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이 논문을 경영학 석사학위논문으로 제출함

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전동규의 석사학위논문을 인준함

2021 년 2 월

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# Abstract

We study the optimal inventory problem for two partially substitutable products where they face different levels of supply reliability. We find that the implications of increasing substitution rate result in a decreased unreliable product (e.g., organic product) order but an increased reliable product (e.g., regular product) order, as the regular product becomes an effective substitute during the organic product's stock-out. We also find that increasing risk of random disruptions and yield uncertainty result in different response strategies. Higher disruption risk pushes the firm to increase the order size of both products until the substitution rate reaches some threshold; after this threshold (i.e., when the substitution rate is very high), the retailer only increases the regular order size and reduces the organic order size for the increasing disruption risks. However, higher yield uncertainty only increases the organic order size whereas the regular order size decreases or remains unchanged.

**Keyword :** Inventory management, supply disruptions, yield uncertainty, substitution
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# 1. Introduction

As the supply chains expand across the globe and become more complex, the need for measures to address supply chain risks is increasingly important. An important aspect of supply chain risks is supply-side uncertainty, which has gained increasing attention from both researchers and practitioners in recent several decades. In particular, food industry supply chain is highly exposed to the risk of supply uncertainties and easily affected by unexpected incidents such as natural disasters or global pandemics. For example, the recent lockdowns, closed borders, and international trade restrictions caused by COVID-19 have taken a heavy toll especially on the global food supply chains (Clapp, 2020). In addition, such supply risks amplify the risk of worldwide food supply shortages and food-price spike due to disruptions in production and distribution across the globe (Reinhart and Subbaraman, 2020).

One good application of this is the growing organic and/or fair trade food industry where its supply faces considerable disruption risk and yield uncertainty. Organic and fair trade supply chains bear the brunt of such supply risks since they are vulnerable due to several intrinsic properties of the industry. First, there is a relatively smaller number of reliable suppliers within the organic/fair trade supply system compared to that of conventional products. Moreover, production costs for organic/fair trade products are typically higher due to a greater labor input per unit of output and because economies of scale cannot be achieved due to a relatively small output volumes. Ortiz-Miranda and Moragues-Faus (2015) argue that fair trade supply chains could not circumvent the problems of indebtedness of smallholders since they were under the influence of a 'simple reproduction squeeze' resulted from high costs of production and consumption, low yields, high interest rates, and low farm-gate prices. Another challenge in organic/fair trade supply chain is that it has more complex stages in their marketing channels to be processed and delivered to end-customers, which imposes more difficulties on the efficient inventory management. In addition, De Ponti et al. (2012) suggested in their study that organic crops are harvested on average 80% of conventional crops, and standard deviation of organic crop yields accounts for 21%, i.e., average relative yield of organic products being 79%. Due to such intrinsic supply uncertainties, organic/fair trade products suffer from frequent stock out and consumers often choose to substitute their organic demands with conventional counterparts. Despite such supply risks, however, consumers are purchasing more organic and fair trade products these days, if available, as they seek to enhance their personal health and immunity and care for the environment. Accordingly, global organic/fair trade product sales surpassed the USD 100 billion in 2018 and are expected to reach USD 150 billion within the next several years (Ecovia Intelligence, 2020). Hence, forging an efficient inventory system for seamless supply of organic products into the market is a critical issue for retailers to fulfill the increased demands for organic products. Thus, in this paper, we examine the optimal joint inventory problem of two partially substitutable (e.g., organic and regular) products where one product faces greater degree of supply uncertainties than the other.

Specifically, we study a firm facing two different types of supply uncertainties: supply disruptions and yield uncertainty. Supply disruption represents the state in which the product supplier is unavailable at random points in time for a random duration. In particular, the supply disruption follows a stochastic process with a two-state (ON and OFF) markov chain where the OFF state implies the state of supply disruption. Snyder (2014) states that disruptions can have strong impacts on supply chain operations and are inevitable in a supply chain. In addition, it may come from a variety of sources, including strikes, machine/facility breakdowns or maintenance, and natural disasters or man-made catastrophe. In the literature, classical EOQ model with supply disruptions (a.k.a. EOQD) was first introduced by Parlar and Berkin (1991), whose model was revised and improved by Berk and Arreola-Risa (1994). The model proposed by Berk and Arreola-Risa (1994) can be optimized numerically but cannot be solved in closed-from, whereas the models we propose in this paper can be solved in closed-form, which can be usefully utilized in more richer and sophisticated models. Another supply uncertainty we address in this paper is yield uncertainty. Random yield at the supplier is critical to the retailers as the actual amount of items received from the supplier might fall short of the amount requisitioned, which results in a service failure and increased lost sales cost at the retailer. The classical paper which addressed random yields in inventory management was done by Silver (1976) and Parlar and Perry (1995), details of which will be discussed in the next section.

Another interesting context we address is partial substitution between the products. Due to the highly volatile and easily-depleted supply conditions within the organic food supply chain, consumers who are willing to buy the organic products are apt to confront the circumstances where the organic products are already stocked out on the shelf. In these situations, consumers have an option to choose whether to purchase the substitutable product (i.e. regular product) or leave the store, which incurs a penalty cost to the retailer. Thus, determining the optimal ordering quantities for each product is a very important issue for retailers when making inventory decisions. In this regard, we discuss the concept of partial substitution between the products in case one product is stock-out and cannot be replenished in due course resulting from the supplier disruptions.

This paper presents EOQ-based inventory models under single-product and two-product case.

In a single-product case, both disruptions and random yields are considered when formulating the expected inventory cost function. In a two-product case, we develop an inventory model that incorporates supplier disruptions and yield uncertainty in an unreliable product (i.e., organic product), as well as partial substitution between the products. We assume that there is no supply-side uncertainty for a reliable product (i.e., regular product) as there are sufficient number of suppliers in the market from which retailers can secure the consistent and stable supply of the products. We aim to derive the optimal order sizes in closed-from and compare them with the classical EOQ solution or the variant solutions presented in the past literature.

Our primary focus is to derive the optimal joint inventory policy and obtain insights on the impacts of product substitution and supply uncertainties. Our analysis shows that substitution rate has a significant impact on the manager's inventory decisions. We find that, as the substitution rate increases, the firm decreases the order quantity of organic product while increasing the order quantity of regular product. This is because the regular product emerges as an effective substitute for the organic product, and thus the firm increases the degree of risk-pooling via substitution. We also identify the impact of supply uncertainties on the inventory decisions, and find that the response strategies against supply disruptions and yield uncertainty vary. Specifically, whereas the optimal strategy is to increase the order quantities of both products as disruption risk increases until some threshold substitution rate, it is optimal to increase the organic order quantity only to face the increasing yield uncertainty. When the substitution rate is very high, however, it is optimal to increase only the regular order size and reduce the organic order size for the increasing disruption risks.

## 2. Literature review

Recently, there has been increasing attention on the problem of designing and effectively managing the supply chain of perishable products (e.g., fresh produce). A good example in this stream of research was done by Blackburn and Scudder (2009), whose research consider the product value deterioration as a challenge for the effective management of the fresh produce supply chain. They argue that the appropriate model to minimize the loss in value in the supply chain is a hybrid strategy of a responsive model and an efficient model. Cai and Zhou (2014) address the supply chain of a firm that produces a wide array of fresh products to supply two potential markets, an export market and a local market. Export market is more profitable compared to the local market, but export market involves a high risk of decaying during the delivery. They study the situation when the delivery service to the export market can be disrupted, and the optimal supply chain policies in the perishable food production industry. Soto-Silva et al. (2016) build upon this stream of research by considering an efficient management of fresh fruit supply chain, which suffers from comparatively long lead times as well as severe supply and demand uncertainties. They attempt to investigate the operations research methods that have been applied to the fresh fruit supply chain and gain better understandings of the modeling methods employed in these settings. Asian et al. (2019) examine the value of sharing economy (SE) in affording small-size organic farmers ways to overcome a set of challenges (e.g., increasing transportation costs, relatively lower yields, and market barriers) they face and improve their competitiveness in the market by sharing an SE-based cooperative platform. They develop a two-stage optimization model that jointly addresses the production-inventory planning and pricing problems of multiple competing organic food supply chains using a cooperative game theory approach. One of the latest research regarding organic food supply chains was conducted by Tundys and Wiśniewski (2020) They attempt to clarify ideas of alternative short food supply chains for organic products and to explore the activities that can generate economic benefits in individual markets using a computer simulation method.

The inventory management under random disruptions at suppliers has been extensively studied by many authors. Parlar and Berkin (1991) is one of the earliest studies that incorporate disruptions at the supplier into classical inventory models. Their model has been referred to as *EOQ with disruptions (EOQD)*. They study the classical EOQ inventory system in which the supplier faces sporadic failures for an interval of random duration, during which consumer demands are lost if the retailer has no inventory for selling. The underlying features of their study are zero-inventory ordering (ZIO) policy and no backorders. It is assumed that the decision maker (the retailer) is aware of the availability status of the supplier at any time. They formulate the total expected cost as a function of order quantity using renewal reward theorem. Their cost function contained two errors which made their model incorrect. A correct version of model was presented by Berk and Arreola-Risa (1994), whose model was shown to be pseudoconvex and unimodal but not solved in closed-form.

Snyder (2014) extends the model introduced by Berk and Arreola-Risa (1994) by developing a tight approximation for the model. He shows that the approximate cost function is convex and can be solved in closed form, which resembles the classical EOQ optimal order quantity in its structural form, but with a few added terms, emphasizing the relationship with the classical EOQ solution. Qi et al. (2009) study the EOQ model with disruptions considering not only the external disruptions (i.e., supplier disruptions) but also the internal disruptions (i.e., retailer disruptions) which destroys the inventory at the retailer. Their cost function is shown to be quasi-convex and thus can be numerically optimized to get the optimal order size using any standard method such as bisection search. They also introduce an approximation of the cost function and present a closed form expression for the optimal order quantity from the supplier to the retailer.

Weiss and Rosenthal (1992) formulate an EOQ-like inventory system with possible disruptions in either supply or demand and derive the optimal ordering quantities in each case. The underlying feature of this work is that the point at which a disruption can occur is known at the beginning whereas the disruption length is random. Parlar and Perry (1995) extend the classical $EOQD$ by developing an inventory model with the reorder point as a decision variable and assuming that the availability state of the system is observed at a cost unlike Parlar and Berkin (1991). This study analyzes the problem with both deterministic and random yields at the supplier and aims to determine the optimal order quantity, reorder point, and review interval that minimize the total expected cost. Parlar and Perry (1996) build on the literature by considering one, two, or multiple suppliers in the system and by setting reorder point as a nonnegative decision variable. Gürler and Parlar (1997) primarily focus on developing and generalizing the two-supplier inventory model with more general forms of the ON (available) and OFF (unavailable) periods at the suppliers. Güllu et al. (1997) study a periodic review inventory model where supply is either available or totally unavailable with given probabilities in a certain period. They assume that availability and unavailability probabilities are nonstationary over time and independent from one period to another. Building upon Parlar and Berkin (1991), Arreola-Risa and DeCroix (1998) address an inventory system with random supply disruptions and partial backorders. During stockouts, a fraction of demands is backordered and the remaining fraction is lost. They derive explicit expressions for $(s, S)$ models with supply disruptions.

Chopra et al. (2007) considers a single period inventory problem with two distinct suppliers, where one supplier is prone to both delay (recurrent) risks and disruption risks but cheaper and the other is completely reliable but more expensive. Schmitt and Snyder (2012) extend the analysis to the infinite-horizon system, developing models for both single-supplier and two-supplier case. Hu and Kostamis (2015) focus on exploring an effective strategy to mitigate the serious impacts of supply disruptions in a supply chain. They adopt a practice of multiple sourcing strategy that involves both reliable and unreliable suppliers. Unlike the papers cited above, Ross et al. (2008) discuss a firm that experiences random demand and disruptions, where the parameters of demand and disruptions are time-varying. There have been a few articles that have considered stochastic demand with supply disruptions, details of which are omitted in this paper. Please see Vakharia and Yenipazarli (2009), Atan and Snyder (2012), and Snyder et al. (2016) for further details on supply chain disruptions.

The modeling inventory decisions when the actual amount received is uncertain due to the fluctuations in supplier's yields has been one of the focuses of operations management research and largely discussed by many authors. Silver (1976) is one of the earliest studies that extends the classical EOQ formulation to include the case where the actual quantity received from the supplier does not match the quantity ordered. Henig and Gerchak (1990) aims to provide qualitative implications of yield uncertainty for lot sizing in a general periodic review inventory system. Their analysis involves two cases: one concerning the manner in which the random yield is dependent on the lot size (input level) without any assumptions, and another assuming that the yield (actual quantity received) is a random multiple of the lot size, which is referred to as the *stochastically proportional yield model*. They provide analyses of single-period, finite-horizon, and infinite-horizon models with attempts to provide theoretical underpinnings of modeling with variable yield. Parlar and Perry (1995) presents a stochastic inventory model under both disruptions and random yields in which the actual quantity received is a random function of the quantity ordered. The analyses under both deterministic yield and random yield are presented and they discuss an example with yield distributed as a beta random variable.

Agrawal and Nahmias (1997) consider an inventory model to determine optimal order sizes and the optimal number of suppliers when the amount received from the supplier is random. They present a critical trade-off between small order sizes from many suppliers which help reduce the yield uncertainty and increased fixed costs as a penalty for ordering from too many suppliers. They also explore the cases of suppliers being identical and nonidentical. Dada et al. (2007) discuss the problem of a newsvendor ordering from either reliable or unreliable suppliers. Reliable supplier is defined as the supplier that can fulfill the amounts requisitioned, and unreliable supplier delivers an amount less than the amount ordered with some probability. They provide results that indicate the unreliability effect in three perspectives: newsvendor, customers, and suppliers. The major finding is that, in case where a newsvendor orders from multiple reliable or unreliable suppliers, the aggregate quantity ordered is higher from the perspective of the newsvendor, while the service level provided is lower from the perspective of end-consumers. From the perspective of suppliers, it is found as a general rule that cost takes precedence over reliability when it comes to indexing suppliers for selection.

Snyder and Shen (2019) introduce an EOQ model with yield uncertainty, whose optimal order size does not depend on the distribution of random yield, but only on its first two moments. Schmitt and Snyder (2012) highlight the importance of addressing multiple time periods when modeling inventory systems under disruptions. They use a model similar to that of Chopra et al. (2007), presenting infinite-horizon inventory models under two systems: a system with one unreliable supplier facing both random disruptions and yield uncertainty, and another system with both unreliable supplier and a perfectly reliable, but more expensive, supplier. While many previous literature on imperfect supply has paid attention on proportional yield problem, Skouri et al. (2014) study a somewhat idealized but simple setting where entire supply batches may be disrupted; this problem can be referred to as binary policy (also known as "all or nothing" policy), which is a special case of the proportional yield problem.

Many authors have conducted studies on inventory management regarding substitution between products on a variety of distinct contexts. Pasternack and Drezner (1991) is one of the earliest studies that consider a single-period inventory system with two products that can be substituted for each other when one item is out of stock and the other is available. This paper assumes that the substitution occurs with a probability of one should the need arise, but at a different revenue level. Drezner et al. (1995) present an EOQ model considering substitution between products at a given unit cost. They provide analyses under three cases: no substitution, full substitution, and partial substitution. Gurnani and Drezner (2000) suggest a deterministic nested substitution problem where multiple products can be substituted at a certain cost. They found that that the total cost function, when more than two products are considered, may not be convex in the decision variables. To solve this problem, they employ a sequence of substitutions to new variables which enables the reformulation of a new cost function to be convex and thus can be optimally solved.

Nagarajan and Rajagopalan (2008) consider a scenario with two substitutable products where a retailer should determine the stocking level for the two products. They fix a proportion of unsatisfied customers for a product that will purchase the other item if it is available in stock. They provide analyses of the model in both single-period and multi-period scenarios. Lu et al. (2011) present a supply chain model where two products are fulfilled from two suppliers of different features. They assume a downward substitution between products, that is, a higher-grade product can be substituted for the other lower-grade one when it is unavailable. Among two suppliers, one supplier is subject to random disruptions (failures) whereas the other one is perfectly reliable. Chen et al. (2015) also consider a downward substitution between products, but with stochastic demand for each item. They assume supplier-driven substitution, i.e., the supplier (not the retailer) reacts to the stock-out and takes action. The products considered are perishable and thus a single-period problem is addressed in this paper. Another feature is that it incorporates customer service commitments (i.e., whether or not the firm commits to providing a minimum level of service) into the model to check how optimal inventory levels are affected by service commitments. Salameh et al. (2014) and Maddah et al. (2016) both consider the joint replenishment model with partial substitution between products. In particular, Maddah et al. (2016) present a special case of three substitutatble products in an EOQ framework. We contribute to this literature by considering two products with varying degrees of supply reliability where the demand for unreliable product can be substituted with the more reliable one.

# 3. Model formulation and Results

Consider a risk-neutral retailer that deals with two types of products from independent suppliers with varying levels of reliability. We assume that one type of product (i.e., organic) suffers from random supply disruptions and random yields, while the other product (i.e., regular) is fully reliable. We consider an EOQ-based inventory policy for both product types with no backorders. The retailer aims to minimize the expected total inventory cost by determining the optimal order quantities of each product type.

In a single-product case, we assume that the supplier of the organic product is not perfectly reliable, that is, the supplier may experience both supply disruptions and yield uncertainty. When the supplier experiences disruptions, the retailer must wait until the supplier has recovered to make an order. In a two-product case, we focus on the joint inventory management of both organic and regular product. We assume that organic product can be partially substituted with the regular product (i.e., one-way substitution) in case stock-out occurs for the organic product.

We derive the total expected cost function $\mathbb{E}[TC]$ using the renewal reward theorem (Ross 1996), as follows:

$$\mathbb{E}[TC] = \frac{\mathbb{E}[\textit{cost per cycle}]}{\mathbb{E}[\textit{cycle length}]}.$$

That is, in both cases, we formulate $\mathbb{E}[C]$, the expected cost per cycle at the retailer, and $\mathbb{E}[T]$, the expected order cycle length. Then we apply the renewal reward theorem to derive $\mathbb{E}[TC]$.

## 3.1 Single-product case: Organic product only

Consider an EOQ-based continuous review inventory model with single product. In this case, we deal with the product that is highly volatile in its supply capacities and has relatively unstable supply chain system (e.g., organic blueberry/milk/coffee beans etc.). Figure 1 illustrates a typical inventory curve of one product at the retailer when both random supply disruptions and random yields are considered. $Q$ and $T$ represent the order quantity and the cycle time, respectively. The random yield is captured by $Y$, where $Y$ is a continuous random variable with probability density function $f_Y$ and cumulative distribution function $F_Y$. We consider $Y$ to be bounded from above by 0 since the actual yield $(Q + Y)$ must be less than or equal to the order quantity $Q$. Noting that $Y$ is set to be truncated at $-Q$ since we can't receive a negative amount, we can establish the range of random variable $Y$ as $-Q \leq Y \leq 0$. However, as in the literature (e.g., Snyder and Shen (2019)), we formulate the cost function assuming the range of $Y$ as $[-\infty, \infty]$ to ensure that $Y$ does not depend on the decision variable $Q$. We denote the fixed ordering cost by $k$, holding cost per unit per time by $h$, and stock-out cost by $p$. To focus solely on supply uncertainties, we assume a constant deterministic demand rate (say, per year) denoted by $d$.

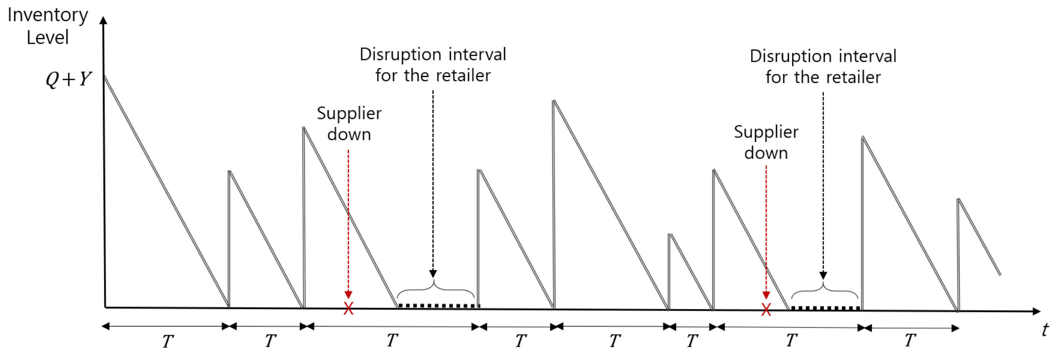


Figure 1: Inventory curve of a single product with supply disruptions and random yields

We consider supply disruption probability in the model, and we define it as the probability that the supplier is in an OFF period when the inventory level at the retailer reaches 0, denoted by $\psi$. When supply disruption occurs, the supplier enters the 'OFF' period. We assume that the 'ON' and 'OFF' intervals at the supplier follow *i.i.d.* exponential distributions with disruption rate $\lambda$ and recovery rate $\mu$, respectively, following Berk and Arreola-Risa (1994), Qi et al. (2009), Snyder (2014), Snyder and Shen (2019). During the OFF period, if the retailer depletes its remaining inventory level to 0, then the retailer disruption starts as in the figure. The length of retailer's downtime interval follows an exponential distribution with recovery rate $\mu$, regardless of the actual time of supplier's disruption due to the memoryless property. Using a two-state continuous-time Markov chain (CTMC), the disruption probability $\psi$ can be expressed as follows (e.g., Berk and Arreola-Risa (1994) and Snyder and Shen (2019)):

$$\psi = \frac{\lambda}{\lambda+\mu}\big[1 - e^{-(\lambda+\mu)t}\big].$$

Note that $\psi$ defined here is a disruption probability that does not consider the random yields (i.e., $Y = 0$). It is assumed that the yield (actual amount received) is always equal to the order quantity $Q$, and thus the cycle time from the beginning of an inventory cycle until the inventory level at the retailer hits 0 is always $Q/d$. Incorporating yield uncertainty, the cycle time becomes a random variable as a function of $Y$, $(Q+Y)/d$. Accordingly, the probability that the supplier is in an OFF interval when the retailer's inventory level hits 0 is also a random variable that depends on $Y$. This implies that the disruption probability $\psi$ may stochastically vary in each cycle. Since we aim to derive the expected total cost function, we obtain the expected disruption probability, denoted by $\hat{\psi}$, as follows:

$$\hat{\psi} \;=\; \int_{-\infty}^{\infty} \frac{\lambda}{\lambda+\mu}\Big[1-e^{-\frac{(\lambda+\mu)(Q+Y)}{d}}\Big] f_Y(y)\,dy \;=\; \frac{\lambda}{\lambda+\mu}\Big[1-\int_{-\infty}^{\infty} e^{-\frac{(\lambda+\mu)(Q+Y)}{d}} f_Y(y)\,dy\Big].$$

**Formulation of $\mathbb{E}[C]$.** We now derive the expected cost per inventory cycle, $\mathbb{E}[C]$, by summing over the expected values of ordering, holding, and shortage costs. When $\hat{\psi} = \frac{\lambda}{\lambda+\mu}[1 - \int_{-\infty}^{\infty} e^{-\frac{(\lambda+\mu)(Q+Y)}{d}} f_Y(y)\,dy]$, then $\mathbb{E}[C]$ is formulated as the following:

$$\begin{aligned}
\mathbb{E}[C] &= k + \int_{-\infty}^{\infty} \frac{h}{2d}(Q+Y)^2 f_Y(y)\,dy + \hat{\psi}p\frac{d}{\mu} \\
&= k + \frac{h}{2d}\Big[Q^2\int_{-\infty}^{\infty} f_Y(y)\,dy + 2Q\int_{-\infty}^{\infty} y f_Y(y)\,dy + \int_{-\infty}^{\infty} y^2 f_Y(y)\,dy\Big] + \hat{\psi}p\frac{d}{\mu} \\
&= k + \frac{h}{2d}\Big[Q^2 + 2Q\mathbb{E}[Y] + \mathbb{E}[Y^2]\Big] + \hat{\psi}p\frac{d}{\mu} \\
&= k + \frac{h}{2d}\Big[Q^2 + 2Q\mathbb{E}[Y] + \mathbb{E}[Y]^2 + \mathrm{Var}[Y]\Big] + \hat{\psi}p\frac{d}{\mu} \\
&= k + \frac{h}{2d}\Big[(Q+\mathbb{E}[Y])^2 + \mathrm{Var}[Y]\Big] + \hat{\psi}p\frac{d}{\mu}.
\end{aligned} \tag{1}$$

**Formulation of $\mathbb{E}[T]$.** The expected ordering cycle length, $\mathbb{E}[T]$, can be formulated by first deriving the expected value of each differing cycle duration, and add the expected delay period of the supplier in the event of disruptions at the supplier with a probability of $\hat{\psi}$.

$$\mathbb{E}[T] = \mathbb{E}\Big[\frac{Q+Y}{d}\Big] + \hat{\psi}\frac{1}{\mu} = \frac{Q+\mathbb{E}[Y]}{d} + \hat{\psi}\frac{1}{\mu}. \tag{2}$$

**Formulation of $\mathbb{E}[TC]$.** Combining (1) and (2), we can formulate $\mathbb{E}[TC]$ using renewal reward theorem. When the mean and the variance of Y, $\mathbb{E}[Y]$ and $\mathrm{Var}[Y]$, are known, and when $\hat{\psi}$ is defined as $\frac{\lambda}{\lambda+\mu}[1 - \int_{-\infty}^{\infty} e^{-\frac{(\lambda+\mu)(Q+Y)}{d}} f_Y(y)\,dy]$, we can express $\mathbb{E}[TC]$ as a function of $Q$ as follows:

$$\mathbb{E}[TC] = \frac{k + \frac{h}{2d}[(Q+\mathbb{E}[Y])^2 + \mathrm{Var}[Y]] + \hat{\psi}p\frac{d}{\mu}}{\frac{Q+\mathbb{E}[Y]}{d} + \hat{\psi}\frac{1}{\mu}}. \tag{3}$$

We can see that (3) has exponential terms both at the numerator and the denominator, which renders this function intractable to derive a closed-form solution $Q^*$ due to the complexity of its function and of its first and second derivative. Thus, we derive an approximate closed-form solution $\widetilde{Q}^*$, following Qi et al. (2009) and Snyder (2014). Since ON and OFF periods are exponentially distributed random variables with rate $\lambda$ and $\mu$ their expected durations are given by $1/\lambda$ and $1/\mu$, respectively. In general, the recovery rate $\mu$ is relatively larger than the disruption rate $\lambda$ (i.e., $\mu > \lambda$). For example, if ON cycles last, on average, 3 months, OFF cycles last 0.5 month (2 weeks) on average, then $\lambda = 4$ and $\mu = 24$, and if orders are placed quarterly ($d = 400$, $Q = 100$, $\frac{Q}{d} = 0.25$) and let $Y = -10$, then $e^{-\frac{(\lambda+\mu)(Q+Y)}{d}} = e^{-\frac{(4+24)(100-10)}{400}} = 0.00184 \approx 0$. Not only this case but in most practical settings, $e^{-\frac{(\lambda+\mu)(Q+Y)}{d}}$ is very close to 0 so we can also set $\int_{-\infty}^{\infty} e^{-\frac{(\lambda+\mu)(Q+Y)}{d}} f_Y(y)\, dy \approx 0$ in most realistic instances. We therefore approximate $\hat{\psi} = \frac{\lambda}{\lambda+\mu}[1 - \int_{-\infty}^{\infty} e^{-\frac{(\lambda+\mu)(Q+Y)}{d}} f_Y(y)\, dy]$ using $\hat{\psi} \approx \frac{\lambda}{\lambda+\mu}$. Using this approximation scheme, we obtain the optimal ordering strategy for the single-product case.

**Lemma 1.** *The expected total inventory cost in (3) is strictly convex with respect to Q. Further, the optimal order quantity $\widetilde{Q}^*$ that minimizes expected total cost is given by*

$$\widetilde{Q}^* = \sqrt{\frac{2kd}{h} + Var[Y] + \left(\frac{\hat{\psi}d}{\mu}\right)^2 + \frac{2d^2p\hat{\psi}}{\mu h}} - \frac{\hat{\psi}d}{\mu} - \mathbb{E}[Y]. \tag{4}$$

***Proof*.** Let $G(Q)$ be a total expected cost function $\mathbb{E}[TC]$ and let $\hat{\psi} = \frac{\lambda}{\lambda+\mu}$. Then, the total expected cost in equation (3) can be rewritten as the following:

$$\begin{aligned} G(Q) &= \frac{2dk\mu + hu[(Q+\mathbb{E}[Y])^2 + \mathrm{Var}[Y]] + 2\hat{\psi}pd^2}{2\mu(Q+\mathbb{E}[Y]) + 2\hat{\psi}d} \\ &= \frac{2dk\mu + uh\mathrm{Var}[Y] + 2\hat{\psi}pd^2}{2\mu(Q+\mathbb{E}[Y]) + 2\hat{\psi}d} + \frac{uh(Q+\mathbb{E}[Y])^2}{2\mu(Q+\mathbb{E}[Y]) + 2\hat{\psi}d}. \end{aligned} \tag{5}$$

Then, we get the following second-order condition of (5):

$$\frac{d^2G(Q)}{dQ^2} = \frac{\hat{\psi}^2d^2\mu h + 2\hat{\psi}d^2\mu^2 p + 2dk\mu^3 + h\mu^3\mathrm{Var}[Y]}{[\hat{\psi}d + (Q+\mathbb{E}[Y])\mu]^3}. \tag{6}$$

The reader can easily notice that all terms in equation (6) are positive, and thus $G(Q)$ is strictly convex in Q. Hence, the minimizer $\widetilde{Q}^*$ exists at some $Q$. It follows that the resulting closed-form approximate formula for $Q^*$, denoted by $\widetilde{Q}^*$, which minimizes the total expected cost is the solution to the first-order condition of (5) and can be expressed as follows:

$$\frac{dG(Q)}{dQ} = -\frac{2\mu d^2\hat{\psi}p + 2dk\mu^2 + h\mu^2\mathrm{Var}[Y]) + d^2\hat{\psi}^2 h}{2(\mu(Q+\mathbb{E}[Y]) + \hat{\psi}d)^2} + \frac{h}{2} = 0. \tag{7}$$

Solving (7) with respect to $Q$ gives us the optimal solution for $G(Q)$ as presented in (4). ■

Note that the optimal order size $\widetilde{Q}^*$ depends only on the statistical mean and variance of $Y$, $\mathbb{E}[Y]$ and $\mathrm{Var}[Y]$, rather than its distribution; i.e., neither the distribution type nor the range of $Y$ matter to the optimal order quantity $\widetilde{Q}^*$. Based on the optimal solution (4), we present a following proposition by comparing the special cases of our result with the models in the literature.

**Proposition 1.** *Optimal order quantity $\widetilde{Q}^*$ reduces to each corresponding solution of the following two special cases:*
*(a) When the supplier is never disrupted ($\hat{\psi} = 0$), $\widetilde{Q}^*$ reduces to the solution derived by Snyder and Shen (2019) via the model that only considers random yields; i.e., $Q^* = \sqrt{\frac{2kd}{h} + Var[Y]} - \mathbb{E}[Y]$.*
*(b) When there is no yield uncertainty ($Y = 0$), $\widetilde{Q}^*$ reduces to the approximate closed-form EOQD solution derived by Snyder (2014); i.e., $Q^* = \frac{\sqrt{(\hat{\psi}dh)^2 + 2h\mu(kd\mu + d^2p\hat{\psi})} - \hat{\psi}dh}{h\mu}$, which can be rewritten as $Q^* = \sqrt{\frac{2kd}{h} + \left(\frac{\hat{\psi}d}{\mu}\right)^2 + \frac{2d^2p\hat{\psi}}{\mu h}} - \frac{\hat{\psi}d}{\mu}$.*

***Proof***. To prove proposition 1, it suffices to show that in each case the solution is reducible when the parameters are correspondingly adjusted.
$(a)$ When $\hat{\psi} = 0$, the third and fourth term inside the square root are eliminated and the fifth term outside of the square root is also eliminated to 0. Thus, we get $\widetilde{Q}^* = \sqrt{\frac{2kd}{h} + \mathrm{Var}[Y]} - \mathbb{E}[Y]$, which is equivalent to the solution presented in Snyder and Shen (2019).
$(b)$ When $Y = 0$, it implies the deterministic yield from the order quantity, that is, $\mathbb{E}[Y] = 0$ and $\mathrm{Var}[Y] = 0$. Then we get $\widetilde{Q}^* = \sqrt{\frac{2kd}{h} + \left(\frac{\hat{\psi}d}{\mu}\right)^2 + \frac{2d^2p\hat{\psi}}{\mu h}} - \frac{\hat{\psi}d}{\mu}$, which has the equivalent terms and structure with the approximate EOQD solution ($Q^* = \sqrt{\frac{2kd}{h} + a^2 + b} - a$, for certain constants $a = \frac{\hat{\psi}d}{\mu}$ and $b = \frac{2d^2p\hat{\psi}}{\mu h}$) proposed by Snyder (2014). ■

## 3.2 Two-product case: Both organic and regular products

We now extend the model by incorporating the regular product. Given the sufficient number of large-scale suppliers for the regular products, we assume that they are not subject to any supply uncertainties; i.e., no supply disruption with deterministic yield. As in the previous subsection, the organic product faces supply disruptions based on the *i.i.d.* exponential distributions with recovery rate $\mu$ and disruption rate $\lambda$, and yield uncertainty based on $Y$. During the organic product's stock-out period, we assume that the regular product can serve as a substitute with the substitution rate of $\beta \in [0, 1]$. Figure 2 characterizes a typical inventory diagram for two products, where $Q_o$ and $T_o$ represent the order quantity and the cycle time of the organic product, and $Q_r$ represents the order quantity of the regular product.

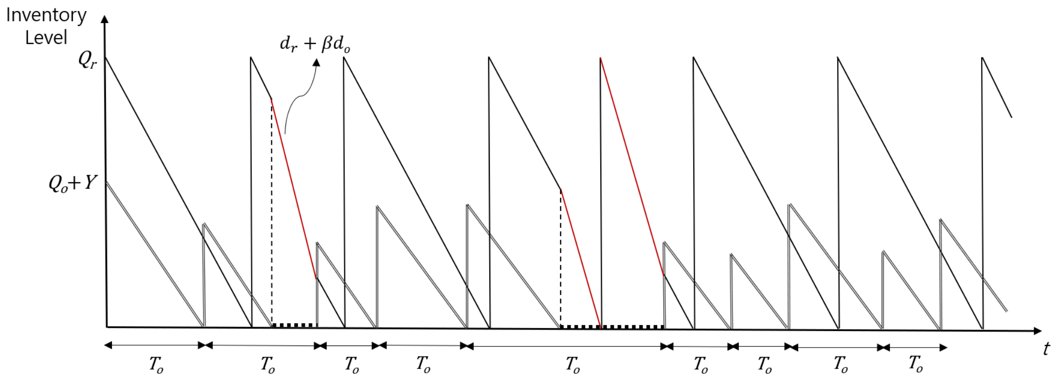


Figure 2: Inventory curves for the two partially substitutable products

We use $h_o$, $h_r$ as holding cost of each product per unit per year, and $p_o$, $p_r$ to denote stock-out cost (penalty cost) per lost sale. We define $h_o \geq h_r$ and $p_o \geq p_r$; the unit holding cost of organic product is greater than or equal to that of regular product, and the same relationship applies to the stock-out cost parameters. Every order placed by the retailer to supplier(s) in each cycle incurs a fixed ordering cost of $k_o$, $k_r$, respectively, and the deterministic consumption rate (demand rate) of each product is $d_o$, $d_r$ units per year, which satisfies $d_o < d_r$. $\beta$ represents a substitution rate from the organic to the regular when the organic product is not available due to the stock-out.

**Formulation of** $\mathbb{E}[TC_o]$. We now derive the total expected cost of organic product, $\mathbb{E}[TC_o]$, which consists of $\mathbb{E}[C_o]$ at numerator and $\mathbb{E}[T_o]$ at denominator by renewal reward theorem. We use $\mathbb{E}[C_o]$ and $\mathbb{E}[T_o]$ to denote the expected cost per cycle and expected cycle length for organic product, respectively. It is worth noting that $\mathbb{E}[C_o]$ and $\mathbb{E}[T_o]$ in this case are almost equivalently formulated as (1) and (2) presented in the single-product case since both cases consider random disruptions and random yields. The only difference comes from the stock-out cost part due to the effect of substitution rate $\beta$. In two product case where regular product can be partially substituted for organic product when it is not available due to its stock-out, the penalty cost for organic is taken into consideration only at a portion of $(1-\beta)$. Thus, the resulting total expected inventory cost function is mostly equivalent to the $\mathbb{E}[TC]$ of equation (3) in its structural form with a slight change at the last term of $\mathbb{E}[C_o]$. The total expected cost of organic product is

$$\mathbb{E}[TC_o] = \frac{k_o + \frac{h_o}{2d_o}[(Q_o + \mathbb{E}[Y])^2 + \mathrm{Var}[Y]] + \hat{\psi} p_o (1-\beta)\frac{d_o}{\mu}}{\frac{Q_o + \mathbb{E}[Y]}{d_o} + \hat{\psi}\frac{1}{\mu}}. \tag{8}$$

**Formulation of** $\mathbb{E}[TC_r]$. To formulate $\mathbb{E}[TC_r]$, total expected cost of regular product, we first formulate $\mathbb{E}[C_r]$, the expected cost per inventory cycle for regular product, and $\mathbb{E}[T_r]$, the expected inventory cycle length, and then apply the renewal reward theorem to derive $\mathbb{E}[TC_r]$ at the retailer. To formulate $\mathbb{E}[C_r]$ we first construct a revised inventory cycle for regular product by defining a fraction of OFF states of the organic supplier over the entire planning horizon. We do this because regular product's inventory flow depends on the state of the organic supplier. That is, when the organic supplier is disrupted and thus organic product is not available, consumers can purchase regular products in substitution for the organic product to some extent $\beta$, which steepens a consumption rate of the regular product from $d_r$ to $d_r+\beta d_0$ during a disrupted period of the organic supplier. Since the organic supplier remains in the OFF state for an exponentially distributed length of time with rate $\mu$ before returning to the ON state, and due to random yields for organic product in each cycle, regular product's inventory cycle length varies in each cycle. Accordingly, inventory cost per cycle also varies in each cycle depending on the supply state of the organic supplier, which renders the formulation of $\mathbb{E}[C_r]$ too complex to derive. Thus, we develop a new inventory cycle pattern for regular product with a slope of $(1-\Theta)d_r + \Theta(d_r + \beta d_0)$, where $\Theta$ is a fraction of disrupted (OFF) states of the organic supplier throughout the entire inventory planning horizon.

We now derive $\Theta$ to formulate $\mathbb{E}[C_r]$. OFF state fraction $\Theta$ can be asymptotically measured by expected duration of OFF state for organic supplier divided by expected cycle length for organic product $\mathbb{E}[t_o]$. This makes sense asymptotically as we assume the inventory system under infinite planning horizon (further reasoning required). In this sense, OFF state fraction $\Theta$ can be derived as follows:

$$\Theta = \frac{\frac{\hat{\psi}}{\mu}}{\frac{Q_o+\mathbb{E}[Y]}{d_o} + \frac{\hat{\psi}}{\mu}} = \frac{\hat{\psi} d_o}{\mu(Q_o + \mathbb{E}[Y]) + \hat{\psi} d_o}.$$

Inventory curve for regular product in Figure 2 can be asymptotically replaced by the one with each cycle having a slope of $(1-\Theta)d_r + \Theta(d_r + \beta d_0)$ over the infinite horizon. The reconfigured slope implies that the bolstered consumption rate for regular product, $d_r + \beta d_0$, occurs only during the organic supplier's disrupted periods, while the usual demand rate, $d_r$, occurs in most of the time periods except for the organic supplier's OFF state intervals. We get the expression for the new slope as $d_r + \Theta\beta d_0$, which is used in the formulation of $t_r$ as $\frac{Q_r}{d_r+\Theta\beta d_o}$. Then, we can derive the expected cost per cycle $\mathbb{E}[C_r]$ and expected cycle length $\mathbb{E}[T_r]$:

$$\mathbb{E}[C_r] = k_r + \frac{Q_r^2 h_r}{2(d_r + \Theta\beta d_o)}, \tag{9}$$

$$\mathbb{E}[T_r] = \frac{Q_r}{d_r + \Theta\beta d_o}. \tag{10}$$

Then, combining (9) and (10) and applying the renewal reward theorem, we can derive $\mathbb{E}[TC_r]$ as a function of $Q_o$ and $Q_r$:

$$\mathbb{E}[TC_r] = \frac{Q_r^2 h_r + 2k_r(d_r + \Theta\beta d_o)}{2Q_r} = \frac{Q_r h_r}{2} + \frac{d_r k_r}{Q_r} + \frac{\beta\hat{\psi} d_o^2 k_r}{Q_r(\mu(Q_o + \mathbb{E}[Y]) + \hat{\psi} d_o)}. \tag{11}$$

**Formulation of** $\mathbb{E}[TC]$. Combining the equation (8) and (11), the total expected inventory cost for both products is

$$\mathbb{E}[TC] = \frac{k_o + \frac{h_o}{2d_o}[(Q_o + \mathbb{E}[Y])^2 + \mathrm{Var}[Y]] + \hat{\psi} p_o (1-\beta)\frac{d_o}{\mu}}{\frac{Q_o + \mathbb{E}[Y]}{d_o} + \hat{\psi}\frac{1}{\mu}} + \frac{Q_r h_r}{2} + \frac{d_r k_r}{Q_r} + \frac{k_r}{Q_r}\Big(\frac{\beta\hat{\psi} d_o^2}{\mu(Q_o + \mathbb{E}[Y]) + \hat{\psi} d_o}\Big). \tag{12}$$

We now obtain the optimal joint inventory policy in the following lemma.

**Lemma 2.** *The expected total inventory cost in (12) is convex in $Q_o$ and $Q_r$. Further, the optimal ordering quantities $\widetilde{Q}_o^*$ and $\widetilde{Q}_r^*$ that minimize the expected total cost are given by*

$$\widetilde{Q}_o^* = \sqrt{\frac{2d_o k_o}{h_o} + Var[Y] + \big(\frac{\hat{\psi} d_o}{\mu}\big)^2 + \frac{2d_o^2 p_o \hat{\psi}}{\mu h_o}\big(1-\beta\big) + \frac{2\beta\hat{\psi} d_o^2 k_r}{\mu h_o \widetilde{Q}_r^*}} - \frac{\hat{\psi} d_o}{\mu} - \mathbb{E}[Y], \tag{13}$$

$$\widetilde{Q}_r^* = \sqrt{\frac{2d_r k_r}{h_r} + \frac{2\beta\hat{\psi} d_o^2 k_r}{h_r(\hat{\psi} d_o + \mu(\widetilde{Q}_o^* + \mathbb{E}[Y]))}}. \tag{14}$$

***Proof*.** We know from calculus that the sum of convex functions is also convex. The first term in (12) is $\mathbb{E}[TC_o]$, which is convex since its second-order condition is positive:

$$\frac{d^2\mathbb{E}[TC_o]}{dQ_o^2} = \frac{\hat{\psi}^2 d_o^2 \mu h_o + 2\hat{\psi} d_o^2 \mu^2 p_o (1-\beta) + 2d_o k_o \mu^3 + h_o \mu^3 \mathrm{Var}[Y]}{[\mu(Q_o + \mathbb{E}[Y]) + \hat{\psi} d_o]^3} > 0. \tag{15}$$

Further, the second term in equation (12) is also convex. To prove convexity of $\mathbb{E}[TC]$, it suffices to show that the remaining (third and fourth) terms of the equation (12) are convex. Let $G(Q_o, Q_r) = \frac{d_r k_r}{Q_r} + \frac{\beta\hat{\psi} d_o^2 k_r}{Q_r(\mu(Q_o + \mathbb{E}[Y]) + \hat{\psi} d_o)}$, where $\hat{\psi} = \frac{\lambda}{\lambda+\mu}$. To prove its convexity, we first establish its Hessian matrix and show that the Hessian matrix is positive semi-definite. Let $H$ be the Hessian matrix and be defined as follows:

$$H = \begin{bmatrix} G_{oo} & G_{or} \\ G_{ro} & G_{rr} \end{bmatrix}$$

where $G_{oo} = \frac{\partial^2 G(Q_o,Q_r)}{\partial Q_o^2}$, $G_{or} = \frac{\partial^2 G(Q_o,Q_r)}{\partial Q_o \partial Q_r}$, $G_{ro} = \frac{\partial^2 G(Q_o,Q_r)}{\partial Q_r \partial Q_o}$, $G_{rr} = \frac{\partial^2 G(Q_o,Q_r)}{\partial Q_r^2}$. To show that $H$ is positive semi-definite, we should show that $|H_1| = G_{oo} \geq 0$, $|H_2| = |H| = G_{oo} \cdot G_{rr} - G_{or}^2 \geq 0$ for all $Q_o$ and $Q_r$. We have

$$G_{oo} = \frac{2\beta d_o^2 k_r \mu^2 \hat{\psi}}{Q_r(\mu(Q_o + \mathbb{E}[Y]) + \hat{\psi} d_o)^3}, \tag{16}$$

$$G_{rr} = \frac{2k_r(\beta d_o^2 \hat{\psi} + d_r(\mu(Q_o + \mathbb{E}[Y]) + \hat{\psi} d_o))}{Q_r^3(\mu(Q_o + \mathbb{E}[Y]) + \hat{\psi} d_o)}, \tag{17}$$

$$G_{or} = \frac{\beta d_o^2 k_r \mu \hat{\psi}}{Q_r^2(\mu(Q_o + \mathbb{E}[Y]) + \hat{\psi} d_o)^2}. \tag{18}$$

We can see from the equation (16) that $|H_1| = G_{oo} \geq 0$ for all $Q_o$ and $Q_r$ since both the numerator and denominator are positive. Further, combining (16)-(18), $|H_2|$ can be expressed as the following:

$$|H_2| = \frac{\beta d_o^2 k_r^2 \mu^2 \hat{\psi}(3\beta d_o^2 \hat{\psi} + 4d_r(\mu(Q_o + \mathbb{E}[Y]) + \hat{\psi} d_o))}{Q_r^4(\mu(Q_o + \mathbb{E}[Y]) + \hat{\psi} d_o)^4}. \tag{19}$$

Note from the equation (19) that $|H_2| = G_{oo} \cdot G_{rr} - G_{or}^2 \geq 0$ for all $Q_o$ and $Q_r$, as both terms at numerator and denominator are positive. Thus, Hessian matrix of $G(Q_o, Q_r)$ is positive semi-definite, which renders $G(Q_o, Q_r)$ a convex function. We therefore conclude that there exists some $(Q_o, Q_r)$ which minimizes the total expected inventory cost $\mathbb{E}[TC]$ in equation (12). It follows that the resulting closed-form formulas for $Q_o^*$ and $Q_r^*$, denoted by $\widetilde{Q}_o^*$ and $\widetilde{Q}_r^*$, respectively, are the solutions to the first-order condition of (12) with respect to $Q_o$ and $Q_r$. Accordingly, we obtain the optimal closed-form solutions $\widetilde{Q}_o^*$ and $\widetilde{Q}_r^*$ as presented in (13) and (14). ■

It is worth noting that both solutions have a similar structural form with the classical *EOQ* solution, but with a few terms added resulting from the supply disruptions, random yields, and substitution between the products. Based on the $\mathbb{E}[TC]$ in (12) and closed-form optimal solutions obtained in Lemma 2, we now analyze the impact of substitution rate $\beta$ on a joint inventory management and compare our results with the solutions proposed in the extant literature to check whether they are reducible.

**Proposition 2.** *If there is no substitution between the organic and regular product ($\beta = 0$), $\widetilde{Q}_o^*$ obtained in Lemma 2 reduces to the optimal solution $\widetilde{Q}^*$ presented in Lemma 1 (single-product case). Further, $\widetilde{Q}_r^*$ reduces to the classical EOQ solution.*

***Proof*.** No substitution ($\beta = 0$) implies that inventory flow of each product becomes independent of each other. Accordingly, we get the equivalent inventory pattern with the one presented in figure 1 for organic product, in which both random supply disruptions and random yields are considered. Meanwhile, we obtain the classical *EOQ* inventory curve for regular product, where there is no supply uncertainty. Hence, when $\beta = 0$ in (13), the fourth term inside the square root, $\frac{2d_o^2 p_o \hat{\psi}}{\mu h_o}(1-\beta)$, becomes $\frac{2d_o^2 p_o \hat{\psi}}{\mu h_o}$. The fifth term inside the square root, $\frac{2\beta\hat{\psi}d_o^2 k_r}{\mu h_o \widetilde{Q}_r^*}$, is eliminated. Thus, $\widetilde{Q}_o^*$ reduces to $\sqrt{\frac{2d_o k_o}{h_o} + \mathrm{Var}[Y] + \left(\frac{\hat{\psi}d_o}{\mu}\right)^2 + \frac{2d_o^2 p_o \hat{\psi}}{\mu h_o}} - \frac{\hat{\psi}d_o}{\mu} - \mathbb{E}[Y]$, which was presented in Lemma 1 as the optimal solution in single-product case (organic only). Next, when $\beta = 0$ in (14), $\widetilde{Q}_r^*$ reduces to the classical *EOQ* solution since the second term inside the square root is eliminated. ■

Note that the substitution between the organic and regular occurs ($\beta \neq 0$) during the stock-out period of the organic product, whose supplier faces two types of supply uncertainties. In the presence of substitution between the products, we derive the optimal solutions in the following two special cases.

**Proposition 3.** *If disruption does not occur ($\hat{\psi} = 0$), the $\widetilde{Q}_o^*$ reduces to the solution of the model that only considers random yield in Snyder and Shen (2019); i.e., $\widetilde{Q}_o^* = \sqrt{\frac{2d_o k_o}{h_o} + Var[Y]} - \mathbb{E}[Y]$. Furthermore, $\widetilde{Q}_r^*$ reduces to the classical EOQ solution.*

***Proof*.** If disruption does not occur, substitution does not occur since there would be no periods of organic stock-out. This implies that each product's inventory cycles have independent flows over the entire planning horizon. Due to their independent relationship, we conclude that $\widetilde{Q}_o^*$ has the equivalent structure and terms with the optimal solution that *EOQ with yield uncertainty* produces. Also, we obtain the classical *EOQ* solution for $\widetilde{Q}_r^*$. ■

**Proposition 4.** *If there is no yield uncertainty ($Y = 0$), the resulting optimal order quantity for organic product $\widetilde{Q}_o^* = \sqrt{\frac{2d_o k_o}{h_o} + \left(\frac{\hat{\psi}d_o}{\mu}\right)^2 + \frac{2\hat{\psi}d_o^2 p_o}{\mu h_o} + \frac{2\beta\hat{\psi}d_o^2}{\mu h_o}\left(\frac{k_r}{\widetilde{Q}_r^*} - p_r\right)} - \frac{\hat{\psi}d_o}{\mu}$ is less than the solution of EOQ with disruptions in Snyder (2014). In contrast, the optimal order quantity for regular product $\sqrt{\frac{2d_r k_r}{h_r} + \frac{2\beta\hat{\psi}d_o^2 k_r}{h_r(\hat{\psi}d_o + \mu\widetilde{Q}_o^*)}}$ is greater than that of the classical EOQ.*

***Proof*.** When $Y = 0$ in (13), $\widetilde{Q}_o^*$ reduces to $\sqrt{\frac{2d_o k_o}{h_o} + \left(\frac{\hat{\psi}d_o}{\mu}\right)^2 + \frac{2d_o^2 p_o \hat{\psi}}{\mu h_o} + \frac{2\beta\hat{\psi}d_o^2}{\mu h_o}\left(\frac{k_r}{\widetilde{Q}_r^*} - p_o\right)} - \frac{\hat{\psi}d_o}{\mu}$ since $\mathrm{Var}[Y]$ and $\mathbb{E}[Y]$ are eliminated. We know that when the supply disruption occurs, the regular product's inventory cycles depend on the flow of the organic product's inventory cycles. Due to the intermittent substitutions throughout the planning horizon, the regular's product's inventory curve does not reduce to the classical *EOQ* inventory pattern. Rather, it increases the ordering amount of regular product to hedge against costly stock-out cost of organic product with a relatively cheaper holding cost of regular product. Thus, when $Y = 0$ in (14), $\widetilde{Q}_r^*$ increases from $\sqrt{\frac{2d_r k_r}{h_r}}$ to $\sqrt{\frac{2d_r k_r}{h_r} + \frac{2\beta\hat{\psi}d_o^2 k_r}{h_r(\hat{\psi}d_o + \mu\widetilde{Q}_o^*)}}$. ■

Proposition 2 shows that, when there is no substitution, the solution for the organic product results in a single-product case model. When there is yield uncertainty only for the organic product, proposition 3 shows that the substitution does not occur since there would be no stock-out for the organic product. In these two cases, the two products' inventory flows become independent over the entire planning horizon. When there is only supply disruptions in proposition 4, we note that $\widetilde{Q}_o^*$ is less than the solution of EOQ with disruptions in Snyder (2014) while $\widetilde{Q}_r^*$ is greater than the classical EOQ. This implies that, due to the possible substitution, the firm orders less organic product while increasing the regular product order.

# 4. Numerical analysis

In this section, we present our computational experiments on the behaviors of the optimal solution and optimal cost under extensive changes of supply risk parameters. Specifically, we aim to demonstrate the effect of substitution rate, supply disruptions, and yield uncertainty on the retailer's optimal inventory decisions.

### 4.1. Effect of substitution rate

We now conduct numerical experiments to check the values of $\widetilde{Q}_o^*$ and $\widetilde{Q}_r^*$ by changing the value of $\beta$ from 0 to 1. We generated 1,000 random instances in which all parameters (except for the $\beta$) were drawn from the following uniform distributions: $k_o \sim U[170, 230]$, $k_r \sim U[120, 180]$, $d_o \sim U[1400, 1600]$, $d_r \sim U[1900, 2100]$, $h_o \sim U[16, 20]$, $h_r \sim U[8, 12]$, $p_o \sim U[8, 12]$, $p_r \sim U[4, 6]$, $\mu \sim U[14, 24]$, $\lambda \sim U[2, 9]$, $\mathbb{E}[Y] \sim U[-60, -20]$, and $\text{Var}[Y] \sim U[100, 1000]$. The above parameter ranges are drawn and adapted from the literature (e.g., Qi et al. (2009), Salameh et al. (2014), Snyder (2014)) whenever applicable. Our numerical experiment demonstrates that substitution rate $\beta$ has a significant impact on a joint inventory management, and contains the following properties on the optimal order sizes of each product ($\widetilde{Q}_o^*$, $\widetilde{Q}_r^*$) and the optimal cost function $\mathbb{E}[TC(\widetilde{Q}_o^*, \widetilde{Q}_r^*)]$ for $\beta \in [0, 1]$:

**Observation 1.** *$\widetilde{Q}_o^*$ is monotone decreasing in $\beta$, whereas $\widetilde{Q}_r^*$ is monotone increasing in $\beta$. Further, $\mathbb{E}[TC(\widetilde{Q}_o^*, \widetilde{Q}_r^*)]$ is monotone decreasing in $\beta$.*

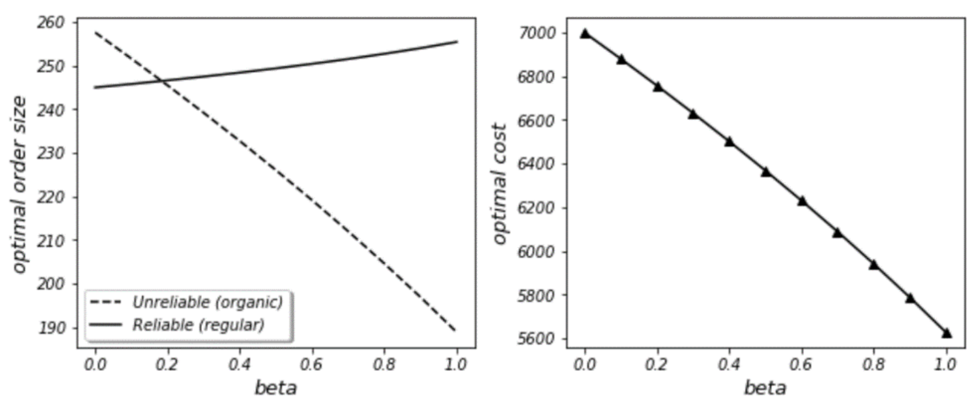


Figure 3: Impact of $\beta$ on the optimal solutions (left) and the optimal cost (right)

Figure 3 graphically illustrates the behaviors of the optimal solutions and the optimal cost with changes in $\beta$ from 0 to 1 when $\lambda = 6$, $\mu = 18$. Observation 1 shows the implication of substitution rate $\beta$ to the joint inventory decisions. We find that, as the substitution rate increases, the order size of regular product increases while the organic order size decreases. This indicates that, as substitution rate increases, the firm can increase the degree of risk-pooling for the organic product, and thus increases the regular product order to hedge against the organic supply shortage. As a result, the expected total inventory cost decreases.

We next examine the impact of changes in key parameters related to supply uncertainties.

### 4.2. Effect of supply disruptions

We first examine the impact of supply disruptions on the optimal inventory decisions by varying the degree of disruption severity. Note that the severity level increases as the recovery rate $\mu$ decreases (i.e., the average disruption period $1/\mu$ increases) and/or the disruption rate $\lambda$ increases (i.e., the disruption frequency increases; or the average ON state period $1/\lambda$ decreases). We find that the following holds for the growing supply disruption risks (i.e., decrease in $\mu$ and/or increase in $\lambda$):

**Observation 2.** *As disruption risk increases, $\widetilde{Q}_o^*$, $\widetilde{Q}_r^*$, and $\mathbb{E}[TC(\widetilde{Q}_o^*, \widetilde{Q}_r^*)]$ increase until substitution rate $\beta$ reaches the threshold $\bar{\beta}$. After the threshold $\bar{\beta}$, the transition occurs, in which $\widetilde{Q}_o^*$ and $\mathbb{E}[TC(\widetilde{Q}_o^*, \widetilde{Q}_r^*)]$ shows the decreasing trend, and only $\widetilde{Q}_r^*$ increases.*

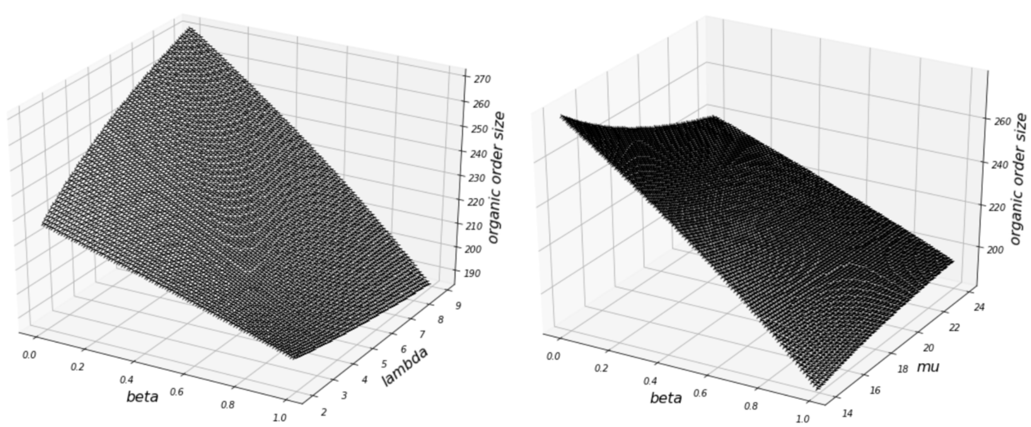


Figure 4: Changes in $\widetilde{Q}_o^*$ as disruption risk increases (increase in $\lambda$, decrease in $\mu$)

Figure 4 graphically represent the changes in the optimal order size of organic product $\widetilde{Q}_o^*$ for the increasing disruption risks (i.e., increase in $\lambda$, decrease in $\mu$). We find that $\widetilde{Q}_o^*$ shows the increasing trend as disruption risk increases until some threshold $\bar{\beta}$. After this threshold (i.e., when the substitution rate $\beta$ is very high), $\widetilde{Q}_o^*$ shows the decreasing trend as disruption rate $\lambda$ increases (left) and recovery rate $\mu$ decreases (right).

### 4.3. Effect of yield uncertainty

We now examine the impact of yield uncertainty by decreasing $\mathbb{E}[Y]$ and increasing $\mathrm{Var}[Y]$, both of which imply the amplifying negative impact of yield uncertainty. For this numerical experiment, we also generated 1,000 random instances in which all parameters (except for the $\mathbb{E}[Y]$ and $\mathrm{Var}[Y]$)

were drawn from the following uniform distributions: $k_o \sim U[170, 230]$, $k_r \sim U[120, 180]$, $d_o \sim U[1400, 1600]$, $d_r \sim U[1900, 2100]$, $h_o \sim U[16, 20]$, $h_r \sim U[8, 12]$, $p_o \sim U[8, 12]$, $p_r \sim U[4, 6]$, $\mu \sim U[14, 24]$, and $\lambda \sim U[2, 9]$.

**Observation 3.** *The following holds for* $\beta \in (0, 1]$*:*
*(a) As* $\mathbb{E}[Y]$ *decreases,* $\widetilde{Q}_o^*$ *increases by* $-\mathbb{E}[Y]$*, whereas* $\widetilde{Q}_r^*$ *and* $\mathbb{E}[TC(\widetilde{Q}_o^*, \widetilde{Q}_r^*)]$ *remain unchanged.*
*(b) As* $Var[Y]$ *increases,* $\widetilde{Q}_o^*$ *and* $\mathbb{E}[TC(\widetilde{Q}_o^*, \widetilde{Q}_r^*)]$ *increase while* $\widetilde{Q}_r^*$ *decreases.*

Figure 5 and 6 graphically illustrate the behaviors of the optimal solutions and the optimal cost with changes in $\mathbb{E}[Y]$ and $\text{Var}[Y]$, respectively, when $\lambda = 6$, $\mu = 18$, $\beta = 0.7$.

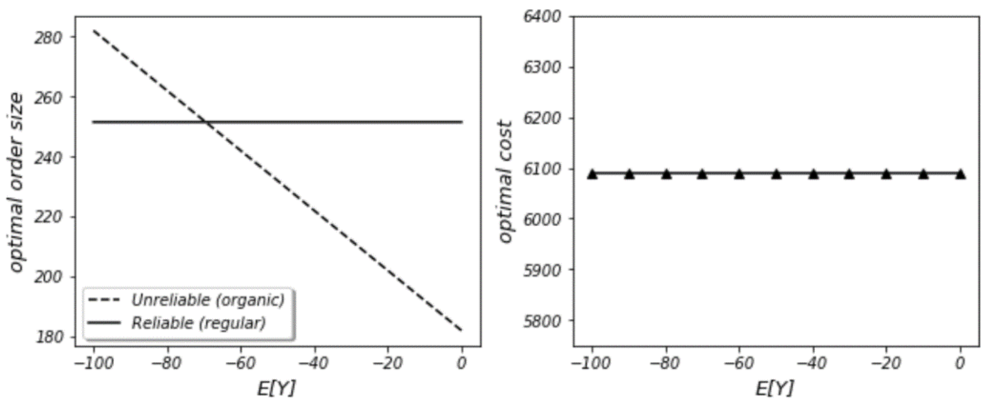


Figure 5: Impact of $\mathbb{E}[Y]$ on the optimal solutions (left) and the optimal cost (right)

We find that in part (a), as $\mathbb{E}[Y]$ decreases, the optimal order size of organic product increases by $-\mathbb{E}[Y] \geq 0$. This suggests that the firm compensates the yield loss by increasing the order size to its exact expected loss amount.

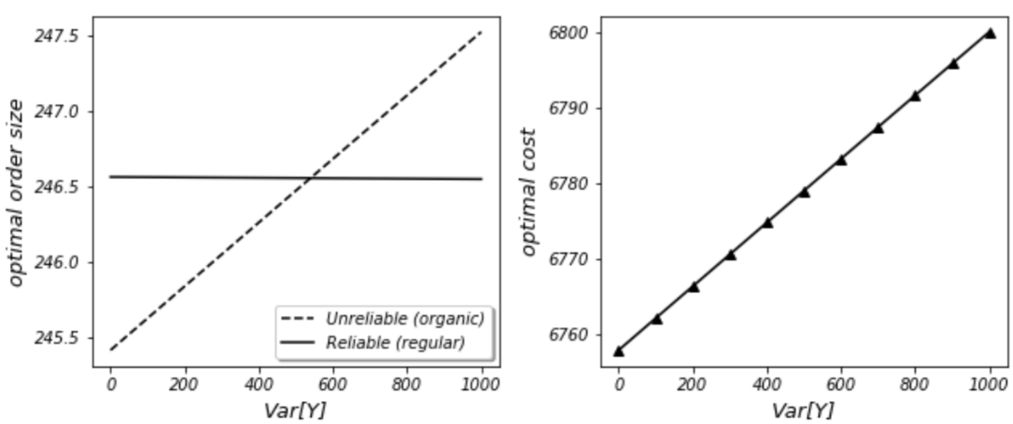


Figure 6: Impact of $\text{Var}[Y]$ on the optimal solutions (left) and the optimal cost (right)

The implication of increase in the yield variance in part (b) is similar, but results in a minor difference in the order size of regular product. That is, as $\mathrm{Var}[Y]$ increases, the optimal order size of organic product increases but the optimal order size of regular product decreases, implying that the firm increases the buffer of the organic product while decreasing the substitute orders.

In sum, we find that the growing supply disruption risks (i.e., decrease in $\mu$, increase in $\lambda$) as well as the growing yield uncertainty (i.e., decrease in $\mathbb{E}[Y]$, increase in $\mathrm{Var}[Y]$) have differing impacts to the optimal inventory decisions, although they both contribute to increased supply uncertainties. The firm attempts to mitigate supply disruption risk by increasing the order sizes of both products until some threshold substitution rate. This is interesting since the substitute product faces no supply uncertainties and partial downward substitution is allowed between products. When the substitution rate is very high, however, the retailer only increases the regular order size and reduces the organic order size for the increasing disruption risks. We also find that when the substitution rate is very high, the firm's optimal total cost may decrease despite the increasing risks of supply disruptions. This is because decrease in cost from lowering the unreliable product (i.e., organic product) order may overweigh the increase in cost from increasing the reliable product (i.e., regular product) order. Moreover, the total expect cost gets its maximum under low substitution rate, high disruption rate, and low recovery rate, as the firm dramatically increases the organic product orders to circumvent the costly organic stock-outs and also increases the substitute product orders as a substitution buffer. We also find that, although the organic supply is still more critical for the firm's efficient inventory management, the firm starts to rely more on the substitute product when the substitution rate is high since it becomes an attractive alternative as the substitution rate increases. In contrast, it is found that yield uncertainty makes the firm to increase only its organic product orders. This implies that the firm handles the risks from random yields in organic product by adjusting its organic product orders. It is worth mentioning that the firm only increases the organic order sizes and even decreases the substitute product orders when the variance of yield uncertainty increases. This can be interpreted as the firm's proactive action to mitigate the organic supply risks by holding more safety stocks for organic product while lowering its dependency on the substitute product.

# 5. Concluding remarks

As the supply chains expand across the globe and become more complex, the need for measures to address supply chain risks is increasingly important. One good application of this is the organic/fair trade food product industry. As the organic and fair trade industry develops rapidly, coordinating the inventory management between the organic and conventional products is becoming critical. We develop an EOQ-based joint inventory model that considers varying degrees of supply uncertainties for two substitutable products to obtain the optimal inventory decisions.

Our analysis shows that the implication of increasing substitution rate results in decreased organic order but increased regular order, as they make regular product a more effective substitute during the organic product's stock-out. In addition, increasing risks of supply disruptions and random yield generally lead to increased total inventory cost. However, we find that their response strategies differ. Higher disruption risk pushes the firm to increase the order size of both products until some threshold substitution rate, but higher yield uncertainty only increases the organic order size whereas the regular order size decreases or remains unchanged. When the substitution rate between the unreliable (e.g., organic) and reliable (e.g., regular) product is very high, however, the firm only increases the regular order size and reduces the organic order size for the increasing disruption risks. In this case, the firm's optimal total cost may decrease despite the increasing risk of supply disruptions.

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# Abstract in Korean

본 연구는 서로 다른 수준의 공급 신뢰성을 가지는 부분적으로 대체 가능한 두 가지 제품(예: 유기농 및 일반 제품)에 대한 최적 재고문제를 다룬다. 상대적으로 공급 불확실성이 큰 제품(유기농 제품)의 재고가 부족하거나 품절인 경우 두 제품 간의 대체율이 높아지면 상대적으로 공급 불확실성이 낮은 제품(일반제품)은 효과적인 대체품으로 작용하게 되어 유기농 제품의 최적주문량은 감소하고 일반제품의 최적주문량은 증가한다. 본 연구에서는 공급붕괴와 수율 불확실성의 위험 증가에 따른 대응전략이 상이함을 제시한다. 공급붕괴의 위험이 커질수록 대체율이 특정 임계치에 도달하기 이전에는 두 제품 모두에 대한 주문량을 높이는 방식으로 대응하는 것이 최적전략인 반면, 대체율이 특정 임계치 보다 큰 경우 즉, 두 제품 간의 대체율이 높은 경우에는 일반제품의 주문량만을 높이고 유기농 제품의 주문량은 감소시키는 것이 최적전략임을 확인하였다. 한편, 수율 불확실성의 경우 공급위험이 커질수록 공급 불확실성이 높은 유기농 제품의 주문량만을 늘리고 상대적으로 공급 리스크가 낮은 일반제품의 주문량은 그대로 유지하거나 오히려 감소시키는 것이 최적 주문전략임을 확인했다.

**주요어 :** 재고관리, 공급붕괴, 수율 불확실성, 대체성
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