이학박사 학위논문

Brownian motions and geodesic flows on finite-volume manifolds with pinched negative curvature

(음수곡률을 갖는 유한 부피 다양체 위의 브라운 운동과

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Abstract

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The geometry of manifolds governs behaviors of the geodesic flow and the Brownian motion on manifolds. Likewise, the geodesic flow and the Brownian motion on manifolds reflect the geometry of manifolds. Thus we can deduce geometric properties of manifolds from behaviors of the geodesic flow and the Brownian motion. In this thesis, we demonstrate such an ensemble of the geodesic flow, the Brownian motion and the geometry of finite-volume manifolds.

First, we establish the central limit theorem of Brownian motions, which arises from the geometry of manifolds: pinched negative curvature and uniformly bounded first derivatives of sectional curvature. We prove the central limit theorem of the Brownian distance and the Green distance of Brownian points by solving a heat equation on the unit tangent bundle to obtain martingales with the same asymptotic distributions. The main ingredient for the solution of the heat equation is foliated Brownian motions and their contraction property. The foliated Brownian motion is a lifted stochastic process of the Brownian motion to the unit tangent bundle.

The last topic is asymptotically harmonic manifolds. We derive a list of characterizations of asymptotically harmonic manifolds with pinched negative curvature. These characterizations reveal that the asymptotic harmonicity is closely related to the Brownian motion. In particular, we relate the asymptotic harmonicity with the central limit theorem of Brownian motions. As we make use of ergodic properties of the geodesic flow and their relation with the Brownian motion, we need additional assumption on the dynamics of the geodesic flow, namely the existence of an equilibrium state for the harmonic potential.

Key words: Foliated Brownian motions, geodesic flow in negatively curved manifolds, thermodynamic formalisms, asymptotically harmonic manifolds
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Contents

Chapter 1

Introduction

The Brownian motion on manifolds attracted interest of many geometers due to its reflection of the geometry since the '70s. In the '80s, it turns out that in negative curvature the Brownian motion is closely related to the geodesic flow. Thus the Brownian motion plays an important role in the research of the geodesic flow and the geometry of negatively curved manifolds.

The main result of this thesis is the central limit theorem of random variables induced by the Brownian distance and the Green distance on a Cartan-Hadamard manifold $\widetilde{\mathcal{M}}$ which is the universal cover of a finite-volume manifold. The central limit theorem admits a characterization of a geometric property, called asymptotic harmonicity, in negatively curvature.

Consider a simply connected complete Riemannian manifold $\widetilde{\mathcal{M}}$ of dimension $d \geq 2$ with pinched negative curvature. That is, $\widetilde{\mathcal{M}}$ has sectional curvature bounded between two negative numbers. We assume that $\widetilde{\mathcal{M}}$ has uniformly bounded first derivatives of the sectional curvature and it is the universal cover of a finite-volume Riemannian manifold $\mathcal{M}$.

The Brownian motion $\widetilde{\mathcal{M}}$ is a diffusion process generated by the Laplace-Beltrami operator $\Delta_{\widetilde{\mathcal{M}}}$ on $\widetilde{\mathcal{M}}$ (See Chapter 3 for the rigorous definition and its properties). Since the Brownian motion $(\widetilde{\omega}_t)_{t\in\mathbb{R}_+}$ on $\widetilde{\mathcal{M}}$ starting from x is transient by negative curvature, the Brownian distance $\mathrm{d}(x, \widetilde{\omega}_t)$ tends to infinity as $t \to \infty$ with probability 1. Moreover, its asymptotic growth is linear ([Gu]): there is $\ell > 0$ such that

$$\ell = \lim_{t\to\infty} \frac{1}{t}\mathrm{d}(x, \widetilde{\omega}_t).$$

Due to the pinched negative curvature, the Green function $\mathrm{G}(x,y)$ on $\widetilde{\mathcal{M}}$, the fundamental solution of $\Delta_{\widetilde{\mathcal{M}}}$, tends to zero as $\mathrm{d}(x,y)\to\infty$. Furthermore, $\mathrm{G}(x,\widetilde{\omega}_t)\to 0$ as $t\to\infty$ exponentially fast with probability 1 ([Kai1]): there exists $h>0$ such that

$$h=\lim_{t\to\infty}-\frac{1}{t}\log\mathrm{G}(x,\widetilde{\omega}_t).$$

The logarithm of the Green function is called the Green distance. In Chapter 5, we prove our main theorem, the central limit theorem for the asymptotic distribution of the Brownian distance and the Green distance.

Theorem (Central limit theorem of Brownian motions)**.** ([K]) There are real numbers $\sigma_{\mathfrak{b}},\sigma_{\mathfrak{K}}$ such that the random variables

$$\frac{1}{\sqrt{t}}\left(\mathrm{d}(x,\widetilde{\omega}_t)-t\ell\right),\ \frac{1}{\sqrt{t}}\left(\log\mathrm{G}(x,\widetilde{\omega}_t)+th\right)$$

asymptotically follow the centered normal distributions $N(0,\sigma_{\mathfrak{b}}^2)$ and $N(0,\sigma_{\mathfrak{K}}^2)$, respectively. Moreover, $\sigma_{\mathfrak{K}}^2\geq 2h$.

While Brownian motions on manifolds with pinched negative curvature has been studied for a long time, the majority of the results holds for either the class of Cartan-Hadamard manifolds with pinched negative curvature or the class of universal covers of compact negatively curved manifolds. Few results are known for the cases in between, especially for universal covers of finite-volume manifolds $\widetilde{\mathcal{M}}$. Our main result is a generalization of the central limit theorem for universal covers of compact negatively curved manifolds proved by F. Ledrappier in [Le6].

We provide in Section 5.3 a lower bound for the expectation of the Gromov product at Brownian points as in [Le6] using the $\mathcal{C}^2$-convergence of the normalized distance functions to the Busemann function in pinched negatively curved manifolds. The lower bound implies the contraction property of the foliated Brownian motion, which plays an important role in the proof of the central limit theorem. Although the resulting lower bound is less sharp than the lower bound obtained in [Le6], it is sufficient to obatain the contraction property. In the course of proving the contraction property, we also show an on-diagonal estimate of heat kernels, i.e. a uniform bound of the heat kernel, on finite-volume manifolds with pinched negative curvature.

As in [Le6], the contraction property of the foliated Brownian motion implies that the leafwise heat equation on the unit tangent bundle for the foliated Laplacian has a solution unique up to additive constant. We construct Martingales from the solutions of the heat equation with the initial conditions of the Busemann function and the logarithm of the Martin kernel of the Brownian motion. We prove that they are asymptotically normal and have the same distributions with the random variables of our interest.

Our second main theorem is a characterization of asymptotically harmonic manifolds with pinched negative curvature. The characterization includes a consequence of the central limit theorem of Brownian motions on finite-volume manifolds with pinched negative curvature. We say that $\widetilde{\mathcal{M}}$ is *asymptotically harmonic* if the mean curvature of the horospheres of $\widetilde{\mathcal{M}}$ is constant. We remark that a manifold whose geodesic spheres have constant mean curvature is called harmonic manifolds. Since horospheres are limit spheres of geodesic spheres (see Section 2.3), we can say that asymptotic harmonicity means having limit spheres of constant mean curvature. For the characterization of asymptotically harmonic manifolds, we need additional assumption on the dynamics of the geodesic flow.

The Martin kernel of the Brownian motion defines a Hölder continuous function F^{BM} on $\mathcal{SM}$. An equilibrium state of F^{BM} is a geodesic flow-invariant Borel probability measure on $\mathcal{SM}$ which maximizes the pressure of F^{BM}. See Chapter 4 for the definition and a necessary and sufficient condition for the existence of equilibrium states. On compact manifolds, an equilibrium state exists for each Hölder continuous function ([F]) while we cannot guarantee for finite-volume manifolds.

In Section 5.4, we prove a characterization of asymptotically harmonic manifolds with pinched negative curvature which are the universal covers of finite-volume manifolds, where the existence of the equilibrium state for the harmonic potential is guaranteed.

Theorem (Characterization of asymptotically harmonic manifolds)**.** Suppose that there is an equilibrium state for the harmonic potential on the unit tangent bundle of $\mathcal{M}$. The following conditions for $\widetilde{\mathcal{M}}$ are equivalent.

1. $\widetilde{\mathcal{M}}$ is asymptotically harmonic.

2. The harmonic measures and the Patterson-Sullivan measures are the same.

3. The Martin kernel is the exponential of the Busemann function.

4. The bottom of the spectrum of the Laplace-Beltrami operator on $\widetilde{\mathcal{M}}$ is $h_{\mathrm{top}}^2/4$.

5. $\sigma_{\mathfrak{K}}^2 = 2h$.

The characterization of asymptotically harmonic manifolds reveal an interplay between the stochastic properties, the geometry and the dynamics of the geodesic flow of $\widetilde{\mathcal{M}}$ and is motivated by works related to Katok's conjecture on measure rigidity.

Conjecture (Katok's conjecture)**.** *If $\mathcal{M}$ is a compact manifold with negative curvature whose Liouville measure on its unit tangent bundle is the measure of maximal entropy for geodesic flow, then $\mathcal{M}$ is a locally symmetric space.*

Katok's conjecture is proved in 2-dimension by A. Katok [Kat1]. However, even though there were attempts to solve the conjecture, it is still open in higher dimension. Instead, the rigidity of asymptotically harmonic manifolds has been established: a compact negatively curved manifold $\mathcal{M}$, whose universal cover $\widetilde{\mathcal{M}}$ is asymptotically harmonic, is a locally symmetric space ([FL], [BFL], [Le4]).

If $\widetilde{\mathcal{M}}$ is asymptotically harmonic, then the Liouville measure on the unit tangent bundle of $\mathcal{M}$ has maximal entropy for the geodesic flow. Hence the rigidity of asymptotically harmonic manifolds with finite volume and pinched negative curvature gives progress on Katok's conjecture in finite-volume manifolds with pinched negative curvature.

Chapter 2

Geometry of negatively curved manifolds

In this chapter, we recall preliminaries of Riemannian geometry. We shall focus on manifolds with negative curvature and their unit tangent bundles.

2.1 Riemannian manifolds

In this section, we recall basic notions in Riemannian geometry and some consequences which are required for the study of Brownian motions and geodesic flows on negatively curved manifolds.

A *Riemannian manifold* $(\mathcal{M}, g)$ is a Hausdorff and second countable $\mathcal{C}^\infty$-manifold $\mathcal{M}$ of dimension d, which is equipped with a symmetric non-degenerate $\mathcal{C}^\infty$-bilinear tensor field g, called a *Riemannian metric* on $\mathcal{M}$. That is, for each $p \in \mathcal{M}$, g_p is an inner product on the tangent space $\mathcal{T}_p\mathcal{M}$ of $\mathcal{M}$ at p and $p \mapsto g_p(X_p, Y_p)$ is a $\mathcal{C}^\infty$-map for any smooth vector fields X, Y on $\mathcal{M}$. We denote $g(\cdot,\cdot)$ by $\langle\cdot,\cdot\rangle_g$ or just $\langle\cdot,\cdot\rangle$ if it causes no confusion. Likewise, we denote the magnitude $\sqrt{g(\mathrm{v},\mathrm{v})}$ of a vector $\mathrm{v} \in \mathcal{T}\mathcal{M}$ by $|\mathrm{v}|_g$ or $|\mathrm{v}|$ if there is no possible confusion.

There is unique a differential d-form $\mathrm{vol} = \mathrm{vol}_{\mathcal{M}}$, called the *volume form* of $(\mathcal{M}, g)$, such that for each point $p \in \mathcal{M}$ and for each positively ordered g_p-orthonormal frame $e_1, \ldots, e_d$ of $\mathcal{T}_p\mathcal{M}$, $\mathrm{vol}(e_1, \cdots, e_d) = 1$. Note that the volume form defines a positive measure on $\mathcal{M}$ and the integration of the volume form on $\mathcal{M}$, which is denoted by $\mathrm{vol}(\mathcal{M})$, is called the volume of the Riemannian manifold $(\mathcal{M}, g)$.

Let $f : \mathcal{M} \to \mathbb{R}$ be a smooth function. A Riemannian metric g defines the *gradient vector field* ∇f of f by $df(v) = g(\nabla f, v)$ for each tangent vector v on $\mathcal{M}$. The Riemannian metric also induces a $(0, 2)$-tensor $\operatorname{Hess} f$, called *Hessian* of f: for any vector fields X, Y on $\mathcal{M}$,

$$\operatorname{Hess} f(X, Y) := \frac{1}{2} g(\nabla_X \nabla f, Y),$$

where L_Z denotes the Lie derivative along a vector field Z. Note that $\operatorname{Hess} f$ is a symmetric tensor.

Given a vector field X, since the Lie derivative $L_X \operatorname{vol}$ of vol along X is a top form, there is a unique smooth function $\operatorname{div} X$, called the *divergence* of a smooth vector field X, satisfying

$$L_X \operatorname{vol} = \operatorname{div} X \operatorname{vol}.$$

Definition 2.1.1 (Laplace-Beltrami operator)**.** The *Laplace-Beltrami operator* $\Delta = \Delta_{\mathcal{M}}$ on $(\mathcal{M}, g)$ is a differential operator on the space $\mathcal{C}^\infty(\mathcal{M})$ of smooth functions on $\mathcal{M}$ given by for $f \in \mathcal{C}^\infty(\mathcal{M})$,

$$\Delta f = \operatorname{div} \nabla f.$$

Note that if $e_1, \ldots, e_d$ form a local orthonormal frame,

$$\Delta f = \sum_{i=1}^{d} \operatorname{Hess} f(e_i, e_i). \tag{2.1}$$

We shall consider a differentiation of a vector field called *covariant derivative*: for smooth vector fields X, Y on $\mathcal{M}$, if $Y \mapsto \nabla_Y X$ is a $(1, 1)$-tensor and $X \mapsto \nabla_Y X$ is a derivation, then we say $\nabla_Y X$ is a covariant derivative of X in the direction of Y. Note that while the value of the Lie derivative $L_Y X$ of a vector field X in the direction of Y at a point p in $\mathcal{M}$ depends on the germ of Y at p, the value of a covariant derivative $\nabla_Y X$ at p does depend only on the value of Y at p.

A Riemannian metric g gives rise to a unique covariant derivative ∇ such that for smooth vector fields X, Y and Z on $\mathcal{M}$,

$$\begin{aligned} \nabla_X Y - \nabla_Y X &= [X, Y], \\ Z g(X, Y) &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y). \end{aligned}$$

It is uniquely determined by the relation called *Koszul formula*:

$$g(\nabla_Y X, Z) = (L_X g)(Y, Z) + d\theta_X(Y, Z),$$

where $\theta_X(Y) := g(X, Y)$.

A *geodesic* γ on $\mathcal{M}$ is a smooth curve with zero acceleration; $\nabla_{\dot\gamma}\dot\gamma = 0$. For every tangent vector v of $\mathcal{M}$, there is a unique geodesic

$$\gamma_{\mathrm{v}} : (-\varepsilon_1(\mathrm{v}), \varepsilon_2(\mathrm{v})) \to \mathcal{M}$$

for some $\varepsilon_1(\mathrm{v}), \varepsilon_2(\mathrm{v}) \in (0, \infty]$ such that $\dot\gamma_{\mathrm{v}}(0) = \mathrm{v}$. Consider the set D_p of tangent vectors v at p whose geodesics γ_{v} are defined on $(-\varepsilon_1(\mathrm{v}), \varepsilon_2(\mathrm{v}))$ with $\varepsilon_2(\mathrm{v}) > 1$. A manifold $\mathcal{M}$ is said to be *complete* if $D_p = \mathcal{T}_p\mathcal{M}$ for every $p \in \mathcal{M}$. We define the *exponential map* $\exp_p : D_p \to \mathcal{M}$ of $\mathcal{M}$ at $p \in \mathcal{M}$ by $\exp_p(\mathrm{v}) = \gamma_{\mathrm{v}}(1)$.

Definition 2.1.2 (Geodesic flow)**.** For a complete Riemannian manifold $\mathcal{M}$, we define a flow $\mathbf{g} = (\mathbf{g}^t)$ on its tangent bundle $\mathcal{T}\mathcal{M}$, called *geodesic flow*:

$$\mathbf{g}^t\mathrm{v} := \dot\gamma_{\mathrm{v}}(t),$$

where $\gamma_{\mathrm{v}}(t) = \exp_p(t\mathrm{v})$ for $\mathrm{v} \in \mathcal{T}_p\mathcal{M}$. Note that the geodesic flow preserves the magnitude of vectors: $\left|\mathbf{g}^t\mathrm{v}\right| = |\mathrm{v}|$.

We define the length $\ell(c)$ of a smooth curve $c : [0, t] \to \mathcal{M}$ on a Riemannian manifold $\mathcal{M}$ by

$$\ell(c) := \int_0^t |\dot c(s)|_g \, ds.$$

It induces a distance $\mathrm{d} = \mathrm{d}_{\mathcal{M}}$ on by the infimum of smooth curves joining two points: for x and y in $\mathcal{M}$,

$$\mathrm{d}(x, y) := \inf_{\substack{c:[0,t]\to\mathcal{M},\\ c(0)=x, c(t)=y}} \ell(c),$$

where the infimum is taken among smooth curves. The distance d is called the *Riemannian distance* of $\mathcal{M}$. A smooth curve on $\mathcal{M}$ is a geodesic if and only if it is locally length-minimizing.

The *Riemannian curvature tensor* R of $(\mathcal{M}, g)$ is a $(1,3)$-tensor

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

for vector fields X, Y and Z on $\mathcal{M}$. For a pair of independent tangent vectors $\mathrm{v}, \mathrm{w} \in \mathcal{T}_p\mathcal{M}$, the *sectional curvature* of the plane spanned by v, w is

$$\operatorname{Sec}_{\mathcal{M}}(\mathrm{v}, \mathrm{w}) := \frac{g(R(\mathrm{v}, \mathrm{w})\mathrm{w}, \mathrm{v})}{|\mathrm{v}|^2 |\mathrm{w}|^2 - g(\mathrm{v}, \mathrm{w})^2}.$$

We say *first derivatives of sectional curvature is uniformly bounded* if

$$\|\nabla \operatorname{Sec}_{\mathcal{M}}\| := \sup_{\substack{e_1, \ldots, e_d, \\ k=1, \ldots, d}} |\nabla_{e_k} \operatorname{Sec}_{\mathcal{M}}(e_1, e_2)|$$

where $e_1, \ldots, e_d$ in the supremum is taken among local frames on $\widetilde{\mathcal{M}}$, is finite.

A *vector field along a curve* γ is a curve J on the tangent bundle $\mathcal{T}\mathcal{M}$ whose curve of base points is γ; $J(t) \in \mathcal{T}_{\gamma(t)}\mathcal{M}$.

Definition 2.1.3 (Jacobi field)**.** A vector field J along a geodesic γ is called *Jacobi field* if it satisfies the linear second order differential equation

$$\nabla_{\dot\gamma} J' + R(J, \dot\gamma)\dot\gamma = 0, \tag{2.2}$$

where $J' := \nabla_{\dot\gamma} J = \frac{d}{dt} J$ as a curve on the smooth manifolds $\mathcal{T}\mathcal{M}$.

Note that given an initial condition $J(0)$ and $J'(0)$, the equation (2.2) has unique solution.

Fix a pair of tangent vector $\mathrm{v}, \mathrm{w} \in \mathcal{T}_p\mathcal{M}$ and let γ be the geodesic with $\dot\gamma(0) = \mathrm{v}$. Then we have a geodesic variation $c(s,t) := \exp_p(t(\mathrm{v} + s\mathrm{w}))$; $c(s, \cdot)$ is a geodesic for every s. Since $c(0, \cdot) = \gamma$, it defines a smooth vector field $J(t) := \frac{\partial c}{\partial s}(0, t)$ along γ. Then it is a solution of the equation (2.2):

$$\begin{aligned}\nabla_{\dot\gamma} J'(t) = \nabla_{\dot\gamma} \frac{\partial^2 c}{\partial t \partial s}(0,t) = \nabla_{\dot\gamma} \frac{\partial^2 c}{\partial s \partial t}(0,t) &= \nabla_{\dot\gamma} \nabla_{\frac{\partial c}{\partial s}} \frac{\partial c}{\partial t}(0,t) \\ = -R(J, \dot\gamma)\dot\gamma(t) + \nabla_{\frac{\partial c}{\partial s}} \nabla_{\dot\gamma} \frac{\partial c}{\partial t}(0,t) &= -R(J, \dot\gamma)\dot\gamma(t).\end{aligned}$$

Therefore, there is a one-to-one correspondence of geodesic variations of γ and Jacobi fields along γ.

Fix a point $p \in \mathcal{M}$. Let $r(y) = r_p(y) := |\exp_p^{-1}(y)|$ and ∂_r be the gradient vector field on $\mathcal{T}_p\mathcal{M} \setminus \{0\}$ of $\mathrm{v} \mapsto |\mathrm{v}|$. For some ball $B(0,\varepsilon)$ of $\mathcal{T}_p\mathcal{M}$, the exponential map $\exp_p$ is a diffeomorphism and r is smooth with

$$\nabla r = (\mathcal{T} \exp_p)\partial_r$$

on $U = \exp_p B(0,\varepsilon) \setminus \{p\}$. Identifying $\mathcal{T}_p\mathcal{M} \setminus \{0\}$ with $(0,\infty) \times \mathcal{S}_p\mathcal{M}$ where $\mathcal{S}_p\mathcal{M} = \{\mathrm{v} \in \mathcal{T}_p\mathcal{M} : |\mathrm{v}| = 1\} \cong \mathbb{S}^{d-1}$, the exponential map $\exp_p$ endows U with the polar coordinate: $(r,\mathrm{v}) \mapsto \exp_p(r\mathrm{v})$. The Riemannian metric g is locally $g = dr^2 + g_r$ where $g_r = \lambda_p(r,\cdot)g_{\mathbb{S}}$ is a one-parameter family of Riemannian metrics on $\mathcal{S}_p\mathcal{M}$.

Theorem 2.1.4. *(Rauch Comparison Theorem) If $-b^2 \leq \mathrm{Sec}_{\mathcal{M}} \leq -a^2$ for some $b > a > 0$, then*

$$\frac{\mathrm{sn}'_{-a^2}(r)}{\mathrm{sn}_{-a^2}(r)} g_r \leq \mathrm{Hess}\, r \leq \frac{\mathrm{sn}'_{-b^2}(r)}{\mathrm{sn}_{-b^2}(r)} g_r,$$

where $\mathrm{sn}_{-a^2}(t) = \frac{1}{a}\sinh(at)$.

If we take a local frame $e_1, \ldots, e_d$ with $e_1 = \nabla r$, it follows from the equation (2.1) and the Rauch comparison theorem that

$$(d-1)\frac{\mathrm{sn}'_{-a^2}(r)}{\mathrm{sn}_{-a^2}(r)} \leq \Delta r \leq (d-1)\frac{\mathrm{sn}'_{-b^2}(r)}{\mathrm{sn}_{-b^2}(r)}. \tag{2.3}$$

2.2 Geometry of tangent bundles

Let $(\mathcal{M}, g)$ be a complete connected Riemannian manifold of dimension d with canonical covariant derivative ∇. We shall define a natural Riemannian metric $g_{\mathcal{S}}$, called the *Sasaki metric*, on the tangent bundle $\mathcal{T}\mathcal{M}$ of $(\mathcal{M}, g)$. If $x = (x^1, \ldots, x^d)$ is a local coordinate on a open set U of $\mathcal{M}$ and $X_i = \frac{\partial}{\partial x^i}$, then we have an induced local coordinate $\hat{x} = (x, \xi)$ on $\mathcal{T}U$: for $\mathrm{v} \in \mathcal{T}_pU$, $\hat{x}(\mathrm{v}) = (x(p), \xi(\mathrm{v}))$, where $\xi(\mathrm{v}) \in \mathbb{R}^d$ with $dx_p\mathrm{v} = \xi^i(\mathrm{v})\, X_i|_p$.

To define the Sasaki metric, we introduce a decomposition $\Xi = \pi^*(\mathcal{T}\mathcal{M}) \oplus \pi^*(\mathcal{T}\mathcal{M})$ of the tangent bundle $\mathcal{T}\mathcal{T}\mathcal{M}$ of $\mathcal{T}\mathcal{M}$ into Whitney sum of two copies of pullback bundles. Note that $\Xi_{\mathrm{v}} = \mathcal{T}_p\mathcal{M} \oplus \mathcal{T}_p\mathcal{M}$ for $\mathrm{v} \in \mathcal{T}_p\mathcal{M}$. For $\mathcal{Z} \in \mathcal{T}\mathcal{T}\mathcal{M}$, let $\mathcal{V}$ be a smooth curve on $\mathcal{T}\mathcal{M}$ with $\dot{\mathcal{V}}(0) = \mathcal{Z}$ and $c = \pi\mathcal{V}$. Then we define the map $\mathcal{I} : \mathcal{T}\mathcal{T}\mathcal{M} \to \Xi$ by

$$\mathcal{I}(\mathcal{Z}) = (\dot{c}(0), \nabla_{\dot{c}}\mathcal{V}(0)).$$

We shall prove that $\mathcal{I}$ is an isomorphism of vector bundles of $\mathcal{TM}$. If $\mathcal{V}$ has the local expression $\hat{x} \circ \mathcal{V}(t) = (\sigma(t), \xi(t))$ where $\sigma(t) = x \circ c(t)$ and $\mathcal{V}(t) = \xi^i(t) X_i\big|_{c(t)}$, then

$$\mathcal{I}(\mathcal{Z}) = \Big(\dot{\sigma}(0),\, \dot{\xi}^k(0) + \Gamma^k_{ij}(0)\sigma^i(0)\xi^j(0)\Big)$$

where Γ^k_{ij} denotes the Christoffel symbol: $\nabla_{X_i} X_j|_{c(t)} = \Gamma^k_{ij}(t) X_k\Big|_{c(t)}$. It follows from the local expression of $\mathcal{I}$ that $\mathcal{I}$ is a well-defined smooth map. Since $\mathcal{Z} = (\dot{\sigma}(0), \dot{\xi}(0))$, $\mathcal{I}$ is fiberwise linear. Thus as Ξ is a vector bundle of rank $2d$, it suffices to show that $\mathcal{I}$ is injective. The injectivity easily follows from the following computations: for $\mathrm{v} \in \mathcal{T}_p\mathcal{M}$,

$$\begin{aligned} \mathcal{I}\left(\frac{\partial}{\partial \xi^i}\bigg|_{\mathrm{v}}\right) &= (0,\, X_i|_p), \\ \mathcal{I}\left(X_i|_{\mathrm{v}}\right) &= \Big(X_i\big|_p, \nabla_{X_i}\mathcal{V}(0)\Big). \end{aligned}$$

Therefore, we identify $\mathcal{T}_{\mathrm{v}}\mathcal{TM}$ with $\mathcal{T}_p\mathcal{M} \oplus \mathcal{T}_p\mathcal{M}$. Note that

$$\ker\left(d\pi|_{\mathrm{v}}\right) = \left\langle \frac{\partial}{\partial \xi^1}\bigg|_{\mathrm{v}}, \ldots, \frac{\partial}{\partial \xi^d}\bigg|_{\mathrm{v}} \right\rangle = 0 \oplus \mathcal{T}_p\mathcal{M}.$$

The geodesic flow $\mathbf{g}$ acts on $\mathcal{TTM}$ by its differential map. Let $\mathrm{v} \in \mathcal{T}_p\mathcal{M}$ and $\mathcal{Z} \in \mathcal{T}_{\mathrm{v}}\mathcal{TM}$ with $\mathcal{Z} = (X, Y) \in \mathcal{T}_p\mathcal{M} \oplus \mathcal{T}_p\mathcal{M}$. Consider a geodesic variation

$$c(s,t) = \exp_p(t(X + sY)).$$

The curve $\mathcal{V}(s) := \frac{\partial c}{\partial t}(s,t)\big|_{t=0}$ on $\mathcal{TM}$ tangent to $\mathcal{Z}$; $\dot{\mathcal{V}}(0) = \mathcal{Z}$. Moreover, $J(t) := \frac{\partial c}{\partial s}(s,t)\big|_{s=0}$ is the Jacobi field along the geodesic γ_{v} with $J(0) = X$ and $J'(0) = Y$. Since $\mathbf{g}^t \circ \mathcal{V}(s) = \frac{\partial c}{\partial t}(s,t)$ and $\pi \circ \mathbf{g}^t \circ \mathcal{V}(s) = c(s,t)$,

$$\begin{aligned} (d\mathbf{g}^t)_{\mathrm{v}}(X,Y) &= \frac{d}{ds}\mathbf{g}^t \mathcal{V}(s)\bigg|_{s=0} \\ &= \left(\frac{\partial c}{\partial s}(0,t), \nabla_{\frac{\partial c}{\partial s}} \frac{\partial c}{\partial t}(0,t)\right) \\ &= \left(\frac{\partial c}{\partial s}(0,t), \nabla_{\frac{\partial c}{\partial t}} \frac{\partial c}{\partial s}(0,t)\right) \\ &= (J(t), J'(t)). \end{aligned}$$

Definition 2.2.1 (Sasaki metric)**.** The *Sasaki metric* on $\mathcal{TM}$ is a Riemannian metric $g_{\mathcal{S}}$ on $\mathcal{TM}$: for $(X_1, Y_1), (X_2, Y_2) \in \mathcal{T}_{\mathrm{v}}\mathcal{TM}$,

$$g_{\mathcal{S}}((X_1, Y_1), (X_2, Y_2)) = g(X_1, X_2) + g(Y_1, Y_2).$$

Tangent bundles are also equipped with a nondegenerate exact 2-form ω, called the *Liouville form*,

$$\omega\left((X_1, Y_1), (X_2, Y_2)\right) = g\left(X_2, Y_1\right) - g\left(Y_2, X_1\right).$$

The Liouville form ω is the exterior derivative of the canonical 1-form α, which is defined by

$$\alpha((X, Y)) = g(\mathrm{v}, X)$$

for $(X, Y) \in \mathcal{T}_{\mathrm{v}}\mathcal{TM}$.

Let $\mathcal{SM} := \{\mathrm{v} \in \mathcal{TM} : g(\mathrm{v}, \mathrm{v}) = 1\}$ denote the unit tangent bundle of $\mathcal{M}$. Since the geodesic flow preserves the magnitude of vectors, $\mathcal{SM}$ is $\mathbf{g}$-invariant. For $\mathrm{v} \in \mathcal{S}_p\mathcal{M}$, its tangent space at v is

$$\mathcal{T}_{\mathrm{v}}\mathcal{SM} = \{(X, Y) \in \mathcal{T}_{\mathrm{v}}\mathcal{TM} = \mathcal{T}_p\mathcal{M} \oplus \mathcal{T}_p\mathcal{M} : g(\mathrm{v}, Y) = 0\}.$$

If J is the Jacobi field with $J(0) = X$ and $J'(0) = Y$ for $(X, Y) \in \mathcal{T}_{\mathrm{v}}\mathcal{M}$, then

$$\begin{aligned} \frac{d}{dt}\left(d\mathbf{g}^t\right)^* \alpha(X, Y) &= \frac{d}{dt}\alpha(J(t), J'(t)) \\ &= \frac{d}{dt} g(\dot{\gamma}_{\mathrm{v}}(t), J(t)) \\ &= g(\dot{\gamma}_{\mathrm{v}}(t), J'(t)) = 0, \end{aligned}$$

where the last identity is due to $g(\dot{\gamma}_{\mathrm{v}}(t), J'(t))$ is constant and $g(\dot{\gamma}_{\mathrm{v}}(0), J'(0)) = 0$. Hence α restricted to $\mathcal{SM}$ is $\mathbf{g}$-invariant, so its derivative $\omega = d\alpha$ is also $\mathbf{g}$-invariant on $\mathcal{SM}$.

The *Liouville measure*, which is a measure on $\mathcal{TM}$ induced by $\alpha \wedge \omega^{d-1}$, is proportional to the Riemannian volume measure $\mathrm{vol}_{\mathcal{SM}}$ of the Sasaki metric on $\mathcal{SM}$ and is $\mathbf{g}$-invariant. The following proposition provides an estimate of the norm of the differential of the geodesic flow for the Sasaki metric on $\mathcal{SM}$.

Proposition 2.2.2. *For* $\mathrm{v}, \mathrm{w} \in \mathcal{S}_p\mathcal{M}$, *let* $R_{\mathrm{v}}(\mathrm{w}) = R(\mathrm{w}, \mathrm{v})\mathrm{v}$ *denote the directional curvature operator. Then*

$$\left\| \left(d\mathbf{g}^t\right)_{\mathrm{v}} \right\| \leq \int_0^t \|\mathrm{Id} - R_{\mathbf{g}^s\mathrm{v}}\| \, ds$$

where $\|\cdot\|$ *is the operator norm.*

We now turn our attention to the unit tangent bundle of negative curvature. In negative curvature, Jacobi fields have convex norm.

Lemma 2.2.3. *Let* $\mathcal{M}$ *have sectional curvature not greater than* $-a^2 < 0$. *For* $\mathrm{v} \in \mathcal{SM}$, *if* J *is a Jacobi field along* γ_{v} *perpendicular to* $\dot{\gamma}_{\mathrm{v}}$, *then*

$$|J|''(t) \geq -a|J|(t),$$

for any t *with* $J(t) \neq 0$. *In particular,* $\mathcal{M}$ *has no conjugate points.*

For $a > 0$, we use the notations $\mathrm{sn}_{-a^2}(t) = \frac{1}{a}\sinh(at)$ and $\mathrm{cs}_{-a^2}(t) = \frac{1}{a}\cosh(at)$. We present Rauch's comparison theorem for Jacobi fields.

Proposition 2.2.4. *For* $\mathrm{v} \in \mathcal{SM}$, *let* J *be a Jacobi field along* γ_{v} *with* $J(0) = 0$, $g(\dot{\gamma}_{\mathrm{v}}(0),\ J'(0)) = 0$ *and* $|J'(0)| = 1$.

1. *If* $\mathrm{Sec}_{\mathcal{M}} \leq -a^2$, *then*

$$|J(t)| \geq \mathrm{sn}_{-a^2}(t) \text{ and } |J|'(t) \geq \frac{\mathrm{cs}_{-a^2}}{\mathrm{sn}_{-a^2}}|J(t)|.$$

2. *If* $0 \geq \mathrm{Sec}_{\mathcal{M}} \geq -b^2$, *then*

$$|J(t)| \leq \mathrm{sn}_{-b^2}(t) \text{ and } |J|'(t) \leq \frac{\mathrm{cs}_{-b^2}}{\mathrm{sn}_{-b^2}}|J(t)|.$$

Lemma 2.2.5. *Suppose that* $\mathcal{M}$ *has non-positive sectional curvature. Then for* $\mathrm{v} \in \mathcal{SM}$ *and a Jacobi field* J *along* γ_{v}, *the following conditions are equivalent:*

1. J *is a parallel vector field along* γ_{v};
2. $|J(t)|$ *is constant on* $\mathbb{R}$;
3. $|J(t)|$ *is bounded on* $\mathbb{R}$.

These conditions implies $g(R(J', \dot{\gamma}_{\mathrm{v}})\dot{\gamma}_{\mathrm{v}}, J') = 0$.

For a unit vector v of a non-positively curved manifold $\mathcal{M}$, a Jacobi field J along γ is said to be *stable* if $|J(t)|$ is bounded on $t > 0$. The following proposition guarantees the existence of stable Jacobi fields.

Proposition 2.2.6. *Let $\mathcal{M}$ be a non-positively curved manifold and let* $\mathrm{v} \in \mathcal{S}_p\mathcal{M}$.

1. *For each $X \in \mathcal{T}_p\mathcal{M}$, there is a unique Jacobi field J_X with $J_X(0) = X$.*
2. *Let (γ_n) be a sequence of unit-speed geodesics with $\gamma_n \to \gamma$ pointwise. If J_n is a Jacobi field along γ_n such that $J_n(0) \to X$ and there is a constant $C > 0$ with $J_n(t_n) \leq C$ for some $t_n \to \infty$, then $J_n \to J_X$ and $J'_n \to J'_X$.*

The uniqueness part ensures us that the space of perpendicular stable Jacobi fields is a $(d-1)$-dimensional vector space.

Proposition 2.2.7. *Let J be a stable Jacobi field along a unit speed geodesic γ on a non-positively curved manifold $\mathcal{M}$ such that J is perpendicular to $\dot{\gamma}$.*

1. *If* $\mathrm{Sec}_{\mathcal{M}} \leq -a^2$, *then for all* $t \geq 0$,

$$|J(t)| \leq |J(0)|e^{-at} \text{ and } |J'(t)| \geq a|J(t)|. \tag{2.4}$$

2. *If* $\mathrm{Sec}_{\mathcal{M}} \geq -b^2$, *then for all* $t \geq 0$,

$$|J(t)| \geq |J(0)|e^{-bt} \text{ and } |J'(t)| \leq b|J(t)|. \tag{2.5}$$

2.3 Cartan-Hadamard manifolds of negative curvature

Given a Riemannian manifold $(\mathcal{M}, g)$, we denote by $\mathrm{p} : \widetilde{\mathcal{M}} \to \mathcal{M}$ its universal cover with the lifted Riemannian metric, which we also denote by g. A *Cartan-Hadamard manifold* is the universal cover $\widetilde{\mathcal{M}}$ of a complete connected Riemannian manifold $\mathcal{M}$ with non-positive sectional curvature; $\mathrm{Sec}_{\mathcal{M}} \leq 0$. It follows from the following theorem that Cartan-Hadamard manifolds are diffeomorphic to Euclidean spaces.

Theorem 2.3.1. *If a complete connected Riemannian manifold $(\mathcal{M}, g)$ has non-positive sectional curvature, then the exponential map $\exp_p : \mathcal{T}_p\mathcal{M} \to \mathcal{M}$ at p is a universal cover of $\mathcal{M}$ for any point p in $\mathcal{M}$.*

Consider a geodesic triangle $\Delta(x, y, z)$ with vertices x, y and z in $\widetilde{\mathcal{M}}$. The comparison triangle of $\Delta(x, y, z)$ in the model space $\mathbb{H}^2_\kappa$ with constant curvature $\kappa \leq 0$ is a geodesic triangle $\Delta_\kappa(\bar{x}, \bar{y}, \bar{z})$ in $\mathbb{H}^2_\kappa$ with vertices $\bar{x}, \bar{y}, \bar{z}$ such that $\mathrm{d}(x, y) = \mathrm{d}_\kappa(\bar{x}, \bar{y})$, $\mathrm{d}(y, z) = \mathrm{d}_\kappa(\bar{y}, \bar{z})$ and $\mathrm{d}(x, z) = \mathrm{d}_\kappa(\bar{x}, \bar{z})$. Note that comparison triangles in the model space of a geodesic triangle are isometric to each other. The following comparison theorem of triangles allows us to estimate geometric quantities of Cartan-Hadamard manifolds.

Proposition 2.3.2. *1. (Theorem II.1A.6 in [BrHa]) If* $\mathrm{Sec}_{\widetilde{\mathcal{M}}} \leq -a^2$*, then for any geodesic triangle* $\Delta(x, y, z)$ *with a comparison triangle* $\Delta_{-a^2}(\bar{x}, \bar{y}, \bar{z})$ *in* $\mathbb{H}^2_{-a^2}$*, the angles at* x, y *and* z *is not greater than the angles at* $\bar{x}, \bar{y}$ *and* $\bar{z}$*, respectively.*

2. (Proposition 1.2.2. in [Bo]) If $\mathrm{Sec}_{\widetilde{\mathcal{M}}} \geq -b^2$*, then for any geodesic triangle* $\Delta(x, y, z)$ *with a comparison triangle* $\Delta_{-a^2}(\bar{x}, \bar{y}, \bar{z})$ *in* $\mathbb{H}^2_{-b^2}$*, the angles at* x, y *and* z *is not less than the angles at* $\bar{x}, \bar{y}$ *and* $\bar{z}$*, respectively.*

For a fixed $p \in \widetilde{\mathcal{M}}$, recall that the exponential map $\exp_p$ induces the polar coordinate on $\widetilde{\mathcal{M}} \setminus \{p\}$:

$$\begin{aligned}(0, \infty) \times \mathcal{S}_p\widetilde{\mathcal{M}} &\to \widetilde{\mathcal{M}} \setminus \{p\} \\ (r, \mathrm{v}) &\mapsto \exp_p r\mathrm{v},\end{aligned}$$

and the Riemannian metric is expressed as

$$g = dr^2 + \lambda_p(r, \mathrm{v}) g_{\mathbb{S}},$$

where λ_p is a smooth function on $\widetilde{\mathcal{M}} \setminus \{p\} = (0, \infty) \times \mathcal{S}_p\widetilde{\mathcal{M}}$. Note that if $\mathrm{Sec}_{\widetilde{\mathcal{M}}} = -a^2$, then $\lambda_p(r, \mathrm{v}) = \mathrm{sn}_{-a^2}(r)$.

We denote by $A_p(r, \mathrm{v})$ the density of the volume element with respect to the polar coordinate at p, i.e., $d\,\mathrm{vol}_{\widetilde{\mathcal{M}}} = A_p(r, \mathrm{v}) dr d\,\mathrm{vol}_{\mathbb{S}^{d-1}}(\mathrm{v})$. Since $\partial_r A_p(r, \mathrm{v}) = A_p(r, \mathrm{v})\Delta r$, by Rauch's comparison theorem, we have the following volume comparison theorem (See Lemma 7.1.2 in [Pe] for the detail).

Theorem 2.3.3. *(Volume comparison) If* $-b^2 \leq \mathrm{Sec}_{\widetilde{\mathcal{M}}} \leq -a^2$ *for some* $b > a > 0$*,*

$$(d-1)\,\mathrm{sn}^d_{-a^2}(r) \leq A_p(r, \cdot) \leq (d-1)\,\mathrm{sn}^d_{-b^2}(r). \tag{2.6}$$

In particular, $\mathrm{vol}_{-a^2} \leq \mathrm{vol}_{\widetilde{\mathcal{M}}} \leq \mathrm{vol}_{-b^2}$.

Let $\widetilde{\mathcal{M}}$ be a Cartan-Hadamard manifold of dimension d. Since $\widetilde{\mathcal{M}}$ is diffeomorphic to the unit open ball in the Euclidean space $\mathbb{R}^d$, we shall define a compactification $\mathrm{cl}(\widetilde{\mathcal{M}})$ of Cartan-Hadamard manifolds homeomorphic to the closed ball in $\mathbb{R}^d$ and then we have a *boundary at infinity* $\partial_\infty\widetilde{\mathcal{M}}$, which is also called the *visual boundary*, homeomorphic to the sphere $\mathbb{S}^{d-1}$.

For x, y, and z in $\widetilde{\mathcal{M}}$, $\mathfrak{b}(x,y,z) = \mathrm{d}(x,z) - \mathrm{d}(y,z)$. Then $\mathfrak{b}$ is a cocycle:

$$\begin{aligned}\mathfrak{b}(x,y,z) - \mathfrak{b}(x,y',z) &= \mathfrak{b}(y',y,z),\\ \mathfrak{b}(x,y,z) - \mathfrak{b}(x',y,z) &= \mathfrak{b}(x,x',z).\end{aligned}$$

And it is Lipschitz continuous in each variable:

$$\begin{aligned}&\left|\mathfrak{b}(x,y,z) - \mathfrak{b}(x',y,z)\right| = |\mathfrak{b}(x,x',z)| \le \mathrm{d}(x,x'),\\ &\left|\mathfrak{b}(x,y,z) - \mathfrak{b}(x,y',z)\right| = |\mathfrak{b}(y',y,z)| \le \mathrm{d}(y',y),\\ &\left|\mathfrak{b}(x,y,z) - \mathfrak{b}(x,y,z')\right| \le |\mathrm{d}(x,z) - \mathrm{d}(x,z')| + |\mathrm{d}(y,z) - \mathrm{d}(y,z')| \le 2\mathrm{d}(z,z').\end{aligned}$$

Fix a point $x_0 \in \widetilde{\mathcal{M}}$. We have an embedding of $\widetilde{\mathcal{M}}$ into the space $\mathcal{C}(\widetilde{\mathcal{M}})$ of continuous functions equipped with compact-open topology: $\mathfrak{b}_{x_0}(z) := \mathfrak{b}(x_0, \cdot, z)$. Since the embedding $\mathfrak{b}_{x_0}$ is a Lipschitz map and $[\mathfrak{b}_{x_0}(z)](x_0) = 0$ for each $z \in \widetilde{\mathcal{M}}$, the image $\mathfrak{b}_{x_0}(\widetilde{\mathcal{M}})$ has compact closure in $\mathcal{C}(\widetilde{\mathcal{M}})$ due to Arzela-Ascoli theorem. So we identify $\widetilde{\mathcal{M}}$ with its image $\mathfrak{b}_{x_0}(\widetilde{\mathcal{M}})$, we define the Busemann compactification $\mathrm{cl}(\widetilde{\mathcal{M}})$ of $\widetilde{\mathcal{M}}$ as the space of equivalence classes of sequences:

Definition 2.3.4 (Busemann compactification)**.** The *Busemann compactification* of $\widetilde{\mathcal{M}}$ is a set of equivalence classes of sequences in $\widetilde{\mathcal{M}}$ defined by

$$\mathrm{cl}(\widetilde{\mathcal{M}}) := \left\{(z_n)_{n=1}^{\infty} : \mathfrak{b}_{x_0}(z_n) \text{ converges in } \mathcal{C}^{\infty}(\widetilde{\mathcal{M}})\right\} / \sim$$

where $(z_n) \sim (w_n)$ if and only if $\lim\limits_{n\to\infty} \mathfrak{b}_{x_0}(z_n) = \lim\limits_{n\to\infty} \mathfrak{b}_{x_0}(w_n)$.

Then identifying a point x in $\widetilde{\mathcal{M}}$ with the class of the constant sequence $(x_n)_{n=1}^{\infty}$ s.t. $x_n = x$, $\widetilde{\mathcal{M}}$ is an open dense set of $\mathrm{cl}(\widetilde{\mathcal{M}})$. We define the boundary of $\widetilde{\mathcal{M}}$ by the space of the equivalence classes of sequences that diverge in $\widetilde{\mathcal{M}}$.

Definition 2.3.5 (Boundary at infinity)**.** We define the *boundary at infinity* of $\widetilde{\mathcal{M}}$ by

$$\partial_\infty\widetilde{\mathcal{M}} = \mathrm{cl}(\widetilde{\mathcal{M}}) \setminus \widetilde{\mathcal{M}}.$$

$\partial_\infty\widetilde{\mathcal{M}}$ is also called the *Busemann boundary* or the *visual boundary* of $\widetilde{\mathcal{M}}$.

For a sequence (z_n) converges to $\xi \in \partial_\infty \widetilde{\mathcal{M}}$, we denote the limit function $\lim_{n\to\infty} b(x, y, z_n)$ by

$$\mathfrak{b}(x, y, \xi) = [\mathfrak{b}_x(\xi)](y), \tag{2.7}$$

which is called the *Busemann function* at ξ.

Definition 2.3.6 (Horosphere)**.** A *horosphere* $\mathcal{H}_\xi$ based at ξ is a level set of Busemann function;

$$\mathcal{H}_\xi = \mathcal{H}_\xi(x_0, R) := [\mathfrak{b}_{x_0}(\xi)]^{-1}(R). \tag{2.8}$$

Since $\mathfrak{b}_{x_0}(z_n)$ converges to $\mathfrak{b}_{x_0}(\xi)$ uniformly on compact sets and the level set $\mathcal{H}_{z_n}(x_0, R) := [\mathfrak{b}_{x_0}(z_n)]^{-1}(\mathrm{d}(x_0, z_n) - R)$ is a sphere, the horosphere $\mathcal{H}_\xi$ is the limit sphere of the sequence of sphere $\mathcal{H}_{z_n}$. A sub-level set $[\mathfrak{b}_{x_0}(\xi)]^{-1}(R, \infty)$ is called a *horoball* at ξ.

The boundary at infinity of a Cartan-Hadamard manifold $\widetilde{\mathcal{M}}$ is equivalent to the set of asymptotic classes of unit speed geodesic rays. Two unit speed geodesic rays $\gamma_1, \gamma_2 : [0, \infty) \to \widetilde{\mathcal{M}}$ are said to be *asymptotic* if

$$\sup\{\mathrm{d}\left(\gamma_1(t), \gamma_2(t)\right) : t \in [0, \infty)\} < \infty.$$

Note that given a point $p \in \widetilde{\mathcal{M}}$, any asymptotic class contains a geodesic ray generated by a vector in $\mathcal{S}_p\widetilde{\mathcal{M}}$. Since two geodesic rays generated by distinct vectors in $\mathcal{S}_p\widetilde{\mathcal{M}}$ are not asymptotic at all, the space of asymptotic classes of geodesic rays coincides with the unit tangent space $\mathcal{S}_p\widetilde{\mathcal{M}}$ which is homeomorphic to $\mathbb{S}^{d-1}$. Hence we identify $\mathcal{S}\widetilde{\mathcal{M}}$ with $\widetilde{\mathcal{M}} \times \partial_\infty \widetilde{\mathcal{M}}$ via the following homeomorphism:

$$\begin{aligned} \mathcal{S}\widetilde{\mathcal{M}} &\to \widetilde{\mathcal{M}} \times \partial_\infty \widetilde{\mathcal{M}} \\ \mathrm{v} &\mapsto (p, \mathrm{v}^+) \end{aligned} \tag{2.9}$$

where p is the base point of v and v^+ is the asymptotic class of γ_{v}.

For any geodesic ray γ, $\mathfrak{b}_x(\gamma(t))$ converges in $\mathcal{C}^\infty(\widetilde{\mathcal{M}})$ and $\mathfrak{b}_x(\gamma_1(t))$ and $\mathfrak{b}_x(\gamma_2(t))$ converge to the same Busemann function if and only if γ_1 and γ_2 are asymptotic. Therefore $\partial_\infty \widetilde{\mathcal{M}}$ and the asymptotic classes of geodesic rays coincide, so $\partial_\infty \widetilde{\mathcal{M}}$ is homeomorphic to $\mathbb{S}^{d-1}$. However, it is not smoothly attached in general. Instead, it has a Hölder structure.

Proposition 2.3.7. *(Proposition 2.1 in [AS]) If $-b^2 \le \mathrm{Sec}_{\widetilde{\mathcal{M}}} \le -a^2$ for some $b > a > 0$, then $\partial_\infty \widetilde{\mathcal{M}}$ has a well-defined Hölder structure; the natural map*

$$\mathcal{S}_x \widetilde{\mathcal{M}} \to \partial_\infty \widetilde{\mathcal{M}} \to \mathcal{S}_y$$

is a/b-Hölder continuous with respect to the angle metrics.

Proof. Let $\mathrm{v}, \mathrm{w} \in \mathcal{S}_x \widetilde{\mathcal{M}}$ and we denote by θ their angle. Then by the comparison theorem (Proposition 2.3.2) and the law of cosine,

$$\mathrm{d}_t - 2t \le \frac{2}{b} \log \theta + C(b),$$

where $\mathrm{d}_t := \mathrm{d}(\gamma_{\mathrm{v}}(t), \gamma_{\mathrm{w}}(t))$, for every sufficiently large t. If θ_t is the angle of the geodesic triangle $\Delta(y, \gamma_{\mathrm{v}}(t), \gamma_{\mathrm{w}}(t))$ at y, then

$$\frac{2}{a} \log \theta_t \le \mathrm{d}_t - 2t + C(a, \mathrm{d}(x, y))$$

for large enough t. Therefore we have $\theta_t \le C'(a, b, \mathrm{d}(x, y)) \theta^{\frac{a}{b}}$. □

For $\xi, \eta \in \partial_\infty \widetilde{\mathcal{M}}$, let z_n, w_n be sequences in $\widetilde{\mathcal{M}}$ converging to ξ and η, respectively. Then for each $x \in \widetilde{\mathcal{M}}$, the following limit exists:

$$(\xi|\eta)_x := \lim_{n\to\infty} \mathrm{d}(x, z_n) + \mathrm{d}(x, w_n) - \mathrm{d}(z_n, w_n),$$

which is called the *Gromov product* of ξ, η with respect to x. For each sufficiently small $\tau > 0$, $\mathrm{d}_\infty^{x,\tau}(\cdot, \cdot) := \exp[-\tau(\cdot|\cdot)_x]$ defines a distance on $\partial_\infty \widetilde{\mathcal{M}}$.

For $x \in \widetilde{\mathcal{M}}$ and $\xi \in \partial_\infty \widetilde{\mathcal{M}}$, by the convexity of distance in $\widetilde{\mathcal{M}}$, there is a unique unit speed geodesic $\gamma_{x,\xi}$ whose asymptotic class is ξ and $\gamma_{x,\xi}(0) = x$. We denote the tangent vector $\dot\gamma_{x,\xi}(0)$ by v_x^ξ. The following proposition ensures uniform $\mathcal{C}^2$-convergence of Busemann functions on compact sets and horospheres are $\mathcal{C}^1$-submanifolds of $\widetilde{\mathcal{M}}$.

Proposition 2.3.8. *Fix $x \in \widetilde{\mathcal{M}}$ and let $z_n \in \widetilde{\mathcal{M}}$ be with $z_n \to \xi$ as $n \to \infty$. Let $f_n := \mathfrak{b}_x(z_n)$ and $f := \mathfrak{b}_x(\xi)$. Then for each $y \in \widetilde{\mathcal{M}}$ and $X \in \mathcal{T}_y \widetilde{\mathcal{M}}$, we have the following uniform convergences on compact sets:*

$$\lim_{n\to\infty} \nabla f_n(y) = \mathrm{v}_y^\xi \text{ and } \lim_{n\to\infty} \nabla_X \nabla f_n(y) = J_X'(0)$$

where J_X is the stable Jacobi field along $\gamma_{y,\xi}$ with $J_X(0) = X$. In particular, Busemann functions are $\mathcal{C}^2$-functions.

Let $\widetilde{\mathcal{M}}$ be an Hadamard manifold with $-b^2 \le \mathrm{Sec}_{\widetilde{\mathcal{M}}} \le -a^2 < 0$. Then the geodesic flow $\mathbf{g}$ on the unit tangent bundle $\mathcal{S}\widetilde{\mathcal{M}}$ is an Anosov flow, i.e., there is a constant $C > 0$ and the tangent bundle of $\mathcal{S}\widetilde{\mathcal{M}}$ has the decomposition

$$\mathcal{TSM} = \widetilde{E}^s \oplus \widetilde{E}^c \oplus \widetilde{E}^u$$

into $\mathbf{g}$-invariant distributions such that $\widetilde{E}^c := \mathbb{R}\mathcal{X}$ where $\mathcal{X}$ is the generator of $\mathbf{g}$, called the *geodesic spray*, and for each $\mathrm{v} \in \mathcal{S}_p\widetilde{\mathcal{M}}$,

$$\begin{aligned} \widetilde{E}^s_{\mathrm{v}} &:= \left\{\mathcal{Z} : |(d\mathbf{g}^t)_{\mathrm{v}}\mathcal{Z}| \le C\exp(-at)|\mathcal{Z}|\right\}, \\ \widetilde{E}^u_{\mathrm{v}} &:= \left\{\mathcal{Z} : |(d\mathbf{g}^{-t})_{\mathrm{v}}\mathcal{Z}| \le C\exp(-at)|\mathcal{Z}|\right\}. \end{aligned}$$

From Proposition 2.2.6, it follows that

$$\begin{aligned} \widetilde{E}^s_{\mathrm{v}} &= \left\{(X,Y) \in \mathcal{T}_p\widetilde{\mathcal{M}} \oplus \mathcal{T}_p\widetilde{\mathcal{M}} : X \perp \mathrm{v},\, Y = J'_{\mathrm{v},X}(0)\right\}, \\ \widetilde{E}^u_{\mathrm{v}} &= \left\{(X,Y) \in \mathcal{T}_p\widetilde{\mathcal{M}} \oplus \mathcal{T}_p\widetilde{\mathcal{M}} : X \perp \mathrm{v}, Y = -J'_{-\mathrm{v},X}(0)\right\}, \end{aligned}$$

where $J_{\mathrm{v},Z}(t)$ is the stable Jacobi field along the geodesic $\gamma_{\mathrm{v}}(t)$ with $J_{\mathrm{v},Z}(0) = Z$. Note that $J(t) := J_{-\mathrm{v},X}(-t)$ is the unstable Jacobi fields along γ_{v} with $J(0) = X$.

Thus both distributions $\widetilde{E}^s$ and $\widetilde{E}^u$ are $(d-1)$-dimensional distributions on $\mathcal{S}\widetilde{\mathcal{M}}$. We call $\widetilde{E}^s$ and $\widetilde{E}^u$ the *stable distribution* and the *unstable distribution*, respectively. We also consider d-dimensional distributions $\widetilde{E}^{cs}$ and $\widetilde{E}^{cu}$, called the *central stable distribution* and the *central unstable distribution*, respectively, given by $\widetilde{E}^{cs}_{\mathrm{v}} = \widetilde{E}^s_{\mathrm{v}} \oplus \widetilde{E}^c_{\mathrm{v}}$ and $\widetilde{E}^{cu}_{\mathrm{v}} = \widetilde{E}^u_{\mathrm{v}} \oplus \widetilde{E}^c_{\mathrm{v}}$.

Definition 2.3.9 (Dynamical foliations)**.** We define dynamical foliations on $\mathcal{S}\widetilde{\mathcal{M}}$ for the geodesic flow. For $\mathrm{v} \in \mathcal{S}_x\widetilde{\mathcal{M}}$, we denote unit normal bundles of horospheres perpendicular to v by

$$\begin{aligned} \widetilde{\mathcal{W}}^s(\mathrm{v}) &:= \left\{\nabla\mathfrak{b}_x(\mathrm{v}^+)(y) : \mathfrak{b}(x,y,\mathrm{v}^+) = 0\right\}, \\ \widetilde{\mathcal{W}}^u(\mathrm{v}) &:= \left\{\nabla\mathfrak{b}_x(\mathrm{v}^-)(y) : \mathfrak{b}(x,y,\mathrm{v}^-) = 0\right\}, \end{aligned}$$

where $\mathrm{v}^\pm := \lim_{t\to\infty} \gamma_{\mathrm{v}}(\pm t)$. We call $\widetilde{\mathcal{W}}^s(\mathrm{v})$ and $\widetilde{\mathcal{W}}^u(\mathrm{v})$ the *stable leaf* and the *unstable leaf* of v, respectively.

The stable leaves and the unstable leaves form foliations $\widetilde{\mathcal{W}}^s$ and $\widetilde{\mathcal{W}}^u$, called the *stable foliation* and the *unstable foliation* for the geodesic flow, respectively.

Note that due to the convexity,

$$\widetilde{\mathcal{W}}^s(\mathrm{v}) = \left\{\mathrm{w} : \lim_{t\to\infty} \mathrm{d}(\gamma_{\mathrm{v}}(t), \gamma_{\mathrm{w}}(t)) = 0\right\},$$
$$\widetilde{\mathcal{W}}^u(\mathrm{v}) = \left\{\mathrm{w} : \lim_{t\to\infty} \mathrm{d}(\gamma_{\mathrm{v}}(-t), \gamma_{\mathrm{w}}(-t)) = 0\right\}.$$

It follows from Proposition 2.3.8 that they are $\mathcal{C}^1$-submanifolds of $\mathcal{S}\widetilde{\mathcal{M}}$ and $\mathcal{T}\widetilde{\mathcal{W}}^s(\mathrm{v}) = \widetilde{E}^s_{\mathrm{v}}$ and $\mathcal{T}\widetilde{\mathcal{W}}^u(\mathrm{v}) = \widetilde{E}^u_{\mathrm{v}}$. Therefore it turns out that the stable and the unstable distributions are integrable and stable and unstable leaves are their integral submanifolds.

The central stable distribution and the central unstable distribution are also integrable with the integral manifolds $\widetilde{\mathcal{W}}^{cs}(\mathrm{v})$ and $\widetilde{\mathcal{W}}^{cu}(\mathrm{v})$:

$$\widetilde{\mathcal{W}}^{cs}(\mathrm{v}) := \{\mathrm{w} : \mathrm{v}^+ = \mathrm{w}^+\},$$
$$\widetilde{\mathcal{W}}^{cu}(\mathrm{v}) := \{\mathrm{w} : \mathrm{v}^- = \mathrm{w}^-\}.$$

We call $\widetilde{\mathcal{W}}^{cs}(\mathrm{v})$ and $\widetilde{\mathcal{W}}^{cu}(\mathrm{v})$ the *central stable leaf* and the *central unstable leaf* of v, respectively. The central unstable leaf of v is the unions of stable leaves of vectors w asymptotic to v. Likewise, the central unstable leaf of v is the union of unstable leaves of vectors w negatively asymptotic to v. With the identification (2.9), $\widetilde{\mathcal{W}}^{cs}(\mathrm{v}) = \widetilde{\mathcal{M}} \times \{\mathrm{v}^+\}$.

For $0 \le k \le 2d-1$, the *Grassmannian bundle* $\mathrm{Gr}_k(\mathcal{T}\mathcal{S}\widetilde{\mathcal{M}}) \to \mathcal{S}\widetilde{\mathcal{M}}$ of rank k of the tangent bundle $\mathcal{T}\mathcal{S}\widetilde{\mathcal{M}} \to \mathcal{S}\widetilde{\mathcal{M}}$ of the unit tangent bundle is a fiber bundle over the unit tangent bundle whose fiber at $\mathrm{v} \in \mathcal{S}\widetilde{\mathcal{M}}$ is the Grassmannian manifold $\mathrm{Gr}_k(\mathcal{T}_{\mathrm{v}}\mathcal{S}\widetilde{\mathcal{M}})$ of k-dimensional subspace in the tangent space $\mathcal{T}_{\mathrm{v}}\mathcal{S}\widetilde{\mathcal{M}}$ at v. We endow $\mathrm{Gr}_k(\mathcal{T}\mathcal{S}\widetilde{\mathcal{M}})$ with a distance by measuring the Hausdorff distance between unit spheres in subspaces: if $E, F \in \mathrm{Gr}_k(\mathcal{T}\mathcal{S}\widetilde{\mathcal{M}})$ are subspaces of tangent spaces and $\mathbb{S}_E$ and $\mathbb{S}_F$ are the unit spheres in E and F, respectively, then

$$\mathrm{d}_{\mathrm{Gr}}(E, F) := \max_{\mathrm{v}_0 \in \mathbb{S}_E, \mathrm{w}_0 \in \mathbb{S}_F} \left\{\min_{\mathrm{w} \in \mathbb{S}_F} \mathrm{d}_{\mathcal{S}}(\mathrm{v}_0, \mathrm{w}), \min_{\mathrm{v} \in \mathbb{S}_E} \mathrm{d}_{\mathcal{S}}(\mathrm{v}, \mathrm{w}_0)\right\}.$$

Note that the stable and the unstable distributions are sections of the Grassmannian bundle $\mathrm{Gr}_{d-1}(\mathcal{T}\mathcal{S}\widetilde{\mathcal{M}})$ of rank $d-1$. Likewise, the central stable and the central unstable distributions are sections of $\mathrm{Gr}_d(\mathcal{T}\mathcal{S}\widetilde{\mathcal{M}})$. Hence we can measure regularities of these distributions with respect to d_{Gr}.

Theorem 2.3.10. *If the first derivatives of the sectional curvature of $\widetilde{\mathcal{M}}$ is uniformly bounded, that is, $\|\nabla \operatorname{Sec}_{\widetilde{\mathcal{M}}}\| < +\infty$, then the distributions $\widetilde{E}^s$ and $\widetilde{E}^u$ are Hölder continuous sections of the Grassmannian bundle $\mathrm{Gr}_{d-1}(\mathcal{T}\mathcal{S}\widetilde{\mathcal{M}})$ of rank $d-1$.*

Since the geodesic spray $\mathcal{X}$ is perpendicular to $\widetilde{E}^s$ and $\widetilde{E}^u$, the central stable and the central unstable distributions are also Hölder continuous. Uniformly boundedness of the first derivatives of sectional curvature is necessary. There is a finite-volume complete Riemannian surface with pinched negative curvature whose stable distribution is not Hölder continuous (see [BB]).

Let $\mathrm{p} : \widetilde{\mathcal{M}} \to \mathcal{M}$ be the universal cover of a complete finite-volume Riemannian manifold $\mathcal{M}$ with $-b^2 \leq \operatorname{Sec}_{\mathcal{M}} \leq -a^2$ for some $b > a > 0$. Then the group Γ of deck transformations is a *lattice* in $\widetilde{\mathcal{M}}$, that is, Γ acts isometrically and properly discontinuously on $\widetilde{\mathcal{M}}$ and the quotient by Γ has finite volume. Due to the following proposition, the action of Γ on $\widetilde{\mathcal{M}}$ is free.

Proposition 2.3.11. *(Corollary 6.2.4. in [Pe]) The fundamental group Γ of $\mathcal{M}$ is a torsion-free group. Thus it acts freely on $\widetilde{\mathcal{M}}$ by deck transformations.*

Definition 2.3.12 (Dirichlet domain)**.** Fix a point $x \in \widetilde{\mathcal{M}}$. The *Dirichlet domain* for Γ with center x is

$$\mathcal{M}(x_0) = \{y : \mathrm{d}(x_0, y) \leq \mathrm{d}(\gamma x_0, y)\, \forall \gamma \in \Gamma\}.$$

Note that $\mathcal{M}(x_0)$ is indeed a fundamental domain for Γ; $\cup_{\gamma\in\Gamma}\gamma\mathcal{M}(x_0)$ and each intersection $\mathcal{M}(x_0) \cap \gamma\mathcal{M}(x_0)$ is a subset of boundary of $\mathcal{M}(x_0)$ and has zero volume.

Given an isometry γ of $\widetilde{\mathcal{M}}$, the *displacement function* of γ is a function

$$\rho_\gamma : \widetilde{\mathcal{M}} \to [0, \infty), \rho_\gamma(x) = \mathrm{d}(x, \gamma x),$$

which gives a characterization of isometries.

Definition 2.3.13. Let γ be an isometry of $\widetilde{\mathcal{M}}$.

1. If there exists $x \in \widetilde{\mathcal{M}}$ such that $\rho_\gamma(x) = 0$, γ is called *elliptic*;
2. If there is $x \in \widetilde{\mathcal{M}}$ such that $\rho_\gamma(x) = \inf \rho_\gamma$, γ is called *loxodromic*;
3. If $\rho_\gamma(x) > \inf \rho_\gamma$ for any x, γ is called *parabolic.*

Note that γ is elliptic if and only if γ has a fixed point in $\widetilde{\mathcal{M}}$, hence elements of Γ are not elliptic.

Proposition 2.3.14. *(Lemma 6.2.7 in [Pe], Proposition 3.3 in [Ba]) If an isometry γ has positive displacement, i.e., $t_\gamma := \inf \rho_\gamma > 0$, then there is unique a geodesic $c(t) = c_\gamma(t)$ such that $(\gamma \circ c)(t) = c(t + t_\gamma)$. In particular, γ is loxodromic.*

Since an isometry γ sends geodesics to geodesics and the boundary at infinity is the set of asymptotic classes of geodesics, the action of γ by isometry on $\widetilde{\mathcal{M}}$ extends to an action on $\partial_\infty \widetilde{\mathcal{M}}$ by homeomorphism; For each isometry γ of $\mathcal{M}$ and $\xi \in \partial_\infty \widetilde{\mathcal{M}}$, if a geodesic $c(t)$ in $\widetilde{\mathcal{M}}$ represents ξ, we define $\gamma\xi$ as the asymptotic class of $(\gamma \circ c)(t)$. This action is well-defined since an isometry preserves asymptotic classes of geodesics.

For a non-elliptic γ, an orbit $\{\gamma^n x\}_{n \in \mathbb{Z}}$ of γ is unbounded (Proposition 3.2 in [Ba]), it has accumulation points in $\partial_\infty \widetilde{\mathcal{M}}$ (Lemma 6.2 in [EO]). Any orbit of γ has the same accumulation points and each accumulation point is fixed by γ. Fix points in $\partial_\infty \widetilde{\mathcal{M}}$ allows us classifying non-elliptic isometries of $\widetilde{\mathcal{M}}$.

Proposition 2.3.15. *Let γ be a non-elliptic isometry of $\widetilde{\mathcal{M}}$.*

1. *(Theorem 6.5 in [EO]) If γ is parabolic, then it fixes a single point in $\partial_\infty \widetilde{\mathcal{M}}$ and if loxodromic, then it fixes a pair of points.*

2. *(Proposition 7.8 in [EO]) If γ is parabolic with fixed point $\xi \in \partial_\infty \widetilde{\mathcal{M}}$, then it leaves horospheres based at ξ fixed.*

3. *(Proposition 6.7 in [EO]) If γ is loxodromic with two fixed points ξ_+ and ξ_- in $\partial_\infty \widetilde{\mathcal{M}}$, then for every x,*

$$\lim_{n \to \pm\infty} \gamma^n x = \xi_\pm.$$

Since Γ is torsion-free, every element of Γ is either parabolic or loxodromic. A point at infinity ξ is called a *parabolic (conical) fixed point* if it is a fixed point of a parabolic (loxodromic) element of Γ. We denote the set of parabolic (conical) fixed points of Γ by $\partial_p \Gamma$ $(\partial_c \Gamma)$.

Since $\mathcal{M}$ is a finite-volume manifold with pinched negative curvature, by Margulis-Gromov lemma, the number of ends of $\mathcal{M}$ is finite and there is a one-to-one correspondence between ends and Γ-equivalence classes of parabolic fixed points:

Theorem 2.3.16. *(Theorem 3.1 in [E]) Fix a point x_0 in $\widetilde{\mathcal{M}}$. Let $\mathcal{M}_0$ be the Dirichlet domain for Γ with center x_0.*

1. *There are only finitely many geodesic rays $c_1, \ldots, c_N : [0, \infty) \to \widetilde{\mathcal{M}}$ staying in $\mathcal{M}_0$. Thus the closure of $\mathcal{M}_0$ in $\mathrm{cl}\widetilde{\mathcal{M}}$ contains only finitely many points $\xi_1, \ldots, \xi_N$ at infinity and $\xi_k = \lim\limits_{t\to\infty} c_k$.*

2. *For each ξ, any elements of the stabilizer group $\mathrm{Stab}_\Gamma(\xi)$ has the same fixed points.*

3. *If $\mathcal{H}$ is a horosphere based at ξ_k then $\mathrm{Stab}_\Gamma(\xi_k)$ acts co-compactly on $\mathcal{H}$.*

Furthermore, there exists a finite set of horoballs $\widetilde{\mathcal{B}}_1, \ldots, \widetilde{\mathcal{B}}_N$ in $\widetilde{\mathcal{M}}$ whose projections $\mathcal{B}_1, \ldots, \mathcal{B}_N$ to $\mathcal{M}$ are disjoint and the complement of their union $\mathcal{M}_c = \mathcal{M} \setminus \cup_{k=1}^N \mathcal{B}_k$ is compact.

The Γ-equivalence class of ξ_k is called a *cuspidal point* of $\mathcal{M}$ and a projected horoball $\mathcal{B}_k$ is called a *cuspidal region* of ξ_k.

Chapter 3

Brownian motions in negative curvature

In this chapter, we introduce a construction of the Brownian motion on manifolds. We recall properties of the Brownian motion on negatively curved manifolds and the lifted process on the unit tangent bundle, called the foliated Brownian motion.

3.1 Brownian motions on manifolds

In this section, we construct Brownian motions on Riemannian manifolds in two different ways. First, we want to define the Brownian motion as a diffusion process generated by the Laplace-Beltrami operator. So we embed manifolds into Euclidean spaces and exploit the theory of stochastic differential equations. The second construction makes use of heat kernels as the transition density of Brownian motions. We refer to Chapter 3 and 4 of [Hs] for the proofs and details.

Let $\mathcal{M}$ be a connected Riemannian manifold of dimension $d \geq 2$. By Nash's embedding theorem, there is a Riemannian isometric smooth embedding of $\mathcal{M}$ into a Euclidean space $\mathbb{R}^N$. Thus we consider $\mathcal{M}$ as a closed Riemmanian smooth submanifolds of $\mathbb{R}^N$. Let $(\xi_\alpha)_{\alpha=1}^N$ be an orthonormal basis of $\mathbb{R}^N$. By translating this basis, we identify the basis with a parallel orthonormal frame field on $\mathbb{R}^N$. For each $x \in \mathcal{M}$, we denote the orthogonal projection of $\mathbb{R}^N = \mathcal{T}_x\mathbb{R}^N$ onto $\mathcal{T}_x\mathcal{M}$ by $P(x)$. Then the Laplace-Beltrami operator on $\mathcal{M}$

satisfies the equation

$$\Delta_{\mathcal{M}} f = \sum_{\alpha=1}^{N} P_\alpha(x)^2 f,$$

for every $f \in \mathcal{C}^2(\mathcal{M})$ where $P_\alpha(x) = P(x)\xi_\alpha$. With the identity, the Laplace-Beltrami operator extends to an elliptic operator on $\mathbb{R}^N$.

If $x = (x^1, \ldots, x^N)$ is the standard coordinate of $\mathbb{R}^N$ with respect to the basis (ξ_α) and $f^\alpha(x) = x^\alpha$ are the coordinate functions, then the functions

$$\begin{aligned} a^{\alpha\beta} &:= \Delta_{\mathcal{M}}(f^\alpha f^\beta) - f^\beta \Delta_{\mathcal{M}} f^\alpha - f^\alpha \Delta_{\mathcal{M}} f^\beta, \\ b^\alpha &:= \Delta_{\mathcal{M}} f^\alpha, \end{aligned}$$

smoothly extends to functions $\widetilde{a}^{\alpha\beta}, \widetilde{b}^\alpha$ on $\mathbb{R}^N$. Consider a second order elliptic differential operator on $\mathbb{R}^N$

$$\widetilde{\Delta} := \frac{1}{2} \sum_{\alpha,\beta} \widetilde{a}^{\alpha\beta} \frac{\partial}{\partial x^\alpha \partial x^\beta} + \sum_\alpha \widetilde{b}^\alpha \frac{\partial}{\partial x^\alpha}.$$

Then for each $f \in \mathcal{C}^\infty(\mathcal{M})$ and $\widetilde{f} \in \mathcal{C}^\infty(\mathbb{R}^N)$ with $\widetilde{f}\Big|_{\mathcal{M}} = f$,

$$\widetilde{\Delta}\widetilde{f} = \Delta_{\mathcal{M}} f.$$

Note that since the matrix-valued function $a = (a^{\alpha\beta})$ on $\mathcal{M}$ is symmetric and positive definite, we may assume that $\widetilde{a}$ on $\mathbb{R}^N$ is also symmetric and positive definite. We fix a symmetric matrix-valued function $\widetilde{\sigma}$ on $\mathbb{R}^N$ such that $\widetilde{a} = \widetilde{\sigma}^2$.

The sample path space of $\mathcal{M}$, denoted by $W(\mathcal{M})$, is the set of continuous paths $\omega : [0, T) \to \mathcal{M}$ for some $0 < T \leq \infty$ such that $\omega_t = \omega(t)$ diverges as $t \to T$ if $T < \infty$. We denote by $e(\omega)$ the upper bound T of the domain of ω. Consider the one-point compactification $\widehat{\mathcal{M}} = \mathcal{M} \cup \{\partial_\infty\}$ of $\mathcal{M}$ and correspond any path ω with ω_t diverges as $t \to e(\omega)$ to the path $\widehat{\omega} : [0, +\infty] \to \widehat{\mathcal{M}}$ such that $\widehat{\omega}_t = \omega_t$ if $t < e(\omega)$ and $\widehat{\omega}_t = \partial_\infty$ otherwise. This admits an identification of $W(\mathcal{M})$ and the subset $\{\widehat{\omega}_\infty = \partial_\infty\}$ of $C\left([0, +\infty], \widehat{\mathcal{M}}\right)$. We equip $W(\mathcal{M})$ with the topology induce by the compact-open topology of $C\left([0, +\infty], \widehat{\mathcal{M}}\right)$.

Let $\mathscr{B} = \mathscr{B}(W(\mathcal{M}))$ be the Borel σ-algebra of $W(\mathcal{M})$. For each $t \in \mathbb{R}_+$, the projection map $X_t : W(\mathcal{M}) \to \mathcal{M}$ at t is the evaluation map at time t: $X_t(\omega) := \omega_t$. Consider a σ-subalgebra $\mathscr{B}_t := \sigma\{X_s : 0 \leq s \leq t\}$ of $\mathscr{B}$ generated by X_s, for $s \in [0, t]$. Then $(\mathscr{B}_t)_{t \geq 0}$ forms a filtration of $\mathscr{B}$ and the family of projection maps defines a stochastic process adapted to the filtration $(\mathscr{B}_t)_{t\geq 0}$ forms a filtration of $\mathscr{B}$ and the family of projection maps defines a stochastic process.

Definition 3.1.1 (Canonical process)**.** The *canonical process* on $\mathcal{M}$ is a stochastic process $X : W(\mathcal{M}) \times \mathbb{R}_+ \to \mathcal{M}$ on the filtered measurable space $(W(\mathcal{M}), (\mathscr{B}_t)_{t\geq 0})$.

The following theorem allows the construction of Brownian motions on manifolds using stochastic differential equations.

Theorem 3.1.2. *For each Borel probability measure μ on $\mathcal{M}$, there is a unique Borel probability measure $\mathbb{P} = \mathbb{P}_\mu$ on $W(\mathbb{R}^N)$ such that the canonical process X on the filtered probability space $(W(\mathbb{R}^N), (\mathscr{B}(W(\mathbb{R}^N))_t)_{t\geq 0}, \mathbb{P})$ is a solution of the stochastic differential equation*

$$dX_t^\alpha = \widetilde{\sigma}^{\alpha\beta}(X_t)dW_t^\beta + \widetilde{b}^\alpha(X_t)dt,$$

with initial distribution $\mathbb{P} \circ X_0^{-1} = \mu$, where $W_t = (W_t^1, \ldots, W_t^N)$ be an N-dimensional Euclidean Brownian motion. Moreover, $X_t \in \mathcal{M}$, $\mathbb{P}$-a.e.

Due to the last statement, we consider X as the canonical process on the filtered measurable space $(W(\mathcal{M}), (\mathscr{B}_t)_{t\geq 0})$ with a probability measure $\mathbb{P}_\mu|_{W(\mathcal{M})}$ which we also denote by $\mathbb{P}_\mu$.

Definition 3.1.3 (Brownian motion)**.** The *Brownian motion* on $\mathcal{M}$ with initial distribution μ is the canonical process X on $(W(\mathcal{M}), (\mathscr{B}_t)_{t\geq 0}, \mathbb{P}_\mu)$.

For each $x \in \mathcal{M}$ we denote the probability measure $\mathbb{P}_{\delta_x}$ whose initial distribution is the point mass δ_x at x by $\mathbb{P}_x$. By Itô's formula, for any $f \in \mathcal{C}^\infty(\mathcal{M})$,

$$M_t^f := f(X_t) - f(X_0) - \int_0^t \Delta_{\mathcal{M}} f(X_s)ds \tag{3.1}$$

is a semi-martingale defined up to a stopping time $e(X)$. For $f, h \in \mathcal{C}^\infty(\mathcal{M})$, the quadratic variation of two semi-martingales M^f and M^h is

$$\langle M^f, M^h\rangle_t = [\Delta_{\mathcal{M}}(fh) - h\Delta_{\mathcal{M}} f - f\Delta_{\mathcal{M}} h](X_t) = \langle \nabla f, \nabla h\rangle(X_t). \tag{3.2}$$

We denote by $\mathcal{D}^{1,2}(\mathbb{R}_+, \mathcal{M})$ the space of continuous functions $u : \mathbb{R}_+ \times \mathcal{M}$ such that $u(t, x)$ is $\mathcal{C}^1$ in $t \in (0, \infty)$ and $\mathcal{C}^2$ in $x \in \mathcal{M}$. We define differential operators on $\mathcal{D}^{1,2}(\mathbb{R}, \mathcal{M})$ by

$$\partial_t u(t,x) = \frac{\partial}{\partial t} u(t,x),$$
$$\Delta_x u(t,x) = \Delta_{\mathcal{M}}[x \mapsto u(t,x)].$$

We denote by $L_{\mathcal{M}} u = \partial_t u - \Delta u$.

Definition 3.1.4 (Heat kernel)**.** A *heat kernel* $\wp$ on $\mathcal{M}$ is a fundamental solution of $L_{\mathcal{M}}$; $\wp : \mathbb{R} \times \mathcal{M} \times \mathcal{M}$ such that

$$\partial_t \wp(t,x,y) - \Delta_x \wp(t,x,y) = 0,$$
$$\lim_{t \to 0} \int_{\mathcal{M}} f(y) \wp(t,\cdot,y) d\,\mathrm{vol}(y) = f(\cdot),$$

where the last limit is in the space $\mathcal{C}_b(\mathcal{M})$ of bounded continuous functions equipped with compact-open topology, for each $f \in \mathcal{C}_b(\mathcal{M})$.

The following theorem guarantees the existence of heat kernels.

Theorem 3.1.5. *(Theorem 4.1.4, 4.1.5 and 4.1.6 of [Hs]) There exists a heat kernel $\wp_{\mathcal{M}} \in \mathcal{C}^\infty((0,\infty) \times \mathcal{M} \times \mathcal{M})$ such that*

1. $\wp_{\mathcal{M}}(t,x,y) = \wp_{\mathcal{M}}(t,y,x)$*;*
2. *(Champman-Kolomogorov equation)*

$$\wp(t,x,y) = \int_{\mathcal{M}} \wp_{\mathcal{M}}(t,x,z) \wp_{\mathcal{M}}(t,z,y) d\,\mathrm{vol}(z); \tag{3.3}$$

3. $\int_{\mathcal{M}} \wp_{\mathcal{M}}(t,x,y) d\,\mathrm{vol}(y) \leq 1$*;*
4. *It is minimal in the sense that for any heat kernel $\wp$ on $\mathcal{M}$, $\wp_{\mathcal{M}}(t,x,y) \leq \wp(t,x,y)$.*

Furthermore, it is the transition density of Brownian motion on $\mathcal{M}$: for every $O \in \mathscr{B}(\mathcal{M})$ and $t > 0$,

$$\mathbb{P}_x[X_t \in O, t < e] = \int_O \wp(t,x,y) d\,\mathrm{vol}(y).$$

Before we prove the theorem, we need to solve the Dirichlet problem for the heat equation on a bounded domain with smooth boundary of $\mathcal{M}$.

Proposition 3.1.6. *If $D \subset \mathcal{M}$ is a relatively compact domain with smooth boundary, then there is a continuous function $\wp_D(t,x,y)$ on $(0,\infty) \times \overline{D} \times \overline{D}$ which is $\mathcal{C}^\infty$ on $(0,\infty) \times D \times D$ such that*

1. *$\partial_t \wp_D(t,x,y) = \Delta_x \wp_D(t,x,y)$ and for every $f \in \mathcal{C}_b(D)$,*

$$u_f(t,y) := \int_D \wp_D(t,x,y) d\,\mathrm{vol}(x)$$

is the unique solution of the Dirichlet problem for heat equation on D: $L_{\mathcal{M}} u_f = 0$ on $(0,\infty) \times \overline{D}$, $u_f = 0$ on $(0,\infty) \times \partial D$ and $\lim\limits_{t \to 0} u_f(t,x) = f(x)$;

2. *$\wp_D(t,x,y) = \wp_D(t,x,y)$ for each $x, y \in D$ and $\wp(t,x,y) = 0$ if $y \in \partial D$;*
3. *$\wp_D(s+t,x,y) = \int_D \wp_D(s,x,z)\wp_D(z,y) d\,\mathrm{vol}(z)$;*
4. *$\int_D \wp_D(t,x,y) d\,\mathrm{vol}(y) \leq 1$ and the inequality is strict if $\mathcal{M} \setminus \overline{D} \neq \emptyset$.*

Note that for every bounded domain D of $\mathcal{M}$, $\Delta_D := \Delta_{\mathcal{M}} : L^2(D) \to L^2(D)$ is a closed self-adjoint operator with discrete spectrum $\mathrm{Spec}(\Delta_D)$ which consists of eigenvalues with finite multiplicity. If we write

$$\mathrm{Spec}(\Delta_D) = \{\lambda_0^D = 0 \leq \lambda_1^D \leq \cdots \leq \lambda_k^D \leq \cdots\}$$

where eigenvalues are repeated with its multiplicity, then there are eigenfunctions $\phi_k^D \in \mathcal{C}^\infty(D)$ of λ_k^D with $\phi_k^D = 0$ on ∂D which form an orthonormal basis of $L^2(D)$. Then we have the Sturm-Liouville decompostion of the Dirichlet heat kernel

$$\wp_D(t,x,y) = \sum_{k=0}^{\infty} e^{-\lambda_k^D t} \phi_k^D(x) \phi_k^D(y),$$

where the series in the Right-handed side converges uniformly on compact sets. We recommand [C] for details.

Proof of Theorem 3.1.5. Given a relatively compact smooth domain D of $\mathcal{M}$, we denote by τ_D the first exit time of Brownian motion from D:

$$\tau_D := \inf\{t : X_t \notin D\}.$$

Lemma 3.1.7. *(Proposition 4.1.3 of [Hs]) For every Borel set C in D,*

$$\mathbb{P}_x[X_t \in C, t < \tau_D] = \int_C \wp_D(t,x,y) d\,\mathrm{vol}(y).$$

Let $\{D_n\}_{n=1}^{\infty}$ be a relatively compact smooth exhaustion of $\mathcal{M}$, i.e., each D_n is a relatively compact domain with smooth boundary, $\overline{D_n} \subset D_n$ and $\cup D_n = \mathcal{M}$. For every nonnegative function $f \in \mathcal{C}(\mathcal{M})$,

$$\begin{aligned}&\int_{\mathcal{M}} \left(\wp_{D_{n+1}}(t,x,y) - \wp_{D_n}(t,x,y)\right) f(y) d\operatorname{vol}(y)\\ &= \mathbb{P}_x[f(X_t) : \tau_{D_n} \leq t < \tau_{D_{n+1}}] \geq 0.\end{aligned}$$

Thus $\wp_{D_{n+1}} \geq \wp_{D_n}$ and we define

$$\wp_{\mathcal{M}}(t,x,y) = \lim_{n\to\infty} \wp_{D_n}(t,x,y).$$

This limit is independent of the choice of the exhaustion $\{D_n\}$. □

We say that $\mathcal{M}$ is *stochastically complete* if $\mathbb{P}_x[e = \infty] = 1$ for every $x \in \mathcal{M}$. It means that every Brownian path does not explode in finite time. One can verify that $\mathcal{M}$ is stochastically complete if and only if for any $(t,x) \in \mathbb{R}_+ \times \mathcal{M}$,

$$\int_{\mathcal{M}} \wp_{\mathcal{M}}(t,x,y) d\operatorname{vol}(y) = 1.$$

Note that the minimal heat kernel $\wp_{\mathcal{M}}$ is the unique heat kernel if $\mathcal{M}$ is stochastically complete.

Definition 3.1.8 (Recurrence and transience)**.**

1. A subset K of $\mathcal{M}$ is called recurrent if for every $x \in \mathcal{M}$,

   $$\mathbb{P}_x[\omega : \exists t_n \uparrow e(\omega) \text{ such that } \omega_{t_n} \in K] = 1;$$

   It is called transient if for every $x \in \mathcal{M}$,

   $$\mathbb{P}_x[\omega : \exists T < e(\omega) \text{ such that } \forall t > T,\, \omega_t \notin K] = 1.$$

2. Brownian motion on $\mathcal{M}$ is called recurrent if every nonempty open set is recurrent and is called transient if every compact set is transient.

We characterize recurrence and transience of Brownian motion on $\mathcal{M}$ using the Green function $\mathrm{G}(x,y) := \int_0^\infty \wp(t,x,y)dt$ of $\mathcal{M}$.

Proposition 3.1.9. *(Proposition 4.4.8 of [Hs]) If Brownian motion on $\mathcal{M}$ is recurrent, then $\mathrm{G}_{\mathcal{M}}(x, y) = \infty$ for all $x, y \in \mathcal{M}$. If Brownian motion is transient, then $\mathrm{G}_{\mathcal{M}}$ is locally integrable.*

3.2 Brownian motions in negative curvature

In this section, we study Brownian motion on a Cartan-Hadamard manifold and finite-volume manifold with pinched negative curvature. We begin with the stochastic completeness of Brownian motion on a Cartan-Hadamard manifold.

Proposition 3.2.1. *A Cartan-Hadamard manifold $\widetilde{\mathcal{M}}$ with $-b^2 \leq \mathrm{Sec}_{\widetilde{\mathcal{M}}} \leq 0$ for some $b > 0$ is stochastically complete.*

Let $\mathcal{M}$ be a finite-volume manifold with $-b^2 \leq \mathrm{Sec}_{\mathcal{M}} \leq -a^2$ for some $b > a > 0$ with universal cover $\mathrm{p} : \widetilde{\mathcal{M}} \to \mathcal{M}$. Then $\widetilde{\mathcal{M}}$ is a Cartan-Hadamard manifold with pinched negative curvature. The deck transformation action on $\widetilde{\mathcal{M}}$ of the fundamental group $\Gamma = \pi_1(\mathcal{M})$ is an isometric action. Let $\mathcal{M}_0 \subset \widetilde{\mathcal{M}}$ be a fundamental domain of Γ-action on $\widetilde{\mathcal{M}}$. We identify $\mathcal{M}_0$ with $\mathcal{M}$.

Since Brownian motion is stochastically complete, we may assume that our sample path spaces are $\widetilde{\Omega} = \mathcal{C}(\mathbb{R}_+, \widetilde{\mathcal{M}})$ and $\Omega = \mathcal{C}(\mathbb{R}_+, \mathcal{M})$ instead of $W(\widetilde{\mathcal{M}})$ and $W(\mathcal{M})$.

For $\kappa < 0$, if $\wp_\kappa(t, x, y) = \wp_{\mathbb{H}^d(\kappa)}(t, x, y)$ is the heat kernel on the d-dimensional hyperbolic space $\mathbb{H}^d(\kappa)$ of constant curvature κ, $\wp_\kappa(t, x, y)$ depends only on t and $\rho = \mathrm{d}_{\mathbb{H}^d(\kappa)}(x, y)$:

$$\wp_\kappa(t, \rho) \sim \frac{(1 + \rho + t)^{\frac{d-1}{2} - 1}(1 + \rho)}{t^{d/2}} \exp\left(-\frac{\rho^2}{4t} - \frac{(d-1)\kappa\rho}{2} - \frac{(d-1)^2\kappa^2 t}{4}\right),$$

where $f \sim g$ means that there exists $c > 1$ such that $c^{-1} \leq \left|\frac{f(s)}{g(s)}\right| \leq c$ for each s [DM].

Due to the pinched negative curvature, we have a comparison theorem of the heat kernel on $\widetilde{\mathcal{M}}$.

Proposition 3.2.2. *(Heat kernel comparison theorem, Theorem 4.5.2 of [Hs])*

$$\wp_{-b^2}(t, \mathrm{d}(x, y)) \leq \wp_{\widetilde{\mathcal{M}}}(t, x, y) \leq \wp_{-a^2}(t, \mathrm{d}(x, y)).$$

By the volume comparison and the heat kernel comparison, we have a super-exponential moment of heat kernels. For any $\tau > 0$,

$$\begin{aligned}
&\int_{\widetilde{\mathcal{M}}} e^{\tau r_x(y)} \wp_{\widetilde{\mathcal{M}}}(t,x,y) d\operatorname{vol}_{\widetilde{\mathcal{M}}}(y) \\
&\leq \int_{\mathbb{H}^d(-b^2)} e^{\tau r} \wp_{-a^2}(t,r) d\operatorname{vol}_{-b^2}(r,\mathrm{v}) \\
&\leq C(d,a,t) \int_{\mathbb{S}^{d-1}} \int_0^\infty r^{\frac{d-1}{2}} \exp\left[-\frac{r^2}{4t} + \left(\frac{(d-1)a^2}{2} + \tau\right) r\right] \operatorname{sn}_{-b^2}^{d-1}(r) dr d\operatorname{vol}_{\mathbb{S}^{d-1}}(\mathrm{v}) \\
&\leq C'(d,a,b,t).
\end{aligned}$$

As a corollary of the heat kernel comparison, we have the finite Green function on $\widetilde{\mathcal{M}}$. Thus it follows from Proposition 3.1.9 that Brownian motion on $\widetilde{\mathcal{M}}$ is transient.

For each $x \in \mathcal{M}$ and its lift $\widetilde{x} \in \widetilde{\mathcal{M}}$, we also denote the push-forward measure of $\mathbb{P}_{\widetilde{x}}$ by $\mathbb{P}_x$: for each $U \subset \mathcal{M}$, if $\widetilde{U} := \mathrm{p}^{-1} U$, then

$$\mathbb{P}_x[\omega_t \in U] := \mathbb{P}_{\widetilde{x}}[\widetilde{\omega}_t \in \widetilde{U}].$$

Then the canonical process of $(\Omega, (\mathscr{F}_t(\mathcal{M}))_{0 \leq t \leq \infty}, \mathbb{P}_x)$ is the Brownian motion on $\mathcal{M}$ started from x. Note that the heat kernel $\wp_{\mathcal{M}}$ on $\mathcal{M}$ is

$$\wp_{\mathcal{M}}(t, \mathrm{p}(\tilde{x}), \mathrm{p}(\tilde{y})) = \sum_{\gamma \in \Gamma} \wp_{\widetilde{\mathcal{M}}}(t, \tilde{x}, \gamma \tilde{y}),$$

for each $\tilde{x}, \tilde{y} \in \widetilde{\mathcal{M}}$.

The stationary measure of the Brownian motion is the probability measure which defines the Brownian motion with initial distribution $\mathrm{m} = \frac{1}{\operatorname{vol}(\mathcal{M})} \operatorname{vol}$: for $\omega \in \Omega$, we define a shift map $\mathscr{S}^t \omega_s = \omega_{t+s}$, then the measure

$$\mathbb{P}_{\mathrm{m}} = \int_{\mathcal{M}} \mathbb{P}_x \, d\,\mathrm{m}(x)$$

is invariant under the shift map: by Champman-Kolomogorov equation (3.3),

$$\begin{aligned}
\mathbb{P}_{\mathrm{m}}[\mathscr{S}^{-t}\{\omega : \omega_s \in U\}] = \mathbb{P}_{\mathrm{m}}[\omega_{t+s} \in U] &= \int_{\mathcal{M}} \int_{\widetilde{\mathcal{M}}} \wp_{\widetilde{\mathcal{M}}}(t+s,x,y) \mathbf{1}_{\widetilde{U}}(y) dy d\,\mathrm{m}(x), \\
&= \int_{\mathcal{M}} \int_{\widetilde{\mathcal{M}}} \int_{\widetilde{\mathcal{M}}} \wp_{\widetilde{\mathcal{M}}}(t,x,z) \wp_{\widetilde{\mathcal{M}}}(s,z,y) \mathbf{1}_{\widetilde{U}}(y) dy dz d\,\mathrm{m}(x),
\end{aligned}$$

hence using Fubini type argument,

$$\begin{aligned}\mathbb{P}_{\mathrm{m}}[\mathscr{S}^{-t}\{\omega : \omega_s \in U\}] &= \int_{\mathcal{M}} \int_{\widetilde{\mathcal{M}}} \left[\int_{\widetilde{\mathcal{M}}} \wp_{\widetilde{\mathcal{M}}}(t, x, z) dx \right] \wp_{\widetilde{\mathcal{M}}}(s, z, y) \mathbf{1}_{\widetilde{U}}(y) dy d\,\mathrm{m}(z) \\ &= \int_{\mathcal{M}} \int_{\widetilde{\mathcal{M}}} \wp_{\widetilde{\mathcal{M}}}(s, z, y) dy d\,\mathrm{m}(z) \\ &= \mathbb{P}[\omega_s \in U].\end{aligned}$$

The shift dynamical system on the path space $(\Omega, \mathscr{S}^t, \mathbb{P}_{\mathrm{m}})$ is ergodic since $\mathcal{M}$ is connected.

Let $r(\omega, t) = \mathrm{d}(\widetilde{\omega}_0, \widetilde{\omega}_t)$ where $\widetilde{\omega}$ is a lift of ω. Since the Brownian motion on $\widetilde{\mathcal{M}}$ is transient, $r(\omega, t) \to \infty$ as $t \to \infty$, $\mathbb{P}_x$-a.e. Moreover, the rate of escape is linear in t, $\mathbb{P}_x$-a.e.

Definition 3.2.3 (Linear drift)**.** There exists a positive constant ℓ, called *the linear drift* of the Brownian motion, such that for every $x \in \widetilde{\mathcal{M}}$ and for $\mathbb{P}_{\mathrm{p}(x)}$-a.s. $\omega \in \Omega$ or $\mathbb{P}_x$-a.s. $\widetilde{\omega} \in \widetilde{\Omega}$,

$$\ell = \lim_{t\to\infty} \frac{1}{t} r(\omega, t) = \lim_{t\to\infty} \frac{1}{t} \mathrm{d}(x, \widetilde{\omega}_t). \tag{3.4}$$

Since r is a sub-additive cocycle, that is, $r(\omega, t + s) \le r(\omega, t) + r(\mathscr{S}^t\omega, s)$ for every $s, t > 0$, by the subadditive ergodic theorem ([Ki]) the limit (3.4) exists and is identical almost everywhere.

By Itô's formula for the distance function $r(y) = r_x(y) := \mathrm{d}(x, y)$, the radial process $r_x(\widetilde{\omega}_t) = \mathrm{d}(x, \widetilde{\omega}_t)$ is written as

$$r_x(\widetilde{\omega}_t) = r_x(\widetilde{\omega}_0) + \beta_t + \int_0^t \Delta r_x(\widetilde{\omega}_s) ds,$$

where $\beta_t = M_t^{r_x}$ is the 1-dimensional Euclidean Brownian motion. Hence, for $\mathbb{P}_x$-a.e. $\widetilde{\omega}$, we have

$$\ell = \lim_{t\to\infty} \frac{1}{t} \int_0^t \Delta_{\mathcal{M}} r_x(\widetilde{\omega}_s) ds.$$

Thus it follows from the Laplacian comparison theorem 2.3 that

$$(d-1)a \le \ell \le (d-1)b.$$

In particular, if $\mathcal{M}$ has constant negative curvature $-a^2$, then $\ell = (d-1)a$.

For $\widetilde{\omega} \in \widetilde{\Omega}$, we define the angular process $\theta_x(\widetilde{\omega}, t)$ by the angular part of $\widetilde{\omega}_t$

in the polar coordinate at x, i.e., the unit vector in $\mathcal{S}_x\widetilde{\mathcal{M}}$ such that

$$\exp_x\left[r_x(\widetilde{\omega}_t)\theta_x(\widetilde{\omega},t)\right] = \widetilde{\omega}_t.$$

The angular process converges almost surely.

Proposition 3.2.4. *([Pr], [Pi]) For every $x \in \widetilde{\mathcal{M}}$ and $\mathbb{P}_x$-a.e. $\widetilde{\omega}$, the limit $\lim_{t\to\infty}\theta(\widetilde{\omega},t)$ exists.*

Proof. Consider a geodesic triangle in $\widetilde{\mathcal{M}}$ with vertices x, $\widetilde{\omega}_j$ and $\widetilde{\omega}_{j+1}$ for a positive integer j. If we denote by ϕ_j the angle of the triangle at x and the radial process $r_x(\widetilde{\omega}_t)$ by r_t, then by comparison,

$$1 - \cos\phi_j \leq \frac{\cosh(a\mathrm{d}(\widetilde{\omega}_j,\widetilde{\omega}_{j+1}))}{\sinh(ar_j)\sinh(ar_{j+1})}.$$

Note that by Itô's formula,

$$\mathrm{d}(\widetilde{\omega}_j,\widetilde{\omega}_{j+1}) = \beta_{j+1} - \beta_j + \int_j^{j+1} \Delta r_x(\widetilde{\omega}_s)ds$$

where β_t is a 1-dimensional Euclidean Brownian motion. Since $\beta_{j+1} - \beta_j \leq (\log j)^{\frac{1}{2}}$ and the integral term is uniformly bounded by Laplacian comparison, if j is sufficiently large, then

$$\begin{aligned} 1 - \cos\phi_j &\leq \frac{C(\log j)^{\frac{1}{2}}}{\sinh\left(\frac{(d-1)a^2 j}{2}\right)\sinh\left(\frac{(d-1)a^2(j+1)}{2}\right)} \\ &\leq C(\log j)^{\frac{1}{2}}\exp\left(-(d-1)a^2 j\right) \end{aligned}$$

for some positive constant $C > 0$. Thus we have $\sum_{j=1}^{\infty}(1-\cos\phi_j) < \infty$. If $\mathrm{d}_{\mathcal{T}_x\widetilde{\mathcal{M}}}$ is the distance on the tangent space $\mathcal{T}_x\widetilde{\mathcal{M}}$, then since

$$\mathrm{d}_{\mathcal{T}_x\widetilde{\mathcal{M}}}(\theta_x(\widetilde{\omega},j),\theta_x(\widetilde{\omega},j+1)) \leq 1 - \cos\phi_j,$$

we have

$$\sum_{j=1}^{\infty}\mathrm{d}_{\mathcal{T}_x\widetilde{\mathcal{M}}}(\theta_x(\widetilde{\omega}_j),\theta_x(\widetilde{\omega}_{j+1})) < \infty.$$

Due to a similar argument, since

$$\sum_{j=1}^{\infty} \mathbb{P}_x \left[\sup_{j \le t \le j+1} \mathrm{d}(\widetilde{\omega}_t, \widetilde{\omega}_j) \ge C(\log j)^{\frac{1}{2}} \right] < \infty,$$

it follows from the Borel-Cantelli lemma that for $\mathbb{P}_x$-a.s. ω,

$$\lim_{j \to \infty} \sup_{j \le t \le j+1} \mathrm{d}_{\mathcal{T}_x \widetilde{\mathcal{M}}}(\theta_x(\widetilde{\omega}, t), \theta_x(\widetilde{\omega}, j+1)).$$

Therefore, $\theta_x(\widetilde{\omega}, t)$ converges a.s. $\square$

Since $r(\widetilde{\omega}, t) \to \infty$ as $t \to \infty$ for $\mathbb{P}_x$-a.e. $\widetilde{\omega}$, the limit $\widetilde{\omega}_\infty := \lim_{t \to \infty} \widetilde{\omega}_t$ exists for $\mathbb{P}_x$-a.e. $\widetilde{\omega}$. In addition, the Brownian path roughly follows the geodesic $\gamma_{\theta(\widetilde{\omega}, \infty)}$.

Proposition 3.2.5. *(Proposition 5 in [Le2]) For each $x \in \widetilde{\mathcal{M}}$ and $\mathbb{P}_x$-a.e. $\widetilde{\omega}$,*

$$\lim_{t \to \infty} \frac{1}{t} \mathrm{d} \left(\widetilde{\omega}_t, \exp_x \left[r(\widetilde{\omega}, t) \theta_x(\widetilde{\omega}, \infty) \right] \right) = 0.$$

Since we can replace $r(\widetilde{\omega}, t)$ by ℓt, We denote i.e.,

Definition 3.2.6 (Harmonic measure)**.** The *harmonic measure* is the family $(\nu_x)_{x \in \widetilde{\mathcal{M}}}$ of the asymptotic distribution ν_x of Brownian paths starting from x, given by for $U \subset \partial_\infty \widetilde{\mathcal{M}}$,

$$\nu_x(U) := \mathbb{P}_x \left[\widetilde{\omega} : \widetilde{\omega}_\infty \in U \right],$$

called harmonic measure at x for $x \in \widetilde{\mathcal{M}}$.

Since the family $(\mathbb{P}_x)$ is Γ-equivariant, $(\nu_x)_{x \in \widetilde{\mathcal{M}}}$ is also Γ-equivariant: for each $\gamma \in \Gamma$,

$$\gamma_* \nu_x = \nu_{\gamma x}.$$

Moreover, $(\nu_x)_{x \in \widetilde{\mathcal{M}}}$ is an absolutely continuous family.

Definition 3.2.7 (Martin kernel)**.** The *Martin kernel* of the Brownian motion on $\widetilde{\mathcal{M}}$ is a function $\mathfrak{K} : \widetilde{\mathcal{M}} \times \widetilde{\mathcal{M}} \times \partial_\infty \widetilde{\mathcal{M}} \to \mathbb{R}$ defined by the Radon-Nikodym derivative of the harmonic measure; for $x, y \in \widetilde{\mathcal{M}}$ and $\xi \in \partial_\infty \widetilde{\mathcal{M}}$,

$$\mathfrak{K}(x, y, \xi) := \frac{d\nu_y}{d\nu_x}(\xi).$$

The Martin kernel is also characterized by the limiting behavior of the Green function.

Theorem 3.2.8. *([AS]) For each sequence (z_n) in $\widetilde{\mathcal{M}}$ with $z_n \to \xi \in \partial_\infty \widetilde{\mathcal{M}}$,*

$$\mathfrak{K}(x, y, \xi) = \lim_{n\to\infty} \frac{\mathrm{G}(y, z_n)}{\mathrm{G}(x, z_n)}.$$

For $\mathrm{v} \in \mathcal{S}_x \widetilde{\mathcal{M}}$, the cone at v of angle θ is

$$C(\mathrm{v}, \theta) := \{\gamma_{\mathrm{w}}(t) : \mathrm{w} \in \mathcal{S}_x \widetilde{\mathcal{M}} \text{ such that } \angle_x(\mathrm{v}, \mathrm{w}) < \theta\}.$$

We have Hölder continuity of the directional derivative of the Martin kernel on the boundary of a cone.

Proposition 3.2.9. *(Lemma 3.2 in [Ha1]) For $\mathrm{v} \in \mathcal{S}_x \widetilde{\mathcal{M}}$, let $C = C(\mathbf{g}^1 \mathrm{v}, \pi/4)$ be the cone at $\mathbf{g}^1\mathrm{v}$ of angle $\pi/4$ and we denote its boundary at infinity by*

$$\partial C = \{\mathrm{w}^+ : \mathrm{w} \in \mathcal{S}_{\gamma_{\mathrm{v}}(1)} \widetilde{\mathcal{M}} \text{ such that } \angle_{\gamma_{\mathrm{v}}(1)}(\mathbf{g}^1\mathrm{v}, \mathrm{w}) < \pi/4\}.$$

Then the directional derivative of the Martin kernel

$$\xi \mapsto \left.\frac{d}{dt}\right|_{t=0} \mathfrak{K}(\gamma_{\mathrm{v}}(0), \gamma_{\mathrm{v}}(t), \xi)$$

is Hölder continuous on ∂C.

Proof. Let $z = \gamma_{\mathrm{v}}(1)$ and $z_\varepsilon = \gamma_{\mathrm{v}}(\varepsilon)$ for $\varepsilon \in [0, 1/2]$. By Cheng-Yau type Harnack inequality ([CY]), for every $y \in C$ and $\varepsilon \in (0, 1/2]$,

$$\frac{|\,\mathrm{G}(y, z_\varepsilon) - \mathrm{G}(y, x)|}{\varepsilon} \le C_0\, \mathrm{G}(y, x). \tag{3.5}$$

By Theorem 6.2 in [AS], for any $\varepsilon \in (0, 1/2)$, the map

$$\varphi_\varepsilon(y) = \frac{\varepsilon^{-1}|\,\mathrm{G}(y, z_\varepsilon) - \mathrm{G}(y, x)| + C_0\, \mathrm{G}(y, x)}{\mathrm{G}(y, x)}$$

is α-Hölder continuous on C for some $\alpha = \alpha(a, b, d)$, i.e., for each $y, y' \in C$ distant away from z, $|\varphi_\varepsilon(y) - \varphi_\varepsilon(y')| \le \angle_z(y, y')^\alpha \varphi_\varepsilon(z)$. From (3.5), it follows that

$$\begin{aligned}|\varphi_\varepsilon(y)-\varphi_\varepsilon(y')| &= \left|\frac{1}{\varepsilon}\left(\frac{\mathrm{G}(y,z_\varepsilon)}{\mathrm{G}(y,x)}-\frac{\mathrm{G}(y,x)}{\mathrm{G}(y,x)}\right)-\frac{1}{\varepsilon}\left(\frac{\mathrm{G}(y',z_\varepsilon)}{\mathrm{G}(y',x)}-\frac{\mathrm{G}(y',x)}{\mathrm{G}(y',x)}\right)\right| \\ &\le 2C_0\angle_z(y,y')^\alpha.\end{aligned}$$

Letting y and y' tend to ξ and η in ∂C, respectively, then

$$\left|\frac{\mathfrak{K}(x,z_\varepsilon,\xi)-\mathfrak{K}(x,x,\xi)}{\varepsilon}-\frac{\mathfrak{K}(x,z_\varepsilon,\eta)-\mathfrak{K}(x,x,\eta)}{\varepsilon}\right|\le 2C_0\angle_z(\xi,\eta)^\alpha.$$

Thus the proposition follows by $\varepsilon\to 0$. ☐

We introduce another invariant of the Brownian motion called the *stochastic entropy* of the Brownian motion denoted by h.

Definition 3.2.10 (Stochastic entropy)**.** The stochastic entropy of the Brownian motion on $\widetilde{\mathcal{M}}$ is a constant h such that for each $x\in\widetilde{\mathcal{M}}$, $\mathbb{P}_x$-a.e. $\widetilde{\omega}$,

$$\begin{aligned}h &= \lim_{t\to\infty} -\frac{1}{t}\log \wp(t,x,\widetilde{\omega}_t) \\ &= \lim_{t\to\infty} -\frac{1}{t}\log \mathrm{G}(x,\widetilde{\omega}_t).\end{aligned}$$

The stochastic entropy was first introduced by V. Kaimanovich in [Kai1] for co-compact manifolds with negative curvature. Kaimanovich proved that the Poisson boundary is trivial if and only if the stochastic entropy is zero. The argument in [Le3] proving the well-definedness of the stochastic entropy in co-compact manifolds with negative curvature easily extends to manifolds with finite volume. Note that $h=(d-1)^2a^2$ when $\mathrm{Sec}_{\mathcal{M}}=-a^2$.

There is another characterization of the stochastic entropy analogous to the definition of the topological entropy as the exponential growth of dynamically separated sets (see [Kai1], [Le3]).

Proposition 3.2.11. *For $x\in\widetilde{\mathcal{M}}$, $T>0$ and $0<\delta<1$,*

$$h=\lim_{T\to\infty}\frac{1}{T}\log N(x,T,\delta),$$

where $N(x,T,\delta):=\inf\{\mathrm{Card}(E):\mathbb{P}_x[d(\widetilde{\omega}_T,E)\le 1]\ge\delta\}$.

Proof. Fix $\varepsilon > 0$. Let

$$\mathscr{C}_{T,x} := \{\widetilde{\omega}_0 = x, \wp(T, \widetilde{\omega}_0, \widetilde{\omega}_T) \le e^{-T(h-\varepsilon)}\},$$
$$\mathscr{D}_{T,x} := \{\widetilde{\omega} : \mathrm{d}(\widetilde{\omega}_t, \gamma_{\theta(\widetilde{\omega},\infty)}(\ell T)) \le \varepsilon T, \wp(T, x, \gamma_{\theta(\widetilde{\omega},\infty)}(\ell T)) \ge e^{-T(h+\varepsilon)}\}.$$

Choose a sufficiently large T such that $1 - \frac{\delta}{2} \le \mathbb{P}_x(\mathscr{C}_{T,x}) = \mathbb{P}_x[\widetilde{\omega}_T \in \pi_T \mathscr{C}_{T,x}]$. We denote by $\mathbb{E}_x$ the expectation with respect to $\mathbb{P}_x$. For each finite set E such that $\mathbb{P}_x[d(\widetilde{\omega}_T, E) \le 1] \ge \delta$,

$$\begin{aligned}\delta \le \mathbb{E}_x[d(\widetilde{\omega}_T, E) \le 1] &= \mathbb{P}_x[\{d(\widetilde{\omega}_T, E) \le 1\} \cap \mathscr{C}_{T,x}] + \mathbb{P}_x[\{d(\widetilde{\omega}_T, E) \le 1\} \setminus \mathscr{C}_{T,x}] \\ &\le e^{-T(h-\varepsilon)} \sum_{y \in E} \operatorname{vol} B(y,1) + 1 - (1 - \frac{\delta}{2}) \\ &\le C e^{-T(h-\varepsilon)} \operatorname{Card}(E) + \frac{\delta}{2},\end{aligned}$$

where $C = \sup_z \operatorname{vol} B(z,1)$. Thus, $\frac{\delta}{2C} e^{T(h-\varepsilon)} \le \operatorname{Card}(E)$ and we have

$$h \le \lim_{T \to \infty} \frac{1}{T} \log N(x, T, \delta).$$

For the converse inequality, Let E be a minimal set satisfying $d(\widetilde{\omega}_T, E) \le 1$ for every $\widetilde{\omega} \in \mathscr{D}_{T,x}$ and $F \subset \{\gamma_{\theta(\widetilde{\omega},\infty)}(\ell T) : \widetilde{\omega} \in \mathscr{D}_{T,x}\}$ a maximal $\frac{1}{2}$-separated set. Note that $\operatorname{Card}(E) \ge N(x, T, \mathbb{P}_x(\mathscr{D}_{T,x}))$ and $\operatorname{Card}(F) \le C' e^{T(h+\varepsilon)}$. For each $f \in F$,

$$N(f) := \{e \in E : \exists\, \widetilde{\omega} \in \mathscr{D}_{T,x} \text{ s.t. } \mathrm{d}(f, \gamma_{\theta(\widetilde{\omega},\infty)}(\ell T)) \le \frac{1}{2}, \mathrm{d}(\widetilde{\omega}_T, e) \le 1\}.$$

Then $\operatorname{Card} N(f) \le e^{C'' \varepsilon T}$. Therefore, we have

$$N(x, T, \mathbb{P}_x(\mathscr{D}_{T,x})) \le \operatorname{Card}(E) \le e^{C'' \varepsilon T} \operatorname{Card}(F) \le C' e^{T[h + (2 + C'')\varepsilon]}.$$

Given δ, for each T large enough, $N(x, T, \delta) \le N(x, T, \mathscr{D}_{T.,x})$. $\square$

The stochastic entropy is related to the spectral information of $\widetilde{\mathcal{M}}$, the bottom of the spectrum $\lambda_0 := \inf \operatorname{Spec}(\Delta_{\widetilde{\mathcal{M}}})$ of the Laplacian on $\widetilde{\mathcal{M}}$. Note that $\lambda_0 \ge (d-1)^2 a^2/4$ ([M]). In particular, $\lambda_0 = (d-1)^2 a^2/4$ if $\widetilde{\mathcal{M}}$ has constant negative curvature $-a^2$. It was proved in Proposition 3 of [Le4] for co-compact manifolds. The proof is valid for pinched negative curvature and even the co-finiteness is not required.

Proposition 3.2.12. *For any Cartan-Hadamard manifold $\widetilde{\mathcal{M}}$, we have*

$$4\lambda_0 \leq h.$$

Proof. Since $\wp(t,x,y)$ is a solution of the heat equation,

$$\begin{aligned}\wp(t,x,y)\log\wp(t,x,y) &= \int_0^t \frac{\partial}{\partial s}\left(\wp(s,x,y)\log\wp(s,x,y)\right)ds\\ &= \int_0^t (1+\log\wp(s,x,y))\frac{\partial}{\partial s}\wp(s,x,y)ds\\ &= \int_0^t (1+\log\wp(s,x,y))\Delta_y\wp(s,x,y)ds.\end{aligned}$$

By applying this equation,

$$\begin{aligned}h &= \lim_{t\to\infty} -\frac{1}{t}\int_{\widetilde{\mathcal{M}}} \wp(t,x,y)\log\wp(t,x,y)d\,\mathrm{vol}(y)\\ &= \lim_{t\to\infty}\frac{1}{t}\int_0^t\int_{\widetilde{\mathcal{M}}}\langle\nabla\log\wp(s,x,y),\nabla\wp(s,x,y)\rangle_g d\,\mathrm{vol}(y)ds\\ &= \lim_{t\to\infty}\frac{4}{t}\int_0^t\int_{\widetilde{\mathcal{M}}}\left\|\nabla\sqrt{\wp(s,x,y)}\right\|^2 d\,\mathrm{vol}(y)ds\\ &\geq \frac{4}{t}\int_0^t \lambda_0 ds = 4\lambda_0.\end{aligned}$$

The inequality is due to Rayleigh's theorem. □

3.3 Foliated Brownian motions

In this section, we introduce a Markov process on the unit tangent bundle of a finite-volume manifold with pinched negative curvature, called foliated Brownian motion. The foliated Brownian motion was first introduced in the way to develop the ergodic theory of foliations (See [CC], [Ga]). It is also used to prove the central limit theorem for the geodesic flow on hyperbolic manifolds ([LeJ], [EFL]). It plays a crucial role in the proof of the central limit theorem of Brownian motion.

Let $\mathcal{M}$ be a finite-volume manifold of dimension d with $-b^2 \leq \mathrm{Sec}_{\widetilde{\mathcal{M}}} \leq -a^2 < 0$ for some $b > a > 0$. We denote by $\mathrm{p} : \widetilde{\mathcal{M}} \to \mathcal{M}$ its universal cover and by Γ the deck transformation group of the universal cover acting isometrically on $\widetilde{\mathcal{M}}$. Recall that we identify the unit tangent bundle $\mathcal{S}\widetilde{\mathcal{M}}$

with $\widetilde{\mathcal{M}} \times \partial_\infty \widetilde{\mathcal{M}}$ in (2.9). With such identification, we have that for $\mathrm{v} = (x, \xi)$, $\widetilde{\mathcal{W}}^{cs}(\mathrm{v}) = \widetilde{\mathcal{M}} \times \{\xi\}$ and $\nabla_y \mathfrak{b}(y, x, \xi) = (y, \xi)$. Furthermore, we shall identify the unit tangent bundle $\mathcal{SM}$ of $\mathcal{M}$ with $\mathcal{M}_0 \times \partial_\infty \widetilde{\mathcal{M}}$. Since $\widetilde{\mathcal{W}}^{cs}(\mathrm{v})$ is diffeomorphic to $\widetilde{\mathcal{M}}$ via $\pi : \mathcal{S}\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$ for $\mathrm{v} \in \mathcal{S}\widetilde{\mathcal{M}}$, we endow central stable leaves of $\widetilde{\mathcal{W}}^{cs}$ with a metric g_s induced from the metric g on $\widetilde{\mathcal{M}}$: for $\mathrm{v} \in \mathcal{S}_x\widetilde{\mathcal{M}}$, define g_s on $\widetilde{E}^{cs}_{\mathrm{v}}$ from g on $\mathcal{T}_x\widetilde{\mathcal{M}}$.

Let $X : \mathcal{S}\widetilde{\mathcal{M}} \to \widetilde{E}^{cs}$ be a section of the central stable distribution which is leafwise $\mathcal{C}^1$, i.e., the restriction $X|_{\widetilde{\mathcal{W}}^{cs}(x,\xi)}$ is $\mathcal{C}^1$ on $\widetilde{\mathcal{W}}^{cs}(x, \xi)$ for each $(x, \xi) \in \mathcal{S}\widetilde{\mathcal{M}}$. We identify the restriction $X|_{\widetilde{\mathcal{W}}^{cs}(x,\xi)}$ with a $\mathcal{C}^1$-vector field X^ξ on $\widetilde{\mathcal{M}}$ for each ξ; $X^\xi(x) := X(x, \xi)$. We define the g_s-divergence div_s by

$$\operatorname{div}_s X(x, \xi) = \operatorname{div} X^\xi(x).$$

Let $u \in \mathcal{C}(\mathcal{S}\widetilde{\mathcal{M}})$ be a leafwise $\mathcal{C}^2$-function; $u|_{\widetilde{\mathcal{W}}^{cs}(\mathrm{v})}$ is a $\mathcal{C}^2$-function on $\widetilde{\mathcal{W}}^{cs}(\mathrm{v})$ for each v. Thus for each $\xi \in \partial_\infty \widetilde{\mathcal{M}}$, $u^\xi(x) := u(x, \xi)$ is $\mathcal{C}^2$ on $\widetilde{\mathcal{M}}$. We define the *foliated Laplacian* Δ_s by

$$\Delta_s u = \operatorname{div}_s \nabla u,$$

where $\nabla u(x, \xi) := \nabla u^\xi(x)$.

Fix a fundamental domain $\mathcal{M}_0 \subset \widetilde{\mathcal{M}}$ of Γ. We identify $\mathcal{M}$, $\mathcal{SM}$ with $\mathcal{M}_0$, $\mathcal{M}_0 \times \partial\widetilde{\mathcal{M}}$, respectively. Note that the central stable leaf $\widetilde{\mathcal{W}}^{cs}(x, \xi) = \widetilde{\mathcal{M}} \times \{\xi\}$ is projected onto

$$\mathcal{W}^{cs}(x, \xi) := \{(y, \gamma^{-1}\xi) \in \mathcal{SM} : y \in \mathcal{M}_0, \gamma \in \Gamma\}.$$

The central stable foliation $\mathcal{W}^{cs} = \{\mathcal{W}^{cs}(\mathrm{v}) : \mathrm{v} \in \mathcal{SM}\}$ of $\mathcal{SM}$ is the collection of the projected stable leaves. Similarly, we define the central stable distribution E^{cs} of $\mathcal{SM}$. The central stable leaves of $\mathcal{W}^{cs}$ inherit the Riemannian metric from g_s on the leaves of $\mathcal{W}^{cs}$ which is also denoted by g_s. We denote the inherited differentials by div_s and Δ_s.

Definition 3.3.1 (Foliated Brownian motion)**.** Let $\mathcal{P}(\mathcal{SM})$ be the space of probability measures on $\mathcal{SM}$. We define a transition semigroup $\mathbf{P} : (0, \infty) \times \mathcal{SM} \to \mathcal{P}(\mathcal{SM})$ by

$$d\,\mathbf{P}[t, \mathrm{v}](\mathrm{w}) = \sum_{\gamma \in \Gamma} \wp(t, x, \gamma y)\, d\delta_{\gamma^{-1}\xi}(\eta)\, d\operatorname{vol}|_{\mathcal{M}_0}(y),$$

for $\mathrm{v} = (x, \xi), \mathrm{w} = (y, \eta) \in \mathcal{SM}$. The transition semigroup defines a unique family $\{\mathbb{P}_{(x,\xi)}\}_{(x,\xi)\in\mathcal{SM}}$ of Borel probability measures on the space

$$\mathcal{S}\Omega := \mathcal{C}(\mathbb{R}_+, \mathcal{SM})$$

of sample paths on $\mathcal{SM}$. The canonical filtration is the collection of the smallest σ-algebras

$$\mathscr{F}_t = \mathscr{F}_t(\mathcal{SM}) := \sigma\{\pi_s : 0 \le s \le t\}$$

for which the projection $\pi_s(\omega) = \omega_s$ on $\mathcal{S}\Omega$ is measurable for each $0 \le s \le t$. The canonical process $Z_t(\omega) = \omega_t$ of the filtered space

$$\left(\mathcal{S}\Omega, \{\mathscr{F}_t\}_{0\le t\le\infty}, \mathbb{P}_{(x,\xi)}\right)$$

is a Markov process which is called the *foliated Brownian motion* for $\mathcal{W}^{cs}$ with initial distribution $\delta_{(x,\xi)}$, for each $(x, \xi) \in \mathcal{SM}$.

We define the Markov operator $\mathcal{Q}^t : \mathcal{C}_b(\mathcal{SM}) \to \mathcal{C}_b(\mathcal{SM})$ on the space of bounded continuous functions on $\mathcal{SM}$ by

$$\mathcal{Q}^t f(\mathrm{v}) := \int_{\mathcal{SM}} f d\,\mathbf{P}[t, \mathrm{v}] = \sum_{\gamma\in\Gamma} \int_{\mathcal{M}_0} f(y, \gamma^{-1}\xi)\wp(t, x, \gamma y) d\,\mathrm{vol}(y). \tag{3.6}$$

Note that the foliated Brownian motion for $\mathcal{W}^{cs}$ is the projected process of a Markov process, called the foliated Brownian motion for $\widetilde{\mathcal{W}}^{cs}$, with the transition semigroup

$$d\,\widetilde{\mathbf{P}}[t, \mathrm{v}](\mathrm{w}) = \wp(t, x, y) d\delta_\xi(\eta) d\,\mathrm{vol}(y). \tag{3.7}$$

Let $\widetilde{\mathcal{Q}}^t$ be the Markov operator on $\mathcal{S}\widetilde{\mathcal{M}}$. For any $f \in \mathcal{C}_b(\mathcal{SM})$ and for each $(x, \xi) \in \mathcal{M}_0 \times \partial\widetilde{\mathcal{M}}$,

$$\mathcal{Q}^t f(x, \xi) = \int_{\widetilde{\mathcal{M}}} \widetilde{f}(y, \xi)\wp(t, x, y) d\,\mathrm{vol}(y) = \widetilde{\mathcal{Q}}^t \widetilde{f}(x, \xi),$$

where $\widetilde{f}$ is the Γ-invariant lift of f to $\mathcal{S}\widetilde{\mathcal{M}}$. Note that the infinitesimal generator of the Markov operator is the foliated Laplacian:

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{Q}^t f = \Delta_s f.$$

L. Garnett proved in [Ga] that the Markov operator $\widetilde{\mathcal{Q}}$ admits an invariant measure $\mathrm{m}^{\widetilde{\mathcal{Q}}}$ on $\mathcal{S}\widetilde{\mathcal{M}}$ of the form

$$d\mathrm{m}^{\widetilde{\mathcal{Q}}}(x,\xi) = d\nu_x(\xi) d\widetilde{\mathrm{m}}(x) = \mathfrak{K}(x_0, x, \xi) d\widetilde{\mathrm{m}}(x) d\nu_{x_0}(\xi),$$

where $\widetilde{m} = \frac{1}{\mathrm{vol}(\mathcal{M}_0)} \mathrm{vol}$ and ν_x is the harmonic measure. We have an induced probability measure $\mathrm{m}^{\mathcal{Q}} := \mathrm{m}^{\widetilde{\mathcal{Q}}}|_{\mathcal{M}_0 \times \partial \widetilde{\mathcal{M}}}$ on $\mathcal{S}\mathcal{M}$. By Γ-equivariance of ν_x,

$$\begin{aligned}\int_{\mathcal{S}\mathcal{M}} \mathcal{Q}^t f d\mathrm{m}^{\mathcal{Q}} &= \frac{1}{\mathrm{vol}(\mathcal{M}_0)} \int_{\mathcal{M}_0} \int_{\partial\widetilde{\mathcal{M}}} \sum_\gamma \int_{\mathcal{M}_0} \widetilde{f}(y, \gamma^{-1}\xi) \wp(t, x, \gamma y) d\,\mathrm{vol}(y) d\nu_x(\xi) d\,\mathrm{vol}(x) \\ &= \frac{1}{\mathrm{vol}(\mathcal{M}_0)} \int_{\mathcal{M}_0} \sum_\gamma \int_{\mathcal{M}_0} \int_{\partial\widetilde{\mathcal{M}}} \widetilde{f}(y, \xi) \wp(t, \gamma^{-1}x, y) d\nu_{\gamma^{-1}x}(\xi) d\,\mathrm{vol}(x) d\,\mathrm{vol}(y).\end{aligned}$$

Since we know $d\nu_{\gamma^{-1}x}(\xi) = \mathfrak{K}(y, \gamma^{-1}x, \xi) d\nu_y(\xi)$, the integrand in the right-handed side is:

$$\begin{aligned}&\sum_\gamma \int_{\mathcal{M}_0} \int_{\partial\widetilde{\mathcal{M}}} \widetilde{f}(y,\xi) \wp(t, \gamma^{-1}x, y) d\nu_{\gamma^{-1}x}(\xi) d\,\mathrm{vol}(x) \\ &= \int_{\partial\widetilde{\mathcal{M}}} \widetilde{f}(y,\xi) \sum_\gamma \int_{M_0} \wp(t, \gamma^{-1}x, y) \mathfrak{K}(y, \gamma^{-1}x, \xi) d\,\mathrm{vol}(x) d\nu_y(\xi) \\ &= \int_{\partial\widetilde{\mathcal{M}}} \widetilde{f}(y,\xi) \int_{\widetilde{\mathcal{M}}} \wp(t, x, y) \mathfrak{K}(y, x, \xi) d\,\mathrm{vol}(x) d\nu_y(\xi) \\ &= \int_{\partial\widetilde{\mathcal{M}}} \widetilde{f}(x,\xi) d\nu_y(\xi).\end{aligned}$$

We used the harmonicity of the Martin kernel in the last equality:

$$\int_{\widetilde{\mathcal{M}}} \wp(t, x, y) \mathfrak{K}(y, x, \xi) d\,\mathrm{vol}(x) = \mathfrak{K}(y, y, \xi) = 1.$$

Therefore, we have the $\mathcal{Q}^t$-invariance of $\mathrm{m}^{\mathcal{Q}}$. The stationary measure of the foliated Brownian motion is $\mathbb{P}_{\mathrm{m}^{\mathcal{Q}}} = \int_{\mathcal{S}\mathcal{M}} \mathbb{P}_{(x,\xi)} d\mathrm{m}^{\mathcal{Q}}(x,\xi)$ and is ergodic for the shift map on $\mathcal{S}\Omega$.

We have an integral expression of the linear drift and the stochastic entropy. Proposition 2.9 and 2.16 in [LS2] prove the same descriptions for the Brownian motion on co-compact negatively curved manifolds. The identities for co-finite manifolds follow in the same way.

Proposition 3.3.2. *The linear drift and the stochastic entropy have integral* *expressions:*

$$\begin{aligned}\ell &= \int_{\mathcal{M}_0}\int_{\partial\widetilde{\mathcal{M}}} \Delta_y \mathfrak{b}(y,x,\xi) d\nu_y(\xi) d\widetilde{\mathrm{m}}(y) \\ &= \int_{\mathcal{M}_0}\int_{\partial\widetilde{\mathcal{M}}} \langle -\nabla_y \mathfrak{b}(y,x,\xi), \nabla_y \log \mathfrak{K}(x,y,\xi)\rangle_g \, d\nu_y(\xi)\, d\widetilde{\mathrm{m}}(y),\end{aligned}$$

and

$$h = \int_{\mathcal{M}_0}\int_{\partial\widetilde{\mathcal{M}}} |\nabla_y \log \mathfrak{K}(x,y,\xi)|^2 \, d\nu_y(\xi)\, d\widetilde{\mathrm{m}}(y).$$

Moreover, $\ell^2 \leq h$.

Proof. We only verify the second equality. The other equalities follow immediately from the same argument as in [LS2].

$$\begin{aligned}\ell &= \int_{\mathcal{M}_0}\int_{\partial\widetilde{\mathcal{M}}} \Delta_y \mathfrak{b}(y,x,\xi) d\nu_y(\xi) d\widetilde{\mathrm{m}}(y) \\ &= \int_{\partial\widetilde{\mathcal{M}}}\int_{\mathcal{M}_0} \Delta_y \mathfrak{b}(y,x,\xi)\mathfrak{K}(x,y,\xi) d\widetilde{\mathrm{m}}(y) d\nu_x(\xi) \\ &= \int_{\partial\widetilde{\mathcal{M}}}\int_{\mathcal{M}_0} \langle -\nabla_y \mathfrak{b}(y,x,\xi), \nabla_y \mathfrak{K}(x,y,\xi)\rangle_g d\widetilde{\mathrm{m}}(y) d\nu_x(\xi) \\ &= \int_{\mathcal{M}_0}\int_{\partial\widetilde{\mathcal{M}}} \langle -\nabla_y \mathfrak{b}(y,x,\xi), \nabla_y \log \mathfrak{K}(x,y,\xi)\rangle d\nu_y(\xi) d\widetilde{\mathrm{m}}(y).\end{aligned}$$

□

Chapter 4

Equilibrium states in negative curvature

Let $\mathcal{M}$ be a finite-volume Cartan-Hadamard manifold of dimension d with pinched negative curvature; $-b^2 \leq \mathrm{Sec}_{\mathcal{M}} \leq -a^2$ for some $b > a > 0$. We also assume that the first derivatives of sectional curvature is uniformly bounded; $\|\nabla \mathrm{Sec}_{\mathcal{M}}\| < +\infty$. We denote by $\mathrm{p} : \widetilde{\mathcal{M}} \to \mathcal{M}$ the universal covering of $\mathcal{M}$ and denote by Γ the group of deck transformations.

In this chapter, we recall basic notions in thermodynamic formalisms for the geodesic flow on $\mathcal{SM}$. Thermodynamic formalism is a study of invariant measures of geodesic flow, called *equilibrium states*. It is defined by variational principle for a dynamical quantity called pressure, which is a generalization of a dynamical invariant, entropy. Hence the equilibrium state is a generalization of a measure of maximal entropy. We shall study the existence, uniqueness and construction of equilibrium states and applications to Liouville measures and harmonic measures.

4.1 Construction of Gibbs measures

In this section, we introduce preliminaries of thermodynamic formalisms in negative curvature. Especially, we construct Gibbs measures and explain how they are related to the existence of equilibrium states. Most results in this section can be found in [PPS].

A *potential* on $\mathcal{SM}$ is a bounded Hölder continuous function $F : \mathcal{SM} \to \mathbb{R}$. We denote the lift of a potential F on $\mathcal{SM}$ to $\mathcal{S}\widetilde{\mathcal{M}}$ by $\widetilde{F}$. We define a line integral of a lifted potential by

$$\int_x^y \widetilde{F} := \int_0^{\mathrm{d}(x,y)} \widetilde{F}(\mathbf{g}^t \mathrm{v}_x^y)dt,$$

for $x, y \in \widetilde{\mathcal{M}}$ where $\mathrm{v}_x^y \in \mathcal{S}_x\widetilde{\mathcal{M}}$ is the unit vector at x pointing y, i.e., $\mathbf{g}^{\mathrm{d}(x,y)}\mathrm{v}_x^y \in \mathcal{S}_y\widetilde{\mathcal{M}}$.

Let $x \in \widetilde{\mathcal{M}}$. The *Poincaré series* of a potential F on $\mathcal{SM}$ is a function $Q_{F,x} : \mathbb{R} \to [0, \infty]$,

$$Q_{F,x}(s) = \sum_{\gamma\in\Gamma} \exp \int_x^{\gamma x} (\widetilde{F} - s).$$

Note that the convergence of $Q_{F,x}(s)$ is independent of the choice of x.

Definition 4.1.1 (Critical exponent)**.** The *critical exponent* δ_F of F is the abscissa of convergence for $Q_{F,x}$, that is,

$$\delta_F = \limsup_{t\to\infty} \frac{1}{n} \log \sum_{n-1<\mathrm{d}(x,\gamma x)\leq n} \exp \int_x^{\gamma x} \widetilde{F}.$$

It is known that $\delta_F > -\infty$ and $\delta_{F\circ\iota} = \delta_F$, where $\iota : \mathcal{S}\widetilde{\mathcal{M}} \to \mathcal{S}\widetilde{\mathcal{M}}$ is the flip map $\iota(\mathrm{v}) = -\mathrm{v}$ (Lemma 3.3 in [PPS]).

Given a potential F, due to pinched negative curvature and the Hölder continuity of F, for each $x, y, z \in \widetilde{\mathcal{M}}$ and $\xi \in \partial_\infty\widetilde{\mathcal{M}}$, the following limit exists:

$$C_F(x, y, \xi) := \lim_{z\to\xi} \int_y^z \widetilde{F} - \int_x^z \widetilde{F}.$$

This map $C_F : \widetilde{\mathcal{M}} \times \widetilde{\mathcal{M}} \times \partial_\infty\widetilde{\mathcal{M}} \to \mathbb{R}$ is called a *Gibbs cocycle* for F. It indeed satisfies a cocycle condition: for any $x, y \in \widetilde{\mathcal{M}}$ and $\xi \in \partial_\infty\widetilde{\mathcal{M}}$,

$$C_F(x, z, \xi) = C_F(x, y, \xi) + C_F(y, z, \xi),\ C_F(x, y, \xi) = -C_F(y, x, \xi).$$

For $s \in \mathbb{R}$, $C_{F-s}(x, y, \xi) = C_F(x, y, \xi) + s\mathfrak{b}(x, y, \xi)$. In particular, when F is a constant potential s, $C_F(x, y, \xi) = -s\mathfrak{b}(x, y, \xi)$. From the Γ-invariance of the lift $\widetilde{F}$, we have the Γ-invariance property of the Gibbs cocycle: for each $x, y \in \widetilde{\mathcal{M}}$ and $\xi \in \partial_\infty\widetilde{\mathcal{M}}$,

$$C_F(\gamma x, \gamma y, \gamma\xi) = C_F(x, y, \xi).$$

Definition 4.1.2 (Patterson-Sullivan density)**.** A *Patterson-Sullivan density* for F is a family $(\mu_x)_{x\in\widetilde{\mathcal{M}}}$ of finite Borel measures absolutely continuous to each other on $\partial_\infty\widetilde{\mathcal{M}}$ satisfying

$$\begin{aligned}\gamma_*\mu_x &= \mu_{\gamma x},\\ d\mu_y(\xi) &= e^{C_{F-\delta_F}(x,y,\xi)}d\mu_x(\xi),\end{aligned}$$

for each $x, y \in \widetilde{\mathcal{M}}$, $\gamma \in \Gamma$.

We have the existence of a Patterson-Sullivan density whenever the critical exponent is finite and the uniqueness if the set $\partial_c\Gamma$ of conical fixed points has positive measure.

Proposition 4.1.3. *Suppose that $\delta_F < +\infty$.*

1. *(Proposition 3.9 in [PPS]) There is a Patterson-Sullivan density for F whose support is $\partial_\infty\widetilde{\mathcal{M}}$.*
2. *(Corollary 5.10, 5.12 in [PPS]) If $\mu_x(\partial_c\Gamma) > 0$, then it is a unique Patterson-Sullivan density up to multiplicative constant.*

Note that if (μ_x) is a non-atomic Patterson-Sullivan density, then it is a unique Patterson-Sullivan density up to multiplicative constant since the set $\partial_p\Gamma = \partial_\infty\widetilde{\mathcal{M}} \setminus \partial_c\Gamma$ of parabolic fixed points is at most countably infinite due to the co-finiteness of Γ.

Fix a point x_0 in $\widetilde{\mathcal{M}}$ and a Γ-fundamental domain $\mathcal{M}_0$. Let $\partial^2_\infty\widetilde{\mathcal{M}}$ be the set of distinct pairs of boundary points. The *Hopf parametrization* at x_0 is a homeomorphism from $\mathcal{S}\widetilde{\mathcal{M}}$ onto the product $\partial^2_\infty\widetilde{\mathcal{M}} \times \mathbb{R}$:

$$\begin{aligned}\mathcal{S}\widetilde{\mathcal{M}} &\to \partial^2_\infty\widetilde{\mathcal{M}} \times \mathbb{R}\\ \mathrm{v} &\mapsto (\mathrm{v}^-, \mathrm{v}^+, \mathfrak{b}(x_0, \pi(\mathrm{v}), \mathrm{v}^+)).\end{aligned}$$

Definition 4.1.4 (Gibbs measure)**.** Let (μ_x) and (μ^ι_x) be Patterson-Sullivan densities for F and $F\circ\iota$, respectively. We define a $\mathbf{g}^t$-invariant and Γ-invariant Borel measure $\widetilde{\mu}_F$ on $\mathcal{S}\widetilde{\mathcal{M}}$ via the Hopf parametrization at x_0:

$$\widetilde{\mu}_F(\mathrm{v}) := e^{C_{F\circ\iota-\delta_F}(x_0,\pi(\mathrm{v}),\mathrm{v}^-)+C_{F-\delta_F}(x_0,\pi(\mathrm{v}),\mathrm{v}^+)}d\mu^\iota_{x_0}(\mathrm{v}^-)d\mu_{x_0}(\mathrm{v}^+)dt.$$

We define the *Gibbs measure* μ_F of F by the push-forward measure of $\widetilde{\mu}_F|_{\mathcal{S}\mathcal{M}_0}$.

Note that the definition of $\widetilde{\mu}_F$ does not depend on the choice of x_0. Due to the Γ-invariance of $\widetilde{\mu}_F$, the Gibbs measure μ_F does not depend on the choice of $\mathcal{M}_0$. It indeed has a Gibbs property:

Proposition 4.1.5. *(Proposition 3.16 in [PPS]): For each compact set $K \in \widetilde{\mathcal{SM}}$, there exist $r > 0$ and $c_{K,r} > 0$ such that for every $T, T' \geq 0$ and for every* v*,*

$$c_{K,r}^{-1} \leq \frac{\mu_F(B(\mathrm{v}, T, T', r)}{\exp \int_{-T'}^{T} (F(\mathbf{g}^t \mathrm{v}) - \delta_F)\, dt} \leq c_{K,r}, \tag{4.1}$$

where $B(\mathrm{v}, T, T', r)$ is the Bowen ball around v*:*

$$B(\mathrm{v}, T, T', r) := \{\mathrm{w} \in \mathcal{SM} : \sup_{t \in [-T', T]} \mathrm{d}(\gamma_{\widetilde{\mathrm{v}}}(t), \gamma_{\widetilde{\mathrm{w}}}(t)) < r, \exists \textit{ a lift } \widetilde{\mathrm{w}} \in \widetilde{\mathcal{SM}}\}.$$

Now we shall recall the definition of equilibrium states. Given a potential F on $\mathcal{SM}$, let $\mathcal{P}_F^{\mathbf{g}}(\mathcal{SM})$ be the space of $\mathbf{g}^t$-invariant Borel probability measures μ such that

$$\int_{\mathcal{SM}} F_- d\mu < \infty,$$

where $F_- := \max\{-F, 0\}$ is the negative part of F. For each $\mu \in \mathcal{P}_F^{\mathbf{g}}(\mathcal{SM})$, the metric pressure of μ for F is a weighted metric entropy:

$$P_F(\mu) := h_\mu(\mathbf{g}^1) + \int_{\mathcal{SM}} F d\mu,$$

where $h_\mu(\mathbf{g}^1)$ is the metric entropy of μ with respect to the time-1 map of $\mathbf{g}$. Note that it is the metric entropy of μ for the zero potential $F = 0$.

The *equilibrium state* of F is a measure $\mu_{\max}$ in $\mathcal{P}_F^{\mathbf{g}}(\mathcal{SM})$ which attains the supremum of metric pressure for F:

$$P_F(\mu_{\max}) = \sup \left\{ P_F(\mu) : \mu \in \mathcal{P}_F^{\mathbf{g}}(\mathcal{SM}) \right\}.$$

The supremum of metric pressure, which is denoted by P_F, is called the *topological pressure* of F. If F is the zero pressure, the topological pressure is the topological entropy. The Gibbs measure determines whether an equilibrium state for F exists or not.

Proposition 4.1.6. *(Theorem 6.1 in [PPS]) Let F be a potential such that $\delta_F < \infty$.*

1. $\delta_F = P_F$.

2. *If the Gibbs measure μ_F on $\mathcal{SM}$ is finite then the normalized Gibbs measure is the unique equilibrium state for F. Furthermore, it is ergodic. Otherwise, there is no equilibrium state for F.*

If the zero potential admits an equilibrium state, its equilibrium state is the measure of maximal entropy, also called the *Bowen-Margulis measure*. The Patterson-Sullivan density for the zero potential is called the *Patterson-Sullivan measures* and the measure class of the Patterson-Sullivan measures for the zero potential is called the *visibility class*.

Two potential F and F' on $\mathcal{SM}$ is said to be *cohomologous* to each other if there is a Hölder continuous functions $\widetilde{G} : \mathcal{S}\widetilde{\mathcal{M}} \to \mathbb{R}$ such that for each $\mathrm{v} \in \mathcal{S}\widetilde{\mathcal{M}}$,

$$\widetilde{F}(\mathrm{v}) - \widetilde{F}'(\mathrm{v}) = \frac{d}{dt}\bigg|_{t=0} \widetilde{G}(\mathbf{g}^t \mathrm{v}).$$

Let F and F' are cohomologous via $\widetilde{G}$. Since

$$\left| \int_x^{\gamma x} F - \int_x^{\gamma x} F' \right| = \left| \widetilde{G}(\mathbf{g}^{\mathrm{d}(x,\gamma x)} \mathrm{v}_x^{\gamma x}) - \widetilde{G}(\mathrm{v}_x^{\gamma x}) \right| \leq \sup_{\mathcal{S}_x \widetilde{\mathcal{M}}} 2|\widetilde{G}|$$

for every $\gamma \in \Gamma$, we have $\delta_F = \delta_{F'}$. We further have

$$C_F(x, y, \xi) - C_{F'}(x, y, \xi) = \widetilde{G}(\mathrm{v}_x^{\xi}) - \widetilde{G}(\mathrm{v}_y^{\xi}).$$

If (μ_x) is a Patterson-Sullivan density for F, then we have a Patterson-Sullivan density (μ'_x) for F from μ_x:

$$d\mu'_x(\xi) := e^{-\widetilde{G}(\mathrm{v}_x^{\xi})} d\mu_x(\xi).$$

Using this Patterson-Sullivan density, we conclude that the Gibbs measure $m_{F'}$ coincides with m_F.

V. Pit and B. Schapira found a necessary and sufficient condition for the finiteness of Gibbs measure in [PS]. One can find the same statement also in [PPS].

Proposition 4.1.7. *A Hölder continuous potential F admits an equilibrium state if and only if for every maximal parabolic subgroup Π of Γ, the following* *series converges:*

$$\sum_{\gamma\in\Pi} \mathrm{d}(x,\gamma x)\exp\int_x^{\gamma x}(\widetilde{F}-P_F).$$

We denote by $\mu_x^{\mathcal{T}}$ the *spherical measure* at x, the push-forward measure of μ_x via the inverse of homeomorphism $\mathcal{S}_x\widetilde{\mathcal{M}} \to \partial_\infty\widetilde{\mathcal{M}}$ for each $x \in \widetilde{\mathcal{M}}$. We have an ergodic theorem for the geodesic flow for spherical measures. We also derive a Gibbs property for spherical measures (see [Le2]).

Proposition 4.1.8. *If a bounded Hölder continuous potential F admits an equilibrium state μ then for every $\phi \in \mathcal{C}_b(\mathcal{SM})$, $x \in \mathcal{M}$ and for $\mu_x^{\mathcal{T}}$-a.e.* v *in $\mathcal{SM}$,*

$$\frac{1}{t}\int_0^t \phi(\mathbf{g}^s\mathrm{v})ds \to \int_{\mathcal{SM}} \phi\, d\mu \ \ \text{as } t\to\infty, \tag{4.2}$$

$$\lim_{t\to\infty} -\frac{1}{t}\log\mu_x^{\mathcal{T}}(B(\mathrm{v},t,0,\varepsilon)) = h_\mu \ \ \text{for some } \varepsilon > 0. \tag{4.3}$$

Proof. Since μ is ergodic, the set G of the vectors for which the convergence (4.2) holds is a union of central stable leaves with $\mu(G) = 1$. Thus for any $x, y \in \mathcal{M}$, the projections $G_x^+ := \{\widetilde{\mathrm{v}}^+ : \mathrm{v} \in \mathcal{S}_x\mathcal{M}\}$ and G_y^+ of fiber onto the boundary at infinity $\partial_\infty\widetilde{\mathcal{M}}$ are identical. Since $\mu_x^{\mathcal{T}}(G\cap\mathcal{S}_x\mathcal{M}) = \mu_x(G_x^+)$, $G\cap\mathcal{S}_x\mathcal{M}$ is a $\mu_x^{\mathcal{T}}$-full set if and only if $G\cap\mathcal{S}_y\mathcal{M}$ is a $\mu_y^{\mathcal{T}}$-full set. Therefore $G\cap\mathcal{S}_x\mathcal{M}$ is a $\mu_x^{\mathcal{T}}$-full set for every $x\in\mathcal{M}$.

From the Gibbs property (4.1) of μ_F, for $\mu_x^{\mathcal{T}}$-a.e. $\mathrm{v}\in G\cap\mathcal{S}_x\mathcal{M}$,

(4.4)

$$\lim_{t\to\infty} -\frac{1}{t}\log\mu(B(\mathrm{v},t,0,\varepsilon)) = P_F - \lim_{t\to\infty}\frac{1}{t}\int_0^t F(\mathbf{g}^s\mathrm{v})ds = P_F - \int F d\mu = h_\mu.$$

A local central stable manifold of $\mathrm{v}\in\mathcal{SM}$ is

$$\mathcal{W}_\varepsilon^{cs}(\mathrm{v}) := \{\mathrm{w} : \mathrm{d}(\mathbf{g}^t\mathrm{v}, \mathbf{g}^t\mathrm{w}) < \varepsilon,\ \forall t\geq 0\}.$$

The spherical measure is a transversal measure, so it can be defined by local central stable manifolds:

$$\mu_x^{\mathcal{T}}(S) = \mu\left(\cup_{\mathrm{w}\in S}\mathcal{W}_\varepsilon^{cs}(\mathrm{w})\right).$$

Since the Bowen ball consists of local central stable manifolds, (4.4) holds when we replace μ with $\mu_x^{\mathcal{T}}$. □

Definition 4.1.9 (Geometric potential)**.** The *geometric potential* F^{su} is a Hölder continuous function on $\mathcal{SM}$ induced by a Γ-invariant function on $\mathcal{S}\widetilde{\mathcal{M}}$ defined by

$$\widetilde{F^{su}}(\mathrm{v}) = -\left.\frac{d}{dt}\right|_{t=0} \log\det d_{\mathrm{v}}\mathbf{g}^t|_{\widetilde{E}^u(\mathrm{v})},$$

where $d_{\mathrm{v}}\mathbf{g}^t : \mathcal{T}_{\mathrm{v}}\mathcal{S}\widetilde{\mathcal{M}} \to \mathcal{T}_{\mathbf{g}^t\mathrm{v}}\mathcal{S}\widetilde{\mathcal{M}}$ is the tangent map of the flow map $\mathbf{g}^t$ at v and $\widetilde{E}^u(\mathrm{v}) = \mathcal{T}_{\mathrm{v}}\widetilde{\mathcal{W}}^u(\mathrm{v})$ is the unstable distribution.

From the pinched negative curvature and uniform bound on the first derivatives of sectional curvature, since the unstable distribution is Hölder continuous (Theorem 2.3.10), $\widetilde{F^{su}}$ is Hölder continuous.

From the identity $\widetilde{\mathcal{W}}^u(\iota\mathrm{v}) = \iota\widetilde{\mathcal{W}}^s(\mathrm{v})$ for the flip map ι which is a Riemannian isometry on $\mathcal{S}\widetilde{\mathcal{M}}$, we have $\det d_{\mathrm{v}}\mathbf{g}^{-t}|_{\widetilde{E}^u(\mathrm{v})} = \det d_{\iota\mathrm{v}}\mathbf{g}^t|_{\widetilde{E}^s(\iota\mathrm{v})}$. Since the geodesic flow preserves the Liouville measure,

$$\det d_{\mathrm{v}}\mathbf{g}^t = \det\left(d_{\mathrm{v}}\mathbf{g}^t|_{\widetilde{E}^u(\mathrm{v})}\right)\det\left(d_{\mathrm{v}}\mathbf{g}^t|_{\widetilde{E}^s(\mathrm{v})}\right) = 1$$

and hence $\det d_{\mathrm{v}}\mathbf{g}^t|_{\widetilde{E}^u(\mathrm{v})} = \det d_{-\mathrm{v}}\mathbf{g}^{-t}|_{\widetilde{E}^u(-\mathrm{v})}$. Therefore it follows that

$$\widetilde{F^{su}} \circ \iota = \widetilde{F^{su}}.$$

Ruelle's inequality holds for a finite manifold with pinched negative curvature and uniformly bounded first derivatives (Theorem 1.1 in [R]), which implies nonpositive pressure of F^{su}: for each $\mu \in \mathcal{P}^{\mathbf{g}}_{F^{su}}(\mathcal{SM})$,

$$P_{F^{su}}(\mu) \leq 0.$$

From Ruelle's inequality and Chapter 7 of [PPS], it follows that the normalized Liouville measure is the equilibrium state for F^{su}.

Theorem 4.1.10. *(Theorem 7.2 in [PPS]) The normalized Liouville measure* $\mathrm{m} = \frac{1}{\mathrm{vol}(\mathcal{SM})}\mathrm{vol}_{\mathcal{SM}}$ *is the equilibrium state for* F^{su} *and* $P_{F^{su}} = 0$.

The measure class determined by the Patterson-Sullivan density for the geometric potential is called the *Lebesgue class*. The spherical measure of F^{su} is the Lebesgue measure on the unit sphere.

4.2 Ergodic properties of Brownian motions

In this section, we discuss the thermodynamic formalisms for the harmonic potential, which arises from the Brownian motion and an equidistribution theorem of Brownian paths. Using such ergodic properties of the Brownian motion, we also provide a characterization of the asymptotic harmonicity as an application of the central limit theorem to the ergodic theory of the geodesic flow on $\mathcal{M}$.

Recall that given $x \in \widetilde{\mathcal{M}}$ we identify $(r, \mathrm{v}) \in (0, \infty) \times \mathcal{S}_x\widetilde{\mathcal{M}}$ with $\exp_x(r\mathrm{v}) \in \widetilde{\mathcal{M}} \setminus \{x\}$ and $g = dr^2 + \lambda_x(r, \mathrm{v})g_{\mathbb{S}}$. Now we denote the density of volume at $z = (r, \mathrm{v})$ with respect to the polar coordinate at x by $A_x(z)$:

$$d\,\mathrm{vol}(z) = A_x(z)dr d\,\mathrm{vol}_{\mathbb{S}}(\mathrm{v}).$$

Note that $A_x(z) = \lambda_x^{d-1}(r, \mathrm{v})$. We denote by $\theta(\widetilde{\omega}, t)$ the unit vector in $\mathcal{S}_x\widetilde{\mathcal{M}}$ such that $\widetilde{\omega}_t = (r, \theta)(\widetilde{\omega}, t) := (r(\widetilde{\omega}, t), \theta(\widetilde{\omega}, t))$.

We introduce another natural potential on $\mathcal{SM}$ induced from the Brownian motion. which we call the *harmonic potential.*

Definition 4.2.1 (Harmonic potential)**.** Define a function $\widetilde{F^{\mathrm{BM}}}$ on $\mathcal{S}\widetilde{\mathcal{M}}$ by

$$\widetilde{F^{\mathrm{BM}}}(\mathrm{v}) = -\left.\frac{d}{dt}\right|_{t=0} \log \mathfrak{K}(\gamma_{\mathrm{v}}(0), \gamma_{\mathrm{v}}(t), \mathrm{v}_+)$$

for $\mathrm{v} \in \mathcal{S}\widetilde{\mathcal{M}}$. Since $\widetilde{F^{\mathrm{BM}}}$ is Γ-invariant, there is an induced function F^{BM} on $\mathcal{SM}$, called the *Harmonic potential.*

Note that the Γ-invariance $\widetilde{F^{\mathrm{BM}}}$ follows from the Γ-invarince of $\mathfrak{K}$:

$$\mathfrak{K}(\gamma x, \gamma y, \gamma\xi) = \mathfrak{K}(x, y, \xi),$$

for each $x, y \in \widetilde{\mathcal{M}}$, $\xi \in \partial_\infty\widetilde{\mathcal{M}}$ and $\gamma \in \Gamma$. Since $\nabla_y \log \mathfrak{K}(x, y, \xi)|_{y=x}$ is smooth in y, by Proposition 3.2.9, $\widetilde{F^{\mathrm{BM}}}$ is Hölder continuous on $\mathcal{S}\widetilde{\mathcal{M}}$, hence F^{BM} is indeed a potential on $\mathcal{SM}$. Note that F^{BM} is cohomologous to $F^{\mathrm{BM}} \circ \iota$ via

$$\widetilde{\Theta}(\mathrm{v}) := \lim_{t\to\infty} \frac{\mathrm{G}(\gamma_{\mathrm{v}}(t), \gamma_{\iota\mathrm{v}}(t)}{\mathrm{G}(\gamma_{\mathrm{v}}(t), \gamma_{\mathrm{v}}(0))\,\mathrm{G}(\gamma_{\iota\mathrm{v}}(0), \gamma_{\iota\mathrm{v}}(t))},$$

which is called the *Naïm kernel* (see [LL]). The limit in the definition of the Naïm kernel exists (see [N], [Kai2]).

Note that F^{BM} has the harmonic measure $(\nu_x)_{x\in\widetilde{\mathcal{M}}}$ as a Patterson-Sullivan density of dimension 0. Since the harmonic measure does not have atom ([KL], [BeHu]), the set Π_Γ of parabolic fixed points on Γ in $\partial\widetilde{\mathcal{M}}$ has countably many points, Π_Γ is a null set for the harmonic measure. As the set of conical fixed points $\Lambda_c\Gamma = \partial\widetilde{\mathcal{M}} \setminus \Pi_\Gamma$ has positive measure with respect to the harmonic measure, the topological pressure of F^{BM} vanishes; $P_{F^{\mathrm{BM}}} = 0$ (Corollary 5.10 of [PPS]). We denote by $\widetilde{\nu}$ the Gibbs measure on $\mathcal{S}\widetilde{\mathcal{M}}$ of F^{BM} and (ν_x). Proposition 4.1.6 for F^{BM} demonstrates that F^{BM} admits an equilibrium state ν on $\mathcal{S}\mathcal{M}$ for F^{BM} if and only if $\widetilde{\nu}\,(\mathcal{S}\mathcal{M}_0)$ is finite and ν agrees with the induced measure on $\mathcal{S}\mathcal{M}$ by $\widetilde{\nu}$. From Proposition 4.1.7 it follows that F^{BM} admits an equilibrium state if and only if for every parabolic subgroup Π of Γ,

$$\sum_{\gamma\in\Pi} \frac{\mathrm{d}(x,\gamma x)}{\mathfrak{K}(x,\gamma x,(\mathrm{v}_x^{\gamma x})_+)} < \infty,$$

where $\mathrm{v}_x^y \in \mathcal{T}_x^1\widetilde{\mathcal{M}}$ such that $\mathbf{g}^{\mathrm{d}(x,y)}\mathrm{v}_x^y \in \mathcal{S}_y\widetilde{\mathcal{M}}$. We shall provide dynamical aspects of Brownian motions using the ergodic theory of ν.

The following theorem demonstrates how dynamical invariants and stochastic invariants are related to each other. We follow the argument in [Le2], but we complete the proof by showing the inequality $h \le \ell h_\nu$ using the idea in [Le1]

Theorem 4.2.2. *If F^{BM} admits an equilibrium state ν, then*

$$h = \ell h_\nu.$$

Proof. Let $x \in \widetilde{\mathcal{M}}$, $\delta \in \left(0,\frac{1}{2}\right)$ and $0 < \varepsilon, \varepsilon'$. We denote for each $T > 0$,

$$\begin{aligned}
\mathscr{C}_T := \{\widetilde{\omega} : \mathrm{d}(\widetilde{\omega}_T, (\ell T, \widetilde{\omega}_\infty)) \le \varepsilon T \text{ and}\\
\mu_x^{\mathcal{T}}\{\mathrm{v} : d_{\ell T}(\mathrm{v}, \theta(\widetilde{\omega},\infty)) \le \varepsilon'\} \le e^{-(\ell h_\nu - \varepsilon)T}\},\\
\mathscr{D}_T := \{\widetilde{\omega} : \mathrm{d}(\widetilde{\omega}_T, (\ell T, \widetilde{\omega}_\infty)) \le \varepsilon T \text{ and}\\
\mu_x^{\mathcal{T}}\{\mathrm{v} : \mathrm{d}\left(\gamma_{\mathrm{v}}(\ell T), \gamma_{\theta(\widetilde{\omega},\infty)}(\ell T)\right) \le \varepsilon'\} \ge e^{-(\ell h_\nu + \varepsilon)T}\}.
\end{aligned}$$

For every T large enough, $\mathbb{P}_x(\mathscr{C}_T) \ge 2\delta$ for some $\varepsilon' > 0$ by Proposition 3.2.5. Thus if we fix a sufficiently large T and choose $E \subset \widetilde{\mathcal{M}}$ with $\operatorname{Card} E = N(x,T,1-\delta)$,

$$\mathbb{P}_x\{\mathrm{d}(\widetilde{\omega}_T, E) \le 1\} \ge 1 - \delta.$$

We note that $E_\infty := \{\theta(\widetilde{\omega},\infty) : \widetilde{\omega} \in \mathscr{C}_T, \mathrm{d}(\widetilde{\omega}_T, E) \leq 1\}$ has the $\mu_x^{\mathcal{T}}$-measure greater than δ and $\{\gamma_{\theta(\widetilde{\omega},\infty)}(\ell T) : \widetilde{\omega} \in \mathscr{C}_T, \mathrm{d}(\widetilde{\omega}_T, E) \leq 1\}$ is covered by balls on the sphere of radius ε' less than $N(x, T, 1-\delta)C^{\varepsilon T}$. ($C$ is the maximal cardinal of covers for the intersection of the sphere of radius ℓT and $(\varepsilon+1)T$ balls by ε' balls on the sphere.) Such ball O in the sphere of radius ε' is the set of base points of vectors in $\mathbf{g}^{\ell T} V$ where $V = \{\mathrm{v} : d_{\ell T}(\mathrm{v}, \mathrm{w}) \leq \varepsilon'\}$ for some w. We conclude that since such V has the $\mu_{\mathcal{SM}}$-measure less than $e^{-(\ell h_\nu - \varepsilon)T}$,

$$\delta \leq \mu_x^{\mathcal{T}}(E_\infty) \leq N(x, T, 1-\delta) e^{-T[\ell h_\nu - \varepsilon - \varepsilon \log C]}.$$

Thus we have $\ell h_\nu \leq \lim_{T\to\infty} \frac{1}{T} \log N(x, T, 1-\delta)$.

Choose a smallest set $E \subset \widetilde{\mathcal{M}}$ such that $\mathrm{d}(\widetilde{\omega}_T, E) \leq 1$ for each $\widetilde{\omega} \in \mathscr{D}_T$ and a maximal ε'-separeted set $F \subset \{\gamma_{\theta(\widetilde{\omega},\infty)}(\ell T) : \widetilde{\omega} \in \mathscr{D}_T\}$. Since $\mathscr{D}_T \subset \{\widetilde{\omega} : \mathrm{d}(\widetilde{\omega}_T, E) \leq 1\}$, $\mathrm{Card}(E) \geq N(x, T, \mathbb{P}_x(\mathscr{D}_T))$ and $\mathrm{Card}(F) \leq C' e^{\ell h_\nu T + \varepsilon T}$. ($C'$ is the maximal number of overlappings.) For every $f \in F$ if we denote

$$N(f) := \{e \in E : \exists \widetilde{\omega} \in \mathscr{D}_T \text{ s.t. } \mathrm{d}(f, \gamma_{\theta(\widetilde{\omega},\infty)}(\ell T)) \leq \varepsilon', \mathrm{d}(e, \widetilde{\omega}_T) \leq 1\}.$$

Since $\cup_{e \in N(f)} B(e, 1) \subset B(f, \varepsilon T + \varepsilon' + 1)$, there exists $C'' > 0$ such that

$$\mathrm{Card}\, N(f) \leq \sup_{e \in E, f \in F} \frac{\mathrm{vol}\,(B(f, \varepsilon T + \varepsilon' + 1))}{\mathrm{vol}(B(e, 1))} \leq e^{C'' \varepsilon T}.$$

Therefore,

$$N(x, T, \mathbb{P}_x(\mathscr{D}_T)) \leq \mathrm{Card}(E) \leq \exp(C'' \varepsilon T)\, \mathrm{Card}\, F \leq e^{T[\ell h_\nu + (1+C'')\varepsilon]}.$$

□

The following proposition proves the equidistribution of Brownian paths with respect to ν. To be precise, the equidistribution means that geodesics which Brownian paths roughly follow are generic for ν. The proof follows the argument for compact manifolds ([Le2]).

Proposition 4.2.3. *Assume that F^{BM} admits an equilibrium state ν. For every $x \in \widetilde{\mathcal{M}}$, for each bounded continuous function $\phi \in \mathcal{C}_b(\mathcal{SM})$ and for $\mathbb{P}_x$-a.e. $\widetilde{\omega}$,*

$$\int \phi \, d\nu = \lim_{t\to\infty} \frac{1}{\ell t} \int_0^{r(\widetilde{\omega},t)} \widetilde{\phi}(\mathbf{g}^s \theta(\widetilde{\omega}, t))\, ds.$$

Proof. For $\mathrm{v},\mathrm{w}\in\mathcal{S}\widetilde{\mathcal{M}}$, let $d_t(\mathrm{v},\mathrm{w})$ be the distance on the geodesic sphere $S(x,t)$ between $\mathbf{g}^t\mathrm{v}$ and $\mathbf{g}^t\mathrm{w}$. Then

$$d_t(\mathrm{v},\mathrm{w})\le d_s(\mathrm{v},\mathrm{w})\frac{\sinh(at)}{\sinh(as)}$$

for every $0<t<s$ due to the curvature upper bound $\sec_{\widetilde{\mathcal{M}}}\le -a^2<0$. Since the Sasaki distance is Hölder equivalent to the distance $\mathrm{d}_0(\mathrm{v},\mathrm{w}):=\sup_{0\le t\le 1}\mathrm{d}\left(\gamma_{\mathrm{v}}(t),\gamma_{\mathrm{w}}(t)\right)$,

$$\left|\int_0^t\widetilde{\phi}\circ\mathbf{g}^s(\mathrm{v})ds-\int_0^t\widetilde{\phi}\circ\mathbf{g}^s(\mathrm{w})ds\right|\le C(a,\phi)\mathrm{d}(\gamma_{\mathrm{v}}(t),\gamma_{\mathrm{w}}(t)).$$

Hence the proposition follows from of Proposition 4.1.8 and Proposition 3.2.5: for $\mathbb{P}_x$-a.e. $\widetilde{\omega}$,

$$\begin{aligned}
&\lim_{t\to\infty}\left|\frac{1}{\ell t}\int_0^{r(\widetilde{\omega},t)}\widetilde{\phi}(\mathbf{g}^s\theta(\widetilde{\omega},t))ds-\int\phi d\nu\right|\\
&\quad\le\lim_{t\to\infty}\frac{1}{\ell t}\left|\int_0^{r(\widetilde{\omega},t)}\widetilde{\phi}(\mathbf{g}^s\theta(\widetilde{\omega},t))ds-\int_0^{\ell t}\widetilde{\phi}(\mathbf{g}^s\theta(\widetilde{\omega},\infty))ds\right|\\
&\quad+\lim_{t\to\infty}\left|\frac{1}{\ell t}\int_0^{\ell t}\widetilde{\phi}(\mathbf{g}^s\theta(\widetilde{\omega},\infty))ds-\int\phi d\nu\right|\\
&\quad=\lim_{t\to\infty}\frac{1}{\ell t}\left|\int_0^{r(\widetilde{\omega},t)}\widetilde{\phi}(\mathbf{g}^s\theta(\widetilde{\omega},t))ds-\int_0^{r(\widetilde{\omega},t)}\widetilde{\phi}(\mathbf{g}^s\theta(\widetilde{\omega},\infty))ds\right|+0\\
&\quad\le\lim_{t\to\infty}\frac{C(a,\phi)}{\ell t}\mathrm{d}(\widetilde{\omega}_t,(r,\theta)(\widetilde{\omega},t))=0.
\end{aligned}$$

We used (4.2) of Proposition 4.1.8 in the equation and Proposition 3.2.5 in the last inequality. □

The equidistribution of Brownian paths provides another stochastic invariant, namely, the exponential growth along Brownian paths. It helps to understand the relation between the harmonic measure class and the Lebesgue measure class. The proof in [Le2] extends to the finite-volume case.

Theorem 4.2.4. *For each $x\in\widetilde{\mathcal{M}}$ and for $\mathbb{P}_{x,}$-a.e. $\widetilde{\omega}$, if there is an equilibrium state ν of F^{BM}, the following limit exists:*

$$\Upsilon=\lim_{t\to\infty}\frac{1}{t}\log A_x(\widetilde{\omega}_t)=-\ell\int F^{su}d\nu.$$

Proof. Let $(d\mathbf{g}^t)_{\mathrm{v}}$ be the tangent map of the flow map $\mathbf{g}^t$ at $\mathrm{v} = (x, \xi) \in \mathcal{S}\widetilde{\mathcal{M}}$. Since the angle between stable distribution $\widetilde{E}^s(\mathrm{v})$ and $(d\mathbf{g}^t)_{\mathrm{v}}(\mathcal{T}_{\mathrm{v}}\mathcal{S}_x\widetilde{\mathcal{M}})$, where $\mathcal{T}_{\mathrm{v}}\mathcal{S}_x\widetilde{\mathcal{M}}$ is the tangent space of the sphere $\mathcal{S}_x\widetilde{\mathcal{M}}$, is bounded away from zero uniformly on v and $t > 0$,

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{t} \log A_{\pi\mathrm{v}}(\pi\mathbf{g}^t\mathrm{v}) &= \lim_{t\to\infty} \frac{1}{t} \log\det d\mathbf{g}^t|_{\mathcal{T}_{\mathrm{v}}\mathcal{S}_x\widetilde{\mathcal{M}}} \\ &= \lim_{t\to\infty} \frac{1}{t} \log\det d\mathbf{g}^t|_{\widetilde{E}^u(\mathrm{v})} \\ &= \lim_{t\to\infty} -\frac{1}{t} \int_0^t F^{su}(\mathbf{g}^s\mathrm{v})ds.\end{aligned}$$

Therefore, by Proposition 4.2.3, for $\mathbb{P}_x$-a.e. $\widetilde{\omega}$,

$$\lim_{t\to\infty} \frac{1}{t} \log A_x(\widetilde{\omega}_t) = \lim_{t\to\infty} -\frac{1}{t} \int_0^{r(\widetilde{\omega},t)} F^{su}(\mathbf{g}^s\theta(\widetilde{\omega}_t))ds = -\ell \int F^{su} d\nu.$$

□

Since $h_\nu \leq h_{\mathrm{top}}$ and $h_\nu + \int F^{su} d\nu \leq P_{F^{su}} = 0$, from Theorem 4.2.2 and Theorem 4.2.4, we have the following theorem as a corollary.

Theorem 4.2.5. *Suppose that F^{BM} admits an equilibrium state ν. Denote the topological entropy of $(\mathcal{S}\mathcal{M}, (\mathbf{g}^t))$ by h_{top}.*

1. *We have $h \leq \ell h_{\mathrm{top}}$. The equality holds if and only if the harmonic measure class and the visibility class coincide.*

2. *We have $h \leq \Upsilon = -\ell \int F^{su}\, d\nu$. The equality holds if and only if the harmonic measure class and the Lebesgue class agree.*

Proof. It follows from Theorem 4.2.2 that $h = \ell h_{\mathrm{top}}$ is equivalent to $h_{\mathrm{top}} = h_\nu$. This equality happens if and only if ν is the measure of maximal entropy.

By Theorem 4.2.4, the equality $\Upsilon = h$ is equivalant to

$$P_{F^{su}}(\nu) = h_\nu + \int F^{su} d\nu = 0,$$

which holds if and only if ν is the equilibrium state for F^{su}. □

Remark 4.2.6. *If there is an equilibrium state of F^{BM}, then since $\ell^2 \leq h \leq \ell h_{\mathrm{top}}$, we have $\ell \leq h_{\mathrm{top}}$ and*

$$4\lambda_0 \leq h \leq \ell h_{\mathrm{top}} \leq h_{\mathrm{top}}^2. \tag{4.5}$$

Chapter 5

Central limit theorem of Brownian motions

Let $\mathcal{M}$ be a finite-volume Riemannian manifold with $-b^2 \leq \mathrm{Sec}_{\mathcal{M}} \leq -a^2 < 0$ and $\|\nabla \mathrm{Sec}_{\mathcal{M}}\| < +\infty$ for some $b \geq a > 0$. We denote by $\mathrm{p} : \widetilde{\mathcal{M}} \to \mathcal{M}$ its universal cover and Γ is the group of deck transformations of the universal cover. We identify $\mathcal{M}$ and $\mathcal{SM}$ with a Γ-fundamental domain $\mathcal{M}_0$ in $\widetilde{\mathcal{M}}$ and $\mathcal{M}_0 \times \partial_\infty \widetilde{\mathcal{M}}$, respectively.

In this chapter, we prove the central limit theorem of random variables

$$\begin{aligned} Y_t^\ell(\widetilde{\omega}) &= \mathrm{d}(x, \widetilde{\omega}_t) - t\ell, \\ Y_t^h(\widetilde{\omega}) &= \log \mathrm{G}(x, \widetilde{\omega}_t) + th. \end{aligned}$$

We know that for almost every path $\widetilde{\omega}$, these random variables asymptotically grow slower than t pointwise. We prove that they asympototically grow as fast as $\sqrt{t}$ in distribution.

Theorem 5.0.1. *There are constants $\sigma_{\mathfrak{b}}, \sigma_{\mathfrak{K}}$ such that the random variables $\frac{1}{\sqrt{t}} Y_t^\ell$ and $\frac{1}{\sqrt{t}} Y_t^h$ asymptotically follow normal distributions $N(0, \sigma_{\mathfrak{b}})$ and $N(0, \sigma_{\mathfrak{K}})$, respectively. More precisely, for every $x \in \widetilde{\mathcal{M}}$,*

$$\mathbb{P}_x \left[\frac{Y_t^\ell}{\sigma_{\mathfrak{b}} \sqrt{t}} \leq r \right], \mathbb{P}_x \left[\frac{Y_t^h}{\sigma_{\mathfrak{K}} \sqrt{t}} \leq r \right] \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{r} \exp\left(-\frac{s^2}{2} \right) ds, \ \textit{as } t \to \infty,$$

where $\mathbb{P}_x$ is the probability measures on the space $\mathcal{C}(\mathbb{R}_+, \widetilde{\mathcal{M}})$ of continuous sample paths which defines the Brownian motion on $\widetilde{\mathcal{M}}$ starting from x.

In the last two sections, we establish characterizations of asymptotically harmonic manifolds with pinched negative curvature as an application of the central limit theorem and we introduce the related problems.

5.1 Leafwise heat equation

In the proof of the central limit theorem of Brownian motions, the heart of the proof is the contraction property on rotationally Hölder spaces of the foliated Brownian motion. Let $\tau > 0$.

Definition 5.1.1 (Rotationally Hölder space)**.** For f in the space $\mathcal{C}_b(\mathcal{SM})$ of bounded continuous functions, the *rotational τ-Hölder norm* of f is

$$\|f\|_{\mathcal{L}^\tau} = \|f\|_\infty + \sup_{x\in\mathcal{M}_0} \sup_{\xi,\eta\in\partial\widetilde{\mathcal{M}}} \frac{|\widetilde{f}(x,\xi) - \widetilde{f}(x,\eta)|}{d_\infty^{x,\tau}(\xi,\eta)}.$$

We define the *rotationally Hölder space* by

$$\mathcal{L}^\tau = \{f \in \mathcal{C}_b(\mathcal{SM}) : \|f\|_{\mathcal{L}^\tau} < \infty\}.$$

The following statement corresponds to the uniqueness of a $\mathcal{Q}^t$-invariant measure for compact negatively curved manifolds (see [Le6]). In [H2], it was shown that the uniqueness for the $(\Delta_s + Y)$-diffusion on compact negatively curved manifolds holds for a stably closed vector field Y on $\mathcal{SM}$ with positive pressure.

Proposition 5.1.2. *For every $\mathcal{Q}^t$-invariant measure η on $\mathcal{SM}$ and for each $f \in \mathcal{L}^\tau$,*

$$\int f d\eta = \int f d\mathrm{m}^{\mathcal{Q}}.$$

Proof. If η is a $\mathcal{Q}^t$-invariant measure on $\mathcal{SM}$, its Γ-invariant lift $\widetilde{\eta}$ to $\mathcal{S}\widetilde{\mathcal{M}}$ is disintegrated into $d\widetilde{\eta}(x,\xi) = d\widetilde{\eta}_x(\xi)d\widetilde{\mathrm{m}}(x)$ over the fibration $\mathcal{S}\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}} \times \partial\widetilde{\mathcal{M}}$ where $\widetilde{\eta}_x$ are the conditional measures on the unit spheres $\mathcal{S}_x\widetilde{\mathcal{M}} = \{x\} \times \partial\widetilde{\mathcal{M}}$ of $\widetilde{\eta}$ ([Ga]). As in the proof of Proposition 4.1.8, we can consider $\widetilde{\eta}_x$ as a probability measure of the union of local leaves; for some sufficiently small $\delta > 0$,

$$\widetilde{\eta}_x(A) := \widetilde{\eta}(\cup_{\mathrm{w}\in A}\mathcal{W}_\delta^{cs}(\mathrm{w}))/\widetilde{\eta}(\cup_{\mathrm{v}\in\mathcal{S}_x\widetilde{\mathcal{M}}}\mathcal{W}_\delta^{cs}(\mathrm{v})).$$

We denote by $\mathbb{E}_x$ the expectation with respect to $\mathbb{P}_x$. From the $\mathcal{Q}^t$-invariance, we have

$$\begin{aligned}\int_{\mathcal{SM}} f d\eta &= \int_{\mathcal{M}_0} \int_{\partial\widetilde{\mathcal{M}}} \mathcal{Q}^t f(x,\xi)\, d\widetilde{\eta}_x(\xi)\, d\widetilde{\mathrm{m}}(x) \\ &= \int_{\mathcal{M}_0} \int_{\partial\widetilde{\mathcal{M}}} \int_{\widetilde{\mathcal{M}}} \wp(t,x,y)\widetilde{f}(y,\xi)\, d\,\mathrm{vol}(y)\, d\widetilde{\eta}_x(\xi)\, d\widetilde{\mathrm{m}}(x) \\ &= \int_{\mathcal{M}_0} \mathbb{E}_x\left[\int_{\partial\widetilde{\mathcal{M}}} \widetilde{f}(\widetilde{\omega}_t,\xi)\, d\widetilde{\eta}_x(\xi)\right] d\widetilde{\mathrm{m}}(x).\end{aligned}$$

Note that given $\varepsilon > 0$, $x \in \mathcal{M}_0$ and $f \in \mathcal{L}^\tau$, there is $\theta_0 = \theta_0(\varepsilon) > 0$ for every $y \in \widetilde{\mathcal{M}}$ and $\xi, \eta \in \partial\widetilde{\mathcal{M}}$ with $\angle_y(\xi,\eta) < \theta_0$, $|\widetilde{f}(y,\xi) - \widetilde{f}(y,\eta)| < \varepsilon$. Given $\xi \in \partial\widetilde{\mathcal{M}}$, we set for $T, \theta > 0$,

$$\begin{aligned}\Upsilon(x,\xi,\theta) &:= \left\{\widetilde{\omega} : \angle_x(\widetilde{\omega}_\infty,\xi) < \frac{\theta}{3}\right\}, \\ \Xi(x,T,\theta) &:= \left\{\widetilde{\omega} : \mathrm{d}(x,\widetilde{\omega}_t) \geq \frac{\ell}{2}t, \angle_x(\theta(\widetilde{\omega},t),\theta(\widetilde{\omega},\infty)) < \frac{\theta}{3},\ \forall t \geq T\right\}.\end{aligned}$$

Then if $\theta \in (0,\theta_0)$ is small enough, than for any x and ξ, $\mathbb{P}_x(\Upsilon(x,\xi,\theta)) < \frac{\varepsilon}{2\|f\|_\infty}$ (by [BeHu]). Choose such a small θ. There is $T_0 = T_0(x,\theta)$ such that if $t > T_0$, $|\widetilde{f}(\widetilde{\omega}_t,\xi) - \widetilde{f}(\mathrm{v}^x_{\widetilde{\omega}_t})| < \frac{\varepsilon}{2\|f\|_\infty}$ for each $\widetilde{\omega} \in \Xi(x,t,\theta) \setminus \Upsilon(x,\xi,\theta)$ and $\mathbb{P}_x(\Xi(x,t,\theta)) > 1 - \frac{\varepsilon}{2\|f\|_\infty}$. Hence if $t > T_0$, then

$$\begin{aligned}&\left|\mathbb{E}_x\left[\int_{\partial\widetilde{\mathcal{M}}} \widetilde{f}(\widetilde{\omega}_t,\xi) - \widetilde{f}(\mathrm{v}^x_{\widetilde{\omega}_t})\, d\widetilde{\eta}_x(\xi)\right]\right| \\ &\leq \int_{\partial\widetilde{\mathcal{M}}} \mathbb{E}_x\left[\left|\widetilde{f}(\widetilde{\omega}_t,\xi) - \widetilde{f}(\mathrm{v}^x_{\widetilde{\omega}_t})\right| \left(\mathbf{1}_{\Xi(x,t,\delta)\setminus\Upsilon(x,\xi,\theta)} + \mathbf{1}_{\Xi(x,t,\delta)\cap\Upsilon(x,\xi,\theta)} + \mathbf{1}_{\Xi(x,t,\delta)^c}\right)\right] d\widetilde{\eta}_x(\xi) \\ &\leq \varepsilon + 2\|f\|_\infty \left(\int_{\partial\widetilde{\mathcal{M}}} \mathbb{P}_x\left[\Upsilon(x,\xi,\theta)\right] d\widetilde{\eta}_x(\xi) + \mathbb{P}_x\left[\Xi(x,t,\delta)^c\right]\right) \\ &< 3\varepsilon.\end{aligned}$$

Since $\phi_t(x) := \mathbb{E}_x\left[\int \widetilde{f}(\widetilde{\omega}_t,\cdot)d\eta_x\right]$ and $\psi_t(x) := \mathbb{E}_x\left[\widetilde{f}(\mathrm{v}^x_{\widetilde{\omega}_t})\right]$ are bounded by $\|f\|_\infty$, it follows that $\phi_t - \psi_t \to 0$ as $t \to \infty$ in $L^1(\mathcal{M}_0,\widetilde{\mathrm{m}})$ and hence $\lim\limits_{t\to\infty} \int \psi_t d\widetilde{\mathrm{m}} = \int f d\eta$. Thus we have

$$\int f d\eta = \lim_{t\to\infty} \int_{\mathcal{M}_0} \mathbb{E}_x\left[\int_{\partial\widetilde{\mathcal{M}}} \widetilde{f}(\widetilde{\omega}_t,\xi)\, d\widetilde{\eta}_x(\xi)\right] d\widetilde{\mathrm{m}}(x) = \lim_{t\to\infty} \int_{\mathcal{M}_0} \mathbb{E}_x\left[\widetilde{f}(\mathrm{v}^x_{\widetilde{\omega}_t})\right] d\widetilde{\mathrm{m}}(x).$$

From the Γ-invariance of $\widetilde{f}$ and the heat kernel, it follows that

$$\begin{aligned}\int_{\mathcal{M}_0} \mathbb{E}_x\left[\widetilde{f}(\mathrm{v}^x_{\widetilde{\omega}_t})\right] d\widetilde{\mathrm{m}}(x) &= \int_{\mathcal{M}_0}\int_{\mathcal{M}_0}\sum_{\gamma\in\Gamma}\wp(t,x,\gamma y)\widetilde{f}(\mathrm{v}^x_{\gamma y})\, d\,\mathrm{vol}(y)\, d\widetilde{\mathrm{m}}(x)\\ &= \int_{\mathcal{M}_0}\int_{\mathcal{M}_0}\sum_{\gamma\in\Gamma}\wp(t,y,\gamma^{-1}x)\widetilde{f}(\mathrm{v}^{\gamma^{-1}x}_y)\, d\,\mathrm{vol}(x)\, d\widetilde{\mathrm{m}}(y)\\ &= \int_{\mathcal{M}_0}\mathbb{E}_y\left[\widetilde{f}(\mathrm{v}^{\widetilde{\omega}_t}_y)\right] d\widetilde{\mathrm{m}}(y).\end{aligned}$$

Letting t tend to infinity, we have

$$\begin{aligned}\int f d\eta &= \int_{\mathcal{M}_0}\mathbb{E}_y\left[\widetilde{f}(y,\widetilde{\omega}_\infty)\right] d\widetilde{\mathrm{m}}(y)\\ &= \int_{M_0}\int_{\partial\widetilde{\mathcal{M}}}\widetilde{f}(y,\xi)\, d\nu_y(\xi)\, d\widetilde{\mathrm{m}}(y).\end{aligned}$$

Therefore, $\int f d\eta = \int f d\mathrm{m}^{\mathcal{Q}}$. □

Definition 5.1.3 (Integration operator)**.** The *integration operator* is an operator $\mathcal{N} : \mathcal{C}_b(\mathcal{SM}) \to \mathcal{C}_b(\mathcal{SM})$ defined for $f \in \mathcal{C}_b$,

$$\mathcal{N}(f) := \int_{\mathcal{SM}} f\, d\mathrm{m}^{\mathcal{Q}}.$$

The Markov operator $\mathcal{Q}^t$ converges to $\mathcal{N}$ on $\mathcal{L}^\tau$. Furthermore the following theorem shows the rate of convergence is exponentially fast. We prove the theorem in Section 5.3.

Theorem 5.1.4. (Contraction property of foliated Brownian motions) $\mathcal{Q}^t : \mathcal{L}^\tau \to \mathcal{L}^\tau$ *defines a one-parameter semigroup of continuous operators for small enough* $\tau > 0$. *Furthermore, there is* $C = C(\tau) > 0$ *such that for every* $t > 0$,

$$\|\mathcal{Q}^t - \mathcal{N}\|_{\mathcal{L}^\tau} \le e^{-Ct}.$$

Given $f \in \mathcal{L}^\tau$, if $\int f d\mathrm{m}^{\mathcal{Q}} = 0$, then the $\mathcal{L}^\tau$-limit of $\int_0^T \mathcal{Q}^t f dt$ exists by the contraction property. The limit $u := \lim_{T\to\infty}\int_0^T \mathcal{Q}^t f dt$ is a weak solution of the leafwise heat equation $\Delta_s u = -f$, thus a strong solution in $\mathcal{L}^\tau$. Since a leafwise harmonic u is $\mathcal{Q}^t$-invariant, the uniqueness also follows from the contraction property (See [Le6] for the detail). Therefore we obtain the following corollary.

Corollary 5.1.5. *For small enough $\tau > 0$ and every $f \in \mathcal{L}^\tau$ with $\int f d\mathrm{m}^{\mathcal{Q}} = 0$, there exists a solution $u \in \mathcal{L}^\tau$ to the leafwise heat equation $\Delta_s u = -f$ which is unique up to additive constants. In addition, u is $\mathcal{C}^2$ along the stable leaves.*

Let $\alpha : \mathcal{SM} \to E^{cs*}$ be a continuous section of the dual bundle E^{cs*} of the central stable distribution E^{cs} of $\mathcal{SM}$ and $\widetilde{\alpha} : \mathcal{S}\widetilde{\mathcal{M}} \to \widetilde{E}^{cs}$ be the lift of α. The section α is called a *leafwise closed 1-form* of class $\mathcal{C}^1$ if $\widetilde{\alpha}|_{\widetilde{\mathcal{W}}^{cs}(\mathrm{v})}$ is a closed 1-form on $\widetilde{\mathcal{W}}^{cs}(\mathrm{v})$ of class $\mathcal{C}^1$ for any $\mathrm{v} \in \mathcal{S}\widetilde{\mathcal{M}}$. For each $(x, \xi) \in \mathcal{S}\widetilde{\mathcal{M}}$, since $\widetilde{\mathcal{W}}^s(x, \xi) = \widetilde{\mathcal{M}} \times \{\xi\}$ is diffeomorphic to $\widetilde{\mathcal{M}}$, there is a 1-form $\widetilde{\alpha}^\xi$ on $\widetilde{\mathcal{M}}$ which agrees with the pull-back of $\widetilde{\alpha}|_{\widetilde{\mathcal{W}}^s(x,\xi)}$. Furthermore, if α is a leafwise closed 1-form of class $\mathcal{C}^1$, then there exists $A^\xi \in \mathcal{C}^1(\widetilde{\mathcal{M}})$ such that $dA^\xi = \widetilde{\alpha}^\xi$. Hence if α is a leafwise closed 1-form of class $\mathcal{C}^1$, we define for each foliated Brownian path $\omega \in \mathcal{S}\Omega$ starting from $(x, \xi) \in \mathcal{SM}$,

$$\int_{\omega_0}^{\omega_t} \alpha := A^\xi(\widetilde{\omega}_t) - A^\xi(\widetilde{\omega}_0)$$

for every $t \geq 0$, where $\widetilde{\omega}$ is a Brownian path on $\mathcal{S}\widetilde{\mathcal{M}}$ such that $(\widetilde{\omega}_t, \xi) \in \mathcal{S}\widetilde{\mathcal{M}}$ is a lift of ω_t.

We denote by δ_s the leafwise co-differential g_s-dual to $-\mathrm{div}_s$, that is, $\delta_s\alpha = -\mathrm{div}_s\alpha^\#$ where $\alpha^\# : \mathcal{SM} \to E^{cs}$ is the continuous section g_s-dual to α. Since

$$\delta_s\widetilde{\alpha}(x, \xi) = -\mathrm{div}_s\widetilde{\alpha}^\#(x, \xi) = -\mathrm{div}\nabla A^\xi(x) = -\Delta A^\xi(x) = -\Delta_s A(x, \xi),$$

by (3.1),

$$\mathbf{X}_t(\omega) = \int_{\omega_0}^{\omega_t} \alpha + \int_0^t \delta_s\alpha(\omega_r)dr = A^\xi(\widetilde{\omega}_t) - A^\xi(\widetilde{\omega}_0) - \int_0^t \Delta A^\xi(\widetilde{\omega}_r)dr \tag{5.1}$$

is a martingale on $(\mathcal{S}\Omega, \{\mathscr{F}_t(\mathcal{SM})\}_{0\leq t\leq\infty}, \mathbb{P}_{\mathrm{m}^{\mathcal{Q}}})$ having the quadratic variation

$$d\langle \mathbf{X}, \mathbf{X}\rangle_t(\omega) = (\Delta(A^\xi)^2 - 2A^\xi\Delta A^\xi)(\widetilde{\omega}_t)dt = 2|\alpha^\#(\omega_t)|^2 dt.$$

If β is a leafwise closed 1-form of class $\mathcal{C}^1$ such that $\delta_s\beta$ is Hölder continuous on $\mathcal{SM}$, applying Corollary 5.1.5, there is $u \in \mathcal{L}^\tau$ such that $\Delta_s u = \delta_s\beta - \int \delta_s\beta d\mathrm{m}^{\mathcal{Q}}$. Hence, due to the equation (5.1) for $\alpha = \beta + du$, we have a martingale

$$\mathbf{X}_t = \int_{\omega_0}^{\omega_t} (\beta+du) + \int_0^t \delta_s(\beta+du)(\omega_r)dr = \int_{\omega_0}^{\omega_t} \beta + u(\omega_t) - u(\omega_0) + t\int \delta_s \beta d\mathrm{m}^{\mathcal{Q}} \tag{5.2}$$

with the quadratic variation $\langle \mathbf{X}, \mathbf{X}\rangle_t(\omega) = 2\int_0^t |\alpha^{\#} + \nabla u|^2(\omega_t)ds$.

5.2 Proof of the central limit theorem of Brownian motions

For $(x,\xi) \in \mathcal{S}\widetilde{\mathcal{M}}$, let $B(x,\xi) := \mathfrak{b}(x,x_0,\xi)$, $K(x,\xi) := \log \mathfrak{K}(x_0,x,\xi)$. Note that

$$\Delta_s B(x,\xi) = \Delta_x \mathfrak{b}(x,x_0,\xi)$$

is Hölder continuous due to uniform bounds of the first derivatives of curvature. On the other hand,

$$\Delta_s K(x,\xi) = -|\nabla_x \log \mathfrak{K}(x_0,x,\xi)|^2$$

is Hölder contiunous by Proposition 3.2.9. By Corollary 5.1.5 for $f = \Delta_s B$, and $\Delta_s K$, there exist $u_{\mathfrak{b}}, u_{\mathfrak{K}} \in \mathcal{L}^\tau$ for which we obtain square-integrable martingales

$$\begin{aligned} \mathbf{B}_t(\omega) &= \mathfrak{b}(\widetilde{\omega}_t, \widetilde{\omega}_0, \xi) - t\ell + u_{\mathfrak{b}}(\omega_t) - u_{\mathfrak{b}}(\omega_0), \\ \mathbf{K}_t(\omega) &= \log \mathfrak{K}(\widetilde{\omega}_0, \widetilde{\omega}_t, \xi) + th + u_{\mathfrak{K}}(\omega_t) - u_{\mathfrak{K}}(\omega_0), \end{aligned}$$

for $\omega \in \mathcal{S}\Omega$ with a lift $(\widetilde{\omega},\xi) \in \mathcal{S}\widetilde{\mathcal{M}}$, by the Itô formula (5.2) for $\beta_{(x,\xi)} = dB_x^\xi$, dK_x^ξ, respectively. Their quadratic variations are

$$\langle \mathbf{B}, \mathbf{B}\rangle_t(\omega) = 2\int_0^t |\nabla B + \nabla u_{\mathfrak{b}}|^2(\omega_s)ds, \tag{5.3}$$

$$\langle \mathbf{K}, \mathbf{K}\rangle_t(\omega) = 2\int_0^t |\nabla K + \nabla u_{\mathfrak{K}}|^2(\omega_s)ds. \tag{5.4}$$

We denote by $\mathbb{E}_{(x,\xi)}$ the expectation with respect to $\mathbb{P}_{(x,\xi)}$. From the equalities (5.3) and (5.4) of quadratic variations,

$$\mathbb{E}_{(x,\xi)}\left[\frac{1}{t}\langle \mathbf{B},\mathbf{B}\rangle_t(\omega)\right] = \frac{2}{t}\int_0^t \mathcal{Q}^s|\nabla B+\nabla u_{\mathfrak{b}}|^2(x,\xi)ds,$$
$$\mathbb{E}_{(x,\xi)}\left[\frac{1}{t}\langle \mathbf{K},\mathbf{K}\rangle_t(\omega)\right] = \frac{2}{t}\int_0^t \mathcal{Q}^s|\nabla K+\nabla u_{\mathfrak{K}}|^2(x,\xi)ds.$$

Due to the ergodicity of $\mathrm{m}^{\mathcal{Q}}$, for $\mathrm{m}^{\mathcal{Q}}$-a.e. (x,ξ),

$$\lim_{t\to\infty}\mathbb{E}_{(x,\xi)}\left[\frac{1}{t}\langle \mathbf{B},\mathbf{B}\rangle_t(\omega)\right] = 2\int |\nabla B+\nabla u_{\mathfrak{b}}|^2 d\mathrm{m}^{\mathcal{Q}}, \tag{5.5}$$

$$\lim_{t\to\infty}\mathbb{E}_{(x,\xi)}\left[\frac{1}{t}\langle \mathbf{K},\mathbf{K}\rangle_t(\omega)\right] = 2\int |\nabla K+\nabla u_{\mathfrak{K}}|^2 d\mathrm{m}^{\mathcal{Q}}. \tag{5.6}$$

Using Markov property, we have

$$\begin{aligned}\mathbb{E}_{(x,\xi)}\left[\frac{1}{t+1}\langle \mathbf{M},\mathbf{M}\rangle_{t+1}\right] &= \mathbb{E}_{(x,\xi)}\left[\frac{t}{t+1}\mathbb{E}_{\omega_1}\left[\frac{1}{t}\langle \mathbf{M},\mathbf{M}\rangle_t\right]\right]\\ &= \frac{t}{t+1}\mathbb{E}_{(x,\xi)}\left[\frac{1}{t}\int_0^t \mathcal{Q}^r F(\omega)dr\right].\end{aligned}$$

for $\mathbf{M}=\mathbf{B}$ or $\mathbf{K}$ and $F = 2|\nabla B+\nabla u_{\mathfrak{b}}|^2$ or $2|\nabla K+\nabla u_{\mathfrak{K}}|^2$, respectively. Given $x\in\widetilde{\mathcal{M}}$, for ν_x-a.e. ξ and $\mathbb{P}_{(x,\xi)}$-a.e. ω,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \mathcal{Q}^r F(\omega)dr = \int F d\mathrm{m}^{\mathcal{Q}}.$$

Hence for each x, there is ξ for which we have the limits (5.5) and (5.6). We denote the square root of the limits by $\sigma_{\mathfrak{b}}$ and $\sigma_{\mathfrak{K}}$, respectively. Note that both of $\sigma_{\mathfrak{b}},\sigma_{\mathfrak{K}}$ are positive since B and K are unbounded while $u_{\mathfrak{b}}$ and $u_{\mathfrak{K}}$ are bounded. We have $\sigma_{\mathfrak{b}},\sigma_{\mathfrak{K}}<\infty$ since both of $2|\nabla B+\nabla u_{\mathfrak{b}}|^2$ and $2|\nabla K+\nabla u_{\mathfrak{K}}|^2$ are bounded. Thus for every x, there is ξ such that the distributions of $\frac{\mathbf{B}_t}{\sigma_{\mathfrak{b}}\sqrt{t}}$ and $\frac{\mathbf{K}_t}{\sigma_{\mathfrak{K}}\sqrt{t}}$ under $\mathbb{P}_{(x,\xi)}$ converge to $N(0,1)$ as $t\to\infty$ due to the following lemma :

Lemma 5.2.1. *([He]) Let $(M_t)_{0\le t\le\infty}$ be a continuous, centered, square-integrable martingale on a filtered probability space with stationary increments. If $M_0=0$ and there is $\sigma>0$ such that*

$$\lim_{t\to\infty}\mathbb{E}\left[\left|\frac{1}{t}\langle M,M\rangle_t-\sigma^2\right|\right]=0, \tag{5.7}$$

then the distribution of $\frac{1}{\sigma\sqrt{t}}M_t$ *is asymptotically normal.*

Let $W_t^\ell(\omega) := \mathrm{d}(\widetilde{\omega}_0, \widetilde{\omega}_t) - t\ell$. Since the distribution of W_t^ℓ under $\mathbb{P}_{(x,\xi)}$ and the distribution of Y_t^ℓ under $\mathbb{P}_x$ coincide, it is enough to show that W_t^ℓ and $\mathbf{B}_t$ have the same $\mathbb{P}_{(x,\xi)}$-distribution. For $\mathbb{P}_{(x,\xi)}$-a.e. ω, since

$$B(\omega_t) - B(\omega_0) - \mathrm{d}(\widetilde{\omega}_0, \widetilde{\omega}_t) = \mathfrak{b}(\widetilde{\omega}_t, \widetilde{\omega}_0, \xi) - \mathrm{d}(\widetilde{\omega}_0, \widetilde{\omega}_t) \to -2(\xi|\widetilde{\omega}_\infty)_{\widetilde{\omega}_0}$$

as $t \to \infty$ and $|(\xi|\widetilde{\omega}_\infty)_{\widetilde{\omega}_0}| < \infty$,

$$\lim_{t\to\infty} \frac{1}{\sigma_{\mathfrak{b}}\sqrt{t}} \left[B(\omega_t) - B(\omega_0) - \mathrm{d}(\widetilde{\omega}_0, \widetilde{\omega}_t)\right] = 0.$$

Hence the distribution of $\frac{1}{\sigma_{\mathfrak{b}}\sqrt{t}}W_t^\ell$ under $\mathbb{P}_{(x,\xi)}$ also converges to the normal distribution since

$$W_t^\ell(\omega) = [\mathrm{d}(\widetilde{\omega}_0, \widetilde{\omega}_t) - B(\omega_t) + B(\omega_0)] - [u_{\mathfrak{b}}(\omega_t) - u_{\mathfrak{b}}(\omega_0)] + \mathbf{B}_t(\omega),$$

and

$$\frac{1}{\sigma_{\mathfrak{b}}\sqrt{t}} |u_{\mathfrak{b}}(\omega_t) - u_{\mathfrak{b}}(\omega_0)| \le \frac{2}{\sigma_{\mathfrak{b}}\sqrt{t}} \|u_{\mathfrak{b}}\|_\infty \to 0, \text{ as } t \to \infty.$$

Let $W_t^h(\omega) := \log \mathrm{G}(\widetilde{\omega}_0, \widetilde{\omega}_t) + th$. Since the $\mathbb{P}_{(x,\xi)}$-distribution of W_t^h and the $\mathbb{P}_x$-distribution of Y_t^h are the same, to verify that $\frac{1}{\sigma_{\mathfrak{K}}\sqrt{t}}W_t^h$ is asymptotically normal, it is sufficient to show that for $\mathbb{P}_{(x,\xi)}$-a.e. ω with a lift $(\widetilde{\omega}, \xi)$ to $\mathcal{S}\widetilde{\mathcal{M}}$,

$$\limsup_{t\to\infty} |\log \mathrm{G}(\widetilde{\omega}_0, \widetilde{\omega}_t) - K(\omega_t) + K(\omega_0)| < \infty. \tag{5.8}$$

Note that for $\mathbb{P}_{(x,\xi)}$-a.e. ω with a lift $(\widetilde{\omega}, \xi)$, $K(\omega_t) - K(\omega_0) = \log \mathfrak{K}(\widetilde{\omega}_0, \widetilde{\omega}_t, \xi)$ and $\widetilde{\omega}_\infty \neq \xi$. We denote by z_t the closest point to $\widetilde{\omega}_0$ on the geodesic ray $[\widetilde{\omega}_t, \xi)$ generated by $(\widetilde{\omega}_t, \xi)$, z_t converges to a point $z_\infty \in \widetilde{\mathcal{M}}$ on the geodesic $(\widetilde{\omega}_\infty, \xi)$ joining two boundary points $\widetilde{\omega}_\infty$ and ξ. We have that for every y on $[\widetilde{\omega}_t, \xi)$,

$$|\log \mathrm{G}(\widetilde{\omega}_0, \widetilde{\omega}_t) - \log \mathfrak{K}(\widetilde{\omega}_0, \widetilde{\omega}_t, \xi)| \le \left|\log \frac{\mathrm{G}(\widetilde{\omega}_0, \widetilde{\omega}_t)}{\mathrm{G}(z_t, \widetilde{\omega}_t)}\right| + \left|\log \frac{\mathrm{G}(z_t, \widetilde{\omega}_t)}{\left(\frac{\mathrm{G}(y, \widetilde{\omega}_t)}{\mathrm{G}(y, z_t)}\right)}\right| + \left|\log \frac{\left(\frac{\mathrm{G}(y, \widetilde{\omega}_t)}{\mathrm{G}(y, z_t)}\right)}{\mathfrak{K}(\widetilde{\omega}_0, \widetilde{\omega}_t, \xi)}\right|.$$

Applying the Harnack inequality to the first term in the right handed side, since $\{\mathrm{d}(\widetilde{\omega}_0, z_t)\}_{t\ge0}$ is bounded, it follows that $\left|\log \frac{\mathrm{G}(\widetilde{\omega}_0, \widetilde{\omega}_t)}{\mathrm{G}(z_t, \widetilde{\omega}_t)}\right| \le C_1$ for some constant $C_1 = C_1(\widetilde{\omega}) > 0$ dependent of $\widetilde{\omega}$ but not t. And by the Ancona inequality ([A]), the second term in the right handed side is also bounded by

$C_2(\widetilde{\omega})$. Letting y tend to ξ, we see that the last term converges to $\left|\log \frac{\mathfrak{K}(z_t,\widetilde{\omega}_t,\xi)}{\mathfrak{K}(\widetilde{\omega}_0,\widetilde{\omega}_t,\xi)}\right|$ which is also bounded by $C_1(\widetilde{\omega})$ due to the Harnack inequality. Therefore we have (5.8) and this completes the proof of Theorem 5.0.1.

We have an explicit formula of the variance $\sigma_{\mathfrak{K}}^2$ of the asymptotic distribution of the random variable $\frac{1}{\sqrt{t}}Y_t^h$.

Theorem 5.2.2. *The variance $\sigma_{\mathfrak{K}}^2$ has an explicit expression:*

$$\sigma_{\mathfrak{K}}^2 = 2h + \int |\nabla u_{\mathfrak{K}}|^2 d\mathrm{m}^{\mathcal{Q}}.$$

In particular, $\sigma_{\mathfrak{K}}^2 \geq 2h$ and the equality holds if and only if $|\nabla \log \mathfrak{K}|^2 = h$.

Proof. We begin with the proof of the integral equation for the foliated Laplacian ([Y]): for every bounded function φ uniformly $\mathcal{C}^2$ on stable leaves,

$$\int 2\langle \nabla \log \mathfrak{K}, \nabla\varphi\rangle d\mathrm{m}^{\mathcal{Q}} = -\int \Delta_s \varphi d\mathrm{m}^{\mathcal{Q}}. \tag{5.9}$$

Consider the function $\Phi(y) := \int_{\partial\widetilde{\mathcal{M}}} \varphi(y,\xi)d\nu_y(\xi) = \int_{\partial\widetilde{\mathcal{M}}} \varphi(y,\xi)\mathfrak{K}(x,y,\xi)d\nu_x(\xi)$. Applying the Laplacian, since $\Delta_y \mathfrak{K}(x,y,\xi) = 0$ we have

$$\begin{aligned}\Delta\Phi(y) &= \int_{\partial\widetilde{\mathcal{M}}} \mathfrak{K}(x,y,\xi)\Delta_s\varphi(y,\xi) + 2\langle \nabla_y\varphi(y,\xi), \nabla_y \mathfrak{K}(x,y,\xi)\rangle d\nu_x(\xi)\\ &= \int_{\partial\widetilde{\mathcal{M}}} \Delta_s\varphi(y,\xi) + 2\langle \nabla_y\varphi(y,\xi), \nabla_y \log\mathfrak{K}(x,y,\xi)\rangle d\nu_y(\xi).\end{aligned}$$

Thus integrating with respect to vol and using Green's formula, since Φ is uniformly $\mathcal{C}^2$,

$$\begin{aligned}&\int_{\mathcal{SM}} \Delta_s\varphi(y,\xi) + 2\langle \nabla_y\varphi(y,\xi), \nabla_y \log\mathfrak{K}(x,y,\xi)\rangle d\mathrm{m}^{\mathcal{Q}}(y,\xi)\\ &= \int_{\mathcal{M}} \Delta\Phi(x) d\,\mathrm{vol}(x)\\ &= \lim_{\varepsilon\to 0}\int_{\mathcal{M}_\varepsilon} \Delta\Phi(x) d\,\mathrm{vol}(x)\\ &= \lim_{\varepsilon\to 0}\int_{\partial\mathcal{M}_\varepsilon} \langle \nabla\Phi, \mathbf{n}_\varepsilon\rangle = 0,\end{aligned}$$

where $\mathcal{M}_\varepsilon = \{x\in\mathcal{M} : inj(x) \geq \varepsilon\}$ and $\mathbf{n}_\varepsilon$ is the unit normal vector on $\partial\mathcal{M}_\epsilon$.

From the integral formula (5.9) for the foliated Laplacian, it follows that

$$\sigma_{\mathfrak{K}}^2 = 2\int_{\mathcal{SM}} |\nabla \log \mathfrak{K}(x,\cdot,\xi) + \nabla u_{\mathfrak{K}}|^2 d\mathrm{m}^{\mathcal{Q}} = 2\int |\nabla \log \mathfrak{K}|^2 + |\nabla u_{\mathfrak{K}}|^2 d\mathrm{m}^{\mathcal{Q}},$$

since $\int \Delta_s u_{\mathfrak{K}} d\mathrm{m}^{\mathcal{Q}} = 0$. Since $\int |\nabla \log \mathfrak{K}|^2 d\mathrm{m}^{\mathcal{Q}} = h$ (Proposition 3.3.2),

$$\sigma_{\mathfrak{K}}^2 - 2h = 2\int |\nabla u_{\mathfrak{K}}|^2 d\mathrm{m}^{\mathcal{Q}} \geq 0,$$

and the equality holds if and only if $u_{\mathfrak{K}}$ is constant. It follows from the equation $\Delta_s u_{\mathfrak{K}} = |\nabla \log \mathfrak{K}|^2 - h$ that $u_{\mathfrak{K}}$ is constant if and only if $|\nabla \log \mathfrak{K}|^2$ is constant and equal to h. □

5.3 Proof of the contraction property

In this section, we prove the contraction property on Hölder spaces of the foliated Brownian motion. For the Hölder semi-norm, we prove a lower bound of the expectation of the Busemann functions at Brownian points which depends only on the dimension and the curvature bounds and linearly on time T. The lower bound follows from the fact that the Laplacian of the Busemann function has the same lower bound with the Laplacian of the distance function due to the Rauch comparison theorem. We also show the Doeblin property of the Brownian motion for the estimate of the uniform norm.

Proposition 5.3.1. *For sufficiently small τ, there exists $C_1 > 0$ such that for each $t > 0$,*

$$\sup_{x\in\mathcal{M}_0} \sup_{\xi,\eta\in\partial\widetilde{\mathcal{M}}} \frac{|\mathcal{Q}^t f(x,\xi) - \mathcal{Q}^t f(x,\eta)|}{d_\infty^{x,\tau}(\xi,\eta)} \leq \|f\|_{\mathcal{L}^\tau} e^{-C_1 t}.$$

Proof. Since we have that

$$\begin{aligned}
\frac{|\mathcal{Q}^t f(x,\xi) - \mathcal{Q}^t f(x,\eta)|}{d_\infty^{x,\tau}(\xi,\eta)} &\leq \int_{\widetilde{\mathcal{M}}} \wp(t,x,y) \frac{\left|\widetilde{f}(y,\xi) - \widetilde{f}(y,\eta)\right|}{d_\infty^{x,\tau}(\xi,\eta)} d\,\mathrm{vol}_{\widetilde{\mathcal{M}}}(y) \\
&\leq \|f\|_{\mathcal{L}^\tau} \int_{\widetilde{\mathcal{M}}} \wp(t,x,y) \frac{d_\infty^{y,\tau}(\xi,\eta)}{d_\infty^{x,\tau}(\xi,\eta)} d\,\mathrm{vol}_{\widetilde{\mathcal{M}}}(y) \\
&= \|f\|_{\mathcal{L}^\tau} \mathbb{E}_x \left[\frac{d_\infty^{\widetilde{\omega}_t,\tau}(\xi,\eta)}{d_\infty^{x,\tau}(\xi,\eta)}\right],
\end{aligned}$$

it is sufficient to find $C_1 > 0$ such that

$$\sup_{x,\xi,\eta} \mathbb{E}_x \left[\frac{d_\infty^{\widetilde{\omega}_t,\tau}(\xi,\eta)}{d_\infty^{x,\tau}(\xi,\eta)} \right] < e^{-C_1 t}.$$

Due to the Markov property of the Brownian motion,

$$\begin{aligned} \sup_{x,\xi,\eta} \mathbb{E}_x \left[\frac{d_\infty^{\widetilde{\omega}_{t+s},\tau}(\xi,\eta)}{d_\infty^{x,\tau}(\xi,\eta)} \right] &= \sup_{x,\xi,\eta} \mathbb{E}_x \left[\frac{d_\infty^{\widetilde{\omega}_s,\tau}(\xi,\eta)}{d_\infty^{x,\tau}(\xi,\eta)} \mathbb{E}_x \left[\left. \frac{d_\infty^{\widetilde{\omega}_{t+s},\tau}(\xi,\eta)}{d_\infty^{\widetilde{\omega}_s,\tau}(\xi,\eta)} \right| \mathscr{F}_s(\widetilde{\mathcal{M}}) \right] \right] \\ &\leq \sup_{x,\xi,\eta} \mathbb{E}_x \left[\frac{d_\infty^{\widetilde{\omega}_t,\tau}(\xi,\eta)}{d_\infty^{x,\tau}(\xi,\eta)} \right] \sup_{x,\xi,\eta} \mathbb{E}_x \left[\frac{d_\infty^{\widetilde{\omega}_s,\tau}(\xi,\eta)}{d_\infty^{x,\tau}(\xi,\eta)} \right]. \end{aligned}$$

Let us write $g(\widetilde{\omega}_t) := (\xi|\eta)_{\widetilde{\omega}_t} - (\xi|\eta)_x$. Applying the Taylor theorem to the function $R \mapsto \exp(-\tau R)$ and substituting $g(\widetilde{\omega}_t)$ for R, we have

$$\frac{d_\infty^{\widetilde{\omega}_t,\tau}(\xi,\eta)}{d_\infty^{x,\tau}(\xi,\eta)} \leq 1 - \tau g(\widetilde{\omega}_t) + \tau^2 \mathrm{d}(x,\widetilde{\omega}_t)^2 e^{2\tau \mathrm{d}(x,\widetilde{\omega}_t)}.$$

By Proposition 3.2.2, for some constant $C_1' > 0$,

$$\sup_x \mathbb{E}_x \left[\mathrm{d}(x,\widetilde{\omega}_t)^2 e^{2\tau \mathrm{d}(x,\widetilde{\omega}_t)} \right] < C_1'. \tag{5.10}$$

Therefore, with (5.10) and Lemma 5.3.2 below, we have

$$\sup_{0\leq t\leq T} \sup_{x,\xi,\eta} \mathbb{E}_x \left[\frac{d_\infty^{\widetilde{\omega}_t,\tau}(\xi,\eta)}{d_\infty^{x,\tau}(\xi,\eta)} \right] \leq 1 - \tau(d-1)a + \tau^2 C_1'.$$

Fix $T \geq 1$ and sufficiently small τ such that $1-\tau(d-1)a+\tau^2 C_1' < 1$. For such small τ, put $C_1 = (1 - a(d-1)\tau + C_1'\tau^2)^{\frac{1}{T}}$ and the inequality follows. □

Lemma 5.3.2. *For every $T \geq 0$,*

$$\inf_{x\in\mathcal{M}_0} \inf_{\xi\neq\eta} \mathbb{E}_x[(\xi|\eta)_{\widetilde{\omega}_T} - (\xi|\eta)_x] \geq (d-1)aT.$$

Proof of Lemma 5.3.2. Due to the equation

$$(\xi|\eta)_x - (\xi|\eta)_y = \frac{1}{2}\mathfrak{b}(x,y,\xi) + \frac{1}{2}\mathfrak{b}(x,y,\eta),$$

it suffices to show that

$$\mathbb{E}_x[\mathfrak{b}(\widetilde{\omega}_T, x, \eta)] \geq (d-1)aT.$$

Choose $z_n \in \widetilde{\mathcal{M}}$ such that $z_n \to \xi$ as $n \to \infty$ and write

$$f_n(y) := \mathfrak{b}(y, x, z_n) = \mathrm{d}(y, z_n) - \mathrm{d}(x, z_n).$$

By the Laplacian comparison (2.3),

$$\begin{aligned} \Delta f_n(y) = \Delta_y \mathrm{d}(y, z_n) &\geq (d-1)\frac{\mathrm{sn}'_{-a^2}(\mathrm{d}(y, z_n))}{\mathrm{sn}_{-a^2}(\mathrm{d}(y, z_n))} \\ &= a(d-1)\coth\left(a\,\mathrm{d}(y, z_n)\right) \end{aligned} \tag{5.11}$$

where $\mathrm{sn}_{-a^2}(t) = \frac{1}{a}\sinh(at)$.

Let $f(y, \xi) = \mathfrak{b}(y, x, \xi)$. Then, since Δ_s is the generator of $\mathcal{Q}^t$ and $\Delta_s f(y, \xi) = \Delta_y f(y, \xi)$,

$$\mathbb{E}_x[\mathfrak{b}(\widetilde{\omega}_T, x, \xi)] = \mathcal{Q}^T f(x, \xi) = \int_0^T \mathcal{Q}^t \Delta_s f(x, \xi) dt = \int_0^T \mathbb{E}_x[\Delta \mathfrak{b}(\widetilde{\omega}_t, x, \xi)] dt.$$

Due to (5.11) and Proposition 2.3.8,

$$\mathbb{E}_x[\mathfrak{b}(\widetilde{\omega}_T, x, \xi)] \geq (d-1)aT$$

for every $x \in \widetilde{\mathcal{M}}$ and every $\xi \in \partial\widetilde{\mathcal{M}}$. □

Let $\wp_{\mathcal{M}}(t, x, y) = \sum_{\gamma \in \Gamma} \wp(t, x, \gamma y)$ for $x, y \in \mathcal{M}_0$ be the heat kernel on $\mathcal{M}$. We have $\lim\limits_{t\to\infty} \wp_{\mathcal{M}}(t, x, y) = \frac{1}{\mathrm{vol}(\mathcal{M})}$, in particular, $\wp_{\mathcal{M}}(t, x, x)$ decreases as $t \to \infty$ (see [CK]). We also have that

$$\begin{aligned} &\frac{1}{\mathrm{vol}(\mathcal{M})}\left(\int_{\mathcal{M}} \left|\wp_{\mathcal{M}}(t, x, y) - \frac{1}{\mathrm{vol}(\mathcal{M})}\right| d\,\mathrm{vol}(y)\right)^2 \\ &\leq \int_{\mathcal{M}} \left|\wp_{\mathcal{M}}(t, x, y) - \frac{1}{\mathrm{vol}(\mathcal{M})}\right|^2 d\,\mathrm{vol}(y) \\ &= \left(\wp_{\mathcal{M}}(2t, x, x) - \frac{1}{\mathrm{vol}(\mathcal{M})}\right). \end{aligned} \tag{5.12}$$

Hence the integral on the left-handed side decreases to zero as t goes to infinity. Indeed, it decays exponentially fast (see [Don]). The following lemma shows that it has uniform exponential decay rate.

Lemma 5.3.3. *There exists a constant $C_2 = C_2(d,b) > 0$ such that for each $x \in \mathcal{M}$,*

$$\int_{\mathcal{M}} \left| \wp_{\mathcal{M}}(t,x,y) - \frac{1}{\mathrm{vol}(\mathcal{M})} \right| d\,\mathrm{vol}(y) \le C_2 e^{-\frac{\lambda_1}{2}t},$$

where $\lambda_1 = \inf\{\lambda > 0 : \lambda \in \mathrm{Spec}(\Delta_{\mathcal{M}})\}$.

Remark 5.3.4. *Since the bottom of the (L^2-)esssential spectrum $\lambda_{ess} :=$ $\inf \mathrm{Spec}_{ess}(\Delta_{\mathcal{M}})$ of the Laplacian is positive ([Dod]) and $\mathrm{Spec}(\Delta_{\mathcal{M}}) \cap [0, \lambda_{ess})$ is discrete ([Don]), the smallest nonzero the spectrum λ_1 is also positive.*

Proof. If we consider $\mathcal{P}^t f(x) := \int (\wp_{\mathcal{M}}(t,x,y) - \mathrm{vol}(\mathcal{M})^{-1}) f(y) d\,\mathrm{vol}(y)$ as a self-adjoint operator acting on the space $L_0^2(\mathcal{M})$ of square-integrable functions with zero integral, $\Delta|_{L_0^2(\mathcal{M})}$ is the generator of $\mathcal{P}^t$ with the bottom of the spectrum λ_1. Therefore the operator norm satisfies

$$\|\mathcal{P}^t\| \le e^{-\frac{\lambda_1 t}{2}} \tag{5.13}$$

for every $t > 0$ (see the proof of Proposition V.1.2 in [EN]).

For every $x \in \mathcal{M}_0$, if we denote $f_t(y) = \wp_{\mathcal{M}}(t,x,y) - \frac{1}{\mathrm{vol}(\mathcal{M})}$, then $f_{t+t_0}(y) = \mathcal{P}^t f_{t_0}(y)$. It follows from (5.12) and (5.13) that

$$\begin{aligned}
\int_{\mathcal{M}} \left| \wp_{\mathcal{M}}(t+t_0,x,y) - \frac{1}{\mathrm{vol}(\mathcal{M})} \right| d\,\mathrm{vol}(y) &\le \left(\mathrm{vol}(\mathcal{M}) \int_{\mathcal{M}} |f_{t+t_0}(y)|^2 \, d\,\mathrm{vol}(y) \right)^{1/2} \\
&\le \|\mathcal{P}^t\| \, \|f_{t_0}\|_2 \le e^{-\frac{\lambda_1 t}{2}} |f_{2t_0}(x)|^{1/2} \\
&= e^{-\frac{\lambda_1 t}{2}} \left| \wp_{\mathcal{M}}(2t_0,x,x) - \frac{1}{\mathrm{vol}(\mathcal{M})} \right|^{1/2}.
\end{aligned}$$

Thus it suffices to prove that the diagonal supremum $\sup\limits_{x\in\mathcal{M}} \wp_{\mathcal{M}}(2t_0,x,x)$ of the heat kernel on $\mathcal{M}$ is finite for some $t_0 > 0$.

Fix $x_0 \in \mathcal{M}_0$ and let $\mathcal{M}_0 = \mathcal{M}(x_0)$ be the Dirichlet domain for Γ with center x_0. In order to estimate the diagonal supremum $\sup\limits_{x\in\mathcal{M}} \wp_{\mathcal{M}}(t,x,x)$ of the heat kernel on $\mathcal{M}$, we shall use the Gaussian upper bound of the heat kernel on $\widetilde{\mathcal{M}}$ (Corollary 5 in [Gr]): there is a constant $C = C(d,b)$ such that for each $t > 1$,

$$\wp(t,x,y) \le C \left(\frac{\mathrm{d}(x,y)^2}{t} \right)^{1+\frac{d}{2}} \exp\left(-\frac{\mathrm{d}(x,y)^2}{4t} - \lambda_0 t \right). \tag{5.14}$$

For a cuspidal point $\xi \in \Pi(\mathcal{M}_0) := \partial\widetilde{\mathcal{M}} \cap \overline{\mathcal{M}}_0$, we denote the cuspidal region of level n based at ξ by

$$\mathcal{H}(\xi, n) := \{y \in \mathcal{M}_0 : \mathfrak{b}(x_0, y, \xi) \geq n\}.$$

Let $x_n = x_n^\xi \in \mathcal{H}(\xi, n)$ be the point in the geodesic ray joining x_0 and ξ with $\mathfrak{b}(x_0, x_n, \xi) = n$. If γ is in the stabilizer Γ_ξ of ξ, then x_0 and γx_0 are in the horosphere of the same level based at ξ. This implies that for every $\gamma \in \Gamma_\xi$,

$$e^{-b(n+1)}\mathrm{d}(x_0, \gamma x_0) \leq \mathrm{d}(x_n, \gamma x_n) \leq e^{-an}\mathrm{d}(x_0, \gamma x_0). \tag{5.15}$$

Applying (5.15) to the Gaussian bound (5.14), for each $\gamma \in \Gamma_\xi$,

$$\begin{aligned} \wp(t, x_n, \gamma x_n) \leq{}& Ce^{-\lambda_0 t}\mathrm{d}(x_0, \gamma x_0)^{d+2} \\ &\times \exp\left(-\frac{\mathrm{d}(x_0, \gamma x_0)^2}{4te^{2b(d-1)(n+1)}} - a(d+2)n\right). \end{aligned} \tag{5.16}$$

We want to show that given $\delta > 0$, there is $t > 0$ such that for every sufficiently large n,

$$\text{the right-hand side of (5.16)} \;\leq e^{-\delta \mathrm{d}(x_0, \gamma x_0)}.$$

To simplify the notation, we put

$$f_{n,\xi}(R) := R^{d+2} \exp\left(-\frac{R^2}{4te^{2b(d-1)(n+1)}} + \delta R\right).$$

Since its derivative is

$$\begin{aligned} f'_{n,\xi}(R) = R^{d+1} &\left(d + 2 - \frac{R^2}{2t}e^{-2b(d+2)(n+1)} + \delta R\right) \\ &\times \exp\left(-\frac{R^2}{4t}e^{-2b(d+2)(n+1)} + \delta R\right), \end{aligned}$$

the positive nonzero extreme point of $f_{n,\xi}$ is

$$R_n := t\delta e^{2b(n+1)} + \sqrt{t^2\delta^2 e^{4b(n+1)} + 2t(d+2)e^{2b(n+1)}}.$$

Thus $f_{n,\xi}$ has the maximum on $\mathbb{R}_+$ at R_n:

$$\begin{aligned} f_{n,\xi}(R) &\leq f_{n,\xi}(R_n) \\ &= R_n^{d+2} \exp\left(-\frac{R_n^2}{4t} + \delta R_n - a(d+2)n\right) \\ &\leq \left(3t\delta e^{2b(n+1)}\right)^{d+2} \exp\left(-\frac{t\delta^2 e^{4b(n+1)}}{2} + 3t\delta^2 e^{2b(n+1)} - a(d+2)n\right) \\ &= \left[(3t\delta) e^{2b(n+1)-an}\right]^{d+2} e^{-\frac{9\delta^2 t}{2}} \exp\left(-\frac{\delta^2 t}{2}(e^{2b(n+1)} - 3)^2\right). \end{aligned}$$

Therefore, there is $N_\xi(\delta, t)$ such that if $n > N_\xi(\delta, t)$, then $f_{n,\xi}(R) \leq C^{-1} t^{-1-\frac{d}{2}} e^{\lambda_0 t}$, hence $\wp(t, x_n, \gamma x_n) \leq e^{-\delta \mathrm{d}(x_0, \gamma x_0)}$. We conclude that

$$\sum_{\gamma \in \Gamma_\xi} \wp(t, x_n, \gamma x_n) \leq \sum_{\gamma \in \Gamma_\xi} e^{-\delta \mathrm{d}(x_0, \gamma x_0)} = Q_{\Gamma_\xi, x_0}(\delta), \tag{5.17}$$

where $Q_{G,x}(\delta) := \sum_{g \in G} e^{-\delta \mathrm{d}(x, gx)}$ denotes the Poincaré series of a discrete group G of isometries on $\widetilde{\mathcal{M}}$. We denote the abscissa of convergence of $Q_{G,x}$, which is called the *critical exponent* of G, by δ_G.

Put $N_\xi := N_\xi(\delta_\Gamma + 1, t)$ and choose N larger than $\max_{\xi \in \Pi(\mathcal{M}_0)} N_\xi$. We define a truncated domain in the fundamental domain $\mathcal{M}_0$ by

$$\mathcal{M}_N := \mathcal{M}_0 \setminus \bigcup_{\xi \in \Pi(\mathcal{M}_0)} \mathcal{H}(\xi, N).$$

Note that $\mathcal{M}_N$ is a pre-compact domain. Take $x_0 \in \mathcal{M}_N$ and $x \in \mathcal{H}(\xi, N)$ for some $\xi \in \Pi(\mathcal{M}_0)$. Then we can replace x by $x_n = x_n^\xi$ for some $n \geq N$: there is $n \geq N$ such that $x \in \mathcal{H}(\xi, n) \setminus \mathcal{H}(\xi, n+1)$ and $d(x, x_n)$ is bounded uniformly on $n \geq N$.

We may assume that given $t > 0$, $g(R) = \left(\frac{R^2}{t}\right)^{1+\frac{d}{2}} \exp\left(-\frac{R^2}{4t}\right)$ is decreasing and $g(R) \leq \exp(-(\delta_\Gamma + 1)R)$ for every $R > 0$. Assume that x_n is on the geodesic ray $[x_0, \xi)$ joining x_0 and ξ and $d(x_0, x_n) = n$. From $\mathrm{d}(x_N, \gamma x_N) - 2(n - N) \leq \mathrm{d}(x_n, \gamma x_n)$, writing $R_n = \mathrm{d}(x_n, \gamma x_n)$ for $n \geq N$, there exists $C' = C'(d) > 1$ such that

$$
\begin{aligned}
g(R_n) &\leq g(R_N - 2(n-N)) \\
&= \left(\frac{(R_N - 2(n-N))^2}{t}\right)^{1+d/2} \exp\left(-\frac{(R_N - 2(n-N))^2}{4t}\right) \\
&\leq C' g(R_N) g_N(2(n-N)),
\end{aligned}
$$

where $g_N(T) := \left(\frac{T}{\sqrt{t}}\right)^{d+2} \exp\left(-\frac{T^2 - R_N T}{4t}\right)$. By the similar computation as in (5.17),

$$
g_N(T) \leq g_N(T_N) \leq \left(\frac{R_N}{\sqrt{t}}\right)^{d+2} \exp\left(\frac{15}{64t} R_N^2\right),
$$

where T_N the critical value of g_N. Thus we have

$$
g(R_n) \leq C' \left(\frac{R_N}{t}\right)^{2d+4} \exp\left(-\frac{1}{16t} R_N^2\right) \leq C'' e^{-(\delta_\Gamma + 1)\mathrm{d}(x_N, \gamma x_N)},
$$

for some $C'' > 1$ independent of N. Then it follows that

$$
\begin{aligned}
P(t,x,x) \leq & C \sum_{\gamma \in \Gamma} \left(\frac{\mathrm{d}(x,\gamma x)^2}{t}\right)^{1+\frac{d}{2}} \exp\left(-\frac{\mathrm{d}(x,\gamma x)^2}{4t} - \lambda t\right) \\
\leq & C Q_{\Gamma_\xi, x_0}(\delta_\Gamma + 1) + C C'' \sum_{\gamma \notin \Gamma_\xi} e^{-(\delta_\Gamma + 1)\mathrm{d}(x_N, \gamma x_N)} \\
\leq & C(1 + C'') \max\left\{ Q_{\Gamma, x_0}(\delta_\Gamma + 1), Q_{\Gamma, x_N^\xi}(\delta_\Gamma) \right\}.
\end{aligned}
$$

Hence we have $\sup_{x \in \mathcal{H}(\xi,N)} P(t,x,x) < \infty$ for every $\xi \in \Pi(\mathcal{M}_0)$. Therefore, since $\Pi(\mathcal{M}_0)$ is a finite set, $\sup_{x \in \mathcal{M}} P(t,x,x) < \infty$. □

We are ready to verify the exponential decay of uniform norm and complete the proof of Theorem 5.1.4. It is enough to show that the exponential decay of the supremum norm since we have already proved the exponential decay of Hölder norm in Proposition 5.3.1.

Proposition 5.3.5. *There exists a constant $C_2 > 0$ such that for every $f \in \mathcal{L}$ and $t > 0$,*

$$
\|\mathcal{Q}^t f - \mathcal{N} f\|_\infty \leq \|f\|_{\mathcal{L}_\tau} e^{-C_2 t}.
$$

Proof. Denote $F_t(x) := \int \mathcal{Q}^t f(x,\xi) d\nu_x(\xi)$.

$$\begin{aligned}
\left|\mathcal{Q}^t f(x,\xi) - \int f d\mathrm{m}^{\mathcal{Q}}\right| &= \left|\mathcal{Q}_t f(x,\xi) - \int \mathcal{Q}^{\frac{t}{2}} f d\mathrm{m}^{\mathcal{Q}}\right| \\
&\le \left|\mathcal{Q}^t f(x,\xi) - \mathcal{Q}^{\frac{t}{2}} F_{\frac{t}{2}}(x)\right| + \left|\mathcal{Q}^{\frac{t}{2}} F_{\frac{t}{2}}(x) - \int \mathcal{Q}^{\frac{t}{2}} f d\mathrm{m}^{\mathcal{Q}}\right| \\
&\le \left|\mathcal{Q}^{\frac{t}{2}} \left(\mathcal{Q}^{\frac{t}{2}} f(x,\xi) - F_{\frac{t}{2}}(x)\right)\right| + \left|\mathcal{Q}^{\frac{t}{2}} F_{\frac{t}{2}}(x) - \int \mathcal{Q}^{\frac{t}{2}} f d\mathrm{m}^{\mathcal{Q}}\right|.
\end{aligned}$$

By Lemma 5.3.3, the last term of the last inequality decays exponentially:

$$\begin{aligned}
\left|\mathcal{Q}^{\frac{t}{2}} F_{\frac{t}{2}}(x) - \int \mathcal{Q}^{\frac{t}{2}} f d\mathrm{m}^{\mathcal{Q}}\right| &= \left|\int_{\mathcal{M}_0} P(t/2,x,y) F_{\frac{t}{2}}(y) d\,\mathrm{vol}(y) - \int_{\mathcal{M}_0} F_{\frac{t}{2}}(y) d\widetilde{m}(y)\right| \\
&\le \|F_{\frac{t}{2}}\|_\infty \int_{\mathcal{M}_0} \left|P(t/2,x,y) - \frac{1}{\mathrm{vol}(\mathcal{M}_0)}\right| d\,\mathrm{vol}(y) \\
&\le \|f\|_\tau e^{-\frac{\lambda_1 t}{4}}.
\end{aligned}$$

For the first term, it follows from Proposition 5.3.1 that

$$\begin{aligned}
\left|\mathcal{Q}^{\frac{t}{2}} \left(\mathcal{Q}^{\frac{t}{2}} f(x,\xi) - F_{\frac{t}{2}}(x,\xi)\right)\right| &\le \sup_{y\in\mathcal{M}_0} \left|\mathcal{Q}^{\frac{t}{2}} f(y,\xi) - F_{\frac{t}{2}}(y,\xi)\right| \\
&\le \sup_{y\in\mathcal{M}_0} \int \left|\mathcal{Q}^{\frac{t}{2}} f(y,\xi) - \mathcal{Q}^{\frac{t}{2}} f(y,\eta)\right| d\nu_y(\eta) \\
&\le \|f\|_\tau e^{-\frac{C_1 t}{2}}.
\end{aligned}$$

□

5.4 Asymptotically harmonic manifolds

In this section, we study characterizations of asymptotically harmonic manifolds with pinched negative curvature and uniformly bounded first derivatives of sectional curvature as a corollary of the central limit theorem of Brownian motions.

Definition 5.4.1 (Asymptotically harmonic manifold)**.** An *asymptotically harmonic manifold* is a complete, connected and simply connected manifold $\mathcal{N}$ with $\mathrm{Sec}_{\mathcal{N}} < 0$ such that every horosphere has constant mean curvature H for some constant H.

Note that $\widetilde{\mathcal{M}}$ is asymptotically harmonic if and only if for each $(x, \xi) \in \mathcal{S}\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}} \times \partial_\infty \widetilde{\mathcal{M}}$,

$$\Delta[\mathfrak{b}_x(\xi)](x) = H,$$

where $[\mathfrak{b}_x(\xi)](y) = \mathfrak{b}(x, y, \xi)$.

Remark 5.4.2. *While we shall focus on the negatively curved case, one can generalize the definition of asymptotic harmonicity for complete simply connected manifolds without conjugate point (see [KP1], [KP2]). For an asymptotically harmonic manifold $\mathcal{N}$ with uniformly bounded curvature tensor and their first derivatives, the following statements are equivalent (see [KP1]):*

1. *$\mathcal{N}$ has rank one;*
2. *The geodesic flow on the unit tangent bundle of $\mathcal{N}$ is an Anosov flow;*
3. *$\mathcal{N}$ is a Gromov hyperbolic space;*
4. *Volume growth in $\mathcal{N}$ is purely exponential and its exponential growth rate is the mean curvature of horospheres.*

It follows from $\Delta[\mathfrak{b}_x(\xi)](x) = -F^{su}(x, \xi)$ that asymptotic harmonicity implies that $F^{su} = -h_{\mathrm{m}}(\mathbf{g}^1)$; since

$$P_{F^{su}} = P_{F^{su}}(\mathrm{m}) = h_{\mathrm{m}}(\mathbf{g}^1) + \int F^{su} d\,\mathrm{m} = 0,$$

$h_{\mathrm{m}}(\mathbf{g}^1) = H$. Every constant potential has the same Gibbs measure, so the Liouville measure and the Bowen-Margulis measure coincide. Therefore we conclude that $H = h_{\mathrm{top}}$, where h_{top} denotes the topological entropy of $(\mathbf{g}^t)$ on $\widetilde{\mathcal{M}}$.

Remark 5.4.3. *It is shown in [CS] that the topological entropy h_{top} coincides with the volume entropy h_{vol} of $\widetilde{\mathcal{M}}$ and the L^2-spectrum of $\Delta_{\widetilde{\mathcal{M}}}$ is*

$$\mathrm{Spec}(\Delta_{\widetilde{\mathcal{M}}}) = \left[h_{\mathrm{top}}^2/4, \infty\right).$$

Combining with the results in Section 4.2, we have some properties of asymptotically harmonic manifolds. It is a generalization to the co-finite case of an elegant work by F. Ledrappier on characterization of co-compact asymptotically harmonic manifolds with negative curvature [Le4].

Proposition 5.4.4. *Suppose that $\mathcal{M}$ admits an equilibrium state for F^{su}. Then the following statements are equivalent:*

1. *for each $x \in \widetilde{\mathcal{M}}$, $\mu_x = \nu_x$;*
2. *for each $x, y \in \widetilde{\mathcal{M}}$ and $\xi \in \partial_\infty \widetilde{\mathcal{M}}$,*
$$\mathfrak{K}(x, y, \xi) = \exp(h_{\mathrm{top}} \mathfrak{b}(x, y, \xi));$$
3. *$\widetilde{\mathcal{M}}$ is asymptotically harmonic;*
4. *$4\lambda_0 = h_{\mathrm{top}}^2$.*

In particular, $\Delta_y \log \mathfrak{K}(x, y, \xi) = |\nabla_y \log \mathfrak{K}(x, y, \xi)|^2 = h_{\mathrm{top}}^2$.

Proof. It is clear that 1 and 2 are equivalent to each other. By Theorem 4.2.5, 1 or 2 implies $h = \ell h_{\mathrm{top}}$. Since 2 also implies that $|\nabla_y \log \mathfrak{K}(x, y, \xi)|^2 = h_{\mathrm{top}}^2$, we have
$$\ell h_{\mathrm{top}} = h = \int |\nabla \log \mathfrak{K}(x, \cdot, \xi)|^2(x) d\mathrm{m}^{\mathcal{Q}}(x, \xi) = h_{\mathrm{top}}^2.$$
Hence we deduce that $\ell = h_{\mathrm{top}}$ and $h = h_{\mathrm{top}}^2$. Therefore, it follows that
$$\Delta[\mathfrak{b}_x(\xi)] = -h_{\mathrm{top}}^{-1} \Delta \log \mathfrak{K}(x, \cdot, \xi) = h_{\mathrm{top}}^{-1} |\nabla \log \mathfrak{K}(x, \cdot, \xi)|^2 = h_{\mathrm{top}}.$$
Asymptotic harmonicity implies $4\lambda_0 = h_{\mathrm{top}}^2$ by Remark 5.4.3.

Now we suppose that $4\lambda_0 = h_{\mathrm{top}}^2$. It follows from (4.5) that $h = \ell h_{\mathrm{top}} = h_{\mathrm{top}}^2$, which implies coincidence of the visibility class and the harmonic class by Theorem 4.2.5. Now we denote the Radon-Nikodym derivative of ν_x with respect to μ_x by $U(x, \cdot)$: for $\xi \in \partial_\infty \widetilde{\mathcal{M}}$,
$$\frac{d\nu_x}{d\mu_x}(\xi) = U(x, \xi).$$
Then for each $x, y \in \widetilde{\mathcal{M}}$ and $\xi \in \partial_\infty \widetilde{\mathcal{M}}$,
$$\begin{aligned}
&\log U(x, \xi) = \log \frac{d\nu_x}{d\mu_x}(\xi) \\
&= \log \frac{d\nu_x}{d\nu_y}(\xi) + \log \frac{d\nu_y}{d\mu_y}(\xi) + \log \frac{d\mu_y}{d\mu_x}(\xi) \\
&= -\log \mathfrak{K}(x, y, \xi) + \log U(y, \xi) + h_{\mathrm{top}} \mathfrak{b}(x, y, \xi).
\end{aligned}$$

Hence $\log U(x,\xi)$ is smooth in x and Hölder continuous in ξ. Taking the gradient and Laplacian in the variable y, we have

$$\nabla_y \log \mathfrak{K}(x,y,\xi) = \nabla_y \log U(y,\xi) + h_{\text{top}} X(y,\xi); \tag{5.18}$$

$$-|\nabla_y \log \mathfrak{K}(x,y,\xi)|^2 = \Delta_y \log U(x,\xi) - h_{\text{top}} F^{su}(y,\xi), \tag{5.19}$$

where $X(x,\xi) := (x,\xi)$. Comparing the squared modulus of the first equation with the second equation,

$$\begin{aligned} h_{\text{top}}^2 + 2h_{\text{top}}\langle X(y,\xi), \nabla_y \log U(y,\xi)\rangle + |\nabla_y \log U(y,\xi)|^2 \\ = h_{\text{top}} F^{su}(y,\xi) - \Delta_y \log U(y,\xi). \end{aligned} \tag{5.20}$$

If we integrate both sides of (5.20),

$$h_{\text{top}}^2 + \int 2h_{\text{top}}\langle X, \nabla \log U\rangle + |\nabla \log U|^2 d\mathrm{m}^{\mathcal{Q}} = \ell h_{\text{top}}.$$

Due to $h_{\text{top}}^2 = h = \ell h_{\text{top}}$, the equation is reduced to

$$\int 2h_{\text{top}}\langle X, \nabla \log U\rangle + |\nabla \log U|^2 d\mathrm{m}^{\mathcal{Q}} = 0.$$

Note that from (5.18) and Proposition 3.3.2, since $X(y,\xi) = \nabla_y \mathfrak{b}(x,y,\xi)$,

$$\int \langle X, \nabla \log U\rangle d\mathrm{m}^{\mathcal{Q}} = \int \langle X, \nabla \log \mathfrak{K}\rangle d\mathrm{m}^{\mathcal{Q}} - h_{\text{top}} = \ell - h_{\text{top}} = 0.$$

Therefore, $|\nabla \log U| = 0$ and $\log U(\cdot,\xi)$ is a constant function for each ξ. As $\log U(x,\xi)$ is continuous in ξ and there is a dense central stable leaf of $\mathcal{SM}$, we conclude that $\log U$ is constant on $\mathcal{S}\widetilde{\mathcal{M}}$ and $\nu_x = \mu_x$. □

As a corollary of the central limit theorem of Brownian motions, we also have another characterization of asymptotically harmonic manifolds with pinched negative curvature and uniformly bounded first derivatives of sectional curvature.

Proposition 5.4.5. *Suppose that $\mathcal{M}$ admits an equilibrium state for F^{BM}. Then $\widetilde{\mathcal{M}}$ is asymptotically harmonic if and only if $\sigma_{\mathfrak{K}}^2 = 2h$.*

Proof. First we verify that $|\nabla \log \mathfrak{K}|^2 = h$ implies the asymptotic harmonicity of $\mathcal{M}$. It follows from Proposition 3.3.2 and Theorem 4.2.5 that

$$\begin{aligned} h &\le \ell h_{\text{top}} \le \sup_{\mu} -\ell \int F^{\text{BM}} d\mu \\ &= \sup_{\mu} \ell \int \langle X, \nabla \log \mathfrak{K} \rangle d\mu \le \sup_{\mu} \ell \left| \int |\nabla \log \mathfrak{K}|^2 d\mu \right|^{1/2} = \ell \sqrt{h} \le h, \end{aligned}$$

where $X(x,\xi) := (x,\xi)$ and μ in the supremum is taken among invariant measures. Hence inequalities are equalities and ν is the measure of maximal entropy since the second inequality holds if and only if $\mu = \nu$.

Replacing the supremum of integrations by integrations with respect to ν, we have

$$\int \langle X, \nabla \log \mathfrak{K} \rangle d\mu = \left| \int |\nabla \log \mathfrak{K}|^2 d\mu \right|^{1/2},$$

which occurs if and only if $X = \nabla \log \mathfrak{K}$. Therefore we have

$$\text{div} X = \Delta \log \mathfrak{K} = -|\nabla \log \mathfrak{K}|^2 = -h.$$

Conversely, if $\mathcal{M}$ is asymptotically harmonic, by Proposition 5.4.4,

$$\mathfrak{K}(x,y,\xi) = \exp(h_{\text{top}} \mathfrak{b}(x,y,\xi))$$

and $\|\nabla \log \mathfrak{K}\|^2 = h_{\text{top}}^2$. From $h = \ell h_{\text{top}}$ and $\ell = h_{\text{top}}$, it follows that $h = h_{\text{top}}^2 = \|\nabla \log \mathfrak{K}\|^2$. Therefore we have $\sigma_{\mathfrak{K}}^2 = 2h$. This completes the proof. ☐

We have the following corollary from Proposition 5.4.4 and the proof of Proposition 5.4.5.

Proposition 5.4.6. *Suppose that $\mathcal{M}$ admits an equilibrium state for F^{BM}. $\widetilde{\mathcal{M}}$ is asymptotically harmonic if and only if $4\lambda_0 = h$.*

Proof. It is enough to show that if $4\lambda_0 = h$ then $\widetilde{\mathcal{M}}$ is asymptotically harmonic. For each $\lambda < \lambda_0$, since $\Delta + \lambda$ is coerceive, it has finite Green function and the Martin boundary of $\Delta + \lambda$ coincides with $\partial_\infty \widetilde{\mathcal{M}}$ and is equipped with a Hölder structure (see [A], [S]). If $\mathfrak{K}_\lambda(x,y,\xi)$ is the Martin kernel of $\Delta + \lambda$,

$$h - 4\lambda = \int |2\nabla \log \mathfrak{K}_\lambda - \nabla \log \mathfrak{K}|^2 d\text{m}^{\mathfrak{Q}}.$$

From the equation, it follows that $\int |2\nabla \log \mathfrak{K}_\lambda - \nabla \log \mathfrak{K}|^2 d\text{m}^{\mathfrak{Q}} \to 0$ as $\lambda \to \lambda_0$. We shall show that $|\nabla \log \mathfrak{K}|^2 = 4\lambda_0$. Fix a point $x \in \widetilde{\mathcal{M}}$. There is an increasing sequence $\lambda_n \to \lambda_0$ such that for $\nu_x^{\mathcal{T}}$- a.e. $\mathrm{v} = (x, \xi) \in \mathcal{S}_x\widetilde{\mathcal{M}}$ and for every $R > 0$,

$$\lim_{n\to\infty} \int_{B(x,R)} |2\nabla_y \log \mathfrak{K}_\lambda(x, y, \xi) - \nabla_y \log \mathfrak{K}(x, y, \xi)|^2 d\,\mathrm{vol}(y) = 0.$$

Thus it follows that for $\widetilde{\mathrm{m}^{\mathcal{Q}}}$-a.e. v, $\sqrt{\mathfrak{K}(\gamma_{\mathrm{v}}(0), \cdot, \mathrm{v}_+)}$ is a weak λ_0-eigenfunction of Δ and since $\sqrt{\mathfrak{K}}$ is smooth, $|\nabla_y \log \mathfrak{K}(x, y, \xi)|^2 = 4\lambda_0 = h$. As the proof of Proposition 5.4.5, since

$$h_{\mathrm{top}} \leq \sup_\mu \int \langle X, \nabla \log \mathfrak{K}\rangle d\mu \leq \sup_\mu \left| \int |\nabla \log \mathfrak{K}|^2 \right|^{1/2} \leq \sqrt{h},$$

it follows that $|\nabla \log \mathfrak{K}|^2 = h$ and $\widetilde{\mathcal{M}}$ is asymptotically harmonic. □

5.5 Further study

In this section, we discuss questions derived from the central limit theorem of Brownian motions and the characterization of asymptotically harmonic manifolds.

The first question is about an application of the central limit theorem to the ergodic theory of geodesic flow. We can extend the central limit theorem to the stochastic line integral of leafwise 1-forms on $\mathcal{SM}$ of class $\mathcal{C}^4$.

Let α be a leafwise 1-form of class $\mathcal{C}^4$ on $\mathcal{SM}$ with its lift $\widetilde{\alpha}$ to $\mathcal{S}\widetilde{\mathcal{M}}$. Then for each central stable leaf $\widetilde{\mathcal{W}}^s(x, \xi)$, we identify the restriction $\widetilde{\alpha}|_{\widetilde{\mathcal{W}}^s(x,\xi)}$ to the leaf with a 1-form $\widetilde{\alpha}^\xi$ on $\widetilde{\mathcal{M}}$.

The *orthonormal frame bundle* $\mathcal{O}(\widetilde{\mathcal{M}})$ of $\widetilde{\mathcal{M}}$ is the set of orthogonal transformations $u : \mathbb{R}^d \to \mathcal{T}_x\widetilde{\mathcal{M}}$. Fix an orthonormal basis $e_1, \ldots, e_d$ of $\mathbb{R}^d$, we define a scalarization of a 1-form β on $\widetilde{\mathcal{M}}$ by the $\mathbb{R}^d$-valued 1-form

$$\overline{\beta} := \{\overline{\beta}_k\}_{k=1}^d,$$

where $\overline{\beta}_k(u) := \beta(ue_k)$ for $u \in \mathcal{O}(\widetilde{\mathcal{M}})$. Note that if β is of class $\mathcal{C}^4$, then $\overline{\beta}_k$ is of class $\mathcal{C}^3$ for each $k = 1, \ldots, d$. Fix a d-dimensional Euclidean Brownian motion $w_t := \{w_t^k\}_{k=1,\ldots,d}$. Let $\widetilde{\upsilon}_t$ be a horizontal lift of the Brownian motion $\widetilde{\omega}_t$ on $\widetilde{\mathcal{M}}$ to $\mathcal{O}(\widetilde{\mathcal{M}})$.

The *stochastic line integral* of a leafwise 1-form α is defined by Stratonovich stochastic integration

$$\int_{\omega[0,t]} \alpha = \int_{\widetilde{\omega}[0,t]} \widetilde{\alpha}^{\xi} = \int_0^t \overline{\widetilde{\alpha}^{\xi}}_k(\widetilde{v}_s) \circ w_s^k = \int_0^t \widetilde{\alpha}^{\xi}(\widetilde{v}_s e_k) \circ dw_s^k,$$

if ω_t is lifted to $(\widetilde{\omega}_t, \xi)$. Then the random variable

$$M_t^{\alpha}(\omega) := \int_{\omega[0,t]} \alpha + \int_0^t \delta_s \alpha(\omega_s) ds \tag{5.21}$$

is a Martingale with quadratic variation $2|\alpha(\omega_t)|^2 dt$.

By the same arguement as in Section 5.2, we obtain the following central limit theorem for Martingales of the form (5.21).

Theorem 5.5.1. *Let α be a leafwise 1-form on $\mathcal{SM}$ of class $\mathcal{C}^4$ with $\delta_s \alpha \in \mathcal{L}^{\tau}$ for some τ. Then there is a solution u in $\mathcal{L}^{\tau}$ such that $\Delta_s u = \delta_s \alpha - \int \delta_s \beta d\mathrm{m}^{\mathcal{Q}}$ and the random variable*

$$\int_{\omega[0,t]} \alpha + t \int \delta_s \beta d\mathrm{m}^{\mathcal{Q}} + u(\omega_t) - u(\omega_0)$$

is a martingale with quadratic variation $2|\alpha^{\#} + \nabla u|^2(\omega_t) dt$. Furthermore, the random variable

$$X_t^{\alpha}(\omega) := \frac{1}{\sqrt{t}} \left(\int_{\omega[0,t]} \alpha + t \int \delta_s \alpha d\mathrm{m}^{\mathcal{Q}} \right)$$

is asymptotically centered normal with variance $\sigma^2 = \int 2|\alpha^{\#} + \nabla u|^2 d\mathrm{m}^{\mathcal{Q}}$.

We can ask whether it is possible to prove the central limit theorem of geodesic flow using the central limit theorem as in [LeJ].

Question 1. *Given a function f on $\mathcal{SM}$ of class uniform $\mathcal{C}^4$, can we construct a leafwise 1-form α^f on $\mathcal{SM}$ for which the the Birkhoff random variable*

$$\frac{1}{\sqrt{t}} \left(\int_0^t f(\mathbf{g}^t \mathrm{v}) dt - t \int f d\nu \right)$$

on the probability space $(\mathcal{SM}, \nu)$ and X_t^{α} on $(\mathcal{S}\Omega, \mathrm{m}^{\mathcal{Q}})$ have the same asymptotic distribution?

In results related to the characterization of asymptotically harmonic manifolds, we require the existence of the equilibrium state for the harmonic potential F^{BM} on $\mathcal{SM}$. While there are works for conditions under which the measure of maximal entropy exists and counter-examples ([DPPS2], [DPPS1]), few are known for harmonic potentials. We can ask when the equilibrium state ν for the harmonic potential F^{BM} on $\mathcal{SM}$.

Question 2.
1. *Is there a finite-volume manifold with pinched negative curvature $\mathcal{M}$ and uniformly bounded first derivatives of sectional curvature whose Gibbs measure of F^{BM} on $\mathcal{SM}$ is not finite?*
2. *If it exists, for which family of finite-volume manifolds with pinched negative curvature $\mathcal{M}$ and uniformly bounded first derivatives of sectional curvature, we can guarantee the existence of the equilibrium states for its harmonic potential?*

The last question is related to the rigidity of asymptotically harmonic manifolds. The rigidity of asymptotic harmonicity gives us a partial solution on Katok's conjecture since every asymptotically harmonic manifold has its Liouville measure as a measure of maximal entropy for the geodesic flow on the unit tangent bundle of a finite-volume quotient.

Question 3. *If $\widetilde{\mathcal{M}}$ is an asymptotically harmonic manifold, then is $\widetilde{\mathcal{M}}$ a symmetric space?*

For a compact negatively curved manifold $\mathcal{M}$, it is proved that if $\widetilde{\mathcal{M}}$ is asymptotically harmonic, then it has smooth stable and unstable distributions for the geodesic flow [FL]. Since any contact Anosov flow on a compact manifold with smooth Anosov distribution is $\mathcal{C}^\infty$-conjugate to a reparametrization of the geodesic flow of a locally symmetric space, [BFL] the rigidity of asymptotically harmonic manifolds holds for the co-compact negatively curved case.

We expect that we can generalize [FL] and [BFL] to co-finite manifolds with pinched negative curvature and uniformly bounded first derivatives of sectional curvature.

Bibliography

[A] Ancona, A. (1987). Negatively curved manifolds, elliptic operators, and the Martin boundary. *Ann. of Math.* **125** 495–536.

[AS] Anderson, M.T. and Schoen, R. (1985). Positive harmonic functions on complete manifolds of negative curvature. *Ann. of Math.* **121** 429–461.

[Ba] Ballmann W. (1995). *Lectures on spaces of nonpositive curvature. Oberwolfach Semin* **25**. Birkhäuser/Springer, Basel.

[BeHu] Benoist, Y. and Hulin, D. (2019). Harmonic measures on negatively curved manifolds. *Ann. Inst. Fourier (Grenoble)* **69** 2951–2971.

[Bo] Bowditch, B. H. (1995). Geometrical finiteness with variable negative curvature. *Duke Math. J.* **77** 229–274.

[BFL] Benoist, Y., Foulon, P. and Labourie, F. (1992). Flots d'Anosov á distributions stable et instable différentiables. *J. Amer. Math. Soc.* **5** 33–74.

[BB] Ballmann, W., Brin, M. and Burns, K. (1987). On the differentiability of horocycles and horocycle foliations. *J. Diff. Geom.* **26** 337–347.

[BrHa] Bridson, M.R. and Haeflinger, A. (2013). *Metric spaces of non-positive curvature. Grundlehren Math. Wiss.* **319**. Springer, Berlin.

[CC] Candel, A. and Conlon, L. (2003). *Foliations II Grad. Stud. Math.* **23**. Amer. Math. Soc., Providence, RI.

[CS] Castillon, P. and Sambusetti, A. (2014). On asymptotically harmonic manifolds of negative curvature. *Math. Z.* **277** 1049–1072.

[CK] Chavel, I. and Karp, L. (1991). Large time behavior of the heat kernel: the parabolic λ-potential alternative. *Comment. Math. Helv.* **66** 541–556.

[C] Chavel, I. (1984). *Eigenvalues in Riemannian geometry. Pure and Applied Mathematics* **115**, Academic Press.

[CY] Cheng, S.-Y. and Yau, S.-T. (1975). Differential equations on Riemannian manifolds and their geometric applications. *Comm. Pure. Appl. Math.* **28**, 333–354.

[DPPS1] Dal'bo, F., Peiné, M., P.icaud, J.-C. and Sambusetti, A. (2017). Convergence and counting in infinite measure. *Ann. Inst. Fourier (Grenoble)* **67** 483–520.

[DPPS2] Dal'bo, F., Peiné, M., Picaud, J.-C. and Sambusetti, A. (2009). On the growth of nonuniform lattices in pinched negatively curved manifolds *J. Reine. Angew. Math.* **627** 31–52.

[DM] Davis, E.B. and Mandouvalos, N. (1988). Heat kernel bounds on hyperbolic space and Kleinian groups. *Proc. London Math. Soc.* **52** 182–208.

[Dod] Dodziuk, J. (1987). A lower bound for the first eigenvalue of a finite-volume negatively curved manifold. *Bull. Braz. Math. Soc. (N.S.)* **18** 23–34.

[Don] Donnelly, H. (1987). Essential spectrum and heat kernel. *J. Funct. Anal.* **75** 362-381.

[E] Eberlein, P. (1980). Lattices in spaces of negative curvature. *Ann. Math.* **111** 435–476.

[EO] Eberlein, P. and O'Neill, B. (1973). Visibility manifolds. *Pac. J. Math.* **46** 45–109.

[EN] Engel, K.-J. and Nagel, R. (1999). *One-parameter semigroups for linear evolution equations. Grad. Texts in Math.* **194** Springer, New York.

[EFL] Enriquez, N., Franchi, J. and Le Jan, Y. (2001). Central limit theorem for the geodesic flow associated with a Kleinian group, case $\delta > d/2$. *J. Math. Pures Appl.* **80** 153–175.

[F] Franco, E. (1977). Flows with unique equilibrium states. *Amer. J. Math.* **99** 486–514.

[FL] Foulon, P. and Labourie, F. (1992). Sur les variétés compactes asymptotiquement harmoniques. *Invent. Math.* **69** 375–392.

[Ga] Garnett, L. (1983). Foliations, the ergodic theorem and Brownian motion. *J. Funct. Anal.* **51** 285–311.

[Gu] Guivarc'h, Y. (1981) Mouvement brownien sur les revêtements d'une variété compacte. *C. R. Math. Acad. Sci. Paris.* **292** 851–853.

[Gr] Grigor'yan, A. (1994) Heat kernel upper bounds on a complete noncompact manifold. *Rev. Mat. Iberoam.* **10** 395–452.

[Ha1] Hamenstädt, U. (1990). An explicit description of harmonic measure. *Math. Z.* **205** 287–299.

[H2] Hamenstädt, U. (1997). Harmonic measures for compact negatively curved manifolds. *Acta Math.* **178** 39–107.

[He] Helland, I. S. (1982). Central limit theorems for martingales with discrete or continuous time. *Scand. J. Stat.* **9** 79–94.

[Hs] Hsu, E. P. (2002). *Stochasic analysis on manifolds. Grad. Stud. Math.* **38**. Amer. Math. Soc., Providence, RI.

[Kai1] Kaimanovich, V. A. (1986). Brownian motion and harmonic functions on covering manifolds. An entropic approach. *Dokl. Akad. Nauk* **288** 1045–1049.

[Kai2] Kaimanovich, V. A. (1990). Invariant measures of the geodesic flow and measures at infinity on negatively curved manifolds. *Ann. Inst. H. Poincaré, Phys. Thér.* **53** 361–393.

[Kat1] Katok, A. (1982). Entropy and closed geodesies. *Ergod. Th. & Dynam. Sys.* **2** 339–365.

[Kat2] Katok, A. (1988). Four applications of conformal equivalence to geometry and dynamics. *Ergod. Th. & Dynam. Sys.* **8** 139–152.

[KL] Kifer, Y. and Ledrappier, F. (1990). Hausdorff dimension of harmonic measures on negatively curved manifolds. *Trans. Amer. Math. Soc.* **318** 685–704.

[K] Kim, J. (2021). Central limit theorem of Brownian motions in pinched negative curvature. *Ergod. Th. & Dynam. Sys.* 1–30. doi:10.1017/etds.2021.46

[Ki] Kingman, J. F. C. (1968). The ergodic theory and subadditive stochastic processes. *J. R. Stat. Soc. Ser. B. Stat. Methodol.* **30** 499–510.

[KP1] Knieper, G. and Peyerimhoff, N. (2015). Geometric properties of rank one asymptotically harmonic manifolds. *J. Diff. Geom.* **100** 507–532.

[KP2] Knieper, G. and Peyerimhoff, N. (2016) Harmonic functions on rank one asymptotically harmonic manifolds. *J. Geom. Anal.* **26** 750–781.

[LeJ] Le Jan, Y. (1994). The central limit theorem for the geodesic flow on noncompact manifolds of constant curvature. *Duke Math. J.* **74** 159–175.

[Le1] Ledrappier, F. (1987). Propriété de Poisson et courbure négative. *C. R. Math. Acad. Sci. Paris.* **305** 191–194.

[Le2] Ledrappier, F. (1988). Ergodic properties of Brownian motion on covers of compact negatively-curve manifolds. *Bull. Braz. Math. Soc. (N.S.)* **19** 115–140.

[Le3] Ledrappier, F. (1988). Profil d'entropie dans le cas continu. *Astérisque* **236**. Société Mathématique de France, Paris.

[Le4] Ledrappier, F. (1990). Harmonic measures and Bowen-Margulis measures. *Israel J. Math.* **71** 275–287.

[Le5] Ledrappier, F. (1993). A heat kernel characterization of asymptotic harmonicity. *Proc. Amer. Math. Soc.* **118** 1001–1004.

[Le6] Ledrappier, F. (1995). Central limit theorem in negative curvature. *Ann. Probab.* **23** 1219–1233.

[LL] Ledrappier, F. and Lim, S. (2021). Local limit theorem in negative curvature. *Duke Math. J. Advance publication* 1–97.

[LS1] Ledrappier, F. and Shu, L. (2014). Entropy rigidity of symmetric spaces without focal points. *Trans. Amer. Math. Soc.* **366** 3805–3820.

[LS2] Ledrappier, F. and Shu, L. (2017). Differentiating the stochastic entropy for compact negatively curved spaces under conformal changes. *Ann. Inst. Fourier* **67** 1115–1183.

[M] McKean, H. P. (1970). An upper bound to the spectrum of Δ on a manifold of negative curvature. *J. Diff. Geom.* **4** 359–366.

[N] Naïm, L. (1957). Sur le rôle de frontiére de R. S. Martin dans la théorie du potentiel, *Ann. Inst. Fourier* **7**, 183–281.

[Pe] Petersen, P. (2016). *Riemannian geometry. Grad. Texts in Math.* **171** Springer, New York.

[Pi] Pinsky, M. A. (1978). Stochastic Riemannian geometry. In *Probabilistic Analysis and Related Topics* (A. T. Bharucha-Reid, ed.) **1** 199–236. Academic Press, New York.

[PPS] Paulin, F., Pollicott, M. and Schapira, B. (2015). *Equilibrium states in negative curvature. Astérisque* **373**. Société Mathématique de France, Paris.

[PS] Pit, V. and Schpira, B. (2018) Finiteness of Gibbs measures on noncompact manifolds with pinched negative curvature. *Ann. Inst. Fourier* **68** 457–510.

[Pr] Prat, J.-J. (1975). Étude asymptotique et convergence angulaire du mouvement Brownien sur une variété à courbure négative. *C. R. Math. Acad. Sci. Paris.* **280** 1539–1542.

[R] Riquelme, F. (2018). Ruelle's inequality in negative curvature. *Discrete Contin. Dyn. Syst.* **38** 2809–2825.

[S] Sullivan, D. (1987). Related aspects of positivity in Riemannian geometry. *J. Diff. Geom.* **25** 327–351.

[Y] Yue, C. (1991). Integral formulas for the Laplacian along the unstable foliations to rigidity problems for manifolds of negative curvature. *Ergod. Th. & Dynam. Sys.* **11** 803–819.

국문초록

다양체의 기하학적 성질은 측지흐름과 브라운 운동의 행동의 대부분을 결정하는데, 역으로 브라운 운동과 측지흐름이 다양체의 기하학적 성질을 반영하기도 한다. 따라서 측지흐름과 브라운 운동을 연구함으로써 다양체의 기하학적 성질을 유추하는 것이 가능하다. 본 학위 논문에서는 다양체에서 측지흐름과 브라운운동 그리고 기하학적 성질 사이의 상호작용을 기술하고자한다.

첫번째로 우리는 브라운 운동의 중심 극한 정리를 증명하는데, 이는 다양체의 기하학적 성질인 고른 유계를 갖는 음수 곡률과 곡률의 1계 미분의 고른 유계로 부터 나오는 결과라고 할 수 있다. 브라운 운동의 중심극한 정리는 브라운 운동을 따르는 입자의 시작점으로 부터의 거리와 그린 함수의 로그로 정의되는 그린 거리로 정의되는 확률 변수의 분포가 점근적으로 정규분포를 따른다는 것을 말한다. 다양체 위의 브라운 운동의 단위접속으로 올려진 확률 과정을 엽층화된 브라운 운동이라 부르는데, 엽층화된 브라운 운동의 수축정리를 이용하여 중심극한 정리를 증명할 수 있다.

그 다음으로 두 음수 사이에 갇힌 곡률을 갖는 점근적 조화 다양체의 동치조건을 제시한다. 동치 명제를 통해 브라운 운동과 점근적 조화 다양체가 긴밀하게 연관되어 있음을 알 수 있는데, 특히 중심 극한 정리를 적용하여 그린 거리에 대한 점근적 분포의 분산을 이용한 점근적 조화 다양체의 특성화를 유도할 수 있다. 증명 과정이 브라운 운동의 측지 흐름에 대한 에르고딕 성질을 요구하기 때문에 부가적으로 동역학적 성질인 조화 잠재함수의 평형상태의 존재성에 대한 가정이 필요하다.

주요어휘: 엽층화된 브라운 운동, 음의 곡률을 갖는 다양체의 측지 흐름, 열역학적 형식론, 점근적 조화 다양체

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감사의 글

제가 학위논문을 쓸 수 있도록 가장 큰 도움을 주신 제 지도 교수님이신 임선희 교수님께 감사드립니다. 교수님께 지도 받으면서 연구자로써 많은 배움을 얻을 수 있었습니다. 부족한 저를 이끌어주시고 학문 외적으로도 많은 가르침을 주시는 교수님께 지도를 받을 수 있었던 건 대학원에 입학한 이후 가장 큰 행운이었습니다. 다시 한번 감사의 말씀을 드립니다.

제가 브라운 운동의 연구를 하는데 아낌없는 도움을 주시고 학위논문 심사에도 참여해 주신 François Ledrappier 교수님께 감사드립니다. 저의 파리 방문 때 환대해 주시고 많은 도움을 주셨기에 연구를 무사히 마칠 수 있었습니다. 그리고 저의 박사학위 논문 제출 자격고사에 이어 학위논문 심사에 참여해 주신 김판기 교수님, Otto van Koert 교수님께 감사드립니다. 대학원 초년 차부터 제 질문에 자세히 답변해 주시며 많은 도움을 주시고 제 학위논문 심사에도 참여해 주신 권상훈 교수님께 감사드립니다.

제가 학부시절 전공을 옮겨 수학을 공부하고 나아가 대학원 입학을 결심하는데 큰 도움을 주신 김정래 교수님 그리고 유승진 교수님께 감사드립니다. 대학원 입학 전과 입학하여 학위 과정 중에도 많은 조언을 주셔서 대학원 생활에 큰 힘이 되었습니다. 감사합니다.

서로의 문제를 자기 일처럼 같이 고민해 주고 공부를 함께 해온 연구실 선후배분들 그리고 대학원 동기들에게도 감사의 말씀을 전합니다.

마지막으로 항상 저의 편에서 응원해 주신 저의 부모님과 형 김재윤, 그리고 항상 활력이 되어주는 막내들 티민, 말숙, 보리에게 말로는 다하기 어려운 고마운 마음을 전합니다.