}^{-\xi_j m}|m\rangle_{jj}\langle m|\right)\right)\mathrm{e}^{-ir^\mathsf{T}\Omega(S\hat{r}+\bar{r})}\right)\\
&= \prod_{j=1}^{n}(1-\mathrm{e}^{-\xi_j})\mathrm{Tr}\left(\left(\bigotimes_{j=1}^{n}\left(\sum_{m=0}^{\infty}\mathrm{e}^{-\xi_j m}|m\rangle_{jj}\langle m|\right)\right)\hat{D}_{-S^{-1}r}\right)\mathrm{e}^{ir^\mathsf{T}\Omega^\mathsf{T}\bar{r}}
\end{aligned}$$

Let's evaluate a single-mode characteristic function,

$$\begin{aligned}
\chi^{(1)}(\alpha) :&= \mathrm{Tr}\left[\sum_{m=0}^{\infty} \mathrm{e}^{-\xi_j m} |m\rangle \langle m| \hat{D}_\alpha\right] \\
&= \frac{1}{\pi}\int_{\mathbb{C}} \sum_{m=0}^{\infty} \mathrm{e}^{-\xi_j m} \langle \gamma \mid m\rangle \langle m| \hat{D}_\alpha |\gamma\rangle \, \mathrm{d}^2\gamma \\
&= \frac{1}{\pi}\int_{\mathbb{C}} \sum_{m=0}^{\infty} \mathrm{e}^{-\xi_j m} \mathrm{e}^{-\frac{|\gamma|^2}{2}} \frac{\gamma^{*m}}{\sqrt{m!}} \langle m| \hat{D}_\alpha \hat{D}_\gamma |0\rangle \, \mathrm{d}^2\gamma \\
&= \frac{1}{\pi}\int_{\mathbb{C}} \sum_{m=0}^{\infty} \mathrm{e}^{-\xi_j m} \mathrm{e}^{-\frac{|\gamma|^2}{2}} \frac{\gamma^{*m}}{\sqrt{m!}} \mathrm{e}^{-\frac{|\alpha+\gamma|^2}{2}} \frac{(\alpha+\gamma)^m}{\sqrt{m!}} \mathrm{e}^{\frac{1}{2}(\alpha\gamma^* - \alpha^*\gamma)} \, \mathrm{d}^2\gamma \\
&= \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{e}^{-\frac{1}{2}(|\gamma|^2 + |\alpha+\gamma|^2 + \alpha^*\gamma - \alpha\gamma^*)} \sum_{m=0}^{\infty} \frac{(\gamma^*(\alpha+\gamma)\mathrm{e}^{-\xi_j})^m}{m!} \, \mathrm{d}^2\gamma \\
&= \frac{\mathrm{e}^{-\frac{|\alpha|^2}{2}}}{\pi} \int_{\mathbb{C}} \mathrm{e}^{-(\alpha^*\gamma + \gamma^*\gamma)} \mathrm{e}^{\gamma^*(\alpha+\gamma)\mathrm{e}^{-\xi_j}} \, \mathrm{d}^2\gamma \\
&= \frac{\mathrm{e}^{-\frac{|\alpha|^2}{2}}}{\pi} \int_{\mathbb{C}} \mathrm{e}^{-|\gamma|^2(1-\mathrm{e}^{-\xi_j})} \mathrm{e}^{\alpha\gamma^*\mathrm{e}^{-\xi_j} - \alpha^*\gamma} \, \mathrm{d}^2\gamma
\end{aligned}$$

If we set $\gamma = \frac{(x+iy)}{\sqrt{1-\mathrm{e}^{-\xi_j}}}$ where $x, y \in \mathbb{R}$ then,

$$\frac{\mathrm{e}^{-\frac{|\alpha|^2}{2}}}{\pi} \int_{\mathbb{R}^2} \mathrm{e}^{-\left(\frac{x^2+y^2}{1-\mathrm{e}^{-\xi_j}}\right)(1-\mathrm{e}^{-\xi_j})} \mathrm{e}^{\alpha\left(\frac{x-iy}{\sqrt{1-\mathrm{e}^{-\xi_j}}}\right)\mathrm{e}^{-\xi_j} - \alpha^*\left(\frac{x+iy}{\sqrt{1-\mathrm{e}^{-\xi_j}}}\right)} \frac{1}{\sqrt{1-\mathrm{e}^{-\xi_j}}} dx \frac{1}{\sqrt{1-\mathrm{e}^{-\xi_j}}} dy$$

$$= \frac{\mathrm{e}^{-\frac{|\alpha|^2}{2}}}{\pi(1-\mathrm{e}^{-\xi_j})} \int_{\mathbb{R}^2} \exp\left[-\begin{pmatrix} x & y \end{pmatrix} I_2 \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} \frac{\alpha\mathrm{e}^{-\xi_j} - \alpha^*}{\sqrt{1-\mathrm{e}^{-\xi_j}}} \\ \frac{-i\alpha\mathrm{e}^{-\xi_j} - i\alpha^*}{\sqrt{1-\mathrm{e}^{-\xi_j}}} \end{pmatrix}\right] dxdy.$$

By gaussian integral, we can get

$$\frac{1}{\pi}\int_{\mathbb{C}} \sum_{m=0}^{\infty} \mathrm{e}^{-\xi_j m} \langle \gamma \mid m\rangle \langle m| \hat{D}_\alpha |\gamma\rangle \, \mathrm{d}^2\gamma = \frac{\mathrm{e}^{-\frac{|\alpha|^2}{2}}}{\pi(1-\mathrm{e}^{-\xi_j})} \cdot \frac{\pi}{\sqrt{Det(I)}} \mathrm{e}^{\frac{1}{4}\mathbf{b}^\intercal\mathbf{b}} = \frac{\mathrm{e}^{-\frac{|\alpha|^2}{2}v_j}}{1-\mathrm{e}^{-\xi_j}}$$

where $\mathbf{b} = \begin{pmatrix} \frac{\alpha\mathrm{e}^{-\xi_j} - \alpha^*}{\sqrt{1-\mathrm{e}^{-\xi_j}}} \\ \frac{-i\alpha\mathrm{e}^{-\xi_j} - i\alpha^*}{\sqrt{1-\mathrm{e}^{-\xi_j}}} \end{pmatrix}$ and $v_j = \frac{1+\mathrm{e}^{-\xi_j}}{1-\mathrm{e}^{-\xi_j}}$.

So,

$$\mathrm{Tr}\left[\sum_{m=0}^{\infty} \mathrm{e}^{-\xi_j m} |m\rangle \langle m| \hat{D}_\alpha\right] = \frac{\mathrm{e}^{-\frac{|\alpha|^2}{2}v_j}}{1-\mathrm{e}^{-\xi_j}}.$$

Next, put $\alpha_j = \frac{x_j + ip_j}{\sqrt{2}}$ then,

$$\mathrm{Tr}\left[\sum_{m=0}^{\infty} \mathrm{e}^{-\xi_j m} |m\rangle \langle m| \hat{D}_{\alpha_j}\right] = \frac{\mathrm{e}^{-\frac{1}{2}\frac{(x_j^2+p_j^2)}{2}v_j}}{1-\mathrm{e}^{-\xi_j}}$$

and since $\hat{D}_{-r} = \bigotimes_{j=1}^n \hat{D}_{\alpha_j}$,

$$\mathrm{Tr}\left[\bigotimes_{j=1}^{n}\left(\sum_{m=0}^{\infty} \mathrm{e}^{-\xi_j m} |m\rangle \langle m|\right) \hat{D}_{-r}\right] = \frac{\mathrm{e}^{-\frac{1}{4}r^\intercal\left(\bigoplus_{j=1}^n v_j I_2\right)r}}{\prod_{j=1}^n (1-\mathrm{e}^{-\xi_j})}$$

where $r = (x_1, p_1, \ldots, x_n, p_n) \in \mathbb{R}^{2n}$.

So we have

$$\mathrm{Tr}\left[\bigotimes_{j=1}^{n}\left(\sum_{m=0}^{\infty} \mathrm{e}^{-\xi_j m} |m\rangle \langle m|\right) \hat{D}_{-S^{-1}r}\right] = \frac{\mathrm{e}^{-\frac{1}{4}(S^{-1}r)^\intercal\left(\bigoplus_{j=1}^n v_j I_2\right)S^{-1}r}}{\prod_{j=1}^n (1-\mathrm{e}^{-\xi_j})}.$$

Therefore,

$$\begin{aligned}
\chi_G(r) &= \prod_{j=1}^{n}(1-\mathrm{e}^{-\xi_j})\mathrm{Tr}\left(\left(\bigotimes_{j=1}^{n}\left(\sum_{m=0}^{\infty} \mathrm{e}^{-\xi_j m} |m\rangle_{jj} \langle m|\right)\right) \hat{D}_{-S^{-1}r}\right) \mathrm{e}^{ir^\intercal\Omega^\intercal\bar{r}} \\
&= \mathrm{e}^{-\frac{1}{4}r^\intercal(S^{-1})^\intercal\left(\bigoplus_{j=1}^n v_j I_2\right)S^{-1}r}\mathrm{e}^{ir^\intercal\Omega^\intercal\bar{r}} \\
&= \mathrm{e}^{-\frac{1}{4}r^\intercal(S^{-1})^\intercal\Omega^\intercal\left(\bigoplus_{j=1}^n v_j I_2\right)\Omega S^{-1}r}\mathrm{e}^{ir^\intercal\Omega^\intercal\bar{r}} \\
&= \mathrm{e}^{-\frac{1}{4}r^\intercal\Omega^\intercal\sigma\Omega r}\mathrm{e}^{ir^\intercal\Omega^\intercal\bar{r}}.
\end{aligned}$$

□

Remark 5.4.7. The gaussian state can be written as

$$\varrho_G = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} \chi_G(r)\hat{D}_r dr = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} \mathrm{e}^{-\frac{1}{4}r^\intercal\Omega^\intercal\sigma\Omega r}\mathrm{e}^{ir^\intercal\Omega^\intercal\bar{r}}\hat{D}_r dr$$

So every gaussian states are completely determined by first moment $\bar{r}$, and covariance matrix σ.

Chapter 6

GAUSSIAN QUANTUM CHANNEL

6.1 Gaussian unitary operation

In mathematical terms a quantum channel $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a completely positive trace-preserving map(CPTP map).

Theorem 6.1.1 (Linear displacements on statistical moments)**.** The unitary action $\hat{D}_r^\dagger \varrho \hat{D}_r$ of a linear displacement $\hat{D}_r$ maps the initial first moments $\bar{r}$ and second moments(covariance matrix) σ of the gaussian states ϱ according to

$$\begin{aligned} \bar{r} &\mapsto \bar{r} + r, \\ \sigma &\mapsto \sigma \end{aligned}$$

so that it sends all guassian states into gaussian states.

Proof.

$$
\begin{aligned}
&\mathrm{Tr}\left[\hat{D}_r^\dagger \varrho_G \hat{D}_r \hat{r}\right] \\
&\quad= \prod_{j=1}^n (1-\mathrm{e}^{-\xi_j})\mathrm{Tr}\left[\left(\bigotimes_{j=1}^n \left(\sum_{m=0}^\infty \mathrm{e}^{-\xi_j m}|m\rangle_{jj}\langle m|\right)\right)\hat{S}\hat{D}_{\bar{r}}\hat{D}_r\hat{r}\hat{D}_r^\dagger\hat{D}_{\bar{r}}^\dagger\hat{S}^\dagger\right] \\
&\quad= \prod_{j=1}^n (1-\mathrm{e}^{-\xi_j})\mathrm{Tr}\left[\left(\bigotimes_{j=1}^n \left(\sum_{m=0}^\infty \mathrm{e}^{-\xi_j m}|m\rangle_{jj}\langle m|\right)\right)S\hat{r}+\bar{r}+r\right] \\
&\quad= \bar{r}+r=\bar{r}'
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{Tr}\left[\{(\hat{r}-\bar{r}'),(\hat{r}-\bar{r}')^\mathsf{T}\}\hat{D}_r^\dagger \varrho_G \hat{D}_r\right] \\
&= \prod_{j=1}^n (1-\mathrm{e}^{-\xi_j})\mathrm{Tr}\left[\left(\bigotimes_{j=1}^n \left(\sum_{m=0}^\infty \mathrm{e}^{-\xi_j m}|m\rangle_{jj}\langle m|\right)\right)\hat{S}\hat{D}_{\bar{r}}\hat{D}_r\{(\hat{r}-\bar{r}'),(\hat{r}-\bar{r}')^\mathsf{T}\}\hat{D}_r^\dagger\hat{D}_{\bar{r}}^\dagger\hat{S}^\dagger\right] \\
&= \prod_{j=1}^n (1-\mathrm{e}^{-\xi_j})\mathrm{Tr}\left[\left(\bigotimes_{j=1}^n \left(\sum_{m=0}^\infty \mathrm{e}^{-\xi_j m}|m\rangle_{jj}\langle m|\right)\right)\hat{S}\hat{D}_{\bar{r}}\{(\hat{r}-\bar{r}),(\hat{r}-\bar{r})^\mathsf{T}\}\hat{D}_{\bar{r}}^\dagger\hat{S}^\dagger\right] \\
&= \sigma
\end{aligned}
$$

□

Theorem 6.1.2 (Symplectic transformations on statistical moments)**.** The unitary action $\hat{S}_*^\dagger \varrho \hat{S}_*$ of a quadratic operator $\hat{S}_* = \mathrm{e}^{\frac{i}{2}\hat{r}^\mathsf{T} H_* \hat{r}}$ maps the initial first moments $\bar{r}$ and second moments σ of the gaussian states ϱ according to

$$
\begin{aligned}
\bar{r} &\mapsto S_*\bar{r}, \\
\sigma &\mapsto S_*\sigma S_*^\mathsf{T}
\end{aligned}
$$

where $S_* = \mathrm{e}^{\Omega H_*}$.

so that it sends all guassian states into gaussian states

Proof.

$$\begin{aligned}
&\mathrm{Tr}\left[\hat{S}_*^\dagger \varrho_G \hat{S}_* \hat{r}\right]\\
&\quad = \prod_{j=1}^{n}(1-\mathrm{e}^{-\xi_j})\mathrm{Tr}\left[\left(\bigotimes_{j=1}^{n}\left(\sum_{m=0}^{\infty}\mathrm{e}^{-\xi_j m}|m\rangle_{jj}\langle m|\right)\right)\hat{S}\hat{D}_{\bar{r}}\hat{S}_*\hat{r}\hat{S}_*^\dagger\hat{D}_{\bar{r}}^\dagger\hat{S}^\dagger\right]\\
&\quad = \prod_{j=1}^{n}(1-\mathrm{e}^{-\xi_j})\mathrm{Tr}\left[\left(\bigotimes_{j=1}^{n}\left(\sum_{m=0}^{\infty}\mathrm{e}^{-\xi_j m}|m\rangle_{jj}\langle m|\right)\right)S_*(S\hat{r}+\bar{r})\right]\\
&\quad = S_*\bar{r}
\end{aligned}$$

$$\begin{aligned}
&\mathrm{Tr}\left[\{(\hat{r}-S_*\bar{r}),(\hat{r}-S_*\bar{r})^\mathsf{T}\}\hat{D}_r^\dagger\varrho_G\hat{D}_r\right]\\
&= \prod_{j=1}^{n}(1-\mathrm{e}^{-\xi_j})\mathrm{Tr}\left[\left(\bigotimes_{j=1}^{n}\left(\sum_{m=0}^{\infty}\mathrm{e}^{-\xi_j m}|m\rangle_{jj}\langle m|\right)\right)\hat{S}\hat{D}_{\bar{r}}\hat{S}_*\{(\hat{r}-S_*\bar{r}),(\hat{r}-S_*\bar{r})^\mathsf{T}\}\hat{S}_*^\dagger\hat{D}_{\bar{r}}^\dagger\hat{S}^\dagger\right]\\
&= \prod_{j=1}^{n}(1-\mathrm{e}^{-\xi_j})\mathrm{Tr}\left[\left(\bigotimes_{j=1}^{n}\left(\sum_{m=0}^{\infty}\mathrm{e}^{-\xi_j m}|m\rangle_{jj}\langle m|\right)\right)\hat{S}\hat{D}_{\bar{r}}\{S_*(\hat{r}-\bar{r}),(\hat{r}-\bar{r})^\mathsf{T}S_*^\mathsf{T}\}\hat{D}_{\bar{r}}^\dagger\hat{S}^\dagger\right]\\
&= \prod_{j=1}^{n}(1-\mathrm{e}^{-\xi_j})\mathrm{Tr}\left[\left(\bigotimes_{j=1}^{n}\left(\sum_{m=0}^{\infty}\mathrm{e}^{-\xi_j m}|m\rangle_{jj}\langle m|\right)\right)\{S_*S\hat{r},\hat{r}^\mathsf{T}S^\mathsf{T}S_*^\mathsf{T}\}\right]\\
&= S_*\sigma S_*^\mathsf{T}
\end{aligned}$$

□

Theorem 6.1.3 (Gaussian unitary)**.** A unitary transformation generated by a second-order Hamiltonian(which called *gaussian unitary*), sends all gaussian states into gaussian state.

Proof. We know that both linear displacement and Sympletic transformation send Guassian states into gaussian states. First, one may write some Hamiltonian for n modes as

$$\hat{H}' = \frac{1}{2}\hat{D}_{-\bar{r}}\hat{r}^\mathsf{T}H\hat{r}\hat{D}_{\bar{r}},$$

where $\hat{D}_{\bar{r}} = \mathrm{e}^{i\bar{r}^\mathsf{T}\Omega\hat{r}}$ is a Weyl operator with displacement $\bar{r}$ and H is a $2n \times 2n$ postive symmetric matrix. The equation above may be exponentiated to obtain a relationship between unitary operations:

$$\mathrm{e}^{i\hat{H}'} = \hat{D}_{-\bar{r}}\mathrm{e}^{\frac{i}{2}\hat{r}^\mathsf{T}H\hat{r}}\hat{D}_{\bar{r}}.$$

Since $\hat{S}\hat{r}\hat{S}^\dagger = S\hat{r}$ where $S = \mathrm{e}^{\Omega H}$ and $\hat{S} = \mathrm{e}^{\frac{i}{2}\hat{r}^\mathsf{T}H\hat{r}}$,

$$\hat{S}\mathrm{e}^{i\bar{r}^\mathsf{T}\Omega\hat{r}}\hat{S}^\dagger = \mathrm{e}^{i\bar{r}^\mathsf{T}\Omega S\hat{r}} = \mathrm{e}^{\gamma i}\mathrm{e}^{i\bar{r}^\mathsf{T}S^{\mathsf{T}^{-1}}\Omega\hat{r}} = \hat{D}_{S^{-1}\bar{r}}.$$

So

$$\hat{S}\hat{D}_{\bar{r}} = \hat{D}_{S^{-1}\bar{r}}\hat{S},$$

and we have

$$\mathrm{e}^{i\hat{H}} = \mathrm{e}^{\gamma i}\mathrm{e}^{i\hat{H}'} = \mathrm{e}^{\gamma i}\hat{D}_{-\bar{r}}\hat{D}_{S^{-1}\bar{r}}\hat{S} = \mathrm{e}^{i\bar{r}^\mathsf{T}\Omega S^{-1}\bar{r}/2}\hat{D}_{(S^{-1}-I)\bar{r}}\hat{S}$$

where $\gamma = \bar{r}^\mathsf{T}r$.

Therefore, unitary transformation generated by a second-order Hamiltonian is decomposed as multipication of Linear displacement and Symplectic transformation. So sends gaussian states into gaussian states. □

6.2 Tensor product and partial trace of gaussian states

Theorem 6.2.1 (Tensor product of gaussian states)**.** Given two gaussian states ϱ_A, ϱ_B of any number of modes m and n with $\bar{r}_A$ and $\bar{r}_B$ and σ_A and σ_B, then $\varrho_A \otimes \varrho_B$ is a gaussian state with $\bar{r} = \bar{r}_A \oplus \bar{r}_B$ and $\sigma = \sigma_A \oplus \sigma_B$.

Proof.

$$
\begin{aligned}
&\varrho_A \otimes \varrho_B \\
&= \frac{1}{(2\pi)^{m+n}} \int_{\mathbb{R}^{2(m+n)}} \mathrm{e}^{-\frac{1}{4} r_A^\intercal \Omega_A^\intercal \sigma_A \Omega_A r_A} \mathrm{e}^{-\frac{1}{4} r_B^\intercal \Omega_B^\intercal \sigma_B \Omega_B r_B} \mathrm{e}^{i r_A^\intercal \Omega_A^\intercal \bar{r}_A} \mathrm{e}^{i r_B^\intercal \Omega_B^\intercal \bar{r}_B} \hat{D}_{r_A} \otimes \hat{D}_{r_B} dr \\
&= \frac{1}{(2\pi)^{m+n}} \int_{\mathbb{R}^{2(m+n)}} \mathrm{e}^{-\frac{1}{4}(r_A \oplus r_B)^\intercal \Omega^\intercal (\sigma_A \oplus \sigma_B) \Omega (r_A \oplus r_B)} \mathrm{e}^{i (r_A \oplus r_B)^\intercal \Omega^\intercal (\bar{r}_A \oplus \bar{r}_B)} \hat{D}_{r_A} \otimes \hat{D}_{r_B} dr
\end{aligned}
$$

where $\Omega = \Omega_A \oplus \Omega_B$ and $dr = dr_A dr_B$.

Since

$$
\begin{aligned}
\hat{D}_{r_A} \otimes \hat{D}_{r_B} &= \mathrm{e}^{i r_A^\intercal \Omega_A \hat{r}_A \oplus i r_B^\intercal \Omega_B \hat{r}_B} \\
&= \mathrm{e}^{i (r_A \oplus r_B)^\intercal \Omega (\hat{r}_A \oplus \hat{r}_B)},
\end{aligned}
$$

we have

$$
\varrho_A \otimes \varrho_B = \frac{1}{(2\pi)^{m+n}} \int_{\mathbb{R}^{2(m+n)}} \mathrm{e}^{-\frac{1}{4} r^\intercal \Omega^\intercal \sigma \Omega r} \mathrm{e}^{i r^\intercal \Omega^\intercal \bar{r}} \hat{D}_r dr
$$

where $r = r_A \oplus r_B$, and $\bar{r} = \bar{r}_A \oplus \bar{r}_B$ and $\sigma = \sigma_A \oplus \sigma_B$. □

Remark 6.2.2. By the Fourier-Weyl relation,

$$
\mathrm{Tr}(\varrho) = \mathrm{Tr}\left[\frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} \chi(r) \hat{D}_r \, \mathrm{d}\, r \right] = 1.
$$

Since $\mathrm{Tr}(\varrho) = \lim_{M \to \infty} \sum_{m=0}^{M} \langle \phi_m, \varrho \phi_m \rangle$, where ϕ_m is basis of $L^2(\mathbb{R}^n)$,

$$
\begin{aligned}
\mathrm{Tr}(\varrho) &= \lim_{M \to \infty} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} \chi(r) \sum_{m=0}^{M} \langle \phi_m | \hat{D}_r | \phi_m \rangle \, \mathrm{d}\, r \\
&= \lim_{M \to \infty} \left\langle \chi(r), \frac{1}{(2\pi)^n} \sum_{m=0}^{M} \langle \phi_m | \hat{D}_r | \phi_m \rangle \right\rangle \\
&= 1 = \chi(0) = \left\langle \chi(r), \delta^{2n} \right\rangle.
\end{aligned}
$$

So for all characteristic function of baussian states ϱ_G,

$$
\lim_{M \to \infty} \left\langle \chi_G(r), \frac{1}{(2\pi)^n} \sum_{m=0}^{M} \langle \phi_m | \hat{D}_r | \phi_m \rangle \right\rangle = \left\langle \chi_G(r), \delta^{2n} \right\rangle.
$$

Theorem 6.2.3 (Partial trace of Guassian state)**.** Given a Guassian state ϱ of number of modes $n+m$, with

$\bar{r} = \bar{r}_A \oplus \bar{r}_B$ and $\sigma = \begin{pmatrix} \sigma_A & \sigma_{AB} \\ \sigma_{AB}^{\mathsf{T}} & \sigma_B \end{pmatrix}$

where each $\bar{r}_A$ and $\bar{r}_B$ are $2n$ and $2m$ dimensional real vectors, and each σ_A and σ_B are $2n \times 2n$ and $2m \times 2m$ matrices. Then the partial trace of ϱ : $\mathrm{Tr}_B(\varrho)$ is Guassian state with $\bar{r}_{\mathrm{Tr}_B(\varrho)} = \bar{r}_A$, and $\sigma_{\mathrm{Tr}_B(\varrho)} = \sigma_A$.

Proof.

$$\begin{aligned}
\mathrm{Tr}_B(\varrho) &= \frac{1}{(2\pi)^{m+n}} \lim_{M\to\infty} \left\langle \chi(r)\hat{D}_{r_A}, \sum_{m=0}^{M} \langle m| \hat{D}_{r_B} |m\rangle \right\rangle \\
&= \frac{1}{(2\pi)^{m+n}} \left\langle \chi(r)\hat{D}_{r_A}, (2\pi)^m \delta^{2m}(r_B) \right\rangle
\end{aligned}$$

Since,

$$\begin{aligned}
\varrho &= \frac{1}{(2\pi)^{m+n}} \int_{\mathbb{R}^{2(m+n)}} \mathrm{e}^{-\frac{1}{4} r^{\mathsf{T}} \Omega^{\mathsf{T}} \sigma \Omega r} \mathrm{e}^{i r^{\mathsf{T}} \Omega^{\mathsf{T}} \bar{r}} \hat{D}_r dr \\
&= \frac{1}{(2\pi)^{m+n}} \int_{\mathbb{R}^{2(m+n)}} \exp\left[-\frac{1}{4} (r_A \oplus r_B)^{\mathsf{T}} \Omega^{\mathsf{T}} \begin{pmatrix} \sigma_A & \sigma_{AB} \\ \sigma_{AB}^{\mathsf{T}} & \sigma_B \end{pmatrix} \Omega (r_A \oplus r_B) \right] \\
&\qquad \times \mathrm{e}^{i (r_A \oplus r_B)^{\mathsf{T}} \Omega^{\mathsf{T}} (\bar{r}_A \oplus \bar{r}_B)} \hat{D}_{r_A} \otimes \hat{D}_{r_B} dr,
\end{aligned}$$

we have

$$\mathrm{Tr}_B(\varrho) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} \mathrm{e}^{-\frac{1}{4} r_A^{\mathsf{T}} \Omega_A^{\mathsf{T}} \sigma_A \Omega r_A} \mathrm{e}^{i r_A^{\mathsf{T}} \Omega_A^{\mathsf{T}} \bar{r}_A} \hat{D}_{r_A} dr_A.$$

□

6.3 Gaussian quantum channel

Theorem 6.3.1 (Robertson-Schrodinger uncertainty relation)**.** Let σ be the covariance matrix of gaussian states ϱ, then

$$\sigma + i\Omega \geq 0$$

Proof. Since

$$\begin{aligned} 2\mathrm{Tr}[\varrho(\hat{r}-\bar{r})(\hat{r}-\bar{r})^{\mathsf{T}}] &= \mathrm{Tr}[\varrho\{(\hat{r}-\bar{r}),(\hat{r}-\bar{r})^{\mathsf{T}}\} + \varrho[\hat{r},\hat{r}^{\mathsf{T}}]] \\ &= \mathrm{Tr}[\varrho\{(\hat{r}-\bar{r}),(\hat{r}-\bar{r})^{\mathsf{T}}\}] + \mathrm{Tr}[\varrho[\hat{r},\hat{r}^{\mathsf{T}}]] \\ &= \sigma + i\Omega \end{aligned}$$

What to show is $2\mathrm{Tr}[\varrho(\hat{r}-\bar{r})(\hat{r}-\bar{r})^{\mathsf{T}}] \geq 0$.

$$\begin{aligned} &2\mathrm{Tr}[\varrho(\hat{r}-\bar{r})(\hat{r}-\bar{r})^{\mathsf{T}}] \\ &= 2\prod_{j=1}^{n}(1-\mathrm{e}^{-\xi_j})\mathrm{Tr}\left(\bigotimes_{j=1}^{n}\left(\sum_{m=0}^{\infty}\mathrm{e}^{-\xi_j m}\left|m\right\rangle\left\langle m\right|\right)\hat{S}\hat{D}_{\bar{r}}(\hat{r}-\bar{r})(\hat{r}-\bar{r})^{\mathsf{T}}\hat{D}_{\bar{r}}^{\dagger}\hat{S}^{\dagger}\right) \\ &= 2\prod_{j=1}^{n}(1-\mathrm{e}^{-\xi_j})S\ \mathrm{Tr}\left(\bigotimes_{j=1}^{n}\left(\sum_{m=0}^{\infty}\mathrm{e}^{-\xi_j m}\left|m\right\rangle\left\langle m\right|\right)\hat{r}\hat{r}^{\mathsf{T}}\right)S^{\mathsf{T}} \\ &= \prod_{j=1}^{n}(1-\mathrm{e}^{-\xi_j})S\ \mathrm{Tr}\left[\bigotimes_{j=1}^{n}\left(\sum_{m=0}^{\infty}\mathrm{e}^{-\xi_j m}\left|m\right\rangle\left\langle m\right|\right)\left(\bigoplus_{j=1}^{n}\begin{pmatrix}2\hat{a}_j^*\hat{a}_j+1 & i \\ -i & 2\hat{a}_j^*\hat{a}_j+1\end{pmatrix}\right)\right]S^{\mathsf{T}} \\ &= SMS^{\mathsf{T}} \end{aligned}$$

where $M = \bigoplus_{j=1}^{n}\begin{pmatrix}v_j & i \\ -i & v_j\end{pmatrix}$, $v_j = \frac{1+\mathrm{e}^{-\xi_j}}{1-\mathrm{e}^{-\xi_j}} \geq 1$.

Since for all $\begin{pmatrix}c_1 & c_2\end{pmatrix} \in \mathbb{C}^2$,

$$\begin{pmatrix}c_1^* & c_2^*\end{pmatrix}\begin{pmatrix}v_j & i \\ -i & v_j\end{pmatrix}\begin{pmatrix}c_1 \\ c_2\end{pmatrix} = \left(|c_1|^2+|c_2|^2\right)v_j - 2\,\mathrm{Im}(c_1^*c_2) \geq 0,$$

M is positive Hermitian, so does SMS^{T}. □

Definition 6.3.2 (Gaussian quantum channel)**.** [EW07] We denote by $S_1(\mathcal{H})$ the closed Banach space of trace-class operators on $\mathcal{H}$.
Then a gaussian channel $T : S_1(\mathcal{H}) \to S_1(\mathcal{H})$ is defined as follows

$$T(\varrho) = \mathrm{Tr}_E \left[U(\varrho \otimes \varrho_E)U^\dagger \right], \qquad \varrho \in \mathcal{B}(\mathcal{H}).$$

where ϱ_E are gaussian states, and $U, U^\dagger$ are gaussian unitary. Especially, ϱ_E is environment state of space, which makes a noise.

Theorem 6.3.3 (Complete characterisation of gaussian channel)**.** Given an initial gaussian state of n modes with vector of first moments $\bar{r}$ and covariance matrix σ, the evolution due to a gaussian channel is completely described by two $2n \times 2n$ real matrices X and Y, which act as follows

$$\bar{r} \mapsto X\bar{r} + r'$$
$$\sigma \mapsto X\sigma X^\mathsf{T} + Y,$$

and the matrices X and Y must be such that

$$Y + i\Omega \geq iX\Omega X^\mathsf{T}.$$

Proof. Since an Linear displacement acts as translation operator of the first moment, verifying symplectic transfrom is enough to see complete characterisation of this channel. Here is the m-mode state ϱ_E(Environmental state), so that $\varrho' = \varrho \otimes \varrho_E$ is state with

$$\bar{r}' = \bar{r} \oplus \bar{r}_E$$
$$\sigma' = \sigma \oplus \sigma_E.$$

We apply symplectic transform at ϱ' and then partial trace for environmental part, so that

$$T(\varrho) = \mathrm{Tr}_E[\hat{S}^\dagger(\varrho \otimes \varrho_E)\hat{S}]$$

where $\hat{S}$ is unitary, generated by purely quadratic Hamiltonian, which acts as $\hat{S}^{\dagger}\hat{r}\hat{S} = S\hat{r}$ and $S \in Sp_{2n,\mathbb{R}}$.

Let us

$$S = \begin{pmatrix} X & B \\ C & D \end{pmatrix} \in Sp_{2(n+m),\mathbb{R}}$$

where $X \in M_{2n}$ and $D \in M_{2m}$. Then,

$$S\Omega S^{\mathsf{T}} = \begin{pmatrix} X\Omega_n X^{\mathsf{T}} + B\Omega_m B^{\mathsf{T}} & X\Omega_n C^{\mathsf{T}} + B\Omega_m D^{\mathsf{T}} \\ C\Omega_n X^{\mathsf{T}} + D\Omega_m B^{\mathsf{T}} & C\Omega_n C^{\mathsf{T}} + D\Omega_m D^{\mathsf{T}} \end{pmatrix} = \begin{pmatrix} \Omega_n & 0 \\ 0 & \Omega_m \end{pmatrix}$$

where each Ω_n, Ω_m is n, m dimentional skew symmetric matrix and $\Omega = \Omega_n \oplus \Omega_m$. So,

$$X\Omega_n X^{\mathsf{T}} + B\Omega_m B^{\mathsf{T}} = \Omega_n \tag{6.1}$$

and

$$\begin{aligned} S(\sigma \oplus \sigma_E)S^{\mathsf{T}} &= \begin{pmatrix} X & B \\ C & D \end{pmatrix} \begin{pmatrix} \sigma & 0 \\ 0 & \sigma_E \end{pmatrix} \begin{pmatrix} X^{\mathsf{T}} & C^{\mathsf{T}} \\ B^{\mathsf{T}} & D^{\mathsf{T}} \end{pmatrix} \\ &= \begin{pmatrix} X\sigma X^{\mathsf{T}} + B\sigma_E B^{\mathsf{T}} & X\sigma C^{\mathsf{T}} + B\sigma_E D^{\mathsf{T}} \\ C\sigma X^{\mathsf{T}} + D\sigma_E B^{\mathsf{T}} & C\sigma C^{\mathsf{T}} + D\sigma_E D^{\mathsf{T}} \end{pmatrix}. \end{aligned}$$

Then, by partial trace and setting $B\sigma_E B^{\mathsf{T}} = Y$,

$$\sigma \mapsto X\sigma X^{\mathsf{T}} + B\sigma_E B^{\mathsf{T}} = X\sigma X^{\mathsf{T}} + Y.$$

Since $\sigma_E + i\Omega_m \geq 0$,

$$B\sigma_E B^{\mathsf{T}} + iB\Omega_m B^{\mathsf{T}} = Y + iB\Omega_m B^{\mathsf{T}} \geq 0. \tag{6.2}$$

By (6.1) and (6.2),

$$Y + i\Omega_n \geq iX\Omega_n X^{\mathsf{T}}.$$

□

Theorem 6.3.4. Conversely, Given that maps as follows

$$\bar{r} \mapsto X\bar{r} + r'$$
$$\sigma \mapsto X\sigma X^{\mathsf{T}} + Y,$$

which satisfying

$$Y + i\Omega \geq iX\Omega X^{\mathsf{T}},$$

then we can determine gaussian channels. i.e. Given some σ_E, there are linear displacement and symplectic transform satisfying those conditions.

Proof. Without loss of generosity, we set initial state ϱ as a gaussian state with $\bar{r} \in \mathbb{R}^{2n}$ and $\sigma \in Sp_{2n,\mathbb{R}}$, and ϱ_E is a gaussian state with $\bar{r}_E = \mathbf{0} \in \mathbb{R}^{4n}$ and $\sigma_E = I_{4n}$.

What to show is that there exists symplectic transform satisfying above conditions. Suppose there exists $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{6n,\mathbb{R}}$, where A is $2n \times 2n$, and D is $4n \times 4n$ matrix. If we set a linear displacement as $\bar{r} \to \bar{r} + r'$ then we can say

$$S \begin{pmatrix} \sigma & 0 \\ 0 & I_{4n} \end{pmatrix} S^{\mathsf{T}} = \begin{pmatrix} A\sigma A^{\mathsf{T}} + BB^{\mathsf{T}} & * \\ * & * \end{pmatrix}$$
$$S \begin{pmatrix} \bar{r} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} A\bar{r} \\ * \end{pmatrix}$$

So $A\sigma A^{\mathsf{T}} + BB^{\mathsf{T}} = X\sigma X^{\mathsf{T}} + Y$ and $A\bar{r} = X\bar{r}$ for all $\bar{r} \in \mathbb{R}^{2n}$, then we get $A = X$ and $Y = BB^{\mathsf{T}}$,
so that $BB^{\mathsf{T}} + i\Omega_n - iX\Omega_n X^{\mathsf{T}} \geq 0$.

Since $iX\Omega_n X^{\mathsf{T}} - i\Omega_n$ is skew-symmetric, for all $\mathbf{v} \in \mathbb{R}^{2n}$,

$$\mathbf{v}^{\mathsf{T}}(BB^{\mathsf{T}} + i\Omega_n - iX\Omega_n X^{\mathsf{T}})\mathbf{v} = \mathbf{v}^{\mathsf{T}}(BB^{\mathsf{T}})\mathbf{v} \geq 0,$$

so $Y = BB^{\mathsf{T}} \geq 0$. If $Y = 0$, then $B = 0$. If we show existance of B, C, D in case of $Y > 0$, then we prove it in general case of $Y \geq 0$.

Suppose that $Y > 0$, then there exists $\sqrt{Y} > 0$ such that $(\sqrt{Y})^\intercal = \sqrt{Y}$.
Now, Let $B = \sqrt{Y}O$, where B and O are $2n \times 4n$ matrices which satisfy $OO^\intercal = I_{2n}$.
What to show is there exists O satisfying symplectic condition such that

$$X\Omega_n X^\intercal + \sqrt{Y}O\Omega_{2n}O^\intercal\sqrt{Y} = \Omega_n.$$

Since $\sqrt{Y}^{-1}(\Omega_n - X\Omega_n X^\intercal)\sqrt{Y}^{-1}$ is skew-symmetric, by spectral theory, there exist $R_1 \in O(2n)$ and $R_1' \in M_{2n\times 4n}$ such that

$$R_1(\sqrt{Y}^{-1}(\Omega_n - X\Omega_n X^\intercal)\sqrt{Y}^{-1})R_1^\intercal = \bigoplus_{j=1}^{n} r_j\Omega_1 = R_1'\Omega_{2n}{R_1'}^\intercal.$$

By the hypothesis, $I_{2n} \geq i\sqrt{Y}^{-1}(X\Omega_n X^\intercal - \Omega_n)\sqrt{Y}^{-1}$.
So for all j, $|r_j| \leq 1$. And without loss of generosity, let R_1' is 2×4 matrix, such that

$$R_1' = \begin{pmatrix} \cos\theta_1 & 0 & -\sin\theta_1 & 0 \\ 0 & \cos\theta_1 & 0 & \sin\theta_1 \end{pmatrix},$$

then

$$R_1'(R_1')^\intercal = I_2,$$

$$R_1' \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} (R_1')^\intercal = \begin{pmatrix} 0 & \cos 2\theta_1 \\ -\cos 2\theta_1 & 0 \end{pmatrix},$$

so we can find θ_1 satisfying $\cos 2\theta_1 = r_1$ and then le $O = R_1^\intercal R_1'$. Now, we check there exists $O \in M_{2n\times 4n}$ satisfying $OO^\intercal = I_{2n}$ and symplectic condition.

From now on, we determine $A = X$ and $B = \sqrt{Y}O$.
So if we find out C and D satisfying that $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is symplectic matrix, then we can determine S. Now, what to show is there exist C and D such that

$$A\Omega_n C^\intercal + B\Omega_{2n}D^\intercal = 0 \tag{6.3}$$

$$C\Omega_n C^\intercal + D\Omega_{2n}D^\intercal = \Omega_{2n}. \tag{6.4}$$

Since $\Omega_{2n} - D\Omega_{2n}D^\mathsf{T}$ is skew-symmetric, there exists $R_2 \in O(4n)$ such that

$$R_2(\Omega_{2n} - D\Omega_{2n}D^\mathsf{T})R_2^\mathsf{T} = R_2'\Omega_n {R_2'}^\mathsf{T}$$

where $R_2' \in M_{4n\times 2n}$ and we can get $R_2^\mathsf{T} R_2'$ as C.

By (6.3), $B\Omega_{2n}D^\mathsf{T} = -A\Omega_n C^\mathsf{T}$ and because of our hypothesis that $Y > 0$, $\sqrt{Y}$ is invertible. Finally, we can find out D as

$$\begin{aligned} D^\mathsf{T} &= -\Omega_{2n}O^\mathsf{T}\sqrt{Y}^{-1}B\Omega_{2n}D^\mathsf{T} \\ &= \Omega_{2n}O^\mathsf{T}\sqrt{Y}^{-1}A\Omega_n C^\mathsf{T}. \end{aligned}$$

□

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초 록

양자역학의 도입 이후, 양자정보이론은 수리물리학에서 중요 관심사로 떠올랐다. 특히, bosonic gaussian 양자상태는 물리적 구현이 상대적으로 용이하여 광자를 통한 실험설계가 가능하기 때문에 양자정보에서 다루는 주요 대상 중 하나이다. Gaussian 양자상태와 양자채널을 다루는 많은 논문들이 있지만, 보통 충분한 수학적 엄밀함으로 쓰여 있지는 않다. 이 논문에서는 함수해석과 작용소 이론에 관련된 부분들을 엄밀히 채워보고자 한다.

주요 내용은 Alessio Serafini 책[Ser17]의 접근방식을 따라가지만, 정의역 문제 또는 Hamiltonian의 self-adjointness 등 빠진 부분을 명확히 밝힘으로써 모호한 부분 없이 bosonic gaussian 양자상태와 채널을 분석하도록 할 것이다.

주요어: bosonic gaussian 양자상태, 양자채널, 연속변수, self-adjoint

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