이학박사 학위논문

Regularity results for fully nonlinear equations with oblique boundary conditions and time-dependent tug-of-war games

(사선형 경계조건을 갖는 완전 비선형 방정식 및 시간의존 줄다리기 경기의 정칙성)

2021년 2월

서울대학교 대학원

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지도교수 변 순 식

이 논문을 이학박사 학위논문으로 제출함

2020년 10월
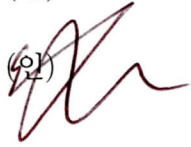
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수리과학부

한정민

한정민의 이학박사 학위논문을 인준함

2020년 12월

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A dissertation

submitted in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

to the faculty of the Graduate School of

Seoul National University

by

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February 2021

Abstract

Regularity results for fully nonlinear equations with oblique boundary conditions and time-dependent tug-of-war games

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In this thesis, we deal with two different types of problems related to nonlinear partial differential equations. One is the oblique derivative problem and the other is the tug-of-war game.

We study fully nonlinear elliptic and parabolic equations in nondivergnece form with oblique boundary conditions in the first part. Our boundary condition is a generalization of the Neumann condition. We derive global Calderón-Zygmund type estimates under a minimal boundary regularity assumption.

In the second part, we study a stochastic two-player zero-sum game which is called tug-of-war. In particular, we consider time-dependent games. We show global Lipschitz type estimates for value functions of such stochastic games. Furthermore, we also investigate their long-time asymptotics and PDE connections as applications.

Key words: Regularity, viscosity solution, fully nonlinear equation, oblique derivative problem, tug-of-war, dynamic programming principle
Student Number: 2014-21193

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Chapter 1

Introduction

This thesis is based on the papers [8, 9, 28, 29]. The aim of this thesis is to study regularity properties for oblique derivative problems and tug-of-war games. More precisely, we first obtain global Calderón-Zygmund type estimates for fully nonlinear elliptic and parabolic equations in nondivergence form with oblique boundary conditions. We also derive regularity results and other properties for value functions of time-dependent tug-of-war games with noise.

Oblique derivative problems for fully nonlinear equations

The Neumann boundary condition is one of the most common types of boundary conditions for partial differential equations along with Dirichlet boundary condition. In the Neumann problems, boundary conditions are given by the normal derivative of solutions. Then one can have the following question: What happens if other directional derivatives are given as boundary data? Oblique derivative problems are considered to deal with such cases.

The word 'oblique' means 'having a sloping direction'. Literally, we consider the case when the boundary condition is given by oblique directional derivatives of solutions. In general, an oblique boundary condition is taking the form of

$$\beta \cdot Du + \gamma u = g, \qquad (1.0.1)$$

where γ and g are real-valued functions defined on the boundary of a given

domain, and β is a vector-valued function defined on the boundary with

$$\beta \cdot \mathbf{n} \geq \delta_0 \qquad\qquad (1.0.2)$$

for some $\delta_0 > 0$ and the inward normal vector $\mathbf{n}$. We note that (1.0.2) represents that the slope vector β makes an angle more than some level with the boundary of the domain. We can see that Neumann condition is the case of $\beta = \mathbf{n}$ and $\gamma = 0$.

Since the oblique boundary condition is a generalization of the Neumann condition, approaches to oblique derivative problems are essentially no different from those of the Neumann case. In these problems, the explicit boundary values of our solution are still unknown. We only know the condition (1.0.1) which our solution satisfies. Thus, as Milakis and Silvestre mentioned in [59], it is wise to think that the oblique boundary condition is a part of the equation.

The theory for oblique derivative boundary value problems has been developing over the past decades. Winzell [79, 80], Lieberman [43, 44, 45, 46, 47, 48] and Ural'tseva [73] presented noteworthy results for oblique derivative problems. We also refer the reader to [26, 49, 60, 68, 21, 20, 57, 58, 61, 69] for further discussion on this topic. Several applications of oblique derivative problems can be found in [19, 4, 25].

The notion of viscosity solutions suggested a new paradigm to study partial differential equations. In particular, it promoted the development of researches on PDEs in nondivergence form. Fundamental properties of viscosity solutions to fully nonlinear elliptic equations were presented in [12, 11, 10, 33] and Wang extended these results to the parabolic case in [75, 76, 77]. For oblique derivative problems, Ishii [32] studied the existence and uniqueness for the elliptic case. In the parabolic case, such issues were covered in [34]. Meanwhile, much progress has also been made on the regularity theory. Milakis and Silvestre [59] established $C^{1,\alpha}$- and $C^{2,\alpha}$-regularity for elliptic Neumann problems. We can find such regularity results for general oblique derivative problems in [42] by Li and Zhang. Chatzigeorgiou and Milakis [16] presented similar estimates for the parabolic case.

Tug-of-war games with noise

Probabilistic approaches for PDEs were first considered by Doob, Kac and Kakutani in [38, 39, 36, 37, 22]. They studied the connection between Brow-

nian motion and Laplacian. Such interpretation opened up a new direction in comprehending PDEs. Since then, various probabilistic views for more general equations have been discussed. Tug-of-war game, which will be considered in this thesis, is also one of these schemes. This game interpretation is closely linked to p-Laplace type equations.

In the study of tug-of-war games, we are interested in the expectation of game outcome, which is called the game value function. A dynamic programming principle (DPP) is a key tool to investigate game values. In many cases, the game value satisfies a DPP which arises from the game settings. Thus, we can examine value functions by dealing with the corresponding DPP. Various issues for game values, such as existence, uniqueness and regularity, can be investigated through studying related DPPs. On the other hand, such approaches using DPP also play an important role to look into the relation between tug-of-war game values and p-Laplace type equations. We can regard the DPP linked to tug-of-war games as a discretization of the p-Laplacian.

For $1 \leq p < \infty$, the p-Laplace operator is defined by

$$\Delta_p u := \operatorname{div}(|Du|^{p-2} Du) = |Du|^{p-2} \left(\Delta u + (p-2) \frac{\langle D^2 u Du, Du \rangle}{|Du|^2} \right). \quad (1.0.3)$$

Here we focus on the term

$$\Delta u + (p-2) \frac{\langle D^2 u Du, Du \rangle}{|Du|^2} =: \Delta_p^N u,$$

not including the scaling factor $|Du|^{p-2}$. The nonlinear term

$$\frac{\langle D^2 u Du, Du \rangle}{|Du|^2} =: \Delta_\infty^N u$$

is called the (normalized) ∞-Laplace operator. We observe that Δ_p^N is represented by a linear combination of Δ and Δ_∞^N. In [64], Peres, Schramm, Sheffield and Wilson dealt with a game interpretation for ∞-Laplacian by using tug-of-war. For a general p-Laplacian, Peres and Sheffield [65] studied the connection with tug-of-war including noise.

A lot of progress has been made on the studies of tug-of-war games and their related problems in the last decade. In [40, 53, 54, 55], several types of mean value characterization for solutions to p-Laplace type equations. We

can also find the existence and uniqueness results for functions satisfying DPP related to elliptic p-Laplace equations in [52, 30]. For regularity issues, we refer to [51, 67, 62] which considered Harnack inequality for game values. Arroyo, Heino and Parviainen established Hölder type estimates for space-varying time-independent games in [2]. Lipschitz type regularity for such games was discussed in [3]. For time-dependence games, which is corresponding to parabolic equations, regularity estimates were presented in [53, 62]. We also refer the reader to [15, 6, 1, 24, 56, 41] for further discussions on tug-of-war games.

The reminder of this thesis is divided into two parts. The first part, Chapter 3, deals with $W^{2,p}$-regularity theory of oblique derivative problems for nondivergence elliptic and parabolic equations. Precisely, we will obtain global Calderón-Zygmund estimates for fully nonlinear elliptic and parabolic equations in nondivergence form with oblique boundary conditions. In both cases, we first establish boundary Hessian estimates for oblique derivative problems when the equation does not contain lower order terms. After that, we consider boundary $W^{1,p}$-regularity for general equations in order to reach the desired regularity. To this end, we get an estimate for boundary data and apply this result to the Dirichlet case. Finally, we obtain the global regularity results by using a standard flattening argument. We note that the preceding results [42, 16] for the model problems are essential to deduce our main results. We also refer to [78] which investigated elliptic Dirichlet problems.

In the second part, Chapter 4, we are devoted to the study of the value function for time-dependent tug-of-war games. First, we prove the existence and uniqueness of value functions. We also show that the value function satisfies a DPP, (4.0.1). We next investigate regularity theory for game values. For the interior regularity, we present Hölder and Lipschitz type estimates for our value function. To do this, we first derive an estimate in the time direction. The method used to show this estimate is motivated from the study on parabolic PDEs (see [35]). On the other hand, in order to establish regularity estimates in the spatial directions, we utilize the cancellation argument introduced in [62, 2, 50, 3]. It is remarkable that we need Hölder type regularity result to derive Lipschitz type estimates. In the boundary case, we establish several estimates when the given boundary data is Lipschitz continuous. For this purpose, we derive proper estimates for the exit time of our games by considering an auxiliary stochastic process. In addition, we provide some applications of our results. We investigate long-time behavior of our value

functions and observe the connection with time-independent games. And we also present uniform convergence results for value functions to solutions of a parabolic p-Laplace type equation when the step size goes to zero.

Chapter 2

Preliminaries

2.1 Oblique derivative problems

2.1.1 Notations

We start this section with some notations, which will be used throughout this dissertation.

1. For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we write $x' = (x_1, x_2, \ldots, x_{n-1})$. We also write $x = (x', x_n)$.

2. $S(n)$ is the set of $n \times n$ symmetric matrices and $||M|| = \sup_{|x| \leq 1} |Mx|$ for every $M \in S(n)$.

3. $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_n > 0\}$.

4. For $x_0 \in \mathbb{R}^n$ and $r > 0$, $B_r(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$. We write $B_r = B_r(0)$ and $B_r^+ = B_r \cap \mathbb{R}^n_+$.

5. We write $B_{r,h} = B_r(-(R-h)e_n)$, where R satisfies $(R-h)^2 + r^2 = R^2$. We also write $B_{r,h}^+ = B_{r,h} \cap \mathbb{R}^+$.

6. We write $T_r = \{(x', 0) \in \mathbb{R}^{n-1} : |x'| < r\}$ and $T_r(x_0') := T_r + x_0'$ where $x_0' \in \mathbb{R}^{n-1}$.

7. $Q_r(x_0, t_0) := B_r(x_0) \times (t_0 - r^2, t_0)$ and $Q_r^+(x_0, t_0) := B_r^+(x_0) \times (t_0 - r^2, t_0)$ for $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$ and $r > 0$. $Q_r = Q_r(0, 0)$ and $Q_r^+ = Q_r^+(0, 0)$.

8. $Q^+_{r,\delta}(x_0, t_0) := B^+_{r-\delta}(x_0) \times (t_0 - r^2 + \delta^2, t_0)$, $Q^+_{r,\delta} = Q^+_{r,\delta}(0,0)$ (only used in Chapter 3).

9. $V^+_{r,h}(x_0, t_0) := B^+_{r,h}(x_0) \times (t_0 - r^2, t_0)$, $V^+_{r,h} = V^+_{r,h}(0,0)$.

10. $V^+_{r,h,\delta}(x_0, t_0) := B^+_{r-\delta, h-\delta h/r}(x_0) \times (t_0 - r^2 + \delta^2, t_0)$, $V^+_{r,h,\delta} = V^+_{r,h,\delta}(0,0)$.

11. $Q^*_r(x_0, t_0) = T_r(x_0) \times (t_0 - r^2, t_0)$, $Q^*_r := Q^*_r(0,0)$.

12. $K^d_r = (-r/2, r/2)^d$ for $r > 0$ and $d = n - 1$ or n, $K^d_r(x_0) = K^d_r + x_0$.

13. For $|\nu| \leq r$, we write $Q^\nu_1 = Q_r(0,0) \cap (\{x_n > -\nu\} \times \mathbb{R})$. We also write $Q^\nu_r(x_0, t_0) = Q^\nu_r + (x_0, t_0)$.

14. $Q^\nu_{r,\delta} = Q_{r,\delta}(0', \nu, 0) \cap (\mathbb{R}^n_+ \times \mathbb{R})$, $Q^\nu_{r,\delta}(x_0, t_0) = Q^\nu_{r,\delta} + (x_0, t_0)$.

15. Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 2$, and $\partial\Omega$ is the boundary of Ω.

16. $\Omega_T = \Omega \times (0, T)$ and $S_T = \partial\Omega \times (0, T)$ for $T > 0$.

17. For $U \subset \mathbb{R}^n \times \mathbb{R}$, $\partial_p U$ is the parabolic boundary of U.

18. For $U \subset \mathbb{R}^n \times \mathbb{R}$, we write $rU := \{(rx, r^2 t) \in \mathbb{R}^n \times \mathbb{R} : (x, t) \in U\}$ and $rU(x, t) := rU + (x, t)$.

19. We denote the time derivative, gradient and Hessian of u by u_t, $Du = (D_1 u, \cdots, D_n u)$, and $D^2 u = (D_{ij} u)$, respectively, where $D_i u = \frac{\partial u}{\partial x_i}$ and $D_{ij} u = \frac{\partial^2 u}{\partial x_i \partial x_j}$ for $1 \leq i, j \leq n$.

20. For each set $U \subset \mathbb{R}^n$ ($U \subset \mathbb{R}^n \times \mathbb{R}$, respectively), $|U|$ is the n-dimensional ($(n+1)$-dimensional, respectively) Lebesgue measure of U.

21. Let U be a set in $\mathbb{R}^n$ ($\mathbb{R}^n \times \mathbb{R}$, respectively) with $|U| \neq 0$, and f be a measurable function on U. Then we write

$$\fint_U f(x)dx = \frac{1}{|U|} \int_U f(x)dx \quad \left(\fint_U f(x,t)dxdt = \frac{1}{|U|} \int_U f(x,t)dxdt \right).$$

22. Let $\Omega \subset \mathbb{R}^n$. If a function u is continuous in Ω, we write $u \in C(\Omega)$. The C-norm of u is given by

$$||u||_{C(\Omega)} := \sup_{x \in \Omega} |u(x)|.$$

If Du $(D^2 u)$ is continuous in Ω, we write $C^1(\Omega)$ $(C^2(\Omega)$, respectively) and

$$||u||_{C^1(\Omega)} := ||u||_{C(\Omega)} + ||Du||_{C(\Omega)},$$
$$||u||_{C^2(\Omega)} := ||u||_{C^1(\Omega)} + ||D^2 u||_{C(\Omega)}.$$

23. If a function u satisfies

$$|u(x) - u(y)| \leq C|x - y|^\alpha$$

for any $x, y \in \Omega$ and some $0 < \alpha \leq 1$ and $C > 0$, we write $u \in C^{0,\alpha}(\Omega)$. The $C^{0,\alpha}$-norm of u is given by

$$||u||_{C^{0,\alpha}(\Omega)} := ||u||_{C(\Omega)} + \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|^\alpha}$$

$$=: ||u||_{C(\Omega)} + [u]_{C^{0,\alpha}(\Omega)}.$$

24. If a function u satisfies that Du is α-Hölder continuous in x, we write $u \in C^{1,\alpha}(\Omega)$.

$$||u||_{C^{1,\alpha}(\Omega)} := ||u||_{C^1(\Omega)} + \sum_{i=1}^{n} \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|D_i u(x) - D_i u(y)|}{|x - y|^\alpha}$$

$$=: ||u||_{C^1(\Omega)} + [u]_{C^{1,\alpha}(\Omega)}.$$

25. If a function u satisfies that $D^2 u$ is α-Hölder continuous in x, we write $u \in C^{2,\alpha}(\Omega)$. The $C^{2,\alpha}$-norm of u is given by

$$||u||_{C^{2,\alpha}(\Omega)} := ||u||_{C^2(\Omega)} + \sum_{i,j=1}^{n} \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|D_{ij} u(x) - D_{ij} u(y)|}{|x - y|^\alpha}$$

$$=: ||u||_{C^2(\Omega)} + [u]_{C^{2,\alpha}(\Omega)}.$$

26. Let $1 \leq p \leq \infty$. If a function u satisfies that

$$\int_\Omega |u(x)|^p dx < \infty,$$

we write $u \in L^p(\Omega)$. The L^p-norm of u is given by

$$||u||_{L^p(\Omega)} := \left(\int_\Omega |u(x)|^p dx \right)^{1/p}.$$

In addition, if a function u satisfies that

$$\operatorname*{ess\,sup}_{x \in \Omega} |u(x)| < \infty,$$

we write $u \in L^\infty(\Omega)$ with its norm

$$||u||_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{x \in \Omega} |u(x)|.$$

27. If a function u satisfies that $u, Du \in L^p(\Omega)$, we write $u \in W^{1,p}(\Omega)$. The $W^{1,p}$-norm of u is given by

$$||u||_{W^{1,p}(\Omega)} := \left(||u||^p_{L^p(\Omega)} + ||Du||^p_{L^p(\Omega)} \right)^{1/p}.$$

Moreover, if a function u satisfies that $u, Du, D^2u \in L^p(\Omega)$, we write $u \in W^{2,p}(\Omega)$. The $W^{2,p}$-norm of u is given by

$$||u||_{W^{2,p}(\Omega)} := \left(||u||^p_{L^p(\Omega)} + ||Du||^p_{L^p(\Omega)} + ||D^2u||^p_{L^p(\Omega)} \right)^{1/p}.$$

28. Let $\Omega \subset \mathbb{R}^n \times \mathbb{R}$. If a function u is continuous in Ω, we write $u \in C(\Omega)$. The C-norm of u is given by

$$||u||_{C(\Omega)} := \sup_{(x,t) \in \Omega} |u(x,t)|.$$

If Du (D^2u and u_t) is continuous in Ω, we write $C^1(\Omega)$ ($C^2(\Omega)$, respectively).

$$||u||_{C^1(\Omega)} := ||u||_{C(\Omega)} + ||Du||_{C(\Omega)},$$

$$||u||_{C^2(\Omega)} := ||u||_{C^1(\Omega)} + ||u_t||_{C(\Omega)} + ||D^2u||_{C(\Omega)}.$$

29. If a function u satisfies

$$|u(x,t) - u(y,s)| \le C(|x-y|^\alpha + |t-s|^{\alpha/2})$$

for any $(x,t), (y,s) \in \Omega$ and some $0 < \alpha \le 1$ and $C > 0$, we write $u \in C^{0,\alpha}(\Omega)$. (i.e, u is $(\alpha/2)$-Hölder continuous in t and α-Hölder continuous in x) The $C^{0,\alpha}$-norm of u is given by

$$||u||_{C^{0,\alpha}(\Omega)} := ||u||_{C(\Omega)} + \sup_{\substack{(x,t),(y,s)\in\Omega \\ (x,t)\neq(y,s)}} \frac{|u(x,t) - u(y,s)|}{|x-y|^\alpha + |t-s|^{\alpha/2}}$$

$$=: ||u||_{C(\Omega)} + [u]_{C^{0,\alpha}(\Omega)}.$$

30. If a function u is $((1+\alpha)/2)$-Hölder continuous in t and Du is α-Hölder continuous in x, we write $u \in C^{1,\alpha}(\Omega)$.

$$||u||_{C^{1,\alpha}(\Omega)} := ||u||_{C^1(\Omega)} + \sup_{\substack{(x,t),(x,s)\in\Omega \\ t\neq s}} \frac{|u(x,t) - u(x,s)|}{|t-s|^{(1+\alpha)/2}}$$

$$+ \sum_{i=1}^n \sup_{\substack{(x,t),(y,s)\in\Omega \\ (x,t)\neq(y,s)}} \frac{|D_i u(x,t) - D_i u(y,s)|}{|x-y|^\alpha + |t-s|^{\alpha/2}}$$

$$=: ||u||_{C^1(\Omega)} + [u]_{C^{1+\alpha}(\Omega)}.$$

31. If a function u satisfies that u_t is $(\alpha/2)$-Hölder continuous in t and $D^2 u$ is α-Hölder continuous in x, we write $u \in C^{2,\alpha}(\Omega)$. The $C^{2,\alpha}$-norm of u is given by

$$||u||_{C^{2,\alpha}(\Omega)} := ||u||_{C^2(\Omega)} + \sup_{\substack{(x,t),(x,s)\in\Omega \\ t\neq s}} \frac{|u_t(x,t) - u_t(x,s)|}{|x-y|^\alpha + |t-s|^{\alpha/2}}$$

$$+ \sum_{i,j=1}^n \sup_{\substack{(x,t),(x,s)\in\Omega \\ (x,t)\neq(y,s)}} \frac{|D_{ij}u(x,t) - D_{ij}u(y,s)|}{|x-y|^\alpha + |t-s|^{\alpha/2}}$$

$$=: ||u||_{C^2(\Omega)} + [u]_{C^{2+\alpha}(\Omega)}.$$

32. Let $1 \leq p \leq \infty$. If a function u satisfies that

$$\int_\Omega |u(x,t)|^p dx dt < \infty,$$

we write $u \in L^p(\Omega)$. The L^p-norm of u is given by

$$||u||_{L^p(\Omega)} := \left(\int_\Omega |u(x,t)|^p dx dt \right)^{1/p}.$$

In addition, if a function u satisfies that

$$\operatorname*{ess\,sup}_{(x,t)\in\Omega} |u(x,t)| < \infty,$$

we write $u \in L^\infty(\Omega)$ with its norm

$$||u||_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{(x,t)\in\Omega} |u(x,t)|.$$

33. If a function u satisfies that $u, Du, \in L^p(\Omega)$, we write $u \in W^{1,p}(\Omega)$. The $W^{1,p}$-norm of u is given by

$$||u||_{W^{1,p}(\Omega)} := \left(||u||_{L^p(\Omega)}^p + ||Du||_{L^p(\Omega)}^p \right)^{1/p}.$$

34. If a function u satisfies that $u, u_t, Du, D^2u \in L^p(\Omega)$, we write $u \in W^{2,p}(\Omega)$. The $W^{2,p}$-norm of u is given by

$$||u||_{W^{2,p}(\Omega)} := \left(||u||_{L^p(\Omega)}^p + ||u_t||_{L^p(\Omega)}^p + ||Du||_{L^p(\Omega)}^p + ||D^2u||_{L^p(\Omega)}^p \right)^{1/p}.$$

2.1.2 Elliptic equations

Second order differentiability

Let $V \subset \Omega$, $M > 0$ and $u \in C(\Omega)$. We define

$\underline{G}_M(u,V) = \{x_0 \in V | \text{there is a concave paraboloid } P \text{ with opening } M \text{ such}$
$\text{that } P(x_0) = u(x_0) \text{ and } P(x) \leq u(x) \text{ for any } x \in V\}$

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and

$$\underline{A}_M(u, V) = V \backslash \underline{G}_M(u, V).$$

Using convex paraboloids, we can define $\overline{G}_M(u, V)$ and $\overline{A}_M(u, V)$ analogously. And we also define

$$G_M(u, V) = \underline{G}_M(u, V) \cap \overline{G}_M(u, V)$$

and

$$A_M(u, V) = \underline{A}_M(u, V) \cap \overline{A}_M(u, V).$$

Now we set

$$\underline{\Theta}(u, V)(x) = \inf\{M > 0 : x \in \underline{G}_M(V)\},$$

$$\overline{\Theta}(u, V)(x) = \inf\{M > 0 : x \in \overline{G}_M(V)\}$$

and

$$\Theta(u, V)(x) = \sup\{\underline{\Theta}(u, V)(x), \overline{\Theta}(u, V)(x)\}.$$

We can see that the above notions are closely related to the second derivatives of functions. To obtain second order regularity results, we need to look at the properties of these sets.

Viscosity solutions

We begin this subsection with introducing the notion of viscosity solution. First, we introduce Pucci extremal operators.

Definition 2.1.1 (Pucci extremal operator). *For any $M \in S(n)$, the Pucci extremal operator $\mathcal{M}^+$ and $\mathcal{M}^-$ are defined as following:*

$$\mathcal{M}^+(\lambda, \Lambda, M) = \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i \text{ and } \mathcal{M}^-(\lambda, \Lambda, M) = \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i$$

where e_i are eigenvalues of M.

Consider
$$L_b^{\pm}(\lambda, \Lambda, b, u) = \mathcal{M}^{\pm}(\lambda, \Lambda, D^2 u) \pm b|Du|$$
for $b > 0$, respectively.

The following notions are also essential in defining viscosity solutions.

Definition 2.1.2. *Let $b \geq 0$ and $0 < \lambda \leq \Lambda$. We define the classes $\underline{S}(\lambda, \Lambda, b, f)$ $(\overline{S}(\lambda, \Lambda, b, f)$, respectively) to be the set of all continuous functions u that satisfy $L_b^+ u \geq f$ $(L_b^- u \leq f)$ in the viscosity sense in Ω. We also define*

$$S(\lambda, \Lambda, b, f) = \overline{S}(\lambda, \Lambda, b, f) \cap \underline{S}(\lambda, \Lambda, b, f)$$

and

$$S^*(\lambda, \Lambda, b, f) = \overline{S}(\lambda, \Lambda, b, |f|) \cap \underline{S}(\lambda, \Lambda, b, -|f|).$$

When $b = 0$, we abbreviate $S, \overline{S}, \underline{S}, S^*(\lambda, \Lambda, 0, f)$ to $S, \overline{S}, \underline{S}, S^*(\lambda, \Lambda, f)$.

Now we can introduce the notion of viscosity solution. First, we consider the case when f is a continuous function.

Definition 2.1.3 (C^2-viscosity solution). *Let $F = F(X, q, r, x)$ be continuous in all variables and $f \in C(\Omega \cup \Gamma)$. A continuous function $u \in C(\Omega \cup \Gamma)$ is called a C^2-viscosity solution of $(3.1.1)$ if the following conditions hold:*

(a) *for all $\varphi \in C^2(\Omega \cup \Gamma)$ touching u by above at $x_0 \in \Omega \cup \Gamma$,*

$$F(D^2\varphi(x_0), D\varphi(x_0), u(x_0), x_0) \geq f(x_0)$$

when $x_0 \in \Omega$ and $\beta(x_0) \cdot D\varphi(x_0) \geq 0$ when $x_0 \in \Gamma$.

(b) *for all $\varphi \in C^2(\Omega \cup \Gamma)$ touching u by below at $x_0 \in \Omega \cup \Gamma$,*

$$F(D^2\varphi(x_0), D\varphi(x_0), u(x_0), x_0) \leq f(x_0)$$

when $x_0 \in \Omega$ and $\beta(x_0) \cdot D\varphi(x_0) \leq 0$ when $x_0 \in \Gamma$.

We can also define viscosity solutions without continuity assumption for f.

Definition 2.1.4 ($W^{2,p}$-viscosity solution). *Let $F = F(X, q, r, x)$ be continuous in X, q, r and measurable in x. Suppose $p > n$ and $f \in L^p(\Omega)$. A continuous function u is called a $W^{2,p}$-viscosity solution for $(3.1.1)$ if the following conditions hold:*

(a) *For all $\varphi \in W^{2,p}(\Omega)$ whenever $\epsilon > 0$, $\mathcal{O}$ is open in $\overline{\Omega}$ and*

$$F(D^2\varphi(x), D\varphi(x), \varphi(x), x) \geq f(x) + \epsilon \quad a.e. \text{ in } \mathcal{O}$$

$$\text{and} \qquad \beta \cdot D\varphi(x) \geq \epsilon \quad \text{a.e. on } \mathcal{O} \cap \partial\Omega,$$

u − φ cannot attain a local minimum in $\mathcal{O}$.

(b) *For all $\varphi \in W^{2,p}(\Omega)$ whenever $\epsilon > 0$, $\mathcal{O}$ is open in $\overline{\Omega}$ and*

$$F(D^2\varphi(x), D\varphi(x), \varphi(x), x) \leq f(x) - \epsilon \quad \text{a.e. in } \mathcal{O}$$

$$\text{and} \qquad \beta \cdot D\varphi(x) \leq -\epsilon \quad \text{a.e. on } \mathcal{O} \cap \partial\Omega,$$

u − φ cannot attain a local maximum in $\mathcal{O}$.

2.1.3 Parabolic equations

Parabolic second order differentiability

Similarly to the elliptic case, we first need to characterize paraboloids to observe second order differentiability.

Definition 2.1.5. *Let $M > 0$. A* convex paraboloid *P with opening M is defined by*

$$P(x,t) = a + l \cdot x + \frac{M}{2}(|x|^2 - t),$$

where $a \in \mathbb{R}$ and $l \in \mathbb{R}^n$. We also define a concave paraboloid *by replacing M with $-M$ in the above definition.*

Let Ω be a bounded domain in $\mathbb{R}^n \times \mathbb{R}$, $U \subset \Omega$ be an open subset of Ω, $M > 0$, and $u \in C(\Omega)$. For $s \in \mathbb{R}$, we use the following notation $U_s = \{(x,t) \in U : t \leq s\}$ temporarily. Next we define 'good set' and 'bad set'. Let $\underline{G}_M(u, U)$ be the set of points $(x_0, t_0) \in U$ which satisfy that there is a concave paraboloid P with opening M such that $P(x_0, t_0) = u(x_0, t_0)$ and $P(x,t) \leq u(x,t)$ for any $(x,t) \in U_{t_0}$, and $\underline{A}_M(u, U) = U \backslash \underline{G}_M(u, U)$. Analogously, we can define $\overline{G}_M(u, U)$ and $\overline{A}_M(u, U)$ by using a convex paraboloid as a barrier. In addition, we denote by

$$G_M(u, U) = \underline{G}_M(u, U) \cap \overline{G}_M(u, U),$$

$$A_M(u, U) = \underline{A}_M(u, U) \cup \overline{A}_M(u, U).$$

Roughly speaking, A_M can be understood to be a set of points with 'bad Hessian'. Thus, we need to obtain uniform estimates for its measure to

establish $W^{2,p}$-theory, which will be our main purpose investigated in Section 3.3.

Viscosity solutions

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\Gamma \subset \partial\Omega$ and $T > 0$. Consider the following problem

$$\begin{cases} F(D^2u, Du, u, x, t) - u_t = f & \text{in } \Omega_T, \\ \beta \cdot Du = 0 & \text{on } \Gamma_T = \Gamma \times (0, T). \end{cases} \quad (2.1.1)$$

As in the previous subsection, we can define a viscosity solution for (2.1.1) as follows.

Definition 2.1.6 (C^2-viscosity solution). *Let F be continuous in all variables and $f \in C(\Omega_T \cup \Gamma_T)$. A continuous function $u \in C(\Omega_T \cup \Gamma_T)$ is called a viscosity solution of* (2.1.1) *if the following conditions hold:*

(a) for all $\varphi \in C^2(\Omega_T \cup \Gamma_T)$ touching u by above at $(x_0, t_0) \in \Omega_T \cup \Gamma_T$,

$$F(D^2\varphi(x_0, t_0), D\varphi(x_0, t_0), u(x_0, t_0), x_0, t_0) - \varphi_t(x_0, t_0) \geq f(x_0, t_0)$$

when $(x_0, t_0) \in \Omega_T$ and $\beta(x_0, t_0) \cdot D\varphi(x_0, t_0) \geq 0$ when $(x_0, t_0) \in \Gamma_T$.

(b) for all $\varphi \in C^2(\Omega_T \cup \Gamma_T)$ touching u by below at $(x_0, , t_0) \in \Omega_T \cup \Gamma_T$,

$$F(D^2\varphi(x_0, t_0), D\varphi(x_0, t_0), u(x_0, t_0), x_0, t_0) - \varphi_t(x_0, t_0) \leq f(x_0, t_0)$$

when $(x_0, t_0) \in \Omega_T$ and $\beta(x_0, t_0) \cdot D\varphi(x_0, t_0) \leq 0$ when $(x_0, t_0) \in \Gamma_T$.

Definition 2.1.7 ($W^{2,p}$-viscosity solution). *Let F be continuous in X and measurable in x. Suppose $p > n+1$ and $f \in L^p(\Omega_T)$. A continuous function u is called a $W^{2,p}$-viscosity solution for* (3.1.1) *if the following conditions hold:*

(a) For all $\varphi \in W^{2,p}(\Omega_T)$ whenever $\epsilon > 0$, $\mathcal{O}$ is open in $\Omega_T \cup \Gamma_T$ and

$$F(D^2\varphi(x, t), D\varphi(x, t), \varphi(x, t), x, t) - \varphi_t(x, t) \geq f(x, t) + \epsilon \quad \text{a.e. in } \mathcal{O}$$

$$\text{and} \quad \beta \cdot D\varphi(x, t) \geq \epsilon \quad \text{a.e. on } \mathcal{O} \cap \Gamma_T,$$

$u - \varphi$ cannot attain a local minimum in $\mathcal{O}$.

(b) For all $\varphi \in W^{2,q}(\Omega_T)$ whenever $\epsilon > 0$, $\mathcal{O}$ is open in $\Omega_T \cup \Gamma_T$ and

$$F(D^2\varphi(x,t), D\varphi(x,t), \varphi(x,t), x, t) - \varphi_t(x,t) \leq f(x,t) - \epsilon \quad a.e. \ in \ \mathcal{O}$$

$$and \quad \beta \cdot D\varphi(x,t) \leq -\epsilon \quad a.e. \ on \ \mathcal{O} \cap \Gamma_T,$$

$u - \varphi$ cannot attain a local maximum in $\mathcal{O}$.

Note that if a function u satisfies the condition (a) ((b), respectively) in the above definition, we say that $F(D^2u, Du, u, x, t) - u_t \geq (\leq)f$ in the viscosity sense.

Next we introduce Pucci's operator and the class S for the parabolic case.

Definition 2.1.8. *For any $M \in S(n)$, the Pucci extremal operator $\mathcal{M}^+$ and $\mathcal{M}^-$ are defined as follows:*

$$\mathcal{M}^+(\lambda, \Lambda, M) = \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i \ and \ \mathcal{M}^-(\lambda, \Lambda, M) = \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i$$

where e_i are eigenvalues of M. For $b \geq 0$ and u be a continuous function in the viscosity sense, we also write

$$L^\pm(\lambda, \Lambda, b, u) = \mathcal{M}^\pm(\lambda, \Lambda, D^2u) \pm b|Du| - u_t.$$

Next, we present an important concept to understand viscosity solutions.

Definition 2.1.9. *Let $\Omega \subset \mathbb{R}^n \times \mathbb{R}$, $b \geq 0$ and $0 < \lambda \leq \Lambda$. We define the classes $\underline{S}(\lambda, \Lambda, b, f)$ $(\overline{S}(\lambda, \Lambda, b, f)$, respectively) to be the set of all continuous functions u that satisfy $L^+u \geq f(L^-u \leq f)$ in the viscosity sense in Ω. We also define*

$$S(\lambda, \Lambda, b, f) = \overline{S}(\lambda, \Lambda, b, f) \cap \underline{S}(\lambda, \Lambda, b, f)$$

and

$$S^*(\lambda, \Lambda, b, f) = \overline{S}(\lambda, \Lambda, b, |f|) \cap \underline{S}(\lambda, \Lambda, b, -|f|).$$

When $b = 0$, we abbreviate $S(\lambda, \Lambda, 0, f)$ to $S(\lambda, \Lambda, f)$.

2.2 Time-dependent tug-of-war games

We start with several notations, which will be used throughout Chapter 4.

2.2.1 Notations

1. Let $k \geq 1$. For each $x = (x_1, \cdots, x_k), y = (y_1, \cdots, y_k) \in \mathbb{R}^k$,

$$\langle x, y \rangle := \sum_{i=1}^{k} x_i y_i.$$

2. $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ is the n-dimensional unit sphere.

3. For $\nu \in S^{n-1}$, $B_\epsilon^\nu = \{x \in B_\epsilon(0) : \langle x, \nu \rangle = 0\}$.

4. For each set $U \subset \mathbb{R}^n$, $\text{dist}(x, U) = \inf\{|x - y| : y \in U\}$ is the distance from x to U.

5. For each $\epsilon > 0$, we write $O_\epsilon = \{x \in \mathbb{R}^n \backslash \overline{\Omega} : \text{dist}(x, \partial\Omega) < \epsilon\}$ and $I_\epsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) < \epsilon\}$. We also write $\Gamma_\epsilon = O_\epsilon \cup I_\epsilon \cup \partial\Omega$ and $\Omega_\epsilon = \overline{\Omega} \cup O_\epsilon$.

6. For $\epsilon > 0$, $I_{\epsilon,T} = \{(x, t) \in \Omega \times [\frac{\epsilon^2}{2}, T] : \text{dist}(x, \partial\Omega) < \epsilon\} \cup \left(\Omega \times (0, \frac{\epsilon^2}{2})\right)$, $O_{\epsilon,T} = \{(x, t) \in (\mathbb{R}^n \backslash \overline{\Omega}) \times (0, T] : \text{dist}(x, \partial\Omega) < \epsilon\} \cup \left(\Omega_\epsilon \times (-\frac{\epsilon^2}{2}, 0)\right)$ and $\Gamma_{\epsilon,T} = I_{\epsilon,T} \cup O_{\epsilon,T} \cup \partial_p \Omega_T$. We write $\Omega_{\epsilon,T} = \overline{\Omega}_T \cup O_{\epsilon,T}$.

7. For $\{A_i\}_{i \in I} \subset \mathbb{R}$, we write

$$\text{midrange}_{i \in I} A_i = \frac{1}{2}\left(\sup_{i \in I} A_i + \inf_{i \in I} A_i\right).$$

8. For each $(n-1)$-dimensional set $U \subset \mathbb{R}^n$, $\mathcal{L}^{n-1}(U)$ is the $(n-1)$-dimensional Lebesgue measure of U.

9. Let U be an $(n-1)$-dimensional set in $\mathbb{R}^n$ with $\mathcal{L}^{n-1}(U) \neq 0$, and f be a measurable function on U. Then we write

$$\fint_U f(x) d\mathcal{L}^{n-1}(x) = \frac{1}{\mathcal{L}^{n-1}(U)} \int_U f(x) d\mathcal{L}^{n-1}(x).$$

10. Let $r, \epsilon > 0$ be given numbers. We write $Q_{r,\epsilon} = B_{r+\epsilon}(0) \times (-r^2 - \frac{\epsilon^2}{2}, 0)$ (only used in Chapter 4).

11. $\Sigma_a = \{(x, z, t, s) : x, z \in B_{ar}(0), -ar^2 < t < 0, |t - s| < \frac{\epsilon^2}{2}\}$.

12. $\Lambda_{t,\epsilon} = B_{r+\epsilon}(0) \times (t - \frac{\epsilon^2}{2}, t]$.

13. $\mathbf{R}_\nu = \{M \in \mathbf{O}(n) : Me_1 = \nu\}$, where $\mathbf{O}(n)$ is the orthogonal group in dimension n and $e_1 = (1, 0, \cdots, 0)$.

14. We abbreviate

$$\sup_{\substack{\nu_x, \nu_z \in S^{n-1} \\ (P_{\nu_x}, P_{\nu_z}) \in \mathbf{R}_{\nu_x} \times \mathbf{R}_{\nu_z}}}$$

to

$$\sup_{\nu_x, \nu_z \in S^{n-1}}.$$

2.2.2 Background knowledge

The time-dependent tug-of-war game

We introduce here the stochastic two-player zero-sum game which will be considered in Chapter 4.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $T > 0$ and $\alpha, \beta \in (0, 1)$ be fixed numbers with $\alpha + \beta = 1$. We also consider a function $F \in C(\Gamma_{\epsilon, T})$. From now on, we will use the symbol u_ϵ to denote a function satisfying the DPP (4.0.1) in Ω_T for given F.

Our game setting is as follows. There is a token located at a point $(x_0, t_0) \in \Omega_T$. Players will move it at each turn according to the outcome of the following processes. We write locations of the token as $(x_1, t_1), (x_2, t_2), \cdots$ and denote by $Z_j = (x_j, t_j)$ for our convenience.

When $Z_j \in \Omega \backslash I_\epsilon$, Player I and II choose some vectors $\nu_j^{\mathrm{I}}, \nu_j^{\mathrm{II}} \in \partial B_\epsilon$. First, players compete to move token with a fair coin toss. Next, they have one more stochastic process to determine how to move the token. The winner of first coin toss, Player $i \in \{\mathrm{I}, \mathrm{II}\}$ can move the token to direction of the chosen vector ν_j^i with probability α. Otherwise, the token is moved uniformly random in the $(n-1)$-ball perpendicular to ν_j^i. After these processes are finished, t_j is changed by $t_{j+1} = t_j - \epsilon^2/2$.

If $Z_j \in \Gamma_\epsilon$, the game progresses in the same way as above with probability $1 - \delta_\epsilon(Z_j)$. On the other hand, with probability $\delta_\epsilon(Z_j)$, the game is over and Player II pays Player I payoff $F(Z_j)$.

We denote by τ the number of total turns until end of the game. One can observe that τ must be finite in our setting since the game ends when $t \leq 0$.

Now we give mathematical construction for this game. Let $\xi_0, \xi_1, \cdots$ be iid random variables to have a uniform distribution $U(0,1)$. This process $\{\xi_j\}_{j=0}^{\infty}$ is independent of $\{Z_j\}_{j=0}^{\infty}$.

Define $\tilde{C} := \{0,1\}$. We set random variables $c_0, c_1, \cdots \in \tilde{C}$ as follows:

$$c_j = \begin{cases} 0 & \text{when } \xi_{j-1} \leq 1 - \delta_\epsilon(Z_{j-1}), \\ 1 & \text{when } \xi_{j-1} > 1 - \delta_\epsilon(Z_{j-1}) \end{cases}$$

for $j \geq 1$ and $c_0 = 0$. Then we can write the stopping time τ by

$$\tau := \inf\{j \geq 0 : c_{j+1} = 1\}.$$

In our game, each player chooses their strategies by using past data (history). We can write a history as the following vector

$$\big((c_0, Z_0), (c_1, Z_1), \cdots, (c_j, Z_j)\big).$$

Then, the strategy of Player i can be defined by a Borel measurable function as $\mathcal{S}_i = \{S_i^j\}_{j=1}^{\infty}$ with

$$S_i^j : \{(c_0, Z_0)\} \times \bigcup_{k=1}^{j-1} (\tilde{C} \times \Omega_{\epsilon, T}) \to \partial B_\epsilon(0)$$

for any $j \in \mathbb{N}$.

Next we define a probability measure $\mathbb{P}_{S_{\mathrm{I}}, S_{\mathrm{II}}}^{Z_0}$ natural product σ-algebra of the space of all game trajectories for any starting point $Z_0 \in \Omega_{\epsilon, T}$. By Kolmogorov's extension theorem, we can construct the measure to the family of transition densities

$$\begin{aligned} &\pi_{S_{\mathrm{I}}, S_{\mathrm{II}}}((c_0, Z_0), (c_1, Z_1), \cdots, (c_j, Z_j); C, A_{j+1}) \\ &= (1 - \delta_\epsilon(Z_j)) \pi_{S_{\mathrm{I}}, S_{\mathrm{II}}}^{local}((Z_0, Z_1, \cdots, Z_j); A_{j+1}) \mathbb{I}_0(C) \mathbb{I}_{c_j}(\{0\}) \\ &\quad + \delta_\epsilon(Z_j) \mathbb{I}_{Z_j}(A_j) \mathbb{I}_1(C) \mathbb{I}_{c_j}(\{0\}) + \mathbb{I}_{Z_j}(A_j) \mathbb{I}_{c_j}(\{1\}) \end{aligned}$$

for $A_n = A \times \{t_n\}$ (A is any Borel set in $\mathbb{R}^n$ and $n \geq 0$) and $C \subset \tilde{C}$, where

$$\begin{aligned} &\pi_{S_{\mathrm{I}}, S_{\mathrm{II}}}^{local}(Z_0, Z_1, \cdots, Z_j; A_{j+1}) \\ &= \frac{1}{2}\bigg[\alpha\big(\mathbb{I}_{(x_j + \nu_{j+1}^I, t_{j+1})}(A_{j+1}) + \mathrm{I}_{(x_j + \nu_{j+1}^{II}, t_{j+1})}(A_{j+1})\big) \end{aligned}$$

$$+ \frac{\beta}{\omega_{n-1}\epsilon^{n-1}}\left(\mathcal{L}^{n-1}(B_\epsilon^{\nu_{j+1}^{\mathrm{I}}}(Z_j) \cap A_{j+1}) + \mathcal{L}^{n-1}(B_\epsilon^{\nu_{j+1}^{\mathrm{II}}}(Z_j) \cap A_{j+1})\right)\Bigg].$$

Here, $\omega_{n-1} = \mathcal{L}^{n-1}(B_1^{n-1})$ where B_1^{n-1} is the $(n-1)$-dimensional unit ball and

$$\mathbb{I}_z(B) = \begin{cases} 0 & \text{when } z \notin B, \\ 1 & \text{when } z \in B. \end{cases}$$

Finally, for any starting point $Z_0 = (x_0, t_0) \in \Omega_T$, we define value functions u_{I} and u_{II} of this game for Player I and II by

$$u_{\mathrm{I}}(Z_0) = \sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} \mathbb{E}^{Z_0}_{S_{\mathrm{I}}, S_{\mathrm{II}}}[F(Z_\tau)]$$

and

$$u_{\mathrm{II}}(Z_0) = \inf_{S_{\mathrm{II}}} \sup_{S_{\mathrm{I}}} \mathbb{E}^{Z_0}_{S_{\mathrm{I}}, S_{\mathrm{II}}}[F(Z_\tau)],$$

respectively.

Chapter 3

Regularity results for oblique derivative problems

In this chapter we are concerned with regularity theory for fully nonlinear equations with oblique boundary conditions. We deal with the elliptic case in Section 3.1 and the parabolic case in Section 3.2. In each case, we provide a global Carlderón-Zygmund type estimate.

3.1 $W^{2,p}$-regularity for elliptic problems

3.1.1 Hypotheses and main results

In this section, we establish global $W^{2,p}-$regularity theory for elliptic oblique derivative problems. We consider the following problem

$$\begin{cases} F(D^2u, Du, u, x) = f & \text{in } \Omega, \\ \beta \cdot Du = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1.1}$$

Here, $\Omega \subset \mathbb{R}^n$ is a bounded domain with its boundary $\partial\Omega$, β is a given vector-valued function on $\partial\Omega$ with $||\beta||_{L^\infty(\partial\Omega)} \leq 1$ and $F = F(X, q, r, x)$ is a function on $S(n) \times \mathbb{R}^n \times \mathbb{R} \times \Omega$.

We assume that F is convex in X, continuous in X, q, r and x, and satisfies

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the following structure condition

$$
\begin{aligned}
\mathcal{M}^-(\lambda, \Lambda, X_1 - X_2) &- b|q_1 - q_2| - c|r_1 - r_2| \\
&\leq F(X_1, q_1, r_1, x) - F(X_2, q_2, r_2, x) \\
&\leq \mathcal{M}^+(\lambda, \Lambda, X_1 - X_2) + b|q_1 - q_2| + c|r_1 - r_2|
\end{aligned}
\tag{3.1.2}
$$

for fixed $0 < \lambda \leq \Lambda$ and $b, c > 0$, and $X_1, X_2 \in S(n)$, $q_1, q_2 \in \mathbb{R}^n$, $r_1, r_2 \in \mathbb{R}$ and $x \in \Omega$.

Next we introduce the following definition in order to measure the oscillation of F in the variable x.

Definition 3.1.1. *Let $F : S(n) \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$ and $x_0 \in \Omega$. For $x \in \Omega$, We define*

$$
\psi_F(x, x_0) := \sup_{X \in S(n) \setminus \{0\}} \frac{|F(X, 0, 0, x) - F(X, 0, 0, x_0)|}{||X||}.
$$

We will assume that F has small oscillation in the L^n-sense to obtain $W^{2,p}$-regularity.

We now state the main result in this subsection.

Theorem 3.1.2. *Let Ω be a bounded C^3-domain and $\mathbf{n}$ be the inward unit normal vector of $\partial\Omega$. Assume that u is a $W^{2,p}$-viscosity solution of (3.1.1) where $F(X, q, r, x)$ is convex in X, continuous in x and satisfies structure condition (3.1.2) with $F(0, 0, 0, x) \equiv 0$, $\beta \in C^2(\partial\Omega)$ with $\beta \cdot \mathbf{n} \geq \delta_0$ for some $\delta_0 > 0$ and $f \in L^p(\Omega) \cap C(\Omega)$ for $n < p < \infty$. Then there exist two constants*

$$
\epsilon_0 = \epsilon_0(n, p, \lambda, \Lambda, \delta_0, ||\beta||_{C^2(\partial\Omega)})
$$

and

$$
C = C(n, p, \lambda, \Lambda, \delta_0, ||\beta||_{C^2(\partial\Omega)}, b, c, r_0, \operatorname{diam}(\Omega))
$$

if

$$
\left(\fint_{B_r(x_0) \cap \Omega} \psi(x_0, x)^n \, dx \right)^{1/n} \leq \epsilon_0
$$

for any $x_0 \in \Omega$ and $0 < r < r_0$, then $u \in W^{2,p}(\Omega)$ with the estimate

$$
||u||_{W^{2,p}(\Omega)} \leq C(||u||_{L^\infty(\Omega)} + ||f||_{L^p(\Omega)}).
$$

3.1.2 Auxiliary results

We introduce some useful lemmas in order to proceed with our discussion.

We first mention a well-known lemma from [13, Lemma 4.2] to be used for our work.

Lemma 3.1.3 (Calderón-Zygmund decomposition)**.** *Assume that A and B are measurable sets and $A \subset B \subset Q_1$. Suppose that there exists an $\epsilon \in (0,1)$ such that $|A| \leq \epsilon$ and for any dyadic cube Q and its predecessor $\tilde{Q}$, $|A \cap Q| > \epsilon |Q| \implies \tilde{Q} \subset B$. Then, $|A| \leq \epsilon |B|$.*

Strong (p,p)-estimate is also necessary to derive our desired result. We can find the proof in [71, Theorem 1].

Proposition 3.1.4 (Strong (p,p)-estimate)**.** *The maximal operator M is defined as follows:*

$$M(f)(x) = \sup_{\rho > 0} \fint_{B_\rho(x)} |f(x)| dx.$$

Then, for any $f \in L^p(\mathbb{R}^n)$ where $1 < p < \infty$,

$$||M(f)||_{L^p(\mathbb{R}^n)} \leq C(n,p)||f||_{L^p(\mathbb{R}^n)}.$$

The following measure theoretic property can be found in several references, for example, [13].

Proposition 3.1.5. *Let g be a nonnegative and measurable function in Ω and μ_g be its distribution function, that is,*

$$\mu_g(t) = |\{x \in \Omega : g(x) > t\}| \quad for \ t > 0.$$

Let $\eta > 0$ and $M > 1$ be constants. Then, for $0 < p < \infty$,

$$g \in L^p(\Omega) \iff \sum_{k \geq 1} M^{pk} \mu_g(\eta M^k) =: S < \infty$$

and

$$C^{-1} S \leq ||g||_{L^p(\Omega)}^p \leq C(|\Omega| + S).$$

In [13], there are shown several properties of A_M and G_M in the interior case. We can also find corresponding results for boundary estimates in [78].

Lemma 3.1.6. *[78] Assume that $u \in \overline{S}(\lambda, \Lambda, f)$ in $B^+_{12\sqrt{n}h_1^{-1}, 12\sqrt{n}} \subset \Omega$, $u \in C(\Omega)$ and $||u||_{L^\infty(\Omega)} \leq 1$. Then there exist universal constants $M > 1$ and $0 < \sigma < 1$ such that $||f||_{L^n(B^+_{12\sqrt{n}h_1^{-1}, 12\sqrt{n}})} \leq 1$ implies*

$$|\underline{G}_M(u, \Omega) \cap (Q_1^{n-1} \times (0,1) + x_0)| \geq 1 - \sigma$$

for any $x_0 \in B^+_{9\sqrt{n}h_1^{-1}, 9\sqrt{n}} \cup T_{9\sqrt{n}h_1^{-1}}$.

Lemma 3.1.7. *[78] Let $u \in S(\lambda, \Lambda, f)$ in $B^+_{12\sqrt{n}h_1^{-1}, 12\sqrt{n}} \subset \Omega \subset \mathbb{R}^n_+$ and u be continuous in Ω. Assume that $||u||_{L^\infty(\Omega)} \leq 1$. Then there exist universal constants C, μ such that $||f||_{L^n(B^+_{12\sqrt{n}h_1^{-1}, 12\sqrt{n}})} \leq 1$ implies*

$$|A_t(u, \Omega) \cap (Q_1^{n-1} \times (0,1) + x_0)| \leq Ct^{-\mu}$$

for any $x_0 \in B^+_{9\sqrt{n}h_1^{-1}, 9\sqrt{n}} \cup T_{9\sqrt{n}h_1^{-1}}$ and $t > 1$.

3.1.3 Boundary $W^{2,p}$-estimates

The purpose of this subsection is to obtain boundary $W^{2,p}$-regularity for elliptic oblique derivative problems.

Consider the following problem

$$\begin{cases} F(D^2u, x) = f & \text{in } B_1^+, \\ \beta \cdot Du = 0 & \text{on } T_1. \end{cases} \tag{3.1.3}$$

We note that structure condition (3.1.2) can be replaced by the following uniform ellipticity condition with constants λ and Λ for this problem, that is,

$$\lambda||X_2|| \leq F(X_1 + X_2, q, r, x) - F(X_1, q, r, x) \leq \Lambda||X_2||$$

for any $X_1, X_2 \in S(n)$, $X_2 \geq 0$, $q \in \mathbb{R}^n$, $r \in \mathbb{R}$ and $x \in \Omega$.

Let us state the main theorem in this subsection.

Theorem 3.1.8. *Let u be a C^2-viscosity solution of (3.1.3) where $F(X, x)$ is uniformly elliptic with λ and Λ, convex in X, continuous in X and x, $F(0, x) = 0$, $\beta \in C^2(T_1)$ with $\beta \cdot \mathbf{n} \geq \delta_0$ for some $\delta_0 > 0$, and $f \in L^p(B_1^+) \cap C(B_1^+)$ for $n < p < \infty$. Then there exist two constants $\epsilon_0 =$*

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$\epsilon_0(n, p, \lambda, \Lambda, \delta_0, ||\beta||_{C^2(T_1)})$ *and* $C = C(n, p, \lambda, \Lambda, \delta_0, ||\beta||_{C^2(T_1)})$ *such that*

$$\left(\fint_{B_r(x_0) \cap B_1^+} \psi(x_0, x)^n \, dx \right)^{1/n} \leq \epsilon_0$$

for any $x_0 \in B_1^+$ *and* $r > 0$ *implies* $u \in W^{2,p}(B_{1/2}^+)$ *and we have the estimate*

$$||u||_{W^{2,p}(B_{1/2}^+)} \leq C(||u||_{L^\infty(B_1^+)} + ||f||_{L^p(B_1^+)}).$$

We first introduce the following lemmas in [42]. These results play an important role in establishing regularity results for oblique derivative problems.

Lemma 3.1.9 (ABP maximum principle). *[42] Let* $\Omega \subset B_1$ *and* u *satisfy*

$$\begin{cases} u \in S^*(\lambda, \Lambda, b, f) & in \ \Omega, \\ \beta \cdot Du = g & on \ \Gamma. \end{cases} \qquad (3.1.4)$$

Suppose that there exists $\xi \in \partial B_1$ *such that* $\beta \cdot \xi \geq \delta_0$. *Then*

$$||u||_{L^\infty(\Omega)} \leq ||u||_{L^\infty(\partial\Omega \setminus \Gamma)} + C(||g||_{L^\infty(\Gamma)} + ||f||_{L^n(\Omega)}),$$

where C *only depends on* n, λ, Λ, b *and* δ_0.

Lemma 3.1.10. *[42] Let* u *satisfy* (3.1.4). *Then for any* $\Omega' \subset\subset \Omega \cup \Gamma$, $u \in C^{0,\alpha}(\overline{\Omega}')$ *and*

$$||u||_{C^{0,\alpha}(\overline{\Omega}')} \leq C(||u||_{L^\infty(\Omega)} + ||f||_{L^n(\Omega)} + ||g||_{L^\infty(\Gamma)}),$$

where $0 < \alpha < 1$ *only depends on* n, λ, Λ, b *and* δ_0, *and* C *depends also on* Ω' *and* Ω.

In [42], the case $b = 0$ was only considered. We can extend Lemma 3.1.9 and 3.1.10 to the case of $S(\lambda, \Lambda, b, f)$ with $b \neq 0$, since key ideas of the proof of these theorems are ABP maximum principle and Harnack inequality. In this case, the universal constant C also depends on b.

Meanwhile, interior and boundary $C^{1,1}$-estimates for model problems are necessary to establish $W^{2,p}$-estimates. For the interior case, once can find $C^{1,1}$-estimate in [13]. We refer to the following results in [42] for oblique derivative problems.

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Lemma 3.1.11. *[42] Let F be convex, u be a viscosity solution of*

$$\begin{cases} F(D^2 u) = f & in \ \Omega, \\ \beta \cdot Du = g & on \ \Gamma \subset \partial\Omega, \end{cases} \tag{3.1.5}$$

and $0 < \alpha < \tilde{\alpha}$, where $0 < \tilde{\alpha} < 1$ is a constant depending only on n, λ, Λ and δ_0. Suppose that $\Gamma \in C^{1,\alpha}$, $\beta, g \in C^{1,\alpha}(\overline{\Gamma})$ and $f \in C^{0,\alpha}(\overline{\Omega})$. Then for any $\Omega' \subset\subset \Omega \cup \Gamma$, $u \in C^{2,\alpha}(\overline{\Omega}')$ and

$$||u||_{C^{2,\alpha}(\overline{\Omega}')} \leq C(||u||_{L^\infty(\Omega)} + ||f||_{C^{0,\alpha}(\overline{\Omega})} + ||g||_{C^{1,\alpha}(\overline{\Gamma})} + |F(0)|),$$

where C only depends on $n, \lambda, \Lambda, \delta_0, \alpha, ||\beta||_{C^{1,\alpha}(\overline{\Gamma})}, \Omega'$ and Ω.

Now we fix small enough $h_0 > 0$ such that

$$\beta(x) \cdot \mathbf{n}(y) < 0 \tag{3.1.6}$$

for any $x \in T_1$ and $y \in \partial B^+_{1,h_0} \backslash T_1$ throughout this subsection. Then we can obtatin the following approximation lemma for solutions of (3.1.3).

Lemma 3.1.12. *Let $0 < \epsilon < 1$, h_0 be a constant satisfying (3.1.6) and u be a C^2-viscosity solution of (3.1.3). Assume that $||u||_{L^\infty(B^+_{1,h_0})} \leq 1$ and $||\psi(\cdot, 0)||_{L^n(B^+_{1,h_0})} \leq \epsilon$. Then, there exists a function $h \in C^2(\overline{B}^+_{\frac{3}{4}, \frac{3}{4}h_0})$ such that $u - h \in S(\varphi)$, $||h||_{C^2(\overline{B}^+_{\frac{3}{4}, \frac{3}{4}h_0})} \leq C$ and*

$$||u - h||_{L^\infty(B^+_{\frac{3}{4}, \frac{3}{4}h_0})} + ||\varphi||_{L^n(B^+_{\frac{3}{4}, \frac{3}{4}h_0})} \leq C(\epsilon^\gamma + ||f||_{L^n(B^+_{1,h_0})})$$

for some $0 < \gamma = \gamma(n, \lambda, \Lambda, \delta_0) < 1$ and $C = C(n, \lambda, \Lambda, \delta_0, ||\beta||_{C^2(T_1)})$. Here, $\varphi = f - F(D^2 h, \cdot)$.

Proof. We consider a function h which is a solution of

$$\begin{cases} F(D^2 h, 0) = 0 & in \ B^+_{\frac{7}{8}, \frac{7}{8}h_0}, \\ h = u & on \ \partial B^+_{\frac{7}{8}, \frac{7}{8}h_0} \backslash T_{\frac{7}{8}}, \\ \beta \cdot Dh = 0 & on \ T_{\frac{7}{8}}. \end{cases} \tag{3.1.7}$$

Then by Lemma 3.1.10, for some $\alpha_1 = \alpha_1(n, \lambda, \Lambda, \delta_0)$ and $C = C(n, \lambda, \Lambda, \delta_0)$,

$$||u||_{C^{0,\alpha_1}(B^+_{\frac{7}{8}, \frac{7}{8}h_0})} \leq C(1 + ||f||_{L^n(B^+_{1,h_0})}) \tag{3.1.8}$$

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and by Lemma 3.1.11 with a proper scaling,

$$||h||_{L^\infty(B^+_{\frac{7}{8}-\delta,(\frac{7}{8}-\delta)h_0})} + \delta||Dh||_{L^\infty(B^+_{\frac{7}{8}-\delta,(\frac{7}{8}-\delta)h_0})} + \delta^2||D^2h||_{L^\infty(B^+_{\frac{7}{8}-\delta,(\frac{7}{8}-\delta)h_0})}$$
$$\leq C$$

for some constant C depending on $n, \lambda, \Lambda, \delta_0$ and $||\beta||_{C^2(T_1)}$.

Let $w = u - h$. Observe that w satisfies

$$
\begin{cases}
w \in S(\lambda/n, \Lambda, \varphi) & \text{in } B^+_{\frac{7}{8},\frac{7}{8}h_0}, \\
w = 0 & \text{on } \partial B^+_{\frac{7}{8},\frac{7}{8}h_0} \backslash T_{\frac{7}{8}}, \\
\beta \cdot Dw = 0 & \text{on } T_{\frac{7}{8}}.
\end{cases}
\tag{3.1.9}
$$

By the ABP estimate (Lemma 3.1.9), we can see

$$||w||_{L^\infty(B^+_{\frac{7}{8}-\delta,(\frac{7}{8}-\delta)h_0})} \leq C(||\varphi||_{L^n(B^+_{\frac{7}{8}-\delta,(\frac{7}{8}-\delta)h_0})} + ||w||_{L^\infty(\partial B^+_{\frac{7}{8}-\delta,(\frac{7}{8}-\delta)h_0} \backslash T_{\frac{7}{8}-\delta})})$$
$$\leq C(||f||_{L^n(B^+_{\frac{7}{8}-\delta,(\frac{7}{8}-\delta)h_0})} + ||F(D^2h, \cdot)||_{L^n(B^+_{\frac{7}{8}-\delta,(\frac{7}{8}-\delta)h_0})}$$
$$+ ||w||_{L^\infty(\partial B^+_{\frac{7}{8}-\delta,(\frac{7}{8}-\delta)h_0} \backslash T_{\frac{7}{8}-\delta})})$$

for some $C = C(n, \lambda, \Lambda, \delta_0)$. Since $h \in C^2(B^+_{\frac{7}{8}-\delta})$, we also derive that

$$||F(D^2h, \cdot)||_{L^n(B^+_{\frac{7}{8}-\delta,(\frac{7}{8}-\delta)h_0})} \leq ||\psi(\cdot, 0)||_{L^n(B^+_{\frac{7}{8}-\delta,(\frac{7}{8}-\delta)h_0})}||D^2h||_{L^\infty(B^+_{\frac{7}{8}-\delta,(\frac{7}{8}-\delta)h_0})}$$
$$\leq C\delta^{-2}\epsilon$$

where $C = C(n, \lambda, \Lambda, \delta_0, ||\beta||_{C^2(T_1)})$.

On the other hand, we already know that $w \equiv 0$ on $\partial B^+_{\frac{7}{8},h_0} \backslash T_{\frac{7}{8}}$ and $u \in C^{0,\alpha_1}(\overline{B}^+_{\frac{7}{8},h_0})$. We can also obtain a global Hölder regularity for h by using [42, Corollary 3.1] and (3.1.8). Combining these results, we can derive that

$$||w||_{L^\infty(\partial B^+_{\frac{7}{8}-\delta,(\frac{7}{8}-\delta)h_0} \backslash T_{\frac{7}{8}-\delta})} \leq C\delta^{\alpha_2}(1 + ||f||_{L^n(B^+_{1,h_0})})$$

for some $\alpha_2 \in (0, \alpha_1)$ and $C = C(n, \lambda, \Lambda, \delta_0)$. Thus, if we put $\gamma = \delta^{\frac{\alpha_2}{2+\alpha_2}}$,

$$||w||_{L^\infty(\partial B^+_{\frac{7}{8}-\delta,(\frac{7}{8}-\delta)h_0})} \leq C\{||f||_{L^n(B^+_{1,h_0})} + \delta^{-2}\epsilon + \delta^{\alpha_2}(1 + ||f||_{L^n(B^+_{1,h_0})})\}$$

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$$\leq C(\epsilon^\gamma + ||f||_{L^n(B^+_{1,h_0})})$$

for some constant C depending only on $n, \lambda, \Lambda, \delta_0$ and $||\beta||_{C^2(T_1)}$. Then the proof is completed by choosing $\delta = 1/8$. $\qquad\square$

Next we show following lemmas which give us information about power decay of $|A_M(u,\Omega)|$.

Lemma 3.1.13. *Let* $0 < \epsilon_0 < 1$, $B^+_{14\sqrt{n}h_1^{-1},14\sqrt{n}} \subset \Omega \subset \mathbb{R}^n_+$ *and* u *be a* C^2-*viscosity solution of*

$$\begin{cases} F(D^2 u, x) = f & in \ B^+_{14\sqrt{n}h_1^{-1},14\sqrt{n}}, \\ \beta \cdot Du = 0 & on \ T_{14\sqrt{n}h_1^{-1}}, \end{cases} \qquad (3.1.10)$$

where $h_1 = h_1(\delta_0)$ *is a small constant satisfying* (3.1.6) *for any* $x \in T_{14\sqrt{n}h_1^{-1}}$ *and* $y \in \partial B^+_{14\sqrt{n}h_1^{-1},14\sqrt{n}} \setminus T_{14\sqrt{n}h_1^{-1}}$. *Assume that*

$$||f||_{L^n(B^+_{14\sqrt{n}h_1^{-1},14\sqrt{n}})}, ||\psi(\cdot,0)||_{L^n(B^+_{14\sqrt{n}h_1^{-1},14\sqrt{n}})} \leq \epsilon < 1$$

for some ϵ *depending on* $n, \epsilon_0, \lambda, \Lambda, \delta_0$ *and* $||\beta||_{C^2(T_{14\sqrt{n}h_1^{-1}})}$. *Then,*

$$G_1(u,\Omega) \cap (Q_2^{n-1} \times (0,2) + \tilde{x}_0) \neq \varnothing$$

for some $\tilde{x}_0 \in B^+_{9\sqrt{n}h_1^{-1},9\sqrt{n}} \cup T_{9\sqrt{n}h_1^{-1}}$ *implies*

$$|G_M(u,\Omega) \cap (Q_1^{n-1} \times (0,1) + x_0)| \geq 1 - \epsilon_0,$$

where $x_0 \in B^+_{9\sqrt{n}h_1^{-1},9\sqrt{n}} \cup T_{9\sqrt{n}h_1^{-1}}$ *and* M *is a constant depending only on* $n, \lambda, \Lambda, \delta_0$ *and* $||\beta||_{C^2(T_{14\sqrt{n}h_1^{-1}})}$.

Proof. Let $x_1 \in G_1(u,\Omega) \cap (Q_2^{n-1} \times (0,2) + \tilde{x}_0)$. Then, for every $x \in \Omega$ and some affine function L,

$$|(u - L)(x)| \leq |x - x_1|^2/2.$$

Define $\tilde{u}(x) = (u - L)(x)/C(n,\delta_0)$ so that $||\tilde{u}||_{L^\infty(B^+_{14\sqrt{n}h_1^{-1},14\sqrt{n}})} \leq 1$ and $|\tilde{u}(x)| \leq |x|^2$ in $\Omega \setminus B^+_{14\sqrt{n}h_1^{-1},14\sqrt{n}}$. Observe that $||L||_{C^1(B^+_{14\sqrt{n}})}$ is uniformly bounded and depending only on n, δ_0 in this case.

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Set $\tilde{F}(D^2\tilde{u}, x) = F(CD^2\tilde{u}, x)/C(n, \delta_0)$ and $\tilde{f}(x) = f(x)/C(n, \delta_0)$. Then we can observe that the elliptic constants of F and $\tilde{F}$ are the same and $\tilde{u}$ satisfies

$$\begin{cases} \tilde{F}(D^2\tilde{u}, x) = \tilde{f} & \text{in } B^+_{14\sqrt{n}h_1^{-1}, 14\sqrt{n}}, \\ \beta \cdot D\tilde{u} = -\beta \cdot DL/C(n, \delta_0) & \text{on } T_{14\sqrt{n}h_1^{-1}}. \end{cases} \qquad (3.1.11)$$

Now consider a function $\tilde{h} \in C(\overline{B}^+_{13\sqrt{n}h_1^{-1}, 13\sqrt{n}})$ such that

$$\begin{cases} \tilde{F}(D^2\tilde{h}, 0) = 0 & \text{in } B^+_{13\sqrt{n}h_1^{-1}, 13\sqrt{n}}, \\ \tilde{h} = \tilde{u} & \text{on } \partial B^+_{13\sqrt{n}h_1^{-1}, 13\sqrt{n}} \backslash T_{13\sqrt{n}h_1^{-1}}, \\ \beta \cdot D\tilde{h} = -\beta \cdot DL/C(n, \delta_0) & \text{on } T_{13\sqrt{n}h_1^{-1}}. \end{cases} \qquad (3.1.12)$$

Observe that $\beta \cdot DL \in C^2(T_{14\sqrt{n}h_1^{-1}})$ since $\beta \in C^2(T_{14\sqrt{n}h_1^{-1}})$ and DL is a constant vector. Hence, we can obtain C^2-estimate for $\tilde{h}$. By using again Lemma 3.1.10 and Lemma 3.1.11, we deduce that

$$\begin{aligned} &||\tilde{u}||_{C^{0,\alpha_1}(B^+_{13\sqrt{n}h_1^{-1}, 13\sqrt{n}})} \\ &\leq C(1 + ||f||_{L^n(B^+_{14\sqrt{n}h_1^{-1}, 14\sqrt{n}})} + ||DL||_{L^\infty(B^+_{14\sqrt{n}h_1^{-1}, 14\sqrt{n}})}) \end{aligned}$$

for some $\alpha_1 = \alpha_1(n, \lambda, \Lambda, \delta_0)$ and

$$\begin{aligned} &||\tilde{h}||_{C(B^+_{(13\sqrt{n}-\delta)h_1^{-1}, 13\sqrt{n}-\delta})} + \delta||D\tilde{h}||_{C(B^+_{(13\sqrt{n}-\delta)h_1^{-1}, 13\sqrt{n}-\delta})} \\ &\qquad + \delta^2||D^2\tilde{h}||_{C(B^+_{(13\sqrt{n}-\delta)h_1^{-1}, 13\sqrt{n}-\delta})} \\ &\leq C(||\tilde{h}||_{L^\infty(B^+_{(13\sqrt{n}-\delta)h_1^{-1}, 13\sqrt{n}-\delta})} + ||\beta \cdot DL||_{C^2(T_{(13\sqrt{n}-\delta)h_1^{-1}})}) \end{aligned}$$

for some C depending on $n, \lambda, \Lambda, \delta_0$ and $||\beta||_{C^2(T_{14\sqrt{n}h_1^{-1}})}$. Observe that

$$\begin{aligned} ||\beta \cdot DL||_{C^2(T_{14\sqrt{n}h_1^{-1}})} &\leq C(n, \delta_0)||\beta||_{C^2(T_{14\sqrt{n}h_1^{-1}})} \\ &\leq C(n, \delta_0, ||\beta||_{C^2(T_{14\sqrt{n}h_1^{-1}})}) \end{aligned}$$

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and

$$||\tilde{h}||_{L^\infty(B^+_{(13\sqrt{n}-\delta)h_1^{-1},13\sqrt{n}-\delta})}$$
$$\leq ||\tilde{h}||_{L^\infty(\partial B^+_{(13\sqrt{n}-\delta)h_1^{-1},13\sqrt{n}-\delta}\setminus T_{(13\sqrt{n}-\delta)h_1^{-1})}}$$
$$+ C(n,\lambda,\Lambda,\delta_0)||\beta \cdot DL||_{L^\infty(T_{(13\sqrt{n}-\delta)h_1^{-1}})}$$
$$\leq ||\tilde{u}||_{L^\infty(\partial B^+_{13\sqrt{n}h_1^{-1}}\setminus T_{13\sqrt{n}h_1^{-1}})} + C(n,\lambda,\Lambda,\delta_0)$$
$$+ C(n,\lambda,\Lambda,\delta_0)(1 + ||f||_{L^n(B^+_{14\sqrt{n}h_1^{-1},14\sqrt{n}})} + ||DL||_{L^\infty(B^+_{14\sqrt{n}h_1^{-1},14\sqrt{n}})})\delta^{\alpha_2}$$
$$\leq C(n,\lambda,\Lambda,\delta_0)$$

for some $\alpha_2 \in (0,\alpha_1)$. We used the similar argument in the proof of Lemma
3.1.12 to estimate $||\tilde{h}||_{L^\infty(\partial B^+_{(13\sqrt{n}-\delta)h_1^{-1},13\sqrt{n}-\delta}\setminus T_{(13\sqrt{n}-\delta)h_1^{-1}})}$. Then we see that

$$||D^2\tilde{h}||_{L^\infty(B^+_{(13\sqrt{n}-\delta)h_1^{-1},13\sqrt{n}-\delta})} \leq \delta^{-2}C(n,\lambda,\Lambda,\delta_0,||\beta||_{C^2(T_{14\sqrt{n}h_1^{-1}})})$$

and therefore

$$||D^2\tilde{h}||_{L^\infty(B^+_{12\sqrt{n}h_1^{-1},12\sqrt{n}})} \leq C(n,\lambda,\Lambda,\delta_0,||\beta||_{C^2(T_{14\sqrt{n}h_1^{-1}})}).$$

This implies

$$A_N(\tilde{h}, B^+_{12\sqrt{n}h_1^{-1},12\sqrt{n}}) \cap (Q_1^{n-1} \times (0,1) + x_0) = \varnothing$$

for sufficiently large $N = N(n,\lambda,\Lambda,\delta_0,||\beta||_{C^2(T_{14\sqrt{n}h_1^{-1}})}) > 1$. Now we extend
$\tilde{h}$ to $\dot{h}$ such that $\dot{h}$ is continuous in Ω, $\dot{h} = \tilde{u}$ in $\Omega\setminus B^+_{13\sqrt{n}h_1^{-1},13\sqrt{n}}$, and $||\tilde{u} - \dot{h}||_{L^\infty(\Omega)} = ||\tilde{u} - \tilde{h}||_{L^\infty(B^+_{12\sqrt{n}h_1^{-1},12\sqrt{n}})}$. Then

$$||\tilde{u} - \dot{h}||_{L^\infty(\Omega)} \leq ||\tilde{u}||_{L^\infty(B^+_{12\sqrt{n}h_1^{-1},12\sqrt{n}})} + ||\tilde{h}||_{L^\infty(B^+_{12\sqrt{n}h_1^{-1},12\sqrt{n}})} \leq C_0$$

for some $C_0 = C_0(n,\lambda,\Lambda,\delta_0,||\beta||_{C^2(T_{14\sqrt{n}h_1^{-1}})})$ and thus $|\dot{h}(x)| \leq C_0 + |x|^2$ in

$\Omega \backslash B^+_{12\sqrt{n}h_1^{-1}, 12\sqrt{n}}$. Hence, for some $M_0 \geq N$, we see

$$A_{M_0}(\dot{h}, \Omega) \cap (Q_1^{n-1} \times (0,1) + x_0) = \varnothing.$$

Consider $w = \tilde{u} - \dot{h}$. We can see that

$$\begin{cases} w \in S(\lambda/n, \Lambda, \tilde{f} - \tilde{F}(D^2\tilde{h}, \cdot)) & \text{in } B^+_{13\sqrt{n}h_1^{-1}, 13\sqrt{n}}, \\ w = 0 & \text{on } \partial B^+_{13\sqrt{n}h_1^{-1}, 13\sqrt{n}} \backslash T_{13\sqrt{n}h_1^{-1}}, \\ \beta \cdot Dw = 0 & \text{on } T_{13\sqrt{n}h_1^{-1}}. \end{cases} \qquad (3.1.13)$$

By ABP maximum principle and the definition of w,

$$\|w\|_{L^\infty(\Omega)} = \|w\|_{L^\infty(B^+_{13\sqrt{n}h_1^{-1}, 13\sqrt{n}})} \leq C(\epsilon^\gamma + \|f\|_{L^n(B^+_{14\sqrt{n}h_1^{-1}, 14\sqrt{n}})}) \leq C\epsilon^\gamma$$

for some $0 < \gamma = \gamma(n, \lambda, \Lambda, \delta_0) < 1$ and

$$C = C(n, \lambda, \Lambda, \delta_0, \|\beta\|_{C^2(T_{14\sqrt{n}h_1^{-1}})}).$$

Set $\tilde{w} = w/C\epsilon^\gamma$. We observe that $\tilde{w}$ satisfies the hypothesis of Lemma 3.1.7 and therefore

$$|A_t(\tilde{w}, \Omega) \cap (Q_1^{n-1} \times (0,1) + x_0)| \leq Ct^{-\mu}$$

for any $x_0 \in B^+_{9\sqrt{n}h_1^{-1}, 9\sqrt{n}} \cup T_{9\sqrt{n}h_1^{-1}}$ and $t > 1$. Since

$$A_{2M_0}(\tilde{u}, \Omega) \subset A_{M_0}(w, \Omega) \cup A_{M_0}(\dot{h}, \Omega)$$

and

$$A_{M_0}(\dot{h}, \Omega) \cap (Q_1^{n-1} \times (0,1) + x_0) = \varnothing,$$

we have

$$\begin{aligned} |A_{2M_0}(\tilde{u}, \Omega) \cap (Q_1^{n-1} \times (0,1) + x_0)| &\leq |A_{M_0}(w, \Omega) \cap (Q_1^{n-1} \times (0,1) + x_0)| \\ &= |A_{M_0/C\epsilon^\gamma}(\tilde{w}, \Omega) \cap (Q_1^{n-1} \times (0,1) + x_0)| \\ &\leq C(M_0/C\epsilon^\gamma)^{-\mu} \\ &\leq \epsilon_0 \end{aligned}$$

by choosing $M = 2CM_0$ and ϵ sufficiently small. We finish the proof. $\qquad \square$

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Lemma 3.1.14. *Let $0 < \epsilon_0 < 1$ and u be a C^2-viscosity solution of* (3.1.10).
*Assume that $||u||_{L^\infty(B^+_{14\sqrt{n}h_1^{-1},14\sqrt{n}})} \le 1$ and $||f||_{L^n(B^+_{14\sqrt{n}h_1^{-1},14\sqrt{n}})} \le \epsilon$. Extend f
to zero outside $B^+_{14\sqrt{n}h_1^{-1},14\sqrt{n}}$ and let*

$$\left(\fint_{B_r(x_0)\cap B^+_{14\sqrt{n}h_1^{-1},14\sqrt{n}}} \psi(x_0,x)^n \, dx \right)^{1/n} \le \epsilon$$

for any $x \in B^+_{14\sqrt{n}h_1^{-1},14\sqrt{n}}$, $r > 0$ and some

$$\epsilon = \epsilon(n,\epsilon_0,\lambda,\Lambda,\delta_0,||\beta||_{C^2(T_{14\sqrt{n}h_1^{-1}})}) > 0.$$

Then, for
$$A := A_{M^{k+1}}(u, B^+_{14\sqrt{n}h_1^{-1},14\sqrt{n}}) \cap (Q_1^{n-1} \times (0,1)),$$

$$B := (A_{M^k}(u, B^+_{14\sqrt{n}h_1^{-1},14\sqrt{n}})\cap(Q_1^{n-1} \times (0,1)))$$
$$\cup \{x \in Q_1^{n-1} \times (0,1) : M(f^n) \ge (c_0M^k)^n\}$$

*where $k \in \mathbb{N}_0$, $M > 1$ only depends on $n,\lambda,\Lambda,\delta_0,||\beta||_{C^2(T_{14\sqrt{n}h_1^{-1}})}$ and c_0 also
depends on ϵ_0, we have*
$$|A| \le \epsilon_0|B|.$$

Proof. By definition of A and B, we can check that $A \subset B \subset Q_1^{n-1} \times (0,1)$.
And since $B \subsetneq Q_1^{n-1} \times (0,1)$ by Lemma 3.1.6, we can use Lemma 3.1.13 and
observe that $|A| \le \epsilon_0$. Thus, it is sufficient to obtain that for any dyadic cube
Q and its predecessor $\tilde{Q}$,

$$|A \cap Q| > \epsilon_0|Q| \quad \Rightarrow \quad \tilde{Q} \subset B$$

by Lemma 3.1.3.

Let $Q = (Q_{1/2^i}^{n-1} \times (0,1/2^i)) + x_0$ and $\tilde{Q} = (Q_{1/2^{i-1}}^{n-1} \times (0,1/2^{i-1})) + \tilde{x}_0$.
Assume that $|A \cap Q| > \epsilon_0|Q|$ and $\tilde{Q} \not\subseteq B$. Then one can find a point $x_1 \in
\tilde{Q} \cap G_{M^k}(u, B^+_{14\sqrt{n}h_1^{-1},14\sqrt{n}})$ with $M(f^n)(x_1) < (c_0M^k)^n$.

First, assume that $x_{0,n} \le 8\sqrt{n}/2^i$. We consider a proper transforma-
tion $Ty = (x_0', 0) + 2^{-i}y$ and define $\tilde{u}(y) = 2^{2i}M^{-k}u(Ty)$, $\tilde{\beta}(y) = \beta(Ty)$,
$\tilde{F}(X,y) = M^{-k}F(M^kX,Ty)$ and $\tilde{f}(y) = M^{-k}f(Ty)$. Then we can see that

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$\tilde{u}$ is a viscosity solution of

$$\begin{cases} \tilde{F}(D^2\tilde{u}, y) = \tilde{f} & \text{in } B^+_{14\sqrt{n}h_1^{-1}, 14\sqrt{n}}, \\ \tilde{\beta} \cdot D\tilde{u} = 0 & \text{on } T_{14\sqrt{n}h_1^{-1}}, \end{cases} \qquad (3.1.14)$$

since $B^+_{14\sqrt{n}h_1^{-1}, 14\sqrt{n}} + (x_0', 0) \subset B^+_{14\sqrt{n}h_1^{-1}, 14\sqrt{n}}$.

Note that $\tilde{\beta} \in C^2(T_{14\sqrt{n}h_1^{-1}})$ and $\tilde{F}$ has the same elliptic constant of F and $||\psi_{\tilde{F}}||_{L^n(B^+_{13\sqrt{n}h_1^{-1}, 13\sqrt{n}})} \leq C\epsilon$ for some universal C, as $\psi_{\tilde{F}}(y, 0) = \psi_F(Ty, (x_0', 0))$.

Now we can deduce that

$$||\tilde{f}||_{L^n(B^+_{14\sqrt{n}h_1^{-1}, 14\sqrt{n}})} = \left(\int_{B^+_{14\sqrt{n}h_1^{-1}, 14\sqrt{n}}} |\tilde{f}(y)|^n \, dy \right)^{1/n}$$
$$\leq C(n, \delta_0)c_0$$
$$\leq \epsilon,$$

by taking c_0 sufficiently small, where we have used Proposition 3.1.4 in the second inequality.

Again, since $\tilde{Q} \cap G_{M^k}(u, B^+_{14\sqrt{n}h_1^{-1}, 14\sqrt{n}}) \neq \varnothing$, we observe that

$$T^{-1}\tilde{Q} \cap G_1(\tilde{u}, T^{-1}B^+_{14\sqrt{n}h_1^{-1}, 14\sqrt{n}}) \neq \varnothing.$$

We also see that $|T^{-1}\tilde{x}_0| < 9\sqrt{n}$ from $|x_0 - \tilde{x}_0| < \sqrt{n}/2^i$. Therefore, by Lemma 3.1.13,

$$|T^{-1}Q \cap G_M(\tilde{u}, T^{-1}B^+_{14\sqrt{n}h_1^{-1}, 14\sqrt{n}})| \geq 1 - \epsilon_0,$$

which implies

$$|T^{-1}Q \cap G_{M^{k+1}}(u, B^+_{14\sqrt{n}h_1^{-1}, 14\sqrt{n}})| \geq (1 - \epsilon_0)|Q|,$$

but this is a contradiction.

Next, we consider the case $x_{0,n} > 8\sqrt{n}/2^i$. In this case, we can check that $B_{8\sqrt{n}/2^i}(x_0 + e_n/2^{i+1}) \subset B^+_{8\sqrt{n}}$. Define $T : B_{8\sqrt{n}} \to B_{8\sqrt{n}/2^i}(x_0 + e_n/2^{i+1})$ as

$$Ty = x_0 + \frac{e_n}{2^{i+1}} + \frac{y}{2^{i+1}}.$$

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Now we define $\tilde{u}(y) = 2^{2(i+1)} M^{-k} u(Ty)$, $\tilde{F}(X,y) = M^{-k} F(M^k X, Ty)$ and $\tilde{f}(y) = M^{-k} u(Ty)$. Then observe that $\tilde{F}(D^2 \tilde{u}, y) = \tilde{f}(y)$ in $B_{8\sqrt{n}}$. By applying [13, Lemma 7.11] to $\tilde{u}$, we can conclude the proof. $\qquad\square$

Proof of Theorem 3.1.8. Fix a point $x_0 \in B_{1/2} \cap \{x \geq 0\}$. When $x_0 \in T_{1/2}$, consider a fixed number $r \in \left(0, \frac{1-|x_0|}{14\sqrt{n}} h_1\right)$ (h_1 is the constant as in Lemma 3.1.13) and define

$$K = \frac{\epsilon r}{\epsilon r^{-1} ||u||_{L^\infty(B^+_{14r\sqrt{n}}(x_0))} + ||f||_{L^n(B^+_{14r\sqrt{n}}(x_0))}}$$

where $\epsilon = \epsilon(n, \epsilon_0, \lambda, \Lambda, p, \delta_0, ||\beta||_{C^2(T_1)})$ is the same as in Lemma 3.1.13 and $0 < \epsilon_0 < 1$ is to be determined. We also define $\tilde{u}(y) = Kr^{-2} u(ry + x_0)$, $\tilde{f}(y) = Kf(ry + x_0)$, $\tilde{\beta}(y) = \beta(ry + x_0)$ and $\tilde{F}(X,y) = KF(K^{-1}X, ry + x_0)$. Then, $\tilde{u}$ is a viscosity solution of (3.1.14). Observe that F and $\tilde{F}$ have the same elliptic constants, $||\tilde{u}||_{L^\infty(B^+_{14\sqrt{n}h_1^{-1}, 14\sqrt{n}})} \leq 1$, $||\psi_{\tilde{F}}||_{L^n(B^+_{14\sqrt{n}h_1^{-1}, 14\sqrt{n}})} \leq \epsilon$, $\tilde{\beta} \in C^2(T_{14\sqrt{n}h_1^{-1}})$ and $||\tilde{f}||_{L^n(B^+_{14\sqrt{n}h_1^{-1}, 14\sqrt{n}})} \leq \epsilon$. Therefore, we can apply Lemma 3.1.14 to $\tilde{u}$. Let M and c_0 be the same constants and $\epsilon_0 = (2M^p)^{-1}$. Then we obtain by using a similar argument in the proof of [78, Theorem 2.2],

$$||D^2 \tilde{u}||_{L^p(B^+_{1/2})} \leq C,$$

that is,

$$||D^2 u||_{L^p(B^+_{r/2}(x_0))} \leq C(||u||_{L^\infty(B^+_1)} + ||f||_{L^p(B^+_1)}),$$

where $C = C(n, \lambda, \Lambda, p, \delta_0, ||\beta||_{C^2(T_1)}) > 0$.

On the other hand, if $x_0 \in B^+_{1/2}$, we can apply the results of interior estimates, like as in [13, Theorem 7.1]. Combining the interior and boundary estimates, we get the desired results by using a standard covering argument. $\qquad\square$

3.1.4 Boundary $W^{1,p}$-estimates

In this subsection, we extend the regularity result in Section 3.1.3 for equations involving ingredients $F(X, q, r, x)$.

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We now consider the following problem

$$
\begin{cases}
F(D^2u, Du, u, x) = f & \text{in } B_1^+, \\
\beta \cdot Du = 0 & \text{on } T_1.
\end{cases}
\tag{3.1.15}
$$

By means of the structure condition (3.1.2), we can see that u is also a viscosity solution of

$$
\begin{cases}
F(D^2u, 0, 0, x) = \tilde{f} & \text{in } B_1^+, \\
\beta \cdot Du = 0 & \text{on } T_1
\end{cases}
\tag{3.1.16}
$$

for some function $\tilde{f}$ with

$$
|\tilde{f}| \leq |f| + b|Du| + c|u|.
\tag{3.1.17}
$$

We already know that $W^{2,p}$-norm of u is estimated by L^∞-norm of u and L^n-norm of $\tilde{f}$ by Theorem 3.1.8. Therefore, it is essential in obtaining $W^{1,p}$-estimate for u to show our desired result.

We first show the following approximation lemma.

Lemma 3.1.15. *Let $n < p < \infty$ and $0 \leq \nu \leq 1$. Assume that F is continuous in x and satisfies (3.1.2) with $F(0,0,0,x) = 0$ and $\beta \in C^2(T_2)$ with $\beta \cdot \mathbf{n} \geq \delta_0$ for some $\delta_0 > 0$. Then, for every $\rho > 0$, $\varphi \in C(\partial B_1(0', \nu))$ with $\|\varphi\|_{C(\partial B_1(0', \nu))} \leq C_1$ for some $C_1 > 0$ and $g \in C^{0,\alpha}(T_2)$ with $0 < \alpha < 1$ and $\|g\|_{C^\alpha(T_2)} \leq C_2$ for some $C_2 > 0$, there exists a positive number $\delta = \delta(\rho, n, \lambda, \Lambda, \delta_0, p, \alpha, C_1, C_2) < 1$ such that*

$$
\|\psi(0, \cdot)\|_{L^p(B_2^+)}, \|f\|_{L^p(B_2^+)}, b, c \leq \delta
$$

implies the following: if u and v satisfy

$$
\begin{cases}
F(D^2u, Du, u, x) = f & \text{in } B_1(0', \nu) \cap \mathbb{R}_+^n, \\
u = \varphi & \text{on } \overline{\partial B_1(0', \nu) \cap \mathbb{R}_+^n}, \\
\beta \cdot Du = g & \text{on } B_1(0', \nu) \cap T_1
\end{cases}
$$

and

$$
\begin{cases}
F(D^2v, 0, 0, 0) = 0 & \text{in } B_{\frac{7}{8}}(0', \nu) \cap \mathbb{R}_+^n, \\
v = u & \text{on } \overline{\partial B_{\frac{7}{8}}(0', \nu) \cap \mathbb{R}_+^n}, \\
\beta \cdot Du = g & \text{on } B_{\frac{7}{8}}(0', \nu) \cap T_1,
\end{cases}
$$

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then $||u - v||_{L^\infty(B_{\frac{7}{8}}(0',\nu) \cap \mathbb{R}^n_+)} \leq \rho$.

We will use following facts to prove Lemma 3.1.15. These results can be
found in [78, Proposition 1.5] and [42, Theorem 3.1], respectively. For the
proof of Proposition 3.1.16, see [14, Theorem 3.8].

Proposition 3.1.16. *For $k \in \mathbb{N}$, let $\Omega_k \subset \Omega_{k+1}$ be an increasing sequence
of domains and $\Omega := \cup_{k \geq 1}\Omega_k$. Let $p > n$ and F, F_k be measurable in x and
satisfy structure condition (3.1.2). Assume that $f \in L^p(\Omega)$, $f_k \in L^p(\Omega_k)$ and
that $u_k \in C(\Omega_k)$ are $W^{2,p}$-viscosity subsolutions (supersolutions, respectively)
of $F_k(D^2u_k, Du_k, u_k, x) = f_k$ in Ω_k. Suppose that $u_k \to u$ locally uniformly
in Ω and for $B_r(x_0) \subset \Omega$ and $\varphi \in W^{2,p}(B_r(x_0))$*

$$||(s - s_k)^+||_{L^p(B_r(x_0))} \to 0 \qquad \left(||(s - s_k)^-||_{L^p(B_r(x_0))} \to 0\right) \qquad (3.1.18)$$

where $s(x) = F(D^2\varphi, D\varphi, u, x) - f(x)$ and $s_k(x) = F(D^2\varphi_k, D\varphi_k, u_k, x) - f_k(x)$. Then u is an $W^{2,p}$-viscosity subsolution (supersolution) of

$$F(D^2u, Du, u, x) = f(x) \quad in \ \Omega.$$

*Moreover, if F and f are continuous, then u is an C^2-viscosity subsolution
(supersolution) provided that (3.1.18) holds for $\varphi \in C^2(B_r(x_0))$.*

Lemma 3.1.17. *Suppose that $\Gamma \in C^2$ and $\beta \in C^2(\overline{\Gamma})$. Let u and v satisfy*

$$\begin{cases} F(D^2u) \geq f_1 & in \ \Omega, \\ \beta \cdot Du + \gamma u \geq g_1 & on \ \Gamma \end{cases}$$

and

$$\begin{cases} F(D^2v) \leq f_2 & in \ \Omega, \\ \beta \cdot Dv + \gamma v \leq g_2 & on \ \Gamma. \end{cases}$$

Then

$$\begin{cases} u - v \in \underline{S}(\lambda/n, \Lambda, f_1 - f_2) & in \ \Omega \\ \beta \cdot D(u - v) + \gamma(u - v) \geq g_1 - g_2 & on \ \Gamma. \end{cases}$$

Proof of Lemma 3.1.15. We will show this lemma by contradiction. Suppose
not. Then there exists a number $\rho_0 > 0$ so that for any $F_k, f_k, b_k, c_k, \psi_{F_k}$ with

$$||\psi_{F_k}(0, \cdot)||_{L^p(B_2^+)}, ||f_k||_{L^p(B_2^+)}, b_k, c_k \leq \delta_k \to 0$$

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as $k \to \infty$, if u_k and v_k satisfy

$$
\begin{cases}
F_k(D^2 u_k, Du_k, u_k, x) = f_k & \text{in } \overline{B_1(0', \nu_k)} \cap \mathbb{R}^n_+, \\
u_k = \varphi_k & \text{on } \overline{\partial B_1(0', \nu_k)} \cap \mathbb{R}^n_+, \\
\beta \cdot Du_k = g_k & \text{on } B_1(0', \nu_k) \cap T_1
\end{cases}
\tag{3.1.19}
$$

and

$$
\begin{cases}
F_k(D^2 v_k, 0, 0, 0) = 0 & \text{in } B_{\frac{7}{8}}(0', \nu_k) \cap \mathbb{R}^n_+, \\
v_k = u_k & \text{on } \overline{\partial B_{\frac{7}{8}}(0', \nu_k)} \cap \mathbb{R}^n_+, \\
\beta \cdot Dv_k = g_k & \text{on } B_{\frac{7}{8}}(0', \nu_k) \cap T_1,
\end{cases}
\tag{3.1.20}
$$

then $||u - v||_{L^\infty(B_{\frac{7}{8}}(0', \nu) \cap \mathbb{R}^n_+)} > \rho_0$. Here, $\varphi_k \in C(\partial B_1(0', \nu_k))$ and $g_k \in C^{0,\alpha}(T_2)$ satisfy $||\varphi_k||_{L^\infty(\partial B_1(0', \nu_k))} \leq C_1$ and $||g_k||_{C^{0,\alpha}(T_2)} \leq C_2$, respectively.

We assumed that $F_k(X, q, r, x)$ are Lipschitz in X, q, r from the condition (3.1.2), hence there exists a subsequence F_{k_i} and a function F_∞ such that $F_{k_i}(\cdot, \cdot, \cdot, 0)$ converges to $F_\infty(\cdot)$ uniformly on compact subsets of $S(n) \times \mathbb{R}^n \times \mathbb{R}$ by Arzelà-Ascoli theorem. Now we use Lemma 3.1.9 and get

$$
\begin{aligned}
&||u_k||_{L^\infty(B_1(0', \nu_k) \cap \mathbb{R}^n_+)} \\
&\leq ||\varphi_k||_{L^\infty(\partial B_1(0', \nu_k) \cap \mathbb{R}^n_+)} \\
&\qquad + C(n, \lambda, \Lambda, \delta_0)(||f_k||_{L^n(B_1(0', \nu_k) \cap \mathbb{R}^n_+)} + ||g_k||_{L^\infty(T_1)}) \\
&\leq C(C_1, C_2, n, \lambda, \Lambda, \delta_0)
\end{aligned}
$$

for sufficiently large k. We can also see that u_k satisfy boundary Hölder regularity by Lemma 3.1.10, that is, for any $0 < \delta < 1$,

$$
||u_k||_{C^{0,\alpha_1}(\overline{B_{1-\eta}(0', \nu_k) \cap \mathbb{R}^n_+})} \leq C(C_1, C_2, n, \lambda, \Lambda, \delta_0) \eta^{-\alpha_1}
\tag{3.1.21}
$$

for some $\alpha_1 = \alpha_1(n, \lambda, \Lambda, \delta_0)$ and sufficiently large k.

Suppose that there exists a number ν_∞ and a subsequence $\{\nu_{k_i}\}$ such that $\nu_{k_i} \to \nu_\infty$ as $i \to \infty$. We can assume that this subsequence is monotone. If ν_{k_i} is decreasing, we can check that

$$
B_1(0', \nu_\infty) \cap \mathbb{R}^n_+ \subset B_1(0', \nu_{k_i}) \cap \mathbb{R}^n_+
$$

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for any i. Thus, we can observe that

$$||u_{k_i}||_{C^{0,\alpha_1}(\overline{B_{15/16}(0',\nu_\infty)\cap\mathbb{R}_+^n})} \le C(C_1, C_2, n, \lambda, \Lambda, \delta_0) \tag{3.1.22}$$

by using the result (3.1.21). Meanwhile, if ν_{k_i} is increasing, there exists a number i_0 such that

$$B_{31/32}(0', \nu_{k_i}) \cap \mathbb{R}_+^n \supset B_{15/16}(0', \nu_\infty) \cap \mathbb{R}_+^n \quad \text{for} \quad i \ge i_0.$$

Then we can also deduce (3.1.22) for some proper subsequence u_{k_i}.

Hence, we can see that there is a subsequence u_{k_i} and a function u_∞ such that u_{k_i} uniformly converge to u_∞ in $\overline{B_{15/16}(0', \nu_\infty) \cap \mathbb{R}_+^n}$.

On the other hand, we see that for $\phi \in C^2(B_2^+)$,

$$|F_{k_i}(D^2\phi, D\phi, u_{k_i}, x) - f_{k_i}(x) - F_\infty(D^2\phi, 0, 0, 0)|$$
$$\le c_{k_i} C(C_1, C_2, n, \lambda, \Lambda, \delta_0) + b_{k_i}|D\phi| + \psi_{F_{k_i}}(0, x)|D^2\phi|$$
$$+ |f_{k_i}| + |(F_{k_i} - F_\infty)(D^2\phi, 0, 0, 0)|$$

and thus,

$$\lim_{i\to\infty} ||F_{k_i}(D^2\phi, D\phi, u_{k_i}, x) - f_{k_i}(x) - F_\infty(D^2\phi, 0, 0, 0)||_{L^p(B_r(x_0))} = 0.$$

for any ball $B_r(x_0) \subset B_{15/16}(0, \nu_\infty) \cap \mathbb{R}_+^n$. And we can observe that there exists a function $g_\infty \in C^{0,\alpha}(T_1)$ by Arzelà-Ascoli theorem, since g_k are uniformly bounded and equicontinuous on T_1. Therefore, we can deduce that u_∞ satisfies

$$\begin{cases} F_\infty(D^2 u_\infty, 0, 0, 0) = 0 & \text{in } B_{15/16}(0', \nu_\infty) \cap \mathbb{R}_+^n, \\ \beta \cdot Du_\infty = g_\infty & \text{on } B_{15/16}(0', \nu_\infty) \cap T_1, \end{cases} \tag{3.1.23}$$

in the viscosity sense by Proposition 3.1.16 and [42, Proposition 2.1].

Now consider $w_{k_i} := u_\infty - v_{k_i}$ for each i. Then w_{k_i} satisfies

$$\begin{cases} w_{k_i} \in S(\lambda/n, \Lambda, 0) & \text{in } B_{\frac{7}{8}}(0', \nu_\infty) \cap \mathbb{R}_+^n, \\ w_{k_i} = u_\infty - u_{k_i} & \text{on } \overline{\partial B_{\frac{7}{8}}(0', \nu_\infty) \cap \mathbb{R}_+^n}, \\ \beta \cdot Dw_{k_i} = g_\infty - g_{k_i} & \text{on } B_{\frac{7}{8}}(0', \nu_\infty) \cap T_1. \end{cases} \tag{3.1.24}$$

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by Lemma 3.1.17. Using Lemma 3.1.9, we observe that

$$||w_{k_i}||_{L^\infty(B_{\frac{7}{8}}(0',\nu_\infty)\cap\mathbb{R}^n_+)}$$
$$\leq ||u_\infty - u_{k_i}||_{L^\infty(\partial B_{\frac{7}{8}}(0',\nu_\infty))} + C(n,\lambda,\Lambda,\delta_0)||g_\infty - g_{k_i}||_{L^\infty(B_{\frac{7}{8}}(0',\nu_\infty)\cap T_1)}$$
(3.1.25)

and the right-hand side of (3.1.25) tends to zero as $i \to \infty$, that is, w_{k_i} converge uniformly to zero. It implies that v_{k_i} converge uniformly to u_∞ in $\overline{B_{\frac{7}{8}}(0',\nu_\infty)} \cap \mathbb{R}^n_+$. But this contradicts our assumptions, and therefore we can complete the proof. □

Now we can establish $W^{1,p}$-estimates for viscosity solutions of the problem (3.1.15).

Theorem 3.1.18. *Let $n < p < \infty$. Assume that F is convex in X and continuous in x satisfies the structure condition (3.1.2) with $F(0,0,0,x) = 0$ and u be a C^2-viscosity solution of (3.1.15) where $f \in L^p(B_1^+) \cap C(B_1^+)$ and $\beta \in C^2(T_1)$ with $\beta \cdot \mathbf{n} \geq \delta_0 > 0$. Then, there exists a constant $\epsilon_0 = \epsilon_0(n,\lambda,\Lambda,p,\delta_0,\alpha)$ such that*

$$\left(\fint_{B_r(x_0)\cap B_1^+} \psi(x_0,x)^p \, dx \right)^{1/p} \leq \epsilon_0$$

for any $x_0 \in B_1^+$ and $r \leq r_0$ for some $r_0 > 0$ implies $u \in C^{1,\alpha}(\overline{B}_{1/2}^+)$ with $\alpha = \alpha(n,\lambda,\Lambda,p) \in (0,1)$ and

$$||u||_{C^{1,\alpha}(\overline{B}_{1/2}^+)} \leq C(||u||_{L^\infty(B_1^+)} + ||f||_{L^p(B_1^+)})$$

for some $C = C(n,\lambda,\Lambda,b,c,p,\delta_0,||\beta||_{C^2(T_1)},r_0)$.

Proof. Let $n \leq p' < p$. Fix $y \in T_{1/2}$ and set $d = \min\{1/2, r_0\}$. First we rescale the equation. Choose a constant σ such that

$$\sigma \leq \frac{d}{2}, \quad \sigma b \leq \frac{\delta}{64MC(n)}, \quad \sigma^2 c \leq \frac{\delta}{64(M+1)C(n)}$$

where δ is from Lemma 3.1.15 and M is to be determined and $C(n)$ is a universal constant.

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Now we define

$$K = K(y)$$

$$:= ||u||_{L^\infty(B_d^+(y))} + \frac{1}{\epsilon_0} \sup_{r \leq d} \left[r^{1-\alpha} \left(r^{-n} \int_{B_r^+(y)} |f(x)|^{p'} \, dx \right)^{\frac{1}{p'}} \right]$$

where $0 < \alpha < 1$ is a constant to be chosen later. Then we see that

$$K(y) \leq ||u||_{L^\infty(B_1^+)} + C(n, \epsilon_0) \left[M(f^p)(y) \right]^{\frac{1}{p}} < \infty$$

for any y. Set $\tilde{u}(x) = u(\sigma x)/K$, $\tilde{f}(x) = \sigma^2 f(\sigma x)/K$, $\tilde{\beta}(x) = \beta(\sigma x)$ and

$$\tilde{F}(X, q, r, x) = \frac{\sigma^2}{K} F(K\sigma^{-2}X, K\sigma^{-1}q, Kr, \sigma x).$$

Then we observe that $\tilde{u}$ satisfies

$$\begin{cases} \tilde{F}(D^2\tilde{u}, D\tilde{u}, \tilde{u}, x) = \tilde{f} & \text{in } B_2^+, \\ \tilde{\beta} \cdot D\tilde{u} = 0 & \text{on } T_2. \end{cases} \qquad (3.1.26)$$

We check that $\tilde{F}$ satisfies (3.1.2) with $b_{\tilde{F}} = \sigma b$, $c_{\tilde{F}} = \sigma^2 c$ and

$$r^{1-\alpha} \left(r^{-n} \int_{B_r^+} |\tilde{f}(x)|^{p'} \, dx \right)^{\frac{1}{p'}} \leq \epsilon_0 \sigma^{1+\alpha}$$

for any $r \in (0, 2)$. We can also deduce $||\psi_{\tilde{F}}(0, \cdot)||_{L^{p'}(B_1^+)} \leq \delta$ by choosing ϵ_0 small enough since $\psi_{\tilde{F}}(0, x) = \psi_F((0', y_n), \sigma x)$.

Now we deduce boundary $C^{1,\alpha}$-estimates. We use induction to establish this regularity result. It is sufficient to show that there exist some constants $\mu, C_3 > 0$, $0 < \alpha < 1$ and a sequence of linear functions $l_k(x) = a_k + b_k \cdot x$ for $k \geq -1$ such that

(i) $||\tilde{u} - l_k||_{L^\infty(B_{\mu^k}^+)} \leq \mu^{k(1+\alpha)}$

(ii) $|a_{k-1} - a_k| + \mu^{k-1}|b_{k-1} - b_k| \leq 4C_3\mu^{(k-1)(1+\alpha)}$

(iii) $\beta(0) \cdot b_k = 0$.

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We set $l_{-1} = l_0 = 0$ and choose $\mu \leq 1/4$ with $6C_3||\beta||_{C^2(T_1)}\mu^2 \leq \mu^{1+\alpha}$ and

$$M = 4C_3 \sum_{i=0}^{\infty} \left(\frac{1}{4}\right)^{i\alpha} \geq 4C_3 \sum_{i=0}^{\infty} \mu^{i\alpha}. \qquad (3.1.27)$$

For $k = 0$, we can check that all of these conditions are satisfied. Now assume that (i), (ii) and (iii) hold for some $k > 0$. We show that these are also satisfied for $k + 1$.

Define

$$v_k(x) = \frac{(\tilde{u} - l_k)(\mu^k x)}{\mu^{k(1+\alpha)}}.$$

Then we observe that v_k is a viscosity solution

$$\begin{cases} F_k(D^2 v_k, D v_k, v_k, x) = f_k + g_k & \text{in } B_2^+, \\ \overline{\beta} \cdot D v_k = -(\overline{\beta} \cdot b_k)/\mu^{k\alpha} & \text{on } T_2, \end{cases} \qquad (3.1.28)$$

where

$$F_k(X, q, r, x) = \mu^{k(1-\alpha)}\tilde{F}(\mu^{k(\alpha-1)}X, \mu^{k\alpha}q, \mu^{k(\alpha+1)}r, \mu^k x),$$

$$g_k(x) = F_k(D^2 v_k, D v_k, v_k, x) \\ - F_k(D^2 v_k, D v_k + \mu^{-k\alpha}b_k, v_k + \mu^{-k(1+\alpha)}l_k(\mu^k x), x),$$

$$f_k(x) = \mu^{k(1-\alpha)}\tilde{f}(\mu^k x)$$

and

$$\overline{\beta}(x) = \tilde{\beta}(\mu^k x).$$

Observe that $\psi_{F_k}(0, x) = \psi_{\tilde{F}}(0, \mu^k x)$ and $\tilde{F}$ satisfies the structure condition (3.1.2) with $b_{F_k} = \mu^k b_{\tilde{F}}$ and $c_{F_k} = \mu^{2k} c_{\tilde{F}}$.

For any $x \in B_1^+$,

$$|g_k(x)| = |F_k(D^2 v_k, D v_k, v_k, x) \\ - F_k(D^2 v_k, D v_k + \mu^{-k\alpha}b_k, v_k + \mu^{-k(1+\alpha)}l_k(\mu^k x), x)| \\ \leq b_{F_k} \cdot \mu^{-k\alpha}|b_k| + c_{F_k} \cdot \mu^{-k(1+\alpha)}|l_k(\mu^k x)|.$$

We already know that $|a_{k-1} - a_k| + \mu^{k-1}|b_{k-1} - b_k| \leq 4C_3\mu^{(k-1)(1+\alpha)}$ by assumption. By using this and (3.1.27), we can check that $|a_k|, |b_k| \leq M$ and

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thus we obtain that $||l_k||_{L^\infty(B_1^+)} \leq 2M$. Therefore,

$$|g_k(x)| \leq b_{F_k} \cdot \mu^{-k\alpha}M + c_{F_k} \cdot \mu^{-k(\alpha+1)} \cdot 2M \leq \mu^{k(1-\alpha)}\frac{\delta}{16}.$$

Now we deduce that

$$||f_k||_{L^{p'}(B_1^+)} + ||g_k||_{L^{p'}(B_1^+))} \leq \frac{\delta}{2} + \frac{\delta}{16}\mu^{k(1-\alpha)} \leq \delta.$$

On the other hand, we see that $v_k \in S^*(\lambda, \Lambda, 1, |f_k| + |g_k| + \mu^{2k}c_{\tilde{F}})$ when k is large enough. Therefore by Lemma 3.1.10,

$$||v_k||_{C^{0,\alpha_0}(B_1^+)}$$
$$\leq ||v_k||_{L^\infty(\partial B_1^+)} + C(n, \lambda, \Lambda, \delta_0) \times$$
$$\qquad (||f_k||_{L^n(B_1^+)} + ||g_k||_{L^n(B_1^+)} + \mu^{2k}c_{\tilde{F}} + \mu^{-k\alpha}||\overline{\beta} \cdot b_k||_{L^\infty(B_1^+)})$$
$$\leq 1 + C(n, \lambda, \Lambda, \delta_0)(\delta + \mu^{(1-\alpha)k}|b_k|)$$
$$\leq C(n, \lambda, \Lambda, \delta_0, C_3)$$

for some $\alpha_0 = \alpha_0(n, \lambda, \Lambda, \delta_0)$. Note that we have used $||v_k||_{L^\infty(B_1^+)} \leq 1$, $\beta \in C^2(T_1)$, $\beta(0) \cdot b_k = 0$ and $|b_k| \leq 6C_3$ to obtain the last inequality.

Consider $h \in C(\overline{B}_{\frac{7}{8}}^+)$ such that

$$\begin{cases} F_k(D^2h, 0, 0, 0) = 0 & \text{in } B_{\frac{7}{8}}^+, \\ h = v_k & \text{on } \partial B_{\frac{7}{8}}^+ \backslash T_{\frac{7}{8}}, \\ \overline{\beta} \cdot Dh = -(\overline{\beta} \cdot b_k)/\mu^{k\alpha} & \text{on } T_{\frac{7}{8}}. \end{cases} \qquad (3.1.29)$$

By Lemma 3.1.11, we see that

$$||h||_{C^2(\overline{B}_{\frac{3}{4}}^+)} \leq C_*(1 + \mu^{-k\alpha}||\overline{\beta} \cdot b_k||_{C^2(T_{\frac{7}{8}})})$$

for some constant C_* which only depends on $n, \lambda, \Lambda, ||\beta||_{C^2(T_1)}$ and δ_0. Set $C_* = C_3$. Then we see that

$$||h||_{C^2(\overline{B}_{\frac{3}{4}}^+)} \leq C_3(1 + \mu^{-k\alpha}||\overline{\beta} \cdot b_k||_{C^2(T_{\frac{7}{8}})})$$
$$\leq C_3(1 + \mu^{-k\alpha}|b_k| \cdot \mu^k||\beta||_{C^2(T_1)})$$

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since $\beta(0) \cdot b_k = 0$. Then we have

$$||h||_{C^2(\overline{B}_{\frac{3}{4}}^+)} \leq C_3(1 + 6C_3\mu^{1-\alpha}||\beta||_{C^2(T_1)}) \leq 2C_3.$$

We can also observe that

$$||v_k - h||_{L^\infty(B_{\frac{3}{4}}^+)} \leq \rho$$

by Lemma 3.1.15 to v_k and h with $\rho = C_3\mu^2$.
Set $\bar{l}(x) = h(0) + Dh(0) \cdot x$. Then,

$$\begin{aligned}
||v_k - \bar{l}||_{L^\infty(B_{2\mu}^+)} &\leq ||v_k - h||_{L^\infty(B_{2\mu}^+)} + ||h - \bar{l}||_{L^\infty(B_{2\mu}^+)} \\
&\leq C_3\mu^2 + \frac{1}{2}C_3(2\mu)^2 \\
&\leq \mu^{1+\alpha}.
\end{aligned}$$

Note that the last inequality is deduced by $6C_3||\beta||_{C^2(T_1)}\mu^2 \leq \mu^{1+\alpha}$.
Since $|(v_k - \bar{l})(x)| \leq \mu^{1+\alpha}$ for any $x \in B_{2\mu}^+$, we see that

$$|\tilde{u}(x) - l_k(x) - \mu^{k(1+\alpha)}\bar{l}(\mu^{-k}(x))| \leq \mu^{(k+1)(1+\alpha)}$$

for any $x \in B_{2\mu^{k+1}}^+$. We denote l_{k+1} by

$$l_{k+1}(x) = l_k(x) + \mu^{k(1+\alpha)}\bar{l}(\mu^{-k}(x)).$$

We have shown that the condition (i) still holds. And in this case, we can
also observe that the second condition is satisfied because

$$\begin{aligned}
|a_k - a_{k+1}| + \mu^k|b_k - b_{k+1}| &= \mu^{k(1+\alpha)}(|h(0)| + |Dh(0)|) \\
&\leq \mu^{k(1+\alpha)}||h||_{C^2(\overline{B}_{\frac{3}{4}}^+)} \\
&\leq 4C_3\mu^{k(1+\alpha)}.
\end{aligned}$$

Finally, we also check that

$$\beta(0) \cdot b_{k+1} = \beta(0) \cdot (b_k + \mu^{k\alpha}Dh(0)) = \beta(0) \cdot \mu^{k\alpha}Dh(0) = 0,$$

since
$$\beta(0) \cdot Dh(0) = \overline{\beta}(0) \cdot Dh(0) = -(\overline{\beta}(0) \cdot b_k)/\mu^{k\alpha} = 0.$$

Hence, as k tends to ∞, there exists a linear function l such that

$$|l(0)|, |Dl(0)| \leq C_4 K(y) \qquad (3.1.30)$$

and

$$||u - l||_{L^\infty(B_r^+(y))} \leq C_4 r^{1+\alpha} K(y) \qquad (3.1.31)$$

for any $y \in T_{1/2}$, small number r and some universal constant $C_4 = 4C_3$.
Since

$$K(y) \leq ||u||_{L^\infty(B_1^+)} + \epsilon_0^{-1} \sup_{r \leq d} \left(r^{1-\alpha-\frac{n}{p}} ||f||_{L^p(B_1^+)} \right), \qquad (3.1.32)$$

we get $u|_{\overline{T}_{1/2}} \in C^{1,\alpha}(T_{1/2})$ with

$$||u||_{C^{1,\alpha}(\overline{T}_{1/2})} \leq C(||u||_{L^\infty(B_1^+)} + ||f||_{L^p(B_1^+)}) \qquad (3.1.33)$$

for some $C = C(n, \lambda, \Lambda, \delta_0, ||\beta||_{C^2(T_1)})$ by choosing $\alpha = 1 - n/p$.

By proper scaling, we have $u|_{\overline{T}_{2/3}} \in C^{1,\alpha}(\overline{T}_{2/3})$. From [59, Proposition 2.2], we can deduce that u satisfies the assumption of [78, Theorem 3.1]. Therefore, by combining (3.1.33) with and [78, Theorem 3.1], we can complete the proof. $\qquad \square$

By means of the above result, we get a boundary $W^{1,p}$-estimate for solutions of (3.1.15).

Corollary 3.1.19. *Let $n < p < \infty$ and u be a viscosity solution of (3.1.15). Then, under the assumption of Theorem 3.1.18, $u \in W^{1,p}(B_{1/4}^+)$ and*

$$||u||_{W^{1,p}(B_{1/4}^+)} \leq C(||u||_{L^\infty(B_1^+)} + ||f||_{L^p(B_1^+)})$$

for some $C = C(n, \lambda, \Lambda, b, c, p, \delta_0, ||\beta||_{C^{1,\alpha}(T_1)}, r_0)$.

3.1.5   Global estimates

We are now ready to prove the global regularity result, Theorem 3.1.2.

Proof of Theorem 3.1.2. We use a flattening argument for applying to Corollary 3.1.19. If $\partial\Omega \in C^3$, for any $x_0 \in \partial\Omega$ there exists a neighborhood $N(x_0)$ and a C^3-diffeomorphism

$$\Psi : U(x_0) \to B_1 \cap \mathbb{R}^n_+$$

such that $\Psi(x_0) = 0$. Then for $\tilde{u} = u \circ \Psi^{-1}$, it is known that $\tilde{u}$ is a $W^{2,p}$-viscosity solution of

$$\begin{cases} \tilde{F}(D^2\tilde{u}, D\tilde{u}, \tilde{u}, x) = \tilde{f} & \text{in } B_1^+, \\ \tilde{\beta} \cdot D\tilde{u} = 0 & \text{on } T_1. \end{cases}$$

where $\tilde{f} = f \circ \Psi^{-1}$, $\tilde{\beta} = (\beta \circ \Psi^{-1}) \cdot (D\Psi \circ \Psi^{-1})^t$ and

$$\begin{aligned}
\tilde{F}(D^2\tilde{\varphi}, D\tilde{\varphi}, \tilde{u}, x) &= F(D^2\varphi, D\varphi, u, x) \circ \Psi^{-1} \\
&= F(D\Psi^T \circ \Psi^{-1} D^2\tilde{\varphi} D\Psi \circ \Psi^{-1} + (D\tilde{\varphi}\partial_{i,j}\Psi \circ \Psi^{-1})_{1\leq i,j\leq n}, \\
&\qquad D\varphi D\Psi \circ \Psi^{-1}, \tilde{u}, \Psi^{-1}(x))
\end{aligned}$$

for $\tilde{\varphi} \in W^{2,p}(B_1^+)$ and $\varphi = \tilde{\varphi} \circ \Psi (\in W^{2,p}(U(x_0)))$. We observe that $\psi_{\tilde{F}}(x, x_0) \leq C(\Psi)\psi_F(\Psi^{-1}(x), \Psi^{-1}(x_0))$ and $\tilde{F}$ is uniformly elliptic with constants $\lambda C(\Psi)$, $\Lambda C(\Psi)$ where $C(\Psi)$ is a uniform constant depending only on Ψ. (see [78]) And we also check that $\tilde{\beta} \in C^2$ since $\Psi, \Psi^{-1} \in C^3$. Now by using Corollary 3.1.19 and covering argument, we get a boundary estimate.

Finally, we obtain the following global regularity result to combine interior estimate (see [14, 72]) and Corollary 3.1.19. $\qquad\square$

3.2 $W^{2,p}$-regularity for parabolic problems

3.2.1 Hypotheses and main results

We now study the following parabolic oblique boundary value problem

$$\begin{cases} F(D^2u, Du, u, x, t) - u_t = f & \text{in } \Omega_T, \\ \beta \cdot Du = 0 & \text{on } S_T, \\ u(\cdot, 0) = 0 & \text{in } \Omega, \end{cases} \tag{3.2.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with $n \geq 2$ and $T > 0$.

We always assume the following conditions: $F(X, q, r, x, t)$ is convex in

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X, continuous in X, q, r, x and t, and satisfies

$$
\begin{aligned}
\mathcal{M}^-(\lambda, \Lambda, X_1 - X_2) &- b|q_1 - q_2| - c|r_1 - r_2| \\
&\leq F(X_1, q_1, r_1, x, t) - F(X_2, q_2, r_2, x, t) \quad\quad (3.2.2) \\
&\leq \mathcal{M}^+(\lambda, \Lambda, X_1 - X_2) + b|q_1 - q_2| + c|r - s|
\end{aligned}
$$

for fixed $0 < \lambda \leq \Lambda$ and $b, c > 0$, $M, N \in S(n)$, and any $q_1, q_2 \in \mathbb{R}^n$, $r_1, r_2 \in \mathbb{R}$ and $(x, t) \in \mathbb{R}^n \times \mathbb{R}$.

Similarly to the elliptic case, we consider an oscillation function $\boldsymbol{\psi}_F$.

Definition 3.2.1. *Let $F : S(n) \times \mathbb{R}^n \times \mathbb{R} \times \Omega_T \to \mathbb{R}$ and $(x_0, t_0) \in \Omega_T$. For $(x, t) \in \Omega_T$, We define*

$$
\boldsymbol{\psi}_F((x, t), (x_0, t_0)) := \sup_{X \in S(n) \backslash \{0\}} \frac{|F(X, 0, 0, x, t) - F(X, 0, 0, x_0, t_0)|}{||X||},
$$

Theorem 3.2.2. *Let Ω be a bounded C^3-domain with $T > 0$ and $\mathbf{n}$ be the inward unit normal vector to $\partial\Omega$. Assume that u is a viscosity solution of (3.2.1), where $F(X, q, r, x, t)$ is convex in X, continuous in x and t, and satisfies the structure condition (3.2.2) with $F(0, 0, 0, x, t) = 0$, $\beta \in C^2(S_T)$ with $\beta \cdot \mathbf{n} \geq \delta_0$ for some $\delta_0 > 0$ and $f \in L^p(\Omega_T) \cap C(\Omega_T)$ for $n + 2 < p < \infty$. Then there exists ϵ_0 depending on $n, p, \lambda, \Lambda, \delta_0$ and $||\beta||_{C^2(S_T)}$ such that if*

$$
\left(\fint_{Q_r(x_0, t_0) \cap \Omega_T} \boldsymbol{\psi}_F((x_0, t_0), (x, t))^p \, dx dt \right)^{1/p} \leq \epsilon_0
$$

for any $(x_0, t_0) \in \Omega_T$ and $0 < r < r_0$, then $u \in W^{2,p}(\Omega_T)$ with the global $W^{2,p}$-estimate

$$
||u||_{W^{2,p}(\Omega_T)} \leq C(||u||_{L^\infty(\Omega_T)} + ||f||_{L^p(\Omega_T)})
$$

for some C depending only on $n, p, \lambda, \Lambda, \delta_0, b, c, r_0, ||\beta||_{C^2(S_T)}, T$ and $\mathrm{diam}(\Omega)$.

3.2.2 Auxiliary results

We present some geometric and analytic tools which will be used in this section.

We first introduce a parabolic Calderón-Zygmund decomposition. The following definition and lemma can be found in [31].

Definition 3.2.3. *Given $m \in \mathbb{N}$, and a dyadic cube K of Q, the set $\overline{K}^m$
is obtained by stacking m copies of its predecessor $\overline{K}$. More precisely, if the
predecessor $\overline{K}$ has the form $L \times (a, b)$, then $\overline{K}^m = L \times (b, b + m(b - a))$.*

Lemma 3.2.4 (Calderón-Zygmund decomposition). *[31] Let $m \in \mathbb{N}$. Con-
sider two subsets A and B of a cube Q. Assume that $|A| \leq \delta|Q|$ for some
$\delta \in (0, 1)$. Assume also the following: for any dyadic cube $K \subset Q$,*

$$|K \cap A| > \delta|K| \Rightarrow \overline{K}^m \subset B.$$

Then $|A| \leq \frac{m+1}{m}\delta|B|$.

The following properties are parabolic counterparts of Proposition 3.1.4
and 3.1.5, which can be found in [75].

Proposition 3.2.5 (Strong (p, p)-estimate). *Let f be a locally integrable
function in $\mathbb{R}^n \times \mathbb{R}$ and Ω be a bounded domain in $\mathbb{R}^n \times \mathbb{R}$. The maximal
operator M is defined as follows:*

$$M(f)(x, t) = \sup_{\rho > 0} \fint_{Q_\rho(x,t)} |f(x, t)| dx dt.$$

Then

$$\|M(f)\|_{L^p(\Omega)} \leq C(n, p)\|f\|_{L^p(\Omega)},$$

whenever $f \in L^p(\Omega)$ for $1 < p < \infty$.

Proposition 3.2.6. *Let f be a nonnegative and measurable function in a
domain $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ and μ_f be its distribution function, that is,*

$$\mu_f(\lambda) = |\{(x, t) \in \Omega : f(x, t) > \lambda\}| \quad \text{for } \lambda > 0.$$

Let $\eta > 0$ and $M > 1$ be given constants. Then, for $0 < p < \infty$,

$$f \in L^p(\Omega) \iff \sum_{k \geq 1} M^{pk} \mu_f(\eta M^k) =: S < \infty$$

and

$$C^{-1}S \leq \|f\|_{L^p(\Omega)}^p \leq C(|\Omega| + S),$$

where C is a universal constant.

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Next we focus on parabolic Hessian estimates. In [75], we can find the following interior and boundary estimates with slight modifications.

Lemma 3.2.7. *[75] Let $u \in \overline{S}(f)$ in $\Omega \times (0,1]$ for some domain $\Omega \in \mathbb{R}^n$. If $||u||_{L^\infty(\Omega_1)} \leq 1$, then for any $\Omega' \subset\subset \Omega$ and $\tau \neq 1$,*

$$|\underline{A}_s(u, \Omega \times (0,1]) \cap (\Omega' \times (0, \tau])|$$
$$\leq C(n, \lambda, \Lambda, \Omega, \tau, d(\Omega', \partial\Omega)) \frac{(1 + ||f||_{L^{n+1}(\Omega \times (0,1])})^\mu}{s^\mu},$$

where μ is universal.

Lemma 3.2.8. *[75] Let $u \in \overline{S}(f)$ in $\Omega = K_4^{n-1} \times (0,2) \times (0,2]$. Suppose $||u||_{L^\infty(\Omega)} \leq 1$. Then*

$$|\underline{A}_s(u, \Omega) \cap (K_2^{n-1} \times (0,1) \times (0,1])| \leq C(n, \lambda, \Lambda) \frac{(1 + ||f||_{L^{n+1}(\Omega)})^\mu}{s^\mu}.$$

By using scaling argument, we can derive the next lemma as a direct consequence of the above results.

Lemma 3.2.9. *Let $\Omega = B_{12\sqrt{n}}^+ \times (0,13]$, $0 < r \leq 1$, and $(x_0, t_0) \in T_{12\sqrt{n}} \times (0,13]$ such that $r\Omega(x_0, t_0) = B_{12r\sqrt{n}}^+ \times (t_0, t_0 + 13r^2] \subset \Omega$. Assume that $u \in \overline{S}(f)$ in $r\Omega(x_0, t_0)$, $u \in C(\Omega)$ and $||u||_{L^\infty(\Omega)} \leq 1$.*
Then there exist universal constants $M > 1$ and $0 < \sigma < 1$ such that if

$$\left(\fint_{r\Omega(x_0,t_0)} |f(x,t)|^{n+1} dx dt \right)^{\frac{1}{n+1}} \leq 1,$$

then we have

$$\frac{|\underline{G}_M(u, \Omega) \cap (K_r^{n-1} \times (0,r) \times (0,r^2) + (x_1, t_1))|}{|K_r^{n-1} \times (0,r) \times (0,r^2)|} \geq 1 - \sigma, \qquad (3.2.3)$$

whenever $(x_1, t_1) \in (B_{9\sqrt{n}}(x_0) \cap \{x_n \geq 0\}) \times [t_0, t_0 + 10r^2]$.

Proof. Fix $0 < \sigma < 1$. If $t_0 \leq 15 - 5\sigma r^2$, we can choose a large constant M satisfying (3.2.3) by Lemma 3.2.7 and 3.2.8.
Consider the case $t_0 > 15 - 5\sigma r^2$. Observe that

$$|\underline{A}_t(u, \Omega_T) \cap ((K_r^{n-1} \times (0,r) \times (0,r^2) + (x_0, t_0)|$$

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$$\leq \left| \underline{A}_t(u, \Omega_T) \cap \left((K_1^{n-1} \times (0,1) \times \left(t_0 - 1, T - \frac{\sigma}{2} \right] + (x_0, 0) \right) \right|$$

$$+ \left| (K_1^{n-1} \times (0,1) \times \left(T - \frac{\sigma}{2}, t_0 \right) + (x_0, 0)) \right|$$

$$\leq C(n, \lambda, \Lambda, \sigma) t^{-\mu} + \frac{\sigma}{2}.$$

Then we can obtain the desired result by choosing M sufficiently large such
that

$$C(n, \lambda, \Lambda, \sigma) M^{-\mu} \leq \frac{\sigma}{2},$$

and it gives the desired result. $\qquad \square$

The next lemma shows that if there is a point with opening 1, then the
density of 'good sector' is guaranteed large enough.

Lemma 3.2.10. *Under the same hypotheses as in Lemma 3.2.9, we further
assume that $u \in S^*(f)$ in $r\Omega(x_0, t_0)$, $u \in C(\Omega)$, and*

$$G_1(u, \Omega) \cap (K_{3r}^{n-1} \times (0, 3r) \times (r^2, 10r^2) + (\tilde{x}_1, \tilde{t}_1)) \neq \varnothing$$

for some $(\tilde{x}_1, \tilde{t}_1) \in (B_{9r\sqrt{n}}(x_0) \cap \{x_n \geq 0\}) \times [t_0, t_0 + 5r^2]$.
  Then there exist universal constants $M > 1$ and $0 < \sigma < 1$ such that if

$$\left(\fint_{r\Omega(x_0, t_0)} |f(x, t)|^{n+1} dx dt \right)^{\frac{1}{n+1}} \leq 1,$$

then we have

$$\frac{|G_M(u, \Omega) \cap (K_r^{n-1} \times (0, r) \times (0, r^2) + (x_1, t_1))|}{|K_r^{n-1} \times (0, r) \times (0, r^2)|} \geq 1 - \sigma$$

for any $(x_1, t_1) \in (B_{9r\sqrt{n}}(x_0) \cap \{x_n \geq 0\}) \cap [t_0, \tilde{t}_1]$.

Proof. Let $(x_2, t_2) \in G_1(u, \Omega) \cap (K_{3r}^{n-1} \times (0, 3r) \times (r^2, 10r^2) + (\tilde{x}_1, \tilde{t}_1))$. By the
definition of G_1, we have paraboloids with opening 1 touching u at (x_2, t_2)
from above and below. Then we can find a linear function (on x) L such that

$$|u(x, t) - L(x)| \leq \frac{1}{2}(|x - x_2|^2 - (t - t_2)).$$

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Define $v(x, t) = (u(x, t) - L(x))/C$ where $C = C(n)$ is a constant so that

$$||v||_{L^\infty(B^+_{12r\sqrt{n}}(x_0) \times (t_0, t_2))} \leq 1$$

and

$$|v(x, t)| \leq |x|^2 + t_2 - t \qquad \text{in } (B^+_{12\sqrt{n}} \backslash B^+_{12r\sqrt{n}}(x_0)) \times (0, t_2). \qquad (3.2.4)$$

Now we can see that $v \in S^*(f/C)$ in $B^+_{12r\sqrt{n}} \times [t_0, t_2)$. Then we have

$$\frac{|G_M(v, B^+_{12r\sqrt{n}} \times (t_0, t_1)) \cap (K^{n-1}_r \times (0, r) \times (0, r^2) + (x_1, t_1))|}{|K^{n-1}_r \times (0, r) \times (0, r^2)|} \geq 1 - \sigma.$$

by Lemma 3.2.9.

Combining the above estimate with (3.2.4), we observe that

$$\frac{|G_N(v, \Omega) \cap (K^{n-1}_r \times (0, r) \times (0, r^2) + (x_1, t_1))|}{|K^{n-1}_r \times (0, r) \times (0, r^2)|} \geq 1 - \sigma$$

for some $N \geq M$. We also deduce that

$$G_M(v, \Omega) = G_{MC(n)}(u, \Omega),$$

and this completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Using the Calderón-Zygmund decomposition, Lemma 3.2.4, we can prove
the following result.

Lemma 3.2.11. *Under the same hypotheses as in Lemma 3.2.9, we further
assume that $u \in \overline{S}(f)$ in $r\Omega(x_0, t_0)$, $u \in C(\Omega)$, $||u||_{L^\infty(\Omega)} \leq 1$ and*

$$\left(\fint_{r\Omega(x_0, t_0)} |f(x, t)|^{n+1} dx dt \right)^{\frac{1}{n+1}} \leq 1.$$

Extend f by zero outside $r\Omega(x_0, t_0)$ and define

$$A := A_{M^{k+1}}(u, \Omega) \cap (K^{n-1}_r \times (0, r) \times (t_1, t_1 + r^2)),$$

$$B := (A_{M^k}(u, \Omega) \cap (K^{n-1}_r \times (0, r) \times (t_1, t_1 + r^2))$$

$$\cup \{(x,t) \in K_r^{n-1} \times (0,r)times(t_0, t_0 + r^2) : M(|f|^{n+1})(x,t) \geq (c_0 M^k)^{n+1}\}$$

for any $k \in \mathbb{N}_0$ and $t_1 \in [t_0, t_0 + 5r^2]$.

Then $|A| \leq 2\sigma|B|$, where $c_0 = c_0(n)$, $0 < \sigma < 1$ and $M > 1$ are universal.

Proof. By the definition of A and B, we know that

$$A \subset B \subset K_r^{n-1} \times (0,r) \times (t_1, t_1 + r^2).$$

We also have $|A| \leq \sigma|K_r^{n-1} \times (0,r) \times (t_1, t_1 + r^2)|$ from Lemma 3.2.9. Now we prove that for any dyadic cube $K \subset Q$,

$$|K \cap A| > \sigma|K| \quad \text{implies} \quad \overline{K}^m \subset B$$

for some $m \in \mathbb{N}$.

Let

$$K = (K_{r/2^i}^{n-1} \times (0, r/2^i) \times (0, r^2/2^{2i})) + (x_2, t_2)$$

be a dyadic cube with its predecessor

$$\tilde{K} = (K_{r/2^{i-1}}^{n-1} \times (0, r/2^{i-1}) \times (0, r^2/2^{2(i-1)})) + (\tilde{x}_2, \tilde{t}_2)$$

for some $i \geq 1$. Suppose that K satisfies $|K \cap A| > \sigma|K|$ but $\overline{K}^m \not\subset B$ for any m. Then there is a point $(x_3, t_3) \in \overline{K}^1 \backslash B$, that is,

$$(x_3, t_3) \in \overline{K}^1 \cap G_{M^k}(u, \Omega) \quad \text{and} \quad M(|f|^{n+1})(x_3, t_3) < (c_0 M^k)^{n+1}.$$

Now we define a transformation T by

$$T(y,s) = (\tilde{x}_2 + 2^{-i}y, \tilde{t}_2 + 2^{-2i}s),$$

and set $\tilde{u}(y,s) = 2^{2i}M^{-k}u(T(y,s))$ and $\tilde{f}(y,s) = M^{-k}f(T(y,s))$. Since $\overline{K}^1 \subset K_r^{n-1} \times (0,r) \times (t_1, t_1 + r^2)$, we can observe that

$$(r/2^i)\Omega(\tilde{x}_2, \tilde{t}_2) \subset r\Omega(x_1, t_1)$$

and $\tilde{u} \in S^*(\tilde{f})$ in $r\Omega(0,0)$. We also see that $|\tilde{x}_2 - x_3| < 2^{-(i-1)}r$ and thus

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$B^{+}_{12r\sqrt{n}/2^i}(\tilde{x}_2) \subset K^{n}_{28r\sqrt{n}/2^i}(x_3)$. Now we have the following estimate

$$\left(\fint_{r\Omega(0,0)} |\tilde{f}(x,t)|^{n+1} dxdt \right)^{\frac{1}{n+1}} \leq c_0 C(n) \leq 1$$

by direct calculations. Note that we chose some sufficiently small c_0 in order
to obtain the last inequality.

On the other hand, since $(x_3,t_3) \in \overline{K}^1 \cap G_{M^k}(u,\Omega)$, we have

$$G_1(\tilde{u}, T^{-1}\Omega) \cap (K^{n-1}_{3r} \times (0,3r) \times (4r^2, 13r^2)) \neq \varnothing$$

and then the hypothesis of Lemma 3.2.10 is satisfied in Ω. Since $x_{2,n} \geq \tilde{x}_{2,n}$,
$t_{2,n} \geq \tilde{t}_{2,n}$ and $|x_2 - \tilde{x}_2| \leq r\sqrt{n}/2^i$, we have

$$(2^i(x_2 - \tilde{x}_2), 2^{2i}(t_2 - \tilde{t}_2)) \in (B_{9r\sqrt{n}} \cap \{x_n \geq 0\}) \times [0, 3r^2].$$

Thus, we get the following quantity

$$\frac{|G_M(\tilde{u}, T^{-1}\Omega) \cap (K^{n-1}_r \times (0,r) \times (0,r^2) + (2^i(x_2 - \tilde{x}_2), 2^{2i}(t_2 - \tilde{t}_2)))|}{|K^{n-1}_r \times (0,r) \times (0,r^2)|}$$

is less than $1 - \sigma$. From this estimate, we have

$$|G_{M^{k+1}}(u,\Omega) \cap K| \geq (1-\sigma)|K|,$$

and it contradicts our assumption. Therefore, we can conclude the proof. $\quad\square$

Finally, we get the estimate for the density of 'bad sector'.

Corollary 3.2.12. *Under the same hypotheses as in Lemma 3.2.9, we further
assume that $u \in S^*(f)$ in $r\Omega(x_0,t_0)$, $u \in C(\Omega)$, and $\|u\|_{L^\infty(\Omega)} \leq 1$. Then
there exist universal constants C and μ such that if*

$$\left(\fint_{r\Omega(x_0,t_0)} |f(x,t)|^{n+1} dxdt \right)^{\frac{1}{n+1}} \leq 1,$$

then we have

$$\frac{|A_s(u,\Omega) \cap (K^{n-1}_r \times (0,r) \times (0,r^2) + (x_1,t_1))|}{|K^{n-1}_r \times (0,r) \times (0,r^2)|} \leq Cs^{-\mu}$$

for any $(x_1, t_1) \in (B_{9r\sqrt{n}}(x_0) \cap \{x_n \geq 0\}) \times [t_0, t_0 + 5r^2]$.

Proof. Without loss of generality, we can assume that $(x_0, t_0) = (0, 0)$. Let

$$\alpha_k = \frac{|A_{M^k}(u, \Omega) \cap (K_r^{n-1} \times (0, r) \times (t_1, t_1 + r^2))|}{|K_r^{n-1} \times (0, r) \times (t_1, t_1 + r^2)|},$$

$$\beta_k = \frac{|\{(x, t) \in K_r^{n-1} \times (0, r) \times (t_0, t_0 + r^2) : M(|f|^{n+1})(x, t) \geq (c_0 M^k)^{n+1}\}|}{|K_r^{n-1} \times (0, r) \times (t_1, t_1 + r^2)|}.$$

By Lemma 3.2.11, we have $\alpha_{k+1} \leq 2\sigma(\alpha_k + \beta_k)$ for any $k \geq 0$. Then it can be derived directly that

$$\alpha_k \leq (2\sigma)^k + \sum_{i=0}^{k-1} (2\sigma)^{k-i} \beta_i.$$

On the other hand, we can also obtain

$$\beta_i \leq C(c_0 M^i)^{-(n+1)} \frac{||f||_{L^{n+1}}^{n+1}}{r^{n+2}} \leq CM^{-(n+1)i}$$

by using Proposition 3.2.5. Thus, we can observe that

$$\alpha_k \leq (2\sigma)^k + C\sum_{i=0}^{k-1} (2\sigma)^{k-i} M^{-(n+1)i} \leq (1 + Ck) \max\{2\sigma, M^{-(n+1)}\}^k,$$

and the right-hand side is estimated by $C(n)M^{-\mu k}$ for some sufficiently small μ. We now finish the proof. $\square$

3.2.3 Boundary $W^{2,p}$-estimates

As in the previous section, we first consider $W^{2,p}$-regularity for the following problem

$$\begin{cases} F(D^2 u, x, t) - u_t = f & \text{in } Q_1^+, \\ \beta \cdot Du = 0 & \text{on } Q_1^*. \end{cases} \tag{3.2.5}$$

in order to obtain the desired regularity results.

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We are interested in the case when F is slightly perturbed in x and t from

$$\begin{cases} F(D^2u) - u_t = f & \text{in } Q_1^+, \\ \beta \cdot Du = 0 & \text{on } Q_1^*. \end{cases} \tag{3.2.6}$$

If a solution of the model equation is regular enough to have $C^{1,1}$-regularity, we can expect that the solution of (3.2.5) enjoys the required $W^{2,p}$-regularity.

The following is the main theorem of this subsection.

Theorem 3.2.13. *Let u be a viscosity solution of (3.2.5) where $F(X, x, t)$ is uniformly elliptic with λ and Λ, convex in X, continuous in X, x and t and $F(0, x, t) = 0$, $\beta \in C^2(\overline{Q}_1^*)$ with $\beta \cdot \mathbf{n} \geq \delta_0$ for some $\delta_0 > 0$, and $f \in L^p(Q_1^+) \cap C(Q_1^+)$ for $n+1 < p < \infty$. Then there exist ϵ_0 and C depending on $n, p, \lambda, \Lambda, \delta_0$ and $||\beta||_{C^2(\overline{Q}_1^*)}$ such that*

$$\left(\fint_{B_r(x_0,t_0) \cap Q_1^+} \psi((x_0, t_0), (x, t))^{n+1} \, dx dt \right)^{\frac{1}{n+1}} \leq \epsilon_0$$

for any $(x_0, t_0) \in Q_1^+$ and $r > 0$ implies $u \in W^{2,p}(Q_{\frac{1}{2}}^+)$, and we have the estimate

$$||u||_{W^{2,p}(Q_{\frac{1}{2}}^+)} \leq C(||u||_{L^\infty(Q_1^+)} + ||f||_{L^p(Q_1^+)}). \tag{3.2.7}$$

To prove Theorem 3.2.13, we use $C^{1,1}$-regularity results for solutions of (3.2.6), which is indeed, there are results for this problem proved in [16]. We refer to a series of lemmas, Lemma 3.2.14 - 3.2.16, from [16] to present their modifications as follows.

Lemma 3.2.14. *Let $f \in C(\overline{Q}_1^+)$, $g \in C(\overline{Q}_1^*)$, and $u \in C(\overline{Q}_1^+)$ satisfy*

$$\begin{cases} u \in S^*(\lambda, \Lambda, f) & \text{in } Q_1^+, \\ \beta \cdot Du = g & \text{on } Q_1^*. \end{cases} \tag{3.2.8}$$

Suppose that there exists $\xi \in Q_1^*$ such that $\beta \cdot \xi \geq \delta_0$. Then

$$||u||_{L^\infty(Q_1^+)} \leq ||u||_{L^\infty(\partial_p Q_1^+ \setminus Q_1^*)} + C(||g||_{L^\infty(Q_1^*)} + ||f||_{L^{n+1}(Q_1^+)})$$

where C only depends on n, λ, Λ and δ_0.

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Lemma 3.2.15. *Let $f \in C(\overline{Q}_1^+)$, $g \in C(\overline{Q}_1^*)$, and $u \in C(\overline{Q}_1^+)$ satisfy*

$$\begin{cases} u \in S^*(\lambda, \Lambda, f) & \text{in } Q_1^+, \\ \beta \cdot Du = g & \text{on } Q_1^*. \end{cases} \qquad (3.2.9)$$

Then $u \in C^{0,\alpha}(Q_{\frac{1}{2}}^+)$ and

$$||u||_{C^{0,\alpha}(\overline{Q}_{\frac{1}{2}}^+)} \le C(||u||_{L^\infty(Q_1^+)} + ||f||_{L^{n+1}(Q_1^+)} + ||g||_{L^\infty(Q_1^*)})$$

where $0 < \alpha < 1$ and $C > 1$ depend only on n, λ, Λ and δ_0.

Lemma 3.2.16. *Let F be convex, $u \in C(Q_1^+ \cup Q_1^*)$ be a viscosity solution of*

$$\begin{cases} F(D^2u) - u_t = f & \text{in } Q_1^+, \\ \beta \cdot Du = g & \text{on } Q_1^*, \end{cases} \qquad (3.2.10)$$

and $0 < \alpha < \tilde{\alpha}$, where $0 < \tilde{\alpha} < 1$ is a constant depending only on n, λ, Λ and δ_0. Suppose that $\beta, g \in C^{1,\alpha}(\overline{Q}_1^*)$ and $f \in C^{0,\alpha}(\overline{Q}_1^+)$. Then $u \in C^{2,\alpha}(\overline{Q}_{1/4}^+)$ and

$$||u||_{C^{2,\alpha}(\overline{Q}_{1/4}^+)} \le C(||u||_{L^\infty(Q_1^+)} + ||f||_{C^{0,\alpha}(\overline{Q}_1^+)} + ||g||_{C^{1,\alpha}(\overline{Q}_{\frac{1}{2}}^*)})$$

where C only depends on $n, \lambda, \Lambda, \delta_0, \alpha$ and $||\beta||_{C^{1,\alpha}(\overline{Q}_{\frac{1}{2}}^*)}$.

Remark 3.2.17. *Lemma 3.2.14 and 3.2.15 still hold if $S^*(\lambda, \Lambda, f)$ is replaced with $S^*(\lambda, \Lambda, b, f)$, since b only influences the dependency of constant C.*

Next we state and prove the following global Hölder estimate for model problems which will be used later in Lemma 3.2.21.

Lemma 3.2.18. *Let $u \in C(\overline{V}_{1,h_0}^+)$ be a viscosity solution of*

$$\begin{cases} F(D^2u) - u_t = 0 & \text{in } V_{1,h_0}^+, \\ \beta \cdot Du = 0 & \text{on } Q_1^*, \\ u = \varphi & \text{on } \partial_p V_{1,h_0}^+ \backslash Q_1^*, \end{cases} \qquad (3.2.11)$$

where $\beta \in C^2(\overline{Q}_1^*), \varphi \in C^{0,\alpha}(\partial_p V_{1,h_0}^+ \backslash Q_1^*)$ for some $0 < \alpha < 1$ and $h_0 > 0$ is sufficiently small with

$$\beta(x,t) \cdot \mathbf{n}(y) < 0 \text{ for any } (x,t) \in Q_1^* \text{ and } y \in \partial B_{1,h_0}^+ \backslash T_1. \qquad (3.2.12)$$

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Then $u \in C^{0,\frac{\alpha}{2}}(\overline{V}^+_{1,h_0})$ and

$$||u||_{C^{0,\frac{\alpha}{2}}(\overline{V}^+_{1,h_0})} \leq C||\varphi||_{C^{0,\alpha}(\partial_p V^+_{1,h_0} \backslash Q^*_1)},$$

where C only depends on n, λ, Λ and δ_0.

To prove the above lemma, we need the following one which can be shown by using the results of [17, Theorem 7] and [34, Lemma 4.1, Lemma 4.3] and the arguments in the proof of [42, Theorem 3.1]. For Neumann problems, see [16, Proposition 11].

Lemma 3.2.19. *Let $\Omega \subset \mathbb{R}^n$ be bounded, $T > 0$, $\beta \in C^2(\overline{\Gamma})$ with $\Gamma \subset \Omega_T$, and u, v satisfy*

$$\begin{cases} F(D^2 u) - u_t \geq f_1 & in \ \Omega_T, \\ \beta \cdot Du \geq g_1 & on \ \Gamma, \end{cases}$$

and

$$\begin{cases} F(D^2 v) - v_t \leq f_2 & in \ \Omega_T, \\ \beta \cdot Dv \leq g_2 & on \ \Gamma \end{cases}$$

in the viscosity sense, respectively. Then

$$\begin{cases} u - v \in \underline{S}(\lambda/n, \Lambda, f_1 - f_2) & in \ \Omega_T, \\ \beta \cdot D(u - v) \geq g_1 - g_2 & on \ \Gamma. \end{cases}$$

Proof of Lemma 3.2.18. For each $(x_1, t_1) \in \partial_p V^+_{1,h_0} \backslash Q^*_1$, consider

$$w_1(x,t) = \varphi(x_1, t_1) + ||\varphi||_{C^{0,\alpha}(\partial_p V^+_{1,h_0} \backslash Q^*_1)} \Psi(x,t)$$

and

$$w_2(x,t) = \varphi(x_1, t_1) - ||\varphi||_{C^{0,\alpha}(\partial_p V^+_{1,h_0} \backslash Q^*_1)} \Psi(x,t),$$

where

$$\Psi(x,t) := K_1((\mathbf{n}(x_1, t_1) \cdot (x - x_1) + K_2(t_1 - t))^{\frac{\alpha}{2}}),$$

$K_1 > 0$ only depends on $n, \lambda, \Lambda, \delta_0$ and $K_2 = (\sqrt{1 + \lambda/2n} - 1)/2$.

Then we can check that

$$\begin{cases} F(D^2 \Psi) - \Psi_t \leq 0 & in \ V^+_{1,h_0}, \\ \beta \cdot D\Psi \leq 0 & on \ Q^*_1. \end{cases}$$

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Since u is a viscosity solution of (3.2.11), we can observe that

$$
\begin{cases}
u - w_1 \in \underline{S}(\lambda/n, \Lambda, 0) & \text{in } V_{1,h_0}^+, \\
\beta \cdot D(u - w_1) \geq 0 & \text{on } Q_1^*. \\
u - w_1 \leq 0 & \text{on } \partial_p V_{1,h_0}^+ \backslash Q_1^*
\end{cases}
$$

and

$$
\begin{cases}
u - w_2 \in \overline{S}(\lambda/n, \Lambda, 0) & \text{in } V_{1,h_0}^+, \\
\beta \cdot D(u - w_2) \leq 0 & \text{on } Q_1^*. \\
u - w_2 \geq 0 & \text{on } \partial_p V_{1,h_0}^+ \backslash Q_1^*
\end{cases}
$$

from Lemma 3.2.19. Then by ABP maximum principle, we have $w_2 \leq u \leq w_1$. This implies

$$
\begin{aligned}
|u(x,t) - \varphi(x_1,t_1)| &\leq \|\varphi\|_{C^{0,\alpha}(\partial_p V_{1,h_0}^+ \backslash Q_1^*)} \Psi(x,t) \\
&\leq K_1 \|\varphi\|_{C^{0,\alpha}(\partial_p V_{1,h_0}^+ \backslash Q_1^*)} (\mathbf{n}(x_1,t_1) \cdot (x - x_1) + K_2(t_1 - t))^{\frac{\alpha}{2}} \\
&\leq C \|\varphi\|_{C^{0,\alpha}(\partial_p V_{1,h_0}^+ \backslash Q_1^*)} (|x - x_1|^{\frac{\alpha}{2}} + |t - t_1|^{\frac{\alpha}{4}})
\end{aligned}
$$

for some constant C depending on n, λ, Λ and δ_0. This implies boundary Hölder regularity. Now we can complete the proof by combining this estimate with Lemma 3.2.15. $\qquad\square$

Remark 3.2.20. *We have assumed that $\beta \in C^2(\overline{\Gamma})$ in Lemma 3.2.18 and 3.2.19, as [34, Lemma 4.1, Lemma 4.3] hold under this assumption.*

We now fix $h_0 = h_0(n, \delta_0) > 0$ given in Lemma 3.2.18.

Lemma 3.2.16 and Lemma 3.2.18 enable us to prove a useful approximation lemma below.

Lemma 3.2.21. *Let $0 < \epsilon < 1$ and u be a viscosity solution of (3.2.5). Assume that $\|u\|_{L^\infty(V_{1,h_0}^+)} \leq 1$ and $\|\psi((\cdot, \cdot), (0,0))\|_{L^{n+1}(V_{1,h_0}^+)} \leq \epsilon$. Then, there exists a function $h \in C^2(\overline{V}_{\frac{3}{4}, \frac{3}{4}h_0}^+)$ such that $u - h \in S(\varphi)$, $\|h\|_{C^2(\overline{V}_{\frac{3}{4}, \frac{3}{4}h_0}^+)} \leq C$ and*

$$
\|u - h\|_{L^\infty(V_{\frac{3}{4}, \frac{3}{4}h_0}^+)} + \|\varphi\|_{L^{n+1}(V_{\frac{3}{4}, \frac{3}{4}h_0}^+)} \leq C(\epsilon^\gamma + \|f\|_{L^{n+1}(V_{1,h_0}^+)})
$$

for some $0 < \gamma = \gamma(n, \lambda, \Lambda, \delta_0) < 1$ and $C = C(n, \lambda, \Lambda, \delta_0, \|\beta\|_{C^2(\overline{Q}_1^*)})$. Here, $\varphi(x,t) = f(x,t) - F(D^2 h(x,t), x, t) + F(D^2 h(x,t), 0, 0)$.

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Proof. Let h be a solution of

$$\begin{cases} F(D^2h, 0, 0) - h_t = 0 & \text{in } V^+_{\frac{7}{8}, \frac{7}{8}h_0}, \\ h = u & \text{on } \partial_p V^+_{\frac{7}{8}, \frac{7}{8}h_0} \backslash Q^*_{\frac{7}{8}}, \\ \beta \cdot Dh = 0 & \text{on } Q^*_{\frac{7}{8}}. \end{cases} \qquad (3.2.13)$$

Applying Lemma 3.2.15 to u, we can obtain

$$||u||_{C^{0,\alpha_1}(V^+_{\frac{7}{8}, \frac{7}{8}h_0})} \leq C(1 + ||f||_{L^{n+1}(V^+_{1,h_0})}) \qquad (3.2.14)$$

for some $\alpha_1 = \alpha_1(n, \lambda, \Lambda, \delta_0)$ and $C = C(n, \lambda, \Lambda, \delta_0)$. We also have

$$||h||_{L^\infty(V^+_{\frac{7}{8}, \frac{7}{8}h_0, \delta})} + \delta ||Dh||_{L^\infty(V^+_{\frac{7}{8}, \frac{7}{8}h_0, \delta})}$$
$$+ \delta^2(||h_t||_{L^\infty(V^+_{\frac{7}{8}, \frac{7}{8}h_0, \delta})} + ||D^2h||_{L^\infty(V^+_{\frac{7}{8}, \frac{7}{8}h_0, \delta})}) \leq C,$$

where C is a constant depending on $n, \lambda, \Lambda, \delta_0$ and $||\beta||_{C^2(\overline{Q^*_1})}$ by means of
Lemma 3.2.16 with scaling.

We set $w = u - h$. Then, w satisfies

$$\begin{cases} w \in S(\lambda/n, \Lambda, \varphi) & \text{in } V^+_{\frac{7}{8}, \frac{7}{8}h_0}, \\ w = 0 & \text{on } \partial_p V^+_{\frac{7}{8}, \frac{7}{8}h_0} \backslash Q^*_{\frac{7}{8}}, \\ \beta \cdot Dw = 0 & \text{on } Q^*_{\frac{7}{8}}. \end{cases} \qquad (3.2.15)$$

Apply Lemma 3.2.8 to w, we get

$$||w||_{L^\infty(V^+_{\frac{7}{8}, \frac{7}{8}h_0, \delta})} \leq C(||\varphi||_{L^{n+1}(V^+_{\frac{7}{8}, \frac{7}{8}h_0, \delta})} + ||w||_{L^\infty(\partial_p V^+_{\frac{7}{8}, \frac{7}{8}h_0, \delta} \backslash Q^*_{\frac{7}{8}})})$$
$$\leq C(||f||_{L^{n+1}(V^+_{\frac{7}{8}, \frac{7}{8}h_0, \delta})} + ||F(D^2h, \cdot, \cdot) - F(D^2, 0, 0)||_{L^{n+1}(V^+_{\frac{7}{8}, \frac{7}{8}h_0, \delta})}$$
$$+ ||w||_{L^\infty(\partial_p V^+_{\frac{7}{8}, \frac{7}{8}h_0, \delta} \backslash Q^*_{\frac{7}{8}})})$$

for some $C = C(n, \lambda, \Lambda, \delta_0)$. Observe that

$$||F(D^2h, \cdot, \cdot) - F(D^2h, 0, 0)||_{L^{n+1}(V^+_{\frac{7}{8}, \frac{7}{8}h_0, \delta})}$$
$$\leq ||\boldsymbol{\psi}((\cdot, \cdot), (0, 0))||_{L^{n+1}(V^+_{\frac{7}{8}, \frac{7}{8}h_0, \delta})} ||D^2h||_{L^\infty(V^+_{\frac{7}{8}, \frac{7}{8}h_0, \delta})}$$

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$$\leq C\delta^{-2}\epsilon,$$

where $C = C(n, \lambda, \Lambda, \delta_0, ||\beta||_{C^2(\overline{Q}_1^*)})$. We have used $h \in C^2(\overline{V}_{\frac{7}{8}, \frac{7}{8}h_0}^+)$ in the first inequality.

Meanwhile, we can see that $w \equiv 0$ on $\partial_p V_{\frac{7}{8}, \frac{7}{8}h_0}^+ \backslash Q_{\frac{7}{8}}^*$ and $u \in C^{0,\alpha_1}(V_{\frac{7}{8}, \frac{7}{8}h_0}^+)$. Then we also obtain a global Hölder regularity for h by combining Lemma 3.2.18 with (3.2.14). Now we get

$$||w||_{L^\infty(\partial_p V_{\frac{7}{8}, \frac{7}{8}h_0, \delta}^+ \backslash Q_{\frac{7}{8}}^*)} \leq C\delta^{\alpha_2}(1 + ||f||_{L^{n+1}(V_{1,h_0}^+)})$$

for some $\alpha_2 \in (0, \alpha_1)$ and $C = C(n, \lambda, \Lambda, \delta_0)$. Thus, if we put $\gamma = \delta^{\frac{\alpha_2}{2+\alpha_2}}$,

$$||w||_{L^\infty(\partial_p V_{\frac{7}{8}, \frac{7}{8}h_0, \delta}^+)} \leq C\{||f||_{L^{n+1}(V_{1,h_0}^+)} + \delta^{-2}\epsilon + \delta^{\alpha_2}(1 + ||f||_{L^{n+1}(V_{1,h_0}^+)})\}$$
$$\leq C(\epsilon^\gamma + ||f||_{L^{n+1}(V_{1,h_0}^+)})$$

for some constant C depending only on $n, \lambda, \Lambda, \delta_0$ and $||\beta||_{C^2(Q_1^*)}$. This completes the proof. $\square$

The following lemmas give us useful information about solutions of (3.2.5) in the viscosity sense.

Lemma 3.2.22. *Let* $0 < \epsilon_0 < 1$, $\Omega = B_{14\sqrt{n}h_1^{-1}, 14\sqrt{n}}^+ \times (0, 15]$, $r \leq 1$, *and* u *be a viscosity solution of*

$$\begin{cases} F(D^2u, x, t) - u_t = f & in \ \Omega, \\ \beta \cdot Du = 0 & on \ S := T_{14\sqrt{n}h_1^{-1}} \times (0, 15], \end{cases} \quad (3.2.16)$$

where $h_1 = h_1(n, \delta_0)$ *is a small constant satisfying (3.2.12) for any* $(x, t) \in S$ *and* $y \in \partial B_{14\sqrt{n}h_1^{-1}, 14\sqrt{n}}^+ \backslash T_{14\sqrt{n}h_1^{-1}}$. *Consider a point* $(x_0, t_0) \in S$ *with* $r\Omega(x_0, t_0) \subset \Omega$. *Assume that*

$$\left(\fint_{r\Omega(x_0,t_0)} |f(x,t)|^{n+1}dxdt\right)^{\frac{1}{n+1}} + \left(\fint_{r\Omega(x_0,t_0)} |\psi((x,t),(x_0,t_0))|^{n+1}dxdt\right)^{\frac{1}{n+1}} \leq \epsilon$$

for some $\epsilon < 1$ *depending on* $n, \epsilon_0, \lambda, \Lambda, \delta_0$ *and* $||\beta||_{C^2(Q_{14\sqrt{n}h_1^{-1}}^*)}$. *Then,*

$$G_1(u, \Omega) \cap (K_{3r}^{n-1} \times (0, 3r) \times (r^2, 10r^2) + (\tilde{x}_1, \tilde{t}_1)) \neq \varnothing \quad (3.2.17)$$

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for some $(\tilde{x}_1, \tilde{t}_1) \in (B^+_{9r\sqrt{n}h_1^{-1}, 9r\sqrt{n}}(x_0) \cup T_{9r\sqrt{n}}(x_0)) \times [t_0 + 2r^2, t_0 + 5r^2]$ *implies*

$$\frac{|G_M(u, \Omega) \cap (K_r^{n-1} \times (0, r) \times (0, r^2) + (x_1, t_1))|}{|K_r^{n-1} \times (0, r) \times (0, r^2)|} \geq 1 - \epsilon_0,$$

where $(x_1, t_1) \in (B^+_{9r\sqrt{n}h_1^{-1}, 9r\sqrt{n}}(x_0) \cup T_{9r\sqrt{n}}(x_0)) \times [t_0 + 2r^2, \tilde{t}_1]$ *and* M *is a constant depending only on* $n, \lambda, \Lambda, \delta_0$ *and* $||\beta||_{C^2(Q^*_{14\sqrt{n}h_1^{-1}})}$.

Proof. From (3.2.17), there exists a point (x_2, t_2) such that

$$(x_2, t_2) \in G_1(u, \Omega) \cap (K_{3r}^{n-1} \times (0, 3r) \times (r^2, 10r^2) + (\tilde{x}_1, \tilde{t}_1)).$$

By the definition of G_1, we can find a linear function L such that

$$|u(x, t) - L(x)| \leq \frac{1}{2}(|x - x_2|^2 - (t - t_2))$$

for any $(x, t) \in B^+_{14\sqrt{n}h_1^{-1}, 14\sqrt{n}} \times (0, t_2)$. Let $\tilde{u}(x, t) = (u(x, t) - L(x))/C(n)$ with $\tilde{u}$ satisfying $||\tilde{u}||_{L^\infty(B^+_{14r\sqrt{n}h_1^{-1}, 14r\sqrt{n}}(x_0) \times (t_0, t_2))} \leq 1$ and

$$|\tilde{u}(x, t)| \leq |x|^2 - (t - t_2) \text{ in } (B^+_{14\sqrt{n}h_1^{-1}, 14\sqrt{n}} \setminus B^+_{14r\sqrt{n}h_1^{-1}, 14r\sqrt{n}}(x_0)) \times [0, t_2].$$

Here we can check that

$$||L||_{C^1(B^+_{14r\sqrt{n}h_1^{-1}, 14r\sqrt{n}}(x_0) \times [0, t_2])} \leq C(n) + ||u||_{L^\infty(\Omega)},$$

and thus $|DL|$ is uniformly bounded and depending only on n and $||u||_{L^\infty(\Omega)}$ in this case.

Next we define $\tilde{F}(D^2\tilde{u}, x, t) = F(CD^2\tilde{u}, x, t)/C(n)$, $\tilde{f}(x, t) = f(x, t)/C(n)$. We see that the elliptic constants of F and $\tilde{F}$ are the same and $\tilde{u}$ is a viscosity solution of

$$\begin{cases} \tilde{F}(D^2\tilde{u}, x, t) - \tilde{u}_t = \tilde{f} & \text{in } r\Omega(x_0, t_0), \\ \beta \cdot D\tilde{u} = -\beta \cdot DL/C(n) & \text{on } rS(x_0, t_0). \end{cases} \qquad (3.2.18)$$

Set $\Omega' = B^+_{14\sqrt{n}h_1^{-1}, 14\sqrt{n}h_1} \times (1, 15]$, $\Omega'' = B^+_{13\sqrt{n}h_1^{-1}, 13\sqrt{n}} \times (2, 15]$ and $S' = T^+_{14\sqrt{n}} \times (1, 15]$. We also write $\Omega'_\delta = B^+_{(14-\delta)\sqrt{n}h_1^{-1}, (14-\delta)\sqrt{n}} \times (1 + \delta^2, 15]$. Consider

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a function $\tilde{h} \in C(r\Omega'(x_0, t_0))$ which solves

$$
\begin{cases}
\tilde{F}(D^2\tilde{h}, 0, 0) - \tilde{h}_t = 0 & \text{in } r\Omega'(x_0, t_0), \\
\tilde{h} = \tilde{u} & \text{on } \partial_p(r\Omega'(x_0, t_0))\backslash rS'(x_0, t_0), \qquad (3.2.19) \\
\beta \cdot D\tilde{h} = -\beta \cdot DL/C(n) & \text{on } rS'(x_0, t_0)
\end{cases}
$$

in the viscosity sense. Since $\beta \in C^2(rS(x_0, t_0))$ and DL is a constant vector, $\beta \cdot DL \in C^2(rS(x_0, t_0))$. Then we can derive that

$$
||\tilde{u}||_{C(r\Omega'(x_0,t_0))} + r[\tilde{u}]_{C^{0,\alpha_1}(r\Omega'(x_0,t_0))}
$$
$$
\leq C(1 + r^{\frac{n}{n+1}}||f||_{L^{n+1}(r\Omega(x_0,t_0))} + r||\beta \cdot DL||_{L^\infty(r\Omega(x_0,t_0))})
$$

for some $\alpha_1 = \alpha_1(n, \lambda, \Lambda, \delta_0) \in (0, 1)$ and $C = C(n, \lambda, \Lambda, \delta_0) > 0$ by Lemma 3.2.15. On the other hand, applying Lemma 3.2.14 and 3.2.16 to $\tilde{h}$, we also have

$$
||\tilde{h}||_{C(r(\Omega'_\delta)(x_0,t_0))} + r\delta||D\tilde{h}||_{C(r(\Omega'_\delta)(x_0,t_0))}
$$
$$
+ (r\delta)^2(||\tilde{h}_t||_{C(r(\Omega'_\delta)(x_0,t_0))} + ||D^2\tilde{h}||_{C(r(\Omega'_\delta)(x_0,t_0))})
$$
$$
\leq C(||\tilde{h}||_{L^\infty(r\Omega'(x_0,t_0))} + r||\beta \cdot DL||_{C(rS(x_0,t_0))} + r^2||D\beta \otimes DL||_{C(rS(x_0,t_0))}
$$
$$
+ r^{2+\alpha}[D\beta \otimes DL]_{C^{0,\alpha}(rS(x_0,t_0))})
$$

for any $\alpha \in (0, 1)$ and some C depending only on $n, \lambda, \Lambda, \delta_0$ and $||\beta||_{C^2(rS(x_0,t_0))}$. Next, we observe that

$$
||\beta \cdot DL||_{C(rS(x_0,t_0))} + r||D\beta \otimes DL||_{C(rS(x_0,t_0))} + r^{1+\alpha}[D\beta \otimes DL]_{C^{0,\alpha}(rS(x_0,t_0))})
$$
$$
\leq C(n, ||\beta||_{C^2(rS(x_0,t_0))})
$$

and

$$
||\tilde{h}||_{L^\infty(r(\Omega'_\delta)(x_0,t_0))}
$$
$$
\leq ||\tilde{h}||_{L^\infty(\partial_p(r(\Omega'_\delta)(x_0,t_0)))\backslash rS'(x_0,t_0))} + C(n, \lambda, \Lambda, \delta_0)r||\beta \cdot DL||_{L^\infty(rS(x_0,t_0))}
$$
$$
\leq ||\tilde{u}||_{L^\infty(\partial_p(r\Omega'(x_0,t_0))\backslash rS'(x_0,t_0))} + C(n, \lambda, \Lambda, \delta_0, ||\beta||_{C^2(rS(x_0,t_0))})r
$$
$$
+ C(n, \lambda, \Lambda, \delta_0)(1 + r^{\frac{n}{n+1}}||f||_{L^{n+1}(r\Omega(x_0,t_0))} + r||DL||_{L^\infty(r\Omega(x_0,t_0))})\delta^{\alpha_2}
$$
$$
\leq C(n, \lambda, \Lambda, \delta_0, ||\beta||_{C^2(rS(x_0,t_0))})
$$

for any $0 < \delta < 2$ and some $\alpha_2 \in (0, \alpha_1)$. We have used a similar argument

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for the second inequality in the proof of Lemma 3.2.21. Hence, we get

$$||D^2\tilde{h}||_{L^\infty(r(\Omega'_\delta)(x_0,t_0)))} + ||\tilde{h}_t||_{L^\infty(r(\Omega'_\delta)(x_0,t_0)))}$$
$$\leq \delta^{-2}C(n,\lambda,\Lambda,\delta_0,||\beta||_{C^2(rS(x_0,t_0))})$$

for any $0 < \delta < 2$ and therefore

$$||D^2\tilde{h}||_{L^\infty(r\Omega''(x_0,t_0))} + ||\tilde{h}_t||_{L^\infty(r\Omega''(x_0,t_0))} \leq C(n,\lambda,\Lambda,\delta_0,||\beta||_{C^2(rS(x_0,t_0))}).$$

Then the above estimate leads to

$$A_N(\tilde{h}, r\Omega''(x_0,t_0)) \cap (Q_r^{n-1} \times (0,r) \times (0,r^2) + (x_1,t_1)) = \varnothing$$

for any $(x_1,t_1) \in (B_{9r\sqrt{n}} \cap \{x_n \geq 0\}) \times [t_0 + 2r^2, \tilde{t}_1]$ and a sufficiently large $N = N(n,\lambda,\Lambda,\delta_0,||\beta||_{C^2(rS(x_0,t_0))})$.

Extend $\tilde{h}|_{r\Omega''(x_0,t_0)}$ to H with the property that H is continuous in Ω_{t_2}, where

$$\Sigma_s := \{(x,t) \in \Sigma : t \leq s\} \quad \text{for } \Sigma \in \mathbb{R}^n \times \mathbb{R},$$

$H = \tilde{u}$ in $\Omega_{t_2}\backslash(r\Omega'(x_0,t_0))_{t_2}$, and

$$||\tilde{u} - H||_{L^\infty(\Omega_{t_2})} = ||\tilde{u} - \tilde{h}||_{L^\infty((r\Omega''(x_0,t_0))_{t_2})}.$$

Then we have

$$||\tilde{u} - H||_{L^\infty(\Omega_{t_2})} \leq ||\tilde{u}||_{L^\infty((r\Omega''(x_0,t_0))_{t_2})} + ||\tilde{h}||_{L^\infty((r\Omega''(x_0,t_0))_{t_2})} \leq C_0$$

for some $C_0 = C_0(n,\lambda,\Lambda,\delta_0,||\beta||_{C^2(rS(x_0,t_0))})$. From this, we see that

$$|H(x,t)| \leq C_0 + |x|^2 - (t - t_2) \quad \text{in} \quad \Omega_{t_2}\backslash(r\Omega''(x_0,t_0))_{t_2}.$$

It can be obtained directly that

$$A_{M_0}(H,\Omega) \cap (K_r^{n-1} \times (0,r) \times (0,r^2) + (x_1,t_1)) = \varnothing$$

for some $M_0 \geq N$.

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Define $w = \tilde{u} - H$. Then w satisfies

$$
\begin{cases}
w \in S(\lambda/n, \Lambda, \tilde{f} - \tilde{F}(D^2\tilde{h}, \cdot, \cdot) + \tilde{h}_t) & \text{in } r\Omega'(x_0, t_0), \\
w = 0 & \text{on } \partial_p(r\Omega'(x_0, t_0))_{t_2} \backslash (rS'(x_0, t_0))_{t_2}, \\
\beta \cdot Dw = 0 & \text{on } rS'(x_0, t_0).
\end{cases}
$$
$$(3.2.20)$$

From Lemma 3.2.14, we can derive

$$||w||_{L^\infty(\Omega_{t_2})} = ||w||_{L^\infty((r\Omega''(x_0,t_0))_{t_2})} \le C(\epsilon^\gamma + ||f||_{L^{n+1}(r\Omega(x_0,t_0))}) \le C\epsilon^\gamma$$

for some $\gamma \in (0, 1)$ depending only on $n, \lambda, \Lambda, \delta_0$ and $C > 1$ which also depends on $||\beta||_{C^2(rS(x_0,t_0))}$.

Now write $\tilde{w} = w/C\epsilon^\gamma$. Since $\tilde{w}$ satisfies the assumptions of Corollary 3.2.12, it holds that

$$\frac{|A_s(\tilde{w}, \Omega) \cap (K_r^{n-1} \times (0, r) \times (0, r^2) + (x_1, t_1))|}{|K_r^{n-1} \times (0, r) \times (0, r^2)|} \le Cs^{-\mu}.$$

We also check that

$$A_{2M_0}(\tilde{u}, \Omega) \subset A_{M_0}(w, \Omega) \cup A_{M_0}(H, \Omega),$$

$$A_{M_0}(H, \Omega) \cap (K_r^{n-1} \times (0, r) \times (0, r^2) + (x_1, t_1)) = \varnothing.$$

This implies

$$
\begin{aligned}
|A_{2M_0}(\tilde{u}, \Omega) &\cap (K_r^{n-1} \times (0, r) \times (0, r^2) + (x_1, t_1))| \\
&\le |A_{M_0}(w, \Omega) \cap (K_r^{n-1} \times (0, r) \times (0, r^2) + (x_1, t_1))| \\
&= |A_{M_0/C\epsilon^\gamma}(\tilde{w}, \Omega) \cap (K_r^{n-1} \times (0, r) \times (0, r^2) + (x_1, t_1))| \\
&\le C(M_0/C\epsilon^\gamma)^{-\mu}|K_r^{n-1} \times (0, r) \times (0, r^2)| \\
&\le \epsilon_0|K_r^{n-1} \times (0, r) \times (0, r^2)|
\end{aligned}
$$

for $M = 2CM_0$ and a sufficiently small ϵ. Then we get the desired result. $\square$

Lemma 3.2.23. *Let* $0 < \epsilon_0 < 1$, $\Omega = B^+_{14\sqrt{n}h_1^{-1}, 14\sqrt{n}} \times (0, 15]$, $r \le 1$, *and* u *be a viscosity solution of* (3.2.16). *Assume that* $||u||_{L^\infty(r\Omega(x_0,t_0))} \le 1$ *and*

$$\left(\fint_{r\Omega(x_0,t_0)} |f(x, t)|^{n+1} dx dt \right)^{\frac{1}{n+1}} \le \epsilon$$

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for some $\epsilon > 0$ depending only on $n, \epsilon_0, \lambda, \Lambda, \delta_0, ||\beta||_{C^2(rS(x_0,t_0))}$.
 Extend f to zero outside $r\Omega(x_0, t_0)$ and let

$$\left(\fint_{Q_r(x_1,t_1) \cap r\Omega(x_0,t_0)} \boldsymbol{\psi}((x_1,t_1),(x,t))^{n+1} \, dxdt \right)^{\frac{1}{n+1}} \leq \epsilon$$

for any $(x_1, t_1) \in r\Omega(x_0, t_0)$, $r > 0$. Then, for

$$A := A_{M^{k+1}}(u, \Omega) \cap (K_r^{n-1} \times (0, r) \times (t_0 + 2r^2, t_0 + 3r^2)),$$

$$B := (A_{M^k}(u, \Omega) \cap (K_r^{n-1} \times (0, r) \times (t_0 + 2r^2, t_0 + 3r^2)) \, \cup$$
$$\{(x,t) \in K_r^{n-1} \times (0, r) \times (t_0 + 2r^2, t_0 + 3r^2) : M(|f|^{n+1})(x,t) \geq (c_0 M^k)^{n+1}\},$$

where $k \in \mathbb{N}_0$, $M > 1$ only depends on $n, \lambda, \Lambda, \delta_0, ||\beta||_{C^2(rS(x_0,t_0))}$ and c_0 also depends on ϵ_0, we have
$$|A| \leq 2\epsilon_0 |B|.$$

Proof. First of all, we observe that

$$A \subset B \subset K_r^{n-1} \times (0, r) \times (t_0 + 2r^2, t_0 + 3r^2).$$

We also have $B \subsetneq K_r^{n-1} \times (0, r) \times (t_0 + 2r^2, t_0 + 3r^2)$ by Lemma 3.2.10. Thus, applying Lemma 3.2.22 to u, we obtain $|A| \leq 2\epsilon_0$. Then we only need to show that for any parabolic dyadic cube K and its predecessor $\tilde{K}$,

$$|A \cap K| > \epsilon_0 |K| \quad \Rightarrow \quad \overline{K}^1 \subset B$$

by means of Lemma 3.2.4.
 We define

$$K = (K_{r/2^i}^{n-1} \times (0, r/2^i) \times (0, r^2/2^{2i})) + (x_1, t_1)$$

and
$$\tilde{K} = (K_{r/2^{i-1}}^{n-1} \times (0, r/2^{i-1})) \times (0, r^2/2^{2(i-1)})) + (\tilde{x}_1, \tilde{t}_1).$$

Suppose that $|A \cap K| > \epsilon_0 |K|$ and $\overline{K}^1 \nsubseteq B$. There exists a point $(x_2, t_2) \in \overline{K}^1 \cap G_{M^k}(u, \Omega)$ with $M(|f|^{n+1})(x_2, t_2) < (c_0 M^k)^{n+1}$.

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First, assume that $x_{1,n} \leq 8r\sqrt{n}/2^i$. Consider a linear transformation

$$T(y,s) = (x_1', 0, t^*) + (2^{-i}y, 2^{-2i}s),$$

where $t^* = t_1 - 2^{1-2i}r^2$. Now we set

$$\tilde{u}(y,s) = 2^{2i}M^{-k}u(T(y,s)),$$

$$\tilde{\beta}(y) = \beta(T(y,s)),$$

$$\tilde{F}(X,y) = M^{-k}F(M^kX, T(y,s))$$

and

$$\tilde{f}(y) = M^{-k}f(T(y,s)).$$

Then $\tilde{u}$ is a viscosity solution of

$$\begin{cases} \tilde{F}(D^2\tilde{u}, y, s) - \tilde{u}_t = \tilde{f} & \text{in } r\Omega(0,0), \\ \tilde{\beta} \cdot D\tilde{u} = 0 & \text{on } rS(0,0), \end{cases} \tag{3.2.21}$$

since $(r/2^i)\Omega(x_1', 0, t^*) \subset r\Omega(x_0, t_0)$. Observe that $\tilde{\beta} \in C^2(rS(0,0))$ and $\tilde{F}$ has the same elliptic constant of F. Let

$$\boldsymbol{\psi}_{\tilde{F}}((y,s),(0,0)) = \boldsymbol{\psi}_F(T(y,s),(x_1', 0, t^*)).$$

Then we also have $||\boldsymbol{\psi}_{\tilde{F}}||_{L^{n+1}(r\Omega'(0,0))} \leq C\epsilon$ for some $C = C(n) > 0$

In addition, we obtain

$$||\tilde{f}||_{L^{n+1}(r\Omega(0,0))} = \left(\int_{r\Omega(0,0)} |\tilde{f}(y,s)|^{n+1} \, dyds \right)^{\frac{1}{n+1}}$$
$$\leq C(n)c_0$$
$$\leq \epsilon$$

by using Proposition 3.2.5 and choosing c_0 small enough.

One the other hand, we have

$$T^{-1}\overline{K}^1 \cap G_1(\tilde{u}, T^{-1}(r\Omega(0,0))) \neq \varnothing$$

by the assumption $\overline{K}^1 \cap G_{M^k}(u, \Omega) \neq \varnothing$. And since $|x_1 - \tilde{x}_1| < r\sqrt{n}/2^i$, we observe that $|T^{-1}\tilde{x}_1| < 9r\sqrt{n}$. Consequently, applying Lemma 3.2.22 to $\tilde{u}$,

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we get

$$\frac{|T^{-1}K \cap G_M(\tilde{u}, T^{-1}(r\Omega(0,0)))|}{|T^{-1}K|} \geq 1 - \epsilon_0.$$

Then it follows immediately that

$$\frac{|K \cap G_{M^{k+1}}(u, r\Omega(0,0))|}{|K|} \geq 1 - \epsilon_0.$$

This leads to a contradiction.

Now we consider the interior case $x_{1,n} > 8r\sqrt{n}/2^i$. Observe that

$$Q_{8r\sqrt{n}/2^i}(x_1 + re_n/2^{i+1}, t_1) \subset Q^+_{8r\sqrt{n}h_1^{-1}, 8r\sqrt{n}}(x_0, t_0)$$

in this case. Again, set $T : Q_{8r\sqrt{n}} \to Q_{8r\sqrt{n}/2^i}(x_1 + re_n/2^{i+1}, t_1)$ such that

$$T(y,s) = \left(x_1 + \frac{re_n}{2^{i+1}} + \frac{y}{2^{i+1}}, t_1 + \frac{s}{2^{2(i+1)}} \right).$$

and we write

$$\tilde{u}(y,s) = 2^{2(i+1)} M^{-k} u(T(y,s)),$$

$$\tilde{F}(X,y,s) = M^{-k} F(M^k X, T(y,s))$$

and

$$\tilde{f}(y,s) = M^{-k} u(T(y,s)).$$

We can check that $\tilde{u}$ is a solution of

$$\tilde{F}(D^2\tilde{u}, y, s) - \tilde{u}_t = \tilde{f}(y,s) \qquad \text{in } Q_{8r\sqrt{n}}$$

in the viscosity sense. Applying [75, Corollary 5.2] to $\tilde{u}$, we can also deduce
our desired result. □

Proof of Theorem 3.2.13. We fix $(x_0, t_0) \in Q^+_{2/3} \cup Q^*_{2/3}$. If $(x_0, t_0) \in Q^*_{2/3}$, let
r be a fixed number in $\left(0, \min \left\{ \frac{1-|x_0|}{14\sqrt{n}} h_1, \sqrt{-\frac{t_0}{15}} \right\} \right)$ and we set

$$K = \frac{\epsilon r^{\frac{n+2}{n+1}}}{\epsilon r^{-1} ||u||_{L^\infty(r\Omega(x_0,t_0))} + ||f||_{L^{n+1}(r\Omega(x_0,t_0))}}.$$

Here, $\Omega = B^+_{14\sqrt{n}h_1^{-1}, 14\sqrt{n}} \times (0, 15]$ with $h_1 = h_1(\delta_0)$ as in Lemma 3.2.22

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and $\epsilon = \epsilon(n, \epsilon_0, \lambda, \Lambda, p, \delta_0, ||\beta||_{C^2(\overline{Q_1^*})})$ is a constant as in Lemma 3.2.22 with
$\epsilon_0 \in (0, 1)$ to be chosen later.

Let

$$\tilde{u}(y, s) = Kr^{-2}u(ry + x_0, r^2s + t_0),$$

$$\tilde{f}(y, s) = Kf(ry + x_0, r^2s + t_0),$$

$$\tilde{\beta}(y, s) = \beta(ry + x_0, r^2s + t_0),$$

and

$$\tilde{F}(X, y) = KF(K^{-1}X, ry + x_0, r^2s + t_0).$$

Then, $\tilde{u}$ is a solution of

$$\begin{cases} \tilde{F}(D^2\tilde{u}, y, s) - \tilde{u}_t = \tilde{f} & \text{in } \Omega, \\ \tilde{\beta} \cdot D\tilde{u} = 0 & \text{on } S := T^+_{14\sqrt{n}h_1^{-1}} \times (0, 15] \end{cases} \qquad (3.2.22)$$

in the viscosity sense. It can be checked without difficulty that F and $\tilde{F}$ have
the same elliptic constants, $\tilde{\beta} \in C^2(S)$, $||\tilde{u}||_{L^\infty(\Omega)} \leq 1$,

$$||\psi_{\tilde{F}}||_{L^{n+1}(\Omega)} \leq C(n)\epsilon_0 \leq \epsilon,$$

$$||\tilde{f}||_{L^{n+1}(\Omega)} \leq Kr^{-\frac{n+2}{n+1}}||f||_{L^{n+1}(r\Omega(x_0, t_0))} \leq \epsilon$$

for a sufficiently small ϵ_0. Thus, the assumption of Lemma 3.2.23 is satisfied.
Set

$$\alpha_k = |A_{M^k}(u, \Omega) \cap (K_1^{n-1} \times (0, 1) \times (2, 3))|,$$

$$\beta_k = |\{(x, t) \in K_1^{n-1} \times (0, 1) \times (2, 3) : M(|f|^{n+1})(x, t) \geq (c_0 M^k)^{n+1}\}|$$

and choose $\epsilon_0 = 1/(4M^p)$. By direct calculation, we have

$$\alpha_k \leq (2\epsilon_0)^k + \sum_{i=0}^{k-1}(2\epsilon_0)^{k-i}\beta_i.$$

We also observe that

$$||M(|f|^{n+1})||_{L^{\frac{p}{n+1}}} \leq C(n, p)$$

by Proposition 3.2.5, and this implies

$$\sum_{i=0}^{\infty} M^{pk}\alpha_k \leq C(n,p).$$

Using Proposition 3.2.6, we discover

$$||\tilde{u}_t||_{L^p(Q^+_{\frac{1}{2}}(0,-\frac{1}{8}))} + ||D^2\tilde{u}||_{L^p(Q^+_{\frac{1}{2}}(0,-\frac{1}{8}))} \leq C,$$

that is,

$$||u_t||_{L^p(Q^+_{r/2}(x_0,t_0-\frac{r^2}{8}))} + ||D^2u||_{L^p(Q^+_{r/2}(x_0,t_0-\frac{r^2}{8}))} \leq C(||u||_{L^\infty(Q^+_1)} + ||f||_{L^p(Q^+_1)}),$$

where $C = C(n,\lambda,\Lambda,p,r,\delta_0,||\beta||_{C^2(\overline{Q}^*_1)}) > 0$.

Besides, when $(x_0,t_0) \in Q^+_{2/3}$, we can apply the results of interior estimates, like as in [75, Theorem 5.6]. Combining the interior and boundary estimates, we get

$$||u_t||_{L^p(Q^+_{\frac{1}{2}}(0,-\frac{1}{8}))} + ||D^2u||_{L^p(Q^+_{\frac{1}{2}}(0,-\frac{1}{8}))} \leq C,$$

where $C = C(n,\lambda,\Lambda,p,\delta_0,||\beta||_{C^2(\overline{Q}^*_1)}) > 0$.

We also need to establish proper regularity results in $Q^+_{\frac{1}{2}} \times [-1/8,0)$. For these estimates, we extend F and β such that our assumptions are satisfied. Then we can obtain the estimate (3.2.7). □

3.2.4 Boundary $W^{1,p}$-estimates

We have obtained $W^{2,p}$-regularity for solutions of (3.2.5) in the previous subsection. Here, we extend this regularity to the case when the function F also contains ingredients q and r.

Let u be a viscosity solution of the following problem

$$\begin{cases} F(D^2u, Du, u, x, t) - u_t = f & \text{in } Q^+_1, \\ \beta \cdot Du = 0 & \text{on } Q^*_1, \end{cases} \tag{3.2.23}$$

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and assume that this u also solves

$$\begin{cases} F(D^2u, 0, 0, x, t) - u_t = \tilde{f} & \text{in } Q_1^+, \\ \beta \cdot Du = 0 & \text{on } Q_1^* \end{cases} \tag{3.2.24}$$

for some function $\tilde{f}$ in the viscosity sense.

By virtue of the structure condition (3.2.2). we have (3.1.17) like as in the elliptic case. Thus, we need to obtain $W^{1,p}$-regularity for u in order to reach our goal. The following theorem provides the type of estimates which we want to derive.

Theorem 3.2.24. *Let $n+2 < p < \infty$. Assume that F satisfies the structure condition (3.2.2) with $F(0, 0, 0, x, t) = 0$ and u be a viscosity solution of (3.2.23) where $f \in L^p(Q_1^+) \cap C(\overline{Q}_1^+)$ and $\beta \in C^2(\overline{Q}_1^*)$. Then, there exists a constant $\epsilon_0 = \epsilon_0(n, \lambda, \Lambda, p, \delta_0, \alpha)$ such that if*

$$\left(\fint_{Q_r(x_0, t_0) \cap Q_1^+} \boldsymbol{\psi}((x_0, t_0), (x, t))^p \, dxdt \right)^{1/p} \leq \epsilon_0$$

for any $(x_0, t_0) \in Q_1^+$ and $r \leq r_0$ for some $r_0 > 0$, then $u \in C^{1,\alpha}(\overline{Q}_{\frac{1}{2}}^+)$ with $\alpha = \alpha(n, p, \lambda, \Lambda) \in (0, 1)$ and we have the estimate

$$||u||_{C^{1,\alpha}(\overline{Q}_{\frac{1}{2}}^+)} \leq C(||u||_{L^\infty(Q_1^+)} + ||f||_{L^p(Q_1^+)}) \tag{3.2.25}$$

for some $C = C(n, \lambda, \Lambda, b, c, p, ||\beta||_{C^2(\overline{Q}_1^*)}, r_0)$.

Several steps are needed to prove the above theorem. We first establish $W^{1,p}$-regularity for Dirichlet problems, Theorem 3.2.28. Next, we derive $C^{1,\alpha}$-regularity on the flat boundary for the oblique boundary problem (3.2.23). Comparison estimates like Lemma 3.2.26 and 3.2.29 will be utilized to obtain these regularity results.

We now introduce a useful building block in this section. One can find its proof in [18, Theorem 6.1].

Proposition 3.2.25. *For $k \in \mathbb{N}$, let $\Omega_k \subset \Omega_{k+1}$ be an increasing sequence of domains in $\mathbb{R}^n \times \mathbb{R}$ and $\Omega := \cup_{k \geq 1} \Omega_k$. Let $p > n+1$ and F, F_k be continuous and measurable in x and t, and satisfy structure condition (3.2.2). Assume that $f \in L^p(\Omega)$, $f_k \in L^p(\Omega_k)$ and that $u_k \in C(\Omega_k)$ are viscosity subsolutions*

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(supersolutions, respectively) of

$$F_k(D^2 u_k, Du_k, u_k, x, t) - (u_k)_t = f_k \quad in \ \Omega_k.$$

Suppose that $u_k \to u$ locally uniformly in Ω and that for any cylinders $Q_r(x_0, t_0) \subset \Omega$ and $\varphi \in C^2(Q_r(x_0, t_0))$,

$$||(s - s_k)^+||_{L^p(Q_r(x_0,t_0))} \to 0 \qquad \left(||(s - s_k)^-||_{L^p(Q_r(x_0,t_0))} \to 0 \right) \qquad (3.2.26)$$

where

$$s(x, t) = F(D^2\varphi, D\varphi, u, x, t) - f(x, t),$$

$$s_k(x, t) = F(D^2\varphi_k, D\varphi_k, u_k, x, t) - f_k(x, t).$$

Then u is a viscosity subsolution (supersolution) of

$$F(D^2 u, Du, u, x, t) - u_t = f \quad in \ \Omega.$$

$W^{1,p}$-regularity for Dirichlet problems

Before proving Theorem 3.2.24, we need to establish $W^{1,p}$-regularity for Dirichlet boundary problems. For the elliptic case, we refer to [78].

First we introduce a global Hölder estimate. This can be obtained by using the interior regularity [75, Theorem 4.19] and the boundary regularity [76, Theorem 2.5, Theorem 2.17].

Lemma 3.2.26. *Let $n + 1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$ be a C^2-domain. Suppose that $u \in S^*(\lambda, \Lambda, b, f)$ in Ω_T satisfies $u = \varphi$ on $\partial_p \Omega_T$ where $f \in L^p(\Omega_T)$ and $\varphi \in C^{0,\beta}(\partial_p \Omega_T)$ with $\beta \in (0, 1)$. Then $u \in C^{0,\alpha}(\overline{\Omega}_T)$ for some $\alpha = \alpha(n, \lambda, \Lambda, b, p, \beta) \in (0, 1)$ with the estimate*

$$||u||_{C^{0,\alpha}(\overline{\Omega}_T)} \le C(||u||_{L^\infty(\Omega_T)} + ||\varphi||_{C^{0,\beta}(\partial_p \Omega_T)} + ||f||_{L^{n+1}(\Omega_T)}) \qquad (3.2.27)$$

for some $C = C(n, \lambda, \Lambda, b, p, T, \mathrm{diam}(\Omega))$.

Then we can show the following compactness lemma. (See [78, Proposition 3.2] for the elliptic case)

Lemma 3.2.27. *Let $n + 1 < p < \infty$ and $0 \le \nu \le 1$. Assume that F satisfies (3.2.2) with $F(0, 0, 0, x, t) \equiv 0$. Then, for every $\rho > 0$, $\varphi \in C^{0,\gamma}(\partial_p Q_1^\nu)$ with $||\varphi||_{L^\infty(\partial_p Q_1^\nu)} \le C_1$ for some $C_1 > 0$, there exists a positive number*

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$\delta = \delta(\rho, n, \lambda, \Lambda, p, \gamma, C_1) < 1$ *such that if*

$$||\boldsymbol{\psi}((0,0), (\cdot, \cdot))||_{L^p(Q_1^\nu)} + ||f||_{L^p(Q_1^\nu)} + b + c \leq \delta,$$

then for any u and v solving

$$\begin{cases} F(D^2u, Du, u, x, t) - u_t = f & in \ Q_1^\nu, \\ u = \varphi & on \ \partial_p Q_1^\nu, \end{cases}$$

and

$$\begin{cases} F(D^2v, 0, 0, 0, 0) - v_t = 0 & in \ Q_1^\nu, \\ v = \varphi & on \ \partial_p Q_1^\nu, \end{cases}$$

in the viscosity sense, respectively, we have $||u - v||_{L^\infty(Q_1^\nu)} \leq \rho$.

Proof. Suppose not. Then there exists $\rho_0 > 0$ such that if u_k and v_k are viscosity solutions to

$$\begin{cases} F_k(D^2u_k, Du_k, u_k, x, t) - (u_k)_t = f_k & in \ Q_1^{\nu_k}, \\ u_k = \varphi_k & on \ \partial_p Q_1^{\nu_k}, \end{cases}$$

and

$$\begin{cases} F_k(D^2v_k, 0, 0, 0, 0) - (v_k)_t = 0 & in \ Q_1^{\nu_k}, \\ v_k = \varphi_k & on \ \partial_p Q_1^{\nu_k}, \end{cases}$$

respectively, then $||u_k - v_k||_{L^\infty(Q_1^{\nu_k})} > \rho_0$ for every $F_k, f_k, b_k, c_k, \boldsymbol{\psi}_{F_k}$ with

$$||\boldsymbol{\psi}_{F_k}((0,0), (\cdot, \cdot))||_{L^p(Q_1^{\nu_k})}, ||f_k||_{L^p(Q_1^{\nu_k})}, b_k, c_k \leq \delta_k \to 0 \quad as \ k \to \infty$$

and $\varphi_k \in C^{0,\gamma}(\partial_p Q_1^{\nu_k})$ with $||\varphi_k||_{C^{0,\gamma}(\partial_p Q_1^{\nu_k})} \leq C_1$.

Combining Arzelà-Ascoli theorem with (3.2.2), we see that there is a subsequence F_{k_i} and a function F_∞ such that $F_{k_i}(\cdot, \cdot, \cdot, 0, 0)$ converges uniformly to $F_\infty(\cdot)$ on compact subsets of $S(n) \times \mathbb{R}^n \times \mathbb{R}$. Hence, by ABP maximum principle, we have

$$||u_k||_{L^\infty(Q_1^{\nu_k})}$$
$$\leq ||\varphi_k||_{L^\infty(\partial_p Q_1^{\nu_k})} + C(n, \lambda, \Lambda)(||f_k||_{L^{n+1}(Q_1^{\nu_k})} + c_{F_k}||u_k||_{L^\infty(Q_1^{\nu_k})})$$

and $||v_k||_{L^\infty(Q_1^{\nu_k})} \leq ||\varphi_k||_{L^\infty(\partial_p Q_1^{\nu_k})}$. Then for sufficiently large k, we can see

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that $||u_k||_{L^\infty(Q_1^{\nu_k})}, ||v_k||_{L^\infty(Q_1^{\nu_k})} \leq C(C_1)$. Here we can also obtain

$$||u_k||_{C^{0,\alpha}(Q_1^{\nu_k})}, ||v_k||_{C^{0,\alpha}(Q_1^{\nu_k})} \leq C(C_1, n, \lambda, \Lambda, b, p)$$

for some $\alpha = \alpha(n, \lambda, \Lambda, b, p, \gamma)$ by using Lemma 3.2.26.

Assume there is a subsequence $\{\nu_{k_i}\} \subset \{\nu_k\}$ and a number $0 \leq \nu_\infty \leq 1$ such that $\nu_{k_i} \to \nu_\infty$ as $i \to \infty$. It is sufficient to consider the case of monotone subsequences. When $\{\nu_{k_i}\}$ is decreasing, then $Q_1^{\nu_\infty} \subset Q_1^{\nu_k}$ for every i. Thus we can observe that there are functions u_∞, v_∞ such that u_{k_i}, v_{k_i} converge uniformly to u_∞, v_∞ on $Q_1^{\nu_\infty}$ by using Arzelà-Ascoli theorem directly. For increasing subsequences, we consider an extension of φ_k to $(B_1 \cap \{-\nu_\infty \leq x_n \leq -\nu_k\}) \times (-1, 0)$ with

$$||\varphi_k||_{C^{0,\gamma}((B_1 \cap \{-\nu_\infty \leq x_n \leq -\nu_k\}) \times (-1,0))} \leq C_1.$$

Then we can also deduce the uniform convergence for increasing subsequences.

Now we have functions $u_\infty, v_\infty \in C(\overline{Q}_1^{\nu_\infty})$ and $\varphi \in C(\partial_p Q_1^{\nu_\infty})$ such that

$$u_{k_i} \to u_\infty, \ v_{k_i} \to v_\infty \quad \text{uniformly on } \overline{Q}_1^{\nu_\infty}$$

and

$$u_\infty = v_\infty = \varphi_\infty \quad \text{on } \partial_p Q_1^{\nu_\infty}.$$

We first observe that v_∞ solves

$$\begin{cases} F_\infty(D^2 v_\infty, 0, 0, 0, 0) - (v_\infty)_t = 0 & \text{in } Q_1^{\nu_\infty}, \\ v_\infty = \varphi_\infty & \text{on } \partial_p Q_1^{\nu_\infty} \end{cases} \qquad (3.2.28)$$

in the viscosity sense. On the other hand, for u_∞, we see that

$$|F_{k_i}(D^2\phi, D\phi, u_{k_i}, x, t) - f_{k_i}(x, t) - F_\infty(D^2\phi, 0, 0, 0, 0)|$$
$$\leq c_{k_i} C(C_1) + b_{k_i}|D\phi| + \psi_{F_{k_i}}((0,0), (x,t))|D^2\phi|$$
$$+ |f_{k_i}| + |(F_{k_i} - F_\infty)(D^2\phi, 0, 0, 0, 0)|$$

for a test function $\phi \in C^2(Q_1^{\nu_\infty})$. Therefore, we can check that

$$||F_{k_i}(D^2\phi, D\phi, u_{k_i}, x, t) - f_{k_i}(x, t) - F_\infty(D^2\phi, 0, 0, 0, 0)||_{L^p(Q_r(x_0, t_0))} \to 0$$

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as $i \to \infty$ for any $Q_r(x_0, t_0) \subset Q_1^{\nu_\infty}$. Now applying Proposition 3.2.25 to u_∞, we derive that u_∞ is also a solution of (3.2.28) in the viscosity sense. Then we can get a contradiction because (3.2.28) has a unique solution. □

We next consider the following problems

$$\begin{cases} F(D^2u, 0, 0, 0, 0) - u_t = 0 & \text{in } Q_1, \\ u = \varphi_1 & \text{on } \partial_p Q_1 \end{cases} \qquad (3.2.29)$$

and

$$\begin{cases} F(D^2u, 0, 0, 0, 0) - u_t = 0 & \text{in } Q_1^+, \\ u = \varphi_2 & \text{on } \partial_p Q_1^*, \end{cases} \qquad (3.2.30)$$

where F is uniformly elliptic, $\varphi_1 \in C(\partial_p Q_1)$ and $\varphi_2 \in C(\partial_p Q_1^+) \cap C^{1,\gamma}(Q_1^*)$. For (3.2.29), we can find the following estimate in [76, Theorem 4.8]:

$$||u||_{C^{1,\alpha}(\overline{Q}_{\frac{1}{2}})} \leq C||u||_{L^\infty(Q_1)}$$

for some α and C depending only on n, λ and Λ. On the other hand, by using [76, Theorem 2.1], we can also derive the following boundary estimates for (3.2.30)

$$||u||_{C^{1,\alpha}(\overline{Q}_{\frac{1}{2}}^+)} \leq C(||u||_{L^\infty(Q_1^+)} + ||\varphi_2||_{C^{1,\gamma}(Q_1^*)})$$

for some α and C depending only on n, λ and Λ. That is, F satisfies interior and boundary $C^{1,\alpha}$-estimates. Furthermore, we can obtain

$$||u||_{C^{1,\alpha}(\overline{Q}_{\frac{1}{2}}^\nu)} \leq C(||u||_{L^\infty(Q_1^\nu)} + ||\varphi_2||_{C^{1,\gamma}(Q_1^*)}) \qquad (3.2.31)$$

for some α and C depending only on n, λ and Λ by a using proper scaling argument.

Now we prove $W^{1,p}$-regularity for parabolic Dirichlet boundary problems. In the elliptic case, the corresponding result can be found in [78, Theorem 3.1].

Theorem 3.2.28. *Let $n+1 < p < \infty$. Assume that F satisfies the structure condition* (3.2.2) *with $F(0, 0, 0, x, t) = 0$ and u is a viscosity solution of*

$$\begin{cases} F(D^2u, Du, u, x, t) - u_t = f & \text{in } Q_1^+, \\ u = \varphi & \text{on } Q_1^*, \end{cases} \qquad (3.2.32)$$

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where $f \in L^p(Q_1^+) \cap C(\overline{Q}_1^+)$ and $\varphi \in C^{1,\gamma}(\overline{Q}_1^*)$. Then, the followings hold:

(i) *For any $p > n + 2$, there exists a constant $\epsilon_0 = \epsilon_0(n, \lambda, \Lambda, p, \alpha, \overline{\alpha})$ such that if*

$$\left(\fint_{Q_r(x_0,t_0) \cap Q_1^+} \psi((x_0,t_0),(x,t))^p \, dxdt \right)^{1/p} \leq \epsilon_0$$

for any $(x_0, t_0) \in Q_1^+$ and $r \leq r_0$ for some $r_0 > 0$, then $u \in C^{1,\alpha}(\overline{Q}_{\frac{1}{2}}^+)$ with $\alpha < \min\{1 - \frac{n+2}{p}, \overline{\alpha}(1 - \gamma), \gamma\}$ and we have the estimate

$$||u||_{C^{1,\alpha}(\overline{Q}_{\frac{1}{2}}^+)} \leq C(||u||_{L^\infty(Q_1^+)} + ||\varphi||_{C^{1,\gamma}(Q_1^*)} + ||f||_{L^p(Q_1^+)}) \qquad (3.2.33)$$

for some $C = C(n, \lambda, \Lambda, b, c, p, r_0)$.

(ii) *For any $p \leq n + 2$, there exists a constant $\epsilon_0 = \epsilon_0(n, \lambda, \Lambda, p)$ such that if*

$$\left(\fint_{Q_r(x_0,t_0) \cap Q_1^+} \psi((x_0,t_0),(x,t))^p \, dxdt \right)^{1/p} \leq \epsilon_0$$

for any $(x_0, t_0) \in Q_1^+$ and $r \leq r_0$ for some $r_0 > 0$, then $u \in W^{1,q}(B_{\frac{1}{2}}^+)$ for any $q < p_{n+2}^* := (n + 2)p/(n + 2 - p)$ $((n+2)_{n+2}^* := \infty)$ and we have the estimate

$$||u||_{W^{1,q}(Q_{\frac{1}{2}}^+)} \leq C(||u||_{L^\infty(Q_1^+)} + ||\varphi||_{C^{1,\gamma}(Q_1^*)} + ||f||_{L^p(Q_1^+)}) \qquad (3.2.34)$$

for some $C = C(n, \lambda, \Lambda, b, c, p, q, r_0)$.

Proof. We fix $(y, s) \in Q_{\frac{1}{2}}^+$, $n + 1 < p' < p$ and $d = \min\{\frac{1}{2}, r_0\}$. Consider a number $\sigma > 0$ with

$$\sigma \leq \frac{d}{2}, \quad \sigma b \leq \frac{\delta}{32MC(n)}, \quad \sigma^2 c \leq \frac{\delta}{32(M+1)C(n)},$$

where δ is the constant in Lemma 3.2.27, $C(n)$ is a constant and M is to be determined.

We first consider the case $y_n < \sigma/2$. Let

$$K = K(y, s)$$
$$:= ||u||_{L^\infty(Q_d(y,s) \cap Q_1^+)} + ||\varphi||_{C^{1,\gamma}(Q_d(y,s) \cap Q_1^*)}$$

$$+\frac{1}{\epsilon_0}\sup_{r\leq d}\left[r^{1-\alpha}\left(r^{-(n+2)}\int_{Q_r(y,s)\cap Q_1^+}|f(x,t)|^{p'}\,dxdt\right)^{\frac{1}{p'}}\right]$$

for some $0 < \alpha < 1$ and $\epsilon_0 > 0$ to be determined. One can check that

$$K(y,s)\leq ||u||_{L^\infty(Q_1^+)}+||\varphi||_{C^{1,\gamma}(Q_1^*)}+C(n,\epsilon_0)\big[M(f^p)(y,s)\big]^{\frac{1}{p}}<\infty$$

for any (y,s). Now we define

$$\tilde{u}(x,t)=\frac{1}{K}u(\sigma x+y,\sigma^2 t+s),$$

$$\tilde{f}(x,t)=\frac{\sigma^2}{K}f(\sigma x+y,\sigma^2 t+s),$$

$$\tilde{F}(X,q,r,x,t)=\frac{\sigma^2}{K}F(K\sigma^{-2}M,K\sigma^{-1}q,Kr,\sigma x+y,\sigma^2 t+s),$$

$$\tilde{\varphi}(x,t)=\frac{1}{K}\varphi(\sigma x+y,\sigma^2 t+s)$$

and $\nu=y_n/\sigma$. Then $\tilde{u}$ solves the following problem

$$\begin{cases} \tilde{F}(D^2\tilde{u},D\tilde{u},\tilde{u},x,t)-\tilde{u}_t=\tilde{f} & \text{in } Q_2^\nu, \\ \tilde{u}=\tilde{\varphi} & \text{on } Q_2\cap\{x_n=-\nu\} \end{cases}$$

in the viscosity sense. We observe that $\tilde{F}$ satisfies (3.2.2) with $b_{\tilde{F}}=\sigma b$, $c_{\tilde{F}}=\sigma^2 c$ and

$$r^{1-\alpha}\left(r^{-(n+2)}\int_{Q_\nu^r}|\tilde{f}(x,t)|^{p'}\,dxdt\right)^{\frac{1}{p'}}\leq\epsilon_0\sigma^{1+\alpha}$$

for any $r\in(0,2)$. And since

$$\boldsymbol{\psi}_{\tilde{F}}((0,0),(x,t))=\boldsymbol{\psi}_F((y,s),(\sigma x+y,\sigma^2 t+s)),$$

we also obtain

$$||\boldsymbol{\psi}_{\tilde{F}}((0,0),(x,t))||_{L^{p'}(Q_1^\nu)}\leq C(n)\epsilon_0<\delta$$

by taking sufficiently small ϵ_0.

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Next we prove $C^{1,\alpha}$-regularity for the case $p > n + 2$. It is sufficient to show that there exist universal constants $\mu, C_1, C_2, K(C_2) > 0$, $0 < \alpha, \beta < 1$ and functions $l_{k,s}(x) = a_{k,s} + b_{k,s} \cdot x$, $\varphi_k(x,t) = \mu^{-k(1+\alpha)}(\tilde{\varphi} - l_{k,s})(\mu^k x, \mu^{2k} t)$ for each $k \geq -1$ satisfying the followings:

(i) $||\tilde{u} - l_{k,s}||_{L^\infty(Q_\nu^{\mu^k})} \leq \mu^{k(1+\alpha)}$.

(ii) $|a_{k-1,s} - a_{k.s}| + \mu^{k-1}|b_{k-1.s} - b_{k,s}| \leq C_2 \mu^{(k-1)(1+\alpha)}$.

(iii) $\dfrac{|(\tilde{u} - l_{k,s})(\nu^k x_1, \mu^{2k} t_1) - (\tilde{u} - l_{k,s})(\nu^k x_2, \mu^{2k} t_2)|}{(|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}})^\beta} \leq K(C_2) C_1 \mu^{k(1+\alpha)}$

for every $(x_1, t_1), (x_2, t_2) \in Q_1 \cap \{x_n \geq -\nu\}$.

(iv) $||\varphi_k||_{C^{1,\gamma}(Q_1 \cap \{x_n = -\nu\mu^{-k}\})} \leq 4$ if $\nu \leq \mu^k$.

We define $l_{-1,s} = l_{0,s} = 0$ and $C_1 = C(n, \lambda, \Lambda, p)$, $\beta = \alpha(n, \lambda, \Lambda, p)$ where C, α are constants as in Lemma 3.2.26 when it is applied to $\tilde{u} \in S^*(\lambda/n, \Lambda, 1, \tilde{f})$ in Q_2^ν. And we also set $C_2 = 5C(n, \lambda, \Lambda)$ and $\overline{\alpha} = \alpha(n, \lambda, \Lambda)$ are constants in (3.2.31). Now we choose $\alpha < \min\{\overline{\alpha}(1 - \gamma), \gamma\}$ and $\mu \leq 1/4$ such that

$$4C_2(2\mu)^{1+\overline{\alpha}} \leq \mu^{1+\alpha} \text{ and } M = 4C_1 \sum_{i=0}^\infty \left(\frac{1}{4}\right)^{i\alpha} \geq 4C_1 \sum_{i=0}^\infty \mu^{i\alpha}. \qquad (3.2.35)$$

We first check that these conditions are true for $k = 0$. It is immediate that (i) and (ii) hold in this case. We observe that $\tilde{u} \in S^*(\lambda/n, \Lambda, 1, \tilde{f} + \frac{\delta}{32C(n)})$ since $\sigma b \leq 1$, $\sigma^2 c \leq \frac{\delta}{32(M+1)C(n)}$ and $|\tilde{u}| \leq 1$. Applying Lemma 3.2.26, we obtain $||\tilde{u}||_{C^{0,\beta}(Q_1^\nu)} \leq 4C_1$. For (iv), we see that

$$||\varphi_0||_{C^{1,\gamma}(Q_1 \cap \{x_n = -\nu\})} = ||\tilde{\varphi}||_{C^{1,\gamma}(Q_1 \cup \{x_n = -\nu\})} \leq 1.$$

Now we assume that (i)-(iv) are satisfied for $k \geq 0$. We need to show that these conditions still hold for $k + 1$. Set

$$v_k(x,t) = \frac{(\tilde{u} - l_{k,s})(\mu^k x, \mu^{2k} t)}{\mu^{k(1+\alpha)}}.$$

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We observe that v_k solves

$$\begin{cases} F_k(D^2 v_k, Dv_k, v_k, x, t) - (v_k)_t = f_k + g_k & \text{in } Q_2^{\frac{\nu}{\mu^k}}, \\ v_k = \varphi_k & \text{on } Q_2 \cap \{x_n = -\frac{\nu}{\mu^k}\} \end{cases}$$
$$(3.2.36)$$

in the viscosity sense. Here

$$F_k(X, q, r, x, t) = \mu^{k(1-\alpha)} \tilde{F}(\mu^{k(\alpha-1)} X, \mu^{k\alpha} q, \mu^{k(\alpha+1)} r, \mu^k x, \mu^{2k} t),$$

$$\begin{aligned} g_k(x, t) =& F_k(D^2 v_k, Dv_k, v_k, x, t) \\ & - F_k(D^2 v_k, Dv_k + \mu^{-k\alpha} b_{F_k}, v_k + \mu^{-k(1+\alpha)} l_{k,s}(\mu^k x), x, t), \end{aligned}$$

and

$$f_k(x, t) = \mu^{k(1-\alpha)} \tilde{f}(\mu^k x, \mu^{2k} t).$$

We see that

$$\boldsymbol{\psi}_{F_k}((0,0), (x,t)) = \boldsymbol{\psi}_{\tilde{F}}((0,0), (\mu^k x, \mu^{2k} t))$$

and $\tilde{F}$ satisfies (3.2.2) with $b_{F_k} = \mu^k b_{\tilde{F}}$ and $c_{F_k} = \mu^{2k} c_{\tilde{F}}$. On the other hand, we also have

$$\begin{aligned} |g_k(x, t)| =& |F_k(D^2 v_k, Dv_k, v_k, x, t) \\ & - F_k(D^2 v_k, Dv_k + \mu^{-k\alpha} b_{F_k}, v_k + \mu^{-k(1+\alpha)} l_{k,s}(\mu^k x), x, t)| \\ \leq & b_{F_k} \cdot \mu^{-k\alpha} |b_{k,s}| + c_{F_k} \cdot \mu^{-k(\alpha+1)} |l_{k,s}(\mu^k x)| \end{aligned}$$

for any $(x, t) \in Q_1^{\frac{\nu}{\mu^k}}$. Therefore, we obtain $|a_{k,s}|, |b_{k,s}| \leq M/2$ from condition (ii) and (3.2.35), and this implies $||l_k||_{L^\infty(Q_1^{\frac{\nu}{\mu^k}})} \leq M$. Now we have

$$|g_k(x, t)| \leq b_{F_k} \cdot \mu^{-k\alpha} M + c_{F_k} \cdot \mu^{-k(\alpha+1)} M \leq \mu^{k(1-\alpha)} \frac{\delta}{16}$$

and this yields

$$||f_k + g_k||_{L^{p'}(Q_1^{\frac{\nu}{\mu^k}})} \leq ||f_k||_{L^{p'}(Q_1^{\frac{\nu}{\mu^k}})} + ||g_k||_{L^{p'}(Q_1^{\frac{\nu}{\mu^k}})} \leq \frac{\delta}{2} + \frac{\delta}{16} \mu^{k(1-\alpha)} \leq \delta.$$

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Furthermore, we can also observe that

$$||v_k||_{C^{0,\beta}(Q_1 \cap \{x_n \geq \nu\mu^{-k}\})} = ||v_k||_{L^\infty(Q_1 \cap \{x_n \geq \nu\mu^{-k}\})} + [v_k]_{C^{0,\beta}(Q_1 \cap \{x_n \geq \nu\mu^{-k}\})}$$
$$\leq 1 + K(C_2)C_1 =: C_0$$

since we assumed that (i) and (iii) hold.

Next let $h \in C(\overline{Q_{\frac{1}{2}}^{\frac{\nu}{\mu^k}}})$ be a solution of

$$
\begin{cases}
F_k(D^2h, 0, 0, 0, 0) - h_t = 0 & \text{in } Q_1^{\frac{\nu}{\mu^k}}, \\
h = v_k & \text{on } \partial_p Q_1^{\frac{\nu}{\mu^k}},
\end{cases}
\tag{3.2.37}
$$

in the viscosity sense. Then Lemma 3.2.27 leads to

$$||v_k - h||_{L^\infty(Q_1^{\frac{\nu}{\mu^k}})} \leq \rho \tag{3.2.38}$$

for $\rho = C_2(2\mu)^{1+\overline{\alpha}}$. Meanwhile, we can also obtain

$$||h||_{C^{1,\overline{\alpha}}(\overline{Q_{\frac{1}{2}}^{\frac{\nu}{\mu^k}}})} \leq C_2 \tag{3.2.39}$$

from (3.2.31).

Now we define $\bar{l}(x) = h(0,0) + Dh(0,0) \cdot x$. It can be checked without difficulty that

$$||v_k - \bar{l}||_{L^\infty(Q_{2\mu}^{\frac{\nu}{\mu^k}})} \leq ||v_k - h||_{L^\infty(Q_{2\mu}^{\frac{\nu}{\mu^k}})} + ||h - \bar{l}||_{L^\infty(Q_{2\mu}^{\frac{\nu}{\mu^k}})}$$
$$\leq \frac{1}{2}\mu^{1+\alpha} + C_2(2\mu)^{1+\overline{\alpha}}$$
$$\leq \mu^{1+\alpha}$$

and this yields

$$|\tilde{u}(x,t) - l_{k,s}(x) - \mu^{k(1+\alpha)}\bar{l}(\mu^{-k}(x))| \leq \mu^{(k+1)(1+\alpha)}$$

for every $(x,t) \in Q_{2\mu^{k+1}}^{\nu}$. Hence, we see that (i) is satisfied if we set

$$l_{k+1,s}(x) = l_{k,s}(x) + \mu^{k(1+\alpha)}\bar{l}(\mu^{-k}(x)).$$

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Moreover, we also observe that

$$|a_{k,s} - a_{k+1,s}| + \mu^k |b_{k,s} - b_{k+1,s}| = \mu^{k(1+\alpha)}(|h(0,0)| + |Dh(0,0)|)$$
$$\leq \mu^{k(1+\alpha)} ||h||_{C^{1,\bar{\alpha}}(\overline{Q}_{\frac{1}{2}}^{\frac{\nu}{\mu^k}})}$$
$$\leq C_2 \mu^{k(1+\alpha)}$$

and this leads that (ii) also holds for $k+1$.

It remains to show that (iii) and (iv) are still true. For (iv), we need to derive that $||\varphi_{k+1}||_{C^{1,\gamma}(Q_1 \cap \{x_n = -\nu\mu^{-(k+1)}\})} \leq 4$. Since

$$\varphi_{k+1}(x,t) = \frac{(\tilde{\varphi} - l_{k+1,s})(\mu^{k+1}x, \mu^{2(k+1)}t)}{\mu^{(k+1)(1+\alpha)}},$$

we have

$$D\varphi_{k+1}(x,t) = \frac{(D\tilde{\varphi} - Dl_{k+1,s})(\mu^{k+1}x, \mu^{2(k+1)}t)}{\mu^{(k+1)\alpha}}.$$

Recall that we assumed that $||\varphi_k||_{C^{1,\gamma}(Q_1 \cap \{x_n = -\nu\mu^{-k}\})} \leq 4$ and we also deduced that $||\varphi_{k+1}||_{L^\infty(Q_1 \cap \{x_n = -\nu\mu^{-(k+1)}\})} \leq 1$. Then we can deduce that

$$||D\varphi_{k+1}||_{L^\infty(Q_1 \cap \{x_n = -\nu\mu^{-(k+1)}\})}$$
$$= \left|\left| \frac{D\varphi_k(\mu x, \mu^2 t) - Dh(0,0))}{\mu^\alpha} \right|\right|_{L^\infty(Q_1 \cap \{x_n = -\nu\mu^{-(k+1)}\})}$$
$$\leq \left|\left| \frac{D\varphi_k(\mu x, \mu^2 t) - D\varphi_k(0', -\nu/\mu, 0))}{\mu^\alpha} \right|\right|_{L^\infty(Q_1 \cap \{x_n = -\nu\mu^{-(k+1)}\})}$$
$$+ \left|\left| \frac{D\varphi_k(0', -\nu/\mu, 0)) - Dh(0,0))}{\mu^\alpha} \right|\right|_{L^\infty(Q_1 \cap \{x_n = -\nu\mu^{-(k+1)}\})}$$
$$\leq \mu^{-\alpha}(2\mu + C_2\mu^{\bar{\alpha}})$$
$$\leq (2 + C_2)\mu^{\bar{\alpha} - \alpha}$$

and

$$[D\varphi_{k+1}]_{C^{0,\gamma}(Q_1 \cap \{x_n = -\nu\mu^{-(k+1)}\})} \leq \mu^{(k+1)(\gamma - \alpha)}[D\tilde{\varphi}]_{C^{0,\gamma}(Q_1 \cap \{x_n = -\nu\})}$$
$$\leq \mu^{(k+1)(\gamma - \alpha)}.$$

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Now we obtain the following estimate

$$||\varphi_k||_{C^{1,\gamma}(Q_1 \cap \{x_n = -\nu\mu^{-k}\})} \leq 1 + (2 + C_2)\mu^{\bar{\alpha} - \alpha} + 4\mu^{(k+1)(\gamma - \alpha)} \leq 4$$

and thus (iv) is also true.

Finally, we prove that (iii) is satisfied for $k + 1$. We already know that $v_k - \tilde{l} \in S^*(\lambda/n, \Lambda, b_k, f_k + g_k + \delta/8)$ in $Q_{2\mu}^{\frac{\mu}{\nu^k}}$. By Lemma 3.2.26, we have

$$||v_k - \tilde{l}||_{C^{0,\beta}(\overline{Q}_\mu^{\frac{\nu}{\mu^k}})} \leq C_1\mu^{-\beta}(2\mu^{1+\alpha} + 2\delta\mu^{2 - \frac{n+2}{p'}} + \mu^\gamma[\varphi_k - \tilde{l}]_{C^{0,\gamma}(Q_\mu \cap \{x_n = -\frac{\nu}{\mu^k}\})}).$$

Now we choose sufficiently small δ with $2\delta \leq \mu^{\alpha + \frac{n+2}{p'} - 1}$. Since $h = v = \varphi$ on $Q_1 \cap \{x_n = -\frac{\nu}{\mu^k}\}$, we obtain

$$\left|\varphi_k\left(x', -\frac{\nu}{\mu^k}, t\right) - \tilde{l}\left(x', -\frac{\nu}{\mu^k}\right)\right| \leq C_2\left|\left(x', -\frac{\nu}{\mu^k}, t\right)\right|^{1+\bar{\alpha}}.$$

This implies
$$||\varphi_k - \tilde{l}||_{L^\infty(Q_\mu \cap \{x_n = -\frac{\nu}{\mu^k}\})} \leq 4C_2\mu^{1+\bar{\alpha}}$$

if $\nu/\mu^k \leq \mu$. (Note that $Q_\mu \cap \{x_n = -\frac{\nu}{\mu^k}\} = \varnothing$ when $\nu/\mu^k > \mu$) Furthermore, we also derive from the above estimate that

$$|(\varphi_k - \tilde{l})(x_1, t_1) - (\varphi_k - \tilde{l})(x_2, t_2)|$$
$$= |(\varphi_k - \tilde{l})(x_1, t_1) - (\varphi_k - \tilde{l})(x_2, t_2)|^\gamma |(\varphi_k - \tilde{l})(x_1, t_1) - (\varphi_k - \tilde{l})(x_2, t_2)|^{1-\gamma}$$
$$\leq (4 + C_2)^\gamma(|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}})^\gamma \cdot (8C_2)^{1-\gamma}\mu^{(1+\bar{\alpha})(1-\gamma)}$$

for any $(x_1, t_1), (x_2, t_2) \in Q_\mu \cap \{x_n = -\frac{\nu}{\mu^k}\}$. Now we deduce that

$$||v_k - \tilde{l}||_{C^{0,\beta}(\overline{Q}_\mu^{\frac{\nu}{\mu^k}})}$$
$$\leq C_1\mu^{-\beta}(2\mu^{1+\alpha} + 2\delta\mu^{2 - \frac{n+2}{p'}} + \mu^\gamma(4 + C_2)^\gamma(8C_2)^{1-\gamma}\mu^{(1+\bar{\alpha})(1-\gamma)})$$
$$\leq C_1\mu^{-\beta}(3 + (4 + C_2)^\gamma(8C_2)^{1-\gamma})\mu^{1+\alpha}$$

and this leads to

$$|(\tilde{u} - l_{k+1,s})(\mu^{k+1}x_1, \mu^{2(k+1)}t_1) - (\tilde{u} - l_{k+1,s})(\mu^{k+1}x_2, \mu^{2(k+1)}t_2)|$$

$$\leq K(C_2)C_1\mu^{(k+1)(1+\alpha)}(|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}})^\beta$$

with $K(C_2) = 3 + (4 + C_2)^\gamma(8C_2)^{1-\gamma}$. Hence, we can conclude that (i)-(iv) are satisfied for $k + 1$.

Therefore, we can always find a linear function l_s with

$$|l_s(0)|, |Dl_s(0)| \leq CK(y, s) \tag{3.2.40}$$

and

$$||u - l_s||_{L^\infty(Q_r(y,s)\cap Q_1^+)} \leq Cr^{1+\alpha}K(y, s) \tag{3.2.41}$$

for any $(y, s) \in Q_{\frac{1}{2}}^{\frac{\sigma}{2}}$ and sufficiently small $r > 0$.

Next, in the case $y_n \geq \sigma/2$, we can refer to the interior $C^{1,\alpha}$-regularity in [18, Lemma 7.4]. Thus, we get the estimate (3.2.33) in the case $p > n + 2$ with $p' = n + 2$ and $\alpha < 1 - \frac{n+2}{p}$ since

$$K(y, s)$$
$$\leq ||u||_{L^\infty(Q_d(y,s)\cap Q_1^+)} + ||\varphi||_{C^{1,\gamma}(Q_d(y,s)\cap Q_1^*)} + \epsilon_0^{-1}\sup_{r\leq d}\left(r^{1+\alpha-\frac{n+2}{p}}||f||_{L^p(Q_1^+)}\right).$$

Besides, we also see that (3.2.40) and (3.2.41) are also satisfied for almost every $(y, s) \in Q_{\frac{1}{2}}^+$ when $n + 1 < p \leq n + 2$. Then we get

$$\frac{|u(y + x, s + t) - u(y, s)|}{|x| + |t|^{\frac{1}{2}}} \leq CK(y, s)$$

for some $C > 0$ and almost every $(y, s) \in Q_{\frac{1}{2}}^+$ and $(x, t) \in Q_r\backslash\{(0, 0)\}$ such that $(y + x, s + t) \in Q_1^+$. Write

$$I_{(x,t)}(y, s) := \frac{|u(y + x, s + t) - u(y, s)|}{|x| + |t|^{\frac{1}{2}}}.$$

It is straightforward to check that

$$||I_{(x,t)}||_{L^q(Q_{\frac{1}{2}}^+)} \leq C||K(\cdot, \cdot)||_{L^q(Q_{\frac{1}{2}}^+)} \leq C(||u||_{L^\infty(Q_1^+)} + ||\varphi||_{C^{1,\gamma}(Q_1^*)} + J)$$

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for any $q \in (p', (n+2)p'/[n+2-p'(1-\alpha)])$, where

$$J = \left\{ \int_{Q_{\frac{1}{2}}^+} \sup_{r \le d} \left[r^{q(1-\alpha)} \left(r^{-(n+2)} \int_{Q_r(y,s) \cap Q_1^+} |f(x,t)|^{p'} \, dxdt \right)^{\frac{q}{p'}} \right] dyds \right\}^{\frac{1}{q}}.$$

As in the proof of [18, Lemma 7.4], we obtain $J \le C||f||_{L^p(Q_1^+)}$ for some
$C = C(n, p, p')$, and then

$$\sup_{|(x,t)| < r} ||I_{(x,t)}||_{L^q(Q_{\frac{1}{2}}^+)} \le C(||u||_{L^\infty(Q_1^+)} + ||\varphi||_{C^{1,\gamma}(Q_1^*)} + ||f||_{L^p(Q_1^+)})$$

for any $p' \le q < p^*$ with proper p' and α. This leads to the second assertion.
□

Proof of Theorem 3.2.24

In this subsection, we give the proof of Theorem 3.2.24. As we mentioned
before, $W^{1,p}$-regularity for Dirichlet problems is necessary to show Theorem
3.2.24.

Similarly to the previous subsection, we first prove a compactness lemma
for problems with oblique boundary data.

Lemma 3.2.29. *Let $n+1 < p < \infty$ and $0 \le \nu \le 1$. Assume that F satisfies
(3.2.2) with $F(0,0,0,x,t) \equiv 0$ and $\beta \in C^2(Q_2^*)$ with $\beta \cdot \mathbf{n} \ge \delta_0$ for some
$\delta_0 > 0$. Then, for every $\rho > 0$, $\varphi \in C(\partial_p Q_1)$ with $||\varphi||_{L^\infty(\partial_p Q_1)} \le C_1$ for some
$C_1 > 0$ and $g \in C^{0,\alpha}(\overline{Q}_2^*)$ with $0 < \alpha < 1$ and $||g||_{C^{0,\alpha}(\overline{Q}_2^*)} \le C_2$ for some
$C_2 > 0$, there exists a positive number $\delta = \delta(\rho, n, \lambda, \Lambda, \delta_0, p, C_1, C_2) < 1$ such
that if*

$$||\psi((0,0),(\cdot,\cdot))||_{L^p(Q_2^+)} + ||f||_{L^p(Q_2^+)} + b + c \le \delta,$$

then for any u and v solving

$$\begin{cases} F(D^2u, Du, u, x, t) - u_t = f & in \ Q_1^+, \\ u = \varphi & on \ \partial_p Q_1^+ \backslash Q_1^*, \\ \beta \cdot Du = g & on \ Q_1^*, \end{cases}$$

and

$$\begin{cases} F(D^2v, 0, 0, 0, 0) - v_t = 0 & in\ Q_{\frac{3}{4}}^+, \\ v = u & on\ \partial_p Q_{\frac{3}{4}}^+ \backslash Q_{\frac{3}{4}}^*, \\ \beta \cdot Du = g & on\ Q_{\frac{3}{4}}^* \end{cases}$$

in the viscosity sense, respectively, we have $||u - v||_{L^\infty(Q_{\frac{3}{4}}^+)} \leq \rho.$

Proof. Assume that there is a number $\rho_0 > 0$ such that if u_k and v_k solve

$$\begin{cases} F_k(D^2u_k, Du_k, u_k, x, t) - (u_k)_t = f_k & in\ Q_1^+, \\ u_k = \varphi_k & on\ \partial_p Q_1^+ \backslash Q_1^*, \\ \beta \cdot Du_k = g_k & on\ Q_1^*, \end{cases} \tag{3.2.42}$$

and

$$\begin{cases} F_k(D^2v_k, 0, 0, 0, 0) - (v_k)_t = 0 & in\ Q_{\frac{3}{4}}^+, \\ v_k = u_k & on\ \partial_p Q_{\frac{3}{4}}^+ \backslash Q_{\frac{3}{4}}^*, \\ \beta \cdot Dv_k = g_k & on\ Q_{\frac{3}{4}}^* \end{cases} \tag{3.2.43}$$

in the viscosity sense, respectively, then $||u_k - v_k||_{L^\infty(Q_{\frac{3}{4}}^+)} > \rho_0$ for any $F_k, f_k,$ b_k, c_k, ψ_{F_k} with

$$||\psi_{F_k}((0,0), (\cdot, \cdot))||_{L^p(Q_2^+)}, ||f_k||_{L^p(Q_2^+)}, b_k, c_k \leq \delta_k \to 0 \quad as\ k \to \infty.$$

We also assume that $\varphi_k \in C(\partial_p Q_1)$ with $||\varphi_k||_{L^\infty(\partial_p Q_1)} \leq C_1$ and $g_k \in C^{0,\alpha}(\overline{Q}_2^*)$ with $||g_k||_{C^{0,\alpha}(\overline{Q}_2^*)} \leq C_2$ for each k, respectively.

From the structure condition (3.2.2), we can find a subsequence F_{k_i} and a function F_∞ so that $F_{k_i}(\cdot, \cdot, \cdot, 0, 0)$ converges uniformly to $F_\infty(\cdot)$ on compact subsets of $S(n) \times \mathbb{R}^n \times \mathbb{R}$ by using Arzelà-Ascoli theorem. Then for any $\delta_1 \in (0, 1)$ and sufficiently large k, it follows from Lemma 3.2.14 that

$$||u_k||_{L^\infty(Q_1^+)} \leq ||\varphi_k||_{L^\infty(\partial_p Q_1^+ \backslash Q_1^*)}$$
$$+ C(n, \lambda, \Lambda, \delta_0)(||f_k||_{L^{n+1}(Q_1^+)} + ||g_k||_{L^\infty(Q_1^*)} + c_{F_k}||u_k||_{L^\infty(Q_1^+)})$$

and this implies

$$||u_k||_{L^\infty(Q_1^+)} \leq C(C_1, C_2, n, \lambda, \Lambda, \delta_0).$$

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Moreover, applying Lemma 3.2.15 to u_k, we have

$$
\begin{aligned}
&||u_k||_{C^{0,\alpha_1}(Q_{1,\delta_1}^+)} \\
&\leq C(||u_k||_{L^\infty(Q_1^+)} + ||f_k||_{L^{n+1}(Q_1^+)} + ||g_k||_{L^\infty(Q_1^*)})\delta_1^{-\alpha_1} \qquad (3.2.44) \\
&\leq C(C_1, C_2, n, \lambda, \Lambda, \delta_0)\delta_1^{-\alpha_1},
\end{aligned}
$$

where $\alpha_1 \in (0,1)$ only depends on n, λ, Λ and δ_0. Then we obtain

$$
||u_{k_i}||_{C^{0,\alpha_1}(Q_{\frac{7}{8}}^+)} \leq C(C_1, C_2, n, \lambda, \Lambda, \delta_0) \qquad (3.2.45)
$$

from (3.2.44). Therefore, there exists a subsequence $\{u_{k_i}\} \subset \{u_k\}$ and a function u_∞ with u_{k_i} converging uniformly to u_∞ in $Q_{\frac{7}{8}}^+$.

Similarly as in the proof of Lemma 3.2.27, we take a test function $\phi \in C^2(\overline{Q}_{\frac{7}{8}}^+)$. Then we can observe that

$$
\lim_{i \to \infty} ||F_{k_i}(D^2\phi, D\phi, u_{k_i}, x, t) - f_{k_i}(x, t) - F_\infty(D^2\phi, 0, 0, 0, 0)||_{L^p(Q_r(x_0, t_0))} = 0
$$

for any $Q_r(x_0, t_0) \subset Q_{\frac{7}{8}}^+$. Since $\{g_k\} \subset C^{0,\alpha}(\overline{Q}_1^*)$ are uniformly bounded and equicontinuous on Q_1^*, we can find a function $g_\infty \in C^{0,\alpha}(Q_1^*)$ by Arzelà-Ascoli theorem. Thus by Proposition 3.2.25 and [16, Proposition 31], we have

$$
\begin{cases}
F_\infty(D^2 u_\infty, 0, 0, 0, 0) - (u_\infty)_t = 0 & \text{in } Q_{\frac{7}{8}}^+, \\
\beta \cdot Du_\infty = g_\infty & \text{on } Q_{\frac{7}{8}}^*
\end{cases} \qquad (3.2.46)
$$

in the viscosity sense.

Set $w_{k_i} := u_\infty - v_{k_i}$. Then we observe that

$$
\begin{cases}
w_{k_i} \in S(\lambda/n, \Lambda, 0) & \text{in } Q_{\frac{3}{4}}^+, \\
w_{k_i} = u_\infty - u_{k_i} & \text{on } \partial_p Q_{\frac{3}{4}}^+ \backslash Q_{\frac{3}{4}}^*, \\
\beta \cdot Dw_{k_i} = g_\infty - g_{k_i} & \text{on } Q_{\frac{3}{4}}^*
\end{cases} \qquad (3.2.47)
$$

in the viscosity sense by means of Lemma 3.2.19. Applying Lemma 3.2.8 to

w_{k_i}, we get

$$||w_{k_i}||_{L^\infty(Q_{\frac{3}{4}}^+)}$$
$$\leq ||u_\infty - u_{k_i}||_{L^\infty(\partial_p Q_{\frac{3}{4}}^+ \backslash Q_{\frac{3}{4}}^*))} + C(n, \lambda, \Lambda, \delta_0)||g_\infty - g_{k_i}||_{L^\infty(Q_{\frac{3}{4}}^+)} \tag{3.2.48}$$

and this converges to zero as $i \to \infty$. This shows that v_{k_i} converges uniformly to u_∞ on $Q_{\frac{3}{4}}^+$. But it is a contradiction since we already have assumed $||u - v||_{L^\infty(Q_{\frac{3}{4}}^+)} > \rho_0$. $\qquad\square$

Proof of Theorem 3.2.24. We show that $u|_{\overline{Q}_{\frac{2}{3}}^*} \in C^{1,\alpha}(\overline{Q}_{\frac{2}{3}}^*)$ with

$$||u|_{\overline{Q}_{\frac{2}{3}}^*}||_{C^{1,\alpha}(\overline{Q}_{\frac{2}{3}}^*)} \leq C(||u||_{L^\infty(Q_1^+)} + ||f||_{L^{n+1}(Q_1^+)}) \tag{3.2.49}$$

for some $0 < \alpha = \alpha(n, p) < 1$ and $C = C(n, p, \lambda, \Lambda, \delta_0, ||\beta||_{C^2(\overline{Q}_1^*)}) > 0$. Then Theorem 3.2.24 is obtained by using the results of Theorem 3.2.28.

Let $p' \in (n+1, p)$, $(y, s) \in \overline{Q}_{\frac{2}{3}}^*$, $d = \min\{\frac{1}{3}, r_0\}$, and we choose $\sigma > 0$ such that

$$\sigma \leq \frac{d}{2}, \quad \sigma b \leq \frac{\delta}{32MC(n)}, \quad \sigma^2 c \leq \frac{\delta}{32(M+1)C(n)}.$$

Here, δ is the same as in Lemma 3.2.29, $C(n)$ is universal and M will be chosen later.

Define

$$K = K(y, s)$$
$$= ||u||_{L^\infty(Q_d(y,s) \cap Q_1^+)} + \frac{1}{\epsilon_0} \sup_{r \leq d} \left[r^{1-\alpha} \left(r^{-(n+2)} \int_{Q_r(y,s) \cap Q_1^+} |f(x, t)|^{p'} dxdt \right)^{\frac{1}{p'}} \right],$$

where $0 < \alpha < 1$ is to be determined. Observe that for any (y, s),

$$K(y, s) \leq ||u||_{L^\infty(Q_1^+)} + C(n, \epsilon_0) \left[M(f^p)(y, s) \right]^{\frac{1}{p}} < \infty.$$

Let

$$\tilde{u}(x, t) = \frac{1}{K} u(\sigma x + y, \sigma^2 t + s),$$

$$\tilde{f}(x, t) = \frac{\sigma^2}{K} f(\sigma x + y, \sigma^2 t + s),$$

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$$\tilde{\beta}(x,t) = \beta(\sigma x + y, \sigma^2 t + s),$$

$$\tilde{F}(X, q, r, x, t) = \frac{\sigma^2}{K} F(K\sigma^{-2}X, K\sigma^{-1}q, Kr, \sigma x + y, \sigma^2 t + s).$$

Then $\tilde{u}$ is a viscosity solution of

$$\begin{cases} \tilde{F}(D^2\tilde{u}, D\tilde{u}, \tilde{u}, x, t) - \tilde{u}_t = \tilde{f} & \text{in } Q_2^+, \\ \tilde{\beta} \cdot D\tilde{u} = 0 & \text{on } Q_2^*. \end{cases} \tag{3.2.50}$$

We can see that $\tilde{F}$ also satisfies (3.2.2) with $b_{\tilde{F}} = \sigma b$, $c_{\tilde{F}} = \sigma^2 c$,

$$r^{1-\alpha} \left(r^{-(n+2)} \int_{Q_r^+} |\tilde{f}(x,t)|^{p'} \, dxdt \right)^{\frac{1}{p'}} \le \epsilon_0 \sigma^{1+\alpha}$$

for every $r \in (0,2)$. and

$$\|\psi_{\tilde{F}}((0,0),(\cdot,\cdot))\|_{L^{p'}(Q_1^+)} \le \delta$$

for small ϵ_0.

Now we establish $C^{1,\alpha}$-regularity. To this end, we need to show that there are some universal constants $\mu, C_1 > 0$, $0 < \alpha < 1$ and linear functions $l_{k,s}(x) = a_{k,s} + b_{k,s} \cdot x$ for each $k \ge -1$ such that

(i) $\|\tilde{u} - l_{k,s}\|_{L^\infty(Q_{\mu^k}^+)} \le \mu^{k(1+\alpha)}$.

(ii) $|a_{k-1,s} - a_{k,s}| + \mu^{k-1}|b_{k-1,s} - b_{k,s}| \le 2C_1 \mu^{(k-1)(1+\alpha)}$.

(iii) $\beta(0,0) \cdot b_{k,s} = 0$.

Let $l_{-1,s} = l_{0,s} = 0$ and consider a fixed number $\mu \le 1/4$ such that

$$6C_1\|\beta\|_{C^2(\overline{Q_1^*})}\mu^2 \le \mu^{1+\alpha}$$

and

$$M = 4C_1 \sum_{i=0}^{\infty} \left(\frac{1}{4} \right)^{i\alpha} \ge 4C_1 \sum_{i=0}^{\infty} \mu^{i\alpha}. \tag{3.2.51}$$

We use induction to prove that the above conditions are satisfied for every k. It can be checked without difficulty when $k = 0$. Next we show that (i)-(iii)

are still satisfied for $k + 1$ under the assumption that these conditions hold
for $k > 0$.

Let
$$v_k(x, t) = \frac{(\tilde{u} - l_{k,s})(\mu^k x, \mu^{2k} t)}{\mu^{k(1+\alpha)}}.$$

Then v_k is a viscosity solution of

$$\begin{cases} F_k(D^2 v_k, D v_k, v_k, x, t) - (v_k)_t = f_k + g_k & \text{in } Q_2^+, \\ \beta_k \cdot D v_k = -(\beta_k \cdot b_{k,s})/\mu^{k\alpha} & \text{on } Q_2^*, \end{cases} \qquad (3.2.52)$$

where

$$F_k(X, q, r, x, t) = \mu^{k(1-\alpha)} \tilde{F}(\mu^{k(\alpha-1)} X, \mu^{k\alpha} q, \mu^{k(\alpha+1)} r, \mu^k x, \mu^{2k} t),$$

$$\begin{aligned} g_k(x, t) =& F_k(D^2 v_k, D v_k, v_k, x, t) \\ &- F_k(D^2 v_k, D v_k + \mu^{-k\alpha} b_{F_k}, v_k + \mu^{-k(1+\alpha)} l_{k,s}(\mu^k x), x, t), \end{aligned}$$

$$f_k(x, t) = \mu^{k(1-\alpha)} \tilde{f}(\mu^k x, \mu^{2k} t)$$

and

$$\beta_k(x, t) = \beta(\mu^k x, \mu^{2k} t).$$

We remark that $\boldsymbol{\psi}_{F_k}((0,0), (x,t)) = \boldsymbol{\psi}_{\tilde{F}}((0,0), (\mu^k x, \mu^{2k} t))$ and $\tilde{F}$ satisfies
(3.2.2) with $b_{F_k} = \mu^k b_{\tilde{F}}$ and $c_{F_k} = \mu^{2k} c_{\tilde{F}}$.

As in the proof of Theorem 3.2.32, we can observe that

$$|g_k(x, t)| \leq b_{F_k} \cdot \mu^{-k\alpha} M + c_{F_k} \cdot \mu^{-k(\alpha+1)} M \leq \mu^{k(1-\alpha)} \frac{\delta}{16}.$$

This implies

$$||f_k + g_k||_{L^{p'}(B_1^+)} \leq ||f_k||_{L^{p'}(Q_1^+)} + ||g_k||_{L^{p'}(Q_1^+)} \leq \frac{\delta}{2} + \frac{\delta}{16} \mu^{k(1-\alpha)} \leq \delta.$$

On the other hand, we see that $v_k \in S^*(\lambda/n, \Lambda, b_{F_k}, |f_k| + |g_k| + \mu^{2k} c_{\tilde{F}})$.
Note that $b_{F_k} \leq 1$ if k is sufficiently large. Therefore by Lemma 3.2.8,

$$\begin{aligned} ||v_k||_{C^{0,\alpha_0}(Q_1^+)} \leq& ||v_k||_{L^\infty(\partial_p Q_1^+)} + C(n, \lambda, \Lambda, \delta_0)(||f_k||_{L^{n+1}(Q_1^+)} \\ &+ ||g_k||_{L^{n+1}(Q_1^+)} + \mu^{2k} c_{\tilde{F}} + \mu^{-k\alpha} ||\beta_k \cdot b_k||_{L^\infty(Q_1^*)}) \end{aligned}$$

$$\leq 1 + C(n, \lambda, \Lambda, \delta_0)(\delta + \mu^{(1-\alpha)k}|b_{F_k}|)$$
$$\leq C(n, \lambda, \Lambda, \delta_0, C_1)$$

for some $\alpha_0 = \alpha_0(n, \lambda, \Lambda, \delta_0) \in (0, 1)$. Note that we have used $||v_k||_{L^\infty(Q_1^+)} \leq 1$, $\beta \in C^2(\overline{Q}_1^*)$ and $|b_{k,s}| \leq C(C_1)$ to obtain the last inequality.

Now let $h \in C(\overline{Q}_{\frac{7}{8}}^+)$ be a solution of

$$
\begin{cases}
F_k(D^2 h, 0, 0, 0, 0) - h_t = 0 & \text{in } Q_{\frac{7}{8}}^+, \\
h = v_k & \text{on } \partial_p Q_{\frac{7}{8}}^+ \backslash Q_{\frac{7}{8}}^*, \\
\beta_k \cdot Dh = -(\beta_k \cdot b_{k,s})/\mu^{k\alpha} & \text{on } Q_{\frac{7}{8}}^*
\end{cases}
\tag{3.2.53}
$$

in the viscosity sense. Applying Lemma 3.2.16 to h, we see that

$$||h||_{C^2(\overline{Q}_{\frac{3}{4}}^+)} \leq C_*(1 + \mu^{-k\alpha}||\overline{\beta} \cdot b_{k,s}||_{C^2(Q_{\frac{7}{8}}^*)}) \tag{3.2.54}$$

for some $C_* = C_*(n, \lambda, \Lambda, \delta_0, ||\beta||_{C^2(Q_2^*)})$. Set $C_1 = C_*$. Then we see that

$$||h||_{C^2(\overline{Q}_{\frac{3}{4}}^+)} \leq C_1(1 + \mu^{-k\alpha}||\overline{\beta} \cdot b_{k,s}||_{C^2(Q_{\frac{7}{8}}^*)})$$
$$\leq C_1(1 + \mu^{-k\alpha}|b_{k,s}| \cdot \mu^k ||\beta||_{C^2(\overline{Q}_1^*)})$$

since $\beta(0, 0) \cdot b_{k,s} = 0$. Therefore, we have

$$||h||_{C^2(\overline{Q}_{\frac{3}{4}}^+)} \leq C_1(1 + 6C_3\mu^{1-\alpha}||\beta||_{C^2(\overline{Q}_1^*)}) \leq 2C_1.$$

We also have

$$||v_k - h||_{L^\infty(Q_{\frac{3}{4}}^+)} \leq \rho$$

by applying Lemma 3.2.29 to v_k and h with $\rho = 4C_1\mu^2$.

Define $\bar{l}(x) = h(0, 0) + Dh(0, 0) \cdot x$. Then we can obtain

$$||v_k - \bar{l}||_{L^\infty(Q_{2\mu}^+)} \leq ||v_k - h||_{L^\infty(Q_{2\mu}^+)} + ||h - \bar{l}||_{L^\infty(Q_{2\mu}^+)}$$
$$\leq 4C_1\mu^2 + \frac{1}{2}C_1(2\mu)^2$$
$$\leq \mu^{1+\alpha}$$

and this leads to

$$|\tilde{u}(x,t) - l_{k,s}(x) - \mu^{k(1+\alpha)}\bar{l}(\mu^{-k}x)| \le \mu^{(k+1)(1+\alpha)}$$

for any $(x,t) \in Q^+_{2\mu^{k+1}}$. Therefore, we see that the first condition is satisfied
if we set

$$l_{k+1,s}(x) = l_{k,s}(x) + \mu^{k(1+\alpha)}\bar{l}(\mu^{-k}x).$$

Moreover, we also observe that

$$\begin{aligned}
|a_{k,s} - a_{k+1,s}| + \mu^k|b_{k,s} - b_{k+1,s}| &= \mu^{k(1+\alpha)}(|h(0,0)| + |Dh(0,0)|) \\
&\le \mu^{k(1+\alpha)}||h||_{C^2(Q^+_{\frac{3}{4}})} \\
&\le 2C_1\mu^{k(1+\alpha)}.
\end{aligned}$$

Now the condition (ii) is proved. Finally, we can also check that

$$\beta(0,0) \cdot b_{k+1,s} = \beta(0,0) \cdot (b_{k,s} + \mu^{k\alpha}Dh(0,0)) = \beta(0,0) \cdot \mu^{k\alpha}Dh(0,0) = 0$$

since

$$\beta(0,0) \cdot Dh(0,0) = \overline{\beta}(0,0) \cdot Dh(0,0) = -(\overline{\beta}(0,0) \cdot b_{k,s})/\mu^{k\alpha} = 0.$$

Hence, we can always find a linear function l_s with

$$|l_s(0)|, |Dl_s(0)| \le C_1 K(y,s) \tag{3.2.55}$$

and

$$||u - l_s||_{L^\infty(Q_r(y,s) \cap Q^+_1)} \le r^{1+\alpha}K(y,s) \tag{3.2.56}$$

for any $(y,s) \in \overline{Q}^*_{\frac{2}{3}}$ and sufficiently small $r > 0$. This implies $u|_{\overline{Q}^*_{\frac{2}{3}}} \in C^{1,\alpha}(\overline{Q}^*_{\frac{2}{3}})$
with (3.2.49) by choosing $\alpha < 1 - \frac{n+2}{p}$ and $p' = n + 2$. Then we can get the
desired result by employing Theorem 3.2.28. $\qquad\square$

Remark 3.2.30. *We remark that the induction argument in the proof works
well only for $(y,s) \in Q^*_{\frac{2}{3}}$, not for $(y,s) \in Q^+_{\frac{2}{3}}$. One can observe that the
condition that $\beta(0,0) \cdot b_{k,s} = 0$ may break down when $y_n > 0$. On the other
hand, we have used the result of Theorem 3.2.28 in the above proof. Note that
it is required that boundary data $\varphi \in C^{1,\alpha}(Q^*_1)$ for some $0 < \alpha < 1$ in order*

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to employ Theorem 3.2.28. However, for $n+1 < p \leq n+2$, we cannot assert that $u|_{\overline{Q}^*_{\frac{2}{3}}} \in C^{1,\alpha}(\overline{Q}^*_{\frac{2}{3}})$ for any $\alpha \in (0,1)$. Therefore, in this case, we cannot obtain the desired results in this way.

Thanks to Theorem 3.2.24, we get $W^{2,p}$-estimate for viscosity solutions of (3.2.23) directly.

Corollary 3.2.31. *Let $n + 2 < p < \infty$ and u be a viscosity solution of (3.2.23). Then, under the assumption of Theorem 3.2.24, $u \in W^{2,p}(Q^+_{1/4})$ and*

$$||u||_{W^{2,p}(Q^+_{1/4})} \leq C(||u||_{L^\infty(Q^+_1)} + ||f||_{L^p(Q^+_1)})$$

for some $C = C(n, \lambda, \Lambda, b, c, p, ||\beta||_{C^2(\overline{Q}^*_1)})$.

3.2.5 Global estimates

We can establish the global $W^{2,p}$-regularity for (3.2.1) by using Corollary 3.2.31.

Proof of Theorem 3.2.2. First, we get the following interior $W^{2,p}$-estimate

$$||u||_{W^{2,p}(Q)} \leq C(||u||_{L^\infty(Q)} + ||f||_{L^p(Q)})$$

for any $Q \subset\subset \Omega_T$ from [75, Theorem 5.9]. Thus, it is sufficient to consider the boundary case.

We are going to use a flattening argument in order to get a boundary estimate. For any $x_0 \in \partial\Omega$, we can find a neighborhood $N(x_0)$ of x_0 and a C^3-diffeomorphism $\Psi : U(x_0) \to B^+_1$ with $\Psi(x_0) = 0$ since $\partial\Omega$ is C^3. Then for each $t_0 \in (0, T]$, we define $\Psi_{t_0} : U(x_0) \times (t_0 - 1, t_0) \to Q^+_1$ such that

$$\Psi_{t_0}(x, t) = (\Psi(x), t - t_0).$$

Fix $t_0 \in (0, T]$ and let $\tilde{u} = u \circ \Psi_{t_0}^{-1}$. Then $\tilde{u}$ is a solution of

$$\begin{cases} \tilde{F}(D^2\tilde{u}, D\tilde{u}, \tilde{u}, x, t) - \tilde{u}_t = \tilde{f} & \text{in } Q^+_1, \\ \tilde{\beta} \cdot D\tilde{u} = 0 & \text{on } Q^*_1 \end{cases}$$

in the viscosity sense, where $\tilde{f} = f \circ \Psi_{t_0}^{-1}$, $\tilde{\beta} = (\beta \circ \Psi_{t_0}^{-1}) \cdot (D\Psi_{t_0} \circ \Psi_{t_0}^{-1})^t$ and

$$\tilde{F}(D^2\tilde{\varphi}, D\tilde{\varphi}, \tilde{u}, x, t) = F(D^2\varphi, D\varphi, u, x, t) \circ \Psi_{t_0}^{-1}$$

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$$= F(D\Psi_{t_0}^T \circ \Psi_{t_0}^{-1} D^2\tilde\varphi D\Psi_{t_0} \circ \Psi_{t_0}^{-1} + (D\tilde\varphi \partial_{i,j}\Psi_{t_0} \circ \Psi_{t_0}^{-1})_{1\le i,j\le n},$$
$$D\varphi D\Psi_{t_0} \circ \Psi_{t_0}^{-1}, \tilde u, \Psi_{t_0}^{-1}(x,t))$$

for $\tilde\varphi \in W^{2,p}(Q_1^+)$ and $\varphi = \tilde\varphi \circ \Psi_{t_0} \in W^{2,p}(U(x_0) \times (t_0-1, t_0))$. Here, we also note that we extended u by zero when $t < 0$.

Now we can see that there exists a uniform constant $C(\Psi)$ such that

$$\boldsymbol{\psi}_{\tilde F}((x,t),(x_0,t_0)) \le C(\Psi)\boldsymbol{\psi}_F((\Psi^{-1}(x,t)),(\Psi^{-1}(x_0,t_0)))$$

and $\tilde F$ is uniformly elliptic with constants $\lambda C(\Psi)$, $\Lambda C(\Psi)$, see [78]. In addition, we also have $\tilde\beta = (\beta \circ \Psi_{t_0}^{-1}) \cdot (D\Psi_{t_0} \circ \Psi_{t_0}^{-1})^t \in C^2$ since $\Psi, \Psi^{-1} \in C^3$ and $\beta \in C^2(S_T)$. Therefore, we can obtain the boundary estimate, thanks to Corollary 3.2.31 along with a standard covering argument. This completes the proof. $\qquad\square$

Chapter 4

Regularity results for time-dependent tug-of-war games

In this chapter, we establish regularity theory for value functions of time-dependent tug-of-war games introduced in Section 2.2. For the interior case, we show a Lipschitz type estimate, Theorem 4.2.1. After that, we deal with the boundary regularity in Section 4.3. Besides, we also observe the existence and uniqueness, long-time behavior and uniform convergence of game values. We remark that our tug-of-war game is closely linked to the following normalized parabolic p-Laplace equation

$$(n + p)u_t = \Delta_p^N u.$$

In studying value functions of tug-of-war games, it is unavoidable to introduce the following DPP

$$
\begin{aligned}
&u_\epsilon(x, t) \\
&= \frac{1 - \delta_\epsilon(x, t)}{2} \times \\
&\quad \left[\sup_{\nu \in S^{n-1}} \left\{ \alpha u_\epsilon\left(x + \epsilon\nu, t - \frac{\epsilon^2}{2}\right) + \beta \fint_{B_\epsilon^\nu} u_\epsilon\left(x + h, t - \frac{\epsilon^2}{2}\right) d\mathcal{L}^{n-1}(h) \right\} \right. \\
&\quad \left. + \inf_{\nu \in S^{n-1}} \left\{ \alpha u_\epsilon\left(x + \epsilon\nu, t - \frac{\epsilon^2}{2}\right) + \beta \fint_{B_\epsilon^\nu} u_\epsilon\left(x + h, t - \frac{\epsilon^2}{2}\right) d\mathcal{L}^{n-1}(h) \right\} \right] \\
&\quad + \delta_\epsilon(x, t)F(x, t).
\end{aligned}
\tag{4.0.1}
$$

Here, $0 < \alpha, \beta < 1$ with $\alpha + \beta = 1$, $F : \Gamma_{\epsilon,T} \to \mathbb{R}$ is continuous and $\delta_\epsilon \in C(\Omega_{\epsilon,T})$ is defined by

$$\delta_\epsilon(x,t) = \begin{cases} 0 & \text{in } \Omega_T \backslash I_{\epsilon,T}, \\ \min\left\{1, 1 - \frac{\text{dist}(x,\partial\Omega)}{\epsilon}\right\} \times \min\left\{1, 1 - \frac{\sqrt{2t}}{\epsilon}\right\} & \text{in } I_{\epsilon,T}, \\ 1 & \text{in } O_{\epsilon,T}. \end{cases}$$

We recall that

$$I_{\epsilon,T} = \left\{(x,t) \in \Omega \times \left[\frac{\epsilon^2}{2}, T\right] : \text{dist}(x,\partial\Omega) < \epsilon\right\} \cup \left(\Omega \times \left(0, \frac{\epsilon^2}{2}\right)\right),$$

$$O_{\epsilon,T} = \left\{(x,t) \in (\mathbb{R}^n\backslash\overline{\Omega}) \times (0,T] : \text{dist}(x,\partial\Omega) < \epsilon\right\} \cup \left(\Omega_\epsilon \times \left(-\frac{\epsilon^2}{2}, 0\right)\right),$$

$$\Gamma_{\epsilon,T} = I_{\epsilon,T} \cup O_{\epsilon,T} \cup \partial_p\Omega_T$$

and

$$\Omega_{\epsilon,T} = \overline{\Omega}_T \cup O_{\epsilon,T}.$$

As we will see later, (4.0.1) represents the 'law' which value functions satisfy. Therefore, we can consider that this DPP plays a similar role to the 'equation' in PDE theory (in fact, (4.0.1) also includes the boundary condition since it contains the term $\delta_\epsilon(x,t)F(x,t)$.

For convenience, we introduce here a notation. For any $\mathcal{L}^{n-1}$-measurable function f defined on $\Omega_{\epsilon,T}$, we define

$$\mathscr{A}_\epsilon f(x,t;\nu) = \alpha f(x + \epsilon\nu, t) + \beta \fint_{B_\epsilon^\nu} f(x+h,t)d\mathcal{L}^{n-1}(h)$$

for each $(x,t) \in \Omega_T$ and $\nu \in S^{n-1}$. Recall that

$$\operatorname*{midrange}_{i \in I} A_i = \frac{1}{2}\left(\sup_{i \in I} A_i + \inf_{i \in I} A_i\right).$$

Then (4.0.1) can be written as

$$u_\epsilon(x,t) = (1 - \delta_\epsilon(x,t))\operatorname*{midrange}_{\nu \in S^{n-1}} \mathscr{A}_\epsilon u_\epsilon\left(x, t - \frac{\epsilon^2}{2}; \nu\right) + \delta_\epsilon(x,t)F(x,t).$$

We call such a function u_ϵ a *solution* to (4.0.1).

4.1 The existence and uniqueness of game values

Before establishing regularity theory for game values, we investigate the existence and uniqueness of game values for time-dependent tug-of-war games. We first prove the existence and uniqueness of a function satisfying (4.0.1) with continuous boundary data F. After that, we show that this function is the value function of our tug-of-war game.

In (4.0.1), the value of $u_\epsilon(x,t)$ is determined by values of the function in $B_\epsilon(x) \times \{t - \epsilon^2/2\}$. And we also see that this DPP contains integral terms for the function at time $t - \epsilon^2/2$. Thus, we have to consider the measurability for the function u_ϵ, more precisely, for strategies of our game. In general, existence of measurable strategies is not guaranteed (for example, see [52, Example 2.4]). But we can avoid this problem under our setting. The function δ_ϵ plays an important role in this issue.

We begin this section by observing a basic property of the operator $\mathscr{A}_\epsilon$.

Proposition 4.1.1. *Let $u \in C(\overline{\Omega}_{\epsilon,T})$. Then $\mathscr{A}_\epsilon u(x,t;\nu)$ is continuous with respect to each variable in $\overline{\Omega}_{T,\epsilon} \times \partial B_\epsilon(0)$.*

Proof. For any $(x,t),(y,s) \in \Omega_T$, let us define a parabolic distance by

$$d((x,t),(y,s)) = |x - y| + |t - s|^{1/2}.$$

We write the modulus of continuity of a function f with respect to the distance d by ω_f.

For fixed $|\nu| = \epsilon$, we can see that for any $x, y \in \overline{\Omega}$,

$$\left| \alpha u\left(x + \epsilon\nu, t - \frac{\epsilon^2}{2}\right) - \alpha u\left(y + \epsilon\nu, t - \frac{\epsilon^2}{2}\right) \right| \leq \alpha\omega_u(|x - y|)$$

and

$$\left| \beta \fint_{B_\epsilon^\nu} u\left(x + h, t - \frac{\epsilon^2}{2}\right) d\mathcal{L}^{n-1}(h) - \beta \fint_{B_\epsilon^\nu} u\left(y + h, t - \frac{\epsilon^2}{2}\right) d\mathcal{L}^{n-1}(h) \right|$$
$$\leq \beta \fint_{B_\epsilon^\nu} \left| u\left(x + h, t - \frac{\epsilon^2}{2}\right) - u\left(y + h, t - \frac{\epsilon^2}{2}\right) \right| d\mathcal{L}^{n-1}(h)$$

$$\leq \beta \omega_u(|x - y|).$$

Thus, we get

$$|\mathscr{A}_\epsilon u(x, t; \nu) - \mathscr{A}_\epsilon u(y, t; \nu)| \leq \omega_u(|x - y|).$$

Next, for any $t, s > 0$, we also calculate that

$$\left| \alpha u\left(x + \epsilon \nu, t - \frac{\epsilon^2}{2}\right) - \alpha u\left(x + \epsilon \nu, s - \frac{\epsilon^2}{2}\right) \right| \leq \alpha \omega_u(|t - s|^{1/2}),$$

$$\left| \beta \fint_{B_\epsilon^\nu} u\left(x + h, t - \frac{\epsilon^2}{2}\right) d\mathcal{L}^{n-1}(h) - \beta \fint_{B_\epsilon^\nu} u\left(x + h, s - \frac{\epsilon^2}{2}\right) d\mathcal{L}^{n-1}(h) \right|$$
$$\leq \beta \fint_{B_\epsilon^\nu} \left| u\left(x + h, t - \frac{\epsilon^2}{2}\right) - u\left(x + h, s - \frac{\epsilon^2}{2}\right) \right| d\mathcal{L}^{n-1}(h)$$
$$\leq \beta \omega_u(|t - s|^{1/2})$$

and hence

$$|\mathscr{A}_\epsilon u(x, t; \nu) - \mathscr{A}_\epsilon u(x, s; \nu)| \leq \omega_u(|t - s|^{1/2}).$$

Finally, for any $\nu, \chi \in S^{n-1}$,

$$W(x, t, \nu) - W(x, t, \chi)$$
$$= \alpha \left[u\left(x + \epsilon \nu, t - \frac{\epsilon^2}{2}\right) - u\left(x + \epsilon \chi, t - \frac{\epsilon^2}{2}\right) \right]$$
$$+ \beta \left[\fint_{B_\epsilon^\nu} u\left(x + h, t - \frac{\epsilon^2}{2}\right) d\mathcal{L}^{n-1}(h) - \fint_{B_\epsilon^\chi} u\left(x + h, t - \frac{\epsilon^2}{2}\right) d\mathcal{L}^{n-1}(h) \right].$$

Combining the above results, we see that

$$|W(x, t, \nu) - W(x, t, \chi)|$$
$$\leq \alpha \epsilon \omega_u(\epsilon|\nu - \chi|)$$
$$+ \beta \fint_{B_\epsilon^\nu} \left| u\left(x + h, t - \frac{\epsilon^2}{2}\right) - u\left(x + Ph, t - \frac{\epsilon^2}{2}\right) \right| d\mathcal{L}^{n-1}(h)$$

where $P : \nu^\perp \to \chi^\perp$ is a rotation satisfying $|h - Ph| \leq C|h||\nu - \chi|$. Here, we

check that

$$\fint_{B_\epsilon^\nu} \left| u\left(x+h, t-\frac{\epsilon^2}{2}\right) - u\left(x+Ph, t-\frac{\epsilon^2}{2}\right) \right| d\mathcal{L}^{n-1}(h) \le \omega_u(|h-Ph|)$$

$$\le \omega_u(C\epsilon|\nu-\chi|).$$

Therefore, we obtain

$$|\mathscr{A}_\epsilon u_\epsilon(x,t;\nu) - \mathscr{A}_\epsilon u_\epsilon(x,t;\chi)| \le \omega_u(C\epsilon|\nu-\chi|).$$

Now we can conclude the proof to combine above results. $\qquad\square$

For convenience, we write that

$$Tu(x,t) = (1-\delta_\epsilon(x,t)) \operatorname*{midrange}_{\nu\in S^{n-1}} \mathscr{A}_\epsilon u\left(x, t-\frac{\epsilon^2}{2}; \nu\right) + \delta_\epsilon(x,t)F(x,t).$$

$$(4.1.1)$$

Next we observe that the operator T preserves continuity and monotonicity.

Lemma 4.1.2. *For any $u \in C(\overline{\Omega}_{\epsilon,T})$, Tu is also in $C(\overline{\Omega}_{\epsilon,T})$. Furthermore, for any $u, v \in C(\overline{\Omega}_{\epsilon,T})$ with $u \le v$, it holds that*

$$Tu \le Tv.$$

Proof. By the definition of T, we can check that $u \le v$ implies $Tu \le Tv$ without difficulty.

Next we need to show that $Tu \in C(\overline{\Omega}_{\epsilon,T})$ if $u \in C(\overline{\Omega}_{\epsilon,T})$. When $(x,t) \in \overline{O}_{\epsilon,T}$, we see that $Tu = u = F \in C(\overline{O}_{\epsilon,T})$ by assumption. We need to consider the case of $\overline{I}_{\epsilon,T}$ and $\Omega_T \backslash I_{\epsilon,T}$.

First assume that $(x,t), (y,s) \in \Omega_T \backslash I_{\epsilon,T}$. Observe that

$$\left| \operatorname*{midrange}_{\nu\in S^{n-1}} \mathscr{A}_\epsilon u(x,t;\nu) - \operatorname*{midrange}_{\nu\in S^{n-1}} \mathscr{A}_\epsilon u(y,s;\nu) \right|$$

$$\le \frac{1}{2} \left| \sup_{\nu\in S^{n-1}} \mathscr{A}_\epsilon u(x,t;\nu) - \sup_{\nu\in S^{n-1}} \mathscr{A}_\epsilon u(y,s;\nu) \right|$$

$$+ \frac{1}{2} \left| \inf_{\nu\in S^{n-1}} \mathscr{A}_\epsilon u(x,t;\nu) - \inf_{\nu\in S^{n-1}} \mathscr{A}_\epsilon u(y,s;\nu) \right|.$$

Since

$$\left| \sup_{\nu \in S^{n-1}} \mathscr{A}_\epsilon u(x,t;\nu) - \sup_{\nu \in S^{n-1}} \mathscr{A}_\epsilon u(y,s;\nu) \right| \le \sup_{\nu \in S^{n-1}} \left| \mathscr{A}_\epsilon u(x,t;\nu) - \mathscr{A}_\epsilon u(y,s;\nu) \right|$$

and

$$\left| \inf_{\nu \in S^{n-1}} \mathscr{A}_\epsilon u(x,t;\nu) - \inf_{\nu \in S^{n-1}} \mathscr{A}_\epsilon u(y,s;\nu) \right| \le \sup_{\nu \in S^{n-1}} \left| \mathscr{A}_\epsilon u(x,t;\nu) - \mathscr{A}_\epsilon u(y,s;\nu) \right|,$$

we get

$$\left| \operatorname*{midrange}_{\nu \in S^{n-1}} \mathscr{A}_\epsilon u(x,t;\nu) - \operatorname*{midrange}_{\nu \in S^{n-1}} \mathscr{A}_\epsilon u(y,s;\nu) \right|$$
$$\le \sup_{\nu \in S^{n-1}} \left| \mathscr{A}_\epsilon u(x,t;\nu) - \mathscr{A}_\epsilon u(y,s;\nu) \right|$$
$$\le \omega_u(d((x,t),(y,s))).$$

We used the result of Proposition 4.1.1 in the last inequality. Thus, Tu is also continuous in $\Omega_T \backslash I_{\epsilon,T}$.

When $(x,t),(y,s) \in \overline{I}_{\epsilon,T}$,

$$\left| (1-\delta_\epsilon(x,t)) \operatorname*{midrange}_{\nu \in S^{n-1}} \mathscr{A}_\epsilon u(x,t;\nu) - (1-\delta_\epsilon(y,s)) \operatorname*{midrange}_{\nu \in S^{n-1}} \mathscr{A}_\epsilon u(y,s;\nu) \right|$$
$$\le (1-\delta_\epsilon(x,t)) \left| \operatorname*{midrange}_{\nu \in S^{n-1}} \mathscr{A}_\epsilon u(x,t;\nu) - \operatorname*{midrange}_{\nu \in S^{n-1}} \mathscr{A}_\epsilon u(y,s;\nu) \right|$$
$$+ \left| \delta_\epsilon(x,t) - \delta_\epsilon(y,s) \right| \cdot \left| \operatorname*{midrange}_{\nu \in S^{n-1}} \mathscr{A}_\epsilon u(y,s;\nu) \right|$$
$$\le \omega_u(d((x,t),(y,s))) + \frac{3}{\epsilon} \|u\|_{L^\infty(\overline{\Omega}_{\epsilon,T})} d((x,t),(y,s))$$

because

$$|\delta_\epsilon(x,t) - \delta_\epsilon(y,s)| \le \frac{3}{\epsilon} d((x,t),(y,s)).$$

Similarly, we can also calculate

$$|\delta_\epsilon(x,t)F(x,t) - \delta_\epsilon(y,s)F(y,s)|$$
$$\le \omega_F(d((x,t),(y,s))) + \frac{3}{\epsilon} \|F\|_{L^\infty(\overline{\Gamma}_{\epsilon,T})} d((x,t),(y,s)).$$

Combining above results, we obtain the continuity of Tu in $I_{\epsilon,T}$.

Finally, we need to check the coincidence of the function value on $\partial I_{\epsilon,T}$. Observe that $\partial I_{\epsilon,T}$ can be decomposed by two disjoint connected sets $\partial_p \Omega_T$ and $\partial_p(\Omega_T \backslash I_{\epsilon,T})$ of $\mathbb{R}^n \times \mathbb{R}$. Then we can observe that

$$\lim_{O_{\epsilon,T} \ni (y,s) \to (x,t)} Tu(y,s) = \lim_{I_{\epsilon,T} \ni (y,s) \to (x,t)} Tu(y,s) = Tu(x,t)$$

for any $(x,t) \in \partial_p \Omega_T$ and

$$\lim_{\Omega_T \backslash I_{\epsilon,T} \ni (y,s) \to (x,t)} Tu(y,s) = \lim_{I_{\epsilon,T} \ni (y,s) \to (x,t)} Tu(y,s) = Tu(x,t)$$

for any $(x,t) \in \partial_p(\Omega_T \backslash I_{\epsilon,T})$ by using the above calculation. Thus we obtain the continuity of Tu and the proof is finished. $\qquad\square$

Since T preserves continuity, we do not need to worry about the measurability issue. Therefore, for any continuous function u, Tu is well-defined at every point in Ω_T.

Now we can obtain the existence and uniqueness of these functions.

Theorem 4.1.3. *Let $F \in C(\Gamma_{\epsilon,T})$. Then the bounded function u_ϵ satisfying* (4.0.1) *with boundary data F exists and is unique.*

Proof. We get the desired result via an argument similar to the proof of [52, Theorem 5.2]. We can see the existence of these functions without difficulty since the operator T is well-defined inductively for any continuous boundary data.

For uniqueness, consider two functions u and v satisfying $Tu = u$, $Tv = v$ with boundary data F. We see that $u(\cdot,t) = v(\cdot,t)$ when $0 < t \le \epsilon^2/2$ by definition of T. Then we can also get the same result when $\epsilon^2/2 < t \le \epsilon^2$ because past data of u and v still coincide. Repeating this process, we obtain $u(x,t) = v(x,t)$ for any $(x,t) \in \Omega_T$ and hence the uniqueness is proved. $\quad\square$

We look into the relation between functions satisfying (4.0.1) and values for parabolic tug-of-war games here.

Theorem 4.1.4. *The value functions of tug-of-war game with noise u_I and u_{II} with payoff function F coincide with the function u_ϵ.*

Proof. We need to deduce that

$$u_\epsilon \le u_I \qquad \text{and} \qquad u_{II} \le u_\epsilon$$

since $u_{\mathrm{I}} \leq u_{\mathrm{II}}$ by the definition of value functions.

First, we show the latter inequality. Let $Z_0 \in \Omega_T$ and denote by S_{II}^0 a strategy for Player II such that

$$\mathscr{A}_\epsilon u_\epsilon(Z_j; \nu_j^{\mathrm{II}}) = \inf_{\nu \in S^{n-1}} \mathscr{A}_\epsilon u_\epsilon(Z_j; \nu)$$

for $j \geq 0$. Note that this S_{II}^0 exists since $\mathscr{A}_\epsilon u_\epsilon$ is continuous on ν by Proposition 4.1.1. Measurability of such strategies can be shown by using [70, Theorem 5.3.1].

Next we fix an arbitrary strategy S_I for Player I. Define

$$\Phi(c, x, t) = \left\{ \begin{array}{ll} u_\epsilon(x, t) & \text{when } c = 0, \\ F(x, t) & \text{when } c = 1. \end{array} \right.$$

for any $(x, t) \in \Omega_{\epsilon, T}$. Then we have

$$\mathbb{E}_{S_{\mathrm{I}}, S_{\mathrm{II}}^0}^{Z_0}[\Phi(c_{j+1}, Z_{j+1})|(c_0, Z_0), \cdots, (c_j, Z_j)]$$
$$\leq \frac{1 - \delta_\epsilon(Z_j)}{2}\left[\mathscr{A}_\epsilon u_\epsilon(x_j, t_{j+1}; \nu_{j+1}^{\mathrm{I}}) + \mathscr{A}_\epsilon u_\epsilon(x_j, t_{j+1}; \nu_{j+1}^{\mathrm{II}})\right] + \delta_\epsilon(Z_j)F(Z_j)$$
$$\leq (1 - \delta_\epsilon(Z_j))\operatorname*{midrange}_{\nu \in S^{n-1}} \mathscr{A}_\epsilon u_\epsilon(x_j, t_{j+1}; \nu) + \delta_\epsilon(Z_j)F(Z_j)$$
$$= \Phi(c_j, Z_j).$$

Hence, we can see that $M_k = \Phi(c_k, Z_k)$ is a supermartingale in this case. Since the game ends in finite steps, we can obtain

$$u_{\mathrm{II}}(Z_0) = \inf_{S_{\mathrm{II}}} \sup_{S_{\mathrm{I}}} \mathbb{E}_{S_{\mathrm{I}}, S_{\mathrm{II}}}^{Z_0}[F(Z_\tau)] \leq \sup_{S_{\mathrm{I}}} \mathbb{E}_{S_{\mathrm{I}}, S_{\mathrm{II}}^0}^{Z_0}[F(Z_\tau)]$$
$$= \sup_{S_{\mathrm{I}}} \mathbb{E}_{S_{\mathrm{I}}, S_{\mathrm{II}}^0}^{Z_0}[\Phi(c_{\tau+1}, Z_{\tau+1})] \leq \sup_{S_{\mathrm{I}}} \mathbb{E}_{S_{\mathrm{I}}, S_{\mathrm{II}}^0}^{Z_0}[\Phi(c_0, Z_0)]$$
$$= u_\epsilon(Z_0)$$

by using optional stopping theorem.

Now we can also derive that $u_\epsilon \leq u_{\mathrm{I}}$ by using a similar argument. Then we get the desired result. $\qquad \square$

4.2 Interior estimates

In this section, we prove interior Lipschitz regularity of game values. Since we showed the relation between game values and (4.0.1) in the previous section, we only need to investigate the properties of functions satisfying this DPP.

We first state our main theorem.

Theorem 4.2.1. *Let* $\bar{Q}_{2r} \subset \Omega_T \backslash I_{\epsilon,T}$, $0 < \alpha < 1$ *and* $\epsilon > 0$ *be small. Suppose that* u_ϵ *satisfies* (4.0.1) *with boundary data* $F \in L^\infty(\Gamma_{\epsilon,T})$. *Then for any* $x, z \in B_r(0)$ *and* $-r^2 < t, s < 0$,

$$|u_\epsilon(x,t) - u_\epsilon(z,s)| \leq C||F||_{L^\infty(\Gamma_{\epsilon,T})}(|x - z| + |s - t|^{\frac{1}{2}} + \epsilon),$$

where $C > 0$ *is a constant which only depends on* r, α *and* n.

The proof of Theorem 4.2.1 is divided into two parts. In the first part, we provide an estimate for the function u_ϵ with respect to t. Precisely, it shows a relation between the oscillation of u_ϵ in time direction and the oscillation in spatial direction. Next, we concentrate on proving regularity results with respect to x. We first obtain Hölder type estimate and then turn to Lipschitz estimate. Comparison arguments play a key role in the proof of the main theorem.

4.2.1 Regularity with respect to time

In this subsection, we investigate regularity for the value function u_ϵ with respect to t. The aim of this section is to prove Lemma 4.2.2 below. This lemma provides some information about a relation between the oscillation in a time slice and that in the whole cylinder.

We use a comparison argument in the proof of the lemma. We will first find an appropriate function $\bar{v}$ ($\underline{v}$, respectively) which plays a similar role as a supersolution (subsolution, respectively) in PDE theory. After that, we will deduce the desired result by estimating the difference of those functions. The method used here is motivated by [35, Lemma 4.3]. Our proof may be regarded as a discrete version of this lemma.

From now on, we fix $0 < r < 1$ and $T > 0$. Since we only consider interior regularity, it is sufficient to show the regularity result in a cylinder Q_r with proper translation. We still use the notation Ω_T after the translation.

Lemma 4.2.2. *Let* $\bar{Q}_{2r} \subset \Omega_T \backslash I_{\epsilon,T}$, $-r^2 < s < t < 0$ *and* u_ϵ *satisfies* (4.0.1) *with boundary data* $F \in L^\infty(\Gamma_{\epsilon,T})$ *for given* $0 < \alpha < 1$. *Then, for given* $\epsilon > 0$, u_ϵ *satisfies the estimate*

$$|u_\epsilon(x,t) - u_\epsilon(x,s)| \leq 18 \sup_{-r^2 < \tau < 0} \operatorname*{osc}_{\Lambda_{\tau,\epsilon}} u_\epsilon$$

for any $x \in B_r$.

Proof. We set

$$A = \sup_{-r^2 < \tau < 0} \operatorname*{osc}_{\Lambda_{\tau,\epsilon}} u_\epsilon$$

and

$$\bar{v}_c(x,t) = c + 7r^{-2}At + 2r^{-2}A|x|^2,$$

where $c \in \mathbb{R}$. Define

$$\bar{c} = \inf\{c \in \mathbb{R} : \bar{v}_c \geq u_\epsilon \text{ in } \Lambda_{-r^2,\epsilon}\}$$

and we write $\bar{v} = \bar{v}_{\bar{c}}$. Then for any $\eta > 0$, we can always choose $(x_\eta, t_\eta) \in \Lambda_{-r^2,\epsilon}$ so that

$$u_\epsilon(x_\eta, t_\eta) \geq \bar{v}(x_\eta, t_\eta) - \eta.$$

In this case, there would be some accumulation points $(\bar{x}, \bar{t}) \in \bar{\Lambda}_{-r^2,\epsilon}$ as $\eta \to 0$. Furthermore, $\bar{x}$ must satisfy $|\bar{x}| \leq r$, since if not,

$$2A \leq \bar{v}(x_\eta, t_\eta) - \bar{v}(0, t_\eta) \leq u_\epsilon(x_\eta, t_\eta) - u_\epsilon(0, t_\eta) + \eta \leq A + \eta$$

for any $\eta > 0$, then it is a contradiction when $A > 0$.

Now we compare $\operatorname{midrange}_{\nu \in S^{n-1}} \mathscr{A} \bar{v}(x, \nu, t - \epsilon^2/2)$ with $\bar{v}(x,t)$. First, observe that

$$\operatorname*{midrange}_{\nu \in S^{n-1}} \mathscr{A}\bar{v}\left(x, \nu, t - \frac{\epsilon^2}{2}\right)$$

$$\leq \alpha \operatorname*{midrange}_{\nu \in S^{n-1}} \bar{v}\left(x + \epsilon\nu, t - \frac{\epsilon^2}{2}\right) + \beta \sup_{\nu \in S^{n-1}} \fint_{B_\epsilon^{e_1}} \bar{v}\left(x + P_\nu h, t - \frac{\epsilon^2}{2}\right) d\mathcal{L}^{n-1}(h)$$

for some $P_\nu \in \mathbf{R}_\nu$. We see that

$$\fint_{B_\epsilon^{e_1}} |x + P_\nu h|^2 d\mathcal{L}^{n-1}(h) = \fint_{B_\epsilon^\nu} (|x|^2 + 2\langle x, P_\nu h\rangle + |P_\nu h|^2) d\mathcal{L}^{n-1}(h)$$

$$\leq |x|^2 + \epsilon^2$$

for any $\nu \in S^{n-1}$. Next we need to show that

$$\operatorname*{midrange}_{\nu \in S^{n-1}} |x + \epsilon \nu|^2 \leq |x|^2 + \epsilon^2.$$

Observe that

$$\begin{aligned}
\sup_{\kappa \in B_\epsilon} |x + \kappa|^2 &= \sup_{\nu \in S^{n-1}} \sup_{-\epsilon \leq a \leq \epsilon} |x + a\nu|^2 \\
&= \sup_{\nu \in S^{n-1}} \sup_{-\epsilon \leq a \leq \epsilon} (a^2 + 2a\langle x, \nu \rangle + |x|^2).
\end{aligned}$$

Since $a^2 + 2a\langle x, \nu \rangle + |x|^2$ is convex in a, we observe that

$$\sup_{-\epsilon \leq a \leq \epsilon} (a^2 + 2a\langle x, \nu \rangle + |x|^2) = \epsilon^2 + 2\epsilon|\langle x, \nu \rangle| + |x|^2.$$

We also see that there is a unit vector μ so that

$$\sup_{\nu \in S^{n-1}} (\epsilon^2 + 2\epsilon|\langle x, \nu \rangle| + |x|^2) = |x + \epsilon\mu|^2,$$

as S^{n-1} is compact. Then we get

$$\operatorname*{midrange}_{\nu \in S^{n-1}} |x + \epsilon \nu|^2 \leq \frac{1}{2}(|x + \epsilon\mu|^2 + |x - \epsilon\mu|^2) = |x|^2 + \epsilon^2.$$

Therefore, we discover

$$\begin{aligned}
\operatorname*{midrange}_{\nu \in S^{n-1}} &\mathscr{A}\bar{v}\left(x, \nu, t - \frac{\epsilon^2}{2}\right) \\
&\leq \bar{c} + 7r^{-2}A\left(t - \frac{\epsilon^2}{2}\right) + 2r^{-2}A\{\alpha(|x|^2 + \epsilon^2) + \beta(|x|^2 + \epsilon^2)\} \\
&\leq \bar{c} + 7r^{-2}At + 2r^{-2}A|x|^2 - \frac{3}{2}r^{-2}A\epsilon^2 = \bar{v}(x, t) - \frac{3}{2}r^{-2}A\epsilon^2.
\end{aligned}$$

Thus,

$$\operatorname*{midrange}_{\nu \in S^{n-1}} \mathscr{A}\bar{v}\left(x, \nu, t - \frac{\epsilon^2}{2}\right) \leq \bar{v}(x, t) \tag{4.2.1}$$

for all $(x, t) \in Q_r$.

Let $M = \sup_{Q_{r,\epsilon} \setminus \Lambda_{-r^2,\epsilon}} (u_\epsilon - \bar{v})$ and suppose $M > 0$. In this case, we see that $u_\epsilon \leq \bar{v} + M$ in $Q_{r,\epsilon}$. For any $\eta' > 0$, we can choose a point $(x_{\eta'}, t_{\eta'}) \in Q_{r,\epsilon}$ such that

$$u_\epsilon(x_{\eta'}, t_{\eta'}) > \bar{v}(x_{\eta'}, t_{\eta'}) + M - \eta'.$$

We have to show that $(x_{\eta'}, t_{\eta'})$ must be in $\bar{Q}_r$ for any sufficiently small $\eta' > 0$. By the definition of M, $t_{\eta'} > -r^2$. Note that we cannot assert this when $M \leq 0$. On the other hand, for any $|x| \geq r$,

$$\bar{v}(x, t) - \bar{v}(0, t) \geq 2A.$$

We also observe that $u_\epsilon(x, t) - u_\epsilon(0, t) \leq A$. Hence it is always true that

$$(u_\epsilon - \bar{v})(x, t) \leq (u_\epsilon - \bar{v})(0, t).$$

Thus, $(x_{\eta'}, t_{\eta'}) \in \bar{Q}_r$. Then we obtain that

$$\operatorname*{midrange}_{\nu \in S^{n-1}} \mathscr{A} \left\{ \bar{v}\left(x_{\eta'}, \nu, t_{\eta'} - \frac{\epsilon^2}{2}\right) + M \right\} \geq \operatorname*{midrange}_{\nu \in S^{n-1}} \mathscr{A} u_\epsilon \left(x_{\eta'}, \nu, t_{\eta'} - \frac{\epsilon^2}{2}\right)$$
$$= u_\epsilon(x_{\eta'}, t_{\eta'})$$
$$> \bar{v}(x_{\eta'}, t_{\eta'}) + M - \eta'.$$

In the first inequality, we have used that $\bar{v} + M \geq u_\epsilon$ in $Q_{r,\epsilon}$. Therefore,

$$\operatorname*{midrange}_{\nu \in S^{n-1}} \mathscr{A} \bar{v}\left(x_{\eta'}, \nu, t_{\eta'} - \frac{\epsilon^2}{2}\right) > \bar{v}(x_{\eta'}, t_{\eta'}) - \eta' \tag{4.2.2}$$

for any $\eta' > 0$. We combine (4.2.1) with (4.2.2) to discover that $A = 0$, and so $\bar{v} = u_\epsilon = \bar{c}$. If u_ϵ is not a constant function, then we have a contradiction to $A > 0$. Hence $M \leq 0$ and therefore $u_\epsilon \leq \bar{v}$ in $Q_{r,\epsilon}$.

On the other hand, consider

$$\underline{v}(x, t) = \underline{c} - 7r^{-2}At - 2r^{-2}A|x|^2,$$

where

$$\underline{c} = \sup\{c \in \mathbb{R} : \underline{v}_c \leq u_\epsilon \text{ in } \Lambda_{-r^2,\epsilon}\}.$$

Following the above procedure, we can show that $u_\epsilon \geq \underline{v}$ in $Q_{r,\epsilon}$. For arbitrary

$\eta > 0$, we can choose $(\bar{x}_\eta, \bar{t}_\eta), (\underline{x}_\eta, \underline{t}_\eta) \in \bar{\Lambda}_{-r^2,\epsilon}$ such that

$$u_\epsilon(\bar{x}_\eta, \bar{t}_\eta) \geq \bar{v}(\bar{x}_\eta, \bar{t}_\eta) - \eta$$

and

$$u_\epsilon(\underline{x}_\eta, \underline{t}_\eta) \leq \bar{v}(\underline{x}_\eta, \underline{t}_\eta) + \eta.$$

Then

$$\bar{v}(\bar{x}_\eta, \bar{t}_\eta) - \underline{v}(\underline{x}_\eta, \underline{t}_\eta) \leq \operatorname*{osc}_{\Lambda_{t,\epsilon}} u_\epsilon + 2\eta,$$

and hence

$$\bar{c} - \underline{c} \leq 3A + \frac{7}{2}r^{-2}A\epsilon^2 \leq 7A.$$

Therefore, we obtain

$$\operatorname*{osc}_{Q_r} u_\epsilon \leq \sup_{Q_r} \bar{v} - \inf_{Q_r} \underline{v} \leq \bar{c} - \underline{c} + 7A + 4A \leq 18A.$$

This completes the proof. □

Remark 4.2.3. *We showed in the proof of Lemma 4.2.2 that the oscillation of u_ϵ in time direction is uniformly estimated by the oscillation of u_ϵ in spatial direction on $(\epsilon^2/2)$-time slices. Note that an $(\epsilon^2/2)$-time slice $\Lambda_{t,\epsilon}$ shrinks to $B_r \times \{t\}$ as $\epsilon \to 0$ for any t. Thus, we can see that regularity for u_ϵ with respect to t almost depends on the regularity with respect to x provided ϵ is small enough.*

4.2.2 Hölder regularity

The aim of this subsection is to obtain a Hölder type estimate for u_ϵ. This result will be essentially used to prove Lipschitz regularity with respect to x in the next section.

We will use a comparison argument arising from game interpretations for obtaining regularity results in spatial direction. This argument plays an important role in obtaining the desired estimate. Several regularity results for functions satisfying various time-independent DPPs were proved by calculations based on this argument (see [50, 2, 3]). It was proved in [62] that functions satisfying another time-dependent DPP have Hölder regularity. Our proof differs from that in [62] due to the difference of the setting of DPP.

Our argument depends on the distance between two points. If two points

are relatively far away, we will consider 'multidimensional DPP'(For a more detailed explanation, see [50]). We divide the argument into two subcases. For each case, we will get the desired estimate by choosing proper behavior of an auxiliary function. In addition, we can derive our estimate by direct calculation when two points are close enough.

Lemma 4.2.4. *Let* $\bar{B}_{2r}(0) \times [-2r^2 - \epsilon^2/2, \epsilon^2/2] \subset \Omega_T \backslash I_{\epsilon,T}$, $0 < \alpha < 1$ *and* $\epsilon > 0$ *is small. Suppose that* u_ϵ *satisfies* (4.0.1) *with boundary data* $F \in L^\infty(\Gamma_{\epsilon,T})$. *Then for any* $0 < \delta < 1$,

$$|u_\epsilon(x,t) - u_\epsilon(z,s)| \le C\|u_\epsilon\|_{L^\infty(\overline{\Omega}_{\epsilon,T})}(|x-z|^\delta + \epsilon^\delta),$$

whenever $x, z \in B_r(0)$, $-r^2 < t < 0$, $|t - s| < \epsilon^2/2$ *and* $C > 0$ *is a constant which only depends on* r, δ, α *and* n.

Proof. First, we can assume that $\|u_\epsilon\|_{L^\infty(\overline{\Omega}_{\epsilon,T})} \le r^\delta$ by scaling. Let us construct an auxiliary function. Define

$$f_1(x,z) = C|x-z|^\delta + M|x+z|^2, \tag{4.2.3}$$

$$f_2(x,z) = \begin{cases} C^{2(N-i)}\epsilon^\delta & \text{if } (x,z) \in A_i \\ 0 & \text{if } |x-z| > N\epsilon/10 \end{cases} \tag{4.2.4}$$

and

$$g(t,s) = \max\{M(|t-r^2|^{\delta/2} - r^\delta), M(|s-r^2|^{\delta/2} - r^\delta)\} \tag{4.2.5}$$

where $N = N(r, \delta, \alpha, n) \in \mathbb{N}$, $C = C(r, \delta, \alpha, n) > 1$ and $M = M(r) > 1$ are constants to be determined, and

$$A_i = \{(x,z) \in \mathbb{R}^{2n} : (i-1)\epsilon/10 < |x-z| \le i\epsilon/10\}$$

for $i = 0, 1, ..., N$.

Now we define

$$H(x,z,t,s) = f_1(x,z) - f_2(x,z) + g(t,s). \tag{4.2.6}$$

We first show that

$$|u_\epsilon(x,t) - u_\epsilon(z,s)| \le C(|x-z|^\delta + \epsilon^\delta)$$

for every $x, z \ (x \neq z) \in B_{2r}(0)$, $-2r^2 < t < 0$ and $|t - s| < \epsilon^2/2$. To this end, choose M sufficiently large so that

$$u_\epsilon(x, t) - u_\epsilon(z, s) - H(x, z, t, s) \leq C^{2N}\epsilon^\delta + C\epsilon^\delta \qquad \text{in} \quad \Sigma_2\backslash\Sigma_1.$$

So, if we prove that

$$u_\epsilon(x, t) - u_\epsilon(z, s) - H(x, z, t, s) \leq C^{2N}\epsilon^\delta + C\epsilon^\delta \qquad \text{in} \quad \Sigma_1\backslash\Upsilon \qquad (4.2.7)$$

where $\Upsilon = \{(x, z, t, s) \in \mathbb{R}^{2n} \times \mathbb{R}^2 : x = z, \ -r^2 < t < 0, \ |t - s| < \epsilon^2/2\}$, then it is shown that Lemma 4.2.4 holds in $\Sigma_2\backslash\Upsilon$. Since we can obtain this estimate for $u_\epsilon(z, s) - u_\epsilon(x, t)$, we have

$$|u_\epsilon(x, t) - u_\epsilon(z, s)| \leq C^{2N}\epsilon^\delta + C\epsilon^\delta + H(x, z, t, s) \qquad \text{in} \quad \Sigma_2\backslash\Upsilon.$$

Now we can assume that $z = -x$ by proper scaling and transformation, and then we get

$$|u_\epsilon(x, t) - u_\epsilon(-x, s)| \leq C|x|^\delta + C'\epsilon^\delta$$

for some universal constant $C' > 0$. It gives the result of Lemma 4.2.4.

Suppose that (4.2.7) is not true. Then

$$K := \sup_{(x,z,t,s)\in\Sigma_1\backslash\Upsilon} (u_\epsilon(x, t) - u_\epsilon(z, s) - H(x, z, t, s)) > C^{2N}\epsilon^\delta + C\epsilon^\delta. \quad (4.2.8)$$

Let $\eta > 0$. We can choose $(x', z', t', s') \in \Sigma_1\backslash\Upsilon$ such that

$$u_\epsilon(x', t') - u_\epsilon(z', s') - H(x', z', t', s') \geq K - \eta.$$

Recall the DPP (4.0.1). Using this together with the previous inequality, we know that

$$K \leq u_\epsilon(x', t') - u_\epsilon(z', s') - H(x', z', t', s') + \eta$$

$$\leq \frac{1}{2}\left[\sup_{\nu_{x'},\nu_{z'}\in S^{n-1}}\left\{\mathscr{A}u_\epsilon\left(x', \nu_{x'}, t' - \frac{\epsilon^2}{2}\right) - \mathscr{A}u_\epsilon\left(z', \nu_{z'}, s' - \frac{\epsilon^2}{2}\right)\right\}\right.$$
$$+ \inf_{\nu_{x'},\nu_{z'}\in S^{n-1}}\left\{\mathscr{A}u_\epsilon\left(x', \nu_{x'}, t' - \frac{\epsilon^2}{2}\right) - \mathscr{A}u_\epsilon\left(z', \nu_{z'}, s' - \frac{\epsilon^2}{2}\right)\right\}\right]$$
$$- H(x', z', t', s') + 2\eta.$$

Let

$$[\mathbf{I}] = \frac{1}{2} \sup_{\nu_{x'},\nu_{z'}\in S^{n-1}} \left\{ \mathscr{A} u_\epsilon\left(x',\nu_{x'},t'-\frac{\epsilon^2}{2}\right) - \mathscr{A} u_\epsilon\left(z',\nu_{z'},s'-\frac{\epsilon^2}{2}\right)\right\}$$

and

$$[\mathbf{II}] = \frac{1}{2} \inf_{\nu_{x'},\nu_{z'}\in S^{n-1}} \left\{ \mathscr{A} u_\epsilon\left(x',\nu_{x'},t'-\frac{\epsilon^2}{2}\right) - \mathscr{A} u_\epsilon\left(z',\nu_{z'},s'-\frac{\epsilon^2}{2}\right)\right\}.$$

We see that

$$u_\epsilon(x',t') - u_\epsilon(z',s')$$
$$= \operatorname*{midrange}_{\nu_{x'}\in S^{n-1}} \mathscr{A} u_\epsilon\left(x',\nu_{x'},t'-\frac{\epsilon^2}{2}\right) - \operatorname*{midrange}_{\nu_{z'}\in S^{n-1}} \mathscr{A} u_\epsilon\left(z',\nu_{z'},s'-\frac{\epsilon^2}{2}\right)$$
$$\leq [\mathbf{I}] + [\mathbf{II}] + \eta.$$

By the definition of $\mathscr{A}$, we see that

$$[\mathbf{I}] = \frac{1}{2} \sup_{\nu_{x'},\nu_{z'}\in S^{n-1}} \left[\alpha\left\{ u_\epsilon\left(x+\epsilon\nu_{x'},t'-\frac{\epsilon^2}{2}\right) - u_\epsilon\left(z'+\epsilon\nu_{z'},s'-\frac{\epsilon^2}{2}\right)\right\} \right.$$
$$\left. + \beta \fint_{B_\epsilon^{\nu_{x'}}} \left\{ u_\epsilon\left(x'+P_{\nu_{x'}}h,t'-\frac{\epsilon^2}{2}\right) - u_\epsilon\left(z'+P_{\nu_{z'}}h,s'-\frac{\epsilon^2}{2}\right)\right\} d\mathcal{L}^{n-1}(h)\right].$$

Now we estimate $[\mathbf{I}]$ (and $[\mathbf{II}]$) by H-related terms. Let

$$[\mathbf{III}] = \alpha H(x+\epsilon\nu_x, z+\epsilon\nu_z, t, s) + \beta \fint_{B_\epsilon^{e_1}} H(x+P_{\nu_x}h, z+P_{\nu_z}h, t, s) d\mathcal{L}^{n-1}(h).$$

Recall $f(x,z) = f_1(x,z) - f_2(x,z)$ and $H(x,z,t,s) = f(x,z) + g(t,s)$. Then we see that

$$H(x+\epsilon\nu_x, z+\epsilon\nu_z, t, s) = f(x+\epsilon\nu_x, z+\epsilon\nu_z) + g(t,s)$$

and

$$\fint_{B_\epsilon^{e_1}} H(x+P_{\nu_x}h, z+P_{\nu_z}h, t, s) d\mathcal{L}^{n-1}(h)$$

$$= \fint_{B_\epsilon^{e_1}} \left\{ f(x + P_{\nu_x}h, z + P_{\nu_z}h) + g(t, s) \right\} d\mathcal{L}^{n-1}(h)$$

$$= \fint_{B_\epsilon^{e_1}} f(x + P_{\nu_x}h, z + P_{\nu_z}h) d\mathcal{L}^{n-1}(h) + g(t, s).$$

Then we can write [**III**] as

$$\alpha f(x + \epsilon\nu_x, z + \epsilon\nu_z) + \beta \fint_{B_\epsilon^{e_1}} f(x + P_{\nu_x}h, z + P_{\nu_z}h) d\mathcal{L}^{n-1}(h) + g(t, s).$$

Here we define an operator T as

$$T f(x, z, P_{\nu_x}, P_{\nu_z})$$
$$= \alpha f(x + \epsilon\nu_x, z + \epsilon\nu_z) + \beta \fint_{B_\epsilon^{e_1}} f(x + P_{\nu_x}h, z + P_{\nu_z}h) d\mathcal{L}^{n-1}(h).$$

Since

$$u_\epsilon(y, t) - u_\epsilon(\tilde{y}, \tilde{t}) \leq K + H(y, \tilde{y}, t, \tilde{t}) = K + f(y, \tilde{y}) + g(t, \tilde{t}) \qquad (4.2.9)$$

by the definition of K, we obtain that

$$[\mathbf{I}] \leq \frac{1}{2} \sup_{\nu_{x'}, \nu_{z'} \in S^{n-1}} \left[\alpha \left\{ K + H\left(x' + \epsilon\nu_{x'}, z' + \epsilon\nu_{z'}, t' - \frac{\epsilon^2}{2}, s' - \frac{\epsilon^2}{2} \right) \right\} \right.$$
$$\left. + \beta \fint_{B_\epsilon^{e_1}} \left\{ K + H\left(x' + P_{\nu_{x'}}h, z' + P_{\nu_{z'}}h, t' - \frac{\epsilon^2}{2}, s' - \frac{\epsilon^2}{2} \right) \right\} d\mathcal{L}^{n-1}(h) \right]$$
$$\leq \frac{1}{2} \left[K + \sup_{\nu_{x'}, \nu_{z'} \in S^{n-1}} T f(x', z', P_{\nu_{x'}}, P_{\nu_{z'}}) + g\left(t' - \frac{\epsilon^2}{2}, s' - \frac{\epsilon^2}{2} \right) \right].$$

Next we have to estimate [**II**]. Choose $\rho_{x'}, \rho_{z'} \in S^{n-1}$ so that

$$\inf_{\nu_{x'}, \nu_{z'} \in S^{n-1}} T f(x', z', P_{\nu_{x'}}, P_{\nu_{z'}}) \geq T f(x', z', P_{\rho_{x'}}, P_{\rho_{z'}}) - 2\eta.$$

Then we calculate that

$$[\mathbf{II}] \leq \frac{1}{2} \left[\alpha \left\{ u_\epsilon\left(x + \epsilon\rho_{x'}, t' - \frac{\epsilon^2}{2} \right) - u_\epsilon\left(z' + \epsilon\rho_{z'}, t' - \frac{\epsilon^2}{2} \right) \right\} \right.$$

$$+ \beta \fint_{B_\epsilon^{e_1}} \left\{ u_\epsilon \left(x' + P_{\rho_{x'}} h, t' - \frac{\epsilon^2}{2} \right) - u_\epsilon \left(z' + P_{\rho_{z'}} h, t' - \frac{\epsilon^2}{2} \right) \right\} d\mathcal{L}^{n-1}(h) \Bigg]$$

$$\leq \frac{1}{2} \Bigg[K + \alpha f(x' + \epsilon \rho_{x'}, z' + \epsilon \rho_{z'})$$

$$+ \beta \fint_{B_\epsilon^{e_1}} f(x' + P_{\rho_{x'}} h, z' + P_{\rho_{z'}} h) d\mathcal{L}^{n-1}(h) + g\left(t' - \frac{\epsilon^2}{2}, s' - \frac{\epsilon^2}{2} \right) \Bigg]$$

$$\leq \frac{1}{2} \Bigg[K + Tf(x', z', P_{\rho_{x'}}, P_{\rho_{z'}}) + g\left(t' - \frac{\epsilon^2}{2}, s' - \frac{\epsilon^2}{2} \right) \Bigg]$$

$$\leq \frac{1}{2} \Bigg[K + \inf_{\nu_{x'}, \nu_{z'} \in S^{n-1}} Tf(x', z', P_{\nu_{x'}}, P_{\nu_{z'}}) + g\left(t' - \frac{\epsilon^2}{2}, s' - \frac{\epsilon^2}{2} \right) \Bigg] + \eta.$$

We used (4.2.9) again in the second inequality.

Combining the estimate for [**I**] and [**II**], we obtain

$$K \leq u_\epsilon(x', t') - u_\epsilon(z', s') - H(x', z', t', s') + \eta$$

$$\leq K + \operatorname*{midrange}_{\nu_{x'}, \nu_{z'} \in S^{n-1}} Tf(x', z, P_{\nu_{x'}}, P_{\nu_{z'}}) + g\left(t' - \frac{\epsilon^2}{2}, s' - \frac{\epsilon^2}{2} \right)$$

$$- H(x', z', t', s') + 2\eta.$$

Since η is arbitrarily chosen, if we show that

$$\operatorname*{midrange}_{\nu_{x'}, \nu_{z'} \in S^{n-1}} Tf(x', z', P_{\nu_{x'}}, P_{\nu_{z'}}) + g\left(t' - \frac{\epsilon^2}{2}, s' - \frac{\epsilon^2}{2} \right) < H(x', z', t', s'),$$

that is,

$$\operatorname*{midrange}_{\nu_{x'}, \nu_{z'} \in S^{n-1}} Tf(x', z', P_{\nu_{x'}}, P_{\nu_{z'}}) - f(x', z')$$

$$< g(t', s') - g\left(t' - \frac{\epsilon^2}{2}, s' - \frac{\epsilon^2}{2} \right), \tag{4.2.10}$$

then the proof is completed.

Now we need to estimate (4.2.10). Without loss of generality, we assume that $t' \geq s'$. Then we see that

$$g(t', s') - g\left(t' - \frac{\epsilon^2}{2}, s' - \frac{\epsilon^2}{2} \right)$$

$$= M(|s' - r^2|^{\delta/2} - r^\delta) - M\left(\left|s' - \frac{\epsilon^2}{2} - r^2\right|^{\delta/2} - r^\delta\right)$$

$$= M|s' - r^2|^{\delta/2} - M\left|s' - \frac{\epsilon^2}{2} - r^2\right|^{\delta/2}.$$

Note that

$$M|s' - r^2|^{\delta/2} - M\left|s' - \frac{\epsilon^2}{2} - r^2\right|^{\delta/2} \geq M\left\{r^\delta - \left(r^2 + \frac{\epsilon^2}{2}\right)^{\frac{\delta}{2}}\right\}$$

and

$$\left(r^2 + \frac{\epsilon^2}{2}\right)^{\frac{\delta}{2}} \leq r^\delta + \left(\frac{\epsilon^2}{2}\right)^{\frac{\delta}{2}} \leq r^\delta + \epsilon^\delta$$

for $0 < \delta \leq 1$. We also deduce that

$$M|s' - r^2|^{\delta/2} - M\left|s' - \frac{\epsilon^2}{2} - r^2\right|^{\delta/2} \geq -M\frac{\delta}{2}|s' - r^2|^{\frac{\delta}{2}-1}\frac{\epsilon^2}{2} \geq -Mr^{\frac{\delta}{2}-1}\epsilon^2,$$

since $h(t) = |t|^{\delta/2}$ is concave.

Therefore, we see that

$$g(t', s') - g\left(t' - \frac{\epsilon^2}{2}, s' - \frac{\epsilon^2}{2}\right) \geq \min\{-M\epsilon^\delta, -M\tilde{C}(r)\epsilon^2\} =: \sigma.$$

To establish (4.2.10), we will distinguish several cases. And from now on, we will write (x, z, t, s) instead of (x', z', t', s') in our calculations for convenience.

Case $|x - z| > N\epsilon/10$

In this case, $f(x, z) = f_1(x, z)$ as $f_2(x, z) = 0$. Thus we can write (4.2.10) as

$$\operatorname*{midrange}_{\nu_x, \nu_z \in S^{n-1}} Tf_1(x, z, P_{\nu_x}, P_{\nu_z}) - f_1(x, z) < \sigma. \qquad (4.2.11)$$

For any $\eta > 0$, we can choose some vectors $\nu_x, \nu_z \in S^{n-1}$ and related rotations $P_{\nu_x} \in \mathbf{R}_{\nu_x}$, $P_{\nu_z} \in \mathbf{R}_{\nu_z}$ so that

$$\sup_{h_x, h_z \in S^{n-1}} Tf_1(x, z, P_{h_x}, P_{h_z}) \leq Tf_1(x, z, P_{\nu_x}, P_{\nu_z}) + \eta.$$

Hence if we find some unit vectors μ_x, μ_z and rotations P_{μ_x}, P_{μ_z} such that

$$\operatorname*{midrange}_{h_x, h_z \in S^{n-1}} Tf_1(x, z, P_{h_x}, P_{h_z})$$

$$\leq \frac{1}{2}\{Tf_1(x, z, P_{\nu_x}, P_{\nu_z}) + Tf_1(x, z, P_{\mu_x}, P_{\mu_z}) + \eta\},$$

then we obtain (4.2.11) by showing

$$\frac{1}{2}\{Tf_1(x, z, P_{\nu_x}, P_{\nu_z}) + Tf_1(x, z, P_{\mu_x}, P_{\mu_z})\} - f_1(x, z) < \sigma - \eta. \quad (4.2.12)$$

Denote $\mathbf{v} = \frac{x-z}{|x-z|}$, $y_V = \langle y, \mathbf{v} \rangle$ and $y_{V_\perp} = y - y_V \mathbf{v}$. Then y is orthogonally decomposed into $y_V \mathbf{v}$ and $y_{V_\perp}$. By using Taylor expansion, we know that for any h_x and h_z,

$$\begin{aligned} f_1&(x + \epsilon h_x, z + \epsilon h_z) \\ &= f_1(x, z) + C\delta|x - z|^{\delta-1}(h_x - h_z)_V \epsilon + 2M\langle x + z, h_x + h_z \rangle \epsilon \\ &\quad + \frac{1}{2}C\delta|x - z|^{\delta-2}\{(\delta - 1)(h_x - h_z)_V^2 + |(h_x - h_z)_{V^\perp}|^2\}\epsilon^2 \\ &\quad + M|h_x + h_z|^2\epsilon^2 + \mathcal{E}_{x,z}(\epsilon h_x, \epsilon h_z), \end{aligned}$$

where $\mathcal{E}_{x,z}(h_x, h_z)$ is the second-order error term. Now we estimate the error term by Taylor's theorem as follows:

$$|\mathcal{E}_{x,z}(\epsilon h_x, \epsilon h_z)| \leq C|(\epsilon h_x, \epsilon h_z)^t|^3(|x - z| - 2\epsilon)^{\delta-3}$$

if $|x - z| > 2\epsilon$. Thus if we choose $N \geq \frac{100C}{\delta}$, we get

$$|\mathcal{E}_{x,z}(\epsilon h_x, \epsilon h_z)| \leq 10|x - z|^{\delta-2}\epsilon^2.$$

Now we establish (4.2.11). We first consider a small constant $0 < \Theta < 4$ to be determined later and we divide again this case into two separate subcases. In the first subsection, we consider the case when ν_x, ν_z are in almost opposite directions and nearly parallel to the vector $x - z$. Otherwise, it is covered in the second subsection. In each case, we will choose proper rotations and investigate changes in the value of the auxiliary function f_1. The concavity of f_1 plays a key role in both cases.

Case $(\nu_x - \nu_z)_V^2 \geq (4 - \Theta)$

Observe that

$$\underset{\nu_x, \nu_z \in S^{n-1}}{\text{midrange}} Tf_1(x, z, P_{\nu_x}, P_{\nu_z})$$

$$\leq \frac{1}{2}\Big\{Tf_1(x, z, P_{\nu_x}, P_{\nu_z}) + Tf_1(x, z, -P_{\nu_x}, -P_{\nu_z}) + \eta\Big\}$$

and

$$\frac{1}{2}\Big\{Tf_1(x, z, P_{\nu_x}, P_{\nu_z}) + Tf_1(x, z, -P_{\nu_x}, -P_{\nu_z})\Big\} - f_1(x, z)$$

$$= \frac{\alpha}{2}\Big\{f_1(x + \epsilon\nu_x, z + \epsilon\nu_z) + f_1(x - \epsilon\nu_x, z - \epsilon\nu_z) - 2f_1(x, z)\Big\}$$

$$+ \frac{\beta}{2}\Big\{\fint_{B_\epsilon^{e_1}} f_1(x + P_{\nu_x}h, z + P_{\nu_z}h)d\mathcal{L}^{n-1}(h)$$

$$+ \fint_{B_\epsilon^{e_1}} f_1(x - P_{\nu_x}h, z - P_{\nu_z}h)d\mathcal{L}^{n-1}(h) - 2f_1(x, z)\Big\}.$$

We first estimate the α-term. Using the Taylor expansion of f_1 and the above estimates, we get

$$f_1(x + \epsilon\nu_x, z + \epsilon\nu_z) + f_1(x - \epsilon\nu_x, z - \epsilon\nu_z) - 2f_1(x, z)$$

$$= C\delta|x - z|^{\delta-2}\big\{(\delta - 1)(\nu_x - \nu_z)_V^2 + |(\nu_x - \nu_z)_{V^\perp}|^2\big\}\epsilon^2 + 2M|\nu_x + \nu_z|^2\epsilon^2$$

$$+ \mathcal{E}_{x,z}(\epsilon\nu_x, \epsilon\nu_z) + \mathcal{E}_{x,z}(-\epsilon\nu_x, -\epsilon\nu_z)$$

$$\leq C\delta|x - z|^{\delta-2}\big\{(\delta - 1)(4 - \Theta) + \Theta\big\}\epsilon^2 + 2M(2\epsilon)^2 + 20|x - z|^{\delta-2}\epsilon^2$$

$$\leq \big[C\delta|x - z|^{\delta-2}\{(\delta - 1)(4 - \Theta) + \Theta\} + 8M + 20|x - z|^{\delta-2}\big]\epsilon^2.$$

And note that

$$|P_{\nu_x}h - P_{\nu_z}h| \leq |\nu_x + \nu_z|, \tag{4.2.13}$$

for some proper P_{ν_x}, P_{ν_z} and for any $h \in B_1^{e_1}$(see [3, Appendix A]), to see that

$$\fint_{B_\epsilon^{e_1}} f_1(x + P_{\nu_x}h, z + P_{\nu_z}h)d\mathcal{L}^{n-1}(h) - f_1(x, z)$$

$$= \fint_{B_\epsilon^{e_1}} \Big[C\delta |x - z|^{\delta-1} (P_{\nu_x} h - P_{\nu_z} h)_V + 2M \langle x + z, P_{\nu_x} h + P_{\nu_z} h \rangle$$

$$+ \frac{C}{2} |x - z|^{\delta-2} \big\{ (\delta - 1)(P_{\nu_x} h - P_{\nu_z} h)_V^2 + |(P_{\nu_x} h - P_{\nu_z} h)_{V^\perp}|^2 \big\}$$

$$+ M |P_{\nu_x} h + P_{\nu_z} h|^2 + \mathcal{E}_{x,z}(h_x, h_z) \Big] d\mathcal{L}^{n-1}(h)$$

$$= \frac{1}{2} \fint_{B_\epsilon^{e_1}} \Big[C |x - z|^{\delta-2} \big\{ (\delta - 1)(P_{\nu_x} h - P_{\nu_z} h)_V^2 + |(P_{\nu_x} h - P_{\nu_z} h)_{V^\perp}|^2 \big\}$$

$$+ 2M |P_{\nu_x} h + P_{\nu_z} h|^2 + 2\mathcal{E}_{x,z}(h_x, h_z) \Big] d\mathcal{L}^{n-1}(h)$$

$$\leq \frac{1}{2} \big\{ |x - z|^{\delta-2} (C\Theta + 20) + 8M \big\} \epsilon^2.$$

The last inequality follows from $|\nu_x + \nu_z|^2 \leq \Theta$. In the same way, it is also obtained

$$\fint_{B_\epsilon^{e_1}} f_1(x - P_{\nu_x} h, z - P_{\nu_z} h) d\mathcal{L}^{n-1}(h) - f_1(x, z)$$

$$\leq \frac{1}{2} \big\{ |x - z|^{\delta-2} (C\Theta + 20) + 8M \big\} \epsilon^2.$$

These estimates give

$$\frac{1}{2} \big\{ T f_1(x, z, \nu_x, \nu_z) + T f_1(x, z, -\nu_x, -\nu_z) \big\} - f_1(x, z)$$

$$\leq \frac{\alpha}{2} \Big[C\delta |x - z|^{\delta-2} \big\{ (\delta - 1)(4 - \Theta) + \Theta \big\} + 8M + 20|x - z|^{\delta-2} \Big] \epsilon^2$$

$$+ \frac{\beta}{2} \big\{ C\Theta |x - z|^{\delta-2} + 8M + 20|x - z|^{\delta-2} \big\} \epsilon^2$$

$$\leq \Big[\frac{C}{2} \big\{ \Theta + \alpha\delta(\delta - 1)(4 - \Theta) \big\} + 10 \Big] |x - z|^{\delta-2} \epsilon^2 + 4M\epsilon^2.$$

Observe that $\Theta + \alpha\delta(\delta - 1)(4 - \Theta) < 0$ if $\Theta < 4\alpha\delta(1 - \delta)/\{1 - \alpha\delta(\delta - 1)\}$. Then we can choose sufficiently large C depending only on r, δ, α and n so that

$$\operatorname*{midrange}_{\nu_x, \nu_z \in S^{n-1}} T f_1(x, z, P_{\nu_x}, P_{\nu_z}) - f_1(x, z) < -M\tilde{C}\epsilon^2.$$

Thus, we get (4.2.11).

Case $(\nu_x - \nu_z)_V^2 \leq (4 - \Theta)$

It is clear that $|\nu_x - \nu_z|_V < 2 - \Theta/4$ in this case. Furthermore, we check that

$$\operatorname*{midrange}_{\nu_x, \nu_z \in S^{n-1}} Tf_1(x, z, P_{h_x}, P_{h_z})$$

$$\leq \frac{1}{2}\{Tf_1(x, z, P_{\nu_x}, P_{\nu_z}) + Tf_1(x, z, P_{-\mathbf{v}}, P_{\mathbf{v}})\} + \eta. \tag{4.2.14}$$

Now we estimate the right hand side. By the DPP, it can be written as

$$\frac{1}{2}\{Tf_1(x, z, P_{\nu_x}, P_{\nu_z}) + Tf_1(x, z, P_{-\mathbf{v}}, P_{\mathbf{v}})\} - f_1(x, z)$$

$$= \frac{\alpha}{2}\{f_1(x + \epsilon\nu_x, z + \epsilon\nu_z) + f_1(x - \epsilon\mathbf{v}, z + \epsilon\mathbf{v}) - 2f_1(x, z)\}$$

$$+ \frac{\beta}{2}\left\{ \fint_{B_\epsilon^{e1}} f_1(x + P_{\nu_x}h, z + P_{\nu_z}h)d\mathcal{L}^{n-1}(h) \right.$$

$$\left. + \fint_{B_\epsilon^{e1}} f_1(x + P_{-\mathbf{v}}h, z + P_{\mathbf{v}}h)d\mathcal{L}^{n-1}(h) - 2f_1(x, z) \right\}.$$

We will continue in a similar way to the previous case. For the α-term, we deduce that

$$f_1(x + \epsilon\nu_x, z + \epsilon\nu_z) + f_1(x - \epsilon\mathbf{v}, z + \epsilon\mathbf{v}) - 2f_1(x, z)$$

$$= \frac{1}{2}\left[C\delta|x - z|^{\delta-1}\{(\nu_x - \nu_z)_V - 2\}\epsilon + 2M\langle x + z, \nu_x + \nu_z\rangle\epsilon \right.$$

$$+ \frac{C}{2}\delta|x - z|^{\delta-2}\{(\delta - 1)((\nu_x - \nu_z)_V^2\epsilon^2 + (2\epsilon)^2) + |(\nu_x - \nu_z)_{V^\perp}|^2\epsilon^2\}$$

$$\left. + 4M\epsilon^2 + M|\nu_x + \nu_z|^2\epsilon^2 + \mathcal{E}_{x,z}(\epsilon\nu_x, \epsilon\nu_z) + \mathcal{E}_{x,z}(-\epsilon\mathbf{v}, \epsilon\mathbf{v})\right]$$

$$\leq \frac{1}{2}\left\{-\frac{\Theta}{4}C\delta|x - z|^{\delta-1}\epsilon + 8M\epsilon r + 2C\delta|x - z|^{\delta-2}\epsilon^2 + 20|x - z|^{\delta-2}\epsilon^2 + 2M\epsilon^2\right\}.$$

Then we see that

$$2C\delta|x - z|^{\delta-2}\epsilon^2 + 20|x - z|^{\delta-2}\epsilon^2 + 2M\epsilon^2$$

$$\leq \frac{10}{N}(2C\delta + 20 + 2M \operatorname{diam}(\Omega)^{2-\delta})|x - z|^{\delta-1}\epsilon$$
$$\leq \delta^2|x - z|^{\delta-1}\epsilon$$

for sufficiently large C and $N \geq 100C/\delta$, since $|x - z| > N\epsilon/10$ and Ω is bounded. Thus,

$$f_1(x + \nu_x, z + \nu_z) + f_1(x - \epsilon\mathbf{v}, z + \epsilon\mathbf{v}) - 2f_1(x, z)$$
$$\leq \left\{\frac{\delta}{2}|x - z|^{\delta-1}\left(\delta - C\frac{\Theta}{4}\right) + 4Mr\right\}\epsilon.$$

Next, we estimate the β-term. By a direct calculation, we see that

$$\fint_{B_\epsilon^{e_1}} \{f_1(x + P_{\nu_x}h, z + P_{\nu_z}h) + f_1(x + P_{-\mathbf{v}}h, z + P_{\mathbf{v}}h) - 2f_1(x, z)\}d\mathcal{L}^{n-1}(h)$$

$$= \fint_{B_\epsilon^{e_1}} \{f_1(x + P_{\nu_x}h, z + P_{\nu_z}h) - f_1(x, z)\}d\mathcal{L}^{n-1}(h)$$

$$+ \fint_{B_\epsilon^{e_1}} \{f_1(x + P_{-\mathbf{v}}h, z + P_{\mathbf{v}}h) - f_1(x, z)\}d\mathcal{L}^{n-1}(h)$$

$$\leq \fint_{B_\epsilon^{e_1}} \left[\frac{C}{2}\delta|x - z|^{\delta-2}\{(\delta - 1)(P_{\nu_x}h - P_{\nu_z}h)_V^2 + |(P_{\nu_x}h - P_{\nu_z}h)_{V^\perp}|^2\}\right.$$

$$\left. + M|P_{\nu_x}h + P_{\nu_z}h|^2 + \mathcal{E}_{x,z}(h_x, h_z)\right]d\mathcal{L}^{n-1}(h)$$

$$+ \fint_{B_\epsilon^{e_1}} \left[\frac{C}{2}\delta|x - z|^{\delta-2}(2h)^2 + \mathcal{E}_{x,z}(-\epsilon\mathbf{v}, \epsilon\mathbf{v})\right]d\mathcal{L}^{n-1}(h)$$

$$\leq C\delta|x - z|^{\delta-2}(2\epsilon)^2 + M(2\epsilon)^2 + 20|x - z|^{\delta-2}\epsilon^2,$$

we have used (4.2.13) for the last inequality. Now we observe that

$$C\delta|x - z|^{\delta-2}(2\epsilon)^2 + M(2\epsilon)^2 + 20|x - z|^{\delta-2}\epsilon^2 \leq 2\delta^2|x - z|^{\delta-1}\epsilon.$$

Therefore β-term is estimated by $2\delta^2|x - z|^{\delta-1}\epsilon$.

Combining these estimates, we conclude

$$\frac{1}{2}\{Tf_1(x, z, P_{\nu_x}, P_{\nu_z}) + Tf_1(x, z, P_{-\mathbf{v}}, P_{\mathbf{v}})\} - f_1(x, z)$$

$$\leq \frac{\alpha}{2}\left\{\frac{\delta}{2}|x - z|^{\delta-1}\left(\delta - C\frac{\Theta}{4}\right) + 4Mr\right\}\epsilon + \beta\delta^2|x - z|^{\delta-1}\epsilon$$

$$\leq -M\tilde{C}\epsilon^2$$

for sufficiently large $C = C(r, \delta, \alpha, n)$. Combining this with (4.2.14), we obtain (4.2.10).

Case $0 < |x - z| \leq N\epsilon/10$

We observe that

$$
\begin{aligned}
|f_1(x &+ h_x, z + h_z) - f_1(x, z)| \\
&= C(|x - z + h_x - h_z|^\delta - |x - z|^\delta) + M(|x + z + h_x + h_z|^2 - |x + z|^2) \\
&\leq C|h_x - h_z|^\delta + 2M|x + z| \, |h_x + h_z| + M|h_x + h_z|^2 \\
&\leq 2C\epsilon^\delta + 8Mr\epsilon + 4M\epsilon^2 \\
&\leq 3C\epsilon^\delta
\end{aligned}
$$

for any $x, z \in B_r$ and $h_x, h_z \in B_\epsilon$ if $C = C(r, \delta)$ is sufficiently large. Therefore, we see that

$$
\begin{aligned}
\sup_{h_x, h_z \in S^{n-1}} &Tf_1(x, z, P_{h_x}, P_{h_z}) - f_1(x, z) \\
&= \sup_{h_x, h_z \in S^{n-1}} \left[\alpha\{f_1(x + h_x, z + h_z) - f_1(x, z)\} + \right. \\
&\qquad \left. \beta \fint_{B_\epsilon^{e_1}} \{f_1(x + P_{h_x}h, z + P_{h_z}h) - f_1(x, z)\} d\mathcal{L}^{n-1}(h) \right] \\
&\leq 3\alpha C\epsilon^\delta + 3\beta C\epsilon^\delta = 3C\epsilon^\delta
\end{aligned}
$$

and

$$
\begin{aligned}
\sup_{h_x, h_z \in S^{n-1}} Tf(x, z, P_{h_x}, P_{h_z}) &= \sup_{h_x, h_z \in S^{n-1}} T(f_1 - f_2)(x, z, P_{h_x}, P_{h_z}) \\
&\leq \sup_{h_x, h_z \in S^{n-1}} Tf_1(x, z, P_{h_x}, P_{h_z}).
\end{aligned}
\tag{4.2.15}
$$

By the assumption, we can find $i \in \{1, 2, \cdots, N\}$ such that

$$(i - 1)\frac{\epsilon}{10} < |x - z| \leq i\frac{\epsilon}{10}.$$

We deduce that

$$\inf_{h_x, h_z \in S^{n-1}} Tf(x, z, P_{h_x}, P_{h_z})$$

$$\leq \sup_{h_x, h_z \in S^{n-1}} Tf_1(x, z, P_{h_x}, P_{h_z}) - \sup_{h_x, h_z \in S^{n-1}} Tf_2(x, z, P_{h_x}, P_{h_z})$$

$$\leq \sup_{h_x, h_z \in S^{n-1}} Tf_1(x, z, P_{h_x}, P_{h_z}) - \alpha C^{2(N-i+1)} \epsilon^\delta$$

$$= \sup_{h_x, h_z \in S^{n-1}} Tf_1(x, z, P_{h_x}, P_{h_z}) - \alpha \left(C^2 - \frac{2}{\alpha} \right) C^{2(N-i)} \epsilon^\delta - 2C^{2(N-i)} \epsilon^\delta$$

$$\leq \sup_{h_x, h_z \in S^{n-1}} Tf_1(x, z, P_{h_x}, P_{h_z}) - 2f_2(x, z) - 8C\epsilon^\delta.$$

The last inequality is obtained if C is large. Therefore, we calculate that

$$\operatorname*{midrange}_{\nu_x, \nu_z \in S^{n-1}} Tf(x, z, P_{h_x}, P_{h_z}) \leq \sup_{h_x, h_z \in S^{n-1}} Tf_1(x, z, P_{h_x}, P_{h_z}) - f_2(x, z) - 4C\epsilon^\delta$$

$$\leq f_1(x, z) + 3C\epsilon^\delta - f_2(x, z) - 4C\epsilon^\delta$$

$$\leq f(x, z) - C\epsilon^\delta,$$

and then we get (4.2.10) for choosing $C = C(r, \delta, \alpha, n)$ sufficiently large.

Case $|x - z| = 0$

According to the results in the previous sections, we observe that

$$|u_\epsilon(x, t) - u_\epsilon(z, s)| \leq C_1 \|u_\epsilon\|_{L^\infty(\overline{\Omega}_{\epsilon, T})} (|x - z|^\delta + \epsilon^\delta),$$

for any x, z $(x \neq z) \in B_r(0)$, $-r^2 < t < 0$, $|t - s| < \epsilon^2/2$ and some $C_1 = C_1(r, \delta, \alpha, n) > 0$.

Fix $x \in B_r(0)$ and $t, s \in (-r^2, 0)$ with $|t - s| < \epsilon^2/2$. Then we can choose a point $y \in B_\epsilon(x)$ and deduce that

$$|u_\epsilon(x, t) - u_\epsilon(x, s)| \leq |u_\epsilon(x, t) - u_\epsilon(y, s)| + |u_\epsilon(y, s) - u_\epsilon(x, s)|$$

$$\leq C_1 \|u_\epsilon\|_{L^\infty(\overline{\Omega}_{\epsilon, T})} (|x - y|^\delta + \epsilon^\delta)$$

$$\leq 2C_1 \|u_\epsilon\|_{L^\infty(\overline{\Omega}_{\epsilon, T})} \epsilon^\delta.$$

Now set $C = 2C_1$. Then we can conclude the proof of this lemma. $\qquad \square$

For any $x \in B_r$ and $-r^2 < s < t < 0$, consider a cylinder $B_{\sqrt{t-s}}(x) \times [s,t]$. Applying Lemma 4.2.4, we find that

$$\underset{B_{\sqrt{t-s}}(x) \times \left(\tau - \frac{\epsilon^2}{2}, \tau\right)}{\operatorname{osc}} u_\epsilon \leq C(r,\delta,\alpha,n) \|u_\epsilon\|_{L^\infty(\overline{\Omega}_{\epsilon,T})} (|t-s|^{\frac{\delta}{2}} + \epsilon^\delta)$$

for any $\tau \in (s,t)$. Then we obtain the following estimate

$$|u_\epsilon(x,t) - u_\epsilon(x,s)| \leq C(r,\delta,\alpha,n) \|u_\epsilon\|_{L^\infty(\overline{\Omega}_{\epsilon,T})} (|t-s|^{\frac{\delta}{2}} + \epsilon^\delta)$$

by virtue of Lemma 4.2.2.

Combining this and Lemma 4.2.4, we get the desired regularity.

Theorem 4.2.5. *Let $\bar{Q}_{2r} \subset \Omega_T \backslash I_{\epsilon,T}$, $0 < \delta, \alpha < 1$ and $\epsilon > 0$ is small. Suppose that u_ϵ satisfies (4.0.1) with boundary data $F \in L^\infty(\Gamma_{\epsilon,T})$. Then for any $x, z \in B_r(0)$ and $-r^2 < t, s < 0$,*

$$|u_\epsilon(x,t) - u_\epsilon(z,s)| \leq C \|u_\epsilon\|_{L^\infty(\overline{\Omega}_{\epsilon,T})} (|x-z|^\delta + |s-t|^{\frac{\delta}{2}} + \epsilon^\delta),$$

where $C > 0$ is a constant which only depends on r, δ, α and n.

4.2.3 Lipschitz regularity

We will prove Lipschitz type regularity for the function u_ϵ in this subsection. In the previous section, we utilized the concavity on the distance of two points of the auxiliary function to get the result. In order to prove Lipschitz estimate, the auxiliary function is also needed to have this property. However, we no longer have the strong concavity that was helpful in the proof there. Therefore, we need to build the proof in a different manner in several places.

For this reason, we will construct other (concave) auxiliary function for proving Lipschitz estimate. This causes some difficulties compared to the Hölder case. As in the proof of Lemma 4.2.4, we will distinguish two subcases. More delicate calculations are needed when two points are sufficiently far apart. Note that we will exploit the Hölder regularity result here. In the case that two points are sufficiently close, the proof is quite similar to the previous section.

Lemma 4.2.6. *Let $\bar{B}_{2r}(0) \times [-2r^2 - \epsilon^2/2, \epsilon^2/2] \subset \Omega_T \backslash I_{\epsilon,T}$, $0 < \alpha < 1$ and $\epsilon > 0$ is small. Suppose that u_ϵ satisfies (4.0.1) with boundary data*

$F \in L^\infty(\Gamma_{\epsilon,T})$. *Then,*

$$|u_\epsilon(x,t) - u_\epsilon(z,s)| \leq C||u_\epsilon||_{L^\infty(\overline{\Omega}_{\epsilon,T})}(|x-z| + \epsilon),$$

whenever $x, z \in B_r(0)$, $-r^2 < t < 0$ *and* $|t - s| < \epsilon^2/2$ *and* $C > 0$ *is a constant which only depends on* r, α *and* n.

Proof. We can expect that $|x - z|$ will play the same role as f_1 in the Hölder case. But for a Lipschitz type estimate, we cannot deduce the desired result by using that function $|x - z|$. Therefore, we need to define a new auxiliary function $\omega : [0, \infty) \rightarrow [0, \infty)$. First define

$$\omega(t) = t - \omega_0 t^\gamma \qquad 0 \leq t \leq \omega_1 := (2\gamma\omega_0)^{-1/(\gamma-1)},$$

where $\gamma \in (1, 2)$ is a constant and $\omega_0 > 0$ will be determined later. Observe that

$$\omega'(t) = 1 - \gamma\omega_0 t^{\gamma-1} \in [1/2, 1] \qquad \text{for } 0 \leq t \leq \omega_1$$

and

$$\omega''(t) = -\gamma(\gamma - 1)\omega_0 t^{\gamma-2} < 0 \qquad \text{for } 0 \leq t \leq \omega_1.$$

Then we can construct ω to be increasing, strictly concave and C^2 in $(0, \infty)$.

Assume that $||u_\epsilon||_{L^\infty(\overline{\Omega}_{\epsilon,T})} \leq r$ by scaling as in the previous section, and we define

$$f_1(x, z) = C\omega(|x - z|) + M|x + z|^2.$$

Consider the functions f_2 and g for $\delta = 1$ as (4.2.4) and (4.2.5), respectively. Now we set again the auxiliary function H by

$$H(x, z, t, s) = f_1(x, z) - f_2(x, z) + g(t, s)$$

and let

$$f(x, z) = f_1(x, z) - f_2(x, z).$$

As in the previous section, we will first deduce that

$$|u_\epsilon(x,t) - u_\epsilon(z,s)| \leq C(|x-z| + \epsilon) \qquad \text{in} \quad \Sigma_2 \backslash \Upsilon.$$

We can choose M sufficiently large so that

$$u_\epsilon(x,t) - u_\epsilon(z,s) - H(x, z, t, s) \leq C^{2N}\epsilon + C\epsilon \qquad \text{in} \quad \Sigma_2 \backslash \Sigma_1.$$

Thus, for proving the lemma, it is sufficient to show that

$$u_\epsilon(x,t) - u_\epsilon(z,s) - H(x,z,t,s) \leq C^{2N}\epsilon + C\epsilon \qquad \text{in} \quad \Sigma_1 \backslash \Upsilon.$$

Suppose not. Then

$$K := \sup_{(x,z,t,s) \in \Sigma_1 \backslash \Upsilon} (u_\epsilon(x,t) - u_\epsilon(z,s) - H(x,z,t,s)) > C^{2N}\epsilon + C\epsilon.$$

In this case, we can choose $(x', z', t', s') \in \Sigma_1 \backslash \Upsilon$ such that

$$u_\epsilon(x',t') - u_\epsilon(z',s') - H(x',z',t',s')) \geq K - \eta \tag{4.2.16}$$

for any $\eta > 0$.

Similarly as in Section 4.2.2, we need to establish (4.2.10) in order to prove Lemma 4.2.6. The only difference is the right-hand side of the inequality. In this case, it is sufficient to deduce that the left-hand side of (4.2.10) is less than $\sigma = \min\{-M\epsilon, -M\tilde{C}\epsilon^2\}$, where $\tilde{C}$ only depends on r.

We use again the notation (x, z, t, s) instead of (x', z', t', s').

Case $|x - z| > N\epsilon/10$

For the same reason as in the previous section, we shall deduce (4.2.11). To do this, it is sufficient to show (4.2.12) for any $\eta > 0$ and some $P_{\nu_x} \in \mathbf{R}_{\nu_x}$, $P_{\nu_z} \in \mathbf{R}_{\nu_z}$.

Now we calculate the Taylor expansion of f_1. We see

$$\begin{aligned}
f_1(x + \epsilon h_x, &z + \epsilon h_z) - f_1(x, z) \\
&\leq C\omega'(|x - z|)(h_x - h_z)_V \epsilon + 2M\langle x + z, h_x + h_z\rangle\epsilon \\
&\quad + \frac{1}{2}C\omega''(|x - z|)(h_x - h_z)_V^2 \epsilon^2 + \frac{1}{2}C\frac{\omega'(|x - z|)}{|x - z|}|(h_x - h_z)_{V^\perp}|^2\epsilon^2 \\
&\quad + (4M + 10|x - z|^{\gamma-2})\epsilon^2
\end{aligned} \tag{4.2.17}$$

for any $h_x, h_z \in \mathbb{R}^n$. Then we check that

$$|\mathcal{E}_{x,z}(h_x, h_z)| \leq C|(h_x, h_z)^t|^3(|x - z| - 2\epsilon)^{\gamma-3} \leq C|(h_x, h_z)^t|^3(|x - z| - 2\epsilon)^{\gamma-3}$$

if $|x - z| > 2\epsilon$ and $h_x, h_z \in B_\epsilon$, because for the third derivatives it holds $D^3_{(x,z)}\omega(|x - z|) \leq C|x - z|^{\gamma-3}$ for some constant $C > 0$. Thus if we choose

$N \geq \frac{100C}{\delta}$, we get
$$|\mathcal{E}_{x,z}(h_x, h_z)| \leq 10|x - z|^{\gamma-2}\epsilon^2.$$

For estimating α-term in $Tf_1(x, z, P_{\nu_x}, P_{\nu_z})$, we can use (4.2.17) directly. On the other hand, more observations about P_{ν_x}, P_{ν_z} are needed to estimate β-term. First we see that

$$
\begin{aligned}
f_1(x &+ \epsilon P_{\nu_x}\zeta, z + \epsilon P_{\nu_z}\zeta) - f_1(x, z) \\
&= C\omega'(|x - z|)(P_{\nu_x}\zeta - P_{\nu_z}\zeta)_V\epsilon + 2M\langle x + z, P_{\nu_x}\zeta + P_{\nu_z}\zeta\rangle\epsilon \\
&\quad + \frac{1}{2}C\omega''(|x - z|)(P_{\nu_x}\zeta - P_{\nu_z}\zeta)_V^2\epsilon^2 + \frac{1}{2}C\frac{\omega'(|x - z|)}{|x - z|}|(P_{\nu_x}\zeta - P_{\nu_z}\zeta)_{V^\perp}|^2\epsilon^2 \\
&\quad + M|P_{\nu_x}\zeta + P_{\nu_z}\zeta|^2\epsilon^2 + \mathcal{E}_{x,z}(\epsilon h_x, \epsilon h_z)
\end{aligned}
$$

from (4.2.17). Due to rotational symmetry, integral over the first-order terms is zero. Note that $\omega'' < 0$ and (4.2.13) to see that

$$
\begin{aligned}
\fint_{B_\epsilon^{e1}} &f_1(x + P_{\nu_x}h, z + P_{\nu_z}h)d\mathcal{L}^{n-1}(h) - f_1(x, z) \\
&\leq \frac{C}{2}\frac{\omega'(|x - z|)}{|x - z|}|\nu_x + \nu_z|^2\epsilon^2 + (4M + 10|x - z|^{\gamma-2})\epsilon^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
Tf_1(x, z, &P_{\nu_x}, P_{\nu_z}) - f_1(x, z) \\
&\leq \alpha C\omega'(|x - z|)(\nu_x - \nu_z)_V\epsilon + 2\alpha M\langle x + z, \nu_x + \nu_z\rangle\epsilon \\
&\quad + \frac{\alpha}{2}C\omega''(|x - z|)(\nu_x - \nu_z)_V^2\epsilon^2 \\
&\quad + \frac{1}{2}C\frac{\omega'(|x - z|)}{|x - z|}(\alpha|(\nu_x - \nu_z)_{V^\perp}|^2 + \beta|\nu_x + \nu_z|^2)\epsilon^2 \\
&\quad + (4M + 10|x - z|^{\gamma-2})\epsilon^2.
\end{aligned}
$$

Now we set $\Theta = |x - z|^s$ for some $s \in (0, 1]$ to be chosen later. In order to deduce (4.2.11), we divide again this case into two separate subcases.

$(\nu_x - \nu_z)_V^2 \geq 4 - \Theta$

Consider two rotations P_{ν_x}, P_{ν_z} which satisfy (4.2.13). Observe that

$$
\begin{aligned}
\frac{1}{2}\{Tf_1(x, z, P_{\nu_x}, P_{\nu_z}) &+ Tf_1(x, z, -P_{\nu_x}, -P_{\nu_z})\} - f_1(x, z) \\
&\leq \frac{\alpha}{2} C\omega''(|x - z|)(\nu_x - \nu_z)_V^2 \epsilon^2 \\
&+ \frac{1}{2} C \frac{\omega'(|x - z|)}{|x - z|}(\alpha|(\nu_x - \nu_z)_{V^\perp}|^2 + \beta|\nu_x + \nu_z|^2)\epsilon^2 \\
&+ (4M + 10|x - z|^{\gamma - 2})\epsilon^2.
\end{aligned}
\tag{4.2.18}
$$

Since $\Theta \leq 1$ for sufficiently small r and $\frac{1}{2} \leq \omega' \leq 1$ and $\omega'' < 0$,

$$
\begin{aligned}
\frac{1}{2}\{Tf_1(x, z, P_{\nu_x}, P_{\nu_z}) &+ Tf_1(x, z, -P_{\nu_x}, -P_{\nu_z})\} - f_1(x, z) \\
&\leq \frac{3}{2}\alpha C\omega''(|x - z|)\epsilon^2 + \frac{C}{2}\frac{1}{|x - z|}(\alpha|(\nu_x - \nu_z)_{V^\perp}|^2 + \beta|\nu_x + \nu_z|^2)\epsilon^2 \\
&+ (4M + 10|x - z|^{\gamma - 2})\epsilon^2.
\end{aligned}
$$

We know that $|(\nu_x - \nu_z)_{V^\perp}|^2 \leq \Theta$ by the assumption and we also see

$$
|\nu_x + \nu_z|^2 = 4 - |(\nu_x - \nu_z)|^2 \leq 4 - |(\nu_x - \nu_z)_V|^2 \leq \Theta.
$$

Thus,

$$
\begin{aligned}
\frac{1}{2}\{Tf_1(x, z, P_{\nu_x}, P_{\nu_z}) &+ Tf_1(x, z, -P_{\nu_x}, -P_{\nu_z})\} - f_1(x, z) \\
&\leq \left\{\frac{3}{2}\alpha C\omega''(|x - z|) + \frac{C}{2}\frac{\Theta}{|x - z|} + 4M + 10|x - z|^{\gamma - 2}\right\}\epsilon^2.
\end{aligned}
$$

By the definition of ω, $\omega''(|x - z|) = -\gamma(\gamma - 1)\omega_0|x - z|^{\gamma - 2}$ if $|x - z| < \omega_1$. Choosing $\gamma = 1 + s$. Since $|x - z| < 1$, we get

$$
\begin{aligned}
\frac{1}{2}\{Tf_1(x, z, P_{\nu_x}, P_{\nu_z}) &+ Tf_1(x, z, -P_{\nu_x}, -P_{\nu_z})\} - f_1(x, z) \\
&\leq \left[C\left\{-\frac{3}{2}\alpha s(s + 1)\omega_0 + 11\right\}|x - z|^{s - 1} + 4M\right]\epsilon^2.
\end{aligned}
$$

Note that if $|x - z| < \omega_1$ (See the definition of ω),

$$-\frac{3}{2}\alpha s(s+1)\omega_0 + 11 < 0$$

for sufficiently large ω_0. Now we select $C = C(r, \alpha, n)$ sufficiently large so that

$$\left[C\left\{-\frac{3}{2}\alpha s(s+1)\omega_0 + 11\right\}|x - z|^{s-1} + 4M\right]\epsilon^2 \leq -M\tilde{C}\epsilon^2$$

then we get (4.2.10).

Case $(\nu_x - \nu_z)_V^2 < (4 - \Theta)$

Consider two rotations $P_{\mathbf{v}}$ and $P_{-\mathbf{v}}$ as follows: The first column vectors of $P_{-\mathbf{v}}$ and $P_{\mathbf{v}}$ are $\mathbf{v}$ and $-\mathbf{v}$, respectively. And other column vectors are the same. Then we observe,

$$
\begin{aligned}
Tf_1(x, z, P_{-\mathbf{v}}, P_{\mathbf{v}}) &- f_1(x, z) \\
&\leq -2\alpha C\omega'(|x - z|)\epsilon + 2\alpha C\omega''(|x - z|)\epsilon^2 + (4M + 10|x - z|^{\gamma-2})\epsilon^2 \\
&\leq -2\alpha C\omega'(|x - z|)\epsilon + (4M + 10|x - z|^{\gamma-2})\epsilon^2,
\end{aligned}
$$

and thus

$$
\begin{aligned}
\frac{1}{2}\{Tf_1(x, z, &P_{\nu_x}, P_{\nu_z}) + Tf_1(x, z, P_{-\mathbf{v}}, P_{\mathbf{v}})\} - f_1(x, z) \\
&\leq \alpha C\omega'(|x - z|)\{(\nu_x - \nu_z)_V - 2\}\epsilon + 2\alpha M\langle x + z, \nu_x + \nu_z\rangle\epsilon \\
&\quad + \frac{1}{2}C\frac{\omega'(|x - z|)}{|x - z|}\{\alpha|(\nu_x - \nu_z)_{V^\perp}|^2 + \beta|\nu_x + \nu_z|^2\}\epsilon^2 \\
&\quad + (4M + 10|x - z|^{\gamma-2})\epsilon^2.
\end{aligned}
$$

Set

$$\kappa = \frac{|(\nu_x - \nu_z)_{V^\perp}|^2}{\Theta}.$$

Then $1 < \kappa \leq \frac{4}{\Theta}$ by the assumption. Observe that

$$|(\nu_x - \nu_z)_V| \leq \sqrt{|\nu_x - \nu_z|^2 - \kappa\Theta} \leq \sqrt{4 - \kappa\Theta} \leq 2 - \frac{\kappa}{4}\Theta$$

and hence
$$|(\nu_x - \nu_z)_{V^\perp}| \leq 4(2 - (\nu_x - \nu_z)_V).$$

On the other hand, we have

$$
\begin{aligned}
|\nu_x + \nu_z|^2 &= 4 - |\nu_x - \nu_z|^2 \\
&\leq 4 - (\nu_x - \nu_z)_V^2 \\
&\leq 4(2 - (\nu_x - \nu_z)_V).
\end{aligned}
$$

We observe that

$$\frac{1}{2}\{Tf_1(x, z, P_{\nu_x}, P_{\nu_z}) + Tf_1(x, z, P_{-\mathbf{v}}, P_{\mathbf{v}})\} - f_1(x, z)$$

$$\leq 2\alpha M\langle x + z, \nu_x + \nu_z\rangle\epsilon + (2 - (\nu_x - \nu_z)_V)C\omega'(|x - z|)\left(-\alpha + \frac{20}{N}\right)\epsilon$$

$$+ (4M + 10|x - z|^{\gamma-2})\epsilon^2,$$

as $|x - z| > N\epsilon/10$.

Next we estimate $M\langle x + z, \nu_x + \nu_z\rangle\epsilon$. We already know that u_ϵ satisfies Hölder type estimate for any exponent $\delta \in (0, 1)$ by Theorem 4.2.5. Now by the counter assumption (4.2.16),

$$u_\epsilon(x, t) - u_\epsilon(z, s) - C\omega(|x - z|) - M|x + z|^2 - g(t, s) \geq K - \eta > 0.$$

Then we see

$$M|x + z|^2 < u_\epsilon(x, t) - u_\epsilon(z, s) \leq C_{u_\epsilon}(|x - z|^{1/2} + \epsilon^{1/2}).$$

Note that C_{u_ϵ} is a constant depending only on r, α and n. Thus, we obtain that

$$
\begin{aligned}
|x + z| &< \sqrt{\frac{C_{u_\epsilon}}{M}}(|x - z|^{1/2} + \epsilon^{1/2})^{1/2} \\
&\leq \sqrt{\frac{C_{u_\epsilon}}{M}}\left[|x - z|^{1/4} + \frac{1}{2}|x - z|^{-1/4}\epsilon^{1/2} + o(\epsilon^{1/2})\right] \\
&\leq \sqrt{\frac{C_{u_\epsilon}}{M}}\left[|x - z|^{1/4} + \frac{1}{2}\left(\frac{10}{N}\right)^{1/4}\epsilon^{1/4} + o(\epsilon^{1/2})\right].
\end{aligned}
$$

Hence we observe

$$M\langle x + z, \nu_x + \nu_z\rangle\epsilon \leq 2M|x + z|\epsilon$$

$$\leq 2\sqrt{MC_{u_\epsilon}}|x - z|^{1/4}\epsilon + \sqrt{MC_{u_\epsilon}}\left(\frac{10}{N}\right)^{1/4}\epsilon^{5/4} + o(\epsilon^{3/2})$$

$$\leq 3\sqrt{MC_{u_\epsilon}}|x - z|^{1/4}\epsilon$$

since $\sqrt{MC_{u_\epsilon}}(10/N)^{1/4}\epsilon^{5/4} + o(\epsilon^{3/2})$ is bounded by $\sqrt{MC_{u_\epsilon}}|x - z|^{1/2}\epsilon$. Therefore, if we choose $\gamma = 1 + s = 5/4$,

$$\frac{1}{2}\{Tf_1(x, z, P_{\nu_x}, P_{\nu_z}) + Tf_1(x, z, P_{-\mathbf{v}}, P_{\mathbf{v}})\} - f_1(x, z)$$

$$\leq 6\alpha\sqrt{MC_{u_\epsilon}}|x - z|^s\epsilon + C\omega'(|x - z|)\times$$

$$\left[-\alpha\kappa\frac{|x - z|^s}{4} + \frac{5}{N}\left\{\alpha|(\nu_x - \nu_z)_{V^\perp}|^2 + \frac{\beta}{n + 1}|(\rho_x - \rho_z)_{V^\perp}|^2\right\}\right]\epsilon$$

$$+ (4M + 10|x - z|^{s-1})\epsilon^2.$$

Note that

$$(4M + 10|x - z|^{s-1})\epsilon^2 \leq (4M + 10)\frac{10}{N}|x - z|^s\epsilon.$$

Then

$$\frac{1}{2}\{Tf_1(x, z, P_{\nu_x}, P_{\nu_z}) + Tf_1(x, z, P_{-\mathbf{v}}, P_{\mathbf{v}})\} - f_1(x, z)$$

$$\leq 6\alpha\sqrt{MC_{u_\epsilon}}|x - z|^s\epsilon + (2 - (\nu_x - \nu_z)_V)C\omega'(|x - z|)\left(-\alpha + \frac{20}{N}\right)\epsilon$$

$$+ (4M + 10)\frac{10}{N}|x - z|^s\epsilon.$$

Since we already know that $\kappa\Theta/4 \leq 2 - (\nu_x - \nu_z)_V$ and $\omega' \in [1/2, 1]$, we see that

$$\frac{1}{2}\{Tf_1(x, z, P_{\nu_x}, P_{\nu_z}) + Tf_1(x, z, P_{-\mathbf{v}}, P_{\mathbf{v}})\} - f_1(x, z)$$

$$\leq \left[6\alpha\sqrt{MC_{u_\epsilon}} + C\left(-\frac{\alpha}{8} + \frac{5}{N}\right)\frac{|(\nu_x - \nu_z)_{V^\perp}|^2}{|x - z|^s} + (4M + 10)\frac{10}{N}\right]|x - z|^s\epsilon$$

$$\leq \left[6\alpha\sqrt{MC_{u_\epsilon}} + C\left(-\frac{\alpha}{8} + \frac{5}{N}\right) + (4M + 10)\frac{10}{N}\right]|x - z|^s\epsilon.$$

Fix $N > 100/\alpha$ and choose $C = C(r, \alpha, n)$ large enough so that

$$6\alpha\sqrt{MC_{u_\epsilon}} + C\left(-\frac{\alpha}{8} + \frac{5}{N}\right) + (4M + 10)\frac{10}{N} < 0.$$

Then we conclude that

$$\frac{1}{2}\left\{Tf_1(x, z, P_{\nu_x}, P_{\nu_z}) + Tf_1(x, z, P_{-\mathbf{v}}, P_{\mathbf{v}})\right\} - f_1(x, z)$$

$$\leq \frac{N}{10}\left[6\alpha\sqrt{MC_{u_\epsilon}} + C\left(-\frac{\alpha}{8} + \frac{5}{N}\right) + (4M + 10)\frac{10}{N}\right]|x - z|^{s-1}\epsilon^2$$

$$\leq -M\tilde{C}\epsilon^2,$$

since $|x - z| > N\epsilon/10$. Now we obtained the desired result.

Case $0 < |x - z| \leq N\epsilon/10$

It is quite similar to the Hölder case. First, we see that for any $x, z \in B_r$ and $h_x, h_z \in S^{n-1}$,

$$|f_1(x + \epsilon h_x, z + \epsilon h_z) - f_1(x, z)|$$

$$\leq C\left|\omega(|x + \epsilon h_x - z - \epsilon h_z|) - \omega(|x - z|)\right|$$

$$\quad + M\left||x + \epsilon h_x + z + \epsilon h_z|^2 - |x + z|^2\right|$$

$$\leq C\left(\left||x + \epsilon h_x - z - \epsilon h_z| - |x - z|\right| + \omega_0\left||x + \epsilon h_x - z - \epsilon h_z|^\gamma - |x - z|^\gamma\right|\right)$$

$$\quad + M\left||x + \epsilon h_x + z + \epsilon h_z|^2 - |x + z|^2\right|$$

$$\leq 2C\epsilon + 2C\omega_0\gamma(2r)^{\gamma-1}(2\epsilon) + 8Mr\epsilon + 4M\epsilon^2.$$

Then we can choose a constant $C > 0$ such that

$$|f_1(x + \epsilon h_x, z + \epsilon h_z) - f_1(x, z)| \leq 20C\epsilon.$$

As in the previous section,

$$\sup_{h_x, h_z \in S^{n-1}} Tf_1(x, z, P_{h_x}, P_{h_z}) - f_1(x, z)$$

$$= \sup_{h_x, h_z \in S^{n-1}}\left[\alpha\{f_1(x + \epsilon h_x, z + \epsilon h_z) - f_1(x, z)\}\right.$$

$$+ \beta \fint_{B_\epsilon^{e_1}} \{f_1(x + P_{h_x}h, z + P_{h_z}h) - f_1(x, z)\} d\mathcal{L}^{n-1}(h) \Bigg]$$

$$\leq 20\alpha C\epsilon + 20\beta C\epsilon = 20C\epsilon$$

and note that (4.2.15) is still valid here. We can find $i \in \{1, 2, \cdots, N\}$ such that $(i-1)\frac{\epsilon}{10} < |x - z| \leq i\frac{\epsilon}{10}$ as in the previous section. Now, if C is large enough,

$$\inf_{h_x, h_z \in S^{n-1}} Tf(x, z, P_{h_x}, P_{h_z})$$

$$\leq \sup_{h_x, h_z \in S^{n-1}} Tf_1(x, z, P_{h_x}, P_{h_z}) - \sup_{h_x, h_z \in S^{n-1}} Tf_2(x, z, P_{h_x}, P_{h_z})$$

$$\leq \sup_{h_x, h_z \in S^{n-1}} Tf_1(x, z, P_{h_x}, P_{h_z}) - \alpha C^{2(N-i+1)}\epsilon$$

$$= \sup_{h_x, h_z \in S^{n-1}} Tf_1(x, z, P_{h_x}, P_{h_z}) - \alpha \left(C^2 - \frac{2}{\alpha} \right) C^{2(N-i)}\epsilon - 2C^{2(N-i)}\epsilon$$

$$\leq \sup_{h_x, h_z \in S^{n-1}} Tf_1(x, z, P_{h_x}, P_{h_z}) - 2f_2(x, z) - 50C\epsilon.$$

Therefore, we calculate that

$$\operatorname*{midrange}_{h_x, h_z \in S^{n-1}} Tf(x, z, P_{h_x}, P_{h_z})$$

$$\leq \sup_{h_x, h_z \in S^{n-1}} Tf_1(x, z, P_{h_x}, P_{h_z}) - f_2(x, z) - 25C\epsilon$$

$$\leq f_1(x, z) + 20C\epsilon - f_2(x, z) - 25C\epsilon.$$

We finally choose a large constant $C > M$ depending only on r, α and n to obtain (4.2.10).

Case $|x - z| = 0$

Similar to the previous section, we already showed that

$$|u_\epsilon(x, t) - u_\epsilon(z, s)| \leq C_2 ||u_\epsilon||_{L^\infty(\overline{\Omega}_\epsilon, T)}(|x - z| + \epsilon),$$

for any x, z $(x \neq z) \in B_r(0)$, $-r^2 < t < 0$, $|t - s| < \epsilon^2/2$ and some $C_2 = C_2(r, \alpha, n) > 0$. Then we can obtain the desired result by using the same argument as in Section 4.2.2. $\qquad \square$

Now Lemma 4.2.2 and Lemma 4.2.6 yield the Lipschitz type regularity in the whole cylinder. We remark here that if the boundary data F is bounded, u_ϵ satisfies

$$||u_\epsilon||_{L^\infty(\Omega_T)} \leq ||F||_{L^\infty(\Gamma_{\epsilon,T})}$$

(see [55, 62]). Then we can complete the proof of Theorem 4.2.1.

4.3 Boundary estimates

We now consider regularity for functions u_ϵ satisfying (4.0.1) near the boundary.

For boundary estimates, we need to consider a suitable boundary regularity condition. To this end, we introduce a boundary regularity condition for the domain Ω.

Definition 4.3.1 (Exterior sphere condition). *We say that a domain Ω satisfies an exterior sphere condition if for any $y \in \partial\Omega$, there exists $B_\delta(z) \subset \mathbb{R}^n \backslash \Omega$ with $\delta > 0$ such that $y \in \partial B_\delta(z)$.*

Throughout this section, we always assume that Ω satisfies Definition 4.3.1 and $\Omega \subset B_R(z)$ for some $R > 0$. We also assume that the boundary data F satisfies

$$|F(x,t) - F(y,s)| \leq L(|x-y| + |t-s|^{1/2}) \tag{4.3.1}$$

for any $(x,t), (y,s) \in \Gamma_{\epsilon,T}$ and some $L > 0$.

Let $y \in \partial\Omega$ and take $z \in \mathbb{R}^n \backslash \Omega$ with $B_\delta(z) \subset \mathbb{R}^n \backslash \Omega$ and $y \in \partial B_\delta(z)$. We consider a time-independent tug-of-war game. Assume that the rules to move the token are the same as that of the original game, but of course, we do not consider the time parameter t in this case. We also assume that the token cannot escape outside $\overline{B}_R(z)$ and the game ends only if the token is located in $\overline{B}_\delta(z)$. Now we fix specific strategies for both players. For each $k = 0, 1, \ldots$, assume that Player I and II takes the vector $\nu_k^{\mathrm{I}} = -\frac{x_k - z}{|x_k - z|}$ and $\nu_k^{\mathrm{II}} = \frac{x_k - z}{|x_k - z|}$, respectively. We write these strategies for Player I, II as S_{I}^z and S_{II}^z. On the other hand, we need to define strategies and random processes when $B_\epsilon(x_k) \backslash B_R(z) \neq \varnothing$. In this case, x_{k+1} is defined by $x_k + \epsilon\nu_k^{\mathrm{I}}$ if Player I wins coin toss twice and

$$x_k + \mathrm{dist}(x_k, \partial B_R(z))\nu_k^{\mathrm{II}} = z + R\frac{x_k - z}{|x_k - z|}$$

if Player II wins coin toss twice. When random walk occurs, x_{k+1} is chosen uniformly in $B_\epsilon^{\nu_k^I}(x) \cap B_R(z)$.

We denote by

$$\tau^* = \inf\{k : x_k \in \overline{B}_\delta(z)\}.$$

The following lemma gives an estimate for the the stopping time τ^*.

Lemma 4.3.2. *Under the setting as above, we have*

$$\mathbb{E}_{S_I^z, S_{II}^z}^{x_0}[\tau^*] \leq \frac{C(n, \alpha, R/\delta)(\text{dist}(\partial B_\delta(y), x_0) + o(1))}{\epsilon^2}$$

for any $x_0 \in \Omega \subset B_R(z) \backslash \overline{B}_\delta(z)$. Here $o(1) \to 0$ as $\epsilon \to 0$ and $\lceil x \rceil$ means the least integer greater than or equal to $x \in \mathbb{R}$.

Proof. Set $g_\epsilon(x) = \mathbb{E}_{S_I^z, S_{II}^z}^x[\tau^*]$. Then we observe that g_ϵ satisfies the following DPP

$$g_\epsilon(x) = \frac{1}{2}\left[\left\{\alpha g_\epsilon(x + \rho_x \epsilon \nu_x) + \beta \fint_{B_\epsilon^{\nu_x}(x) \cap B_R(z)} g_\epsilon(y) d\mathcal{L}^{n-1}(y)\right\} \right.$$
$$\left. + \left\{\alpha g_\epsilon(x - \epsilon \nu_x) + \beta \fint_{B_\epsilon^{\nu_x}(x) \cap B_R(z)} g_\epsilon(y) d\mathcal{L}^{n-1}(y)\right\}\right] + 1,$$

where $\rho_x = \min\{1, \epsilon^{-1} \text{dist}(x, \partial B_R(z))\}$ and $\nu_x = (x - z)/|x - z|$. Note that $\rho_x = 1$ for any $x \in B_{R-\epsilon}(z) \backslash \overline{B}_\delta(z)$. Next we define $v_\epsilon = \epsilon^2 g_\epsilon$. It is straightforward that

$$v_\epsilon(x) = \frac{\alpha}{2}\left(v_\epsilon(x + \rho_x \epsilon \nu_x) + v_\epsilon(x - \epsilon \nu_x)\right)$$
$$+ \beta \fint_{B_\epsilon^{\nu_x}(x) \cap B_R(z)} v_\epsilon(y) d\mathcal{L}^{n-1}(y) + \epsilon^2. \tag{4.3.2}$$

From the definition of v_ϵ and (4.3.2), we observe that the function v_ϵ is rotationally symmetric, that is, v_ϵ is a function of $r = |x - z|$. If we denote by $v_\epsilon(x) = V(r)$, the DPP (4.3.2) can be represented by

$$V(r) = \frac{\alpha}{2}\left(V(r + \rho_r \epsilon) + V(r - \epsilon)\right)$$
$$+ \beta \fint_{B_\epsilon^{\nu_x}(x) \cap B_R(z)} V(|y - z|) d\mathcal{L}^{n-1}(y) + \epsilon^2, \tag{4.3.3}$$

where $\rho_r = \min\{1, \epsilon^{-1}(R - r)\}$.

Now we can deduce that (4.3.3) has a connection to the following problem

$$
\begin{cases}
\frac{1-\alpha}{2r}\frac{n-1}{n+1}w' + \frac{\alpha}{2}w'' = -1 & \text{when } r \in (\delta, R + \epsilon), \\
w(\delta) = 0, \\
w'(R + \epsilon) = 0
\end{cases}
$$

by using Taylor expansion. Note that if we set $v(x) = w(|x|)$,

$$
\frac{1 - \alpha}{2r}\frac{n - 1}{n + 1}w' + \frac{\alpha}{2}w'' = -1
$$

can be transformed by

$$
\Delta_p^N v = -2(p + n),
$$

where $p = (1 + n\alpha)/(1 - \alpha)$ (for the definition of Δ_p^N, see the next section). On the other hand, we have

$$
w(r) =
\begin{cases}
-\frac{n+1}{2\alpha+n-1}r^2 + c_1 r^{\frac{2\alpha n - n + 1}{(n+1)\alpha}} + c_2 & \text{when } \alpha \neq \frac{n-1}{2n}, \\
-\frac{n}{n-1}r^2 + c_1 \log r + c_2 & \text{when } \alpha = \frac{n-1}{2n}
\end{cases}
$$

by direct calculation. Here

$$
c_1 =
\begin{cases}
\frac{2(n+1)^2\alpha}{(2\alpha+n-1)(2\alpha n - n + 1)}(R + \epsilon)^{\frac{n + 2\alpha - 1}{(n+1)\alpha}} & \text{when } \alpha \neq \frac{n-1}{2n}, \\
\frac{2n}{n-1}(R + \epsilon)^2 & \text{when } \alpha = \frac{n-1}{2n}
\end{cases}
$$

is positive if $\alpha \geq \frac{n-1}{2n}$ and negative otherwise. We extend this function to the interval $(\delta - \epsilon, R + \epsilon]$.

Observe that

$$
\frac{\alpha}{2}\big(w(r + \epsilon) + w(r - \epsilon)\big) + \beta \fint_{B_\epsilon^{\nu_x}(x)} w(|y - z|)d\mathcal{L}^{n-1}(y)
$$

$$
= w(r) - \frac{n + 1}{2\alpha + n - 1}\left(\alpha + \frac{n - 1}{n + 1}\beta\right)\epsilon^2 + o(\epsilon^2)
$$

$$
\leq w(r) - \left[\frac{n + 1}{2\alpha + n - 1}\left(\alpha + \frac{n - 1}{n + 1}\beta\right) - \eta\right]\epsilon^2
$$

for some $\eta > 0$ when $\alpha \neq \frac{n-1}{2n}$ (we can also obtain a similar estimate if

$\alpha = \frac{n-1}{2n}$). Set

$$c_0 := \frac{n+1}{2\alpha + n - 1}\left(\alpha + \frac{n-1}{n+1}\beta\right) - \eta > 0.$$

Then we have

$$\mathbb{E}^{x_0}_{S^z_{\mathrm{I}}, S^z_{\mathrm{II}}}[v(x_k) + kc_0\epsilon^2 | x_0, \ldots, x_{k-1}]$$
$$= \alpha\left(v(x_{k-1} + \epsilon\nu_{x_{k-1}}) + v(x_{k-1} - \epsilon\nu_{x_{k-1}})\right)$$
$$+ \beta \fint_{B^{\nu_{x_{k-1}}}_{\epsilon}(x_{k-1})} v(y-z)d\mathcal{L}^{n-1}(y) + kc_0\epsilon^2$$
$$\leq v(x_{k-1}) + (k-1)c_0\epsilon^2,$$

if $B_\epsilon(x_{k-1}) \subset B_R(z)\backslash\overline{B}_{\delta-\epsilon}(z)$. The same estimate can be derived in the case $B_\epsilon(x_{k-1})\backslash B_R(z) \neq \varnothing$ since w is an increasing function of r and it implies

$$v(x + \rho_x\epsilon\nu_x) \leq v(x + \epsilon\nu_x)$$

and

$$\fint_{B^{\nu_x}_{\epsilon}(x)\cap B_R(y)} v(y-z)d\mathcal{L}^{n-1}(y) \leq \fint_{B^{\nu_x}_{\epsilon}(x)} v(y-z)d\mathcal{L}^{n-1}(y).$$

Now we see that $v(x_k) + kc_0\epsilon^2$ is a supermartingale. By the optional stopping theorem, we have

$$\mathbb{E}^{x_0}_{S^z_{\mathrm{I}}, S^z_{\mathrm{II}}}[v(x_{\tau^*\wedge k}) + (\tau^* \wedge k)c_0\epsilon^2] \leq v(x_0). \tag{4.3.4}$$

We also check that
$$0 \leq -\mathbb{E}^{x_0}_{S^z_{\mathrm{I}}, S^z_{\mathrm{II}}}[v(x_{\tau^*})] \leq o(1),$$

since $x_{\tau^*} \in \overline{B}_\delta(z)\backslash\overline{B}_{\delta-\epsilon}(z)$.

Meanwhile, it can be also observed that $w' > 0$ is a decreasing function in the interval $(\delta, R + \epsilon)$ and thus

$$w' \leq \frac{2(n+1)}{2\alpha + n - 1}\delta\left[\left(\frac{R+\epsilon}{\delta}\right)^{\frac{n+2\alpha-1}{(n+1)\alpha}} - 1\right] \qquad \text{in } (\delta, R + \epsilon).$$

From the above estimate, we have

$$0 \leq w(x_0) \leq C(n, \alpha, R/\delta) \operatorname{dist}(\partial B_\delta(y), x_0). \qquad (4.3.5)$$

Finally, combining (4.3.5) with (4.3.4) and passing to a limit with k, we have

$$c_0 \epsilon^2 \mathbb{E}_{S_I^z, S_{II}^z}^{x_0}[\tau^*] \leq w(x_0) - \mathbb{E}_{S_I^z, S_{II}^z}^{x_0}[w(x_{\tau^*})]$$
$$\leq C(n, \alpha, R/\delta) \operatorname{dist}(\partial B_\delta(y), x_0) + o(1)$$

and it gives our desired estimate. $\qquad \square$

By means of Lemma 4.3.2, we can deduce following boundary regularity results. First, we give an estimate for u_ϵ on the lateral boundary.

Theorem 4.3.3. *Assume that Ω satisfies the exterior sphere condition and F satisfies (4.3.1). Then for the value function u_ϵ with boundary data F, we have*

$$|u_\epsilon(x, t) - u_\epsilon(y, s)|$$
$$\leq C(n, \alpha, R, \delta, L)(K + K^{1/2}) + L(|x - y| + |t - s|^{1/2} + 2\delta), \qquad (4.3.6)$$

where $K = \min\{|x - y|, t\} + \epsilon$ and R, δ are the constants in Lemma 4.3.2 for every $(x, t) \in \Omega_T$ and $(y, s) \in O_{\epsilon, T}$.

Proof. We first consider the case $t = s$. Set $N = \lceil 2t/\epsilon^2 \rceil$. Since Ω satisfies the exterior sphere condition, we can find a ball $B_\delta(z) \subset \mathbb{R}^n \backslash \Omega$ such that $y \in \partial B_\delta(z)$. Assume that Player I takes a strategy S_I^z of pulling towards z.

We estimate the expected value for the distance $|x_\tau - x_0|$ under the game setting. Let θ be the angle between ν and $x - z$. And we assume that $x = 0$ and $z = (0, \cdots, 0, r \sin \theta, -r \cos \theta)$ by using a proper transformation. Then the following term

$$\alpha|x + \epsilon\nu - z| + \beta \fint_{B_\epsilon^{\nu_x}(x)} |\tilde{x} - z| d\mathcal{L}^{n-1}(\tilde{x})$$

can be written as

$$A(\theta)$$
$$= \alpha\sqrt{(r \sin \theta)^2 + (r \cos \theta + \epsilon)^2} + \beta \fint_{T_\epsilon} \sqrt{(y - r \sin \theta)^2 + (r \cos \theta)^2} d\mathcal{L}^{n-1}(y)$$

$$= \alpha\sqrt{r^2 + 2r\epsilon\cos\theta + \epsilon^2} + \beta \fint_{T_\epsilon} \sqrt{r^2 - 2ry_{n-1}\sin\theta + |y|^2} d\mathcal{L}^{n-1}(y)$$

$$=: \alpha A_1(\theta) + \beta A_2(\theta),$$

where $r = |x - z|$ and $T_\epsilon = \{x = (x_1, \ldots, x_n) \in B_\epsilon(0) : x_n = 0\}$. Observe that A_1 is decreasing in the interval $(0, \pi)$. (Thus, A_1 has the maximum at $\theta = 0$ in $[0, \pi]$) On the other hand, we have

$$A_2'(\theta) = -\fint_{T_\epsilon} \frac{ry_{n-1}\cos\theta}{\sqrt{r^2 - 2ry_{n-1}\sin\theta + |y|^2}} d\mathcal{L}^{n-1}(y)$$

and this function is a symmetric function about $\theta = \pi/2$. We also check that $A_2' < 0$ in $(0, \pi/2)$. Thus, we verify that A_2 has a maximum at $\theta = 0, \pi$ in $[0, \pi]$ and $\theta(0) = \theta(\pi)$. This leads to the following estimate

$$\sup_{\nu \in S^{n-1}} \left[\alpha|x + \epsilon\nu - z| + \beta \fint_{B_\epsilon^\nu(x)} |\tilde{x} - z| d\mathcal{L}^{n-1}(\tilde{x}) \right]$$

$$= \alpha(|x - z| + \epsilon) + \beta \fint_{B_\epsilon^{\nu_x}(x)} |\tilde{x} - z| d\mathcal{L}^{n-1}(\tilde{x}), \tag{4.3.7}$$

where $\nu_x = (x - z)/|x - z|$.

Therefore, we have

$$\mathbb{E}_{S_{\mathrm{I}}^z, S_{\mathrm{II}}}^{(x_0, t)}[|x_k - z| |(x_0, t_0), \ldots, (x_{k-1}, t_{k-1})]$$

$$\leq \frac{1 - \delta_\epsilon(x_{k-1}, t_{k-1})}{2} \left[\alpha(|x_{k-1} - z| - \epsilon) + \beta \fint_{B_\epsilon^{\nu_{x_{k-1}}}(x_{k-1})} |\tilde{x} - z| d\mathcal{L}^{n-1}(\tilde{x}) \right]$$

$$+ \frac{1 - \delta_\epsilon(x_{k-1}, t_{k-1})}{2} \left[\alpha(|x_{k-1} - z| + \epsilon) + \beta \fint_{B_\epsilon^{\nu_{x_{k-1}}}(x_{k-1})} |\tilde{x} - z| d\mathcal{L}^{n-1}(\tilde{x}) \right]$$

$$+ \delta_\epsilon(x_{k-1}, t_{k-1})|x_{k-1} - z|$$

$$= |x_{k-1} - z|$$

$$+ \beta(1 - \delta_\epsilon(x_{k-1}, t_{k-1})) \left(\fint_{B_\epsilon^{\nu_{x_{k-1}}}(x_{k-1})} |\tilde{x} - z| d\mathcal{L}^{n-1}(\tilde{x}) - |x_{k-1} - z| \right).$$

We also observe that

$$0 < \beta(1 - \delta_\epsilon(x_{k-1}, t_{k-1})) < 1,$$

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$$|x_{k-1} - z| \le |\tilde{x} - z| \le \sqrt{(x_{k-1} - z)^2 + \epsilon^2} \qquad \text{for } x \in B_\epsilon^{\nu_{x_{k-1}}},$$

and

$$0 < \sqrt{a^2 + \epsilon^2} - a < \frac{\epsilon^2}{2a} \qquad \text{for } a > 0.$$

Therefore,

$$\mathbb{E}^{(x_0,t)}_{S_{\mathrm{I}}^z, S_{\mathrm{II}}}[|x_k - z| \,|\, (x_0, t_0), \ldots, (x_{k-1}, t_{k-1})] \le |x_{k-1} - z| + C\epsilon^2$$

for some $C = C(n, \delta) > 0$. This yields that

$$M_k = |x_k - z| - Ck\epsilon^2$$

is a supermartingale.

Applying the optional stopping theorem and Jensen's inequality to M_k, we derive that

$$
\begin{aligned}
\mathbb{E}^{(x_0,t)}_{S_{\mathrm{I}}^z, S_{\mathrm{II}}}&[|x_\tau - z| + |t_\tau - t|^{1/2}] \\
&= \mathbb{E}^{(x_0,t)}_{S_{\mathrm{I}}^z, S_{\mathrm{II}}}\left[|x_\tau - z| + \epsilon\sqrt{\frac{\tau}{2}}\right] \\
&\le |x_0 - z| + C\epsilon^2 \mathbb{E}^{(x_0,t)}_{S_{\mathrm{I}}^z, S_{\mathrm{II}}}[\tau] + C\epsilon\big(\mathbb{E}^{(x_0,t)}_{S_{\mathrm{I}}^z, S_{\mathrm{II}}}[\tau]\big)^{1/2}.
\end{aligned}
\tag{4.3.8}
$$

Next we need to obtain estimates for $\mathbb{E}^{(x_0,t)}_{S_{\mathrm{I}}^z, S_{\mathrm{II}}}[\tau]$. To do this, we use the result in Lemma 4.3.2. We can check that the exit time τ of the original game is bounded by τ^* because the expected value of $|x_k - z|$ for given $|x_{k-1} - z|$ is maximized when Player II chooses the strategy S_{II}^z from (4.3.7). Thus, we have

$$
\begin{aligned}
\mathbb{E}^{(x_0,t)}_{S_{\mathrm{I}}^z, S_{\mathrm{II}}}[\tau] &\le \min\{\mathbb{E}^{x_0}_{S_{\mathrm{I}}^z, S_{\mathrm{II}}^z}[\tau^*], N\} \\
&\le \min\left\{\frac{C(n, \alpha, R/\delta)(\mathrm{dist}(\partial B_\delta(z), x_0) + \epsilon)}{\epsilon^2}, N\right\}
\end{aligned}
$$

for any strategy S_{II} for Player II. We also see that

$$\mathrm{dist}(x_0, \partial B_\delta(z)) \le |x_0 - y|.$$

This and (4.3.8) imply

$$\mathbb{E}^{(x_0,t)}_{S_{\mathrm{I}}^z,S_{\mathrm{II}}}[|x_\tau - z| + |t_\tau - t|^{1/2}]$$
$$\leq |x_0 - y| + C\min\{|x_0 - y| + \epsilon, \epsilon^2 N\} + C\min\{|x_0 - y| + \epsilon, \epsilon^2 N\}^{1/2},$$

where C is a constant depending on n, α, R and δ. Therefore, we get

$$|\mathbb{E}^{(x_0,t)}_{S_{\mathrm{I}}^z,S_{\mathrm{II}}}[F(x_\tau, t_\tau)] - F(z,t)|$$
$$\leq L(|x_0 - y| + C(n, \alpha, R, \delta)\min\{|x_0 - y| + \epsilon, \epsilon^2 N\}$$
$$+ C(n, \alpha, R/\delta)\min\{|x_0 - y| + \epsilon, \epsilon^2 N\}^{1/2})$$

and this yields

$$u_\epsilon(x_0, t) = \sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} \mathbb{E}^{(x_0,t)}_{S_{\mathrm{I}},S_{\mathrm{II}}}[F(x_\tau, t_\tau)]$$
$$\geq \inf_{S_{\mathrm{II}}} \mathbb{E}^{(x_0,t)}_{S_{\mathrm{I}}^z,S_{\mathrm{II}}}[F(x_\tau, t_\tau)]$$
$$\geq F(z,t) - L\{C(n,\alpha,R,\delta)(K + K^{1/2}) + |x_0 - y|\}$$
$$\geq F(y,t) - C(n,\alpha,R,\delta,L)(K + K^{1/2}) - L(|x_0 - y| + 2\delta)$$

for $K = \min\{|x_0 - y| + \epsilon, \epsilon^2 N\}$. Note that we can also derive the upper bound for $u_\epsilon(x_0, t)$ by taking the strategy where Player II pulls toward to z.

Meanwhile, in the case of $t \neq s$, we have

$$|u_\epsilon(x,t) - u_\epsilon(y,s)|$$
$$\leq |u_\epsilon(x,t) - u_\epsilon(y,t)| + |u_\epsilon(y,t) - u_\epsilon(y,s)|$$
$$\leq C(n,\alpha,R/\delta,L)(K + K^{1/2}) + L(|x - y| + 2\delta) + L|t - s|^{1/2},$$

where $K = \min\{|x_0 - y| + \epsilon, \epsilon^2 N\}$ and $N = \lceil 2t/\epsilon^2 \rceil$. This gives our desired estimate. $\qquad\square$

We can also derive the following result on the initial boundary.

Theorem 4.3.4. *Assume that Ω satisfies the exterior sphere condition and F satisfies (4.3.1). Then for the value function u_ϵ with boundary data F, we have*

$$|u_\epsilon(x,t) - u_\epsilon(y,s)| \leq C(|x - y| + t^{1/2} + \epsilon) \qquad (4.3.9)$$

for every $(x,t) \in \Omega_T$ and $(y,s) \in \Omega \times (-\epsilon^2/2, 0]$. The constant C only depends on n and L.

Proof. Set $(x,t) = (x_0, t_0)$ and $N = \lceil 2t/\epsilon^2 \rceil$. As in the above lemma, we also estimate the expected value of the distance between y and the exit point x_τ. Consider the case that Player I chooses a strategy of pulling to y. When $|x_{k-1} - y| \geq \epsilon$, we have

$$\mathbb{E}^{(x_0,t_0)}_{S^y_{\mathrm{I}}, S_{\mathrm{II}}}[|x_k - y|^2 | (x_0, t_0), \ldots, (x_{k-1}, t_{k-1})]$$

$$\leq (1 - \delta_\epsilon(x_{k-1}, t_{k-1})) \times$$

$$\left[\frac{\alpha}{2} \{(|x_{k-1} - y| + \epsilon)^2 + (|x_{k-1} - y| - \epsilon)^2\} \right.$$

$$\left. + \beta \fint_{B^{\nu_{x_{k-1}}}_\epsilon (x_{k-1})} |\tilde{x} - y|^2 d\mathcal{L}^{n-1}(\tilde{x}) \right] + \delta_\epsilon(x_{k-1}, t_{k-1})|x_{k-1} - y|^2$$

$$\leq \alpha(|x_{k-1} - y|^2 + \epsilon^2) + \beta(|x_{k-1} - y|^2 + C\epsilon^2)$$

$$\leq |x_{k-1} - y|^2 + C\epsilon^2$$

for some constant $C > 0$ which is independent of ϵ. We recall the notation $\nu_{x_{k-1}} = (x_{k-1} - z)/|x_{k-1} - z|$ here. Otherwise, we also see that

$$\mathbb{E}^{(x_0,t_0)}_{S^y_{\mathrm{I}}, S_{\mathrm{II}}}[|x_k - y|^2 | (x_0, t_0), \ldots, (x_{k-1}, t_{k-1})]$$

$$\leq (1 - \delta_\epsilon(x_{k-1}, t_{k-1})) \left[\frac{\alpha}{2}(|x_{k-1} - y| + \epsilon)^2 + \beta \fint_{B^{\nu_{x_{k-1}}}_\epsilon (x_{k-1})} |\tilde{x} - y|^2 d\mathcal{L}^{n-1}(\tilde{x}) \right]$$

$$+ \delta_\epsilon(x_{k-1}, t_{k-1})|x_{k-1} - y|^2,$$

and then we get the same estimate as above since

$$(|x_{k-1} - y| + \epsilon)^2 \leq 2(|x_{k-1} - y|^2 + \epsilon^2).$$

Therefore, we see that

$$M_k = |x_k - y|^2 - Ck\epsilon^2$$

is a supermartingale.

Now we obtain

$$\mathbb{E}^{(x_0,t)}_{S^y_{\mathrm{I}}, S_{\mathrm{II}}}[|x_\tau - y|^2] \leq |x_0 - y|^2 + C\epsilon^2 \mathbb{E}^{(x_0,t)}_{S^y_{\mathrm{I}}, S_{\mathrm{II}}}[\tau]$$

by using the optional stopping theorem. Since $\tau < \lceil 2t/\epsilon^2 \rceil$, the right-hand side term is estimated by $|x_0 - y|^2 + C(t + \epsilon^2)$. Applying Jensen's inequality, we get

$$
\begin{aligned}
\mathbb{E}^{(x_0,t)}_{S^y_{\mathrm{I}},S_{\mathrm{II}}}[|x_\tau - y|] &\leq \left(\mathbb{E}^{(x_0,t)}_{S^y_{\mathrm{I}},S_{\mathrm{II}}}[|x_\tau - y|^2] \right)^{\frac{1}{2}} \\
&\leq \left(|x_0 - y|^2 + C(t + \epsilon^2) \right)^{\frac{1}{2}} \\
&\leq |x_0 - y| + C(t^{1/2} + \epsilon).
\end{aligned}
$$

From the above estimate, we deduce that

$$
\begin{aligned}
u_\epsilon(x_0, t) = \sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} \mathbb{E}^{(x_0,t)}_{S_{\mathrm{I}},S_{\mathrm{II}}}[F(x_\tau, t_\tau)] \\
\geq F(y, t) - L \, \mathbb{E}^{(x_0,t)}_{S^y_{\mathrm{I}},S_{\mathrm{II}}}[|x_\tau - y| + |t - t_\tau|^{1/2}] \\
\geq F(y, t) - C(|x_0 - y| + t^{1/2} + \epsilon).
\end{aligned}
$$

The upper bound can be derived in a similar way, and then we get the estimate (4.3.9). $\qquad\square$

4.4 Applications

4.4.1 Long-time asymptotics

In PDE theory, the study of asymptotic behavior of solutions of parabolic equations as time goes to infinity has drawn a lot of attention. We will have a similar discussion for our value function u_ϵ when the boundary data F does not depend on t in $\Gamma_\epsilon \times (\epsilon^2, \infty)$. The heuristic idea in this section can be summarized as follows. Assume that we start the game at (x_0, t_0) for sufficiently large t_0. Then we can expect that the probability of the game ending in the initial boundary would be close to zero, that is, the game finishes on the lateral boundary in most cases. Since we assumed that F is independent of t for $t > \epsilon^2$, we may consider this game as something like a time-independent game with the same boundary data. Thus, it is reasonable to guess that the value function of the time-dependent game converges that of the corresponding time-independent game. We refer the reader to [7] which contains a detailed discussion of asymptotic behaviors for value functions of evolution problems. Moreover, long-time asymptotics for related PDEs can

be found in [5, 74, 66, 63].

To observe the asymptotic behavior of value functions, we first need to obtain the following comparison principle. Since it can be shown in a straight-forward manner by using the DPP (4.0.1), we omit the proof. One can find similar results in [52, Theorem 5.3].

Lemma 4.4.1. *Let u and v be functions satisfying (4.0.1) with boundary data F_u and F_v, respectively. Suppose that $F_u \leq F_v$ in $\Gamma_{\epsilon,T}$. Then,*

$$u \leq v \qquad \text{in } \Omega_{\epsilon,T}.$$

Now we state the main result of this section.

Theorem 4.4.2. *Let Ω be a bounded domain. Consider functions $\psi \in C(\Gamma_\epsilon)$ and $\varphi \in C(\Gamma_{\epsilon,T} \cap \{t \leq 0\})$, and define a function $F \in C(\Omega_{\epsilon,T})$ as follows:*

$$F(x,t) = \begin{cases} \psi(x) & \text{in } \Gamma_\epsilon \times (\epsilon^2, T], \\ \varphi(x, \epsilon^2/2) + \frac{2t(\psi(x) - \varphi(x,\epsilon^2/2))}{\epsilon^2} & \text{in } \Gamma_\epsilon \times (\frac{\epsilon^2}{2}, \epsilon^2], \\ \varphi(x,t) & \text{in } \Omega_\epsilon \times [-\frac{\epsilon^2}{2}, \frac{\epsilon^2}{2}]. \end{cases} \quad (4.4.1)$$

Assume that u_ϵ is the function satisfying (4.0.1) with boundary data F. Then we have

$$\lim_{T \to \infty} u_\epsilon(x, T) = U_\epsilon(x)$$

where U_ϵ is the function satisfying the following DPP

$$U_\epsilon(x)$$
$$= (1 - \overline{\delta_\epsilon}(x)) \operatorname*{midrange}_{\nu \in S^{n-1}} \left[\alpha U_\epsilon(x + \epsilon\nu) + \beta \fint_{B_\epsilon^\nu} U_\epsilon(x + h) d\mathcal{L}^{n-1}(h) \right] \quad (4.4.2)$$
$$+ \overline{\delta_\epsilon}(x) \psi(x)$$

in Ω_ϵ with boundary data ψ where

$$\overline{\delta_\epsilon}(x) := \lim_{t \to \infty} \delta_\epsilon(x,t) = \begin{cases} 0 & \text{in } \Omega \backslash I_\epsilon, \\ 1 - \operatorname{dist}(x, \partial\Omega)/\epsilon & \text{in } I_\epsilon, \\ 1 & \text{in } O_\epsilon. \end{cases}$$

Remark 4.4.3. *We can find the existence and uniqueness of value functions under different setting in [52], which is related to the normalized p-Laplace operator for $p \geq 2$. In that paper, the existence of measurable strategies is*

shown without regularization. Thus, we do not have to consider a "regularized function" such as $\varphi(x, \epsilon^2/2) + 2t(\psi(x) - \varphi(x, \epsilon^2/2))/\epsilon^2$ in that case. Meanwhile, for the time-independent version of our settings, results for these issues are shown in [30].

Proof of Theorem 4.4.2. We will set some proper barrier functions $\underline{u}, \overline{u}$ such that

$$\underline{u} \leq u_\epsilon \leq \overline{u}$$

and show the coincidence for the limits of two barrier functions as $t \to \infty$. In our proof, the uniqueness result for elliptic games is essential. The motivation of this proof is from [2, Proposition 3.3].

Let $\underline{\varphi}, \overline{\varphi}$ be constants defined by

$$\underline{\varphi} = \min\{\inf_{\Gamma_\epsilon} \psi, \inf_{\Omega_\epsilon} \varphi\} \text{ and } \overline{\varphi} = \max\{\sup_{\Gamma_\epsilon} \psi, \sup_{\Omega_\epsilon} \varphi\},$$

respectively. We consider $\underline{u}, \overline{u}$ be functions satisfying (4.0.1) with boundary data $\underline{F}$ and $\overline{F}$, where

$$\underline{F}(x,t) = \begin{cases} \psi(x) & \text{in } \Gamma_\epsilon \times (\epsilon^2, T] \ , \\ \underline{\varphi} + 2t(\psi(x) - \underline{\varphi})/\epsilon^2 & \text{in } \Gamma_\epsilon \times (\frac{\epsilon^2}{2}, \epsilon^2], \\ \underline{\varphi} & \text{in } \Omega_\epsilon \times [-\frac{\epsilon^2}{2}, \frac{\epsilon^2}{2}], \end{cases}$$

and

$$\overline{F}(x,t) = \begin{cases} \psi(x) & \text{in } \Gamma_\epsilon \times (\epsilon^2, T], \\ \overline{\varphi} + 2t(\psi(x) - \overline{\varphi})/\epsilon^2 & \text{in } \Gamma_\epsilon \times (\frac{\epsilon^2}{2}, \epsilon^2], \\ \overline{\varphi} & \text{in } \Omega_\epsilon \times [-\frac{\epsilon^2}{2}, \frac{\epsilon^2}{2}], \end{cases}$$

respectively. Note that $\underline{F}$ and $\overline{F}$ are continuous in $\overline{\Gamma_{\epsilon,T}}$ and have constant initial data.

By Lemma 4.4.1, we have $\underline{u} \leq u_\epsilon \leq \overline{u}$. Thus it is sufficient to show that $\lim_{t\to\infty} \underline{u}(\cdot, t), \lim_{t\to\infty} \overline{u}(\cdot, t)$ exist and satisfy the limiting DPP (4.4.2). First we see that

$$||\underline{u}||_{L^\infty(\Omega_{\epsilon,T})} \leq ||\underline{F}||_{L^\infty(\Gamma_{\epsilon,T})} \text{ and } ||\overline{u}||_{L^\infty(\Omega_{\epsilon,T})} \leq ||\overline{F}||_{L^\infty(\Gamma_{\epsilon,T})}$$

by using the DPP of $\underline{u}$ and $\overline{u}$. Thus, these functions are uniformly bounded.

Next, we prove monotonicity of sequences $\{\underline{u}(x, t+j\epsilon^2/2)\}_{j=0}^\infty$ and $\{\overline{u}(x, t+$

$j\epsilon^2/2)\}_{j=0}^{\infty}$ for any $(x,t) \in \Omega_\epsilon \times (-\epsilon^2/2, 0]$. Without loss of generality, we only consider the case $\underline{u}$. Let (x_0, t_0) be a point in $\Omega \times (-\epsilon^2/2, 0]$ and denote by

$$a_j = \underline{u}(x_0, t_0 + j\epsilon^2/2)$$

for simplicity. For any $(x_0, t_0) \in \Omega \times (-\epsilon^2/2, 0]$, we can derive that

$$\underline{\psi} = a_0 = a_1 \le a_2$$

by direct calculation and

$$
\begin{aligned}
a_3 &= \left(1 - \delta_\epsilon\left(x_0, t_0 + \frac{3\epsilon^2}{2}\right)\right) \operatorname*{midrange}_{\nu \in S^{n-1}} \mathscr{A}_\epsilon u(x_0, t_0 + \epsilon^2; \nu) \\
&\quad + \delta_\epsilon\left(x_0, t_0 + \frac{3\epsilon^2}{2}\right) F\left(x_0, t_0 + \frac{3\epsilon^2}{2}\right) \\
&\ge (1 - \delta_\epsilon(x_0, t_0 + \epsilon^2)) \operatorname*{midrange}_{\nu \in S^{n-1}} \mathscr{A}_\epsilon u\left(x_0, t_0 + \frac{\epsilon^2}{2}; \nu\right) \\
&\quad + \delta_\epsilon(x_0, t_0 + \epsilon^2) F(x_0, t_0 + \epsilon^2) = a_2
\end{aligned}
$$

since $\delta_\epsilon(x_0, t_0 + \epsilon^2) = \delta_\epsilon(x_0, t_0 + 3\epsilon^2/2)$ and $F(x_0, t_0 + \epsilon^2) \le F(x_0, t_0 + 3\epsilon^2/2)$.

Next, assume that $a_k \ge a_{k-1}$ for some $k \ge 4$. Note that $F(x,t) = \psi(x)$ for $x \in \Gamma_\epsilon$ and

$$\delta_\epsilon\left(x_0, t_0 + \frac{k\epsilon^2}{2}\right) = \delta_\epsilon\left(x_0, t_0 + \frac{(k-1)\epsilon^2}{2}\right)$$

in this case. Then, we see

$$
\begin{aligned}
a_{k+1} &= \left(1 - \delta_\epsilon\left(x_0, t_0 + \frac{(k+1)\epsilon^2}{2}\right)\right) \operatorname*{midrange}_{\nu \in S^{n-1}} \mathscr{A}_\epsilon u\left(x_0, t_0 + \frac{k\epsilon^2}{2}; \nu\right) \\
&\quad + \delta_\epsilon\left(x_0, t_0 + \frac{(k+1)\epsilon^2}{2}\right) \psi(x_0) \\
&\ge \left(1 - \delta_\epsilon\left(x_0, t_0 + \frac{k\epsilon^2}{2}\right)\right) \operatorname*{midrange}_{\nu \in S^{n-1}} \mathscr{A}_\epsilon u\left(x_0, t_0 + \frac{(k-1)\epsilon^2}{2}; \nu\right) \\
&\quad + \delta_\epsilon\left(x_0, t_0 + \frac{k\epsilon^2}{2}\right) \psi(x_0) = a_k.
\end{aligned}
$$

Therefore, $\{a_j\}$ is increasing for any $(x_0, t_0) \in \Omega \times (-\epsilon^2/2, 0]$. It is also

possible to obtain $\{\overline{u}(x, t + j\epsilon^2/2)\}$ is decreasing by using similar arguments. Therefore, we obtain $\{\underline{u}(x, t + j\epsilon^2/2)\}$ and $\{\overline{u}(x, t + j\epsilon^2/2)\}$ converges for any $(x, t) \in \Omega_\epsilon \times (-\epsilon^2/2, 0]$ by applying the monotone convergence theorem.

Now we show that U_ϵ satisfies the DPP (4.4.2). Fix $-\epsilon^2/2 \le t_1 < 0$ arbitrary and write

$$\underline{U_{t_1}}(x) = \lim_{j \to \infty} \underline{u}(x, t_1 + j\epsilon^2/2)$$

for $x \in \Omega$. By definition of $\underline{u}$, we see that

$$\underline{U_{t_1}}(x) = (1 - \overline{\delta_\epsilon}(x)) \lim_{j \to \infty} \left[\underset{\nu \in S^{n-1}}{\mathrm{midrange}} \, \mathscr{A}_\epsilon \underline{u}\left(x, t_1 + \frac{j\epsilon^2}{2}; \nu\right) \right] + \overline{\delta_\epsilon}(x)\psi(x).$$

Therefore, it is sufficient to show that

$$\lim_{j \to \infty} \sup_{\nu \in S^{n-1}} \mathscr{A}_\epsilon \underline{u}\left(x, t_1 + \frac{j\epsilon^2}{2}; \nu\right) = \sup_{\nu \in S^{n-1}} \tilde{\mathscr{A}}_\epsilon \underline{U_{t_1}}(x; \nu) \tag{4.4.3}$$

and

$$\lim_{j \to \infty} \inf_{\nu \in S^{n-1}} \mathscr{A}_\epsilon \underline{u}\left(x, t_1 + \frac{j\epsilon^2}{2}; \nu\right) = \inf_{\nu \in S^{n-1}} \tilde{\mathscr{A}}_\epsilon \underline{U_{t_1}}(x; \nu) \tag{4.4.4}$$

where

$$\tilde{\mathscr{A}}_\epsilon v(x; \nu) = \alpha v(x + \epsilon\nu) + \beta \fint_{B_\epsilon^\nu} v(x + h) d\mathcal{L}^{n-1}(h). \tag{4.4.5}$$

These equalities can be derived by the argument in the proof of [2, Proposition 3.3]. First, we get (4.4.3) from monotonicity of $\{\underline{u}(x, t_1 + j\epsilon^2/2)\}$. On the other hand, by means of the monotonicity of $\{\underline{u}(x, t_1 + j\epsilon^2/2)\}$ and continuity of $\mathscr{A}_\epsilon \underline{u}(x, t; \cdot)$, we can show the existence of a vector $\tilde{\nu} \in S^{n-1}$ satisfying

$$\mathscr{A}_\epsilon \underline{u}\left(x, t_1 + \frac{j\epsilon^2}{2}; \tilde{\nu}\right) \le \lim_{j \to \infty} \inf_{\nu \in S^{n-1}} \tilde{\mathscr{A}}_\epsilon \underline{U_{t_1}}(x; \nu) \qquad \text{for any } j \ge 0.$$

Now (4.4.4) is obtained by the monotone convergence theorem. Thus, we deduce that $\underline{U_{t_1}}$ satisfies the DPP (4.4.2) for every $-\epsilon^2/2 \le t_1 < 0$. By

uniqueness of solutions to (4.4.2), [2, Theorem 3.7], we can deduce that

$$\lim_{t \to \infty} \underline{u}(x, t) = U_\epsilon(x).$$

We can prove the same result for $\overline{u}$ by repeating the above steps. Combining these results with $\underline{u} \leq u_\epsilon \leq \overline{u}$, we get

$$\lim_{t \to \infty} u_\epsilon(x, t) = U_\epsilon(x)$$

and then we can finish the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

We finish this section by proving a corollary. One can apply the above theorem with Theorem 4.2.1. This coincides with the result for elliptic case, [3, Theorem 1.1].

Corollary 4.4.4. *Let $\bar{B}_{2r} \subset \Omega \backslash I_\epsilon$ and $\epsilon > 0$ be small. Suppose that U_ϵ satisfies (4.4.2). Then for any $x, y \in B_r(0)$,*

$$|U_\epsilon(x) - U_\epsilon(y)| \leq C(|x - y| + \epsilon),$$

where $C > 0$ is a constant which only depends on r, n and $||\psi||_{L^\infty(\Gamma_\epsilon)}$.

Proof. Let $r > 0$ with $\bar{B}_{2r} \subset \Omega \backslash I_\epsilon$ and $x, y \in B_r(0)$. By Theorem 4.4.2, for any $\eta > 0$, we can find some large $t > 0$ such that

$$|u_\epsilon(x, t) - U_\epsilon(x)| < \eta \quad \text{and} \quad |u_\epsilon(y, t) - U_\epsilon(y)| < \eta,$$

where u_ϵ is a function satisfying (4.0.1). And by Theorem 4.2.1, we know that

$$|u_\epsilon(x, t) - u_\epsilon(y, t)| \leq C(|x - y| + \epsilon),$$

where C is a constant depending on r, n and $||F||_{L^\infty(\Gamma_{\epsilon,T})}$. (Here, F is a boundary data as in Theorem 4.4.2)

Then we have

$$|U_\epsilon(x) - U_\epsilon(y)| \leq |U_\epsilon(x) - u_\epsilon(x, t)| + |u_\epsilon(x, t) - u_\epsilon(y, t)| + |u_\epsilon(y, t) - U_\epsilon(y)|$$
$$< C(|x - y| + \epsilon) + 2\eta.$$

Since we can choose η arbitrarily small, we obtain

$$|u_\epsilon(x, t) - u_\epsilon(y, t)| \leq C(|x - y| + \epsilon)$$

for some $C = C(n, p, \Omega, ||\psi||_{L^\infty(\Gamma_\epsilon)}) > 0$ since we can estimate

$$||F||_{L^\infty(\Gamma_{\epsilon,T})} \leq ||\psi||_{L^\infty(\Gamma_\epsilon)}$$

by choosing proper boundary data F. $\square$

4.4.2   Uniform convergence as $\epsilon \to 0$

The objective of this section is to study behavior of u_ϵ when ϵ tends to zero. This issue has been studied in several preceding papers (see [40, 52, 62, 2]). Those results show that there is a close relation between value functions of tug-of-war games and certain types of PDEs. Now we will establish that there is a convergence theorem showing that u_ϵ converge to the unique viscosity solution of the following Dirichlet problem for the normalized parabolic p-Laplace equation

$$\begin{cases} (n+p)u_t = \Delta_p^N u & \text{in } \Omega_T, \\ u = F & \text{on } \partial_p \Omega_T \end{cases} \tag{4.4.6}$$

as $\epsilon \to 0$. Here, p satisfies $\alpha = (p-1)/(p+n)$ and $\beta = (n+1)/(p+n)$.

Now we introduce the notion of viscosity solutions for (4.4.6). Note that we need to consider the case when the gradient vanishes. Here we use semicontinuous extensions of operators in order to define viscosity solutions. For these extensions, we refer the reader to [23, 27] for more details.

Definition 4.4.5 (Viscosity solution). *A function $u \in C(\Omega_T)$ is a viscosity solution to (4.4.6) if the following conditions hold:*

(a) for all $\varphi \in C^2(\Omega_T)$ touching u from above at $(x_0, t_0) \in \Omega_T$,

$$\begin{cases} \Delta_p^N \varphi(x_0, t_0) \geq (n+p)\varphi_t(x_0, t_0) & \text{if } D\varphi(x_0, t_0) \neq 0, \\ \lambda_{\max}((p-2)D^2\varphi(x_0, t_0)) \\ \qquad + \Delta\varphi(x_0, t_0) \geq (n+p)\varphi_t(x_0, t_0) & \text{if } D\varphi(x_0, t_0) = 0. \end{cases}$$

(b) for all $\varphi \in C^2(\Omega_T)$ touching u from below at $(x_0, t_0) \in \Omega_T$,

$$\begin{cases} \Delta_p^N \varphi(x_0, t_0) \leq (n+p)\varphi_t(x_0, t_0) & \text{if } D\varphi(x_0, t_0) \neq 0, \\ \lambda_{\min}((p-2)D^2\varphi(x_0, t_0)) \\ \qquad + \Delta\varphi(x_0, t_0) \leq (n+p)\varphi_t(x_0, t_0) & \text{if } D\varphi(x_0, t_0) = 0. \end{cases}$$

Here, the notation $\lambda_{\max}(X)$ and $\lambda_{\min}(X)$ mean the largest and the smallest eigenvalues of a symmetric matrix X.

The following Arzelà-Ascoli criterion will be used to obtain the main result in this section. It is essentially the same proposition as [62, Lemma 5.1]. We can find the proof of this criterion for elliptic version in [55, Lemma 4.2].

Lemma 4.4.6. *Let $\{u_\epsilon : \overline{\Omega}_T \to \mathbb{R}, \epsilon > 0\}$ be a set of functions such that*

(a) *there exists a constant $C > 0$ so that $|u_\epsilon(x,t)| < C$ for every $\epsilon > 0$ and every $(x,t) \in \overline{\Omega}_T$.*

(b) *given $\eta > 0$, there are constants r_0 and ϵ_0 so that for every $\epsilon > 0$ and $(x,t), (y,s) \in \overline{\Omega}_T$ with $d((x,t), (y,s)) < r_0$, it holds*

$$|u_\epsilon(x,t) - u_\epsilon(y,s)| < \eta.$$

Then, there exists a uniformly continuous function $u : \overline{\Omega}_T \to \mathbb{R}$ and a subsequence $\{u_{\epsilon_i}\}$ such that u_{ϵ_i} uniformly converge to u in $\overline{\Omega}_T$, as $i \to \infty$.

Now we can describe the relation between functions satisfying (4.0.1) and solutions to the normalized parabolic p-Laplace equation.

Theorem 4.4.7. *Assume that Ω satisfies the exterior sphere condition and $F \in C(\Gamma_{\epsilon,T})$ satisfies (4.3.1). Let u_ϵ denote the solution to (4.0.1) with boundary data F for each $\epsilon > 0$. Then, there exist a function $u : \overline{\Omega}_{\epsilon,T} \to \mathbb{R}$ and a subsequence $\{\epsilon_i\}$ such that*

$$u_{\epsilon_i} \to u \qquad \text{uniformly in} \quad \overline{\Omega}_T$$

and the function u is a unique viscosity solution to (4.4.6).

Remark 4.4.8. *The uniqueness of solutions to (4.4.6) can be found in [62, Lemma 6.2].*

Proof. First we check that there is a subsequence $\{u_{\epsilon_i}\}$ with u_{ϵ_i} converge uniformly to u on $\overline{\Omega}_T$ for some function u. By using the definition of u_ϵ, we have

$$||u_\epsilon||_{L^\infty(\Omega_T)} \leq ||F||_{L^\infty(\Omega_T)} < \infty$$

for any $\epsilon > 0$. Hence, u_ϵ are uniformly bounded. By means of Theorem 4.2.1, Theorem 4.3.3 and Theorem 4.3.4, we know that $\{u_\epsilon\}$ are equicontinuous. Therefore, we can find a subsequence $\{u_{\epsilon_i}\}_{i=1}^\infty$ converging uniformly to a function $u \in C(\overline{\Omega}_T)$ by Lemma 4.4.6.

Now we need to show that u is a viscosity solution to (4.4.6). On the parabolic boundary, we see that

$$u(x,t) = \lim_{i\to\infty} u_{\epsilon_i}(x,t) = F(x,t)$$

for any $(x,t) \in \partial_p \Omega_T$.

Next we prove that u satisfies

$$(n+p)u_t = \Delta_p^N u \qquad \text{in } \Omega_T$$

in the viscosity sense. Without loss of generality, it suffices to show that u satisfies condition (a) in Definition 4.4.5.

Fix $(x,t) \in \Omega_T$. Then there is a small number $R > 0$ such that

$$Q := (x_0, t_0) + B_R(0) \times (-R^2, 0) \subset\subset \Omega_T.$$

We also assume that $\epsilon > 0$ satisfies $Q \subset \Omega_T \backslash I_{\epsilon,T}$. Suppose that a function $\varphi \in C^2(Q)$ touches u from below at (x,t). Then we observe that

$$\inf_Q (u - \varphi) = (u - \varphi)(x,t) \le (u - \varphi)(z,s)$$

for any $(z,s) \in Q$. Since u_ϵ converge uniformly to u, for sufficiently small $\epsilon > 0$, there is a point $(x_\epsilon, t_\epsilon) \in Q$ such that

$$\inf_Q (u_\epsilon - \varphi) \le (u_\epsilon - \varphi)(z,s)$$

for any $(z,s) \in Q$. We also check that $(x_\epsilon, t_\epsilon) \to (x,t)$ as $\epsilon \to 0$.

Recall (4.1.1). Since $(x_\epsilon, t_\epsilon) \in \Omega_T \backslash I_{\epsilon,T}$, we have

$$Tu(x,t) = \operatorname*{midrange}_{\nu \in S^{n-1}} \mathscr{A}_\epsilon u\left(x, t - \frac{\epsilon^2}{2}; \nu\right).$$

We also set $\psi = \varphi + (u_\epsilon - \varphi)(x_\epsilon, t_\epsilon)$ and observe that $u_\epsilon \ge \psi$ in Q. Now it

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can be checked that

$$u_\epsilon(x_\epsilon, t_\epsilon) = Tu_\epsilon(x_\epsilon, t_\epsilon) \geq T\psi(x_\epsilon, t_\epsilon)$$

and

$$T\psi(x_\epsilon, t_\epsilon) = T\varphi(x_\epsilon, t_\epsilon) + (u_\epsilon - \varphi)(x_\epsilon, t_\epsilon)$$

Therefore,

$$u_\epsilon(x_\epsilon, t_\epsilon) \geq T\varphi(x_\epsilon, t_\epsilon) + (u_\epsilon - \varphi)(x_\epsilon, t_\epsilon)$$

and this implies

$$0 \geq T\varphi(x_\epsilon, t_\epsilon) - \varphi(x_\epsilon, t_\epsilon). \tag{4.4.7}$$

On the other hand, by the Taylor expansion, we observe that

$$\frac{1}{2}\left[\varphi\left(x + \epsilon\nu, t - \frac{\epsilon^2}{2}\right) + \varphi\left(x - \epsilon\nu, t - \frac{\epsilon^2}{2}\right)\right]$$

$$= \varphi(x, t) - \frac{\epsilon^2}{2}\varphi_t(x, t) + \frac{\epsilon^2}{2}\langle D^2\varphi(x, t)\nu, \nu\rangle + o(\epsilon^2)$$

and

$$\fint_{B_\epsilon^\nu}\varphi\left(x + h, t - \frac{\epsilon^2}{2}\right)d\mathcal{L}^{n-1}(h)$$

$$= \varphi(x, t) - \frac{\epsilon^2}{2}\varphi_t(x, t) + \frac{\epsilon^2}{2(n+1)}\Delta_{\nu^\perp}\varphi(x, t) + o(\epsilon^2)$$

where

$$\Delta_{\nu^\perp}\varphi(x, t) = \sum_{i=1}^{n-1}\langle D^2\varphi(x, t)\nu_i, \nu_i\rangle$$

with $\nu_1, \cdots, \nu_{n-1}$ the orthonormal basis for the space $\nu^\perp$ for $\nu \in S^{n-1}$.

We already know that $\mathscr{A}_\epsilon\varphi$ is continuous with respect to ν in Proposition 4.1.1. Therefore, there exists a vector $\nu_{\min} = \nu_{\min}(\epsilon)$ minimizing $\mathscr{A}_\epsilon\varphi(x_{\epsilon,\eta}, t_\epsilon; \cdot)$. Then we can calculate

$$T\varphi(x_\epsilon, t_\epsilon)$$

$$\geq \frac{\alpha}{2}\left\{\varphi\left(x_\epsilon+\nu_{\min}, t_\epsilon-\frac{\epsilon^2}{2}\right)+\varphi\left(x_\epsilon-\nu_{\min}, t_\epsilon-\frac{\epsilon^2}{2}\right)\right\}$$

$$+\beta \fint_{B_\epsilon^{\nu_{\min}}}\varphi\left(x_\epsilon+h, t_\epsilon-\frac{\epsilon^2}{2}\right)d\mathcal{L}^{n-1}(h)$$

$$\geq \varphi(x_\epsilon, t_\epsilon)-\frac{\epsilon^2}{2}\varphi_t(x_\epsilon, t_\epsilon)$$

$$+\frac{\beta}{2(n+1)}\epsilon^2\left\{\Delta_{\nu_{\min}^\perp}\varphi(x_\epsilon, t_\epsilon)+(p-1)\langle D^2\varphi(x_\epsilon, t_\epsilon)\nu_{\min}, \nu_{\min}\rangle\right\}.$$

Then by (4.4.7), we observe that

$$\frac{\epsilon^2}{2}\varphi_t(x_\epsilon, t_\epsilon)\geq \frac{\beta\epsilon^2}{2(n+1)}\left\{\Delta_{\nu_{\min}^\perp}\varphi(x_\epsilon, t_\epsilon)+(p-1)\langle D^2\varphi(x_\epsilon, t_\epsilon)\nu_{\min}, \nu_{\min}\rangle\right\}.$$

$$(4.4.8)$$

Suppose that $D\varphi(x, t)\neq 0$. Since $(x_\epsilon, t_\epsilon)\to(x, t)$ as $\epsilon\to 0$, it can be seen that

$$\nu_{\min}\to -\frac{D\varphi(x, t)}{|D\varphi(x, t)|}=:-\mu$$

as $\epsilon\to 0$. We also check that

$$\Delta_{(-\mu)^\perp}\varphi(x, t)+(p-1)\langle D^2\varphi(x_\epsilon, t_\epsilon)(-\mu), (-\mu)\rangle = \Delta_p^N\varphi(x, t).$$

Now we divide both side in (4.4.8) by ϵ^2 and let $\epsilon\to 0$. Since $Q_R\subset\Omega_T$, it can be seen that $\delta_\epsilon(x_\epsilon, t_\epsilon)\epsilon^{-2}\to 0$ as $\epsilon\to 0$. Hence, we deduce

$$\varphi_t(x, t)\geq \frac{1}{n+p}\Delta_p^N\varphi(x, t).$$

Next consider the case $D\varphi(x, t)=0$. Observe that

$$\Delta_{\nu_{\min}^\perp}\varphi(x_\epsilon, t_\epsilon)+(p-1)\langle D^2\varphi(x_\epsilon, t_\epsilon)\nu_{\min}, \nu_{\min}\rangle$$
$$=\Delta\varphi(x_\epsilon, t_\epsilon)+(p-2)\langle D^2\varphi(x_\epsilon, t_\epsilon)\nu_{\min}, \nu_{\min}\rangle.$$

For $p\geq 2$, we see

$$(p-2)\langle D^2\varphi(x_\epsilon, t_\epsilon)\nu_{\min}, \nu_{\min}\rangle\geq (p-2)\lambda_{\min}(D^2\varphi(x_\epsilon, t_\epsilon)).$$

We already know that $(x_\epsilon, t_\epsilon)\to(x, t)$ as $\epsilon\to 0$ and the map $z\mapsto\lambda_{\min}(D^2\varphi(z))$

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is continuous. Therefore, it turns out

$$\varphi_t(x,t) \geq \frac{1}{n+p}\{\Delta\varphi(x,t) + (p-2)\lambda_{\min}(D^2\varphi(x,t))\} \qquad (4.4.9)$$

by similar calculation in the previous case.

For $1 < p < 2$, by using similar argument in the previous case and

$$(p-2)\langle D^2\varphi(x_\epsilon, t_\epsilon)\nu_{\min}, \nu_{\min}\rangle \geq (p-2)\lambda_{\max}(D^2\varphi(x_\epsilon, t_\epsilon))$$
$$= \lambda_{\min}((p-2)D^2\varphi(x_\epsilon, t_\epsilon)),$$

we also obtain the inequality (4.4.9).

We can also prove the reverse inequality to consider a function φ touching u from above and a vector $\nu_{\max}$ maximizing $\mathscr{A}_\epsilon\varphi(x_\epsilon, t_\epsilon; \cdot)$ and to do similar calculation again as above. $\qquad \square$

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국문초록

이 학위 논문에서는 비선형 편미분방정식과 관련된 두 가지 유형의 문제를 다룬다. 구체적으로 우리는 사선형 문제와 줄다리기 경기에 대하여 해의 정칙성을 중점적으로 탐구한다.

먼저, 우리는 논문의 전반부에서 사선형 미분경계조건이 주어진 비발산 완전 비선형 타원형 및 포물형 방정식을 연구한다. 사선형 경계조건은 노이만 경계조건의 일반화라고 할 수 있는데, 우리의 목표는 이러한 문제에 대해 경계의 C^3-정칙성 가정 하에서 칼데론-지그문트 유형의 가늠을 이끌어내는 것이다.

한편, 후반부에서 우리는 줄다리기 경기로 불리우는 2인 영합 확률경기, 그 중에서도 시간 의존형 경기에 대해 탐구한다. 우리는 이러한 확률경기의 결과값에 대한 립쉬츠 유형의 가늠을 얻는다. 또, 이러한 결과의 응용으로서 우리는 결과값 함수의 장기간 점근적 행동과 편미분방정식과의 연관성에 대해서 연구한다.

감사의 글

대학원 생활을 마무리하는 시점에서 지난 7년을 되돌아보니 여러 가지 감정이 교차하는 것을 느낍니다. 그 동안 정말 많은 일이 있었고, 또 많은 분들의 도움을 받았습니다. 제가 무사히 학위과정을 마칠 수 있도록 도움을 주신 분들께 지면을 빌려 감사의 말씀을 전하고자 합니다.

먼저 부족한 저를 제자로 받아주시고 성심성의껏 지도해주신 변순식 교수님께 감사드립니다. 제가 연구실에 처음 들어왔을 때가 어렴풋이 기억이 나는데, 사실 교수님의 연구분야에 대해 잘 모르는 채로 연구실에 들어온 터라 처음에는 교수님의 지도를 따라가지 못하기도 했고 많이 헤매기도 하였습니다. 가끔씩은 공부가 잘 되지 않아 힘들었던 적도 있었습니다. 그러나 그럴 때마다 교수님께서 격려와 조언을 해 주신 덕에 마음을 다잡고 연구를 진행해 나갈 수 있었습니다. 교수님의 지도 하에 공부하면서 수학적 지식이나 통찰 외에도 학문을 대하는 자세를 배울 수 있었고, 학문 외적으로도 많은 경험을 쌓을 수 있었습니다.

Mikko Parviainen 교수님께도 감사의 말씀을 드리고 싶습니다. 처음에 제가 줄다리기 경기에 대해 흥미를 느껴 메일을 드렸을 때 친절한 답변과 함께 관련 자료들도 많이 소개하여 주셨고, 그 덕분에 해당 주제에 대한 연구를 시작할 수 있었습니다. 이후에도 저의 연구에 대해 많은 조언을 아끼지 않아 주셨는데, 이 또한 제게 큰 도움이 되었습니다. 한편으로 제가 잠시 핀란드에서 방문학생 신분으로 체류하였던 적이 있었는데, 빨리 현지 적응을 마치고 연구에 매진할 수 있도록 배려해 주셨는데, 이 또한 감사드립니다.

또한 바쁘신 와중에도 귀중한 시간을 할애하여 박사학위 논문심사를 맡아 주신 이기암 교수님, 김판기 교수님, 옥지훈 교수님께도 감사의 말씀을 드립니다.

연구실에서 보낸 6년은 매일같이 같은 공간에서 동료분들과 동고동락하는 시간이었습니다. 동료분들과 함께 공부하면서 많이 배우기도 했고 때로는 학문적인 자극을 받기도 했습니다. 연구의 진척이 없을 때에는 서로 고민을 털어 놓으면서 심리적으로 위안을 받은 적도 있었는데, 그럴 때마다 비슷한 고민을 하는 사람들이 옆에 있다는 것이 얼마나 축복받은 일인지 느낄 수 있었습니다. 짧지 않은 시간이었지만, 좋은 동료분들을 만난 덕분에 학위과정을 무탈하게 마칠 수 있었던 것 같습니다. 한 발짝 앞에서 제게 여러 방면으로 많은 도움을 주셨던 유승진 교수님, 조유미 박사님, 장윤수 교수님, 김유찬 교수님, 신필수 교수님, 이미경 교수님, 소형석 박사님, 박정태 박사님, 오제한 교수님, 윤영훈 교수님; 학문적으로 참신한 시각을 제공해 주셨던 고은경 교수님, Karthik Adimurthi 박사님; 연구실 생활을 하면서 같이 열심히 공부한 민규, 원태형, 남

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