이학 박사 학위논문

Rotor-routing action on spanning trees and harmonic cycle

(생성 나무 위의 로터 라우팅 작용과 조화 사이클)

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Rotor-routing action on spanning trees and harmonic cycle

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Abstract

Rotor-routing action on spanning trees and harmonic cycle

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In this thesis, we investigate the relation between the harmonic cycles of a two dimensional complex and the critical group of its underlying graph. The harmonic space of a cell complex is defined to be the kernel of the combinatorial Laplacian and is naturally isomorphic to the homology group by combinatorial Hodge theory. The critical group of a graph is a finite abelian group which is related to the chip-firing game and has the cardinality equal to the number of spanning trees. For two-dimensional cell complexes obtained by adding an additional edge to an acyclization of a graph, Kim and Kook found a combinatorial formula for the generator of one-dimensional harmonic space over real coefficients, using spanning trees of the given graph. We introduce the refined version of the formula for an integral generator, by tracking the trace of a chip in the action of the critical group on spanning trees.

Key words: spanning tree, the critical group, the chip-firing game, harmonic space
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Chapter 1

Introduction

The *Laplacian matrix* L of a graph G is first introduced and studied by Kirchhoff [Kir47]. Its explicit definition is given by

$$\begin{aligned} L_{v,v} &= \deg(v), \\ L_{v,w} &= -(\#\text{ of edges between } v \text{ and } w). \quad (v \neq w) \end{aligned}$$

Kirchhoff used the Laplacian matrix for analyzing electric circuits, and stated Kirchhoff's theorem, or the matrix-tree theorem, which shows that any cofactor of the Laplacian matrix is equal to the number of spanning trees. After then, the work of Fiedler [Fie73] relating the smallest eigenvalue of the Laplacian and the connectivity of the graph drew attention and accelerated the studies about the graph Laplacian and its spectrum. This thesis is going to focus on two objects that are derived from the graph Laplacian; harmonic cycle and the critical group.

Eckmann [Eck44] formulated the combinatorial Laplacian of a simplicial complex, as a high dimensional analogue of the graph Laplacian. Given a chain complex $\mathcal{C} = \{(C_i, \partial_i)\}_i$, the *i-combinatorial Laplacian* matrix Δ_i

is the operator on C_i defined to be

$$\Delta_i = \partial_{i+1}\partial_{i+1}^t + \partial_i^t\partial_i,$$

where ∂_i^t is the transpose of the boundary map ∂_i. There have been a lot of studies about the combinatorial Laplacians, such as a high dimensional generalization of the matrix-tree theorem [DKM09], or the spectrum of the combinatorial Laplacians [HJ13],[KRS00].

Eckmann [Eck44] proved the discrete version of Hodge theorem, which states that the kernel of the combinatorial Laplacian Δ_i is naturally isomorphic to the i-th homology group $H_i = \ker\partial_i \,/\, \operatorname{im}\partial_{i+1}$. The kernel of Δ_i is called an *i-harmonic space* of $\mathcal{C}$ and denoted by $\mathcal{H}_i$. The elements of $\mathcal{H}_i$ are called *i-harmonic cycles.*

In particular, Kim and Kook [KK19] found a combinatorial interpretation of the coefficients of harmonic cycles under certain conditions. Under the conditions that $\operatorname{rank} H_{i-1} = 0$ and $\operatorname{rank} H_i = 1$, there is a combinatorial formula for computing the generator of i-harmonic space; the standard harmonic cycle λ is defined to be

$$\lambda := \sum_U w(C_U)C_U, \tag{1.0.1}$$

where the summation runs over all cycle-trees U with the unique cycle C_U, and $w(\cdot)$ is the winding number; see [KK18], [KK19] for details. In particular, for a 2-dimensional complex X of the form $X = \mathcal{A}(G) \sqcup \widetilde{e}$, where $\mathcal{A}(G)$ is an acyclization of a graph G and $\widetilde{e} = \{s, x\}$ is an additional edge not in G, the above formula can be written with the sum over the set of spanning trees of G. More precisely, the standard harmonic cycle

$\lambda \in C_1(X)$ can be obtained by the following formula

$$\lambda = \sum_{T \in \mathcal{T}(G)} \left(\gamma_T(x,s) + [\widetilde{e}]\right),$$

where $\gamma_T(x,s)$ is the unique directed path from x to s in the spanning tree T and $\widetilde{e}$ is oriented with $\widetilde{e} = (s,x)$. The formula provides a nice combinatorial way to compute the generator of the harmonic space $\mathcal{H}_1$ over the real coefficients. Since the coefficients in λ are not necessarily relatively prime in general, it may fail to generate the harmonic space over the integer coefficient. Hence one might ask the following natural question:

Question 1.1. Can we find a partition $\mathcal{T}(G) = B_1 \sqcup \cdots \sqcup B_k$ such that for each i, the sum

$$\sum_{T \in B_i} \left(\gamma_T(x,s) + [\widetilde{e}]\right)$$

is an integral generator of $\mathcal{H}_1(X;\mathbb{Z})$?

It turns out that the size of each block B_i is equal to the order of some element in the *critical group* of the graph G, which is importantly studied in this thesis. The critical group $\mathcal{K}(G)$ of a graph G is the finite abelian group defined to be

$$\mathcal{K}(G) := {}^{\ker \partial_0}\!\Big/_{\operatorname{im} L},$$

where ∂_0 is the 0-boundary map that assigns each vertex of G the value 1, and L is the Laplacian matrix of G.

In a combinatorial point of view, the critical group is related to the *chip-firing game*, which is a solitary game on a graph concerning dynamics of the number of chips on each vertex. For each vertex, an integer that represents the number of chips is assigned. If a vertex is 'fired', then it sends a chip to each one of its neighbors. Under the rule of the chip-firing game, the critical group consists of chip configurations with total number of chips zero up to the equivalent relation defined upon the firing moves. Good references for background on the critical groups and the chip-firing games include [Big99], [Kli18], and [CP18].

The chip-firing game is first considered by Spencer, as a balancing game on 1-dimensional grid with N chips are placed at the origin [Spe86]. Björner, Lovász, and Shor followed the work as a generalization of the domain to arbitrary graphs [BLS91]. In [Big99], Biggs interpreted the chip-firing game as a 'dollar game' and studied the structure of the critical group.

The critical group have been broadly studied in various literature. Several different names are used to refer the critical group in various contexts, e.g. the Jacobian group, the Picard group, and the sandpile group, etc. Lorenzini summarizes these approaches in [Lor08]:

- Lorenzini used the name the *group of components* and studied the group in the perspective of arithmetic geometry in [Lor91].
- In the context of physics, Dhar considered piles of sand and its dynamics [Dha90]. The group is called the *sandpile group*.
- In the perspective of algebraic curves, Bacher, La Harpe, and Nagnibeda regarded a graph as a discrete analogue of a Riemann surface and called the group as the *Picard group*. They also showed that the group is isomorphic to a group called the *Jacobian group*.

An important property of the critical group is that the cardinality is equal to the number of spanning trees of the graph, as a simple consequence of the matrix-tree theorem. Explicit bijections between the critical group and the set of spanning trees are established in [MD92],[CL03], under the fixed labeling on the edges. Recently, there have been some studies on the free transitive group action of the critical group on the set of spanning trees, including the rotor-routing action [Hol+08], the Bernadi process [Ber06] and the cycle-cocycle reversal [BBY19].

In particular, we provide an answer to Question 1.1 using the rotor-routing action of $\mathcal{K}(G)$, given that the graph G is planar. More precisely, we have the following theorem.

Theorem 1.2. *Let G be a planar graph. and let $X = \mathcal{A}(G) \sqcup \widetilde{e}$ where $\widetilde{e} = \{s, x\}$ be an edge not in G. Let $c_x \in \mathcal{K}(G)$ be an element that has values $1,-1$ at the vertices x, s, respectively, and has value 0 elsewhere. For any orbit $\mathcal{O}$ in $\mathcal{T}(G)$ under the rotor-routing action of c_x, the sum*

$$\sum_{T \in \mathcal{O}} (\gamma_T(x, s) + [\widetilde{e}]) \tag{1.0.2}$$

is an integral generator of $\mathcal{H}_1(X)$.

The key idea is tracking the movement of the chip while iterating the rotor-routing operation on spanning trees, and using the reversibility of directed cycles, which is studied by Chan, Church, and Grochow [CCG14].

The rest of the thesis is organized as follows. In Chapter 2, we introduce the basic notions on graphs, harmonic spaces, and critical groups. In Chapter 3, we focus on the rotor-routing action of $\mathcal{K}(G)$ on the set $\mathcal{T}(G)$, and introduce the main result of the thesis.

Chapter 2

The harmonic space and the critical group

2.1 Basic definitions

We first give some basic definitions in graph theory.

A *simple graph* (resp. *multigraph*) is a pair $G = (V, E)$, where V is a set whose elements are called *vertices* (singular: *vertex*) of G, and E is a set (resp. multiset) of unordered pairs of vertices, which are called *edges* of G. We call both simple graph and multigraph a *graph* without distinguishing.

A graph is *finite* if its vertex set is finite. For an edge $e = \{v, w\}$, the vertices v and w are called the *endpoints* of e, and the edge e is said to be *incident* on v and w. The *degree* of a vertex v is the number of edges incident to v and denoted by $\deg(v)$. Two vertices v and w are said to be *adjacent* if there is an edge incident to both v and w. A *loop* is an edge $e = \{v, w\}$ where $v = w$.

A *directed graph* or *digraph* is a variation of a graph whose edge set consists of *ordered* pairs of vertices, rather than unordered ones. We call an ordinary graph an *undirected* graph to distinguish it from a directed one. For an (directed) edge $e = (v, w)$, the vertices v and w are called the *tail* and *head* of e, respectively. we say that an edge $e = (v, w)$ is *outgoing from* v and *incoming to* w. For a vertex v, its *indegree* (resp. *outdegree*) is the number of edges that are incoming to (resp. outgoing from) v, and denoted by $\mathrm{indeg}(v)$ (resp. $\mathrm{outdeg}(v)$). In particular, we say a vertex v is a *sink* if its outdegree is zero. An undirected graph can be naturally regarded as a directed graph, by replacing each undirected edge $e = \{v, w\}$ with a pair of directed edges (v, w) and (w, v).

A *walk* (resp. a *directed walk*) from a vertex v to a vertex w is an alternating sequence of vertices and edges

$$v_1(= v), e_1, v_2, e_2, \dots, v_{k-1}, e_{k-1}, v_k(= w)$$

where $e_i = \{v_i, v_{i+1}\}$ (resp. $e_i = (v_i, v_{i+1})$) for each i. If there is a condition such that every $v_1, \dots, v_k$ are distinct, then the walk is called a *(directed) path*. If there is a condition such that every $v_1, \dots, v_k$ are distinct but $v_1 = v_k$, then the walk is called a *(directed) cycle*. A graph is called *connected* if there exists a path between any pair of vertices. A graph is called *acyclic* if it is has no cycles.

A *subgraph* of a graph $G = (V(G), E(G))$ is a graph $H = (V(H), E(H))$ where $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

For the rest of this thesis, we only consider finite connected graphs without loops.

2.2 Spanning trees and the matrix-tree theorem

A graph is called a *tree* if it is connected and acyclic. For a graph $G = (V, E)$, a *spanning tree* of G is a subgraph of G which is a tree and contains all vertices of G. Equivalently, a spanning tree is a maximal connected acyclic subgraph of G. We will denote by $\mathcal{T}(G)$ the set of spanning trees of G, and $\tau(G)$ the number of spanning trees of G.

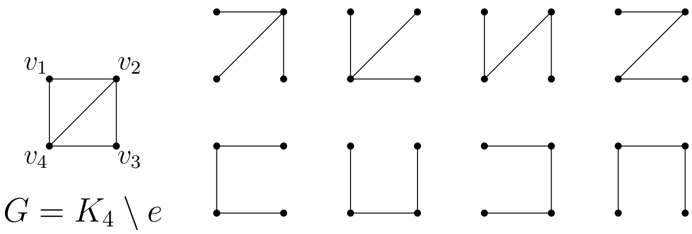


Figure 2.1: All 8 spanning trees of the graph $G = K_4 \backslash e$, where $e = \{v_1, v_3\}$.

Kirchhoff's theorem, which is also known as the *matrix-tree theorem* provides an elegant formula that counts the number of spanning trees using the determinant of a matrix. Let $n = |V|$ and $m = |E|$. Fix an orientation on each edge. The *(signed) incidence matrix* ∂ of G is an $n \times m$ matrix whose rows and columns are indexed by the vertices and the edges of G, respectively, where the entries are defined by

$$\partial_{v,e} = \begin{cases} 1, & \text{if } v \text{ is the head of } e, \\ -1, & \text{if } v \text{ is the tail of } e, \\ 0, & \text{otherwise.} \end{cases}$$

A subset of the columns of ∂ is linearly independent if and only if the subgraph with the corresponding set of edges is acyclic. Therefore, a spanning tree can be alternatively defined by the subset of the columns of ∂ that is a basis for $\operatorname{im}\partial$. Consequently, the number of edges of a spanning tree is equal to the rank of ∂, which is $n-1$.

The *Laplacian matrix* of G is an $n \times n$ matrix $L := \partial\partial^t$. Equivalently, its entries are defined by

$$\begin{aligned} L_{v,v} &= \deg(v), \\ L_{v,w} &= -(\#\text{ of edges between } v \text{ and } w). \quad (v \neq w) \end{aligned}$$

It is known that L has eigenvalue 0 with the multiplicity 1. Namely, the Laplacian matrix of a graph has $n-1$ nonzero eigenvalues.

Theorem 2.1 (Matrix-tree theorem, [Kir47]). *Let L be the Laplacian matrix of a graph G and $\lambda_1, \ldots, \lambda_{n-1}$ be the nonzero eigenvalues of L. Then*

$$\tau(G) = \frac{\lambda_1 \cdots \lambda_{n-1}}{n}.$$

An alternative form of the matrix-tree theorem uses the *reduced Laplacian matrix* of G. For a vertex v, the *reduced Laplacian matrix* of G (with respect to v) is the $(n-1)\times(n-1)$ matrix $\widetilde{L}_v$ obtained from L by deleting the row and column corresponding to v. It is well known that regardless of the choice of v, reduced Laplacians have the same determinants. We omit the subscript and simply write $\widetilde{L}$ if there is no need to consider the choice of the vertex. The alternative form of the matrix-tree theorem states that the determinant of the reduced Laplacian is equal to the number of spanning trees, i.e.,

$$\tau(G) = \det(\widetilde{L}).$$

Example 2.2. Let G be a graph in Figure 2.1, which is obtained by deleting one edge in the complete graph K_4. Its incidence matrix and the Laplacian matrix are

$$\partial = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_{1,2} & e_{1,4} & e_{2,3} & e_{2,4} & e_{3,4} \end{array} \\ \begin{pmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}, \quad L = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix} \end{array},$$

where $e_{i,j}$ is the edge from v_i to v_j for $i < j$. One can see that every 3×3 principal minor of L is equal to $\tau(G) = 8$.

2.3 Chain complexes and the combinatorial Hodge theory

In this section, we will review basic preliminaries regarding chain complexes and introduce the combinatorial Hodge theory.

A *chain complex* is an algebraic tool for computation of topological invariants of several objects, such as a graph or its generalizations.

Definition 2.3. Let R be a ring with identity. A *chain complex (over R)* is a collection $\mathcal{C} = \{(C_i, \partial_i)\}_{i\in\mathbb{Z}}$ such that for each i,

- C_i is a free R-module of finite rank, called an *i-chain group*. An element in C_i is called an *i-chain*.
- ∂_i is a homomorphism $\partial_i : C_i \longrightarrow C_{i-1}$, called an *$i$-boundary operator*.
- The composition $\partial_i \circ \partial_{i+1}$ is equal to the zero map, or equivalently, $\operatorname{im} \partial_{i+1} \subseteq \ker \partial_i$.

We fix a basis $\mathcal{B}$ of the free module C_i and represent an i-chain in C_i as a formal sum $\sum_{\sigma\in\mathcal{B}} c_\sigma[\sigma]$ using the bracket. We call an element in $\ker \partial_i$ an i-*cycle*. Note that the i-boundary operator ∂_i is also called the i-*boundary matrix*, as it can be represented by an $r_{i-1} \times r_i$ matrix, where $r_i = \operatorname{rank} C_i$. Thus we can omit the composition symbol $\circ$ and simply write $\partial_{i-1}\partial_i$ for the composition $\partial_{i-1} \circ \partial_i$. We also use the notations $C_i(\mathcal{C})$, $C_i(\mathcal{C}; R)$ for C_i and $\partial_{i,\mathcal{C}}$ for ∂_i if we need to represent the complex $\mathcal{C}$ and the base ring R explicitly.

In this thesis, we assume that a chain complex $\mathcal{C}$ is *finite dimensional*, i.e., for some non-negative integer d, the i-chain group C_i is trivial for $i > d$. The maximal index i such that C_i is nontrivial is called the *dimension* of $\mathcal{C}$.

Given a graph $G = (V, E)$, its 1-dimensional chain complex structure $\mathcal{C}_G$ is given as follows: The chain groups C_0 and C_1 are defined to be the free R-modules generated by V and E, respectively. The 1-boundary operator ∂_1 is defined to be the signed incidence matrix ∂ of G. Note that a 1-cycle in $\ker \partial_1$ corresponds to a formal sum of cycles in G. Let $C_{-1} = R$ and $C_i = 0$ for $i \neq -1, 0, 1$. The 0-boundary operator ∂_0 is defined to be the homomorphism induced by a map $v \mapsto 1$ for all $v \in V$, i.e., it can be represented as a $1 \times |V|$ matrix whose entries are all one. The following sequence summarizes the structure of $\mathcal{C}_G$.

$$C_1 \xrightarrow{\partial_1 = \partial} C_0 \xrightarrow{\partial_0} C_{-1} \xrightarrow{0} 0$$

Now consider an object X obtained from a graph G by ‘filling in’ some cycles. By augmenting the chain complex $\mathcal{C}_G$, we can obtain the 2-dimensional chain complex that stores the structural data of X. The augmentation is done by the following construction: Let $Z = \{z_1, \ldots, z_k\}$

be the set of cycles which are selected to be filled in. Let $C_2 := R^Z$ and define the 2-boundary map $\partial_2 : C_2 \to C_1$ by the homomorphism induced by the identity map $z_i \mapsto z_i$. Since Z is contained in $\ker \partial_1$, the condition $\operatorname{im} \partial_2 \subseteq \ker \partial_1$ is met and we have a 2-dimensional chain complex $\mathcal{C}_X$. Similarly we can augment the chain complex $\mathcal{C}_G$ to an arbitrary dimension.

Remark 2.4. The object X described in the above paragraph is an instance of the *cell complex*, which is a high dimensional generalization of a graph. Topologically, a cell complex is constructed by 'attaching' cells under a certain rule, where each cell is homeomorphic to an open disk. In above construction, the *i-dimensional cells* (or *i-cells*) correspond to the generators of the i-chain group C_i. We denote by X_i the set of i-cells of X. An *i-skeleton* of X is a union $\bigsqcup_{j \le i} X_j$. In this thesis, we identify a cell complex X with the augmented chain complex $\mathcal{C}_X$ in the above paragraph, rather than using the rigorous definition of cell complexes. See [Hat00] for the detailed definition of a cell complex.

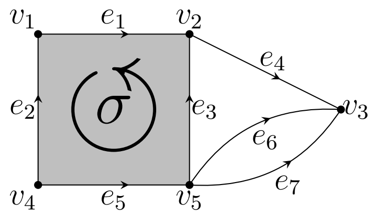


Figure 2.2: A cell complex with one 2-cell, seven 1-cells and five 0-cells.

Example 2.5. Figure 2.2 shows a cell complex X, where $X_2 = \{\sigma\}$, $X_1 = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and $X_0 = \{v_1, v_2, v_3, v_4, v_5\}$. The orientation of each cell is represented in the arrow. The boundary matrices of X are

$$\partial_2 = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{array} \begin{array}{c} \sigma \\ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \end{array}, \qquad \partial_1 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 & -1 \end{pmatrix} \end{array},$$

$$\partial_0 = \emptyset \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array}.$$

Given a cell complex X, the quotient module $H_i(\mathcal{C}; R) = \ker \partial_i \,/\, \operatorname{im} \partial_{i+1}$ is called the *i-homology group*. The *i-Betti number* of X is defined by $\beta_i = \operatorname{rank}_R H_i(X; R)$. Betti numbers gives intuitive information of the topological structure of the given complex. For example, β_0 is equal to the number of connected components minus one, and β_1 counts the 'one-dimensional holes'. For the cell complex in Figure 2.2, its 0- and 1-Betti numbers are $\beta_0 = 0$ and $\beta_1 = 1$.

Remark 2.6. In algebraic topology, the common choice of the base ring R is the integer ring $\mathbb{Z}$ or the filed of real numbers $\mathbb{R}$. The main difference in these choices is the existence of the *torsion* part in the quotient modules. Over the real coefficients, each chain group C_i is a real vector space, and a quotient of C_i by its subspace is still a vector space with no torsion. On the other hand, the computation on the integer coefficients involves a torsion part when we deal with a quotient module.

Given a cell complex X, we define the *i-coboundary operator* by the transpose $\partial_i^t : C_{i-1} \to C_i$ of the i-boundary matrix ∂_i. An element in $\ker \partial_{i+1}^t$ is called an *i-cocycle*. The *i-combinatorial Laplacian* of X is the operator

$$\Delta_i := \partial_{i+1}\partial_{i+1}^t + \partial_i^t\partial_i : C_i \to C_i.$$

Combinatorial Hodge theory states that the kernel of Δ_i over the real coefficients is naturally isomorphic to the i-homology group $H(X;\mathbb{R})$.

Theorem 2.7 (Combinatorial Hodge theory, [Fri98])**.** *Let X be a cell complex. The i-chain group $C_i(X;\mathbb{R})$ with the real coefficients can be decomposed into*

$$C_i(X;\mathbb{R}) = \ker \Delta_i \oplus \operatorname{im} \partial_{i+1} \oplus \operatorname{im} \partial_i^t, \tag{2.3.1}$$

as a real vector space. Moreover, $\ker \Delta_i = \ker \partial_{i+1}^t \cap \ker \partial_i$ *and* $\ker \Delta_i$ *is isomorphic to $H_i(X;\mathbb{R})$ as a vector space.*

Proof. First we show that $\ker \Delta_i = \ker \partial_{i+1}^t \cap \ker \partial_i$. Let $z \in \ker \Delta_i$. Taking the standard inner product $\langle \cdot, z\rangle$ to the both sides of the identity $\Delta_i z = 0$, we have

$$\begin{aligned}
\langle \Delta_i z, z\rangle &= \langle \partial_{i+1}\partial_{i+1}^t z, z\rangle + \langle \partial_i^t\partial_i z, z\rangle \\
&= \langle \partial_{i+1}^t z, \partial_{i+1}^t z\rangle + \langle \partial_i z, \partial_i z\rangle \\
&= 0.
\end{aligned}$$

Therefore we have $\partial_{i+1}^t z = 0$ and $\partial_i z = 0$, yielding that $\ker \Delta_i \subseteq \ker \partial_{i+1}^t \cap \ker \partial_i$.

Conversely, let $z \in \ker \partial_{i+1}^t \cap \ker \partial_i$. Then $\Delta_i z = \partial_{i+1}(\partial_{i+1}^t z) + \partial_i^t(\partial_i z) = 0$ immediately shows $\ker \Delta_i \supseteq \ker \partial_{i+1}^t \cap \ker \partial_i$.

The decomposition $C_i(X;\mathbb{R}) = \ker \partial_i \oplus \operatorname{im} \partial_i^t$ follows from the elementary fact that $(\ker M)^{\perp} = \operatorname{im} M$ for a matrix M. Hence it suffices to show $\ker \partial_i = \ker \Delta_i \oplus \operatorname{im} \partial_{i+1}$ to prove (2.3.1). Indeed, since the orthogonal complement of $\operatorname{im} \partial_{i+1}$ is equal to $\ker \partial_{i+1}^t$ and $\ker \Delta_i = \ker \partial_{i+1}^t \cap \ker \partial_i$, we have $\ker \partial_i = \ker \Delta_i \oplus \operatorname{im} \partial_{i+1}$.

An isomorphism between $\ker \Delta_i$ and $H(X;\mathbb{R})$ is obtained from

$$H_i(X;\mathbb{R}) = {}^{\ker \partial_i}\Big/_{\operatorname{im} \partial_{i+1}} = {}^{\ker \Delta_i \oplus \operatorname{im} \partial_{i+1}}\Big/_{\operatorname{im} \partial_{i+1}} \cong \ker \Delta_i.$$

□

The kernel of Δ_i is called an *i-harmonic space* of X, and denoted by $\mathcal{H}_i(X;\mathbb{R})$ (or simply $\mathcal{H}_i(X)$, $\mathcal{H}_i$). An element of $\mathcal{H}_i(X)$ is called an *i-harmonic cycle*. Equivalently, an i-harmonic cycle is an element which is an i-cycle and an i-cocycle at the same time.

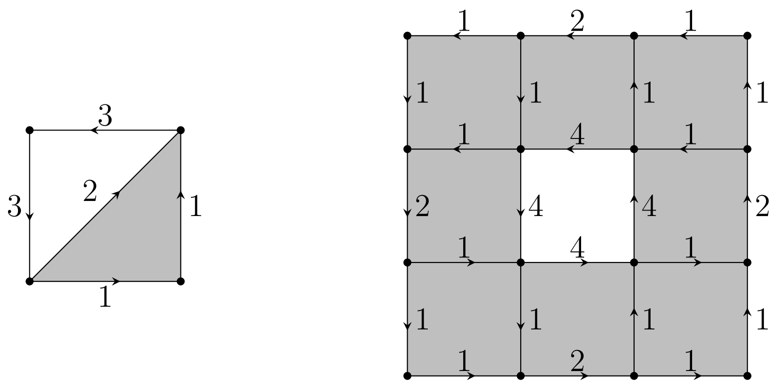


Figure 2.3: Examples of 1-harmonic cycles.

Example 2.8. Figure 2.3 shows two examples of 1-chains that are 1-harmonic cycles, where the numbers represents the coefficients of the edges. That these chains are indeed harmonic cycles can be checked by showing these chains are in $\ker \partial_1 \cap \ker \partial_2^t = \mathcal{H}_1$. For each vertex, the net amount of incoming and outgoing 'flow' is zero, and hence these 1-chains are 1-cycles. On the other hand, for each face, the net amount of 'flow' along its boundary is zero, where the flow of the opposite direction of the boundary is counted as negative. Therefore these 1-chains are also 1-cocycles.

Example 2.9. Let $\widetilde{W}_n$ be a 2-dimensional complex obtained from the wheel graph W_n by 'filling in' the all inner triangles but one; See Figure 2.4. Since the first Betti number of $\widetilde{W}_n$ is equal to one, its 1-harmonic space $\mathcal{H}_1(\widetilde{W}_n)$ has rank 1. Figure 2.4 describes integral generators of $\mathcal{H}_1(\widetilde{W}_n)$ for $n = 3, 4, 5, 6$. If n is odd, the coefficients in an integral generator of $\mathcal{H}_1(\widetilde{W}_n)$ are from the *Fibonacci numbers* f_n, which is defined by the recurrence $f_{n+2} = f_{n+1} + f_n$ with the initial conditions $f_0 = 1$ and $f_1 = 1$. If n is even, the coefficients are from the *Lucas numbers* l_n, which has the same recurrence $l_{n+2} = l_{n+1} + l_n$ with the Fibonacci numbers but has the different initial conditions $l_0 = 2$ and $l_1 = 1$.

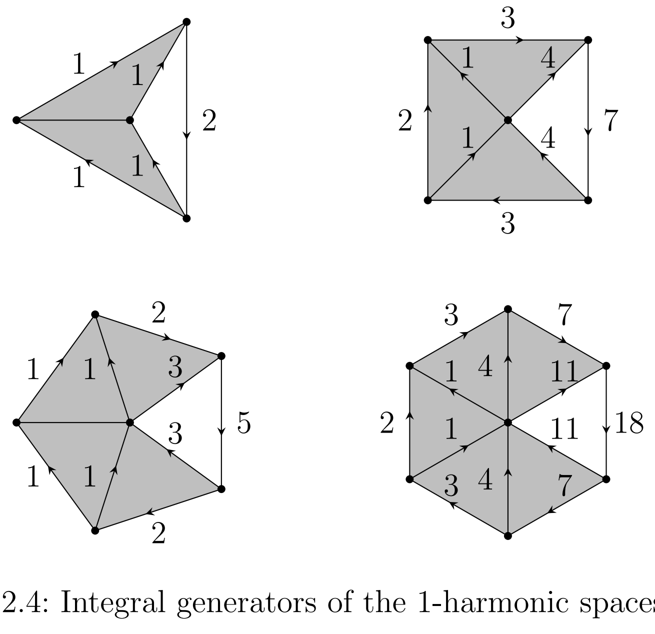


Figure 2.4: Integral generators of the 1-harmonic spaces of $\widetilde{W}_n$.

Definition 2.10. Given a graph G, an *acyclization* of G is a 2-dimensional chain complex $\mathcal{A}(G)$ whose 1-skeleton is equal to G and $H_i(\mathcal{A}(G)) = 0$ for all $0 \leq i \leq 2$.

Algebraically, the construction of $\mathcal{A}(G)$ is equivalent to declaring a basis $\{z_1, \ldots, z_g\}$ for $\ker \partial_1$ as 2-cells. The main objective of this thesis is to study the 1-harmonic space $\mathcal{H}_1(X)$ of the complex $X = \mathcal{A}(G) \sqcup \widetilde{e}$ with $\mathbb{Z}$ coefficients, where $\widetilde{e}$ is a new edge that was not originally in G. Note that $\mathcal{H}_1(X)$ is independent on the choice of the acyclization $\mathcal{A}(G)$. Topologically, X is homotopy equivalent to the one-dimensional sphere S^1, so its 1-Betti number is equal to 1. Therefore, its 1-harmonic space $\mathcal{H}_1(X)$ is one-dimensional.

Interestingly, a generator of $\mathcal{H}_1(X)$ can be computed in a combinatorial way using the spanning trees of G [KK19]. For a spanning tree T of G and vertices $v, w \in V(G)$, we denote by $\gamma_T(v, w)$ the unique path from v to w in T, and naturally identify it with a 1-chain in $C_1(X)$.

Theorem 2.11 ([KK19])**.** *Let $\mathcal{A}(G)$ be an acyclization of a graph G. Let $X = \mathcal{A}(G) \sqcup \widetilde{e}$ be a 2-dimensional complex obtained by adding an additional edge $\widetilde{e} = \{s, x\}$ to $\mathcal{A}(G)$. Define a 1-chain $\lambda \in C_1(X)$ by*

$$\lambda = \sum_{T \in \mathcal{T}(G)} \left(\gamma_T(x, s) + [\widetilde{e}]\right)$$

where $\widetilde{e}$ is oriented with $\widetilde{e} = (s, x)$. Then λ is a 1-harmonic cycle of X.

The harmonic cycle λ defined in Theorem 2.11 is called the *standard harmonic cycle* of X, and is a generator of $\mathcal{H}_1(X; \mathbb{R})$. An example of the standard harmonic cycle is described in Figure 2.5; for each spanning tree of G, the unique directed path $\gamma_T(x, s)$ is drawn in red arrows. Summing

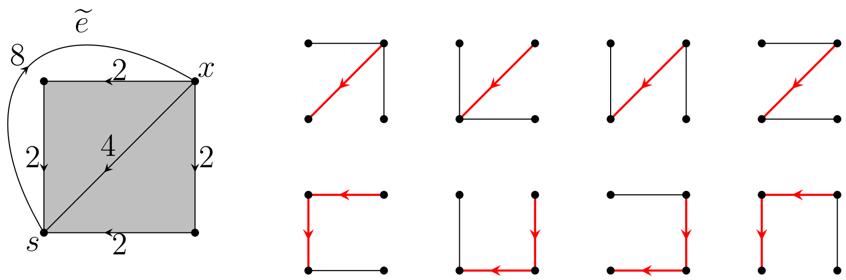


Figure 2.5: The standard harmonic cycle of the complex $\mathcal{A}(G) \sqcup \widetilde{e}$.

up all directed cycles $\gamma_T(x,s) + [\widetilde{e}]$ over all spanning trees T, we have the standard harmonic cycle λ of X. Note that the standard harmonic cycle is not necessarily a generator of $\mathcal{H}_1(X;\mathbb{Z})$, as its coefficients are not necessarily coprime. We end this section with the following natural question, which will be answered in Chapter 3.

Question 2.12. Can we find a partition $\mathcal{T}(G) = B_1 \sqcup \cdots \sqcup B_k$ such that for each i, the sum

$$\sum_{T \in B_i} (\gamma_T(x,s) + [\widetilde{e}])$$

is an integral generator of $\mathcal{H}_1(X;\mathbb{Z})$?

2.4 Chip-firing game and the critical group of a graph

The *critical group* $\mathcal{K}(G)$ is a finite abelian group associated to a graph G. This group has been studied in various fields and called in several different names, e.g. the *Jacobian group*, the *Picard group*, and the *sandpile group*, etc. In order to define $\mathcal{K}(G)$, we first give an explanation of an elementary combinatorial game on a graph, which is related to the definition of the critical group and has a lot of interesting properties.

Let $G = (V, E)$ be a graph. The *chip-firing game* on G is a solitary game associated with integers assigned to its vertices. A *chip configuration* is an integer valued function $c : V \longrightarrow \mathbb{Z}$. Intuitively, we can think of a chip configuration as assigning some number of *chips* (or *dept*, if the integer is negative.) to each vertex of G. The *firing move* at a vertex v is the transition such that the vertex v sends a chip to each one of its neighbors. More precisely, let c, c' be chip configurations where c' is obtained from c by *firing* a vertex v. Then we have

- $c'(v) = c(v) - \deg(v)$,
- $c'(w) = c(w) + (\#\text{ of edges between } v \text{ and } w). \quad (v \neq w)$

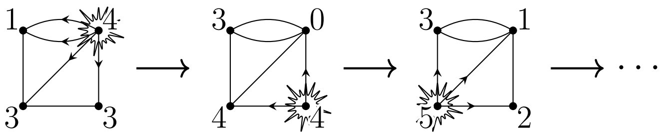


Figure 2.6: A description of the chip-firing game on a graph

The dynamics of the chip-firing game can be formally summarized by the Laplacian matrix L of G: Regard chip configurations c, c' as vectors in the 0-chain group $C_0(G; \mathbb{Z})$. Then the matrix $L = \partial_1 \partial_1^t$ is an operator on $C_0(G; \mathbb{Z})$ and the firing move at a vertex v is equivalent to subtracting the column of L indexed by v, i.e., $c' = c - L\mathbf{e}_v$, where $\mathbf{e}_v$ is the standard basis vector having 1 at the index v and 0 elsewhere.

We say that two chip configurations c_1 and c_2 are *firing-equivalent* if one can be obtained from the other by a sequence of firing moves and the inverses of the firing moves. Equivalently, c_1 and c_2 are firing-equivalent if

$c_1 \equiv c_2$ modulo $\operatorname{im} L$. This relation is clearly an equivalent relation. Note that two firing-equivalent chip configurations have the same total number of chips, since firing-moves do not add or remove chips.

Definition 2.13. The *critical group* $\mathcal{K}(G)$ of G is a quotient group

$$\mathcal{K}(G) := {}^{\ker \partial_0}\big/_{\operatorname{im} L}.$$

Intuitively, the critical group $\mathcal{K}(G)$ consists of firing-equivalence classes in G whose total number of chips are zero. Since $L = \partial_1 \partial_1^t$ and $\operatorname{im} L \subseteq \operatorname{im} \partial_1 \subseteq \ker \partial_0$, the quotient is well-defined. What is interesting is that $\mathcal{K}(G)$ is finite and its cardinality is equal to the number of spanning trees of G.

Proposition 2.14. *Let $\widetilde{L}$ be a reduced Laplacian of G. The critical group is isomorphic to the cokernel of $\widetilde{L}$, i.e.,*

$$\mathcal{K}(G) \cong \operatorname{coker} \widetilde{L} = {}^{\mathbb{Z}^{n-1}}\big/_{\operatorname{im} \widetilde{L}}.$$

In particular, the cardinality of $\mathcal{K}(G)$ is equal to $\det(\widetilde{L}) = \tau(G)$.

Sketch of proof. Let s be the vertex such that $\widetilde{L}$ is obtained from L by deleting the row and column corresponding to s. For $v \neq s$, let $c_v = [v] - [s]$. In other words, c_v is a chip configuration whose value at s is -1 and has value 1 at v and 0 elsewhere. Then the set of chip configurations $\{c_v : v \neq s\}$ is an integral basis for $\ker \partial_0$. Then the map $c_v \mapsto \mathbf{e}_v \in \mathbb{Z}^{n-1}$ induces an isomorphism between $\mathcal{K}(G)$ and $\operatorname{coker} \widetilde{L}$, where $\mathbf{e}_v$ is the standard basis vector. The argument of the cardinality follows from the matrix-tree theorem and the fact that the cokernel of a non-singular integer matrix M has the cardinality equal to $|\det(M)|$. $\square$

Remark 2.15. The isomorphic form $\mathbb{Z}^{n-1} / \operatorname{im} \widetilde{L}$ of $\mathcal{K}(G)$ is indeed commonly used to describe a variant form of the chip-firing game, also known as the *dollar game*. A vertex is designated as the 'sink' or 'bank', and chip configurations are restricted to have non-negative values on non-sink vertices. Under this setting, there are a lot of fascinating combinatorial properties, including the connection between the Tutte polynomial of the graph and a set of representatives of $\mathcal{K}(G)$; see [Big99], [Mer05], [CP18], or [Kli18] for details.

Example 2.16. Let $G = K_4 \setminus e$ be the graph in Figure 2.1. Its reduced Laplacian matrix with respect to v_1 is

$$\widetilde{L}_{v_1} = \begin{array}{c} \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccc} v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{pmatrix} \end{array}.$$

The critical group $\mathcal{K}(G)$ is then obtained by computing the cokernel of $\widetilde{L}_{v_1}$, which is $\mathbb{Z} / 8\mathbb{Z}$.

Example 2.17. The cycle graph C_n and the complete graph K_n has the critical groups $\mathcal{K}(C_n) \simeq \mathbb{Z} / n\mathbb{Z}$ and $\mathcal{K}(K_n) \simeq (\mathbb{Z} / n\mathbb{Z})^{n-2}$, respectively. See [Lor91], [Mer92].

The following lemma provides a relation between the critical group $\mathcal{K}(G)$ and an integral generator of the 1-harmonic space $\mathcal{H}_1(X)$, where $X = \mathcal{A}(G) \sqcup \widetilde{e}$ is defined in the previous section.

Lemma 2.18. *Let $\mathcal{A}(G)$ be an acyclization of a graph G. Let X be a 2-dimensional complex obtained from $\mathcal{A}(G)$ by adding an edge $\widetilde{e} = \{s, x\}$.*

Let h be an integral generator of $\mathcal{H}_1 = \mathcal{H}_1(X;\mathbb{Z})$. Then the coefficient of $[\widetilde{e}] = [s,x]$ in h is equal to the order of the element $\partial_1[\widetilde{e}] = [x]-[s]$ in the critical group $\mathcal{K}(G)$ up to sign.

Proof. We first show that the coefficient of $[\widetilde{e}]$ in h is equal to the cardinality of the quotient group $\ker\partial_1 \,/\, (\operatorname{im}\partial_2 \oplus \mathcal{H}_1)$ up to sign. By the definition of X, we can decompose $\ker\partial_1$ into $\ker\partial_1 = \operatorname{im}\partial_2 \oplus \langle z_{\widetilde{e}}\rangle$ as a $\mathbb{Z}$-module, where $z_{\widetilde{e}}$ is a cycle in which the coefficient of $[\widetilde{e}]$ is 1. By definition, the coefficient of $[\widetilde{e}]$ is zero in every elements in $\operatorname{im}\partial_2$. Therefore we have

$$\ker\partial_1 \Big/ (\operatorname{im}\partial_2 \oplus \mathcal{H}_1) \cong (\operatorname{im}\partial_2 \oplus \langle z_{\widetilde{e}}\rangle) \Big/ (\operatorname{im}\partial_2 \oplus \langle h\rangle),$$

and its cardinality is clearly equal to the absolute value of the coefficient of $\widetilde{e}$ in h.

Now consider the element $[x]-[s] \in C_0(X) = C_0(G)$. Clearly $[x]-[s]$ is in $\operatorname{im}\partial_1 = \ker\partial_0$, so it belongs to $\mathcal{K}(G)$. Consider the quotient group $\operatorname{im}\partial_1 \,/\, \partial_1(\ker\partial_2^t)$. Again by the definition of X, an integral basis of $\ker\partial_2^t$ can be chosen with the columns of the $(|E(G)|+1)\times(|V(G)|+1)$ matrix

$$\left(\begin{array}{ccc|c} & & & 0 \\ & \partial_{1,G}^t & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & 1 \end{array}\right),$$

where the last row is indexed by $\widetilde{e}$. Thus the space $\partial_1(\ker\partial_2^t)$ is generated by the columns of $\operatorname{im}\partial_1\partial_{1,G}^t$ and $\partial_1[\widetilde{e}] = [x]-[s]$. Indeed the matrix $\partial_1\partial_{1,G}^t$ is the Laplacian matrix of G, and hence the quotient $\operatorname{im}\partial_1 \,/\, \partial_1(\ker\partial_2^t)$ is equal to the critical group $\mathcal{K}(G)$ modded out by the subgroup generated by $[x]-[s]$. The cardinality is clearly equal to $\tau(G)$ divided by the order of $[x]-[s]$ in $\mathcal{K}(G)$.

We now compare the cardinalities of two quotients $\ker\partial_1 \,/\, (\operatorname{im}\partial_2 \oplus \mathcal{H}_1)$ and $\operatorname{im}\partial_1 \,/\, \partial_1(\ker\partial_2^t)$. Note that the maps ∂_1 and ∂_2^t induce isomorphisms

$$\operatorname{im}\partial_1 \Big/ \partial_1\left(\ker\partial_2^t\right) \xleftarrow[\partial_1]{\sim} C_1 \Big/ \left(\ker\partial_1 + \ker\partial_2^t\right) \xrightarrow[\partial_2^t]{\sim} \operatorname{im}\partial_2^t \Big/ \partial_2^t\left(\ker\partial_1\right).$$

On the other hand, ∂_2^t induces another isomorphism

$$\ker\partial_1 \Big/ \left(\operatorname{im}\partial_2^t \oplus \mathcal{H}_1\right) \xrightarrow[\partial_2^t]{\sim} \partial_2^t\left(\ker\partial_1\right) \Big/ \operatorname{im}\partial_2^t\partial_2.$$

Also, there is a natural isomorphism

$$\operatorname{im}\partial_2^t \Big/ \partial_2^t(\ker\partial_1) \cong \left(\operatorname{im}\partial_2^t \Big/ \operatorname{im}\partial_2^t\partial_2\right) \Big/ \left(\partial_2^t\left(\ker\partial_1\right) \Big/ \operatorname{im}\partial_2^t\partial_2\right).$$

Since $\operatorname{im}\partial_2^t \,/\, \operatorname{im}\partial_2^t\partial_2$ is isomorphic to the critical group $\mathcal{K}(G)$ via the isomorphisms

$$\mathcal{K}(G) \xleftarrow[\partial_1]{\sim} C_1 \Big/ \left(\ker\partial_1 \oplus \operatorname{im}\partial_1^t\right) \xrightarrow[\partial_2^t]{\sim} \operatorname{im}\partial_2^t \Big/ \operatorname{im}\partial_2^t\partial_2 \tag{2.4.1}$$

and its cardinality is equal to $\tau(G)$, we have the desired equality. □

The isomorphic forms of $\mathcal{K}(G)$ in (2.4.1) are well-known as different names. The quotient group $C_1(G;\mathbb{Z}) \,/\, (\ker\partial_1 \oplus \operatorname{im}\partial_1^t)$ is called the *cut-flow group* of G, as it is quotiented by the sum of the *cut space* $\operatorname{im}\partial_1^t$ and the *flow space* $\ker\partial_1$. The quotient group $\operatorname{im}\partial_2^t \,/\, \operatorname{im}\partial_2^t\partial_2$ is called the *co-critical group* of G, where ∂_2 is the 2-boundary map of the acyclization of G. Though it is defined on an acyclization of G, the co-critical group is independent of the choice of the acyclization. In particular, for a planar graph G, its co-critical group is the same with the critical group of its planar dual graph G^*; see Section 3.4 or [DKM15], [Kli18] for details.

Remark 2.19. The quotient group $\operatorname{im} \partial_1 \,/\, \partial_1 (\ker \partial_2^t)$ we used in the proof of Lemma 2.18 is indeed a group of the firing-equivalence classes under a variation of the chip-firing game. We regard $\widetilde{e} = \{s, x\}$ as a 'secret route', and re-define the firing rules of the game as follows:

- Chips can move along edges in G, via the firing moves in G.
- The vertex x can send a chip to s along the 'secret route' $\widetilde{e}$, and vice versa.

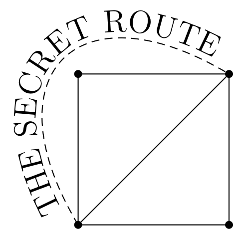


Figure 2.7: A variation of the chip-firing game with the secret route.

The modified firing-rule is then governed by the columns of the Laplacian as well as the vector $\partial_1[\widetilde{e}]$. In other words, the quotient $\operatorname{im} \partial_1 \,/\, \partial_1 (\ker \partial_2^t)$ consists of the firing-equivalent classes of chip configurations with total chips zero. For example, consider a graph $K_4 \setminus e$ with the new edge $\widetilde{e}$ attached as described in Figure 2.7. The firing moves under the modified rules correspond to subtracting the columns of the matrix

$$\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \left(\begin{array}{rrrr|r} 2 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & -1 \\ 0 & -1 & 2 & -1 & 0 \\ -1 & -1 & -1 & 3 & 1 \end{array} \right),$$

where the vertices are labeled with $v_1, \ldots, v_4$ in clockwise way starting from the top-left corner. The classical firing moves at each vertex on G corresponds to subtracting the first four columns, which are those of the Laplacian matrix L of G. The firing move at v_4 sending a single chip to v_2 along $\widetilde{e}$ corresponds to subtracting the last column of the matrix.

Chapter 3

Rotor-routing action on spanning trees

3.1 Definition of the rotor-routing action

Since the cardinality of the critical group $\mathcal{K}(G)$ is equal to the number of spanning trees of G, one might ask for a *natural* bijection between $\mathcal{K}(G)$ and $\mathcal{T}(G)$. By 'natural' we mean that the bijection is invariant under the automorphisms of G.

Wagner [Wag00] formalized the question as follows. Let $f : G \to H$ be a graph isomorphism. Then f induces two bijections $f_{\mathcal{K}}$ and $f_{\mathcal{T}}$:

$$f_{\mathcal{K}} : \mathcal{K}(G) \to \mathcal{K}(H),$$
$$f_{\mathcal{T}} : \mathcal{T}(G) \to \mathcal{T}(H).$$

The goal is to construct a map $\psi_G : \mathcal{K}(G) \to \mathcal{T}(G)$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{K}(G) \arrow[r, "\psi_G"] \arrow[d, "f_{\mathcal{K}}"] & \mathcal{T}(G) \arrow[d, "f_{\mathcal{T}}"] \\
\mathcal{K}(H) \arrow[r, "\psi_H"] & \mathcal{T}(H)
\end{tikzcd}$$

However, the answer is clearly 'no', since there is no distinguished element in $\mathcal{T}(G)$ which might correspond to the identity element of $\mathcal{K}(G)$. Wagner explicitly found the graphs of which there is no natural bijection between the critical group and the set of spanning trees [Wag00]. Instead, we might expect a nice action of $\mathcal{K}(G)$ on the set $\mathcal{T}(G)$.

The *rotor-routing action* is one of free transitive actions of $\mathcal{K}(G)$ on the set $\mathcal{T}(G)$, introduced by Holroyd, Levine, Mészáros, Peres, Propp, and Wilson [Hol+08]. To define the action, we need to fix an additional structure on a given graph.

Definition 3.1. A *ribbon graph* G is a graph together with a choice of cyclic ordering of the edges incident to each vertex.

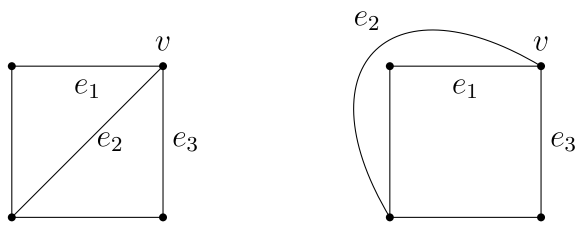


Figure 3.1: Two different ribbon graph structures on the graph $K_4 \setminus e$.

The ribbon graph structure determines how the graph is embedded onto a surface. Figure 3.1 describes two different ribbon graph structures on the same graph $K_4 \setminus e$. For each vertex, fix a cyclic ordering on the vertex to be oriented clockwise. In the left and right ribbon graphs, the cyclic orderings of edges $\{e_1, e_2, e_3\}$ on the vertex v is (e_1, e_2, e_3) and (e_1, e_3, e_2), respectively.

Given a ribbon graph, let us imagine a situation that a *rotor* is attached to each vertex that points an outgoing edge from the vertex. Each rotor can be activated by a *chip*, which causes the rotor to rotate and point toward the next edge in the cyclic ordering. Then the chip is *routed* along this edge. The mechanism is described in a formal way as follows.

Definition 3.2. Let G be a ribbon graph. A *rotor configuration* ρ is an assignment to each non-sink vertex v of an edge $\rho(v)$ that is outgoing from v. We call the edge $\rho(v)$ the *rotor* at v. A *(single chip) rotor state* is a pair (ρ, v) consisting of a rotor configuration ρ and a vertex v, which represents the location of the chip.

Remark 3.3. The definition of a rotor configuration is in fact defined on directed graphs, since it assigns each vertex an *outgoing* edge. We regard undirected graphs as directed ones, by replacing each undirected edge $\{v, w\}$ by a pair of directed edges (v, w) and (w, v).

The *rotor-routing operation* is an action on a rotor state (ρ, v), obtaining a new state (ρ^+, v^+) defined as follows:

- $\rho^+(w) = \rho(w)$ for $w \neq v$, and $\rho^+(v)$ is the next edge following $\rho(v)$ in the cyclic ordering at v.
- v^+ is the head of the edge $\rho^+(v)$.

Figure 3.2 shows an example of a rotor-routing operation. At each vertex, the cyclic ordering of its incident edges is clockwise. The location of the chip is represented by a circle.

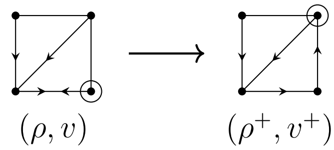


Figure 3.2

If a rotor state (ρ', v') can be obtained from (ρ, v) by iterating the rotor-routing operation for a finite number of times, i.e., there is a finite sequence of rotor states $\sigma = ((\rho_1, v_1), \ldots, (\rho_n, v_n))$ such that $(\rho_1, v_1) = (\rho, v)$, $(\rho_n, v_n) = (\rho', v')$, and $(\rho_i, v_i) = (\rho^+_{i-1}, v^+_{i-1})$, then we write $(\rho, v) \overset{\sigma}{\rightsquigarrow} (\rho', v')$, or simply $(\rho, v) \rightsquigarrow (\rho', v')$.

We say that a rotor configuration ρ (or a rotor state (ρ, v)) is *acyclic* if it has no directed cycles. Given an undirected graph G, fix a vertex s. As mentioned in Remark 3.3, we regard each edge $\{v, w\}$ as a pair of directed edges (v, w) and (w, v), but we remove the edges outgoing edges from s so that s is a *sink*. By directing all edges in a spanning tree T of G toward the sink s, there is a natural one-to-one correspondence between the set of spanning trees in $\mathcal{T}(G)$ and the set of acyclic rotor configurations in G. We denote by ρ_T the acyclic configuration corresponding to a spanning tree T.

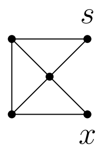
We now describe the rotor-routing action of $\mathcal{K}(G)$ on $\mathcal{T}(G)$. Note that $\mathcal{K}(G)$ is generated by the chip configurations c_x whose value at the sink s is -1 and has value 1 at a single non-sink vertex x and 0 elsewhere. Therefore it is sufficient to describe an action $T \mapsto c_x \cdot T$. The action is defined as follows:

1. Place a chip on the vertex x in ρ_T, so that we have a rotor state (ρ_T, x).

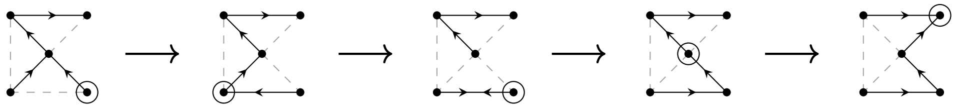
2. Iterate the rotor-routing operation on (ρ_T, x) until the chip arrives

the sink s.

3. The resulting rotor state (ρ', s) is acyclic, and hence $\rho' = \rho_{T'}$ for some spanning tree T'. Define $c_x \cdot T := T'$.



(a) A graph G



(b) The rotor-routing action on a spanning tree of G

Figure 3.3

Example 3.4. Let G be a graph in Figure 3.3(a). We assume that for each vertex in G, the cyclic ordering on the vertex is clockwise oriented with respect to the embedding of the figure. Figure 3.3(b) describes a rotor-routing action on a spanning tree of G. Starting from x, the chip arrives the sink s in four moves, resulting the new spanning tree.

Holroyd, Levine, Mészáros, Peres, Propp, and Wilson showed that the action is indeed well-defined, and is a free transitive action on $\mathcal{T}(G)$.

Theorem 3.5 ([Hol+08])**.** *The action $T \mapsto c_x \cdot T$ satisfies the followings:*

- *The chip arrives the sink s in the finite number of moves, i.e., the process eventually terminates.*

- *The resulting state (ρ', s) is acyclic.*

- *The action of c_x on $\mathcal{T}(G)$ is a permutation on $\mathcal{T}(G)$.*

In other words, the rotor-routing action is a well-defined free transitive action of $\mathcal{K}(G)$ on $\mathcal{T}(G)$.

Holroyd et al. proved Theorem 3.5 by introducing the notion of *unicycles.* we say that a rotor configuration ρ is a *unicycle* if it has a unique directed cycle $C(\rho)$. We also say that a rotor state (ρ, v) is a *unicycle* if ρ is a unicycle and $v \in C(\rho)$. The following lemma shows a simple but important property of unicycles.

Lemma 3.6 ([Hol+08])**.** *Let G be a sink-free directed graph. The rotor-routing operation is a permutation on the set of unicycles of G.*

Proof. Since the number of unicycles of G is finite, it is sufficient to show the following two properties:

(a) If (ρ, v) is a unicycle, then (ρ^+, v^+) is also a unicycle.

(b) Given a unicycle (ρ, v), there exists a unicycle (ρ', v') such that $(\rho'^+, v'^+) = (\rho, v)$.

Let (ρ, v) be a unicycle. Clearly there exists a directed cycle in ρ', since each vertex in ρ' must have outdegree one. We have to show that the uniqueness of the cycle and that v^+ is contained in that cycle. Since ρ and ρ' only differ by the rotor at v, the set of edges $\{\rho(w)\}_{w \neq v} = \{\rho'(w)\}_{w \neq v}$ has no directed cycles. Therefore the directed cycle must contain the edge $\rho^+(v)$, and such a cycle is unique and contains v^+. Hence, (a) is proved.

Given a unicycle (ρ, v), define a rotor state (ρ', v') as follows:

- v' is the predecessor on v in the directed cycle $C(\rho)$.

- $\rho'(w) = \rho(w)$ for $w \neq v'$ and $\rho'(v')$ is the predecessor of $\rho(v')$ in the cyclic ordering at v'.

Clearly the rotor-routing operation maps (ρ', v') to (ρ, v). It remains to show that (ρ', v') is a unicycle. Clearly (ρ', v') has a directed cycle, as each vertex in ρ' has outdegree one. Suppose a directed cycle in ρ' does not contains v'. Then the cycle is also in ρ, since ρ agrees with ρ' except at v'. But v' is in the unique cycle $C(\rho)$ of ρ, which is a contradiction. Hence ρ' has a unique cycle that contains v', and (ρ', v') is a unicycle. Thus (b) is proved. $\square$

Proof of Theorem 3.5. We first show that the chip eventually arrives the sink s. Suppose the sink s is never visited. Since we assume that G is finite, there is a vertex u that is visited infinitely often. If there exists an edge from u to some vertex w, then w is also visited infinitely many times. But since we assume that G is connected, there exists a directed path from u to s. Inducting along the path, we can conclude that s is eventually visited, which contradicts the assumption.

Next we show that (ρ', s) is acyclic. Let $\widetilde{G} = G \sqcup \widetilde{e}$ be a sink-free digraph where $\widetilde{e}$ is an additional edge from the sink s to x. Then the rotor-routing operations in G and $\widetilde{G}$ are equivalent, except when the chip is on s; in G, the operation terminates since s has no outgoing edges, while in $\widetilde{G}$ the chip moves to x along $\widetilde{e}$ without changing the states of rotors, i.e., $(\rho^+, s^+) = (\rho, x)$. Moreover, we have a natural bijection between the set of acyclic rotor states in G and the set of unicycle states in $\widetilde{G}$ whose cycle contains s. We can assume that the rotor-routing operation is performed in $\widetilde{G}$. By Lemma 3.6, a unicycle is mapped to a unicycle by the rotor-routing operation. Since (ρ, x) is a unicycle in $\widetilde{G}$, the resulting state (ρ', s) is also a unicycle in $\widetilde{G}$. Since the cycle $C(\rho')$ contains the sink s, the state (ρ', s) is acyclic in G.

It remains to show that the action c_x is a permutation on $\mathcal{T}(G)$. Since $\mathcal{T}(G)$ is finite, it is enough to show surjectivity. Let $T \in \mathcal{T}(G)$ be a spanning tree and $\rho = \rho_T$ be its corresponding acyclic rotor configuration. Consider the rotor state (ρ, s) in $\widetilde{G}$. By Lemma 3.6, the inverse of the rotor-routing operation on (ρ, s) is well-defined. Iterate the inverse of the rotor-routing operation on (ρ, s) until the next time the chip arrives the sink s and a unicycle (ρ', s) is obtained. Applying the rotor-routing operation on (ρ', s) once, we have the unicycle (ρ', x). Since the unique cycle of ρ' contains the sink s, the configuration ρ' is acyclic in G. Furthermore, the edge $\widetilde{e}$ is ignored during the iteration of the rotor-routing operation on (ρ', x) in $\widetilde{G}$ until the state (ρ, s) is obtained. Therefore the rotor-routing operations on (ρ', x) until the state (ρ, s) is obtained are the same in G and $\widetilde{G}$. Therefore, in G, the action c_x on $\mathcal{T}(G)$ is surjective. $\square$

Remark 3.7. As in the proof of Theorem 3.5, it is convenient to assume that the rotor-routing operation is performed in the augmented graph $\widetilde{G} = G \sqcup \widetilde{e}$, rather than in G itself. We will frequently recall this argument when we deal with the action of c_x on $\mathcal{T}(G)$.

Lemma 3.6 gives an equivalence relation $\leftrightsquigarrow$ on the set of unicycles in a sink-free directed graph: $(\rho, v) \leftrightsquigarrow (\rho', v')$ if and only if one can be obtained from another by iterating the rotor-routing operations. For a graph with a unique sink, we can still define an equivalence relation $(\rho, v) \leftrightsquigarrow (\rho', v')$ in G, on the union of the set of acyclic rotor states and the set of unicycles, by the argument in Remark 3.7. Furthermore, we can state Lemma 3.6 in a different way.

Lemma 3.8. *Let G be a graph with a unique sink s. During the action $T \mapsto c_x \cdot T$, each occurring rotor state is either acyclic or a unicycle.*

3.2 Rotor-routing on planar graphs and reversibility of cycles

In this section, we introduce the notion of the *reversibility* of directed cycles and some nice properties of the rotor-routing process on planar graphs, which are studied by Chan, Church, and Grochow [CCG14].

For a directed cycle C, we denote by $\overline{C}$ the reversal of C, the same cycle with oppositely directed edges. Similarly we denote by $\overline{P}$ the reversal of the directed path P. If (ρ, v) is a unicycle with a cycle $C = C(\rho)$, then we denote by $\overline{\rho}$ the configuration obtained by reversing C and leaving all other rotors unchanged.

We say that a directed cycle of length two in a rotor configuration is *flat* if it is just a round-trip on the same edge. Note that not all directed cycles of length two are flat, since there can be multiple edges between two vertices. The directed cycle C is flat if and only if its reversal $\overline{C}$ is equal to C itself.

(a) A flat cycle

(b) A non-flat cycle of length 2

Figure 3.4

Definition 3.9 ([CCG14])**.** Let (ρ, v) be a unicycle with the directed cycle $C = C(\rho)$. We say that C is *reversible* if $(\rho, v) \leftrightsquigarrow (\overline{\rho}, v)$.

The reversibility of a cycle is indeed well-defined; it does not depend on the choice of (ρ, v) and indeed is a property of cycle itself.

Proposition 3.10 ([CCG14]). *If (ρ, v) and (ρ', v') are unicycles with $C = C(\rho) = C(\rho')$, then $(\rho, v) \leftrightsquigarrow (\overline{\rho}, v)$ if and only if $(\rho', v') \leftrightsquigarrow (\overline{\rho'}, v')$.*

Chan, Church, and Grochow studied the relation between the reversibility of cycles and the sink-independence of the rotor-routing actions. They also proved that the planarity is a necessary and sufficient condition for a graph that all cycles are reversible, as well as the condition that the rotor-routing action is sink-independent.

Proposition 3.11 ([CCG14]). *A ribbon graph G is planar if and only if all cycles on G are reversible.*

Theorem 3.12 ([CCG14]). *The rotor-routing action of $\mathcal{K}(G)$ on $\mathcal{T}(G)$ is independent of the choice of sink if and only if G is planar.*

Another important characterization of planar graphs is that every cycle in a planar graph is *separating*, i.e., the *left side* and the *right side* of a directed cycle can be distinguished, unless the cycle is flat. More formally, let C be a non-flat directed cycle in a planar graph G. we can partition the set of the vertices not in C into two sets $L_C \sqcup R_C$ according to the relative position with respect to C; The set L_C (resp. R_C) consists of vertices $w \notin C$ such that there is a path P from w to some vertex v in C, where $P \cap C = \{v\}$ and the last edge is on the left (resp. right) side of C at v.

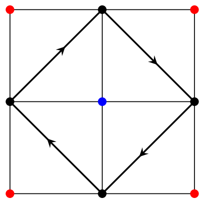


Figure 3.5: Vertices are separated by a directed cycle C, drawn in thick arrow. Red and blue represents the vertices in L_C and R_C, respectively.

In the aspect of the reversibility, the sets L_C and R_C are distinguished by the number of visits.

Lemma 3.13 ([CCG14]). *Let G be an undirected planar graph and assume the cyclic ordering of edges at each vertex is given clockwise. Let (ρ, v) be a unicycle in G, with a directed cycle C. During the rotor-routing process from (ρ, v) to $(\overline{\rho}, v)$,*

- *each vertex $w \in L_C$ is visited 0 times.*
- *each vertex $w \in R_C$ is visited $\deg_G(w)$ times.*

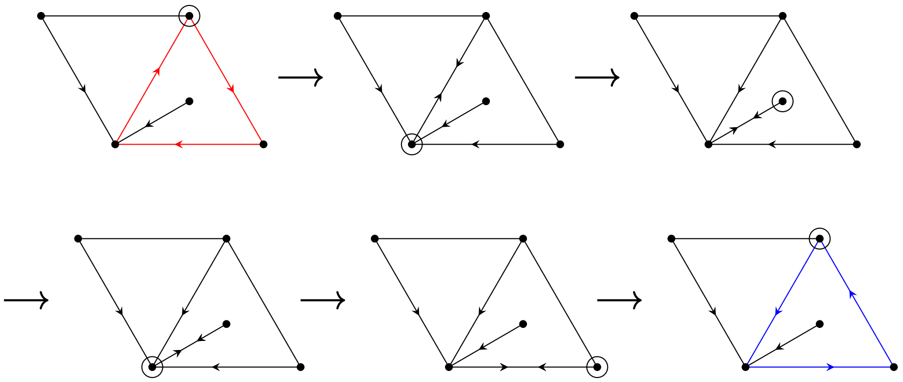


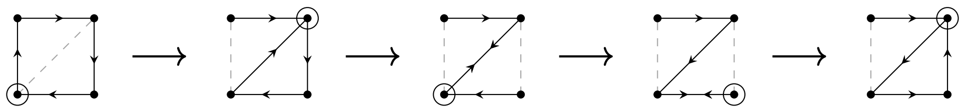
Figure 3.6: An example of Lemma 3.13. During the red cycle C is reversed to blue cycle $\overline{C}$, the vertex in L_C is never visited, while the vertex in R_C is visited exactly once.

3.3 Chip-trace of rotor-routing process and harmonic cycles

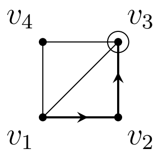
We now investigate the relation between the rotor-routing action on a planar graph G and an integral generator of the 1-harmonic space of the complex $X = \mathcal{A}(G) \sqcup \widetilde{e}$, where $\widetilde{e} = \{s, x\}$ is an edge not in G.

Definition 3.14. Let σ be a sequence of rotor states in a graph G such that $(\rho, v) \overset{\sigma}{\rightsquigarrow} (\rho', v')$ for some rotor states (ρ, v) and (ρ', v'). The *chip-trace* of σ is the 1-chain $\mathrm{tr}(\sigma) \in C_1(G; \mathbb{Z})$ generated by the directed walk from v to v' that the chip traverses during σ.

Example 3.15. In the sequence of rotor states in Figure 3.7(a), the directed walk generated by the chip visits vertices v_1, v_3, v_1, v_2, v_3 in order. The chip-trace $\mathrm{tr}(\sigma)$ is the 1-chain $[v_1, v_3] + [v_3, v_1] + [v_1, v_2] + [v_2, v_3]$. Canceling out the terms $[v_1, v_3] + [v_3, v_1]$, we have $\mathrm{tr}(\sigma) = [v_1, v_2] + [v_2, v_3]$.



(a) A sequence σ of rotor-routing operations on the graph $K_4 \setminus e$.



(b) The chip-trace of σ.

Figure 3.7

It turns out that the trace of the chip during the rotor-routing action of $c_x \in \mathcal{K}(G)$ is related to an integral generator of the 1-harmonic space of the complex $X = \mathcal{A}(G) \sqcup \widetilde{e}$. For a spanning tree $T \in \mathcal{T}(G)$, we consider the trace of the chip during the iteration of the rotor-routing action of c_x on T until the rotor configuration comes back to T itself. We regard the action is performed on the augmented graph $\widetilde{G} = G \sqcup \widetilde{e}$ so that if the chip arrives the sink s, in the next step it moves to x along $\widetilde{e}$ without changing the rotor configuration. (See Remark 3.7.)

Theorem 3.16. *Let G be a graph with unique sink s and let $T \in \mathcal{T}(G)$ be a spanning tree of G. Let σ be the minimal nonempty sequence of rotor states such that*

$$(\rho_T, x) \overset{\sigma}{\rightsquigarrow} (\rho_T, x).$$

Then $\operatorname{tr}(\sigma)$ is an integral generator of $\mathcal{H}_1(X)$, where $X = \mathcal{A}(G) \sqcup \widetilde{e}$.

Proof. We first show that $\operatorname{tr}(\sigma)$ belongs to both $\ker \partial_1$ and $\ker \partial_2^t$. Clearly $\operatorname{tr}(\sigma) \in \ker \partial_1$, since it is a closed walk from x to itself. Note that as a $\mathbb{Z}$-module, $\ker \partial_2^t$ is decomposed into

$$\ker \partial_2^t = \operatorname{im} \partial_{1,G}^t \oplus \langle [\widetilde{e}] \rangle ,$$

so it is sufficient to show that the restriction $\operatorname{tr}(\sigma)|_G$ of $\operatorname{tr}(\sigma)$ to the edges in G belongs to $\operatorname{im} \partial_{1,G}^t$. Define a 0-chain $c \in C_0(G; \mathbb{Z})$ as follows. For each non-sink vertex v, define $c(v)$ by the whole number of full turns of the rotor at v during the sequence σ, and define $c(s) := 0$. For a non-sink vertex v, each edge outgoing from v is traversed exactly $c(v)$ times in σ. Therefore the value of $\operatorname{tr}(\sigma)|_G$ at $[v, w]$ is equal to $c(v) - c(w)$, which gives a formula $\operatorname{tr}(\sigma)|_G = -\partial_{1,G}^t(c)$. Hence we have $\operatorname{tr}(\sigma) \in \ker \partial_2^t$, and $\operatorname{tr}(\sigma) \in \mathcal{H}_1(X)$.

The coefficient of $[\widetilde{e}]$ in $\operatorname{tr}(\sigma)$ is equal to the number of times the chip arrives the sink s, which is equal to the order of c_x in the critical group $\mathcal{K}(G)$. By Lemma 2.18, $\operatorname{tr}(\sigma)$ must be an integral generator of $\mathcal{H}_1(G)$. $\square$

For planar graphs, we can say more things about Theorem 3.16, using the reversibility of directed cycles. We will give an answer to Question 2.12 for planar graphs with Theorem 3.21. In order to introduce the theorem, we need a sequence of lemmas.

Lemma 3.17. *Let G be a planar graph with unique sink s. Let (ρ, v) be a unicycle on G with the non-flat cycle $C = C(\rho)$. Let σ be the sequence of rotor states such that $(\rho, v) \overset{\sigma}{\rightsquigarrow} (\overline{\rho}, v)$. If $s \in L_C$, then the chip-trace* $\mathrm{tr}(\sigma)$ *is equal to the reversal $\overline{C}$ of the cycle C.*

Proof. Since the vertices in C and R_C are non-sink vertices, we can assume that the rotor-routing process σ is done on the undirected graph. Let w be a vertex. If w is in C, then the number of times a vertex $w \in C$ is visited by the chip is equal to 1 plus the number of edges at w on the right of C, since σ reverses the cycle C. If w is in R_C, then by Lemma 3.13, the vertex w is visited exactly $\deg_G(w)$ times. Therefore, the edges incident to some vertex in R_C are traversed exactly once for each direction by the chip, and is cancelled out in $\mathrm{tr}(\sigma)$. The edges incident to some vertex in L_C are never visited in σ, and the edges in C is traversed once in the opposite direction of C, thus we have the conclusion. □

Before we state and prove the next lemma, we make a notation of the *restriction* of a directed path (or cycle): For a directed path (or cycle) P and vertices $v, w \in P$, we denote by $P|_v^w$ the restriction of P from v to w.

Lemma 3.18. *Let $(\rho_0, x_0), (\rho_1, x_1), \ldots$ be a finite sequence of rotor states on a planar graph G with the sink s such that*

- *(ρ_0, x_0) is acyclic, and (ρ_i, x_i) is a unicycle with a cycle $C_i = C(\rho_i)$ for $i \geq 1$.*

- $(\rho_{i+1}, x_{i+1}) = (\overline{\rho_i}^+, x_i^+)$ *for* $i \geq 1$.
- $(\rho_1, x_1) = (\rho_0^+, x_0^+)$.

For $i \geq 0$*, let* γ_i *be the unique directed path from* x_i *to* s *in* ρ_0*. Then for each* $i \geq 1$*, the following statements hold:*

(a) γ_i *passes by* x_0 *and there exists a vertex* $y_i \in \gamma_i|_{x_i}^{x_0}$ *such that* γ_{i-1} *and* γ_i *meets at* y_i *and* $\gamma_{i-1}|_{y_i}^{s} = \gamma_i|_{y_i}^{s}$.

(b) $\rho_i(w) = \rho_0(w)$ *for vertices* $w \notin \gamma_{i-1}|_{x_{i-1}}^{x_0}$*. The edges in* $\gamma_{i-1}|_{x_{i-1}}^{x_0}$ *is directed in the opposite direction of* $\gamma_{i-1}|_{x_{i-1}}^{x_0}$ *in* ρ_i*, and* $\rho_i(x_{i-1})$ *is directed toward* x_i.

(c) The cycle C_i *is of the form* $\rho_i(x_{i-1}) \cup \gamma_i|_{x_i}^{y_i} \cup \overline{\gamma_{i-1}}|_{y_i}^{x_{i-1}}$*, and either* C_i *is a flat cycle or* $s \in L_{C_i}$*. In other words,* s *is not visited in* rev_i*, where* rev_i *is the sequence of the rotor states such that* $(\rho_i, x_i) \overset{\mathrm{rev}_i}{\rightsquigarrow} (\overline{\rho_i}, x_i)$.

(d) $\mathrm{tr}(\sigma_i) = \overline{\gamma_i}|_{x_0}^{x_i}$*, where* σ_i *is the sequence of rotor states such that* $(\rho_0, x_0) \overset{\sigma_i}{\rightsquigarrow} (\overline{\rho_i}, x_i)$.

Proof. We use an induction on i. Suppose the lemma holds for $1, \ldots, i$, and consider the rotor state $(\rho_{i+1}, x_{i+1}) = (\overline{\rho_i}^+, x_i^+)$. By the hypotheses on (b) and (c), $\overline{\rho_i}$ agrees with ρ_0 except that $\gamma_i|_{x_i}^{x_0}$ is reversed and $\overline{\rho_i}(x_i)$ is directed toward x_{i-1}. Since $\overline{\rho_i}$ and ρ_{i+1} differ only by the rotor at x_i and the head of $\rho_{i+1}(x_i)$ is x_{i+1} by definition, (b) is proved for $i+1$.

Since both γ_i and γ_{i+1} end with s, two paths must meet at some vertex y_{i+1}. It is clear that two paths $\gamma_i|_{y_{i+1}}^{s}$ and $\gamma_{i+1}|_{y_{i+1}}^{s}$ are the same, since ρ_0 is acyclic. Suppose y_{i+1} is not in $\gamma_i|_{x_i}^{x_0}$. For any vertex w, we consider the walk starting from w, along the directed edges in ρ_{i+1}. If the walker visits some vertex in $\gamma_i|_{x_i}^{x_0}$, then by (b), the walker reaches x_{i+1} and arrives the sink along γ_{i+1}. If the walker does not visit any vertex in $\gamma_i|_{x_i}^{x_0}$, then again by (b), the walk is the same with the one performed in ρ_0, and it eventually arrives the sink. Since the directed walk starting from any vertex arrives the sink, ρ_{i+1} must be acyclic, but it contradicts our assumption. Therefore, y_{i+1} must lie in $\gamma_i|_{x_i}^{x_0}$. and (a) is proved for $i+1$.

By (a) and (b), C_{i+1} is of the form $\rho_{i+1}(x_i) \cup \gamma_{i+1}|_{x_{i+1}}^{y_{i+1}} \cup \overline{\gamma_i}|_{y_{i+1}}^{x_i}$. Suppose C_{i+1} is not flat. Consider a walk starting from x_i, along the directed edges $\overline{\rho_i}(x_i), \ldots, \overline{\rho_1}(x_1)$ and $\rho_0(x_0)$. Along the walk, the vertices $x_i, x_{i-1}, \ldots, x_0$ are visited in this order, and the walk ends at the head of $\rho_0(x_0)$. Denote by x_{-1} the head of $\rho_0(x_0)$. We claim that x_j is not in $R_{C_{i+1}}$ for each $-1 \le j < i$.

For $j = i-1$, suppose $x_{i-1} \in R_{C_{i+1}}$. Consider the cyclic ordering at x_i. The edge $\overline{\rho_i}(x_i)$ must be followed by $\rho_{i+1}(x_i)$, which is the outgoing edge from x_i in C_{i+1}. But since $x_{i-1} \in R_{C_{i+1}}$, the incoming edge to x_i in C_{i+1} is located between $\overline{\rho_i}(x_i)$ and $\rho_{i+1}(x_i)$ in the cyclic ordering at x_i, which is a contradiction. Hence the vertex x_{i-1} is not in $R_{C_{i+1}}$. Suppose x_j is in $R_{C_{i+1}}$ for some $-1 \le j < i-1$. Choose j to be a maximal such an index. Then x_{j+1} must lie in C_{i+1}. The vertex x_{j+2} cannot be in $R_{C_{i+1}}$ by the choice of j, hence either $x_{j+2} \in C_{i+1}$ or $x_{j+2} \in L_{C_{i+1}}$. If $x_{j+2} \in L_{C_{i+1}}$, then the incoming edge of x_{j+1} in C_{i+1} contradicts that $\rho_{j+2}(x_{j+1})$ is the next edge of $\overline{\rho_{j+1}(x_j)} = \overline{\rho_{j+1}}(x_{j+1})$ in the cyclic ordering of edges at x_{j+1}. Therefore x_{j+2} must be in C_{i+1}, and the edge $\rho_{j+2}(x_{j+1})$ and the path $C_{i+1}|_{x_{j+2}}^{x_{j+1}}$ forms a cycle. Repeating this argument, the vertices $x_{j+1}, \ldots, x_i$ are contained in C_{i+1}, see Figure 3.8. Let C' be the cycle defined by $C' = C_{i+1}|_{x_i}^{x_{i-1}} \cup \rho_i(x_{i-1})$. If C' is flat, then the outgoing edge of x_i in C' is equal to $\overline{\rho_i(x_{i-1})} = \overline{\rho_i}(x_i)$. But the outgoing edge of x_i in C_{i+1} is $\rho_{i+1}(x_i)$, hence we have $\rho_{i+1}(x_i) = \overline{\rho_i}(x_i)$, which is the case that the degree of x_i is 1. Therefore $x_{i+1} = x_{i-1}$ and C_{i+1} is a flat cycle formed by the unique edge incident to x_i, contradicting the assumption. Now suppose C' is not flat. The incoming and outgoing edges of x_i in C' are $\rho_i(x_{i-1})$ and $\rho_{i+1}(x_i)$, respectively. Since these two edges are consecutive in the cyclic order on x_i, there is no edge on the left of C' incident to x_i. But the outgoing edge of x_{i-1} in C_{i+1} is on the left of C', which is contradiction since $C_{i+1}|_{x_{i-1}}^{x_i}$ must come back to x_i without visiting $C_{i+1}|_{x_i}^{x_{i-1}}$.

Therefore $x_{-1}, \ldots, x_i$ must be in either C_{i+1} or $L_{C_{i+1}}$. In particular, x_{-1} must be in $L_{C_{i+1}}$, since C_{i+1} do not contains $\gamma_0|_{x_{-1}}^{s}$, by the hypotheses on (a) and (c). Therefore, the sink s is also contained in $L_{C_{i+1}}$, by the existence of the path $\gamma_0|_{x_k}^{s}$, where k is the smallest index such that $x_k \in C_{i+1}$.

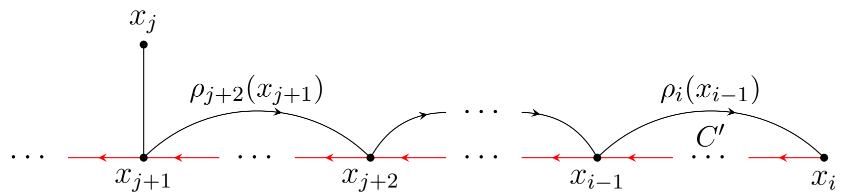


Figure 3.8: A description of the proof of Lemma 3.18 (c). The red arrows represent C_{i+1}.

It remains to show (d). Let σ_{i+1} be the sequence of the rotor states such that $(\rho_0, x_0) \overset{\sigma_{i+1}}{\rightsquigarrow} (\overline{\rho_{i+1}}, x_{i+1})$, which clearly exists by (c). Note that σ_{i+1} can be divided into two parts $\widetilde{\sigma}_i$ and rev_{i+1}, where $(\rho_0, x_0) \overset{\widetilde{\sigma}_i}{\rightsquigarrow} (\rho_{i+1}, x_{i+1})$. By the induction hypothesis, the chip-trace of $\widetilde{\sigma}_i$ is equal to $\overline{\gamma_i}|_{x_0}^{x_i} \cup \rho_{i+1}(x_i)$. By (c), the sequence rev_{i+1} does not visit s. By Lemma 3.17, the chip-trace $\mathrm{tr}(\mathrm{rev}_{i+1})$ is equal to the reversal of C_{i+1}, i.e., $\overline{\rho_{i+1}(x_i)} \cup \gamma_i|_{x_i}^{y_{i+1}} \cup \overline{\gamma_{i+1}}|_{y_{i+1}}^{x_{i+1}}$. (See Figure 3.9.) Therefore the chip-trace of σ_{i+1} is equal to $\mathrm{tr}(\sigma_{i+1}) = \mathrm{tr}(\widetilde{\sigma}_i) + \mathrm{tr}(\mathrm{rev}_{i+1}) = \overline{\gamma_{i+1}}|_{x_0}^{x_{i+1}}$.

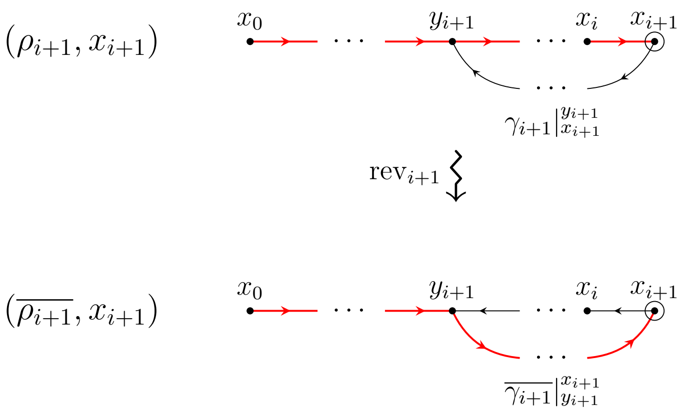


Figure 3.9: Description of the changes in the chip-trace during the rotor-routing process in the sequence rev_{i+1}. The red arrows represents the trace of the chip so far.

□

Lemma 3.19. *Let T and T' be spanning trees of a planar graph G such that $T' = c_x \cdot T$. Then we have $\mathrm{tr}(\sigma) = \gamma_{T'}(x, s)$, where $(\rho_T, x) \overset{\sigma}{\rightsquigarrow} (\rho_{T'}, s)$.*

Proof. Let $(\rho_0, x_0) = (\rho_T, x)$. We recursively define a sequence of rotor states as follows:

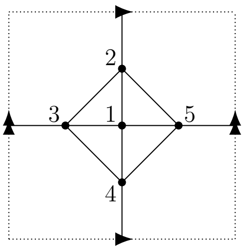
- If (ρ_i^+, x_i^+) is acyclic, then define $(\rho_{i+1}, x_{i+1}) = (\rho_i^+, x_i^+)$.
- If (ρ_i^+, x_i^+) is a unicycle, then define $(\rho_{i+1}, x_{i+1}) = (\overline{\rho_i^+}, x_i^+)$.

By (c) and (d) of Lemma 3.18, each (ρ_i, x_i) can be obtained by iterating rotor-routing process starting from (ρ_0, x_0). Since the chip eventually reaches the sink s, we have $(\rho_n, x_n) = (\rho_{T'}, s)$ for some positive integer n. We use an induction on i to show that the chip-trace of the sequence of rotor states σ_i such that $(\rho_0, x_0) \overset{\sigma_i}{\rightsquigarrow} (\rho_i, x_i)$ is equal to the unique directed path P_i from x_0 to x_i in ρ_i.

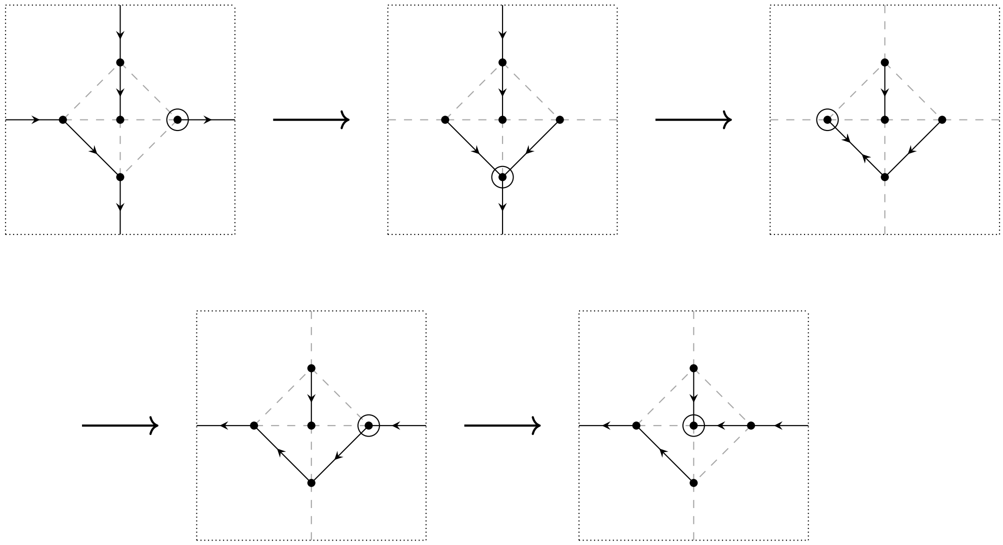
Suppose the claim is true for $1, \ldots, i$. Consider the case (ρ_i^+, x_i^+) is acyclic, so that $(\rho_{i+1}, x_{i+1}) = (\rho_i^+, x_i^+)$. Since ρ_i and ρ_{i+1} differ only by the rotor at x_i, the directed path $P_{i+1} = P_i \cup \rho_{i+1}(x_i)$ is the unique path from x_0 to x_i in ρ_{i+1}. The sequence σ_{i+1} is just an extension of σ_i by joining (ρ_{i+1}, x_{i+1}) at the last step. Therefore $\mathrm{tr}(\sigma_{i+1})$ is equal to P_{i+1}.

Now consider the case (ρ_i^+, x_i^+) is a unicycle, so that $(\rho_{i+1}, x_{i+1}) = (\overline{\rho_i^+}, x_i^+)$. Let $j \leq i$ be the largest index such that ρ_j is acyclic. We divide σ_{i+1} into two parts σ_j and σ' by $(\rho_0, x_0) \overset{\sigma_j}{\rightsquigarrow} (\rho_j, x_j) \overset{\sigma'}{\rightsquigarrow} (\rho_{i+1}, x_{i+1})$. By (d) of Lemma 3.18, the chip-trace of σ' is equal to the reversal of the unique directed path γ from x_{i+1} to x_j in ρ_j. Together with the induction hypothesis for j, we have $\mathrm{tr}(\sigma_{i+1}) = \mathrm{tr}(\sigma_j) + \mathrm{tr}(\sigma') = P_j + \overline{\gamma}$. Since ρ_j is acyclic, there is no nontrivial cycle in the walk $P_j + \overline{\gamma}$. Therefore, the chip-trace $\mathrm{tr}(\sigma_{i+1})$ is indeed a path from x_0 to x_{i+1}, which is in ρ_{i+1} by (b) of Lemma 3.18. $\square$

Remark 3.20. Lemma 3.19 may fail on a non-planar graph. Consider the embedding of the complete graph K_5 in a torus described in Figure 3.10(a). The cyclic ordering at each vertex is clockwise with respect to the embedding. Figure 3.10(b) shows a sequence of rotor states occurring in the action of c_5 on a spanning tree, with a choice of sink $s = 1$. The chip-trace of the sequence is $[5,4] + [4,3] + [3,5] + [5,1]$, which is different from the unique directed path from 5 to 1 in the resulting tree.



(a) An embedding of the complete graph K_5 in a torus.



(b) Iteration of the rotor-routing operations on a spanning tree of K_5.

Figure 3.10

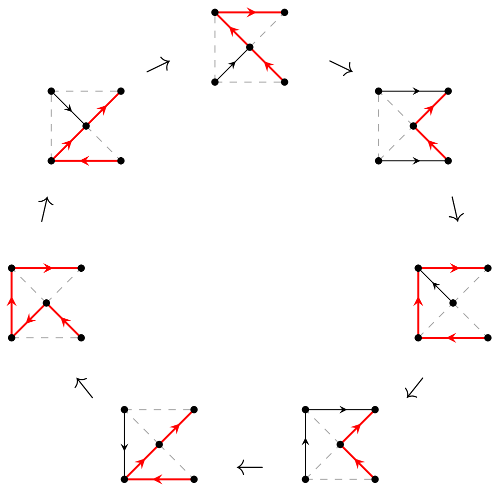
Theorem 3.21. *Let G be a planar graph and let $X = \mathcal{A}(G) \sqcup \widetilde{e}$. For any orbit $\mathcal{O}$ in $\mathcal{T}(G)$ under the action of $c_x \in \mathcal{K}(G)$, the sum*

$$\sum_{T \in \mathcal{O}} (\gamma_T(x, s) + [\widetilde{e}]) \tag{3.3.1}$$

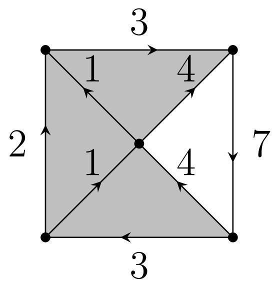
is an integral generator of $\mathcal{H}_1(X)$.

Proof. By Lemma 3.19, the sum in (3.3.1) is equal to the chip-trace of the sequence of rotor states that starts from and ends with (ρ_T, x) for some $T \in \mathcal{O}$. By Theorem 3.16, it is an integral generator of $\mathcal{H}_1(X)$. □

Example 3.22. Figure 3.11 describes an example of Theorem 3.21. Let $X = \widetilde{W}_4$. Each orbit under the rotor-routing action of c_x contains seven spanning trees. An integral generator of $\mathcal{H}_1(X)$ can be obtained by using the unique directed path in each spanning tree in an orbit.



(a) An orbit in $\mathcal{T}(G)$ under the action of $c_x \in \mathcal{K}(G)$.



(b) An integral generator of $\mathcal{H}_1(X)$.

Figure 3.11

3.4 Rotor-routing on the planar dual graph

For a planar graph $G = (V, E)$, its *(planar) dual graph* $G^* = (V^*, E^*)$ is defined as follows:

- The vertices of G^* are the faces of G (including the infinite face).
- For each edge in G, connect two vertices in G^* that correspond to the two faces in G having the edge in their boundaries in common.

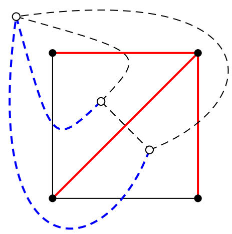


Figure 3.12: A spanning tree of the graph $G = K_4 \setminus e$ and its dual in G^*

It is an elementary fact that the dual G^* is also planar. By definition, the edge sets E and E^* can be naturally identified. We denote $e^* \in E^*$ by the edge of G^* that corresponds to the edge $e \in E$ of G. It is well known that there is a natural bijection between $\mathcal{T}(G)$ and $\mathcal{T}(G^*)$, by taking the set complement. In Figure 3.12, the dual graph G^* of the graph $G = K_4 \setminus e$ is drawn in dashed lines. A spanning tree of G is drawn in red, and its corresponding tree of G^* is drawn in blue.

As well as G and G^* have the same number of spanning trees, their critical groups $\mathcal{K}(G)$ and $\mathcal{K}(G^*)$ are isomorphic [CR00]. In order to show this fact, we consider another isomorphic form of the $\mathcal{K}(G)$.

Definition 3.23. The *cut-flow group* $\mathcal{E}(G)$ of a graph G is defined by

$$\mathcal{E}(G) = {C_1(G;\mathbb{Z})}\Big/{\left(\ker\partial_1 \oplus \operatorname{im}\partial_1^t\right)},$$

where $C_1(G;\mathbb{Z})$ is the 1-chain group of G over the integer coefficients and ∂_1 is the 1-boundary map.

As mentioned in the proof of Lemma 2.18, it can be easily shown that the map ∂_1 induces an isomorphism between $\mathcal{E}(G)$ and $\mathcal{K}(G)$. Therefore it is equivalent to show that $\mathcal{E}(G) \simeq \mathcal{E}(G^*)$ to prove $\mathcal{K}(G) \simeq \mathcal{K}(G^*)$.

First we identify $C_1(G^*;\mathbb{Z})$ with $C_1(G;\mathbb{Z})$ by choosing a compatible orientation on E^*, with respect to a fixed orientation on E. Let $e \in E$ be an edge of G oriented by $e = (u, v)$. Let $e^* \in E^*$ be the corresponding edge of e in G^*. The two endpoints of e^* are two faces a, b of G that share e in their boundaries. Without loss of generality, let a be the face that appears 'before' e in the cyclic ordering on the tail u of e. We orient e^* with $e^* = (b, a)$, so that a is the head of e^* and b be the tail of e^*.

Then the following proposition completes the proof of $\mathcal{E}(G) \simeq \mathcal{E}(G^*)$.

Proposition 3.24. *Let G^* be the dual of the planar graph G. Let $C_1(G;\mathbb{Z})$ and $C_1(G^*;\mathbb{Z})$ are identified by the above argument. Then we have*

$$\ker\partial_{1,G} = \operatorname{im}\partial_{1,G^*}^t, \quad \operatorname{im}\partial_{1,G}^t = \ker\partial_{1,G^*}.$$

The isomorphism can be alternatively shown by identifying $\mathcal{K}(G^*)$ with the *co-critical group* $\mathcal{K}^*(G)$ of G. The co-critical group $\mathcal{K}^*(G)$ is defined to be

$$\mathcal{K}^*(G) := {\operatorname{im}\partial_2^t}\Big/{\operatorname{im}\partial_2^t\partial_2} = {C_2(X;\mathbb{Z})}\Big/{\operatorname{im}\partial_2^t\partial_2},$$

where X is the acyclization of G by filling in the finite faces of G. (Assume that the faces of X are oriented counterclockwise.) By definition, there is an isomorphism between $\mathcal{K}(G^*)$ and $\mathcal{K}^*(G)$ induced by the restriction of the domain to the finite faces. Then the isomorphism between $\mathcal{K}(G)$ and $\mathcal{K}(G^*)$ that we constructed in the above is equivalent to the isomorphism between $\mathcal{K}(G)$ and $\mathcal{K}^*(G)$ defined by

$$\mathcal{K}(G) \xleftarrow[\partial_1]{\sim} \mathcal{E}(G) \xrightarrow[\partial_2^t]{\sim} \mathcal{K}^*(G),$$

which is mentioned earlier in (2.4.1).

Example 3.25. Figure 3.13 describes an example of the isomorphism between the critical groups of the graph $G = K_4 \setminus e$ and its dual G^*. On each corner, a representative of each isomorphic image of an element in the corresponding group is drawn.

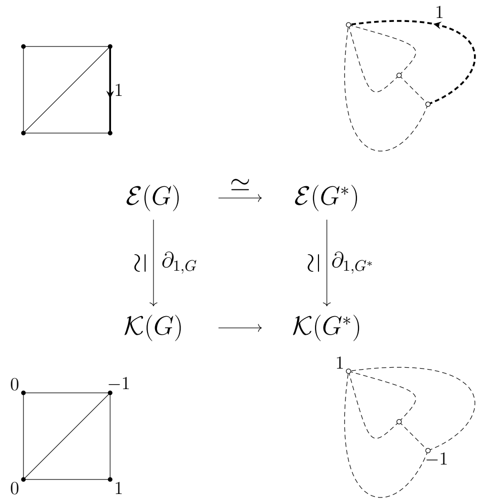


Figure 3.13: The isomorphism between $\mathcal{K}(G)$ and $\mathcal{K}(G^*)$

M.Baker conjectured that the rotor-routing action is compatible with the planar duality. More formally, he asked whether $(c \cdot T)^* = c^* \cdot T^*$ holds, where $\cdot$ denotes the rotor-routing action and $T^* \in \mathcal{T}(G^*)$ and $c^* \in \mathcal{K}(G^*)$ are the corresponding dual elements of $T \in \mathcal{T}(G)$ and $c \in \mathcal{K}(G)$, respectively. The conjecture is affirmatively proved by himself and Yao Wang in [BW18], and independently by Chan, Glass, Macauley, Perkinson, Werner, and Yang in [Cha+15].

Theorem 3.26. *(Chan, Glass, Macauley, Perkinson, Werner, and Yang) Let the cyclic orderings on the vertices in a planar graph G and its dual G^* are clockwise and counterclockwise, respectively. Then the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{K}(G) \times \mathcal{T}(G) & \longrightarrow & \mathcal{T}(G) \\ \downarrow & & \downarrow \\ \mathcal{K}(G^*) \times \mathcal{T}(G^*) & \longrightarrow & \mathcal{T}(G^*) \end{array},$$

where the vertical maps are induced from the planar duality and the horizontal maps are the rotor-routing actions.

Note that the cyclic ordering of G^* is given counterclockwise. Still the same arguments in the previous sections can be applied without problems.

Example 3.27. In Figure 3.14, the rotor-routing action of the elements $c \in \mathcal{K}(G)$ and $c^* \in \mathcal{K}(G^*)$ in Figure 3.13 on the spanning trees in Figure 3.12 are described in each row respectively. The sinks are chosen with the vertices that has -1 value on c and c^*, respectively. Explicitly, the sink of G is chosen with the vertex at the upper right corner and the sink of G^* is chosen with the vertex at the lower right corner. Note that the cyclic orderings on the vertices in G and G^* are clockwise and counterclockwise,

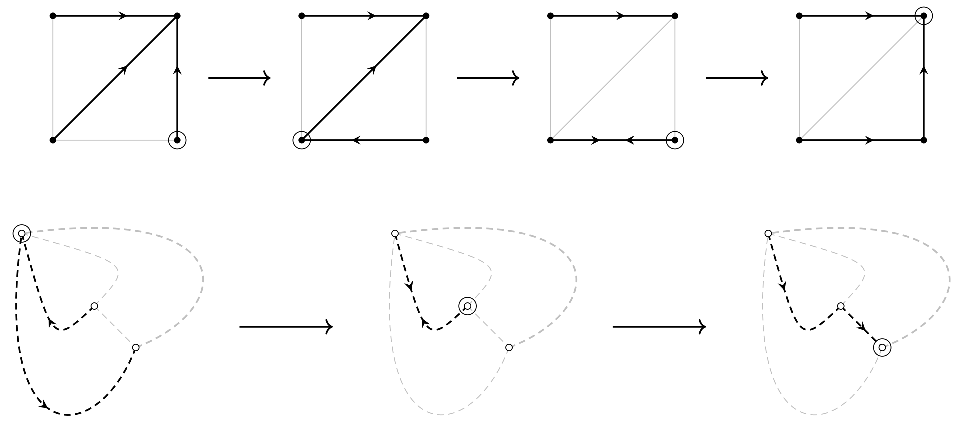


Figure 3.14: Compatibility of the rotor-routing action with respect to the planar dual

respectively. One can observe that the two resulting trees are dual to each other.

Using this compatibility of the rotor-routing action, we investigate another combinatorial formula of an integral generator of 1-harmonic space of the 2-dimensional complex of more general classes.

Let G be a planar graph. Let X be the 2-dimensional complex obtained from G by 'filling in' all the finite faces but one. We denote by f_0 the unique finite face that is not filled in and f_∞ the infinite face. Topologically, its first Betti number β_1 is equal to one, hence there is a unique integral generator of the 1-harmonic space $\mathcal{H}_1(X)$ up to sign. Note that this condition is more general setting than those in the previous section; the complex of the form $X = \mathcal{A}(G) \sqcup \widetilde{e}$ satisfies the given condition.

The following proposition characterizes the 1-cocycles of X in the perspective of the planar dual.

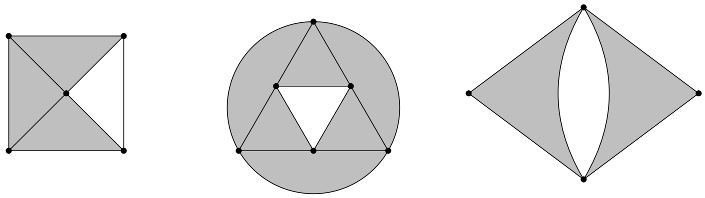


Figure 3.15: Planar graphs whose all but one finite faces are filled in.

Proposition 3.28. *Let d be a 1-chain in $C_1(X) = C_1(G)$ and let $d^* \in C_1(G^*)$ be the corresponding dual element of d. Then those followings are equivalent:*

(a) $d \in \ker \partial^t_{2,X}$*, i.e., d is a 1-cocycle of X.*

(b) For each face f in G that is filled in, the coefficient of d in $\partial_{1,G^*} d^*$ is zero.

Sketch of proof. The set of 2-cells in X can be naturally embedded in the vertex set V^* of G^*. Via this embedding, the value of a 2-cell in $\partial^t_{2,X} d$ and $\partial_{1,G^*} d^*$ are the same. □

Equivalently, a 1-chain is a 1-cocycle if and only if its dual is a formal sum of directed paths from f_0 to f_∞ in G^*.

Corollary 3.29. *For a spanning tree $T \in \mathcal{T}(G)$, there exists a unique (up to scalar multiplication) nonzero 1-cocycle $\zeta_T \in \ker \partial^t_{2,X}$ whose support is contained in $E \setminus T$.*

Proof. Let $T^* \in \mathcal{T}(G^*)$ be the corresponding dual tree of T. The directed path $\gamma_{T^*}(f_0, f_\infty)$ in T^* is the unique element that satisfies (b) in Proposition 3.28 (up to scalar multiplication). The proof is done by defining ζ_T to be the dual of $\gamma_{T^*}(f_0, f_\infty)$. □

Let $c \in \mathcal{K}(G)$ be the element whose corresponding dual c^* has value 1 and -1 at f_0 and f_∞, respectively, and 0 elsewhere. Equivalently, c^* is the chip configuration with a single chip on f_0, where the sink is f_∞. Figure 3.16 describes an example of the choice of c; the chip configuration c (resp. c^*) is drawn in the second (resp. third) figure, together with one of the representatives of its corresponding element in the cut-flow group $\mathcal{E}(G)$ (resp. $\mathcal{E}(G^*)$). By Theorem 3.12, the choice of sink does not matter in the rotor-routing action. Hence the action of c can be simply described by decomposing it into the sum of three chip configurations that has a value 1 on a vertex and the value -1 at another vertex which will be the sink, and 0 elsewhere.

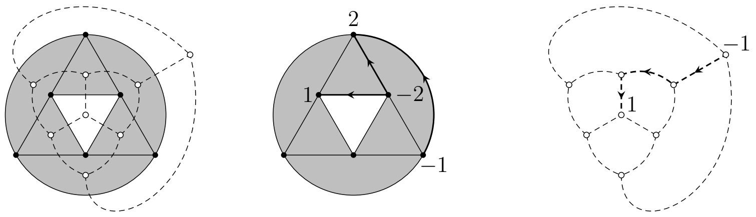


Figure 3.16: A chip configuration c whose dual c^* has a single chip on f_0 with the sink f_∞.

Then we have the dual version of Theorem 3.21.

Theorem 3.30. *Let X be a 2-dimensional complex obtained by filling in all but one finite faces of a planar graph G. Let $c \in \mathcal{K}(G)$ be the chip configuration whose dual $c^* \in \mathcal{K}(G^*)$ has a single chip on the empty face f_0 where the sink is the infinite face f_∞. Then for any orbit $\mathcal{O}$ in $\mathcal{T}(G)$*

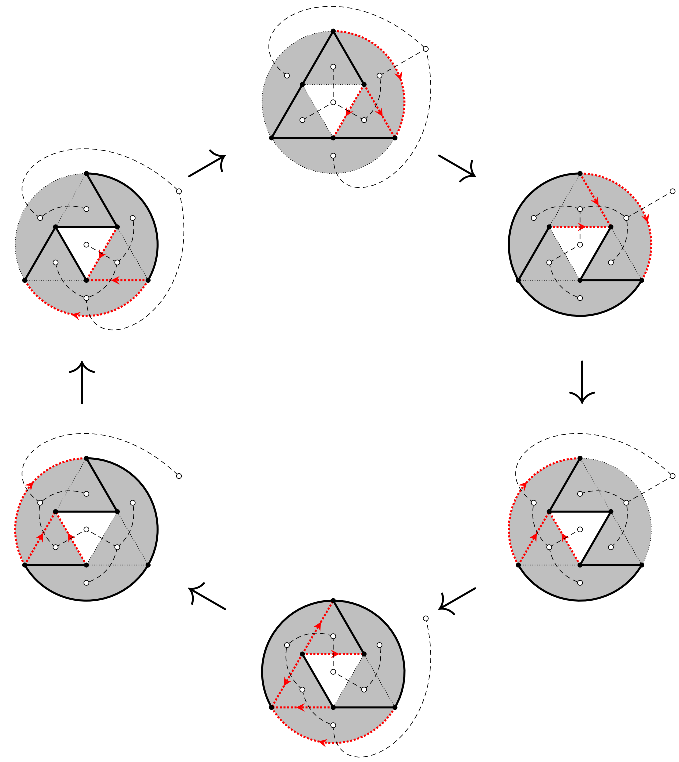
under the action of c, the sum

$$\sum_{T \in \mathcal{O}} \zeta_T$$

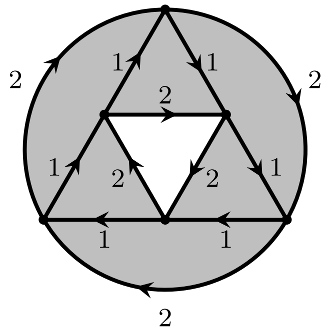
is an integral generator of $\mathcal{H}_1(X)$.

Proof. Let $\mathcal{H}_1^* = \mathcal{H}_1^*(X)$ be the image of $\mathcal{H}_1(X)$ under the identification map $C_1(X;\mathbb{Z}) \longrightarrow C_1(G^*;\mathbb{Z})$. By Propositions 3.24 and 3.28, $\mathcal{H}_1^*$ is equal to $\mathrm{im}^t_{1,G^*} \cap K^*$, where K^* is a submodule of $C_1(G^*;\mathbb{Z})$ generated by directed paths from f_0 to f_∞. Let X^* be the complex obtained by adding the new edge $\widetilde{e}^* = \{f_0, f_\infty\}$ to the acyclization $\mathcal{A}(G^*)$ of G^*. Fix an orientation $\widetilde{e}^* = (f_\infty, f_0)$. For an element $k \in K^*$, additionally assigning $\widetilde{e}^*$ the value $\partial_{1,G^*} f_\infty = -\partial_{1,G^*} f_0$ defines a map $K^* \longrightarrow \ker \partial_{1,X^*}$. Then the image of $\mathcal{H}_1^*$ under this map is the 1-harmonic space $\mathcal{H}_1(X^*)$ of X^*. By Theorem 3.21, the sum $\sum_{T^* \in \mathcal{O}^*} (\gamma_{T^*}(f_0, f_\infty) + [\widetilde{e}^*])$ is an integral generator of $\mathcal{H}_1(X^*)$ for any orbit $\mathcal{O}^*$ in $\mathcal{T}(G^*)$ under the action of c^*. Consequently the sum $\sum_{T^* \in \mathcal{O}^*} \gamma_{T^*}(f_0, f_\infty)$ is an integral generator of $\mathcal{H}_1^*$. By the duality, we can conclude that the sum $\sum_{T \in \mathcal{O}} \zeta_T$ is an integral generator of $\mathcal{H}_1(X)$ for any orbit $\mathcal{O}$ in $\mathcal{T}(G)$ under the action of c. □

Example 3.31. Figure 3.17 describes an example of Theorem 3.30 where X is the second complex in Figure 3.15. An orbit $\mathcal{O}$ in $\mathcal{T}(G)$ under the action of the element $c \in \mathcal{K}(G)$ is drawn in Figure 3.17(a). For each tree T in the orbit, its corresponding cocycle ζ_T is drawn in red dotted arrow. The sum $\sum_{T \in \mathcal{O}} \zeta_T$ coincides with an integral generator of $\mathcal{H}_1(X)$, which is described in Figure 3.17(b).



(a) An orbit in $\mathcal{T}(G)$ under the action of $c \in \mathcal{K}(G)$.



(b) An integral generator of $\mathcal{H}_1(X)$.

Figure 3.17

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국문초록

본 학위논문에서는 이차원 복합체의 조화 사이클과 그래프의 크리티컬 군 사이의 관계에 대해 알아본다. 세포 복합체의 조화 공간은 조합적 라플라시안의 핵으로 정의되며, 조합적 호지 이론에 의해 호몰로지 군과 동형이다. 그래프의 크리티컬 군은 칩 발사 게임과 관련이 있는 유한 생성 아벨 군으로 생성 나무의 개수와 같은 크기를 가진다. 그래프의 비순환화에 간선을 더해 만들어진 이차원 복합체에 대해, 해당 그래프의 생성 나무를 이용한 실수 계수의 조화 공간의 생성자에 대한 조합적 공식이 알려져 있다. 본 학위 논문에서는 정수 계수 생성자에 해당하는 개선된 공식을 크리티컬 군의 생성 나무 위의 작용에서의 칩의 자취를 추적함으로써 제시한다.

주요어휘: 생성 나무, 크리티컬 군, 칩 발사 게임, 조화 공간
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