creative commons

C O M M O N S D E E D

저작자표시-비영리-변경금지 2.0 대한민국

이용자는 아래의 조건을 따르는 경우에 한하여 자유롭게

- 이 저작물을 복제, 배포, 전송, 전시, 공연 및 방송할 수 있습니다.

다음과 같은 조건을 따라야 합니다:

저작자표시. 귀하는 원저작자를 표시하여야 합니다.

비영리. 귀하는 이 저작물을 영리 목적으로 이용할 수 없습니다.

변경금지. 귀하는 이 저작물을 개작, 변형 또는 가공할 수 없습니다.

- 귀하는, 이 저작물의 재이용이나 배포의 경우, 이 저작물에 적용된 이용허락조건을 명확하게 나타내어야 합니다.
- 저작권자로부터 별도의 허가를 받으면 이러한 조건들은 적용되지 않습니다.

저작권법에 따른 이용자의 권리는 위의 내용에 의하여 영향을 받지 않습니다.

이것은 이용허락규약(Legal Code)을 이해하기 쉽게 요약한 것입니다.

Disclaimer

dCollection

Ph.D. DISSERTATION

Index Coding With Erroneous Side Information and Multiple Senders

보조 정보 오류 및 다중 송신기 하에서 인덱스 부호화에
관한 연구

BY

KIM JAE-WON

FEBRUARY 2020

DEPARTMENT OF ELECTRICAL AND
COMPUTER ENGINEERING
COLLEGE OF ENGINEERING
SEOUL NATIONAL UNIVERSITY

Abstract

In this dissertation, three main contributions are given as i) index coding with erroneous side information, ii) code equivalences between network codes with link errors and index codes with side information errors, and iii) index coding with multiple senders and extension to a cellular network.

First, index coding with erroneous side information is studied. Although side information is a crucial part of index coding, the existence of side information errors was not considered. Since side information is stored in memory devices and there are errors in those devices, it is important to consider side information errors to utilize index coding in a realistic scenario. Dealing with side information errors, an encoding method based on the proposed fitting matrix is introduced and a decoding procedure based on the syndrome decoding is proposed. Some bounds for the optimal index codelength with side information errors are proposed and a special graph called a δ_s-cycle is found. It is proved that the existence of a δ_s-cycle is a necessary and sufficient condition for reducing index codelength. In addition, the results on erroneous side information are generalized for a scenario considering both side information errors and channel errors.

Second, code equivalences between network codes with link errors and index codes with side information errors are studied. There is a code equivalence between network codes and index codes for a given network coding instance. However, a code equivalence between them for a given index coding instance was not studied. To complete code equivalences between them, a code equivalence for a given index coding instance is proposed. In order to find the valid corresponding network coding instance for a given index coding instance, the index coding instance has to be modified and a method converting the index coding instance into the corresponding network coding instance is proposed. Furthermore, code equivalence results are generalized considering link errors and side information errors.

Third, index coding with multiple senders is studied and it is extended for a cellular network. In general, one sender is considered for index coding. However, there are lots of scenarios, where messages are distributed in multiple senders. Thus, index coding with multiple senders has to be studied. An encoding method based on the proposed fitting matrix is studied and a necessary and sufficient condition for reducing index codelength with multiple senders is proved. Since all receivers do not belong to coverage of all senders in reality, index coding for a cellular network is studied.

keywords: Cellular network, fitting matrix, index coding, link errors, multiple senders, network coding, side information, side information errors.

student number: 2014-21663

Contents

List of Tables

List of Figures

Chapter 1

INTRODUCTION

1.1 Background

Index coding has attracted significant attention in various research areas since it was first introduced by Birk and Kol [1]. Due to its relevance to various topics in information theory, lots of research has been done on index coding even though it was first considered for the satellite communication systems. The conventional index coding instance consists of one sender and several receivers. A sender has all messages and each receiver wants a subset of the messages and also has a subset of the messages as side information. In this case, the purpose of index coding is exploiting side information to reduce the number of transmissions in an error-free broadcast channel. Bar-Yossef *et al.* proved that the optimal codelength of linear index codes is equal to the parameter $minrk$ of the fitting matrix of the side information graph and suggested an optimal construction method for the linear index codes [2]. In addition, it was proved that there is a nonlinear index code which outperforms the optimal linear index codes [3]. Random index coding was studied for infinitely long message length [4]. In order to find the optimal index code, a lot of index coding schemes have been researched [2]–[9].

In addition to researches on finding index coding schemes, there are some researches on finding the relationship between index coding and other problems. In [10], it was proved that any network coding instance can be transformed to the corresponding index coding instance and a solution for the network coding instance exists if and only if a solution for the corresponding index coding instance exists. It was studied that topological interference management (TIM) can be performed using index coding [11]. Furthermore, it was shown that there is a duality between distributed storage and index coding [12], [13].

Most index coding problems assume that the sender knows the side information graph and each receiver has some subsets of messages as side information. Recently, Kao *et al.* researched a case where the sender only knows the probability distribution of side information in the receivers [14]. The index coding problems where a form of side information in each receiver is a linear combination of messages were studied [15], [16]. While most of the index coding problems are studied in an error-free broadcast channel, there has been some work on the index codes with channel errors [17]-[18]. In particular, Dau *et al.* introduced error correcting index codes (ECIC) in the erroneous broadcast channel and algebraically analyzed them [17]. There has also been research on capacity analysis and application of index codes with side information over the additive white Gaussian noise (AWGN) channel [19], [20]. In [18], Byrne and Calderini studied index coding instances with channel errors and coded side information and extended the results of [17]. Furthermore, functional index coding instances were introduced in [21]. As we can see from the previous works, index coding problems have been generalized further and become more realistic.

1.2 Overview of Dissertation

This dissertation is organized as follows.

In Chapter 2, the basic concepts of index coding are introduced. An Index coding instance and the corresponding side information graph are described in Section 2.1. In Section 2.2, the optimal linear index codelength is given based on fitting matrices and the minimum rank of them.

In Chapter 3, index coding with erroneous side information is studied. The problem formulation and some results are presented in Section 3.1. In Section 3.2, an encoding procedure based on the proposed fitting matrix and a decoding procedure based on the syndrome decoding are proposed. The graphical properties and bounds for the optimal codelength of the index codes with side information errors are derived in Section 3.3. Many properties are generalized considering both side information errors and channel errors in Section 3.4.

In Chapter 4, code equivalences between network codes and index codes are proposed for both a given network coding instance and a given index coding instance. In Section 4.1, the problem formulation and some results are presented. The main results on a code equivalence between a network code and an index code for a given index coding instance are derived in Section 4.2. Then, the code equivalence results are generalized to erroneous cases in Section 4.3.

In Chapter 5, index coding with multiple senders is studied and it is extended for a cellular network. In Section 5.1, the problem setup is introduced and the encoding method of linear index codes based on the fitting matrix is studied in Section 5.2. Next, some properties of linear index codes with multiple senders are studied in Section 5.3. Then, index coding in a cellular network is studied in Section 5.4.

Finally, the concluding remarks are given in Chapter 6.

1.3 Notations

In this section, some notations used in the dissertation are described. Let $\mathbb{F}_q$ be the finite field of size q, where q is a power of prime and $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. Let $Z[n] = \{1, 2, \dots, n\}$ for a positive integer n. For a vector $\mathbf{x} \in \mathbb{F}_q^n$, $\mathrm{wt}(\mathbf{x})$ denotes the Hamming weight of $\mathbf{x}$. Let $\mathbf{x}_D$ be a subvector $(x_{i_1}, x_{i_2}, \dots, x_{i_{|D|}})$ of a vector $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{F}_q^n$ for a subset $D = \{i_1, i_2, \dots, i_{|D|}\} \subseteq Z[n]$, where $i_1 < i_2 < \dots < i_{|D|}$. I also introduce a submatrix A_D of $A \in {\mathbb{F}_q}^{n\times N}$, that is, the matrix consisting of $|D|$ rows of A as

$$A_D = \begin{pmatrix} A_{i_1} \\ A_{i_2} \\ \vdots \\ A_{i_{|D|}} \end{pmatrix} \tag{1.1}$$

where A_i is the ith row of A.

Chapter 2

Preliminaries

In this chapter, some preliminaries of index coding are introduced. First, an index coding instance and its side information graph are described. Then, the conventional fitting matrix for the index coding problem is introduced and an encoding method based on the fitting matrices is given.

2.1 Index Coding Instance

The conventional index coding problem consists of one sender and m receivers. One sender has all messages $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{F}_q^n$ and it broadcasts encoded mes-

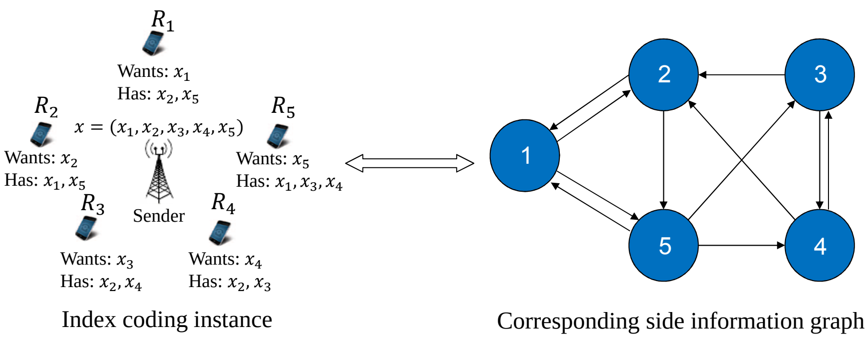

Figure 2.1: An index coding instance and its corresponding side information graph.

sages through the error-free broadcast channel. Each receiver R_i for $i \in Z[m]$ wants some messages and also has some messages as side information. The purpose of index coding is exploiting side information in order to reduce codelength. If the number of receivers and the number of messages are the same and every receiver wants one message with the same index, a unipartite side information graph $\mathcal{G}$ can describe a given index coding instance. A node in $\mathcal{G}$ means both a receiver and a message. A directed edge from one node (say 1) to another node (say 2) means that receiver 1 has x_2 as side information. Fig. 2.1 shows an example for an index coding instance and its side information graph. A side information graph shows the wanted messages and side information of all receivers.

If the number of receivers is different from the number of messages or there is a receiver wanting more than one messages, a side information graph $\mathcal{G}$ becomes bipartite, where each receiver i wants to receive one message $x_{f(i)}$. There are two types of nodes, that is, user (receiver) nodes and packet (message) nodes. Then, a directed edge from a receiver node to a message node means that the receiver has the message as side information and a directed edge from a message node to a receiver node means that the receiver wants to receive the message. Fig. 2.2 shows an example for a bipartite side information graph $\mathcal{G}$. In Fig. 2.2, receiver 3 wants to receive message 1 and has messages 2 and 3 as side information. In the index coding problem, it is assumed that a sender knows a side information graph $\mathcal{G}$.

2.2 Fitting Matrix

It was proved that the optimal codelength of scalar linear index codes can be found from the minimum rank of fitting matrices [2]. Assume that the number of receivers is m, the number of messages is n, and each receiver wants one message. From results of [22], the conventional fitting matrix for the single sender index coding problem is easily deduced as in the following definition.

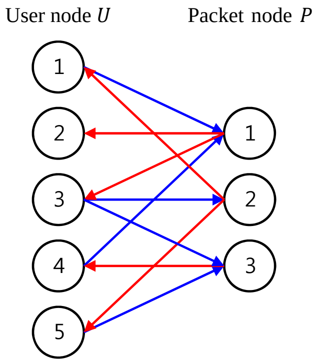

Figure 2.2: An example of a directed bipartite side information graph.

Definition 2.1. *An $n \times m$ matrix F^c over $\mathbb{F}_q$ fits $\mathcal{G}$ if and only if the followings hold.*

1. $F^c_{f(k),k} = 1$ *for* $k \in Z[m]$.

2. $F^c_{i,k} = 0$, *where* x_i *is not known to receiver* k *and* $k \in Z[m]$.

In addition, it was studied in [22] that the minimum rank of fitting matrices is the optimal scalar linear index codelength for the single sender index coding problem for any m, n, and finite field $\mathbb{F}_q$. It is noted that each column of F^c represents each receiver and an optimal generator matrix can be obtained from a fitting matrix having the minimum rank after deleting linearly dependent columns. Thus, the conventional scalar linear index coding problem can be solved by finding a fitting matrix having the minimum rank. However, the conventional fitting matrix F^c cannot give a solution for a general index coding problem with the vector case or the nonlinear case or side information errors or channel errors.

Chapter 3

Index Coding With Erroneous Side Information

In this chapter, new index coding problems with erroneous side information are presented, where each receiver has erroneous side information. In the conventional index coding, it is assumed that every receiver can exploit its side information directly because there is no side information error. However, there is always a possibility of memory errors in the receivers, which causes side information errors. Since there are some instances where side information is attained by sending messages through the broadcast channel, channel errors also cause erroneous side information. Thus, we have to consider a possibility having erroneous side information in the index coding problem.

There are several applications of index coding with erroneous side information such as TIM. It is known that the interference channels of a receiver in TIM correspond to the messages, which are not the side information of the receiver in index coding. If the interference channels vary due to the moving receivers or are misunderstood as the non-interference channels from the false channel state information in TIM, this corresponds to index coding with erroneous side information. Thus, index coding with erroneous side information can be applied for the effective solutions of TIM.

In this chapter, I propose the encoding and decoding procedures of index codes with side information errors (ICSIE), where each receiver has erroneous side information symbols in the error-free broadcast channel. One of the most important parameters in index coding is the codelength. Thus, the bounds on the optimal codelength of the proposed ICSIE are derived and its crucial graph, called a δ_s-cycle, similar to the cycle in the conventional index coding is proposed. I relate the conventional index codes with the ICSIE by using the proposed bound and compare the cycle of the conventional index coding with the δ_s-cycle of the proposed ICSIE. Furthermore, it is shown that there is a similarity between the generator matrix of the ICSIE and the transpose of the parity check matrix of the error correcting code when the corresponding side information graph is a clique. Finally, the ECIC in [17] is generalized by using the proposed ICSIE, which is called a generalized error correcting index code (GECIC). That is, I consider the more general scenario, where both channel errors and side information errors exist.

3.1 Problem Formulation and Some Results

3.1.1 Problem Formulation for Index Coding With Erroneous Side Information

The conventional index coding is explained before introducing index coding with erroneous side information. I consider the index coding problem, where all information packets are elements in $\mathbb{F}_q$ and each receiver just wants one packet. This scenario is considered because any index coding problems can be converted to the problem of the above scenario if the size of packets is fixed. If the certain receiver wants d packets, we can split the receiver into the d receivers, each of which wants to receive one packet with the same side information. In this scenario, I can describe the conventional index coding with side information problem as follows. There are one sender which has n information packets (messages) as $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{F}_q^n$ and m receivers (or users) $R_1, R_2, \ldots, R_m$, having subvectors of $\mathbf{x}$ as side information. Let $\mathcal{X}_i$ be the set of side information indices of the receiver R_i for $i \in Z[m]$. That is, each receiver R_i already knows the subvector $\mathbf{x}_{\mathcal{X}_i}$. Each receiver R_i wants to receive one element in $\mathbf{x}$, called the wanted packet denoted by $x_{f(i)}$ and it is assumed that $\{f(i)\} \cap \mathcal{X}_i = \phi$. It is assumed that the sender knows the side information graph $\mathcal{G}$ and broadcasts a codeword to receivers through the error-free channel in the conventional index coding problem.

Having side information in each receiver is a crucial part of the index coding problem. I propose a new index coding problem with erroneous side information by changing the side information condition in the conventional index coding problem, that is, each receiver R_i has at most δ_s erroneous side information symbols. In the proposed index coding problem, a sender knows a side information graph $\mathcal{G}$ but does not know which side information is erroneous in each receiver. In addition, each receiver does not know which side information is erroneous. Furthermore, we can also consider additive channel errors in each receiver as in [17]. That is, each receiver receives $\mathbf{y} + \boldsymbol{\epsilon}_i$, where $\mathbf{y}$ is a codeword and $\boldsymbol{\epsilon}_i$ is an additive error vector such that $\mathbf{y}, \boldsymbol{\epsilon}_i \in \mathbb{F}_q^N$ and $\mathrm{wt}(\boldsymbol{\epsilon}_i) \leq \delta_c$.

Definition 3.1. *A generalized error correcting index code with parameters $(\delta_s, \delta_c, \mathcal{G})$ over $\mathbb{F}_q$, denoted by a $(\delta_s, \delta_c, \mathcal{G})$-GECIC is a set of codewords having:*

1. *An encoding function $E : \mathbb{F}_q^n \to \mathbb{F}_q^N$.*
2. *A set of decoding functions $D_1, D_2, \ldots, D_m$ such that $D_i : \mathbb{F}_q^N \times \mathbb{F}_q^{|\mathcal{X}_i|} \to \mathbb{F}_q$ satisfying*

$$D_i(E(\mathbf{x}) + \boldsymbol{\epsilon}_i, \hat{\mathbf{x}}_{\mathcal{X}_i}) = x_{f(i)} \tag{3.1}$$

for all $i \in Z[m]$, $\mathbf{x} \in \mathbb{F}_q^n$, $\boldsymbol{\epsilon}_i \in \mathbb{F}_q^N$ with $\mathrm{wt}(\boldsymbol{\epsilon}_i) \leq \delta_c$, and $\mathrm{wt}(\mathbf{x}_{\mathcal{X}_i} - \hat{\mathbf{x}}_{\mathcal{X}_i}) \leq \delta_s$, where $\hat{\mathbf{x}}_{\mathcal{X}_i}$ is the erroneous side information vector of the receiver R_i.

Here, $\mathrm{wt}(\mathbf{x}_{\mathcal{X}_i} - \hat{\mathbf{x}}_{\mathcal{X}_i}) \leq \delta_s$ means that the maximum number of side information errors is δ_s for each receiver. In this chapter, I consider a linear index code. That is, $E(\mathbf{x}) = \mathbf{x}G$ for all $\mathbf{x} \in \mathbb{F}_q^n$, where $G \in \mathbb{F}_q^{n \times N}$ is a generator matrix of the index code and N denotes the codelength of the $(\delta_s, \delta_c, \mathcal{G})$-GECIC. Let $N_{\mathrm{opt}}^q(\delta_s, \delta_c, \mathcal{G})$ be the optimal codelength of the $(\delta_s, \delta_c, \mathcal{G})$-GECIC over $\mathbb{F}_q$. If $\delta_c = 0$, a $(\delta_s, 0, \mathcal{G})$-GECIC is called a $(\delta_s, \mathcal{G})$-index code with side information errors, denoted by a $(\delta_s, \mathcal{G})$-ICSIE and the optimal codelength $N^q_{\text{opt}}(\delta_s, \delta_c, \mathcal{G})$ is modified to $N^q_{\text{opt}}(\delta_s, \mathcal{G})$. Similarly, for $\delta_s = 0$, we have $(\delta_c, \mathcal{G})$-ECIC and $N^q_{\text{opt}}(\delta_c, \mathcal{G})$ as in [17].

3.1.2 Property of Generator Matrix of GECIC

I find the property of the generator matrix for the proposed $(\delta_s, \delta_c, \mathcal{G})$-GECIC by generalizing Lemma 3.8 in [17]. Let $\mathcal{I}(q, \mathcal{G}, \delta_s)$ be the set of vectors defined by

$$\mathcal{I}(q, \mathcal{G}, \delta_s) = \bigcup_{i \in Z[m]} \mathcal{I}_i(q, \mathcal{G}, \delta_s) \tag{3.2}$$

where $\mathcal{I}_i(q, \mathcal{G}, \delta_s) = \{\mathbf{z} \in \mathbb{F}_q^n : \text{wt}(\mathbf{z}_{\mathcal{X}_i}) \leq 2\delta_s, z_{f(i)} \neq 0\}$. The support set of $\mathcal{I}(q, \mathcal{G}, \delta_s)$ is defined as

$$J(\mathcal{G}, \delta_s) = \bigcup_{i \in Z[m]} \{\{f(i)\} \cup Y_i \cup I_i : Y_i \subseteq \mathcal{Y}_i, I_i \subseteq \mathcal{X}_i\} \tag{3.3}$$

where $|I_i| \leq 2\delta_s$ and $\mathcal{Y}_i = Z[n] \setminus \big(\{f(i)\} \cup \mathcal{X}_i\big)$.

Then, the property of the generator matrix of the $(\delta_s, \delta_c, \mathcal{G})$-GECIC is given in the following theorem.

Theorem 3.1 (Generalization of Lemma 3.8 in [17]). *A matrix G is a generator matrix of the $(\delta_s, \delta_c, \mathcal{G})$-GECIC if and only if*

$$\text{wt}(\mathbf{z}G) \geq 2\delta_c + 1, \textit{ for all } \mathbf{z} \in \mathcal{I}(q, \mathcal{G}, \delta_s). \tag{3.4}$$

Proof. It will be proved in the similar manner as in [17], that is, I use the similar concepts as Hamming spheres of the classical error correcting codes. Here, I find the set of message vectors which should be distinguished by the received codewords. Let $B(\mathbf{x}, \delta_c)$ be the Hamming sphere of the received codeword $\mathbf{y}$ defined by

$$B(\mathbf{x}, \delta_c) = \{\mathbf{y} \in \mathbb{F}_q^N : \mathbf{y} = \mathbf{x}G + \boldsymbol{\epsilon}, \boldsymbol{\epsilon} \in \mathbb{F}_q^N, \text{and } \text{wt}(\boldsymbol{\epsilon}) \leq \delta_c\}. \tag{3.5}$$

First, I prove that the receiver R_i can recover the wanted packet $x_{f(i)}$ if and only if

$$B(\mathbf{x}, \delta_c) \cap B(\mathbf{x}', \delta_c) = \phi \tag{3.6}$$

for every pair $\mathbf{x}$ and $\mathbf{x}' \in \mathbb{F}_q^n$ such that $x_{f(i)} \neq x'_{f(i)}$ and $\mathrm{wt}(\mathbf{x}_{\mathcal{X}_i} - \mathbf{x}'_{\mathcal{X}_i}) \leq 2\delta_s$.

The only difference from Lemma 3.8 in [17] is that I replace the condition $\mathbf{x}_{\mathcal{X}_i} = \mathbf{x}'_{\mathcal{X}_i}$ with $\mathrm{wt}(\mathbf{x}_{\mathcal{X}_i} - \mathbf{x}'_{\mathcal{X}_i}) \leq 2\delta_s$. The side information vectors $\mathbf{x}_{\mathcal{X}_i}$ and $\mathbf{x}'_{\mathcal{X}_i}$ of the receiver R_i satisfy the inequality $\mathrm{wt}(\mathbf{x}_{\mathcal{X}_i} - \mathbf{x}'_{\mathcal{X}_i}) \leq 2\delta_s$ if they can be the same by changing at most δ_s side information symbols, respectively. Now, I prove the above statement as follows.

Necessity: R_i has to recover $x_{f(i)}$ by using side information and the received codeword. Suppose that there are two vectors $\mathbf{x}$ and $\mathbf{x}' \in \mathbb{F}_q^n$ such that $x_{f(i)} \neq x'_{f(i)}$. Since R_i can always recover $x_{f(i)}$, R_i has to distinguish such $\mathbf{x}$ and $\mathbf{x}'$. If $\mathrm{wt}(\mathbf{x}_{\mathcal{X}_i} - \mathbf{x}'_{\mathcal{X}_i}) \geq 2\delta_s + 1$, R_i can distinguish $\mathbf{x}$ and $\mathbf{x}'$ from the side information. However, if $\mathrm{wt}(\mathbf{x}_{\mathcal{X}_i} - \mathbf{x}'_{\mathcal{X}_i}) \leq 2\delta_s$, R_i cannot distinguish $\mathbf{x}$ and $\mathbf{x}'$ from the side information and thus the received codewords of $\mathbf{x}$ and $\mathbf{x}'$ should be distinguished, which corresponds to (3.6).

Sufficiency: For $\mathbf{x}$ and $\mathbf{x}' \in \mathbb{F}_q^n$, consider the cases where $x_{f(i)} \neq x'_{f(i)}$. For these cases, it is easy to note that the sufficiency also holds. Thus, we only have to consider the remaining cases such that $x_{f(i)} = x'_{f(i)}$ to prove the sufficiency. In fact, for such $\mathbf{x}$ and $\mathbf{x}'$, R_i does not need to distinguish $\mathbf{x}$ and $\mathbf{x}'$ because $x_{f(i)} = x'_{f(i)}$ and R_i is only interested in $x_{f(i)}$. Thus, there is no restriction on such $\mathbf{x}$ and $\mathbf{x}'$ for R_i to recover $x_{f(i)}$.

Let $\mathbf{z} = \mathbf{x} - \mathbf{x}'$. Since each receiver R_i has to recover $x_{f(i)}$, (3.6) should be satisfied for all $i \in Z[m]$. That is, the matrix G corresponds to the generator matrix of the $(\delta_s, \delta_c, \mathcal{G})$-GECIC if and only if $\mathrm{wt}(\mathbf{z}G) \geq 2\delta_c + 1$ for all $\mathbf{z} \in \mathcal{I}(q, \mathcal{G}, \delta_s)$. $\square$

Here are several remarks regarding the above theorem.

Remark 3.1. *It is obvious that G is a generator matrix of the $(\delta_s, \delta_c, \mathcal{G})$-GECIC if and only if*

$$\mathrm{wt}(\Sigma_{i\in K} z_i G_i) \geq 2\delta_c + 1 \tag{3.7}$$

for all $K \in J(\mathcal{G}, \delta_s)$ and all $z_i \in \mathbb{F}_q^*$.

Remark 3.2. *Since the receiver R_i is only interested in $x_{f(i)}$, it is possible to have $B(\mathbf{x}, \delta_c) \cap B(\mathbf{x}', \delta_c) \neq \phi$ for $\mathrm{wt}(\mathbf{x}_{\mathcal{X}_i} - \mathbf{x}'_{\mathcal{X}_i}) \leq 2\delta_s$ and $x_{f(i)} = x'_{f(i)}$. It means that R_i does not need to distinguish $\mathbf{x}$ and $\mathbf{x}'$ because $x_{f(i)} = x'_{f(i)}$.*

Remark 3.3. *For a $(\delta_s, \mathcal{G})$-ICSIE, the inequality $\mathrm{wt}(\mathbf{z}G) \geq 2\delta_c + 1$ becomes $\mathbf{z}G \neq \mathbf{0}$. If the side information is assumed to be erased, we have*

$$\mathcal{I}_i(q, \mathcal{G}, \delta_s) = \{\mathbf{z} \in \mathbb{F}_q^n : \mathrm{wt}(\mathbf{z}_{\mathcal{X}_i}) \leq \delta_s, z_{f(i)} \neq 0\}. \tag{3.8}$$

I provide an example for the aforementioned theorem.

Example 3.1. *Let $q = 2$, $m = n = 4$, $\delta_s = 1$, $f(i) = i$, and $\mathcal{X}_i = Z[4] \setminus \{i\}$ for all $i \in Z[4]$. In general, the uncoded case is the worst case in the error-free channel, that is, the codelength is $n = 4$. However, we can construct a $(\delta_s = 1, \mathcal{G})$-ICSIE with codelength 3. From Theorem 3.1, it is clear that $\mathcal{I}(q, \mathcal{G}, \delta_s = 1)$ includes all vectors in $\mathbb{F}_2^4$ except $(1, 1, 1, 1)$ and $(0, 0, 0, 0)$. Assume that we have a 4×3 matrix G as*

$$G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{3.9}$$

Then, we have $\mathbf{z}G \neq \mathbf{0}$ for all $\mathbf{z} \in \mathcal{I}(q, \mathcal{G}, \delta_s = 1)$. Thus, the above matrix G is a generator matrix of the $(\delta_s = 1, \mathcal{G})$-ICSIE.

From Lemma 8.2 in [17] and Theorem 3.1, a relation between an ECIC and a GECIC can be deduced. In [17], $\Gamma = \{(m, n, \mathcal{X}, f)\}$ is defined as a set of index coding instances and an ECIC with δ_c is said to be static under the set Γ if it satisfies all instances in Γ. Now, I define Γ_{δ_s} for a side information graph $\mathcal{G}$ as a set of all instances, each of which is constructed by deleting $\min(2\delta_s, |\mathcal{X}_i|)$ outgoing edges from each receiver R_i in $\mathcal{G}$. Then, we have the following lemma.

Lemma 3.1. *For a side information graph $\mathcal{G}$, an ECIC with δ_c static under Γ_{δ_s} is equivalent to a $(\delta_s, \delta_c, \mathcal{G})$-GECIC.*

Proof. Let $\mathcal{I}(\Gamma_{\delta_s}) = \cup_{\bar{\mathcal{G}}}\mathcal{I}(q, \bar{\mathcal{G}}, 0)$, where $\bar{\mathcal{G}}$ is constructed by deleting $\min(2\delta_s, |\mathcal{X}_i|)$ outgoing edges from each receiver R_i in $\mathcal{G}$. From Lemma 8.2 in [17] and Theorem 3.1, two problems are equivalent if $\mathcal{I}(\Gamma_{\delta_s}) = \mathcal{I}(q, \mathcal{G}, \delta_s)$. Since each $\bar{\mathcal{G}}$ represents the vector $\mathbf{z} \in \mathcal{I}_i(q, \mathcal{G}, \delta_s)$ by selecting at most $2\delta_s$ non-zero elements of $\mathbf{z}_{\mathcal{X}_i}$, we have $\mathcal{I}(\Gamma_{\delta_s}) = \mathcal{I}(q, \mathcal{G}, \delta_s)$. □

From Lemma 3.1, we can further infer that a GECIC is equivalent to an ECIC for a modified index coding instance. First, I modify $\mathcal{G}$ to $\tilde{\mathcal{G}}$, where there are $\binom{|\mathcal{X}_i|}{2\delta_s}$ corresponding receivers of R_i in $\mathcal{G}$ wanting the same $x_{f(i)}$ with $\mathcal{X}_i^* = \mathcal{X}_i \setminus I_i$ such that $I_i \subseteq \mathcal{X}_i$ and $|I_i| = 2\delta_s$ in $\tilde{\mathcal{G}}$. Then, we have the following lemma.

Lemma 3.2. *For a side information graph $\mathcal{G}$, a $(\delta_c, \tilde{\mathcal{G}})$-ECIC is equivalent to a $(\delta_s, \delta_c, \mathcal{G})$-GECIC.*

Proof. We can easily notice that a $(\delta_c, \tilde{\mathcal{G}})$-ECIC is equivalent to an ECIC with δ_c static under Γ_{δ_s} and thus to a $(\delta_s, \delta_c, \mathcal{G})$-GECIC from Lemma 3.1. □

3.2 Encoding and Decoding of $(\delta_s, \mathcal{G})$-ICSIE

In this section, I propose the encoding and decoding procedures of the proposed index code with erroneous side information in the error-free broadcast channel, that is, the ICSIE.

3.2.1 Encoding Procedure

In general, design of the index codes is to find a generator matrix with the minimum codelength for the given side information graph, called the optimal index codes. In fact, any linearly dependent equations of the message packets can be generated by their minimum set of linearly independent equations. Thus, design of the optimal $(\delta_s, \mathcal{G})$-ICSIE corresponds to finding the minimum number of linearly independent equations of message packets, whose generator matrix satisfies the property in Theorem 3.1 with no channel error. Thus, we have the following remark for the $(\delta_s, \mathcal{G})$-ICSIE.

Remark 3.4. *If a generator matrix G of the $(\delta_s, \mathcal{G})$-ICSIE has rank less than or equal to the codelength N, the matrix deleting any dependent columns from G can also be its generator matrix. Thus, the generator matrix G_{opt} of the optimal $(\delta_s, \mathcal{G})$-ICSIE should have the rank $N^q_{\text{opt}}(\delta_s, \mathcal{G})$.*

I propose the optimal construction method of the $(\delta_s, \mathcal{G})$-ICSIE, which is similar to that of the conventional index code in [2]. First, I generalize two well known definitions, the fitting matrices and their minimum rank for the given side information graph in [2]. From Lemma 3.2, I can find a generalized fitting matrix for $\mathcal{G}$ through a fitting matrix for $\tilde{\mathcal{G}}$ as in the following definition.

Definition 3.2. *Let $T_i = \{\mathbf{i}_1, \ldots, \mathbf{i}_{\binom{|\mathcal{X}_i|}{2\delta_s}}\}$ for $i \in Z[m]$, where $\mathbf{i}_j$ denotes the set of chosen indices from $\mathcal{X}_i$ with cardinality $2\delta_s$ for $|\mathcal{X}_i| \geq 2\delta_s$ and otherwise, $T_i = \{\mathbf{i}_1\} = \{\mathcal{X}_i\}$. An $n \times \Sigma_{i \in Z[m]} \binom{|\mathcal{X}_i|}{2\delta_s}$ matrix A_g is said to be a generalized fitting matrix for $\mathcal{G}$ if A_g satisfies the followings:*

1. $A_g = [A^{(i)}_{ab}]$ *consists of* m *disjoint* $n \times \binom{|\mathcal{X}_i|}{2\delta_s}$ *submatrices* $A^{(i)}$ *for* $i \in Z[m]$.

2. *For* $i \in Z[m]$ *and* $b \in Z\big[\binom{|\mathcal{X}_i|}{2\delta_s}\big]$, $A^{(i)}_{ab} = 0$ *for* $a \in \mathbf{i}_b$ *and* $A^{(i)}_{ab}$ *can take any value of* $\mathbb{F}_q$ *for* $a \in \mathcal{X}_i \setminus \mathbf{i}_b$.

3. $A^{(i)}_{f(i)b} = 1$ *for* $b \in Z\big[\binom{|\mathcal{X}_i|}{2\delta_s}\big]$ *and* $i \in Z[m]$.

4. $A^{(i)}_{ab} = 0$ *for* $a \in \mathcal{Y}_i$, $b \in Z\big[\binom{|\mathcal{X}_i|}{2\delta_s}\big]$, *and* $i \in Z[m]$.

Definition 3.3. $minrk_q(\delta_s, \mathcal{G}) = \min\{rk_q(A_g) : A_g$ *fits* $\mathcal{G}\}$, *where* $rk_q(A_g)$ *denotes the rank of* A_g *over* $\mathbb{F}_q$.

By using the generalized fitting matrices and their minimum rank, the optimal codelength of the proposed ICSIE can be given in the following theorem, which corresponds to generalization of Theorem 1 in [2] for the conventional index coding with $\delta_s = 0$.

Theorem 3.2 (Generalization of Theorem 1 in [2]). $N^q_{\text{opt}}(\delta_s, \mathcal{G}) = minrk_q(\delta_s, \mathcal{G})$.

Proof. Since the $(\delta_s, \mathcal{G})$-ICSIE is equivalent to the conventional index code with $\delta_s = 0$ for a modified side information graph $\tilde{\mathcal{G}}$ from Lemma 3.2, a generator matrix of the conventional index code for $\tilde{\mathcal{G}}$ satisfies the condition of the $(\delta_s, \mathcal{G})$-ICSIE in Theorem 3.1 and vice versa. Thus, A_g can be a generator matrix of the $(\delta_s, \mathcal{G})$-ICSIE because A_g is the conventional fitting matrix for $\tilde{\mathcal{G}}$. Since A_g can be a generator matrix of the $(\delta_s, \mathcal{G})$-ICSIE and $N^q_{\text{opt}}(\delta_s = 0, \tilde{\mathcal{G}}) = minrk_q(\delta_s, \mathcal{G})$ from Lemma 3.5 in [17], we have $N^q_{\text{opt}}(\delta_s, \mathcal{G}) = minrk_q(\delta_s, \mathcal{G})$. □

Thus, the optimal generator matrix of the $(\delta_s, \mathcal{G})$-ICSIE can be constructed from the generalized fitting matrix whose rank is $minrk_q(\delta_s, \mathcal{G})$ by removing all dependent columns. There is an example for construction of a generator matrix in Example 3.1 as follows.

Example 3.2. *Let* $q = 2$, $m = n = 4$, $\delta_s = 1$, $f(i) = i$, *and* $\mathcal{X}_i = Z[4] \setminus \{i\}$ *for all* $i \in Z[4]$. *From Theorem 3.2,* $N^q_{\text{opt}}(\delta_s, \mathcal{G})$ *can be found by* $minrk_q(\delta_s, \mathcal{G})$. *A matrix* A_g *which fits* $\mathcal{G}$ *is described as*

$$A_g = \begin{pmatrix} A^{(1)} & A^{(2)} & A^{(3)} & A^{(4)} \end{pmatrix} \tag{3.10}$$

where

$$A^{(1)} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix}, A^{(2)} = \begin{pmatrix} 0 & 0 & * \\ 1 & 1 & 1 \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix} \tag{3.11}$$

$$A^{(3)} = \begin{pmatrix} 0 & 0 & * \\ 0 & * & 0 \\ 1 & 1 & 1 \\ * & 0 & 0 \end{pmatrix}, A^{(4)} = \begin{pmatrix} 0 & 0 & * \\ 0 & * & 0 \\ * & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \tag{3.12}$$

and $*$ *denotes any value of* 0 *or* 1. *In order to minimize the rank of* A_g, *the value* 0 *or* 1 *is selected for* $*$ *in* A_g *and the dependent columns are removed. Then, one of the optimal generator matrices* G_{opt} *is given as*

$$G_{\text{opt}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{3.13}$$

3.2.2 Decoding Procedure

I propose the decoding procedure of the $(\delta_s, \mathcal{G})$-ICSIE similar to that of the ECIC in [17]. That is, we can consider the decoding procedure which is similar to the syndrome decoding of the classical linear error correcting code. First, I find the syndrome related to the side information errors, which is used to find the correct side information. This procedure is different from that of the ECIC because the side information errors exist in the proposed ICSIE.

In order to introduce the decoding procedure, I assume the followings:

1. Each receiver receives a codeword $\mathbf{y} = \mathbf{x}G$ through the error-free channel.
2. The receiver R_i has a side information vector $\hat{\mathbf{x}}_{\mathcal{X}_i}$ for $i \in Z[m]$, where the number of erroneous side information symbols is less than or equal to δ_s.
3. The receiver R_i only wants to recover $x_{f(i)}$ for $i \in Z[m]$.

In addition, I define the following notations:

1. $\tilde{\mathbf{x}}_{\delta_s} = \mathbf{x}_{\mathcal{X}_i} - \hat{\mathbf{x}}_{\mathcal{X}_i}$, where $\mathbf{x}_{\mathcal{X}_i}$ is a correct side information vector of R_i.
2. $H^{(i)}$ is a matrix whose rows form a basis of the dual of span$(\{G_j\}_{j\in\{f(i)\}\cup\mathcal{Y}_i})$.
3. $H_e^{(i)}$ is a matrix whose rows form a basis of the dual of span$(\{G_j\}_{j\in\mathcal{Y}_i})$.

Remark 3.5. *If we choose $H_e^{(i)}$ so that its rows span the orthogonal complement of span$(\{G_j\}_{j\in\mathcal{Y}_i})$, then automatically, rows of $H_e^{(i)}$ do not span the orthogonal complement of span($G_{f(i)}$) because $G_{f(i)}$ does not belong to span$(\{G_j\}_{j\in\mathcal{Y}_i})$ in the case of the ICSIE by Theorem 3.1.*

Then, the decoding procedure of the $(\delta_s, \mathcal{G})$-ICSIE for each receiver R_i is described in Algorithm 3.1. We need the following theorem for $\mathbf{p}_i$ in (3.15) of the proposed decoding procedure.

Theorem 3.3. *Let η_i be a subset of $\mathcal{X}_i$ with $|\eta_i| \le \delta_s$. Let $\mathbf{p}_i$ be a solution of $\mathbf{s}_i = H^{(i)}\mathbf{p}_i$, where $\mathbf{p}_i \in$ span$(\{G_j\}_{j\in\eta_i})$. Then, $\mathbf{p}_i$ is given as $\mathbf{p}_i = \tilde{\mathbf{x}}_{\delta_s} G_{\mathcal{X}_i} + \mathbf{k}$, where $\mathbf{k} \in$span$(\{G_j\}_{j\in\mathcal{Y}_i})$.*

Algorithm 3.1 Decoding procedure for R_i

Input: $\mathbf{y}$, $\hat{\mathbf{x}}_{\mathcal{X}_i}$, and G

Output: $x_{f(i)}$

Step 1) Compute the syndrome

$$\mathbf{s}_i = H^{(i)}(\mathbf{y} - \hat{\mathbf{x}}_{\mathcal{X}_i} G_{\mathcal{X}_i})^\intercal = H^{(i)}(\tilde{\mathbf{x}}_{\delta_s} G_{\mathcal{X}_i})^\intercal. \tag{3.14}$$

Step 2) Find one solution $\mathbf{p}_i$ that satisfies $\mathbf{s}_i = H^{(i)}\mathbf{p}_i{}^\intercal$ under the condition that $\mathbf{p}_i$ is a linear combination of rows of $G_{\mathcal{X}_i}$, where the number of linearly combined rows in $G_{\mathcal{X}_i}$ is less than or equal to δ_s.

Step 3) Make the following equation

$$\tilde{\mathbf{y}} = \mathbf{y} - \hat{\mathbf{x}}_{\mathcal{X}_i} G_{\mathcal{X}_i} - \mathbf{p}_i = x_{f(i)} G_{f(i)} + (\mathbf{x}_{\mathcal{Y}_i} - \mathbf{b}) G_{\mathcal{Y}_i} \tag{3.15}$$

where $\mathbf{b} \in \mathbb{F}_q^{|\mathcal{Y}_i|}$.

Step 4) Find $x_{f(i)}$ by multiplying the matrix $H_e^{(i)\intercal}$ on both sides of (3.15).

Proof. We can find a solution $\mathbf{p}_i$ for $\mathbf{s}_i = H^{(i)}\mathbf{p}_i^\intercal$, under the condition that $\mathbf{p}_i$ is a linear combination of rows of $G_{\mathcal{X}_i}$, where the number of linearly combined rows in $G_{\mathcal{X}_i}$ is less than or equal to δ_s because there exist at least one such $\mathbf{p}_i$ due to $\tilde{\mathbf{x}}_{\delta_s} G_{\mathcal{X}_i}$. Moreover, if we find a solution $\mathbf{p}_i$ under the condition mentioned above, then we have

$$\mathbf{p}_i = \tilde{\mathbf{x}}_{\delta_s} G_{\mathcal{X}_i} + \mathbf{k} \tag{3.16}$$

due to the property of the generator matrix. Specifically, from

$$\mathbf{s}_i = H^{(i)}(\tilde{\mathbf{x}}_{\delta_s} G_{\mathcal{X}_i})^\intercal = H^{(i)}\mathbf{p}_i^\intercal \tag{3.17}$$

we have $H^{(i)}(\tilde{\mathbf{x}}_{\delta_s} G_{\mathcal{X}_i} - \mathbf{p}_i)^\intercal = 0$. Thus, $\tilde{\mathbf{x}}_{\delta_s} G_{\mathcal{X}_i} - \mathbf{p}_i = aG_{f(i)} - \mathbf{b}G_{\mathcal{Y}_i}$ and

$$aG_{f(i)} - \mathbf{b}G_{\mathcal{Y}_i} - \tilde{\mathbf{x}}_{\delta_s} G_{\mathcal{X}_i} + \mathbf{p}_i = \mathbf{0} \tag{3.18}$$

where $a \in \mathbb{F}_q$ and $\mathbf{b} \in \mathbb{F}_q^{|\mathcal{Y}_i|}$. Then, we can easily check that LHS of (3.18) is $\mathbf{x}G$ such that $\mathrm{wt}(\mathbf{x}_{\mathcal{X}_i}) \leq 2\delta_s$. Since RHS of (3.18) is zero, a should be zero by Theorem 3.1. Therefore, $\mathbf{p}_i = \tilde{\mathbf{x}}_{\delta_s} G_{\mathcal{X}_i} + \mathbf{b}G_{\mathcal{Y}_i}$. $\square$

Using Theorem 3.3, (3.15) can be given as $\tilde{\mathbf{y}} = x_{f(i)}G_{f(i)} + (\mathbf{x}_{\mathcal{Y}_i} - \mathbf{b})G_{\mathcal{Y}_i}$ and from Step 4), we have $\tilde{\mathbf{y}}H_e^{(i)\mathsf{T}} = x_{f(i)}G_{f(i)}H_e^{(i)\mathsf{T}}$ due to Remark 3.5. Thus, $x_{f(i)}$ can easily be obtained.

Remark 3.6. *An interesting fact of this decoding procedure is that we can decode $x_{f(i)}$ even if we do not know the exact $\tilde{\mathbf{x}}_{\delta_s}G_{\mathcal{X}_i}$ as in the following example.*

Example 3.3. *Let $q = 2, m = n = 9, \delta_s = 1$, and $f(i) = i$ for $i \in Z[9]$. Suppose that $\mathcal{X}_i = Z[9] \setminus \{i\}$ for $i \in Z[8]$ and $\mathcal{X}_9 = \{2, 3, 5, 6, 7, 8\}$. It is easy to check that one of the possible generator matrices of the above setting is given as*

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}. \tag{3.19}$$

For the message vector $\mathbf{x} = (1, 1, 1, 1, 0, 0, 0, 0, 1)$, we have the received codeword $\mathbf{y} = \mathbf{x}G = (0, 1, 1, 0, 1, 0)$ in the error-free channel. In this case, I focus on the decoding procedure of the receiver R_9. I assume that $\hat{\mathbf{x}}_{\mathcal{X}_9} = (1, 1, 0, 0, 0, 1)$. That is, the receiver R_9 has erroneous side information $\hat{x}_8$.

The decoding procedure is described as follows:

1. *We compute $\mathbf{y} - \hat{\mathbf{x}}_{\mathcal{X}_9}G_{\mathcal{X}_9} = (1, 1, 1, 0, 1, 1)$.*

2. *Then, we can make $H^{(9)}$ from $G_{\{9\}\cup\mathcal{Y}_9}$ as*

$$H^{(9)} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3.20}$$

3. *Also, we can make $H_e^{(9)}$ as*

$$H_e^{(9)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3.21}$$

4. *Compute the syndrome as*

$$\mathbf{s}_9 = H^{(9)}(\mathbf{y} - \hat{\mathbf{x}}_{\mathcal{X}_9} G_{\mathcal{X}_9})^\intercal = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}. \tag{3.22}$$

5. *Find a solution $\mathbf{p}_9$ for $H^{(9)}\mathbf{p}_9^\intercal = \mathbf{s}_9$ under the decoding condition.*

   Then, we have $\mathbf{p}_9 = (0,0,0,1,1,1)$ and $(0,0,1,1,1,0)$. In fact, we need just one of two solutions. Choosing the first solution for $\mathbf{p}_9$ means that the receiver R_9 decides $\hat{x}_8$ as the erroneous side information while choosing the second solution means that $\hat{x}_7$ is decided as the erroneous side information.

6. *If $\hat{x}_8$ is chosen as the erroneous side information, then (3.15) in Algorithm 3.1 becomes*

$$\mathbf{y} - \hat{\mathbf{x}}_{\mathcal{X}_9} G_{\mathcal{X}_9} - (0,0,0,1,1,1) = (1,1,1,1,0,0). \tag{3.23}$$

$$(1,1,1,1,0,0) = x_9 G_9 + (\mathbf{x}_{\mathcal{Y}_i} - \mathbf{b}) G_{\mathcal{Y}_9}. \tag{3.24}$$

   Multiplying $H_e^{(9)\intercal}$ on both sides leads to $(1,0,0,0) = x_9(1,0,0,0)$. Thus, $x_9 = 1$, which is the correct value.

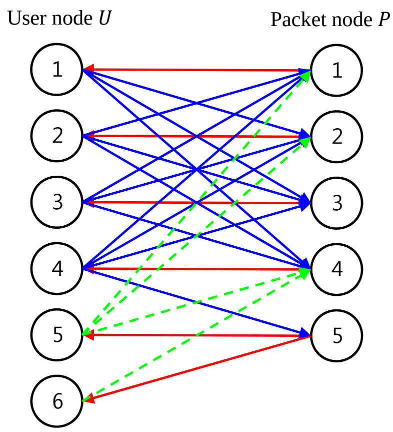

Figure 3.1: A side information graph $\mathcal{G}$ in Example 3.4 for description of a δ_s-cycle.

7. *If $\hat{x}_7$ is chosen as the erroneous side information, then (3.15) in Algorithm 3.1 becomes*

$$\mathbf{y} - \hat{\mathbf{x}}_{\mathcal{X}_9} G_{\mathcal{X}_9} - (0,0,1,1,1,0) = (1,1,0,1,0,1). \tag{3.25}$$

$$(1,1,0,1,0,1) = x_9 G_9 + (\mathbf{x}_{\mathcal{Y}_i} - \mathbf{b}) G_{\mathcal{Y}_9}. \tag{3.26}$$

Multiplying $H_e^{(9)\mathsf{T}}$ on both sides leads to $(1,0,0,0) = x_9(1,0,0,0)$. *Thus,* $x_9 = 1$, *which is also the correct value.*

3.3 Properties and Bounds for Codelength of $(\delta_s, \mathcal{G})$-ICSIE

In this section, I introduce a new type of graphs, called a δ_s-cycle for encoding of the index codes and derive some bounds for the optimal codelength of the $(\delta_s, \mathcal{G})$-ICSIE.

3.3.1 δ_s-cycle

First, I define a δ_s-cycle in $\mathcal{G}$ and generalize the generalized independent set and the generalized independence number of $\mathcal{G}$ in [17]. Let Φ be the set of subsets of $Z[n]$ defined by

$$\Phi = \{B \subseteq Z[n] \big| |\mathcal{X}_i \cap B| \geq 2\delta_s + 1 \text{ for all } i \in Z[m] \text{ s.t. } f(i) \in B\} \tag{3.27}$$

for a side information graph $\mathcal{G}$ of the $(\delta_s, \delta_c, \mathcal{G})$-GECIC.

Definition 3.4. *A subgraph $\mathcal{G}'$ of $\mathcal{G}$ is called a δ_s-cycle if the set of packet node indices of $\mathcal{G}'$ is an element of Φ (say B) and the set of user node indices of $\mathcal{G}'$ consists of $i \in Z[m]$ such that $f(i) \in B$ and its edges consist of the corresponding edges in $\mathcal{G}$. The graph $\mathcal{G}$ is said to be δ_s-acyclic if there is no δ_s-cycle in $\mathcal{G}$.*

There is an example for a δ_s-cycle in $\mathcal{G}$ as follows.

Example 3.4. *Let $\delta_s = 1$ and a side information graph $\mathcal{G}$ is given in Fig. 3.1. Then, a subgraph $\mathcal{G}'$ which consists of users 1 to 4, packets 1 to 4, and corresponding solid edges is a δ_s-cycle because $B = \{1, 2, 3, 4\} \in \Phi$. However, $\mathcal{G}$ is not a δ_s-cycle because R_6 only has x_5 as side information so that $|\mathcal{X}_6 \cap \{1, 2, 3, 4, 5\}| = 1 < 3$. Although $|\mathcal{X}_5 \cap \{1, 2, 3, 4, 5\}| = 3$, $\mathcal{G}$ cannot be a δ_s-cycle because of R_6.*

Definition 3.5. *A set of packet node indices of a δ_s-cycle is called a δ_s-cycle induced set. Two δ_s-cycles are said to be disjoint if their δ_s-cycle induced sets are disjoint.*

Definition 3.6 (Generalization of Definition 4.1 in [17])**.** *A subset Q of $Z[n]$ is called a δ_s-generalized independent set in $\mathcal{G}$ if every nonempty subset K of Q belongs to $J(\mathcal{G}, \delta_s)$.*

Definition 3.7 (Generalization of Definition 4.2 in [17])**.** *Let $\gamma(\mathcal{G})$ be the largest size of a δ_s-generalized independent set in $\mathcal{G}$, which is called the δ_s-generalized independence number.*

In the following lemma, I will show the relationship between a δ_s-generalized independent set and a δ_s-acyclic graph $\mathcal{G}$.

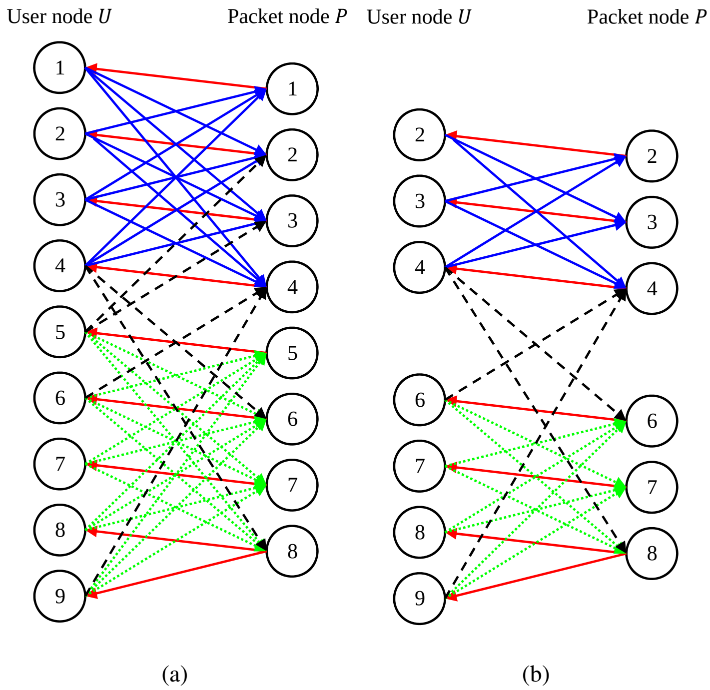

Figure 3.2: The side information graphs of Example 3.5: (a) The side information graph $\mathcal{G}$. (b) The side information graph $\mathcal{G}'$ after removing packet nodes 1 and 5.

Lemma 3.3. *Let $Z[n]$ be a set of the packet node indices in $\mathcal{G}$. Then, $Z[n]$ is a δ_s-generalized independent set if and only if the side information graph $\mathcal{G}$ is δ_s-acyclic.*

Proof. $Necessity$: Let $\mathcal{G}'$ be a subgraph of $\mathcal{G}$, where packet nodes of $\mathcal{G}'$ are a subset of packet nodes of $\mathcal{G}$ and the sets of user nodes and edges of $\mathcal{G}'$ are determined by $\mathcal{G}$. Suppose that $\mathcal{G}$ is δ_s-cyclic. Then, we can make a subgraph $\mathcal{G}'$ such that each receiver has side information symbols whose number is larger than or equal to $2\delta_s + 1$. Let $Q \subseteq Z[n]$ be the set of packet node indices of $\mathcal{G}'$. Then, $Q \notin J(\mathcal{G}', \delta_s)$ and Q is not also included in $J(\mathcal{G}, \delta_s)$. It contradicts the assumption and thus $\mathcal{G}$ is δ_s-acyclic.

$Sufficiency$: If $\mathcal{G}$ is δ_s-acyclic, every subgraph $\mathcal{G}'$ is also δ_s-acyclic. That is, every nonempty subset $Q \subseteq Z[n]$ belongs to $J(\mathcal{G}, \delta_s)$ because there exists at least one receiver, whose number of side information symbols is less than or equal to $2\delta_s$. Specifically, there is at least one receiver R_i in $\mathcal{G}$ whose number of side information symbols is less than or equal to $2\delta_s$ because $\mathcal{G}$ is δ_s-acyclic. Thus, $Z[n]$ belongs to $J(\mathcal{G}, \delta_s)$. In fact, any subset of $Z[n]$ containing $f(i)$ belongs to $J(\mathcal{G}, \delta_s)$. Now, consider the subgraph $\mathcal{G}'$ of $\mathcal{G}$ obtained by removing the packet node $f(i)$. Since $\mathcal{G}'$ is also δ_s-acyclic, there also exists at least one receiver R_j in $\mathcal{G}'$ whose number of side information symbols is less than or equal to $2\delta_s$. It means that any subset of $Z[n] \setminus \{f(i)\}$ containing $f(j)$ belongs to $J(\mathcal{G}, \delta_s)$. Through the similar ways, we can note that $Z[n]$ is a δ_s-generalized independent set. $\square$

The important theorem for a δ_s-cycle is given as follows.

Theorem 3.4. *$\Phi = \phi$ if and only if $N_{\text{opt}}^q(\delta_s, \mathcal{G}) = n$ for the $(\delta_s, \mathcal{G})$-ICSIE.*

Proof. $Necessity$: From Lemma 3.3, there is an equivalence between a δ_s-generalized independent set and a δ_s-acyclic graph. If the set of packet node indices of $\mathcal{G}$, $Z[n]$, is the δ_s-generalized independent set, $\mathcal{I}(q, \mathcal{G}, \delta_s)$ is the set of all vectors in $\mathbb{F}_q^n$ except for the all zero vector. Thus, if $\Phi = \phi$, $Z[n]$ is the δ_s-generalized independent set. Thus, all rows of a generator matrix of the $(\delta_s, \mathcal{G})$-ICSIE should be linearly independent from Theorem 3.1. Then, we have $N_{\text{opt}}^q(\delta_s, \mathcal{G}) = n$.

$Sufficiency$: Suppose that $\Phi \neq \phi$. Then, I prove that the codelength can be reduced by at least one. I choose one δ_s-cycle $\mathcal{G}'$ in $\mathcal{G}$ and let $|B|$ be the number of packet nodes in $\mathcal{G}'$. Let $\mathbf{x} \in \mathbb{F}_q^n$ consist of two parts as $\mathbf{x}' \in \mathbb{F}_q^{|B|}$ related with $\mathcal{G}'$ and the remaining packets $\mathbf{x}'' \in \mathbb{F}_q^{n-|B|}$. If we encode $\mathbf{x}' \in \mathbb{F}_q^{|B|}$ as $(x_1' + x_2', x_2' + x_3', \ldots, x_{|B|-1}' + x_{|B|}')$ whose length is $|B| - 1$ and send $\mathbf{x}''$ in the uncoded form, it satisfies the property of the generator matrix in Theorem 3.1 as follows.

The generator matrix G' of this code except the uncoded part is given as

$$G' = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ & & & & 1 \end{pmatrix}. \tag{3.28}$$

We can easily find that any $|B| - 1$ rows of G' are linearly independent and $|B|$ rows of G' are linearly dependent. Thus, if there is no vector whose Hamming weight is $|B|$ in $\mathcal{I}(q, \mathcal{G}', \delta_s)$, G' satisfies the property of the generator matrix. Since $|\mathcal{X}_i'| \geq 2\delta_s + 1$ for every user i in $\mathcal{G}'$ and $|B| \geq 2\delta_s + 2$, there is no vector whose Hamming weight is $|B|$ in $\mathcal{I}(q, \mathcal{G}', \delta_s)$. Since there is the generator matrix G' whose codelength is $|B| - 1$, $N_{\text{opt}}^q(\delta_s, \mathcal{G}) \neq n$. It contradicts the assumption and thus $\Phi = \phi$. $\square$

Remark 3.7. *From Theorem 3.4, we can conclude that a δ_s-cycle is very crucial for index coding with erroneous side information because the existence of a δ_s-cycle is a necessary and sufficient condition for a possibility to reduce the codelength of the $(\delta_s, \mathcal{G})$-ICSIE.*

Next, I compare the δ_s-cycle of the ICSIE with the cycle of the conventional index code in the following remark, where we can find similarities between the δ_s-cycle and the cycle.

Remark 3.8. *The δ_s-cycle of the ICSIE has very similar properties as those of cycles in the conventional index coding problem with respect to the index codelength.*

The cycle properties of the conventional index coding problem in [23] are given as:

1. *(Theorem 1 in [23]) If $\mathcal{G}$ is acyclic, $N^q_{\text{opt}}(0, \mathcal{G}) = n$.*

2. *(Lemma 2 in [23]) If a packet node has one outgoing edge, we say that the packet node is a unicast packet node. That is, there is only one user who wants this packet. We can reduce the codelength by the number of disjoint cycles which only consist of unicast packet nodes.*

3. *(Lemma 3 in [23]) If all distinct cycles of $\mathcal{G}$ are edge-disjoint and involve only unicast packets, then $N^q_{\text{opt}}(0, \mathcal{G}) = n - C(\mathcal{G})$, where $C(\mathcal{G})$ is defined as the largest number of edge-disjoint cycles in $\mathcal{G}$ that involve only unicast packets.*

I compare properties of the δ_s-cycle in the ICSIE with those of the cycle in the conventional index coding problem [23]. If $\delta_s = 0$, there is at least one cycle in a 0-cycle because every receiver node has at least one outgoing edge. Then, from Theorem 3.4, the corresponding properties of the δ_s-cycle of the $(\delta_s, \mathcal{G})$-ICSIE are given as follows:

1. *$\mathcal{G}$ is δ_s-acyclic if and only if $N^q_{\text{opt}}(\delta_s, \mathcal{G}) = n$.*

2. *Let β be the maximum number of vertex disjoint δ_s-cycles in $\mathcal{G}$. Then, the optimal codelength can be reduced by the number of disjoint δ_s-cycles, that is, $N^q_{\text{opt}}(\delta_s, \mathcal{G}) \leq n - \beta$. This is given in the proof of the sufficiency of Theorem 3.4.*

3. *The last property is given in the corollary below.*

Note that the cycle property 1) of the conventional index coding is the sufficient condition but the corresponding property of the ICSIE is the necessary and sufficient condition.

Corollary 3.1. *Suppose that there is no δ_s-cycle in a subgraph $\mathcal{G}'$ constructed by removing one packet node from each of β δ_s-cycles and the corresponding edges. Then, $N^q_{\text{opt}}(\delta_s, \mathcal{G}) = n - \beta$.*

Proof. It is trivial that $n - \beta$ can be an upper bound for $N_{\text{opt}}^{q}(\delta_s, \mathcal{G})$ from the above property 2) and Theorem 3.4. Thus, it is enough to show that $n - \beta$ can be a lower bound. If we remove one packet node and the corresponding edges from each δ_s-cycle, the optimal codelength for the resulting graph is given as $N_{\text{opt}}^{q}(\delta_s, \mathcal{G}') = n - \beta$ because $\mathcal{G}'$ is δ_s-acyclic. Also, an index code for $\mathcal{G}$ satisfies the property of the generator matrix for $\mathcal{G}'$ if we just substitute values of the removed packet nodes with 0. Thus, we can say $N_{\text{opt}}^{q}(\delta_s, \mathcal{G}') \leq N_{\text{opt}}^{q}(\delta_s, \mathcal{G})$. That is, $n - \beta = N_{\text{opt}}^{q}(\delta_s, \mathcal{G})$. □

An example for the previous corollary is provided as follows.

Example 3.5. *Let $q = 2, m = 9, n = 8$, and $\delta_s = 1$. A side information graph $\mathcal{G}$ is shown in Fig. 3.2(a), where the maximum number of disjoint δ_s-cycles is 2. I denote two δ_s-cycle induced sets as B_1 and B_2, that is, $B_1 = \{1, 2, 3, 4\}$ and $B_2 = \{5, 6, 7, 8\}$. The edges for B_1 and B_2 are represented by solid and dotted lines, respectively. If we remove the packet nodes 1 and 5 and the corresponding edges from two δ_s-cycles, there is no δ_s-cycle in the resulting side information graph as shown in Fig. 3.2(b). Then, the optimal codelength is 6 from Corollary 3.1.*

Remark 3.9. *When $|\mathcal{X}_i| = 2\delta_s + 1$ for all $i \in Z[m]$, we have another example for Corollary 3.1.*

3.3.2 Clique

In this section, I consider side information graphs, where each receiver knows all information except the wanted packet, that is, $\mathcal{X}_i = Z[n] \setminus \{f(i)\}$ for all $i \in Z[m]$. Since each receiver wants one symbol, we can consider these side information graphs as cliques. In the conventional index coding problem, the number of disjoint cliques is used as an upper bound of the optimal codelength and there are many heuristic algorithms to find the codelength based on the number of disjoint cliques. Therefore, I discuss cliques in the $(\delta_s, \mathcal{G})$-ICSIE problem. It is clear that cliques in index coding with erroneous side information are different from those of the conventional index coding, where cliques can be covered by one transmission for any fields.

A set of vectors is said to be $(2\delta_s + 1)$-linearly independent if any $2\delta_s + 1$ vectors in the vector set are linearly independent. Since all vectors whose Hamming weight is less than or equal to $2\delta_s + 1$ belong to $\mathcal{I}(q, \mathcal{G}, \delta_s)$ when $\mathcal{G}$ is a clique, we have the following observation.

Observation 3.1. *In the $(\delta_s, \mathcal{G})$-ICSIE problem, finding the optimal codelength for cliques is equivalent to finding the minimum dimension of the set of n vectors which is $(2\delta_s + 1)$-linearly independent.*

In general, $N^q_{\text{opt}}(\delta_s, \mathcal{G})$ goes to $2\delta_s + 1$ as the size of the finite field goes to infinity. If the size of a clique is less than or equal to $2\delta_s + 1$, it is easy to check that $N^q_{\text{opt}}(\delta_s, \mathcal{G}) = n$. Thus, I consider the clique of size $n > 2\delta_s + 1$.

Theorem 3.5. *There are some special cases of cliques for the $(\delta_s, \mathcal{G})$-ICSIE whose optimal codelength can be found as:*

1. *When $n = 2\delta_s + 2$, $N^q_{\text{opt}}(\delta_s, \mathcal{G}) = 2\delta_s + 1$ over $\mathbb{F}_q$.*
2. *When $\delta_s = 1$, $N^q_{\text{opt}}(\delta_s, \mathcal{G})$ is the minimum value of N satisfying $2^{N-1} \geq n$ over $\mathbb{F}_2$.*
3. *If there are N and n satisfying $N \leq \frac{q+1}{q}(2\delta_s + 1) - 1$ and $n \leq N + 1$ over $\mathbb{F}_q$, $N^q_{\text{opt}}(\delta_s, \mathcal{G})$ is the minimum value of N.*
4. *When $n = 3\delta_s + 3$, $N^q_{\text{opt}}(\delta_s, \mathcal{G}) = 3\delta_s + 1$ over $\mathbb{F}_2$.*

Proof. 1. It is directly proved from Corollary 3.1 because $|\mathcal{X}_i| = 2\delta_s + 1$ for all $i \in Z[m]$.

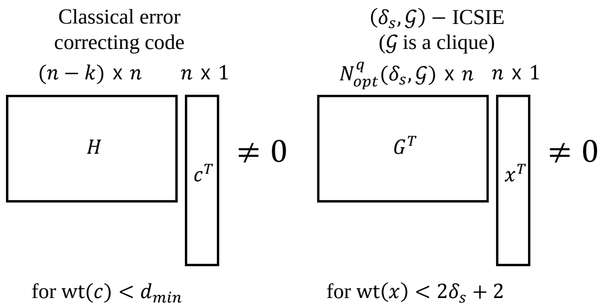

Figure 3.3: Comparison between the classical error correcting code and the $(\delta_s, \mathcal{G})$-ICSIE.

2. Let $Ind_q(N, k)$ be the maximal cardinality of the k-linearly independent subset of $\mathbb{F}_q^N$. We have $Ind_2(N, 3) = 2^{N-1}$ from [24], where $N \geq 3$. Thus, we can think N as a codelength and $k = 2\delta_s + 1$. To achieve the codelength N, the number of messages should be less than or equal to $Ind_q(N, k)$.

3. From Theorem 2 in [25], for $2 \leq k \leq N$, we have $Ind_q(N, k) = N + 1$ if and only if $\frac{q}{q+1}(N + 1) \leq k$.

4. From [24], we have $Ind_2(N, N - m) = N + 2$, where $N = 3m + i, i = 0, 1,$ and $m \geq 2$. Since $N - m = 2\delta_s + 1$, we have $i = 1$ and $m = \delta_s$. Then, for $N = 3\delta_s + 1$ and $\delta_s \geq 2$, we have $Ind_2(N, 2\delta_s + 1) = N + 2$. If $\delta_s = 1$, $N^q_{\text{opt}}(\delta_s, \mathcal{G}) = 3\delta_s + 1$ by 2) when $n = 3\delta_s + 3$.

□

Remark 3.10. *Construction of a generator matrix of each case in Theorem 3.5 is also well defined. Specifically, we can simply attain a generator matrix of case 1) as in proof of Theorem 3.4 and a matrix whose rows consist of any vectors having odd weight in* $\mathbb{F}_2^N$ *can be a generator matrix in the above case 2). We can find a generator matrix of case 3), which consists of* N*-tuple unit vectors and the all one vector. A generator matrix of case 4) consists of* N*-tuple unit vectors and two vectors* $(1, ..., 1, 0, ..., 0)$ *and*

$(0, ..., 0, 1, ..., 1)$*, where the number of* 1 *in the two vectors is* $2\delta_s + 1$*. For all cases, we just select* n *vectors among them as rows of a generator matrix.*

Next, I show how to construct a generator matrix of the $(\delta_s, \mathcal{G})$-ICSIE from a parity check matrix H of an (n, k, d_{min}) classical error correcting code in the following proposition.

Proposition 3.1. *Let* $\mathcal{G}$ *be the clique of size* n *and* $\bar{H}$ *be the matrix having the smallest* $n-k$ *among* H *of* (n, k, d_{min}) *classical error correcting codes with* $d_{min} \geq 2\delta_s + 2$*. Then,* $\bar{H}^\intercal$ *becomes the optimal generator matrix of the* $(\delta_s, \mathcal{G})$*-ICSIE.*

Proof. In the $(\delta_s, \mathcal{G})$-ICSIE problem, where a side information graph $\mathcal{G}$ is the clique of size n, it is clear that any $2\delta_s + 1$ row vectors of a generator matrix are linearly independent by Observation 3.1. That is, $\mathbf{x}G \neq \mathbf{0}$ for any $\mathbf{x}$ such that $\mathrm{wt}(\mathbf{x}) \leq 2\delta_s + 1$. Then, we can easily check that an $n \times (n-k)$ matrix $H^\intercal$ can be a generator matrix of a $(\delta_s, \mathcal{G})$-ICSIE if $d_{min} \geq 2\delta_s + 2$ as shown in Fig. 3.3. Thus, when a side information graph $\mathcal{G}$ is the clique of size n, the optimal codelength of the $(\delta_s, \mathcal{G})$-ICSIE is the minimum value of $n-k$ for an (n, k, d_{min}) classical error correcting code satisfying $d_{min} \geq 2\delta_s + 2$. $\square$

Remark 3.11. *One of examples is the Reed-Solomon code when* n *divides* $q-1$*. In this case, the optimal codelength of the* $(\delta_s, \mathcal{G})$*-ICSIE is* $n - k = d_{min} - 1 = 2\delta_s + 1$*.*

Remark 3.12. *Even if* $\mathcal{G}$ *is not a clique, we can regard the parity check matrix of the classical error correcting code as the transpose of the generator matrix of the* $(\delta_s, \mathcal{G})$*-ICSIE when* d_{min} *of the error correcting code is larger than the maximum weight of vectors in* $\mathcal{I}(q, \mathcal{G}, \delta_s)$*.*

3.3.3 Lower Bounds for the Optimal Codelength

It is not difficult to check the following corollary and observations for the $(\delta_s, \mathcal{G})$-ICSIE.

Corollary 3.2. *Let $S = \{j \in Z[n] | \exists i \in Z[m]$ s.t. $f(i) = j$ and $|\mathcal{X}_i| \leq 2\delta_s\}$. Then, we have $N^q_{\text{opt}}(\delta_s, \mathcal{G}) \geq |S| + 1$ for $n > |S|$.*

Proof. It is clear that $\mathcal{I}_i(q, \mathcal{G}, \delta_s) = \{\mathbf{z} \in \mathbb{F}^n_q : z_{f(i)} \neq 0\}$, that is, $G_{f(i)}$ does not belong to span$(\{G_j\}_{j \in Z[n] \setminus f(i)})$ for $f(i) \in S$. Thus, the corollary is obvious. □

Remark 3.13. *Thus, having less than or equal to $2\delta_s$ side information symbols is the same as not having side information in index coding with erroneous side information.*

Observation 3.2. *For the given $(\delta_s, \mathcal{G})$-ICSIE problem, let $\mathcal{G}'$ be an edge-induced subgraph obtained by deleting some outgoing edges of user nodes of $\mathcal{G}$. Then, $N^q_{\text{opt}}(\delta_s, \mathcal{G}) \leq N^q_{\text{opt}}(\delta_s, \mathcal{G}')$.*

Observation 3.3. *If $\delta'_s \leq \delta_s$, $N^q_{\text{opt}}(\delta'_s, \mathcal{G}) \leq N^q_{\text{opt}}(\delta_s, \mathcal{G})$.*

Now, the relationship of the optimal codelength between the conventional index code and the proposed ICSIE is given in the following theorem.

Theorem 3.6. *Suppose that the $(0, \bar{\mathcal{G}})$-IC problem is constructed by deleting any min$(2\delta_s, |\mathcal{X}_i|)$ outgoing edges from each receiver R_i in the $(\delta_s, \mathcal{G})$-ICSIE problem. That is, each receiver of $\bar{\mathcal{G}}$ has max$(0, |\mathcal{X}_i| - 2\delta_s)$ side information symbols and then it becomes the conventional index coding problem. Then, $N^q_{\text{opt}}(0, \bar{\mathcal{G}}) \leq N^q_{\text{opt}}(\delta_s, \mathcal{G})$.*

Proof. From Lemma 3.2, a generator matrix of the $(\delta_s, \mathcal{G})$-ICSIE problem can be a generator matrix of the $(0, \bar{\mathcal{G}})$-IC problem because $\mathcal{I}(q, \bar{\mathcal{G}}, 0) \subseteq \mathcal{I}(q, \mathcal{G}, \delta_s)$. Specifically, for a vector $\mathbf{z}' \in \mathcal{I}_i(q, \bar{\mathcal{G}}, 0)$, $\text{wt}(\mathbf{z}'_{\mathcal{X}_i}) \leq 2\delta_s$ since $\text{wt}(\mathbf{z}'_{\mathcal{X}'_i})$ should be zero, where

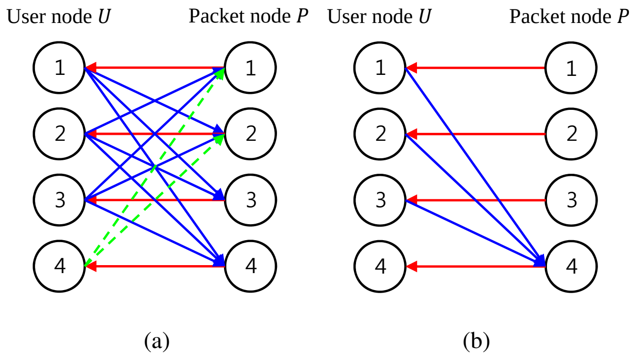

Figure 3.4: The bipartite side information graphs of Example 3.6: (a) The bipartite side information graph $\mathcal{G}$. (b) The bipartite side information graph $\bar{\mathcal{G}}$.

$\mathcal{X}_i'$ is a set of side information indices of R_i for $\bar{\mathcal{G}}$. Thus, $\mathcal{I}_i(q, \bar{\mathcal{G}}, 0) \subseteq \mathcal{I}_i(q, \mathcal{G}, \delta_s)$. Since it is true for all $i \in Z[m]$, $\mathcal{I}(q, \bar{\mathcal{G}}, 0) \subseteq \mathcal{I}(q, \mathcal{G}, \delta_s)$. □

Remark 3.14. *In general, if we reduce δ_s, we can have a lower bound from Observation 3.3. Similarly, if we delete outgoing edges of user nodes, we can have an upper bound from Observation 3.2. However, if we reduce δ_s to 0 and delete min$(2\delta_s, |\mathcal{X}_i|)$ outgoing edges of each receiver R_i, we can have a lower bound as in Theorem 3.6. Thus, the worst case of the resulting $(0, \bar{\mathcal{G}})$-IC problems can be a lower bound for the corresponding $(\delta_s, \mathcal{G})$-ICSIE problem.*

Example 3.6. *Let $q = 2, n = m = 4, \delta_s = 1, f(i) = i$, and $\mathcal{G}$ as shown in Fig. 3.4(a). Then, we have $N^q_{\text{opt}}(1, \mathcal{G}) = 4$. If we delete two outgoing edges from each receiver, there is a side information graph $\bar{\mathcal{G}}$ as shown in Fig. 3.4(b). In the conventional index coding problem, $N^q_{\text{opt}}(0, \bar{\mathcal{G}}) = n$ if a side information graph $\bar{\mathcal{G}}$ is acyclic [23]. Since the graph in Fig. 3.4(b) is acyclic, $N^q_{\text{opt}}(0, \bar{\mathcal{G}}) = 4$. From Theorem 3.6, $N^q_{\text{opt}}(1, \mathcal{G}) = 4$ because $4 = N^q_{\text{opt}}(0, \bar{\mathcal{G}}) \leq N^q_{\text{opt}}(1, \mathcal{G}) \leq 4 = n$.*

Example 3.7. *Let $q = 2, n = m = 4, \delta_s = 1, f(i) = i$, and $\mathcal{X}_i = Z[4] \setminus \{i\}$ for all $i \in Z[4]$ as in Example 3.1. If we delete two outgoing edges from each receiver, the* *corresponding graph has at least one cycle because each receiver has one outgoing edge. In this case, we can reduce the codelength by at least one because all cycles in the graph consist of unicast packets [23]. Then, the worst case of the corresponding graph* $\bar{\mathcal{G}}$ *has* $N^q_{\text{opt}}(0, \bar{\mathcal{G}}) = 3$. *Thus,* $3 \leq N^q_{\text{opt}}(1, \mathcal{G}) \leq 4$. *In fact, we have* $3 \leq N^q_{\text{opt}}(1, \mathcal{G}) = 3$ *because there is a generator matrix of the* $(\delta_s, \mathcal{G})$*-ICSIE given by*

$$G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{3.29}$$

It is easy to derive the following lower bound for the optimal codelength.

Theorem 3.7. $N^q_{\text{opt}}(\delta_s, \mathcal{G}) \geq \gamma(\mathcal{G})$.

Proof. From the definition of $\gamma(\mathcal{G})$, the corresponding $\gamma(\mathcal{G})$ rows of a generator matrix of the $(\delta_s, \mathcal{G})$-ICSIE should be linearly independent. □

3.4 Generalized Error Correcting Index Codes

I generalize many properties of the ECIC in [17] by considering the side information errors. That is, I describe the properties of the $(\delta_s, \delta_c, \mathcal{G})$-GECIC similar to those of the $(\delta_c, \mathcal{G})$-ECIC in [17] by using the properties of the $(\delta_s, \mathcal{G})$-ICSIE. Some notations of the ECIC in [17] are changed for consistency.

Proposition 3.2. *Properties of the* $(\delta_c, \mathcal{G})$*-ECIC in [17] can be generalized to those of the* $(\delta_s, \delta_c, \mathcal{G})$*-GECIC as:*

1. *Generalization of Lemma 3.8 in [17];*

   Theorem 3.1

2. *Generalization of Proposition 4.6 in [17];*

   $N_{\text{opt}}^{q}(\delta_s, \delta_c, \mathcal{G}) \leq l_q[N_{\text{opt}}^{q}(\delta_s, \mathcal{G}), 2\delta_c + 1]$, *where* $l_q[a, b]$ *denotes the minimum codelength for the dimension* a *and* $d_{min} = b$.

3. *Generalization of Theorem 5.1 in [17];*

   $N_{\text{opt}}^{q}(\delta_s, \delta_c, \mathcal{G}) \geq N_{\text{opt}}^{q}(\delta_s, \mathcal{G}) + 2\delta_c$.

4. *Generalization of the property of* $\gamma(\mathcal{G})$ *in [17];*

   Assume that $m = n$ *and* $f(i) = i$ *for all* $i \in Z[n]$ *so that the side information graph* $\mathcal{G}$ *can be represented as the unipartite form. Then,* $\gamma(\mathcal{G}) = \delta_s$*-MAIS*$(\mathcal{G})$*, where* δ_s*-MAIS*$(\mathcal{G})$ *denotes the maximum size of a* δ_s*-acyclic induced subgraph of* $\mathcal{G}$.

5. *Generalization of Theorem 4.3 in [17];*

   $N_{\text{opt}}^{q}(\delta_s, \delta_c, \mathcal{G}) \geq l_q[\gamma(\mathcal{G}), 2\delta_c + 1]$.

Proof. From Lemma 3.2, all generalization except 4) can be easily proved by the same methods as in [17] if we replace the conventional index code with the $(\delta_s, \mathcal{G})$-ICSIE. In the case of 4), I already prove an equivalence between a δ_s-generalized independent set and a δ_s-acyclic graph in Lemma 3.3. □

Remark 3.15. *By 4) of Proposition 3.2, we can think that a* δ_s*-cycle corresponds to a cycle in the conventional index coding.*

Now, I introduce some properties of the $(\delta_s, \delta_c, \mathcal{G})$-GECIC.

Theorem 3.8. *Suppose that the* $(\delta_c, \bar{\mathcal{G}})$*-ECIC problem is constructed by deleting any min*$(2\delta_s, |\mathcal{X}_i|)$ *outgoing edges from each receiver* R_i *in the* $(\delta_s, \delta_c, \mathcal{G})$*-GECIC problem. That is, each receiver of* $\bar{\mathcal{G}}$ *has max*$(0, |\mathcal{X}_i| - 2\delta_s)$ *side information symbols and then it* *becomes the conventional error correcting index coding problem. Then,* $N^q_{\mathrm{opt}}(\delta_c, \bar{\mathcal{G}}) \leq N^q_{\mathrm{opt}}(\delta_s, \delta_c, \mathcal{G})$.

Proof. The proof is similar to that of Theorem 3.6 and thus I omit it. $\square$

Theorem 3.9. *Let* $\hat{\mathcal{G}}$ *be an edge-induced subgraph of* $\mathcal{G}$*, which is obtained by deleting all outgoing edges of all users in* $\mathcal{G}$*, that is, none of the receivers have any side information in* $\hat{\mathcal{G}}$*. Then,* $N^q_{\mathrm{opt}}(\delta_s, \delta_c, \mathcal{G}) = N^q_{\mathrm{opt}}(\delta_s, \delta_c, \hat{\mathcal{G}})$ *if* $\Phi = \phi$.

Proof. If $\Phi = \phi$, $Z[n]$ is a δ_s-generalized independent set. Then, $\mathcal{I}(q, \mathcal{G}, \delta_s) = \mathcal{I}(q, \hat{\mathcal{G}}, \delta_s)$ and thus $N^q_{\mathrm{opt}}(\delta_s, \delta_c, \mathcal{G}) = N^q_{\mathrm{opt}}(\delta_s, \delta_c, \hat{\mathcal{G}})$. $\square$

In Section 3.2.2, the decoding procedure of the $(\delta_s, \mathcal{G})$-ICSIE was introduced in Algorithm 3.1 and the decoding procedure of the $(\delta_s, \delta_c, \mathcal{G})$-GECIC can also be derived similarly. Since the decoding procedure of the $(\delta_s, \delta_c, \mathcal{G})$-GECIC is similar to Algorithm 1, I just enumerate some differences as follows.

1. Each receiver receives a codeword $\mathbf{y}' = \mathbf{y} + \boldsymbol{\epsilon}$, where $\boldsymbol{\epsilon}$ is the error with $\mathrm{wt}(\boldsymbol{\epsilon}) \leq \delta_c$.

2. The input of the decoding procedure $\mathbf{y}$ is changed to $\mathbf{y}'$.

3. $H_e^{(i)}$ is a matrix whose rows form a basis of the dual of span$(\{G_j\}_{j \in \mathcal{Y}_i})$ and do not form a basis of the dual of span$(G_{f(i)})$.

4. Equation (3.14) in Algorithm 3.1 is changed to $\mathbf{s}_i = H^{(i)}(\mathbf{y}' - \hat{\mathbf{x}}_{\mathcal{X}_i} G_{\mathcal{X}_i})^{\mathsf{T}}$.

5. Step 2) in Algorithm 3.1: Find a solution having syndrome $\mathbf{s}_i$ in the set $\{\mathbf{p}_i + \hat{\boldsymbol{\epsilon}} | \mathbf{p}_i$ is a linear combination of at most δ_s rows of $G_{\mathcal{X}_i}$ and $\mathrm{wt}(\hat{\boldsymbol{\epsilon}}) \leq \delta_c\}$.

6. Equation (3.15) in Algorithm 3.1 is changed to

$$\tilde{\mathbf{y}} = \mathbf{y}' - \hat{\mathbf{x}}_{\mathcal{X}_i} G_{\mathcal{X}_i} - \mathbf{p}_i - \hat{\boldsymbol{\epsilon}} = x_{f(i)} G_{f(i)} + (\mathbf{x}_{\mathcal{Y}_i} - \mathbf{b}) G_{\mathcal{Y}_i}. \tag{3.30}$$

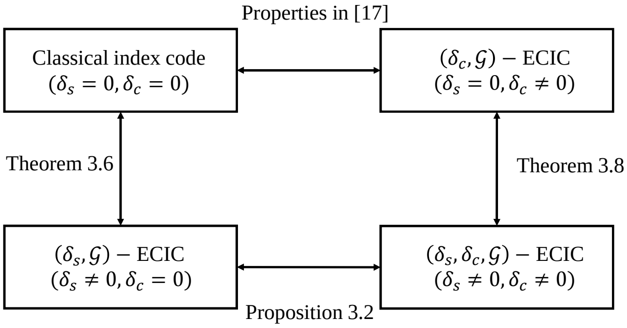

Figure 3.5: Relationship of several index coding problems for the optimal codelength.

It is enough to show validity of (3.30) in order to prove successful decoding of each receiver. We can see the following equations in the decoding procedure of the $(\delta_s, \delta_c, \mathcal{G})$-GECIC as

$$\mathbf{s}_i = H^{(i)}(\mathbf{y}' - \hat{\mathbf{x}}_{\mathcal{X}_i} G_{\mathcal{X}_i})^\intercal = H^{(i)}(\tilde{\mathbf{x}}_{\delta_s} G_{\mathcal{X}_i} + \boldsymbol{\epsilon})^\intercal = H^{(i)}(\mathbf{p}_i + \hat{\boldsymbol{\epsilon}})^\intercal \tag{3.31}$$

$$H^{(i)}(\tilde{\mathbf{x}}_{\delta_s} G_{\mathcal{X}_i} + \boldsymbol{\epsilon} - \mathbf{p}_i - \hat{\boldsymbol{\epsilon}})^\intercal = \mathbf{0} \tag{3.32}$$

$$\tilde{\mathbf{x}}_{\delta_s} G_{\mathcal{X}_i} + \boldsymbol{\epsilon} - \mathbf{p}_i - \hat{\boldsymbol{\epsilon}} = aG_{f(i)} - \mathbf{b}G_{\mathcal{Y}_i}. \tag{3.33}$$

Then, by Theorem 3.1, $a = 0$ and we have

$$-\mathbf{p}_i - \hat{\boldsymbol{\epsilon}} = -\tilde{\mathbf{x}}_{\delta_s} G_{\mathcal{X}_i} - \boldsymbol{\epsilon} - \mathbf{b}G_{\mathcal{Y}_i}. \tag{3.34}$$

Since $\mathbf{y}' = x_{f(i)} G_{f(i)} + \mathbf{x}_{\mathcal{X}_i} G_{\mathcal{X}_i} + \mathbf{x}_{\mathcal{Y}_i} G_{\mathcal{Y}_i} + \boldsymbol{\epsilon}$, we have (3.30).

In Fig. 3.5, I show the relationship between the proposed index codes and several index coding problems, specifically in terms of the optimal codelength.

Chapter 4

Code Equivalences Between Network Codes With Link Errors and Index Codes With Side Information Errors

Network coding was introduced in [26] to improve the throughput gain of terminals in a network structure, where a source node transmits information to terminal nodes through links and internal nodes. In order to improve the throughput gain, some internal nodes encode their incoming symbols, which is called network coding. In [27], it was proved that a linear network code for multicast in a network can achieve the max-flow bound. For multicast cases, there exist some algorithms to construct network codes achieving the maxflow-mincut capacity for a single source [28], [29].

In contrast to an error-free link case, a network code dealing with erroneous data on links was also studied, referred to as a network code with link errors (NCLE) in this chapter. As erroneous data on links in a network are considered, the number of overall link errors in a network which network codes can overcome was studied [30], [31].

In this chapter, I focus on a code equivalence between network coding and index coding [10] in which their equivalence was introduced and a corresponding index coding instance was derived for a given network coding instance. It was also shown that any network codes can be converted to the corresponding index codes and vice versa. However, the code equivalence between two problems for a given index coding instance was not presented in [10]. In [32], they showed an equivalence between network computation and functional index coding for a given network coding instance and also suggested their relation for a given index coding instance with the corresponding models for both a network coding instance and an index coding instance, called the equivalent index coding instance and network coding instance, respectively. However, their model of the corresponding network coding instance for a given index coding instance is actually the given index coding instance itself, which means that finding a network code in the corresponding network coding instance is the same as finding an index code in the given index coding instance. The same problem exists in [33]. To solve an index coding problem by using the meaningful corresponding network coding problem, I propose a different model of the corresponding network coding instance for a given index coding instance.

In this chapter, I show a new code equivalence between a network code and an index code for a given index coding instance. For a given index coding instance, I modify a given side information graph by adding receivers and messages and introducing hyper-links in the network in order to derive the corresponding network coding instance in a similar manner as in [10]. Then, I show that an index code achieving the maximum acyclic induced subgraph (MAIS) bound [2] can be converted to the corresponding network code and vice versa. Moreover, the equivalent index coding instance of a given network coding instance does not contain the receiver $\hat{t}_{\text{all}}$, which was given in the earlier studies [10], [32]. In [33], it was noted that an equivalence between secure network and index coding can be achieved without $\hat{t}_{\text{all}}$. In this chapter, I analyze why $\hat{t}_{\text{all}}$ can be removed in detail. Then, the code equivalence results for both a given network coding instance and a given index coding instance are generalized to erroneous cases. From the fact that an NCLE and an ICSIE are equivalent, I derive the relationship between the conventional network code with error-free links and an NCLE.

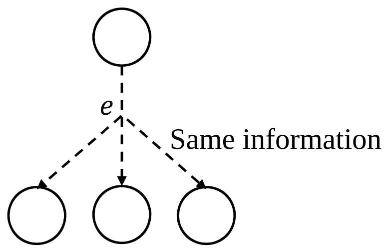

Figure 4.1: The description for a hyper-link.

4.1 Problem Formulation and Some Results

4.1.1 Network Codes

I introduce a network coding instance with a network structure $\mathbb{G} = (V, E, \mathcal{F})$ described by a directed acyclic graph, where V and E denote the sets of nodes and links in $\mathbb{G}$, respectively and $\mathcal{F}$ is a function of terminals.

1. $\bar{S} \subseteq V$ denotes a set of source nodes in $\mathbb{G}$, where source nodes do not have incoming links.

2. S denotes a set of source messages, that is, $\bar{s} \in \bar{S}$ has some source messages in $\{s|s \in S\}$.

3. $T \subseteq V$ denotes a set of terminal nodes in $\mathbb{G}$, where terminal nodes do not have outgoing links.

4. $\mathcal{F}$ denotes a function of the terminal nodes in $\mathbb{G}$, which indicates a set of indices of each terminal's desired messages.

5. $e \in E$ is said to be a hyper-link if e is a broadcast link which transmits the same information to multiple nodes as shown in Fig. 4.1.

6. For an outgoing link $e \in E$ from a node $u \in V$, $\mathrm{In}(e)$ denotes a set of incoming links of u.

7. In the case of $u \in \bar{S}$, $\mathrm{In}(e)$ denotes a set of messages that u has.

8. $\mathrm{In}(v)$ denotes a set of incoming links of v for which $v \in V \setminus \bar{S}$.

9. When it is straightforward, I regard s, e, and t as some indices.

In this network coding instance, I assume the followings:

1. Each message is one symbol in $\mathbb{F}_q$.

2. Each link carries one symbol in $\mathbb{F}_q$ without any errors.

3. X_s denotes an element of a message vector $\mathbf{X}_S \in \mathbb{F}_q^{|S|}$.

4. X_e denotes a symbol on a link e for which $e \in E$.

5. For a set $A \subseteq S$, $\mathbf{X}_A$ denotes a sub-vector of $\mathbf{X}_S$ and for a set $B \subseteq E$, $\mathbf{X}_B$ denotes a vector consisting of $|B|$ symbols of the corresponding links.

Definition 4.1. *A network code for $\mathbb{G}$ over $\mathbb{F}_q$, denoted by a $\mathbb{G}$-NC consists of:*

1. *A local encoding function $F_e : \mathbb{F}_q^{|\mathrm{In}(e)|} \to \mathbb{F}_q$ for $e \in E$*

2. *A decoding function $D_t : \mathbb{F}_q^{|\mathrm{In}(t)|} \to \mathbb{F}_q^{|\mathcal{F}(t)|}$ so that $D_t(\mathbf{X}_{\mathrm{In}(t)}) = \mathbf{X}_{\mathcal{F}(t)}$ for $t \in T$.*

The above encoding functions of links are local functions and the global function $\bar{F}_e$ is defined as $\bar{F}_e(\mathbf{X}_S) = F_e(\mathbf{X}_{\mathrm{In}(e)})$.

4.1.2 Index Codes

An index coding instance is described as follows:

1. There are one sender which has n information messages as $\mathbf{Y} = (Y_1, \ldots, Y_n) \in \mathbb{F}_q^n$ and m receivers (or users) $R_1, R_2, \ldots, R_m$, having sub-vectors of $\mathbf{Y}$ as side information.

2. Let $\mathcal{X}_i$ be the set of side information indices of a receiver R_i for $i \in Z[m]$.

3. Each receiver R_i wants to receive some elements in $\mathbf{Y}$, referred to as the wanted messages denoted by $\mathbf{Y}_{f(i)}$, where $f(i)$ represents the set of indices of the wanted messages of R_i and $f(i) \cap \mathcal{X}_i = \phi$.

4. A side information graph $\mathcal{G}$ shows the wanted messages and side information of all receivers and the sender knows $\mathcal{G}$. A side information graph is a bipartite graph which consists of message nodes and receiver nodes. A directed edge from a message node to a receiver node means that the receiver wants to receive that message. Conversely, a directed edge from a receiver node to a message node means that the receiver has that message as side information.

5. The sender transmits encoded messages to receivers through an error-free broadcast channel.

Definition 4.2. *An index code for $\mathcal{G}$ over $\mathbb{F}_q$, denoted by a $\mathcal{G}$-IC is a set of codewords having:*

1. *An encoding function $\hat{F} : \mathbb{F}_q^n \to \mathbb{F}_q^N$, where N is the codelength*

2. *A set of decoding functions $\hat{D}_1, \hat{D}_2, \ldots, \hat{D}_m$ such that $\hat{D}_i : \mathbb{F}_q^N \times \mathbb{F}_q^{|\mathcal{X}_i|} \to \mathbb{F}_q^{|f(i)|}$ satisfying*

$$\hat{D}_i(\hat{F}(\mathbf{Y}), \mathbf{Y}_{\mathcal{X}_i}) = \mathbf{Y}_{f(i)} \tag{4.1}$$

for all $i \in Z[m]$, $\mathbf{Y} \in \mathbb{F}_q^n$.

Let $\mathcal{I}(q, \mathcal{G})$ be a set of vectors defined by

$$\mathcal{I}(q, \mathcal{G}) = \bigcup_{i \in Z[m]} \mathcal{I}_i(q, \mathcal{G}) \tag{4.2}$$

where $\mathcal{I}_i(q, \mathcal{G}) = \{\mathbf{Z} \in \mathbb{F}_q^n | \mathbf{Z}_{\mathcal{X}_i} = \mathbf{0}, \mathbf{Z}_{f(i)} \neq \mathbf{0}\}$.

Since $\mathcal{I}(q,\mathcal{G})$ is a set of vectors which are differences of two confusable message vectors, we have the following observation.

Observation 4.1. *A $\mathcal{G}$-IC is valid if and only if*

$$\hat{F}(\mathbf{Y}) \neq \hat{F}(\mathbf{Y}') \text{ for all } \mathbf{Y} - \mathbf{Y}' \in \mathcal{I}(q,\mathcal{G}). \tag{4.3}$$

4.1.3 Code Equivalence for Given Network Coding Instance

For a given network coding instance, the corresponding index coding instance except $\hat{t}_{\text{all}}$ is given as follows [10]:

1. A sender has a message $\mathbf{Y} = (\mathbf{Y}_S, \mathbf{Y}_E)$ and there are $|E| + |T|$ receivers, each of which is a corresponding receiver R_e of a link or R_t of a terminal in the given network coding instance.
2. For $e \in E$, R_e can be described as $\mathcal{X}_e = \text{In}(e)$ and $f(e) = \{e\}$.
3. For $t \in T$, R_t can be described as $\mathcal{X}_t = \text{In}(t)$ and $f(t) = \mathcal{F}(t)$.

From the above relation, we have the following observations.

Observation 4.2. *In the corresponding side information graph $\mathcal{G}$, there is no cycle in a subgraph which consists of $\{R_e | e \in E\}$ and $\{Y_e | e \in E\}$ because a given network graph is directed acyclic.*

Observation 4.3. *If there is a path from a source node to a terminal node which wants to receive source messages of that source node in $\mathbb{G}$, there is a cycle consisting of receivers and messages related with that path in the corresponding side information graph $\mathcal{G}$.*

In fact, the existence of $\hat{t}_{\text{all}}$ in [10] originates from a directed acyclic network structure. Thus, an identical result can be obtained even if we remove $\hat{t}_{\text{all}}$ in the corresponding index coding instance. In [33], the authors also omitted $\hat{t}_{\text{all}}$ and the detailed explanation for the redundancy of $\hat{t}_{\text{all}}$ in this chapter is given in the following proposition.

Proposition 4.1. *For a given network coding instance, the modeling of the corresponding index coding instance in [10] obtains an identical result even if the receiver $\hat{t}_{\text{all}}$ is removed, that is, $\hat{t}_{\text{all}}$ is redundant.*

Proof. From Observation 4.2, we can see that there is no cycle in a subgraph which consists of $\{R_e | e \in E\}$ and $\{Y_e | e \in E\}$. Since this subgraph is acyclic, the optimal codelength for the subgraph is $|E|$, meaning that every vector $\mathbf{Z} \in \mathbb{F}_q^{|S|+|E|}$ such that $\mathbf{Z}_S = \mathbf{0}$ and $\mathbf{Z}_E \neq \mathbf{0}$ belongs to $\mathcal{I}(q, \mathcal{G})$. In [10], $\hat{t}_{\text{all}}$ wants to receive $\mathbf{Y}_E$ and has $\mathbf{Y}_S$ as side information. Then, $\mathcal{I}_{\hat{t}_{\text{all}}}(q, \mathcal{G})$ is the set of all vectors in $\mathbb{F}_q^{|E|+|S|}$ such that $\mathbf{Z}_S = \mathbf{0}$ and $\mathbf{Z}_E \neq \mathbf{0}$. Since every vector in $\mathcal{I}_{\hat{t}_{\text{all}}}(q, \mathcal{G})$ is already included in $\mathcal{I}(q, \mathcal{G})$, the receiver $\hat{t}_{\text{all}}$ can be removed from the corresponding index coding instance. $\square$

In [10], the corresponding index code for a given network code is defined as follows:

1. The index codelength is $|E|$ and $\hat{F}(\mathbf{Y}) = (Y_B(e) : e \in E)$.

2. $Y_B(e) = Y_e + \bar{F}_e(\mathbf{Y}_S)$.

3. If e is not an outgoing link of a source node, R_e can recover Y_e from $Y_B(e) - \bar{F}_e(\mathbf{Y}_S)$, where $\bar{F}_e(\mathbf{Y}_S)$ is obtained from the fact that R_e can have $\{\bar{F}_{e'}(\mathbf{Y}_S) | e' \in \text{In}(e)\}$ using its side information and $\hat{F}(\mathbf{Y})$. If e is an outgoing link of a source node, R_e can obtain $\bar{F}_e(\mathbf{Y}_S)$ using its side information.

4. R_t can recover $\mathbf{Y}_{f(t)}$ using D_t from the fact that R_t can have $\{\bar{F}_{e'}(\mathbf{Y}_S) | e' \in \text{In}(t)\}$ using its side information and $\hat{F}(\mathbf{Y})$.

Also, the corresponding network code for a given index code with codelength $|E|$ is defined as follows:

1. $\boldsymbol{\sigma} \in \mathbb{F}_q^{|E|}$ is a realization of the index codeword.

2. For $e \in E$, F_e is defined as a function whose output is $X_e = \hat{D}_e(\boldsymbol{\sigma}, (X_{e'} : e' \in \mathrm{In}(e)))$.

3. For $t \in T$, D_t is defined as a function whose output is $\hat{D}_t(\boldsymbol{\sigma}, (X_{e'} : e' \in \mathrm{In}(t)))$.

In [10], it was proved that for a given network coding instance, a $\mathbb{G}$-NC exists if and only if the corresponding $\mathcal{G}$-IC exists. Although I introduce a hyper-link, it is easy to note that the code equivalence result does not change. Thus, we have the following theorem.

Theorem 4.1 (Theorem 1 in [10])**.** *For a given network coding instance, a $\mathbb{G}$-NC exists if and only if the corresponding $\mathcal{G}$-IC exists.*

From Theorem 4.1, Observation 4.2, and Observation 4.3, we have the following observation.

Observation 4.4. *Since the optimal index codelength of the corresponding index coding instance is larger than or equal to $|E|$, the above code equivalence for a given network coding intance is the code equivalence between a network code and the corresponding index code achieving the MAIS bound [2], where the maximum acyclic subgraph consists of $\{R_e|e \in E\}$ and $\{Y_e|e \in E\}$.*

4.2 Code Equivalence for Given Index Coding Instance

For a given index coding instance, we can construct the corresponding network coding instance as in [32], [33]. However, their corresponding network coding instance is actually the same as the given index coding instance. Specifically, although they showed the code convertibility, finding network codes is actually the same as finding index codes for the given index coding instance in their models. Thus, their models do not give any code diversities. To deal with this problem, I use the model defined in [10] to derive the corresponding network coding instance for a given index coding instance. Since I use the model in [10], my goal is to find a code equivalence between an index code achieving the MAIS bound and the corresponding network code.

In [10], they studied a code equivalence for a given network coding instance. Although they did not show a code equivalence for a given index coding instance, it is expected that the similar approach is valid for a given index coding instance. However, some index coding instances cannot be directly converted to corresponding network coding instances as in [10] considering the validity of the corresponding network structure which will be explained in this section. Thus, I propose a new method converting a given index coding instance into the corresponding network coding instance. It is necessary to modify a given index coding instance to use the previous relationship between two coding instances [10]. Specifically, some receivers and messages are added to the given index coding instance, where $\mathcal{G}$ becomes $\mathcal{G}'$. For simplicity, I consider an index coding scenario with $m = n$, that is, the number of receivers is the same as that of messages.

Now, I explain how to make $\mathcal{G}'$ and the corresponding network coding instance with $\mathbb{G}$. In order to make the corresponding network coding instance using the same relationship between two coding instances in the previous section, it is necessary to determine which messages are related with links and source messages for a given side information graph $\mathcal{G}$. From Observations 4.2 and 4.3, I first classify messages in $\mathcal{G}$ as follows:

1. Choose any maximum acyclic subgraph $\hat{\mathcal{G}}$ of $\mathcal{G}$.

2. Let $\mathbf{Y}_{\hat{E}}$ be messages in $\hat{\mathcal{G}}$.

3. Let $\{R_{\hat{e}}|\hat{e} \in \hat{E}\}$ be receivers in $\hat{\mathcal{G}}$.

4. Let $\mathbf{Y}_S$ and $\{R_t | t \in T\}$ be the remaining messages and receivers in $\mathcal{G}$, respectively.

Next, I modify $\mathcal{G}$ to $\mathcal{G}'$ and determine $\mathbf{Y}_E$ in $\mathcal{G}'$ in order to ensure the validity of the corresponding network structure. Specifically, I determine $\mathbf{Y}_E$ by adding several messages to $\mathbf{Y}_{\hat{E}}$ and determine $\{R_e | e \in E\}$ by adding a number of receivers to $\{R_{\hat{e}} | \hat{e} \in \hat{E}\}$. Also, we have to determine which message in $\mathbf{Y}_E$ is related with a hyper-link. If we try to map a given index coding instance to the corresponding network coding instance as in [10] without using hyper-links, the resulting topology of the corresponding network coding instance may not be valid. Before choosing $\mathbf{Y}_E$ and hyper-links, I classify such problematic cases based on the outgoing edges of the receivers in $\mathcal{G}$.

There are six problematic cases based on message nodes and four problematic cases based on receiver nodes as in the following claim.

Claim 4.1. *The problematic cases based on the outgoing edges of receivers in a side information graph $\mathcal{G}$ can be classified into the following ten cases, which should be modified to make the modified side information graph $\mathcal{G}'$ or are related with hyper-links.*

1. *Y_s has one incoming edge from $\{R_t | t \in T\}$ for $s \in S$.*

2. *Y_s has more than one incoming edge from $\{R_t | t \in T\}$ for $s \in S$.*

3. *Y_s has one incoming edge from $\{R_{e'} | e' \in \hat{E}\}$ and one incoming edge from $\{R_t | t \in T\}$ for $s \in S$.*

4. *Y_e has one incoming edge from $\{R_{e'} | e' \in \hat{E}\}$ and one incoming edge from $\{R_t | t \in T\}$ for $e \in \hat{E}$.*

5. *Y_e has more than one incoming edge from $\{R_{e'} | e' \in \hat{E}\}$ for $e \in \hat{E}$.*

6. *Y_e has more than one incoming edge from $\{R_t | t \in T\}$ for $e \in \hat{E}$.*

7. *R_e has one outgoing edge to $\{Y_s|s \in S\}$ and one outgoing edge to $\{Y_{e'}|e' \in \hat{E}\}$ for $e \in \hat{E}$.*

8. *R_t has one outgoing edge to $\{Y_s|s \in S\}$ for $t \in T$.*

9. *R_t has more than one outgoing edge to $\{Y_s|s \in S\}$ for $t \in T$.*

10. *R_t has one outgoing edge to $\{Y_s|s \in S\}$ and one outgoing edge to $\{Y_{e'}|e' \in \hat{E}\}$ for $t \in T$.*

For the cases 3), 4), 7), and 10) of Claim 4.1, I restrict the number of edges as two, that is, one for each type of edges. For the cases with the unrestricted number of edges, they are interpreted as the combination of the above cases. Thus, I only consider the cases with the restricted number of edges and modify $\mathcal{G}$ to $\mathcal{G}'$ by solving the above ten cases sequentially.

In contrast to the ten problematic cases in Claim 4.1, there are ten cases which do not need to be modified for $\mathcal{G}'$ as in the following claim. In these cases, I do not consider situations reduced to the above problematic cases.

Claim 4.2. *The other ten cases well converted by the model in [10].*

1. *Y_e has one incoming edge from $\{R_{e'}|e' \in \hat{E}\}$ for $e \in \hat{E}$.*

2. *Y_e has one incoming edge from $\{R_t|t \in T\}$ for $e \in \hat{E}$.*

3. *Y_s has one incoming edge from $\{R_{e'}|e' \in \hat{E}\}$ for $s \in S$.*

4. *Y_s has more than one incoming edge from $\{R_{e'}|e' \in \hat{E}\}$ for $s \in S$.*

5. *R_e has one outgoing edge to $\{Y_s|s \in S\}$ for $e \in \hat{E}$.*

6. *R_e has more than one outgoing edge to $\{Y_s|s \in S\}$ for $e \in \hat{E}$.*

7. *R_e has one outgoing edge to $\{Y_{e'}|e' \in \hat{E}\}$ for $e \in \hat{E}$.*

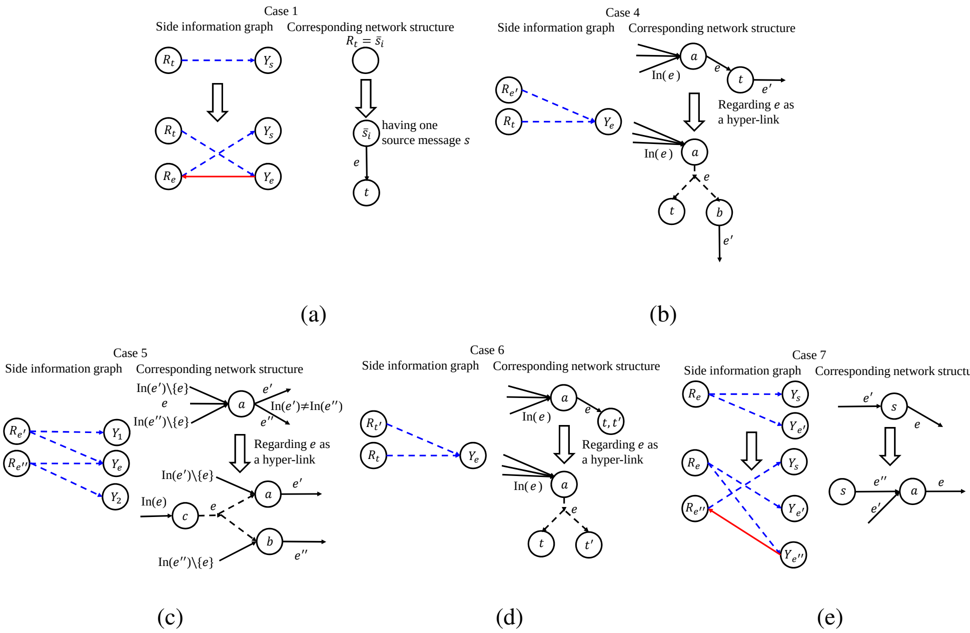

Figure 4.2: Modifications of the problematic cases in Claim 4.1: (a) Case 1). (b) Case 4). (c) Case 5). (d) Case 6). (e) Case 7).

8. *R_e has more than one outgoing edge to $\{Y_{e'}|e' \in \hat{E}\}$ for $e \in \hat{E}$.*

9. *R_t has one outgoing edge to $\{Y_{e'}|e' \in \hat{E}\}$ for $t \in T$.*

10. *R_t has more than one outgoing edge to $\{Y_{e'}|e' \in \hat{E}\}$ for $t \in T$.*

For example, the case 1) in Claim 4.2 can be described as e being an incoming edge of e', which does not violate the network structure. Thus, the case 1) does not need to be modified. The other cases in Claim 4.2 are easily verified that they do not violate the network structure when they are mapped to the corresponding network coding instance.

At this point, I suggest how to solve the ten problematic cases in Claim 4.1 so that the corresponding network coding instance is valid as in Fig. 4.2.

1. The case 1) is described as R_t having Y_s as side information for $t \in T$ and $s \in S$, implying that the terminal and the source are identical. To address this,

I add a new link-related receiver R_e having Y_s as side information and wanting the corresponding message Y_e in $\mathcal{G}$. In addition, I delete the incoming edge of Y_s from R_t and add a new edge from R_t to Y_e, after which we have the corresponding network structure as shown in Fig. 4.2(a).

2. The case 2) can be regarded as several cases of the case 1).

3. The case 3) can be solved similarly to the case 1).

4. The case 4) indicates that the terminal node is the intermediate node, that is, R_t and $R_{e'}$ have Y_e as side information as in Fig. 4.2(b). However, it can be solved if e is a hyper-link. If we assume that e is a hyper-link, we can obtain the valid network structure as in Fig. 4.2(b).

5. The case 5) can be a problem when two receivers having Y_e as the side information have different side information as in Fig. 4.2(c). This situation can be modified by a method similar to that in the case 4). If Y_1 or Y_2 is Y_s for $s \in S$, it is needed to consider the case 7).

6. The case 6) is described as R_t and $R_{t'}$ have Y_e as side information, which means that the terminals t and t' are identical. Also, it can be solved if we regard e as a hyper-link as in Fig. 4.2(d).

7. The case 7) indicates that the source node is the intermediate node, that is, R_e has Y_s and $Y_{e'}$ as side information. This can be modified by adding a new link-related receiver $R_{e''}$ having Y_s as side information and the corresponding message $Y_{e''}$. I delete the edge from R_e to Y_s and add a new edge from R_e to $Y_{e''}$. Accordingly, we have the corresponding network structure as shown in Fig. 4.2(e).

8. The case 8) can be solved similarly to the case 1).

9. The case 9) can be regarded as several cases of the case 1).

10. The case 10) can be solved similarly to the case 1).

By solving the above problematic cases sequentially and modifying $\mathcal{G}$ to $\mathcal{G}'$, the valid corresponding network coding instance can be derived from any index coding instance. The main idea is introducing hyper-links and modifying $\mathcal{G}$ by adding some links originated from source nodes in the perspective of network codes. Once the corresponding network coding instance with $\mathbb{G}$ is derived, I show the code equivalence between a $\mathbb{G}$-NC and a $\mathcal{G}'$-IC for a given index coding instance as in the following theorem.

Theorem 4.2. *For a given side information graph $\mathcal{G}$, a $\mathcal{G}'$-IC with codelength $|E|$ exists if and only if the corresponding $\mathbb{G}$-NC exists.*

Proof. By solving the problematic cases in Claim 4.1, we can determine $\{R_e | e \in E\}$ and $\mathbf{Y}_E$. Since we can have the valid network coding instance, a $\mathcal{G}'$-IC with the codelength $|E|$ exists if and only if the corresponding $\mathbb{G}$-NC exists by Theorem 4.1. □

My main concern is the code equivalence between a $\mathcal{G}$-IC and $\mathbb{G}$-NC. However, Theorem 4.2 shows the code equivalence between a $\mathcal{G}'$-IC and a $\mathbb{G}$-NC. Thus, we have the following corollary.

Corollary 4.1. *For a given side information graph $\mathcal{G}$, a $\mathcal{G}$-IC with the codelength $|\hat{E}|$ exists if and only if the corresponding $\mathbb{G}$-NC exists.*

Proof. Using the problematic cases in Claim 4.1, I prove the corollary, where the same notations for the above cases are used.

$Necessity$: Assume that a $\mathcal{G}$-IC with codelength $|\hat{E}|$ exists. For the cases related with hyper-links, that is, the cases except 1) and 7) in Claim 4.1, it is easy to note that hyper-links do not affect the code equivalence result. Thus, I only consider the cases 1) and 7).

For the case 1), I add one more symbol $Y_e + Y_s$ to the $\mathcal{G}$-IC. Then, in $\mathcal{G}'$, R_t has Y_e as side information and does not have Y_s as side information. However, R_t can obtain Y_s because $Y_e + Y_s$ is transmitted. Thus, R_t can recover what R_t wants from the $\mathcal{G}$-IC, which is the same situation as before. Trivially, R_e can obtain Y_e from $Y_e + Y_s - Y_s$ because R_e has Y_s as side information in $\mathcal{G}'$.

For the case 7), it is similarly solved as the case 1). I add one more symbol $Y_s + Y_{e''}$ to the $\mathcal{G}$-IC. Then, in $\mathcal{G}'$, R_e has $Y_{e''}$ as side information and does not have Y_s as side information. However, R_e can obtain Y_s because $Y_s + Y_{e''}$ is transmitted. Thus, R_e can recover what R_e wants from the $\mathcal{G}$-IC, which is the same situation as before. Also, $R_{e''}$ can recover $Y_{e''}$ from $Y_s + Y_{e''} - Y_s$.

Thus, a $\mathcal{G}'$-IC with codelength $|E|$ exists if a $\mathcal{G}$-IC with codelength $|\hat{E}|$ exists using additional transmissions described as the above because the number of additional transmissions is $|E| - |\hat{E}|$. Consequently, the corresponding $\mathbb{G}$-NC exists by Theorem 4.2.

$Sufficiency$: Suppose that a $\mathbb{G}$-NC exists. Then, we have a $\mathcal{G}'$-IC with codelength $|E|$, that is, $\hat{F}(\mathbf{Y}) = (Y_B(e) : e \in E)$, where $Y_B(e) = Y_e + \bar{F}_e(\mathbf{Y}_S)$. In fact, selecting $|\hat{E}|$ components of the given index code is sufficient for making a $\mathcal{G}$-IC with codelength $|\hat{E}|$.

For the case 1), in $\mathcal{G}$, R_t can obtain what R_t wants even if we do not transmit $Y_e + \bar{F}_e(\mathbf{Y}_S)$ among $\hat{F}(\mathbf{Y})$. In $\mathcal{G}'$, R_t needs $\bar{F}_e(Y_s)$ related to e for decoding what R_t wants. Although R_t in $\mathcal{G}$ does not have Y_e as side information, R_t can obtain $\bar{F}_e(Y_s)$ because R_t has Y_s as side information. Thus, R_t in $\mathcal{G}$ can use the decoding method in $\mathcal{G}'$ even though we do not transmit $Y_B(e)$. For the case 7), it is similarly proved as the case 1) because R_e in $\mathcal{G}$ has Y_s as side information. Thus, $Y_B(e'')$ is not needed for making the $\mathcal{G}$-IC from the $\mathcal{G}'$-IC.

By removing $\{Y_B(e) | e \text{ is an added link in } \mathcal{G}'\}$ from $\mathcal{G}'$-IC, we can make $\mathcal{G}$-IC with codelength $|\hat{E}|$. $\square$

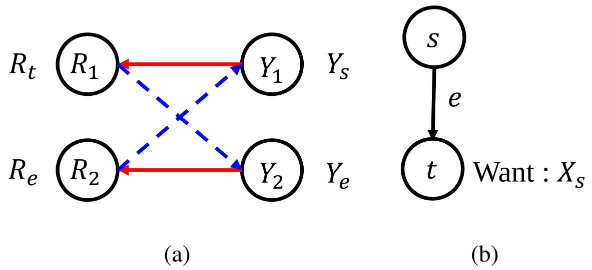

Figure 4.3: The description of Example 4.1: (a) A given side information graph $\mathcal{G}$. (b) The corresponding network coding instance $\mathbb{G}$.

From Corollary 4.1, any corresponding network codes can ensure existence of $\mathcal{G}$-IC achieving the MAIS bound. The following simple example shows that a non-trivial index code achieving the MAIS bound can be found from a trivial network code.

Example 4.1. *Suppose that a given side information graph $\mathcal{G}$ is given in Fig. 4.3(a). Then, the corresponding network coding instance $\mathbb{G}$ is given in Fig. 4.3(b). I choose Y_2 as Y_e and Y_1 as Y_s. In Fig. 4.3(b), $\bar{F}_e(X_s) = X_e$ is trivially X_s. Then, a $\mathcal{G}$-IC with codelength* 1 *is defined as $Y_e + \bar{F}_e(Y_s) = Y_e + Y_s = Y_1 + Y_2$.*

Although the corresponding network code is trivial, it results in a non-trivial index code. If there is no network code for $\mathbb{G}$, it is noted that there is no index code for $\mathcal{G}$ achieving the MAIS bound.

From the code equivalence result, we can reduce a given side information graph to the side information graph of the smaller size that every message is included in at least one cycle. It is because the following proposition holds.

Proposition 4.2. *If Y_1 does not form a cycle in $\mathcal{G}$, Y_1 can be removed to derive the corresponding network coding instance.*

Proof. Y_1 is selected for the link component because Y_1 is included in any maximum acyclic subgraph of $\mathcal{G}$. Then, the link corresponding to Y_1 is not included in any path from a source node to a terminal node which wants to receive messages of the source

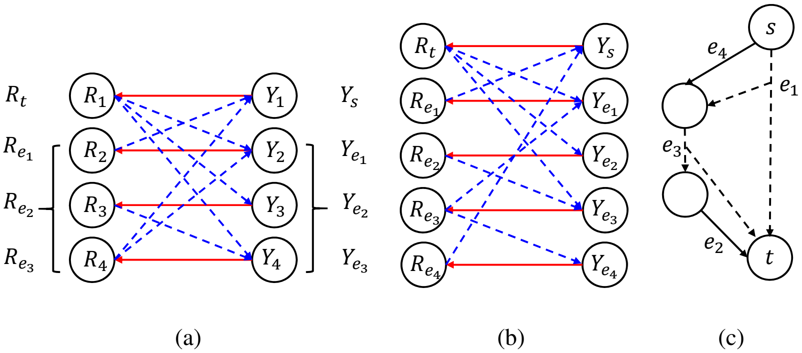

Figure 4.4: The description of Example 4.2: (a) A given side information graph $\mathcal{G}$. (b) A modified side information graph $\mathcal{G}'$. (c) The corresponding network coding instance $\mathbb{G}$.

node in $\mathbb{G}$ from Observation 4.3. Thus, we do not need to consider these meaningless scenarios because we can treat them as the uncoded forms in the perspective of index codes and dummy links carrying 0s in the perspective of network codes. $\square$

There is another example for Corollary 4.1.

Example 4.2. *Suppose that a given side information graph $\mathcal{G}$ is given in Fig. 4.4(a). Then, a modified side information graph $\mathcal{G}'$ and the corresponding network coding instance are shown in Fig. 4.4(b) and Fig. 4.4(c), respectively. Assume that $q = 2$.*

Since $\mathcal{G}$ cannot be directly converted into the corresponding network coding instance, we first find the maximum acyclic subgraph $\hat{\mathcal{G}}$ of $\mathcal{G}$ and determine $\mathbf{Y}_E$ and the corresponding receivers as in Fig. 4.4(a), where $\hat{\mathcal{G}}$ consists of R_2, R_3, R_4 and Y_2, Y_3, Y_4. Subsequently, we can make a modified side information graph by solving one case with the case 7) and two cases with the case 4) in Claim 4.1. Then, it is noted that e_1 and e_3 are hyper-links. An index code for $\mathcal{G}$ with codelength $|\hat{E}| = 3$ is $(Y_s + Y_{e_1}, Y_{e_1} + Y_{e_2}, Y_{e_2} + Y_{e_3})$. Then, every receiver can recover what it wants. For example, R_t can obtain Y_s from $Y_s + Y_{e_1} + Y_{e_1}$. With Theorem 4.2 and Corollary 4.1, we can find an index code for $\mathcal{G}'$ as $(Y_s + Y_{e_1}, Y_{e_1} + Y_{e_2}, Y_{e_2} + Y_{e_3}, Y_s + Y_{e_4})$ and a *network code for $\mathbb{G}$ as follows:*

1. *Let* $\boldsymbol{\sigma} = \mathbf{0}$.
2. $X_{e_1} = \hat{D}_{e_1}(\mathbf{0}, (X_{e'} : e' \in \mathrm{In}(e_1))) = (Y_s + Y_{e_1} = 0) + (Y_s = X_s) = X_s$.
3. $X_{e_2} = (Y_{e_2} + Y_{e_3} = 0) + (Y_{e_3} = X_{e_3}) = X_{e_3}$.
4. $X_{e_3} = (Y_{e_1} + Y_{e_2} = 0) + (Y_{e_2} + Y_{e_3} = 0) + (Y_{e_1} = X_{e_1}) = X_{e_1}$.
5. $X_{e_4} = (Y_s + Y_{e_4} = 0) + (Y_s = X_s) = X_s$.
6. $D_t = (Y_s + Y_{e_1} = 0) + (Y_{e_1} = X_{e_1}) = X_{e_1} = X_s$.

Similarly, we can find an index code for $\mathcal{G}$ from a network code for $\mathbb{G}$. Suppose that we have a network code for $\mathbb{G}$ such that $X_{e_1} = X_s$, $X_{e_2} = X_{e_3} = X_{e_4} = 0$. Then, an index code for $\mathcal{G}'$ is $(Y_{e_1} + Y_s, Y_{e_2}, Y_{e_3}, Y_{e_4})$ and an index code for $\mathcal{G}$ is $(Y_{e_1} + Y_s, Y_{e_2}, Y_{e_3})$. Even though the corresponding network code seems to be useless, an optimal index code for $\mathcal{G}$ can be found from that network code.

Remark 4.1. *In general, a given side information graph can be converted to several distinct modified side information graphs but any modified side information graph can be re-converted to the original side information graph. Thus, for a given index coding instance, there are several distinct corresponding network coding instances but each corresponding network coding instance can be re-converted to the original index coding instance.*

Remark 4.2. *Although there are many types of modified index coding instances, the code equivalence in Corollary 4.1 holds for every modified side information graph. Specifically, it is noted that if there is a $\mathcal{G}$-IC with the codelength $|\hat{E}|$, there is a $\mathcal{G}'$-IC with the codelength $|E|$ for every modified side information graph and the corresponding $\mathbb{G}$-NC. Thus, it is recommended to select a coding instance whose solution can be easily found.*

4.3 Generalization to Erroneous Cases

As index coding with erroneous side information was studied in Chapter 3 and there are code equivalences between a network code and an index code for both a given network coding instance and a given index coding instance, it is natural to consider which network codes are related with index codes with side information errors. In this section, I define network codes with link errors and index codes with side information errors and show their code equivalences.

4.3.1 Network Codes With Link Errors

In order to provide a code equivalence between an NCLE and an ICSIE, a generalized network coding scenario is considered, where each internal node can resolve its erroneous incoming symbols.

For this scenario, if we know the probability distribution of the link errors, the throughput gain can be improved by assigning suitable error resistance capabilities to the internal nodes in a network structure. That is, large error resistance capabilities of internal nodes for the vulnerable links can improve throughput gain of an entire network because error propagation may be moderated. In this perspective, I introduce a new network code which deals with erroneous data on links.

In the same model in Section 4.1, I assume the followings:

1. At the ends of the links, errors may occur due to transmissions through links, referred to as link errors and source nodes may have erroneous source symbols.
2. $\boldsymbol{\delta} = (\delta_{e_1}, ..., \delta_{e_{|E|}}, \delta_{t_1}, ..., \delta_{t_{|T|}})$ is a vector whose elements correspond to the error resistance capability for each outgoing link from the node and terminal in $E \cup T$.
3. For an index set A, $\tilde{\mathbf{X}}_A$ denotes vectors with erroneous symbol elements.

Next, I explain $\boldsymbol{\delta}$ in detail. Now, I consider a network code capable of resolving some link errors. Assume that $e \in E$ is an outgoing link from a node $u \in V$ and there are less than or equal to δ_e symbol errors in the incoming links of u. If an encoding function F_e can make a correct encoded outgoing symbol X_e from incoming symbols with less than or equal to δ_e symbol errors, then F_e and e are said to have an error resistance capability δ_e. When u is a source node, incoming symbols of u denote the source messages possessed by u, meaning that F_e can resolve up to δ_e erroneous message symbols. Similarly, the decoding function D_t of a terminal $t \in T$ and t are said to have an error resistance capability δ_t if D_t can correctly recover a decoded vector $\mathbf{X}_{\mathcal{F}(t)}$ from incoming symbols with less than or equal to δ_t symbol errors. If we only consider error resistance capabilities of terminals, it reduces to error correcting network codes. In such a case, a network code with link errors is summarized as follows.

Definition 4.3. *Let $\boldsymbol{\delta} = (\delta_{e_1}, ..., \delta_{e_{|E|}}, \delta_{t_1}, ..., \delta_{t_{|T|}})$ be a vector whose elements correspond to the error resistance capability for each outgoing link and terminal in $E \cup T$. Then, a network code with link errors with parameters $(\boldsymbol{\delta}, \mathbb{G})$ over $\mathbb{F}_q$, denoted by a $(\boldsymbol{\delta}, \mathbb{G})$-NCLE consists of:*

1. *A local encoding function $F_e : \mathbb{F}_q^{|\mathrm{In}(e)|} \to \mathbb{F}_q$ for $e \in E$*

2. *A decoding function $D_t : \mathbb{F}_q^{|\mathrm{In}(t)|} \to \mathbb{F}_q^{|\mathcal{F}(t)|}$ for $t \in T$*

3. *Satisfying $F_e(\mathbf{X}_{\mathrm{In}(e)}) = F_e(\tilde{\mathbf{X}}_{\mathrm{In}(e)})$ and $D_t(\mathbf{X}_{\mathrm{In}(t)}) = D_t(\tilde{\mathbf{X}}_{\mathrm{In}(t)})$ for any $e \in E$ and $t \in T$, where $\mathrm{wt}(\mathbf{X}_{\mathrm{In}(e)} - \tilde{\mathbf{X}}_{\mathrm{In}(e)}) \leq \delta_e$ and $\mathrm{wt}(\mathbf{X}_{\mathrm{In}(t)} - \tilde{\mathbf{X}}_{\mathrm{In}(t)}) \leq \delta_t$.*

Note that the error resistance capabilities are defined for encoding and decoding functions, that is, encoding functions for the outgoing links of one node can have different error resistance capabilities despite the fact that they have identical erroneous incoming symbols.

4.3.2 Index Codes With Side Information Errors

I introduce index codes with side information errors slightly different from those of Chapter 3. In the same model in Section 4.1, I assume the followings:

1. Each receiver R_i may have erroneous side information as $\tilde{\mathbf{Y}}_{\mathcal{X}_i}$ for $i \in Z[m]$.

2. Let $\boldsymbol{\delta}_s = (\delta_s^{(1)}, ..., \delta_s^{(m)})$ be a vector whose elements correspond to the side information error resistance capability of each receiver.

Next, a $(\boldsymbol{\delta}_s, \mathcal{G})$-index code with side information errors is introduced. I consider an index code which can overcome arbitrary side information errors for each receiver, where each receiver does not know which side information is erroneous. Specifically, each receiver R_i has a side information error resistance capability $\delta_s^{(i)}$ such that the receiver can decode the wanted messages even though less than or equal to $\delta_s^{(i)}$ symbols of side information are erroneous. Then, the $(\boldsymbol{\delta}_s, \mathcal{G})$-index code with side information errors is described as follows.

Definition 4.4. *Let $\boldsymbol{\delta}_s = (\delta_s^{(1)}, ..., \delta_s^{(m)})$ be the vector of side information error resistance capabilities. An index code with side information errors with parameters $(\boldsymbol{\delta}_s, \mathcal{G})$ over $\mathbb{F}_q$, denoted by a $(\boldsymbol{\delta}_s, \mathcal{G})$-ICSIE is a set of codewords having:*

1. *An encoding function $\hat{F} : \mathbb{F}_q^n \to \mathbb{F}_q^N$*

2. *A set of decoding functions $\hat{D}_1, \hat{D}_2, \dots, \hat{D}_m$ such that $\hat{D}_i : \mathbb{F}_q^N \times \mathbb{F}_q^{|\mathcal{X}_i|} \to \mathbb{F}_q^{|f(i)|}$ satisfying*

$$\hat{D}_i(\hat{F}(\mathbf{Y}), \tilde{\mathbf{Y}}_{\mathcal{X}_i}) = \mathbf{Y}_{f(i)} \tag{4.4}$$

   for all $i \in Z[m]$, $\mathbf{Y} \in \mathbb{F}_q^n$, and $\mathrm{wt}(\mathbf{Y}_{\mathcal{X}_i} - \tilde{\mathbf{Y}}_{\mathcal{X}_i}) \le \delta_s^{(i)}$.

Then, the optimal codelength of a $(\boldsymbol{\delta}_s, \mathcal{G})$-ICSIE is $N_{\text{opt}}^q(\boldsymbol{\delta}_s, \mathcal{G})$.

4.3.3 Code Equivalence Between NCLE and ICSIE for Given Network Coding Instance

For a given network coding instance, the only change of the corresponding index coding instance in Section 4.1 is existence of errors. The corresponding index coding instance is summarized as follows:

1. A sender has a message $\mathbf{Y} = (\mathbf{Y}_S, \mathbf{Y}_E)$ and there are $|E| + |T|$ receivers, each of which is a corresponding receiver R_e of a link or R_t of a terminal in the given network coding instance.
2. For $e \in E$, R_e can be described as $\mathcal{X}_e = \text{In}(e)$, $f(e) = \{e\}$, and $\delta_s^{(e)} = \delta_e$.
3. For $t \in T$, R_t can be described as $\mathcal{X}_t = \text{In}(t)$, $f(t) = \mathcal{F}(t)$, and $\delta_s^{(t)} = \delta_t$.

Similarly, encoding functions of the corresponding index code for a given network code is defined as follows:

1. The index codelength is $|E|$ and $\hat{F}(\mathbf{Y}) = (Y_B(e) : e \in E)$.
2. $Y_B(e) = Y_e + \bar{F}_e(\mathbf{Y}_S)$.

Also, the corresponding network code for a given index code with codelength $|E|$ is defined as follows:

1. $\boldsymbol{\sigma} \in \mathbb{F}_q^{|E|}$ is a realization of the ICSIE.
2. For $e \in E$, F_e is defined as a function whose output is $X_e = \hat{D}_e(\boldsymbol{\sigma}, (\tilde{X}_{e'} : e' \in \text{In}(e)))$.
3. For $t \in T$, D_t is defined as a function whose output is $\hat{D}_t(\boldsymbol{\sigma}, (\tilde{X}_{e'} : e' \in \text{In}(t)))$.

Then, we have the following theorem.

Theorem 4.3. *For a given network coding instance, a $(\boldsymbol{\delta}, \mathbb{G})$-NCLE exists if and only if the corresponding $(\boldsymbol{\delta}_s, \mathcal{G})$-ICSIE with codelength $|E|$ exists.*

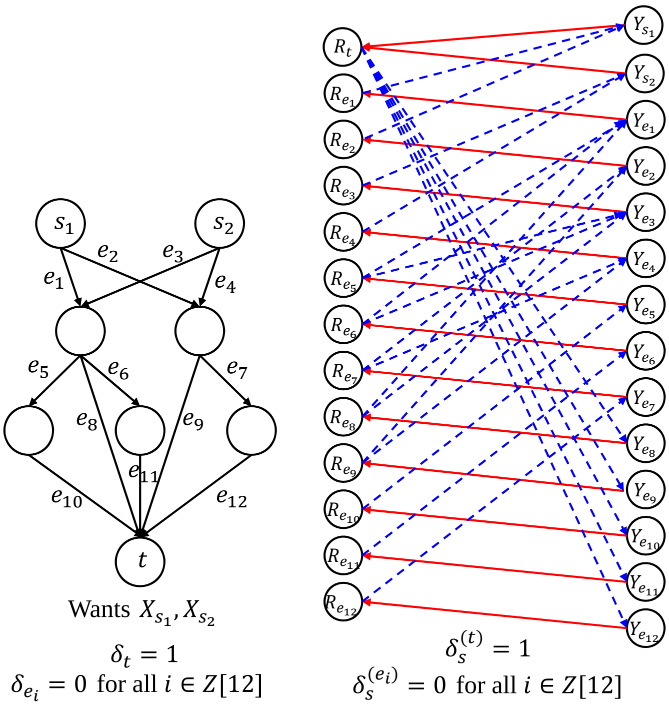

Figure 4.5: An example of a network coding instance and the corresponding index coding instance. (a) A network coding instance (b) The corresponding side information graph with $\boldsymbol{\delta}_s$.

Proof. I focus on how existence of errors affects the decoding procedures of the defined network codes and index codes.

$Necessity$: For a given $(\boldsymbol{\delta}, \mathbb{G})$-NCLE, the corresponding ICSIE is $(Y_B(e) : e \in E)$. For $e \in E$ such that e is not an outgoing link of a source node, R_e has $\tilde{\mathbf{Y}}_{\mathrm{In}(e)}$ as side information. Then, for $e' \in \mathrm{In}(e)$, R_e can compute $Y_B(e') - \tilde{Y}_{e'}$, which may be an erroneous value of $\bar{F}_{e'}(\mathbf{Y}_S)$. From erroneous values of $\{\bar{F}_{e'}(\mathbf{Y}_S)|e' \in \mathrm{In}(e)\}$, R_e can compute $\bar{F}_e(\mathbf{Y}_S)$ because the maximum number of errors is up to $\delta_s^{(e)}$ and it is the same as δ_e defined for $\bar{F}_e$. Thus, R_e can recover Y_e from $Y_B(e) - \bar{F}_e(\mathbf{Y}_S)$. If e is an outgoing link of a source node, R_e can obtain $\bar{F}_e(\mathbf{Y}_S)$ using its side information because $\delta_e = \delta_s^{(e)}$. For $t \in T$, it is similarly proved using the fact that $\delta_t = \delta_s^{(t)}$.

$Sufficiency$: Since codelength is $|E|$ and Observation 4.2 holds, it is noted that for any valid realizaiton $\boldsymbol{\sigma}$ of a given ICSIE and $\mathbf{Y}_S$, there is unique $\mathbf{Y}_E$. For $e \in E$, X_e is defined as $\hat{D}_e(\boldsymbol{\sigma}, (\tilde{X}_{e'} : e' \in \text{In}(e)))$. For $t \in T$, D_t is defined as $\hat{D}_t(\boldsymbol{\sigma}, (\tilde{X}_{e'} : e' \in \text{In}(t)))$. Without loss of generality, we assume that $\mathbf{X}_S = \mathbf{Y}_S$. For $e \in E$ with less than or equal to δ_e incoming errors, $\hat{D}_e(\boldsymbol{\sigma}, (\tilde{X}_{e'} : e' \in \text{In}(e))) = Y_e$ because $\delta_e = \delta_s^{(e)}$ and Y_e is unique. For $t \in T$ with less than or equal to δ_t incoming errors, $\hat{D}_t(\boldsymbol{\sigma}, (\tilde{X}_{e'} : e' \in \text{In}(t))) = \mathbf{Y}_{\mathcal{F}(t)} = \mathbf{X}_{\mathcal{F}(t)}$ because $\delta_t = \delta_s^{(t)}$. □

Here is an example of Theorem 4.3.

Algorithm 4.1 Decoding procedure for the terminal t

Input: $\tilde{X}_{s_1}, \tilde{X}_{s_1}, \tilde{X}_{s_2}, \tilde{X}_{s_1} + \tilde{X}_{s_2}$, and $\tilde{X}_{s_1} + \tilde{X}_{s_2}$, one of which may be erroneous.

Output: X_{s_1} and X_{s_2}

Step 1) Compare two received symbols of $\tilde{X}_{s_1} + \tilde{X}_{s_2}$.

Step 2) If they are different, determine the received $\tilde{X}_{s_1}$ and $\tilde{X}_{s_2}$ as outputs and skip the following steps.

Step 3) If they are identical, compare two received symbols of $\tilde{X}_{s_1}$.

Step 4) If two received symbols of $\tilde{X}_{s_1}$ are identical, determine the received $\tilde{X}_{s_1}$ as X_{s_1} and X_{s_2} can be obtained from $\tilde{X}_{s_1} + \tilde{X}_{s_2} - \tilde{X}_{s_1}$.

Step 5) If two received symbols of $\tilde{X}_{s_1}$ are different, determine the received $\tilde{X}_{s_2}$ as X_{s_2} and X_{s_1} can be obtained from $\tilde{X}_{s_1} + \tilde{X}_{s_2} - \tilde{X}_{s_2}$.

Example 4.3. *Suppose that a given network coding instance and the corresponding side information graph with* $\boldsymbol{\delta}_s$ *are given as in Fig. 4.5. A network code for Fig. 4.5(a) can be described as follows:*

1. $X_{e_1} = X_{e_2} = X_{e_5} = X_{e_8} = X_{e_{10}} = X_{s_1}$

2. $X_{e_3} = X_{e_4} = X_{e_7} = X_{e_{12}} = X_{s_2}$

3. $X_{e_6} = X_{e_9} = X_{e_{11}} = X_{s_1} + X_{s_2}$

4. D_t is given in Algorithm 4.1.

Then, the corresponding index code for Fig. 4.5(b) can be described as follows:

1. *The transmitted codeword $\hat{F}(\mathbf{Y}) = (Y_B(e) : e \in E)$ consists of 12 components.*

2. *$Y_B(e_i) = Y_{e_i} + Y_{s_1}$ for $i \in \{1, 2, 5, 8, 10\}$*

3. *$Y_B(e_i) = Y_{e_i} + Y_{s_2}$ for $i \in \{3, 4, 7, 12\}$*

4. *$Y_B(e_i) = Y_{e_i} + Y_{s_1} + Y_{s_2}$ for $i \in \{6, 9, 11\}$*

5. *The decoding functions of receivers can be defined as in Theorem 4.3.*

Thus, by finding an NCLE for a given network coding instance, we can find the corresponding ICSIE with codelength $|E|$. Furthermore, if we find an ICSIE with codelength $|E|$ for Fig. 4.5(b), we can obtain the corresponding NCLE for Fig. 4.5(a).

Since the code equivalence between an NCLE and an ICSIE is shown when a network coding instance is given, we can utilize the properties of an ICSIE to derive those of an NCLE. First, I introduce a property of an ICSIE in the following lemma, which is similar to that in Chapter 3.

Lemma 4.1. *Suppose that a $(\mathbf{0}, \tilde{\mathcal{G}})$-IC problem is constructed by deleting any less than or equal to min$(2\delta_s^{(i)}, |\mathcal{X}_i|)$ outgoing edges from each receiver R_i in a $(\boldsymbol{\delta}_s, \mathcal{G})$-ICSIE problem. That is, each receiver of $\tilde{\mathcal{G}}$ has larger than or equal to max$(0, |\mathcal{X}_i| - 2\delta_s^{(i)})$ side information symbols and then it becomes the conventional index coding problem with $\boldsymbol{\delta}_s = \mathbf{0}$. Then, $N_{\text{opt}}^q(\mathbf{0}, \tilde{\mathcal{G}}) \leq N_{\text{opt}}^q(\boldsymbol{\delta}_s, \mathcal{G})$.*

Proof. This is proved similarly to the method in Chapter 3 and thus I omit it here. □

Lemma 4.1 shows the relationship between the conventional index code and an ICSIE. Thus, we can infer that a property between the conventional network code with $\boldsymbol{\delta} = \mathbf{0}$ and an NCLE is derived by Lemma 4.1 as in the following theorem.

Theorem 4.4. *Let E_v be a set of outgoing links of $v \in V \setminus \bar{S}$ and δ_v be $\min\{\delta_e | e \in E_v\}$, where $E_t = \{t\}$ if $t \in T$. There exists the conventional network code with $\boldsymbol{\delta} = \mathbf{0}$ after deleting arbitrary $2\delta_v$ links from $\mathrm{In}(v)$ for all $v \in V \setminus \bar{S}$ in $\mathbb{G}$ if a $(\boldsymbol{\delta}, \mathbb{G})$-NCLE exists for a given network structure $\mathbb{G}$. For the case of $e \in \mathrm{In}(v)$ being a hyper-link, deleting e from v means that e does not transmits a symbol to v.*

Proof. Let $\mathcal{G}$ be the corresponding side information graph of a $(\boldsymbol{\delta}, \mathbb{G})$-NCLE. Instead of deleting arbitrary $2\delta_v$ incoming links for all $v \in V \setminus \bar{S}$ in $\mathbb{G}$, we can change these $2\delta_v$ links as incoming links of dummy nodes with no outgoing link. Let $\tilde{\mathcal{G}}$ be the corresponding side information graph of the modified network structure $\tilde{\mathbb{G}}$ with $\boldsymbol{\delta} = \mathbf{0}$, where dummy nodes are considered as the above and $\bar{\mathbb{G}}$ be the modified network structure with $\boldsymbol{\delta} = \mathbf{0}$, where those incoming links are deleted. From Theorem 4.1, $N^q_{\mathrm{opt}}(\mathbf{0}, \tilde{\mathcal{G}}) = |E|$ if and only if the conventional network code for $\tilde{\mathbb{G}}$ exists. From Lemma 4.1, $N^q_{\mathrm{opt}}(\mathbf{0}, \tilde{\mathcal{G}}) \leq N^q_{\mathrm{opt}}(\boldsymbol{\delta}_s, \mathcal{G})$ and from Theorem 4.3, $N^q_{\mathrm{opt}}(\boldsymbol{\delta}_s, \mathcal{G}) = |E|$ if and only if the $(\boldsymbol{\delta}, \mathbb{G})$-NCLE is feasible. Thus, it is noted that the conventional network code for $\tilde{\mathbb{G}}$ exists if the $(\boldsymbol{\delta}, \mathbb{G})$-NCLE exists. From the conventional network code for $\tilde{\mathbb{G}}$, we can find the conventional network code for $\bar{\mathbb{G}}$ by removing encoding functions of those links. $\square$

4.3.4 Code Equivalence Between NCLE and ICSIE for Given Index Coding Instance

For a code equivalence between an NCLE and an ICSIE for a given index coding instance, it is sufficient to decide error resistance capabilities of added receivers regarding Claim 4.1. For these receivers, let each error resistance capability be 0. Then, we have the following theorem.

Theorem 4.5. *For a given side information graph $\mathcal{G}$ and $\boldsymbol{\delta}_s$, a $(\boldsymbol{\delta}_s, \mathcal{G})$-ICSIE with the codelength $|\hat{E}|$ exists if and only if the corresponding $(\boldsymbol{\delta}, \mathbb{G})$-NCLE exists.*

Proof. It can be similary proved as in Corollary 4.1. From Theorem 4.3 and the fact that $\delta_e = \delta_s^{(e)}$ for $e \in E$ and $\delta_t = \delta_s^{(t)}$ for $t \in T$, it can be directly proved. $\square$

Here is an example for Theorem 4.5.

Example 4.4. *Suppose that a given side information graph $\mathcal{G}$ is given in Fig. 4.4(a) and $\boldsymbol{\delta}_s = (1,0,0,0)$. Then, a modified side information graph $\mathcal{G}'$ and the corresponding network coding instance are shown in Fig. 4.4(b) and Fig. 4.4(c), respectively. Then, $\delta_s^{(t)} = 1$ and $\delta_s^{(e_i)} = 0$ for $i \in Z[4]$ in Fig. 4.4(b). Consequently, $\delta_t = 1$ and $\delta_{e_i} = 0$ for $i \in Z[4]$ in Fig. 4.4(c). I assume $q = 2$.*

A $(\boldsymbol{\delta}_s, \mathcal{G})$-ICSIE with codelength $|\hat{E}| = 3$ is $(Y_s + Y_{e_1}, Y_{e_1} + Y_{e_2}, Y_{e_2} + Y_{e_3})$. Then, every receiver can recover what it wants. For example, R_t can calculate $Y_s + Y_{e_1}$, $Y_s + Y_{e_2}$, and $Y_s + Y_{e_3}$ from the received codeword. Since $\delta_s^{(t)} = 1$ and R_t has $\tilde{Y}_{e_1}$, $\tilde{Y}_{e_2}$, and $\tilde{Y}_{e_3}$ as side information, subtracting them from $Y_s + Y_{e_1}$, $Y_s + Y_{e_2}$, and $Y_s + Y_{e_3}$, respectively results in the true symbol by majority decoding. With Theorem 4.3 and Theorem 4.5, we can find a $(\boldsymbol{\delta}'_s, \mathcal{G}')$-ICSIE as $(Y_s + Y_{e_1}, Y_{e_1} + Y_{e_2}, Y_{e_2} + Y_{e_3}, Y_s + Y_{e_4})$ and a $(\boldsymbol{\delta}, \mathbb{G})$-NCLE as follows:

1. *Let $\boldsymbol{\sigma} = \mathbf{0}$.*
2. $X_{e_1} = \hat{D}_{e_1}(\mathbf{0}, (X_{e'} : e' \in \mathrm{In}(e_1))) = (Y_s + Y_{e_1} = 0) + (Y_s = X_s) = X_s$.
3. $X_{e_2} = (Y_{e_2} + Y_{e_3} = 0) + (Y_{e_3} = X_{e_3}) = X_{e_3}$.
4. $X_{e_3} = (Y_{e_1} + Y_{e_2} = 0) + (Y_{e_2} + Y_{e_3} = 0) + (Y_{e_1} = X_{e_1}) = X_{e_1}$.
5. $X_{e_4} = (Y_s + Y_{e_4} = 0) + (Y_s = X_s) = X_s$.
6. *D_t: calculate $(Y_s + Y_{e_1} = 0) + (\tilde{Y}_{e_1} = \tilde{X}_{e_1})$, $(Y_s + Y_{e_1} = 0) + (Y_{e_1} + Y_{e_2} = 0) + (\tilde{Y}_{e_2} = \tilde{X}_{e_2})$, and $(Y_s + Y_{e_1} = 0) + (Y_{e_1} + Y_{e_2} = 0) + (Y_{e_2} + Y_{e_3} = 0) + (\tilde{Y}_{e_3} = \tilde{X}_{e_3})$. Then, choose the majority value as the output.*

It is similar to that in Example 4.2 and D_t is changed because $\delta_t = 1$. Also, we can find a $(\boldsymbol{\delta}_s, \mathcal{G})$-ICSIE with codelength $|\hat{E}|$ from a $(\boldsymbol{\delta}, \mathbb{G})$-NCLE.

Chapter 5

Index Coding With Multiple Senders and Extension to Cellular Network

Since there are a lot of scenarios where messages are distributed among multiple senders, index coding with multiple senders has attracted significant attention and a lot of researches on multiple senders have been done to find the capacity region. Index coding with multiple senders was introduced [34]. Graph-theoretic approaches to the two sender index coding problem were researched [35]. In [36], partitioned distributed composite coding was studied based on composite coding in [4] for a general multiple sender case. In [37], multi-sender cooperative composite coding was developed and advantages of cooperative composite coding were studied. A method finding optimal linear index codes based on the rank minimization and decodability conditions of receivers was studied in [38].

Existing researches on index coding with multiple senders consider a scenario, where every receiver can receive encoded messages from all senders. That assumption is valid if every receiver belongs to coverage of each sender. However, some receivers do not belong to coverage of all senders in reality. For example, some receivers can receive encoded messages from a subset of senders in a cellular network due to the problem of coverage. Thus, index coding in a cellular network has to be studied to

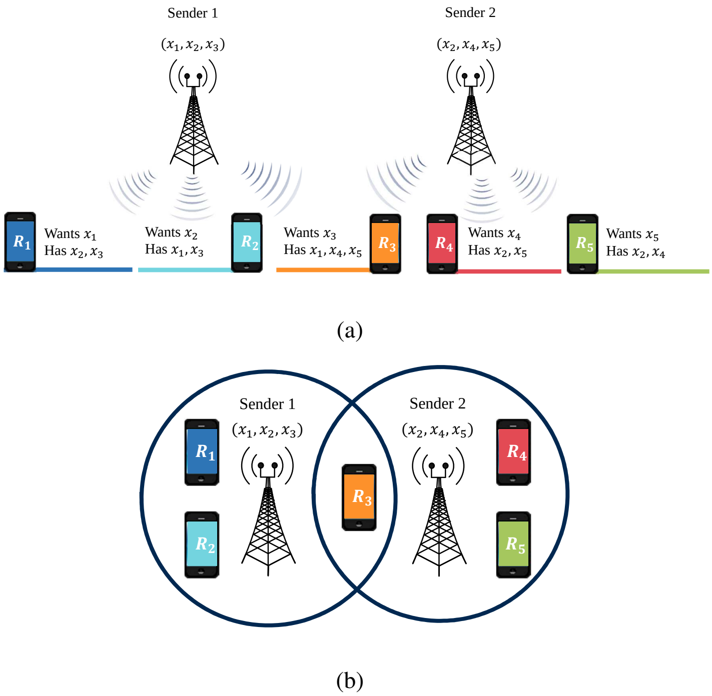

Figure 5.1: A description for index coding with two senders in a cellular network: (a) An index coding instance. (b) Coverage of two senders.

utilize index coding in the realistic scenario. In a cellular network, an index coding instance where multiple senders exist and receivers are restricted to receive encoded messages from some senders has to be considered and this scenario for two senders is depicted in Fig. 5.1.

In Fig. 5.1, receivers 1 and 2 can receive encoded messages from only sender 1, receivers 4 and 5 can receive encoded messages from only sender 2, and receiver 3 can receive all encoded messages. Then, every receiver is satisfied if sender 1 transmits $(x_1 + x_2 + x_3)$ and sender 2 transmits $(x_2 + x_4 + x_5)$. Since receiver 3 can receive all encoded messages, receiver 3 can calculate $(x_1 + x_2 + x_3) - (x_2 + x_4 + x_5) = x_1 + x_3 - x_4 - x_5$. By using its side information, receiver 3 can recover x_3.

In this chapter, I study some properties on linear index codes with multiple senders. Since there are multiple senders, a lot of properties of index codes with the single sender have to be modified. First, a fitting matrix for multiple senders is introduced and an encoding method using the fitting matrix is also studied as in [2]. Then, some properties related with the optimal codelength are studied and whether given side information is critical or not is studied for multiple senders.

Furthermore, linear index coding with multiple senders in a cellular network is studied. In a cellular network with two senders, another type of fitting matrices is introduced and the encoding method based on the fitting matrix is studied to find the optimal codelength. In addition, some properties on the optimal codelength of linear index codes for the cellular network case are studied.

In summary, contrast to the existing researches on multiple senders, my main contributions are given as follows:

1. Compared with [38], an alternative fitting matrix approach for constructing linear index codes is proposed.

2. The necessary and sufficient condition for reducing linear index codelength for multiple senders is studied.

3. The more realistic scenario of a cellular network for linear index coding is considered for the first time.

5.1 Problem Formulation and Some Results

In the linear index coding problem for the case of multiple senders, there are $|S|$ senders and m receivers $R_1, \ldots, R_m$, where $S = \{s_1, s_2, \ldots, s_{|S|}\}$ is a set of senders. In this chapter, I only consider a scalar linear index code. Then, there are n messages represented by $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{F}_q^n$, which are distributed in $|S|$ senders so that $\cup_{i \in Z[|S|]} M_i = Z[n]$, where M_i denotes a set of indices of messages which s_i has for $i \in Z[|S|]$. Let $M = \{M_1, \ldots, M_{|S|}\}$ and $M_c = \{i \mid \exists s_j, s_k \text{ for any } j, k \in Z[|S|] \text{ such that } j \neq k, i \in M_j, \text{ and } i \in M_k\}$.

Each sender broadcasts its encoded messages through the error-free broadcast channel. Each receiver R_i has $\mathbf{x}_{\mathcal{X}_i}$ as side information and wants to receive the wanted message denoted by $x_{f(i)}$, where $f(i)$ is an index of the message that R_i wants to receive and $\mathcal{X}_i$ is a set of side information indices of R_i. Each receiver R_i receives all encoded messages from $|S|$ senders and R_i has to recover $x_{f(i)}$ from the received codeword and $\mathbf{x}_{\mathcal{X}_i}$. I assume that $\{f(i)\} \cap \mathcal{X}_i = \phi$ and let $\mathcal{Y}_i = Z[n] \setminus \{f(i)\} \setminus \mathcal{X}_i$ for $i \in Z[m]$.

Let $\mathcal{G}$ be a bipartite side information graph, where a directed edge from a receiver node to a message node means that the receiver has the message as side information and a directed edge from a message node to a receiver node means that the receiver wants to receive the message. It is assumed that s_1 knows $\mathcal{G}$ and M, s_1 determines encoding procedures of all senders, and each sender can receive its encoding strategy from s_1.

Then, a linear index code for multiple senders is defined as follows.

Definition 5.1. *A linear index code with multiple senders over $\mathbb{F}_q$, denoted by a $(\mathcal{G}, M)$-IC is a set of codewords having:*

1. *Generator submatrices $G^{(i)} \in \mathbb{F}_q^{|M_i| \times N_i}$ for $i \in Z[|S|]$, where N_i denotes the codelength of the sub-codeword $\mathbf{x}_{M_i} G^{(i)}$ generated by s_i.*
2. *Decoding functions $D_j : \mathbb{F}_q^{N_1+N_2+\cdots+N_{|S|}} \times \mathbb{F}_q^{|\mathcal{X}_j|} \to \mathbb{F}_q$ satisfying*

$$D_j(\mathbf{x}_{M_1} G^{(1)}, \ldots, \mathbf{x}_{M_{|S|}} G^{(|S|)}, \mathbf{x}_{\mathcal{X}_j}) = x_{f(j)} \tag{5.1}$$

   for all $j \in Z[m]$, $\mathbf{x} \in \mathbb{F}_q^n$.

It is noted that an $n \times N$ generator matrix G for a $(\mathcal{G}, M)$-IC is constructed by using generator submatrices $G^{(i)}$ for $i \in Z[|S|]$ and the codelength N is $N_1 + \cdots + N_{|S|}$. Let $N_{\text{opt}}^q(\mathcal{G}, M)$ be the optimal codelength of a $(\mathcal{G}, M)$-IC. Since there are more than one sender, another graph representing them is needed as in the following definition [34].

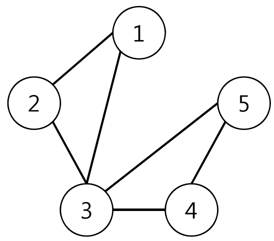

Figure 5.2: A message graph U for $n = 5$, $M_1 = \{1, 2, 3\}$, and $M_2 = \{3, 4, 5\}$.

Definition 5.2. *A message graph U is a unipartite graph of n message nodes, where an undirected edge between any two nodes i and j exists if and only if x_i and x_j are known to the same sender, that is, $i, j \in M_s$ for some $s \in Z[|S|]$ such that $i \neq j$.*

From a message graph U, it is noted that which two messages are contained in the same sender, that is, which two messages can be encoded together. In general, there is the constraint of multiple senders that two messages not connected in U cannot be encoded together. Fig. 5.2 is an example of U for $n = 5$, $M_1 = \{1, 2, 3\}$, and $M_2 = \{3, 4, 5\}$.

We have a well known claim of linear index codes as follows [2].

Claim 5.1. *If G is a generator matrix of a linear index code, $\Sigma_{i \in Z[N]} a_i^k G_i = e_{f(k)} + \Sigma_{j \in \mathcal{X}_k} b_j^k e_j$ for some $a_i^k, b_j^k \in \mathbb{F}_q$, where $k \in Z[m]$, G_i denotes the ith column of G, and e_j denotes the jth standard basis vector.*

5.2 Encoding Procedure of $(\mathcal{G}, M)$-IC

In this section, I first introduce a new type of a fitting matrix for the multiple senders. Then, it is proved that the minimum rank of the proposed fitting matrices is the same as $N_{\text{opt}}^q(\mathcal{G}, M)$.

In [38], they proposed a method to find the optimal codelength of linear index codes. They suggested many strategies for search space reduction and checked decoding conditions of receivers for every generator matrix candidate. Unlike their methods, I consider the fitting matrix which always satisfies decoding conditions of receivers and propose a new method to find the optimal codelength based on the fitting matrix. The fitting matrix for the multiple senders consists of $|S|$ submatrices as shown in the following definition.

Definition 5.3. *The fitting matrix F of size $n \times m|S|$ for a $(\mathcal{G}, M)$-IC is described as follows:*

1. *There are $|S|$ submatrices $F^{(1)}, F^{(2)}, \ldots, F^{(|S|)}$ of size $n \times m$ representing each sender.*
2. *$F_{i,k}^{(j)} = 0$ for $i \notin M_j$, $j \in Z[|S|]$, and $k \in Z[m]$.*
3. *$\Sigma_{j=1}^{|S|} a_j^k F_{f(k),k}^{(j)} = 1$ for $a_j^k \in \mathbb{F}_q$ and $k \in Z[m]$.*
4. *$\Sigma_{j=1}^{|S|} a_j^k F_{i_1,k}^{(j)} = 0$ for the same $a_j^k \in \mathbb{F}_q$ as the above, $i_1 \in \mathcal{Y}_k$, and $k \in Z[m]$.*
5. *For $i_2 \in M_j \cap \mathcal{X}_k$, $j \in Z[|S|]$, and $k \in Z[m]$, $F_{i_2,k}^{(j)}$ is any element of $\mathbb{F}_q$.*

The exact role of each item in Definition 5.3 will be explained later and thus I briefly explain each item. 2) is related with capabilities of senders encoding messages. From 3), 4), and 5), it is noted that we can get the equation in Claim 5.1, which is needed for decoding of R_k.

As mentioned before, the minimum rank of the conventional fitting matrices for the single sender IC problem is known to be the optimal scalar linear index codelength. Similarly, the minimum rank of the fitting matrices for a $(\mathcal{G}, M)$-IC is the optimal scalar linear index codelength as shown in the following theorem.

Theorem 5.1. *Let F' be a fitting matrix having the minimum rank. Then, an optimal generator matrix of a $(\mathcal{G}, M)$-IC is the matrix G' generated by deleting linearly dependent columns in F'.*

Proof. First, I show that a fitting matrix of a $(\mathcal{G}, M)$-IC can be a generator matrix. From 2) of Definition 5.3, the index code corresponding to the fitting matrix can be made for multiple senders. Then, for $k \in Z[m]$, R_k can obtain $x_{f(k)}$ by using its side information and the linear combination of the kth columns of submatrices represented in Definition 5.3, that is, the linear combination of the received codeword components.

Next, I show that the minimum rank of these fitting matrices is the optimal codelength. Let G be an $n \times N$ generator matrix of a $(\mathcal{G}, M)$-IC. From Claim 5.1, it is noted that $\Sigma_{i \in Z[N]} a_i^k G_i = e_{f(k)} + \Sigma_{j \in \mathcal{X}_k} b_j^k e_j$ for some $a_i^k, b_j^k \in \mathbb{F}_q$, where $k \in Z[m]$ and G_i denotes the ith column of G.

By the definition of the fitting matrix F, it is noted that columns whose linear combination is $e_{f(k)} + \Sigma_{j \in \mathcal{X}_k} b_j^k e_j$ can be the kth columns of submatrices of F if each column satisfies 2) of Definition 5.3. Since every column G_i is used for the encoding in one sender, we can classify which column of G belongs to which sender. Let $C_l = \{i | i \in Z[N] \text{ and } G_i \text{ is used for encoding in } s_l\}$ for $l \in Z[|S|]$. Then, we can split $\Sigma_{i \in Z[N]} a_i^k G_i$ into $\Sigma_{i \in C_1} a_i^k G_i + \cdots + \Sigma_{i \in C_{|S|}} a_i^k G_i$. If we make the kth column of $F^{(l)}$ as $\Sigma_{i \in C_l} a_i^k G_i$ for all k and l, it becomes a fitting matrix for a $(\mathcal{G}, M)$-IC and it is obvious that the rank of this fitting matrix is smaller than or equal to N. Since the rank of a generator matrix is the same as codelength, G' is an optimal generator matrix of a $(\mathcal{G}, M)$-IC. $\square$

Now, we have the following corollary specifying the fitting matrix F for a $(\mathcal{G}, M)$-IC.

Corollary 5.1. *An optimal index code can also be found when coefficients in 3) and 4) of Definition 5.3 are modified as follows:*

1. $\Sigma_{j=1}^{|S|} F_{f(k),k}^{(j)} = 1$ *for* $k \in Z[m]$.
2. $\Sigma_{j=1}^{|S|} F_{i_1,k}^{(j)} = 0$ *for* $i_1 \in \mathcal{Y}_k$ *and* $k \in Z[m]$.

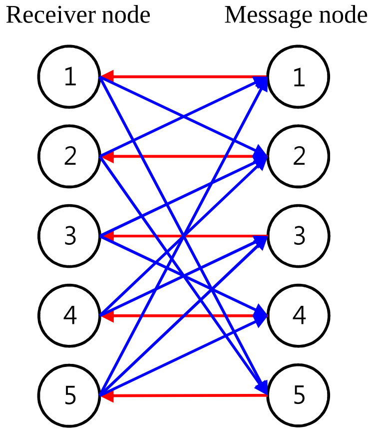

Figure 5.3: A side information graph $\mathcal{G}$ for Example 5.1.

Proof. For the perspective of the minimum rank, nonzero coefficients in 3), 4) of Definition 5.3 do not change the rank of the fitting matrix. For zero coefficients, if we make the corresponding columns as all-zero columns, the rank of the modified fitting matrix becomes smaller than or equal to that of the original one and it is also a fitting matrix for a $(\mathcal{G}, M)$-IC. $\square$

We have the following remark for the encoding procedure based on the fitting matrix.

Remark 5.1. *It is noted that the sum of the ith column of each submatrix of the fitting matrix F for a $(\mathcal{G}, M)$-IC is the ith column of the conventional fitting matrix for the single sender IC problem with $\mathcal{G}$ from Definition 2.1, where $i \in Z[m]$. After choosing linearly independent columns of F, it is noted that selected columns in $F^{(j)}$ are used for the encoding procedure of s_j for $j \in Z[|S|]$.*

From Remark 5.1, it is noted that we can construct a fitting matrix systematically from Definition 5.3. By considering coefficients of 3) and 4) in Definition 5.3 as 1s, we can determine all elements of F as in the following example.

Example 5.1. *Let $m = n = 5$, $M = \{M_1, M_2\}$, $M_1 = \{1, 2, 3\}$, and $M_2 = \{3, 4, 5\}$. A side information graph $\mathcal{G}$ is given in Fig. 5.3. Then, the fitting matrix of a $(\mathcal{G}, M)$-IC is derived as*

$$F = \begin{pmatrix} F^{(1)} & F^{(2)} \end{pmatrix}, \tag{5.2}$$

where

$$F^{(1)} = \begin{pmatrix} 1 & * & 0 & 0 & * \\ * & 1 & * & * & 0 \\ a & b & c & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, F^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -a & -b & 1-c & * & * \\ 0 & 0 & * & 1 & * \\ * & * & 0 & 0 & 1 \end{pmatrix}. \tag{5.3}$$

$*$ denotes any element of $\mathbb{F}_q$ and $a, b, c \in \mathbb{F}_q$. By finding a fitting matrix having the minimum rank, we can find an optimal generator matrix.

Since both a method in [38] and the proposed method based on the fitting matrix can find the optimal codelength, we have the following remark to compare them.

Remark 5.2. *Consider $\mathcal{P}'$ in [38] and assume that $m = n$ and $f(j) = j$ for $j \in Z[m]$. The complexity for solving $\mathcal{P}'$ is $O\Big(q^{\Sigma_{i \in Z[|S|]} \frac{|M_i|^2 + |M_i|}{2}} \times \big(\Sigma_i |M_i|^2 + \Sigma_j \Sigma_i (1 + |\mathcal{Y}_j|)^2 |M_i|\big)\Big)$ from the number of undetermined elements, the rank calculation of lower triangular matrices, and checking decoding conditions.*

The complexity for the proposed method is $O\Big(q^{\Sigma_j \big(\Sigma_i (|\mathcal{X}_j \cap M_i| + |(\mathcal{Y}_j \cup \{j\}) \cap M_i \cap M_c|) - |(\mathcal{Y}_j \cup \{j\}) \cap M_c|\big)} \times |S| n^3\Big)$ from the number of undetermined elements and the rank calculation. It is not easy to compare the above two complexities. However, it is expected that the smaller $|\mathcal{X}_j|$, $|M_c|$, and $|S|$, the lower the complexity for the proposed method based on the fitting matrix. Since the proposed fitting matrix can be used to derive some properties of index codes in the following sections, I consider the encoding procedure based on the fitting matrix in this chapter.

The following proposition shows that we can further simplify the fitting matrix F for a $(\mathcal{G}, M)$-IC if side information of receivers has some properties.

Proposition 5.1. *For a $(\mathcal{G}, M)$-IC, if $\mathcal{X}_i \subset M_j$ and $f(i) \in M_j$ for $i \in Z[m]$ and $j \in Z[|S|]$, the ith columns in the submatrices except $F^{(j)}$ can be assumed as all-zero columns in the perspective of the optimal codelength and the ith column in $F^{(j)}$ is the ith column of the conventional fitting matrix of the single sender IC problem.*

Proof. It is noted that the ith column of $F^{(j)}$ can be transformed into the ith column of the conventional fitting matrix by adding the other ith columns of submatrices. Also, elementary column operations do not change the rank of matrices and the ith column of $F^{(j)}$ can initially be set to the ith column of the conventional fitting matrix from Definition 5.3, $\mathcal{X}_i \subset M_j$, and $f(i) \in M_j$. Since the ith columns in the other submatrices are not needed for decoding of R_i if we set the ith column of $F^{(j)}$ as the above, the ith columns in the other submatrices can be assumed as all-zero columns to minimize the rank of the fitting matrix. □

Thus, if every receiver satisfies the condition in Proposition 5.1, it can be reduced to the single sender IC problem.

5.3 Properties of $(\mathcal{G}, M)$-IC

In this section, I study some properties of a $(\mathcal{G}, M)$-IC through the graphical approach and fitting matrices. First, I introduce a graph which will be used in the graphical approach. In Chapter 3, a 0-cycle is found to be important in an index coding problem for the single sender case. For the multiple sender case, a 0-cycle is also related with a lot of properties of a $(\mathcal{G}, M)$-IC.

A 0-cycle is a subgraph induced by a set of messages and all receivers wanting those messages in which every receiver wanting a message in it has at least one side

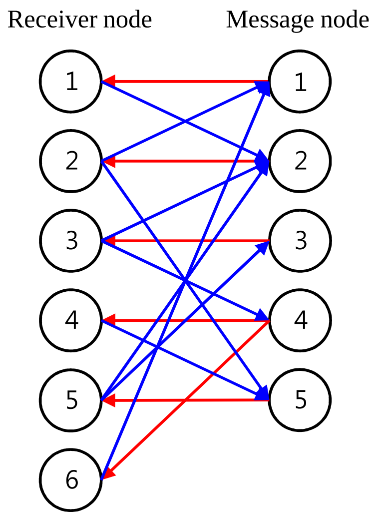

Figure 5.4: A side information graph $\mathcal{G}$ for Example 5.2.

information symbol among those messages. For $m = n$ and $f(i) = i$ for $i \in Z[m]$, a 0-cycle is similar to a partial clique defined in [39]. However, a 0-cycle can be defined for $m \neq n$. Furthermore, k-partial clique must have one receiver whose number of side information symbols is $|B| - k - 1$ but there is no such constraint in a 0-cycle. Now, I define a special type of a 0-cycle for a $(\mathcal{G}, M)$-IC.

Definition 5.4. *Let U' be a subgraph of U, which has message nodes corresponding to a 0-cycle. Then, a 0-cycle is said to be message-connected if and only if there is at least one path between any two message nodes in U'. Otherwise, a 0-cycle is said to be message-disconnected.*

The message connectivity serves a crucial role for index coding with multiple senders because it is related with capabilities of encoding those messages. There is an example for a 0-cycle and a message-connected 0-cycle.

Example 5.2. *Assume that a message graph and M are given in Fig. 5.2 and a side information graph is given in Fig. 5.4. Then, a subgraph of $\mathcal{G}$ induced by message* *nodes $\{1,2,3\}$ and receiver nodes $\{1,2,3\}$ is a 0-cycle because $2 \in \mathcal{X}_1$, $1 \in \mathcal{X}_2$, and $2 \in \mathcal{X}_3$. Furthermore, it is also a message-connected 0-cycle because a subgraph of U induced by $\{1,2,3\}$ has at least one path between any two message nodes. A subgraph of $\mathcal{G}$ induced by message nodes $\{1,2,5\}$ and receiver nodes $\{1,2,5\}$ is a 0-cycle but it is not a message-connected 0-cycle. In addition, a subgraph of $\mathcal{G}$ induced by message nodes $\{3,4,5\}$ and receiver nodes $\{3,4,5,6\}$ is not a 0-cycle because R_6 has x_1 as side information but no side information in $\{x_3, x_5\}$.*

It is said that x_i forms a 0-cycle if there exists at least one 0-cycle containing x_i for $i \in Z[n]$, otherwise it is said that x_i does not form a 0-cycle. The similar argument holds for a message-connected 0-cycle and a message-disconnected 0-cycle.

In the perspective of the optimal codelength, some side information does not help to reduce the codelength, that is, removing the corresponding edges in $\mathcal{G}$ does not increase the optimal codelength. Then, it is said that the side information or edges are not critical. In the following lemmas, some properties of a 0-cycle are given.

Lemma 5.1. *For a message x_{i_1} such that $i_1 \in Z[n]$, x_{i_1} is the same as being sent in the uncoded form if x_{i_1} does not form a 0-cycle.*

Proof. Assume that $R_1, \ldots, R_k$ want to receive x_{i_1}. If x_{i_1} does not form a 0-cycle in $\mathcal{G}$, there are two cases. The first one is that there exists a receiver (say R_1) which does not have any side information. In this case, it is trivially proved.

The other case is that x_{i_1} does not form a 0-cycle but each R_i for $i \in Z[k]$ has at least one side information symbol. Without loss of generality, this case means that each of the side information symbols of R_1 (say one of them is x_{i_2}) does not form a 0-cycle. Then, we can continue the same procedure for $x_{i_2}, x_{i_3}, \ldots, x_{i_a}$. Then, it results in the fact that a receiver R_b wanting x_{i_a} does not have any side information symbols, which means that x_{i_a} is the same as being sent in the uncoded form and having x_{i_a} as side information is not critical. Note that we can always find such R_b and x_{i_a} because

x_{i_1} does not form a 0-cycle and an union of 0-cycles is still a 0-cycle. By repeating this procedure, it is the same that R_1 does not have any side information symbols. $\square$

Lemma 5.2. *For a message x_i such that $i \in Z[n]$, x_i is the same as being sent in the uncoded form if x_i does not form a message-connected 0-cycle.*

Proof. There are two cases. The first one is that x_i does not form a 0-cycle and this case is directly proved from Lemma 5.1. The second case is that x_i only forms a message-disconnected 0-cycle.

Let $T = \{x_i, x_{i_1}, x_{i_2}, \ldots, x_{i_t}\}$ be a message set represented by the maximum 0-cycle including x_i. Then, every message forming a 0-cycle is included in T. Since the other messages are the same as being sent in the uncoded forms, having those messages as side information is not critical. Then, partition T into subsets of messages $T_1, T_2, \ldots, T_k$ based on existence of a path between any two messages in the message graph induced by T. Thus, messages in different groups cannot be encoded together. Since the encoded messages related with T_q do not have any information of messages in T_p for $p, q \in Z[k]$ such that $p \neq q$, having messages in T_q as side information does not help to recover messages in T_p. Thus, receivers wanting messages in T_p do not need to know messages in T_q as side information.

Assume that $x_i \in T_1$. Then, it can be assumed that all side information of receivers wanting messages in T_1 is included in T_1. Thus, we can consider a side information graph induced by T_1. In this graph, let $T^{(1)}$ be a message set represented by the maximum 0-cycle and partition $T^{(1)}$ into subsets of messages $T_1^{(1)}, T_2^{(1)}, \ldots, T_{k_1}^{(1)}$ based on existence of a path between any two messages in the message graph induced by $T^{(1)}$. If $x_i \notin T^{(1)}$, it is proved from Lemma 5.1 because it means that x_i does not form a 0-cycle after deleting uncritical side information. If $x_i \in T^{(1)}$, we assume that $x_i \in T_1^{(1)}$ and continue the same procedure as the above. Then, x_i does not belong to $T^{(j)}$ for some j or x_i belongs to $T^{(h)}$ such that $k_h = 1$ or $T^{(h)}$ is ϕ. However, x_i cannot belong

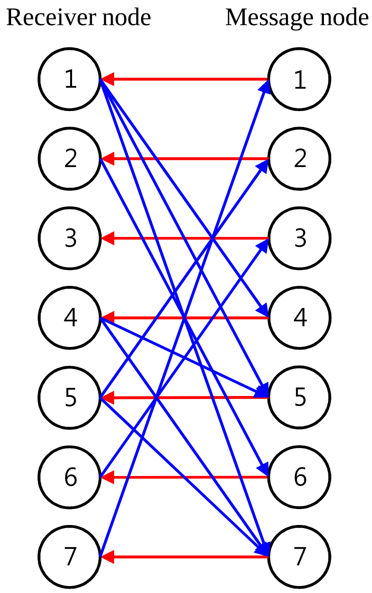

Figure 5.5: A side information graph $\mathcal{G}$ for Example 5.3.

to $T^{(h)}$ for $k_h = 1$ because it is assumed that x_i does not form a message-connected 0-cycle. Thus, we can conclude that x_i is the same as being sent in the uncoded form from Lemma 5.1. $\square$

Lemmas 5.1 and 5.2 provide some conditions that a message is the same as being sent in the uncoded form. There is an example for Lemmas 5.1 and 5.2.

Example 5.3. *Let $M_1 = \{1, 2, 3, 4\}$, $M_2 = \{2, 3, 5, 6\}$, and $M_3 = \{3, 4, 6, 7\}$. A side information graph $\mathcal{G}$ is given in Fig. 5.5. Then, x_2, x_3, and x_6 are the same as being sent in the uncoded forms from Lemma 5.1 because they do not form a 0-cycle. Although subgraphs of $\mathcal{G}$ induced by message nodes $\{1, 4, 5, 7\}$ and $\{1, 5, 7\}$ form 0-cycles, they are message-disconnected 0-cycles because 5 is only connected to $\{2, 3, 6\}$ in U. Since x_5 only forms a message-disconnected 0-cycle, x_5 is the same as being sent in the uncoded form from Lemma 5.2.*

The following two theorems show some cases that side information is not critical.

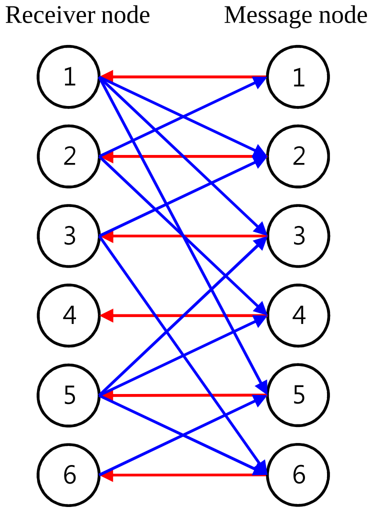

Figure 5.6: A side information graph $\mathcal{G}$ for Example 5.4.

Theorem 5.2. *Assume that there is no message-connected* 0*-cycle containing both* x_b *and a message with an index in* $M_i \cap M_c$ *for* $a \in M_i$*,* $b \in (\cup_{j \in Z[|S|]} M_j) \setminus M_i$*, and* $i \in Z[|S|]$*. Then, receivers wanting* x_a *do not need to know* x_b *as side information.*

Proof. It is divided into two cases that x_b does not form a 0-cycle with a message with an index in $M_i \cap M_c$ or forms a message-disconnected 0-cycle with a message with an index in $M_i \cap M_c$.

For the first case with no 0-cycle, it is again classified into two cases and it is noted that an union of 0-cycles is still a 0-cycle. Then, the first one is that every message with an index in $M_i \cap M_c$ does not form a 0-cycle. Thus, from Lemma 5.1, messages with indices in $M_i \cap M_c$ are the same as being sent in the uncoded forms. If $a \in M_i \cap M_c$, it is trivially proved. For $a \in M_i \setminus M_c$, knowing x_b as side information does not help to recover x_a because there is no path from x_b to x_a in the message graph excluding messages with indices in $M_i \cap M_c$, which means that the encoded messages related with x_b do not have any information of x_a. The second one is that x_b does not form a 0-cycle. Then, x_b is the same as being sent in the uncoded form and thus having x_b as side information is not critical from Lemma 5.1.

For the second case with a message-disconnected 0-cycle, it can be proved by the similar method in the proof of Lemma 5.2. Assume that the maximum 0-cycle including x_b is represented as messages $T = \{x_b, x_{b_1}, x_{b_2}, \ldots, x_{b_t}\}$. If $x_a \notin T$, the theorem is easily proved because x_a is the same as being sent in the uncoded form from Lemma 5.1 and thus we assume that $x_a \in T$. Then, partition T into subsets of messages $T_1, T_2, \ldots, T_k$ based on existence of a path between any two messages in the message graph induced by T as in Lemma 5.2 and assume that T_1 contains all messages having paths with messages of M_i in the message graph induced by T. If $x_b \notin T_1$, it is directly proved because x_a and x_b belong to different groups of T. If $x_b \in T_1$, let $T^{(1)}$ be a message set represented by the maximum 0-cycle in the subgraph induced by T_1. Then, it is similarly proved as in Lemma 5.2.

Specifically, if we continue the same procedure as the above, $x_a \notin T^{(j)}$ or x_a and x_b belong to different groups of $T^{(j)}$ or x_b does not belong to $T^{(j)}$ for some j or x_b belongs to $T^{(h)}$ such that $k_h = 1$ or $T^{(h)}$ is ϕ. However, x_a and x_b cannot belong to $T^{(h)}$ together because it is assumed that x_b does not form a message-connected 0-cycle with messages with indices in $M_i \cap M_c$ and there is no path between x_a and x_b not including messages with indices in $M_i \cap M_c$ in the message graph. Thus, it can be concluded that receivers wanting x_a do not need to know x_b as side information. $\square$

Here is an example for Theorem 5.2.

Example 5.4. *Let $M_1 = \{1, 2, 3\}$, $M_2 = \{3, 4\}$, and $M_3 = \{4, 5, 6\}$. A side information graph $\mathcal{G}$ is given in Fig. 5.6. Although x_5 forms a message-connected 0-cycle from a subgraph induced by $\{x_5, x_6\}$, there is no message-connected 0-cycle having both x_3 and x_5 because x_4 does not form a 0-cycle and there is no path between 3 and 5 in the subgraph of U excluding 4. Then, R_1 does not need to know x_5 as side information from Theorem 5.2.*

Theorem 5.3. *For $a \in M_i \setminus M_c$ and $i \in Z[|S|]$, assume that R_k for $k \in Z[m]$ wants x_a. If every message in $\{x_q | q \in M_i \cap M_c \cap \mathcal{Y}_k\}$ does not form a message-connected 0-cycle, side information of R_k in $(\cup_{j\in Z[|S|]} M_j) \setminus M_i$ is not critical.*

Proof. Since every message in $\{x_q | q \in M_i \cap M_c \cap \mathcal{Y}_k\}$ does not form a message-connected 0-cycle, those messages are the same as being sent in the uncoded forms from Lemma 5.2. Thus, we can assume that R_k knows all messages with indices in $M_i \cap M_c$ as side information. Considering the fitting matrix F and linearly independent columns of F as $F_1, \ldots, F_N$, $\Sigma_z b_z^k F_z = e_a + \Sigma_{y\in\mathcal{X}_k} c_y^k e_y$ for some $b_z^k, c_y^k \in \mathbb{F}_q$ from Claim 5.1. In fact, the kth column of $F^{(i)}$ is given as $e_a + \Sigma_{y\in\mathcal{X}_k} d_y^k e_y$ from the definition of the fitting matrix, where $d_y^k \in \mathbb{F}_q$. Thus, the kth column of $F^{(h)}$ for all $h \in Z[|S|] \setminus \{i\}$ can be deleted because the minimum rank of the modified fitting matrix is smaller than or equal to the minimum rank of the original one and R_k can obtain x_a from the kth column of $F^{(i)}$. It means that side information of R_k in $(\cup_{j\in Z[|S|]} M_j) \setminus M_i$ is not critical. □

From Theorem 5.3, we have the following corollary.

Corollary 5.2. *For R_k such that $f(k) \in M_c$ and $k \in Z[m]$, assume that senders $s_{k_1}, \ldots, s_{k_i}$ have $x_{f(k)}$ and M_c^k is a set of message indices of those senders. If every message in $\{x_q | q \in M_c^k \cap M_c \cap \mathcal{Y}_k\}$ does not form a message-connected 0-cycle, side information of R_k in $(\cup_{j\in Z[|S|]} M_j) \setminus M_p$ for certain $p \in \{k_1, \cdots, k_i\}$ is not critical.*

Proof. It is similar to the proof of Theorem 5.3 and the only difference is that for the kth columns of submatrices of F, it is not determined which kth column of submatrices has the non-zero value in the $f(k)$th position. Thus, if the kth column of $F^{(p)}$ is selected to have the non-zero value in the $f(k)$th position, side information of R_k in $(\cup_{j\in Z[|S|]} M_j) \setminus M_p$ is not critical as in Theorem 5.3. □

From Theorem 5.3 and Corollary 5.2, it is noted that if all receivers satisfy the above conditions, an optimal $(\mathcal{G}, M)$-IC can be obtained by the single sender IC problem with the modified side information graph because side information in the other senders is not critical.

Existence of a 0-cycle is a necessary and sufficient condition for reducing codelength by index coding in the single sender problem as shown in Chapter 3. In the multiple sender case, a message-connected 0-cycle has the same property as shown in the following theorem.

Theorem 5.4. *For a given side information graph $\mathcal{G}$ and a given message graph U, $N^q_{\text{opt}}(\mathcal{G}, M) = n$ if and only if there is no message-connected 0-cycle in $\mathcal{G}$.*

Proof. $Necessity$: Assume that there is a message-connected 0-cycle represented by $\{x_1, x_2, \ldots, x_{n'}\}$. Since there exists at least one path between any pair of message nodes corresponding to the 0-cycle in the subgraph of U, there exists a spanning tree $\mathcal{T}$. Then, a $(\mathcal{G}, M)$-IC with codelength $n-1$ can be constructed by using the index code $\{x_i + x_j : i \text{ and } j \text{ are connected in } \mathcal{T}\}$. From this index code with codelength $n' - 1$, we can get the sum or difference between any two messages x_i, x_j for $i, j \in Z[n']$. Since every receiver wanting one of messages in $\{x_1, x_2, \ldots, x_{n'}\}$ has at least one side information symbol in $\{x_1, x_2, \ldots, x_{n'}\}$, it can recover the wanted message. If we send the remaining messages in $\{x_{n'+1}, \ldots, x_n\}$ as the uncoded forms, every receiver in $\mathcal{G}$ can obtain what it wants and the codelength is $n - 1$.

$Sufficiency$: If there is no message-connected 0-cycle, every message does not form a message-connected 0-cycle. Then, from Lemma 5.2, every message is the same as being sent in the uncoded form. □

Remark 5.3. *For $m = n$, existence of a message-connected cycle in [35] is not a necessary and sufficient condition for reducing codelength. Specifically, existence of a message-connected cycle is a sufficient condition for reducing codelength and is not* *necessary. For example, assume that $M_1 = \{1, 4\}$, $M_2 = \{2, 3, 4\}$, and $f(i) = i$ for $i \in Z[4]$. Let $\mathcal{X}_1 = \{2\}$, $\mathcal{X}_2 = \{3\}$, $\mathcal{X}_3 = \{1\}$, and $\mathcal{X}_4 = \{2\}$. Then, there is no message-connected cycle. However, the entire graph is a message-connected 0-cycle and we can make the index code $(x_1 + x_4, x_2 + x_4, x_2 + x_3)$ with codelength 3 from Theorem 5.4.*

5.4 Extension to Cellular Network

Now, I consider the index coding in a cellular network scenario, where receivers can receive a subset of sub-codewords because some senders cannot cover all receivers. If each receiver belongs to coverage of only one sender, it just reduces to disjoint single sender index coding problems. However, there is a possibility to reduce index codelength more efficiently if some receivers belong to coverage of more than one sender. I first describe index coding in a cellular network as follows.

5.4.1 Problem Description: Two Sender Case

In the linear index coding with a cellular network, the problem setting is similar to that of the linear index coding problem for the multiple sender case except that some receivers are restricted to receive a subset of sub-codewords. For simplicity, I assume that $m = n$, $f(i) = i$ for $i \in Z[m]$, $|S| = 2$, and the field size $q = 2$. Then, $\mathcal{G}$ is a unipartite side information graph as depicted in Chapter 2.

Since a cellular network is assumed, there are three types of receivers based on coverage of senders. Let $R(s_j) = \{i | R_i \text{ can only receive the sub-codeword from } s_j\}$ for $j \in Z[2]$, $R(s_c) = \{i | R_i \text{ can receive the entire codeword}\}$, and $R = \{R(s_1), R(s_2), R(s_c)\}$. It is easily noted that $R(s_j) \subseteq M_j$ for $j \in Z[2]$.

Then, a linear index code for a cellular network is defined as follows.

Definition 5.5. *A linear index code over $\mathbb{F}_2$ for a cellular network with two senders, denoted by a $(\mathcal{G}, M, R)$-IC is a set of codewords having:*

1. *Generator submatrices $G^{(i)} \in \mathbb{F}_2^{|M_i| \times N_i}$ for $i \in Z[2]$, where N_i denotes the codelength of the sub-codeword $\mathbf{x}_{M_i} G^{(i)}$ generated by s_i.*

2. *Decoding functions D_j satisfying*

$$D_j(\mathbf{x}_{M_1} G^{(1)}, \mathbf{x}_{\mathcal{X}_j}) = x_{f(j)} \text{ if } j \in R(s_1) \tag{5.4}$$

$$D_j(\mathbf{x}_{M_2} G^{(2)}, \mathbf{x}_{\mathcal{X}_j}) = x_{f(j)} \text{ if } j \in R(s_2) \tag{5.5}$$

$$D_j(\mathbf{x}_{M_1} G^{(1)}, \mathbf{x}_{M_2} G^{(2)}, \mathbf{x}_{\mathcal{X}_j}) = x_{f(j)} \text{ if } j \in R(s_c) \tag{5.6}$$

for all $j \in Z[m]$, $\mathbf{x} \in \mathbb{F}_2^n$.

It is noted that an $n \times N$ generator matrix G for a $(\mathcal{G}, M, R)$-IC is constructed by using generator submatrices $G^{(i)}$ for $i \in Z[2]$ and the codelength N is $N_1 + N_2$. Let $N^2_{\text{opt}}(\mathcal{G}, M, R)$ be the optimal codelength of a $(\mathcal{G}, M, R)$-IC.

Since receivers with indices in $R(s_i)$ cannot receive the sub-codeword from s_j for $i, j \in Z[2]$ with $i \neq j$, we have the following proposition.

Proposition 5.2. *For a receiver with an index in $R(s_i)$, side information in $M_j \setminus M_i$ is not critical for $i, j \in Z[2]$ with $i \neq j$.*

Proof. Since every receiver with an index in $R(s_i)$ cannot receive encoded messages made from using messages with indices in $M_j \setminus M_i$, those receivers can recover their wanted messages without using side information in $M_j \setminus M_i$. $\square$

Thus, we can assume that every receiver with an index in $R(s_i)$ has side information only in M_i for $i \in Z[2]$.

5.4.2 Encoding Procedure of $(\mathcal{G}, M, R)$-IC

Now, I first introduce a fitting matrix for the cellular network case, which is similar to that of the multiple sender case. Since some receivers receive a subset of sub-codewords, a new parameter instead of the minimum rank of the fitting matrix is needed to represent codelength. The fitting matrix for the cellular network consists of three submatrices as shown in the following definition.

Definition 5.6. *The fitting matrix F of size $n \times (|R(s_1)| + |R(s_2)| + 2|R(s_c)|)$ for a $(\mathcal{G}, M, R)$-IC is described as follows:*

1. *There are three submatrices $F^{(1)}, F^{(2)}, F^{(3)}$ corresponding to three types of receivers $R(s_1)$, $R(s_2)$, and $R(s_c)$, respectively.*
2. *For $F^{(i)}$, each column represents each receiver and it is the same as the column corresponding to that receiver of the conventional fitting matrix of the single sender IC problem, where $i \in Z[2]$ and $F^{(i)}$ is used for encoding in s_i.*
3. *For $F^{(3)}$, each receiver R_k is represented by two columns k_1, k_2 used for encoding in s_1 and s_2.*
4. *$F^{(3)}_{i,k_j} = 0$ for $i \notin M_j$, $j \in Z[2]$, and $k \in R(s_c)$.*
5. *$\Sigma_{j=1}^{2} F^{(3)}_{k,k_j} = 1$ for $k \in R(s_c)$.*
6. *$\Sigma_{j=1}^{2} F^{(3)}_{i_1,k_j} = 0$ for $i_1 \in \mathcal{Y}_k$ and $k \in R(s_c)$.*
7. *For $i_2 \in M_j \cap \mathcal{X}_k$, $j \in Z[2]$, and $k \in R(s_c)$, $F^{(3)}_{i_2,k_j}$ is any element of $\mathbb{F}_2$.*

Let V_i be a vector space spanned by columns of $F^{(i)}$ for $i \in Z[3]$. Unlike the multiple sender case, the optimal codelength of linear index codes in the cellular network is not the minimum rank of the fitting matrices. The following theorem shows the optimal codelength of linear index codes in the cellular network.

Theorem 5.5. *$N^2_{\text{opt}}(\mathcal{G}, M, R)$ is the minimum of $\dim(V_1 + V_2 + V_3) + \dim(V_1 \cap V_2)$.*

Proof. First, it is easily understood that the fitting matrix for the cellular network with two senders can be a generator matrix of a $(\mathcal{G}, M, R)$-IC. From the fitting matrix, the codelength of the index code can be given as $\dim(V_1)+\dim(V_2)+\dim(V_3)-\dim(V_3\cap(V_1+V_2))$ because receivers with indices in $R(s_i)$ have to receive encoded messages by $F^{(i)}$ with codelength $\dim(V_i)$ for $i \in Z[2]$ and receivers with indices in $R(s_c)$ can recover their wanted messages by additionally receiving independent messages (columns of $F^{(3)}$) from $V_1 + V_2$ with codelength $\dim(V_3) - \dim(V_3 \cap (V_1 + V_2))$. Since $\dim(V_1+V_2+V_3) = \dim(V_1+V_2)+\dim(V_3)-\dim(V_3\cap(V_1+V_2))$ from the inclusion-exclusion principle, codelength can be represented as $\dim(V_1 + V_2 + V_3) - \dim(V_1 + V_2) + \dim(V_1) + \dim(V_2) = \dim(V_1 + V_2 + V_3) + \dim(V_1 \cap V_2)$.

Assume that G is a generator matrix of a $(\mathcal{G}, M, R)$-IC. Then, it can be partitioned into two parts based on encoding of each sender. Let $G_v^{(i)}$ be a vector space spanned by columns of $G^{(i)}$ for $i \in Z[2]$, where $G^{(i)}$ is a generator matrix of s_i. Then, the codelength of this index code is $\dim(G_v^{(1)}) + \dim(G_v^{(2)})$. From the similar methods in Claim 5.1 and the proof of Theorem 5.1, it is noted that the fitting matrix for the cellular network can be made from columns of G. Specifically, a vector space V_i can be made from $G_v^{(i)}$ for $i \in Z[2]$ and V_3 can be made from $G_v^{(1)}$ and $G_v^{(2)}$, which means that V_i is a subspace of $G_v^{(i)}$ for $i \in Z[2]$ and V_3 is a subspace of $G_v^{(1)} + G_v^{(2)}$. Similar to Corollary 5.1, we only consider cases satisfying 5), 6) of Definition 5.6 because we can consider columns with zero coefficients as all-zero columns in the perspective of the optimal codelength.

Then, it is noted that $\dim(V_1+V_2+V_3) \le \dim(G_v^{(1)}+G_v^{(2)})$ and $\dim(V_1\cap V_2) \le \dim(G_v^{(1)} \cap G_v^{(2)})$. Thus, $\dim(V_1 + V_2 + V_3) + \dim(V_1 \cap V_2) \le \dim(G_v^{(1)} + G_v^{(2)}) + \dim(G_v^{(1)}\cap G_v^{(2)}) = \dim(G_v^{(1)})+\dim(G_v^{(2)})$, which means that the optimal codelength is the minimum of $\dim(V_1 + V_2 + V_3) + \dim(V_1 \cap V_2)$. $\square$

The example of Proposition 5.2 and the fitting matrix for the cellular network is given as follows.

Example 5.5. *Let* $m = 5$*,* $M = \{M_1, M_2\}$*,* $M_1 = \{1, 2, 3\}$*,* $M_2 = \{3, 4, 5\}$*,* $R(s_1) = \{1, 2\}$*,* $R(s_2) = \{4, 5\}$*, and* $R(s_c) = \{3\}$*. A side information graph* $\mathcal{G}$ *is given in Fig.*

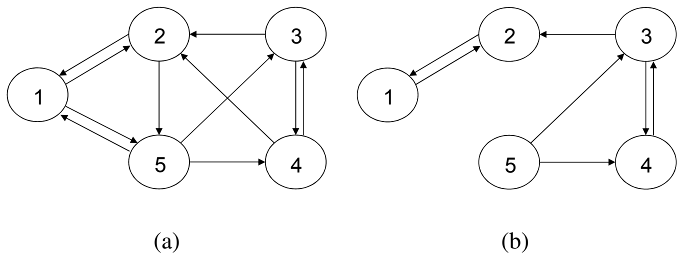

Figure 5.7: A side information graph $\mathcal{G}$ of Example 5.5: (a) A side information graph $\mathcal{G}$. (b) The modified side information graph $\hat{\mathcal{G}}$ from Proposition 5.2.

5.7(a). From Proposition 5.2, $\mathcal{G}$ can be transformed into $\hat{\mathcal{G}}$ as in Fig. 5.7(b). Then, the fitting matrix of a $(\hat{\mathcal{G}}, M, R)$-IC is derived as

$$F = \begin{pmatrix} F^{(1)} & F^{(2)} & F^{(3)} \end{pmatrix}, \tag{5.7}$$

where

$$F^{(1)} = \begin{pmatrix} 1 & * \\ * & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, F^{(2)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ * & * \\ 1 & * \\ 0 & 1 \end{pmatrix}, F^{(3)} = \begin{pmatrix} 0 & 0 \\ * & 0 \\ c & 1+c \\ 0 & * \\ 0 & 0 \end{pmatrix}. \tag{5.8}$$

$*$ denotes any element of $\mathbb{F}_q$ and $c \in \mathbb{F}_2$. By finding a fitting matrix with the minimum value of $\dim(V_1+V_2+V_3)+\dim(V_1 \cap V_2)$, we can find an optimal generator matrix.

As mentioned before, I only consider a side information graph $\hat{\mathcal{G}}$ by deleting uncritical side information from Proposition 5.2 as in Example 5.5.

5.4.3 Properties of $(\mathcal{G}, M, R)$-IC

In this section, I study some properties of a $(\mathcal{G}, M, R)$-IC for a cellular network with two senders. First, we have the following definition describing a new type of a side information graph.

Definition 5.7. *For the cellular network, a subgraph $\mathcal{G}'$ of $\mathcal{G}$ is denoted by $\mathcal{H}_{\mathcal{G}'}$ if every message is side information of at least one receiver in $\mathcal{G}'$.*

There is an example for $\mathcal{H}_{\mathcal{G}'}$.

Example 5.6. *Assume the same situation as in Example 5.5. Then, $\hat{\mathcal{G}}$ is not $\mathcal{H}_{\hat{\mathcal{G}}}$ because no receiver has x_5 as side information. However, a subgraph $\mathcal{G}'$ induced by nodes $\{1, 2, 3, 4\}$ is $\mathcal{H}_{\mathcal{G}'}$ because every message is side information of at least one receiver.*

The following propositions show some relationships between a $(\mathcal{G}, M)$-IC and a $(\mathcal{G}, M, R)$-IC.

Proposition 5.3. *$N^2_{\text{opt}}(\mathcal{G}, M, R) = N^2_{\text{opt}}(\mathcal{G}, M)$ if $\dim(V_1 \cap V_2) = 0$ holds for any fitting matrix of a $(\mathcal{G}, M, R)$-IC.*

Proof. From Proposition 5.1 and Definitions 5.3 and 5.6, it is noted that the fitting matrix for a $(\mathcal{G}, M)$-IC and the fitting matrix for a $(\mathcal{G}, M, R)$-IC have the same form except for column permutation and all-zero columns. Since $N^2_{\text{opt}}(\mathcal{G}, M, R)$ is the minimum value of $\dim(V_1 + V_2 + V_3) + \dim(V_1 \cap V_2)$ and $N^2_{\text{opt}}(\mathcal{G}, M)$ is the minimum value of $\dim(V_1 + V_2 + V_3)$, they are the same if $\dim(V_1 \cap V_2) = 0$. $\square$

Proposition 5.4. *A fitting matrix with $\dim(V_1 \cap V_2) \neq 0$ exists if and only if $\mathcal{H}_{\mathcal{G}'}$ satisfying one of the followings exists, where $\mathcal{H}_{\mathcal{G}'}$ consists of receivers with indices in $R(s_1) \cup R(s_2)$ and contains at least one receiver with an index in $R(s_i)$ for all $i \in Z[2]$.*

1. *A receiver with an index in $R(s_1)$ has a message with an index in $R(s_2)$ as side information or vice versa.*

2. *A receiver with an index in $R(s_1)$ and a receiver with an index in $R(s_2)$ have the same message as side information.*

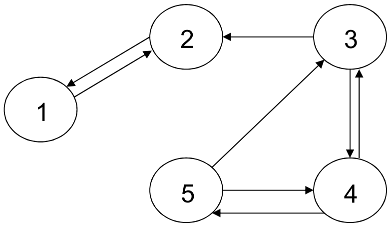

Figure 5.8: A side information graph $\mathcal{G}$ of Example 5.7.

Proof. From the combinatoric point of view, a fitting matrix with $\dim(V_1 \cap V_2) \neq 0$ exists if and only if there exist column vectors such that the sum of a column vectors of $F^{(1)}$ and b column vectors of $F^{(2)}$ is the all-zero vector, where $a, b \neq 0$ and the sum of those a column vectors is not the all-zero vector. First, it is easily noted that there exist column vectors such that the sum of a column vectors of $F^{(1)}$ and b column vectors of $F^{(2)}$ is the all-zero vector, where $a, b \neq 0$ if and only if those $a+b$ receivers (messages) form $\mathcal{H}_{\mathcal{G}'}$. It is because each message has to be known to at least one receiver in order for the element corresponding to each message in the sum of those vectors to be 0.

Next, we deal with the additional condition that there exists a case that the sum of those a column vectors is not the all-zero vector. It is noted that this condition holds if and only if a column corresponding to one of a receivers and a column corresponding to one of b receivers can have 1 in the same position, which reduces to the above two cases. $\square$

There is an example for Propositions 5.3 and 5.4.

Example 5.7. *Let* $m = 5$*,* $M = \{M_1, M_2\}$*,* $M_1 = \{1, 2, 3\}$*,* $M_2 = \{3, 4, 5\}$*,* $R(s_1) = \{1, 2\}$*,* $R(s_2) = \{4, 5\}$*, and* $R(s_c) = \{3\}$*. A side information graph* $\mathcal{G}$ *is given in Fig. 5.8. Although a subgraph* $\mathcal{G}'$ *induced by* $\{1, 2, 4, 5\}$ *forms* $\mathcal{H}_{\mathcal{G}'}$*, 1) and 2) of Proposition 5.4 do not hold. Thus,* $\dim(V_1 \cap V_2) = 0$ *always holds from Proposition 5.4 and* $N^2_{\text{opt}}(\mathcal{G}, M, R) = N^2_{\text{opt}}(\mathcal{G}, M)$ *from Proposition 5.3.*

A cycle in a side information graph $\mathcal{G}$ is known to be important in the single sender index coding problem and there are lots of cycle-cover algorithms to find the suboptimal index codelength. Along with these, I classify which cycle can be used to reduce index codelength in the cellular network.

Theorem 5.6. *In the cellular network, cycles can be classified into the following cases based on a possibility of reducing codelength.*

1. *The cases that codelength cannot be reduced:*
   - *(a) A message-disconnected cycle.*
   - *(b) A message-connected cycle which consists of receivers with indices in $R(s_1) \cup R(s_2)$ and contains at least one receiver with an index in $R(s_i)$ for all $i \in Z[2]$.*
2. *The cases that codelength can be reduced:*
   - *(a) A message-connected cycle consisting of receivers with indices in $R(s_i)$ for $i \in Z[2]$.*
   - *(b) A message-connected cycle containing at least one receiver with an index in $R(s_c)$.*

Proof. Since a message-disconnected cycle is a message-disconnected 0-cycle and $N^2_{\text{opt}}(\mathcal{G}, M, R) \geq N^2_{\text{opt}}(\mathcal{G}, M)$, a message-disconnected cycle cannot reduce codelength from Theorem 5.4. Next, for a message-connected cycle described in 1), assume that a receivers with indices in $R(s_1)$ and b receivers with indices in $R(s_2)$ form a message-connected cycle. Since each receiver can receive encoded messages from only one sender and two subgraphs induced by a receivers and b receivers, respectively are acyclic, $a + b$ transmissions are required.

It is trivial that a message-connected cycle consisting of receivers with indices in $R(s_i)$ for $i \in Z[2]$ can reduce codelength. For a message-connected cycle containing at least one receiver with an index in $R(s_c)$, it is noted that every receiver has one message as side information and one message cannot be side information of two receivers. Since each receiver with an index in $R(s_i)$ for $i \in Z[2]$ of that cycle has side information with an index in M_i, each receiver can recover the wanted message if s_i transmits the sum of the wanted message and its side information.

Similar to the perspective of a message graph in Theorem 5.4, we make a graph consisting of the messages. We add an undirected edge between the wanted message and its side information for each receiver with an index in $R(s_1) \cup R(s_2)$, which represents each transmission of the sum of the wanted message and its side information. Then, there is no cycle in the resulting graph consisting of those receivers with indices in $R(s_1) \cup R(s_2)$ because there is no cycle in the subgraph of $\mathcal{G}$ induced by those receivers.

Since there is no cycle in the resulting graph and there is at least one message with an index in M_c due to message-connectivity, we can make a spanning tree including the other messages with indices in $R(s_c)$ while maintaining the above undirected edges. Then, new connected edges correspond to transmissions of the sum of two messages as same as before by a sender capable of encoding those messages. Then, every receiver can recover its wanted message because receivers with indices in $R(s_c)$ can receive all encoded messages corresponding to edges in the spanning tree. $\square$

Next, I study properties of $N^2_{\text{opt}}(\mathcal{G}, M, R)$ in the cellular network. In Section 5.3, Theorem 5.4 shows that existence of a message-connected 0-cycle is a necessary and sufficient condition for reducing codelength in the index coding problem with multiple senders. However, it does not hold for the cellular network. Furthermore, a receiver with no side information can help to reduce codelength in the cellular network as shown in the following example.

Example 5.8. *Let* $M_1 = \{1, 2, 3\}$*,* $M_2 = \{1, 3\}$*,* $R(s_1) = \{1\}$*,* $R(s_2) = \{3\}$*,* $R(s_c) = \{2\}$*,* $\mathcal{X}_1 = \{2, 3\}$*,* $\mathcal{X}_2 = \phi$*, and* $\mathcal{X}_3 = \{1\}$*. Although a subgraph of* $\mathcal{G}$

induced by $\{1, 3\}$ is a message-connected 0-cycle, it cannot reduce codelength from Theorem 5.6 and thus the optimal codelength for this subgraph is 2. Now, consider R_2 which has no side information. In the index coding problem with multiple senders or single sender, adding a receiver with no side information increases the optimal codelength by one when $m = n$. However, in this example, every receiver can recover its wanted message if s_1 transmits $(x_1 + x_2 + x_3)$ and s_2 transmits $(x_1 + x_3)$. From this example, it is noted that the general condition for $N^2_{\text{opt}}(\mathcal{G}, M, R) = n$ is hard to be found through the conventional properties of index coding.

Now, I discuss some properties of $N^2_{\text{opt}}(\mathcal{G}, M, R)$.

Proposition 5.5. *Assume that $\dim(V_1 \cap V_2) = 0$ always holds. Then, $N^2_{\text{opt}}(\mathcal{G}, M, R) = n$ if and only if there is no message-connected 0-cycle in $\mathcal{G}$.*

Proof. From Proposition 5.3 and Theorem 5.4, it is easily proved. $\square$

Similar to Proposition 5.4, I discuss the condition for $\dim(V_3 \cap (V_1 + V_2)) \neq 0$. If $\dim(V_3 \cap (V_1 + V_2)) \neq 0$, there are some column vectors in the fitting matrix such that the sum of them is the all-zero vector, where u columns of them are in $F^{(3)}$. In the following observation, I classify those u columns in $F^{(3)}$ into some receivers or certain vectors.

Observation 5.1. *Using notations in Definition 5.6, u columns in $F^{(3)}$ correspond to some receivers or certain vectors based on relationships between selected columns in $F^{(3)}$.*

1. *The cases that some columns correspond to some receivers:*

   (a) *If both the k_1th column and the k_2th column are selected for $k \in R(s_c)$, the sum of them corresponds to R_k.*

(b) *If only the k_ith column is selected among the k_1th column and the k_2th column for $i \in Z[2]$ and $k \in R(s_c) \cap (M_i \setminus M_c)$, the k_ith column corresponds to the receiver whose side information corresponds to $M_c \cup (\mathcal{X}_k \cap M_i)$ and wanted message is x_k.*

2. *The case that some columns correspond to certain vectors:*

   (a) *If only the k_jth column is selected among the k_1th column and the k_2th column for $i, j \in Z[2]$ such that $i \neq j$ and $k \in R(s_c) \cap M_i$, the k_jth column is any column with 0s in the positions in $Z[n] \setminus (M_c \cup (\mathcal{X}_k \cap M_j))$.*

   (b) *It is said that a message (say x_d) is covered by the above columns corresponding to certain vectors if one of those columns can have 1 at the dth position.*

The following observation shows a necessary and sufficient condition for $\dim(V_3 \cap (V_1 + V_2)) \neq 0$ from the combinatoric point of view.

Observation 5.2. *A fitting matrix with $\dim(V_3 \cap (V_1 + V_2)) \neq 0$ exists if and only if there are a receivers with indices in $R(s_1) \cup R(s_2)$, b receivers corresponding to receivers in Observation 5.1, and b' columns corresponding to certain vectors in Observation 5.1 for $a \neq 0$ and $b + b' \neq 0$ satisfying one of the following nine cases:*

1. *The cases for $b = 0$:*

   (a) *Each of a messages is known to at least one of a receivers or is covered by b' columns. At this point, at least one of a messages has to be covered by b' columns.*

   (b) *a receivers (messages) form $\mathcal{H}_{\mathcal{G}'}$ and none of a messages is covered by b' columns. Let one of a receivers have x_d as side information such that x_d does not belong to a messages. Then, x_d is covered by b' columns.*

2. *The cases for $b' = 0$ and $a + b$ receivers (messages) form $\mathcal{H}_{\mathcal{G}'}$:*

   (a) *One of a messages is side information of one of b receivers.*

   (b) *One of b messages is side information of one of a receivers.*

   (c) *One of a receivers and one of b receivers have the same message as side information.*

3. *The cases for $b, b' \neq 0$ which are not reduced to the above cases:*

   (a) *$a + b$ receivers (messages) do not form $\mathcal{H}_{\mathcal{G}'}$. Also, all of a messages are known to at least one of $a + b$ receivers and there is at least one of b messages not known to $a + b$ receivers. In this case, those unknown messages are covered by b' columns.*

      i. *One of a messages is side information of one of b receivers.*

      ii. *One of b messages is side information of one of a receivers.*

      iii. *One of a receivers and one of b receivers have the same message as side information.*

      iv. *x_d is covered by b' columns and one of a receivers has x_d as side information, where x_d does not belong to $a + b$ messages.*

The following corollary shows that the cellular network case can be reduced to disjoint index coding problems if $\dim(V_3 \cap (V_1 + V_2)) = 0$ always holds.

Corollary 5.3. *If $\dim(V_3 \cap (V_1 + V_2)) = 0$ always holds, $N^2_{\text{opt}}(\mathcal{G}, M, R)$ is the sum of the optimal codelength of subproblems induced by $R(s_1)$, $R(s_2)$, and $R(s_c)$.*

Proof. Since $\dim(V_3 \cap (V_1 + V_2)) = 0$ always holds, codelength can be represented as $\dim(V_1) + \dim(V_2) + \dim(V_3)$. $\square$

From Corollary 5.3, it is noted that an $R(s_i)$ induced subgraph does not have a message-connected 0-cycle for $i \in Z[2]$ and an $R(s_c)$ induced subgraph does not have a message connected 0-cycle if and only if $N^2_{\mathrm{opt}}(\mathcal{G}, M, R) = n$ given that $\dim(V_3 \cap (V_1 + V_2)) = 0$ always holds.

Chapter 6

Conclusion

In this dissertation, index coding with erroneous side information, code equivalences between network codes with link errors and index codes with side information errors, and index coding with multiple senders and extension to a cellular network were studied.

First, I generalized the index coding problem, where there is a possibility to have erroneous side information in each receiver. The property of the generator matrix and the decoding procedure of the proposed index codes with erroneous side information were suggested, which are based on the idea of Hamming spheres and the syndrome decoding, respectively. I also suggested some bounds for the optimal codelength of the $(\delta_s, \mathcal{G})$-ICSIE and showed the relationship between the conventional index coding and index coding with erroneous side information. In addition, I found a δ_s-cycle of the GECIC, which has similar properties as those of a cycle in the conventional index coding. It was also found that the existence of a δ_s-cycle is crucial in the proposed index coding problem. The proposed ICSIE was also analyzed when a side information graph $\mathcal{G}$ is a clique. Through this, it was found that the generator matrix for the $(\delta_s, \mathcal{G})$-ICSIE corresponds to the transpose of the parity check matrix of the classical error correcting code when the related parameters are properly chosen. It was also shown that the existing bounds and properties for the $(\delta_c, \mathcal{G})$-ECIC can be generalized to those of the $(\delta_s, \delta_c, \mathcal{G})$-GECIC by using the properties of the $(\delta_s, \mathcal{G})$-ICSIE. That is, we can easily derive bounds for the optimal codelength of the $(\delta_s, \delta_c, \mathcal{G})$-GECIC, which is the index code in the more generalized scenario.

Second, a new code equivalence between a network code and an index code was proposed. I showed that the corresponding side information graph does not need the receiver $\hat{t}_{\text{all}}$, which is contained in the previous models [10], [32]. Next, for a given side information graph $\mathcal{G}$, I derived the corresponding network coding instance $\mathbb{G}$ by modifying $\mathcal{G}$ with added receivers and introducing hyper-links in the network. With the proposed method of modifying $\mathcal{G}$, I showed a code equivalence between a network code and an index code for a given index coding instance. I also generalized code equivalence results to erroneous cases for both a given network coding instance and a given index coding instance. In order to provide the code equivalence between an NCLE and an ICSIE, I considered a new type of a network code, referred to as a $(\boldsymbol{\delta}, \mathbb{G})$-NCLE, where the intermediate nodes can resolve incoming errors. From the fact that an NCLE and an ICSIE are equivalent, I derived the relationship between the conventional network code with error-free links and an NCLE.

Third, I studied linear index codes with multiple senders. The fitting matrix for a $(\mathcal{G}, M)$-IC was introduced and I showed that the fitting matrix having the minimum rank can be an optimal generator matrix. Some properties of a $(\mathcal{G}, M)$-IC based on a 0-cycle were studied and whether given side information is critical or not was studied for multiple senders. Furthermore, another index coding scenario was also considered, where each receiver can receive a subset of sub-codewords as in the cellular network. Another type of fitting matrices was introduced and it was proved that the optimal codelength of linear index codes can be found from these fitting matrices by minimizing the value of $\dim(V_1 + V_2 + V_3) + \dim(V_1 \cap V_2)$. In addition, some properties on the optimal codelength for the cellular network were studied.

Bibliography

[1] Y. Birk and T. Kol, "Informed-source coding-on-demand (ISCOD) over broadcast channels," in *Proc. IEEE Conf. on Comput. Commun. (INFOCOM)*, San Francisco, CA, 1998, pp. 1257–1264.

[2] Z. Bar-Yossef, Y. Birk, T. S. Jayram, and T. Kol, "Index coding with side information," *IEEE Trans. Inf. Theory*, vol. 57, no. 3, pp. 1479–1494, 2011.

[3] E. Lubetzky and U. Stav, "Nonlinear index coding outperforming the linear optimum," *IEEE Trans. Inf. Theory*, vol. 55, no. 8, pp. 3544–3551, 2009.

[4] F. Arbabjolfaei, B. Bandemer, Y.-H. Kim, E. Sasoglu, and L. Wang, "On the capacity region for index coding," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, 2013, pp. 962–966.

[5] C. Thapa, L. Ong, and S. J. Johnson, "Interlinked cycles for index coding: Generalizing cycles and cliques," *IEEE Trans. Inf. Theory*, vol. 63, no. 6, pp. 3692–3711, Jun. 2017.

[6] A. Blasiak, R. D. Kleinberg, and E. Lubetzky, "Broadcasting with side information: Bounding and approximating the broadcast rate," *IEEE Trans. Inf. Theory*, vol. 59, no. 9, pp. 5811–5823, Sep. 2013.

[7] F. Arbabjolfaei and Y.-H. Kim, "Local time sharing for index coding," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, 2014, pp. 286–290.

[8] K. Shanmugam, A. G. Dimakis, and M. Langberg, “Local graph coloring and index coding,” in *Proc. IEEE Int.l Symp. Inf. Theory (ISIT)*, 2013, pp. 1152–1156.

[9] S. Kwak, J. So, and Y. Sung, “An extended least difference greedy clique-cover algorithm for index coding,” in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, 2014, pp. 501–505.

[10] M. Effros, S. EI Rouayheb, and M. Langberg, “An equivalence between network coding and index coding,” *IEEE Trans. Inf. Theory*, vol. 61, no. 5, pp. 2478–2487, 2015.

[11] S. A. Jafar, “Topological interference management through index coding,” *IEEE Trans. Inf. Theory*, vol. 60, no. 1, pp. 529–568, 2014.

[12] A. Mazumdar, “On a duality between recoverable distributed storage and index coding,” in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, 2014, pp. 1977–1981.

[13] K. Shanmugam and A. G. Dimakis, “Bounding multiple unicasts through index coding and locally repairable codes,” in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, 2014, pp. 296–300.

[14] D. T. H. Kao, M. A. Maddah-Aliy, and A. S. Avestimehr, “Blind index coding,” in *Proc. IEEE Int. Symp. Information Theory (ISIT)*, Jun. 2015, pp. 2371–2375.

[15] K. W. Shum, M. Dai, and C. W. Sung, “Broadcasting with coded side information,” *2012 IEEE 23rd International Symposium on Personal, Indoor and Mobile Radio Communications (PIMRC)*, vol. 89, no. 94, pp. 9–12, Sep. 2012.

[16] N. Lee, A. G. Dimakis, and R. W. Heath, “Index coding with coded side information,” *IEEE Commun. Lett.*, vol. 19, no. 3, pp. 319–322, 2015.

[17] S. H. Dau, V. Skachek, and Y. M. Chee, “Error correction for index coding with side information,” *IEEE Trans. Inf. Theory*, vol. 59, no. 3, pp. 1517–1531, 2013.

[18] E. Byrne and M. Calderini, "Error correction for index coding with coded side information," *IEEE Trans. Inf. Theory*, vol. 63, no. 6, pp. 3712–3728, 2017.

[19] L. Natarajan, Y. Hong, and E. Viterbo, "Index codes for the Gaussian broadcast channel using quadrature amplitude modulation," *IEEE Commun. Lett.*, vol. 19, no. 8, pp. 1291–1294, 2015.

[20] L. Natarajan, Y. Hong, and E. Viterbo, "Capacity of coded index modulation," in *Proc. IEEE Int. Symp. Information Theory (ISIT)*, Jun. 2015, pp. 596–600.

[21] A. Gupta and B. S. Rajan, "Error-correcting functional index codes, generalized exclusive laws and graph coloring," in *Proc. IEEE Int. Conf. Commun.*, Kuala Lumpur, Malaysia, 2016, pp. 1–7.

[22] S. H. Dau, V. Skachek, and Y. M. Chee, "On the security of index coding with side information," *IEEE Trans. Inf. Theory*, vol. 58, no. 6, pp. 3975–3988, Jun. 2012.

[23] M. J. Neely, A. S. Tehrani, and Z. Zhang, "Dynamic index coding for wireless broadcast networks," *IEEE Trans. Inf. Theory*, vol. 59, no. 11, pp. 7525–7540, 2013.

[24] S. B. Damelin, G. Michalski, G. L. Mullen, and D. Stone, "The number of linearly independent binary vectors with applications to the construction of hypercubes and orthogonal arrays, pseudo (t, m, s)-nets and linear codes," *Monatsh. Math. 141*, pp. 277–288, 2004.

[25] S. B. Damelin, G. Michalski, and G. L. Mullen, "The cardinality of sets of k-independent vectors over finite fields," *Monatsh. Math*, vol. 150, no. 4, pp. 289–295, 2007.

[26] R. Ahlswede, N. Cai, R. Li, and R. Yeung, "Network information flow," *IEEE Trans. Inf. Theory*, vol. 46, no. 4, pp. 1204–1216, Jul. 2000.

[27] N. Cai, R. Li, and R. Yeung, "Linear network coding," *IEEE Trans. Inf. Theory*, vol. 49, no. 2, pp. 371–381, Feb. 2003.

[28] R. Koetter and M. Medard, "An algebraic approach to network coding," *IEEE/ACM Trans. Networking*, vol. 11, no. 5, pp. 782–795, Oct. 2003.

[29] S. Jaggi, P. Sanders, P. A. Chou, M. Effros, S. Egner, K. Jain, and L. M. G. M. Tolhuizen, "Polynomial time algorithms for multicast network code construction," *IEEE Trans. Inf. Theory*, vol. 51, no. 6, pp. 1973–1982, May 2005.

[30] R. W. Yeung and N. Cai, "Network error correction, part 1 and part 2," *Commun. and Inf. Systems*, vol. 6, pp. 19–36, 2006.

[31] Z. Zhang, "Linear network-error correction codes in packet networks," *IEEE Trans. Inf. Theory*, vol. 54, no. 1, pp. 209–218, Jan. 2008.

[32] A. Gupta and B. S. Rajan, "A relation between network computation and functional index coding problems," *IEEE Trans. Commun.*, vol. 65, no. 2, pp. 705–714, Feb. 2017.

[33] L. Ong, B. N. Vellambi, J. Kliewer, and P. L. Yeoh, "An equivalence between secure network and index coding," in *Proc. IEEE Globecom Workshops*, Washington, DC, 2016, pp. 1–6.

[34] L. Ong, C. K. Ho, and F. Lim, "The single-uniprior index-coding problem: The single-sender case and the multi-sender extension," *IEEE Trans. Inf. Theory*, vol. 62, no. 6, pp. 3165–3182, Jun. 2016.

[35] C. Thapa, L. Ong, and S. J. Johnson, "Graph-theoretic approaches to two-sender index coding," in *Proc. IEEE Globecom Workshops (GC Wkshps)*, 2016, pp. 1–6.

[36] P. Sadeghi, F. Arbabjolfaei, and Y.-H. Kim, "Distributed index coding," in *Proc. IEEE Inf. Theory Workshop*, 2016, pp. 330–334.

[37] M. Li, L. Ong, and S. J. Johnson, "Cooperative multi-sender index coding," *IEEE Trans. Inf. Theory*, vol. 65, no. 3, pp. 1725–1739, Mar. 2019.

[38] M. Li, L. Ong, and S. J. Johnson, "Multi-sender index coding for collaborative broadcasting: A rank-minimization approach," *IEEE Trans. Commun.*, vol. 67, no. 2, pp. 1452–1466, Feb. 2019.

[39] Y. Birk and T. Kol, "Coding on demand by an informed source (ISCOD) for efficient broadcast of different supplemental data to caching clients," *IEEE Trans. Inf. Theory*, vol. 52, no. 6, pp. 2825–2830, Jun. 2006.

초록

이 학위 논문에서는, i) 보조 정보에 오류가 존재하는 인덱스 부호화, ii) 링크 오류가 있는 네트워크 부호와 보조 정보 오류가 있는 인덱스 부호의 동치성, iii) 다중 송신기 상황에서의 인덱스 부호화 및 셀룰러 네트워크로의 확장에 대해 연구되었다.

먼저, 보조 정보에 오류가 있는 인덱스 부호화가 연구되었다. 보조 정보가 인덱스 부호화에서 매우 중요한 부분임에도 불구하고 보조 정보 오류는 일반적으로 고려되지 않았다. 보조 정보는 메모리 장치에 저장되고 메모리 장치는 오류를 야기하므로 현실적인 상황에서 인덱스 부호화를 이용하기 위해서는 보조 정보 오류를 고려해야 한다. 보조 정보 오류를 해결하기 위해 제안된 피팅 행렬 기반의 부호화 방법이 제안되었고 신드롬 복호화 기반의 복호 과정이 연구되었다. 최적의 인덱스 부호 길이에 대한 상계 및 하계가 구해졌고 δ_s-cycle이라는 그래프 형태의 존재 유무가 인덱스 부호 길이를 줄일 수 있는 지에 대해 필요충분조건임을 밝혔다. 더불어, 위의 결과들을 보조 정보 오류 뿐만 아니라 채널 오류까지 고려한 상황으로 확장하였다.

두 번째로, 링크 오류가 있는 네트워크 부호와 보조 정보 오류가 있는 인덱스 부호가 서로 동치임을 밝혔다. 기존에는 주어진 네트워크 부호화 상황에서 네트워크 부호와 인덱스 부호가 동치임이 밝혀져 있었다. 하지만 주어진 인덱스 부호화 상황에서 두 부호의 동치성은 밝혀지지 않았다. 두 부호간 동치성을 완성시키기 위해 주어진 인덱스 부호화 상황에서 두 부호의 동치성이 연구되었다. 주어진 인덱스 부호화 상황에 대응되는 네트워크 부호화 상황을 유도하기 위해서 인덱스 부호화 상황을 변형하여 대응되는 네트워크 부호화 상황을 유도하는 방법이 제안되었다.

더불어, 두 부호간 동치성을 링크 오류 및 보조 정보 오류를 고려하여 확장하였다.

마지막으로, 다중 송신기를 고려한 인덱스 부호화 상황이 연구되었고 이를 셀룰러 네트워크로 확장하였다. 일반적으로 인덱스 부호화 문제에서는 하나의 송신기만 가정한다. 하지만, 실제 상황에서는 정보들이 다양한 송신기에 나눠서 저장되어 있는 경우가 빈번하다. 따라서, 다중 송신기 상황에서 인덱스 부호화 문제를 연구할 필요가 있다. 다중 송신기 상황에서 새로운 피팅 행렬을 제안하고 그것을 기반으로 한 부호화 방식이 연구되었다. 또한, 인덱스 부호길이를 줄일 수 있는 필요충분조건을 구하였다. 실제 상황에서는 모든 송신기가 모든 수신기에게 정보를 전달할 수 없으므로 셀룰러 네트워크를 고려한 인덱스 부호화가 연구되었다.

주요어: 셀룰러 네트워크, 피팅 행렬, 인덱스 부호화, 링크 오류, 다중 송신기, 네트워크 부호화, 보조 정보, 보조 정보 오류

학번: 2014-21663

ACKNOWLEGEMENT

Some contents of this dissertation are based on followings and the policy of IEEE is given.

© 2017, IEEE. Reprinted, with permission, from [J.-W. Kim and J.-S. No, "Index coding with erroneous side information,"*IEEE Trans. Inf. Theory*, vol. 63, no. 12, pp. 7687–7697, Dec. 2017.].

© 2019, IEEE. Reprinted, with permission, from [J.-W. Kim and J.-S. No, "Code equivalences between network codes with link errors and index codes with side information errors," *IEEE Access*, vol. 7, pp. 54144–54154, Apr. 2019.].

© 2019, IEEE. Reprinted, with permission, from [J.-W. Kim and J.-S. No, "Linear index coding with multiple senders and extension to a cellular network," *IEEE Trans. Commun.*, vol. 67, pp. 8666–8677, Dec. 2019.].

In reference to IEEE copyrighted material which is used with permission in this thesis, the IEEE does not endorse any of [Seoul National University]'s products or services. Internal or personal use of this material is permitted.

If interested in reprinting/republishing IEEE copyrighted material for advertising or promotional purposes or for creating new collective works for resale or redistribution, please go to http://www.ieee.org/publications_standards/publications/rights/rights_link.html to learn how to obtain a License from RightsLink. If applicable, University Microfilms and/or ProQuest Library, or the Archives of Canada may supply single copies of the dissertation.