이학석사 학위논문

Trace polynomials of words in the free group of rank two

(계수 2 자유군에서의 대각합 다항식)

2020년 8월

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Trace polynomials of words in the free group of rank two

by

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Abstract

Trace polynomials of words in the free group of rank two

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Procesi's theorem guarantees that traces in a two generator subgroup of $\mathrm{SL}(2,\mathbb{C})$ are polynomials in traces of the generators. These polynomials are called trace polynomials and defined for words in the free group of rank two. Let $\mathcal{C}$ denote the set of cyclically reduced words in F_2. Improving Jorgensen's algorithm, we classify all words in $\mathcal{C}$ with the word lengths less than nine via their trace polynomials. Then we check whether they are in $\sim$-equivalence defined from the operation Mirror, Left shift, and Inverse on $\mathcal{C}$. We prove that two words of the same trace polynomials are $\sim$-equivalent when the word lengths are less than nine. We also show, by counterexamples, this result does not hold for the word lengths greater than eight. As a corollary, we verify Wang's conjecture for the word lengths less than nine.

Key words: Trace polynomial, Free group of rank two, Special linear group, Cyclically reduced words
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Contents

Introduction

The theorem of Procesi states that the traces in a two generator subgroup of $\mathrm{SL}(2,\mathbb{C})$ can be expressed as polynomials in traces of generators. Thus, we can consider a natural mapping from words in F_2 to polynomials in $\mathbb{Z}[x, y, z]$, which are called trace polynomials. Wang presented that $\sim$-equivalence defined from operations Left shift, Mirror, and Inverse on cyclically reduced words, implies two words to have the same trace polynomials. The converse of this sentence does not hold. However, a computational scheme was needed to find out the word lengths where the equivalence of the two conditions breaks.

In this thesis, we implement two algorithms that compute trace polynomials of words in F_2. Using the results, we show that cyclically reduced words of the same trace polynomials are $\sim$-equivalent when the word lengths are less than nine. Also, we prove this statement does not hold when the word lengths are greater than or equal to nine by showing counterexamples. The results of this thesis also state that the conjecture of Wang is valid for the word lengths less than nine.

This thesis is structured as follows. In Chapter 1, we introduce the definition and basic properties of the trace polynomial. In Chapter 2, we implement two algorithms that compute trace polynomials of words in F_2. We also compare the time complexity of the two methods. In Chapter 3, we deal with the equivalence relation $\sim$ on cyclically reduced words in F_2. We also consider the trace polynomial as a map taking each equivalence class to a trace polynomial. From the computational results obtained from Chapter 2, we establish a theorem about the injectivity of that map.

Chapter 1

Preliminaries

In this chapter, we first deal with two sequences of polynomials arising from the traces in $\mathrm{SL}(2,\mathbb{C})$. Following this, we derive the notion of trace polynomial of a word in the free group of rank two.

1.1 Traces in $\mathrm{SL}(2,\mathbb{C})$

Let $A \in \mathrm{SL}(2,\mathbb{C})$ and let $\tau = \mathrm{tr}(A)$. Then $A + A^{-1} = \tau I$ by Cayley-Hamilton theorem. For a two by two complex matrix B, we see that

$$\mathrm{tr}(AB) + \mathrm{tr}(A^{-1}B) = \mathrm{tr}\left((A + A^{-1})B\right) = \mathrm{tr}\,A\,\mathrm{tr}\,B. \qquad (1.1)$$

We also see that

$$\begin{aligned}
A^2 &= \tau A - I,\\
A^3 &= (\tau^2 - 1)A - \tau I,\\
A^4 &= (\tau^3 - 2\tau)A - (\tau^2 - 1)I, \text{ and}\\
&\vdots\\
A^n &= \beta_n(\tau)A - \beta_{n-1}(\tau)I, \text{ for all } n \in \mathbb{Z}.
\end{aligned}$$

where $\{\beta_n(x)\}$ are *Chebyshev polynomials* defined by a recurrence relation

$$\beta_{n+1}(x) = x\beta_n(x) - \beta_{n-1}(x) \text{ with } \beta_0 = 0 \text{ and } \beta_1 = 1.$$

One can also see that $\beta_{-n}(x) = -\beta_n(x)$ for all $n \in \mathbb{Z}$. It is straightforward to see that the traces of integer powers of $A \in \mathrm{SL}(2,\mathbb{C})$ are polynomials in the variable $\mathrm{tr}(A) = \tau$. Define a sequence of polynomials $\{\tau_n(x)\}$ by $\tau_n(x) = x\beta_n(x) - 2\beta_{n-1}(x)$ for all $n \in \mathbb{Z}$. For $A \in \mathrm{SL}(2,\mathbb{C})$, it follows that

$$\mathrm{tr}(A^n) = \tau_n(\mathrm{tr}\, A) \text{ for all } n \in \mathbb{Z}.$$

Moreover one can find the recurrence relation given by

$$\tau_{n+1}(x) = x\tau_n(x) - \tau_{n-1}(x).$$

Chebyshev polynomials $\{\beta_n(x)\}$ will be often used to deal with traces in $\mathrm{SL}(2,\mathbb{C})$. We describe related lemmas below.

Lemma 1.1.1. *Let $\beta_n(x)$ be Chebyshev polynomials. Then for $n \in \mathbb{Z}$,*

1. $X^n = \beta_n(\mathrm{tr}\, X)X - \beta_{n-1}(\mathrm{tr}\, X)I$*, for all* $n \in \mathbb{Z}$ *and* $X \in \mathrm{SL}(2,\mathbb{C})$.
2. $\beta_n(x) = \tau_{n+1}(x) - \tau_{n-1}(x)$
3. $\beta_n(2\cos\theta) = \dfrac{\sin n\theta}{\sin\theta}$, *for* $0 \le \theta < \pi$

   where the value of right hand side at $\theta = 0$ *is taken from limit.*

Proof. Use induction on n. □

1.2 Free group of rank two

From now on let F_2 denote the group consisting of all words generated by two alphabets a and b.

reduced word

A word W is said to be *reduced* if it does not contain redundant pairs. Every element of F_2 can be written as a reduced form uniquely. Also, a word W is called a *cyclically reduced word* when every cyclic permutation of W is reduced.

We establish some combinatorial lemmas related to the number of (cyclically) reduced words. The proof of these lemmas gives us basic idea about an algorithm yielding all (cyclically) reduced words of given length.

Lemma 1.2.1.

1. *The number of all reduced words with length* n *is* $a_n = 4 \cdot 3^{n-1} (n \geq 1)$
2. *Let* c_n *denote the number of all reduced words of the form* gWg^{-1} *with reduced* W *with the length* $n-2$ *and* $g \in \{a, b, a^{-1}, b^{-1}\}$. *The number of all cyclically reduced words with the lengths* n *denoted by* b_n, *satisfies* $b_n = a_n - c_n$. *Moreover* c_n *satisfies the following recurrence relation.*

$$c_n = 3 \cdot c_{n-2} + 2 \cdot b_{n-2} (n \geq 3). \tag{1.2}$$

Proof. *1* is straightforward. To prove *2*, consider a word in F_2, which is reduced but not cyclically reduced. Then the first and the last alphabet must be inverses of each other. So it establishes the first equation of the lemma. Now, prove the last equation of the theorem. Let W be a word on length n. Consider a word W' of the lengths $n-2$ made by deleting the first and the last alphabet from W. So, W' is reduced, now there are two cases of the form of W'. Consider all the cases

- W' is cyclically reduced

  In this case, the first and the last alphabets of W cannot be inverses of each other. Without loss of generality, let a and b are both ends of W'. Since W is cyclically reduced, if one fixes the first alphabet of W, then the last is decided naturally. So there are 4 candidate pairs of both ends of W, $(a, a^{-1}), (b, b^{-1}), (a^{-1}, a)$, and (b^{-1}, b) which make W is of the form of $gVg{-}1$ for some $g \in \{a, b, a^{-1}, b^{-1}\}$. However, two of them make W not reduced, so this case is invalid. Thus there are only $2 \cdot b_{n-2}$ many possible forms of W.

- W' is not cyclically reduced

  In this case, the first and the last element of W' must be inverses of each other. So there are 3 candidates of g, which can be the first alphabet of W'. Without loss of generality, let $g = a$. Then the last alphabet of W' is a^{-1} and it gives constraints to the candidates of the starting and the ending element of W. So there are 3 candidate pairs, (a, a^{-1}),

(b, b^{-1}) and (b^{-1}, b) as W is cyclically reduced. Thus it gives $3 \cdot c_{n-2}$ possible forms to W. □

We can find the recurrence relation on b_n with the help of the above two lemmas. For $n \geq 3$,

$$c_n = 3 \cdot c_{n-2} + 2 \cdot b_{n-2} = 2 \cdot a_{n-2} + c_{n-2} = 2 \cdot 4 \cdot 3^{n-3} + c_{n-2}.$$

and it gives

$$b_n = a_n - c_{n-2} - 8 \cdot 3^{n-3} = a_n + (b_{n-2} - a_{n-2}) - 8 \cdot 3^{n-3} = b_{n-2} + 8 \cdot 3^{n-2}$$

More generally one can write b_n as

$$b_n = 3^n + 2 + (-1)^n \ (n \geq 1). \tag{1.3}$$

1.3 Trace polynomial of words in F_2

In this section, we cast a question about the traces in a two generator subgroups of $\mathrm{SL}(2, \mathbb{C})$. Then we introduce the notion of trace polynomials with examples. We also deal with the basic properties of trace polynomials.

1.3.1 Traces in two-generator subgroups of $\mathrm{SL}(2, \mathbb{C})$

In Section 1.1, we saw that the traces of integer powers of $A \in \mathrm{SL}(2, \mathbb{C})$ is given by

$$\mathrm{tr}(A^n) = \tau_n(\mathrm{tr}\, A) \text{ for all } n \in \mathbb{Z}.$$

So the traces in a cyclic subgroup $\langle A \rangle$ of $\mathrm{SL}(2, \mathbb{C})$ are polynomials in the variable $\mathrm{tr}\, A$. This polynomial is not dependent on the generator A. What about the case of two generator subgroups? Let us take an example with two elements A, B in $\mathrm{SL}(2, \mathbb{C})$. Consider $W = AB^2A^3$. Applying Lemma 1.1.1 to each power of A and B, we have

$$\begin{aligned} W &= A(\beta_2(\mathrm{tr}\, B)B - \beta_1(\mathrm{tr}\, B)I)(\beta_3(\mathrm{tr}\, A)A - \beta_2(\mathrm{tr}\, A)I) \\ &= A((\mathrm{tr}\, B)B - I)((\mathrm{tr}\, A^2 - 1)A - I) \\ &= ((\mathrm{tr}\, A)^2 - 1)(\mathrm{tr}\, B)ABA - (\mathrm{tr}\, B)AB - ((\mathrm{tr}\, A)^2 - 1)A^2 + A. \end{aligned}$$

Let x, y, and z denote the trace of A,B, and AB, respectively. Taking the trace both sides, we have

$$\begin{aligned}\operatorname{tr}(W) &= (x^2-1)y\operatorname{tr}(ABA) - y\operatorname{tr}(AB) - (x^2-1)\operatorname{tr}(A^2) + \operatorname{tr}(A)\\ &= (x^2-1)y(xz-y) - yz - (x^2-1)(x^2-2) + x\\ &= -2 - yz + y^2 + x - xyz + 3x^2 - x^2y^2 + x^3yz - x^4\end{aligned}$$

where $\operatorname{tr}(ABA) = \operatorname{tr}(AB)\operatorname{tr}(A) - \operatorname{tr}(B) = xz - y$ is obtained from (1.1). So one can guess that a similar process can be done for all elements in two generator subgroup $\langle A, B\rangle$ of $\mathrm{SL}(2,\mathbb{C})$. At this point, we introduce a theorem presented by Procesi [6].

Theorem 1.3.1. *[6] Let $f : \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \longrightarrow \mathbb{C}$ be a regular function. Moreover, let f be invariant under the diagonal action of $\mathrm{SL}(2,\mathbb{C})$ by conjugation. Then there exists $F \in \mathbb{Z}[x, y, z]$ such that*

$$f(A,B) = F(\operatorname{tr} A, \operatorname{tr} B, \operatorname{tr} AB)$$

for all $A, B \in \mathrm{SL}(2,\mathbb{C})$.

In the above context, *regular* means that the function is a polynomial when its domain $\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})$ is considered as a closed subset of an affine space $\mathbb{C}^4\times\mathbb{C}^4$. Procesi theorem states that the regular function $f(A,B)$ on $\mathrm{SL}(2,\mathbb{C})\times \mathrm{SL}(2,\mathbb{C})$ which is invariant under the diagonal conjugation action of $\mathrm{SL}(2,\mathbb{C})$ can be expressed as a polynomial in three variables $\operatorname{tr} A, \operatorname{tr} B,$ and $\operatorname{tr} AB$. By taking an appropriate regular function, we can answer to our question about the case of two generator subgroups of $\mathrm{SL}(2,\mathbb{C})$.

1.3.2 Definition of the trace polynomial

Let w be a word in F_2. For each $X, Y \in \mathrm{SL}(2,\mathbb{C})$, let $\phi_{X,Y}$ denotes a group homomorphism

$$\begin{gathered}\phi_{X,Y} : F_2 \to \mathrm{SL}(2,\mathbb{C}) \text{ satisfying}\\ \phi_{X,Y}(a) = X \text{ and } \phi_{X,Y}(b) = Y.\end{gathered}$$

Then the map $f_w : \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \longrightarrow \mathbb{C}$ defined by

$$f_w(X,Y) = \mathrm{tr}(\phi_{X,Y}(w))$$

is a regular map. Using the property of traces, we see that f_w satisfies

$$f_w(g^{-1}Xg, g^{-1}Yg) = f_w(X,Y) \text{ for all } g \in \mathrm{SL}(2,\mathbb{C})$$

Thus f_w is invariant under the diagonal conjugation action of $\mathrm{SL}(2,\mathbb{C})$.

Definition of trace polynomials
For $w \in F_2$, there exists a polynomial $P \in \mathbb{Z}[x,y,z]$ satisfying

$$P(\mathrm{tr}\, X, \mathrm{tr}\, Y, \mathrm{tr}\, XY) = \mathrm{tr}(\phi_{X,Y}(w)) \text{ for all } X, Y \in \mathrm{SL}(2,\mathbb{C}). \tag{1.4}$$

We say that P is a *trace polynomial* of w.

We defined the trace polynomial of a word in F_2. Then a natural question arises. Is the trace polynomial of a word unique? It can be shown by using the fact that $(\mathrm{tr}\, A, \mathrm{tr}\, B, \mathrm{tr}\, AB)$ generates the full coordinate $\mathbb{C}^3$, for $A, B \in \mathrm{SL}(2,\mathbb{C})$. The detail occupies below.

Lemma 1.3.2. *[4] Consider a map $\psi : SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) \longrightarrow \mathbb{C}^3$ given by*

$$\psi(A,B) = (\mathrm{tr}\, A, \mathrm{tr}\, B, \mathrm{tr}\, AB), \textit{ for } A, B \in \mathrm{SL}(2,\mathbb{C})$$

Then ψ is surjective.

Proof. Let (x,y,z) be given in $\mathbb{C}^3$. pick any $\mu \in \mathbb{C}$ satisfying following equation

$$\mu^2 - z\mu + 1 = 0$$

Then μ satisfies $\mu + \mu^{-1} = z$ clearly. Consider two matrices A and B such that

$$A = \begin{bmatrix} x & 1 \\ -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & -\mu \\ \mu^{-1} & y \end{bmatrix}.$$

Then it is straight forward to check that $\mathrm{tr}\, A = x$ and $\mathrm{tr}\, B = y$. we also see that

$$\operatorname{tr}(C) = \operatorname{tr}\begin{bmatrix}\mu^{-1} & -x\mu + y \\ 0 & \mu\end{bmatrix} = z.$$

□

From the above lemma, it is straightforward to see that every word in F_2 has a unique polynomial. From now on for each word w, we use $\alpha(w)$ to denote the unique trace polynomial of w.

Corollary 1.3.3. *[4] Let $w \in F_2$ be given. If two polynomials $P, Q \in \mathbb{Z}[x, y, z]$ satisfy*

$$P(\operatorname{tr} A, \operatorname{tr} B, \operatorname{tr} AB) = \operatorname{tr}(\phi_{A,B}(w)) = Q(\operatorname{tr} A, \operatorname{tr} B, \operatorname{tr} AB) \\ \textit{for all } A, B \in \mathrm{SL}(2, \mathbb{C}),$$

then we have $P = Q$.

1.3.3 Basic properties of trace polynomials

In this section, we deal with basic properties of the trace polynomials. Clearly, the trace polynomial of identity matrix is constant two. Since the matrix trace has the cyclic property, so are trace polynomials. That is, for word W in F_2, we have $\alpha(Wa) = \alpha(aW)$ and $\alpha(Wb) = \alpha(bW)$. Using (1.1), we also have

$$\alpha(w_1 w_2) = \alpha(w_1)\,\alpha(w_2) - \alpha(w_1 {w_2}^{-1}) \text{ for all } w_1, w_2 \in \mathrm{SL}(2, \mathbb{C}). \tag{1.5}$$

For example, setting $w_1 = a$ and $w_2 = bab^{-1}$, we can compute $\alpha(abab^{-1})$ as follow

$$\begin{aligned}\alpha(abab^{-1}) &= \alpha(aba)\,\alpha(b^{-1}) - \alpha(abab) \\ &= \alpha(ba^2)\,\alpha(b^{-1}) - \alpha(abab) \\ &= y\,\alpha(ba^2) - \alpha\left((ab)^2\right) \\ &= y(\alpha(ba)\,\alpha(a) - \alpha(b)) - (\alpha(ab)\,\alpha(ab) - \alpha(e)) \\ &= y(xz - y) - (z^2 - 2) = 2 - y^2 - z^2 + xyz.\end{aligned}$$

Similarly one can compute $\alpha(ab^3)$

$$\begin{aligned}\alpha(ab^3) &= \alpha(ab^2)\,\alpha(b) - \alpha(ab)\\ &= (\alpha(ab)\,\alpha(b) - \alpha(a))\,\alpha(b) - \alpha(ab)\\ &= (yz - x)y - z = y^2z - xy - z.\end{aligned}$$

Following these, we introduce a lemma to compute the trace polynomials of F_2-endomorphic images of words.

Lemma 1.3.4. *[2] Let f be an endomorphism on F_2 and let w be a word of F_2. Then the trace polynomial of $f(w)$ is*

$$\alpha(f(w)) = \alpha(w)\Big(\alpha(f(a)), \alpha(f(b)), \alpha(f(ab))\Big). \tag{1.6}$$

Proof. It suffices to show that for fixed $A, B \in \mathrm{SL}(2, \mathbb{C})$,

$$\mathrm{tr}(\phi_{A,B}(f(w))) = R\big(\mathrm{tr}(A), \mathrm{tr}(B), \mathrm{tr}(AB)\big).$$

where R denotes the right hand side of (2.3). Let $X = \phi_{A,B}(f(a))$ and $Y = \phi_{A,B}(f(b))$. From the definition of ϕ, we see that

$$\phi_{X,Y} = \phi_{A,B} \circ f.$$

Using the last equation, we have

$$\begin{aligned}R(\mathrm{tr}(A), \mathrm{tr}(B), \mathrm{tr}(AB)) &= \alpha(w)\Big(\mathrm{tr}(\phi_{A,B}(f(a)), \phi_{A,B}(f(b)), \phi_{A,B}(f(ab))\Big)\\ &= \alpha(w)\Big(\mathrm{tr}\,X, \mathrm{tr}\,Y, \mathrm{tr}\,XY\Big)\\ &= \mathrm{tr}\Big(\phi_{X,Y}(w)\Big)\\ &= \mathrm{tr}\Big(\phi_{A,B}(f(w))\Big).\end{aligned}$$

□

Let us calculate the trace polynomial of aba^3b^{-1}. Consider an endormorphism f on F_2 given by $f(a) = a$ and $f(b) = bab^{-1}$. Then for a word $w = ab^3$, the trce polynomial of $f(w) = aba^3b^{-1}$ is

$$\begin{aligned}
\alpha(f(w)) &= \alpha(ab^3)(\alpha(a), \alpha(bab^{-1}), \alpha(abab^{-1})) \\
&= \alpha(ab^3)(x, x, 2 - y^2 - z^2 + xyz) \\
&= (y^2z - xy - z)(x, x, 2 - y^2 - z^2 + xyz) \\
&= x^2 + y^2 + z^2 - xyz - x^2y^2 - x^2z^2 + x^3yz - 2.
\end{aligned}$$

Is there a general method to compute the trace polynomial of a given word? The answer is affirmative, and we introduce the algorithms in the next chapter.

Chapter 2

Computation of trace polynomials

All words in the free group of rank two have the trace polynomial by Procesi's theorem. In this chapter, we prove the existence of the trace polynomial differently. Using a simple method, we can compute the trace polynomial recursively. We also present a more efficient method that can also calculate the trace polynomial of words.

2.1 Existence of the trace polynomial

In this section, we show the existence of trace polynomial of words recursively without using the theorem of Procesi. To proceed with the proof smoothly, we define the trace polynomials of vectors in $\mathbb{Z}^{2r}$ for a positive integer r.

2.1.1 $2r$-vectors and multiplicative groups

Let r be a positive integer and let $m \in \mathbb{Z}^{2r}$. We say m is a $2r$-*vector*. Let G be a multiplicative group and $m = (m_1, m_2, \ldots, m_{2r-1}, m_{2r})$ be a $2r$-vector for $r > 0$. For given $p, q \in G$, we define $m(p, q) \in G$ by

$$m(p,q) = \prod_{i=1}^{r} p^{m_{2i-1}} q^{m_{2i}}.$$

For example, take $G = \mathrm{SL}(2,\mathbb{C})$ and $m = (1,3,2,4)$. Then for $A, B \in \mathrm{SL}(2,\mathbf{C})$, we have

$$m(A,B) = AB^3A^2B^4 \in \mathrm{SL}(2,\mathbf{C}).$$

Taking G as F_2 with $m = (1,2,0,-2,1,0)$, we see that

$$m(a,b) = ab^2a^0b^{-2}ab^0 = a^2.$$

Following this, we want to define *the trace polynomial of a* $2r$*-vector*. Let $r > 0$ and m be a $2r$-vector. Define $\alpha(m) \in \mathbb{Z}[x,y,z]$ as the trace polynomial of $m(a,b) \in F_2$. From the definition of α, we have

$$\alpha\,(m)(\operatorname{tr} A, \operatorname{tr} B, \operatorname{tr} AB) = \operatorname{tr}\big(m(A,B)\big) \text{ for all } A, B \in \mathrm{SL}(2,\mathbb{C}).$$

Finally, we introduce a function which is useful to compute the trace polynomials. For $r > 0$ and $2r$-vector m, we define the function $\mathbf{B}(m) \in \mathbb{Z}[x,y]$ by

$$\Big(\mathbf{B}(m)\Big)(x,y) = \prod_{i=1}^{r} \beta_{m_{2i-1}}(x)\beta_{m_{2i}}(y).$$

For example, if we take $v = (2,2,1,2)$, then

$$\big(\mathbf{B}(v)\big)(x,y) = \beta_2(x)\beta_2(y)\beta_1(x)\beta_2(y) = x \cdot y \cdot 1 \cdot y = xy^2.$$

It is not difficult to check that for given $r > 0$ and for all $2r$-vector v, the followings hold.

1. $\mathbf{B}(v) = \mathbf{B}(-v)$
2. $\mathbf{B}(v) = 0$ if and only if v admits a zero entry.

2.1.2 Proof of the existence of trace polynomials

A words in F_2 can be represented as $m(a,b)$ for a $2r$-vector m. So it suffices to show that every $2r$-vector has a corresponding trace polynomial for all positive integers r.

Proof. Our goal is to prove this:

For every $r > 0$ and for given $2r$-vector m, there exsits $\alpha(m) \in \mathbb{Z}[x, y, z]$ satisfying

$$\alpha(m)(\operatorname{tr} A, \operatorname{tr} B, \operatorname{tr} AB) = \operatorname{tr}\big(m(A, B)\big) \text{ for all } A, B \in \mathrm{SL}(2, \mathbb{C}).$$

We use induction on r. Assume that $r = 1$. Then $m = (p, q)$ for some $p, q \in \mathbb{Z}$.Take $\alpha(m)$ as

$$\alpha(m) = \beta_p(x)\beta_q(y)z - x\beta_p(x)\beta_{q-1}(y) - y\beta_{p-1}(x)\beta_q(y) + 2\beta_{p-1}(x)\beta_{q-1}(y).$$

We see that

$$\begin{aligned} m(A, B) &= (\beta_p(\operatorname{tr} A)A - \beta_{p-1}(\operatorname{tr} A)I)(\beta_q(\operatorname{tr} B)B - \beta_{q-1}(\operatorname{tr} B)I) \\ &= \beta_p(\operatorname{tr} A)\beta_q(\operatorname{tr} B)AB - \beta_p(\operatorname{tr} A)\beta_{q-1}(\operatorname{tr} B)A \\ &\quad - \beta_{p-1}(\operatorname{tr} A)\beta_q(\operatorname{tr} B)B + \beta_{p-1}(\operatorname{tr} A)\beta_{q-1}(\operatorname{tr} B)I. \end{aligned} \tag{2.1}$$

Taking the trace of the both sides, we get

$$\operatorname{tr}\big(m(A, B)\big) = \alpha(m)(\operatorname{tr} A, \operatorname{tr} B, \operatorname{tr} AB).$$

Assume the induction hypothesis holds for r less than n. Let $r = n$ and $2r$-vector m is given. Let A, B be elements of $\mathrm{SL}(2, \mathbb{C})$. Like the case when $r = 1$, the matrix $m(A, B)$ can be expressed as

$$\begin{aligned} m(A, B) &= \prod_{i=1}^{2r} \Big(\beta_{m_i}(\operatorname{tr}(X_i))X_i - \beta_{m_i-1}(\operatorname{tr}(X_i))I\Big) \\ &= \sum_{\mu} \Big((-1)^{\|\mu\|_1}\mathbf{B}(m - \mu) \cdot (\mathbf{1} - \mu)(A, B)\Big) \end{aligned}$$

where $\mathbf{1}$ is the $2r$-vector with all entries one and X_i denotes

$$X_i = \begin{cases} A, \text{if } i \text{ odd} \\ B, \text{if } i \text{ even} \end{cases}$$

and the summation is extended over all $2r$-vectors in $\{0, 1\}^{2n}$. For each μ, let

$\epsilon = \mathbf{1} - \mu$, then the family of μ is the same as that of ϵ, and we have

$$\begin{aligned} m(A,B) &= \sum_{\epsilon} (-1)^{\|\mathbf{1}-\epsilon\|_1} \mathbf{B}(\mathbf{m} + \epsilon - \mathbf{1}) \epsilon(A,B) \\ &= \sum_{\epsilon} (-1)^{\|\epsilon\|_1} \mathbf{B}(\mathbf{m} + \epsilon - \mathbf{1}) \epsilon(A,B) \end{aligned}$$

Taking the traces on both sides gives

$$\operatorname{tr}\Big(m(A,B)\Big) = \sum_{\epsilon} (-1)^{\|\epsilon\|_1} \mathbf{B}(\epsilon - \mathbf{1} + \mathbf{m}) \operatorname{tr}\Big(\epsilon(A,B)\Big). \tag{2.2}$$

Since $\operatorname{tr}\Big(m(A,B)\Big)$ is a linear sum of traces of $\epsilon(A,B)$'s, it suffices to show that for every $2r$-dimensional vector ϵ in $\{0,1\}^{2r}$ has the trace polynomials. Then the trace of our given matrix can be expressed into a polynomial in x, y, and z.

First, consider the case when all entries of ϵ are 1. For all $A, B \in \mathrm{SL}(2,\mathbb{C})$, we see that

$$\operatorname{tr}\Big(\epsilon(A,B)\Big) = \operatorname{tr}\Big((AB)^r\Big) = \tau_r\Big(\operatorname{tr}(AB)\Big).$$

and it means that $\alpha(\epsilon) = \tau_r(z)$ where τ_n is polynomials defined in Section 1.1. Secondly, we consider the case at least one entry of ϵ is zero. Let j be the index in $\{1, 2, ..., 2r\}$ such that ϵ_j is the first zero appearing in ϵ. Then we can ***reduce*** ϵ into a $2r-2$ vector, which has the trace polynomial by induction hypothesis. Define $(2r-2)$-dimensional vector ϵ' as follow.

- the case when $j = 1$

$$\epsilon'_i = \begin{cases} \epsilon_{i+2}, & \text{if } 1 \le i \le 2r-4 \\ \epsilon_i + \epsilon_1, & \text{if } i = 2r-3 \\ \epsilon_i + \epsilon_2, & \text{if } i = 2r-2 \end{cases}$$

- the case when $j = 2r$

$$\epsilon'_i = \begin{cases} \epsilon_i + \epsilon_{2r-1}, & \text{if } i = 1 \\ \epsilon_i + \epsilon_{2r}, & \text{if } i = 2 \\ \epsilon_i, & \text{if } 3 \leq i \leq 2r-4 \end{cases}$$

- the other cases, that is, $1 < j < 2r$

$$\epsilon'_i = \begin{cases} \epsilon_i, & \text{if } i \leq j \\ \epsilon_{i-1} + \epsilon_{i+1}, & \text{if } i = j \\ \epsilon_{i+2}, & \text{if } i > j+1 \end{cases}$$

Using that m has a zero entry, we collapsed three coordinates into the one. By definition of ϵ', We see that $\operatorname{tr}(\epsilon'(A,B)) = \operatorname{tr}(\epsilon(A,B))$ for all $A, B \in \mathrm{SL}(2,\mathbb{C})$. It means that ϵ has the same trace polynomial with ϵ'. Hence we prove that all indexes ϵ' in the summation (2.1) has the trace polynomial. □

2.2 Algorithms computing trace polynomials

As we saw in the proof in the previous section, the trace polynomial can be expressed into a polynomial-linear sum of the trace polynomial of indexes. Moreover, for each index in the index family, we can compute directly the trace polynomial (when all entries are one), or reduce the index vector into an even-dimensional vector of a smaller degree. Hence, one can find a process yielding the trace polynomial of the given vector. In this thesis, two recursive functions are used to implement this idea. One can easily make polynomial-valued functions using SymPy Library in Python3.

2.2.1 Algorithm 1 : recursive method

For a word w in F_2, there exists the unique polynomial in $\mathbb{Z}[x,y,z]$ whose value at $(\operatorname{tr} A, \operatorname{tr} B, \operatorname{tr} AB) \in \mathbb{C}^3$ is equal to the trace of $w(A,B)$ for all $A, B \in \mathrm{SL}(2,\mathbb{C})$. Let us notice the following two facts. First, we can compute the trace polynomial of reduced words of word length two.

$$\alpha(a^p b^q) = \beta_q(x)\beta_p(y)z - x\beta_p(x)\beta_{q-1}(y) - y\beta_{p-1}(x)\beta_q(y) + 2\beta_{p-1}(x)\beta_{q-1}(y)$$

Secondly, the equation below can be expressed as a recursive summation. It is possible by ***reducing*** each ϵ into a vector with a smaller dimension that has the same trace polynomial with the original one.

$$\alpha\left(m(A,B)\right)(x,y,z) = \sum_{\epsilon} (-1)^{\|\epsilon\|_1} \mathbf{B}(\epsilon - \mathbf{1} + \mathbf{m})(x,y)\,\alpha(\epsilon)(x,y,z) \tag{2.3}$$

Using the above two facts, we describe the pseudocode of algorithm computing the trace polynomial.

Compute the trace polynomial of words in F_2

Function
$\alpha(m)$
Input
m : An $2r$-dimenional vector (representing a word in F_2)
Output
the trace polynomial of m

Initialize;
Check dimension $2r$ of m;
If r is one;
    Get $p = m[0]$ and $q = m[1]$;
    $result =$

$$\beta_p(x)\beta_q(y)z - x\beta_p(x)\beta_{q-1}(y) - y\beta_{p-1}(x)\beta_q(y) + 2\beta_{p-1}(x)\beta_{q-1}(y);$$

    return $result$;
Initialize $epsilons$ as the set of all $2r$ vector with entries zero or one;
Set $\mathbf{1}$ = the integer $2r$-vector with all one entries;
Set sum = 0;
For ϵ in $epsilons$:
    $eps\_sum = \|\epsilon\|_1$

$B = \mathbf{B}(\epsilon - \mathbf{2} + \mathbf{m})(x, y)$;
If all entries of ϵ are one:
$p = \tau_r(z)$;
Else:
$j =$ the smallest index in $1, 2, 3, \ldots, 2r$ such that $\epsilon[j]$ is zero
Initialize *redu_eps* of dimension $(2r - 2)$ as follow.

- the case when $j = 0$

$$redu\_eps[i] = \begin{cases} eps[i+1], & \text{if } 0 \leq i \leq 2n-5 \\ eps[i-1] + m[0], & \text{if } i = 2n-4 \\ eps[i-1] + m[1], & \text{if } i = 2n-3 \end{cases}$$

- the case when $j = 2n - 1$

$$redu\_eps[i] = \begin{cases} eps[i-1] + m[2n-2], & \text{if } i = 0 \\ eps[i-1] + m[2n-1], & \text{if } i = 1 \\ eps[i-1], & \text{if } 2 \leq i \leq 2n-5 \end{cases}$$

- the other cases, that is, $1 < j < 2n$

$$redu\_eps[i] = \begin{cases} eps[i-1], & \text{if } i \leq j \\ eps[i-2] + m[i], & \text{if } i = j \\ eps[i+1], & \text{if } i > j+1 \end{cases}$$

$p = \alpha(redu\_eps)$;
$\text{sum} = \text{sum} + (-1)^{eps\_sum} \cdot B \cdot p$;
return sum;

Let us analyze the above algorithm. Firstly, this algorithm calls itself with reduced input vectors. It must stop because the dimension of input vectors is decreasing. When the epsilon has all entries one, or input vector has dimension two, it returns the trace polynomial directly. Secondly, since each summation extends over the index family, it is obvious that the time complexity of this algorithm on the dimension of ϵ is fairly large. Even though we assume that each entry is bounded, we see that the time complexity of the algorithm is at least $O(2^{(n^2)})$. To see the reason, let n be an even dimension of the input vector. Since the summation proceeds on 2^n many epsilons and since each summand contains the same calculation for a $n-2$ dimensional reduced vector, the time complexity is at least $O(2^n \cdot 2^{n-2} \cdot 2^{n-4} \cdots 2^2 \cdot 2^0) = O(2^{(n^2)})$.

2.2.2 Algorithm 2 : alternating formula

In this section, we are dealing with another method to compute trace polynomial of words in F_2. Is it possible to express the trace polynomial of a vector m of the form below?

$$\sum_{\epsilon}\{\mathbf{B}(\mathbf{m}+\epsilon)(x,y)\cdot f_\epsilon(z)\} \tag{2.4}$$

The answer is "affirmative". Let us look into each summand of 2.3. Firstly, one can see that

$$\begin{aligned}&\deg_x(\alpha(\epsilon)) \le \epsilon_1+\epsilon_3+\cdots+\epsilon_{2r-1} \text{ and}\\ &\deg_y(\alpha(\epsilon)) \le \epsilon_2+\epsilon_4+\cdots+\epsilon_{2r}.\end{aligned}$$

Thus using the fact that $x\beta_n(x) = \beta_{n+1}(x)+\beta_{n-1}(x)$ and the fact that $\mathbf{B}(\mathbf{m}+\epsilon-\mathbf{1})(x,y)$ is a product of r many $\beta_i(x)$ and r many $\beta_i(y)$, we can ***delete*** x and y in each monomial of $\alpha(\epsilon)$. However, we have to admit ϵ to have the entries in $\{-1,0,1\}$. Let us take an example with $\mathbf{m}=(2,2,1,1)$ and $\epsilon=(0,1,1,1)$. The summand for ϵ in (2.3) can be modified as follow.

$$\begin{aligned}\mathbf{B}(\mathbf{m}-\mathbf{1}+\epsilon)\cdot\alpha(\epsilon) &= \mathbf{B}(1,2,1,1)yz-\mathbf{B}(1,2,1,1)x\\ &= (\mathbf{B}(1,3,1,1)+\mathbf{B}(1,1,1,1))z\\ &\quad-(\mathbf{B}(1,2,2,1)+\mathbf{B}(1,2,0,1))\\ &= \mathbf{B}\big(\mathbf{m}+(0,1,0,0)\big)z+\mathbf{B}\big(\mathbf{m}+(0,-1,0,0)\big)z\\ &\quad-\mathbf{B}\big(\mathbf{m}+(0,0,1,0)\big)-\mathbf{B}\big(\mathbf{m}+(0,0,-1,0)\big)\end{aligned}$$

Hence one can see that the trace polynomial can be expressed as of the form 2.4. We want to deal with an explicit formula presented by Jorgensen. It needs some definition to state that.

For positive r, let μ be a $2r$-vector. We say μ is *alternating* if all its entries are in $\{-1, 0, 1\}$ and has even number of non-zero entries appearing with alternating signs. We also define the degree of alternating vector. Let $d(\mu)$ be an alternating vector of dimension $2r$ for some $r > 0$. Let $d(\mu)$ denotes *the degree of* μ defined by

$$d(\mu) = \text{ the degree of } \alpha\Big(\mathbf{1} - \text{abs}(\mu)\Big)(0, 0, z)$$

where $\mathbf{1}$ denotes the $2r$-vector of all one entries and $\text{abs}(\mu)$ means the vector obtained by taking absolute value all entries of μ. In the above context we consider the degree of zero polynomial as zero. From Lemma 1.3.4, we see that the degree is invariant under the cyclic permutation on entries. Let us take an example with alternating vectors $\mu_1 = (0, 1, 0, 0, -1, 0, 1, -1)$ and $\mu_2 = (-1, 1, -1, 0, 0, 0, 1, 0)$. The degree of these vectors are

$$\begin{aligned}
\mathrm{d}(\mu_1) &= deg\Big(\alpha(\mathbf{1} - \text{abs}(\mu_1))\Big) = deg\Big(\alpha(a^2b^2)(0, 0, z)\Big) = deg(2) = 0, \text{ and} \\
\mathrm{d}(\mu_2) &= deg\Big(\alpha(\mathbf{1} - \text{abs}(\mu_2))\Big) = deg\Big(\alpha(bab^2)(0, 0, z)\Big) = deg(-z) = 1.
\end{aligned}$$

Jorgensen presented an equation of the form (2.4) which satisfies $f_\epsilon = f_{-\epsilon}$ and $f_\epsilon = 0$ unless that ϵ is alternating. The former condition is called *symmetric*, and the latter is called *alternating*. He also showed that the equation in his paper [1] is the unique, symmetric, alternating formula of the form (2.4).

Theorem 2.2.1. *[1] Let* $\mathbf{m}$ *be a* $2r$*-vector for some* $r > 0$*. Then*

$$\alpha(\mathbf{m})(x, y, z) = \frac{1}{2}\mathbf{B}(\mathbf{m})(x, y)\cdot\tau_r(z) + \frac{1}{2}\sum_{\mu}\{(-1)^{r-d(\mu)}\cdot\mathbf{B}(\mathbf{m}+\mu)(x, y)\cdot\tau_{d(\mu)}(z)\}$$

where μ *runs through all alternating* $2r$*-vectors.*

Using Jorgensen's theorem, one can implement another algorithm computing the trace polynomials of words in F_2. The calculation of the degrees of an alternating $2r$-vector needs the computation of the trace polynomials of a

$2r$-vector. In Jorgensen's paper [1], there is an exercise about the computation of degrees of alternating vectors. However, the results obtained by the exercise is different from those obtained by definition of the degree. We restate the exercise and prove that.

Lemma 2.2.2. *Let μ be an alternating $2r$-vector with $\mu_1 = 0$. Collapse all alternate subtuples with the length two from μ until it is not possible. Let μ_r be the **reduced** tuple after collapsing and let $\{a_i\}_{i=0}^n$ be the sequence of the number of consecutive zeros in μ_r. Then we have*

$$d(\mu) = \left|\frac{1}{2}\sum_{i=0}^{n}(-1)^i a_i\right|$$

Proof. By definition of the degree, since it is obtained from the trace polynomial of $\mathbf{1} - |\mu|$, the even consecutive non-zero does not effect on the degree. Using this fact, it suffices to prove the following claim
Claim Let $w_1, w_2, \ldots, w_n$ be reduced words in F_2 satisfying

- All syllables of w_i are in $\{a, b\}$, and
- The last alphabet of w_i coincides with the first alphabet of w_{i+1}, for $0 \leq i \leq n-1$.
- $\sum_{i=0}^{n} |w_i|$ is even.

Then we have

$$\alpha(w_1 w_2 \cdots w_n)(0,0,z) = \tau_r(z) \text{ where } r = \frac{1}{2}\Big|\sum_{i=0}^{n}(-1)^i|w_i|\Big|.$$

Proof of claim
Firstly, it is straightforward to show that

$$\alpha(wa^2)(0,0,z) = \alpha(w)(0,0,z) \text{ and}$$
$$\alpha(wb^2)(0,0,z) = \alpha(w)(0,0,z) \text{ for every } w \in F_2.$$

Using this, we can annihilate all the syllables of even power in $\alpha(\cdot)(0,0,z)$. Use induction on n. Consider the case $n=1$. Then w_0 is either $(ab)^r$ or $(ba)^r$. We have

$$\alpha(w_0)(0,0,z) = \alpha((ab)^r)(0,0,z) = \alpha((ba)^r)(0,0,z) = \tau_r(z).$$

Assume the induction hypothesis that for $n=k$. Without loss of generality let us assume that w_k starts with the alphabet a. Otherwise, one can apply an endomorphism switching a and b and it does not change the trace polynomial of $w_1 w_2 \cdots w_{k+1}$.
Let $c = |w_k| - |w_{k+1}|$. We devide all possibilities into four cases.

- $c = 0$
  In this case, the factor $w_k \cdot w_{k+1}$ can be annihilated in $\alpha(w_1 w_2 \ldots w_{k+1})(0,0,z)$.

$$\alpha(w_1 w_2 \ldots w_k w_{k+1})(0,0,z) = \alpha(w_1 w_2 \ldots w_{k-1})(0,0,z) = \tau_r z$$

  where

$$r = \frac{1}{2}\Big|\sum_{i=0}^{k-1}(-1)^i |w_i|\Big| = \frac{1}{2}\Big|\sum_{i=0}^{k+1}(-1)^i |w_i|\Big|.$$

- $c > 0$
  Let $w'_k = abab\ldots$ of word length c. Then for words $w_1, w_2, \ldots, w'_k$, the summation of word lengths is even since the alternating sum is even by induction hypothesis. We apply our assumption to these words.

$$\alpha(w_1 w_2 \ldots w_k w_k + 1)(0,0,z) = \alpha(w_1 w_2 \ldots w'_k)(0,0,z) = \tau_r(z)$$

  where

$$r = \frac{1}{2}\Big|\sum_{i=0}^{k-1}(-1)^i |w_i| + (-1)^k |w'_k|\Big| = \frac{1}{2}\Big|\sum_{i=0}^{k+1}(-1)^i |w_i|\Big|.$$

- $c < 0$ and $|w_k|$ is even
  Let $w'_k = abab\ldots$ of word length $-c$. Applying same idea with previous case with $w_1, w_2 \ldots w'_k$, we prove the assertion.

- $c < 0$ and $|w_k|$ is odd

Let $w'_k = baba\dots$ of word length $-c$. and Let w'_{k-1} be the word obtained by append w'_k behind w_{k-1}. It is reduced because w_{k-1} ends with the syllable a. Then we have

$$\alpha(w_1w_2\dots w_kw_{k+1})(0,0,z) = \alpha(w_1w_2\dots w'_{k-1})(0,0,z) = \tau_r(z)$$

where

$$\begin{aligned} r &= \frac{1}{2}\Big|\sum_{i=0}^{k-2}(-1)^i|w_i| + (-1)^{k-1}|w'_{k-1}|\Big| \\ &= \frac{1}{2}\Big|\sum_{i=0}^{k-2}(-1)^i|w_i| + (-1)^{k-1}(|w_{k-1}| - |w_k| + |w_{k+1})|\Big| \\ &= \frac{1}{2}\Big|\sum_{i=0}^{k+1}(-1)^i|w_i|\Big|. \end{aligned}$$

□

Let us take an example with an alternating vector μ given by

$$\mu = (0,0,1,0,-1,0,0,0,1,-1,0,1,-1,1,-1,1,0,-1,1,-1).$$

Collapse all alternating tuple of length two of μ until it shown. Then we have $\mu_r = (0,0,1,0,-1,0,0,0,0,1,0,-1)$. The absolute value of the alternating sum of consecutive zeros in μ_r is $|2-1+4-1| = 4$. One can also see that the half of this alternating sum equals the degree of μ computed from the definition.

$$\begin{aligned} d(\mu) &= deg\Big(\alpha(\mathbf{1} - \text{abs}(\mu))(0,0,z)\Big) \\ &= deg(z^2-2) = \frac{4}{2} \end{aligned}$$

Now we state the pseudocode of the algorithm using the alternating formula of the trace polynomials. One can refer to full code for Python3 at the next URL:
`https://github.com/mathapple316/trace_polynomials`

Compute Trace polynomials of words using the explicit formula

Function
deg(μ)
Input
μ : an alternating $2r$-vector
Output
the degree of μ

Initialize;
Set $r =$ the half of $length(\mu)$;
If all entries of μ are one:
    return r;
Else:
    While the first entry of μ is zero
        $\mu =$ LeftShift(μ);
Set $i = 1$
While $i < dim$:
    If $\mu[i-1] \cdot \mu[i]$ is -1:
        $\mu[i-1] = 3$;
        $\mu[i] = 3$;
        $i = i + 2$
        continue;
    Else:
        continue;
Set $sgn = 1$ and count = 0; For entry in μ:
    If entry is 3:
        continue;
    Elif $entry$ is zero:
        count = count + sgn;
        continue;
    Else:
        sgn = -sgn;
        continue;
return $\frac{1}{2}\Big|count\Big|$;

Function

α _jorgensen
Input
m : an $2r$-vector
Output
the trace polynomial $\alpha(m)$ obtained by Theorem 2.2.1

Initialize;
Set $r = \dfrac{length(\mu)}{2}$;
$\alpha = \dfrac{1}{2}\tau_r(z) + \mathbf{B}(m)$;
Generate alt_set = the set of all $2r$-vectors with entries in $\{-1, 0, 1\}$;
For μ in alt_set:
    If μ is alternating:
        $d = \deg(\mu)$;
        $\alpha = \alpha + (-1)^{r-d} \cdot \tau_d(z) \cdot \mathbf{B}(m + (\mu))$;
return α;

We generated all alternating vectors with the given dimension. To use Lemma 2.2.2, we apply *LeftShift* on each index vector. Following Jorgensen's explicit formula, we got the result. Let us analyze the previous algorithm. Firstly, we used another way to compute the degree of alternating vectors rather than follow the definition. In this way, the time complexity in the computation of the degree was reduced effectively. If one uses the definition to compute degrees, that is at least exponential time. However, using Lemma 2.2.2, we see that the time complexity of computation degree becomes polynomial time. Secondly, this algorithm is effective than the recursive method. Assume each entry is bounded. Let n be the even dimension of an input vector. We generated all alternating index vectors and computed their degrees. Thus the time complexity is $O(3^n)$. It is still large, but we see that this method is much more effective than the recursive method, which was at least $O(2^{(n^2)})$.

Chapter 3

Trace polynomials of cyclically reduced words in F_2

What information of words can be got back from their trace polynomials? Also, we will see an equivalence relation on cyclically reduced words preserving their trace polynomials. From now on, let $\mathcal{C}$ denotes *the set of all cyclically reduced words in* F_2.

3.1 Properties of trace polynomial

In this section, our goal is to see that the word lengths can be calculated from their trace polynomials. Firstly, we will see that the syllable lengths of input words can be retrieved from trace polynomial whihin error one. Remind that $\mathcal{C}$ denotes the set of all cyclically reduced words in F_2.

For $w \in \mathcal{C}$, define $\mid w \mid_{syl}$ as *the syllable length of* w. Also we define $\mid w \mid_{csyl}$, *the cyclic syllable length of* w by

$$\mid w \mid_{csyl} = 2 \left\lfloor \frac{\mid w \mid_{syl}}{2} \right\rfloor$$

Is is easy to see that $\mid \cdot \mid_{csyl}$ is equals to $\mid \cdot \mid_{syl}$ when $\mid \cdot \mid_{syl}$ is even, and is the same with $\mid \cdot \mid_{syl} - 1$, otherwise. For example, $\mid a^2b^3a^3 \mid_{syl} = 3$, but $\mid a^2b^3a^3 \mid_{csyl} = 2$. The cyclic syllable length is related to the degree of trace polynomial of words in the variable z.

Lemma 3.1.1. *For $w \in \mathcal{C}$, the degree of $\alpha(w)$ in z equals $\frac{1}{2} | w |_{csyl}$.*

Proof. Without loss of generality, It suffices to prove for the case when the leftmost alphabet of w is a. Otherwise, apply an endomorphism taking a to b and b to a. This map does not change the trace polynomial by Lemma1.3.4. First, we prove for all words with even syllable length $2r$ for $r > 0$. Then the cyclic syllable length is equal to the syllable length by definition. Use induction on r. For the case $r = 1$, let $w = a^m b^n$. Then our assertion is trivial because

$$\alpha(a^m b^n) = \beta_m(x)\beta_n(y)z - \beta_m(x)\beta_{n-1}(y)x - \beta_{m-1}(x)\beta_n(y)y + 2\beta_{m-1}(x)\beta_{n-1}(y)$$

and the degree becomes one because $mn \neq 0$. Assume the assertion holds for $r = 1, 2, 3, \ldots k$ and let $w \in \mathcal{C}$ be given as follow.

$$w = a^{m_1} b^{n_1} a^{m_2} b^{n_2} \cdots a^{m_k} b^{n_k} a^{m_{k+1}} b^{n_{k+1}}$$

Set w_0 as the subword of w by taking the first $2k$ consecutive syllables of w. Then

$$\begin{aligned}\alpha(w) &= \alpha(w_0 a^{m_{k+1}} b^{n_{k+1}}) \\ &= \alpha(w_0)\, \alpha(a^{m_{k+1}} b^{n_{k+1}}) - \alpha(w_0 (a^{m_{k+1}} b^{n_{k+1}})^{-1}) \\ &= \alpha(w_0)\, \alpha(a^{m_{k+1}} b^{n_{k+1}}) - \alpha(w_0 b^{-n_{k+1}} a^{-m_{k+1}}).\end{aligned}$$

In the last expression, w_0 and $a^{m_{k+1}} b^{n_{k+1}}$ are the cyclically reduced words with the syllable length two and 2k, respectively. Therefore the degree of the left term in z is $k + 1$ by induction hypothesis. Also, the cyclic property of α tells us that

$$\begin{aligned}\alpha(w_0 b^{-n_{k+1}} a^{-m_{k+1}}) &= \alpha(a^{-m_{k+1}} w_0 b^{-n_{k+1}}) \\ &= \alpha(a^{-m_{k+1}} a^{m_1} b^{n_1} a^{m_2} b^{n_2} \cdots a^{m_k} b^{n_k} b^{-n_{k+1}}) \\ &= \alpha(a^{m_1 - m_{k+1}} b^{n_1} a^{m_2} b^{n_2} \cdots a^{m_k} b^{n_k - n_{k+1}})\end{aligned}$$

and hence the degree in z of the right term is less than or equal to k.

Let us prove for words of the odd syllable lengths. Let $w \in \mathcal{C}$ with $| w |_{syl} = 2k + 1$ for some non-negative integer k. Then cyclic syllable length is $2k$, and without loss of generality, we also assume the leftmost alphabet of w is a. Then we have

$$\begin{aligned}\alpha(w) &= \alpha(a^{m_1}b^{n_1}a^{m_2}b^{n_2}\cdots a^{m_k}b^{n_k}a^{m_k+1}) \\ &= \alpha(a^{m_k+1}a^{m_1}b^{n_1}a^{m_2}b^{n_2}\cdots a^{m_k}b^{n_k}) \\ &= \alpha(a^{m_1+m_{k+1}}b^{n_1}a^{m_2}b^{n_2}\cdots a^{m_k}b^{n_k})\end{aligned}$$

and thus degree of $\alpha(w)$ in z is k. □

Corollary 3.1.2. *Let $w_1, w_2 \in \mathcal{C}$ with $\alpha(w_1) = \alpha(w_2)$. Then $\mid w_1 \mid_{csyl} = \mid w_2 \mid_{csyl}$.*

Corollary 3.1.3. *Let $w = a^{m_1}b^{n_1}a^{m_2}b^{n_2}\cdots a^{m_k}b^{n_k}$ with non-zero $m_i's$ and $n_i's$. Then the coefficient of z^k in $\alpha(w)$ is*

$$\mathbf{B}(m_1, n_1, m_2, n_2, \cdots, m_k, n_k) = \beta_{m_1}(x)\beta_{n_1}(y)\beta_{m_2}(x)\beta_{n_2}(y)\cdots\beta_{m_k}(x)\beta_{n_k}(y)$$

Proof. Let $m = (m_1, n_1, m_2, n_2, \cdots, m_k, n_k)$. Use expression (2.2) to split $\alpha(w)$ as

$$\alpha(w)(x,y,z) = \sum_{\epsilon}\Big((-1)^{\|\epsilon\|_1}\cdot \mathbf{B}(\epsilon - \mathbf{1} + m)(x,y)\cdot \alpha(\epsilon)(x,y,z)\Big).$$

where the summation is extended over all $2k$ vectors in $\{0,1\}^{2k}$. However, there is only one index vector with all one entries. So we need to compute the summand for that index. Thus the coefficient of z^k is

$$(-1)^{\|\mathbf{1}\|_1}\mathbf{B}(\mathbf{1} - \mathbf{1} + m) = \mathbf{B}(m).$$

□

We saw that the syllable lengths of words in $\mathcal{C}$ can be obtained by looking at their trace polynomials. Then for $w \in \mathcal{C}$, is it also possible to find out the word lengths $\mid w \mid$ from $\alpha(w)$? The answer is affirmative.

Theorem 3.1.4. *Let $w \in \mathcal{C}$ with the trace polynomial $\alpha(w) \in \mathbb{Z}[x,y,z]$ of degree k in z. Also, let $f(x,y)$ be the coefficient of z^k in $\alpha(w)$. Then*

$$\mid w \mid = \deg f(x,y) + 2k.$$

Proof. First we claim that there exists

$$w' = a^{m_1} b^{n_1} a^{m_2} b^{n_2} \cdots a^{m_k} b^{n_k} \tag{3.1}$$

with all non-zero m_i, n_i, and $k > 0$ which is $\sim$-equivalent to w.

First, if w is of the form 3.1, there is nothing to prove. Assume that the w starts from a^r and ends with a^s for non-zero r, s. Since w is cyclically reduced, r and s have the same sign. Otherwise, it can be reduced cyclically. Then $\mathrm{Lsh}^{-s}(w)$ becomes our desired w'. It also proves the case when w starts from b^r and ends with b^s. Secondly, the case when w starts from a syllable b^r and end with a syllable a^s can be easily proved by taking $w' = \mathrm{Mir}(w)$. It proves proof of our claim. Then w' have the same word length with w because it is preserved by operations $\mathrm{Lsh}, \mathrm{Inv}, \text{ and } \mathrm{Mir}$.

Corollary 3.1.3 tells us that

$$f(x,y) = \beta_{m_1}(x)\beta_{n_1}(y)\beta_{m_2}(x)\beta_{n_2}(y)\cdots\beta_{m_k}(x)\beta_{n_k}(y)$$

and its degree of the highest monomial is given by

$$\begin{aligned}
\deg f(x,y) &= \sum_{i=1}^{k} \deg \beta_{m_i}(x) + \sum_{i=1}^{k} \deg \beta_{n_i}(y) \\
&= \sum_{i=1}^{k} (\mid m_i \mid -1) + \sum_{i=1}^{k} (\mid n_i \mid -1) \\
&= \sum_{i=1}^{k} (\mid m_i \mid) + \sum_{i=1}^{k} (\mid n_i \mid) - 2k
\end{aligned}$$

where the degree of $\beta_n(x)$ can be easily computed using induction on its definition with the condition $n \neq 0$. Hence we obtain

$$\begin{aligned}
\mid w \mid &= \mid w' \mid \\
&= (\mid m_1 \mid + \mid n_1 \mid) + (\mid m_2 \mid + \mid n_2 \mid) + \cdots + (\mid m_k \mid + \mid n_k \mid) \\
&= \deg f(x,y) + 2k
\end{aligned}$$

□

Corollary 3.1.5. *Let w_1 and w_2 be two elements in $\mathcal{C}$ having the same trace* *polynomials. Then* $\mid w_1 \mid = \mid w_2 \mid$.

We saw that the length of the word in F_2 can be "recovered" from trace polynomial of that. Let us take an example with the word $w = a^3b^{-1}a^3b^2a^2$. The trace polynomial of w is given by

$$\begin{aligned}\alpha(w) = &-x^8y + x^7y^2z + x^7z - x^6y^3 - x^6yz^2 + 7x^6y - 4x^5y^2z - 6x^5z \\ &+ 4x^4y^3 + 4x^4yz^2 - 15x^4y + 4x^3y^2z + 10x^3z \\ &- 4x^2y^3 - 4x^2yz^2 + 11x^2y - xy^2z - 4xz \\ &+ y^3 + yz^2 - 3y\end{aligned}$$

We see that $\mid w \mid = 11$ can be recovered from leading term of $\alpha(w)$ in z, which is the coefficient of z^2.

$$\begin{aligned}\mid w \mid &= 2 \cdot 2 + \deg\,(y^3 - x^6y + 4x^4y - 4x^2y + y) \\ &= 4 + \deg\,(x^6y) = 11.\end{aligned}$$

For the words in $\mathcal{C}$ with specified form, the trace polynomial determines the absolute values of powers of syllables up to permutation. The detailed statement and proof occupy the last part of this section.

Theorem 3.1.6. *[2] [3] For non-zero integers* m_i *and* n_i $(1 \le i \le k)$*, let* w_1 *and* w_2 *be two words so that*

$$\begin{aligned}w_1 &= a^{m_1}b^{n_1}a^{m_2}b^{n_2}\cdots a^{m_k}b^{n_k}, \text{ and} \\ w_2 &= a^{m'_1}b^{n'_1}a^{m'_2}b^{n'_2}\cdots a^{m'_k}b^{n'_k}\end{aligned}$$

If $\alpha(w_1) = \alpha(w_2)$*, then there exist two permutations* σ *and* η *on* $\{1, 2, 3, \cdots, k\}$ *such that*

$$|m'_{\sigma(i)}| = |m_i| \text{ and } |n'_{\eta(i)}| = |n_i| \text{ for } i \in \{1, 2, 3, \cdots, k\}$$

Proof. Corollary 3.1.3 tells us that

$$\prod_{i=1}^{k} \beta_{m_i}(x)\beta_{n_i}(y) = \prod_{i=1}^{k} \beta_{m'_i}(x)\beta_{n'_i}(y)$$

If we apply (3) of Lemma 1.1.1, then we have

$$\prod_{i=1}^{k}\sin(m_i\theta)=\prod_{i=1}^{k}\sin({m'}_i\theta) \text{ and}$$
$$\prod_{i=1}^{k}\sin(n_i\theta)=\prod_{i=1}^{k}\sin({n'}_i\theta) \text{ for all } 0<\theta<\pi.$$

Pick permutations $\sigma\sigma'\eta$ and η' on $\{1,2,3,\ldots,k\}$.
so that

$$|m_{\sigma(i)}|, |m'_{\sigma'(i)}|, |n_{\eta(i)}|, \text{ and } |\eta'_{b'(i)}|$$

are all decreasing sequences. Our claim is

$$|m_{\sigma(i)}|=|m'_{'(i)}| \text{ and } |n_{\eta(i)}|=|\eta'_{b'(i)}| \text{ for all } i=1,2,\cdots,k$$

Without loss of generality, we only prove the left equality. suppose on the contrary that there exist some indexes which broke the left equality. Let q be the maximum of the such indexes. Then we have

$$\prod_{i=q}^{k}\sin(m_i\theta)=\prod_{i=q}^{k}\sin({m'}_i\theta), 0<\theta<\pi$$

Without loss of generality we assume $|m_{\sigma(q)}|>|m'_{\sigma'(q)}|$. So if we set $\theta=\dfrac{\pi}{q}$, then only the left side becomes zero, and it is a contradiction. Hence our desired permutations exist. $\square$

3.2 Trace polynomial and $\sim$ equivalence class on $\mathcal{C}$

Let $w=u_1u_2u_3\cdots u_n\in\mathcal{C}$. Consider the three operations $\mathrm{Lsh},\mathrm{Inv}$, and Mir on $\mathcal{C}$ define as follow.

- $\mathrm{Lsh}(w)=u_2u_3\cdots u_nu_1$,
- $\mathrm{Mir}(w)=u_nu_{n-1}\cdots u_1$, and

- $\mathrm{Inv}(w) = w^{-1} = {u_n}^{-1}{u_{n-1}}^{-1} \cdots {u_1}^{-1}$.

Using these operations, we define a relation $\sim$ on $\mathcal{C}$. For $w, w' \in \mathcal{C}$, we say w is *$\sim$-related* to w' if w' can be obtained by applying finitely many operations to w within the above. It is straightforward to see that $\sim$ is an equivalence relation on $\mathcal{C}$. Moreover, this relation does not change the neighborhood of each alphabet. So the trace polynomials are preserved. For example, set $w_1 = aba^{-1}b^{-1}$ and $w_2 = abab$, and $w_3 = a^4$. Then their equivalence classes are as follow and trace polynomials are preserved in each equivalence class.

$$[w_1] = \{aba^{-1}b^{-1}, ba^{-1}b^{-1}a, a^{-1}b^{-1}ab, ab^{-1}a^{-1}b, b^{-1}a^{-1}ba, a^{-1}bab^{-1}, bab^{-1}a^{-1}\}$$
$$[w_2] = \{abab, baba, a^{-1}b^{-1}a^{-1}b^{-1}, b^{-1}a^{-1}b^{-1}a^{-1}\}$$
$$[w_3] = \{a^4, a^{-4}\}$$

Lemma 3.2.1. *[2] Let w and w' be two elements of $\mathcal{C}$. If $w \sim w'$, then we have $\alpha(w) = \alpha(w')$.*

Proof. It suffices to show that three operations Lsh, Mir, and Inv preserve the trace polynomial. Firstly, the trace polynomial does not changed under Inv because the matrix trace is invariant under the inverse. Let $w = u_1u_2 \cdots u_n \in \mathcal{C}$. We have

$$\begin{aligned}
\alpha(w) &= \alpha(u_1)\,\alpha(u_2u_3\cdots u_n) - \alpha(u_1(u_2u_3\cdots u_n)^{-1}) \\
&= \alpha(u_2u_3\cdots u_n)\,\alpha(u_1) - \alpha((u_2u_3\cdots u_n){u_1}^{-1}) \\
&= \alpha((u_2u_3\cdots u_n)u_1) \\
&= \alpha(\mathrm{Lsh}(w))
\end{aligned}$$

Finally, we check Mir does not change the trace polynomial. Consider an endomorphism f on F_2 defined by $f(a) = a^{-1}$ and $f(b) = b^{-1}$. Let $w = u_1u_2 \cdots u_n$ also be given. Using already shown properties of Mir and Inv and the result of Lemma 1.3.4, we have

$$\begin{aligned}
\alpha(\mathrm{Mir}(w)) &= \alpha\left(\mathrm{Mir}(w)^{-1}\right) \\
&= \alpha({u_1}^{-1}{u_2}^{-2}\cdots{u_n}^{-1}) \\
&= \alpha(f(w)) \\
&= \alpha(W)\Big(\alpha(a^{-1}), \alpha(b^{-1}), \alpha(a^{-1}b^{-1})\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \alpha(W)\Big(\alpha(a), \alpha(b), \alpha(ab)\Big) \\
&= \alpha(W)
\end{aligned}
$$

□

Now we want to find out $\#[w]$, the size of the equivalence class of w by observing the shape of any representative word. To deal with this, we need to define more notions. Let $\mathcal{L}(w)$ denote the Lsh-orbit of w. That is,

$$\mathcal{L}(w) = \{\mathrm{Lsh}^n(w) \mid \ n \geq 1 \ \}$$

Also, let c_w denote the smallest length of subwords v of w where w is same to v raised to a positive power. It can be expressed as

$$c_w = \min\{\mid v \mid \big| \ v : \text{a subword of } w \text{ satisfying } v^n = w \text{ for some } n \geq 1 \ \}$$

The above definition is well-defined because for $w \in \mathcal{C}$ a subword w itself can make w with one power, so the range set is not empty and we have $c_w \leq \mid w \mid$. Using the following lemma, we can compute the size of equivalence classes with a representative.

Lemma 3.2.2. *Let w be a nontrivial element in $\mathcal{C}$. The number of all elements in $[w]$ is given by*

$$\#[w] = 2c\delta$$

where

$$c = \min\{\, c_v \mid \ v \in \mathcal{L}(w)\, \}$$

and

$$\delta = \begin{cases} 1, & \textit{if } w \in \mathcal{L}(Mir(w)) \cup \mathcal{L}(Mir(w^{-1})) \\ 2, & \textit{else} \end{cases}$$

Proof. Let v be a word in $\mathcal{L}(w)$ where $c = c_v$. Then the subword s chosen by taking c alphabets from the left satisfies $s^n = v$ with $n = \dfrac{\mid v \mid}{c}$. Minimality of the length of s guarantees us

$$v, \mathrm{Lsh}(v), \mathrm{Lsh}^2(v), \cdots, \mathrm{Lsh}^{c-1}(v)$$

are mutually different elements of $\mathcal{L}(v)$. Since $\mathrm{Lsh}^c(v) = (s^{-1}v)s = s^{-1}s^n s = v$, we have

$$\mathcal{L}(w) = \mathcal{L}(v) = \{v, \mathrm{Lsh}(v), \mathrm{Lsh}^2(v), \cdots, \mathrm{Lsh}^{c-1}(v)\}.$$

Hence we see $c = \#(\mathcal{L}(w))$. For the remaining part of the proof, let $\mathcal{LI}(w)$ denotes set of all elements in $\mathcal{C}$ obtained by applying finite sequence of Lsh and Inv to w. The claim is

$$\#(\mathcal{LI}(w)) = 2c.$$

That is, we need to show that w' is not in $\mathcal{L}(w)$. Suppose on the contrary that

$$w^{-1} = \mathrm{Lsh}^k(w) \text{ for some } k \geq 1$$

Let α and β be subwords of w obtained by choosing consecutive first $(\mid w \mid -k)$ alphabets and last k alphabets of w, respectively. Then

$$w^{-1} = \beta^{-1}\alpha^{-1} = \mathrm{Lsh}^k(w) = \beta\alpha$$

and we have

$$w = \alpha\beta = \alpha^{-1}\beta^{-1}$$

It follows that $\alpha = \alpha^{-1}$ and $\beta = \beta^{-1}$ because w is reduced. Therefore α and β must be trivial and it is a contradiction. Our last claim is

$$[w] = \mathcal{LI}(w) \text{ if and only if } \mathrm{Mir}(w) \in \mathcal{LI}(w)$$

If [w] is equals to $\mathcal{LI}(w)$, then $\mathrm{Mir}(w)$ is in $\mathcal{LI}(w)$, clearly. Let us assume that $\mathrm{Mir}(w) \in \mathcal{LI}(w)$. It suffices to show that every $w' \in \mathcal{C}$ satisfying $w \sim w'$ can be expressed as

$$a_n a_{n-1} \cdots a_2 a_1 w \text{ with } a_i \in \{\mathrm{Lsh}, \mathrm{Inv}\} \text{ for some } n \geq 0.$$

Let w' be such an element. By definition of $\sim$, w' can be obtained by applying finite sequence of $\mathrm{Lsh}, \mathrm{Inv}, \mathrm{Mir}$to w. Moreover, we can move all Mir's to the far left of operations using the fact that

$$\mathrm{Mir} \circ \mathrm{Inv} = \mathrm{Inv} \circ \mathrm{Mir}$$
$$\text{and}$$
$$\mathrm{Mir} \circ \mathrm{Lsh} = \mathrm{Lsh}^{-1} \circ \mathrm{Mir}$$

For example, a word

$$\begin{aligned} \mathrm{Lsh} \circ \mathrm{Inv} \circ \mathrm{Mir} \circ \mathrm{Lsh}(w) &= \mathrm{Lsh} \circ \mathrm{Mir} \circ \mathrm{Inv} \circ \mathrm{Lsh}(w) \\ &= \mathrm{Mir} \circ \mathrm{Lsh}^{-1} \circ \mathrm{Inv} \circ \mathrm{Lsh}(w) \\ &= \mathrm{Mir} \circ \mathrm{Lsh}^{|w|-1} \circ \mathrm{Inv} \circ \mathrm{Lsh}(w) \end{aligned}$$

holds for all $w \in \mathcal{C}$ Since $Mir^2 = Id$ and remaining factor obtained by removing $\mathrm{Mir}'s$ in $\mathcal{LI}$, we see that $w' \in \mathcal{LI}(w)$. □

Corollary 3.2.3. *Let $w \in \mathcal{C}$ with $\mid w \mid= n$. Then the followings are equivalent.*

1. $\#[w] = 2$
2. $w \in \{a^n, b^n, a^{-n}, b^{-n}\}$

Applying the above lemma to the prime word lengths, we also reduce the candidates of the size of equivalence classes of given word.

Corollary 3.2.4. *Let $w \in \mathcal{C}$ with the prime word length p, then $\#[w]$ is one of $2, 2p,$ and $4p$.*

The above corollary reduce candidates of the word size by constrain *the factor c* in Lemma 3.2.2 with prime factors. The following constrain candidates by specifying *the factor δ* as one. To proceed the corollary, for $w \in \mathcal{C}$ and for $i \in \mathbb{Z}$, let w_i denotes *the $i(\mathrm{mod}\, n)$-th syllable of w from left.* We also define *the absolute form of w* as the word obtained by taking the absolute values for all power terms of all w_i and we denote by $abs(w)$. Let us take an example. If we have $w = ab^5a^{-3}b^4$, then

$$w_{-1} = b^4, w_3 = a^{-3}, \text{ and } w_5 = a^5.$$

we also see that the absolute form of $w = ab^5a^{-3}b^4$ is

$$\mathrm{abs}(w) = ab^5a^3b^4.$$

Corollary 3.2.5. *Let $w \in \mathcal{C}$ with an even syllable length. Assume abs(w) has a unique syllable abs$(w)_i$. If there is $k \in \mathbb{Z}$ such that*

$$w_{i+k} \neq w_{i-k},$$

then we have

$$\#[w] = 4c$$

where c denotes the same value with that of Lemma 3.2.2.

Proof. It suffices to show that $\delta = 2$, that is

$$w \notin \mathcal{L}(\mathrm{Mir}\, w) \cup \mathcal{L}(\mathrm{Mir}\, w^{-1}).$$

Clearly, $w \notin \mathcal{L}(\mathrm{Mir}\, w^{-1})$ by the existence of a unique syllable in abs(w). Suppose that $w \in \mathcal{L}(\mathrm{Mir}\, w)$. Then we have

$$w_{i+k} = w_{i-k} \text{ for all } k > 0$$

and it contradicts to our assumption. Thus we have desired result. □

Let us see an example of the above corollary. Consider $w = a^{-1}b^2ab^{-1}a^{-1}b$. Then abs$(w) = ab^2abab$ has a unique syllable b^2. Also we see that the syllables a^{-1} and a^2 on both sides of b^5 are different. Moreover the factor c in Lemma 3.2.2 is $\mid w \mid = 7$ by Lemma 3.2.4. Hence it follows that

$$\#[w] = 4c = 28$$

3.2.1 Injectivity of α as a map on $\sim$ equivalence classes

Since $\sim$ relation preserves the trace polynomial, there is a natural map from $\mathcal{C}/_\sim$ to $\mathbb{Z}[x, y, z]$ which makes the right triangle of Diagram3.2 commutes.

We will use the same symbol α for this map.

$$\begin{array}{ccccc} \mathcal{C}_n & \overset{j}{\hookrightarrow} & \mathcal{C} & & \\ \downarrow p_n & & \downarrow p & \searrow^{\alpha} & \\ \mathcal{C}_n/\sim & \overset{j}{\hookrightarrow} & \mathcal{C}/\sim & \overset{\alpha}{\longrightarrow} & \mathbb{Z}[x,y,z] \end{array} \tag{3.2}$$

Let $\mathcal{C}_n$ denotes the set of cyclically reduced words with lengths n. Let $\mathcal{C}/\sim$, $\mathcal{C}_n/\sim$ denote the family of all $\sim$-equivalence classes of $\mathcal{C}$, $\mathcal{C}_n$, respectively. Let $w \in \mathcal{C}$ be given. Then all w' which are $\sim$ equivalent to w have the same word length. That is, the $\sim$ equivalence class of w in $\mathcal{C}_n/\sim$ coincides with the $\sim$ equivalence class of w in $\mathcal{C}/\sim$. Formally one can say that the left rectangle in the above diagram commutes where p_n and p are projection map and j is the inclusion map.

We are ready to deal with our main purpose. The $\sim$ relation does not completely decide the trace polynomial of a word. That is, the converse of the Lemma 3.2.1 does not hold. One can see this from the following two words

$$w = aba^2b^{-1}a^{-1}bab^{-1} \text{ and } w' = a^{-1}ba^2b^{-1}abab^{-1}$$

These two words are not $\sim$ equivalent by Corollary 3.2.5. Also, the computation gives

$$\begin{aligned} \alpha(w) = \ & x^4yz - x^3y^2z^2 - x^3y^2 - x^3z^2 + x^3 + 2x^2y^3z + 2x^2yz^3 \\ & - 4x^2yz - xy^4 - 2xy^2z^2 + 4xy^2 - xz^4 + 4xz^2 - 3x = \alpha(w'). \end{aligned}$$

Then our following question is, on which $\mathcal{C}_n$ the converse of the Lemma 3.2.1 holds? It is equivalent to check the injectivity of

$$\alpha : \mathcal{C}_n/\sim \to \mathbb{Z}[x,y,z]$$

for positive integer n. Following the next section, one can see that this map is injective only when n is less than or equal to eight. We state the main result of this thesis as follow.

Theorem 3.2.6.

$$\alpha : \mathcal{C}_n/\sim \to \mathbb{Z}[x,y,z]$$

is injective if and only if $n \leq 8$

Corollary 3.2.7. *Let $w_1, w_2 \in \mathcal{C}$ of word lengths less than or equal to eight. Then the followings are equivalent.*

1. $w_1 \sim w_2$

2. $\alpha(w_1) = \alpha(w_2)$

The sketch of the proof is as follows. Firstly, we classify all trace polynomials of cyclically reduced words with lengths less than nine. Then we count the number of $\sim$ equivalence classes of given $\mathcal{C}_n$ for $n = 1, 2, 3, \ldots, 8$. For each word length larger than eight, we found a counterexample that brokes the injectivity of our maps. Detailed proof occupies Section 3.3.

3.2.2 Conjecture of Wang

We saw that for the word length less than nine, the equivalent relation $\sim$ generated by $\mathrm{Lsh}, \mathrm{Inv}$, and Mir determines whether two words have the same trace polynomial or not. Finally, we introduce a conjecture introduced in a note on trace polynomials of Wang. Remind that $\mathcal{C}$ is the set of all cyclically reduced words of F_2. For two words w_1, w_2 in $\mathcal{C}$, we define the relation $\sim_w$ as below.

$$w_1 \sim_w w_2 \iff \begin{array}{l}\textit{there exist } \sim\textit{-equivalent two words } u_1, u_2 \in \mathcal{C} \textit{ such that} \\ f(u_1) = w_1 \textit{ and } f(u_2) = w_2 \textit{ for some } f \in End(F_2)\end{array}$$

It is straightforward to see that $\sim_w$ is an equivalence relation on $\mathcal{C}$. Clearly, $\sim$-equivalence of two words implies that $\sim_w$-equivalence. Wang suggested in his note that two cyclically reduced words of the same trace polynomial are $\sim_w$ - equivalent. We describe the conjecture below.

Conjecture of Wang. *For two w_1 and w_2 in $\mathcal{C}$, the followings are equivalent.*

1. $\alpha(w_1) = \alpha(w_2)$

2. $w_1 \sim_w w_2$

One can see that *2* implies *1* by using Lemma 1.3.4. Our main results in Corollary 3.2.7 show that the conjecture is true for the word lengths less than nine.

3.3 Proof of the main theorem

This section deals with the proof of the main theorem. The flow is as follows. Firstly, we introduce an algorithm that classifies all words with the word lengths less than nine via trace polynomials. Secondly, using these results, we prove the injectivity of α on $\mathcal{C}_n/_\sim$ for n less than nine. Finally, we show that the injectivity of α is broken by showing counterexamples for each $n \geq 9$.

3.3.1 Classifying words via trace polynomials

In this subsection, we will get the full data about trace polynomials of words with the word length less than nine. We generate all cyclically reduced words first. Define a function *redu_words* yielding reduced, or cyclically reduced words of the given length. Internal logic is simple. For each possible syllable, we compute deficient word length and putting each reduced word of that deficient length behind. Then check whether it is cyclically reduced or not. Then how can we generate all reduced word of smaller word length? Similar to the previous process, using the recursive method, it suffices to declare output only when given word length is one. For example, one can find all reduced words with the word lengths 3 starting with the syllable a^2 by putting all elements in $\mathcal{C}_2$ starting with syllables b, b^{-1}, b^{-2} and b^2. We have

$$\{a^2b^2, a^2b^{-2}, a^2ba, a^ba^{-1}, a^2b^{-1}a, a^2b^{-1}a^{-1}\}$$

Counting only cyclically reduced words, we find all two words starting with the syllable a^2. We leave pseudocode consisting of three functions. One can refer to full code at the next url:
https://github.com/mathapple316/trace_polynomials

Classifying trace polynomials of all words in $\mathcal{C}_n$

Function
redu_words($word\_length, cyc\_flag$)
Input
$word\_length$: word length of output words
cyc_flag : flag determining output words are cyclically reduced or not
Output

set of vectors representing all reduced words of given length.
If cyc_flag is True, then output vectors represent all cyclically reduced wordsof given length.

If n is 1:
    return $\{[1,0],[0,1],[-1,0],[0,-1]\}$;
Initialize $result = \{[n,0],[-n,0],[0,n],[0,-n]\}$;
For i in $\{-n+1,\ldots,n-1\}$:
    For w' in redu_words($n - \mid i \mid, False$):
        If $w'[0]$ is zero:
            new_word = the word obtained by putting w' behind $[i,0]$;
        Else:
            new_word = the word obtained by putting w' behind $[0,i]$;
        If cyc_flag is True and is_cyc_reduced(new_word) is False:
            continue:
        Add new_word into $result$
return $result$;

Function
is_cyc_reduced($word$)
Input
$word$: an even-dimensional integer vector representing a reduced word in F_2
Output
True or False

Initialize;
If $word\_length$ is two:
    return True

If $word[word\_length-1]$ is zero and $word[0] * word[word\_length-1] < 0$:
    return False

Else if $word[0]$ is zero and $word[1] \cdot word[word\_length-1] < 0$:
    return False
Else:
    return True

Function
classify_alpha(*words*)
Input
words : set of cyclically reduced words
Output
Dictionary(HashMap) consisting of object as below
Key : Trace Polynomial
Value : Set of corresponding vectors

Initialize $dict = \{\}$;
For *word* in *words*:
    $new\_\alpha = \alpha(word)$;
    If new_α is Not in key of dicts:
        dict$[new_\alpha] = \{word\}$;
    Else:
        add *word* in the non-empty set dict$[new_alpha]$;
return *dict*;

3.3.2 The sizes of $\sim$-equivalence classes

Prior to prove the main theorem, we need to define some notions for convenience. Firstly, let $\gamma_n(k)$ be the number of equivalence classes in $\mathcal{C}_n/_\sim$ whose size is k. Formally we can say

$$\gamma_n(k) = \#\{C \in \mathcal{C}_n/_\sim \mid \#(C) = k\}$$

Let $n > 0$ be given. We see that

1. $\sum_{k\geq 1} k \cdot \gamma_n(k) = \#\mathcal{C}_n$
2. $\sum_{k\geq 1} \gamma_n(k) = \#(\mathcal{C}_n/_\sim)$
3. $\gamma_n(k) > 0$ only for finitely many k.

4. $\gamma_n(2) = 2$

1, 2 and 3 are straightforward from definition. To see 4, one can use Lem3.2.2. If a class C in $\mathcal{C}_n/_\sim$ has the size 2, then c and δ defined in Lem3.2.2 must be one. So it follows that every word in C has the syllable length one. There are only 4 such words, which are a^n, a^{-n}, b^n and b^{-n}. The equivalence relation $\sim$ divides these words into two different classes. Finally remind the number of cyclically reduced words which is given by

$$\#(\mathcal{C}_n) = 3^n + 2 + (-1)^n$$

in Lem1.3.

3.3.3 The case when n is one of $1, 2, 3, 5$ and 7

Proof. **n=1**

$$\mathcal{C}_1/_\sim = \{[a], [b]\}$$

and clearly $[a]$ and $[b]$ have different trace polynomials.

□

Proof. **n=2**

$$\mathcal{C}_2/_\sim = \{[a^2], [b^2], [ab], [ab^{-1}]\}$$

and from Table 3.1, we see the trace polynomials of each class are mutually different.

□

Proof. **n=3**
from Table 3.2, we see that $\alpha(\mathcal{C}_3) = 6$, and it suffices to show that $\#\mathcal{C}_3/_\sim = 6$. Suppose on the contrary that

$$\#(\mathcal{C}_3/_\sim) = 6 + N \text{ , for } N \geq 1$$

First we count the number of cyclically reduced words with the word length 3. Remind that b_n denotes the number of all cyclically reduced words with the length n, that is, $b_n = \#(\mathcal{C}_n)$. Using the equation 1.3, we obtain

$$b_3 = 3^3 + 2 + (-1)^3 = 28$$

and we have

$$28 = 2\gamma_3(2) + 6\gamma_3(6) + 12\gamma_3(12) = 4 + 6\gamma_3(6) + 12\gamma_3(12)$$

and it gives

$$\gamma_3(6) + 2\gamma_3(12) = 4$$

From our assumption, we have $\gamma_3(2) + \gamma_3(6) + \gamma_3(12) = 6 + N$ and it tells us that

$$\gamma_3(6) + \gamma_3(12) > 4$$

Thus $\gamma_3(12)$ is negative it is a contradiction. □

Proof. **n=5**
From 3.4, we see that the cardinality of $\alpha(\mathcal{C}_5)$ is 20. It need to show that

$$\#(\mathcal{C}_5 /_{\sim}) \leq 20 \tag{3.3}$$

Suppose that $\#(\mathcal{C}_5 /_{\sim}) = 20+N$ for some positive N. Firstly, there are exactly $b_5 = 3^5 + 2 + (-1)^5 = 244$ cyclically reduced words with the word length 5. Moreover, we know that the cardinality of each $\sim$ - equivalence class is one of $5, 10$, and 20 because 5 is prime. Thus we have

$$\begin{aligned} &2\gamma_5(2) + 10\gamma_5(10) + 20\gamma_5(20) = 244, \text{ and} \\ &\gamma_5(2) + \gamma_5(10) + \gamma_5(20) = 20 + N \end{aligned}$$

and we know that $\gamma_5(2) = 2$. Thus we have

$$\gamma_5(20) = 4 - N < 4 \tag{3.4}$$

Consider four elements below in $\mathcal{C}_5$ (See table3.4 with indexes $19, 20, 21$, and 22)

$$\{a^2b^{-1}ab, a^2b^{-1}a^{-1}b, ab^2a^{-1}b^{-1}, aba^{-1}b^2\}$$

For these words, we have $\delta = 2$ by Lemma 3.2.5. Also, each Left-shift's of these words can not be a power of a subword of that and it means that $c = 1$

for all above four words. Thus the cardinality of classes of above four words is $2 \cdot 2 \cdot 5 = 20$. Finally it is straightforward that they are not $\sim$ - equivalent. It contradicts to equation (3.4). □

Proof. **n=7**
we see that the cardinality of $\alpha(\mathcal{C}_7)$ is 106 from Table 3.6. Similar to the case when $n = 5$, It suffices to show that

$$\#(\mathcal{C}_7 /_\sim) \leq 106 \tag{3.5}$$

Suppose not, that is, $\#(\mathcal{C}_7 /_\sim) = 106+N$ with a positive N. From equation1.3, we obtain $b_7 = 2188$. Since word length is prime and since $\gamma_7(2) = 2$, we get

$$\begin{aligned} &14 \cdot \gamma_7(14) + 28 \cdot \gamma_7(28) = 2184, \text{ and} \\ &\gamma_7(14) + \gamma_7(28) = 104 + N \end{aligned}$$

and it gives us $\gamma_7(28)$ is less than 52. Similar to the case when $n = 5$, we will find 52 many $w \in \mathcal{C}_7$ with $\#[w] = 28$ and any two words are not equivalent. See rows of indexes $55 - 106$ in Table 3.6.

All representative words in each row are not equivalent mutually because their trace polynomials are not the same. It remains to show that for each representative word w, $\#[w] = 28$, that is, δ in Lemma 3.2.2 is 2. Similar to the previous case, we see that all these words have at least one unique syllable up to absolute power of alphabets, and we have $\delta = 2$ by Lemma 3.2.5. Hence we found 52 many words of class-size 28, and it contradicts our assumption. □

3.3.4 The case when n is one of $4, 6$ and 8

For the remaining cases, the main idea is same. Since n is not prime, the candidates of k where $\gamma_n(k)$ can be positive are a little more. It needs to remove some candidates of them.

Proof. **n=4**
Similar to previous cases, we need to show that

$$\#(\mathcal{C}_4 /_\sim) \leq 13 = \#\alpha(\mathcal{C}_4)$$

From Lemma 3.2.2, if $\gamma_4(k) \neq 16$, then k must be one of $2, 4, 8$ and 16. Our first claim is $\gamma_4(16) = 0$. If $[w] = 16$ for some $w \in \mathcal{C}_16$, then by Lemma 3.2.2, we obtain that

$$w = (w')^2 \text{ for some } w' \in \mathcal{C}_2$$

Then it is straightforward to check that $w \in \mathcal{L}(\mathrm{Mir}\, w)$ and it means that $\#[w] = 2 \cdot 2 \cdot 2 = 8$. Thus there is no class of size 16 in $\mathcal{C}_4 /_\sim$.

From Table 3.3, As always, suppose that $\mathcal{C}_4 = 84 + N$ for some positive N and we obtain two equations below

$$4\gamma_4(4) + 8\gamma_4(8) = 84 - 2\gamma_4(2) = 80$$
$$\gamma_4(4) + \gamma_4(8) = 13 + N - \gamma_2(2) = 11 + N$$

and we get $\gamma_4(4) = 2 + N$. This time we compute $\gamma_4(4)$ directly. Let w be a word satisfying $\#[w] = 4$. From Lem3.2.2 again, we see that $w = (w')^2$ for some $w' \in \mathcal{C}_2$ and w can not be the fourth power of a cyclically reduced word. So $[w]$ must be one of

$$[abab] \text{ and } [ab^{-1}ab^{-1}].$$

We derived a contradiction. □

Proof. **n=6**
from Table 3.5, we know that it suffices to show that

$$\#(\mathcal{C}_6 /_\sim) \leq \#(\alpha(\mathcal{C}_6)) = 52 \tag{3.6}$$

From 1.3 we know that there are exactly 732 many cyclically reduced words. Compute all possible sizes of equivalence classes in $\mathcal{C}_6$ using Lem3.2.2. we see that if $\gamma_6(k)$ is not zero, then k is in $\{2, 4, 6, 8, 12, 24\}$. First, we can delete 8 from these candidate. Assume $\#C = 8$ for some $C \in \mathcal{C}_6 /_\sim$. Then from Lem3.2.2, we can find $w \in C$ so that

$$w = (w')^3 \text{ for some } W' \in \mathcal{C}_2$$
$$w' \notin \mathcal{L}(\mathrm{Mir}\, w') \cup \mathcal{L}(\mathrm{Mir}\, (w')^{-1})$$

and it is impossible. Because of the word length, such word w' must be in $\mathcal{L}(\mathrm{Mir}\, w')$ and it means that $\gamma_6(8) = 0$. Secondly, we claim that $\gamma_6(4) = 2$. If a class of an word has size 4, then w must be the cubic of an word w' in

$\mathcal{C}_2$. Otherwise it is the sixth power of an word, which implies that the class size is 2. In this case w' must be in $\mathcal{L}(\mathrm{Mir}\, w')$ by the same assertion above and it means that the size of equivalence class is 2, which is a contradiction. Furthermore, let us find all class in $\mathcal{C}_6/_\sim$ of the size six. If w is a representative of such class, then $w = (w')^2$ for some $w' \in \mathcal{C}_3$. Then all possibilities of $[w]$ is

$$\{[a^2b], [ab^2], [a^2b^{-1}], [ab^{-2}]\}$$

and it implies that $\gamma_6(4) = 4$. Suppose on the contrary on 3.6. Assume that $\#(\mathcal{C}_6/_\sim) = 52 + N$ for positive N. Then we have

$$\begin{aligned}\#(\mathcal{C}_6) &= 732\\ &= 2\gamma_6(2) + 4\gamma_6(4) + 6\gamma_6(6) + 8\gamma_6(8) + 12\gamma_6(12) + 24\gamma_6(24)\\ &= 2\cdot 2 + 4\cdot 2 + 6\cdot 4 + 8\cdot 0 + 12\gamma_6(12) + 24\gamma_6(24)\\ &= 36 + 12\gamma_6(12) + 24\gamma_6(24)\end{aligned}$$

and we have

$$\begin{aligned}&\gamma_6(12) + 2\gamma_6(24) = 58 \text{ and}\\ &\gamma_6(12) + \gamma_6(24) = \#(\mathcal{C}_6/_\sim) - (\gamma_6(2) + \gamma_6(4) + \gamma_6(6) + \gamma_6(8))\\ &= (52 + N) - (2 + 2 + 4 + 0)\\ &= 44 + N\end{aligned}$$

and it means that $\gamma_6(24)$ is less than 14. However, the representative words of indexes $39 - 52$ in the Table 3.5 generate different equivalence classes because all they have mutually different trace polynomials. Also, one can check that all equivalence classes generated by these words have size 24 by Lemma 3.2.5. □

Proof. **n=8**
From Lemma 3.2.2, we see that if $8k \neq 0$, then k is one of $2, 4, 8, 16$ and 32. Clearly, $\gamma_8(2) = 2$. Also, if $\#[w] = 4$ then w is a 4-th power of another word $w' \in \mathcal{C}_2$. $w' \in \mathcal{L}(\mathrm{Mir}\, w')$ because $|\, w' \,| = 2$ there are only two classes

$$[(ab)^4], [(ab^{-1})^4]$$

We saw that $\gamma_8(4) = 2$. Let us compute $\gamma_8(8)$. Let C be an equivalence class in $\mathcal{C}_8$ of size eight. In this case, c defined in Lem 3.2.2 is two or four. However, the former case cannot generate an equivalence class of size 8, because every word of the length two is always contained in the orbit of it's Mirror element. Thus we have $c = 4$ and it means that there there is $w \in C$ which which is the square of another word $w' \in \mathcal{C}_4$. Also, w' can not be any power of subwords of itself. Then we can find nine candidates of w'.

$$\{a^3b, a^3b^{-1}, ab^3, a^2b^2, a^2b^{-2}, abab^{-1}, aba^{-1}b^{-1}, ab^{-1}a^{-1}b^{-1}\}$$

Also, all w generated from above candidates of w' are not equivalent to each other and it means that

$$\gamma_8(8) = 9$$

Thus we have

$$\begin{aligned}\#(\mathcal{C}_8) &= \sum_{k\geq 1} k \cdot \gamma_8(k)\\ &= 2\cdot\gamma_8(2) + 4\cdot\gamma_8(4) + 8\cdot\gamma_8(8) + 16\cdot\gamma_8(16) + 32\cdot\gamma_8(32)\\ &= 84 + 16\cdot\gamma_8(16) + 32\cdot\gamma_8(32)\end{aligned}$$

and using 1.3, it follows that

$$\gamma_8(16) + 2\cdot\gamma_8(32) = 405$$

Suppose on the contrary that, α is not injective on our domain $\mathcal{C}_8/_\sim$. Then the number of corresponding trace polynomials must be less than the number of all equivalence classes of word length 8. We have

$$266 = \#\alpha(\mathcal{C}_8/_\sim) < \gamma_8(2) + \gamma_8(4) + \gamma_8(8) + \gamma_8(16) + \gamma_8(32)$$

and it follows that

$$\gamma_8(32) < 152 \tag{3.7}$$

We will find 152 such classes of size 32 manually. Remind that $\sim$ relation preserves cyclic syllable length of words. For each cyclic syllable length four, six, and eight, we find different corresponding classes.

Classes with the cyclic syllable length four

We enumerate all representatives of different classes. We divide all the cases into the number of alphabets a and b of absolute form. It is a reasonable division because these values are preserved by the relation of $\sim$.

- six a's and two b's
  Consider words of the absolute form a^5bab. In this case, a^5 is a unique absolute syllable. Consider a^5 fixed it's signature as "+". If this class is of size 32, then two b must have a different signature, and a can freely determine its signature. Considering the operation Mir, we can find 2 representative words $\{[a^5b^{-1}ab], [a^5b^{-1}a^{-1}b]\}$. Similarly, one can find two representatives whose absolute form is a^4ba^2b. We found 4 representatives in this case.

- five a's and three b's
  Consider words of the absolute form of a^4b^2ab. These two words also have a unique syllable in the absolute form. Also, c in Lem3.2.2 is two regardless of the signature of b, b^2 and a. We found 8 representatives. For the same reason, one can found 8 representatives of the absolute form $a^3b^2a^2b$. There are 16 representative words in this case.

- four a's and four b's
  Similar to the of the absolute form a^4b^2ab, we can find 8 representatives with absolute form a^3bab^3. Also, one can find 2 representative words of e absolute form $a^2ba^2b^3$ and $a^3b^2ab^2$, respectively. The process is the same as the case of the absolute form a^5bab. We found 12 words altogether in this case.

Using symmetry, we know that there are 4 and 16 number of representatives in the case where the absolute forms have "two a's and six b's" and "three a's and five b's" respectively. Totally, we found $4+4+16+16+12 = 52$ many different classes.

Classes with the cyclic syllable length six

Consider six possible absolute forms of representatives below

$$a^3babab,\ a^2ba^2bab,\ a^2b^2abab,\ a^2bab^2ab,\ a^2ba^2bab, \text{ and } ab^2ab^2ab$$

If two words have different absolute form in above, then they can not be equivalent. We will find a lower bound of the number of representatives by counting the number of representative words whose class has size 32, and absolute form is one of above.

- Absolute form a^3babab
  a^3 is the unique syllable of the absolute form. Consider it is fixed with signature "+". To make c in 3.2.2 have two. The representative can not have similarity with pivot considering the word as a cycle of syllables. Then at least one pair of a's and both ends b's should have a different signature. First, if the symmetry is broken in the pair of both ends b's, then the other three alphabets have any signatures. So we found $2{\cdot}2{\cdot}2 = 8$ different representatives. Also, if the symmetry is broken in not the pair of both ends b's but the pair of a's, then both ends b have the same signature, and the center b can have any signature. Thus we find $2 \cdot 2$ many different representatives. Hence, we found 12 in different classes in this case.

- Absolute form ab^3ababa
  Using the completely same process with the case of absolute form a^3babab, one can see that there are at least 12 representatives in this case.

- Absolute form a^2bab^2ab
  In this case, a^2 is a unique syllable in the absolute form. Also, it is symmetric considering it as a cycle with pivot a^2. Then we know that there is the same number of representative words with the case of absolute form a^3babab, which is 12.

- Absolute form a^2b^2abab
  In this case, a^2 is a unique syllable in the absolute form. Consider the absolute form as a cycle of syllables. Then the symmetry is broken in just left b and right b^2 of a^2. Thus if we fix the signature a^2 as positive, then all the other syllables have any signatures. Hence we found $2^5 = 32$ different representative words.

- Absolute form a^2ba^2bab
  Considering a as the pivot, one can apply the same process with the case of the absolute form a^3baba.

- Absolute form ab^2ab^2ab
  Considering b as the pivot, one can apply the same process with the case of the absolute form a^3baba.

Hence we found $12 + 12 + 12 + 12 + 12 + 32 = 92$ different equivalence classes in this case.

Classes with the cyclic syllable length eight

In this case we enumerate all representative of such classes. Because of the fixed cyclic syllable length, all these words have absolute form *abababab*.

$$w_1 = abababa^{-1}b^{-1} \ , \ w_2 = ababa^{-1}bab^{-1}$$
$$w_3 = ababa^{-1}b^{-1}ab^{-1} \ , \ w_4 = ababa^{-1}ba^{-1}b^{-1}$$
$$w_5 = abab^{-1}a^{-1}bab^{-1} \ , \ w_6 = ababa^{-1}b^{-1}a^{-1}b$$
$$w_7 = abab^{-1}a^{-1}b^{-1}ab^{-1} \ , \ w_8 = a^{-1}bab^{-1}ab^{-1}ab^{-1}$$

Using Lem3.2.2, it is straightforward that each class containing these elements has size 32. It remains to show that each class are not $\sim$ - equivalent. First, the set consisting of the number of positive signs and the number of negative signs is preserved by the relation and it devides above classes into three partitions, $\{[w_1], [w_2]\}$, $\{[w_3], [w_4], [w_5], [w_6]\}$, and $\{[w_7], [w_8]\}$. First, $[w_1]$ is not same to $[w_2]$ becuase w_1 and w_2 are not equivalent. Secondly, w_3 w_4 w_5 can not be equivalent to w_6 since only w_6 has five consecutive syllable with a same signature. Also, if we consider a word $w' = b^{-1}aba$, then each class $[w_3]$, $[w_4]$, and $[w_5]$ have only one representative word which starts from the word w', and all these words are different. We saw that all these representative is in different classes. Finally w_7 and w_8 are not equivalent to each other clearly. Because only w_7 has consecutive three syllables with the same signature.

Hence we found 152 many different equivalence classes of size 32 and it contradicts to 3.7. It proves the injectivity of α on $\mathcal{C}_8/_\sim$

□

3.3.5 The case when n is greater than or equal to 9

Proof. We will see a counterexample that breaks the injectivity of α on $\mathcal{C}_n/_\sim$. Let $k \geq 2$. Consider two words u_1 and u_2 in $\mathcal{C}_n$ given by

$$u_1 = ab^k a^{-1} b \text{ and}$$
$$u_2 = a^{-1} b^k ab.$$

Clearly, they are $\sim$ - equivalent, so their trace polynomials coincide. Consider an endomorphism f on F_2 defined by

$$f(a) = a \text{ and } f(b) = bab^{-1}.$$

Then two words w_1 and w_2 defined by

$$w_1 = f(u_1) = aba^kb^{-1}a^{-1}bab^{-1} \text{ and}$$
$$w_2 = f(u_2) = a^{-1}ba^kb^{-1}abab^{-1}.$$

become our desired pair. Firstly, $\alpha(w_1) = \alpha(w_2)$ is straightforward from Lemma 1.3.4. Let us check they are not $\sim$ - equivalent. Suppose on the contrary that

$${w_1}' = (\mathrm{Lsh})^2(w_1) = a^kb^{-1}a^{-1}bab^{-1}ab, \text{ and}$$
$${w_2}' = (\mathrm{Lsh})^2(w_2) = a^kb^{-1}abab^{-1}a^{-1}b.$$

are $\sim$-equivalent to each other. The condition $k \geq 2$ and the fact that both words end with the alphabet b guarantees that each alphabet behind the subword a^kb^{-1} in ${w_1}'$ and ${w_2}'$ must coincide. But they are a^{-1} in w_1' and a in w_2'. It completes our proof. $\square$

Conclusion

This thesis deals with the trace polynomials of words in the free group of rank two. We implemented an algorithm from the formula presented by Jorgensen to compute trace polynomials of words. Using the algorithm, we classified all cyclically reduced words with the lengths less than nine via their trace polynomials.

Using the computational results, we proved that all two words in $\mathcal{C}$ with the same trace polynomials are $\sim$-equivalent when the word lengths are less than nine. For the word lengths greater than eight, we also found the pairs of cyclically reduced words of those lengths, which have the same trace polynomials but are not $\sim$-equivalent to each other.

Wang presented a conjecture that the $\sim_w$-equivalence is an equivalent condition for two words in $\mathcal{C}$ to have the same trace polynomials. Our results proved the validity of this conjecture for the word lengths less than nine. However, the conjecture is still open. For further research, one can look into this conjecture by checking whether two words in $\mathcal{C}$ of the same trace polynomials are endomorphic images of $\sim$-equivalent two words, starting from the word length nine.

List of Tables

Table 3.1: Trace polynomials of the word length 2

Index	**representative** w	$\#[w]$	**trace polynomial**
1	a^2	2	$x^2 - 2$
2	b^2	2	$y^2 - 2$
3	ab^{-1}	4	$xy - z$
4	ab	4	z

Table 3.2: Trace polynomials of the word length 3

Index	**representative** w	$\#[w]$	**trace polynomial**
1	a^3	2	$x^3 - 3x$
2	b^3	2	$y^3 - 3y$
3	$a^{-1}ba^{-1}$	6	$x^2y - xz - y$
4	aba	6	$xz - y$
5	ab^2	6	$-x + yz$
6	ab^{-2}	6	$xy^2 - x - yz$

Table 3.3: Trace polynomials of the word length 4

Index	**representative** w	$\#[w]$	**trace polynomial**
1	a^4	2	$x^4 - 4x^2 + 2$
2	b^4	2	$y^4 - 4y^2 + 2$
3	$abab$	4	$z^2 - 2$
4	$ab^{-1}ab^{-1}$	4	$x^2y^2 - 2xyz + z^2 - 2$
5	$a^{-2}ba^{-1}$	8	$x^3y - x^2z - 2xy + z$
6	a^2ba	8	$x^2z - xy - z$
7	ab^2a	8	$-x^2 + xyz - y^2 + 2$
8	$a^{-1}b^2a^{-1}$	8	$x^2y^2 - x^2 - xyz - y^2 + 2$
9	ab^3	8	$-xy + y^2z - z$
10	$a^{-1}ba^{-1}b^{-1}$	8	$xyz - y^2 - z^2 + 2$
11	$a^{-1}bab^{-1}$	8	$x^2 - xyz + y^2 + z^2 - 2$
12	$a^{-1}bab$	8	$-x^2 + xyz - z^2 + 2$
13	ab^{-3}	8	$xy^3 - 2xy - y^2z + z$

Table 3.4: Trace polynomials of the word length 5

Index	**representative** w	$\#[w]$	**trace polynomial**
1	a^5	2	$x^5 - 5x^3 + 5x$
2	b^5	2	$y^5 - 5y^3 + 5y$
3	$a^{-3}ba^{-1}$	10	$x^4y - x^3z - 3x^2y + 2xz + y$
4	$a^{-3}b^{-1}a^{-1}$	10	$x^3z - x^2y - 2xz + y$
5	$a^{-2}b^{-2}a^{-1}$	10	$-x^3 + x^2yz - xy^2 + 3x - yz$
6	$a^{-2}b^2a^{-1}$	10	$x^3y^2 - x^3 - x^2yz - 2xy^2 + 3x + yz$
7	$a^{-1}b^{-3}a^{-1}$	10	$-x^2y + xy^2z - xz - y^3 + 3y$
8	$a^{-1}b^{-1}a^{-2}b^{-1}$	10	$xz^2 - x - yz$
9	$a^{-1}b^{-1}ab^{-1}a^{-1}$	10	$-x^3 + x^2yz - xy^2 - xz^2 + 3x + yz$
10	$a^{-1}ba^{-2}b$	10	$x^3y^2 - 2x^2yz - xy^2 + xz^2 - x + yz$
11	$a^{-1}b^{-1}a^2b^{-1}$	10	$-x^3 + x^2yz - xz^2 + 3x - yz$
12	$a^{-1}b^3a^{-1}$	10	$x^2y^3 - 2x^2y - xy^2z + xz - y^3 + 3y$
13	ab^4	10	$-xy^2 + x + y^3z - 2yz$
14	$a^{-1}ba^{-1}b^{-2}$	10	$-x^2y + xy^2z + xz - y^3 - yz^2 + 3y$
15	$a^{-1}b^{-1}a^{-1}b^{-2}$	10	$-xz + yz^2 - y$
16	$a^{-1}b^2a^{-1}b^{-1}$	10	$xy^2z - xz - y^3 - yz^2 + 3y$
17	$a^{-1}b^2a^{-1}b$	10	$x^2y^3 - x^2y - 2xy^2z + xz + yz^2 - y$
18	ab^{-4}	10	$xy^4 - 3xy^2 + x - y^3z + 2yz$
19	$a^2b^{-1}ab$	20	$x^2yz - xy^2 - xz^2 + x$
20	$a^{-2}bab^{-1}$	20	$x^3 - x^2yz + xy^2 + xz^2 - 3x$
21	$a^{-1}b^{-1}ab^2$	20	$x^2y - xy^2z + y^3 + yz^2 - 3y$
22	$aba^{-1}b^2$	20	$-x^2y + xy^2z - yz^2 + y$

Table 3.5: Trace polynomials of the word length 6

Index	representative w	$\#[w]$	trace polynomial
1	a^6	2	$x^6 - 6x^4 + 9x^2 - 2$
2	b^6	2	$y^6 - 6y^4 + 9y^2 - 2$
3	$ababab$	4	$z^3 - 3z$
4	$ab^{-1}ab^{-1}ab^{-1}$	4	$x^3y^3 - 3x^2y^2z + 3xyz^2 - 3xy - z^3 + 3z$
5	$a^{-1}b^{-1}a^{-2}b^{-1}a^{-1}$	6	$x^2z^2 - 2xyz + y^2 - 2$
6	$a^{-1}ba^{-2}ba^{-1}$	6	$x^4y^2 - 2x^3yz - 2x^2y^2 + x^2z^2 + 2xyz + y^2 - 2$
7	ab^2ab^2	6	$x^2 - 2xyz + y^2z^2 - 2$
8	$ab^{-2}ab^{-2}$	6	$x^2y^4 - 2x^2y^2 + x^2 - 2xy^3z + 2xyz + y^2z^2 - 2$
9	$a^{-4}ba^{-1}$	12	$x^5y - x^4z - 4x^3y + 3x^2z + 3xy - z$
10	$a^{-4}b^{-1}a^{-1}$	12	$x^4z - x^3y - 3x^2z + 2xy + z$
11	$a^{-3}b^{-2}a^{-1}$	12	$-x^4 + x^3yz - x^2y^2 + 4x^2 - 2xyz + y^2 - 2$
12	$a^{-3}b^2a^{-1}$	12	$x^4y^2 - x^4 - x^3yz - 3x^2y^2 + 4x^2 + 2xyz + y^2 - 2$
13	$a^{-2}b^{-3}a^{-1}$	12	$-x^3y + x^2y^2z - x^2z - xy^3 + 4xy - y^2z + z$
14	$a^{-2}b^{-1}a^{-1}b^{-1}a^{-1}$	12	$x^2z^2 - x^2 - xyz - z^2 + 2$
15	$a^{-2}b^{-1}ab^{-1}a^{-1}$	12	$-x^4 + x^3yz - x^2y^2 - x^2z^2 + 4x^2 + z^2 - 2$
16	$a^{-2}ba^{-1}ba^{-1}$	12	$x^4y^2 - 2x^3yz - 2x^2y^2 + x^2z^2 - x^2 + 3xyz - z^2 + 2$
17	$a^{-2}baba^{-1}$	12	$-x^4 + x^3yz - x^2z^2 + 4x^2 - 2xyz + z^2 - 2$
18	$a^{-2}b^3a^{-1}$	12	$x^3y^3 - 2x^3y - x^2y^2z + x^2z - 2xy^3 + 5xy + y^2z - z$
19	$a^{-1}b^{-4}a^{-1}$	12	$-x^2y^2 + x^2 + xy^3z - 2xyz - y^4 + 4y^2 - 2$
20	$a^{-2}ba^{-2}b^{-1}$	12	$x^3yz - x^2y^2 - x^2z^2 + 2$
			Continued on next page

Trace polynomials of the word length 6 – continued from previous page

Index	**representative** w	$\#[w]$	**trace polynomial**
21	$a^{-2}ba^2b^{-1}$	12	$x^4 - x^3yz + x^2y^2 + x^2z^2 - 4x^2 + 2$
22	$a^{-2}ba^2b$	12	$-x^4 + x^3yz - x^2y^2 - x^2z^2 + 4x^2 + y^2 - 2$
23	$a^{-1}b^4a^{-1}$	12	$x^2y^4 - 3x^2y^2 + x^2 - xy^3z + 2xyz - y^4 + 4y^2 - 2$
24	ab^5	12	$-xy^3 + 2xy + y^4z - 3y^2z + z$
25	$a^{-1}ba^{-1}b^{-3}$	12	$-x^2y^2 + xy^3z - y^4 - y^2z^2 + 4y^2 + z^2 - 2$
26	$a^{-1}b^{-1}a^{-1}b^{-3}$	12	$-xyz + y^2z^2 - y^2 - z^2 + 2$
27	$a^{-1}b^2a^{-1}b^{-2}$	12	$-x^2y^2 + x^2 + xy^3z - y^4 - y^2z^2 + 4y^2 - 2$
28	$a^{-1}b^2ab^2$	12	$-x^2y^2 + xy^3z - y^2z^2 + 2$
29	$a^{-1}b^2ab^{-2}$	12	$x^2y^2 - xy^3z + y^4 + y^2z^2 - 4y^2 + 2$
30	$a^{-1}b^{-1}a^{-1}ba^{-1}b^{-1}$	12	$xyz^2 - xy - y^2z - z^3 + 3z$
31	$a^{-1}b^{-1}ab^{-1}a^{-1}b^{-1}$	12	$-x^2z + xyz^2 - xy - z^3 + 3z$
32	$a^{-1}ba^{-1}b^{-1}a^{-1}b$	12	$x^2y^2z - xy^3 - 2xyz^2 + 2xy + y^2z + z^3 - 3z$
33	$a^{-1}b^{-1}abab^{-1}$	12	$-x^2z + xyz^2 + xy - y^2z - z^3 + 3z$
34	$a^{-1}b^3a^{-1}b^{-1}$	12	$xy^3z - 2xyz - y^4 - y^2z^2 + 4y^2 + z^2 - 2$
35	$a^{-1}ba^{-1}b^{-1}ab^{-1}$	12	$-x^3y + x^2y^2z + x^2z - xy^3 - 2xyz^2 + 4xy + y^2z + z^3 - 3z$
36	$a^{-1}ba^{-1}bab$	12	$-x^3y + x^2y^2z + x^2z - 2xyz^2 + 2xy + z^3 - 3z$
37	$a^{-1}b^3a^{-1}b$	12	$x^2y^4 - 2x^2y^2 - 2xy^3z + 3xyz + y^2z^2 - y^2 - z^2 + 2$
38	ab^{-5}	12	$xy^5 - 4xy^3 + 3xy - y^4z + 3y^2z - z$
39	$a^{-3}ba^{-1}b^{-1}$	24	$x^3yz - x^2y^2 - x^2z^2 + x^2 - xyz + y^2 + z^2 - 2$
40	$a^{-3}bab^{-1}$	24	$x^4 - x^3yz + x^2y^2 + x^2z^2 - 4x^2 + xyz - y^2 - z^2 + 2$
41	$a^{-2}ba^{-1}b^{-2}$	24	$-x^3y + x^2y^2z + x^2z - xy^3 - xyz^2 + 3xy - z$

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Trace polynomials of the word length 6 – continued from previous page

Index	**representative** w	$\#[w]$	**trace polynomial**
42	$a^{-2}b^{-1}a^{-1}b^{-2}$	24	$-x^2z+xyz^2-y^2z+z$
43	$a^{-2}bab^{-2}$	24	$x^3y-x^2y^2z+xy^3+xyz^2-4xy+z$
44	$a^{-2}b^{-1}ab^{-2}$	24	$-x^3y+x^2y^2z-xy^3-xyz^2+3xy+y^2z-z$
45	$a^{-2}b^2a^{-1}b^{-1}$	24	$x^2y^2z-x^2z-xy^3-xyz^2+2xy+z$
46	$a^{-2}b^2ab^{-1}$	24	$x^3y-x^2y^2z+xy^3+xyz^2-3xy-z$
47	$a^{-2}b^2a^{-1}b$	24	$x^3y^3-x^3y-2x^2y^2z+x^2z-xy^3+xyz^2+xy+y^2z-z$
48	$a^{-2}b^2ab$	24	$-x^3y+x^2y^2z-xyz^2+2xy-y^2z+z$
49	$a^{-1}b^{-1}ab^3$	24	$x^2y^2-x^2-xy^3z+xyz+y^4+y^2z^2-4y^2-z^2+2$
50	$a^{-1}b^{-1}ab^{-3}$	24	$-x^2y^2+x^2+xy^3z-xyz-y^2z^2+y^2+z^2-2$
51	$a^{-1}b^{-1}a^{-1}bab^{-1}$	24	$x^2z-xyz^2+y^2z+z^3-3z$
52	$a^{-1}b^{-1}aba^{-1}b$	24	$x^3y-x^2y^2z-x^2z+xy^3+2xyz^2-3xy-y^2z-z^3+3z$

Table 3.6: Trace polynomials of the word length 7

Index	**representative** w	$\#[w]$	**trace polynomial**
1	a^7	2	$x^7 - 7x^5 + 14x^3 - 7x$
2	b^7	2	$y^7 - 7y^5 + 14y^3 - 7y$
3	$a^{-5}ba^{-1}$	14	$x^6y - x^5z - 5x^4y + 4x^3z + 6x^2y - 3xz - y$
4	$a^{-5}b^{-1}a^{-1}$	14	$x^5z - x^4y - 4x^3z + 3x^2y + 3xz - y$
5	$a^{-4}b^{-2}a^{-1}$	14	$-x^5 + x^4yz - x^3y^2 + 5x^3 - 3x^2yz + 2xy^2 - 5x + yz$
6	$a^{-4}b^2a^{-1}$	14	$x^5y^2 - x^5 - x^4yz - 4x^3y^2 + 5x^3 + 3x^2yz + 3xy^2 - 5x - yz$
7	$a^{-3}b^{-3}a^{-1}$	14	$-x^4y + x^3y^2z - x^3z - x^2y^3 + 5x^2y - 2xy^2z + 2xz + y^3 - 3y$
8	$a^{-3}b^{-1}a^{-1}b^{-1}a^{-1}$	14	$x^3z^2 - x^3 - x^2yz - 2xz^2 + 3x + yz$
9	$a^{-3}b^{-1}ab^{-1}a^{-1}$	14	$-x^5 + x^4yz - x^3y^2 - x^3z^2 + 5x^3 - x^2yz + xy^2 + 2xz^2 - 5x - yz$
10	$a^{-3}ba^{-1}ba^{-1}$	14	$x^5y^2 - 2x^4yz - 3x^3y^2 + x^3z^2 - x^3 + 5x^2yz + xy^2 - 2xz^2 + 3x - yz$
11	$a^{-3}baba^{-1}$	14	$-x^5 + x^4yz - x^3z^2 + 5x^3 - 3x^2yz + 2xz^2 - 5x + yz$
12	$a^{-3}b^3a^{-1}$	14	$x^4y^3 - 2x^4y - x^3y^2z + x^3z - 3x^2y^3 + 7x^2y + 2xy^2z - 2xz + y^3 - 3y$
13	$a^{-2}b^{-4}a^{-1}$	14	$-x^3y^2 + x^3 + x^2y^3z - 2x^2yz - xy^4 + 5xy^2 - 3x - y^3z + 2yz$
14	$a^{-2}b^{-1}a^{-3}b^{-1}$	14	$x^3z^2 - 2x^2yz + xy^2 - xz^2 - x + yz$
15	$a^{-2}b^{-1}a^2b^{-1}a^{-1}$	14	$-x^5 + x^4yz - x^3y^2 - x^3z^2 + 5x^3 - x^2yz + xy^2 + xz^2 - 5x + yz$
16	$a^{-2}ba^{-3}b$	14	$x^5y^2 - 2x^4yz - 3x^3y^2 + x^3z^2 + 4x^2yz + 2xy^2 - xz^2 - x - yz$
17	$a^{-2}b^{-1}a^3b^{-1}$	14	$-x^5 + x^4yz - x^3y^2 - x^3z^2 + 5x^3 - x^2yz + 2xy^2 + xz^2 - 5x - yz$
18	$a^{-2}b^4a^{-1}$	14	$x^3y^4 - 3x^3y^2 + x^3 - x^2y^3z + 2x^2yz - 2xy^4 + 7xy^2 - 3x + y^3z - 2yz$

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Trace polynomials of the word length 7 – continued from previous page

Index	representative w	$\#[w]$	trace polynomial
19	$a^{-1}b^{-5}a^{-1}$	14	$-x^2y^3 + 2x^2y + xy^4z - 3xy^2z + xz - y^5 + 5y^3 - 5y$
20	$a^{-2}ba^{-2}b^{-2}$	14	$-x^4y+x^3y^2z+x^3z-x^2y^3-x^2yz^2+3x^2y - 2xz + y$
21	$a^{-2}b^{-1}a^{-2}b^{-2}$	14	$-x^3z+x^2yz^2+x^2y-2xy^2z+2xz+y^3 - 3y$
22	$a^{-1}b^{-2}a^{-2}b^{-2}$	14	$x^3 - 2x^2yz + xy^2z^2 + xy^2 - 3x - y^3z + 2yz$
23	$a^{-1}b^{-2}ab^{-2}a^{-1}$	14	$-x^3y^2 + x^2y^3z - xy^4 - xy^2z^2 + 3xy^2 + x + y^3z - 2yz$
24	$a^{-2}b^2a^{-2}b^{-1}$	14	$x^3y^2z-x^3z-x^2y^3-x^2yz^2+x^2y+2xz + y$
25	$a^{-1}b^{-1}a^{-2}b^{-1}a^{-1}b^{-1}$	14	$xz^3 - 2xz - yz^2 + y$
26	$a^{-1}b^{-1}a^{-2}b^{-1}a^{-1}b$	14	$x^2yz^2 - 2xy^2z - xz^3 + 2xz + y^3 + yz^2 - 3y$
27	$a^{-1}b^{-1}ab^{-1}ab^{-1}a^{-1}$	14	$-x^4y + x^3y^2z + x^3z - x^2y^3 - 2x^2yz^2+3x^2y+2xy^2z+xz^3-4xz-yz^2 + y$
28	$a^{-1}b^{-1}a^{-1}ba^2b$	14	$-x^3z+x^2yz^2+x^2y-2xy^2z-xz^3+4xz + y^3 + yz^2 - 3y$
29	$a^{-2}b^2a^{-2}b$	14	$x^4y^3-x^4y-2x^3y^2z+x^3z-2x^2y^3+x^2yz^2+3x^2y+2xy^2z-2xz+y^3-3y$
30	$a^{-1}b^{-1}a^{-1}ba^{-2}b$	14	$x^3y^2z - x^2y^3 - 2x^2yz^2 + 2x^2y + xz^3 - 2xz + y^3 + yz^2 - 3y$
31	$a^{-1}ba^{-2}ba^{-1}b$	14	$x^4y^3 - 3x^3y^2z - x^2y^3 + 3x^2yz^2 - 2x^2y+2xy^2z-xz^3+2xz-yz^2+y$
32	$a^{-1}b^{-1}a^2b^{-1}a^{-1}b$	14	$-x^4y + x^3y^2z + x^3z - x^2y^3 - 2x^2yz^2 + 5x^2y + xz^3 - 4xz + y^3 + yz^2 - 3y$
33	$a^{-1}b^{-1}a^{-1}b^{-1}a^2b^{-1}$	14	$-x^3z+x^2yz^2-x^2y-xz^3+4xz-yz^2 + y$

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Trace polynomials of the word length 7 – continued from previous page

Index	representative w	$\#[w]$	trace polynomial
34	$a^{-1}b^2a^{-2}b^2$	14	$x^3y^4 - 2x^3y^2 + x^3 - 2x^2y^3z + 2x^2yz - xy^4 + xy^2z^2 + 3xy^2 - 3x + y^3z - 2yz$
35	$a^{-1}b^{-2}a^2b^{-2}$	14	$-x^3y^2 + x^2y^3z - xy^2z^2 + xy^2 + x - y^3z + 2yz$
36	$a^{-1}b^5a^{-1}$	14	$x^2y^5 - 4x^2y^3 + 3x^2y - xy^4z + 3xy^2z - xz - y^5 + 5y^3 - 5y$
37	ab^6	14	$-xy^4 + 3xy^2 - x + y^5z - 4y^3z + 3yz$
38	$a^{-1}ba^{-1}b^{-4}$	14	$-x^2y^3 + x^2y + xy^4z - xy^2z - xz - y^5 - y^3z^2 + 5y^3 + 2yz^2 - 5y$
39	$a^{-1}b^{-1}a^{-1}b^{-4}$	14	$-xy^2z + xz + y^3z^2 - y^3 - 2yz^2 + 3y$
40	$a^{-1}b^{-2}a^{-1}b^{-3}$	14	$x^2y - 2xy^2z + xz + y^3z^2 - yz^2 - y$
41	$a^{-1}b^2a^{-1}b^{-3}$	14	$-x^2y^3 + x^2y + xy^4z - xy^2z + xz - y^5 - y^3z^2 + 5y^3 + yz^2 - 5y$
42	$a^{-1}b^{-1}a^{-1}b^{-2}a^{-1}b^{-1}$	14	$-xz^2 + x + yz^3 - 2yz$
43	$a^{-1}b^{-1}ab^{-1}a^{-1}b^{-2}$	14	$x^3 - 2x^2yz + xy^2z^2 + xz^2 - 3x - yz^3 + 2yz$
44	$a^{-1}ba^{-1}b^{-2}a^{-1}b$	14	$-x^3y^2 + x^2y^3z + 2x^2yz - xy^4 - 2xy^2z^2 + 3xy^2 - xz^2 + x + y^3z + yz^3 - 4yz$
47	$a^{-1}b^{-1}a^{-1}b^2a^{-1}b^{-1}$	14	$xy^2z^2 - xy^2 - xz^2 + x - y^3z - yz^3 + 4yz$
48	$a^{-1}b^4a^{-1}b^{-1}$	14	$xy^4z - 3xy^2z + xz - y^5 - y^3z^2 + 5y^3 + 2yz^2 - 5y$
49	$a^{-1}baba^{-1}b^2$	14	$-x^3y^2 + x^3 + x^2y^3z - 2xy^2z^2 + 2xy^2 + xz^2 - 3x + yz^3 - 2yz$
50	$a^{-1}bab^{-2}ab$	14	$-x^3y^2 + x^3 + x^2y^3z - xy^4 - 2xy^2z^2 + 5xy^2 + xz^2 - 3x + y^3z + yz^3 - 4yz$
51	$a^{-1}ba^{-1}b^2a^{-1}b$	14	$x^3y^4 - x^3y^2 - 3x^2y^3z + 2x^2yz + 3xy^2z^2 - 2xy^2 - xz^2 + x - yz^3 + 2yz$
52	$a^{-1}b^4a^{-1}b$	14	$x^2y^5 - 3x^2y^3 + x^2y - 2xy^4z + 5xy^2z - xz + y^3z^2 - y^3 - 2yz^2 + 3y$

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Trace polynomials of the word length 7 – continued from previous page

Index	**representative** w	$\#[w]$	**trace polynomial**
53	$a^{-1}b^3a^{-1}b^2$	14	$x^2y^5 - 3x^2y^3 + 2x^2y - 2xy^4z + 4xy^2z - xz + y^3z^2 - yz^2 - y$
54	ab^{-6}	14	$xy^6 - 5xy^4 + 6xy^2 - x - y^5z + 4y^3z - 3yz$
55	$a^{-4}ba^{-1}b^{-1}$	28	$x^4yz - x^3y^2 - x^3z^2 + x^3 - 2x^2yz + 2xy^2 + 2xz^2 - 3x$
56	$a^{-4}bab^{-1}$	28	$x^5 - x^4yz + x^3y^2 + x^3z^2 - 5x^3 + 2x^2yz - 2xy^2 - 2xz^2 + 5x$
57	$a^{-3}ba^{-1}b^{-2}$	28	$-x^4y + x^3y^2z + x^3z - x^2y^3 - x^2yz^2 + 4x^2y - xy^2z - 2xz + y^3 + yz^2 - 3y$
58	$a^{-3}b^{-1}a^{-1}b^{-2}$	28	$-x^3z + x^2yz^2 - xy^2z + 2xz - yz^2 + y$
59	$a^{-3}bab^{-2}$	28	$x^4y - x^3y^2z + x^2y^3 + x^2yz^2 - 5x^2y + xy^2z + xz - y^3 - yz^2 + 3y$
60	$a^{-3}b^{-1}ab^{-2}$	28	$-x^4y + x^3y^2z - x^2y^3 - x^2yz^2 + 4x^2y - xz + yz^2 - y$
61	$a^{-3}ba^{-2}b^{-1}$	28	$x^4yz - x^3y^2 - x^3z^2 - x^2yz + xy^2 + xz^2 + x$
62	$a^{-3}b^2a^{-1}b^{-1}$	28	$x^3y^2z - x^3z - x^2y^3 - x^2yz^2 + 2x^2y - xy^2z + 2xz + y^3 + yz^2 - 3y$
63	$a^{-3}b^2ab^{-1}$	28	$x^4y - x^3y^2z + x^2y^3 + x^2yz^2 - 4x^2y + xy^2z - xz - y^3 - yz^2 + 3y$
64	$a^{-3}ba^2b^{-1}$	28	$x^5 - x^4yz + x^3y^2 + x^3z^2 - 5x^3 + x^2yz - xy^2 - xz^2 + 5x$
65	$a^{-3}b^2a^{-1}b$	28	$x^4y^3 - x^4y - 2x^3y^2z + x^3z - 2x^2y^3 + x^2yz^2 + 2x^2y + 3xy^2z - 2xz - yz^2 + y$
66	$a^{-3}b^2ab$	28	$-x^4y + x^3y^2z - x^2yz^2 + 3x^2y - 2xy^2z + xz + yz^2 - y$
67	$a^{-2}ba^{-1}b^{-3}$	28	$-x^3y^2 + x^2y^3z - xy^4 - xy^2z^2 + 4xy^2 + xz^2 - x - yz$
68	$a^{-2}b^{-1}a^{-1}b^{-3}$	28	$-x^2yz + xy^2z^2 - xz^2 + x - y^3z + 2yz$
69	$a^{-2}bab^{-3}$	28	$x^3y^2 - x^3 - x^2y^3z + x^2yz + xy^4 + xy^2z^2 - 5xy^2 - xz^2 + 3x + yz$

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Trace polynomials of the word length 7 – continued from previous page

Index	**representative** w	$\#[w]$	**trace polynomial**
70	$a^{-2}b^{-1}ab^{-3}$	28	$-x^3y^2+x^3+x^2y^3z-x^2yz-xy^4-xy^2z^2+4xy^2+xz^2-3x+y^3z-2yz$
71	$a^{-2}b^2a^{-1}b^{-2}$	28	$-x^3y^2+x^3+x^2y^3z-xy^4-xy^2z^2+4xy^2-3x$
72	$a^{-2}b^2ab^{-2}$	28	$x^3y^2-x^2y^3z+xy^4+xy^2z^2-4xy^2+x$
73	$a^{-2}b^{-1}a^2b^2$	28	$x^4y-x^3y^2z+x^2y^3+x^2yz^2-4x^2y+y$
74	$a^{-2}b^{-1}a^2b^{-2}$	28	$-x^4y+x^3y^2z-x^2y^3-x^2yz^2+4x^2y+y^3-3y$
75	$a^{-2}ba^{-1}b^{-1}a^{-1}b^{-1}$	28	$x^2yz^2-x^2y-xy^2z-xz^3+2xz+y$
76	$a^{-2}bab^{-1}a^{-1}b^{-1}$	28	$x^3z-x^2yz^2+xy^2z+xz^3-3xz-y$
77	$a^{-2}b^{-1}ab^{-1}a^{-1}b^{-1}$	28	$-x^3z+x^2yz^2-xy^2z-xz^3+3xz+yz^2-y$
78	$a^{-2}ba^{-1}ba^{-1}b^{-1}$	28	$x^3y^2z-x^2y^3-2x^2yz^2+x^2y+xy^2z+xz^3-2xz+y$
79	$a^{-2}baba^{-1}b^{-1}$	28	$-x^3z+x^2yz^2+x^2y-xy^2z-xz^3+3xz-y$
80	$a^{-2}b^{-1}aba^{-1}b^{-1}$	28	$x^3z-x^2yz^2-x^2y+2xy^2z+xz^3-3xz-y^3-yz^2+3y$
81	$a^{-2}b^3a^{-1}b^{-1}$	28	$x^2y^3z-2x^2yz-xy^4-xy^2z^2+3xy^2+xz^2-x+yz$
82	$a^{-2}ba^{-1}b^{-1}ab^{-1}$	28	$-x^4y+x^3y^2z+x^3z-x^2y^3-2x^2yz^2+4x^2y+xy^2z+xz^3-3xz-y$
83	$a^{-2}bab^{-1}ab^{-1}$	28	$x^4y-x^3y^2z-x^3z+x^2y^3+2x^2yz^2-4x^2y-xy^2z-xz^3+4xz+y$
84	$a^{-2}ba^{-1}bab^{-1}$	28	$x^4y-x^3y^2z-x^3z+x^2y^3+2x^2yz^2-3x^2y-xy^2z-xz^3+3xz-y$
85	$a^{-2}babab^{-1}$	28	$x^3z-x^2yz^2+xy^2z+xz^3-4xz+y$
86	$a^{-2}b^3ab^{-1}$	28	$x^3y^2-x^3-x^2y^3z+x^2yz+xy^4+xy^2z^2-4xy^2-xz^2+3x-yz$

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Trace polynomials of the word length 7 – continued from previous page

Index	**representative** w	$\#[w]$	**trace polynomial**
87	$a^{-2}bab^{-1}a^{-1}b$	28	$x^4y-x^3y^2z-x^3z+x^2y^3+2x^2yz^2-4x^2y-xz^3+3xz-y^3-yz^2+3y$
88	$a^{-2}baba^{-1}b$	28	$-x^4y+x^3y^2z+x^3z-2x^2yz^2+3x^2y-xy^2z+xz^3-3xz+yz^2-y$
89	$a^{-2}b^3a^{-1}b$	28	$x^3y^4-2x^3y^2-2x^2y^3z+3x^2yz-xy^4+xy^2z^2+2xy^2-xz^2+x+y^3z-2yz$
90	$a^{-2}b^3ab$	28	$-x^3y^2+x^3+x^2y^3z-x^2yz-xy^2z^2+2xy^2+xz^2-3x-y^3z+2yz$
91	$a^{-1}b^{-1}ab^4$	28	$x^2y^3-2x^2y-xy^4z+2xy^2z+y^5+y^3z^2-5y^3-2yz^2+5y$
92	$a^{-1}b^{-1}ab^{-4}$	28	$-x^2y^3+2x^2y+xy^4z-2xy^2z-y^3z^2+y^3+2yz^2-3y$
93	$a^{-1}b^{-2}ab^{-3}$	28	$-x^2y^3+x^2y+xy^4z-xy^2z-y^3z^2+yz^2+y$
94	$a^{-1}b^{-2}ab^3$	28	$x^2y^3-x^2y-xy^4z+xy^2z+y^5+y^3z^2-5y^3-yz^2+5y$
95	$a^{-1}b^{-2}a^{-1}ba^{-1}b^{-1}$	28	$-x^2yz+xy^2z^2+xz^2-x-y^3z-yz^3+3yz$
96	$a^{-1}b^{-2}a^{-1}bab^{-1}$	28	$-x^3+2x^2yz-xy^2z^2-xy^2-xz^2+3x+y^3z+yz^3-3yz$
97	$a^{-1}b^{-1}ab^{-2}a^{-1}b$	28	$-x^3y^2+x^2y^3z+x^2yz-xy^4-2xy^2z^2+4xy^2-x+y^3z+yz^3-3yz$
98	$a^{-1}b^{-1}a^{-1}b^{-2}ab^{-1}$	28	$-x^2yz+xy^2z^2-xy^2+x-yz^3+2yz$
99	$a^{-1}b^{-1}ab^2a^{-1}b$	28	$x^3y^2-x^2y^3z-x^2yz+xy^4+2xy^2z^2-3xy^2-x-y^3z-yz^3+3yz$
100	$a^{-1}b^{-1}ab^{-1}ab^{-2}$	28	$-x^3y^2+x^2y^3z+x^2yz-2xy^2z^2+xy^2+x+yz^3-2yz$
101	$a^{-1}b^{-1}ab^{-1}ab^2$	28	$x^3y^2-x^2y^3z-x^2yz+xy^4+2xy^2z^2-4xy^2+x-y^3z-yz^3+4yz$
102	$a^{-1}b^{-1}a^{-1}b^{-2}ab$	28	$x^2yz-xy^2z^2-x+y^3z+yz^3-3yz$
103	$a^{-1}b^{-1}a^{-1}b^{-1}ab^2$	28	$x^2yz-xy^2z^2+x+y^3z+yz^3-4yz$

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Trace polynomials of the word length 7 – continued from previous page

Index	representative w	$\#[w]$	trace polynomial
104	$a^{-1}b^{-1}a^{-1}bab^2$	28	$-x^2yz + xy^2z^2 + xy^2 - x - y^3z - yz^3 + 3yz$
105	$a^{-1}b^{-1}a^{-1}b^2a^{-1}b$	28	$x^2y^3z - x^2yz - xy^4 - 2xy^2z^2 + 3xy^2 + xz^2 - x + y^3z + yz^3 - 3yz$
106	$a^{-1}b^{-1}aba^{-1}b^2$	28	$x^3y^2 - x^3 - x^2y^3z + xy^4 + 2xy^2z^2 - 4xy^2 - xz^2 + 3x - y^3z - yz^3 + 3yz$

Table 3.7: Trace polynomials of the word length 8

Index	**representative** w	$\#[w]$	**trace polynomial**
1	a^8	2	$x^8 - 8x^6 + 20x^4 - 16x^2 + 2$
2	b^8	2	$y^8 - 8y^6 + 20y^4 - 16y^2 + 2$
3	$abababab$	4	$z^4 - 4z^2 + 2$
4	$ab^{-1}ab^{-1}ab^{-1}ab^{-1}$	4	$x^4y^4 - 4x^3y^3z + 6x^2y^2z^2 - 4x^2y^2 - 4xyz^3 + 8xyz + z^4 - 4z^2 + 2$
5	$a^{-2}b^{-1}a^{-3}b^{-1}a^{-1}$	8	$x^4z^2 - 2x^3yz + x^2y^2 - 2x^2z^2 + 2xyz + z^2 - 2$
6	$a^{-2}ba^{-3}ba^{-1}$	8	$x^6y^2 - 2x^5yz - 4x^4y^2 + x^4z^2 + 6x^3yz + 4x^2y^2 - 2x^2z^2 - 4xyz + z^2 - 2$
7	$a^{-1}b^{-2}a^{-2}b^{-2}a^{-1}$	8	$x^4 - 2x^3yz + x^2y^2z^2 + 2x^2y^2 - 4x^2 - 2xy^3z + 4xyz + y^4 - 4y^2 + 2$
8	$a^{-1}b^2a^{-2}b^2a^{-1}$	8	$x^4y^4 - 2x^4y^2 + x^4 - 2x^3y^3z + 2x^3yz - 2x^2y^4 + x^2y^2z^2 + 6x^2y^2 - 4x^2 + 2xy^3z - 4xyz + y^4 - 4y^2 + 2$
9	ab^3ab^3	8	$x^2y^2 - 2xy^3z + 2xyz + y^4z^2 - 2y^2z^2 + z^2 - 2$
10	$a^{-1}ba^{-1}b^{-1}a^{-1}ba^{-1}b^{-1}$	8	$x^2y^2z^2 - 2xy^3z - 2xyz^3 + 4xyz + y^4 + 2y^2z^2 - 4y^2 + z^4 - 4z^2 + 2$
11	$a^{-1}baba^{-1}bab$	8	$x^4 - 2x^3yz + x^2y^2z^2 + 2x^2z^2 - 4x^2 - 2xyz^3 + 4xyz + z^4 - 4z^2 + 2$
12	$a^{-1}bab^{-1}a^{-1}bab^{-1}$	8	$x^4 - 2x^3yz + x^2y^2z^2 + 2x^2y^2 + 2x^2z^2 - 4x^2 - 2xy^3z - 2xyz^3 + 4xyz + y^4 + 2y^2z^2 - 4y^2 + z^4 - 4z^2 + 2$
13	$ab^{-3}ab^{-3}$	8	$x^2y^6 - 4x^2y^4 + 4x^2y^2 - 2xy^5z + 6xy^3z - 4xyz + y^4z^2 - 2y^2z^2 + z^2 - 2$
14	$a^{-6}ba^{-1}$	16	$x^7y - x^6z - 6x^5y + 5x^4z + 10x^3y - 6x^2z - 4xy + z$
15	$a^{-6}b^{-1}a^{-1}$	16	$x^6z - x^5y - 5x^4z + 4x^3y + 6x^2z - 3xy - z$

Continued on next page

Trace polynomials of the word length 8 – continued from previous page

Index	representative w	$\#[w]$	trace polynomial
16	$a^{-5}b^{-2}a^{-1}$	16	$-x^6+x^5yz-x^4y^2+6x^4-4x^3yz+3x^2y^2-9x^2+3xyz-y^2+2$
17	$a^{-5}b^2a^{-1}$	16	$x^6y^2-x^6-x^5yz-5x^4y^2+6x^4+4x^3yz+6x^2y^2-9x^2-3xyz-y^2+2$
18	$a^{-4}b^{-3}a^{-1}$	16	$-x^5y+x^4y^2z-x^4z-x^3y^3+6x^3y-3x^2y^2z+3x^2z+2xy^3-7xy+y^2z-z$
19	$a^{-4}b^{-1}a^{-1}b^{-1}a^{-1}$	16	$x^4z^2-x^4-x^3yz-3x^2z^2+4x^2+2xyz+z^2-2$
20	$a^{-4}b^{-1}ab^{-1}a^{-1}$	16	$-x^6+x^5yz-x^4y^2-x^4z^2+6x^4-2x^3yz+2x^2y^2+3x^2z^2-9x^2-xyz-z^2+2$
21	$a^{-4}ba^{-1}ba^{-1}$	16	$x^6y^2-2x^5yz-4x^4y^2+x^4z^2-x^4+7x^3yz+3x^2y^2-3x^2z^2+4x^2-4xyz+z^2-2$
22	$a^{-4}baba^{-1}$	16	$-x^6+x^5yz-x^4z^2+6x^4-4x^3yz+3x^2z^2-9x^2+3xyz-z^2+2$
23	$a^{-4}b^3a^{-1}$	16	$x^5y^3-2x^5y-x^4y^2z+x^4z-4x^3y^3+9x^3y+3x^2y^2z-3x^2z+3xy^3-8xy-y^2z+z$
24	$a^{-3}b^{-4}a^{-1}$	16	$-x^4y^2+x^4+x^3y^3z-2x^3yz-x^2y^4+6x^2y^2-4x^2-2xy^3z+4xyz+y^4-4y^2+2$
25	$a^{-3}b^{-1}a^{-2}b^{-1}a^{-1}$	16	$x^4z^2-2x^3yz+x^2y^2-2x^2z^2-x^2+3xyz-y^2+2$
26	$a^{-3}b^{-1}a^2b^{-1}a^{-1}$	16	$-x^6+x^5yz-x^4y^2-x^4z^2+6x^4-2x^3yz+2x^2y^2+2x^2z^2-9x^2+xyz-y^2+2$
27	$a^{-3}ba^{-2}ba^{-1}$	16	$x^6y^2-2x^5yz-4x^4y^2+x^4z^2+6x^3yz+4x^2y^2-2x^2z^2-x^2-3xyz-y^2+2$

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Trace polynomials of the word length 8 – continued from previous page

Index	representative w	$\#[w]$	trace polynomial
28	$a^{-3}ba^2ba^{-1}$	16	$-x^6+x^5yz-x^4y^2-x^4z^2+6x^4-2x^3yz+3x^2y^2+2x^2z^2-9x^2-xyz-y^2+2$
29	$a^{-3}b^4a^{-1}$	16	$x^4y^4-3x^4y^2+x^4-x^3y^3z+2x^3yz-3x^2y^4+10x^2y^2-4x^2+2xy^3z-4xyz+y^4-4y^2+2$
30	$a^{-2}b^{-5}a^{-1}$	16	$-x^3y^3+2x^3y+x^2y^4z-3x^2y^2z+x^2z-xy^5+6xy^3-7xy-y^4z+3y^2z-z$
31	$a^{-2}b^{-2}a^{-1}b^{-2}a^{-1}$	16	$x^4-2x^3yz+x^2y^2z^2+x^2y^2-4x^2-xy^3z+4xyz-y^2z^2+2$
32	$a^{-2}b^{-2}ab^{-2}a^{-1}$	16	$-x^4y^2+x^3y^3z-x^2y^4-x^2y^2z^2+4x^2y^2+x^2-2xyz+y^2z^2-2$
33	$a^{-3}ba^{-3}b^{-1}$	16	$x^5yz-x^4y^2-x^4z^2-2x^3yz+2x^2y^2+2x^2z^2+xyz-y^2-z^2+2$
34	$a^{-2}b^{-1}a^{-1}b^{-1}a^{-1}b^{-1}a^{-1}$	16	$x^2z^3-2x^2z-xyz^2+xy-z^3+3z$
35	$a^{-2}b^{-1}a^{-1}ba^{-1}b^{-1}a^{-1}$	16	$x^3yz^2-2x^2y^2z-x^2z^3+2x^2z+xy^3-2xy+y^2z+z^3-3z$
36	$a^{-2}b^{-1}ab^{-1}ab^{-1}a^{-1}$	16	$-x^5y+x^4y^2z+x^4z-x^3y^3-2x^3yz^2+4x^3y+x^2y^2z+x^2z^3-5x^2z+xyz^2-xy-z^3+3z$
37	$a^{-2}b^{-1}abab^{-1}a^{-1}$	16	$-x^4z+x^3yz^2+x^3y-2x^2y^2z-x^2z^3+5x^2z+xy^3-4xy+y^2z+z^3-3z$
38	$a^{-3}ba^3b^{-1}$	16	$x^6-x^5yz+x^4y^2+x^4z^2-6x^4+2x^3yz-2x^2y^2-2x^2z^2+9x^2-xyz+y^2+z^2-2$
39	$a^{-3}ba^3b$	16	$-x^6+x^5yz-x^4y^2-x^4z^2+6x^4-2x^3yz+2x^2y^2+2x^2z^2-9x^2+xyz-z^2+2$

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Trace polynomials of the word length 8 – continued from previous page

Index	representative w	$\#[w]$	trace polynomial
40	$a^{-2}ba^{-1}b^{-1}a^{-1}ba^{-1}$	16	$x^4y^2z - x^3y^3 - 2x^3yz^2 + 2x^3y - x^2y^2z + x^2z^3 - 2x^2z + 2xy^3 + 3xyz^2 - 5xy - y^2z - z^3 + 3z$
41	$a^{-2}ba^{-1}ba^{-1}ba^{-1}$	16	$x^5y^3 - 3x^4y^2z - 2x^3y^3 + 3x^3yz^2 - 2x^3y + 5x^2y^2z - x^2z^3 + 2x^2z - 4xyz^2 + 4xy + z^3 - 3z$
42	$a^{-2}bab^{-1}aba^{-1}$	16	$-x^5y + x^4y^2z + x^4z - x^3y^3 - 2x^3yz^2 + 6x^3y - x^2y^2z + x^2z^3 - 5x^2z + 2xy^3 + 3xyz^2 - 7xy - y^2z - z^3 + 3z$
43	$a^{-2}bababa^{-1}$	16	$-x^4z + x^3yz^2 - x^3y - x^2z^3 + 5x^2z - 2xyz^2 + 2xy + z^3 - 3z$
44	$a^{-2}b^2a^{-1}b^2a^{-1}$	16	$x^4y^4 - 2x^4y^2 + x^4 - 2x^3y^3z + 2x^3yz - 2x^2y^4 + x^2y^2z^2 + 5x^2y^2 - 4x^2 + 3xy^3z - 4xyz - y^2z^2 + 2$
45	$a^{-2}b^2ab^2a^{-1}$	16	$-x^4y^2 + x^3y^3z - x^2y^2z^2 + 2x^2y^2 + x^2 - 2xy^3z + 2xyz + y^2z^2 - 2$
46	$a^{-2}b^5a^{-1}$	16	$x^3y^5 - 4x^3y^3 + 3x^3y - x^2y^4z + 3x^2y^2z - x^2z - 2xy^5 + 9xy^3 - 8xy + y^4z - 3y^2z + z$
47	$a^{-1}b^{-6}a^{-1}$	16	$-x^2y^4 + 3x^2y^2 - x^2 + xy^5z - 4xy^3z + 3xyz - y^6 + 6y^4 - 9y^2 + 2$
48	$a^{-2}ba^{-2}b^{-3}$	16	$-x^4y^2 + x^3y^3z - x^2y^4 - x^2y^2z^2 + 4x^2y^2 + x^2z^2 - 2xyz + y^2 - 2$
49	$a^{-2}b^{-1}a^{-2}b^{-3}$	16	$-x^3yz + x^2y^2z^2 + x^2y^2 - x^2z^2 - 2xy^3z + 4xyz + y^4 - 4y^2 + 2$
50	$a^{-2}b^2a^{-2}b^{-2}$	16	$-x^4y^2 + x^4 + x^3y^3z - x^2y^4 - x^2y^2z^2 + 4x^2y^2 - 4x^2 + 2$
51	$a^{-2}b^2a^2b^2$	16	$-x^4y^2 + x^3y^3z - x^2y^4 - x^2y^2z^2 + 4x^2y^2 + y^4 - 4y^2 + 2$
52	$a^{-2}b^2a^2b^{-2}$	16	$x^4y^2 - x^3y^3z + x^2y^4 + x^2y^2z^2 - 4x^2y^2 + 2$

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Trace polynomials of the word length 8 – continued from previous page

Index	**representative** w	$\#[w]$	**trace polynomial**
53	$a^{-2}b^{-1}a^{-1}b^{-1}a^{-2}b^{-1}$	16	$x^2z^3-x^2z-2xyz^2+xy+y^2z-z$
54	$a^{-2}b^{-1}ab^{-1}a^{-2}b^{-1}$	16	$-x^4z+x^3yz^2+x^3y-2x^2y^2z-x^2z^3+3x^2z+xy^3+2xyz^2-4xy-y^2z+z$
55	$a^{-2}ba^{-1}ba^{-2}b^{-1}$	16	$x^4y^2z-x^3y^3-2x^3yz^2+x^2y^2z+x^2z^3-x^2z+3xy-z$
56	$a^{-2}baba^{-2}b^{-1}$	16	$-x^4z+x^3yz^2+x^3y-x^2y^2z-x^2z^3+3x^2z-2xy+z$
57	$a^{-2}b^3a^{-2}b^{-1}$	16	$x^3y^3z-2x^3yz-x^2y^4-x^2y^2z^2+2x^2y^2+x^2z^2+2xyz+y^2-2$
58	$a^{-1}b^{-2}a^{-1}b^{-1}a^{-2}b^{-1}$	16	$-x^2z^2+xyz^3-y^2z^2+2$
59	$a^{-2}ba^{-2}b^{-1}a^{-1}b^{-1}$	16	$x^3yz^2-x^3y-x^2y^2z-x^2z^3+x^2z+2xy+z$
60	$a^{-1}b^{-1}a^{-2}b^{-1}a^{-1}b^2$	16	$x^2y^2z^2-x^2z^2-2xy^3z-xyz^3+4xyz+y^4+y^2z^2-4y^2+2$
61	$a^{-1}b^{-1}ab^{-2}ab^{-1}a^{-1}$	16	$-x^4y^2+x^4+x^3y^3z-x^2y^4-2x^2y^2z^2+4x^2y^2+x^2z^2-4x^2+2xy^3z+xyz^3-4xyz-y^2z^2+2$
62	$a^{-2}ba^{-2}b^{-1}ab^{-1}$	16	$-x^5y+x^4y^2z+x^4z-x^3y^3-2x^3yz^2+4x^3y+x^2y^2z+x^2z^3-3x^2z-xy-z$
63	$a^{-1}b^{-2}a^{-1}ba^2b$	16	$x^4-2x^3yz+x^2y^2z^2+2x^2y^2+x^2z^2-4x^2-2xy^3z-xyz^3+4xyz+y^4+y^2z^2-4y^2+2$
64	$a^{-2}ba^{-1}ba^{-2}b$	16	$x^5y^3-3x^4y^2z-2x^3y^3+3x^3yz^2-x^3y+4x^2y^2z-x^2z^3+x^2z+xy^3-2xyz^2-y^2z+z$
65	$a^{-2}baba^{-2}b$	16	$-x^5y+x^4y^2z+x^4z-2x^3yz^2+4x^3y-2x^2y^2z+x^2z^3-3x^2z+2xyz^2-3xy+y^2z-z$

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Trace polynomials of the word length 8 – continued from previous page

Index	**representative** w	$\#[w]$	**trace polynomial**
66	$a^{-2}b^3a^{-2}b$	16	$x^4y^4 - 2x^4y^2 - 2x^3y^3z + 3x^3yz - 2x^2y^4 + x^2y^2z^2 + 5x^2y^2 - x^2z^2 + 2xy^3z - 4xyz + y^4 - 4y^2 + 2$
67	$a^{-1}b^{-2}a^{-1}ba^{-2}b$	16	$-x^4y^2 + x^3y^3z + 2x^3yz - x^2y^4 - 2x^2y^2z^2 + 4x^2y^2 - x^2z^2 + xyz^3 - 4xyz + y^4 + y^2z^2 - 4y^2 + 2$
68	$a^{-1}ba^{-2}ba^{-1}b^2$	16	$x^4y^4 - x^4y^2 - 3x^3y^3z + 2x^3yz - x^2y^4 + 3x^2y^2z^2 - x^2z^2 + 2xy^3z - xyz^3 - y^2z^2 + 2$
69	$a^{-1}b^{-1}a^2b^{-1}a^{-1}b^2$	16	$-x^4y^2 + x^4 + x^3y^3z - x^2y^4 - 2x^2y^2z^2 + 6x^2y^2 + x^2z^2 - 4x^2 + xyz^3 - 4xyz + y^4 + y^2z^2 - 4y^2 + 2$
70	$a^{-1}b^{-2}a^{-1}b^{-1}a^2b^{-1}$	16	$x^4 - 2x^3yz + x^2y^2z^2 + x^2z^2 - 4x^2 - xyz^3 + 4xyz - y^2z^2 + 2$
71	$a^{-1}b^6a^{-1}$	16	$x^2y^6 - 5x^2y^4 + 6x^2y^2 - x^2 - xy^5z + 4xy^3z - 3xyz - y^6 + 6y^4 - 9y^2 + 2$
72	ab^7	16	$-xy^5 + 4xy^3 - 3xy + y^6z - 5y^4z + 6y^2z - z$
73	$a^{-1}ba^{-1}b^{-5}$	16	$-x^2y^4 + 2x^2y^2 + xy^5z - 2xy^3z - xyz - y^6 - y^4z^2 + 6y^4 + 3y^2z^2 - 9y^2 - z^2 + 2$
74	$a^{-1}b^{-1}a^{-1}b^{-5}$	16	$-xy^3z + 2xyz + y^4z^2 - y^4 - 3y^2z^2 + 4y^2 + z^2 - 2$
75	$a^{-1}b^{-2}a^{-1}b^{-4}$	16	$x^2y^2 - x^2 - 2xy^3z + 3xyz + y^4z^2 - 2y^2z^2 - y^2 + 2$
76	$a^{-1}b^2a^{-1}b^{-4}$	16	$-x^2y^4 + 2x^2y^2 - x^2 + xy^5z - 2xy^3z + xyz - y^6 - y^4z^2 + 6y^4 + 2y^2z^2 - 9y^2 + 2$
77	$a^{-1}b^{-1}a^{-1}b^{-3}a^{-1}b^{-1}$	16	$-xyz^2 + xy + y^2z^3 - 2y^2z - z^3 + 3z$
78	$a^{-1}b^{-1}ab^{-1}a^{-1}b^{-3}$	16	$x^3y - 2x^2y^2z + x^2z + xy^3z^2 - 2xy - y^2z^3 + 2y^2z + z^3 - 3z$

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Trace polynomials of the word length 8 – continued from previous page

Index	**representative** w	$\#[w]$	**trace polynomial**
79	$a^{-1}ba^{-1}b^{-3}a^{-1}b$	16	$-x^3y^3 + x^2y^4z + x^2y^2z - xy^5 - 2xy^3z^2 + 4xy^3 + xyz^2 - xy + y^4z + y^2z^3 - 5y^2z - z^3 + 3z$
80	$a^{-1}b^{-1}ab^3ab^{-1}$	16	$x^3y - 2x^2y^2z + x^2z + xy^3z^2 + xy^3 - 4xy - y^4z - y^2z^3 + 5y^2z + z^3 - 3z$
81	$a^{-1}b^3a^{-1}b^{-3}$	16	$-x^2y^4 + 2x^2y^2 + xy^5z - 2xy^3z + xyz - y^6 - y^4z^2 + 6y^4 + 2y^2z^2 - 9y^2 - z^2 + 2$
82	$a^{-1}b^3ab^3$	16	$-x^2y^4 + 2x^2y^2 - x^2 + xy^5z - 2xy^3z + xyz - y^4z^2 + 2y^2z^2 - z^2 + 2$
83	$a^{-1}b^3ab^{-3}$	16	$x^2y^4 - 2x^2y^2 + x^2 - xy^5z + 2xy^3z - xyz + y^6 + y^4z^2 - 6y^4 - 2y^2z^2 + 9y^2 + z^2 - 2$
84	$a^{-1}b^{-2}a^{-1}ba^{-1}b^{-2}$	16	$x^3y - 2x^2y^2z - x^2z + xy^3z^2 + xy^3 + 2xyz^2 - 4xy - y^4z - y^2z^3 + 3y^2z + z$
85	$a^{-1}b^{-2}a^{-1}b^{-1}a^{-1}b^{-2}$	16	$x^2z - 2xyz^2 + xy + y^2z^3 - y^2z - z$
86	$a^{-1}b^4a^{-1}b^{-2}$	16	$-x^2y^4 + 3x^2y^2 - x^2 + xy^5z - 2xy^3z - xyz - y^6 - y^4z^2 + 6y^4 + 2y^2z^2 - 9y^2 + 2$
87	$a^{-1}ba^{-1}b^{-2}ab^{-2}$	16	$-x^3y^3 + x^2y^4z + x^2y^2z - xy^5 - 2xy^3z^2 + 4xy^3 - xy + y^4z + y^2z^3 - 3y^2z - z$
88	$a^{-1}b^{-1}a^{-1}b^{-2}ab^{-2}$	16	$-x^2y^2z + xy^3z^2 - xy^3 + 2xy - y^2z^3 + y^2z + z$
89	$a^{-1}ba^{-1}b^2ab^2$	16	$-x^3y^3 + x^2y^4z + x^2y^2z - 2xy^3z^2 + 3xy + y^2z^3 - y^2z - z$
90	$a^{-1}b^{-1}a^{-1}b^2ab^2$	16	$-x^2y^2z + xy^3z^2 + xy^3 - 2xy - y^4z - y^2z^3 + 3y^2z + z$
91	$a^{-1}b^{-1}a^{-1}b^{-1}a^{-1}ba^{-1}b^{-1}$	16	$xyz^3 - 2xyz - y^2z^2 + y^2 - z^4 + 4z^2 - 2$
92	$a^{-1}b^{-1}a^{-1}b^{-1}ab^{-1}a^{-1}b^{-1}$	16	$-x^2z^2 + x^2 + xyz^3 - 2xyz - z^4 + 4z^2 - 2$

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Trace polynomials of the word length 8 – continued from previous page

Index	**representative** w	$\#[w]$	**trace polynomial**
93	$a^{-1}b^{-1}a^{-1}ba^{-1}ba^{-1}b^{-1}$	16	$x^2y^2z^2 - x^2y^2 - xy^3z - 2xyz^3 + 4xyz + y^2z^2 + z^4 - 4z^2 + 2$
94	$a^{-1}b^{-1}a^{-1}baba^{-1}b^{-1}$	16	$-x^2z^2 + x^2 + xyz^3 - y^2z^2 - z^4 + 4z^2 - 2$
95	$a^{-1}b^{-1}a^{-1}b^3a^{-1}b^{-1}$	16	$xy^3z^2 - xy^3 - 2xyz^2 + 2xy - y^4z - y^2z^3 + 5y^2z + z^3 - 3z$
96	$a^{-1}b^{-1}ab^{-1}ab^{-1}a^{-1}b^{-1}$	16	$-x^3yz + x^2y^2z^2 - x^2y^2 + x^2z^2 - 2xyz^3 + 4xyz + z^4 - 4z^2 + 2$
97	$a^{-1}b^{-1}a^{-1}babab^{-1}$	16	$x^2z^2 - xyz^3 + y^2z^2 + z^4 - 4z^2 + 2$
98	$a^{-1}b^{-1}a^{-1}babab$	16	$-x^2z^2 + xyz^3 - y^2z^2 + y^2 - z^4 + 4z^2 - 2$
99	$a^{-1}ba^{-1}b^{-1}a^{-1}ba^{-1}b$	16	$x^3y^3z - x^2y^4 - 3x^2y^2z^2 + 2x^2y^2 + 2xy^3z + 3xyz^3 - 6xyz - y^2z^2 + y^2 - z^4 + 4z^2 - 2$
100	$a^{-1}b^{-1}abab^{-1}a^{-1}b$	16	$-x^3yz + x^2y^2z^2 + x^2y^2 + x^2z^2 - 2xy^3z - 2xyz^3 + 4xyz + y^4 + 2y^2z^2 - 4y^2 + z^4 - 4z^2 + 2$
101	$a^{-1}b^2a^{-1}b^{-1}a^{-1}b^2$	16	$x^2y^4z - 2x^2y^2z + x^2z - xy^5 - 2xy^3z^2 + 4xy^3 + 2xyz^2 - 3xy + y^4z + y^2z^3 - 3y^2z - z$
102	$a^{-1}b^5a^{-1}b^{-1}$	16	$xy^5z - 4xy^3z + 3xyz - y^6 - y^4z^2 + 6y^4 + 3y^2z^2 - 9y^2 - z^2 + 2$
103	$a^{-1}baba^{-1}b^3$	16	$-x^3y^3 + 2x^3y + x^2y^4z - x^2y^2z - x^2z - 2xy^3z^2 + 2xy^3 + 3xyz^2 - 5xy + y^2z^3 - 2y^2z - z^3 + 3z$
104	$a^{-1}ba^{-1}b^{-1}ab^{-1}a^{-1}b$	16	$-x^4y^2 + x^3y^3z + 2x^3yz - x^2y^4 - 3x^2y^2z^2 + 4x^2y^2 - x^2z^2 + x^2 + 2xy^3z + 3xyz^3 - 8xyz - y^2z^2 - z^4 + 4z^2 - 2$
105	$a^{-1}b^{-1}aba^{-1}bab^{-1}$	16	$x^4 - 2x^3yz + x^2y^2z^2 + x^2y^2 + 2x^2z^2 - 4x^2 - xy^3z - 2xyz^3 + 4xyz + y^2z^2 + z^4 - 4z^2 + 2$
			Continued on next page

Trace polynomials of the word length 8 – continued from previous page

Index	**representative** w	$\#[w]$	**trace polynomial**
106	$a^{-1}bab^{-3}ab$	16	$-x^3y^3+2x^3y+x^2y^4z-x^2y^2z-x^2z-xy^5-2xy^3z^2+6xy^3+3xyz^2-7xy+y^4z+y^2z^3-5y^2z-z^3+3z$
107	$a^{-1}ba^{-1}b^{-1}ab^{-1}ab^{-1}$	16	$-x^4y^2+x^3y^3z+2x^3yz-x^2y^4-3x^2y^2z^2+4x^2y^2-x^2z^2+2xy^3z+3xyz^3-8xyz-y^2z^2+y^2-z^4+4z^2-2$
108	$a^{-1}ba^{-1}ba^{-1}bab$	16	$-x^4y^2+x^3y^3z+2x^3yz-3x^2y^2z^2+2x^2y^2-x^2z^2+x^2+3xyz^3-6xyz-z^4+4z^2-2$
109	$a^{-1}ba^{-1}b^{-1}ab^{-1}ab$	16	$x^4y^2-x^3y^3z-2x^3yz+x^2y^4+3x^2y^2z^2-4x^2y^2+x^2z^2-2xy^3z-3xyz^3+8xyz+y^2z^2+z^4-4z^2+2$
110	$a^{-1}ba^{-1}b^3a^{-1}b$	16	$x^3y^5-2x^3y^3-3x^2y^4z+5x^2y^2z+3xy^3z^2-2xy^3-4xyz^2+4xy-y^2z^3+2y^2z+z^3-3z$
111	$a^{-1}b^2a^{-1}ba^{-1}b^2$	16	$x^3y^5-2x^3y^3+x^3y-3x^2y^4z+4x^2y^2z-x^2z+3xy^3z^2-xy^3-2xyz^2-y^2z^3+y^2z+z$
112	$a^{-1}b^5a^{-1}b$	16	$x^2y^6-4x^2y^4+3x^2y^2-2xy^5z+7xy^3z-4xyz+y^4z^2-y^4-3y^2z^2+4y^2+z^2-2$
113	$a^{-1}b^4a^{-1}b^2$	16	$x^2y^6-4x^2y^4+4x^2y^2-x^2-2xy^5z+6xy^3z-3xyz+y^4z^2-2y^2z^2-y^2+2$
114	ab^{-7}	16	$xy^7-6xy^5+10xy^3-4xy-y^6z+5y^4z-6y^2z+z$
115	$a^{-5}ba^{-1}b^{-1}$	32	$x^5yz-x^4y^2-x^4z^2+x^4-3x^3yz+3x^2y^2+3x^2z^2-4x^2+xyz-y^2-z^2+2$
116	$a^{-5}bab^{-1}$	32	$x^6-x^5yz+x^4y^2+x^4z^2-6x^4+3x^3yz-3x^2y^2-3x^2z^2+9x^2-xyz+y^2+z^2-2$

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Trace polynomials of the word length 8 – continued from previous page

Index	representative w	$\#[w]$	trace polynomial
117	$a^{-4}ba^{-1}b^{-2}$	32	$-x^5y+x^4y^2z+x^4z-x^3y^3-x^3yz^2+5x^3y-2x^2y^2z-3x^2z+2xy^3+2xyz^2-6xy+z$
118	$a^{-4}b^{-1}a^{-1}b^{-2}$	32	$-x^4z+x^3yz^2-x^2y^2z+3x^2z-2xyz^2+xy+y^2z-z$
119	$a^{-4}bab^{-2}$	32	$x^5y-x^4y^2z+x^3y^3+x^3yz^2-6x^3y+2x^2y^2z+x^2z-2xy^3-2xyz^2+7xy-z$
120	$a^{-4}b^{-1}ab^{-2}$	32	$-x^5y+x^4y^2z-x^3y^3-x^3yz^2+5x^3y-x^2y^2z-x^2z+xy^3+2xyz^2-4xy-y^2z+z$
121	$a^{-4}ba^{-2}b^{-1}$	32	$x^5yz-x^4y^2-x^4z^2-2x^3yz+2x^2y^2+2x^2z^2+x^2-2$
122	$a^{-4}b^2a^{-1}b^{-1}$	32	$x^4y^2z-x^4z-x^3y^3-x^3yz^2+2x^3y-2x^2y^2z+3x^2z+2xy^3+2xyz^2-5xy-z$
123	$a^{-4}b^2ab^{-1}$	32	$x^5y-x^4y^2z+x^3y^3+x^3yz^2-5x^3y+2x^2y^2z-x^2z-2xy^3-2xyz^2+6xy+z$
124	$a^{-4}ba^2b^{-1}$	32	$x^6-x^5yz+x^4y^2+x^4z^2-6x^4+2x^3yz-2x^2y^2-2x^2z^2+9x^2-2$
125	$a^{-4}b^2a^{-1}b$	32	$x^5y^3-x^5y-2x^4y^2z+x^4z-3x^3y^3+x^3yz^2+3x^3y+5x^2y^2z-3x^2z+xy^3-2xyz^2-y^2z+z$
126	$a^{-4}b^2ab$	32	$-x^5y+x^4y^2z-x^3yz^2+4x^3y-3x^2y^2z+x^2z+2xyz^2-3xy+y^2z-z$
127	$a^{-3}ba^{-1}b^{-3}$	32	$-x^4y^2+x^3y^3z-x^2y^4-x^2y^2z^2+5x^2y^2+x^2z^2-x^2-xy^3z-xyz+y^4+y^2z^2-4y^2-z^2+2$
128	$a^{-3}b^{-1}a^{-1}b^{-3}$	32	$-x^3yz+x^2y^2z^2-x^2z^2+x^2-xy^3z+3xyz-y^2z^2+y^2+z^2-2$

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Trace polynomials of the word length 8 – continued from previous page

Index	**representative** w	$\#[w]$	**trace polynomial**
129	$a^{-3}bab^{-3}$	32	$x^4y^2 - x^4 - x^3y^3z + x^3yz + x^2y^4 + x^2y^2z^2 - 6x^2y^2 - x^2z^2 + 4x^2 + xy^3z - y^4 - y^2z^2 + 4y^2 + z^2 - 2$
130	$a^{-3}b^{-1}ab^{-3}$	32	$-x^4y^2 + x^4 + x^3y^3z - x^3yz - x^2y^4 - x^2y^2z^2 + 5x^2y^2 + x^2z^2 - 4x^2 - xyz + y^2z^2 - y^2 - z^2 + 2$
131	$a^{-3}ba^{-2}b^{-2}$	32	$-x^5y + x^4y^2z + x^4z - x^3y^3 - x^3yz^2 + 4x^3y - x^2y^2z - 3x^2z + xy^3 + xyz^2 - 2xy + z$
132	$a^{-3}b^{-1}a^{-2}b^{-2}$	32	$-x^4z + x^3yz^2 + x^3y - 2x^2y^2z + 3x^2z + xy^3 - xyz^2 - 3xy + y^2z - z$
133	$a^{-3}b^2a^{-1}b^{-2}$	32	$-x^4y^2 + x^4 + x^3y^3z - x^2y^4 - x^2y^2z^2 + 5x^2y^2 - 4x^2 - xy^3z + y^4 + y^2z^2 - 4y^2 + 2$
134	$a^{-3}b^2ab^{-2}$	32	$x^4y^2 - x^3y^3z + x^2y^4 + x^2y^2z^2 - 5x^2y^2 + x^2 + xy^3z - y^4 - y^2z^2 + 4y^2 - 2$
135	$a^{-3}ba^2b^{-2}$	32	$x^5y - x^4y^2z + x^3y^3 + x^3yz^2 - 5x^3y + x^2y^2z - xy^3 - xyz^2 + 4xy + z$
136	$a^{-3}b^{-1}a^2b^{-2}$	32	$-x^5y + x^4y^2z - x^3y^3 - x^3yz^2 + 5x^3y - x^2y^2z + xy^3 + xyz^2 - 5xy + y^2z - z$
137	$a^{-3}b^2a^{-2}b^{-1}$	32	$x^4y^2z - x^4z - x^3y^3 - x^3yz^2 + x^3y - x^2y^2z + 3x^2z + xy^3 + xyz^2 - xy - z$
138	$a^{-3}ba^{-1}b^{-1}a^{-1}b^{-1}$	32	$x^3yz^2 - x^3y - x^2y^2z - x^2z^3 + 2x^2z - xyz^2 + 2xy + y^2z + z^3 - 3z$
139	$a^{-3}bab^{-1}a^{-1}b^{-1}$	32	$x^4z - x^3yz^2 + x^2y^2z + x^2z^3 - 4x^2z + xyz^2 - xy - y^2z - z^3 + 3z$
140	$a^{-3}b^{-1}ab^{-1}a^{-1}b^{-1}$	32	$-x^4z + x^3yz^2 - x^2y^2z - x^2z^3 + 4x^2z + z^3 - 3z$

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Trace polynomials of the word length 8 – continued from previous page

Index	representative w	$\#[w]$	trace polynomial
141	$a^{-3}ba^{-1}ba^{-1}b^{-1}$	32	$x^4y^2z - x^3y^3 - 2x^3yz^2 + x^3y + x^2z^3 - 2x^2z + xy^3 + 2xyz^2 - xy - y^2z - z^3 + 3z$
142	$a^{-3}baba^{-1}b^{-1}$	32	$-x^4z+x^3yz^2+x^3y-x^2y^2z-x^2z^3+4x^2z - xyz^2 - 2xy + y^2z + z^3 - 3z$
143	$a^{-3}b^{-1}aba^{-1}b^{-1}$	32	$x^4z-x^3yz^2-x^3y+2x^2y^2z+x^2z^3-4x^2z - xy^3 + 3xy - y^2z - z^3 + 3z$
144	$a^{-3}b^3a^{-1}b^{-1}$	32	$x^3y^3z - 2x^3yz - x^2y^4 - x^2y^2z^2 + 3x^2y^2 + x^2z^2 - x^2 - xy^3z + 3xyz + y^4 + y^2z^2 - 4y^2 - z^2 + 2$
145	$a^{-3}ba^{-1}b^{-1}ab^{-1}$	32	$-x^5y + x^4y^2z + x^4z - x^3y^3 - 2x^3yz^2+5x^3y+x^2z^3-4x^2z+xy^3+2xyz^2 - 5xy - y^2z - z^3 + 3z$
146	$a^{-3}bab^{-1}ab^{-1}$	32	$x^5y-x^4y^2z-x^4z+x^3y^3+2x^3yz^2-5x^3y-x^2z^3+5x^2z-xy^3-2xyz^2+4xy + y^2z + z^3 - 3z$
147	$a^{-3}ba^{-1}bab^{-1}$	32	$x^5y-x^4y^2z-x^4z+x^3y^3+2x^3yz^2-4x^3y-x^2z^3+4x^2z-xy^3-2xyz^2+2xy + y^2z + z^3 - 3z$
148	$a^{-3}babab^{-1}$	32	$x^4z-x^3yz^2+x^2y^2z+x^2z^3-5x^2z+xyz^2 + xy - y^2z - z^3 + 3z$
149	$a^{-3}b^3ab^{-1}$	32	$x^4y^2 - x^4 - x^3y^3z + x^3yz + x^2y^4 + x^2y^2z^2-5x^2y^2-x^2z^2+4x^2+xy^3z-2xyz - y^4 - y^2z^2 + 4y^2 + z^2 - 2$
150	$a^{-3}b^2a^2b^{-1}$	32	$x^5y-x^4y^2z+x^3y^3+x^3yz^2-5x^3y+x^2y^2z - xy^3 - xyz^2 + 5xy - z$
151	$a^{-3}b^2a^{-2}b$	32	$x^5y^3-x^5y-2x^4y^2z+x^4z-3x^3y^3+x^3yz^2 + 4x^3y + 4x^2y^2z - 3x^2z + 2xy^3 - xyz^2 - 4xy - y^2z + z$
152	$a^{-3}bab^{-1}a^{-1}b$	32	$x^5y-x^4y^2z-x^4z+x^3y^3+2x^3yz^2-5x^3y+x^2y^2z-x^2z^3+4x^2z-2xy^3-3xyz^2 + 6xy + y^2z + z^3 - 3z$
			Continued on next page

Trace polynomials of the word length 8 – continued from previous page

Index	**representative** w	$\#[w]$	**trace polynomial**
153	$a^{-3}baba^{-1}b$	32	$-x^5y + x^4y^2z + x^4z - 2x^3yz^2 + 4x^3y - 2x^2y^2z + x^2z^3 - 4x^2z + 3xyz^2 - 3xy - z^3 + 3z$
154	$a^{-3}b^3a^{-1}b$	32	$x^4y^4 - 2x^4y^2 - 2x^3y^3z + 3x^3yz - 2x^2y^4 + x^2y^2z^2 + 4x^2y^2 - x^2z^2 + x^2 + 3xy^3z - 5xyz - y^2z^2 + y^2 + z^2 - 2$
155	$a^{-3}b^3ab$	32	$-x^4y^2 + x^4 + x^3y^3z - x^3yz - x^2y^2z^2 + 3x^2y^2 + x^2z^2 - 4x^2 - 2xy^3z + 3xyz + y^2z^2 - y^2 - z^2 + 2$
156	$a^{-3}b^2a^2b$	32	$-x^5y + x^4y^2z - x^3y^3 - x^3yz^2 + 5x^3y - x^2y^2z + 2xy^3 + xyz^2 - 6xy - y^2z + z$
157	$a^{-2}ba^{-1}b^{-4}$	32	$-x^3y^3 + x^3y + x^2y^4z - x^2y^2z - x^2z - xy^5 - xy^3z^2 + 5xy^3 + 2xyz^2 - 4xy - y^2z + z$
158	$a^{-2}b^{-1}a^{-1}b^{-4}$	32	$-x^2y^2z + x^2z + xy^3z^2 - 2xyz^2 + xy - y^4z + 3y^2z - z$
159	$a^{-2}bab^{-4}$	32	$x^3y^3 - 2x^3y - x^2y^4z + 2x^2y^2z + xy^5 + xy^3z^2 - 6xy^3 - 2xyz^2 + 7xy + y^2z - z$
160	$a^{-2}b^{-1}ab^{-4}$	32	$-x^3y^3 + 2x^3y + x^2y^4z - 2x^2y^2z - xy^5 - xy^3z^2 + 5xy^3 + 2xyz^2 - 6xy + y^4z - 3y^2z + z$
161	$a^{-2}b^{-2}a^{-1}b^{-3}$	32	$x^3y - 2x^2y^2z + x^2z + xy^3z^2 + xy^3 - xyz^2 - 3xy - y^4z + 3y^2z - z$
162	$a^{-2}b^2a^{-1}b^{-3}$	32	$-x^3y^3 + x^3y + x^2y^4z - x^2y^2z + x^2z - xy^5 - xy^3z^2 + 5xy^3 + xyz^2 - 5xy - z$
163	$a^{-2}b^{-2}ab^{-3}$	32	$-x^3y^3 + x^3y + x^2y^4z - x^2y^2z - xy^5 - xy^3z^2 + 4xy^3 + xyz^2 - 2xy + y^4z - 3y^2z + z$
164	$a^{-2}b^2ab^{-3}$	32	$x^3y^3 - x^3y - x^2y^4z + x^2y^2z + xy^5 + xy^3z^2 - 5xy^3 - xyz^2 + 4xy + z$

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Trace polynomials of the word length 8 – continued from previous page

Index	representative w	$\#[w]$	trace polynomial
165	$a^{-2}b^{-1}a^{2}b^{3}$	32	$x^4y^2-x^4-x^3y^3z+x^3yz+x^2y^4+x^2y^2z^2-5x^2y^2-x^2z^2+4x^2+y^2-2$
166	$a^{-2}b^{-1}a^{2}b^{-3}$	32	$-x^4y^2+x^4+x^3y^3z-x^3yz-x^2y^4-x^2y^2z^2+5x^2y^2+x^2z^2-4x^2+y^4-4y^2+2$
167	$a^{-2}ba^{-1}b^{-1}a^{-1}b^{-2}$	32	$-x^3yz+x^2y^2z^2+x^2z^2-x^2-xy^3z-xyz^3+3xyz-z^2+2$
168	$a^{-2}b^{-1}a^{-1}b^{-1}a^{-1}b^{-2}$	32	$-x^2z^2+x^2+xyz^3-xyz-y^2z^2+y^2+z^2-2$
169	$a^{-2}bab^{-1}a^{-1}b^{-2}$	32	$-x^4+2x^3yz-x^2y^2z^2-x^2y^2-x^2z^2+4x^2+xy^3z+xyz^3-4xyz+z^2-2$
170	$a^{-2}b^{-1}ab^{-1}a^{-1}b^{-2}$	32	$x^4-2x^3yz+x^2y^2z^2+x^2y^2+x^2z^2-4x^2-xy^3z-xyz^3+3xyz+y^2z^2-y^2-z^2+2$
171	$a^{-2}ba^{-1}ba^{-1}b^{-2}$	32	$-x^4y^2+x^3y^3z+2x^3yz-x^2y^4-2x^2y^2z^2+3x^2y^2-x^2z^2+x^2+xy^3z+xyz^3-5xyz+y^2+z^2-2$
172	$a^{-2}b^{-1}a^{-1}ba^{-1}b^{-2}$	32	$-x^3yz+x^2y^2z^2+x^2y^2+x^2z^2-x^2-2xy^3z-xyz^3+3xyz+y^4+y^2z^2-4y^2-z^2+2$
173	$a^{-2}baba^{-1}b^{-2}$	32	$x^4-2x^3yz+x^2y^2z^2+x^2y^2+x^2z^2-4x^2-xy^3z-xyz^3+5xyz-y^2-z^2+2$
174	$a^{-2}b^{-1}aba^{-1}b^{-2}$	32	$-x^4+2x^3yz-x^2y^2z^2-2x^2y^2-x^2z^2+4x^2+2xy^3z+xyz^3-4xyz-y^4-y^2z^2+4y^2+z^2-2$
175	$a^{-2}b^{3}a^{-1}b^{-2}$	32	$-x^3y^3+2x^3y+x^2y^4z-x^2y^2z-x^2z-xy^5-xy^3z^2+5xy^3+xyz^2-6xy+z$

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Trace polynomials of the word length 8 – continued from previous page

Index	representative w	$\#[w]$	trace polynomial
176	$a^{-2}ba^{-1}b^{-1}ab^{-2}$	32	$-x^4y^2 + x^3y^3z + x^3yz - x^2y^4 - 2x^2y^2z^2 + 4x^2y^2 + xy^3z + xyz^3 - 4xyz + z^2 - 2$
177	$a^{-2}b^{-1}a^{-1}b^{-1}ab^{-2}$	32	$-x^3yz + x^2y^2z^2 - xy^3z - xyz^3 + 3xyz + y^2z^2 - y^2 - z^2 + 2$
178	$a^{-2}bab^{-1}ab^{-2}$	32	$x^4y^2 - x^3y^3z - x^3yz + x^2y^4 + 2x^2y^2z^2 - 4x^2y^2 - x^2 - xy^3z - xyz^3 + 5xyz - z^2 + 2$
179	$a^{-2}b^{-1}ab^{-1}ab^{-2}$	32	$-x^4y^2 + x^3y^3z + x^3yz - x^2y^4 - 2x^2y^2z^2 + 3x^2y^2 + x^2 + 2xy^3z + xyz^3 - 5xyz - y^2z^2 + y^2 + z^2 - 2$
180	$a^{-2}ba^{-1}bab^{-2}$	32	$x^4y^2 - x^3y^3z - x^3yz + x^2y^4 + 2x^2y^2z^2 - 4x^2y^2 - xy^3z - xyz^3 + 5xyz - y^2 - z^2 + 2$
181	$a^{-2}b^{-1}a^{-1}bab^{-2}$	32	$x^3yz - x^2y^2z^2 - x^2y^2 + 2xy^3z + xyz^3 - 4xyz - y^4 - y^2z^2 + 4y^2 + z^2 - 2$
182	$a^{-2}babab^{-2}$	32	$x^3yz - x^2y^2z^2 + x^2 + xy^3z + xyz^3 - 5xyz + y^2 + z^2 - 2$
183	$a^{-2}b^{-1}abab^{-2}$	32	$-x^3yz + x^2y^2z^2 + x^2y^2 - x^2 - 2xy^3z - xyz^3 + 5xyz + y^4 + y^2z^2 - 4y^2 - z^2 + 2$
184	$a^{-2}b^3ab^{-2}$	32	$x^3y^3 - x^3y - x^2y^4z + x^2y^2z + xy^5 + xy^3z^2 - 5xy^3 - xyz^2 + 5xy - z$
185	$a^{-2}b^{-1}a^{-2}ba^{-1}b^{-1}$	32	$x^3yz^2 - 2x^2y^2z - x^2z^3 + x^2z + xy^3 + xyz^2 - 2xy + z$
186	$a^{-2}b^{-1}a^{-2}bab^{-1}$	32	$x^4z - x^3yz^2 - x^3y + 2x^2y^2z + x^2z^3 - 3x^2z - xy^3 - xyz^2 + 3xy - z$
187	$a^{-2}ba^{-1}b^{-2}a^{-1}b^{-1}$	32	$-x^3yz + x^2y^2z^2 + x^2z^2 - xy^3z - xyz^3 + 2xyz + y^2 - 2$
188	$a^{-2}bab^{-2}a^{-1}b^{-1}$	32	$x^3yz - x^2y^2z^2 - x^2 + xy^3z + xyz^3 - 3xyz - y^2 + 2$

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Trace polynomials of the word length 8 – continued from previous page

Index	representative w	$\#[w]$	trace polynomial
189	$a^{-2}b^{-1}ab^{-2}a^{-1}b^{-1}$	32	$-x^3yz+x^2y^2z^2+x^2-xy^3z-xyz^3+2xyz+y^2z^2-2$
190	$a^{-2}b^2a^{-1}b^{-1}a^{-1}b^{-1}$	32	$x^2y^2z^2-x^2y^2-x^2z^2+x^2-xy^3z-xyz^3+3xyz+y^2+z^2-2$
191	$a^{-2}b^2ab^{-1}a^{-1}b^{-1}$	32	$x^3yz-x^2y^2z^2+xy^3z+xyz^3-3xyz-y^2-z^2+2$
192	$a^{-2}b^{-1}a^2bab$	32	$x^4z-x^3yz^2+x^2y^2z+x^2z^3-4x^2z+z$
193	$a^{-2}b^{-1}a^2b^{-1}a^{-1}b^{-1}$	32	$-x^4z+x^3yz^2-x^2y^2z-x^2z^3+4x^2z-xy+y^2z-z$
194	$a^{-2}ba^{-2}b^{-1}a^{-1}b$	32	$x^4y^2z-x^3y^3-2x^3yz^2+x^3y+x^2z^3-x^2z+xy^3+xyz^2-xy-z$
195	$a^{-2}b^2a^{-1}ba^{-1}b^{-1}$	32	$x^3y^3z-x^3yz-x^2y^4-2x^2y^2z^2+2x^2y^2+x^2z^2-x^2+xy^3z+xyz^3-xyz-z^2+2$
196	$a^{-2}b^2aba^{-1}b^{-1}$	32	$-x^3yz+x^2y^2z^2+x^2y^2-xy^3z-xyz^3+2xyz+z^2-2$
197	$a^{-2}b^{-1}aba^2b^{-1}$	32	$-x^4z+x^3yz^2+x^3y-2x^2y^2z-x^2z^3+4x^2z+xy^3+xyz^2-3xy-z$
198	$a^{-2}b^{-1}aba^2b$	32	$x^4z-x^3yz^2-x^3y+2x^2y^2z+x^2z^3-4x^2z-xy^3-xyz^2+4xy+z$
199	$a^{-2}ba^{-1}b^2a^{-1}b^{-1}$	32	$x^3y^3z-x^3yz-x^2y^4-2x^2y^2z^2+2x^2y^2+x^2z^2+xy^3z+xyz^3-2xyz+y^2-2$
200	$a^{-2}bab^2a^{-1}b^{-1}$	32	$-x^3yz+x^2y^2z^2+x^2y^2-x^2-xy^3z-xyz^3+3xyz-y^2+2$
201	$a^{-2}b^{-1}ab^2a^{-1}b^{-1}$	32	$x^3yz-x^2y^2z^2-x^2y^2+x^2+2xy^3z+xyz^3-4xyz-y^4-y^2z^2+4y^2-2$
202	$a^{-2}b^4a^{-1}b^{-1}$	32	$x^2y^4z-3x^2y^2z+x^2z-xy^5-xy^3z^2+4xy^3+2xyz^2-3xy+y^2z-z$

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Trace polynomials of the word length 8 – continued from previous page

Index	representative w	$\#[w]$	trace polynomial
203	$a^{-2}ba^{-1}b^{-2}ab^{-1}$	32	$-x^4y^2 + x^3y^3z + x^3yz - x^2y^4 - 2x^2y^2z^2 + 4x^2y^2 - x^2 + xy^3z + xyz^3 - 3xyz - y^2 + 2$
204	$a^{-2}bab^{-2}ab^{-1}$	32	$x^4y^2 - x^4 - x^3y^3z + x^2y^4 + 2x^2y^2z^2 - 5x^2y^2 - x^2z^2 + 4x^2 - xy^3z - xyz^3 + 4xyz + y^2 - 2$
205	$a^{-2}b^2a^{-1}b^{-1}ab^{-1}$	32	$-x^4y^2 + x^4 + x^3y^3z - x^2y^4 - 2x^2y^2z^2 + 5x^2y^2 + x^2z^2 - 4x^2 + xy^3z + xyz^3 - 3xyz - y^2 - z^2 + 2$
206	$a^{-2}b^2ab^{-1}ab^{-1}$	32	$x^4y^2 - x^3y^3z - x^3yz + x^2y^4 + 2x^2y^2z^2 - 4x^2y^2 + x^2 - xy^3z - xyz^3 + 3xyz + y^2 + z^2 - 2$
207	$a^{-2}b^{-1}a^2ba^{-1}b$	32	$x^5y - x^4y^2z - x^4z + x^3y^3 + 2x^3yz^2 - 4x^3y - x^2y^2z - x^2z^3 + 4x^2z + xy - z$
208	$a^{-2}b^{-1}a^2b^{-1}ab^{-1}$	32	$-x^5y + x^4y^2z + x^4z - x^3y^3 - 2x^3yz^2 + 4x^3y + x^2y^2z + x^2z^3 - 4x^2z + xy^3 - 2xy - y^2z + z$
209	$a^{-2}ba^{-2}b^{-1}ab$	32	$x^5y - x^4y^2z - x^4z + x^3y^3 + 2x^3yz^2 - 4x^3y - x^2z^3 + 3x^2z - xy^3 - xyz^2 + 2xy + z$
210	$a^{-2}b^2a^{-1}bab^{-1}$	32	$x^4y^2 - x^4 - x^3y^3z + x^2y^4 + 2x^2y^2z^2 - 4x^2y^2 - x^2z^2 + 4x^2 - xy^3z - xyz^3 + 2xyz + z^2 - 2$
211	$a^{-2}b^2abab^{-1}$	32	$x^3yz - x^2y^2z^2 - x^2 + xy^3z + xyz^3 - 3xyz - z^2 + 2$
212	$a^{-2}ba^{-1}b^2ab^{-1}$	32	$x^4y^2 - x^3y^3z - x^3yz + x^2y^4 + 2x^2y^2z^2 - 3x^2y^2 - x^2 - xy^3z - xyz^3 + 3xyz - y^2 + 2$
213	$a^{-2}bab^2ab^{-1}$	32	$-x^4 + 2x^3yz - x^2y^2z^2 - x^2y^2 - x^2z^2 + 4x^2 + xy^3z + xyz^3 - 4xyz + y^2 - 2$

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Trace polynomials of the word length 8 – continued from previous page

Index	representative w	$\#[w]$	trace polynomial
214	$a^{-2}b^4ab^{-1}$	32	$x^3y^3 - 2x^3y - x^2y^4z + 2x^2y^2z + xy^5 + xy^3z^2 - 5xy^3 - 2xyz^2 + 6xy - y^2z + z$
215	$a^{-2}ba^{-1}b^{-1}a^2b^{-1}$	32	$-x^5y + x^4y^2z + x^4z - x^3y^3 - 2x^3yz^2 + 5x^3y + x^2z^3 - 4x^2z + xy^3 + xyz^2 - 4xy + z$
216	$a^{-2}ba^{-1}b^{-1}a^2b$	32	$x^5y - x^4y^2z - x^4z + x^3y^3 + 2x^3yz^2 - 5x^3y - x^2z^3 + 4x^2z - xy^3 - xyz^2 + 5xy - z$
217	$a^{-2}bab^{-2}a^{-1}b$	32	$x^4y^2 - x^3y^3z - x^3yz + x^2y^4 + 2x^2y^2z^2 - 5x^2y^2 + x^2 - xyz^3 + 4xyz - y^4 - y^2z^2 + 4y^2 - 2$
218	$a^{-2}b^2a^{-1}b^{-1}a^{-1}b$	32	$x^3y^3z - x^3yz - x^2y^4 - 2x^2y^2z^2 + 3x^2y^2 + x^2z^2 - x^2 + xyz^3 - xyz + y^4 + y^2z^2 - 4y^2 - z^2 + 2$
219	$a^{-2}b^2ab^{-1}a^{-1}b$	32	$x^4y^2 - x^3y^3z - x^3yz + x^2y^4 + 2x^2y^2z^2 - 4x^2y^2 - xyz^3 + 2xyz - y^4 - y^2z^2 + 4y^2 + z^2 - 2$
220	$a^{-2}b^2a^{-1}ba^{-1}b$	32	$x^4y^4 - x^4y^2 - 3x^3y^3z + 2x^3yz - x^2y^4 + 3x^2y^2z^2 - x^2z^2 + x^2 + 2xy^3z - xyz^3 - xyz - y^2z^2 + y^2 + z^2 - 2$
221	$a^{-2}b^2aba^{-1}b$	32	$-x^4y^2 + x^3y^3z + x^3yz - 2x^2y^2z^2 + 2x^2y^2 - xy^3z + xyz^3 - xyz + y^2z^2 - y^2 - z^2 + 2$
222	$a^{-2}bab^2a^{-1}b$	32	$-x^4y^2 + x^3y^3z + x^3yz - 2x^2y^2z^2 + 2x^2y^2 + x^2 - xy^3z + xyz^3 - 2xyz + y^2z^2 - 2$
223	$a^{-2}b^4a^{-1}b$	32	$x^3y^5 - 3x^3y^3 + x^3y - 2x^2y^4z + 5x^2y^2z - x^2z - xy^5 + xy^3z^2 + 3xy^3 - 2xyz^2 + y^4z - 3y^2z + z$

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Trace polynomials of the word length 8 – continued from previous page

Index	representative w	$\#[w]$	trace polynomial
224	$a^{-2}b^2a^{-1}b^{-1}ab$	32	$-2+z^2+4y^2-y^2z^2-y^4+2xyz-xyz^3+4x^2-x^2z^2-5x^2y^2+2x^2y^2z^2+x^2y^4-x^3y^3z-x^4+x^4y^2$
225	$a^{-2}b^2ab^{-1}ab$	32	$-x^4y^2+x^3y^3z+x^3yz-x^2y^4-2x^2y^2z^2+5x^2y^2-x^2+xyz^3-3xyz+y^4+y^2z^2-4y^2-z^2+2$
226	$a^{-2}b^2a^{-1}bab$	32	$-x^4y^2+x^4+x^3y^3z-2x^2y^2z^2+3x^2y^2+x^2z^2-4x^2-xy^3z+xyz^3-xyz+y^2z^2-y^2-z^2+2$
227	$a^{-2}b^2abab$	32	$-x^3yz+x^2y^2z^2-x^2y^2+x^2-xyz^3+3xyz-y^2z^2+y^2+z^2-2$
228	$a^{-2}b^4ab$	32	$-x^3y^3+2x^3y+x^2y^4z-2x^2y^2z-xy^3z^2+2xy^3+2xyz^2-5xy-y^4z+3y^2z-z$
229	$a^{-2}b^3a^{-1}b^2$	32	$x^3y^5-3x^3y^3+2x^3y-2x^2y^4z+4x^2y^2z-x^2z-xy^5+xy^3z^2+4xy^3-xyz^2-4xy+y^4z-3y^2z+z$
230	$a^{-2}b^3ab^2$	32	$-x^3y^3+x^3y+x^2y^4z-x^2y^2z-xy^3z^2+xy^3+xyz^2-xy-y^4z+3y^2z-z$
231	$a^{-1}b^{-1}ab^5$	32	$x^2y^4-3x^2y^2+x^2-xy^5z+3xy^3z-xyz+y^6+y^4z^2-6y^4-3y^2z^2+9y^2+z^2-2$
232	$a^{-1}b^{-1}ab^{-5}$	32	$-x^2y^4+3x^2y^2-x^2+xy^5z-3xy^3z+xyz-y^4z^2+y^4+3y^2z^2-4y^2-z^2+2$
233	$a^{-1}b^{-2}ab^{-4}$	32	$-x^2y^4+2x^2y^2+xy^5z-2xy^3z-y^4z^2+2y^2z^2+y^2-2$
234	$a^{-1}b^{-2}ab^4$	32	$x^2y^4-2x^2y^2-xy^5z+2xy^3z+y^6+y^4z^2-6y^4-2y^2z^2+9y^2-2$
235	$a^{-1}b^{-3}a^{-1}ba^{-1}b^{-1}$	32	$-x^2y^2z+xy^3z^2-y^4z-y^2z^3+4y^2z+z^3-3z$

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Trace polynomials of the word length 8 – continued from previous page

Index	**representative** w	$\#[w]$	**trace polynomial**
236	$a^{-1}b^{-3}a^{-1}bab^{-1}$	32	$-x^3y+2x^2y^2z-x^2z-xy^3z^2-xy^3+3xy+y^4z+y^2z^3-4y^2z-z^3+3z$
237	$a^{-1}b^{-1}ab^{-3}a^{-1}b$	32	$-x^3y^3+x^3y+x^2y^4z-x^2z-xy^5-2xy^3z^2+5xy^3+2xyz^2-5xy+y^4z+y^2z^3-4y^2z-z^3+3z$
238	$a^{-1}b^{-1}a^{-1}b^{-3}ab^{-1}$	32	$-x^2y^2z+x^2z+xy^3z^2-xy^3-xyz^2+2xy-y^2z^3+2y^2z+z^3-3z$
239	$a^{-1}b^{-1}ab^3a^{-1}b$	32	$x^3y^3-x^3y-x^2y^4z+x^2z+xy^5+2xy^3z^2-4xy^3-2xyz^2+2xy-y^4z-y^2z^3+4y^2z+z^3-3z$
240	$a^{-1}b^{-1}ab^{-1}ab^{-3}$	32	$-x^3y^3+x^3y+x^2y^4z-x^2z-2xy^3z^2+xy^3+2xyz^2-xy+y^2z^3-2y^2z-z^3+3z$
241	$a^{-1}b^{-1}ab^{-1}ab^3$	32	$x^3y^3-x^3y-x^2y^4z+x^2z+xy^5+2xy^3z^2-5xy^3-2xyz^2+4xy-y^4z-y^2z^3+5y^2z+z^3-3z$
242	$a^{-1}b^{-1}a^{-1}b^{-3}ab$	32	$x^2y^2z-x^2z-xy^3z^2+xyz^2-xy+y^4z+y^2z^3-4y^2z-z^3+3z$
243	$a^{-1}b^{-1}a^{-1}b^{-1}ab^3$	32	$x^2y^2z-x^2z-xy^3z^2+xyz^2+xy+y^4z+y^2z^3-5y^2z-z^3+3z$
244	$a^{-1}b^{-1}a^{-1}bab^3$	32	$-x^2y^2z+x^2z+xy^3z^2+xy^3-xyz^2-2xy-y^4z-y^2z^3+4y^2z+z^3-3z$
245	$a^{-1}b^{-2}a^{-1}bab^{-2}$	32	$-x^3y+2x^2y^2z-xy^3z^2-xy^3-xyz^2+3xy+y^4z+y^2z^3-3y^2z-z$
246	$a^{-1}b^{-2}a^{-1}b^{-1}ab^{-2}$	32	$x^3y-2x^2y^2z+xy^3z^2+xyz^2-2xy-y^2z^3+y^2z+z$
247	$a^{-1}b^{-2}a^{-1}b^2a^{-1}b^{-1}$	32	$-x^2y^2z+x^2z+xy^3z^2-xy-y^4z-y^2z^3+4y^2z-z$
248	$a^{-1}b^{-2}a^{-1}b^2ab^{-1}$	32	$-x^3y+2x^2y^2z-xy^3z^2-xy^3-xyz^2+4xy+y^4z+y^2z^3-4y^2z+z$

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Trace polynomials of the word length 8 – continued from previous page

Index	representative w	$\#[w]$	trace polynomial
249	$a^{-1}b^{-2}a^{-1}b^2a^{-1}b$	32	$-x^3y^3+x^3y+x^2y^4z+x^2y^2z-x^2z-xy^5-2xy^3z^2+4xy^3-2xy+y^4z+y^2z^3-4y^2z+z$
250	$a^{-1}b^{-2}a^{-1}b^2ab$	32	$x^3y-2x^2y^2z+xy^3z^2+xy^3+xyz^2-3xy-y^4z-y^2z^3+4y^2z-z$
251	$a^{-1}b^{-1}ab^2a^{-1}b^2$	32	$x^3y^3-x^3y-x^2y^4z+xy^5+2xy^3z^2-4xy^3-xyz^2+2xy-y^4z-y^2z^3+3y^2z+z$
252	$a^{-1}b^{-1}ab^{-2}ab^{-2}$	32	$-x^3y^3+x^3y+x^2y^4z-2xy^3z^2+xy^3+xyz^2-xy+y^2z^3-y^2z-z$
253	$a^{-1}b^{-1}ab^{-2}a^{-1}b^2$	32	$-x^3y^3+x^3y+x^2y^4z-xy^5-2xy^3z^2+5xy^3+xyz^2-4xy+y^4z+y^2z^3-4y^2z+z$
254	$a^{-1}b^{-2}ab^2a^{-1}b$	32	$x^3y^3-x^2y^4z-x^2y^2z+xy^5+2xy^3z^2-4xy^3+xy-y^4z-y^2z^3+4y^2z-z$
255	$a^{-1}b^{-1}ab^{-2}ab^2$	32	$x^3y^3-x^3y-x^2y^4z+xy^5+2xy^3z^2-5xy^3-xyz^2+5xy-y^4z-y^2z^3+4y^2z-z$
256	$a^{-1}b^{-2}ab^2a^{-1}b^{-1}$	32	$x^2y^2z-xy^3z^2+y^4z+y^2z^3-4y^2z+z$
257	$a^{-1}b^{-1}a^{-1}b^{-1}a^{-1}bab^{-1}$	32	$x^2z^2-x^2-xyz^3+xyz+y^2z^2-y^2+z^4-4z^2+2$
258	$a^{-1}b^{-1}a^{-1}ba^{-1}b^{-1}ab^{-1}$	32	$-x^3yz+x^2y^2z^2+x^2z^2-x^2-xy^3z-2xyz^3+5xyz+y^2z^2-y^2+z^4-4z^2+2$
259	$a^{-1}b^{-1}a^{-1}bab^{-1}ab^{-1}$	32	$x^3yz-x^2y^2z^2-x^2z^2+xy^3z+2xyz^3-4xyz-y^2z^2+y^2-z^4+4z^2-2$
260	$a^{-1}b^{-1}a^{-1}ba^{-1}bab^{-1}$	32	$x^3yz-x^2y^2z^2-x^2z^2+x^2+xy^3z+2xyz^3-4xyz-y^2z^2-z^4+4z^2-2$

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Trace polynomials of the word length 8 – continued from previous page

Index	**representative** w	$\#[w]$	**trace polynomial**
261	$a^{-1}b^{-1}a^{-1}bab^{-1}a^{-1}b$	32	$x^3yz-x^2y^2z^2-x^2y^2-x^2z^2+x^2+2xy^3z+2xyz^3-4xyz-y^4-2y^2z^2+4y^2-z^4+4z^2-2$
262	$a^{-1}b^{-1}a^{-1}baba^{-1}b$	32	$-x^3yz+x^2y^2z^2+x^2y^2+x^2z^2-x^2-xy^3z-2xyz^3+3xyz+y^2z^2-y^2+z^4-4z^2+2$
263	$a^{-1}b^{-1}a^{-1}b^3a^{-1}b$	32	$x^2y^4z-2x^2y^2z-xy^5-2xy^3z^2+4xy^3+3xyz^2-3xy+y^4z+y^2z^3-4y^2z-z^3+3z$
264	$a^{-1}b^{-1}aba^{-1}b^3$	32	$x^3y^3-2x^3y-x^2y^4z+x^2y^2z+x^2z+xy^5+2xy^3z^2-5xy^3-3xyz^2+6xy-y^4z-y^2z^3+4y^2z+z^3-3z$
265	$a^{-1}b^{-1}ab^{-1}a^{-1}bab^{-1}$	32	$-x^4+2x^3yz-x^2y^2z^2-x^2y^2-2x^2z^2+4x^2+xy^3z+2xyz^3-4xyz-y^2z^2+y^2-z^4+4z^2-2$
266	$a^{-1}b^{-1}aba^{-1}ba^{-1}b$	32	$x^4y^2-x^3y^3z-2x^3yz+x^2y^4+3x^2y^2z^2-3x^2y^2+x^2z^2-x^2-2xy^3z-3xyz^3+7xyz+y^2z^2-y^2+z^4-4z^2+2$

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국문초록

$SL(2,\mathbb{C})$의 생성자가 2개인 부분군에서는, 행렬대각합을 생성자들의 대각합에 대한 다항식으로 나타낼 수 있고, 그 다항식은 생성자들의 선택에 의존하지 않는다. 이 다항식을 대각합 다항식이라 한다. 본 학위논문에서는 Jorgensen의 정리로 대각합 다항식을 계산하는 알고리듬을 구현하였고, 이 결과로 워드길이 8 이하의 모든 순환기약워드들을 분류하였다. 다음으로, 분류한 순환기약워드들이 Mir, LeftShift, Inverse로 정의되는 $\sim$ 동치관계에 있는지를 조사하였다. 이것으로 다음 두 가지를 증명하였다. 먼저 워드길이 8 이하에 대해, 두 순환기약워드가 $\sim$-동치관계인것은 대각합다항식이 같은것과 동치임을 보였다. 다음으로 워드길이 9 이상부터는, 대각합 다항식은 같지만 $\sim$-동치관계는 아닌 두 순환기약워드가 존재함을 보였다. 이것은 워드길이 8이하에 대해, Wang의 추측이 성립한다는 것을 보여준다.

주요어휘: 대각합 다항식, 계수2 자유군, 특수 선형군, 순환 기약 워드
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