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Distributionally Robust Optimization for Inventory Problems

재고관리 문제에 대한 분포 강건 최적화

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서울대학교 대학원

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Abstract

Distributionally Robust Optimization for Inventory Problems

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The inventory problem is a classical problem in the operations management society to decide an optimal order policy under demand uncertainty. A decision maker chooses order quantities over the planning horizon to achieve the company's objective with respect to performance measures. Classical inventory management researches assume that complete information about the probability distribution of random demand is known, however, only limited information of the probability distribution is available in practice. To tackle this difficulty, a decision maker considers an ambiguity set which is a set of candidate distributions that may contain the unknown true distribution, and minimizes the worst-case expected cost over the ambiguity set. This approach is called distributionally robust optimization (DRO) and widely applied to many operations management problems. We adopt the distributionally robust approach to inventory problems to handle distributional ambiguity.

In this dissertation, we consider three different but closely related problems: newsvendor problem, inventory problem, and empty container repositioning problem. For all three problems, we study decision making under demand uncertainty, but limited information about probability distributions of random demand is given. Hence, we adopt the distributionally robust approach and analyze various aspects of distributionally robust models. First, we study the data-driven distributionally robust newsvendor model with a set of distributions close to the empirical distribution in terms of the Wasserstein distance, and derive the closed-form solution of an optimal order quantity. Second, the inventory problem is considered, which is an extension of the newsvendor problem to the multistage setting. In the multistage setting of distributionally robust inventory problems, the decision maker carefully considers time consistency issue. Time consistency means that the optimal policy derived in the first period maintains its optimality through the planning horizon. We analyze the time consistency issue of the distributionally robust inventory model with a Wasserstein ambiguity set. Third, the empty container repositioning problem with foldable containers is considered, which is a practical application of the inventory problem. We propose a mathematical model of the empty container repositioning problem considering the use of foldable containers under demand uncertainty. To tackle the intractability of the multistage stochastic programming formulation, the linear decision rule formulation is proposed for the tractable and distributionally robust approximation of the multistage stochastic programming formulation. We also conduct computational experiments to validate respective models and findings.

Keywords: Distributionally robust optimization, Newsvendor model, Inventory management, Empty container repositioning, Wasserstein distance

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Chapter 1

Introduction

1.1 Inventory Problems

The inventory problem is a classical problem in the operations management society to decide an optimal order policy under demand uncertainty. A decision maker chooses order quantities over the planning horizon to achieve the company's objective with respect to performance measures. As the competition among companies intensifies, inventory control plays an important role in the survival of companies. Improved service quality through efficient and effective inventory management is becoming an important factor in the company's competitiveness. On the other hand, decision making under uncertainty becomes important as the operation of inventory management becomes complex. Classical inventory management researches assume that complete information about probability distribution of random demand is known, however, only limited information of the probability distribution is available in practice. Accordingly, how to incorporate the distributional ambiguity into decision making process is a key consideration in practical applications. To tackle this difficulty, we adopt a distributionally robust approach to solve inventory problems.

In this dissertation, we consider three different but closely related problems: newsvendor problem, inventory problem, and empty container repositioning problem. First, the newsvendor problem is considered. A newsvendor decides an order quantity for a perishable good before the random demand is realized. The newsvendor model is a building block of many operations management problems, such as inventory control, pricing, and supply chain contracts. In the classical newsvendor model, the complete knowledge about the probability distribution of random demand is assumed. However, in reality, only limited information or historical data is given, so we consider the data-driven setting of the newsvendor model. We study the data-driven distributionally robust newsvendor model with a set of distributions close to the empirical distribution in terms of the Wasserstein distance. We derive the closed-form solution of a distributionally robust order quantity.

Second, the inventory problem is considered, which is an extension of the newsvendor problem to the multistage setting. A decision maker decides an optimal order policy through the planning horizon to optimize total costs. There are various practical applications of the inventory problem such as warehouse management, dynamic pricing, and retail management. We study the data-driven distributionally robust inventory model with a similar setting of the newsvendor problem. In the multistage setting of distributionally robust inventory problems, the decision maker carefully considers time consistency issue. Time consistency means that the optimal policy derived in the first period maintains its optimality through the planning horizon for almost every realization of demand. However, time consistency is not guaranteed in the distributionally robust setting, i.e., the optimal policy derived in the first stage does not satisfy the principle of optimality in dynamic programming. Therefore, we investigate the sufficient condition for time consistency based on the monotone non-decreasing optimal base-stock levels.

Third, the empty container repositioning problem with foldable containers is considered, which is a practical application of the inventory problem. Due to the reusable property of containers, empty containers are returned to the depot after used to transport goods. The empty container repositioning problem can be viewed as an inventory problem considering the reusable property of the container. In the shipping industry, due to trade imbalance between continents, there is a shortage of empty containers in export-oriented ports and a surplus of empty containers in import-oriented ports. Therefore, a shipping company repositions empty containers from import-oriented ports to export-oriented ports with considerable costs. The foldable container is developed to reduce repositioning costs and commercialized recently. The demand uncertainty and foldable containers are considered and the linear decision rule formulation is proposed for a tractable approximation of the multistage stochastic programming formulation.

For all three problems, we study decision making under demand uncertainty, but limited information about probability distributions of random demand is given. Hence, we adopt the distributionally robust approach and analyze various aspects of distributionally robust models.

1.2 Distributionally Robust Optimization

One of the key issues in optimization is decision making under uncertainty. When the objective function $\Psi(\cdot, \cdot) : X \times \Xi \to \mathbb{R}$ depends on decision $x \in X$ and parameter $\xi \in \Xi$, deterministic optimization to derive an optimal solution in terms of the objective function is expressed as follows:

$$\min_{x \in X} \Psi(x, \xi)$$

When parameter or data ξ is uncertain, we model ξ as a random variable with probability distribution μ. In stochastic optimization, a decision maker optimizes the decision over the expectation of the objective function Ψ with respect to distribution μ. Stochastic optimization is expressed as follows:

$$\min_{x \in X} \mathbb{E}_{\mu}[\Psi(x, \xi)]$$

In reality, however, the demand distribution is often impossible to be known precisely. To tackle this difficulty, a decision maker considers an *ambiguity set* $\mathcal{M}$ which is a set of candidate distributions that may contain the unknown true distribution, and minimizes the worst-case expected cost over the ambiguity set. This approach is one specific example of *distributionally robust optimization* (DRO) and expressed as follows:

$$\min_{x \in X} \max_{\mu \in \mathcal{M}} \mathbb{E}_{\mu}[\Psi(x, \xi)]$$

There are various representations of distributionally robust optimization. In this dissertation, we focus on minimization of the worst-case expected cost. One important advantage of DRO is its tractability. Optimization under uncertainty often leads to computationally intractable formulations, however, DRO formulations with the properly chosen objective function and ambiguity set can be translated to tractable formulations such as convex programs. Another advantage is that DRO is a generalization of robust optimization and stochastic optimization. If an ambiguity set is singleton, i.e., $\mathcal{M} = \{\mu\}$, DRO reduces to stochastic optimization. When an ambiguity set consists of all probability distributions supported on Ξ, the resulting DRO formulation is equivalent to the robust formulation.

The key consideration of DRO is a selection of an ambiguity set based on prior information of the probability distribution. A moment-based ambiguity set is constructed by a set of distributions that satisfy the given moment information, e.g., mean and variance. Since the pioneering work of Scarf [76] where the distributionally robust newsvendor model with known mean and variance is considered, there is an extensive literature of DRO with a moment-based ambiguity set [26, 35, 106]. Moment information based DRO is tractable in most cases, however, the resulting solutions are often conservative in terms of average performance. An alternative of the moment information based ambiguity set is an ambiguity set that consists of probability distributions close to the given reference distribution with respect to a statistical distance. A statistical distance measures the distance between two probability distributions. The widely used statistical distances for constructing ambiguity sets are the ϕ-divergence [8, 7, 45] and the Wasserstein distance [34, 30, 70]. This approach is well applicable to the data-driven setting, and a statistical distance based ambiguity set centered at the empirical distribution is closely related to the statistical estimation such as the goodness-of-fit test [15].

In this dissertation, we adopt both approaches to suit the problem circumstance. For the newsvendor and inventory problems, we consider the data-driven setting and construct an ambiguity set constructed by a set of probability distributions close to the empirical distribution in terms of the Wasserstein distance. DRO with the Wasserstein distance has several desirable properties such as the out-of-sample performance guarantee and the convergence property [15, 30]. For the empty container repositioning problem, a moment-based ambiguity set with the known first and second moments is considered for the distributionally robust bound on the expectation of positive parts. We discuss further details in the corresponding chapters.

1.3 Research Motivations and Contributions

In this section, we discuss the motivations and contributions of each problem. When we adopt the distributionally robust approach to handle distributional ambiguity, there are several considerations for each problem. First, for the newsvendor problem, the closed-form expression of an optimal order quantity is essential for applications to various operations management problems. Moreover, we can analyze the structure of an optimal order quantity based on the closed-form expression. To decide a desirable order quantity in the data-driven setting, we study the distributionally robust newsvendor model with a Wasserstein ambiguity set on continuous and unbounded support $\Xi = [0, \infty)$ and the Wasserstein order $p \in [1, \infty)$. To the best of our knowledge, a closed-form solution and an explicit characterization of the worst-case distribution are not studied in the general setting.

Second, the multistage setting of DRO is considered for the inventory problem. In the multistage DRO, there exists an undesirable phenomenon called time consistency, which does not occur in the risk-neutral case where the probability distribution is known exactly. The decision maker hopes that an optimal policy computed at the first stage remains optimal at the later stages. However, the multistage DRO with a Wasserstein ambiguity set is time inconsistent in general, and the best thing we can do is to investigate the sufficient condition that the problem is time consistent. Hence, we analyze the time consistency issue of the distributionally robust inventory model with a Wasserstein ambiguity set.

Third, the empty container repositioning problem, which is a specific application of inventory problems, is considered. With the consideration of using foldable containers, the shipping company has to decide which type (standard or foldable) of containers and quantities of containers to satisfy the demand, which makes the decision process complex. The integrated model is needed to handle the introduction of foldable containers. On the other hand, how to incorporate demand uncertainty and handle computational intractability of the multistage stochastic programming is a key consideration for the success of practical applications. In addition, we consider the distributional ambiguity, i.e., the limited information about the probability distribution. Then, we adopt the decision rule based approximation to obtain computationally tractable and distributionally robust solutions.

The main contributions of the dissertation are summarized as follows:

1. For the newsvendor problem,

   - The closed-form expressions of an optimal order quantity and the worst-case distribution for the risk-neutral newsvendor problem are derived with the general support and the general Wasserstein order.
   - For the risk-averse decision, we also consider the Conditional Value-at-Risk (CVaR) objective for the newsvendor model. We derive closed-form solutions for the Wasserstein order $p = 1$ case and propose a tractable formulation to obtain an optimal order quantity for the $p > 1$ case.

2. For the inventory problem,

   - We derive a sufficient condition for weakly time consistent of the distributionally robust inventory problem with the Wasserstein ambiguity set based on the monotonicity of base-stock levels.
   - We discuss the condition that monotone non-decreasing of optimal order quantities is satisfied based on closed-form solutions for the Wasserstein

newsvendor model, which leads to weak time consistency.

- We discuss further details of the DP formulation, such as calculation of base-stock levels and optimality of an (s, S) policy, and conduct numerical experiments to verify desirable properties of distributionally robust solutions.

3. For the empty container repositioning problem,

    - We propose a mathematical model of the empty container repositioning problem considering the use of foldable containers under demand uncertainty.
    - To tackle the intractability of the multistage stochastic programming formulation, we adopt the linear decision rule and distributionally robust bound on the expectation of positive parts, and propose the LDR and RLDR formulations.
    - The proposed formulations are tractable approximations of the multistage stochastic programming formulation and have distributionally robust properties.
    - We show the cost-saving and storage-saving effects of using foldable containers through numerical experiments and simulations.

1.4 Outline of the Dissertation

In this dissertation, three different inventory problems are considered: newsvendor problem, inventory problem, and empty container repositioning problem. In Chapter 2, we study the data-driven distributionally robust newsvendor model with an ambiguity set defined with a set of distributions close to the empirical distribution in terms of the Wasserstein distance. We derive the closed-form expressions of an optimal order quantity for the distributionally robust newsvendor model. In Chapter 3, we consider the multistage inventory problem and analyze the time consistency issue of the distributionally robust inventory model with a Wasserstein ambiguity set. We also discuss further details of the dynamic programming formulation of the inventory problem In Chapter 4, the empty container repositioning problem with foldable containers is studied. To incorporate demand uncertainty in the operational-planning level decisions, we propose a multistage stochastic programming formulation of the empty container repositioning problem. The stochastic programming formulation is computationally intractable in general, so we adopt the factor-based demand model and distributionally robust bound to propose the tractable and robust formulation. Finally, we conclude the dissertation and discuss possible future research directions in Chapter 5.

Chapter 2

Distributionally Robust Newsvendor Model with a Wasserstein Ambiguity Set

2.1 Problem Description and Literature Review

The newsvendor problem is a well-known problem to decide an order quantity considering the trade-off between the risks of *overage* and *underage* under demand uncertainty. It is a building block of many operations management problems, such as inventory control, pricing, supply chain contracts, and retail management. In the classical setting of the newsvendor problem, complete knowledge of demand distribution is assumed. In reality, however, the demand distribution is often impossible to be known precisely. To tackle this difficulty, a decision maker considers an *ambiguity set* which is a set of candidate distributions that may contain the unknown true distribution, and minimizes the worst-case expected cost over the ambiguity set. This approach is called *distributionally robust optimization* (DRO). Various types of ambiguity sets are proposed with several different prior information about demand distribution. In some practical cases, however, historical data is the only information that can be obtained. Therefore, how to construct an ambiguity set with historical data and optimize over the constructed ambiguity set is important to successful operations. In this study, we consider the data-driven distributionally robust newsvendor model with a Wasserstein ambiguity set.

The distributionally robust newsvendor model dates back to Scarf [76], who considered the distributionally robust order quantity with an ambiguity set that contained all distributions with known first and second moments. Gallego and Moon [32] extended Scarf's basic results to various ways with the same ambiguity set. With the development of DRO with moment-based ambiguity sets [12, 26, 35, 106], several extensions were proposed to consider various objective functions and to construct new ambiguity sets considering the shape of distribution, e.g., symmetry/asymmetry and unimodality/multimodality [39, 66, 68, 75, 113, 115]. In most cases, the distributionally robust newsvendor models with moment constraints are tractable, and in some cases closed-form solutions and explicit characterizations of the worst-case distributions are available. However, the assumption that a decision maker has certain information about moments proves to be unrealistic for many operations management problems. For example, historical data of a newly introduced product is not enough to estimate moments. Moreover, decisions based on an inaccurate estimation of moments can lead to highly suboptimal solutions. Even if the decision maker has the exact moment information, the moment-based ambiguity set is constructed with only moment information and other prior information such as the shape of distribution is abandoned. Another shortcoming of the moment-based ambiguity set is that resulting decisions are sometimes overly conservative due to the unrealistic worst-case distribution [104].

An important alternative is DRO with an ambiguity set which contains probability distributions close to the reference distribution in terms of a statistical distance. A statistical distance measures the distance between two probability distributions and several studies used various statistical distances for constructing ambiguity sets, such as ϕ-divergences [7, 8, 45, 96] and Wasserstein distance [30, 70, 107, 114]. However, as Gao and Kleywegt [34] pointed out, in some cases, an ambiguity set with ϕ-divergences fails to include distributions that a decision maker wishes to include. For instance, consider historical data generated from a normal distribution and a ϕ-divergence ambiguity set based on discrete empirical distribution. The ϕ-divergence ambiguity set does not contain the data-generating normal distribution, because probability distributions in the ambiguity set is absolutely continuous with respect to the empirical distribution, that is, the ambiguity set includes only discrete distributions with the same support of the empirical distribution. Gao and Kleywegt [34] also pointed out that ϕ-divergence does not consider the closeness between two points in the support, thus leading to the inclusion of overly conservative or pessimistic distributions.

DRO with an ambiguity set based on the Wasserstein distance not only alleviates the problems mentioned above, but also has several useful properties. The Wasserstein distance captures closeness between two points, which leads to the realistic measurement of distance between two distributions. Furthermore, the Wasserstein ambiguity set contains both discrete and continuous distributions, because the Wasserstein distance between discrete and continuous distributions can be defined (cf. ϕ-divergence). In addition, the Wasserstein ambiguity set with the empirical distribution contains the data-generating distribution with probabilistic guarantees [31]. Desirable properties, e.g., finite sample guarantee, asymptotic consistency, and tractability [15, 30, 31], are proved for the Wasserstein order $p = 1$.

Although some researches have considered the newsvendor models with a Wasserstein ambiguity set, they are used as examples to emphasize theoretical results of DRO, and the models are limited to discrete and bounded support [34] or the Wasserstein order $p = 1$ [30]. To the best of our knowledge, a closed-form solution and an explicit characterization of the worst-case distribution are not studied in the general setting such as the continuous and unbounded support, and the higher Wasserstein order ($p > 1$). In this study, we study the distributionally robust newsvendor model with a Wasserstein ambiguity set on continuous and unbounded support $\Xi = [0, \infty)$ and the Wasserstein order $p \in [1, \infty)$. We consider the reference distribution for an ambiguity set as the empirical distribution.

The main contributions of this chapter are as follows:

- The closed-form expressions of an optimal order quantity and the worst-case distribution for the risk-neutral newsvendor problem are derived with the general support and the general Wasserstein order.

- For the risk-averse decision, we also consider the Conditional Value-at-Risk (CVaR) objective for the newsvendor model. We derive closed-form solutions for the $p = 1$ case and propose a tractable formulation to obtain an optimal order quantity for the $p > 1$ case.

The rest of this chapter is organized as follows. In Section 2.2, we introduce the definition of the Wasserstein distance and strong duality result for the data-driven DRO with a Wasserstein ambiguity set. In Section 2.3, we derive the closed-form solutions of an optimal order quantity and the worst-case distribution for the distributionally robust newsvendor model. We also consider the risk-averse model and analyze the model in Section 2.4. Section 2.5 provides numerical experiments based on the theoretical results, and we summarize this chapter in Section 2.6.

2.2 Distributionally Robust Optimization with the Wasserstein Distance

In this subsection, we introduce the definition of the Wasserstein distance and discuss properties of the Wasserstein ambiguity set in the optimization perspective. We adopt the strong duality result of data-driven DRO with the Wasserstein distance and related definitions from the result of [34].

Let (Ξ, d) be a separable complete metric space (Polish space) and $\mathcal{B}(\Xi)$ be the Borel σ-algebra. Let $\mathcal{P}(\Xi)$ denote a set of Borel measures defined on $(\Xi, \mathcal{B}(\Xi))$. Let $\mathcal{P}_p(\Xi)$ for $p \in [1, \infty)$ denote a set of probability measures with a finite moment of order p for any $x_0 \in \Xi$, i.e., $\mathcal{P}_p(\Xi) := \left\{\mu \in \mathcal{P}(\Xi) : \int_\Xi d(x_0, x)^p \mu(dx) < \infty\right\}$.

Definition 2.1 (Wasserstein distance)**.** *The Wasserstein distance of order p between two probability measures $\mu, \nu \in \mathcal{P}_p(\Xi)$ is defined as*

$$W_p(\mu, \nu) := \left(\min_{\gamma \in \Gamma(\mu,\nu)} \left\{ \int_{\Xi \times \Xi} d(\xi, \zeta)^p d\gamma(\xi, \zeta) \right\} \right)^{\frac{1}{p}},$$

where $\Gamma(\mu, \nu)$ denotes a set of all probability measures on $\mathcal{P}(\Xi \times \Xi)$ with marginals μ and ν.

We use metric d for the definition of the Wasserstein distance as $d(\xi, \zeta) := |\xi - \zeta|$ throughout this study. The Wasserstein distance is motivated by the optimal transport theory whose foundation is rooted back to the Monge's problem [60]. We refer readers to [100] and [4] for further details. The meaning of the Wasserstein distance is the optimal transport cost of moving mass from μ to ν (Figure 2.1). Therefore, the Wasserstein distance represents a distance between two different distributions considering the distance between two points in Ξ with respect to the metric d. The

Wasserstein distance has been attracting attention in recent studies because of several good properties. First, convergence with respect to the Wasserstein distance implies weak convergence. Second, DRO with a Wasserstein ambiguity set can incorporate the data-driven setting and overcome the absolutely continuous support issue of the ϕ-divergence. The Wasserstein distance is widely applied to various studies, e.g., data-driven DRO [34, 30] and machine learning. For the machine learning literature, the Wasserstein distance of order 1 is called the earth mover's distance (EMD) and actively applied to image retrieval [73], generalized adversarial networks (GAN) [6], and regularization [79].

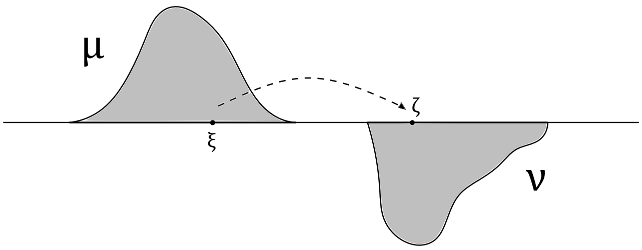

Figure 2.1 Concept of the Wasserstein distance

First, we discuss the properties of the Wasserstein distance, which may be useful in further analysis. The following properties are summarized from various literature [69, 70, 100].

Proposition 2.2 (Properties of the Wasserstein distance)**.** *Let (Ξ, d) be a Polish space. The followings hold.*

- $\Gamma(\mu, \nu)$ *is nonempty, convex and weakly compact.*
- W_p *is finite on* $\mathcal{P}_p(\Xi)$*, so* W_p *is a distance function on* $\mathcal{P}_p(\Xi)$*.* $\mathcal{P}_p(\Xi)$ *equipped with a distance function* W_p *is a metric space.*

- *W_p metrizes the weak convergence in $\mathcal{P}_p(\Xi)$, i.e., μ_k converges to μ weakly if and only if $W_p(\mu_k, \mu) \to 0$.*

- *W_p is continuous on $\mathcal{P}_p(\Xi)$.*

- *The Wasserstein distance is monotone, i.e., if $p_1 \leq p_2$, then $W_{p_1}(\mu, \nu) \leq W_{p_2}(\mu, \nu)$.*

- *W_p is p-convex, i.e., for any $\mu_1, \mu_2 \in \mathcal{P}_p(\Xi)$ and $\lambda \in [0, 1]$,*

$$W_p(\nu, (1-\lambda)\mu_1 + \lambda\mu_2)^p \leq (1-\lambda)W_p(\nu, \mu_1)^p + \lambda W_p(\nu, \mu_2)^p.$$

- *A metric space $(\mathcal{P}_p(\Xi), W_p)$ with the Wasserstein distance is a Polish space.*

We refer to [85] for the definition of p-convexity and [69] for the proof of p-convexity of W_p. We define an ambiguity set based on the Wasserstein distance and discuss the properties of the Wasserstein ambiguity set based on Proposition 2.2.

Definition 2.3 (Wasserstein ambiguity set)**.** *Let ν be the reference distribution. The Wasserstein ambiguity set $\mathcal{M}$ is defined as*

$$\mathcal{M} := \{\mu \in \mathcal{P}_p(\Xi) : W_p(\mu, \nu) \leq \theta\}.$$

The Wasserstein radius θ determines the size of a Wasserstein ambiguity set and a decision maker can control the conservativeness of the model with θ. With the properly chosen radius, the probability that the Wasserstein ambiguity set contains the unknown true distribution is guaranteed [31, 30].

Since $(\mathcal{P}_p(\Xi), W_p)$ is a metric space, we can define a closed ball $\mathcal{B}_\theta(\nu)$ centered at $\nu \in \mathcal{P}_p(\Xi)$ with radius θ as follows:

$$\mathcal{B}_\theta(\nu) := \{\mu \in \mathcal{P}_p(\Xi) : W_p(\mu, \nu) \leq \theta\}$$

Then, $\mathcal{B}_\theta(\nu)$ coincides with Definition 2.3, that is, a Wasserstein ambiguity set is a closed ball in a metric space $(\mathcal{P}_p(\Xi), W_p)$. The following proposition reveals the useful properties of the Wasserstein ambiguity set.

Proposition 2.4. *Let ν be the empirical distribution. Then, the Wasserstein ambiguity set $\mathcal{M}$ is closed, convex, and weakly compact.*

Proof. $\mathcal{M}$ is a closed ball in $(\mathcal{P}_p(\Xi), W_p)$, so $\mathcal{M}$ is closed. To show $\mathcal{M}$ is convex, we use p-convexity of W_p from Proposition 2.2. For any $\mu_1, \mu_2 \in \mathcal{P}_p(\Xi)$ and $\lambda \in [0, 1]$, $(1-\lambda)\mu_1 + \lambda\mu_2 \in \mathcal{P}_p(\Xi)$ and

$$W_p(\nu, (1-\lambda)\mu_1 + \lambda\mu_2)^p \leq (1-\lambda)W_p(\nu, \mu_1)^p + \lambda W_p(\nu, \mu_2)^p \leq (1-\lambda)\theta^p + \lambda\theta^p = \theta^p.$$

The first inequality holds by p-convexity of W_p and the second inequality holds by the definition of $\mathcal{M}$. Therefore, $W_p(\nu, (1-\lambda)\mu_1 + \lambda\mu_2) \leq \theta$ and $(1-\lambda)\mu_1 + \lambda\mu_2 \in \mathcal{M}$, which proves that $\mathcal{M}$ is convex. For the compactness, $\mathcal{M}$ is weakly compact by Banach-Alaoglu theorem (Section 3.15 in [74]) and tightness of empirical distribution ν. For another proof of compactness, since the empirical distribution ν is tight, by Proposition 3 of [70], $\mathcal{M}$ is weakly compact. □

We note that the Wasserstein ambiguity set restricted to contain only normal distributions is not convex when the Wasserstein order $p = 2$ [1].

The properties of the Wasserstein ambiguity set will be used in various analyses. Using the Wasserstein ambiguity set centered at the empirical distribution, DRO with cost function $\Psi : X \times \Xi \to \mathbb{R}$ is expressed as follows:

$$\inf_{x \in X} \sup_{\mu \in \mathcal{M}} \mathbb{E}_{\mu}[\Psi(x, \xi)]. \tag{2.1}$$

From the study of [34], we adopt the strong duality result for data-driven DRO with the Wasserstein distance of order p when the reference distribution is the empirical distribution.

Theorem 2.5 (Strong duality for data-driven DRO, [34]). *Let ν be the empirical distribution with historical data $\{\hat{\xi}^1, \dots, \hat{\xi}^N\}$, i.e., $\nu = \frac{1}{N}\sum_{i=1}^{N} \delta_{\hat{\xi}^i}$ where δ denotes the Dirac measure. Then, the strong dual of (2.1) is*

$$\inf_{x \in X, \lambda \geq 0} \left\{ \lambda \theta^p + \frac{1}{N} \sum_{i=1}^{N} \sup_{\xi \in \Xi} \left[\Psi(x, \xi) - \lambda d(\xi, \hat{\xi}^i)^p \right] \right\}.$$

To avoid the trivial case of the dual formulation in Theorem 2.5, the inner supremum should be finite. To guarantee finiteness of the inner supremum, we need the following definition.

Definition 2.6 (Growth rate, [34]). *The growth rate κ of Ψ is defined as*

$$\kappa := \inf \left\{ \lambda \geq 0 : \int_{\Xi} \sup_{\xi \in \Xi} \left[\Psi(x, \xi) - \lambda d(\xi, \zeta)^p \right] \nu(d\zeta) < \infty \right\}.$$

The growth rate κ is the minimum value of the dual variable λ, which makes the inner supremum finite (see also the definition of steepness of the objective function in Theorem 6.3 and Proposition 6.5 of [30]). The dual variable λ should be greater than or equal to the growth rate of the function Ψ, otherwise, the dual formulation becomes infeasible. We utilize the above results to analyze the distributionally robust newsvendor model with a Wasserstein ambiguity set.

2.3 Distributionally Robust Newsvendor Model

In this section, we consider the data-driven distributionally robust newsvendor model with a risk-neutral decision maker. We consider the general Wasserstein order $p \in [1, \infty)$ and derive the closed-form solution of the optimal order quantity. Then, we characterize the worst-case distribution with perturbations from historical data.

In the newsvendor model, the decision maker sells a single product for a single period. The decision maker decides the order quantity before the random demand $\xi \in \Xi$ is observed. After the demand is realized, the *overage cost* h per unit of unsold goods and the *underage cost* b per unit of shortage are imposed. The objective of the decision maker is to minimize the expected total cost. The newsvendor model can be expressed as follows:

$$\min_{x \geq 0} \mathbb{E}_{\mu}[h(x-\xi)^{+} + b(\xi - x)^{+}]$$

where $X^{+} := \max\{X, 0\}$. In the classical newsvendor problem with the known demand distribution μ, the optimal order quantity is well known as the critical ratio, i.e., the $\frac{b}{h+b}$ quantile of the demand distribution.

However, in practice, the demand distribution is restricted to be known precisely. Although complete knowledge of demand distribution is restricted, historical data can be obtained. Therefore, we propose the data-driven distributionally robust newsvendor model. We consider the support of the demand distribution is $\Xi = [0, \infty)$, and we assume that without loss of generality N historical data $\{\hat{\xi}^{1}, \ldots, \hat{\xi}^{N}\}$ is sorted in nondecreasing order, that is, $\hat{\xi}^{1} \leq \ldots \leq \hat{\xi}^{N}$. Using historical data, we define the empirical distribution $\nu = \frac{1}{N}\sum_{i=1}^{N} \delta_{\hat{\xi}^{i}}$, which is used for constructing a Wasserstein ambiguity set. Then, the data-driven distributionally robust risk-neutral newsvendor model with a Wasserstein ambiguity set is expressed as follows:

$$\min_{x \geq 0} \sup_{\mu \in \mathcal{P}_p(\Xi)} \{\mathbb{E}_\mu[h(x-\xi)^+ + b(\xi - x)^+] : W_p(\mu, \nu) \leq \theta\}. \tag{2.2}$$

Using Theorem 2.5 based on the empirical distribution ν and $d(\xi, \zeta) = |\xi - \zeta|$, the dual reformulation of (2.2) can be expressed as follows:

$$\min_{x \geq 0} \inf_{\lambda \geq 0} \left\{ \lambda \theta^p + \frac{1}{N} \sum_{i=1}^{N} \sup_{\xi \in \Xi} \left[h(x-\xi)^+ + b(\xi - x)^+ - \lambda |\xi - \hat{\xi}^i|^p \right] \right\}. \tag{2.3}$$

To obtain meaningful and simple analysis, especially for the analysis of the inner supremum and derivation of closed-form solutions, we impose a weak restriction on *overage* and *underage* costs.

Assumption 2.7. *The underage cost is greater than or equal to the overage cost, i.e., $b \geq h$.*

The assumption is needed for further analysis, e.g., analysis of the inner supremum in (2.3), the partition of sample points based on x and λ, and feasibility issues in Subsection 2.3.2. In addition, the assumption has real-world meaning: the decision maker considers the underage situation to be more important than the overage situation. In practice, the shortage is more important in many cases because it results not only in a penalty cost but also in loss of goodwill or trust, which may be costly for the decision maker. Therefore, the underage cost is greater than the overage cost for many real-world situations which means that the decision maker tends to order more than the mean of random demand in the newsvendor problem.

We consider two cases: $p = 1$ and $p > 1$. The reason we divide the cases is that the analysis is easier for the Wasserstein order $p = 1$ case. Due to the Kantorovich-Rubinstein duality, calculation of the Wasserstein distance for $p = 1$ is much more tractable. The inner supremum of the dual formulation is first order, which makes the derivation of the explicit form of the inner supremum simple. Even with the difficulty of analysis in the $p > 1$ case, the differentiation can be used for the explicit expression of the inner supremum.

2.3.1 Wasserstein Order $p = 1$

First, we consider the Wasserstein order $p = 1$ for the dual formulation (2.3). We can notice that the growth rate of the newsvendor cost function (Definition 2.6) is b. Therefore, dual variable λ should be greater than or equal to b, i.e., the constraint $\lambda \geq b$ should be added.

To obtain the closed-form solution, we need to characterize equivalent expressions of the inner supremum. For given (x, λ), let $f_i(\xi) := h(x-\xi)^+ + b(\xi - x)^+ - \lambda|\xi - \hat{\xi}^i|$ for $i = 1, \ldots, N$. To analyze the supremum of f_i, we define $N_1(x) := \{1 \leq i \leq N : \hat{\xi}^i < x\}$ and $N_2(x) := \{1 \leq i \leq N : \hat{\xi}^i \geq x\}$ such that N data points are divided into two sets based on x.

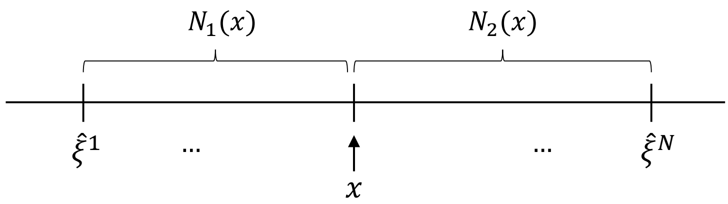

Figure 2.2 Definitions of $N_1(x)$ and $N_2(x)$

Under Assumption 2.7, the supremum of f_i can be derived by simple analysis using strong duality results. One specific instance of f_i is presented in Figure 2.3. There are four cases according to whether $x < \hat{\xi}^i$ or $x \geq \hat{\xi}^i$, and whether $\lambda = b$ or $\lambda > b$. We can see that the supremum of f_i is attained at $\xi = \hat{\xi}^i$ for all four cases. This result can be extended to all instances of f_i, that is, the supremum of f_i is attained at $\xi = \hat{\xi}^i$, which leads to $\sup_{\xi\in\Xi} f_i(\xi) = h(x - \hat{\xi}^i)$ for $i \in N_1(x)$ and $\sup_{\xi\in\Xi} f_i(\xi) = b(\hat{\xi}^i - x)$ for $i \in N_2(x)$.

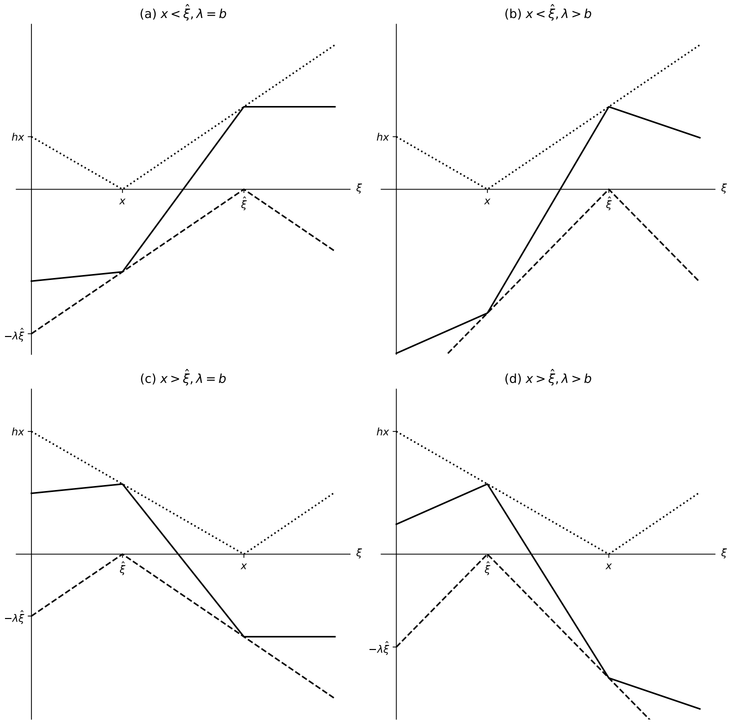

Figure 2.3 Four cases for $f_i(\xi)$. Dotted lines represent $h(x-\xi)^+ + b(\xi - x)^+$ and $-\lambda|\xi - \hat{\xi}^i|$, respectively, and the solid line represents $f_i(\xi)$

Likewise, the supremum of f_i is attained at $\xi = \hat{\xi}^i$, which leads to $\sup_{\xi\in\Xi} f_i(\xi) = h(x - \hat{\xi}^i)$ for $i \in N_1(x)$ and $\sup_{\xi\in\Xi} f_i(\xi) = b(\hat{\xi}^i - x)$ for $i \in N_2(x)$. This result shows that the inner supremum of dual formulation is independent of dual variable λ, which disconnects the linkage between x and λ. Therefore, we obtain the following equivalent expression of the objective function of dual formulation (2.3).

$$
\begin{aligned}
&\min_{x\geq 0,\lambda\geq b} \lambda\theta + \frac{1}{N}\sum_{i=1}^{N}\sup_{\xi\in\Xi}\left[h(x-\xi)^+ + b(\xi-x)^+ - \lambda|\xi-\hat{\xi}^i|\right] \\
&= \min_{x\geq 0,\lambda\geq b} \lambda\theta + \frac{1}{N}\left(\sum_{i\in N_1(x)} h(x-\hat{\xi}^i) + \sum_{i\in N_2(x)} b(\hat{\xi}^i - x)\right) \\
&= b\theta + \min_{x\geq 0}\frac{1}{N}\left(\sum_{i\in N_1(x)} h(x-\hat{\xi}^i) + \sum_{i\in N_2(x)} b(\hat{\xi}^i - x)\right)
\end{aligned}
$$

Since x and λ are separated, the optimal dual variable $\lambda^* = b$. Now, we derive the closed-form solution based on the critical ratio $\frac{b}{h+b}$ in Theorem 2.8.

Theorem 2.8. *Suppose the underage cost is greater than or equal to the overage cost, i.e., $b \geq h$. If $i^* \in \{1,\ldots,N\}$ satisfies $\frac{i^*-1}{N} < \frac{b}{h+b} \leq \frac{i^*}{N}$, then $\hat{\xi}^{i^*}$ is an optimal order quantity and the optimal cost is*

$$
b\theta + \frac{1}{N}\left(\sum_{k=1}^{i^*-1} h(\hat{\xi}^{i^*} - \hat{\xi}^k) + \sum_{k=i^*}^{N} b(\hat{\xi}^k - \hat{\xi}^{i^*})\right).
$$

Proof. Define region i as $(\hat{\xi}^i, \hat{\xi}^{i+1}]$ for $i = 0,\ldots,N$ where $\hat{\xi}^0 := 0$ and $\hat{\xi}^{N+1} := \infty$. For $x \in (\hat{\xi}^i, \hat{\xi}^{i+1}]$, $|N_1(x)| = i$ and $|N_2(x)| = N - i$. Let $g(x) := \frac{1}{N}(\sum_{i\in N_1(x)} h(x - \hat{\xi}^i) + \sum_{i\in N_2(x)} b(\hat{\xi}^i - x))$. Suppose $h\,|N_1(x)| < b\,|N_2(x)|$, then $g(x)$ is nonincreasing as $x \to \hat{\xi}^{i+1}$. Suppose $h\,|N_1(x)| \geq b\,|N_2(x)|$, then $g(x)$ is nonincreasing as $x \to \hat{\xi}^i$.

There exist i^* such that $h(i^*-1) < b(N-i^*+1)$ and $h(i^*) \geq b(N-i^*)$. Then it is optimal to order $\hat{\xi}^{i^*}$. With this order quantity, the optimal cost is $b\theta + g(\hat{\xi}^{i^*})$. □

Remark 2.1 (Sample average approximation). *The optimal order quantity derived in Theorem 2.8 is the $\frac{b}{h+b}$ quantile of the empirical distribution ν, i.e., $x^* = \inf\{q : \nu([0,q]) \geq \frac{b}{b+h}\}$. Under Assumption 2.7, we derive that the distributionally robust solution with respect to the Wasserstein ambiguity set is equivalent to the optimal solution of the data-driven newsvendor model or sample average approximation (SAA) solution. This result coincides with Remark 6.7 in [30].*

The worst-case distribution in $\mathcal{M}$ is an optimal solution of the inner optimization of (2.1). It is important to analyze the closed-form solution of the worst-case distribution because the structure of the distribution affects the conservativeness of the DRO solution. Therefore, the existence conditions and structure of the worst-case distribution in the general distributionally robust optimization with a Wasserstein ambiguity set are studied. We refer to [34] and [30] for more details. We focus on the newsvendor case and propose the explicit characterization of the worst-case distribution based on historical data. The following worst-case distribution in $\mathcal{P}_1(\Xi)$ is the optimal solution of the inner optimization of (2.2).

Proposition 2.9 (Worst-case distribution for $p = 1$). *For each $x \geq 0$, let*

$$\mu^*(x) := \frac{1}{N} \sum_{i \in N_1(x)} \delta_{\hat{\xi}^i} + \frac{1}{N} \sum_{i \in N_2(x)} \delta_{(\hat{\xi}^i + \frac{N\theta}{|N_2(x)|})}.$$

Then, $\mu^*(x)$ is the worst-case distribution for a given x.

Proof. To check if $\mu^*(x)$ is the maximizer of the inner maximization of (2.2) for each x, we have to prove that $\mu^*(x)$ is a feasible distribution in the Wasserstein ambiguity set and satisfies strong duality. First, we show $\mu^*(x)$ satisfies strong duality.

$$\begin{aligned}
&\mathbb{E}_{\mu^*(x)}[h(x-\xi)^+ + b(\xi-x)^+] \\
&= b\theta + \sum_{i\in N_1(x)} h(x-\hat{\xi}^i) + \sum_{i\in N_2(x)} b(\hat{\xi}^i - x) \\
&= \max_{\mu\in\mathcal{P}(\Xi)} \left\{\mathbb{E}_\mu[h(x-\xi)^+ + b(\xi-x)^+] : W_1(\mu,\nu)\le\theta\right\}.
\end{aligned}$$

The first equality holds by the characterization of $\mu^*(x)$ and the second equality holds by strong duality. Hence, $\mu^*(x)$ satisfies strong duality.

To verify $\mu^*(x)$ is a feasible distribution, let $\zeta^i = \hat{\xi}^i$ for $i \in N_1(x)$ and $\zeta^i = \hat{\xi}^i + \frac{N\theta}{|N_2(x)|}$ for $i \in N_2(x)$. Then, by the definition of the Wasserstein distance,

$$\begin{aligned}
&W_1(\mu^*(x),\nu) \\
&= \min\{\frac{1}{N}\sum_{i=1}^N |\hat{\xi}^i - \hat{\zeta}^{\sigma(i)}| : \sigma\in\Pi_N\} \le \frac{1}{N}\sum_{i=1}^N |\hat{\xi}^i - \hat{\zeta}^i| \\
&= \frac{1}{N}\left\{\sum_{i\in N_1(x)} |\hat{\xi}^i - \hat{\xi}^i| + \sum_{i\in N_2(x)} \left|\hat{\xi}^i - \hat{\xi}^i - \frac{N\theta}{|N_2(x)|}\right|\right\} = \theta,
\end{aligned}$$

where Π_N is all permutations of $\{1,\ldots,N\}$. The inequality holds by letting $\sigma(i)=i$. Then, $\mu^*(x)$ is a feasible distribution and inside the Wasserstein ambiguity set. $\square$

We note that the worst-case distribution is constructed by historical data itself for data in $N_1(x)$ and the perturbation of the data in $N_2(x)$. The perturbation depends on historical data and the Wasserstein radius. We also note that the worst-case distribution is not unique. The worst-case distribution implies that the supremum of (2.2) is indeed a maximum.

2.3.2 Wasserstein Order $p > 1$

For the $p > 1$ case, the analysis is more difficult than the $p = 1$ case, but the Wasserstein distance of greater order is stronger by monotonicity (Proposition 2.2) and reflects geometric properties better [100]. When $p > 1$, the growth rate is 0, which leads to $\lambda \geq 0$.

First, we derive the equivalent expression of the inner supremum similar to the $p = 1$ case. We consider the partition of historical data to weaken the dependence of x and λ, which leads to the explicit characterization of the inner supremum. For given (x, λ), we define $f_i(\xi) := h(x - \xi)^+ + b(\xi - x)^+ - \lambda|\xi - \hat{\xi}^i|^p$ for $i = 1, \ldots, N$ to analyze the inner supremum of (2.3). By dividing the intervals according to the positions of (x, λ) and $\hat{\xi}^i$ and analyzing the cases, the maximum of $f_i(\xi)$ is attained at $\xi_r^i := \hat{\xi}^i + (\frac{b}{\lambda p})^{\frac{1}{p-1}}$ or $\xi_l^i := \hat{\xi}^i - (\frac{h}{\lambda p})^{\frac{1}{p-1}}$ based on $f_i(\xi_r^i)$ and $f_i(\xi_l^i)$. The comparison of two values $b(\hat{\xi}^i - x) + (\frac{b}{\lambda p})^{\frac{1}{p-1}} b(\frac{p-1}{p})$ and $h(x - \hat{\xi}^i) + (\frac{h}{\lambda p})^{\frac{1}{p-1}} h(\frac{p-1}{p})$ leads to the following definitions. For given (x, λ), we define $N_1(x, \lambda) := \{1 \leq i \leq N : x - \Delta(\frac{1}{\lambda})^{\frac{1}{p-1}} > \hat{\xi}^i\}$ and $N_2(x, \lambda) := \{1 \leq i \leq N : x - \Delta(\frac{1}{\lambda})^{\frac{1}{p-1}} \leq \hat{\xi}^i\}$, where $\Delta := \frac{1}{h+b}(\frac{1}{p})^{\frac{1}{p-1}}(\frac{p-1}{p})(b^{\frac{p}{p-1}} - h^{\frac{p}{p-1}})$ (Figure 2.4). Under Assumption 2.7, $\Delta \geq 0$ and $\Delta = 0$ when $b = h$. Then, for $i \in N_1(x, \lambda)$, $f_i(\xi_l^i) = h(x - \hat{\xi}^i) + (\frac{h}{\lambda p})^{\frac{1}{p-1}} h(\frac{p-1}{p})$, and for $i \in N_2(x, \lambda)$, $f_i(\xi_r^i) = b(\hat{\xi}^i - x) + (\frac{b}{\lambda p})^{\frac{1}{p-1}} b(\frac{p-1}{p})$.

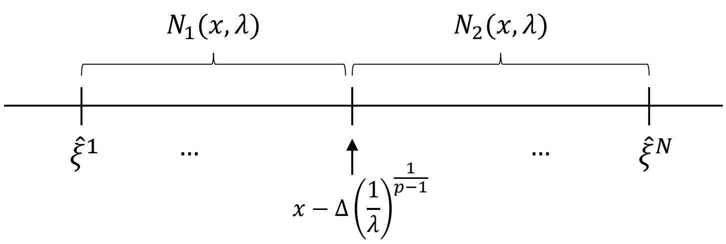

Figure 2.4 Definitions of $N_1(x, \lambda)$ and $N_2(x, \lambda)$

There are two interpretations of $N_1(x,\lambda)$ and $N_2(x,\lambda)$. First, by the definitions, $i \in N_1(x,\lambda)$ represents the samples whose values are less than $x - \Delta(\frac{1}{\lambda})^{\frac{1}{p-1}}$ and $i \in N_2(x,\lambda)$ represents the samples whose values are greater than or equal to $x - \Delta(\frac{1}{\lambda})^{\frac{1}{p-1}}$. Second, the inequality $x - \Delta(\frac{1}{\lambda})^{\frac{1}{p-1}} > \hat{\xi}^i$ is equivalent to $h(x-\hat{\xi}^i) + (\frac{h}{\lambda p})^{\frac{1}{p-1}} h(\frac{p-1}{p}) > b(\hat{\xi}^i - x) + (\frac{b}{\lambda p})^{\frac{1}{p-1}} b(\frac{p-1}{p})$. On the contrary, the inequality $x - \Delta(\frac{1}{\lambda})^{\frac{1}{p-1}} \le \hat{\xi}^i$ is equivalent to $h(x-\hat{\xi}^i) + (\frac{h}{\lambda p})^{\frac{1}{p-1}} h(\frac{p-1}{p}) \le b(\hat{\xi}^i - x) + (\frac{b}{\lambda p})^{\frac{1}{p-1}} b(\frac{p-1}{p})$. With the second interpretation of $N_1(x,\lambda)$ and $N_2(x,\lambda)$, we obtain the explicit expression of the inner supremum.

Proposition 2.10. *Under Assumption 2.7, if $\xi_l^i \ge 0$ for $i \in N_1(x,\lambda)$,*

$$\sup_{\xi\in\Xi} f_i(\xi) = f_i(\xi_l^i) = h(x-\hat{\xi}^i) + \left(\frac{h}{\lambda p}\right)^{\frac{1}{p-1}} h\left(\frac{p-1}{p}\right),$$

and for $i \in N_2(x,\lambda)$,

$$\sup_{\xi\in\Xi} f_i(\xi) = f_i(\xi_r^i) = b(\hat{\xi}^i - x) + \left(\frac{b}{\lambda p}\right)^{\frac{1}{p-1}} b\left(\frac{p-1}{p}\right).$$

In the proposition, ξ_l^i should be nonnegative for $i \in N_1(x,\lambda)$ to attain the maximum of $f_i(\xi)$ at $\xi = \xi_l^i$. The following assumption guarantees that $\xi_l^i \ge 0$ when a dual variable λ is chosen properly.

Assumption 2.11. *For all $i = 1,\ldots,N$, i-th data is greater than or equal to the Wasserstein radius, i.e., $\hat{\xi}^i \ge \theta$.*

Under Assumption 2.11 with properly chosen λ, we will prove that $\xi_l^i \ge 0$ for $i \in N_1(x,\lambda)$ in the proof of Theorem 2.13. Therefore, we can obtain the maximum of $f_i(\xi)$ at $\xi = \xi_l^i$ for $i \in N_1(x,\lambda)$. Even the above assumption is needed for the technical reason, the assumption holds in most of practical instances. The size of the Wasserstein radius to guarantee the probability that a Wasserstein ambiguity set contains the unknown true distribution is $\mathcal{O}(\frac{1}{\sqrt{N}})$ [31, 30]. Hence, except for the extremely small demand, Assumption 2.11 holds in most cases.

The next step is to express the objective function of dual formulation with the explicit form of the inner supremum. By the definitions of $N_1(x,\lambda)$ and $N_2(x,\lambda)$, the objective function of dual formulation (2.3) can be expressed as follows:

$$
\begin{aligned}
&\lambda\theta^p + \frac{1}{N}\left\{\sum_{i\in N_1}\left(h(x-\hat{\xi}^i) + \left(\frac{h}{\lambda p}\right)^{\frac{1}{p-1}} h\left(\frac{p-1}{p}\right)\right) + \sum_{i\in N_2}\left(b(\hat{\xi}^i - x) + \left(\frac{b}{\lambda p}\right)^{\frac{1}{p-1}} b\left(\frac{p-1}{p}\right)\right)\right\}\\
&= \lambda\theta^p + \frac{1}{N}\left\{\sum_{i\in N_1} h\left(x - \Delta\left(\lambda\right)^{-\frac{1}{p-1}} - \hat{\xi}^i\right) + \sum_{i\in N_2} b\left(\hat{\xi}^i - x + \Delta\left(\lambda\right)^{-\frac{1}{p-1}}\right)\right\}\\
&\quad + \frac{1}{N}\Bigg\{\sum_{i\in N_1} h\Delta\left(\frac{1}{\lambda}\right)^{\frac{1}{p-1}} - \sum_{i\in N_2} b\Delta\left(\frac{1}{\lambda}\right)^{\frac{1}{p-1}} + \sum_{i\in N_1}\left(\frac{h}{\lambda p}\right)^{\frac{1}{p-1}} h\left(\frac{p-1}{p}\right)\\
&\quad + \sum_{i\in N_2}\left(\frac{b}{\lambda p}\right)^{\frac{1}{p-1}} b\left(\frac{p-1}{p}\right)\Bigg\}\\
&= \lambda\theta^p + \mathrm{I} + \mathrm{II}\left(\frac{1}{\lambda}\right)^{\frac{1}{p-1}}
\end{aligned}
$$

where $\mathrm{I} := \frac{1}{N}\{\sum_{i\in N_1} h(x - \Delta(\frac{1}{\lambda})^{\frac{1}{p-1}} - \hat{\xi}^i) + \sum_{i\in N_2} b(\hat{\xi}^i - x + \Delta(\frac{1}{\lambda})^{\frac{1}{p-1}})\}$ and $\mathrm{II} := (\frac{|N_1|}{N}h\Delta - \frac{|N_2|}{N}b\Delta + \frac{|N_1|}{N}(\frac{h}{p})^{\frac{1}{p-1}}h(\frac{p-1}{p}) + \frac{|N_2|}{N}(\frac{b}{p})^{\frac{1}{p-1}}b(\frac{p-1}{p}))$. We suppress the dependence of x and λ on N_1 and N_2 for the notational brevity. The first equality holds by adjusting the term $\Delta(\frac{1}{\lambda})^{\frac{1}{p-1}}$ for x to make the similar structure of the objective function for the $p=1$ case. The objective function consists of three parts: $\lambda\theta^p$, the data-driven newsvendor cost based on $x - \Delta(\frac{1}{\lambda})^{\frac{1}{p-1}}$, and the remainder. Now, we derive the optimality condition based on the structure of the objective function.

Lemma 2.12. *Under Assumptions 2.7 and 2.11, the optimal solution (x^*,λ^*) satisfies $x^* - \Delta(\frac{1}{\lambda^*})^{\frac{1}{p-1}} = \hat{\xi}^i$ for some $i = 1,\ldots,N$.*

Proof. We prove the lemma by contradiction. When we suppose the lemma does not hold, there exist i and an optimal solution (x^*, λ^*) such that $\hat{\xi}^i < x^* - \Delta(\frac{1}{\lambda^*})^{\frac{1}{p-1}} < \hat{\xi}^{i+1}$. If we fix λ^*, then the first and third parts of the objective function, $\lambda^*\theta^p + \mathrm{II}\left(\frac{1}{\lambda^*}\right)^{\frac{1}{p-1}}$, are fixed. To see the change of the objective function value as x^* changes, we increase x^* as $x^* - \Delta(\frac{1}{\lambda^*})^{\frac{1}{p-1}}$ remains in the interval $(\hat{\xi}^i, \hat{\xi}^{i+1})$. Then the cost change of the second part, I, is $\frac{i}{N}h - \frac{N-i}{N}b$. By increasing or decreasing x, we can change the total cost downward, which contradicts the assumption that (x^*, λ^*) is optimal. $\square$

Therefore, we can express the optimal solution (x^*, λ^*) as $x^* - \Delta(\frac{1}{\lambda^*})^{\frac{1}{p-1}} = \hat{\xi}^i$ for some $i = 1, \dots, N$ and the corresponding cost $\mathrm{I} = \frac{1}{N}\{\sum_{k=1}^{i-1} h(\hat{\xi}^i - \hat{\xi}^k) + \sum_{k=i}^{N} b(\hat{\xi}^k - \hat{\xi}^i)\}$. By the above lemma, we can assume that $x - \Delta(\frac{1}{\lambda})^{\frac{1}{p-1}} = \hat{\xi}^{i+1}$ for some i. Then $|N_1(x, \lambda)| = i$ and $|N_2(x, \lambda)| = N - i$. To see the independence of x and λ on II, we arrange the equation.

$$\begin{aligned}
\mathrm{II} =& \frac{i}{N}h\Delta - \frac{N-i}{N}b\Delta + \frac{i}{N}\left(\frac{h}{p}\right)^{\frac{1}{p-1}} h\left(\frac{p-1}{p}\right) + \frac{N-i}{N}\left(\frac{b}{p}\right)^{\frac{1}{p-1}} b\left(\frac{p-1}{p}\right) \qquad (2.4)\\
=& \frac{i}{N}\frac{h}{h+b}\left(\frac{1}{p}\right)^{\frac{1}{p-1}}\left(\frac{p-1}{p}\right)\left(b^{\frac{p}{p-1}} - h^{\frac{p}{p-1}}\right)\\
&- \frac{N-i}{N}\frac{b}{h+b}\left(\frac{1}{p}\right)^{\frac{1}{p-1}}\left(\frac{p-1}{p}\right)\left(b^{\frac{p}{p-1}} - h^{\frac{p}{p-1}}\right)\\
&+ \frac{i}{N}\left(\frac{1}{p}\right)^{\frac{1}{p-1}} h^{\frac{p}{p-1}}\left(\frac{p-1}{p}\right) + \frac{N-i}{N}\left(\frac{1}{p}\right)^{\frac{1}{p-1}} b^{\frac{p}{p-1}}\left(\frac{p-1}{p}\right) \qquad (2.5)\\
=& \left(\frac{1}{p}\right)^{\frac{1}{p-1}}\left(\frac{p-1}{p}\right)\left(\frac{1}{h+b}\right)\left(b^{\frac{p}{p-1}}h + h^{\frac{p}{p-1}}b\right) \qquad (2.6)
\end{aligned}$$

The second equality holds by the substituting $\Delta := \frac{1}{h+b}(\frac{1}{p})^{\frac{1}{p-1}}(\frac{p-1}{p})(b^{\frac{p}{p-1}} - h^{\frac{p}{p-1}})$ back to original equation. Then, Equality (2.6) shows that II does not depend on x

and λ. Let $\Lambda := \frac{1}{h+b}(b^{\frac{p}{p-1}}h + h^{\frac{p}{p-1}}b) \geq 0$. Then $\mathrm{II} = (\frac{1}{p})^{\frac{1}{p-1}}(\frac{p-1}{p})\Lambda$ and the objective function of (2.3) can be expressed as follows:

$$
\begin{aligned}
\vartheta(i+1,\lambda) :=& \lambda\theta^p + \left(\frac{1}{p}\right)^{\frac{1}{p-1}}\left(\frac{p-1}{p}\right)\Lambda\left(\frac{1}{\lambda}\right)^{\frac{1}{p-1}} \\
&+ \frac{1}{N}\left(\sum_{k=1}^{i} h(\hat{\xi}^{i+1} - \hat{\xi}^k) + \sum_{k=i+1}^{N} b(\hat{\xi}^k - \hat{\xi}^{i+1})\right)
\end{aligned}
\tag{2.7}
$$

Using the fact $x - \Delta(\frac{1}{\lambda})^{\frac{1}{p-1}} = \hat{\xi}^{i+1}$, the objective function (2.7) can be expressed as the function of $i+1$ and λ. By the above analysis, $\vartheta(i+1,\lambda)$ is separable in $i+1$ and λ, which leads to an optimal solution.

Theorem 2.13. *Under Assumptions 2.7 and 2.11, an optimal order quantity is* $x^* = \hat{\xi}^{i^*} + \Delta p^{\frac{1}{p-1}}\theta(\frac{1}{\Lambda})^{\frac{1}{p}}$, *and an optimal dual variable is* $\lambda^* = \frac{1}{p\theta^{p-1}}(\Lambda)^{\frac{p-1}{p}}$ *where* $i^* \in \{1,\ldots,N\}$ *satisfies* $\frac{i^*-1}{N} < \frac{b}{h+b} \leq \frac{i^*}{N}$. *The optimal cost is*

$$
\theta\Lambda^{\frac{p-1}{p}} + \frac{1}{N}\left(\sum_{k=1}^{i^*-1} h(\hat{\xi}^{i^*} - \hat{\xi}^k) + \sum_{k=i^*}^{N} b(\hat{\xi}^k - \hat{\xi}^{i^*})\right).
$$

Proof. Under the assumptions, Lemma 2.12 holds and the objective function is expressed as (2.7). First, we derive the optimal dual variable λ^*. Since $\vartheta(i,\lambda)$ is a separable function of i and λ, the optimal dual variable λ^* can be derived by partial differentiation in λ.

$$
\frac{\partial\vartheta}{\partial\lambda} = \theta^p - \left(\frac{1}{p}\right)^{\frac{p}{p-1}}\Lambda\left(\frac{1}{\lambda}\right)^{\frac{p}{p-1}} = 0
$$

Then, $\left(\frac{1}{\lambda^*}\right)^{\frac{1}{p-1}} = p^{\frac{1}{p-1}}\theta\left(\frac{1}{\Lambda}\right)^{\frac{1}{p}}$ and $\lambda^* = \frac{1}{p\theta^{p-1}}\Lambda^{\frac{p-1}{p}}$. To obtain the optimal i^*, we express the objective function using λ^*.

$$\begin{aligned}\vartheta(i+1,\lambda^*) =& \frac{1}{p\theta^{p-1}}\Lambda^{\frac{p-1}{p}}\theta^p + \frac{(p-1)\theta}{p}\Lambda^{\frac{p-1}{p}} \\ &+ \frac{1}{N}\left(\sum_{k=1}^{i} h(\hat{\xi}^{i+1}-\hat{\xi}^k) + \sum_{k=i+1}^{N} b(\hat{\xi}^k-\hat{\xi}^{i+1})\right) \\ =& \theta\Lambda^{\frac{p-1}{p}} + \frac{1}{N}\left(\sum_{k=1}^{i} h(\hat{\xi}^{i+1}-\hat{\xi}^k) + \sum_{k=i+1}^{N} b(\hat{\xi}^k-\hat{\xi}^{i+1})\right)\end{aligned}$$

To minimize $\vartheta(i+1,\lambda^*)$ in terms of $i+1$, we refer to the analysis of Theorem 2.8. There exists i^* such that $\frac{i^*-1}{N} < \frac{b}{h+b} \leq \frac{i^*}{N}$ and i^* minimizes $\vartheta(i,\lambda^*)$, that is, $x^* - \Delta(\frac{1}{\lambda^*})^{\frac{1}{p-1}} = \hat{\xi}^{i^*}$. Then, $x^* = \hat{\xi}^{i^*} + \Delta(\frac{1}{\lambda^*})^{\frac{1}{p-1}} = \hat{\xi}^{i^*} + \Delta p^{\frac{1}{p-1}}\theta(\frac{1}{\Lambda})^{\frac{1}{p}}$. Therefore, the optimal solution of (2.3) is (x^*,λ^*) and the optimal cost is $\theta\Lambda^{\frac{p-1}{p}} + \frac{1}{N}\left(\sum_{k=1}^{i^*-1} h(\hat{\xi}^{i^*}-\hat{\xi}^k) + \sum_{k=i^*}^{N} b(\hat{\xi}^k-\hat{\xi}^{i^*})\right)$.

To see that the optimal dual variable λ^* under Assumption 2.11 satisfies $\xi_l^i = \hat{\xi}^i - (\frac{h}{\lambda^* p})^{\frac{1}{p-1}} \geq 0$ for all $i \in N_1$ and Proposition 2.10 holds, we analyze the following inequality.

$$\begin{aligned}&\hat{\xi}^i \geq \left(\frac{h}{\lambda^* p}\right)^{\frac{1}{p-1}} \\ \iff& \lambda^*(\hat{\xi}^i)^{p-1} \geq \frac{h}{p} \\ \iff& \left(\frac{\hat{\xi}^i}{\theta}\right)^{p-1}\left(\left(\frac{1}{h+b}\right)\left(b^{\frac{p}{p-1}}h + h^{\frac{p}{p-1}}b\right)\right)^{\frac{p-1}{p}} \geq h \\ \iff& \left(\frac{\hat{\xi}^i}{\theta}\right)^{p}\left(\frac{1}{h+b}\right)\left(b^{\frac{p}{p-1}}h + h^{\frac{p}{p-1}}b\right) \geq h^{\frac{p}{p-1}}\end{aligned}$$

If $\left(\frac{\hat{\xi}^i}{\theta}\right)^p \geq 1$, then λ^* satisfies $\xi_l^i \geq 0$, because $\left(b^{\frac{p}{p-1}}h + h^{\frac{p}{p-1}}b\right) \geq (h+b)h^{\frac{p}{p-1}}$. In short, if Assumption 2.11 holds, then the optimal dual variable λ^* satisfies $\xi_l^i \geq 0$

for $i \in N_1$. $\square$

Remark 2.2. *If $b = h$, then $\Delta = 0$ and $x^* = \hat{\xi}^{i^*}$. In this case, the optimal order quantity is equal to the SAA solution the same as the $p = 1$ case. The optimal cost is $\theta b^{\frac{p}{p-1}} + \frac{1}{N}\left(\sum_{k=1}^{i^*-1} h(\hat{\xi}^{i^*} - \hat{\xi}^k) + \sum_{k=i^*}^{N} b(\hat{\xi}^k - \hat{\xi}^{i^*})\right)$. The only difference is $\theta b^{\frac{p}{p-1}}$ compared to θb in the $p = 1$ case.*

For the $p > 1$ case, the optimal order quantity is the sum of the SAA solution and $\Delta p^{\frac{1}{p-1}} \theta (\frac{1}{\Lambda})^{\frac{1}{p}}$, where the second part is determined by parameters h, b, and p. If $b > h$, the optimal order quantity x^* is greater than the SAA solution. The first derivative of $\Delta p^{\frac{1}{p-1}} \theta (\frac{1}{\Lambda})^{\frac{1}{p}}$ with respect to $p \geq 2$ is negative, so the optimal order quantity is decreasing in $p \geq 2$ with a fixed set of parameters. The first derivative goes to 0 as p increases to infinity, and $\Delta p^{\frac{1}{p-1}} \theta (\frac{1}{\Lambda})^{\frac{1}{p}}$ goes to $\frac{b-h}{b+h}\theta$ as p increases to infinity. The optimal cost is decreasing in $p > 1$, because $\theta \Lambda^{\frac{p-1}{p}}$ is decreasing in $p > 1$. This result is explained by the monotone property of the Wasserstein distance. According to Proposition 2.2 , $W_{p_1} \leq W_{p_2}$ if $p_1 \leq p_2$, and the ambiguity set becomes smaller as the Wasserstein order p increases. Therefore, the newsvendor model with a higher order is less conservative and the optimal cost is smaller.

We now characterize the worst-case distribution based on $N+1$ points perturbed from historical data. The structure of the worst-case distribution is similar to the $p = 1$ case except for the split of mass for $\hat{\xi}^{i^*}$. The worst-case distribution has mass $\frac{p_0}{N}$ and $\frac{1-p_0}{N}$ at two points perturbed from $\hat{\xi}^{i^*}$, respectively, and mass $\frac{1}{N}$ at $N-1$ points perturbed from historical data (Figure 2.5).

Proposition 2.14 (Worst-case distribution for $p > 1$)**.** *For given optimal order quantity and dual variable (x^*, λ^*), let $\tilde{\xi}^i := \hat{\xi}^i - (\frac{h}{\lambda^* p})^{\frac{1}{p-1}} = \hat{\xi}^i - h^{\frac{1}{p-1}} \theta \left(\frac{1}{\Lambda}\right)^{\frac{1}{p}}$ for*

$i \in N_1(x^*, \lambda^*)$ and $\tilde{\xi}^i := \hat{\xi}^i + (\frac{b}{\lambda^* p})^{\frac{1}{p-1}} = \hat{\xi}^i + b^{\frac{1}{p-1}} \theta \left(\frac{1}{\Lambda}\right)^{\frac{1}{p}}$ for $i \in N_2(x^*, \lambda^*) \setminus \{i^*\}$. Let $\tilde{\xi}^{i_l^*} := \hat{\xi}^{i^*} - h^{\frac{1}{p-1}} \theta \left(\frac{1}{\Lambda}\right)^{\frac{1}{p}}$ and $\tilde{\xi}^{i_r^*} := \hat{\xi}^{i^*} + b^{\frac{1}{p-1}} \theta \left(\frac{1}{\Lambda}\right)^{\frac{1}{p}}$ be two points perturbed from $\hat{\xi}^{i^*}$. Then, $\mu^*(x^*) := \frac{1}{N} \sum \delta_{\tilde{\xi}^i} + \frac{p_0}{N} \delta_{\tilde{\xi}^{i_l^*}} + \frac{1-p_0}{N} \delta_{\tilde{\xi}^{i_r^*}}$ is the worst-case distribution where $p_0 \in [0, 1]$ satisfies $\frac{|N_1(x^*, \lambda^*)| + p_0}{N} = \frac{b}{h+b}$.

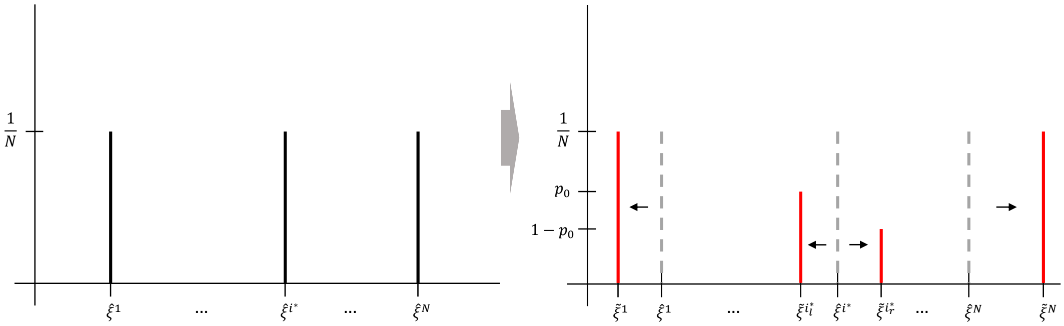

Figure 2.5 Worst-case distribution perturbed from the empirical distribution

Proof. We prove that $\mu^*(x^*)$ is a feasible distribution and satisfies the strong duality similar to the $p = 1$ case. For notational brevity, we suppress dependence on x and λ on N_1 and N_2. First, we will show that points perturbed from data points satisfy the desirable ordering based on x^*, even separation of N_1 and N_2 is based on $x^* - \Delta(\frac{1}{\lambda^*})^{\frac{1}{p-1}}$, that is, $\tilde{\xi}^i < x^*$ for $i \in N_1$ and i_l^*, and $\tilde{\xi}^i \geq x^*$ for $i \in N_2 \setminus \{i^*\}$ and i_r^*. This ordering seems unintuitive when we consider definitions of N_1 and N_2, but the ordering is important for the evaluation of the objective function value. By the definitions of N_1, $\tilde{\xi}^i$ and $\tilde{\xi}^{i_l^*}$, $\tilde{\xi}^i < \hat{\xi}^i < x^*$ for $i \in N_1$ and $\tilde{\xi}^{i_l^*} < \hat{\xi}^{i^*} < x^*$. Since $b^{\frac{1}{p-1}} \geq \Delta p^{\frac{1}{p-1}} = \frac{1}{h+b} \frac{p-1}{p} (b^{\frac{p}{p-1}} - h^{\frac{p}{p-1}})$, $\tilde{\xi}^{i_r^*} = \hat{\xi}^{i^*} + b^{\frac{1}{p-1}} \theta (\frac{1}{\Lambda})^{\frac{1}{p}} \geq \hat{\xi}^{i^*} + \Delta p^{\frac{p}{p-1}} \theta (\frac{1}{\Lambda})^{\frac{1}{p}} = x^*$. Then, $\tilde{\xi}^{i_r^*} \leq \tilde{\xi}^i$ for $i \in N_2 \setminus \{i^*\}$, that is, $\tilde{\xi}^i \geq x^*$ for $i \in N_2 \setminus \{i^*\}$ and i_r^*.

Second, the objective function value with $\mu^*(x^*)$ satisfies the strong duality. The objective function with x^* and $\mu^*(x^*)$ is expressed as follows:

$$
\begin{aligned}
&\mathbb{E}_{\mu^*(x^*)}[h(x^*-\xi)^+ + b(\xi - x^*)^+]\\
=&\frac{1}{N}\sum_{i\in N_1} h(x^*-\tilde{\xi}^i) + \frac{1}{N}\sum_{i\in N_2\setminus\{i^*\}} b(\tilde{\xi}^i - x^*) + \frac{p_0}{N}h(x-\tilde{\xi}^{i^*_l}) + \frac{1-p_0}{N}b(\tilde{\xi}^{i^*_r}-x^*)\\
=&\frac{1}{N}\sum_{i\in N_1} h(\hat{\xi}^{i^*} + \Delta p^{\frac{1}{p-1}}\theta(\frac{1}{\Lambda})^{\frac{1}{p}} - \hat{\xi}^i + h^{\frac{1}{p-1}}\theta(\frac{1}{\Lambda})^{\frac{1}{p}})\\
&+\frac{1}{N}\sum_{i\in N_2\setminus\{i^*\}} b(\hat{\xi}^i + b^{\frac{1}{p-1}}\theta(\frac{1}{\Lambda})^{\frac{1}{p}} - \hat{\xi}^{i^*} - \Delta p^{\frac{1}{p-1}}\theta(\frac{1}{\Lambda})^{\frac{1}{p}})\\
&+\frac{p_0}{N}h(\hat{\xi}^{i^*} + \Delta p^{\frac{1}{p-1}}\theta(\frac{1}{\Lambda})^{\frac{1}{p}} - \hat{\xi}^{i^*} + h^{\frac{1}{p-1}}\theta(\frac{1}{\Lambda})^{\frac{1}{p}})\\
&+\frac{1-p_0}{N}b(\hat{\xi}^{i^*} + b^{\frac{1}{p-1}}\theta(\frac{1}{\Lambda})^{\frac{1}{p}} - \hat{\xi}^{i^*} - \Delta p^{\frac{1}{p-1}}\theta(\frac{1}{\Lambda})^{\frac{1}{p}})\\
=&\frac{1}{N}\sum_{i\in N_1} h(\hat{\xi}^{i^*} - \hat{\xi}^i) + \frac{1}{N}\sum_{i\in N_2} b(\hat{\xi}^i - \hat{\xi}^{i^*})\\
&+\theta\Lambda^{-\frac{1}{p}}\Bigg\{\frac{|N_1|+p_0}{N}h\Delta p^{\frac{1}{p-1}} - \frac{|N_2|-p_0}{N}b\Delta p^{\frac{1}{p-1}}\\
&+\frac{|N_1|+p_0}{N}h^{\frac{p}{p-1}} + \frac{|N_2|-p_0}{N}b^{\frac{p}{p-1}}\Bigg\}\\
=&\frac{1}{N}\sum_{i\in N_1} h(\hat{\xi}^{i^*} - \hat{\xi}^i) + \frac{1}{N}\sum_{i\in N_2} b(\hat{\xi}^i - \hat{\xi}^{i^*}) + \theta\Lambda^{-\frac{1}{p}}\left(\frac{b}{h+b}h^{\frac{p}{p-1}} + \frac{h}{h+b}b^{\frac{p}{p-1}}\right)\\
=&\frac{1}{N}\sum_{i\in N_1} h(\hat{\xi}^{i^*} - \hat{\xi}^i) + \frac{1}{N}\sum_{i\in N_2} b(\hat{\xi}^i - \hat{\xi}^{i^*}) + \theta\Lambda^{\frac{p}{p-1}}
\end{aligned}
$$

The equalities hold by definitions of $\mu^*(x^*)$, $\tilde{\xi}$, and p_0. Therefore, $\mu^*(x^*)$ satisfies the strong duality.

Third, $\mu^*(x^*)$ is a feasible distribution, that is, the Wasserstein distance between $\mu^*(x^*)$ and the empirical distribution ν is less than or equal to the Wasserstein radius θ. By the definition of the Wasserstein distance, $W_p(\mu^*(x^*), \nu)$ can be expressed as follows:

$$
\begin{aligned}
W_p(\mu^*(x^*),\nu)^p = \min_{\gamma_{ij}\geq 0} & \sum_{i=1}^{N+1}\sum_{j=1}^{N}|\tilde{\xi}^i - \hat{\xi}^j|^p \\
\text{s.t.} & \sum_{j=1}^{N}\gamma_{ij} = \begin{cases} \frac{1}{N}, & \text{if } i \in N_1 \cup N_2 \setminus \{i_l^*, i_r^*\} \\ \frac{p_0}{N}, & \text{if } i = i_l^* \\ \frac{1-p_0}{N}, & \text{if } i = i_r^* \end{cases} \\
& \sum_{i=1}^{N+1}\gamma_{ij} = \frac{1}{N} \quad \text{for } j = 1, \ldots, N
\end{aligned}
$$

Then, the following inequality holds.

$$
\begin{aligned}
& W_p(\mu^*(x^*),\nu)^p \\
\leq & \frac{1}{N}\sum_{i\in N_1}|\tilde{\xi}^i - \hat{\xi}^i|^p + \frac{1}{N}\sum_{i\in N_2\setminus\{i^*\}}|\tilde{\xi}^i - \hat{\xi}^i|^p + \frac{p_0}{N}|\tilde{\xi}^{i_l^*} - \hat{\xi}^{i^*}|^p + \frac{1-p_0}{N}|\tilde{\xi}^{i_r^*} - \hat{\xi}^{i^*}|^p \\
= & \frac{1}{N}\sum_{i\in N_1}(h^{\frac{1}{p-1}}\theta(\frac{1}{\Lambda})^p) + \frac{1}{N}\sum_{i\in N_2\setminus\{i^*\}}(b^{\frac{1}{p-1}}\theta(\frac{1}{\Lambda})^p) \\
& + \frac{p_0}{N}(h^{\frac{1}{p-1}}\theta(\frac{1}{\Lambda})^p) + \frac{1-p_0}{N}(b^{\frac{1}{p-1}}\theta(\frac{1}{\Lambda})^p) \\
= & \theta^p\frac{1}{\Lambda}(\frac{|N_1|+p_0}{N}h^{\frac{p}{p-1}} + \frac{|N_2|-p_0}{N}b^{\frac{p}{p-1}}) \\
= & \theta^p\frac{1}{\Lambda}\Lambda = \theta^p
\end{aligned}
$$

The first inequality holds, because $\gamma_{ij} = \frac{1}{N}$ for $i = j, i \neq i_l^*, i_r^*$, $\gamma_{i_l^*,i^*} = \frac{p_0}{N}$, and $\gamma_{i_r^*,i^*} = \frac{1-p_0}{N}$ is a feasible solution of γ_{ij}. The second equality holds by the definitions of p_0 and Λ. Therefore, the Wasserstein distance between $\mu^*(x^*)$ and ν is less than or equal to θ and $\mu^*(x^*)$ is a feasible distribution. □

We derived the closed-form expressions of an optimal order quantity and the worst-case distribution for the general Wasserstein order $p \in [1, \infty)$. The closed-form solutions can be applied to various applications such as multistage inventory control, pricing, and retail management. In the next subsection, we consider the risk measure as an objective for the risk-averse case, rather than an expected cost as the risk-neutral case.

2.4 Risk-averse Newsvendor Model

Although the long-run average performance of the risk-neutral solution outperforms that of the risk-averse solution, risk-averse decisions of the first few periods are important in terms of protection against bankruptcy. Moreover, many decision makers are risk-averse in reality. Therefore, we consider a risk-averse model with the Conditional Value-at-Risk (CVaR) objective. CVaR has several strong points. CVaR is a coherent risk measure and preserves convexity of the newsvendor cost function. Optimization with CVaR is much easier by using the following definition.

Definition 2.15 (CVaR, [72]).

$$\mathrm{CVaR}_\mu^\beta(X) := \inf_{\alpha \in \mathbb{R}} \left\{ \alpha + \frac{1}{1-\beta} \mathbb{E}_\mu[(X-\alpha)^+] \right\}$$

Let $F_\beta(x, \alpha, \mu) := \alpha + \frac{1}{1-\beta} \int (h(x-\xi)^+ + b(\xi-x)^+ - \alpha)^+ \mu(d\xi)$. Then, minimization of the worst-case CVaR of the newsvendor cost function is represented as follows:

$$\begin{aligned} &\min_{x\geq 0} \sup_{\mu \in \mathcal{M}} \mathrm{CVaR}_\mu^\beta(h(x-\xi)^+ + b(\xi-x)^+) \\ =&\min_{x\geq 0} \sup_{\mu \in \mathcal{M}} \inf_{\alpha \in \mathbb{R}} F_\beta(x, \alpha, \mu) \end{aligned} \tag{2.8}$$

Before we utilize Theorem 2.5 directly, we need to change the order of supremum and infimum with a general version of John von Neumann's minimax theorem.

Lemma 2.16 (Sion's minimax theorem). *Let X be a compact, convex subset of a topological vector space and Y be a convex subset of a topological vector space. Let f be a real-valued function on $X \times Y$ such that, $f(\cdot, y)$ is lower-semicontinuous and quasi-convex on X for each $y \in Y$ and $f(x, \cdot)$ is upper-semicontinuous and quasi-*

concave on Y for each $x \in X$. Then,

$$\inf_{x\in X}\sup_{y\in Y} f(x,y) = \sup_{y\in Y}\inf_{x\in X} f(x,y).$$

First, we check the risk-averse newsvendor model satisfies the conditions of Lemma 2.16 to change the order of operations. $F_\beta(x,\alpha,\mu)$ is convex in α [72], and affine in μ. The Wasserstein ambiguity set $\mathcal{M}$ is convex by Proposition 2.4. For fixed x, minimizer of $\inf_{\alpha\in\mathbb{R}} F_\beta(x,\alpha,\mu)$ is achieved in $[0,hx]$ [20, 36]. It can be easily proved that $F_\beta(x,\alpha,\mu)$ is continuous with respect to α and μ. Then,

$$\begin{aligned}
&\sup_{\mu\in\mathcal{M}}\inf_{\alpha\in\mathbb{R}} F_\beta(x,\alpha,\mu) = \sup_{\mu\in\mathcal{M}}\inf_{\alpha\in[0,hx]} F_\beta(x,\alpha,\mu)\\
&= \inf_{\alpha\in[0,hx]}\sup_{\mu\in\mathcal{M}} F_\beta(x,\alpha,\mu) \geq \inf_{\alpha\in\mathbb{R}}\sup_{\mu\in\mathcal{M}} F_\beta(x,\alpha,\mu)\\
&\geq \sup_{\mu\in\mathcal{M}}\inf_{\alpha\in\mathbb{R}} F_\beta(x,\alpha,\mu)
\end{aligned}$$

The second equality holds by Lemma 2.16 and the last inequality holds by minimax inequality.

By the above result and Theorem 2.5, we obtain

$$\begin{aligned}
&\min_{x\geq 0}\sup_{\mu\in\mathcal{M}}\inf_{\alpha\in\mathbb{R}} F_\beta(x,\alpha,\mu) = \min_{x\geq 0}\inf_{\alpha\in\mathbb{R}}\sup_{\mu\in\mathcal{M}} F_\beta(x,\alpha,\mu)\\
&= \min_{x,\lambda\geq 0,\alpha\in\mathbb{R}}\Bigg\{\alpha + \frac{1}{1-\beta}\lambda\theta^p\\
&\qquad + \frac{1}{1-\beta}\frac{1}{N}\sum_{i=1}^{N}\sup_{\xi\in\Xi}\left[\left(h(x-\xi)^+ + b(\xi-x)^+ - \alpha\right)^+ - \lambda|\xi-\hat{\xi}^i|^p\right]\Bigg\}. \qquad (2.9)
\end{aligned}$$

We consider two cases similar to the risk-neutral model: $p=1$ and $p>1$. We utilize a similar analysis of (2.3), which leads to a closed-form solution for the $p = 1$ case and a tractable formulation for the $p > 1$ case.

2.4.1 Wasserstein Order $p = 1$

First, we characterize the explicit form of the inner supremum of (2.9) for $p = 1$. For given (x, λ, α), let $\eta_i(\xi) := (h(x-\xi)^+ + b(\xi - x)^+ - \alpha)^+ - \lambda|\xi - \hat{\xi}^i|$. For the supremum of η_i, we consider the interval of ξ which makes the first part of η_i positive. Otherwise, $\eta_i(\xi)$ would be less than or equal to 0 and the supremum of η_i is 0 when $\xi = \hat{\xi}^i$. We define $N_1(x, \alpha) := \{1 \le i \le N : \hat{\xi}^i < x - \frac{\alpha}{h}\}$ and $N_2(x, \alpha) := \{1 \le i \le N : \hat{\xi}^i \ge x + \frac{\alpha}{b}\}$ (Figure 2.6). Then, $N_1(x, \alpha) \subset N_1(x)$ and $N_2(x, \alpha) \subset N_2(x)$ by the definitions of $N_1(x)$ and $N_2(x)$ in Subsection 2.3.1.

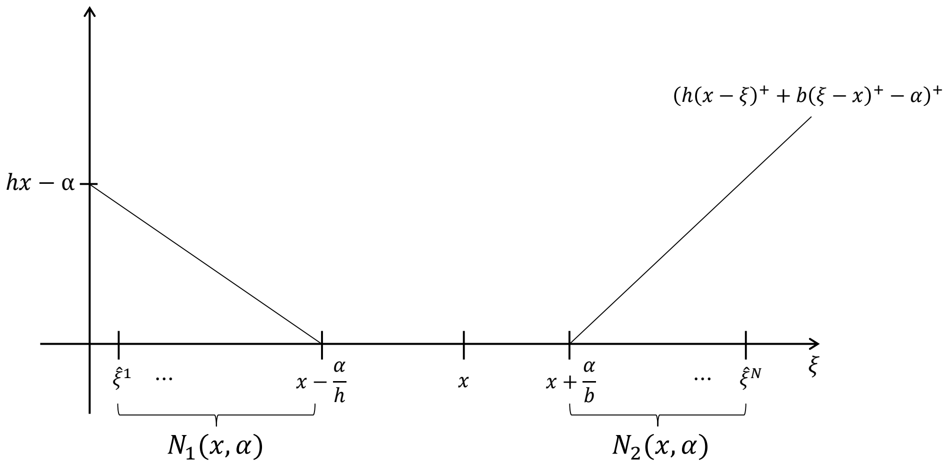

Figure 2.6 Definitions of $N_1(x, \alpha)$ and $N_2(x, \alpha)$

We also assume that Assumption 2.7 holds, i.e., $b \ge h$. By the analysis similar to that of $f_i(\xi)$ in Subsection 2.3.1, for $i \in N_1(x, \alpha)$ the supremum is attained at $\xi = \hat{\xi}^i$ with $\eta_i(\hat{\xi}^i) = h(x - \hat{\xi}^i) - \alpha$, and for $i \in N_2(x, \alpha)$ the supremum is attained at $\xi = \hat{\xi}^i$ with $\eta_i(\hat{\xi}^i) = b(\hat{\xi} - x) - \alpha$. For $i \in N_1(x) \setminus N_1(x, \alpha)$ and $i \in N_2(x) \setminus N_2(x, \alpha)$, the supremum of η_i is 0. Therefore, the linkage between x and λ is disconnected by the analysis of supremum of η_i, which leads to separation of x and λ in the objective function in (2.9). Using the explicit form of the inner supremum, the CVaR objective (2.9) can be expressed as follows:

$$\alpha + \frac{1}{1-\beta}\lambda\theta + \frac{1}{1-\beta}\frac{1}{N}\left(\sum_{i\in N_1(x,\alpha)}\left(h(x-\hat{\xi}^i)-\alpha\right) + \sum_{i\in N_2(x,\alpha)}\left(b(\hat{\xi}^i - x)-\alpha\right)\right) \tag{2.10}$$

Then, $\lambda^* = b$ by the growth rate constraint, and the remaining part can be expressed as follows, which is the same as the data-driven CVaR model where the true distribution is given as the empirical distribution ν, $\mathrm{CVaR}_\nu^\beta(h(x-\xi)^+ + b(\xi-x)^+)$:

$$\min_{x\geq 0,\alpha\in\mathbb{R}} \alpha + \frac{1}{1-\beta}\frac{1}{N}\left(\sum_{i\in N_1(x,\alpha)}\left(h(x-\hat{\xi}^i)-\alpha\right) + \sum_{i\in N_2(x,\alpha)}\left(b(\hat{\xi}^i - x)-\alpha\right)\right) \tag{2.11}$$

The closed-form solutions of the CVaR newsvendor model is characterized by [36]. When distribution function F is given, the inverse distribution function of F is defined as $F^{-1}(q) := \inf\{x\in\mathbb{R} : F(x)\geq q\}$. Then, the closed-form solutions are given as follows:

$$\begin{aligned} x^* &= \frac{h}{h+b}F^{-1}\left(\frac{b(1-\beta)}{h+b}\right) + \frac{b}{h+b}F^{-1}\left(\frac{b+h\beta}{h+b}\right) \\ \alpha^* &= \frac{hb}{h+b}\left(F^{-1}\left(\frac{b+h\beta}{h+b}\right) - F^{-1}\left(\frac{b(1-\beta)}{h+b}\right)\right). \end{aligned}$$

If the true distribution F is given as the empirical distribution ν, the inverse function is written as

$$
\begin{aligned}
F^{-1}\left(\frac{b(1-\beta)}{h+b}\right) &= \inf\left\{x : \nu\left([0,x]\right) \geq \frac{b(1-\beta)}{h+b}\right\} \\
F^{-1}\left(\frac{b+h\beta}{h+b}\right) &= \inf\left\{x : \nu\left([0,x]\right) \geq \frac{b+h\beta}{h+b}\right\}.
\end{aligned}
$$

Therefore, by the definition of the empirical distribution ν, there exist i_1 and i_2 such that

$$
\begin{aligned}
\frac{i_1-1}{N} &< \frac{b(1-\beta)}{h+b} \leq \frac{i_1}{N} \\
\frac{i_2-1}{N} &< \frac{b+h\beta}{h+b} \leq \frac{i_2}{N}.
\end{aligned}
$$

Then the closed-form solutions of (2.9) for $p=1$ can be expressed using i_1 and i_2.

Theorem 2.17. *Under Assumption 2.7, the optimal solutions of the worst-case CVaR newsvendor model are*

$$
\begin{aligned}
x^* &= \frac{h}{h+b}\hat{\xi}^{i_1} + \frac{b}{h+b}\hat{\xi}^{i_2} \\
\alpha^* &= \frac{hb}{h+b}\left(\hat{\xi}^{i_2} - \hat{\xi}^{i_1}\right) \\
\lambda^* &= b.
\end{aligned}
$$

The optimal objective function value is

$$
\frac{hb}{h+b}(\hat{\xi}^{i_2} - \hat{\xi}^{i_1}) + \frac{1}{1-\beta}b\theta + \frac{1}{1-\beta}\frac{1}{N}\left\{\sum_{k=1}^{i_1-1} h(\hat{\xi}^{i_1} - \hat{\xi}^{k}) + \sum_{k=i_2}^{N} b(\hat{\xi}^{k} - \hat{\xi}^{i_2})\right\}.
$$

By the analysis of the inner supremum and separation of x and λ, the optimal order quantity is the same as the data-driven CVaR solution. Theorem 2.17 shows that the equivalence between the SAA solution and the distributionally robust solution with respect to the Wasserstein ambiguity set is extended to the risk-averse case for $p = 1$.

Using the closed-form solutions in Theorem 2.17, $x^* - \frac{\alpha^*}{h} = \hat{\xi}^{i_1}$ and $x^* + \frac{\alpha^*}{b} = \hat{\xi}^{i_2}$, which leads to $N_1(x^*, \alpha^*) = \{1, \ldots, i_1 - 1\}$ and $N_2(x^*, \alpha^*) = \{i_2, \ldots, N\}$ if N samples are sorted in nondecreasing order. The above expression is useful for deriving the worst case distribution.

Proposition 2.18 (Worst-case distribution)**.** *For given $(x^*, \lambda^*, \alpha^*)$, let*

$$\mu^*(x^*, \alpha^*) := \frac{1}{N} \sum_{i \in N \setminus N_2(x^*, \alpha^*)} \delta_{\hat{\xi}^i} + \frac{1}{N} \sum_{i \in N_2(x^*, \alpha^*)} \delta_{(\hat{\xi}^i + \frac{N\theta}{|N_2|})}.$$

Then, $\mu^*(x^*, \alpha^*)$ is the worst-case distribution.

Proof. $\mu^*(x^*, \alpha^*)$ satisfies the strong duality and is a feasible distribution. We omit the detailed proof, because it is similar to the proof of Proposition 2.9. □

2.4.2 Wasserstein Order $p > 1$

To characterize the explicit form of the inner supremum of (2.9) for $p > 1$, we define $\eta_i(\xi) := (h(x - \xi)^+ + b(\xi - x)^+ - \alpha)^+ - \lambda|\xi - \hat{\xi}^i|^p$ for given (x, λ, α) and $i = 1, \ldots, N$. Then, $\sup_{\xi \in \Xi} \eta_i(\xi) \geq 0$ by the definition of η_i. If $h(x - \xi)^+ + b(\xi - x)^+ - \alpha \geq 0$, i.e., $\xi < x - \frac{\alpha}{h}$ or $\xi \geq x + \frac{\alpha}{b}$, then $\eta_i(\xi) = h(x - \xi)^+ + b(\xi - x)^+ - \alpha + \lambda|\xi - \hat{\xi}^i|^p$. Using the analysis of the inner supremum in Subsection 2.3.2, the supremum of η_i may be attained at $\xi_r^i := \hat{\xi}^i + (\frac{b}{\lambda p})^{\frac{1}{p-1}}$ or $\xi_l^i := \hat{\xi}^i - (\frac{h}{\lambda p})^{\frac{1}{p-1}}$. Specifically, for $i \in N_1(x, \lambda)$, $\eta_i(\xi_l^i) = h(x - \hat{\xi}^i) - \alpha + (\frac{h}{\lambda p})^{\frac{1}{p-1}} h(\frac{p-1}{p})$ should be greater than or equal to 0, that is,

$\hat{\xi}^i < x - \frac{\alpha}{h} + (\frac{h}{\lambda p})^{\frac{1}{p-1}} \frac{p-1}{p}$. For $i \in N_2(x,\lambda)$, $\eta_i(\xi_r^i) = b(\hat{\xi}^i - x) - \alpha + (\frac{b}{\lambda p})^{\frac{1}{p-1}} b(\frac{p-1}{p})$ should be greater than or equal to 0, that is, $\hat{\xi}^i \geq x + \frac{\alpha}{b} - (\frac{b}{\lambda p})^{\frac{1}{p-1}} \frac{p-1}{p}$.

Using the above result, we need definitions of index sets considering interactions of both (x,λ) and (x,α). We define $N_1(x,\lambda,\alpha) := \{1 \leq i \leq N : \hat{\xi}^i < x - \Delta(\frac{1}{\lambda})^{\frac{1}{p-1}},\ \hat{\xi}^i < x - \frac{\alpha}{h} + (\frac{h}{\lambda p})^{\frac{1}{p-1}} \frac{p-1}{p}\}$ and $N_2(x,\lambda,\alpha) := \{1 \leq i \leq N : \hat{\xi}^i \geq x - \Delta(\frac{1}{\lambda})^{\frac{1}{p-1}},\ \hat{\xi}^i \geq x + \frac{\alpha}{b} - (\frac{b}{\lambda p})^{\frac{1}{p-1}} \frac{p-1}{p}\}$. For $i \in N_1(x,\lambda,\alpha)$, the supremum of η_i is attained at ξ_l^i with $\eta_i(\xi_l^i) = h(x - \hat{\xi}^i) - \alpha + (\frac{h}{\lambda p})^{\frac{1}{p-1}} h(\frac{p-1}{p})$. For $i \in N_2(x,\lambda,\alpha)$, the supremum of η_i is attained at ξ_r^i with $\eta_i(\xi_r^i) = b(\hat{\xi}^i - x) - \alpha + (\frac{b}{\lambda p})^{\frac{1}{p-1}} b(\frac{p-1}{p})$. If $i \notin N_1(x,\lambda,\alpha)$ and $i \notin N_2(x,\lambda,\alpha)$, then supremum of η_i is 0.

Using the explicit form of the inner supremum, we derive the equivalent expression of (2.9) for $p > 1$. We suppress the dependence of (x,λ,α) on N_1 and N_2 for notational convenience.

$$\min_{x,\lambda\geq 0,\alpha\in\mathbb{R}} \alpha + \frac{1}{1-\beta}\lambda\theta^p + \frac{1}{1-\beta}\frac{1}{N}\left(\sum_{i\in N_1}\left(h(x-\hat{\xi}^i) - \alpha + \left(\frac{h}{\lambda p}\right)^{\frac{1}{p-1}} h\frac{p-1}{p}\right)\right.$$
$$\left. + \sum_{i\in N_2}\left(b(\hat{\xi}^i - x) - \alpha + \left(\frac{b}{\lambda p}\right)^{\frac{1}{p-1}} b\frac{p-1}{p}\right)\right). \tag{2.12}$$

The objective function of (2.12) without $\frac{1}{1-\beta}\lambda\theta^p$ can be expressed as the CVaR objective. Let $\bar{\xi}^i := \hat{\xi}^i - (\frac{h}{\lambda p})^{\frac{1}{p-1}} \frac{p-1}{p}$ for $i \in N_1$ and $\bar{\xi}^i := \hat{\xi}^i + (\frac{b}{\lambda p})^{\frac{1}{p-1}} \frac{p-1}{p}$ for $i \in N_2$. Then, we define

$$\mu(x,\lambda) := \frac{1}{N}\sum_{i\in N_1}\delta_{\bar{\xi}^i} + \frac{1}{N}\sum_{i\in N_2}\delta_{\bar{\xi}^i} + \frac{1}{N}\sum_{i\in N\setminus(N_1\cup N_2)}\delta_{\hat{\xi}^i}.$$

Using the definition of $\mu(x,\lambda)$, (2.12) can be expressed as follows:

$$\min_{x,\lambda\geq 0,\alpha\in\mathbb{R}} \frac{1}{1-\beta}\lambda\theta^p + \mathrm{CVaR}^{\beta}_{\mu(x,\lambda)}\left(h(x-\xi)^+ + b(\xi-x)^+\right) \tag{2.13}$$

However, there exists dependence between x and λ, which makes the further analysis complex. In this case, it is difficult to derive closed-form solutions, but we can obtain a tractable formulation using the analysis of the inner supremum.

Theorem 2.19. *Under Assumption 2.7, the optimal order quantity of (2.12) is determined by the following formulation:*

$$\begin{aligned}
\min_{x,\lambda\geq 0,y,\alpha\in\mathbb{R}} \quad & \alpha + \frac{1}{1-\beta}\lambda\theta^p + \frac{1}{1-\beta}\frac{1}{N}\sum_{i=1}^{N} y_i \\
\text{s.t.} \quad & y_i \geq h(x-\hat{\xi}^i) - \alpha + \left(\frac{h}{\lambda p}\right)^{\frac{1}{p-1}} h\frac{p-1}{p}, && \forall i = 1,\dots,N, \\
& y_i \geq b(\hat{\xi}^i - x) - \alpha + \left(\frac{b}{\lambda p}\right)^{\frac{1}{p-1}} b\frac{p-1}{p}, && \forall i = 1,\dots,N, \\
& y_i \geq 0, && \forall i = 1,\dots,N, \\
& \alpha \leq hx.
\end{aligned}$$

For the risk-averse decision, we consider the CVaR objective for the newsvendor model. We derive the closed-form solutions for the $p = 1$ case, and propose a tractable formulation to obtain the optimal order quantity for the $p > 1$ case. In the next section, numerical experiments are conducted to verify the risk-aversion of the CVaR solution using Theorem 2.17 and 2.19.

2.5 Computational Experiments

We conduct numerical experiments to compare the Wasserstein model with other data-driven models such as the ϕ-divergence model [8]. We also compare with the Scarf's moment-based model [76] in terms of convergence of order quantities and total costs to those with the true underlying distribution. Then, we compare the risk-averse solution with the risk-neutral solution in terms of the worst cost with pessimistic demand realization.

2.5.1 Out-of-sample Performance

One important aspect when analyzing a distributionally robust solution is out-of-sample performance, i.e., the average performance of the distributionally robust solution over test samples from the data-generating distribution. We compare the out-of-sample performance of distributionally robust solutions for Wasserstein models and ϕ-divergence models. The ϕ-divergence models are another data-driven distributionally robust approach based on the ϕ-divergence, such as KL-divergence and χ^2-distance [8].

Let $\mu = (\mu_1, \dots, \mu_N)$ and $\nu = (\nu_1, \dots, \nu_N)$ be probability distributions defined on N points. The ϕ-divergence model is given by

$$\min_{x \geq 0} \sup_{\mu \in \mathcal{M}_\phi(\nu)} \mathbb{E}_\mu [h(x-\xi)^+ + b(\xi - x)^+], \tag{2.14}$$

where $\mathcal{M}_\phi(\nu) := \{\mu \in \mathcal{P}(\Xi) : I_\phi(\mu, \nu) \leq \rho\}$ and $I_\phi(\mu,\nu) = \sum_{i=1}^N \nu_i \phi(\frac{\mu_i}{\nu_i})$, i.e., the ϕ-divergence ambiguity set based on the empirical distribution ν. ϕ-divergence is only defined between the empirical distribution and distributions which are absolutely continuous to the empirical distribution. Hence, the ϕ-divergence ambiguity set consists of distributions whose supports are the same as the empirical distribution. From the strong duality of the ϕ-divergence model [8], the newsvendor model with the ϕ-divergence ambiguity set is expressed as follows:

$$\min_{x\geq 0,\lambda\geq 0,\eta} \eta + \rho\lambda + \lambda \sum_{i=1}^{N} \frac{1}{N}\phi^* \left(\frac{h(x-\hat{\xi}^i)^+ + b(\hat{\xi}^i - x)^+ - \eta}{\lambda} \right) \tag{2.15}$$

where ϕ^* is the conjugate of ϕ. In this experiment, we use KL-divergence and χ^2-distance for ϕ-divergence. The ϕ-divergence function of KL-divergence is $\phi(t) = t\log t - t + 1$ and the conjugate function is $\phi^*(s) = e^s - 1$. The ϕ-divergence function of χ^2-distance is $\phi(t) = \frac{1}{t}(t-1)^2$ and the conjugate function is $\phi^*(s) = 2 - 2\sqrt{1-s}$ for $s \leq 1$. For both cases, the resulting dual formulations are tractable [8].

We will compare the empirical out-of-sample performance of distributionally robust solutions. We set $h = 1$ and $b \in \{1, 3, 9, 19\}$ to set resulting critical ratio as $\{0.50, 0.75, 0.90, 0.95\}$, respectively. We generate $N \in \{50, 500\}$ samples from a normal distribution with two different parameters: mean $m = 100$ and standard deviation $s \in \{20, 40\}$ for different coefficient of variation CV $= \frac{s}{m}$ as $\{0.2, 0.4\}$. We let the Wasserstein radius $\theta = 1$, which is large enough to guarantee probability bounds [31, 30]. We let the radius of ϕ-divergence models $\rho = 0.5$. Let p-Wasserstein denote the Wasserstein newsvendor models of the Wasserstein order p for $p = 1, 2$. $p = 1$ and $p = 2$ are the widely used orders for the Wasserstein distance. The Wasserstein and ϕ-divergence solutions are derived based on the N historical data generated from a normal distribution of mean 100 and standard deviations 20 and 40. Then, the demand d is realized and the total cost is calculated based on the order quantities and the realized demand as $\mathrm{TC}(x^*, d) = h(x^* - d)^+ + b(d - x^*)^+$. We generate 500 demand realizations for d to evaluate average of the realized total cost $\mathrm{TC}(x^*, d)$ from the same normal distribution. The average of the realized total cost represents the empirical out-of-sample performance of distributionally robust solutions. We conduct 100 iterations to generate data sets and calculate average and maximum of simulated costs when the optimal solutions are implemented. x_{avg} denotes the average of optimal order quantities of each model. c_{avg} and c_{max} denote average and maximum of empirical out-of-sample performances over 100 iterations. The results are summarized in Table 2.1.

Table 2.1 Empirical out-of-sample performance of optimal order quantities for Wasserstein and ϕ-divergence models when CV = 0.2 and CV = 0.4

			1-Wasserstein			2-Wasserstein			KL-divergence			χ^2-distance		
CV	b	N	x_{avg}	c_{avg}	c_{max}	x_{avg}	c_{avg}	c_{max}	x_{avg}	c_{avg}	c_{max}	x_{avg}	c_{avg}	c_{max}
0.2	1	50	98.91	16.18	17.85	98.91	16.18	17.85	98.91	16.18	17.85	99.89	16.60	20.81
	1	500	99.77	15.93	17.47	99.77	15.93	17.47	99.77	15.93	17.47	100.81	16.43	19.16
	3	50	113.08	25.82	30.34	113.66	25.80	30.05	120.06	26.99	33.75	122.22	28.43	44.44
	3	500	113.31	25.40	28.63	113.89	25.40	28.53	121.74	27.30	29.89	131.74	33.95	56.02
	9	50	124.17	36.07	44.32	125.50	35.90	42.45	136.12	39.89	58.76	135.56	39.66	58.84
	9	500	125.64	35.09	39.22	126.98	35.16	38.99	145.39	46.35	64.02	150.23	50.74	79.70
	19	50	132.02	42.59	54.00	134.09	42.43	51.32	140.85	45.43	64.65	139.97	45.10	63.97
	19	500	132.80	41.39	47.27	134.86	41.54	47.17	155.51	55.98	86.27	156.39	56.80	87.61
0.4	1	50	97.81	32.36	35.71	97.81	32.36	35.71	98.28	32.33	35.71	99.83	33.04	40.20
	1	500	99.54	31.86	34.94	99.54	31.86	34.94	100.16	31.88	35.05	101.61	32.85	38.32
	3	50	126.17	51.64	60.68	126.74	51.62	60.39	140.47	54.04	67.75	141.85	55.24	80.42
	3	500	126.63	50.80	57.26	127.20	50.80	57.16	143.61	54.65	59.84	163.47	67.91	112.04
	9	50	148.34	72.15	88.63	149.67	71.94	86.75	172.24	79.79	117.49	169.45	78.10	112.70
	9	500	151.29	70.19	78.43	152.62	70.22	78.16	191.20	93.11	143.84	200.46	101.48	159.41
	19	50	164.04	85.18	108.00	166.11	84.92	105.15	181.71	90.86	129.31	179.58	89.78	127.18
	19	500	165.59	82.78	94.54	167.66	82.82	94.39	210.99	111.95	173.48	212.79	113.60	175.21

The optimal order quantities of ϕ-divergence models are larger than those of Wasserstein models for most cases, and the gaps between order quantities increase as b increases. Therefore, the gaps between average simulated costs are considerable, which reflects that the out-of-sample performances of the Wasserstein solutions are better than those of the ϕ-divergence solutions. The optimal order quantities for the ϕ-divergence models are sensitive to coefficient of variation, that is, the decision-maker considering the ϕ-divergence model overly orders when the variance is large. Another reason for the gaps is that the ϕ-divergence ambiguity set cannot contain normal distributions which are data-generating distributions and the resulting ambiguity set becomes unrealistic. In contrast, the Wasserstein ambiguity set contains the unknown true distribution with a certain probability [30]. In summary, optimal order quantities for the Wasserstein models have better out-of-sample performances than those for the ϕ-divergence models.

2.5.2 Convergence Property

One important property of Wasserstein DRO is convergence property, i.e., as the sample size N increases, the distributionally robust solution converges to the true optimal solution with the complete knowledge of the probability distribution. According to the result of Theorem 2.8, for $p = 1$, an optimal order quantity with the Wasserstein ambiguity set is equivalent to the SAA solution. If we choose the Wasserstein radius proportional to $\frac{1}{\sqrt{N}}$ for probability guarantee [31], then the ambiguity set shrinks to the empirical distribution as the sample size N goes to infinity. We choose the Wasserstein radius $\theta_N = \frac{10}{\sqrt{N}}$ to control the conservativeness of the Wasserstein models. Then, the optimal solution of the Wasserstein newsvendor model converges to the optimal solution with true distribution as the sample size N increases, even if the decision maker does not know the true distribution. Furthermore, with θ_N, the objective function value of the Wasserstein models converges to the optimal cost with true distribution as N goes to infinity.

In this experiment, we compare the convergence property of the Wasserstein solutions compared to the moment-based DRO model, specifically Scarf's model. Scarf's model is given by

$$\min_{x \geq 0} \sup_{\mu \in \mathcal{M}(m,s)} \mathbb{E}_{\mu}[h(x-\xi)^{+} + b(\xi - x)^{+}], \tag{2.16}$$

where $\mathcal{M}(m,s)$ denotes the moment-based ambiguity set with known first and second moments, i.e., a set of probability distributions with mean m and standard deviation s. In our setting, the decision maker knows only about historical data. Hence, m and s are estimated by sample mean and sample standard deviation, respectively. Scarf's solution is $x_{\text{Scarf}} = m + \frac{s}{2}\tau$, where $\tau = \sqrt{\frac{b}{h}} - \sqrt{\frac{h}{b}}$ and the optimal objective function value is $s\sqrt{bh}$.

In the data-driven approach, the decision maker can update the order quantity based on the realized data. For example, we consider the repeated setting of the newsvendor problem, that is, a decision maker decides the order quantity repeatedly through the planning horizon. The decision maker decides x_N based on N samples and demand is realized after the decision is implemented. The realized demand can be used to make a decision x_{N+1} with $N+1$ samples. Likewise, order quantities are updated with realized demand as sample size N increases. We conduct numerical experiments by updating order decisions based on the realized demand data until $N = 5,000$. We generated demand samples from a normal distribution of mean 100 and standard deviation 20, and the experimental setting is similar to that of Subsection 2.5.1. The updated order quantities and objective function values are shown in Figures 2.7 and 2.8, respectively. The black horizontal line represents the optimal solution when the underlying true distribution is known.

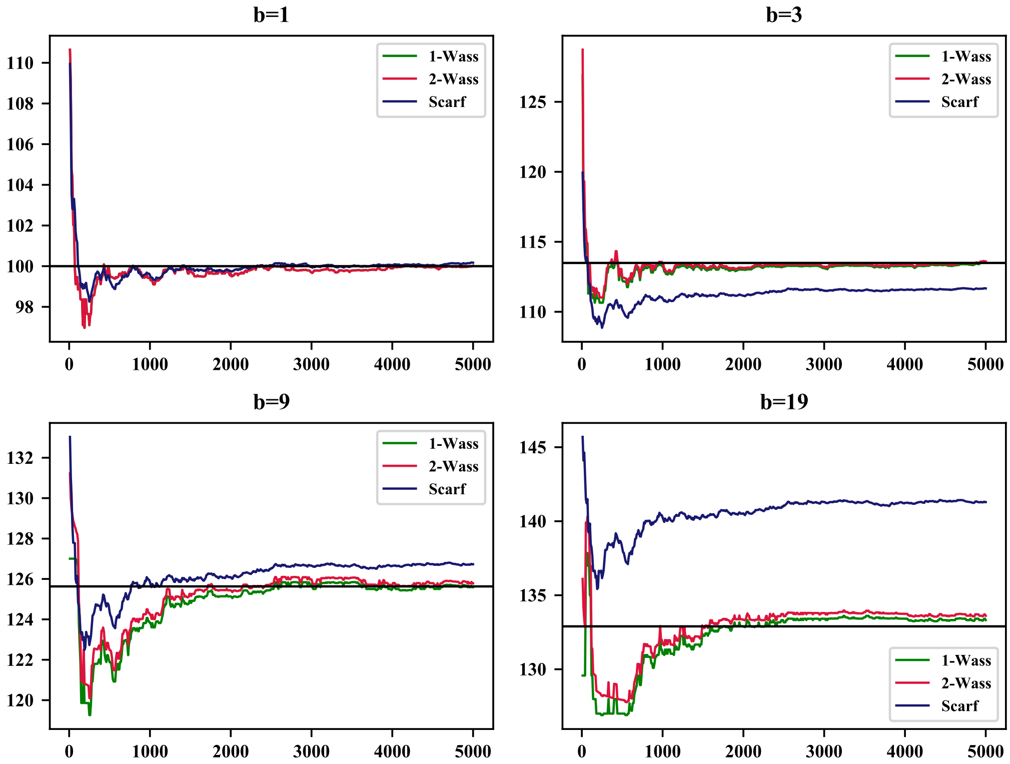

Figure 2.7 Convergence of optimal order quantities to true optimal order quantities as sample size (N) increases

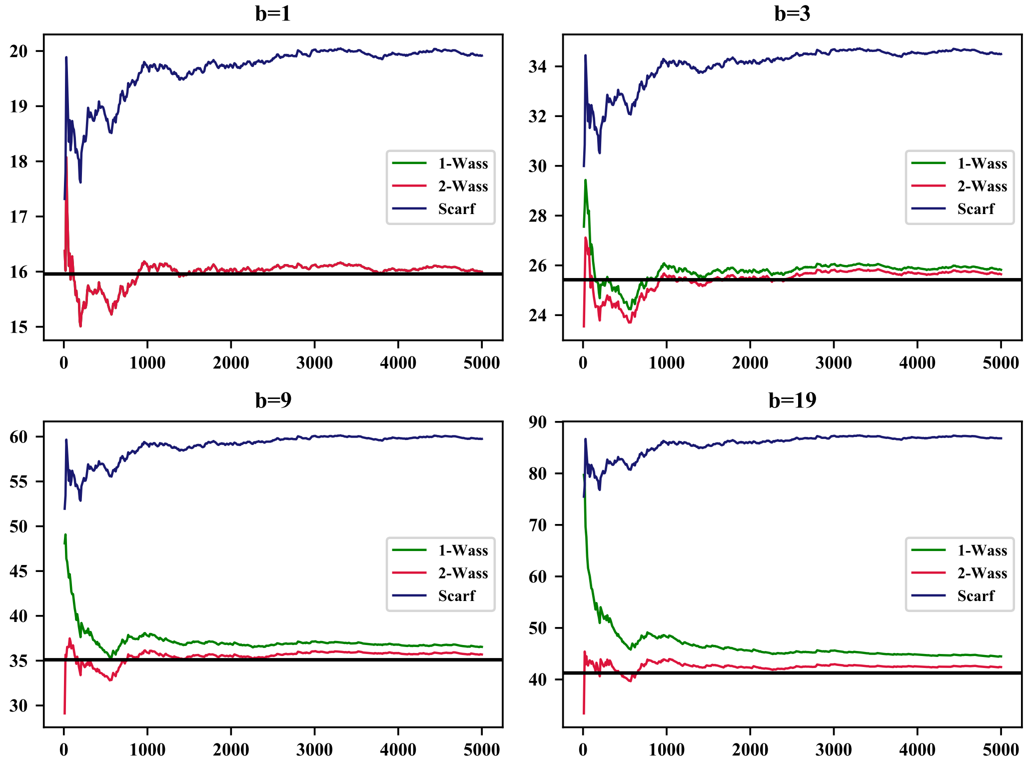

Figure 2.8 Convergence of objective function values to true optimal cost as sample size (N) increases

When $b = 1$, the Wasserstein solution is equal to the median and Scarf's solution is equal to the sample mean, because $\tau = 0$. Therefore, Scarf's solution behaves similarly to that of the Wasserstein model, because samples are generated from a normal distribution and mean and median of a normal distribution are equal. In other cases ($b = 3, 9, 19$), the Wasserstein solutions converge to the true optimal solution, whereas the Scarf's solutions do not. Moreover, the Wasserstein radius θ_N decreases as N increases, which leads to the convergence of objective function values of the Wasserstein model. This convergence result reflects one of the important advantages of the Wasserstein model.

2.5.3 Risk-aversion of the CVaR Solution

In some cases, it is important to reduce the risk of extremely large costs caused by the pessimistic demand realization. To verify such a risk-aversion property of the CVaR model, the following experiment is conducted. Risk-neutral and CVaR solutions are derived based on the N historical data generated from a normal distribution of mean 100 and standard deviation 20. Then, the demand d is realized and the total cost is calculated based on the order quantities and the realized demand as $\text{TC}(x^*, d) = h(x^* - d)^+ + b(d - x^*)^+$. The experimental setting is the same as that described in Subsection 2.5.1 such that we generate 500 demand realizations for d to evaluate the realized total cost from the same normal distribution. We let $\beta \in \{0.2, 0.5, 0.9\}$ for the CVaR coefficient.

The optimal solution x^* and the average and maximum of realized total costs (TC_{avg}, TC_{max}) are summarized in Table 2.2 and 2.3 for $p = 1$ and $p = 2$, respectively. The CVaR model becomes more risk averse as β increases. Hence, the optimal order quantity tends to increase as β increases, but there are exceptions because of the data-driven setting. The closed-form solution of the CVaR model depends on historical data, so the order quantity may decrease as β increases, especially for small values of b. The most noticeable result is that, in most cases, the maximum cost of the risk-averse model is less than that of the risk-neutral model, and the difference becomes significant when b and β are large. It means that the CVaR model prefers less risk of extremely large loss, even the average cost would be increased. When b is large, the order quantity of the risk-averse model is larger than that of the risk-neutral model to decrease the extremely large underage cost. Therefore, more cautious and concerned decision makers would prefer the risk-averse model.

Table 2.2 Optimal solutions and simulation results for risk-neutral and CVaR models with different values of β when the Wasserstein $p = 1$

		risk-neutral			CVaR : $\beta = 0.2$			CVaR : $\beta = 0.5$			CVaR : $\beta = 0.9$		
b	N	x^*	$\mathrm{TC_{avg}}$	$\mathrm{TC_{max}}$	x^*	$\mathrm{TC_{avg}}$	$\mathrm{TC_{max}}$	x^*	$\mathrm{TC_{avg}}$	$\mathrm{TC_{max}}$	x^*	$\mathrm{TC_{avg}}$	$\mathrm{TC_{max}}$
1	50	96.44	15.82	66.89	97.64	15.68	65.69	99.87	15.56	66.20	100.02	15.56	66.35
1	500	99.42	15.57	65.75	99.66	15.56	65.99	99.21	15.57	65.54	99.12	15.58	65.44
3	50	111.22	25.14	156.32	110.71	25.18	157.85	114.42	25.27	146.74	117.09	25.79	138.72
3	500	112.23	25.11	153.30	112.60	25.12	152.18	115.98	25.54	142.06	118.41	26.20	134.76
9	50	121.91	35.57	372.76	123.95	35.21	354.40	127.47	35.27	322.70	129.03	35.58	308.74
9	500	125.92	35.13	336.68	126.66	35.18	330.01	128.23	35.40	315.88	130.76	36.15	293.10
19	50	130.98	41.03	614.63	129.99	41.28	633.40	129.73	41.37	638.36	132.01	40.92	595.13
19	500	131.73	40.95	600.33	131.28	40.99	608.91	132.74	40.88	581.18	139.61	43.47	450.73

Table 2.3 Optimal solutions and simulation results for risk-neutral and CVaR models with different values of β when the Wasserstein $p = 2$

		risk-neutral			CVaR : $\beta = 0.2$			CVaR : $\beta = 0.5$			CVaR : $\beta = 0.9$		
b	N	x^*	$\mathrm{TC_{avg}}$	$\mathrm{TC_{max}}$	x^*	$\mathrm{TC_{avg}}$	$\mathrm{TC_{max}}$	x^*	$\mathrm{TC_{avg}}$	$\mathrm{TC_{max}}$	x^*	$\mathrm{TC_{avg}}$	$\mathrm{TC_{max}}$
1	50	96.44	15.82	66.89	97.79	15.67	65.54	99.87	15.56	66.20	100.02	15.56	66.35
1	500	99.42	15.57	65.75	99.66	15.56	65.99	99.46	15.57	65.78	99.20	15.58	65.53
3	50	111.80	25.11	154.59	111.36	25.13	155.92	115.23	25.40	144.30	118.92	26.37	133.24
3	500	112.81	25.13	151.57	113.25	25.16	150.24	116.79	25.72	139.61	120.23	26.89	129.30
9	50	123.25	35.29	360.76	125.44	35.13	341.00	129.35	35.67	305.79	133.24	37.30	270.80
9	500	127.25	35.24	324.68	128.15	35.39	316.60	130.27	35.96	297.58	136.26	39.25	243.61
19	50	133.05	40.89	575.40	132.30	40.89	589.54	132.65	40.88	582.89	138.54	42.85	471.08
19	500	133.80	40.97	561.10	133.89	40.98	559.39	135.66	41.53	525.76	146.13	47.91	326.80

2.6 Summary

In this chapter, we considered a Wasserstein ambiguity set for the data-driven distributionally robust newsvendor model. To incorporate a wide range of random demand and the Wasserstein distance, we considered continuous and unbounded support $\Xi = [0, \infty)$ and the general Wasserstein order $p \in [1, \infty)$. We derived the closed-form expressions of the optimal order quantity and the worst-case distribution for the risk-neutral model. Esfahani and Kuhn [30] also discussed the equivalence between a closed-form solution and the SAA solution when $p = 1$, but we extended the closed-form analysis to the general $p > 1$. We analyzed the structure of an optimal order quantity based on the closed-form expressions, which is characterized by the sum of the SAA solution and the value determined by the parameters. We also considered the risk-averse model with the CVaR objective, and derived the closed-form solution for the $p = 1$ case, and proposed a tractable formulation to obtain the optimal order quantity for the $p > 1$ case. We conducted numerical experiments to verify the out-of-sample performance of distributionally robust solutions and the convergence results of the Wasserstein models. The Wasserstein solutions showed better out-of-sample performance and convergence properties, which is an important advantage when applied to practical circumstances. The risk aversion of the CVaR model was analyzed in terms of the possibility of extremely large cost caused by the pessimistic realization.

Chapter 3

Distributionally Robust Inventory Model with a Wasserstein Ambiguity Set

3.1 Problem Description and Literature Review

The inventory problem is a classical problem in the operations management society to decide an optimal order policy under demand uncertainty. A decision maker chooses order quantities over a planning horizon to minimize total costs, which consist of purchase, holding, and penalty costs. There are various practical applications of the inventory problem, such as warehouse management, dynamic pricing, and retail management. In classical analysis with the complete knowledge of demand distribution, a base-stock policy is an optimal policy for the inventory model with an independent demand process. In a base-stock policy, there exists a base-stock level for each period, and the decision maker orders up to the base-stock level when the initial inventory level is less than the base-stock level. However, in practice, the exact demand distribution is unknown, and typically only partial information of the probability distribution or historical data is available. A famous remedy for this difficulty is distributionally robust optimization where minimization with respect to the worst-case expected cost over a set of candidate distributions is considered. We denote the set of candidate distributions as an ambiguity set. One of the most important considerations for practical success of DRO is how to construct an ambiguity set with prior information of distributions. In this study, we consider the data-driven setting of the inventory model, i.e., the only information we can obtain is historical data, which is prevalent in real-world operations. With historical data, we construct the ambiguity set based on the Wasserstein distance for the data-driven distributionally robust inventory model.

Since the pioneering work of Scarf [76] which studied the distributionally robust newsvendor model with a known mean and variance, there have been extensive studies on distributionally robust inventory models with ambiguity sets constructed with moment information [44, 62, 33, 78]. Most of the researches analyzed distributionally robust versions of dynamic programming formulations and proved that a base-stock policy is optimal for the distributionally robust inventory model with moment information. However, one drawback of the moment information based DRO is that its solution is overly conservative in terms of average cases to be applied in practical problems [104]. Another shortcoming is that the assumption of exactly known moment information is unrealistic, because all moments are estimated from historical data in the data-driven setting. A possible alternative of the moment information based ambiguity set is an ambiguity set that consists of probability distributions close to the given reference distribution with respect to a statistical distance.

The Wasserstein distance is the optimal transport cost for moving mass from one distribution to the other distribution. The Wasserstein distance can be defined as the distance between a discrete distribution and a continuous distribution, so it can be used to overcome shortcomings of the ϕ-divergence. The ambiguity set, constructed with distributions close to the empirical distribution with respect to the Wasserstein distance, can contain continuous distributions. In addition, the Wasserstein ambiguity set with the empirical distribution contains the data-generating distribution with a probabilistic guarantee based on the measure concentration results [31]. This property leads to desirable properties of the data-driven solutions discussed in [15] and [30], e.g., out-of-sample performance guarantee, convergence property, and tractability. We will discuss these properties specifically in Section 3.4.3. With several advantages of the Wasserstein distance, we study the distributionally robust inventory model with the Wasserstein ambiguity set.

On the other hand, the decision maker carefully considers time consistency issue in the multistage setting of distributionally robust inventory problems [87, 109]. Time consistency means that the optimal policy derived in the first period maintains its optimality through the planning horizon for almost every realization of demand. To discuss time consistency of the inventory problem, we consider two different formulations in viewpoint of possibilities of reoptimization after demand realization for each period: *multistage static formulation* based on stochastic optimization and *distributionally robust dynamic programming formulation* [82, 109]. The decision maker cannot reoptimize after demand realization in the multistage static formulation, while the optimal policy can be reoptimized with demand realization in the dynamic programming formulation. We follow the definition of time consistency established by Xin and Goldberg [109] (see Definition 3.1). A policy is called time consistent if the policy is optimal for both formulations, i.e., the policy remains to be optimal for almost every realization. The inventory problem is *weakly time consistent* if there exists at least one time consistent policy, and *strongly time consistent* if every optimal policy is time consistent. When the demand distribution is specified exactly, the stochastic optimization formulation and the dynamic programming formulation are equivalent, and the inventory problem is strongly time consistent. However, in the multistage DRO, the problem is time inconsistent in general, that is, even weak time consistency does not hold for this case.

In this chapter, we discuss a sufficient condition for weak time consistency of the distributionally robust inventory model with a Wasserstein ambiguity set. Xin and Goldberg [109] considered the moment-based ambiguity set with known first and second moments, and discussed the time consistency issue of the distributionally robust inventory model. We study a similar issue of the data-driven model with a Wasserstein ambiguity set. First, we show that the Wasserstein ambiguity set does not satisfy rectangularity, which is a key property for time consistency, and the distributionally robust inventory model with a Wasserstein ambiguity set is time inconsistent in general. To investigate the sufficient condition, we derive the equivalent form of the multistage static formulation and propose a newsvendor policy to decompose the multistage problem into single stage problems. The newsvendor policy is specified by optimal order quantities of newsvendor models, so we present closed-form solutions of the Wasserstein newsvendor model. We show that when optimal order quantities are monotone non-decreasing, the newsvendor policy becomes optimal for both formulations, and the inventory model is weakly time consistent in spirit of the result of [109]. Moreover, we discuss further details about dynamic programming formulations, such as computation of optimal base-stock levels and optimality of an (s, S) policy with the non-zero fixed order cost. Finally, we conduct numerical experiments to verify desirable properties of Wasserstein DRO.

Our contributions can be summarized as follows:

- We derive a sufficient condition for weakly time consistent of the distributionally robust inventory problem with the Wasserstein ambiguity set based on monotonicity of base-stock levels.

- We present closed-form solutions for the Wasserstein newsvendor model and discuss the condition that monotone non-decreasing of optimal order quantities is satisfied, which leads to weak time consistency.

- We discuss further details of the dynamic programming formulation, such as calculation of base-stock levels and optimality of an (s, S) policy, and we conduct numerical experiments to verify desirable properties of distributionally robust solutions.

The rest of this chapter is organized as follows. In Section 3.2, we formulate the distributionally robust inventory model and present a definition of time consistency. In Section 3.3, we present the sufficient condition for weak time consistency based on the monotonicity of optimal order quantities for the newsvendor policy. We discuss further details about the dynamic programming formulation in Section 3.4 and conduct numerical experiments to validate the advantages of Wasserstein DRO in Section 3.5. We conclude the chapter in Section 3.6.

3.2 Distributionally Robust Inventory Model and Time Consistency

We consider a finite T planning horizon for the inventory problem. Let $(D_1, \ldots, D_T)$ denote random demand over T periods and $D_{[t]} := (D_1, \ldots, D_t)$ denote the demand process up to period t. Let $(d_1, \ldots, d_T)$ denote a particular realization of the random demand. y_t represents the inventory level at the beginning of period t with the initial inventory level y_1 given.

The sequence of events for period t is described as follows: A decision maker observes the inventory level y_t at the beginning of period t. Before demand D_t is realized, the decision maker orders up to x_t with zero lead time, that is, the order quantity is $x_t - y_t$ with a unit purchase cost c_t and an inventory level after the order is x_t. The inventory level and order decision are dependent on the past demand realizations, i.e., x_t and y_t are functions of $d_{[t-1]}$ for $t = 1, \ldots, T$. After demand D_t is realized, the holding cost is incurred for unsold goods with the unit holding cost h_t. Excess demand is backordered and the penalty cost is incurred for shortages with the unit backorder cost b_t. Then, the inventory level at the beginning of period $t+1$ is $y_{t+1} = x_t - D_t$. We assume $b_t > c_t$ and $b_t, c_t, h_t \geq 0$ for $t = 1, \ldots, T$. To make the solutions nontrivial in the later analysis, we also assume $b_t - c_t + c_{t+1} \geq 0$ for $t = 1, \ldots, T$.

A policy $\pi = \big(x_1, x_2(d_{[1]}), \ldots, x_T(d_{[T-1]})\big)$ is defined as a sequence of order decisions for each period. A policy π is said to be feasible if the following two types of constraints are satisfied.

$$x_t(d_{[t-1]}) \geq y_t(d_{[t-1]}), \quad \forall t = 1, \dots, T, \tag{3.1}$$

$$y_{t+1}(d_{[t]}) = x_t(d_{[t-1]}) - d_t, \quad \forall t = 1, \dots, T-1. \tag{3.2}$$

Constraint (3.1) represents the order quantity should be nonnegative and Constraint (3.2) represents the balance equation. Let Π denote the set of feasible policies. For a given feasible policy $\pi \in \Pi$, $(x_1, \dots, x_T)$ and $(y_1, \dots, y_T)$ are determined through demand realizations, that is, we can write $\{x_t^\pi(d_{[t-1]})\}$, $\{y_t^\pi(d_{[t-1]})\}$ for $t = 1, \dots, T$ as functions of π and $d_{[t]}$. In some cases, we suppress the dependence on π and $d_{[t]}$ for notational brevity, but we write the notation when we have to clarify the meaning.

Let $\Xi := \Xi_1 \times \cdots \times \Xi_T$ be the support of demand process $(D_1, \dots, D_T)$, where $\Xi_t = \mathbb{R}_+, t = 1, \dots, T$ denote the support of random demand D_t. Let $\mathcal{B}(\Xi)$ denote Borel σ-algebra and μ denote probability distribution (measure) of demand process $(D_1, \dots, D_T)$ defined on $(\Xi, \mathcal{B}(\Xi))$. In the classical inventory problem, we assume that probability distribution μ is known. The risk-neutral inventory model is expressed as follows:

$$\begin{aligned} \min_{\pi \in \Pi} \mathbb{E}_\mu \Bigg[\sum_{t=1}^{T} \Big\{ & c_t\big(x_t^\pi(D_{[t-1]}) - y_t^\pi(D_{[t-1]})\big) \\ & + h_t\big(x_t^\pi(D_{[t-1]}) - D_t\big)^+ + b_t\big(D_t - x_t^\pi(D_{[t-1]})\big)^+ \Big\} \Bigg] \end{aligned} \tag{3.3}$$

where $X^+ = \max\{X, 0\}$ and $\mathbb{E}_\mu[\cdot]$ is the expectation with respect to distribution μ. If the demand process is *stagewise independent*, i.e., D_t is independent of past demand $D_{[t-1]}$, then the optimal policy is a base-stock policy [116, 93]. Even though most of demand process is correlated in practice, many previous studies assume the independent demand process for simplicity of analysis and derivation of meaningful results. We will discuss further details later.

In practice, however, the complete knowledge of probability distribution is prohibited. Therefore, we assume that the only information we can obtain is historical data. We adopt the distributionally robust approach to the inventory problem with historical data. Distributionally robust optimization (DRO) considers an *ambiguity set* $\mathcal{M}$ defined as a set of probability distributions that may include the unknown true distribution, and minimizes the worst-case expected cost over the ambiguity set. We consider the ambiguity set as a set of distributions close to the empirical distribution in terms of the Wasserstein distance.

In the multistage setting of distributionally robust optimization, two different formulations are considered in viewpoint of possibilities of reoptimization after demand realization [82, 109]. At the beginning of period t, the decision maker observed the demand realizations up to period $t-1$, $d_{[t-1]}$. If the decision maker can reoptimize and revise the policy based on $d_{[t-1]}$ considering costs from period t to period T, then the decision determined at period t, $x_t^{\pi}(d_{[t-1]})$, may be different from the decision determined at period 1. Likewise, the availability of reoptimization after demand realization leads to different formulations.

First, we consider the formulation when the decision maker cannot reoptimize and revise after demand realization. The policy determined at the beginning of the planning horizon is implemented through the periods. In this case, optimization is performed over feasible policies and the formulation is expressed as follows.

$$\min_{\pi\in\Pi}\sup_{\mu\in\mathcal{M}}\mathbb{E}_{\mu}\Bigg[\sum_{t=1}^{T}\big\{c_t\big(x_t^{\pi}(D_{[t-1]})-y_t^{\pi}(D_{[t-1]})\big) \\ +h_t\big(x_t^{\pi}(D_{[t-1]})-D_t\big)^{+}+b_t\big(D_t-x_t^{\pi}(D_{[t-1]})\big)^{+}\big\}\Bigg] \tag{3.4}$$

We call (3.4) the *multistage static formulation* [82, 109]. Now we consider the case that the decision maker can reoptimize the policy based on the realized demand. The decision made at period t depends on the inventory level y_t and demand realizations $d_{[t-1]}$.

$$V_t(y_t,d_{[t-1]})=\min_{x_t\geq y_t}\Bigg\{c_t(x_t-y_t)+ \\ \sup_{\mu\in\mathcal{M}}\mathbb{E}_{\mu|d_{[t-1]}}\big[h_t(x_t-D_t)^{+}+b_t(D_t-x_t)^{+}+V_{t+1}(x_t-D_t)\big]\Bigg\}, \tag{3.5}$$

for $t=1,\dots,T$ where $V_t(\cdot,\cdot)$ is the cost-to-go functions. We call (3.5) the *dynamic programming formulation*. The total cost over the planning horizon is $V_1(y_1)$ when we suppress $d_{[0]}$. The optimal policy of the dynamic programming formulation is given by

$$x_t^{\pi}(d_{[t-1]})\in\arg\min_{x_t\geq y_t}\Bigg\{c_t(x_t-y_t)+ \\ \sup_{\mu\in\mathcal{M}}\mathbb{E}_{\mu|d_{[t-1]}}\big[h_t(x_t-D_t)^{+}+b_t(D_t-x_t)^{+}+V_{t+1}(x_t-D_t)\big]\Bigg\}, \tag{3.6}$$

w.p.1 (in terms of the reference distribution) for all $t=1,\dots,T$.

The natural question is that the optimal policy of the multistage static formulation (3.4) is still optimal for the remaining period $t=2,\dots,T$ in terms of reoptimization with demand realizations up to period $t-1$. Specifically, the decision maker wants the optimal policy π^* of Problem (3.4) to be optimal for (3.5), that is, under demand realization $d_{[t-1]}$, $x_t^{\pi^*}(d_{[t-1]})$ is an optimal solution of $V_t(y_t, d_{[t-1]})$ for $t = 1, \ldots, T$. This question leads to the concept of time consistency [87, 109]. The time inconsistent optimal policy of the multistage static formulation would not be preferred by decision makers, because the optimality of the policy might be collapsed during the implementation through the planning horizon. In other words, the decision maker may not respond effectively to the specific realizations of demand process with the time inconsistent policy. Therefore, time consistency is important in terms of a decision maker's implementation and adaptation through demand realizations.

There are several different viewpoints and definitions of time consistency based on communities and modeling perspectives, especially for finance and risk analysis with dynamic risk measure [19, 18, 23, 81]. Considering a stochastic programming community and optimization perspective, a decision maker concerns the optimal policy and optimality of the policy through demand realizations. Therefore, we adopt the policy-centered definition of time consistency from [109] (see also Section 6.8.5 of [85]).

Definition 3.1 (Time consistency)**.** *Let $\pi \in \Pi$ be a feasible policy. If π is optimal for both the multistage static formulation (3.4) and the dynamic programming formulation (3.5), then π is a time consistent policy. Problem (3.4) is weakly time consistent if there exists at least one time consistent policy. If the sets of optimal policies of (3.4) and (3.5) are equivalent, i.e., every optimal policy is time consistent, then Problem (3.4) is strongly time consistent.*

However, the optimal policies of (3.4) and (3.5) are different in general for the distributionally robust setting. A famous example of strongly time consistent problem is the risk neutral formulation. When $\mathcal{M}$ is singleton, i.e., $\mathcal{M} = \{\mu\}$, Problem (3.4) reduces to the risk-neutral formulation.

$$\min_{\pi\in\Pi} \mathbb{E}_\mu\Big[\sum_{t=1}^{T}\big\{c_t(x_t-y_t)+h_t(x_t-D_t)^+ + b_t(D_t-x_t)^+\big\}\Big] \tag{3.7}$$

$$= \min_{\pi\in\Pi} \mathbb{E}_\mu\Big[\mathbb{E}_{\mu|D_1}\big[\sum_{t=1}^{T}\{c_t(x_t-y_t)+h_t(x_t-D_t)^+ + b_t(D_t-x_t)^+\}\big]\Big] \tag{3.8}$$

$$= \min_{\pi\in\Pi} \mathbb{E}_\mu\Big[\mathbb{E}_{\mu|D_1}\big[\cdots\mathbb{E}_{\mu|D_{[T-1]}}\big[\sum_{t=1}^{T}\{c_t(x_t-y_t)+h_t(x_t-D_t)^+ + b_t(D_t-x_t)^+\}\big]\big]\Big] \tag{3.9}$$

where $\mathbb{E}_{\mu|D_{[t-1]}}$ is conditional expectation given $D_{[t-1]}$. The first equality holds by the law of iterated expectations $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]]$ and the second equality holds by applying the law $T-1$ times. The expectation operator has a decomposability property, that is, Problem (3.7) is equivalent to Problem (3.9).

By strict monotonicity of the expectation operator, the interchangeability of minimization and expectation operations holds [83]. Thus, Problem (3.9) can be represented in the nested form as follows.

$$\begin{aligned}\min_{x_1\geq y_1} c_1(x_1-y_1) &+ \mathbb{E}_\mu\Big[h_1(x_1-D_1)^+ + b_1(D_1-x_1)^+ + \min_{x_2\geq y_2} c_2(x_2-y_2)+ \\ &\mathbb{E}_{\mu|D_{[1]}}\big[\cdots + \min_{x_T\geq y_T} c_T(x_T-y_T) + \mathbb{E}_{\mu|D_{[T-1]}}\big[h_T(x_T-D_T)^+ + b_T(D_T-x_T)^+\big]\big]\Big]\end{aligned} \tag{3.10}$$

Considering the problem at period T and backward through periods $t = T-$

$1, \ldots, 1$, Problem (3.10) is equivalent to the dynamic programming formulations.

$$V_t(y_t, d_{[t-1]}) = \min_{x_t \geq y_t} c_t(x_t - y_t) + \mathbb{E}_{\mu|D_{[t-1]}}[h_t(x_t - D_t)^+ + b_t(D_t - x_t)^+ + V_{t+1}(x_t - D_t)], \quad (3.11)$$

for $t = 1, \ldots, T$ where $V_t(\cdot, \cdot)$ is the cost-to-go functions and $V_{T+1}(\cdot, \cdot) \equiv 0$. Then, Problem (3.7) and Problem (3.11) are equivalent, that is, the risk neutral problem is strongly time consistent. The decomposability and interchangeability play key roles for equivalence between multistage static and dynamic programming formulations, which leads to time consistency.

However, the optimization in Problem (3.11) needs the evaluation of conditional expectation given $d_{[t-1]}$, which makes the DP recursions difficult to solve. In addition, the cost-to-go function $V_t(y_t, d_{[t-1]})$ depends on current inventory level y_t and demand realizations $d_{[t-1]}$ whose possibilities increase dramatically as t increases. By these difficulties, the computational burden of Problem (3.11) could be tremendous. For the sake of simplicity, we assume that the demand process is stagewise independent, that is, the joint distribution of the demand process is expressed in the product form $\mu = \mu_1 \times \cdots \times \mu_T$ and conditional expectations are replaced by unconditional expectations. The information of demand realizations $d_{[t-1]}$ is concentrated in y_t and decision x_t only depends on y_t. Therefore, the dynamic programming formulation (3.11) can be written as

$$V_t(y_t) = \min_{x_t \geq y_t} c_t(x_t - y_t) + \mathbb{E}_{\mu_t}[h_t(x_t - D_t)^+ + b_t(D_t - x_t)^+ + V_{t+1}(x_t - D_t)], \quad (3.12)$$

for $t = 1, \ldots, T$, where $V_t(\cdot)$ is the cost-to-go functions and $V_{T+1}(\cdot) \equiv 0$. To see a base-stock policy is optimal for the dynamic programming formulation, we consider the recursion from period T to period 1. If $V_{t+1}(\cdot)$ is convex, then $V_t(\cdot)$ is also convex because expectation and minimization preserve convexity. There exists unconstrained minimizer $x_t^* \in \mathbb{R}$, because the cost function is convex and continuous. The base-stock policy for period t with the base-stock level x_t^* is optimal. Since $V_{T+1}(\cdot)$ is zero and convex, the base-stock policy is optimal for period T. By the recursion, the base-stock policy with the base-stock levels $x_t^*, t = 1, \ldots, T$ is an optimal policy of Problem (3.12). This argument is applied similarly for the distributionally robust version of the dynamic programming formulation. We refer to Chapter 3 of [85] and Section 3 of [84] for further details about the risk neutral formulation.

Unlike the risk neutral case, decomposability which is an important characteristic for time consistency does not hold in the distributionally robust setting. For simple notation, let $Z = Z^\pi(D_{[T]})$ denote the objective function of Problem (3.4), i.e.,

$$
\begin{aligned}
Z^\pi(D_{[T]}) := \sum_{t=1}^{T} \Big\{ & c_t\big(x_t^\pi(D_{[t-1]}) - y_t^\pi(D_{[t-1]})\big) \\
& + h_t\big(x_t^\pi(D_{[t-1]}) - D_t\big)^+ + b_t\big(D_t - x_t^\pi(D_{[t-1]})\big)^+ \Big\}. \qquad (3.13)
\end{aligned}
$$

Then, the following inequality holds for the distributionally robust setting.

$$
\sup_{\mu \in \mathcal{M}} \mathbb{E}_\mu[Z] \leq \sup_{\mu \in \mathcal{M}} \mathbb{E}_\mu \Big[\sup_{\mu \in \mathcal{M}} \mathbb{E}_{\mu|D_1} \big[\cdots \sup_{\mu \in \mathcal{M}} \mathbb{E}_{\mu|D_{[T-1]}}[Z] \big] \Big] \qquad (3.14)
$$

Xin and Goldberg [109] show that the inequality in (3.14) can hold strictly in a two-stage example when the moment-based ambiguity set is considered. To take advantage of a decomposability property, Shapiro [82] define a rectangular set of an ambiguity set in the multistage DRO. Shapiro [82] define the rectangularity as a decomposability property holds, that is, the inequality in (3.14) holds as an equality.

Definition 3.2 (Rectangular set). *Let $\mathcal{Z}$ be a set of objective functions. We define a set of probability distributions $\widehat{\mathcal{M}}$ as a rectangular set with respect to $\mathcal{M}$ and $\mathcal{Z}$ if*

$$\sup_{\mu \in \widehat{\mathcal{M}}} \mathbb{E}_\mu[Z] = \sup_{\mu \in \mathcal{M}} \mathbb{E}_\mu \Big[\sup_{\mu \in \mathcal{M}} \mathbb{E}_{\mu|D_1} \big[\cdots \sup_{\mu \in \mathcal{M}} \mathbb{E}_{\mu|D_{[T-1]}}[Z] \big] \Big], \ \forall Z \in \mathcal{Z}. \tag{3.15}$$

We define the set $\mathcal{M}$ is rectangular with respect to $\mathcal{Z}$ if (3.15) holds for $\mathcal{M} = \widehat{\mathcal{M}}$.

We refer to Theorem 2.1 and Corollary 2.2 of [82] for the existence of a rectangular set $\widehat{\mathcal{M}}$ and conditions for rectangularity of $\mathcal{M}$, respectively. Using Definition 3.2, [87] discuss time consistency of multistage distributionally robust inventory models when the ambiguity set is rectangular. By the assumption of rectangularity, DRO models in Shapiro and Xin [87] are always weakly time consistent, but they show that there may exist infinitely many time inconsistent policies, which means that the models are not always strongly time consistent. Even though the rectangular set of $\mathcal{M}$ exists, the construction of the rectangular set, $\widehat{\mathcal{M}}$, is not intuitive and easy to implement in practice.

For the simplicity of analysis, we assume that demand process is stagewise independent. The ambiguity set over the planning horizon is expressed as a set of the direct product of distributions for each period.

$$\mathcal{M} := \{\mu = \mu_1 \times \cdots \times \mu_T, \ \mu_t \in \mathcal{M}_t, \ \forall t = 1, \ldots, T\} \tag{3.16}$$

where $\mathcal{M}_t$ is an ambiguity set of D_t which consists of probability distributions defined on $(\Xi_t, \mathcal{B}(\Xi_t))$. We consider ambiguity sets $\mathcal{M}_t,\ t = 1, \ldots, T$ as the Wasserstein ambiguity set. However, the product form does not guarantee a decomposability property, which implies that the product form may not be a rectangular set. Xin and Goldberg [109] discuss the non-rectangularity of a moment-based ambiguity set with the stagewise independence assumption. We will discuss the non-rectangularity of a Wasserstein ambiguity set in Section 3.3.

With the stagewise independent demand, the dynamic programming formulation (3.5) is expressed as follows:

$$V_t(y_t) = \min_{x_t \geq y_t} c_t(x_t - y_t) + \sup_{\mu_t \in \mathcal{M}_t} \mathbb{E}_{\mu_t}[h_t(x_t - D_t)^+ + b_t(D_t - x_t)^+ + V_{t+1}(x_t - D_t)], \tag{3.17}$$

for $t = 1, \ldots, T$ where $V_{T+1}(\cdot) \equiv 0$. In Section 3.3, we will discuss the optimality of a base-stock policy for (3.17) when the ambiguity set $\mathcal{M}$ is constructed by the Wasserstein distance. Then, we will derive the sufficient condition of weak time consistency of the distributionally robust inventory model with a Wasserstein ambiguity set.

3.3 Sufficient Condition for Weak Time Consistency

In this section, we first define the Wasserstein ambiguity set for the data-driven setting of the inventory model. Then, we provide a simple example that shows inequality (3.14) holds with strict inequality, and the Wasserstein ambiguity set is non-rectangular. The distributionally robust inventory model is time inconsistent in general, so we will investigate the sufficient condition for weak time consistency.

We assume that N independent sample paths, $\hat{d}^i := (\hat{d}^i_1, \ldots, \hat{d}^i_T)$ for $i = 1, \ldots, N$, are given as follows:

$$\begin{bmatrix} \hat{d}^1_1 & \cdots & \hat{d}^1_T \\ \vdots & \ddots & \vdots \\ \hat{d}^N_1 & \cdots & \hat{d}^N_T \end{bmatrix} \tag{3.18}$$

Each row represents the sample path of the demand process $(D_1, \ldots, D_T)$ and each column is used to construct the empirical distribution for each period. The t-th column of the data matrix is used to construct $\nu_t := \frac{1}{N}\sum_{i=1}^N \delta_{\hat{d}^i_t}$. The Wasserstein ambiguity set for period t, $\mathcal{M}_t$, and the Wasserstein ambiguity set for the demand process, $\mathcal{M}$, are defined as follows based on Definition 2.3:

$$\mathcal{M}_t := \{\mu_t \in \mathcal{P}_p(\Xi_t) : W_p(\mu_t, \nu_t) \leq \theta_t\},$$

$$\mathcal{M} := \{\mu = \mu_1 \times \cdots \times \mu_T, \mu_t \in \mathcal{M}_t, t = 1, \ldots, T\}.$$

θ_t represents the Wasserstein radius for period t, and $\mathcal{M}$ is expressed in the product form according to stagewise independence.

The following example shows that inequality (3.14) can hold strictly, which means that the Wasserstein ambiguity set $\mathcal{M}$ is non-rectangular. This example is closely related to Example 1 of [109].

Example 3.1. *Consider $T = 2$ and $\Xi_1 = \{0, 1\}$, $\Xi_2 = \{0, 1, 2\}$. Let $\nu_1 = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$ with $\theta_1 = 0$, that is, $\mathcal{M}_1 = \{\nu_1\}$. We assume D_1 and D_2 are independent. We fix a policy π as $y_1 = 0, x_1^\pi(y_1) = 1, x_2^\pi(y_2) = 0$ if $y_2 = 0$, and $x_2^\pi(y_2) = 2$ if $y_2 = 1$. Then, inequality (3.14) is written as follows:*

$$\sup_{\mu \in \mathcal{M}} \mathbb{E}_\mu[Z^\pi(D_1, D_2)] \leq \sup_{\mu_1 \in \mathcal{M}_1} \mathbb{E}_{\mu_1}\left[\sup_{\mu_2 \in \mathcal{M}_2} \mathbb{E}_{\mu_2|D_1}[Z^\pi(D_1, D_2)]\right].$$

The left hand side can be expressed as

$$\begin{aligned}\sup_{\mu \in \mathcal{M}} \mathbb{E}_\mu[Z^\pi(D_1, D_2)] &= \sup_{\mu_2 \in \mathcal{M}_2} \mathbb{E}_{\nu_1 \times \mu_2}[Z^\pi(D_1, D_2)] \\ &= \sup_{\mu_2 \in \mathcal{M}_2} \mathbb{E}_{\mu_2}\left[\frac{1}{2} Z^\pi(0, D_2) + \frac{1}{2} Z^\pi(1, D_2)\right].\end{aligned}$$

By the simple analysis using Theorem 2.5, we obtain the following result.

$$\begin{aligned}&\sup_{\mu_2 \in \mathcal{M}_2} \mathbb{E}_{\mu_2}\left[Z^\pi(0, D_2) + Z^\pi(1, D_2)\right] \\ &= \begin{cases} 2c_1 + h_1 + c_2 + 2b_2 & \text{if } \theta \geq 1 \\ 2c_1 + h_1 + c_2 + (1-\theta)h_2 + (1+\theta)b_2 & \text{if } \theta \leq 1 \end{cases}\end{aligned}$$

The right hand side can be expressed as,

$$\sup_{\mu_1 \in \mathcal{M}_1} \mathbb{E}_{\mu_1}\left[\sup_{\mu_2 \in \mathcal{M}_2} \mathbb{E}_{\mu_2|D_1}[Z^\pi(D_1, D_2)]\right]$$
$$= \frac{1}{2} \sup_{\mu_2 \in \mathcal{M}_2} \mathbb{E}_{\mu_2}[Z^\pi(0, D_2)] + \frac{1}{2} \sup_{\mu_2 \in \mathcal{M}_2} \mathbb{E}_{\mu_2}[Z^\pi(1, D_2)].$$

By the same analysis for the left hand side,

$$\sup_{\mu_2 \in \mathcal{M}_2} \mathbb{E}_{\mu_2}[Z^\pi(0, D_2)] = \begin{cases} c_1 + h_1 + c_2 + 2h_2 & \text{if } \theta \geq 1 \\ c_1 + h_1 + c_2 + (1+\theta)h_2 & \text{if } \theta \leq 1 \end{cases}$$

$$\sup_{\mu_2 \in \mathcal{M}_2} \mathbb{E}_{\mu_2}[Z^\pi(1, D_2)] = \begin{cases} c_1 + 2b_2 & \text{if } \theta \geq 1 \\ c_1 + (1+\theta)b_2 & \text{if } \theta \leq 1 \end{cases}$$

$$\sup_{\mu_2 \in \mathcal{M}_2} \mathbb{E}_{\mu_2}[Z^\pi(0, D_2)] + \sup_{\mu_2 \in \mathcal{M}_2} \mathbb{E}_{\mu_2}[Z^\pi(1, D_2)]$$
$$= \begin{cases} 2c_1 + h_1 + c_2 + 2h_2 + 2b_2 & \text{if } \theta \geq 1 \\ 2c_1 + h_1 + c_2 + (1+\theta)h_2 + (1+\theta)b_2 & \text{if } \theta \leq 1 \end{cases}$$

Therefore, inequality (3.14) holds with strict inequality.

Example 3.1 implies that the equality for (3.14) may not hold and the Wasserstein ambiguity set is non-rectangular. A decomposition property does not hold for the Wasserstein ambiguity set and the distributionally robust inventory problem is time inconsistent in general. In other words, Problem (3.4) does not satisfy weak time consistency in general and the best we can do is investigate the sufficient condition of the problem being weakly time consistent. Before we move on to the sufficient condition, we derive the optimality of a base-stock policy for the DP formulation.

We recall the DP formulation (3.17) with the Wasserstein ambiguity set $\mathcal{M}_t, t = 1, \ldots, T$.

$$V_t(y_t) = \min_{x_t \geq y_t} \{c_t(x_t - y_t) + \sup_{\mu_t \in \mathcal{M}_t} \mathbb{E}_{\mu_t}[h_t(x_t - D_t)^+ + b_t(D_t - x_t)^+ + V_{t+1}(x_t - D_t)]\}$$

for $t = 1, \ldots, T$, where $V_t(\cdot)$ are the cost-to-go functions and $V_{T+1}(\cdot) \equiv 0$. Let $g_t(x_t, d_t) := h_t(x_t - d_t)^+ + b_t(d_t - x_t)^+ + V_{t+1}(x_t - d_t)$. If $V_{t+1}(\cdot)$ is convex, then $\mathbb{E}_{\mu_t}[g_t(x_t, D_t)]$ is convex for given μ_t. Since $\mathcal{M}_t$ is convex by Proposition 2.4 and pointwise supremum over a convex set preserves convexity, $\sup_{\mu_t \in \mathcal{M}_t} \mathbb{E}_{\mu_t}[g_t(x_t, D_t)]$ is convex. Therefore, $V_t(\cdot)$ is convex and the DP formulation (3.17) is a convex problem. The unconstrained version of (3.17) has an optimal solution and the base-stock policy whose base-stock level is the unconstrained minimizer is optimal.

Now we propose the equivalent formulation of the multistage static formulation (3.4) to derive the sufficient condition.

$$\begin{aligned}
&\min_{\pi \in \Pi} \sup_{\mu \in \mathcal{M}} \mathbb{E}_\mu \left[\sum_{t=1}^{T} \{c_t(x_t - y_t) + h_t(x_t - D_t)^+ + b_t(D_t - x_t)^+\} \right] \\
=&\min_{\pi \in \Pi} \sup_{\mu \in \mathcal{M}} \mathbb{E}_\mu \left[\sum_{t=1}^{T} \{c_t(x_t - (x_{t-1} - D_{t-1}) + h_t(x_t - D_t)^+ + b_t(D_t - x_t)^+\} \right] \\
=&\min_{\pi \in \Pi} \sup_{\mu \in \mathcal{M}} \mathbb{E}_\mu \left[\sum_{t=1}^{T} \{(c_t - c_{t+1})x_t + h_t(x_t - D_t)^+ + b_t(D_t - x_t)^+ + c_{t+1}D_t\} \right] - c_1 y_1 \\
=&\min_{\pi \in \Pi} \sup_{\mu \in \mathcal{M}} \mathbb{E}_\mu \left[\sum_{t=1}^{T} \{c_t x_t + h_t(x_t - D_t)^+ + b_t(D_t - x_t)^+ - c_{t+1}(x_t - D_t)\} \right] - c_1 y_1 \\
=&\min_{\pi \in \Pi} \sup_{\mu \in \mathcal{M}} \mathbb{E}_\mu \left[\sum_{t=1}^{T} \{c_t x_t + (h_t - c_{t+1})(x_t - D_t)^+ + (b_t + c_{t+1})(D_t - x_t)^+\} \right] - c_1 y_1
\end{aligned} \tag{3.19}$$

where $c_{T+1} = 0$. The first equality holds by the balance constraint (3.2) and the last equality holds by $x_t - D_t = (x_t - D_t)^+ - (D_t - x_t)^+$. The structure of (3.19) is similar to the multistage static formulation with the moment-based ambiguity set M_t which contains distributions with mean m_t for $t = 1, \ldots, T$. By the similar structure of the multistage static formulation, we can follow the analysis of [109] which considers the moment-based ambiguity set with the first and second moments information. Moreover, the part that affects the optimal solution of (3.19) is the same as the newsvendor cost function with the with unit holding cost $h'_t = h_t - c_{t+1}$ and unit backorder cost $b'_t = b_t + c_{t+1}$, i.e., $c_t x_t + (h_t - c_{t+1})(x_t - D_t)^+ + (b_t + c_{t+1})(D_t - x_t)^+$, except for the fact that x_t is linked through periods by a feasible policy π. Based on the connection to the newsvendor cost function, we propose the newsvendor policy inspired by the lower bound proposed by [109] to get the lower bound of (3.19).

3.3.1 Newsvendor Policy

A newsvendor policy aims to decompose multistage decision process into single stage decisions. The newsvendor policy is conducted as follows: for each period t, the decision maker orders up to x_t and the demand D_t is realized. Then, the decision maker salvages the leftover goods if $x_t - D_t > 0$ or satisfies the shortage with an emergency order if $x_t - D_t < 0$ with unit cost c_{t+1}. For the next period, the initial inventory is 0 and the decision maker orders up to x_{t+1}, and repeats the process. The process for the first few periods is shown in Figure 3.1. By the newsvendor policy, the multistage decision is decomposed by the iteration of single stage decisions, because the decision for each period is not affected by the previous decisions.

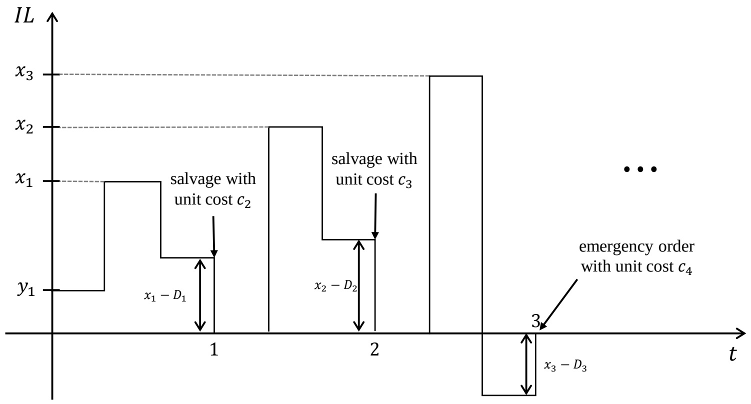

Figure 3.1 Newsvendor policy

The t-period cost of the newsvendor policy is defined as

$$\begin{aligned}
&c_t x_t + h_t(x_t - D_t)^+ + b_t(D_t - x_t)^+ - c_{t+1}(x_t - D_t) \\
=&c_t x_t + (h_t - c_{t+1})(x_t - D_t)^+ + (b_t + c_{t+1})(D_t - x_t)^+
\end{aligned}$$

where c_{t+1} is the salvage or emergency order cost at period t. A decision in period t does not depend on past decisions $x_{[t-1]}$ and realizations $D_{[t-1]}$. The t-period cost is equal to the newsvendor cost function with unit holding cost $h'_t = h_t - c_{t+1}$ and unit backorder cost $b'_t = b_t + c_{t+1}$. Hence, the optimal order quantity, x^*_t, is equivalent to the optimal solution of the newsvendor problem. This is why we call this policy the newsvendor policy. We will discuss the closed-form solution of an optimal order quantity for the newsvendor problem with the Wasserstein ambiguity set in Section 3.3.2. It is worth mentioning that the newsvendor policy is not a feasible policy for Problem (3.4) in general, and we will discuss the condition that the newsvendor policy becomes feasible.

For a distributionally robust setting of the newsvendor policy, we define

$$\begin{aligned}
\vartheta_t :=& \min_{x_t \ge 0} \sup_{\mu_t \in \mathcal{M}_t} \mathbb{E}_{\mu_t}\big[c_t x_t + h_t(x_t - D_t)^+ + b_t(D_t - x_t)^+ - c_{t+1}(x_t - D_t)\big] \\
=& \min_{x_t \ge 0} \sup_{\mu_t \in \mathcal{M}_t} \mathbb{E}_{\mu_t}\big[c_t x_t + (h_t - c_{t+1})(x_t - D_t)^+ + (b_t + c_{t+1})(D_t - x_t)^+\big], \\
x_t^* \in& \arg\min_{x_t \ge 0} \sup_{\mu_t \in \mathcal{M}_t} \mathbb{E}_{\mu_t}\big[c_t x_t + (h_t - c_{t+1})(x_t - D_t)^+ + (b_t + c_{t+1})(D_t - x_t)^+\big]
\end{aligned} \tag{3.20}$$

where x_t^* denotes the optimal order quantity for the newsvendor policy. ϑ_t and x_t^* only depend on period t, i.e., they depend on the Wasserstein ambiguity set $\mathcal{M}_t$ and parameters (c_t, c_{t+1}, h_t, b_t), not on past decisions. We note that the optimal solution, x_t^*, is equivalent to the optimal order quantity of the distributionally robust newsvendor model. The decision maker has a chance to salvage or place an emergency order by adopting the newsvendor policy. Therefore, the total cost of implementing the newsvendor policy is lower than the optimal objective function value of the static formulation (3.19).

Lemma 3.3. *For a fixed feasible policy $\pi \in \Pi$ and given $k \ge 0$, for any $\mu_1 \in \mathcal{M}_1, \ldots, \mu_k \in \mathcal{M}_k$, there exist $\mu_{k+1} \in \mathcal{M}_{k+1}, \ldots, \mu_T \in \mathcal{M}_T$ such that,*

$$\mathbb{E}_{\otimes_{j=1}^T \mu_j}[c_t x_t(y_t) + (h_t - c_{t+1})(x_t(y_t) - D_t)^+ + (b_t + c_{t+1})(D_t - x_t(y_t))^+] \ge \vartheta_t,$$

where $\otimes_{j=1}^T \mu_j$ represents a product measure generated by $(\mu_1, \ldots, \mu_T)$.

Proof. We refer to [109] for proof with the cost function $(c_t - c_{t+1})x_t + h_t(x_t - D_t)^+ + b_t(D_t - x_t)^+$ and the moment-based ambiguity set. Our proof is similar to the proof of [109], but we include a specific proof for completeness.

We will prove the lemma by induction. For notational convenience, we define $\psi_t(x_t, D_t) := c_t x_t + (h_t - c_{t+1})(x_t - D_t)^+ + (b_t + c_{t+1})(D_t - x_t)^+$ and let $\mathbb{E}_i[\cdot]$ denote $\mathbb{E}_{\otimes_{j=1}^i \mu_j}[\cdot]$. The induction hypothesis is written as follows: There exist $\mu_{k+1} \in \mathcal{M}_{k+1}, \ldots, \mu_{k+n} \in \mathcal{M}_{k+n}$ such that $\mathbb{E}_{k+n}[\psi_t(x_t(y_t), D_t)] \geq \vartheta_t$, $\forall t = k+1, \ldots, k+n$.

For $n = 1$, by Jensen's inequality and stagewise independence, $\forall \mu_{k+1} \in \mathcal{M}_{k+1}$,

$$\mathbb{E}_{k+1}\left[\psi_{k+1}\left(x_{k+1}(y_{k+1}), D_{k+1}\right)\right] \geq \mathbb{E}_{\mu_{k+1}}\left[\psi_{k+1}\left(\mathbb{E}_k\left[x_{k+1}(y_{k+1})\right], D_{k+1}\right)\right].$$

By weak compactness of the Wasserstein ambiguity set (Proposition 2.4), there exists $\mu_{k+1} = \mu^*_{k+1}(\mathbb{E}_k\left[x_{k+1}(y_{k+1})\right])$ such that

$$\begin{aligned}
& \mathbb{E}_{\mu_{k+1}}\left[\psi_{k+1}\left(\mathbb{E}_k\left[x_{k+1}(y_{k+1})\right], D_{k+1}\right)\right] \\
= & \sup_{\mu_{k+1} \in \mathcal{M}_{k+1}} \mathbb{E}_{\mu_{k+1}}\left[\psi_{k+1}\left(\mathbb{E}_k\left[x_{k+1}(y_{k+1})\right], D_{k+1}\right)\right] \\
\geq & \min_{x_{t+1} \geq 0} \sup_{\mu_{k+1} \in \mathcal{M}_{k+1}} \mathbb{E}_{\mu_{k+1}}[\psi_{k+1}(x_k, D_k)] = \vartheta_{k+1}.
\end{aligned}$$

Assuming that the induction hypothesis holds for n, the inequality holds for the $n+1$ case with similar arguments, which completes the proof. □

We note that the above lemma was first proposed by [109] for the moment-based ambiguity set. The above result also holds for the Wasserstein ambiguity set and Lemma 3.3 shows that the objective function value of (3.19) is bounded below by the optimal cost of the newsvendor policy, $\sum_{t=1}^{T} \vartheta_t - c_1 y_1$. If the newsvendor policy is feasible, then it will be an optimal policy for the static formulation (3.19). One specific condition that guarantees feasibility of the newsvendor policy is monotonicity of the newsvendor solutions x_t^*. When the sequence $\{x_t^*\}$ is monotone non-decreasing, i.e., $x_t^* \leq x_{t+1}^*$ for $t = 1, \ldots, T-1$, a base-stock policy with the base-stock level x_t^* is equivalent to the newsvendor policy. More specifically, with the base-stock policy, the decision maker always orders up to x_t^*, because the initial inventory level is less than or equal to the base-stock level, i.e., $y_t \leq x_t^*$ for all $t = 1, \ldots, T$. In this case, the newsvendor policy is feasible and optimal for Problem (3.4) by Lemma 3.3. Moreover, the newsvendor policy is optimal for the DP formulation (3.17).

Lemma 3.4. *When the sequence $\{x_t^*\}$ is monotone non-decreasing, i.e., $x_t \leq x_{t+1}$ for $t = 1, \ldots, T-1$, the base-stock policy with base-stock levels x_t^* is optimal for the dynamic programming formulation (3.17).*

Proof. We prove the lemma by induction. The induction hypothesis is written as follows: If the decision maker implements a base-stock policy with base-stock levels $(x_1^*, \ldots, x_{t-1}^*)$, then the optimal base-stock level for period t is x_t^* and $V_t(y_t) = \vartheta_t - c_t(x_{t-1}^* - D_{t-1}) + \sum_{k=t+1}^{T} \vartheta_k$ for $y_t \leq x_t^*$.

For $t = T$, we first check if x_T^* is the optimal base-stock level for period T when a base-stock policy with $(x_1^*, \ldots, x_{T-1}^*)$ is implemented. For $y_T = x_{T-1}^* - D_{T-1} \leq x_T^*$,

$$
\begin{aligned}
V_T(y_T) = \min_{x_T \geq y_T} \sup_{\mu_T \in \mathcal{M}_T} & \mathbb{E}_{\mu_T}[c_T(x_T - y_T) + h_T(x_T - D_T)^+ \\
& + b_T(D_T - x_T)^+ + V_{T+1}(x_T - D_T)] \\
= \min_{x_T \geq y_T} \sup_{\mu_T \in \mathcal{M}_T} & \mathbb{E}_{\mu_T}[c_T x_T + (h_T - c_{T+1})(x_T - D_T)^+ \\
& + (b_T + c_{T+1})(D_T - x_T)^+ + V_{T+1}(x_T - D_T)] - c_T(x_{T-1}^* - D_{T-1}).
\end{aligned}
$$

The equality holds because $V_{T+1}(\cdot) = 0$ and $c_{T+1} = 0$. x_T^* is an unconstrained optimizer of $V_T(y_T)$ and $x_T^* \geq x_{T-1}^* \geq y_T$. Hence, x_T^* is an optimal base-stock level for period T and $V_T(y_T) = \vartheta_T - c_T(x^*_{T-1} - D_{T-1})$ for $y_T \leq x^*_T$.

Assume that the induction hypothesis holds for period $t+1$. We will prove that the induction hypothesis also holds for period t. The cost-to-go function for period t is

$$V_t(y_t) = \min_{x_t \geq y_t} \sup_{\mu_t \in \mathcal{M}_t} \mathbb{E}_{\mu_t}[c_t(x_t - y_t) + h_t(x_t - D_t)^+ + b_t(D_t - x_t)^+ + V_{t+1}(x_t - D_t)].$$

If $x_t - D_t \leq x^*_{t+1}$, $V_{t+1}(x_t - D_t) = \vartheta_{t+1} - c_{t+1}(x_t - D_t) + \sum_{k=t+2}^{T} \vartheta_k$. Then,

$$\begin{aligned}
V_t(y_t) = \min_{x_t \geq y_t} \sup_{\mu_t \in \mathcal{M}_t} \mathbb{E}_{\mu_t}\Bigg[&c_t x_t + h_t(x_t - D_t)^+ + b_t(D_t - x_t)^+ + \vartheta_{t+1} \\
&- c_{t+1}(x_t - D_t) + \sum_{k=t+2}^{T} \vartheta_k\Bigg] - c_t(x^*_{t-1} - D_{t-1}) \\
= \min_{x_t \geq y_t} \sup_{\mu_t \in \mathcal{M}_t} &\mathbb{E}_{\mu_t}[c_t x_t + (h_t - c_{t+1})(x_t - D_t)^+ + (b_t + c_{t+1})(D_t - x_t)^+] \\
&+ \sum_{k=t+1}^{T} \vartheta_k - c_t(x^*_{t-1} - D_{t-1}).
\end{aligned}$$

Thus, if $x_t - D_t \leq x^*_{t+1}$, x^*_t is an optimal base-stock level for period t and $V_t(y_t) = \vartheta_t - c_t(x^*_{t-1} - D_{t-1}) + \sum_{k=t+1}^{T} \vartheta_k$. If $x_t - D_t > x^*_{t+1}$, $V_{t+1}(x_t - D_t) \geq V_t(x^*_{t+1})$.

$$\begin{aligned}
V_t(y_t) &\geq \min_{x_t \geq y_t} \sup_{\mu_t \in \mathcal{M}_t} \mathbb{E}_{\mu_t}[c_t(x_t - y_t) + h_t(x_t - D_t)^+ + b_t(D_t - x_t)^+ + V_{t+1}(x^*_{t+1})] \\
&= \min_{x_t \geq y_t} \sup_{\mu_t \in \mathcal{M}_t} \mathbb{E}_{\mu_t}[c_t x_t + h_t(x_t - D_t)^+ + b_t(D_t - x_t)^+ - c_{t+1} x^*_{t+1}] \\
&\qquad + \sum_{k=t+1}^{T} \vartheta_k - c_t(x^*_{t-1} - D_{t-1})
\end{aligned}$$

$$
\begin{aligned}
&\geq \min_{x_t \geq y_t} \sup_{\mu_t \in \mathcal{M}_t} \mathbb{E}_{\mu_t}[c_t x_t + h_t(x_t - D_t)^+ + b_t(D_t - x_t)^+ - c_{t+1}(x_t - D_t)] \\
&\qquad\qquad + \sum_{k=t+1}^{T} \vartheta_k - c_t(x_{t-1}^* - D_{t-1}) \\
&\geq \sum_{k=t}^{T} \vartheta_k - c_t(x_{t-1}^* - D_{t-1}).
\end{aligned}
$$

In this case, the total cost is dominated by the cost with x_t^*. Therefore, the optimal solution cannot be attained in $x_t > x_{t+1}^* + D_t$ which completes the proof. □

Now we are ready to derive the sufficient condition for the weak time consistency based on monotonicity. By Lemmas 3.3 and 3.4, the following theorem holds with the monotone condition.

Theorem 3.5. *Let x_t^* be an optimal solution of ϑ_t and $\{x_t^*\}$ be a monotone non-decreasing sequence, i.e., $x_t^* \leq x_{t+1}^*$ for $t = 1, \ldots, T-1$. Then the base-stock policy with base-stock levels x_t^* is optimal for both the multistage-static formulation and the dynamic programming formulation, that is, the base-stock policy is time consistent and the problem is weakly time consistent.*

We note that our result holds with the general ambiguity set compared to the result of [109] with the moment-based ambiguity set. The lemmas and theorem also hold for the ambiguity set which has a compactness property. The next question is when the sequence $\{x_t^*\}$ is monotone non-decreasing. To figure out this question, we should know the optimal order quantity of the newsvendor model with a Wasserstein ambiguity set.

3.3.2 Distributionally Robust Newsvendor Model with Unit Purchase Cost

In this subsection, we derive the closed-form solution for the newsvendor model with unit purchase cost. Through this subsection, notations are expressed for a single stage problem. We refer to Chapter 2 for further details about the analysis of the distributionally robust newsvendor model with a Wasserstein ambiguity set.

Let $\Xi = [0, \infty)$ denote the support of the random demand D. N independent samples, $\{\hat{d}^1, \ldots, \hat{d}^N\}$, sorted in non-decreasing order are given, i.e., $\hat{d}^1 \leq \ldots \leq \hat{d}^N$. Using the data, we define the empirical distribution $\nu := \frac{1}{N}\sum_{i=1}^{N} \delta_{\hat{d}^i}$. Then, the distributionally robust newsvendor model with a Wasserstein ambiguity set is expressed as follows:

$$\min_{x \geq 0} \sup_{\mu \in \mathcal{P}_p(\Xi)} \left\{ \mathbb{E}_\mu[cx + h(x-D)^+ + b(D-x)^+] : W_p(\mu, \nu) \leq \theta \right\}. \tag{3.21}$$

Using Theorem 2.5, the dual reformulation of (3.21) can be expressed as follows:

$$\min_{x \geq 0} \inf_{\lambda \geq 0} \left\{ \lambda\theta^p + \frac{1}{N}\sum_{i=1}^{N} \sup_{d \in \Xi} \left[cx + h(x-d)^+ + b(d-x)^+ - \lambda|d - \hat{d}^i|^p \right] \right\}. \tag{3.22}$$

To make the analysis of the inner supremum of (3.22) possible and simple, we adopt Assumption 2.7 which is reasonable in many applications. Based on the similar analysis of Chapter 2, we present closed-form expressions for the optimal order quantity of the newsvendor model with unit purchase cost.

Theorem 3.6 (Wasserstein order $p = 1$)**.** *Under Assumption 2.7, there exists $i^* \in \{1, \ldots, N\}$ such that $\frac{i^*-1}{N} < \frac{b-c}{h+b} \leq \frac{i^*}{N}$. Then, $\hat{d}^{i^*}$ is an optimal order quantity of*

(3.22) and the optimal cost is

$$b\theta + c\hat{d}^{i^*} + \frac{1}{N}\left\{\sum_{k=1}^{i^*-1} h(\hat{d}^{i^*} - \hat{d}^k) + \sum_{j=i^*}^{N} b(\hat{d}^k - \hat{d}^{i^*})\right\}.$$

Theorem 3.7 (Wasserstein order $p > 1$)**.** *Assume that $\hat{d}^i \geq \theta$ for all i and Assumption 2.7 holds. The optimal dual variable $\lambda^* = \frac{1}{p\theta^{p-1}}(\Lambda)^{\frac{p-1}{p}}$ and the optimal order quantity $x^* = \hat{d}^{i^*} + \Delta p^{\frac{1}{p-1}}\theta(\frac{1}{\Lambda})^{\frac{1}{p}}$ is an optimal order quantity, where $\Lambda := \frac{1}{h+b}(b^{\frac{p}{p-1}}(h+c) + h^{\frac{p}{p-1}}(b-c)) \geq 0$ and $\Delta := \frac{1}{h+b}(\frac{1}{p})^{\frac{1}{p-1}}(\frac{p-1}{p})(b^{\frac{p}{p-1}} - h^{\frac{p}{p-1}})$. The optimal cost is*

$$\theta\Lambda^{\frac{p-1}{p}} + c\hat{d}^{i^*} + \frac{1}{N}\left\{\sum_{k=1}^{i^*-1}(\hat{d}^{i^*} - \hat{d}^k) + \sum_{k=i^*}^{N}(\hat{d}^k - \hat{d}^{i^*})\right\}.$$

Remark 3.1 (Structure of optimal order quantities)**.** *When the Wasserstein order $p = 1$, the optimal order quantity derived in Theorem 3.6 is the $\frac{b-c}{h+b}$ quantile of the empirical distribution ν, i.e., $x^* = \inf\{q : \nu([0,q]) \geq \frac{b-c}{b+h}\}$. In other words, the distributionally robust solution with respect to the Wasserstein ambiguity set is equivalent to the optimal solution of the data-driven newsvendor model or the sample average approximation (SAA) solution. This result was first noted by Remark 6.7 in [30]. When $p > 1$, the closed-form solution is the sum of the data-driven solution and the value determined by the parameters.*

The optimal order quantities in Theorems 3.6 and 3.7 can be used to calculate the optimal order quantities of the newsvendor policy, x_t^* for $t = 1, \ldots, T$. However, the optimal solution depends on historical data, so it is difficult to guarantee the monotonicity of x_t^* in general. Nevertheless, there exists a special case based on the i.i.d. demand process where monotonicity is guaranteed.

Remark 3.2. *When the demand process is i.i.d., the whole data in (3.18) can be used to construct the Wasserstein ambiguity set. More specifically, we construct the Wasserstein ambiguity set $\mathcal{M}_1$ with all the data in (3.18) and $\mathcal{M}_t = \mathcal{M}_1$ for all $t = 1, \ldots, T$. If the critical ratio is monotone non-decreasing, i.e., for $t = 1, \ldots, T-1$*

$$\frac{b_t - c_t + c_{t+1}}{h_t + b_t} \leq \frac{b_{t+1} - c_{t+1} + c_{t+2}}{h_{t+1} + b_{t+1}},$$

then $(x_1^*, x_2^*, \ldots, x_T^*)$ is monotone non-decreasing and the inventory problem is weakly time consistent. Moreover, if $c_t = c, h_t = h, b_t = b$ for all $t = 1, \ldots, T$ and $V_{T+1}(y) = -cy$, then the myopic policy is optimal for each period t. In other words, the critical ratio and the base-stock level x^* are the same for all periods, where we define $x^* \in$ $\arg\min_{x \geq 0} \sup_{\mu \in \mathcal{M}_1} \mathbb{E}_\mu[h(x - D)^+ + b(D - x)^+ + cD]$.

Except for the special case such as Remark 3.2, it is difficult to discuss time consistency of Problem (3.4). Moreover, without monotonicity, it is very challenging to analyze the multistage static formulation and its optimal solutions. Therefore, many previous researches focus on the dynamic programming formulation and the structure of its optimal policy. In the next section, we also focus on the dynamic programming formulation (3.17) and discuss about the optimal policy.

3.4 Further Analysis About the Dynamic Programming Formulation

In this section, we discuss further details about the DP formulation. First, we present the dual formulation of the DP formulation which can be used to compute the base-stock levels. Second, we consider the non-zero fixed order cost and derive the optimality of an (s, S) policy for the distributionally robust inventory model based on the K-convexity. Finally, we discuss the desirable properties of Wasserstein DRO, such as the out-of-sample performance guarantee and a convergence property.

3.4.1 Computation of Base-Stock Levels for Dynamic Programming Formulations

As we showed in Section 3.3, the base-stock policy is optimal for the DP formulation (3.17). When the optimal order quantities for the newsvendor policy are monotone non-decreasing, we can compute the optimal base-stock levels with closed-form solutions for the newsvendor model given in Section 3.3.2. However, it is difficult to obtain the closed-form expression for the base-stock levels in more generalized cases. For computation of base-stock levels, we present the dual formulation using the strong duality in Theorem 2.5.

$$
\begin{aligned}
V_t(y_t) &= \min_{x_t \geq y_t} \sup_{\mu_t \in \mathcal{M}_t} \mathbb{E}_{\mu_t}[c_t(x_t - y_t) + h_t(x_t - D_t)^+ + b_t(D_t - x_t)^+ + V_{t+1}(x_t - D_t)] \\
&= \min_{x_t \geq y_t} \inf_{\lambda \geq 0} \lambda \theta^p + c_t(x_t - y_t) \\
&\quad + \frac{1}{N} \sum_{i=1}^{N} \sup_{d_t \in \Xi_t} \left[h_t(x_t - D_t)^+ + b_t(D_t - x_t)^+ + V_{t+1}(x_t - D_t) - \lambda |d_t - \hat{d}_t^i|^p \right]
\end{aligned}
$$

Using auxiliary variables, the dual formulation is written as follows:

$$
\begin{aligned}
V_t(y_t) = \min_{x_t,\lambda} \quad & \lambda\theta^p + c_t(x_t - y_t) + \frac{1}{N}\sum_{i=1}^{N} z_i \\
\text{s.t.} \quad & z_i \geq h_t(x_t - d_t)^+ + V_{t+1}(x_t - d_t) - \lambda|d_t - \hat{d}_t^i|^p, \\
& \qquad\qquad\qquad\qquad \forall i = 1,\ldots,N,\ d_t \in \Xi_t, \\
& z_i \geq b_t(d_t - x_t)^+ + V_{t+1}(x_t - d_t) - \lambda|d_t - \hat{d}_t^i|^p, \\
& \qquad\qquad\qquad\qquad \forall i = 1,\ldots,N,\ d_t \in \Xi_t, \\
& x_t \geq y_t, \\
& \lambda \geq 0.
\end{aligned}
$$

The above formulation is the semi-infinite program if the support Ξ_t is continuous or unbounded. There are several numerical methods to solve the semi-infinite program, such as discretization methods (see [71, 57] and references therein). For the convex semi-infinite program, there are further studies about the KKT conditions and duality [80].

When the support Ξ_t is discrete and finite, e.g., the support for the binomial distribution, the number of constraints is finite and the above formulation can be solved using commercial solvers. Even though the formulation is tractable, the decision variable x_t is continuous and unbounded. Therefore, discretization and truncation are needed to calculate a finite number of $V_t(y_t)$. By solving the formulation recursively through backward in periods from T to 1, the optimal base-stock levels x_t^* can be calculated.

3.4.2 Non-zero Fixed Order Cost

In this subsection, we consider the case where the non-zero fixed order cost K occurs when an order is placed. We assume that the fixed order cost is the same for all periods. In this case, an (s, S) policy is proved to be optimal for many inventory models. For an (s, S) policy, the decision maker orders up to S when the inventory position is less than or equal to the reorder point s. Since the study of Scarf [77], K-convexity is the key concept used to prove the optimality of the (s, S) policy. We refer to [90] for the definition of K-convexity and properties of K-convex functions. We will show that the (s, S) policy is also optimal for the distributionally robust inventory model with a Wasserstein ambiguity set. We note that Klabjan et al. [46] showed that a state-dependent (s, S) policy is optimal for the inventory model with an ambiguity set constructed by a goodness-of-fit test. Bertsimas et al. [15] discussed the connection between the data-driven DRO and a goodness-of-fit test, so our result can be viewed as the extension of the result of [46].

The DP formulation with the fixed order cost is written as follows:

$$\begin{aligned} V_t(y_t) = \min_{x_t \geq y_t} \sup_{\mu_t \in \mathcal{M}_t} \mathbb{E}_{\mu_t}[K\mathbb{I}(x_t - y_t) + c_t(x_t - y_t) + h_t(x_t - D_t)^+ \\ + b_t(D_t - x_t)^+ + V_{t+1}(x_t - D_t)], \end{aligned} \tag{3.23}$$

for $t = 1, \ldots, T$ where $\mathbb{I}(x) = 1$ if $x > 0$ and $\mathbb{I}(x) = 0$ otherwise. For notational convenience, we define $g_t(x_t, d_t) := c_t x_t + h_t(x_t - D_t)^+ + b_t(D_t - x_t)^+ + V_{t+1}(x_t - D_t)$ and $f_t(x_t) := \sup_{\mu_t \in \mathcal{M}_t} \mathbb{E}_{\mu_t}[g_t(x_t, D_t)]$. Then the following lemma shows that K-convexity is preserved under maximization over the Wasserstein ambiguity set.

Lemma 3.8. *If $V_{t+1}(y_{t+1})$ is continuous and K-convex, then:*

(a) $g_t(x_t, d_t)$ *and* $f_t(x_t)$ *is* K*-convex in* x_t*.*

(b) Let S_t^* *be the smallest global minimizer of* $f_t(x_t)$ *and* s_t^* *be the largest* $x \leq S_t^*$ *such that* $f_t(x) = K + f_t(S_t^*)$*. Then an* (s_t^*, S_t^*) *policy is optimal for period* t*.*

(c) $V_t(y_t)$ *is continuous and* K*-convex.*

Proof. (a) Since $V_{t+1}(y_{t+1})$ is K-convex, $g_t(x_t, d_t)$ is K-convex in x_t. For any $x_t \leq x_t', \lambda \in [0, 1]$,

$$f_t((1-\lambda)x_t + \lambda x_t') = \sup_{\mu_t \in \mathcal{M}_t} \mathbb{E}_{\mu_t}[g_t((1-\lambda)x_t + \lambda x_t', D_t)]$$

$$\leq \sup_{\mu_t \in \mathcal{M}_t} \mathbb{E}_{\mu_t}[(1-\lambda)g_t(x_t, D_t) + \lambda g_t(x_t', D_t) + \lambda K]$$

$$\leq (1-\lambda) \sup_{\mu_t \in \mathcal{M}_t} \mathbb{E}_{\mu_t}[g_t(x_t, D_t)] + \lambda \sup_{\mu_t \in \mathcal{M}_t} \mathbb{E}_{\mu_t}[g_t(x_t', D_t)] + \lambda K$$

$$= (1-\lambda) f_t(x_t) + \lambda f_t(x_t') + \lambda K$$

The first inequality holds by the definition of K-convexity. Then $f_t(x_t)$ is K-convex.

(b) With the definition of $f_t(x_t)$, the cost-to-go function is expressed as $V_t(y_t) = \min_{x_t \geq y_t} K\mathbb{I}(x_t - y_t) + f_t(x_t) - c_t y_t$. Since $f_t(x_t)$ is K-convex, an (s_t^*, S_t^*) policy is optimal for period t by the similar analysis of the risk-neutral inventory model with the fixed order cost.

(c) By (b), the cost-to-go function is expressed as

$$V_t(y_t) = \begin{cases} K - c_t y_t + f_t(S_t^*), & \text{if } y_t \le s_t^* \\ f_t(y_t) + c_t y_t, & \text{if } y_t > s_t^*. \end{cases}$$

By the definition of s_t^* and K-convexity of $f_t(x_t)$, $V_t(y_t)$ is continuous and K-convex.

□

The lemma shows that K-convexity of $V_{t+1}(\cdot)$ is preserved through backward in periods if $V_t(\cdot)$ is K-convex. It implies that the optimality of the (s, S) policy is also preserved.

Theorem 3.9. *If the cost-to-function $V_{T+1}(\cdot)$ is convex, e.g., $V_{T+1}(\cdot) = 0$, then the (s, S) policy is optimal for the DP formulation (3.23).*

The above result also holds for the distributionally robust inventory model with the general ambiguity set which has appropriate properties, such as convexity and compactness.

3.4.3 Desirable Properties of Wasserstein DRO

In this subsection, we discuss the desirable properties of the data-driven Wasserstein DRO introduced in [30] and [15] which advocate advantages of the distributionally robust approach. First, we discuss the out-of-sample performance guarantee of the distributionally robust solution based on the measure concentration result (see Theorem 2 of [31]). Then, the convergence property of optimal objective function values as the number of samples N increases is introduced.

The out-of-sample performance is the expected cost of a distributionally robust solution $\hat{x}_t^N$ with respect to the true data-generating distribution, i.e., $\mathbb{E}_{\mu_t}[g_t(x_t, D_t) - c_t y_t]$ where $g_t(x_t, d_t) := c_t x_t + h_t(x_t - D_t)^+ + b_t(D_t - x_t)^+ + V_{t+1}(x_t - D_t)$. The concept of the out-of-sample performance is equivalent to the generalization error in the machine learning theory. However, in practice, the true distribution is unknown and the out-of-sample performance cannot be computed exactly. Therefore, the distributionally robust cost function $V_t(y_t)$ becomes a certificate of the out-of-sample performance with properly chosen radius and there exists performance guarantees based on the measure concentration result [31, 30].

$$\mathbb{P}\left(\mathbb{E}_{\mu_t}[g_t(\hat{x}_t^N, D_t) - c_t y_t] \leq V_t(y_t)\right) \geq 1 - \alpha \tag{3.24}$$

The probability that the out-of-sample performance is less than or equal to $V_t(y_t)$ is called the reliability and depends on a significance α and the Wasserstein radius θ. This probability is closely related to the probability that a Wasserstein ambiguity set contains the unknown true distribution. In Section 3.5.3, we conduct numerical experiments to analyze the reliability in terms of the Wasserstein radius θ.

Another important property of Wasserstein DRO is a convergence property, that is, the optimal objective function value and solutions of Wasserstein DRO converge to those of the risk-neutral formulation with the known true distribution as the sample size N goes to ∞ (see Theorem 3.6 of [30]). If we choose the Wasserstein radius θ_N properly depending on the sample size N [31], then the Wasserstein ambiguity set shrinks to the true data-generating distribution as N increases. This convergence property is essential for practical success of the data-driven approach.

3.5 Computational Experiments

3.5.1 Monotonicity

We discussed the sufficient condition for weak time consistency based on the monotone non-decreasing sequence of base-stock levels in Section 3.3. More specifically, when the optimal order quantities of the newsvendor policy, x_t^*, are monotone non-decreasing, the base-stock policy with x_t^* is optimal and the problem is weakly time consistent. In Remark 3.2, we discussed the case where the sequence x_t^* is monotone non-decreasing based on the i.i.d. demand process with the non-decreasing critical ratio $\frac{b_t - c_t + c_{t+1}}{h_t + b_t}$.

In this experiment, we consider the inventory model with the demand process of increasing mean. We assume $c_t = c, h_t = h, b_t = b$ for all $t = 1, \ldots, T$ and $c_{T+1} = c$ such that critical ratio is the same for all periods with $\frac{b}{h+b}$. Letting $c_{T+1} = c$ means that we assume $V_{T+1}(y) = -cy$ for the dynamic programming formulation. We will investigate the frequency of monotonicity and show that when the demand process has an increasing mean and the number of samples is large enough, we can expect the inventory problem is weakly time consistent with high probability.

We consider the planning horizon of 20 periods and two Wasserstein orders $p = 1, 2$. We set $c = 1, h = 1$ and $b \in \{1, 3, 9, 19\}$ to set resulting critical ratio for the entire period as $\{0.50, 0.75, 0.90, 0.95\}$, respectively. We generate $N \in \{50, 100, 500\}$ samples from normal distributions with an increasing mean and constant standard deviation: mean $m_t = 100 + k(t-1)$ for $t = 1, \ldots, T$ where $k \in \{2, 5, 8, 10\}$ is a step size and standard deviation $\sigma = 20$. We set the Wasserstein radius $\theta = 0.1$ for all periods. We conduct 100 iterations for each case and calculate the frequency that the monotone non-decreasing condition of x_t^* holds, i.e., $x_1^* \leq \ldots \leq x_T^*$ when x_t^* denotes the optimal order quantity of the newsvendor policy (see (3.20)). The results are summarized in Table 3.1. The step size for 1-Wass and 2-Wass represent the step size k of the demand process for the Wasserstein order $p = 1$ and $p = 2$, respectively.

Table 3.1 Frequencies of monotone non-decreasing order quantities x_t^* with constant standard deviation for 100 iterations

		Step size for 1-Wass				Step size for 2-Wass			
b	N	2	5	8	10	2	5	8	10
1	50	0	2	38	61	0	2	38	61
1	100	0	20	79	99	0	20	79	99
1	500	13	99	100	100	13	99	100	100
3	50	0	1	18	51	0	1	18	51
3	100	0	14	62	89	0	14	62	89
3	500	5	94	100	100	5	94	100	100
9	50	0	0	8	25	0	0	8	25
9	100	0	1	36	73	0	1	36	73
9	500	2	80	100	100	2	80	100	100
19	50	0	0	2	2	0	0	2	2
19	100	0	0	18	47	0	0	18	47
19	500	0	55	99	100	0	55	99	100

First, the frequencies increase drastically as the step size increases, because the discrepancies of means become significant and the resulting samples tend to be monotone. When the sample size N increases, the samples become stable and the sequence x_t^* is monotone non-decreasing for most cases. Therefore, all 100 iterations are monotone non-decreasing cases when the step size $k = 10$ and the sample size $N = 500$. On the other hand, the frequencies decrease as the critical ratio increases. When the critical ratio is high, the optimal solution comes from the right extreme of the empirical distribution and monotonicity can be broken due to the extreme data. One noticeable thing is that the results are the same for the Wasserstein order $p = 1$

and $p = 2$, because the difference of optimal solutions of the Wasserstein order $p = 1$ and $p = 2$ is $2\Delta\theta(\frac{1}{\Lambda})^{\frac{1}{2}}$ which is constant depending on parameters and monotonicity does not change depending on the Wasserstein order p (see Section 3.3.2).

In the above experiment, the standard deviation is constant for all periods, so the coefficient of variation (CV), $\frac{\sigma}{m_t}$, decreases as t increases. In other words, the variability of the demand process decreases which leads to less variable samples. To keep the variability constant, we consider the constant coefficient of variation over periods, i.e., $\sigma_t = m_t \times \text{CV}$. We consider $\text{CV} \in \{0.2, 0.4\}$ and $N \in \{100, 500\}$. The results are summarized in Table 3.2. Since the standard deviations are larger than those of the first experiment, the frequencies decrease and are less than 50 when $\text{CV} = 0.4$. The impact of large variability is critical to monotonicity of x_t^*, which leads to time inconsistency of the inventory problem.

Table 3.2 Frequencies of monotone non-decreasing order quantities x_t^* with constant coefficient of variation for 100 iterations

			Step size for 1-Wass				Step size for 2-Wass			
CV	b	N	2	5	8	10	2	5	8	10
0.2	1	100	0	2	6	12	0	2	6	12
	1	500	1	69	95	94	1	69	95	94
	3	100	0	0	6	24	0	0	6	24
	3	500	5	73	96	97	5	73	96	97
	9	100	0	3	10	13	0	3	10	13
	9	500	0	54	89	91	0	54	89	91
	19	100	0	0	3	6	0	0	3	6
	19	500	1	37	76	81	1	37	76	81
0.4	1	100	0	0	0	0	0	0	0	0
	1	500	0	2	19	32	0	2	19	32
	3	100	0	0	1	1	0	0	1	1
	3	500	0	18	27	46	0	18	27	46
	9	100	0	0	0	1	0	0	0	1
	9	500	0	10	20	34	0	10	20	34
	19	100	0	0	0	0	0	0	0	0
	19	500	0	4	23	24	0	4	23	24

When the mean of a demand process is increasing significantly, the sample size is large, and the variability is low, the optimal order quantities of the newsvendor policy are monotone non-decreasing in most cases. For this circumstance, the inventory problem tends to be weakly time consistent, which is a desirable situation for decision makers. The above experiments show that the sufficient condition we have derived is not limited and is likely to be satisfied in practice.

3.5.2 Conservativeness

One criticism of the distributionally robust solution is its conservativeness, that is, the average performance of the distributionally robust solution is quite poor in terms of optimal costs. Therefore, the previous researches try to find a less conservative approach while maintaining distributionally robust properties. One advantage of the Wasserstein distance approach is that its solutions are often less conservative than those of the moment-based approach. We will compare the objective function values of the Wasserstein model and the Scarf model [76].

The Scarf model considers the moment-based ambiguity set with known mean and standard deviation. In the data-driven setting, the decision maker does not know moment information exactly. Therefore, mean m and standard deviation s are estimated by the sample mean and sample standard deviation from historical data, respectively. To make calculations simple, we consider the weakly time consistent case based on the monotone condition. Hence, the experimental setting should satisfy both monotone conditions (see Remark 3.2 and [109]).

We assume the demand process is i.i.d. and the whole historical data are used to construct the ambiguity set and ambiguity sets are the same for all period, i.e.,

$\mathcal{M}_t = \mathcal{M}_1$ for all $t = 1, \ldots, T$. We set $c_t = c, h_t = h$ and $b_t = b + 0.1(t-1)$ to make critical ratio monotone increasing. In this case, the newsvendor policies of the Wasserstein model and the Scarf model are optimal for the respective problem. We refer to Section 3.3 for the newsvendor policy of the Wasserstein model. When $\tau_t = \frac{b_t - h_t - 2(c_t - c_{t+1})}{h_t + b_t}$ for $t = 1, \ldots, T$, the optimal order quantity for the newsvendor policy of the Scarf model is given by $x_t^{\mathrm{SC}} = m + s\frac{\tau_t}{\sqrt{1-\tau_t^2}}$ and the optimal t-period cost $\vartheta_t^{\mathrm{SC}} = c_t m + \sqrt{(h_t + c_t - c_{t+1})(b_t - c_t + c_{t+1})}s$. The total cost is expressed as $\mathrm{TC}_{\mathrm{Scarf}} = \sum_{t=1}^{T} \vartheta_t^{\mathrm{SC}} - c_1 y_1$ where y_1 is initial inventory level. We will compare the optimal total cost of the newsvendor policy for each model.

We consider the planning horizon $T = 20$ and Wasserstein orders $p = 1$ and $p = 2$ with the Wasserstein radius $\theta = 0.1$ similar to Section 3.5.1. We set $c = 1, h = 1$ and $b \in \{3, 9, 19\}$ to satisfy the monotone condition for both models and let the initial inventory level $y_1 = 0$. We generate $N \in \{10, 50, 500\}$ samples from normal distributions with mean $m = 100$ and standard deviation $s \in \{20, 40\}$ to set resulting $\mathrm{CV} \in \{0.2, 0.4\}$. Three different sample sizes represent small, medium, and big data, respectively. We conduct 100 iterations and derive statistics such as average and maximum of the objective function values. The results are summarized in Table 3.3. 1-Wass and 2-Wass represent the Wasserstein models with Wasserstein orders $p = 1$ and $p = 2$, respectively. $\mathrm{TC}_{\mathrm{avg}}$ and $\mathrm{TC}_{\mathrm{max}}$ denote average and maximum of optimal objective function values of each model, respectively. The performance gap in Table 3.3 is defined as $(\mathrm{TC}_{\mathrm{Scarf}} - \mathrm{TC}_{\mathrm{Wass}})/\mathrm{TC}_{\mathrm{Scarf}} \times 100$ for each iteration.

There exist significant performance gaps between two models in terms of the objective function value. The difference in the ambiguity sets leads to the performance gap, that is, the Wasserstein ambiguity set is less conservative than the Scarf

Table 3.3 Objective function values of Wasserstein and Scarf models

			1-Wass		2-Wass		Scarf		Gap_1(%)		Gap_2(%)	
CV	b	N	TC_{avg}	TC_{max}	TC_{avg}	TC_{max}	TC_{avg}	TC_{max}	avg	max	avg	max
0.2	3	10	2568.90	2686.48	2563.44	2681.02	2794.28	2919.26	8.06	9.73	8.26	9.92
	3	50	2567.49	2620.27	2562.03	2614.81	2794.53	2853.46	8.12	8.96	8.32	9.16
	3	500	2566.62	2582.02	2561.17	2576.56	2794.21	2810.74	8.14	8.36	8.34	8.56
	9	10	2745.87	2907.59	2730.59	2892.30	3263.52	3425.26	15.85	18.68	16.32	19.14
	9	50	2743.06	2805.85	2727.78	2790.56	3263.57	3344.06	15.95	17.48	16.42	17.94
	9	500	2741.88	2759.31	2726.60	2744.03	3263.05	3287.27	15.97	16.30	16.44	16.77
	19	10	2874.29	3074.87	2841.54	3042.12	3789.71	3992.67	24.14	27.77	25.01	28.61
	19	50	2877.00	2954.18	2844.26	2921.43	3789.54	3894.21	24.08	26.25	24.94	27.11
	19	500	2876.19	2894.68	2843.44	2861.93	3788.79	3822.27	24.09	24.59	24.95	25.45
0.4	3	10	3127.89	3363.05	3122.43	3357.60	3588.55	3838.52	12.83	15.29	12.98	15.44
	3	50	3125.07	3230.63	3119.61	3225.17	3589.06	3706.91	12.93	14.17	13.08	14.32
	3	500	3123.35	3154.14	3117.89	3148.68	3588.42	3621.48	12.96	13.28	13.11	13.43
	9	10	3469.84	3793.28	3454.56	3777.99	4527.04	4850.53	23.34	27.20	23.68	27.53
	9	50	3464.23	3589.80	3448.95	3574.51	4527.15	4688.13	23.48	25.56	23.81	25.89
	9	500	3461.86	3496.73	3446.58	3481.44	4526.10	4574.53	23.51	23.94	23.85	24.28
	19	10	3706.67	4107.83	3673.93	4075.09	5579.42	5985.35	33.55	38.18	34.14	38.75
	19	50	3712.11	3866.46	3679.36	3833.71	5579.08	5788.42	33.46	36.24	34.05	36.82
	19	500	3710.47	3747.45	3677.73	3714.71	5577.59	5644.53	33.48	34.05	34.06	34.63

ambiguity set. Moreover, the construction of the Scarf model is based on the estimation of moments, which leads to instability when the sample size is small. Both models become stable and the difference between average and maximum of the total costs decrease as the sample size grows. However, the Scarf model is significantly conservative and the performance gaps become noticeable when the critical ratio and coefficient of variation are high. The performance results show that the Wasserstein model is less conservative and more applicable in practice especially for the data-driven setting. One interesting observation is that the Wasserstein ambiguity set with the Wasserstein order $p = 2$ is less conservative, because the 2-Wasserstein distance is stronger than 1-Wasserstein distance, i.e., $W_1(\mu, \nu) \leq W_2(\mu, \nu)$. Hence, the 2-Wasserstein ambiguity set is smaller than the 1-Wasserstein ambiguity set, which leads to less conservative costs.

3.5.3 Out-of-sample Performance Guarantee

An important property of the Wasserstein DRO is the out-of-sample performance guarantee. The out-of-sample performance of the distributionally robust solution represents the expected cost of the solution with the true distribution, i.e., $\mathbb{E}_{\mu}[Z(\hat{x}_N)]$ where μ is the true distribution and $\hat{x}_N$ is the distributionally robust solution with N samples. However, the true distribution μ is unknown and the out-of-sample performance is impossible to be computed exactly. Hence, we generate test samples and approximate the out-of-sample performance through simulations. For the Wasserstein solution $\hat{x}_N$, the probability that the out-of-sample performance is less than the distributionally robust total cost is guaranteed with certain probability (see inequality (3.24)). We denote this probability as the reliability of $\hat{x}_N$. In the following simulation experiment, we analyze the reliability in terms of the Wasserstein radius θ.

The experimental setting is similar to Section 3.5.2. The same parameters are used to maintain monotone conditions and generate samples from a normal distribution with mean $m = 100$ and standard deviation $s = 20$. In this case, optimal order quantities of the newsvendor policy are monotone, and the base-stock policy with the order quantities is optimal. We generate another 100 independent sample paths to compute the out-of-sample performance. More specifically, we calculate the out-of-sample performance by the simulated cost of the base-stock policy with each sample path and average over the samples. We compare the simulated cost with the distributionally robust and compute the empirical probability that the distributionally robust optimal cost is a certificate of the out-of-sample performance. We consider various values of radii $\theta \in \{10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}, 0.2, 0.5, 1, 10\}$ to analyze the impact of the Wasserstein radius. Figure 3.2 shows the reliability as a function of the Wasserstein radius θ for the Wasserstein order $p = 1$ and $p = 2$. The reliability is increasing in the Wasserstein radius θ and grows rapidly to 1 when θ increases from 0.1 to 1, which implies that the critical radius may be contained in the interval. As the radius increases, the Wasserstein ambiguity set may contain the unknown true distribution, i.e., $\mu \in \mathcal{M}$. Hence, the distributionally robust optimal cost may be the certificate of the out-of-sample performance and the reliability is increasing. When the sample size is large, the reliability becomes 1 with the smaller radius, which is consistent with measure concentration results [31].

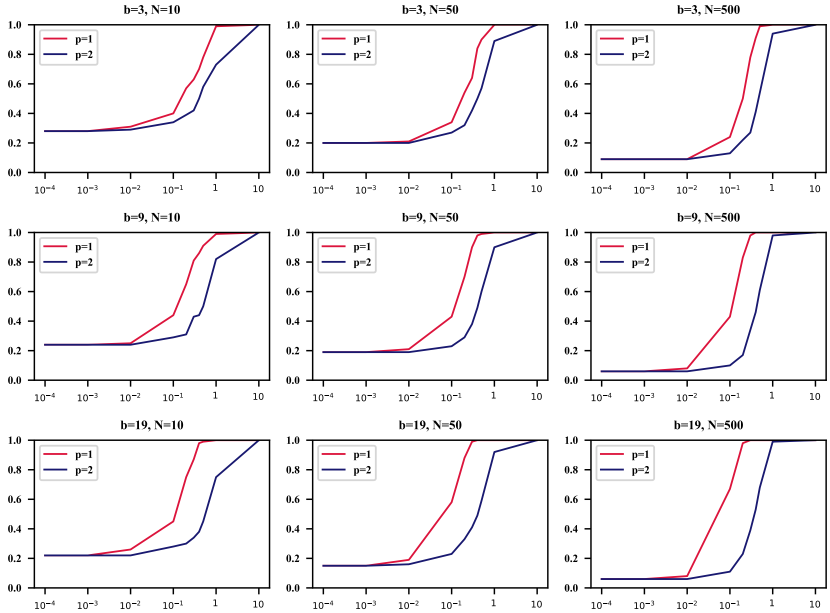

Figure 3.2 Reliability as a function of the Wasserstein radius θ for 100 iterations

3.5.4 Convergence Property

One desirable property of the data-driven solution is the convergence property, i.e., the optimal objective function value and solutions of the Wasserstein DRO converges to those of the risk-neutral formulation with the known true distribution as the sample size N goes to ∞. The decision maker hopes that the Wasserstein ambiguity set goes close to the true distribution as N increases. If we choose the Wasserstein radius θ_N properly depending on the sample size N [31], then the Wasserstein ambiguity set shrinks to the true distribution and the distributionally robust optimal costs converges to the true optimal costs as $N \to \infty$.

In the data-driven setting, the decision maker can update the Wasserstein ambiguity set and optimal decisions according to the realized demand. The decision maker derives optimal decisions based on N samples and $N+1$th demand is realized after the decision is made. Then, the next period decision is updated with $N+1$ samples. We conduct numerical experiments by updating decisions until $N = 10^4$ to investigate the convergence property. We consider the true distribution as a binomial distribution $Bin(n,p)$ with $n = 100, p = 0.5$ to compute the optimal costs of the risk-neutral formulation using dynamic programming. Using the discrete and bounded distribution, the dynamic programming formulation can be computed efficiently. The other parameter setting is similar to that of Section 3.5.2. We choose the Wasserstein radius $\theta_N = \frac{10}{\sqrt{N}}$ such that $\theta_N \to 0$ as $N \to \infty$ and the ambiguity set shrinks to the true distribution (see Section 7.2.4 of [30]). The updated optimal objective function values as the sample size N increases are shown in Figure 3.3. The black horizontal line represents the true optimal cost when the underlying true distribution is known. Figure 3.3 shows that the optimal costs of Wasserstein

DRO converge rapidly to the black horizontal line as N increases. When the Wasserstein order $p = 2$, the optimal costs converge more quickly because the Wasserstein distance is monotone and the Wasserstein ambiguity set with the higher order is smaller. This convergence result shows the desirable property of the Wasserstein distance approach.

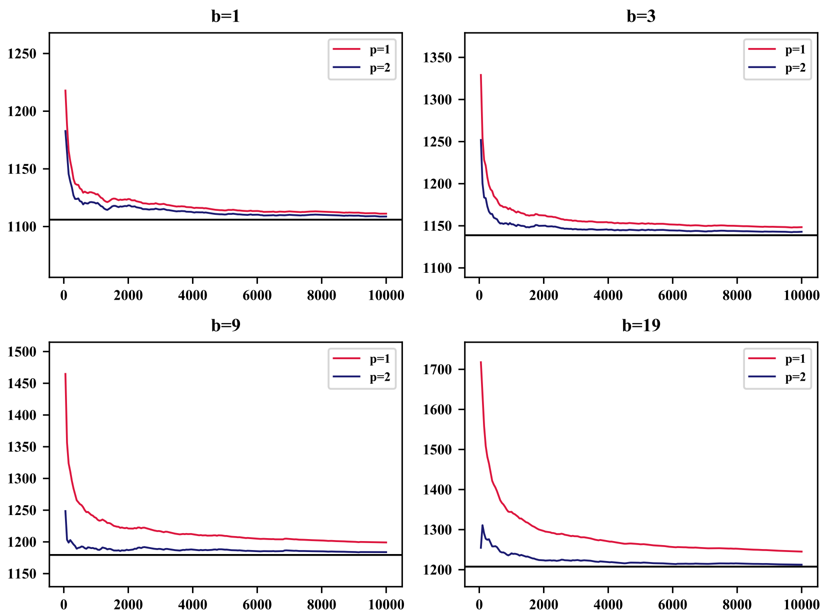

Figure 3.3 Convergence of optimal objective function values to true optimal costs as sample size (N) increases

3.6 Summary

In this chapter, we studied the data-driven distributionally robust inventory model with a Wasserstein ambiguity set centered at the empirical distribution. We adopted the policy-centered definition of time consistency from [109] and discussed time consistency of the inventory problem. We showed that the Wasserstein ambiguity set is non-rectangular, and the Wasserstein inventory model is time inconsistent in general. We derived the sufficient condition for weak time consistency based on the monotone non-decreasing optimal base-stock levels computed by closed-form solutions of the Wasserstein newsvendor model. We also investigated the condition that base-stock levels are monotone non-decreasing when the demand process is i.i.d. Further details about the dynamic programming formulation were analyzed such as computation of optimal base-stock levels, optimality of an (s, S) policy with non-zero fixed order cost, and desirable properties of Wasserstein DRO. We conducted numerical experiments to show that the derived sufficient condition is likely to be satisfied in practice when the demand process increases. We also compared the conservativeness of Wasserstein solutions to that of moment-based solutions to show less conservative optimal costs. The desirable properties of the Wasserstein inventory model such as the out-of-sample performance guarantee and convergence property were validated by numerical experiments.

Our research is based on the assumption of stagewise independent demand process. Time consistency analysis can be extended to the inventory model with correlated demand process. For this direction, the price of correlations should be considered based on the extended representation of the Wasserstein ambiguity set for demand process [2].

Chapter 4

Empty Container Repositioning with Foldable Containers

4.1 Problem Description and Literature Review

Since the 1970s, the volume of maritime transport has increased sharply because of an increase in worldwide trade. 17.1% of world seaborne trade is transported in the form of containers in 2017 [99]. Because of the reusable nature of containerization, the container is returned to the port in an empty state after being used for transporting goods. The time delay between container use and return creates problematic and inconsistent demand and supply for the container. In addition, the extreme imbalance in intercontinental container-shipping volume increases the demand-supply mismatch of empty containers, which contributes to a shortage of empty containers in export dominant ports and a surplus of empty containers in import dominant ports (Figure 4.1). To satisfy demand for empty containers, it is necessary to reposition empty containers from surplus ports to deficit ports. On average, 20% of total container movements by ocean transportation are empty. Repositioning of empty containers is non-value added transportation with enormous transportation costs and nonprofitable consumption of vessel capacity. Each time an empty container is repositioned, an opportunity cost is incurred for shipping one loaded container, which

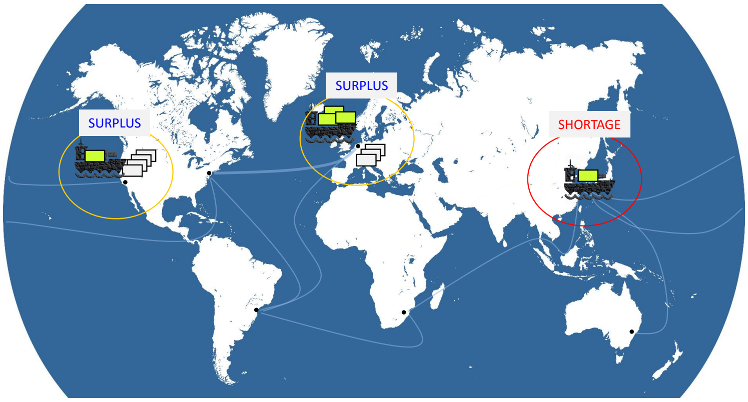

Figure 4.1 Empty containers imbalance between continents

cuts the shipping company's profitability. Therefore, it is important for the shipping company to reposition containers efficiently and effectively. The empty container repositioning (ECR) problem is to determine an optimal repositioning schedule to satisfy demand for empty containers. The key decision is when and how many empty containers are repositioned to a destined port.

There are many efforts to reduce the cost of repositioning empty containers, one of which is a foldable container. Foldable containers can be transported in a package such that a few folded containers occupy the same volume as one standard container. Despite high purchase costs and the additional folding and unfolding costs of foldable containers, foldable containers have advantages in the reduction of repositioning costs and saving in terms of storage space and vessel capacity. There are two leading companies developing foldable containers: Holland Container Innovations (HCI) of the Netherlands and the Korea Railroad Research Institute. HCI presents a foldable container called 4FOLD and received ISO certification in 2013 [42]. 4FOLD has been used by 15 shipping companies involving APL, the world's third-largest shipping company, with 20% savings of total operation cost on average. HCI shows that folding and unfolding of 4FOLD can be done with standard lifting equipment by a two-person team less than four minutes, which makes the adoption of foldable containers affordable. One successful example of adoption of foldable containers is the Shanghai-Los Angeles-Chicago route served by APL [41]. The exports in Chicago are much less than imports, which incurs huge amount of empty repositioning. APL adopts 4FOLD containers in this route and folded empty container are packed in Chicago and transported to Shanghai via Los Angeles. APL saves approximately 20% of the total operating costs with the reduction coming from hinterland transport and handling costs at the terminal.

The introduction of foldable containers affects the operational-level planning decisions at shipping companies, which makes the decision process very complex. The flow of loaded containers is a source of empty containers at the destined port, and the vessel capacity is shared by loaded and empty containers. It interconnects two different decisions: the transportation quantity to satisfy demand and the number of empty containers to reposition. When we only consider standard containers to satisfy the demand, the amount of loaded containers is equal to the demand. With the consideration of using foldable containers in operational decisions, the shipping company has to decide which type (standard or foldable) of containers and quantities of containers to satisfy the demand. The quantity and ratio of containers transported in one period affect the number of returned empty containers in the future period. Therefore, existing studies cannot be applied to the integrated model of standard and foldable containers.

Furthermore, in the competitive shipping industry, operational-level planning requires decision making under uncertainty. Although, the shipping company knows the exact demand in the long-term contracts and establishes plan based on demand information, decision makers face many short-term uncertainties, such as weather, port congestion, and demand fluctuation, during implementation. Moreover, uncertainties may prevent planned decisions from being implemented, which may lead to suboptimal decisions. Uncertainties can lead to serious operational failures in the shipping industry. Among the various uncertainties, the uncertainty of customer demand is the most influential; therefore, we consider the uncertainty of demand in this study.

A famous approach for dynamic decision making under uncertainty is multistage stochastic programming in which the uncertainty is characterized by a known probability distribution of parameters. Transportation and repositioning decisions in the ECR problem are considered wait-and-see decisions. Hence, the ECR problem can be modeled with a multistage stochastic programming framework. However, data estimation, such as demand forecasting with historical data, is difficult in practice. It is impossible to achieve complete knowledge about distributions of uncertainties. Moreover, in general, multistage stochastic program is computationally intractable [86]; therefore, we utilize a robust optimization framework for which only limited information is required. Adjustable robust optimization, proposed by [11], enables dynamic decision making under uncertainty in a robust optimization framework.

In this section, we consider the ECR problem with the adoption of foldable containers. The shipping company decides the type and quantity of empty containers in terms of two different decisions, transporting and repositioning. We focus on the operational planning of ocean transportation between multiple ports over the planning horizon. We also consider demand uncertainty, which leads to the multistage stochastic programming formulation. To tackle the intractability of the multistage formulation, we adopt the concept of adjustable robust optimization. Then, we show that the robust formulation gives a tractable approximation of the stochastic programming formulation.

The empty container repositioning (ECR) problem has attracted considerable attention in academia. Many researchers have considered various situations and proposed solution methodologies. Wang and Meng [103] provided a recent review of container liner fleet deployment, and Lee and Song [51] conducted an extensive review of ocean container transport. Song and Dong [95] discussed the main causes of empty container repositioning and solutions to the ECR problems. Kuzmicz and Pesch [49] addressed various aspects and solutions of ECR problems in the context of Eurasian transportation.

Several researchers have considered decision making under uncertainty in ECR problems through stochastic programming with recourse, inventory control-based policies, and robust optimization. Crainic et al. [25] proposed dynamic deterministic formulations for the empty container allocation problem and extended it to a two-stage stochastic programming formulation under the uncertainties of demand and supply. Cheung and Chen [24] proposed a two-stage stochastic network formulation of the ECR problem under uncertainties of demand, supply, and capacity. They utilized the stochastic quasi-gradient method and an approximation procedure to obtain solutions. Song [94] provided an optimal policy for empty container repositioning with uncertain demand that is similar to the optimal policy for inventory control. The structures of the optimal policy were characterized using the Markov decision process. Li et al. [55] derived the optimal threshold-type policies called

(U, D) policy in a single port case with demand uncertainty and they extended to multi-port case Li et al. [54]. Using the convexity of the cost function, they proposed a heuristic to obtain policies. Lam et al. [50] considered a dynamic stochastic model of the container allocation problem and proposed an approximate dynamic programming approach. Di Francesco et al. [27] proposed a stochastic programming model with uncertain data for empty container repositioning and solved the model using multi-scenario optimization. Erera et al. [29] modeled the empty repositioning problem using a robust optimization framework. They considered interval uncertainty of forecast values and proposed the concept of a recoverable plan similar to the concept of an adjustable robust counterpart. They showed that the problem modeled using the recoverable plan is polynomially solvable. Long et al. [56] proposed a two-stage stochastic programming model for empty container repositioning and solved the program with the Sample Average Approximation (SAA). They utilized the scenario aggregation to handle an extremely large number of scenarios. Shu et al. [89] proposed a two-stage robust optimization model considering both loaded and empty containers. They discussed the complexities of the formulations based on an l_p-norm uncertainty set. Except [29] and [89], most of the previous studies assumed the full distribtuional knowledge of uncertainties which is limited in practice. [29] and [89] utilized the robust optimization framework to tackle this difficulty, however, their models were limited to two-stage decisions.

The foldable container is a newly commercialized technology and the studies of ECR with the use of foldable containers is recently emerging. Konings and Thijs [48] analyzed the economic effects of introducing foldable containers into ocean transport systems, and they discussed the technical and logistical conditions for the successful use of foldable containers. Konings [47] discussed an economical analysis on the adoption of foldable containers considering relevant costs. Shintani et al. [88] discussed the economic impact of using foldable containers in hinterland repositioning of empty containers. They analyzed several strategies of hinterland transportation in which foldable containers were used. Myung [64] extended the results of [88] by offering efficient solution methods and obtained analytical solutions using a network formulation. Moon et al. [61] proposed mathematical models considering foldable and standard containers in maritime transport. They developed heuristic algorithms to solve the proposed models. Then, Myung and Moon [65] proved that the model in [61] can be reduced to a network flow model that can be solved in polynomial time. Moon and Hong [63] developed a mathematical model with standard and foldable containers and proposed a linear programming-based genetic algorithm and a hybrid genetic algorithm to solve the model and they obtained near-optimal solutions. Wang et al. [102] considered the ship type decision problem with the use of foldable containers in empty container repositioning. They proposed a network flow model and addressed an exact algorithm based on a revised network simplex algorithm. To the best of our knowledge, there were no studies considering uncertainty into the model of foldable containers.

In this research, we utilize the concept of adjustable robust optimization to obtain a tractable approximation for a multistage stochastic programming formulation with demand uncertainty. Ben-Tal et al. [11] proposed the concept of adjustable robust counterparts for which decisions can be adjusted dynamically as uncertainty is realized over time. However, they showed that the adjustable robust counterpart is NP-hard, so they proposed the concept of an affinely adjustable robust counterpart (AARC) where adjustable decisions are restricted to affine functions of uncertainty. Then, Ben-Tal et al. [10] applied AARC to a supply chain problem named the retailer-supplier flexible commitment problem. Chen and Sim [20] proposed a tractable deterministic approximation for the goal-driven stochastic optimization model using a piecewise linear decision rule. For this approximation, they developed upper bounds for the expectation of positive parts, which are shown in the objective function of the model. See and Sim [78] proposed the use of [20]' s upper bounds to deal with a multiperiod inventory-management problem. They developed a piecewise linear decision rule named truncated linear decision rule, which extends the result of the linear decision rule.

In summary, contributions of this chapter are threefold. First, we propose a mathematical model of the ECR problem considering the use of foldable containers under uncertainty. To the best of our knowledge, it is the first study to address this topic. Second, we propose the tractable robust formulations with limited information on demand distribution, because it is very difficult to estimate demand distribution precisely with historical data. The robust formulations are used to approximate the multistage stochastic programming formulation. Third, we show the cost-saving and storage-saving effects of using foldable containers through the practical-scaled numerical experiments.

4.2 Multistage Stochastic Programming Formulation

In this section, we define the ECR problem concretely based on the cycle of container flows. We present assumptions of the problem and notations, and propose the deterministic formulation. Then we regard demand as a random variable and propose a multistage stochastic programming formulation to incorporate uncertainties. However, the proposed stochastic formulation is computationally intractable in general. We adopt the distributionally robust approach to approximate the multistage stochastic programming formulation in Section 4.3.

4.2.1 Cycle of Container Flows

To understand the ECR problem, container flow must be understood. A consignor sends cargo to the consignee by ocean transport, which is referred to as *demand* in the ECR literature. To meet *demand*, the shipping company sends empty containers to the consignor. The consignor fills the empty containers with cargo and sends these loaded containers to the port. The shipping company transports the containers via an ocean-transport vessel to the destination port where the consignee receives them. The consignee takes the newly arrived cargo out of the containers and sends the emptied containers back to the depot of the shipping company. The empty containers, upon return to the port, are referenced as the *supply* in the ECR problem.

Container flows of this problem are shown in Figure 4.2.

1. At the beginning of period t, the customer demand from port i to port j occurs. We aggregate the consignors located near port i and denote them as demand occurred at port i. We aggregate consignees similarly.

2. The shipping company sends empty containers from the depot at port i to the consignor's site to satisfy the demand. The empty containers are packed at the consignor's site and returned to port i. It takes v_i periods until empty containers are sent to the consignor's site and the loaded containers are returned to port i.

3. The shipping company transports loaded containers to port j via the vessel with given transportation time τ_{ij}.

4. Cargo is delivered to the consignee and emptied at the consignee's site. Then, the emptied containers are returned to the depot at port j. It takes v_j periods until the loaded containers are transported to the consignee's site and the emptied containers are returned to the port.

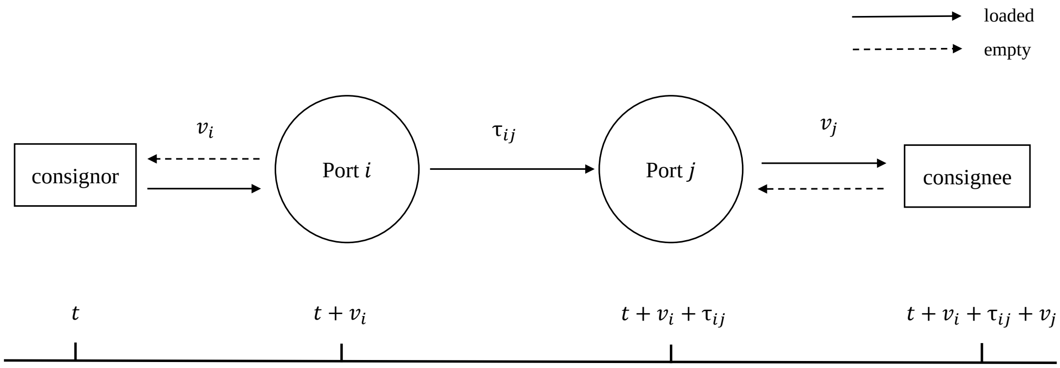

Figure 4.2 Cycle of container flows

Because of the container flow and reusable property of containers, supplied empty containers are stacked at the import dominant port. To meet the demand at the export dominant port, the shipping company has to reposition empty containers. Empty repositioning consumes the vessel capacity which would be used for loaded containers. Therefore, effective and efficient repositioning plan is crucial to the profit of the shipping company.

4.2.2 Assumptions and Nomenclature

In the ECR problem, the shipping company considers both transporting loaded containers and repositioning of empty containers over a planning horizon. Because empty repositioning incurs costs and consumes vessel capacity, foldable containers are used to reduce repositioning costs. Therefore, the shipping company decides the type and quantity of containers to satisfy the demand. According to the transportation decision, the shipping company also decides the type and quantity of repositioned containers to mitigate the trade imbalance. The objective of this problem is to minimize the total operating cost over the planning horizon which consists of transportation, repositioning, holding, penalty, folding, and unfolding costs. One of the challenging decisions is when and how many foldable containers are used, which makes the problem complex.

The assumptions of this problem are as follows:

- The ECR is undertaken at multiple ports on a finite discrete horizon of T periods. The vessel route is not considered, and containers can be repositioned to any port during any period according to given vessel capacity.

- The vessel capacity K_{ijt} from port i to port j in period t, which is shared by loaded and empty containers, is given. The vessel capacity can represent the vessel schedule predetermined by the shipping company.

- The transportation lead time from port i to port j and the inland transportation time (devanning time) at port i are given.

- Demand can be satisfied by both standard and foldable containers. When foldable containers stored at ports are used to satisfy demand, the unfolding operation must be done before transported to the consignor's site.
- Supplied empty containers and repositioned empty containers are two sources of empty containers.
- Unsatisfied demand is satisfied with a short-term lease and incurs a penalty cost.
- Folding and unfolding of foldable containers can be executed only in ports.
- Foldable containers are repositioned in a folded state to occupy less of the vessel capacity than standard containers do.
- Supplied foldable containers from customers are delivered in the unfolded state and used to satisfy demand. Excess foldable containers are stored at a port after being folded.
- The returned container after being emptied is the only source of container supply.

The vessel capacity represents the number of loaded and empty containers in twenty-foot equivalent unit (TEU) that can be transported from port i to port j in period t. The capacity is given by the shipping company in advance according to the fleet schedule. For example, if $K_{ijt} = 0$, no vessel is available vessel in period t to transport containers from port i to port j. Therefore, the vessel capacity can characterize the vessel schedule determined by the shipping company, which leads to the generalization of the first assumption. We assume the containers can be transported to any port during any period, but we can consider the vessel schedule with the vessel capacity.

With the introduction of foldable containers, additional facility and manpower are needed. Because customers may have a negative reaction to additional investments at their sites, folding and unfolding operations are limited to ports. Therefore, customers receive empty foldable containers in an unfolded state, which makes the customers indifferent about the choice of using standard or foldable containers. Although the customer may be concerned about the strength of the foldable container, for this study, the strength of the foldable container is assumed to be the same as a standard one. Hence, the demand can be satisfied by either standard or foldable containers.

According to the assumptions, we propose model formulations of the ECR problem considered. In the next section, we present a deterministic and a multistage stochastic programming formulation. Then, we introduce a robust formulation using a linear decision rule and show that the robust formulation is a tractable approximation of the multistage stochastic formulation of the ECR problem.

The notations for the parameters are as follows:

P	ports
T	periods
C_{ij}^S	unit transportation cost of a standard container from port i to port j
C_{ij}^F	unit transportation cost of a foldable container from port i to port j
R_{ij}^S	unit repositioning cost of a standard container from port i to port j

R_{ij}^F unit repositioning cost of a foldable container from port i to port j

H_i^S holding cost of a standard container per unit per period at port i

H_i^F holding cost of a foldable container per unit per period at port i

P_i^S penalty cost of a standard container per unit per period at port i

P_i^F penalty cost of a foldable container per unit per period at port i

K_{ijt} vessel capacity (TEUs) from port i to port j in period t

N number of foldable containers used to build one folded pack

FC_i unit folding cost of a foldable container at port i

UC_i unit unfolding cost of a foldable container at port i

τ_{ij} transportation time from port i to port j

ν_i inland transportation time (or devanning time) at port i

D_{ijt} demand for transporting containers from port i to port j in period t

Decision variables used in this model are as follows:

r_{ijt}^S repositioning quantity of standard containers from port i to port j in period t

r_{ijt}^F repositioning quantity of foldable containers from port i to port j in period t

x_{ijt}^S number of standard containers used to satisfy demand from port i to port j in period t

x_{ijt}^F number of foldable containers used to satisfy demand from port i to port j in period t

z_{it}^S inventory level of standard containers at port i at the end of period t

z_{it}^F inventory level of foldable containers at port i at the end of period t

The total operating cost consists of consists of transportation, repositioning, holding, penalty, folding, and unfolding costs. The transportation costs consist of expenses incurred during container flow. In other words, costs which are directly involved to the delivery between consignors and consignees: empty container movement to the consignors, ocean transportation by the vessel, and delivery and return from consignees. Recent information technology enables the shipping company to valuate the exact transportation cost per unit delivery. The repositioning cost is the handling cost of transporting empty containers to mitigate the trade imbalance. The holding cost is incurred when empty containers are stored at the depot in ports. The penalty cost is related to the short-term leasing cost incurred to meet unsatisfied demand. Folding and unfolding costs are incurred when folding and unfolding operations are executed at ports.

4.2.3 Deterministic Formulation

First, we investigate the balance equation at port i in period t using notations defined above. Then we propose the deterministic formulation of the ECR problem with foldable containers and discuss the details of the formulation.

Figure 4.3 represents inflows and outflows at port i in period t in a time-space expanded network of the ECR model. According to the container flow and repositioning operation, two different sources of empty containers, $x_{ji,t-\nu_j-\tau_{ji}-\nu_i}$ and $r_{ji,t-\tau_{ji}}$, are transported from port j.

Using the above notation, we explain the balance equation of port i in period t. A balance equation is presented in Figure 4.4. Three types of inflows were considered: repositioning quantities from other ports, number of supplied containers after use,

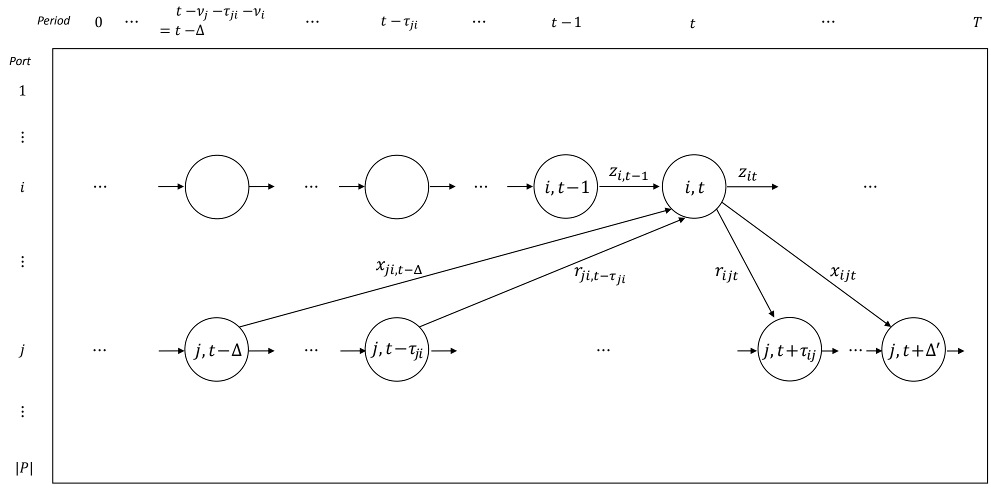

Figure 4.3 Inflows and ouflows at port i in period t in a time-space expanded network of the ECR model

and inventory from the last period. It takes τ_{ji} periods to reposition empty containers from port j to port i. It takes $v_j+\tau_{ji}+v_i$ periods to finish one cycle of container flows such that supplied containers are returned after one cycle. Three types of outflows were considered: repositioning quantities to other ports, number of empty containers used to satisfy customer demand, and inventory amount. Figure 4.4 presents the balance equation of standard containers, which is the same for foldable containers.

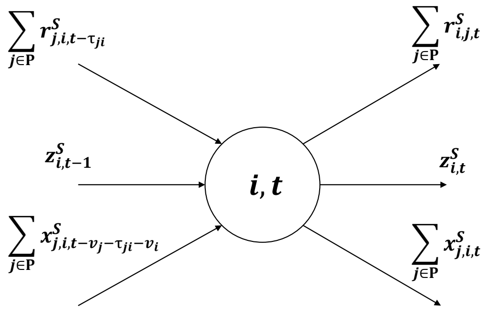

Figure 4.4 Balance equation of standard containers

First, we consider the deterministic demand. The model we developed is similar to that of [61] and [98]. An explanation of this deterministic formulation helps in understanding the stochastic model. Let $\mathrm{RC_{DET}}$ denote total repositioning and transportation costs and $\mathrm{HC_{DET}}$ denote total holding and penalty costs and $\mathrm{FC_{DET}}$ denote total folding and unfolding costs.

$$\mathrm{RC_{DET}} = \sum_{t=1}^{T}\sum_{i\in P}\sum_{j\in P}\left(R_{ij}^S r_{ijt}^S + C_{ij}^S x_{ijt}^S + R_{ij}^F r_{ijt}^F + C_{ij}^F x_{ijt}^F\right)$$
$$\mathrm{HC_{DET}} = \sum_{t=1}^{T}\sum_{i\in P}\left(H_i^S (z_{it}^S)^+ + H_i^F (z_{it}^F)^+ + P_i^S (z_{it}^S)^- + P_i^F (z_{it}^F)^-\right)$$
$$\begin{aligned}\mathrm{FC_{DET}} = \sum_{t=1}^{T}\sum_{i\in P}\Big(&FC_i\big(\sum_{j\in P}(x_{ji,t-\nu_i-\nu_j-\tau_{ji}}^F - x_{ijt}^F)\big)^+ \\ &+ UC_i\big(\sum_{j\in P}(x_{ijt}^F - x_{ji,t-\nu_i-\nu_j-\tau_{ji}}^F)\big)^+\Big)\end{aligned}$$

The deterministic formulation is as follows:

$$\mathrm{TC_{DET}} = \tag{4.1}$$
$$\min \quad \mathrm{RC_{DET}} + \mathrm{HC_{DET}} + \mathrm{FC_{DET}} \tag{4.2}$$
$$\text{s.t.} \quad z_{it}^S = z_{i,t-1}^S + \sum_{j\in P} r_{ji,t-\tau_{ji}}^S - \sum_{j\in P} r_{ijt}^S + \sum_{j\in P} x_{ji,t-\nu_i-\nu_j-\tau_{ji}}^S - \sum_{j\in P} x_{ijt}^S, \quad \forall i \in P, t = 1,\ldots,T \tag{4.3}$$
$$z_{it}^F = z_{i,t-1}^F + \sum_{j\in P} r_{ji,t-\tau_{ji}}^F - \sum_{j\in P} r_{ijt}^F + \sum_{j\in P} x_{ji,t-\nu_i-\nu_j-\tau_{ji}}^F - \sum_{j\in P} x_{ijt}^F, \quad \forall i \in P, t = 1,\ldots,T \tag{4.4}$$
$$x_{ijt}^S + x_{ijt}^F = D_{ijt}, \qquad \forall i,j \in P, t = 1,\ldots,T \tag{4.5}$$
$$r_{ijt}^S + \frac{1}{N} r_{ijt}^F + x_{ij,t-\nu_i}^S + x_{ij,t-\nu_i}^F \le K_{ijt}, \qquad \forall i,j \in P, t = 1,\ldots,T \tag{4.6}$$

$$r^S_{ijt}, r^F_{ijt} \geq 0, x^S_{ijt}, x^F_{ijt} \geq 0, \qquad \forall i, j \in P, t = 1, \ldots, T \qquad (4.7)$$

The objective function represents total operating cost, including repositioning, transportation, inventory holding, penalty, and folding/unfolding costs. Constraints (4.3) and (4.4) represent balance equations for standard and foldable containers, respectively. Constraint (4.5) represents that the demand is satisfied with standard and foldable containers from the depot at the port. Constraint (4.6) is a capacity constraint for the vessel. It shows that when repositioning foldable containers, $1/N$ unit of capacity is used. Empty foldable containers consume less capacity, which leads to more available capacity for loaded containers that is a value-added activity for the shipping company. Constraint (4.7) is a non-negativity constraint.

Although repositioning and transportation decisions are based on the number of containers, the above formulation is a linear program. Because hundreds or thousands of containers are usually used, solutions that are rounded up are very close to the optimal solution. Moreover, the formulation does not have any binary variables. In many cases, rounding binary variables makes the optimal solution of a linear program highly suboptimal. Fortunately, the ECR formulation does not contain any binary variables and the quantity of containers is over hundreds, which makes the linear program formulation of the ECR problem reasonable.

4.2.4 Multistage Stochastic Programming Formulation

We regard demand as a random variable to incorporate uncertainties into the model. Multistage stochastic programming formulation can be proposed with random demand. We denote stochastic demand as $\tilde{d}_{ijt}$ and assume that stochastic demand,

$\tilde{d}_{ijt}$, is realized dynamically over the planning horizon. At the beginning of period, $\tilde{d}_{ijt}$ is realized, and then the shipping company makes transportation and repositioning decisions based on the demand realization and past information. Let $\mathrm{RC}_{\mathrm{STOC}}$ denote total expected repositioning and transportation costs and $\mathrm{HC}_{\mathrm{STOC}}$ denote total expected holding and penalty costs and $\mathrm{FC}_{\mathrm{STOC}}$ denote total expected folding and unfolding costs.

$$
\begin{aligned}
\mathrm{RC}_{\mathrm{STOC}} &= \mathbb{E}\big[\sum_{t=1}^{T}\sum_{i\in P}\sum_{j\in P}\big(R^S_{ij}r^S_{ijt}(\omega)+C^S_{ij}x^S_{ijt}(\omega)+R^F_{ij}r^F_{ijt}(\omega)+C^F_{ij}x^F_{ijt}(\omega)\big)\big]\\
\mathrm{HC}_{\mathrm{STOC}} &= \mathbb{E}\big[\sum_{t=1}^{T}\sum_{i\in P}\big(H^S_i(z^S_{it}(\omega))^+ + H^F_i(z^F_{it}(\omega))^+ + P^S_i(z^S_{it}(\omega))^- + P^F_i(z^F_{it}(\omega))^-\big)\big]\\
\mathrm{FC}_{\mathrm{STOC}} &= \mathbb{E}\big[\sum_{t=1}^{T}\sum_{i\in P}\big(FC_i\big(\sum_{j\in P}(x^F_{ji,t-\nu_i-\nu_j-\tau_{ji}}(\omega)-x^F_{ijt}(\omega))\big)^+\\
&\qquad + UC_i\big(\sum_{j\in P}(x^F_{ijt}(\omega)-x^F_{ji,t-\nu_i-\nu_j-\tau_{ji}}(\omega))\big)^+\big)\big]
\end{aligned}
$$

The multistage stochastic programming formulation is as follows:

$$
\mathrm{TC}_{\mathrm{STOC}} = \tag{4.8}
$$

$$
\min \quad \mathrm{RC}_{\mathrm{STOC}} + \mathrm{HC}_{\mathrm{STOC}} + \mathrm{FC}_{\mathrm{STOC}} \tag{4.9}
$$

$$
\begin{aligned}
\text{s.t.} \quad z^S_{it}(\omega) &= z^S_{i,t-1}(\omega)+\sum_{j\in P}r^S_{ji,t-\tau_{ji}}(\omega)-\sum_{j\in P}r^S_{ijt}(\omega)+\sum_{j\in P}x^S_{ji,t-\nu_i-\nu_j-\tau_{ji}}(\omega)\\
&\quad -\sum_{j\in P}x^S_{ijt}(\omega), \qquad \forall i\in P, t=1,\ldots,T \qquad (4.10)\\
z^F_{it}(\omega) &= z^F_{i,t-1}(\omega)+\sum_{j\in P}r^F_{ji,t-\tau_{ji}}(\omega)-\sum_{j\in P}r^F_{ijt}(\omega)+\sum_{j\in P}x^F_{ji,t-\nu_i-\nu_j-\tau_{ji}}(\omega)\\
&\quad -\sum_{j\in P}x^F_{ijt}(\omega), \qquad \forall i\in P, t=1,\ldots,T \qquad (4.11)
\end{aligned}
$$

$$x^S_{ijt}(\omega) + x^F_{ijt}(\omega) = D_{ijt}(\omega), \qquad \forall i, j \in P, t = 1, \ldots, T \quad (4.12)$$

$$r^S_{ijt}(\omega) + \frac{1}{N} r^F_{ijt}(\omega) + x^S_{ij,t-\nu_i}(\omega) + x^F_{ij,t-\nu_i}(\omega) \leq K_{ijt},$$
$$\forall i, j \in P, t = 1, \ldots, T \quad (4.13)$$

$$r^S_{ijt}(\omega) = r^S_{ijt}(\xi), r^F_{ijt}(\omega) = r^F_{ijt}(\xi), \quad \forall \xi \in \Omega^t(\omega), t = 1, \ldots, T, \omega \in \Omega \quad (4.14)$$

$$x^S_{ijt}(\omega) = x^S_{ijt}(\xi), x^F_{ijt}(\omega) = x^F_{ijt}(\xi), \quad \forall \xi \in \Omega^t(\omega), t = 1, \ldots, T, \omega \in \Omega \quad (4.15)$$

$$r^S_{ijt}(\omega), r^F_{ijt}(\omega) \geq 0, x^S_{ijt}(\omega), x^F_{ijt}(\omega) \geq 0, \qquad \forall i, j \in P, t = 1, \ldots, T \quad (4.16)$$

$\omega \in \Omega$ represents a possible realization or scenario of random demand over T periods where Ω represents a set of all possible realizations or scenarios. The objective function contains expectations over all possible realizations, which reflects the risk-neutral decision making of the shipping company. Decision variables depend on the possible realization of demand. Constraints (4.14) and (4.15) represent non-anticipativity constraints where $\Omega^t(\omega)$ denotes a set of scenarios whose demand is the same as ω until t period. A non-anticipativity constraint means that the decisions only depend on the past realizations and do not depend on the future. Therefore, if any two different demand scenarios have the same demand history until period t, the decision on the subsequent period must be the same for both demand scenarios. For example, suppose that ω_1 and ω_2 have the same demand history until period t, i.e., $d_{ijk}(\omega_1) = d_{ijk}(\omega_2)$ for $k = 1, \ldots, t$. Then, the decisions on period t should be the same, that is, $x_{ijt}(\omega_1) = x_{ijt}(\omega_2)$ and $r_{ijt}(\omega_1) = r_{ijt}(\omega_2)$ for both standard and foldable cases.

The presented formulation accounts for decision making under demand uncertainty; however, in general the optimal solution is difficult to obtain ([86]) because the evaluation of $\mathbb{E}[(\cdot)^+]$ in the multistage setting is extremely difficult. Moreover, the formulation requires complete knowledge of the distribution of demand uncertainty, which is needed for the evaluation of expectation in the objective function. In practice, it is difficult to estimate the distribution precisely from historical data. Therefore, in many practical circumstances, the first and second moments are estimated based on past data to reach the best possible forecast of future demand. With estimations of the first and second moments, we need a tractable and distributionally robust approach to handle the expectation of positive parts $\mathbb{E}[(\cdot)^+]$. To incorporate those practical conditions, we adopt an adjustable robust optimization technique similar to that of [10]; it requires only limited information on distributions and is computationally tractable when using a linear decision rule.

4.3 Affinely Adjustable Robust Formulation

We consider multistage decision making under uncertainty, which means that decisions are made after observing past data realization. Decisions can represent wait-and-see decisions which depend on a portion of uncertain data. This adjustability can be represented by adjustable robust counterpart which was proposed by [11]. Therefore, we adopt the adjustable robust optimization technique and the concept of a linear decision rule. For this direction, we need to make two preparations before proposing an adjustable robust counterpart. First, we will introduce the factor-based demand model which represents affine parameterizations of uncertainty based on uncertain factors. Second, we adopt upper bounds to the expectations of the positive parts which appear at the objective function of the multistage formulation. The following contents are extended from the results of [78].

4.3.1 Factor-Based Demand Model

For utilizing the concept of the adjustable robust counterpart, we utilize a factor-based demand model similar to the model of [78]. A factor-based demand model represents the uncertain demand which is affinely dependent on uncertain factors. We need a specific assumption of uncertain factors for tractability.

Assumption 4.1. *The uncertain factors $\tilde{z} = \left\{\tilde{z}_{ijt}\right\}_{(i,j\in P,t=1,\ldots,T)}$ are zero mean random variables with a covariance matrix Σ. Uncertain factors $\tilde{z}$ are distributed in the conic quadratic representable support set, $\mathcal{W}$.*

The support set $\mathcal{W}$ is conic quadratic representable if $\mathcal{W}$ is represented by a quadratic cone or a second-order cone, e.g., $\mathcal{W} = \left\{z \in \mathbb{R}^n \mid z_1 \geq \sqrt{z_2^2 + \cdots + z_n^2}\ \right\}$.

$\mathcal{W}$ would be intervals, polyhedrons, or ellipsoids. This assumption is essential for the tractability of the formulation. Without this assumption, the robust counterpart over $\mathcal{W}$ would be intractable.

Under Assumption 4.1, we can express the factor-based demand as follows:

$$d_{ijt}(\tilde{z}) = d^0_{ijt} + \sum_{i' \in P} \sum_{j' \in P} \sum_{k=1}^{T} d^k_{i'j'ijt} \tilde{z}_{i'j'k} \quad \forall i, j \in P, t = 1, \ldots, T \tag{4.17}$$

$$d^k_{i'j'ijt} = 0 \quad \forall i, j \in P, t = 1, \ldots, T, k \geq t+1 \tag{4.18}$$

For example, consider demand for two ports and two periods. Then, $d_{121}(\tilde{z}) = d^0_{1,2,1} + d^1_{1,2,1,2,1}\tilde{z}_{1,2,1} + d^1_{2,1,1,2,1}\tilde{z}_{2,1,1}$ and $d_{122}(\tilde{z}) = d^0_{1,2,2} + d^1_{1,2,1,2,2}\tilde{z}_{1,2,1} + d^2_{1,2,1,2,2}\tilde{z}_{1,2,2} + d^1_{2,1,1,2,2}\tilde{z}_{2,1,1} + d^2_{2,1,1,2,2}\tilde{z}_{2,1,2}$ which are affine functions of $\tilde{z}$. $d_{211}(\tilde{z})$ and $d_{212}(\tilde{z})$ can be represented similarly.

Equation (4.17) shows that the uncertain demand is affinely dependent on uncertain factors $\tilde{z}_{ijt}$. As uncertain factors are realized dynamically, Equation (4.18) shows that the uncertain demand is depend only on the realized uncertain factors. See and Sim [78] showed that many demand models, such as the independently distributed demand, ARMA(p, q) demand process, and any other demand models characterized by random factors, can be expressed as a factor-based model.

4.3.2 Bound on Expectations of Positive Parts

One of the most difficult things in the multistage stochastic programming formulation is the evaluation of the expectation in the objective function. It requires the complete knowledge of distribution, which is restricted in practice. Even if the distribution is known precisely, the evaluation of the expectation is computationally intractable in multistage case. Therefore, we use upper bounds similar to those of [76] with assumption of limited information about the distribution of the uncertain factors. The upper bounds provide tight bounds on expectations of positive parts $\mathbb{E}[(\cdot)^+]$ with distributionally robust properties. Expectations of positive parts $\mathbb{E}[(\cdot)^+]$ appear in the objective function of the multistage stochastic programming formulation, for example, holding and penalty costs $\mathbb{E}[H_i^S(z_{it}^S)^+ + P_i^S(z_{it}^S)^-]$. Hence, we need the bound on $\mathbb{E}[(\cdot)^+]$ with distributionally robust and tractable properties. [20] proposed the bounds in the form of affine functions of random factors. Therefore, we adopt the results of [20].

Theorem 4.2 ([20])**.** *Under Assumption 4.1 on uncertain factors, the following functions,* $\Pi^i(y_o, \boldsymbol{y}), i \in \{1,2,3\}$ *are the upper bounds of* $\mathbb{E}\big[(y_o + \boldsymbol{y}'\tilde{\boldsymbol{z}})^+\big]$ *where* $x^+ = \max\{x, 0\}$*:*

1. $\Pi^1(y_0, \boldsymbol{y}) := \big(y_0 + \max_{\tilde{z} \in \mathcal{W}} \tilde{\boldsymbol{z}}'\boldsymbol{y}\big)^+$
2. $\Pi^2(y_0, \boldsymbol{y}) := y_0 + \big(-y_0 + \max_{\tilde{z} \in \mathcal{W}} \tilde{\boldsymbol{z}}'(-\boldsymbol{y})\big)^+$
3. $\Pi^3(y_0, \boldsymbol{y}) := \frac{1}{2} y_0 + \frac{1}{2}\sqrt{y_0^2 + \boldsymbol{y}'\boldsymbol{\Sigma}\boldsymbol{y}}$

Proof. We refer the reader to [20] for the proof. $\square$

Remark 4.1 ([20])**.** *The first bound in Theorem 4.2 is derived from the positive part of support of the uncertain factors. The second bound is derived from the negative part of support of the uncertain factors. The third bound is derived from the covariance of the uncertain factors.*

Remark 4.2. *Chen and Sim [20] proposed five upper bounds of expectation of positive parts. However, the last two bounds require the estimation of forward and back-*

ward deviations defined by [21], which may reflect a new concept for practitioners. Therefore, for the simplicity and applicability in the shipping industry, we omit the last two bounds.

Theorem 4.2 shows that the three bounds are upper bounds on the expectations of positive parts, respectively. Then, Chen and Sim [20] integrated these bounds for better bound.

Theorem 4.3 ([20])**.** *Let*

$$\begin{aligned}\Pi(y_0, \boldsymbol{y}) := \min_{y_{i0}, \boldsymbol{y_i}} & \sum_{i=1}^{3} \Pi^i(y_{i0}, \boldsymbol{y_i}) \\ \text{s.t.} & \sum_{i=1}^{3} y_{i0} = y_0, \\ & \sum_{i=1}^{3} \boldsymbol{y_i} = \boldsymbol{y}.\end{aligned}$$

$\Pi(y_0, \boldsymbol{y})$ *is a better upper bound of the expectation of positive parts than the three bounds from Theorem 4.2, that is,*

$$\mathbb{E}\big[(y_o + \boldsymbol{y'}\tilde{\boldsymbol{z}})^+\big] \leq \Pi(y_0, \boldsymbol{y}) \leq \min_{i=1,2,3} \Pi^i(y_0, \boldsymbol{y})$$

Proof. We refer the reader to [20] for the proof. □

The epigraph form of the bound in Theorem 4.3, $\Pi(y_0, \boldsymbol{y}) \leq M$, is

$$
\begin{aligned}
& \exists\ y_{i0} \in \mathbb{R}, \boldsymbol{y_i} \in \mathbb{R}^N, r_i \in \mathbb{R}, i = 1,2,3 \\
\text{s.t.}\quad & r_1 + r_2 + r_3 \leq M \\
& y_{10} + \max_{\tilde{z} \in \mathcal{W}} \tilde{\boldsymbol{z}}' \boldsymbol{y_1} \leq r_1 \\
& 0 \leq r_1 \\
& \max_{\tilde{z} \in \mathcal{W}} \tilde{\boldsymbol{z}}'(-\boldsymbol{y_2}) \leq r_2 \\
& y_{20} \leq r_2 \\
& \frac{1}{2} y_{30} + \frac{1}{2}\sqrt{y_{30}^2 + \boldsymbol{y_3}' \boldsymbol{\Sigma} \boldsymbol{y_3}} \leq r_3 \\
& y_{10} + y_{20} + y_{30} = y_0 \\
& \boldsymbol{y_1} + \boldsymbol{y_2} + \boldsymbol{y_3} = \boldsymbol{y}
\end{aligned}
$$

where N is the dimension of uncertain factors $\tilde{z}$.

Remark 4.3. *Under Assumption 4.1, the bound in Theorem 4.3 is a second-order cone program (SOCP), which is computationally tractable and solved efficiently with a commercial solver. With this bound, we can approximate the objective function of the multistage stochastic programming formulation.*

Theorem 4.3 shows that the integration of the three bounds from Theorem 4.2 generates a better bound than the three upper bounds provided separately. Hence, we adopt the bound from Theorem 4.3 to propose the adjustable robust counterparts.

4.3.3 Linear Decision Rule Formulation

In this subsection, we explain the LDR and propose a robust formulation based on the LDR. We show that the LDR formulation is a second-order cone program which is computationally tractable and can be solved using commercial solvers. Then, we show that the LDR formulation is a tractable approximation of a multistage stochastic programming formulation for the ECR problem.

Ben-Tal et al. [11] showed that the adjustable robust counterpart is NP-hard, so they proposed an affinely adjustable robust counterpart (AARC) for tractability. The idea of AARC is to restrict decisions to affine functions of uncertainties. An LDR is based on the same concept of an AARC, which means that the repositioning decisions are restricted to affine functions of random factors.

$$r^S_{ijt}(\tilde{z}) = r^{S,0}_{ijt} + \sum_{i',j' \in P} \sum_{k=1}^{t} r^{S,k}_{i'j'ijt} \tilde{z}_{i'j'k}, \quad \forall i,j \in P,\ t = 1,\ldots,T \tag{4.19}$$

$$r^F_{ijt}(\tilde{z}) = r^{F,0}_{ijt} + \sum_{i',j' \in P} \sum_{k=1}^{t} r^{F,k}_{i'j'ijt} \tilde{z}_{i'j'k}, \quad \forall i,j \in P,\ t = 1,\ldots,T \tag{4.20}$$

$$r^{S,k}_{i'j'ijt} = 0, \quad \forall i,j \in P,\ t = 1,\ldots,T,\ k = t+1,\ldots,T \tag{4.21}$$

$$r^{F,k}_{i'j'ijt} = 0, \quad \forall i,j \in P,\ t = 1,\ldots,T,\ k = t+1,\ldots,T \tag{4.22}$$

For example, consider repositioning decisions of standard containers for two ports and two periods. Then, $r^S_{121}(\tilde{z}) = r^{S,0}_{1,2,1} + r^{S,1}_{1,2,1,2,1}\tilde{z}_{1,2,1} + r^{S,1}_{2,1,1,2,1}\tilde{z}_{2,1,1}$ and $r^S_{122}(\tilde{z}) = r^{S,0}_{1,2,2} + r^{S,1}_{1,2,1,2,2}\tilde{z}_{1,2,1} + r^{S,2}_{1,2,1,2,2}\tilde{z}_{1,2,2} + r^{S,1}_{2,1,1,2,2}\tilde{z}_{2,1,1} + r^{S,2}_{2,1,1,2,2}\tilde{z}_{2,1,2}$ which are affine functions of $\tilde{z}$. $r^S_{211}(\tilde{z})$ and $r^S_{212}(\tilde{z})$ can be represented similarly.

Equations (4.19) and (4.20) show that the repositioning decisions for standard and foldable containers are restricted to affine functions of $\tilde{z}$, respectively. Equations

(4.21) and (4.22) show the non-anticipativity of the repositioning decisions. Other decision variables, x_{ijt}, are also affinely dependent on random factors $\tilde{z}$. We omit non-anticipativity constraints for brevity.

$$x_{ijt}^{S}(\tilde{z}) = x_{ijt}^{S,0} + \sum_{i',j' \in P} \sum_{k=1}^{t} x_{i'j'ijt}^{S,k} \tilde{z}_{i'j'k}, \qquad \forall i,j \in P,\ t = 1, \ldots, T \tag{4.23}$$

$$x_{ijt}^{F}(\tilde{z}) = x_{ijt}^{F,0} + \sum_{i',j' \in P} \sum_{k=1}^{t} x_{i'j'ijt}^{F,k} \tilde{z}_{i'j'k}, \qquad \forall i,j \in P,\ t = 1, \ldots, T \tag{4.24}$$

We can approximate the expected repositioning and transporting costs by using the LDR for repositioning and transporting decisions. For example, consider repositioning and transporting costs of the standard container case. The foldable container case can be expressed similarly.

$$\begin{aligned}
&\mathbb{E}[R_{ij}^{S} r_{ijt}^{S}(\tilde{z}) + C_{ij}^{S} x_{ijt}^{S}(\tilde{z})] \\
&\leq \mathbb{E}[R_{ij}^{S}(r_{ijt}^{S,0} + \sum_{i',j' \in P} \sum_{k=1}^{t} r_{i'j'ijt}^{S,k} \tilde{z}_{i'j'k}) + C_{ij}^{S}(x_{ijt}^{S,0} + \sum_{i',j' \in P} \sum_{k=1}^{t} x_{i'j'ijt}^{S,k} \tilde{z}_{i'j'k})] \\
&= R_{ij}^{S} r_{ijt}^{S,0} + C_{ij}^{S} x_{ijt}^{S,0}
\end{aligned}$$

The first inequality holds by the LDR and the second equality holds by the zero-mean assumption of uncertain factors. We can approximate the expected holding and penalty costs, and the folding and unfolding costs by the LDR and the bound from Theorem 4.3. For example, consider the holding and penalty costs of the foldable container case.

$$
\begin{aligned}
&\mathbb{E}[H_i^S(z_{it}^S(\tilde{z}))^+ + P_i^S(z_{it}^S(\tilde{z}))^-] \\
&\leq \mathbb{E}[H_i^S(z_{it}^{S,0} + \sum_{i',j' \in P}\sum_{k=1}^{t} z_{i'j'it}^{S,k}\tilde{z}_{i'j'k})^+ + P_i^S(z_{it}^{S,0} + \sum_{i',j' \in P}\sum_{k=1}^{t} z_{i'j'it}^{S,k}\tilde{z}_{i'j'k})^-] \\
&\leq H_i^S\Pi(z_{it}^{S,0}, \boldsymbol{z}_{i,t}^S) + P_i^S\Pi(-z_{it}^{S,0}, -\boldsymbol{z}_{i,t}^S)
\end{aligned}
$$

The first inequality holds by the LDR and the second inequality holds by the bound from Theorem 4.3. Expected folding and unfolding costs can be approximated similarly.

Using the LDR and bounds from Theorem 4.3, we propose the LDR formulation for the ECR problem. Let $\mathrm{RC}_{\mathrm{LDR}}$ denote total repositioning and transportation costs using the LDR and $\mathrm{HC}_{\mathrm{LDR}}$ denote total holding and penalty costs using the LDR and Theorem 4.3. Let $\mathrm{FC}_{\mathrm{LDR}}$ denote total folding and unfolding costs using the LDR and Theorem 4.3.

$$
\begin{aligned}
\mathrm{RC}_{\mathrm{LDR}} =& \sum_{t=1}^{T}\sum_{i\in P}\sum_{j\in P}\big(R_{ij}^S r_{ijt}^{S,0} + C_{ij}^S x_{ijt}^{S,0} + R_{ij}^F r_{ijt}^{F,0} + C_{ij}^F x_{ijt}^{F,0}\big) \\
\mathrm{HC}_{\mathrm{LDR}} =& \sum_{t=1}^{T}\sum_{i\in P}\big(H_i^S\Pi(z_{it}^{S,0}, \boldsymbol{z}_{i,t}^S) + H_i^F\Pi(z_{it}^{F,0}, \boldsymbol{z}_{i,t}^F) + P_i^S\Pi(-z_{it}^{S,0}, -\boldsymbol{z}_{i,t}^S) \\
&+ P_i^F\Pi(-z_{it}^{F,0}, -\boldsymbol{z}_{i,t}^F)\big) \\
\mathrm{FC}_{\mathrm{LDR}} =& \sum_{t=1}^{T}\sum_{i\in P}\big(FC_i\Pi\big(\sum_{j\in P}(x_{ji,t-\nu_i-\nu_j-\tau_{ji}}^{F,0} - x_{ijt}^{F,0}), \sum_{j\in P}(\boldsymbol{x}_{ji,t-\nu_i-\nu_j-\tau_{ji}}^F - \boldsymbol{x}_{ijt}^F)\big) \\
&+ UC_i\Pi\big(\sum_{j\in P}(x_{ijt}^{F,0} - x_{ji,t-\nu_i-\nu_j-\tau_{ji}}^{F,0}), \sum_{j\in P}(\boldsymbol{x}_{ijt}^F - \boldsymbol{x}_{ji,t-\nu_i-\nu_j-\tau_{ji}}^F)\big)
\end{aligned}
$$

The LDR formulation is as follows:

$$
\begin{aligned}
\mathrm{TC}_{\mathrm{LDR}} = & \\
\min \quad & \mathrm{RC}_{\mathrm{LDR}} + \mathrm{HC}_{\mathrm{LDR}} + \mathrm{FC}_{\mathrm{LDR}} && (4.25)\\
\text{s.t.} \quad & z_{it}^{S,0} = z_{i,t-1}^{S,0} + \sum_{j\in P} r_{ji,t-\tau_{ji}}^{S,0} - \sum_{j\in P} r_{ijt}^{S,0} + \sum_{j\in P} x_{ji,t-\nu_i-\nu_j-\tau_{ji}}^{S,0} - \sum_{j\in P} x_{ijt}^{S,0}, \\
& && \forall i \in P, t = 1,\ldots,T \quad (4.26)\\
& z_{i'j'it}^{S,k} = z_{i'j'i,t-1}^{S,k} + \sum_{j\in P} r_{i'j'ji,t-\tau_{ji}}^{S,k} - \sum_{j\in P} r_{i'j'ijt}^{S,k} + \sum_{j\in P} x_{i'j'ji,t-\nu_i-\nu_j-\tau_{ji}}^{S,k} \\
& \qquad - \sum_{j\in P} x_{i'j'ijt}^{S,k}, && \forall i', j', i \in P, t = 1,\ldots,T, k \le t \quad (4.27)\\
& z_{it}^{F,0} = z_{i,t-1}^{F,0} + \sum_{j\in P} r_{ji,t-\tau_{ji}}^{F,0} - \sum_{j\in P} r_{ijt}^{F,0} + \sum_{j\in P} x_{ji,t-\nu_i-\nu_j-\tau_{ji}}^{F,0} - \sum_{j\in P} x_{ijt}^{F,0}, \\
& && \forall i \in P, t = 1,\ldots,T \quad (4.28)\\
& z_{i'j'it}^{F,k} = z_{i'j'i,t-1}^{F,k} + \sum_{j\in P} r_{i'j'ji,t-\tau_{ji}}^{F,k} - \sum_{j\in P} r_{i'j'ijt}^{F,k} + \sum_{j\in P} x_{i'j'ji,t-\nu_i-\nu_j-\tau_{ji}}^{F,k} \\
& \qquad - \sum_{j\in P} x_{i'j'ijt}^{F,k}, && \forall i', j', i \in P, t = 1,\ldots,T, k \le t \quad (4.29)\\
& x_{ijt}^{S,0} + x_{ijt}^{F,0} = d_{ijt}^{0}, && \forall i, j \in P, t = 1,\ldots,T \quad (4.30)\\
& x_{i'j'ijt}^{S,k} + x_{i'j'ijt}^{F,k} = d_{i'j'ijt}^{k}, && \forall i', j', i, j \in P, t = 1,\ldots,T, k \le t \quad (4.31)\\
& r_{ijt}^{S,0} + \frac{1}{N} r_{ijt}^{F,0} + x_{ij,t-\nu_i}^{S,0} + x_{ij,t-\nu_i}^{F,0} \\
& + \sum_{i',j'\in P} \sum_{k=1}^{t} \big(r_{i'j'ijt}^{S,k} + \frac{1}{N} r_{i'j'ijt}^{F,k} + x_{i'j'ij,t-\nu_i}^{S,k} + x_{i'j'ij,t-\nu_i}^{F,k} \big) \tilde{z}_{ijk} \le K_{ijt}, \\
& && \forall i, j \in P, t = 1,\ldots,T, \tilde{z} \in \mathcal{W} \quad (4.32)
\end{aligned}
$$

The objective function includes the upper bounds of the expectations of positive parts. Because the bounds are second-order cones, the objective function is a second-order cone. Moreover, the bounds are derived only with mean, support, and covariance of uncertainties. Hence, the above formulation does not need any distributional assumptions. All constraints, except Constraint (4.32), are linear and Constraint (4.32) can be transformed to a robust counterpart under Assumption 1. If the uncertain factors have interval or ellipsoidal uncertainty, then the transformed robust counterpart is computationally tractable. Hence, if we assume that Constraint (4.32) can be transformed to be tractable, then the LDR formulation is computationally tractable. From this, we can obtain the following result.

Theorem 4.4. $\mathrm{TC}_{\mathrm{STOC}} \leq \mathrm{TC}_{\mathrm{LDR}}$, *where* $\mathrm{TC}_{\mathrm{STOC}}$ *is the optimal expected cost of the multistage stochastic programming formulation, and* $\mathrm{TC}_{\mathrm{LDR}}$ *is the optimal expected cost under the linear decision rule.*

Proof. The proof is similar to that of [78]. We refer the reader to the electronic companion of [78].

First, we will show that the inventory levels of standard and foldable containers are expressed as affine functions of random factors. We only show the proof of the standard container case, because the foldable container case is the same. Note that constraint (4.10) is $z^S_{it}(\tilde{z}) = z^S_{i,t-1}(\tilde{z}) + \sum_{j\in P} r^S_{ji,t-\tau_{ji}}(\tilde{z}) - \sum_{j\in P} r^S_{ijt}(\tilde{z}) + \sum_{j\in P} x^S_{ji,t-\nu_i-\nu_j\tau_{ji}}(\tilde{z}) - \sum_{j\in P} x^S_{ijt}(\tilde{z})$. By summation over period t and using the linear decision rule, we can obtain

$$
\begin{aligned}
z_{it}^{S}(\tilde{z}) = z_{i0}^{S,0} &+ \sum_{\tau=1}^{t}\sum_{j\in P} r_{ji,\tau-\tau_{ji}}^{S,0} - \sum_{\tau=1}^{t}\sum_{j\in P} r_{ij,\tau}^{S,0} + \sum_{\tau=1}^{t}\sum_{j\in P} x_{ji,t-\nu_i-\nu_j-\tau_{ji}}^{S,0} - \sum_{\tau=1}^{t}\sum_{j\in P} x_{ij,\tau}^{S,0} \\
&+ \sum_{i',j',j\in P}\sum_{\tau=1}^{t}\sum_{k=1}^{t} r_{i'j'ji,\tau-\tau_{ji}}^{S,k}\tilde{z}_{i'j'k} - \sum_{i',j',j\in P}\sum_{\tau=1}^{t}\sum_{k=1}^{t} r_{i'j'ij,\tau}^{S,k}\tilde{z}_{i'j'k} \\
&+ \sum_{i',j',j\in P}\sum_{\tau=1}^{t}\sum_{k=1}^{t} x_{i'j'ji,t-\nu_i-\nu_j-\tau_{ji}}^{S,k}\tilde{z}_{i'j'k} - \sum_{i',j',j\in P}\sum_{\tau=1}^{t}\sum_{k=1}^{t} x_{i'j'ij\tau}^{S,k}\tilde{z}_{i'j'k} \\
= z_{it}^{S,0} &+ \sum_{i',j'\in P} z_{i'j'ijt}^{S,k}\tilde{z}_{i'j'k}
\end{aligned}
$$

Hence, $z_{it}^{S,0}$ is also an affine function of random factors and constraints (4.26) and (4.27) are derived. The linear decision rule solution is a feasible solution of the multistage stochastic programming formulation. Under the linear decision rule and by Theorem 4.3, we obtain

$$
\begin{aligned}
&\mathbb{E}\Big[\sum_{i,j\in P}\big(R_{ij}^{S}r_{ijt}^{S}(\tilde{z}) + C_{ij}^{S}x_{ijt}^{S}(\tilde{z}) + R_{ij}^{F}r_{ijt}^{F}(\tilde{z}) + C_{ij}^{F}x_{ijt}^{F}(\tilde{z})\big)\Big] \\
&+\mathbb{E}\Big[\sum_{j\in P} H_i^{S}\big(z_{it}^{S}(\tilde{z})\big)^{+} + H_i^{F}\big(z_{it}^{F}(\tilde{z})\big)^{+} + P_i^{S}\big(z_{it}^{S}(\tilde{z})\big)^{-} + P_i^{F}\big(z_{it}^{F}(\tilde{z})\big)^{-}\Big] \\
&+\mathbb{E}\Big[\sum_{i\in P} FC_i\big(\sum_{j\in P}(x_{ji,t-\nu_i-\nu_j-\tau_{ji}}^{F}(\omega) - x_{ijt}^{F}(\omega))^{+}\big)\Big] \\
&+\mathbb{E}\Big[\sum_{i\in P} UC_i\big(\sum_{j\in P}(x_{ijt}^{F}(\omega) - x_{ji,t-\nu_i-\nu_j-\tau_{ji}}^{F}(\omega))^{+}\big)\Big] \\
\le &\sum_{i,j\in P}\big(R_{ij}^{S}r_{ijt}^{S,0} + C_{ij}^{S}x_{ijt}^{S,0} + R_{ij}^{F}r_{ijt}^{F,0} + C_{ij}^{F}x_{ijt}^{F,0}\big) \\
&+\sum_{i\in P}\big(H_i^{S}\Pi(z_{it}^{S,0}, \boldsymbol{z}_{i,t}^{S}) + H_i^{F}\Pi(z_{it}^{F,0}, \boldsymbol{z}_{i,t}^{F})\big) \\
&+\sum_{i\in P}\big(P_i^{S}\Pi(-z_{it}^{S,0}, -\boldsymbol{z}_{i,t}^{S}) + P_i^{F}\Pi(-z_{it}^{F,0}, -\boldsymbol{z}_{i,t}^{F})\big)
\end{aligned}
$$

$$+\sum_{i\in P}\big(FC_i\Pi\big(\sum_{j\in P}(x^{F,0}_{ji,t-\nu_i-\nu_j-\tau_{ji}}-x^{F,0}_{ijt}),\sum_{j\in P}(\boldsymbol{x}^F_{ji,t-\nu_i-\nu_j-\tau_{ji}}-\boldsymbol{x}^F_{ijt})\big)$$
$$+UC_i\Pi\big(\sum_{j\in P}(x^{F,0}_{ijt}-x^{F,0}_{ji,t-\nu_i-\nu_j-\tau_{ji}}),\sum_{j\in P}(\boldsymbol{x}^F_{ijt}-\boldsymbol{x}^F_{ji,t-\nu_i-\nu_j-\tau_{ji}})\big)\big).$$

Therefore, we conclude $\mathrm{TC}_{\mathrm{STOC}}\leq\mathrm{TC}_{\mathrm{LDR}}$. □

Theorem 4.4 shows that the LDR formulation is a tractable approximation of the multistage stochastic programming formulation. The optimal solution of the multistage stochastic programming formulation is difficult to obtain; however, the optimal solution of the LDR formulation can be used. We show the analysis of LDR formulation performance and a comparison against a benchmark in the numerical experiments section.

4.3.4 Restricted Linear Decision Rule Formulation

Although the LDR formulation is computationally tractable, it has a lot of decision variables because each decision is dependent on all possible uncertainties. For example, each repositioning decision, $\boldsymbol{r}_{ijt}$, has $|\mathrm{P}|\times|\mathrm{P}|\times|\mathrm{T}|+1$ variables in the LDR formulation. For a simpler formulation, we propose a restricted linear decision rule (RLDR) similar to the idea of [5]. A restricted linear decision rule means that decision rules are restricted to affine functions of the uncertain factors $\tilde{\boldsymbol{z}}_{ij}$ rather than all possible realizations of the uncertain factors. For example, the repositioning decision from port i to port j would not be affected by random factors $\{\tilde{z}_{k,l}\}_{k,l\neq i,j}$. Therefore, the RLDR may perform well in practice despite the additional assumptions required. The RLDR is as follows:

$$\begin{aligned}
r^S_{ijt}(\tilde{z}) &= r^{S,0}_{ijt} + \sum_{k=1}^{t} r^{S,k}_{ijt} \tilde{z}_{ijk} && \forall i,j \in P,\ t = 1,\ldots,T \\
r^F_{ijt}(\tilde{z}) &= r^{F,0}_{ijt} + \sum_{k=1}^{t} r^{F,k}_{ijt} \tilde{z}_{ijk} && \forall i,j \in P,\ t = 1,\ldots,T \\
x^S_{ijt}(\tilde{z}) &= x^{S,0}_{ijt} + \sum_{k=1}^{t} x^{S,k}_{ijt} \tilde{z}_{ijk} && \forall i,j \in P,\ t = 1,\ldots,T \\
x^F_{ijt}(\tilde{z}) &= x^{F,0}_{ijt} + \sum_{k=1}^{t} x^{F,k}_{ijt} \tilde{z}_{ijk} && \forall i,j \in P,\ t = 1,\ldots,T
\end{aligned}$$

For example, consider repositioning decisions of standard containers for two ports and three periods. Then, $r^S_{121}(\tilde{z}) = r^{S,0}_{1,2,1} + r^{S,1}_{1,2,1}\tilde{z}_{1,2,1}$, $r^S_{122}(\tilde{z}) = r^{S,0}_{1,2,2} + r^{S,1}_{1,2,2}\tilde{z}_{1,2,1} + r^{S,2}_{1,2,2}\tilde{z}_{1,2,2}$, and $r^S_{123}(\tilde{z}) = r^{S,0}_{1,2,3} + r^{S,1}_{1,2,3}\tilde{z}_{1,2,1} + r^{S,2}_{1,2,3}\tilde{z}_{1,2,2} + r^{S,3}_{1,2,3}\tilde{z}_{1,2,3}$ which are affine functions of $\tilde{z}_{12}$. In the opposite direction, $r^S_{211}(\tilde{z}) = r^{S,0}_{2,1,1} + r^{S,1}_{2,1,1}\tilde{z}_{2,1,1}$, $r^S_{212}(\tilde{z}) = r^{S,0}_{2,1,2} + r^{S,1}_{2,1,2}\tilde{z}_{2,1,1} + r^{S,2}_{2,1,2}\tilde{z}_{2,1,2}$, and $r^S_{213}(\tilde{z}) = r^{S,0}_{2,1,3} + r^{S,1}_{2,1,3}\tilde{z}_{2,1,1} + r^{S,2}_{2,1,3}\tilde{z}_{2,1,2} + r^{S,3}_{2,1,3}\tilde{z}_{2,1,3}$ which are affine functions of $\tilde{z}_{21}$.

To utilize the RLDR, an additional assumption on demand is needed. Random demand $d_{ijt}(\tilde{z})$ should be a function of $\tilde{\boldsymbol{z}}_{ij}$ rather than of $\tilde{\boldsymbol{z}}$. In other words, $d_{ijt}(\tilde{z})$ depends only on the random factors related to the (i,j) pair. Therefore, $d_{ijt}(\tilde{z})$ can be represented as follows:

$$d_{ijt}(\tilde{z}) = d^0_{ijt} + \sum_{k=1}^{t} d^k_{ijt}\tilde{z}_{ijk}.$$

Without this assumption, the RLDR would be infeasible. Therefore, we assume the above assumption in the RLDR formulation.

Using the RLDR and bounds from Theorem 4.3, we propose the RLDR formulation of the ECR problem. Let $\mathrm{RC}_{\mathrm{RLDR}}$ denote total repositioning and transportation costs using the RLDR and $\mathrm{HC}_{\mathrm{RLDR}}$ denote total holding and penalty costs using the RLDR and Theorem 4.3. Let $\mathrm{FC}_{\mathrm{RLDR}}$ denote total folding and unfolding costs using the RLDR and Theorem 4.3.

$$
\begin{aligned}
\mathrm{RC}_{\mathrm{RLDR}} =& \sum_{t=1}^{T} \sum_{i \in P} \sum_{j \in P} \big(R^S_{ij} r^{S,0}_{ijt} + C^S_{ij} x^{S,0}_{ijt} + R^F_{ij} r^{F,0}_{ijt} + C^F_{ij} x^{F,0}_{ijt} \big) \\
\mathrm{HC}_{\mathrm{RLDR}} =& \sum_{t=1}^{T} \sum_{i \in P} \big(H^S_i \Pi(z^{S,0}_{it}, \boldsymbol{z}^S_{i,t}) + H^F_i \Pi(z^{F,0}_{it}, \boldsymbol{z}^F_{i,t}) + P^S_i \Pi(-z^{S,0}_{it}, -\boldsymbol{z}^S_{i,t}) \\
&+ P^F_i \Pi(-z^{F,0}_{it}, -\boldsymbol{z}^F_{i,t}) \big) \\
\mathrm{FC}_{\mathrm{RLDR}} =& \sum_{t=1}^{T} \sum_{i \in P} \big(FC_i \Pi\big(\sum_{j \in P} (x^{F,0}_{ji,t-\nu_i-\nu_j-\tau_{ji}} - x^{F,0}_{ijt}), \sum_{j \in P} (\boldsymbol{x}^F_{ji,t-\nu_i-\nu_j-\tau_{ji}} - \boldsymbol{x}^F_{ijt}) \big) \\
&+ UC_i \Pi\big(\sum_{j \in P} (x^{F,0}_{ijt} - x^{F,0}_{ji,t-\nu_i-\nu_j-\tau_{ji}}), \sum_{j \in P} (\boldsymbol{x}^F_{ijt} - \boldsymbol{x}^F_{ji,t-\nu_i-\nu_j-\tau_{ji}}) \big)
\end{aligned}
$$

The RLDR formulation is as follows:

$$\mathrm{TC}_{\mathrm{RLDR}} =$$

$$\min \quad \mathrm{RC}_{\mathrm{RLDR}} + \mathrm{HC}_{\mathrm{RLDR}} + \mathrm{FC}_{\mathrm{RLDR}} \tag{4.33}$$

$$\text{s.t.} \quad z^{S,0}_{it} = z^{S,0}_{i,t-1} + \sum_{j \in P} r^{S,0}_{ji,t-\tau_{ji}} - \sum_{j \in P} r^{S,0}_{ijt} + \sum_{j \in P} x^{S,0}_{ji,t-\nu_i-\nu_j-\tau_{ji}} - \sum_{j \in P} x^{S,0}_{ijt}, \quad \forall i \in P, t = 1, \ldots, T \tag{4.34}$$

$$z^{S,k}_{j,i,i,t} = z^{S,k}_{j,i,i,t-1} + r^{S,k}_{j,i,t-\tau_{ji}} + x^{S,k}_{j,i,t-\nu_i-\nu_j-\tau_{ji}}, \ \forall i, j \in P, t = 1, \ldots, T, k \le t \tag{4.35}$$

$$z^{S,k}_{i,j,i,t} = z^{S,k}_{i,j,i,t-1} - r^{S,k}_{i,j,t} - x^{S,k}_{i,j,t}, \qquad \forall i, j \in P, t = 1, \ldots, T, k \le t \tag{4.36}$$

$$z_{it}^{F,0} = z_{i,t-1}^{F,0} + \sum_{j \in P} r_{ji,t-\tau_{ji}}^{F,0} - \sum_{j \in P} r_{ijt}^{F,0} + \sum_{j \in P} x_{ji,t-\nu_i-\nu_j-\tau_{ji}}^{F,0} - \sum_{j \in P} x_{ijt}^{F,0}, \quad \forall i \in P, t = 1, \ldots, T \quad (4.37)$$

$$z_{j,i,i,t}^{F,k} = z_{j,i,i,t-1}^{F,k} + r_{j,i,t-\tau_{ji}}^{F,k} + x_{j,i,t-\nu_i-\nu_j-\tau_{ji}}^{F,k}, \quad \forall i, j \in P, t = 1, \ldots, T, k \leq t \quad (4.38)$$

$$z_{i,j,i,t}^{F,k} = z_{i,j,i,t-1}^{F,k} - r_{i,j,t}^{F,k} - x_{i,j,t}^{F,k}, \quad \forall i, j \in P, t = 1, \ldots, T, k \leq t \quad (4.39)$$

$$x_{ijt}^{S,0} + x_{ijt}^{F,0} = d_{ijt}^{0}, \quad \forall i, j \in P, t = 1, \ldots, T \quad (4.40)$$

$$x_{ijt}^{S,k} + x_{ijt}^{F,k} = d_{ijt}^{k}, \quad \forall i, j \in P, t = 1, \ldots, T, k \leq t \quad (4.41)$$

$$r_{ijt}^{S,0} + \frac{1}{N} r_{ijt}^{F,0} + x_{ij,t-\nu_i}^{S,0} + x_{ij,t-\nu_i}^{F,0} + \sum_{k=1}^{t} \left(r_{ijt}^{S,k} + \frac{1}{N} r_{ijt}^{F,k} + x_{ij,t-\nu_i}^{S,k} + x_{ij,t-\nu_i}^{F,k} \right) \tilde{z}_{ijk} \leq K_{ijt}, \quad \forall i, j \in P, t = 1, \ldots, T, \tilde{z} \in W \quad (4.42)$$

Fewer decision variables of the RLDR formulation are used than in the LDR formulation. Hence, we can obtain the result shown in Theorem 4.5.

Theorem 4.5. $\mathrm{TC}_{\mathrm{STOC}} \leq \mathrm{TC}_{\mathrm{LDR}} \leq \mathrm{TC}_{\mathrm{RLDR}}$, *where* $\mathrm{TC}_{\mathrm{RLDR}}$ *is the optimal expected cost under the RLDR.*

Proof. The first inequality comes from Theorem 4.4. The second inequality is true because the RLDR formulation is a formulation that adds constraints to the LDR formulation, and the objective function of the RLDR formulation is larger than that of the LDR formulation. □

Theorem 4.5 shows that the RLDR formulation is also a tractable approximation. However, the expected cost of the RLDR formulation is worse than that of the LDR formulation. Despite the worse RLDR performance, the size of the RLDR formulation is much smaller than that of the LDR formulation. Therefore, the RLDR formulation is easy to handle and has competitive advantages in practice. We analyzed the performance and computation time for the validity of the RLDR formulation and compare the RLDR against a benchmark.

4.4 Computational Experiments

In this section, we present the numerical analysis of the proposed formulations based on the expected operating costs, computation time, and optimality gap against the benchmark. We define expected value given perfect information (EV|PI) for the benchmark against the proposed models. Then, we conduct simulations for further analysis such as cost-saving effects of foldable containers, cost ratio of total operating costs, and container storage at depots of ports. The following results were solved using Xpress software version 7.9 on a PC with an Intel(R) Core(TM) i5-6600 CPU 3.30 GHz with 32.00GB of RAM.

For validating the performance of the proposed model, we need a benchmark for comparison. However, it is difficult to obtain an optimal solution of a multistage stochastic programming formulation, so we utilize an alternative of $\mathrm{TC}_{\mathrm{STOC}}$ which is possible to calculate. Therefore, we define expected value given perfect information (EV|PI) as follows:

$$\mathrm{EV}|\mathrm{PI} = \mathbb{E}_D[\mathrm{TC}_{\mathrm{DET}}|D] \approx \frac{1}{K}\sum_{k=1}^{K}\left(\mathrm{TC}_{\mathrm{DET}}|D_k\right)$$

EV|PI represents the expected value of the total operating costs given the information of demand. We generate K samples of demand scenarios and calculate $\mathrm{TC}_{\mathrm{DET}}$ for each demand scenario. Then, we calculate EV|PI as an expectation over K samples. EV|PI can be an alternative of $\mathrm{TC}_{\mathrm{STOC}}$, because it is similar to multistage scenario generation approach. EV|PI would be less than the objective function value, because it is calculated based on the complete knowledge about future demand. Therefore, it can be used as a benchmark for comparing the performances of the robust formulations.

In this experiment, we utilize the demand process proposed by [37], which can be represented as the factor-based demand model. To compare performances of the LDR and RLDR formulations, we assume that the demand only depends on random factors $\tilde{\boldsymbol{z}}_{ij}$. Then, the demand is expressed as follows:

$$\begin{aligned} d_{ijt}(\tilde{z}) =& d_{ijt}^0 + \sum_{k=1}^{t} d_{ijt}^k \tilde{z}_{ijk} \\ =& \mu_{ijt}^0 + \sum_{k=1}^{t-1} \alpha \tilde{z}_{ijk} + \tilde{z}_{ijt} \qquad \forall i, j \in P,\ t = 1, \ldots, T \end{aligned}$$

In this demand process, $d_{ijt}^k = \alpha$ for $k = 1, \ldots, t-1$ and $d_{ijt}^t = 1$. We assume that $\tilde{z}_{ijt}$ are independent uniformly distributed random variables in $[-\bar{z}_{ij}, \bar{z}_{ij}]$. Supports of uncertain factors, $\bar{z}_{ij}$, are listed in Table 4.8. When $\alpha = 1$, the demand process is a random walk, and when $\alpha = 0$, the demand process is a stationary i.i.d. process. In this experiment, we use two different values of $\alpha \in \{0, 0.25\}$.

4.4.1 Experimental Setting

We consider a numerical example of five ports and 20 planning periods. The five ports represent Ningbo (NB), Shanghai (SH), Busan (BS), Vancouver (VC), and Los Angeles (LA), respectively, which are five major ports in the North America (NA)-Asia shipping network provided by Maersk [59]. We consider the NA-Asia instance, because the trade imbalance between NA and Asia is highly significant. We generated instances for numerical experiments by aggregating several vessel schedules of Maersk, the world's largest shipping company. The transportation time between Ningbo and Shanghai is 1 to 3 days, and the transportation time between Busan and Shanghai or Ningbo is 2 to 3 days according to the announced schedule. The transportation time to cross the Pacific Ocean is 10 to 20 days, depending on the ports and schedule. Therefore, we set a base period of 4 days to set the transportation time between Asian ports as 1 period and the transportation time to cross the Pacific Ocean as 4 periods or 16 days (Table 4.2). It is worth mentioning that actual travel time is not linearly dependent on the distance. We assume that the inland transportation time ν is 1 base period and $N = 4$, which means that four folded foldable containers are used to build one pack such as 4FOLD of HCI. The demand parameters shown in Table 4.1 are determined by referring to the monthly cargo volume data for each port. To reflect the trade imbalance, we set the mean of demand from the export dominant ports to the import dominant ports to be double for the return direction. We assume that the number of supplied containers determined before the beginning of the planning horizon and the initial inventory of containers are given. We also assume that the initial inventory of foldable containers are one over ten of that of standard containers.

Table 4.1 Mean of demand process, μ_{ij}^0

From-to	NB	SH	BS	VC	LA
NB	-	50	50	300	400
SH	50	-	50	300	400
BS	100	100	-	200	300
VC	150	150	100	-	100
LA	200	200	150	100	-

For simplicity, we assume that the parameters are the same over the planning horizon, for example, $\mu_{ijt}^0 = \mu_{ij}^0$ for all t. We let $H_i^S = 0.2$, $H_i^F = 0.1$, $P_i^S = 2$, $P_i^F = 4$, $FC_i = 0.1$, and $UC_i = 0.1$ for all five ports. The unit holding cost of foldable containers is the half of that of standard containers, because foldable containers are stored with folded state. The penalty cost of foldable containers is twofold that of standard containers, because the purchase cost and leasing cost of foldable containers are much expensive. The transportation cost also is not linearly dependent on the travel time or travel distance. There is a lot of demand between NA-Asia compared to the demand between Asian ports. The shipping company can enjoy the advantages of economies of scale in transportation between NA-Asia. Moreover, transportation costs include inland transportation costs and loading and unloading costs which must occur when the container is transported. These costs represent a considerable portion of transportation costs. For the above reasons, the unit transportation cost between ports does not depend on the actual travel time or travel distance. Therefore, we set unit transportation costs between Asian ports as 1 and between Asia and NA as 2 (Tables 4.3 and 4.4). The transportation time between ports, other cost parameters, capacity, and supports of uncertain factors are listed in Tables 4.2 - 4.8. $[-\bar{z}_{ij}, \bar{z}_{ij}]$ is the interval of uniform distribution of $\tilde{z}_{ijt}$, so we can calculate the standard deviation with $\bar{z}_{ij}$. We use these parameters as a baseline, and vary the cost parameters, such as holding, transportation, and repositioning costs. Finally, we generate 10,000 samples for calculating the benchmark EV|PI.

Table 4.2 Transportation time between ports, τ_{ij}

From-to	NB	SH	BS	VC	LA
NB	-	1	1	4	4
SH	1	-	1	4	4
BS	1	1	-	4	4
VC	4	4	4	-	2
LA	4	4	4	2	-

Table 4.3 Unit transportation cost of standard containers, C_{ij}^S

From-to	NB	SH	BS	VC	LA
NB	-	1	1	2	2
SH	1	-	1	2	2
BS	1	1	-	2	2
VC	2	2	2	-	1
LA	2	2	2	1	-

Table 4.4 Unit transportation cost of foldable containers, C_{ij}^F

From-to	NB	SH	BS	VC	LA
NB	-	1	1	2	2
SH	1	-	1	2	2
BS	1	1	-	2	2
VC	2	2	2	-	1
LA	2	2	2	1	-

Table 4.5 Unit repositioning cost of standard containers, R_{ij}^S

From-to	NB	SH	BS	VC	LA
NB	-	0.8	0.8	1.6	1.6
SH	0.8	-	0.8	1.6	1.6
BS	0.8	0.8	-	1.6	1.6
VC	1.6	1.6	1.6	-	0.8
LA	1.6	1.6	1.6	0.8	-

Table 4.6 Unit repositioning cost of foldable containers, R_{ij}^F

From-to	NB	SH	BS	VC	LA
NB	-	0.4	0.4	0.8	0.8
SH	0.4	-	0.4	0.8	0.8
BS	0.4	0.4	-	0.8	0.8
VC	0.8	0.8	0.8	-	0.4
LA	0.8	0.8	0.8	0.4	-

Table 4.7 Capacity, K_{ij}

From-to	NB	SH	BS	VC	LA
NB	-	300	300	1200	1600
SH	300	-	300	1200	1600
BS	400	400	-	800	1200
VC	600	600	400	-	400
LA	800	800	600	400	-

Table 4.8 Support of uncertain factors, $\bar{z}_{ij}$

From-to	NB	SH	BS	VC	LA
NB	-	5	5	10	20
SH	5	-	5	10	20
BS	8	8	-	8	10
VC	8	8	8	-	8
LA	10	10	8	8	-

4.4.2 Computational Results

The computational results with various holding costs and the values for α are summarized in Table 4.9. We set the holding cost H_i^S as {0.2, 0.1, 0.04} for a resulting P_i^S/H_i^S ratio of {10, 20, 50}. The structure of the optimal solutions of inventory models often depends on the ratio P_i^S/H_i^S. The ECR model is similar to inventory models, which makes the experiments with varying holding costs meaningful. The performance gap of the LDR formulation presented in Table 4.9 is calculated by $(\text{TC}_{\text{LDR}} - \text{EV}|\text{PI})/\text{EV}|\text{PI} \times 100$. The gap of the RLDR formulation is calculated similarly.

The expected costs show that the result of Theorem 4.5 holds, and the performance gap shows that the expected costs of both the LDR and RLDR formulations are very close to the EV|PI. Bertsimas et al. [16] showed that the LDR can be optimal in specific conditions such as convex objective functions and the box uncertainty set. There are a few studies about an optimality and a performance guarantee of the LDR in the multistage setting [13, 14, 28]. The tight performance gap from Table 4.9 would be justified by the theoretical results, even though the LDR and RLDR formulations are based on the multistage setting and bounds from Theorem 4.3. The performance gap can be interpreted as the price of robustness, which means that additional cost is incurred to obtain distributionally robust properties. The computation time of the LDR formulation is over 10,000 seconds, which might seem unreasonable. However, the length of time between the decisions on the first period and the next uncertainty realization is one base period or 4 days, which is sufficient for updating the data and solving the LDR formulation. Therefore, the LDR formulation can be used in the rolling horizon manner, that is, the formulation over the entire planning horizon is solved and only the first decision is implemented. Then, the realized uncertainty data is updated and the formulation is solved with the updated data. The computation time of the RLDR formulation is much less than that of the LDR formulation, because the number of variables are quite small. The smaller computation time of the RLDR formulation offers a competitive advantage in practice. For the instance size of this experiment, the sample average approximation approach can be utilized to calculate the policy of the multistage stochastic programming formulation. However, in reality, the complete knowledge of the probability distribution is often prohibited and the sample average approximation cannot be applied. Moreover, the distributionally robust approach can be remedy for the optimizer's curse in stochastic optimization [92].

To analyze the cost-saving effect for the use of foldable containers, we compare the computational results against the results of using only standard containers. The

Table 4.9 Computational results with different holding costs

		Expected Cost			Time (s)		Gap (%)	
α	HC	EV|PI	LDR	RLDR	LDR	RLDR	LDR	RLDR
0	0.2	193141	193252	193451	11418.9	1815.9	0.057	0.161
	0.1	161704	161832	162109	14423.8	2713.2	0.079	0.250
	0.04	142437	142552	142779	11216.9	2173.1	0.081	0.240
0.25	0.2	193146	193696	194362	14344.5	2795.5	0.285	0.629
	0.1	161715	162614	163217	12368.6	2361.5	0.556	0.929
	0.04	142446	143236	143761	12268.0	2532.0	0.554	0.923

computational results for using only standard containers at various holding costs are summarized in Table 4.10. We compared two cases and show the findings in Table 4.11. The gap shows that in our experiments, at most 11.43% cost savings is realized when foldable containers are used in maritime transport. The cost-saving effect decreases as the holding cost H_i decreases because the considerable cost saving using foldable containers occurs in the holding cost part. However, the cost-saving effect might be overestimated because the expected total cost represents the expected operating costs over the planning horizon. The operating costs do not include fixed or purchase costs of the foldable containers, which may be very costly. Nevertheless, a considerable cost saving may be realized for using commercialized foldable containers.

The shipping industry is highly affected by crude oil prices, because transportation and repositioning costs are proportional to crude oil prices. The computational results with various repositioning and transportation costs (TC) are summarized in Table 4.12. We denote three different parameters as {1,2,4} such that transportation and repositioning costs are once, twice, and four times the costs in Tables 4.3 and 4.5, respectively. The computational results for using both standard and foldable

Table 4.10 Computational results using standard containers only with different holding costs

		Expected Cost			Time (s)		Gap (%)	
α	HC	EV|PI	LDR	RLDR	LDR	RLDR	LDR	RLDR
0	0.2	218071	218164	218280	746.2	200.0	0.043	0.096
	0.1	174037	174128	174243	673.9	231.4	0.053	0.119
	0.04	147616	147698	147803	605.1	227.5	0.056	0.127
0.25	0.2	218077	218700	219069	777.1	193.2	0.286	0.455
	0.1	174045	174697	175070	751.8	162.8	0.375	0.589
	0.04	147626	148256	148616	605.6	217.2	0.427	0.671

Table 4.11 Comparisons between using standard containers only and using standard and foldable containers both

		Standard & Foldable			Standard			Gap (%)		
α	HC	EV|PI	LDR	RLDR	EV|PI	LDR	RLDR	EV|PI	LDR	RLDR
0	0.2	193141	193252	193451	218071	218164	218280	11.43	11.42	11.37
	0.1	161704	161832	162109	174037	174128	174243	7.09	7.06	6.96
	0.04	142437	142552	142779	147616	147698	147803	3.51	3.48	3.40
0.25	0.2	193146	193696	194362	218077	218700	219069	11.43	11.43	11.28
	0.1	161715	162614	163217	174045	174697	175070	7.08	6.92	6.77
	0.04	142446	143236	143761	147626	148256	148616	3.51	3.39	3.27

containers and using only standard containers are summarized in Tables 4.12 and 4.13. Comparisons between the two cases are represented in Table 4.14. With varying transportation and repositioning costs, we observe a tight performance gap and a significant cost-saving effect using foldable containers.

The adoption of foldable containers leads to a new type of cost, folding and unfolding costs. The folding and unfolding operations need additional labor, which makes the introduction of folding containers difficult in areas where labor costs are high. The computational results with various folding and unfolding costs (FC) are summarized in Tables 4.15 and 4.16. We use four different parameters for FC as

Table 4.12 Computational results with different transport/repositioning costs

		Expected Cost			Time (s)		Gap (%)	
α	TC	EV|PI	LDR	RLDR	LDR	RLDR	LDR	RLDR
0	1	193141	193252	193451	11536.8	1767.9	0.057	0.161
	2	321818	321953	322311	10777.9	2361.2	0.042	0.153
	4	577061	577234	577530	10460.1	2364.2	0.030	0.081
0.25	1	193146	193696	194362	14163.4	2795.6	0.285	0.629
	2	321837	322947	323685	13390.6	2443.9	0.345	0.574
	4	577101	578322	578915	14216.3	2548.8	0.212	0.314

Table 4.13 Computational results using standard containers only with different transport/repositioning costs

		Expected Cost			Time (s)		Gap (%)	
α	TC	EV|PI	LDR	RLDR	LDR	RLDR	LDR	RLDR
0	1	218071	218164	218280	808.1	193.8	0.043	0.096
	2	347883	347956	348031	696.7	221.0	0.021	0.042
	4	603707	603893	603931	896.3	230.6	0.031	0.037
0.25	1	218077	218700	219069	767.7	193.3	0.286	0.455
	2	347901	348544	348898	613.0	204.2	0.185	0.287
	4	603828	604801	604936	985.9	192.7	0.161	0.183

{0.05, 0.1, 0.2, 0.4} with holding cost HC=0.1. The lower folding cost represents the case when the additional labor cost of folding/unfolding operations is small. The changes in total costs are relatively small compared to the changes in the HC or TC cases, because most of the total operating costs are transportation costs, and the portion of folding and unfolding costs is small. However, the cost-saving effects decrease as folding and unfolding costs increase, because the utilization of foldable containers may decrease when folding and unfolding costs are high. We note in Table 4.16 that the total costs of the standard container case have not changed, because the total costs are not affected by folding and unfolding costs. For further analysis, we compare the transportation and repositioning quantities of foldable containers

Table 4.14 Comparisons between using standard containers only and using standard and foldable containers both

α	TC	Standard & Foldable EV|PI	LDR	RLDR	Standard EV|PI	LDR	RLDR	Gap (%) EV|PI	LDR	RLDR
0	1	193141	193252	193451	218071	218164	218280	11.43	11.42	11.37
	2	321818	321953	322311	347883	347956	348031	7.49	7.47	7.39
	4	577061	577234	577530	603707	603893	603931	4.41	4.41	4.37
0.25	1	193146	193696	194362	218077	218700	219069	11.43	11.43	11.28
	2	321837	322947	323685	347901	348544	348898	7.49	7.34	7.23
	4	577101	578322	578915	603828	604801	604936	4.43	4.38	4.30

with various folding and unfolding costs in Subsection 4.4.3.

Table 4.15 Computational results with different folding/unfolding costs

α	FC	Expected Cost EV|PI	LDR	RLDR	Time (s) LDR	RLDR	Gap (%) LDR	RLDR
0	0.05	160909	161021	161301	12616.2	2637.4	0.070	0.244
	0.1	161704	161832	162109	14423.8	2713.2	0.079	0.250
	0.2	163294	163436	163681	10641.1	2128.0	0.087	0.237
	0.4	166273	166394	166598	12961.6	2633.5	0.073	0.196
0.25	0.05	160918	161760	162368	12722.9	2818.0	0.523	0.901
	0.1	161715	162614	163217	12368.6	2361.5	0.556	0.929
	0.2	163305	164268	164813	14135.0	2730.1	0.590	0.923
	0.4	166283	167175	167676	18427.4	2927.7	0.536	0.838

In summary, we observed that the LDR and RLDR formulations perform closely to that of the EV|PI, which reflects the applicability in practice. The performance gap is tight and endurable for robustness and tractability. In addition, the computation time of the RLDR formulation is shorter than that of the LDR formulation, which reflects a competitive advantage in practice. Finally, we observed that operating costs can be reduced significantly by using foldable containers.

Table 4.16 Comparisons between using standard containers only and using standard and foldable containers both

		Standard & Foldable			Standard			Gap (%)		
α	FC	EV|PI	LDR	RLDR	EV|PI	LDR	RLDR	EV|PI	LDR	RLDR
0	0.05	160909	161021	161301	174037	174128	174243	8.16	8.14	8.02
	0.1	161704	161832	162109	174037	174128	174243	7.63	7.60	7.49
	0.2	163294	163436	163681	174037	174128	174243	6.58	6.54	6.45
	0.4	166273	166394	166598	174037	174128	174243	4.67	4.65	4.59
0.25	0.05	160918	161760	162368	174037	174128	174243	8.15	7.65	7.31
	0.1	161715	162614	163217	174037	174128	174243	7.62	7.08	6.76
	0.2	163305	164268	164813	174037	174128	174243	6.57	6.00	5.72
	0.4	166283	167175	167676	174037	174128	174243	4.66	4.16	3.92

4.4.3 Simulation Results

The optimal solutions of the LDR and RLDR formulations are optimal affine policies, that is, we obtain optimal parameters of affine policies. To see the obtained optimal policies perform well on actual uncertainty realizations, it is necessary to implement the policies with uncertainty realizations and analyze the results. Based on the affine policies, the actual decisions such as repositioning and transportation quantities are calculated when the uncertainties are realized. Therefore, we implement the affine policies from the LDR and RLDR formulations with actual uncertainty realizations and compare the results with EV|PI. To evaluate the total operating costs of implementing optimal policies, we conduct simulations with scenarios of uncertainty realizations. We use the same samples for calculating EV|PI with varying holding costs and values of α. We calculate the transporting and repositioning decision based on the optimal policies and uncertainty realization. Since the decisions of the LDR and RLDR formulations are the number of containers, we round transporting and repositioning quantities to be integers. The average of total operating costs over

10,000 samples and gap using foldable containers are summarized in Table 4.17. The gap shows the savings in operating costs using foldable containers when implementing optimal policies.

Table 4.17 Comparisons of average total operating costs over simulations with different holding costs

		Standard & Foldable			Standard			Gap (%)		
α	HC	EV|PI	LDR	RLDR	EV|PI	LDR	RLDR	EV|PI	LDR	RLDR
0	0.2	193141.0	193779.8	193869.4	218071.0	218144.4	218218.7	11.43	11.17	11.16
	0.1	161704.0	162186.2	162366.5	174036.6	174111.7	174185.3	7.09	6.85	6.79
	0.04	142436.6	143012.4	143035.5	147616.0	147681.8	147744.3	3.51	3.16	3.19
0.25	0.2	193146.3	193961.9	194371.9	218076.6	218454.4	218622.0	11.43	11.21	11.09
	0.1	161714.6	162591.6	163030.8	174045.0	174446.0	174620.9	7.08	6.80	6.64
	0.04	142446.4	143311.4	143569.8	147626.0	147994.8	148163.4	3.51	3.16	3.10

The cost ratio over total operating costs are presented in Figure 4.5. RC represents average repositioning and transportation costs over samples and HC represents average holding and penalty costs over samples. FC represents folding and unfolding costs over samples. LDR and LDR_S represent the total operating cost and the cost ratio of the LDR formulation when using standard and foldable containers and using only standard containers, respectively. RLDR and RLDR_S are defined similarly. Figure 4.5 shows that even though the folding and unfolding costs are added, the significant savings of holding and penalty costs lead to reduction in the total operating costs. As unit holding cost decreases, the proportion of holding costs in total operating costs decreases and the cost-saving effect diminishes. One of the most influential advantages of using foldable containers is the saving in holding costs.

Another important advantage is the saving of storage space at the port. Empty standard containers occupy substantial space at the port, which causes port conges-

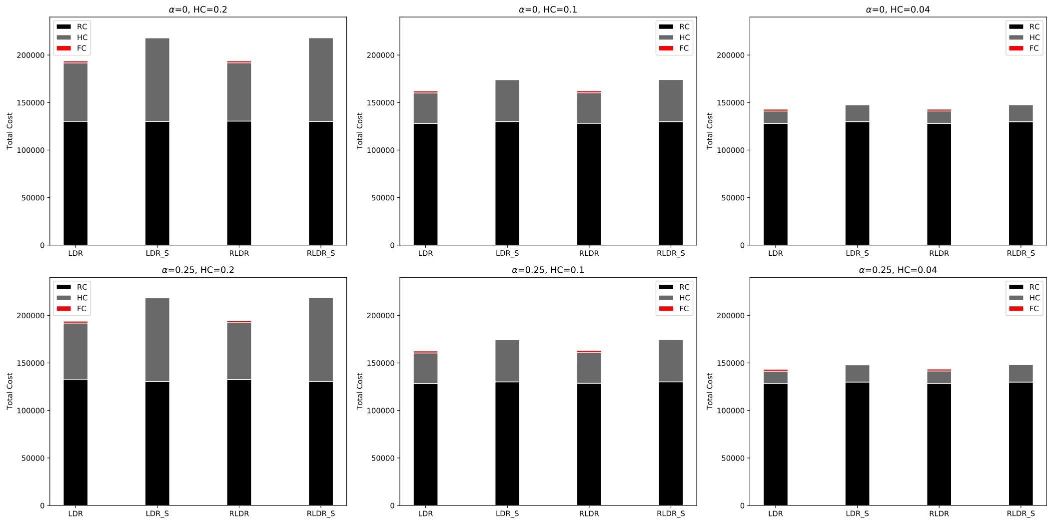

Figure 4.5 Average cost ratio over total operating costs

tion and operations delays. Foldable containers are stored in the folded state at the port, which leads to considerable storage saving. The average storage at each port is presented in Figure 4.6. Although the number of empty containers is not meaningfully reduced, the space taken for storage at the port diminishes substantially. The decline in import dominant ports such as Vancouver and Los Angeles is particularly notable, because supplied empty containers are typically stacked in import dominant ports. The saving in port storage leads to mitigation of port congestion and unnecessary operations that can not be captured by the cost-saving effect.

The utilization of foldable containers is highly influenced by the additional costs of using foldable containers, such as folding and unfolding costs. When using foldable containers, folding and unfolding operations are required with additional labor. Therefore, decisions about operations of foldable containers can change a lot depending on the folding and unfolding costs. We conduct simulations over various folding

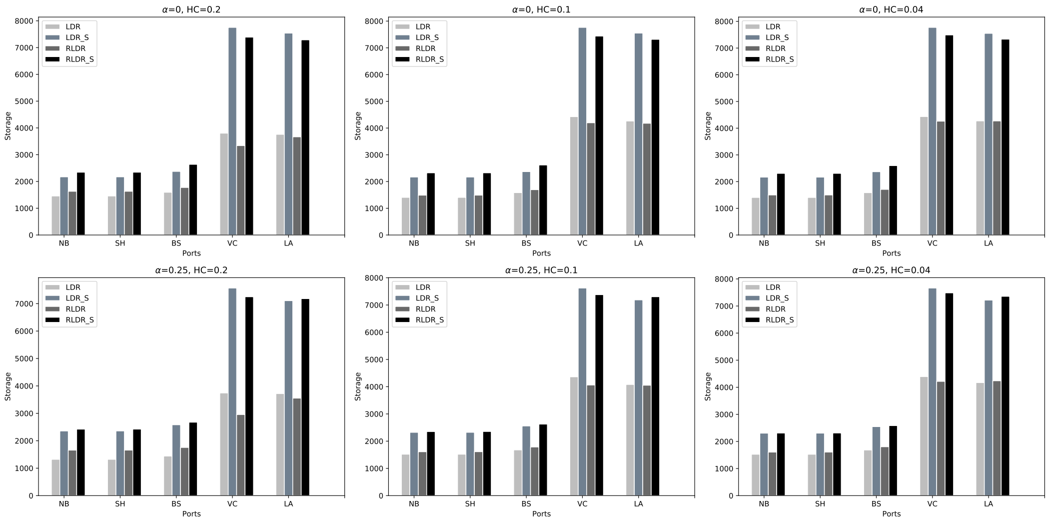

Figure 4.6 Average storage of empty containers at ports

and unfolding costs and summarize transportation and repositioning quantities of foldable containers between NA and Asia in Table 4.18. TQ (RQ) represents total transportation (repositioning) quantities of foldable containers over the planning horizon, respectively. We compare two cases, NA to Asia (from VC, LA to NB, SH, BS) and Asia to NA (from NB, SH, BS to VC, LA), for analyzing decisions between imbalanced ports. Since the ports on the NA side are import-oriented ports, empty containers are stocked at NA ports and foldable containers are repositioned from NA to Asia. However, as the folding and unfolding costs increase, the repositioning of foldable containers decreases drastically. Because of insufficiencies of empty containers at Asian ports, repositioning of foldable containers does not occur and foldable containers are used to transport goods from Asia to NA. In this case, the utilization of foldable containers decreases as folding and unfolding costs increase. Therefore, it is clear that decisions about foldable containers are influenced by folding/unfolding costs, which may affect the adoption and active utilization of foldable containers.

Table 4.18 Total transportation and repositioning quantities of foldable containers between NA-Asia network

		LDR				RLDR			
		NA to Asia		Asia to NA		NA to Asia		Asia to NA	
α	FC	TQ	RQ	TQ	RQ	TQ	RQ	TQ	RQ
0	0.05	47.1	2535.3	10900.8	0.0	0.0	2544.5	10950.8	0.0
	0.1	366.6	2531.0	10897.6	0.0	164.0	2356.5	10764.6	0.0
	0.2	392.5	2503.0	10873.7	0.0	99.0	2128.3	10523.4	0.0
	0.4	0.0	293.4	8706.1	0.0	0.0	0.0	8464.5	0.0
0.25	0.05	27.3	2832.5	11292.7	0.0	0.0	2817.5	11290.1	0.0
	0.1	331.8	2823.3	11252.4	0.0	183.0	2501.9	10921.4	0.0
	0.2	339.7	2632.1	11028.4	0.0	161.0	1976.7	10355.7	0.0
	0.4	0.0	2.3	8402.6	0.0	0.0	0.0	8439.1	0.0

4.5 Summary

In this chapter, we consider the ECR problem with foldable containers under demand uncertainty. For incorporating demand uncertainty into the decision-making process, we propose a multistage stochastic programming formulation. However, in general, a multistage stochastic formulation is computationally intractable; therefore, we adopt the adjustable robust optimization technique and propose a tractable formulation with the LDR and RLDR. The two robust formulations are tractable approximations of a multistage stochastic programming formulation and have distributionally robust properties. Hence, we evaluate the performances of proposed formulations and compare the results with EV|PI benchmark. Furthermore, we validate the advantages of using foldable containers by showing the cost-saving and storage-saving effects through simulations with scenarios. We expect our model to serve as a bridge for analyzing the advantages of foldable containers under uncertainties.

Chapter 5

Conclusions

5.1 Summary and Contributions

Inventory management is essential to the successful operation of the company. As the competition among companies intensifies, inventory management plays an important role in the survival of companies. One of the most important and practical considerations in inventory management is that the information about the probability distribution of random demand is limited. Therefore, distributionally robust optimization is utilized to handle the distributional ambiguity.

In this dissertation, we considered three inventory problems: newsvendor problem, inventory problem, and empty container repositioning problem. For the newsvendor problem, we studied the data-driven distributionally robust newsvendor model with a Wasserstein ambiguity set. To incorporate a wide range of random demand and the Wasserstein distance, we considered continuous and unbounded support $\Xi = [0, \infty)$ and the general Wasserstein order $p \in [1, \infty)$. We derived the closed-form expressions of an optimal order quantity and the worst-case distribution based on the data-driven newsvendor solution. We also considered the CVaR objective to derive risk-averse decisions.

For the inventory problem, we analyzed the data-driven distributionally robust inventory model with a Wasserstein ambiguity set centered at the empirical distribution. We adopted the policy-centered definition of time consistency and discussed time consistency of the inventory problem. We derived the sufficient condition for weak time consistency based on the monotone non-decreasing optimal base-stock levels computed by closed-form solutions of the Wasserstein newsvendor model. We also investigated the condition that base-stock levels are monotone non-decreasing when the demand process is i.i.d. Further details about the dynamic programming formulation were analyzed such as computation of optimal base-stock levels, optimality of an (s, S) policy with non-zero fixed order cost, and desirable properties of Wasserstein DRO.

For the empty container repositioning problem, we proposed a mathematical model of the empty container repositioning problem considering the use of foldable containers under demand uncertainty. To tackle the intractability of the multistage stochastic programming formulation, we adopted the linear decision rule and distributionally robust bound on the expectation of positive parts, and proposed the LDR and RLDR formulations. The proposed formulations are tractable approximations of the multistage stochastic programming formulation and have distributionally robust properties. The distributionally robust approach becomes more important and is drawing attention from academia and practice, which leads to a growing need to study phenomena related to applications of DRO. We analyzed various aspects of applications of distributionally robust optimization to inventory problems, which may affect the successful implementation of distributionally robust decisions.

5.2 Future Research

In this dissertation, we analyzed basic and abstract models such as the newsvendor model and inventory model. There are various applications of the basic models, which lead to several directions for future research. In addition, there exist several practical considerations in the empty container repositioning with foldable containers.

The results of the newsvendor model can serve as a building block for more general applications. There are several extensions of the newsvendor model, such as multi-item setting, risk-averse models with various risk measures, and pricing. Furthermore, the closed-form of a newsvendor order quantity can be applied to various applications, for example, pricing, supply chain contract, and many other operations management problems. For further research, the Wasserstein ambiguity set with the bounding of the shape of distributions, e.g., symmetry or unimodality, could be considered to incorporate prior information on distributions.

The inventory model is a building block for many operations management problems with the multistage setting, so the results of this paper can be applied to various applications. For further research, analysis about time consistency of inventory problems can be extended to other multistage problems, such as warehouse management, dynamic pricing, and retail management. When decision makers face practical problems with the multistage setting, they carefully give attention to the implementation of the optimal decision considering time consistency. Moreover, the optimality of a base-stock policy or an (s, S) policy can be extended to distributionally robust inventory models with other ambiguity sets which have suitable properties similar to the Wasserstein ambiguity set. Because the conditions used in the proof of optimality are quite general, it can be applied to a variety of extensions.

For further research for the empty container repositioning problem, we intend to extend our study to account for the pooling effects of foldable containers. In inventory management models, a centralized system with one large distribution center leads to lower variability than a decentralized system with many distribution centers. Likewise, we can view empty containers as inventory and repositioning decisions of empty containers as ordering decisions. There may be risk pooling effects similar to those in inventory management models, that is, the variability of operating costs may decrease because multiple folded containers act like one standard container. On the other hand, because of the tremendous operating costs of hinterland transport, the use of foldable containers in hinterland transport proves to be another interesting topic. In addition, port congestion and empty container movement with trucks are important topics in maritime logistics. Therefore, the use of foldable containers in hinterland transport will be influential to shipping companies and container terminals.

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국문초록

재고관리는 운영 관리 분야에서 전통적인 문제로, 수요의 불확실성 하에서 최적의 주문 정책을 결정하는 문제다. 의사결정자는 성과 척도로 표현되는 회사의 목적을 달성하기 위해 계획 기간 동안의 주문량을 선택한다. 전통적인 재고 관리 연구에서는 수요의 확률 분포에 대한 정확한 정보를 알고 있다고 가정하지만, 현실에서는 확률 분포에 대한 제한된 정보만 이용가능하다. 이러한 어려움을 해결하기 위해 의사결정자는 알려지지 않은 수요의 확률 분포를 포함할 수 있는 후보 분포들의 집합인 모호성 집합을 고려하고, 이 집합 위에서 최악의 평균 비용을 최소화한다. 이 접근방법을 분포 강건 최적화라고 하며, 많은 운영 관리 문제에 널리 적용되고 있다. 재고관리 문제의 분포에 대한 정보 부족을 다루기 위해 분포 강건 방법을 활용한다.

본 논문에서는 신문가판원 문제, 재고관리 문제, 공컨테이너 재배치 문제 등 세 가지 서로 밀접하게 관련된 문제를 고려한다. 세 가지 문제 모두 수요의 불확실성 하에서 의사결정을 연구하지만, 수요의 확률 분포에 대한 제한된 정보만 주어진다. 이에 따라 분포 강건 방법을 적용하고 분포 강건 모형들의 다양한 측면을 분석한다. 첫째, 데이터로부터 만들어진 경험적 분포로부터 Wasserstein 거리 기준으로 가까운 확률 분포들을 고려한 데이터 기반의 분포 강건 신문가판원 모형을 연구하고, 최적 주문량의 닫힌 형태의 표현을 도출한다. 둘째, 신문가판원 모형이 다단계 문제로 확장된 재고관리 문제를 고려한다. 분포 강건 재고 문제의 다단계 특성에서 의사결정자가 신중하게 고려해야할 사항은 시간 일관성이다. 시간 일관성은 첫 시점에 도출한 최적 정책이 계획 기간 동안 최적성을 유지해야 한다는 것을 의미한다. Wasserstein 모호성 집합을 고려한 분포 강건 재고 모형의 시간 일관성을 분석하고자 한다. 셋째, 접이식 컨테이너를 고려한 공컨테이너 재배치 문제를 고려하는데, 이는 재고 관리 문제의 현실적인 응용 문제이

다. 수요 불확실성 하에서 접이식 컨테이너의 사용을 고려한 공컨테이너 재배치 문제의 수리적 모형을 제안한다. 다단계 추계 계획의 계산 복잡도 문제를 해결하기 위해 선형 결정 규칙에 기반한 수리 모형을 제시하는데, 이는 다단계 추계 계획의 계산가능하며 분포 강건한 근사가 된다. 또한 각각의 모형과 연구 결과를 검증하기 위해 수치 실험을 진행한다.

주요어: 분포 강건 최적화, 신문가판원 모형, 재고관리, 공컨테이너 재배치, Wasserstein 거리

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