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이학 박사 학위논문

Conformal Field Theory
and
Schramm-Loewner Evolution
in
A Doubly Connected Domain

(이중연결영역에서의 등각장론과 슈람-뢰브너 전개)

2021년 8월

서울대학교 대학원

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이 논문을 이학 박사 학위논문으로 제출함

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변성수의 이학 박사 학위논문을 인준함

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Conformal Field Theory
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A dissertation
submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
to the faculty of the Graduate School of
Seoul National University

by

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August 2021

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Abstract

Conformal Field Theory
and
Schramm-Loewner Evolution
in
A Doubly Connected Domain

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The aim of this thesis is to develop a constructive conformal field theory of Gaussian free field in a doubly connected domain and its implementations to the theory of annulus Schramm-Loewner evolution (SLE). In particular, we derive certain types of BPZ equations based on Eguchi-Ooguri's version of Ward's equations, which describe the Virasoro fields in terms of the Lie derivative operators and the Teichmüller modular parameter. As applications, we find Coulomb gas solutions to the null-vector equations for annulus SLE partition functions and construct a large class of martingale-observables for variants of annulus SLE processes.

Key words: Conformal field thoery, Schramm-Loewner evolution, Gaussian free field, doubly connected domain, partition functions, martingale-observables
Student Number: 2015-20264

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Chapter 1

Introduction and overview

In 1999, Schramm proposed his celebrated tool to study critical phenomenon in two dimensional statistical mechanics, now known as Schramm-Loewner evolution (SLE) [115]. For the last two decades since then, SLE theory has successfully produced rigorous proofs of several remarkable conjectures in statistical physics. Among these, conformal invariance of scaling limits of some important critical lattice models has been proved by means of SLE.

Such results have been obtained by the fundamental fact that certain discrete observables converge to conformally covariant martingale processes. For instance, in [123], Smirnov used Cardy's observables to prove conformal invariance of scaling limit of critical site percolation in a hexagonal lattice, see Figure 1.1. Schramm and Sheffield made use of a certain bosonic observable to prove that the level lines of discrete Gaussian free field (GFF) with specific height gaps converges to SLE(4), [118]. Chelkak, Duminil-Copin, Hongler, Kemppaninen and Smirnov proved existence of scaling limit of FK Ising model interfaces and its conformal invariance by virtue of certain parafermionic observables, see [34] and references therein. In addition, some geometric properties of SLE curves have been obtained by means of *SLE martingale-observables*. Examples include the study of Hausdorff dimensions (Beffara's observables), and left passage probabilities (Schramm's observables) of SLE, see [24, 116].

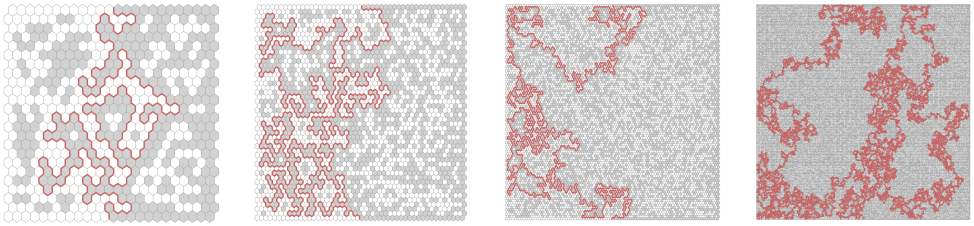
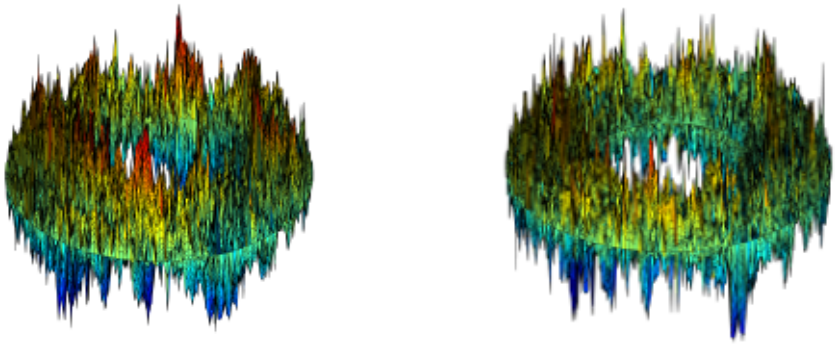


Figure 1.1: Critical percolation interfaces and its convergence to SLE(6)

Consequently, there have been several attempts to find a systematic way of constructing SLE martingale-observables. In the physics literature, in particular in the context of conformal field theory (CFT) [25,41], it is well known that under the insertion of ratio of the *one-leg operator* Ψ to its correlation function $\mathbf{E}\,\Psi$, the correlators in a certain class of fields are martingale-observables for SLEs, see e.g., [16, 17, 19, 86].

In mathematics, a rigorous development of analytic CFT of GFF in a simply connected domain (with understanding of conformal fields as objects in a bosonic Fock space and of a stress energy tensor as a Lie derivative operator) and its implementations to chordal SLE theory were achieved by Kang and Makarov in their monograph [77]. Moreover, their CFT framework has been extended on a compact Riemann surface [78]. Based on the building blocks developed in [77, 78], several versions of field theories in various conformal setups have been constructed to produce martingale-observables for variants of SLEs. To name a few, these include radial SLE [76], dipolar SLE [74, 79], SLE($\kappa, \boldsymbol{\rho}$) with several force points [75] and multiple SLEs [5]. Some applications of Kang-Markarov CFT to SLE theory can be found in [6, 98] for example. We refer to [109] for a recent review on the relation between CFT and SLE theory.

In short, this thesis contributes to the above question of constructing CFT of GFFs and its implementations to SLE theory in a *doubly connected domain*, with a goal to serve as a solid cornerstone to develop a theory in a general multiply connected domain (or on a compact Riemann surface). In a



(a) ER (b) Dirichlet

Figure 1.2: Graphs of discrete GFFs with ER and Dirichlet boundary conditions

doubly connected domain, Zhan has made important achievements on basic properties of SLE, see e.g., [132, 133]. In particular, in [139], Zhan introduced the annulus SLE(κ, Λ) with two marked points, where $\Lambda := \kappa(\log Z)'$ is the drift function of driving process of SLE and derived certain parabolic PDE for the annulus SLE partition function Z (a version of commutation relation for annulus SLEs, cf. [46]), which provides a necessary condition for the reversibility of SLE trace. In physics, Hagendorf, Bernard and Bauer used certain bosonic observable to propose the partition functions for annulus SLE with $\kappa = 4$ associated with GFFs, see [63]. Hagendorf presented physical arguments on how to produce martingale-observables for the Dirichlet boundary case in [62].

To construct a rigorous CFT in a doubly connected domain, the basic fields we consider are one-parameter family of GFFs whose boundary conditions are given by *weighted combination* of Dirichlet and excursion reflected (ER) boundary conditions, see Figure 1.2. In the analysis literature, the Green's function with ER boundary condition appears under the name of *modified Green's function*, which was introduced by Koebe in his classical work [83] on classifying multiply connected domains. In probability theory,

the associated stochastic process, ER Brownian motion, was introduced by Lawler and Drenning [42, 92]. An important feature of the ER Green's function is that the average of its normal derivative on the inner boundary component vanishes, namely, it satisfies *zero period condition*, see e.g., [37, 38]. By combining this property with zero Dirichlet boundary condition under the weight indexed by χ (see (3.2.5)), we consider the one-parameter family of Green's functions interpolating Dirichlet Green's function ($\chi = 0$) with ER one ($\chi = \infty$) and define associated GFFs. Such interpolation is natural not only from analytic/probabilistic point of view but also from conformal field theoretical viewpoint since it turns out that this is the only way of interpolating these two boundary conditions which produces conformal fields satisfying Eguchi-Ooguri type equation, a key ingredient to build precise relation between CFT and SLE, see Section 6.2.

One of main motivations to introduce such general boundary conditions comes from the fact that this interpolation gives rise to infinitely many linearly independent solutions to Belavin-Polyakov-Zamolodchikov (BPZ) type equation. This confirms the prediction that contrary to a simply connected domain where BPZ equation has a finite dimensional solution space (which was shown in the series of papers [52–55]), in a doubly connected domain the solution space is no longer finite dimensional space. We also refer to [87, 88, 110] for the analysis of solutions to various equations arising from CFT with a concrete understanding of the underlying algebraic structure.

The statistical fields satisfying BPZ type equations, thus producing SLE martingale-observables form the OPE family $\mathcal{F}_{\boldsymbol{\beta}} \equiv \mathcal{F}_{(b),\boldsymbol{\beta}}$ generated by central and background charge modifications $\Phi_{\boldsymbol{\beta}} \equiv \Phi_{(b),\boldsymbol{\beta}}$ of the GFFs under OPE multiplication. Here, b is a real parameter related to the central charge c as $c = 1 - 12b^2$ and to the SLE parameter κ as $b = \sqrt{\kappa/8} - \sqrt{2/\kappa}$. On the other hand, we require that for a divisor $\boldsymbol{\beta} = \sum \beta_j \cdot q_j$ called a background charge, the total sum of charges should vanish, i.e., $\sum \beta_j = 0$. On a complex torus (of genus one), this requirement is known as the *neutrality condition* in the physics literature, see e.g., [41]. In order to derive BPZ equations, we

need two main ingredients, the first is *Ward's equation* and the second is the construction of the one-leg operator Ψ satisfying *level two degeneracy equation*.

Whereas Ward's equations in a simply connected domain describe the Virasoro fields in terms of Lie derivative operators within correlations, the statement of those in a doubly connected domain contains the derivative of correlation functions with respect to the modular (Teichmüller) parameter as well. These forms of Ward's equations are well known in the physics literature, e.g., Ward's equations in a doubly connected domain [63] and Eguchi-Ooguri's version of Ward's equations on a complex torus of genus one [50]. We derive them not from the path integral formalism but from the functional equations of classical special functions such as pseudo-addition theorem of Weierstrass zeta function. See [78] for a mathematical derivation of Eguchi-Ooguri's version of Ward's equations.

For given another divisor $\boldsymbol{\tau} = \sum_j \tau_j \cdot \xi_j$ satisfying $a + \int \boldsymbol{\tau} = 0$, we construct the one-leg operator $\Psi_{\boldsymbol{\beta}} \equiv \Psi_{\boldsymbol{\beta}}(p, \boldsymbol{\xi})$ from the GFF as the modified multi-vertex field (or the OPE exponential) with the charge $a = \sqrt{2/\kappa}$ at given marked point p. With this specific choice of the charge at p, the one-leg operator $\Psi_{\boldsymbol{\beta}}$ satisfies the desired level two degeneracy equation.

Following the set-up in the physics literature, we set

$$\Lambda \equiv \Lambda_{\boldsymbol{\beta}}(\xi, \boldsymbol{\xi}) = \kappa\, \partial_\xi \log \mathbf{E}\, \Psi_{\boldsymbol{\beta}}(\xi, \boldsymbol{\xi}),$$

see [75] for this from a viewpoint of Radon-Nikodym derivative (or coordinate change [119]) between the laws of SLEs. Here, the marked boundary points $p, \boldsymbol{q}, \boldsymbol{\xi}$ play different roles. More precisely, the point p corresponds to the starting point of $\mathrm{SLE}(\kappa, \Lambda)$, whereas $\boldsymbol{\xi}$ (resp., $\boldsymbol{q}$) are force points having non-trivial (resp., trivial) conformal dimensions. We consider a collection of all correlation functions of fields in the OPE family $\mathcal{F}_{\boldsymbol{\beta}}$ under the insertion of the properly normalized one-leg operator. Then by means of BPZ-Cardy equations, we show that this collection forms martingale-observables for $\mathrm{SLE}(\kappa, \Lambda)$, see Theorem C below. For the specific case $\chi = 0$, which corre-

sponds to CFT with Dirichlet boundary condition, this generalizes previous results of Izyurov and Kytölä [69] that 1-and 2-point function of GFF with piecewise Dirichlet boundary condition are martingale-observables for associated SLE$(4, \Lambda)$. Here, the neutrality condition coincides with the requirement that jumps of the 1-point function on the boundary should add up to zero. However, the correlation function $Z_{\boldsymbol{\beta}} := \mathbf{E}\,\Psi_{\boldsymbol{\beta}}$ of this type of SLE(κ, Λ) with one force point does not satisfies the null-vector equation for the annulus SLE partition function (introduced in [139]) unless $\kappa = 4$.

In order to find explicit solutions to null-vector equation for generic $\kappa > 0$ and to construct a class of SLE(κ, Λ) martingale-observables associated with these solutions, we apply the *method of screening* (also known as the *Coulomb gas* or *Dotsenko-Fateev* integrals). If $\kappa > 4$, we can deform a Pochhammer contour in this methodology so that the solution can be represented in terms of Euler type integral. In the case $\kappa \leq 4$, we perform a proper analysis on an analytic continuation (with respect to κ) of the Pochhammer contour integral to obtain a non-trivial real-valued solution.

Beyond the method of screening, we present further constructions of SLE martingale-observables, which include the implementations of non-atomic background charges and the use of periodization (or chiral bosonization) of conformal fields. These allow us to construct CFT for continuum interface curves in statistical physics such as discrete GFF, loop-erased random walk and critical Ising interfaces. In the end, based on the theory developed in this thesis, most of the well-known annulus SLE martingale-observables are presented in terms of CFT correlation functions and some geometric properties of certain annulus SLEs such as hitting/left passage probabilities are obtained.

Chapter 2

Formulation of main results

This chapter is devoted to provide a more detailed overview of the thesis and to present main results.

2.1 Notation

Throughout the present thesis, we use the following symbols and notations: let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$; for $r > 0$ let $\mathbb{A}_r = \{z \in \mathbb{C} : e^{-r} < |z| < 1\}$, $\mathbb{T}_r = \{z \in \mathbb{C} : |z| = e^{-r}\}$, $\mathbb{S}_r = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < r\}$, and $\mathbb{R}_r = \{z \in \mathbb{C} : \operatorname{Im} z = r\}$. We denote by $\mathcal{C}_r$ the cylinder $\mathbb{S}_r/\langle z \mapsto z + 2\pi\rangle$, and by D_r a doubly connected domain with modulus r. A subset K of a doubly connected domain D is called a hull in D if $D \setminus K$ is a doubly connected domain and K has positive distance from one boundary component of D. For any hull K of D, let us define $\operatorname{cap}_D(K) := \operatorname{mod}(D) - \operatorname{mod}(D \setminus K)$ as the capacity of K in D, where $\operatorname{mod}(\cdot)$ denotes the modulus of a doubly connected domain. Following the conventions in [133, 139], we write $\Theta(r, z) := \theta(\frac{z}{2\pi}, \frac{ir}{\pi})$, where $\theta(z, r)$ is a Jacobi theta function, see [33, p.63]. For a non-negative parameter χ, which will be used to describe the boundary condition, it is convenient to write

$$\Theta_\chi(r, z) := \Theta(r, z) \exp\left(\frac{z^2}{4(r + \chi)}\right)$$

for the modified theta function. We denote by ζ_r the Weierstrass zeta function with basic periods $(2\pi, 2ir)$. In the sequel we sometimes omit the modular parameter r, e.g., $\Theta_\chi(\cdot) \equiv \Theta_\chi(r, \cdot)$, etc.

2.2 Basic setup

Let $\Phi_{(0)}$ be GFF in D_r whose boundary condition is given by a weighted combination of Dirichlet and ER boundary conditions, see Section 3.3 for the precise definition of GFF and Subsections 3.2.2, 5.1.1 for further details on boundary conditions. For a real parameter b, we define the *central charge modification* $\Phi_{(b)}$ of GFF by adding a non-random pre-pre-Schwarzian form as follows:

$$\Phi_{(b)} := \Phi_{(0)} - 2b \arg w',$$

where w is a conformal map from D_r onto $\mathcal{C}_r$. We remark that $\arg w'$ does not depend on the choice of a conformal map.

Given marked points $\boldsymbol{q} = \{q_k\}$, we call a divisor $\boldsymbol{\beta} = \sum_k \beta_k \cdot q_k$ a *background charge* if it satisfies the neutrality condition

$$\int \boldsymbol{\beta} = \sum_k \beta_k = 0.$$

This requirement follows from an extension of Gauss-Bonnet theorem, see [78]. We define the *background charge modification* $\Phi_{\boldsymbol{\beta}} \equiv \Phi_{\boldsymbol{\beta},(b)}$ associated with $\boldsymbol{\beta}$ as

$$\Phi_{\boldsymbol{\beta}} := \Phi_{(b)} + \sum \beta_k \arg \Big\{ \Theta_\chi(w_k - w_z)\Theta_\chi(\bar{w}_k - w_z) \Big\},$$

where $w_z = w(z)$ and $w_k = w(q_k)$.

Let us denote by $\mathcal{F}_{\boldsymbol{\beta}} \equiv \mathcal{F}_{\boldsymbol{\beta},(b)}$ the OPE family of the $\Phi_{\boldsymbol{\beta}}$, the algebra (over $\mathbb{C}$) spanned by the generators 1, mixed derivatives of $\Phi_{\boldsymbol{\beta}}$, those of OPE exponentials $e^{*\alpha\Phi_{\boldsymbol{\beta}}}$ ($\alpha \in \mathbb{C}$), and their holomorphic/anti-holomorphic/mixed parts (with certain neutrality conditions) under the OPE multiplication $*$.

We now introduce the (covering) annulus SLE with N-force points $\boldsymbol{q} = \{q_1, \cdots q_N\}$. Similarly as in a simply connected domain, the annulus SLE(κ, Λ) can be described by the associated Loewner's differential equation [133]. Let ξ be a (real-valued) continuous function defined on $[0, T)$ for some $0 < T < r$. By definition, the Loewner kernel $S(r, z)$ on $\mathbb{A}_r = \{z : e^{-r} < |z| < 1\}$ is given by

$$S(r, z) := \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{e^{2nr} + z}{e^{2nr} - z}.$$

For each $z \in \mathbb{A}_r$, let $g_t(z)$ be the solution (which exists up to the first time $\tau_z \in (0, T]$ that $g_t(z)$ hits ξ_t) of the annulus Loewner equation

$$\partial_t g_t(z) = g_t(z) S(r - t, g_t(z)/e^{i\xi_t}), \quad g_0(z) = z.$$

For each time $t \in [0, T)$, let $K_t := \{z \in \mathbb{A}_r : \tau_z \le t\}$. Then K_t is a hull in $\mathbb{A}_r$ and has capacity t. Moreover, g_t is a conformal map from $(\mathbb{A}_r \setminus K_t, \mathbb{T}_r)$ onto $(\mathbb{A}_{r-t}, \mathbb{T}_{r-t})$.

It is convenient to describe annulus SLE in the covering space $\mathbb{S}_r$, where the Loewner kernel H is expressed in terms of Θ as

$$H(r, z) := -i\, S(r, e^{iz}) = 2\, \partial_z \log \Theta(r, z).$$

For each $z \in \mathbb{S}_r$, we denote by $\tilde{g}_t(z)$ the solution of the equation

$$\partial_t \tilde{g}_t(z) = H(r - t, \tilde{g}_t(z) - \xi_t), \quad \tilde{g}_0(z) = z,$$

and set $\tilde{K}_t := \{z \in \mathbb{S}_r : \tau_z \le t\}$. Then $\tilde{g}_t$ is a conformal map from $(\mathbb{S}_r \setminus \tilde{K}_t; \mathbb{R}_r)$ onto $(\mathbb{S}_{r-t}; \mathbb{R}_{r-t})$. Moreover for each $z \in \mathbb{S}_r \setminus \tilde{K}_t$, $e^{i\tilde{g}_t(z)} = g_t(e^{iz})$, and $K_t = \{e^{iz} \in \mathbb{A}_r : z \in \tilde{K}_t\}$. In particular, $\tilde{K}_t$ is 2π-periodic and for each $n \in \mathbb{Z}$, we have $\tilde{g}_t(z + 2n\pi) = \tilde{g}_t(z) + 2n\pi$. We call that ξ generates a trace γ if

$$\gamma_t = \gamma(t) := \lim_{z\to\xi_t} \tilde{g}_t^{-1}(z).$$

It was proved by Zhan in [133] that $\xi_t := \sqrt{\kappa}\, B_t$ $(t < r)$ a.s. generates continuous trace, where B_t is a standard one-dimensional Brownian motion.

Suppose Λ is a real-valued $C^{0,1}$ function on $(0,\infty) \times (\mathbb{C} \setminus \{z \in 2\pi\mathbb{Z} + 2ir\mathbb{Z}\})^N$. Let ξ_t be the solution, which exists up to the first time T such that $\xi_t - \tilde{g}_t(q_j) \in 2\pi\mathbb{Z}$, of the SDE

$$d\xi_t = \sqrt{\kappa}\, dB_t + \Lambda(r-t, \xi_t - \tilde{g}_t(q_1), \cdots, \xi_t - \tilde{g}_t(q_N))\, dt, \quad \xi_0 = p.$$

Applying the Girsanov theorem, it can be shown that up to time T, ξ_t a.s. generates continuous trace, see [139, Section 3.2] for further details. We call the trace generated by ξ_t the (covering) annulus SLE(κ, Λ) trace started from p with force points $q_1. \cdots, q_N$. Furthermore if Λ is 2π-periodic in each space variable, ξ_t does not depend on the choice of points $p, q_1, \cdots q_N$ on the covering space, which allows us to define SLE(κ, Λ) only using the geometry of the base space, the annulus $\mathbb{A}_r$ with marked points $e^{2\pi i p}, e^{2\pi i q_1}, \cdots, e^{2\pi i q_N}$.

A non-random (conformal) field f in a planar domain D (or on a Riemann surface in general) is an assignment of smooth function $(f\|\phi) : \phi U \to \mathbb{C}$ for each local chart ϕ. By definition, a non-random field M is a *martingale-observable* for annulus SLE(κ, Λ) if for any $z_1, \cdots, z_n$, the process

$$M_t(z_1, \cdots, z_n) := (M\|\tilde{g}_t^{-1})(z_1, \cdots, z_n), \quad (t < T)$$

(stopped when any z_j or any q_k is swallowed by the Loewner hull $\tilde{K}_t$) is a local martingale on the annulus SLE probability space.

2.3 Main results

In this section, we present main results of this thesis, part of which can be found in [28].

For a smooth vector field v, let us denote by $\mathcal{L}_v^+$ (resp., $\mathcal{L}_v^-$) the $\mathbb{C}$-linear

(resp., anti $\mathbb{C}$-linear) part of Lie derivative operator $\mathcal{L}_v$ with respect to v, i.e.,

$$\mathcal{L}_v^{\pm} := \frac{\mathcal{L}_v \mp i\mathcal{L}_{iv}}{2}.$$

We define a *stress tensor* $A_{\boldsymbol{\beta}} \equiv A_{\boldsymbol{\beta},(b)}$ by

$$A_{\boldsymbol{\beta}} := -\frac{1}{2} J_{(0)} \odot J_{(0)} + \Big(ib\,\partial - \mathbf{E} J_{\boldsymbol{\beta}} \Big) J_{(0)},$$

where

$$J_{(0)} = \partial \Phi_{(0)}, \quad J_{\boldsymbol{\beta}} = \partial \Phi_{\boldsymbol{\beta}}.$$

Here $\odot$ denotes the Wick's product, see e.g., [77, Section 1.2] and [71, 122]. The Virasoro field $T_{\boldsymbol{\beta}} \equiv T_{\boldsymbol{\beta},(b)}$ is given by

$$T_{\boldsymbol{\beta}} := -\frac{1}{2} J_{\boldsymbol{\beta}} * J_{\boldsymbol{\beta}} + ib\,\partial J_{\boldsymbol{\beta}} = A_{\boldsymbol{\beta}} + \mathbf{E} T_{\boldsymbol{\beta}}.$$

The definition of Virasoro field in general and the derivation of this special form for GFFs can be found in Sections 5.3 and 5.6.

We derive the following form of Ward's equation, which describe the stress tensor $A_{\boldsymbol{\beta}}$ (or the Virasoro field $T_{\boldsymbol{\beta}}$) in terms of Lie derivative operators and the differentiation operator with respect to the Teichmüller parameter.

Theorem A. *For any string $\mathcal{X}$ of fields in the OPE family $\mathcal{F}_{\boldsymbol{\beta}}$, we have*

$$2\,\mathbf{E} A_{\boldsymbol{\beta}}(\zeta) \mathcal{X} = \Big(\mathcal{L}^+_{v_\zeta} + \mathcal{L}^-_{v_{\bar{\zeta}}} \Big) \mathbf{E}\mathcal{X} + \partial_r \mathbf{E}\mathcal{X},$$

where all fields are evaluated in the identity chart of $\mathcal{C}_r$ and the Loewner vector field v_ζ is given by

$$(v_\zeta \| \mathrm{id}_{\bar{\mathcal{C}}_r})(z) := H(r, \zeta - z).$$

In order to clarify complex analytic/geometric function theoretic meaning of Ward's equation, let us draw an immediate consequence of Theorem A.

Example. (*Hadamard's variational formula*) As a simplest non-trivial ex-

ample, let us consider the case $\boldsymbol{\beta} = \mathbf{0}$ and $\mathcal{X} = \Phi(z_1)\Phi(z_2)$. Then Ward's equation is equivalent to the following functional relation of the Green's function G in the cylinder $\mathcal{C}_r$:

$$\begin{aligned} 0 &= 4\partial_\zeta G(\zeta, z_1)\partial_\zeta G(\zeta, z_2) \\ &\quad + (v_\zeta(z_1)\partial_{z_1} + v_\zeta(z_2)\partial_{z_2} + v_\zeta(\bar{z}_1)\bar{\partial}_{z_1} + v_\zeta(\bar{z}_2)\bar{\partial}_{z_2} + \partial_r)G(z_1, z_2). \end{aligned}$$

Letting $\zeta \to p \in \mathbb{R}$, this equation gives rise to Hadamard's variational formula for annulus Loewner chains:

$$\frac{d}{dt} G_{\Omega_t}(z_1, z_2)\Big|_{t=0} = -P(z_1)P(z_2),$$

where $\Omega_t := \mathcal{C}_r \backslash \tilde{K}_t$ and P is the Poisson kernel given by

$$P(z) := \frac{\partial}{\partial n}\Big|_{\zeta=p} G(\zeta, z).$$

Such variational formula was extensively studied by Izyurov and Kytölä [69]. We also refer to [64, 85] and references therein for variants of Hadamard's formula in a planar domain.

To define the one-leg operator Ψ or the modified multi-vertex fields (OPE exponentials) in general, it is convenient to introduce the formal (1-point) bosonic fields $\Phi^{\pm}_{(0)}$ in D_r. Although they are not Fock space fields, they can be interpreted as the "holomorphic part" and the "anti-holomorphic part" of the GFF $\Phi_{(0)}$ in the following sense:

$$\Phi_{(0)} = \Phi^+_{(0)} + \Phi^-_{(0)}, \quad \Phi^-_{(0)} = \overline{\Phi^+_{(0)}}.$$

Given divisors $\boldsymbol{\sigma} = \sum_{j=1}^n \sigma_j \cdot z_j$, $\boldsymbol{\sigma}_* = \sum_{j=1}^n \sigma_{j*} \cdot z_j$, we set

$$\Phi_{(0)}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] := \sum_{j=1}^n \Big(\sigma_j \Phi^+_{(0)}(z_j) - \sigma_{*j}\Phi^-_{(0)}(z_j)\Big).$$

Then $\Phi_{(0)}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]$ is a well-defined Fock space field if and only if the following neutrality condition (NC_0) holds:

$$\sum_{j=1}^{n} (\sigma_j + \sigma_{*j}) = 0.$$

Given a double divisor $(\boldsymbol{\sigma}, \boldsymbol{\sigma}_*)$ for $z_j \in \mathcal{C}_r$ satisying the neutrality condition (NC_0), we define the *multi-vertex field* $\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] \equiv \mathcal{O}_{\boldsymbol{\beta},(b)}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]$ by

$$\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] := \frac{C_{(b)}[\boldsymbol{\sigma} + \boldsymbol{\beta}/2, \boldsymbol{\sigma}_* + \boldsymbol{\beta}/2]}{C_{(b)}[\boldsymbol{\beta}/2, \boldsymbol{\beta}/2]} \, e^{\odot i \Phi_{(0)}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]},$$

where the *Coulomb gas correlation function* $C_{(b)}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]$ is a differential of conformal dimensions

$$(\lambda_j, \lambda_{*j}) = (\sigma_j^2/2 - b\,\sigma_j, \sigma_{*j}^2/2 - b\,\sigma_{*j})$$

at each z_j and its evaluation in the identity chart of $\mathcal{C}_r$ is given by

$$\begin{aligned} C_{(b)}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] = \Theta'(0)^{\frac{1}{2}\sum_j (\sigma_j^2 + \sigma_{*j}^2)} \exp\Big(- \frac{(\sum \sigma_j z_j + \sigma_{*j} \bar{z}_j)^2}{4(r+\chi)} \Big) \prod_j \Theta(z_j - \bar{z}_j)^{\sigma_j \sigma_{*j}} \\ \times \prod_{j<k} \Theta(z_j - z_k)^{\sigma_j \sigma_k} \Theta(\bar{z}_j - z_k)^{\sigma_{*j} \sigma_k} \Theta(z_j - \bar{z}_k)^{\sigma_j \sigma_{*k}} \Theta(\bar{z}_j - \bar{z}_k)^{\sigma_{*j} \sigma_{*k}}. \end{aligned}$$

See Section 5.5 for further details on Coulomb gas formalism.

Beyond the central charge modification parameter b, we consider another real parameter a. Given a divisor $\boldsymbol{\tau} = \sum \tau_j \cdot \xi_j$ satisfying

$$a + \sum \tau_j = 0,$$

we define the *one-leg operator* $\Psi \equiv \Psi_{\boldsymbol{\beta}}[\boldsymbol{\tau}]$ by

$$\Psi \equiv \Psi_{\boldsymbol{\beta}}(z, \boldsymbol{\xi}) := \mathcal{O}_{\boldsymbol{\beta}}\Big[a \cdot z + \frac{\boldsymbol{\tau}}{2}, \frac{\boldsymbol{\tau}}{2}\Big].$$

We obtain the following form of BPZ equation, a fundamental functional

relation in CFT.

Theorem B. *Suppose that* $2a(a+b)=1$. *Then for any* $\mathcal{X}\in\mathcal{F}_{\boldsymbol{\beta}}$, *we have*

$$\frac{1}{a^2}\partial_z^2\mathbf{E}\Psi_{\boldsymbol{\beta}}[\boldsymbol{\tau}]\mathcal{X}=\Big(\mathcal{L}^+_{v_z}+\mathcal{L}^-_{v_{\bar z}}+\partial_r+2h_{1,2}\frac{\zeta_r(\pi)}{\pi}+2\mathbf{E}T_{\boldsymbol{\beta}}\Big)\mathbf{E}\Psi_{\boldsymbol{\beta}}[\boldsymbol{\tau}]\mathcal{X},$$

where all fields are evaluated in the identity chart of $\mathcal{C}_r$. *Here, Lie derivative operators do not apply to* z *and* $h_{1,2}=a^2/2-ab$.

To connect CFT with SLE theory, we choose real parameters a and b in terms of SLE parameter κ as

$$a=\sqrt{2/\kappa},\qquad b=\sqrt{\kappa/8}-\sqrt{2/\kappa}.$$

Now we consider $\mathrm{SLE}(\kappa,\Lambda)$, where $\Lambda\equiv\Lambda_{\boldsymbol{\beta}}$ is defined by

$$\Lambda_{\boldsymbol{\beta}}(p,\boldsymbol{\xi}):=\kappa\frac{\partial_\xi|_{\xi=p}\mathbf{E}\Psi_{\boldsymbol{\beta}}(\xi,\boldsymbol{\xi})}{Z_{\boldsymbol{\beta}}[\boldsymbol{\tau}]},\quad Z_{\boldsymbol{\beta}}[\boldsymbol{\tau}]:=\mathbf{E}\Psi_{\boldsymbol{\beta}}(p,\boldsymbol{\xi}).$$

Insertion of the one-leg operator produces an operator that changes the values of Fock space fields. See [77, Section 2.3] for its probabilistic counterpart: insertion of Wick's exponential $e^{\odot\Phi(f)}$ of the GFF in application to a test function f results in the change of the law of $\Phi(f)$ in terms of the Green potential. We now consider the following insertion procedure:

$$\widehat{\mathbf{E}}[\mathcal{X}]:=\frac{\mathbf{E}\Psi_{\boldsymbol{\beta}}(p,\boldsymbol{\xi})\mathcal{X}}{Z_{\boldsymbol{\beta}}[\boldsymbol{\tau}]}.$$

By virtue of BPZ-Cardy type equations (Proposition 7.3.1), we obtain the following theorem.

Theorem C. *For any string* $\mathcal{X}$ *of fields in the OPE family* $\mathcal{F}_{\boldsymbol{\beta}}$, *a non-random field* $M=\widehat{\mathbf{E}}[\mathcal{X}]$ *is a martingale-observable for* $\mathrm{SLE}(\kappa,\Lambda)$.

Before moving on to further generalizations of our CFT framework, let us present basic examples of SLE martingale-observables.

Example. (*Bosonic observables*) Let $\boldsymbol{\beta} = 0$ and $M(z) := \widehat{\mathbf{E}}\Phi(z)$. Then M is evaluated as

$$M(z) = -2b \arg w' + 2a \arg \Theta_\chi(r, w_z) + \sum \tau_k \arg \Theta_\chi(r, w_z - w_k)\Theta_\chi(w_z - \bar{w}_k).$$

In particular when $\chi = 0$, the harmonic function M satisfies piecewise Dirichlet boundary conditions having additional jump of $2\pi a$ at p and $2\pi\tau_k$ at ξ_k's.

For $b = 0$ (thus when $\kappa = 4$) the associated bounded martingale M_t plays an important role for the study of scaling limits of discrete GFF interface [118] and GFF/SLE couplings [48,69]. Further applications of such observables are presented in Proposition 7.4.1 (left passage probability) and 8.1.4 (hitting probability of the inner boundary component).

Example. (*Vertex observables*) Let $\tilde{\boldsymbol{\tau}} = \sum \tilde{\tau}_j \cdot \tilde{\xi}_j$ be a divisor satisfying $a + \int \tilde{\boldsymbol{\tau}} = 0$. Consider the vertex observable M given by

$$M = \widehat{\mathbf{E}}\mathcal{O}_{\boldsymbol{\beta}}[\tilde{\boldsymbol{\tau}} - \boldsymbol{\tau}] = \frac{Z_{\boldsymbol{\beta}}[\tilde{\boldsymbol{\tau}}]}{Z_{\boldsymbol{\beta}}[\boldsymbol{\tau}]}.$$

This martingale-observable provides Radon-Nikodym derivative between the laws of two annulus SLEs associated with partition functions $Z_{\boldsymbol{\beta}}[\tilde{\boldsymbol{\tau}}]$ and $Z_{\boldsymbol{\beta}}[\boldsymbol{\tau}]$. Similar martingale-observables also play an important role in the study of Lawler-Schramm-Werner's restriction property [90], see Subsection 7.4.3.

Even though the family of SLE martingale-observables we construct in Theorem C is quite large, we further extend our theory, mainly to develop a version of CFT for annulus SLEs with following important properties: the periodicity; the reversibility. As a consequence, we construct martingale-observables for various continuum interface curves in statistical physics known to have conformal invariance, see Chapter 8.

We first emphasize that except the ER case when $\chi = \infty$, the drift function Λ is not 2π-periodic with respect to space variables. To construct martingale-observables for annulus SLE with 2π-periodic drift function, we

use the weighted summation of the chiral conformal fields, see [126] for a similar idea on compact Riemann surfaces.

For a weight function $\omega : \mathbb{Z} \to \mathbb{R}_+$, set

$$\Psi^{\omega}_{\boldsymbol{\beta}}(p, \boldsymbol{\xi}) := \sum_{n \in \mathbb{Z}} \omega(n) \Psi_{\boldsymbol{\beta}}(p + 2n\pi, \boldsymbol{\xi}).$$

Here we assume that ω is suitably chosen so that the summation above converges. For instance, when $\chi < \infty$, we allow any weight function ω of polynomial type. We define

$$Z^{\omega}_{\boldsymbol{\beta}}(p, \boldsymbol{\xi}) := \mathbf{E}\Psi^{\omega}_{\boldsymbol{\beta}}(p, \boldsymbol{\xi}), \quad \Lambda^{\omega} \equiv \Lambda^{\omega}_{\boldsymbol{\beta}}(p, \boldsymbol{\xi}) := \kappa\, \partial_p \log Z^{\omega}_{\boldsymbol{\beta}}(p, \boldsymbol{\xi}).$$

Extending Theorem C, we obtain the following. Notice that Theorem C is recovered by letting $\omega(n) = \mathbb{1}_{\{n=0\}}$.

Theorem D. *For any string $\mathcal{X}$ of fields in the OPE family $\mathcal{F}_{\boldsymbol{\beta}}$, a non-random field*

$$M = \widehat{\mathbf{E}}[\mathcal{X}] := \frac{\mathbf{E}\Psi^{\omega}_{\boldsymbol{\beta}}(p, \boldsymbol{\xi})\mathcal{X}}{Z^{\omega}_{\boldsymbol{\beta}}[\boldsymbol{\tau}]}.$$

is a martingale-observable for $\mathrm{SLE}(\kappa, \Lambda^{\omega})$.

Example. As a simplest example, let us consider the case that $\boldsymbol{\beta} = \mathbf{0}$, $\boldsymbol{\tau} = -a \cdot q$ for $q \in \mathbb{R}_r$ and $\omega(n) \equiv 1$. Then up to multiplicative constant the partition functions $Z \equiv Z_{\boldsymbol{\beta}}[\boldsymbol{\tau}]$, $Z^{\omega} \equiv Z^{\omega}_{\boldsymbol{\beta}}[\boldsymbol{\tau}]$ are given by ($x = p - \mathrm{Re} q$)

$$Z(x) = \Theta_I(r, x)^{-\frac{2}{\kappa}} e^{-\frac{x^2}{2\kappa(r+\chi)}}, \quad Z^{\omega}(x) = \Theta_I(r, x)^{-\frac{2}{\kappa}} \Theta_I\Big(\frac{\kappa}{2}(r + \chi), x + \pi\Big).$$

Thus we have

$$\Lambda(x) = -H_I(r, x) + \frac{x}{r + \chi}, \quad \Lambda^{\omega}(x) = -H_I(r, x) - \frac{\kappa}{2} H_I\Big(\frac{\kappa}{2}(r + \chi), x + \pi\Big).$$

Notice here that Λ^{ω} is 2π-periodic, whereas Λ is not. For definitions of Θ_I, H_I, see Section 3.1.

In particular when $\chi = 0$, the associated $\mathrm{SLE}(\kappa, \Lambda^{\omega})$ trace ends at $\{q +$

$2n\pi : n \in \mathbb{Z}\}$. Then by virtue of vertex observables in the previous example, one can see that for each $m \in \mathbb{Z}$,

$$M_m(x) := \frac{Z(x-2m\pi)}{Z^{\omega}(x)} = \exp\Big(\frac{(x-2m\pi)^2}{2\kappa\, r}\Big)\Theta_I\Big(\frac{\kappa\, r}{2}, x+\pi\Big)^{-1}$$

gives rise to the probability that $\mathrm{SLE}(\kappa, \Lambda^{\omega})$ trace ends at $q + 2m\pi$.

We now present a way of constructing martingale-observables for the annulus SLE with two marked points, which satisfies the local commutation relation studied in [139]. The notion of commuting SLEs was first introduced in [46] and in the presence of two marked points, one can think of it as a weaker form of reversibility. Here for simplicity, we focus on *chordal type* annulus $\mathrm{SLE}(\kappa, \Lambda)$ with one force point q starting from p, where both of the marked points lie on the same boundary component, see Section 7.7 for general cases. The drift function Λ is defined in terms of the *annulus SLE partition function* $Z \equiv Z(r, \cdot)$ by $\Lambda = \kappa\, Z'/Z$. It was shown by Zhan [139] that to define a reversible $\mathrm{SLE}(\kappa, \Lambda)$, the associated partition function Z should satisfy the null-vector equation:

$$\partial_r Z = \frac{\kappa}{2} Z'' + H\, Z' + h_{1,2}(\kappa) H'\, Z + C(r) Z, \quad h_{1,2}(\kappa) := \frac{6-\kappa}{2\kappa}, \tag{2.3.1}$$

where $C(r)$ is a constant depending only on the modular parameter r.

To develop CFT for such chordal type annulus SLE, we use the method of *screening*. Due to the reversibility of annulus $\mathrm{SLE}(\kappa, \Lambda)$, the partition function (i.e., the correlator of corresponding one-leg operator) should have the same conformal dimensions at p and q. In the case of chordal SLE in the upper-half plane, it can be achieved by the *effective one-leg operator* with the total charge $2b - a$ (including the background charge $2b$) placed at q and with the charge a placed at p. For more details we refer the reader to [75, 76]. However, this choice of partition function cannot be realized in the annulus because the neutrality conditions depend on the connectivity. See [78] for the neutrality conditions on a compact Riemann surface of genus g. Instead we

consider

$$\Psi_\beta(p,q) := C(\kappa) \oint_{\mathcal{P}(p,q)} \mathcal{O}_{\boldsymbol{\beta}}[a\cdot p + a\cdot q + s\cdot\zeta,\, \mathbf{0}]\, d\zeta, \qquad (s=-2a)$$

where the integration contour $\mathcal{P}(p,q)$ stands for the Pochhammer contour entwining p and q, see Figure 7.3 in Section 7.7. Here $C(\kappa)$ is a universal constant depending only on κ, see (7.7.9) and the background charge is given as $\boldsymbol{\beta} := \beta\cdot q_1 - \beta\cdot q_2$ $(q_2 - q_1 = 2\pi)$ for some $\beta \in \mathbb{R}$. With this choice of $s=-2a$, the OPE exponential $\mathcal{O}_{\boldsymbol{\beta}}[a\cdot p + a\cdot z + s\cdot\zeta,\, \mathbf{0}]$ is a 1-differential with respect to the "screening" variable ζ and it satisfies the neutrality condition (NC_0). Clearly, Ψ_β has the same conformal dimensions both at p and q. For each $\kappa > 0$, we have the following expression (up to multiplicative constant) of $Z_\beta \equiv Z_\beta(r, p-q) := \mathbf{E}\Psi_\beta(p,q)$ in terms of Jacobi's theta function:

$$Z_\beta = \Theta(p-q)^{\frac{2}{\kappa}} \oint_{\mathcal{P}(p,q)} \Theta(p-\zeta)^{-\frac{4}{\kappa}} \Theta(\zeta-q)^{-\frac{4}{\kappa}} \exp\Big(-\frac{(\Sigma - 2\beta\pi)^2}{4(r+\chi)}\Big)\, d\zeta,$$

where $\Sigma = a\,(p+q-2\zeta)$. We remark that for $\kappa > 4$, this Pochhammer contour integral is simplified to the integration along the interval between p, q, see (7.7.18). With this choice of one-leg operator $\Psi_\beta(p,q)$, we find Coulomb gas solutions for annulus SLE partition functions and construct a family of martingale-observables for chordal type $\mathrm{SLE}(\kappa,\Lambda)$.

Theorem E. *For each $\kappa > 0$, the partition function $Z_\beta := \mathbf{E}\,\Psi_\beta(\cdot, q)$ is a non-trivial (real-valued) solution of the null-vector equation* (2.3.1)*. Moreover, for any string $\mathcal{X}$ of fields in the OPE family $\mathcal{F}_{\boldsymbol{\beta}}$, a non-random field*

$$M = \widehat{\mathbf{E}}\mathcal{X} := \frac{\mathbf{E}\Psi_\beta(p,q)\mathcal{X}}{\mathbf{E}\Psi_\beta(p,q)}$$

is a martingale-observable for chordal type $\mathrm{SLE}(\kappa,\Lambda)$.

An important extension of Theorem C, but not discussed in this section, is based on the use of non-atomic background charges, see Section 7.5. Applications of our CFT to statistical physics models will be presented in

Chapter 8.

2.4 Organization of the thesis

The rest of this thesis is organized as follows.

In Chapter 3, we review basic properties of GFF and present its correlation function (i.e., Green's function in a planar domain with several different boundary conditions) in terms of well-known special functions.

Chapter 4 is devoted to give a brief exposition of variants of SLE processes and their evolutionary descriptions.

Chapter 5 contains a brief summary of basic concepts in CFT and calculus of correlation functions in conformal Fock space fields.

In Chapter 6, we prove Eguchi-Ooguri equation and their version of Ward's equation, Theorem A.

Chapter 7 provides the proof of BPZ equation (Theorem B) and establishes the relation between CFT and SLE (Theorems C, D, E).

In Chapter 8, we present a way to construct martingale-observables for continuum interface curves and provide some important examples.

In Chapter 9, some applications of Ward's identities are presented in the context of random matrix theory.

Chapter 3

Gaussian free field

In this chapter we discuss the basic field in our CFT framework, the Gaussian free field (GFF). From a probabilistic point of view, GFF is well known as a natural 2-dimensional time analog of Brownian motion (see e.g., [121] and [77, Lecture 1] for more details), which also plays a crucial role in the theory of random surfaces [120]. Rather, we aim to give a comprehensive account for its correlation functions in general planar domains (or on Riemann surfaces) from a viewpoint of geometric function theory.

3.1 Special functions

Let us compile definitions and basic properties of some special functions which are useful to represent the Loewner vector field and the correlation functions of Fock space fields.

3.1.1 Elliptic functions

Following the conventions in [133, 139], let us define

$$\Theta(r,z)=\theta\Big(\frac{z}{2\pi},\frac{ir}{\pi}\Big),\qquad \Theta_I(r,z)=\theta_2\Big(\frac{z}{2\pi},\frac{ir}{\pi}\Big). \tag{3.1.1}$$

where

$$\theta(z,\tau) := \frac{1}{i}\sum_{n\in\mathbb{Z}}(-1)^n e^{\pi i\tau(n+\frac{1}{2})^2}e^{(2n+1)\pi iz}, \tag{3.1.2}$$

$$\theta_2(z,\tau) := \sum_{n\in\mathbb{Z}}(-1)^n e^{\pi i\tau n^2}e^{2n\pi iz} \tag{3.1.3}$$

are Jacobi's theta functions (see [33, Chapter V]). Note that while $\Theta(r,\cdot)$ is odd function, $\Theta_I(r,\cdot)$ is even function. By definition, $\Theta(r,\cdot)$ is holomorphic function on $\mathbb{C}$ vanishing only on $2\pi\mathbb{Z}+2ir\mathbb{Z}$ and related to Θ_I as

$$\begin{aligned}\Theta_I(r,z) &= i\,\Theta(r,z-ir)\exp(-r/4-iz/2)\\ &= \frac{1}{i}\,\Theta(r,z+ir)\exp(-r/4+iz/2).\end{aligned} \tag{3.1.4}$$

By the infinite product representation of theta functions (cf. [33, p.69]), we have

$$\Theta(r,z) = 2c(r)\,e^{-\frac{r}{4}}\sin\left(\frac{z}{2}\right)\prod_{n=1}^{\infty}\left(1-e^{-2nr}e^{iz}\right)\left(1-e^{-2nr}e^{-iz}\right), \tag{3.1.5}$$

$$\Theta_I(r,z) = c(r)\prod_{n=1}^{\infty}\left(1-e^{-(2n-1)r}e^{iz}\right)\left(1-e^{-(2n-1)r}e^{-iz}\right), \tag{3.1.6}$$

where the universal constant $c(r)$ is given as

$$c(r) = \prod_{n=1}^{\infty}\left(1-e^{-2nr}\right). \tag{3.1.7}$$

Thus one can easily observe the asymptotic behaviors of theta functions

$$\lim_{r\to\infty} e^{r/4}\Theta(r,z) = 2\sin\left(\frac{z}{2}\right), \quad \lim_{r\to\infty} e^{r/4}\Theta'(r,0) = 1 \tag{3.1.8}$$

and

$$\lim_{r\to\infty}\Theta_I(r,z) = 1. \tag{3.1.9}$$

It is well known that theta functions have the quasi-periodicities

$$\Theta(r, z+2\pi) = -\Theta(r,z), \quad \Theta(r, z+2ir) = -\Theta(r,z)\exp(r-iz) \tag{3.1.10}$$

$$\Theta_I(r, z+2\pi) = \Theta_I(r,z), \quad \Theta_I(r, z+2ir) = -\Theta_I(r,z)\exp(r-iz) \tag{3.1.11}$$

and solve the heat equations

$$\partial_r \Theta = \Theta'', \qquad \partial_r \Theta_I = \Theta_I''. \tag{3.1.12}$$

Indeed, theta functions $\Theta(r,\cdot)$ and $\Theta_I(r,\cdot)$ can be represented in terms of fundamental solutions of the heat equation as

$$\Theta(r,z) = \sqrt{\frac{\pi}{r}} \sum_{n\in\mathbb{Z}} (-1)^n \exp\Big(-\frac{(z-\pi+2n\pi)^2}{4r}\Big); \tag{3.1.13}$$

$$\Theta_I(r,z) = \sqrt{\frac{\pi}{r}} \sum_{n\in\mathbb{Z}} \exp\Big(-\frac{(z-\pi+2n\pi)^2}{4r}\Big), \tag{3.1.14}$$

see e.g., [108, (20.2.13),(20.13.4)].

Due to Landen's transformation formula (see [108, Eq. (20.7.16)]), we have (up to multiplicative constant)

$$\Theta(2r, 2z) = \Theta(r,z)\Theta(r, z+\pi). \tag{3.1.15}$$

On the other hand, the transformation of nome is given by (up to multiplicative constant)

$$\begin{aligned} \Theta(r,z) &= \Theta(2r,z)\Theta_I(2r,z) \\ &= \exp(iz/2)\Theta(2r,z)\Theta(2r, z+2ir), \end{aligned} \tag{3.1.16}$$

see [108, Eq. (20.7.11)]. Combining above two identities, we obtain the du-

plication formula: up to multiplicative constant,

$$\begin{aligned}\Theta(r,2z) &= \Theta(r,z)\Theta(r,z+\pi)\Theta_I(r,z)\Theta_I(r,z+\pi)\\ &= \exp(iz)\Theta(r,z)\Theta(r,z+\pi)\Theta(r,z+ir)\Theta(r,z+\pi+ir),\end{aligned} \tag{3.1.17}$$

see [108, Eq. (20.7.10)].

The addition formula is given by

$$\frac{(\Theta\Theta_I(r,x))^2+(\Theta\Theta_I(r,x+\pi))^2}{\Theta(r,\pi)^2\Theta_I(r,\pi)} = \Theta_I(r,2x+\pi). \tag{3.1.18}$$

Notice that by (3.1.8), this identity in the limit $r\to\infty$ corresponds to the trigonometric identity $\sin^2(\cdot)+\cos^2(\cdot)=1$.

By definition, the Weierstrass zeta function ζ_r and elliptic function $\wp_r$ with basic periods $(2\pi, 2ir)$ are given by

$$\zeta_r(z) := \frac{1}{z} + \sum_{\eta}{}^{*}\Big(\frac{1}{z-\eta}+\frac{1}{\eta}+\frac{z}{\eta^2}\Big), \tag{3.1.19}$$

$$\wp_r(z) := \frac{1}{z^2} + \sum_{\eta}{}^{*}\Big(\frac{1}{(z-\eta)^2}-\frac{1}{\eta^2}\Big), \tag{3.1.20}$$

where the sum $\sum_\eta^*$ is taken over all η of the form $2\pi m+2irn$ for $(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}$. Note that ζ_r is holomorphic except for $2\pi\mathbb{Z}+2ir\mathbb{Z}$, and satisfies the following quasi-periodicity: for any integers m,n

$$\zeta_r(z+2m\pi+2nir) = \zeta_r(z)+4m\pi\,\zeta_r(\pi)+4nir\,\zeta_r(ir). \tag{3.1.21}$$

By definition, ζ_r and $\wp_r$ are related as

$$\zeta_r'(z) = -\wp_r(z). \tag{3.1.22}$$

The Legendre identity (see [33, Chap IV. Theorem 2]) is written as

$$\frac{\zeta_r(\pi)}{2\pi} - \frac{\zeta_r(ir)}{2ir} = \frac{1}{4r}. \tag{3.1.23}$$

Using the infinite sum representations of special functions above, one can easily observe that

$$\wp_{2r}(2ir) = \frac{1}{\pi}\Big(\zeta_r(\pi) - 2\zeta_{2r}(\pi)\Big) \tag{3.1.24}$$

and

$$\frac{\zeta_r(\pi)}{\pi} = -\frac{1}{3}\frac{\Theta'''(r,0)}{\Theta'(r,0)} = -\frac{1}{3}\partial_r \log \Theta'(r,0) = \frac{1}{12} - \frac{1}{2}\sum_{k=1}^{\infty} \sinh^{-2}(kr). \tag{3.1.25}$$

It is well known that

$$\zeta_r(\pi) + \zeta_r(ir) + \zeta_r(\pi + ir) = 0. \tag{3.1.26}$$

On the other hand, half-periods of $\wp_r$ satisfies

$$\wp_r(\pi) + \wp_r(ir) + \wp_r(\pi + ir) = 0, \quad \wp_r(\pi) > \wp_r(ir) > \wp_r(\pi + ir). \tag{3.1.27}$$

Moreover

$$\wp_r'(\pi) = \wp_r'(ir) = \wp_r'(\pi + ir) = 0. \tag{3.1.28}$$

The pseudo-addition theorem of Weierstrass zeta function is given as follows: for $x + y + z = 0$,

$$\big(\zeta_r(x) + \zeta_r(y) + \zeta_r(z)\big)^2 + \zeta_r'(x) + \zeta_r'(y) + \zeta_r'(z) = 0, \tag{3.1.29}$$

see [33, p.57].

The Loewner kernel $H(r,\cdot)$ in $\mathbb{S}_r$ is given by

$$H(r,z) := -i\,\mathrm{P.V.}\lim_{n\to\infty}\sum_{n\in 2\mathbb{Z}} \frac{e^{nr} + e^{iz}}{e^{nr} - e^{iz}}, \tag{3.1.30}$$

and can be represented in terms of special functions as

$$H(r,z) := 2\frac{\Theta'(r,z)}{\Theta(r,z)} = 2\zeta_r(z) - \frac{2}{\pi}\zeta_r(\pi)z. \tag{3.1.31}$$

Therefore it is easy to observe that $H(r,\cdot)$ is a meromorphic (odd) function on $\mathbb{C}$ having simple poles at $2\pi m+2irn, (m,n\in\mathbb{Z})$ with residue 2. In particular, we have the following asymptotic behavior of H:

$$H(r,z)=\frac{2}{z}-\frac{2\zeta_r(\pi)}{\pi}z+O(z^3),\qquad \text{as } z\to 0. \tag{3.1.32}$$

The quasi-periodicity (3.1.21) of ζ_r is rewritten as

$$H(r,z+2\pi)=H(r,z),\quad H(r,z+2ir)=H(r,z)-2i. \tag{3.1.33}$$

Moreover, $H(r,\cdot)$ takes real values on $\mathbb{R}\setminus\{2\pi n : n\in\mathbb{Z}\}$ and $\operatorname{Im} H(r,\cdot)\equiv -1$ on $\mathbb{R}_r$. In particular, by (3.1.23), we have

$$H(r,\pi)=0,\quad H(r,ir)=-i,\quad H(r,\pi+ir)=-i. \tag{3.1.34}$$

Note that by (3.1.15) and (3.1.16) we have

$$H(2r,2z)=H(r,z)/2+H(r,z+\pi)/2 \tag{3.1.35}$$

and

$$\begin{aligned} H(r,z)&=H(2r,z)+H_I(2r,z)\\ &=H(2r,z)+H(2r,z+2ir)+i, \end{aligned} \tag{3.1.36}$$

where the function H_I is given as

$$H_I(r,z):=2\frac{\Theta_I'(r,z)}{\Theta_I(r,z)}=H(r,z+ir)+i. \tag{3.1.37}$$

It also follows from (3.1.8) and (3.1.9) that

$$\lim_{r\to\infty}H(r,z)=\cot\left(\frac{z}{2}\right),\qquad \lim_{r\to\infty}H_I(r,z)=0. \tag{3.1.38}$$

On the other hand, by (3.1.25),

$$\lim_{r\to\infty}\frac{\zeta_r(\pi)}{\pi}=\frac{1}{12}. \tag{3.1.39}$$

By the heat equation of Theta function (3.1.12), we have

$$\partial_r H(r,x)=(H''+H'H)(r,x). \tag{3.1.40}$$

The pseudo-addition theorem (3.1.29) is rewritten as

$$\begin{aligned}H(z-w)(H(z)-H(w))=&\frac{1}{2}H^2(z-w)+\frac{1}{2}(H(z)-H(w))^2\\&+H'(z-w)+H'(z)+H'(w)+\frac{6}{\pi}\zeta_r(\pi).\end{aligned} \tag{3.1.41}$$

Differentiating with respect to z,w we obtain

$$H(z+w)=H(z)+H(w)+\frac{H''(z)-H''(w)}{H'(z)-H'(w)}. \tag{3.1.42}$$

In particular, by (3.1.34),

$$\frac{1}{2}(H(z+\pi)-H(z))^2+H'(\pi)+H'(z+\pi)+H'(z)+\frac{6}{\pi}\zeta_r(\pi)=0 \tag{3.1.43}$$

and

$$\frac{1}{2}(H_I(z)-H(z))^2+H'(ir)+H_I'(z)+H'(z)+\frac{6}{\pi}\zeta_r(\pi)=0. \tag{3.1.44}$$

It also follows from (3.1.41) and (3.1.37) that

$$\begin{aligned}H(z-w)(H_I(z)-H_I(w))=&\frac{1}{2}H^2(z-w)+\frac{1}{2}(H_I(z)-H_I(w))^2\\&+H'(z-w)+H_I'(z)+H_I'(w)+\frac{6}{\pi}\zeta_r(\pi)\end{aligned} \tag{3.1.45}$$

and

$$H_I(z-w)(H_I(z)-H(w)) = \frac{1}{2}H_I^2(z-w) + \frac{1}{2}(H_I(z)-H(w))^2 + H_I'(z-w) + H_I'(z) + H'(w) + \frac{6}{\pi}\zeta_r(\pi). \tag{3.1.46}$$

In particular, we have

$$\frac{1}{2}(H_I(z+\pi)-H(z))^2 + H'(\pi) + H_I'(z+\pi) + H_I'(z) + \frac{6}{\pi}\zeta_r(\pi) = 0. \tag{3.1.47}$$

To describe correlation functions with weighted boundary conditions indexed by $\chi \in [0,\infty]$, it is convenient to introduce the modified theta function

$$\Theta_\chi(r,z) := \Theta(r,z)\exp\Big(\frac{z^2}{4(r+\chi)}\Big). \tag{3.1.48}$$

In the sequel, we also write

$$H_\chi(r,z) := 2\log\Theta_\chi'(r,z) = H(r,z) + \frac{z}{r+\chi}. \tag{3.1.49}$$

In particular, we denote

$$\tilde{\Theta} := \Theta_0, \quad \tilde{H} := H_0. \tag{3.1.50}$$

An important feature of the function $\tilde{\Theta}(r,\cdot)$ is that on $\mathbb{R}_r$, its values lie on the positive imaginary axis. Note also that the function $H_\chi(r,\cdot)$ is real-valued on $\mathbb{R}_r$ if and only if $\chi = 0$.

By (3.1.10), we have

$$\Theta_\chi(r,z+2\pi) = -\Theta_\chi(r,z)\exp\Big(\frac{\pi}{r+\chi}(z+\pi)\Big), \tag{3.1.51}$$

$$\Theta_\chi(r,z+2ir) = -\Theta_\chi(r,z)\exp\Big(\frac{\chi}{r+\chi}(r-iz)\Big). \tag{3.1.52}$$

On the other hand, it follows from (3.1.33) that

$$H_\chi(r, z + 2\pi) = H_\chi(r, z) + \frac{2\pi}{r + \chi}, \tag{3.1.53}$$

$$H_\chi(r, z + 2ir) = H_\chi(r, z) - \frac{2\chi}{r + \chi} i. \tag{3.1.54}$$

Note in particular that $\tilde{\Theta}, \tilde{H}$ are (anti-) periodic with respect to $2ir$:

$$\tilde{\Theta}(r, z + 2ir) = -\tilde{\Theta}(r, z), \quad \tilde{H}(r, z + 2ir) = \tilde{H}(r, z). \tag{3.1.55}$$

3.1.2 Schottky-Klein prime function

In this section, we recall a useful special function called *Schottky-Klein (S-K) prime function* in multiply connected domains. Here we follow notations in [40, 103].

We call a domain D *circular* if it is a finitely connected domain all of whose boundary components are circles. We assume that the outer boundary C_0 is a unit circle and denote by C_j $(j = 1, \cdots, g)$ each boundary component. To specify a circular domain, let $\delta_j \in \mathbb{C}$ and $r_j \in \mathbb{R}$ be centres and radii of C_j's, i.e., C_j is given as

$$|\zeta - \delta_j|^2 = (\zeta - \delta_j)(\bar{\zeta} - \bar{\delta}_j) = r_j^2. \tag{3.1.56}$$

We define Möbius maps ϕ_j, $j = 0, 1, \cdots, g$ as

$$\phi_j(\zeta) := \bar{\delta}_j + \frac{r_j^2}{\zeta - \delta_j}, \quad \phi_0(\zeta) := \frac{1}{\zeta}. \tag{3.1.57}$$

Note that ϕ_j correspond to the conjugation map on the circle C_j, i.e.,

$$\bar{\zeta} = \phi_j(\zeta) \quad \text{for} \quad \zeta \in C_j.$$

For each j, define Möbius maps θ_j as

$$\theta_j(\zeta) := \bar{\phi}_j(\zeta^{-1}) = \delta_j + \frac{r_j^2 \zeta}{1 - \bar{\delta}_j \zeta}. \tag{3.1.58}$$

Let C_j' be a circle obtained by reflection of C_j with respect to C_0, i.e.,

$$C_j' = \{1/\bar{\zeta} : \zeta \in C_j\}.$$

Then it is easy to see that the image of C_j' under θ_j is C_j.

By definition, the *Schottky group* $\mathfrak{S}$ is the infinite free group of mappings generated by compositions of basic Möbius maps θ_j and their inverses θ_j^{-1}. Let us denote by $\mathfrak{S}''$ the set of all mapping in the group $\mathfrak{S}$ excluding the identity and all inverse maps. The *Schottky-Klein (S-K) prime function* $\omega(\zeta, z)$ is defined by

$$\omega(\zeta, z) := (\zeta - z)\,\tilde{\omega}(\zeta, z), \tag{3.1.59}$$

$$\tilde{\omega}(\zeta, z) := \prod_{\tilde{\theta} \in \mathfrak{S}''} \frac{(\tilde{\theta}(\zeta) - z)(\tilde{\theta}(z) - \zeta)}{(\tilde{\theta}(\zeta) - \zeta)(\tilde{\theta}(z) - z)}. \tag{3.1.60}$$

We refer to [15] for the convergence of the infinite product. It is also well known that the S-K prime function ω satisfies following transformation properties

$$\omega(\zeta, z) = -\omega(z, \zeta), \quad \bar{\omega}(\zeta^{-1}, z^{-1}) = -\frac{1}{\zeta z}\omega(\zeta, z). \tag{3.1.61}$$

For instance, in the annulus $\mathbb{A}_r$, $\mathfrak{S}$ has one generator

$$\theta_1(\zeta) = e^{-2r}\zeta$$

and $\mathfrak{S}''$ is given by

$$\mathfrak{S}'' = \{\theta_1^{(k)} | k \in \mathbb{N}\}.$$

Therefore, S-K prime function ω in $\mathbb{A}_r$ is represented as

$$\omega(\zeta, z) = (\zeta - z) \prod_{k=1}^{\infty} \frac{(e^{-2rk}\zeta - z)(e^{-2rk}z - \zeta)}{(e^{-2rk}\zeta - \zeta)(e^{-2rk}z - z)}. \tag{3.1.62}$$

3.2 Representation of Green's functions

3.2.1 Dirichlet and Excursion-reflected boundary conditions

Let D be a bounded and $(g+1)$-connected planar domain. Suppose D is bounded by $g+1$ smooth Jordan curves C_j, $(j = 0, 1 \cdots, g)$, where C_0 is taken as the outermost boundary. The Green's function G_D is a unique function such that $G_D(\zeta, z) + \log|\zeta - z|$ is harmonic with respect to ζ throughout the region D including z and satisfies proper boundary conditions. We mainly focus on the following two boundary conditions.

- **Dirichlet boundary condition**:

$$G_D(\zeta, z) = 0 \quad \text{for} \quad \zeta \in C_j, \quad j = 0, 1, \cdots g. \tag{3.2.1}$$

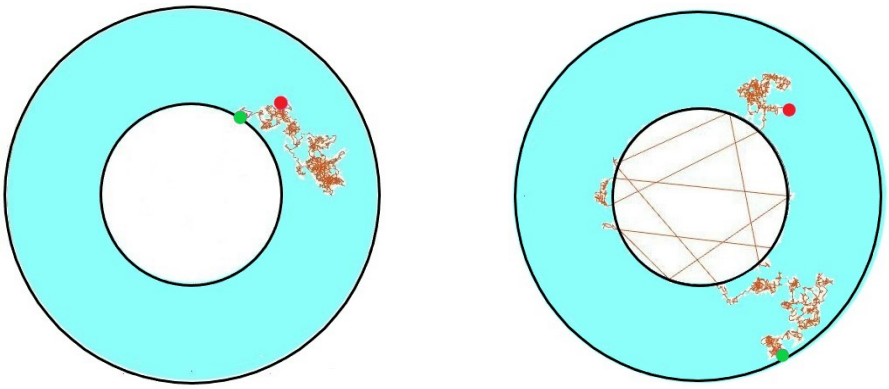
- **ER boundary condition**:

$$G_D(\zeta, z) = \begin{cases} 0 & \text{for} \quad \zeta \in C_0. \\ \\ \gamma_j(z) & \text{for} \quad \zeta \in C_j, \quad j = 1, \cdots g \end{cases} \tag{3.2.2}$$

  where $\gamma_j(z)$'s are functions depending only on z. In addition, it satisfies *zero period condition*

$$\oint_{C_j} \frac{\partial G_D}{\partial n_z} \, ds_z = 0, \quad \text{for} \quad j = 1, \cdots g, \tag{3.2.3}$$

  where ds_z denotes the arclength measure and $\partial/\partial n_z$ denotes the normal derivative.



(a) killed BM (b) ERBM

Figure 3.1: Brownian motions

In the analysis literature, the Green's function with ER boundary condition is often called as a *modified Green's function*. This was introduced by Koebe in his classical work [83] on characterizing canonical multiply connected domains. Later it was rediscovered in the theory of fluid dynamics [99] and now it is also referred to a *hydrodynamic Green's function*, see e.g., [36].

In probability theory, the associated stochastic process, ER Brownian motion, was introduced by Lawler and Drenning, see [42, 92]. An excursion reflected Brownian motion (ERBM) B_D^{ER} in a multiply connected domain D is a strong Markov process stopped at the outermost boundary of D that acts like a Brownian motion when it is away from ∂D. Also it reflects on the inner boundary components of D at a random point chosen according to the harmonic measure from ∞ whenever it hits the inner boundary. It is well known that ERBM is conformally invariant and so is the Green's function for ERBM. We also remark that ERBM is a special case of a more general process called *Brownian motion with darning* [35], which can be constructed using Dirichlet form theory. See also [27] for another generalization of ERBM, called *obliquely reflected Brownian motion* (ORBM).

In particular, in a doubly connected domain when $g = 1$ the Green's function G_r vanishes on the outer boundary component and satisfies

$$\begin{cases} G_r(\zeta, z) = 0 & \text{for Dirichlet b.c.} \\ \oint_\gamma \dfrac{\partial G_r}{\partial n_z}\, ds_z = 0 & \text{for ER b.c.} \end{cases} \tag{3.2.4}$$

on the inner boundary component. Extending (3.2.4), we introduce the weighted combination of Dirichlet and ER boundary conditions: for $\chi \in [0, \infty]$,

$$\frac{1}{1+\chi} G_r(\zeta, z) + \frac{\chi}{1+\chi}\Big(\frac{1}{2\pi}\oint_\gamma \frac{\partial G_r}{\partial n_z} ds_z\Big) = 0. \tag{3.2.5}$$

One may realize (3.2.5) as an analogue of the well-known Robin boundary condition, in which the Neumann condition is replaced by the ER one.

Let us now recall the relation between Green's functions with two different boundary conditions. From now on, we write G_D for Green's function with Dirichlet boundary condition and G_D^{ER} for ER one. We denote by $\{h_j | j = 0, 1, \cdots, g\}$ a set of harmonic functions satisfying

$$h_j(\zeta) = \begin{cases} 1 & \text{on} \quad C_j \\ 0 & \text{on} \quad C_k, \quad k \neq j. \end{cases} \tag{3.2.6}$$

Note that $h_j(\zeta)$ stands for the probability that a Brownian motion starting at ζ exits D through C_j. Therefore, it immediately follows that

$$h_0 + h_1 + \cdots + h_g \equiv 1. \tag{3.2.7}$$

On the other hand, it is well known that

$$h_j(\zeta) = -\frac{1}{2\pi}\oint_{C_j} \frac{\partial G_D(\zeta, z)}{\partial n_z} ds_z. \tag{3.2.8}$$

Due to [42, Proposition 5.2] and [40, Eq. (45)], we have

$$G_D(\zeta, z) = G_D^{ER}(\zeta, z) - \sum_{j=1}^{g} \gamma_j(z)\, h_j(\zeta). \tag{3.2.9}$$

Let us denote by P_{kj} the periods of h_j with respect to circuits around C_k given as

$$P_{kj} = \frac{1}{2\pi} \oint_{C_k} \frac{\partial h_j(z)}{\partial n_z} ds_z. \tag{3.2.10}$$

Then the *period matrix* $\boldsymbol{P} = [P_{kj}]_{k,j=1}^{g}$ is real, symmetric and positive-definite. We write $\boldsymbol{h} = (h_1, \cdots, h_g)^T$. Then by [22, Eq. (6)], the relation (3.2.9) is rewritten as

$$G_D(\zeta, z) = G_D^{ER}(\zeta, z) - \Big\langle \boldsymbol{h}(\zeta), \boldsymbol{P}^{-1}\boldsymbol{h}(z) \Big\rangle, \tag{3.2.11}$$

where $\langle \cdot, \cdot \rangle$ is standard inner product.

3.2.2 Green's function in a doubly connected domain

Let us consider a doubly connected domain, i.e., $g = 1$. By generalized Riemann mapping theorem, we can take the annulus $\mathbb{A}_r$ as our reference domain. It is well known that the Dirichlet Green's function G_r on $\mathbb{A}_r$ is given by

$$\begin{aligned} G_r(\zeta, z) = &-\frac{\log|\zeta| \log|z|}{r} \\ &- \log\Big| \frac{z-\zeta}{1-\zeta\overline{z}} \prod_{k=1}^{\infty} \frac{(1-e^{-2rk}\zeta z^{-1})(1-e^{-2rk} z\zeta^{-1})}{(1-e^{-2rk}\zeta\bar{z})(1-e^{-2rk}\zeta^{-1}\bar{z}^{-1})} \Big|, \end{aligned} \tag{3.2.12}$$

see [65]. Also, it is easy to see that in $\mathbb{A}_r$,

$$h_1(\zeta) = -\frac{\log|\zeta|}{r}, \quad \boldsymbol{P} = P_{11} = \frac{1}{r}.$$

Therefore by (3.2.11), ER Green's function G_r^{ER} on $\mathbb{A}_r$ is given by

$$G_r^{ER}(\zeta, z) = -\log\Big|\frac{z-\zeta}{1-\zeta\bar{z}}\prod_{k=1}^{\infty}\frac{(1-e^{-2rk}\zeta z^{-1})(1-e^{-2rk}z\zeta^{-1})}{(1-e^{-2rk}\zeta\bar{z})(1-e^{-2rk}\zeta^{-1}\bar{z}^{-1})}\Big|. \tag{3.2.13}$$

Note that one can also obtain (3.2.13) from [42, Proposition 5.2 and Lemma 5.5].

In the cylinder $\mathcal{C}_r$, such Green's functions G_r are represented in terms of Jacobi theta function as

$$G_r(\zeta, z) = \begin{cases} \log\Big|\dfrac{\Theta(r, \zeta-\bar{z})}{\Theta(r, \zeta-z)}\Big| & \text{for ER b.c.} \\ \log\Big|\dfrac{\Theta(r, \zeta-\bar{z})}{\Theta(r, \zeta-z)}\Big| - \dfrac{\operatorname{Im}\zeta \operatorname{Im} z}{r} & \text{for Dirichlet b.c.} \end{cases} \tag{3.2.14}$$

It is then straightforward to see that the Green's function

$$G_r^{\chi}(\zeta, z) := \log\Big|\frac{\Theta_\chi(r, \zeta-\bar{z})}{\Theta_\chi(r, \zeta-z)}\Big| = \log\Big|\frac{\Theta(r, \zeta-\bar{z})}{\Theta(r, \zeta-z)}\Big| - \frac{\operatorname{Im}\zeta \operatorname{Im} z}{r+\chi} \tag{3.2.15}$$

satisfies the weighted boundary condition (3.2.5).

3.2.3 Green's function in a multiply connected domain

In [37, 38], the authors express ER Green's function (modified Green's function) defined in a circular domain in terms of S-K prime function as

$$G_D^{ER}(\zeta, z) = -\frac{1}{2}\log\Big|\frac{\omega(\zeta, z)\,\bar{\omega}(1/\zeta, 1/z)}{\omega(\zeta, 1/\bar{z})\,\bar{\omega}(1/\zeta, \bar{z})}\Big| = -\log\Big|\frac{\omega(\zeta, z)}{z\,\omega(\zeta, 1/\bar{z})}\Big|. \tag{3.2.16}$$

For example, in case of $g = 1$, one can recover (3.2.13) by combining (3.1.62) and (3.2.16).

On the other hand, by (3.2.12), the Green's function with Dirichlet boundary condition can be represented as

$$G_D(\zeta, z) = -\log\Big|\frac{\omega(\zeta, z)}{z\omega(\zeta, \bar{z}^{-1})}\Big| - \Big\langle \boldsymbol{h}(\zeta), \boldsymbol{P}^{-1}\boldsymbol{h}(z)\Big\rangle. \tag{3.2.17}$$

Remark. In next section, we will explain that Green's functions have alternative representations obtained by Schottky double relation:

$$G_D^{ER}(\zeta,z)=-\log\Big|\frac{\Theta(\mathcal{A}(z)-\mathcal{A}(\zeta^*)-e)\Theta(\mathcal{A}(z^*)-\mathcal{A}(\zeta)-e)}{\Theta(\mathcal{A}(z)-\mathcal{A}(\zeta)-e)\Theta(\mathcal{A}(z^*)-\mathcal{A}(\zeta^*)-e)}\Big|$$

and

$$\begin{aligned}G_D(\zeta,z)=&-\log\Big|\frac{\Theta(\mathcal{A}(z)-\mathcal{A}(\zeta^*)-e)\Theta(\mathcal{A}(z^*)-\mathcal{A}(\zeta)-e)}{\Theta(\mathcal{A}(z)-\mathcal{A}(\zeta)-e)\Theta(\mathcal{A}(z^*)-\mathcal{A}(\zeta^*)-e)}\Big|\\&-\pi\Big\langle(\mathrm{Im}\tau)^{-1}\mathrm{Im}[\mathcal{A}(\zeta^*)-\mathcal{A}(\zeta)],\mathrm{Im}[\mathcal{A}(z^*)-\mathcal{A}(z)]\Big\rangle.\end{aligned}$$

We also refer to [85] on the solution of Dirichlet boundary problem in a multiply connected domain in terms of a *quasiclassical tau-function*.

3.2.4 Green's function on a compact Riemann surface

In this subsection, we recall a representation of Bipolar Green's function on compact Riemann surfaces in terms of Riemann theta functions, see [78, Appendix A] for further details.

Let $M\equiv M_g$ be a compact Riemann surface of genus g. We denote by p,q for distinct marked points of M. By definition, the bipolar Green's function $z\mapsto G_{p,q}(z)$ is determined (up to additive constants) by the harmonicity and the asymptotic expansion near marked points:

$$G_{p,q}(z)=+\log\frac{1}{|z-p|}+O(1)\quad(z\to p);\tag{3.2.18}$$

$$G_{p,q}(z)=-\log\frac{1}{|z-q|}+O(1)\quad(z\to q)\tag{3.2.19}$$

in some/any chart.

• $g=1$. Let $\mathbb{T}_r:=\mathbb{C}/\Lambda$ be a complex torus of genus one, where $\Lambda=2\pi\,\mathbb{Z}+2ir\,\mathbb{Z}$ is the group generated by $z\mapsto z+2\pi, z\mapsto z+2ir$. A bipolar

Green's function for $\mathbb{T}_r$ is given by

$$G_{p,q}(z) = \log \left| \frac{\Theta(r, z-q)}{\Theta(r, z-p)} \right| - \frac{\operatorname{Im}(p-q)\operatorname{Im} z}{2r}. \tag{3.2.20}$$

• $g > 1$. For a compact Riemann surface M of genus g, let us fix a canonical basis $\{a_j, b_j\}$ of the homology $H_1 = H_1(M)$ satisfying $a_j \cdot b_k = \delta_{jk}$. Then there is a basis $\{dv_j\}$ of the space of all holomorphic 1-differentials on M which is uniquely determined by the equations

$$\oint_{a_k} dv_j = \delta_{jk}. \tag{3.2.21}$$

Recall that the period matrix $\tau = \{\tau_{jk}\}$ is defined as

$$\tau_{jk} = \oint_{b_k} dv_j. \tag{3.2.22}$$

It is well known that the period matrix is symmetric and $\mathrm{Im}\tau > 0$.

Now let us consider the lattice $\Lambda = \mathbb{Z}^g + \tau\mathbb{Z}^g$ in $\mathbb{C}^g$ and denote by $\mathbb{T}_\Lambda$ the Jacobi variety of M, i.e.,

$$\mathbb{T}_\Lambda \equiv \mathbb{T}^g_\Lambda := \mathbb{C}^g/\Lambda. \tag{3.2.23}$$

For each $p_0 \in M$, let us define the *Abel-Jacobi* map $\mathcal{A}$ by

$$\begin{aligned} \mathcal{A}_{p_0} \equiv \mathcal{A} : & M \to \mathbb{T}_\Lambda \\ & p \mapsto \int_{p_0}^{p} (dv_1, \cdots, dv_g)^T. \end{aligned} \tag{3.2.24}$$

By definition, the Riemann theta function $\Theta(\cdot|\tau)$ associated with τ is given by

$$\Theta \equiv \Theta(Z|\tau) := \sum_{N \in \mathbb{Z}^g} e^{2\pi i (Z\cdot N + \frac{1}{2}\tau N \cdot N)}, \quad Z \in \mathbb{C}^g.$$

Note that theta function is even, entire function on $\mathbb{C}^g$ and has following

periodicity properties: for $N \in \mathbb{Z}^g$,

$$\Theta(Z+N) = \Theta(Z), \quad \Theta(Z+\tau N) = e^{-2\pi i(Z\cdot N + \frac{1}{2}\tau N\cdot N)}\Theta(Z). \tag{3.2.25}$$

The bipolar Green's function $G_{p,q}$ is given by (up to an additive constant)

$$\begin{aligned} G_{p,q}(z) =& \log\Big|\frac{\Theta(\mathcal{A}(z)-\mathcal{A}(q)-e)}{\Theta(\mathcal{A}(z)-\mathcal{A}(p)-e)}\Big| \\ & -2\pi\Big\langle (\mathrm{Im}\tau)^{-1}\mathrm{Im}[\mathcal{A}(p)-\mathcal{A}(q)], \mathrm{Im}\,\mathcal{A}(z)\Big\rangle, \end{aligned} \tag{3.2.26}$$

where the point $e \in \mathbb{T}_\Lambda$ is chosen to be an arbitrary element of the theta divisor, i.e., $\Theta(e) = 0$ such that neither of maps

$$z \mapsto \begin{cases} \Theta(\mathcal{A}(z)-\mathcal{A}(p)-e) \\ \Theta(\mathcal{A}(z)-\mathcal{A}(q)-e) \end{cases}$$

is identically zero. Note that the expression (3.2.26) depends on the choice of theta divisor e, but not the choice of base point p_0 of Abel-Jacobi map.

3.2.5 Riemann-Hilbert boundary conditions

For a parameter $\beta \in (-1/2, 1/2)$, we say a function F defined in a domain $\bar{D}$ satisfies RH_β boundary condition (b.c.) on $l \subset \partial D$ if

$$\Big[\cos(\beta\pi)\partial_n - \sin(\beta\pi)\partial_\tau\Big]F(\cdot) = 0 \quad \text{on } \ l \subset \partial D. \tag{3.2.27}$$

Here ∂_n is the (inwards) normal derivative and ∂_τ is the tangential one. In particular, the case $\beta = 0$ corresponds to Neumann b.c.

In this subsection, we present the Green's function in simply/doubly connected domains with Dirichlet and Riemann-Hilbert mixed boundary conditions in terms of well-known special functions.

• $g = 0$. For given $\beta \in (-1/2, 1/2)$, let

$$f_\beta(z) := \frac{e^{(\beta+1/2)z}}{e^z - 1} \tag{3.2.28}$$

and write

$$F_\beta(z) := -\frac{e^{(\beta+1/2)z}}{\beta + 1/2} {}_2F_1\Big(1, \beta + \frac{1}{2}; \beta + \frac{3}{2}; e^z\Big), \tag{3.2.29}$$

where ${}_2F_1$ is the hypergeometric function. In particular for $\beta = 0$,

$$F_0(z) = \log\Big(\frac{1 - e^{z/2}}{1 + e^{z/2}}\Big). \tag{3.2.30}$$

On the other hand, it follows from

$$\frac{d}{dz} {}_2F_1(a, b; a + 1; z) = \frac{a}{z}\Big[\frac{1}{(1-z)^b} - {}_2F_1(a, b; a + 1; z)\Big]$$

that $F'_\beta(z) = f_\beta(z)$. Notice here that since $f_\beta(z) = -f_{-\beta}(-z)$, we have $F_\beta(z) = F_{-\beta}(-z)$ up to additive constant.

Let us denote by $G_\beta(z_1, z_2)$ the Green's function in $\mathbb{S}$ having zero Dirichlet b.c. on $\mathbb{R}$, which satisfies RH b.c on $\mathbb{R}_\pi$. Then it is easy to check that G_β can be expressed in terms of the function F_β as

$$G_\beta(z_1, z_2) = \mathrm{Re}\,[F_\beta(z_1 - \bar{z}_2) - F_\beta(z_1 - z_2)]. \tag{3.2.31}$$

In particular, for the Neumann b.c. case, we have

$$G_0(z_1, z_2) = \log\Big|\frac{\tanh_4(z_1 - \bar{z}_2)}{\tanh_4(z_1 - z_2)}\Big|. \tag{3.2.32}$$

We emphasize here that G_β satisfies the asymmetric relation

$$G_\beta(z_1, z_2) = G_{-\beta}(z_2, z_1). \tag{3.2.33}$$

• $g = 1$. Let us define

$$f_\beta(r,z) := \frac{\Theta'(r,0)}{\Theta(r,\pi-2\beta\pi)}\frac{\Theta(r,z+\pi-2\beta\pi)}{\Theta(r,z)}. \tag{3.2.34}$$

Due to quasi-periodicities (3.1.10) of theta function, we have

$$f_\beta(r,z+2\pi) = f_\beta(r,z), \quad f_\beta(r,z+2ir) = e^{i(2\beta-1)\pi} f_\beta(r,z). \tag{3.2.35}$$

Also since $\Theta(r,\cdot)$ is an odd function,

$$f_\beta(r,z) = -f_{-\beta}(r,-z). \tag{3.2.36}$$

On the other hand, f_β satisfies

$$f_\beta(r,z) = \frac{1}{z} + \frac{H(r,\pi-2\beta\pi)}{2} + O(z), \quad \text{as } z \to 0. \tag{3.2.37}$$

Thus one can observe that f_0 has an alternative expression

$$f_0(r,z) = \frac{H(2r,z) - H_I(2r,z)}{2}. \tag{3.2.38}$$

Note that $f_\beta(r,\cdot)$ is a conformal map from the cylinder $\mathcal{C}_r$ to the upper half plane minus a slit whose argument is $(1/2-\beta)\pi$.

For given $w \in \mathcal{C}_r$, a meromorphic function

$$g_\beta(z) := f_\beta(r, z-\bar{w}) - f_\beta(r, z-w) \tag{3.2.39}$$

satisfies following properties:

- $g_\beta(z+2\pi) = g_\beta(z)$, $g_\beta(r,z+2ir) = e^{i(2\beta-1)\pi} g_\beta(r,z)$;
- $g_\beta(r,\cdot)$ has simple poles at w (resp., $\bar{w}$) with residue -1 (res., 1);
- $g_\beta(r,\cdot)$ maps $\mathbb{R}$ to $i\mathbb{R}$;
- $g_\beta(r,\cdot)$ maps $\mathbb{R}_r$ to a line passing through 0 with angle β with $\mathbb{R}$.

All of these properties easily follows from (3.2.35) and (3.2.37). For instance, to check the last property, observe that for $x \in \mathbb{R}$,

$$\begin{aligned} g_\beta(x+ir) &= \overline{f_\beta(r, x-ir-w)} - f_\beta(r, x+ir-w) \\ &= e^{i(2\beta-1)\pi}\overline{f_\beta(r, x+ir-w)} - f_\beta(r, x+ir-w) \\ &= -e^{i\beta\pi}\Big(e^{i\beta\pi}\overline{f_\beta(r, x+ir-w)} + e^{-i\beta\pi} f_\beta(r, x+ir-w)\Big) \in e^{i\beta\pi}\mathbb{R}. \end{aligned}$$

We denote by $F_\beta(r, \cdot)$ an antiderivative of $f_\beta(r, \cdot)$, i.e., $\partial_z F_\beta(r, z) = f_\beta(r, z)$. To our knowledge, for general β, there is no known expression for $F_\beta(r, z)$ in terms of well-known special functions. On the other hand, by (3.2.38), we have

$$F_0(r, z) = \log\Big(\frac{\Theta(2r, z)}{\Theta_I(2r, z)}\Big) \tag{3.2.40}$$

up to additive constant.

We now present Green's function G_β in $\mathcal{C}_r$ with zero Dirichlet b.c. on $\mathbb{R}$ which satisfies RH_β condition on $\mathbb{R}_r$. The associated (non-symmetric) stochastic process is called *obliquely reflected Brownian motion* (ORBM) and we refer to [27] and references therein. We remark that as pointed out by the authors [27], ORBM gives a natural bridge between reflected Brownian motion ($\beta = 0$) and ERBM ($\beta \to \pm 1/2$).

We claim that Green's function G_β in $\mathcal{C}_r$ with zero Dirichlet b.c. on $\mathbb{R}$ which satisfies RH_β condition on $\mathbb{R}_r$ is expressed as

$$G_\beta(z_1, z_2) = \mathrm{Re}\,[F_\beta(r, z_1 - \bar{z}_2) - F_\beta(r, z_1 - z_2)]. \tag{3.2.41}$$

In particular, by (3.2.40), Green's function G_0 with Dirichlet-Neumann mixed boundary condition is given by

$$G_0(z_1, z_2) = \log\Big|\frac{\Theta/\Theta_I(2r, z_1 - \bar{z}_2)}{\Theta/\Theta_I(2r, z_1 - z_2)}\Big|. \tag{3.2.42}$$

To show (3.2.41), it suffices to check the followings:

1. $G_\beta(z_1, z_2) + \log|z_1 - z_2|$ is harmonic function in both variables;

2. $G_\beta(z_1, z_2) = 0$ if z_1 or z_2 is on $\mathbb{R}$;

3. $G_\beta(\cdot, z_2)$ satisfies RH_β condition on $\mathbb{R}_r$.

All of these requirements easily follow from the properties presented above. We remark that G_β satisfies the asymmetric relation (3.2.33).

3.3 Variants of GFFs

3.3.1 GFF in a multiply connected domain

Let D be a multiply connected domain bounded by $g+1$ smooth Jordan curves C_j, $j = 0, 1 \cdots, g$. As before let C_0 be the outermost boundary component. Let $\mathcal{C} = \mathcal{C}(\boldsymbol{r}, \boldsymbol{\delta})$ be a circular domain, where

$$\boldsymbol{r} = (r_1, \cdots, r_g), \quad r_j \in \mathbb{R}, \quad \boldsymbol{\delta} = (\delta_1, \cdots, \delta_g), \quad \delta_j \in \mathbb{D}$$

are the parameters specifying the domain. By generalized Riemann mapping theorem, there is a conformal map f from D onto some $\mathcal{C}$ which maps C_0 to $\mathbb{T}$.

The Gaussian free field Φ^{ER} in D with ER boundary condition is a Gaussian field indexed by the "*ER energy space*" $\mathcal{E}^{ER}(D)$,

$$\Phi_D^{ER} : \mathcal{E}^{ER}(D) \to L^2(\Omega, \mathbf{P}).$$

Here, the Hilbert space $\mathcal{E}^{ER}(D_r)$ is defined as the completion of test functions $f \in C_0^\infty(D_r)$ with respect to the norm

$$\|f\|^2_{\mathcal{E}^{ER}} = \iint 2G_D^{ER}(\zeta, z) f(\zeta) \overline{f(z)}\, dA(\zeta)\, dA(z),$$

where dA is the area measure.

We denote by $\Phi^{ER} \equiv \Phi_D^{ER}$ the GFF in D with ER boundary condition.

By definition, 2-point correlation functions of Φ^{ER} is given as

$$\mathbf{E}[\Phi_D^{ER}(\zeta)\Phi_D^{ER}(z)] = 2G_D^{ER}(\zeta, z) = -2\log\Big|\frac{\omega(f(\zeta), f(z))}{f(z)\,\omega(f(\zeta), 1/f(\bar{z}))}\Big|, \quad (3.3.1)$$

where ω is the S-K function (3.1.59).

Similarly, GFF $\Phi \equiv \Phi_D$ with zero Dirichlet boundary condition is defined as above, and its 2-point correlation function is given by

$$\mathbf{E}[\Phi_D(\zeta)\Phi_D(z)] = 2G_D(\zeta, z) \quad (3.3.2)$$

For example, in a cylinder $\mathcal{C}_r$, by (3.2.14),

$$\mathbf{E}[\Phi_{\mathcal{C}_r}(\zeta)\Phi_{\mathcal{C}_r}(z)] = 2\log\Big|\frac{\Theta(r, \zeta - \bar{z})}{\Theta(r, \zeta - z)}\Big| - 2\frac{\operatorname{Im}\zeta \operatorname{Im} z}{r}. \quad (3.3.3)$$

In a general circular domain $\mathcal{C}$, it follows from (3.2.17) that

$$\mathbf{E}[\Phi_{\mathcal{C}}(\zeta)\Phi_{\mathcal{C}}(z)] = -2\log\Big|\frac{\omega(\zeta, z)}{z\omega(\zeta, \bar{z}^{-1})}\Big| - 2\Big\langle \boldsymbol{h}(\zeta), \boldsymbol{P}^{-1}\boldsymbol{h}(z)\Big\rangle. \quad (3.3.4)$$

3.3.2 GFF on a compact Riemann surface

On a compact Riemann surface $M \equiv M_g$ of genus g, the Gaussian free field Φ is defined as a bi-variant Fock space field $(z, z_0) \mapsto \Phi(z, z_0)$. The correlation functions of Φ are given by bipolar Green's function, i.e.,

$$\mathbf{E}\Phi(\zeta, z)\Phi(\tilde{\zeta}, \tilde{z}) = 2\Big(G_{\zeta,z}(\tilde{\zeta}) - G_{\zeta,z}(\tilde{z})\Big) = 2\Big(G_{\tilde{\zeta},\tilde{z}}(\zeta) - G_{\tilde{\zeta},\tilde{z}}(z)\Big).$$

For example, in the genus one case, by (3.2.20),

$$\begin{aligned}\mathbf{E}\Phi_{\mathbb{T}}(\zeta, z)\Phi_{\mathbb{T}}(\tilde{\zeta}, \tilde{z}) = 2\log\Big|\frac{\Theta(r, \tilde{\zeta} - z)\,\Theta(r, \tilde{z} - \zeta)}{\Theta(r, \tilde{\zeta} - \zeta)\,\Theta(r, \tilde{z} - z)}\Big| \\ - \frac{\operatorname{Im}(\zeta - z)\operatorname{Im}(\tilde{\zeta} - \tilde{z})}{r}.\end{aligned} \quad (3.3.5)$$

Also, in the case of higher genus, by (3.2.26),

$$\begin{aligned}\mathbf{E}\Phi(\zeta,z)\Phi(\tilde{\zeta},\tilde{z}) = \log\Big|\frac{\Theta(\mathcal{A}(\tilde{\zeta})-\mathcal{A}(z)-e)\Theta(\mathcal{A}(\tilde{z})-\mathcal{A}(\zeta)-e)}{\Theta(\mathcal{A}(\tilde{\zeta})-\mathcal{A}(\zeta)-e)\Theta(\mathcal{A}(\tilde{z})-\mathcal{A}(z)-e)}\Big|^2 \\ -4\pi\Big\langle(\mathrm{Im}\tau)^{-1}\mathrm{Im}[\mathcal{A}(\zeta)-\mathcal{A}(z)],\mathrm{Im}[\mathcal{A}(\tilde{\zeta})-\mathcal{A}(\tilde{z})]\Big\rangle.\end{aligned} \tag{3.3.6}$$

3.3.3 Schottky double construction of GFF

As a Fock space field, the GFF in a multiply connected domain with g holes can be constructed from the GFF on a compact Riemann surface of genus g via the Schottky double. For variants of Schottky double constructions, see [60, 61].

• $g = 1$ *case.* For a cylinder $\mathcal{C}_r$ the associated Schottky double is a complex torus $\mathbb{T}_r$ of genus one. Then it follows from

$$\begin{aligned}\mathbf{E}\Phi_{\mathcal{C}_r}(\zeta)\Phi_{\mathcal{C}_r}(z) &= 2\log\Big|\frac{\Theta(r,\zeta-\bar{z})}{\Theta(r,\zeta-z)}\Big| - 2\frac{\mathrm{Im}\,\zeta\,\mathrm{Im}\,z}{r} \\ &= \frac{1}{2}\mathbf{E}\Phi_{\mathbb{T}_r}(\zeta,\bar{\zeta})\Phi_{\mathbb{T}_r}(z,\bar{z})\end{aligned} \tag{3.3.7}$$

that the Schottky double relation of GFFs holds:

$$\Phi_{\mathcal{C}_r}(z) = \frac{1}{\sqrt{2}}\Phi_{\mathbb{T}_r}(z,\bar{z}).$$

• $g \geq 2$ *case.* To describe the Schottky doubly relation for the higher genus cases, it is convenient to choose the reference domain as the circular one $\mathcal{C} = \mathcal{C}(\boldsymbol{r},\boldsymbol{\delta})$. Recall that Möbius maps ϕ_j, θ_j, $(j = 0, 1, \cdots, g)$ are given as

$$\phi_j(\zeta) := \bar{\delta}_j + \frac{r_j^2}{\zeta-\delta_j}, \quad \phi_0(\zeta) := \frac{1}{\zeta};$$

$$\theta_j(\zeta) := \overline{\phi}_j(1/\zeta) = \delta_j + \frac{r_j^2\zeta}{1-\bar{\delta}_j\zeta}.$$

We write $C_j' = \{1/\bar{\zeta} : \zeta \in C_j\}$ for the reflection of C_j with respect to C_0.

Note that the image of C_j' under θ_j is C_j. Moreover

$$\theta_j(\zeta) = 1/\bar{\zeta} \quad \zeta \in C_j'.$$

Let us denote by F for the region of complex plane exterior to the $2g$ circles $\{C_j, C_j' | j = 1, \cdots, g\}$, which is called the *fundamental region* associated with the Schottky group generated by the Möbius maps $\{\theta_j | j = 1, \cdots, g\}$. Here two halves of F (one is reflection of the other through the unit circle) can be interpreted as two sides of a symmetric compact Riemann surface $\widehat{\mathcal{C}}$ known as *Schottky double* of $\mathcal{C}$. Now we take the circle C_j (which is identified with C_j') as a j-th a-cycle of $\widehat{\mathcal{C}}$. On the other hand, for any $\zeta \in C_j'$, let us take any line joining ζ and $\theta_j(\zeta) = \zeta^*$ as a j-th b-cycle of $\widehat{\mathcal{C}}$. Let $\{dv_j | j = 0, 1, \cdots, g\}$ be the basis of the space of all holomorphic 1-differentials on $\widehat{\mathcal{C}}$ which is uniquely determined by the equations

$$\oint_{a_k} dv_j = \delta_{jk}.$$

We aim to express the *integral of first kind* $\{v_j | j = 0, 1, \cdots, g\}$ in terms of the well-known harmonic functions. Let $\{h_j(\zeta) | j = 0, 1, \cdots, g\}$ be the set of harmonic functions in $\mathcal{C}$ given as (3.2.6). Recall that the period matrix $\boldsymbol{P}$ of h_j is given by

$$\boldsymbol{P} = [P_{kj}]_{k,j=1}^{g}, \quad P_{kj} = \frac{1}{2\pi} \oint_{C_k} \frac{\partial h_j(z)}{\partial n_z} \, ds_z.$$

Note that for any j,

$$h_j(\zeta) = 0, \quad \zeta \in C_0.$$

Thus we extend h_j to fundamental region F as

$$\tilde{h}_j(\zeta) = \begin{cases} h_j(\zeta) & \text{on} \quad \overline{\mathcal{C}} \\ -h_j(\zeta^*) & \text{on} \quad F \backslash \mathcal{C}. \end{cases}$$

Now let $\widehat{v}_j$ be the holomorphic function on F whose imaginary part is $\tilde{h}_j$. Note that by definition,

$$\widehat{v}_j(\zeta^*) = \overline{\widehat{v}_j(\zeta)}, \quad \zeta \in \mathcal{C}.$$

Then it is well known that

$$v_k = -\frac{1}{2\pi}\boldsymbol{P}^{-1}(\widehat{v}_1, \cdots, \widehat{v}_g)^T,$$

which leads

$$\operatorname{Im} v_k = -\frac{1}{2\pi}\boldsymbol{P}^{-1}(\tilde{h}_1, \cdots, \tilde{h}_g)^T.$$

Moreover, the period matrix $\tau = \{\tau_{jk}\}$, $\tau_{jk} = \oint_{b_k} dv_j$ is represented

$$\tau = \frac{i}{\pi}\boldsymbol{P}^{-1},$$

which implies

$$(\mathrm{Im}\tau)^{-1} = \pi\boldsymbol{P}. \tag{3.3.8}$$

See [40] for more details.

Proposition 3.3.1. *GFFs satisfy the Schottky double relation*

$$\Phi_{\mathcal{C}}(z) = \frac{1}{\sqrt{2}}\Phi_{\widehat{\mathcal{C}}}(z, z^*). \tag{3.3.9}$$

Proof. It suffices to show that

$$\mathbf{E}\Phi_{\mathcal{C}}(\zeta)\Phi_{\mathcal{C}}(z) = \frac{1}{2}\mathbf{E}\Phi_{\widehat{\mathcal{C}}}(\zeta, \zeta^*)\Phi_{\widehat{\mathcal{C}}}(z, z^*), \tag{3.3.10}$$

equivalently,

$$G_{\mathcal{C}}(\zeta, z) = \frac{1}{2}(G_{\zeta,\zeta^*}(z) - G_{\zeta,\zeta^*}(z^*)). \tag{3.3.11}$$

By the harmonicity of Green's function, all we need to check is that the right-hand side of (3.3.11) vanishes whenever $\zeta \in C_j$. In other words, by

(3.3.6), all we need to show is for $\zeta \in C_j$,

$$\begin{aligned}&\frac{1}{2}\log\Big|\frac{\Theta(\mathcal{A}(z)-\mathcal{A}(\zeta^*)-e)\Theta(\mathcal{A}(z^*)-\mathcal{A}(\zeta)-e)}{\Theta(\mathcal{A}(z)-\mathcal{A}(\zeta)-e)\Theta(\mathcal{A}(z^*)-\mathcal{A}(\zeta^*)-e)}\Big|\\ &=\pi\Big\langle(\mathrm{Im}\tau)^{-1}\mathrm{Im}[\mathcal{A}(\zeta^*)-\mathcal{A}(\zeta)],\mathrm{Im}[\mathcal{A}(z^*)-\mathcal{A}(z)]\Big\rangle.\end{aligned}$$

On the other hand, for any $\zeta \in \mathcal{C}$, we have

$$\begin{aligned}\mathcal{A}(\zeta^*)-\mathcal{A}(\zeta)=&\Big(\int_\zeta^{\zeta^*} dv_1,\cdots,\int_\zeta^{\zeta^*} dv_g\Big)^T\\ =&\Big(v_1(\zeta^*)-v_1(\zeta),\cdots,v_g(\zeta^*)-v_g(\zeta)\Big)^T\\ =&\Big(\overline{v_1(\zeta)}-v_1(\zeta),\cdots,\overline{v_g(\zeta)}-v_g(\zeta)\Big)^T\\ =&-2i\,\mathrm{Im}\Big(v_1(\zeta),\cdots,v_g(\zeta)\Big)^T,\end{aligned}\tag{3.3.12}$$

which leads to

$$\mathcal{A}(\zeta^*)-\mathcal{A}(\zeta)=\frac{i}{\pi}\boldsymbol{P}^{-1}\boldsymbol{h}(\zeta)=\tau\,\boldsymbol{h}(\zeta).\tag{3.3.13}$$

Therefore for $\zeta \in C_j$, we have

$$\mathcal{A}(\zeta^*)-\mathcal{A}(\zeta)=\tau e_j,\quad e_j=(0,\cdots,1,\cdots,0)^T,$$

and

$$\pi\Big\langle(\mathrm{Im}\tau)^{-1}\mathrm{Im}[\mathcal{A}(\zeta^*)-\mathcal{A}(\zeta)],\mathrm{Im}[\mathcal{A}(z^*)-\mathcal{A}(z)]\Big\rangle=\pi\Big\langle e_j,\mathrm{Im}[\mathcal{A}(z)-\mathcal{A}(z^*)]\Big\rangle.$$

On the other hand, due to (3.2.25), we obtain

$$\begin{aligned}\frac{\Theta(\mathcal{A}(z)-\mathcal{A}(\zeta^*)-e)}{\Theta(\mathcal{A}(z)-\mathcal{A}(\zeta)-e)}&=\frac{\Theta(\mathcal{A}(z)-\mathcal{A}(\zeta)-e+\mathcal{A}(\zeta^*)-\mathcal{A}(\zeta))}{\Theta(\mathcal{A}(z)-\mathcal{A}(\zeta)-e)}\\ &=\frac{\Theta(\mathcal{A}(z)-\mathcal{A}(\zeta)-e+\tau e_j)}{\Theta(\mathcal{A}(z)-\mathcal{A}(\zeta)-e)}\\ &=\exp\Big[-2\pi i\Big(\langle\mathcal{A}(z)-\mathcal{A}(\zeta),e_j\rangle+\frac{1}{2}\tau_{jj}\Big)\Big]\end{aligned}$$

and

$$\frac{\Theta(\mathcal{A}(z^*)-\mathcal{A}(\zeta)-e)}{\Theta(\mathcal{A}(z^*)-\mathcal{A}(\zeta^*)-e)}=\exp\Big[2\pi i\Big(\langle\mathcal{A}(z^*)-\mathcal{A}(\zeta),e_j\rangle+\frac{1}{2}\tau_{jj}\Big)\Big].$$

Therefore we obtain

$$\frac{1}{2}\log\Big|\frac{\Theta(\mathcal{A}(z)-\mathcal{A}(\zeta^*)-e)\Theta(\mathcal{A}(z^*)-\mathcal{A}(\zeta)-e)}{\Theta(\mathcal{A}(z)-\mathcal{A}(\zeta)-e)\Theta(\mathcal{A}(z^*)-\mathcal{A}(\zeta^*)-e)}\Big|=\pi\Big\langle e_j,\mathrm{Im}[\mathcal{A}(z)-\mathcal{A}(z^*)]\Big\rangle,$$

which completes the proof.

□

Corollary 3.3.2. *We have*

$$\log\Big|\frac{z\omega(\zeta,\bar{z}^{-1})}{\omega(\zeta,z)}\Big|^2=\log\Big|\frac{\Theta(\mathcal{A}(z)-\mathcal{A}(\zeta^*)-e)\Theta(\mathcal{A}(z^*)-\mathcal{A}(\zeta)-e)}{\Theta(\mathcal{A}(z)-\mathcal{A}(\zeta)-e)\Theta(\mathcal{A}(z^*)-\mathcal{A}(\zeta^*)-e)}\Big| \tag{3.3.14}$$

and

$$\Big\langle\boldsymbol{h}(\zeta),\boldsymbol{P}^{-1}\boldsymbol{h}(z)\Big\rangle=\pi\Big\langle(\mathrm{Im}\tau)^{-1}\mathrm{Im}[\mathcal{A}(\zeta^*)-\mathcal{A}(\zeta)],\mathrm{Im}[\mathcal{A}(z^*)-\mathcal{A}(z)]\Big\rangle. \tag{3.3.15}$$

Proof. Due to (3.3.4) and (3.3.6), it is suffice to show (3.3.15). Note that for any $\zeta\in\mathcal{C}$, by (3.3.12), we have

$$\mathrm{Im}\,[\mathcal{A}(\zeta^*)-\mathcal{A}(\zeta)]=\frac{1}{\pi}\boldsymbol{P}^{-1}\boldsymbol{h}(\zeta).$$

Therefore, the right-hand side of (3.3.15) is given by

$$\frac{1}{\pi}\Big\langle(\mathrm{Im}\tau)^{-1}\boldsymbol{P}^{-1}\boldsymbol{h}(\zeta),\boldsymbol{P}^{-1}\boldsymbol{h}(z)\Big\rangle,$$

which is equal to left-hand side of (3.3.15) by the relation (3.3.8). □

Chapter 4

Schramm-Loewner evolution

Schramm-Loewner evolution (SLE) is a conformally invariant random collection of curves (indexed by $\kappa > 0$), which satisfies domain Markov property. Schramm introduced SLE in a simply connected domain as a possible candidate for the scaling limits of loop-erased random walk [115]. It has been proved that SLE(κ) is the scaling limit for variants of critical models such as loop-erased random walk ($\kappa = 2$), uniform spanning tree ($\kappa = 8$) [95], critical Ising interface ($\kappa = 3$) and its random cluste representation ($\kappa = 16/3$) [34], critical percolation in hexagonal lattice ($\kappa = 6$) [123], discrete GFF and harmonic explorer ($\kappa = 4$) [117, 118]. Moreover, some of above results have been extended to multiple interfaces [67, 68, 82, 129] as well as to multiply connected domains [67, 137].

In the upper-half plane, SLE is the Loewner chain derived from solving Loewner's equation, where the driving function ξ_t is given by time changed Brownian motion $\sqrt{\kappa}\, B_t$. The SLE curve is almost surely a continuous path [114], see also [26, 73, 100] for more about Hölder continuity. This random curve satisfies the reversibility, see [105,136]. It was also discovered by Schramm and Rohde [114] that SLE trace reveals a phase transition, namely, it is simple for $\kappa \in (0, 4)$, self-intersecting but non-self-crossing for $\kappa \in [4, 8)$, and space-filling for $\kappa \in [8, \infty)$. Moreover, it was proved by Beffara [24] that the Hausdorff dimension of SLE(κ) trace is almost surely $\min(1 + \kappa/8, 2)$. The

duality relation conjectured by Duplantier using CFT [49] describes the exterior boundary of the hull of SLE(κ) for $\kappa > 4$ in terms of SLE($16/\kappa$). This was proved independently by Zhan [135] and Dubédat [47]. For a comprehensive account of SLE, we refer the reader to [80, 89, 91, 127, 128] and references therein.

In a doubly connected domain, Zhan has made important achievements on basic properties of SLE [132, 133]. He introduced annulus SLE_κ whose driving process is a time-changed standard Brownian motion and showed that this SLE trace connects two distinct boundary components. In [134], he derived a partial differential equation for the ending point distribution. To describe scaling limits of the interface models connecting given marked boundary points, it is of particular interest to investigate annulus SLE satisfying reversibility. Here, one has to distinguish two cases. When two marked boundary points are on the same boundary component, the associated SLE process is called the *chordal type annulus SLE*, otherwise it is called the *crossing type annulus SLE*. In particular, in [139], Zhan considered the annulus SLE(κ, Λ) with two marked points, where $\Lambda := \kappa(\log Z)'$ is the drift function of driving process of SLE and introduced certain parabolic PDE for Z, a necessary condition for the reversibility of SLE trace. Here, the function Z is called *annulus SLE partition function*. We remark that such equation for the chordal type annulus SLE also appears in Dubédat's works on commuting SLEs, see [43, 46]. The null-vector equation also naturally arises during the Itô's calculus when one describes the scaling limit of the critical lattice model in doubly connected domains. In general, such descriptions have been obtained by the fundamental fact that certain discrete observables converge to conformally covariant martingale processes. For instance, in [67], Izyurov proved that critical Ising interface connecting two marked boundary points in the annulus converges toward SLE($3, \Lambda$) by virtue of certain parafermionic observables. Here, due to the null-vector equation for $\kappa = 3$, the continumm limit of the discrete observable becomes (local) martingale in the SLE probability space.

Since the description of SLE(κ, Λ) is given by the local growth of the curves via Loewner's equation, the SLE(κ, Λ) trace connects two marked boundary points, or it ends at a random point on the inner boundary component. Nevertheless, in [139], Zhan used Feynman-Kac formula to describe a particular solution of null-vector equation in the crossing case with $\kappa \leq 4$, where its associated SLE trace is reversible. Moreover, in [138], he showed that these SLEs satisfy the generalized restriction property.

On the other hand, Lawler took a different (*global*) approach to define SLEs in multiply connected domains for $\kappa \leq 4$ by describing a probability measure on simple curves connecting two marked points, where it is required that these random curves should satisfy generalized restriction property, see [94]. In addition, Lawler showed that in the annulus, the crossing annulus SLE partition function (the total mass of a positive measure on crossing SLE curves) satisfies the null-vector equation.

4.1 Loewner equations in a multiply connected domain

4.1.1 Standard domain and canonical mapping

Let D be a multiply connected domain bounded by $g+1$ smooth Jordan curves C_j, $j = 0, 1 \cdots, g$. Now let us consider a marked boundary point q on C_0.

We define *complex ER Green's function* by

$$2G_D^{ER+}(\zeta, z) = G_D^{ER}(\zeta, z) + i\tilde{G}_D^{ER}(\zeta, z), \tag{4.1.1}$$

where $\tilde{G}$ is the harmonic conjugate of the ER Green's function. We denote by

$$2H_D^{ER}(\zeta, q) = \frac{\partial G(\zeta, z)}{\partial n_z}\Big|_{z=q} \tag{4.1.2}$$

the ER Poisson kernel. The *complex ER Poisson kernel* $2H_D^{ER+}$ is defined by

$$2H_D^{ER+}(\zeta, q) = H_D^{ER}(\zeta, q) + i\tilde{H}_D^{ER}(\zeta, q), \tag{4.1.3}$$

where $\tilde{H}$ is the harmonic conjugate of the ER Poisson kernel.

From now on, we call the upper-half plane minus a finite number of horizontal slits a *chordal standard domain*. It follows from classical results in complex analysis that a multiply connected domain D with connectivity $g+1$ is conformally equivalent to a chordal standard domain with g horizontal slits. Note that the geometric description of standard domain requires $3g$ real parameters, three for each slits, i.e., one imaginary component, and two real components determining the endpoints of each slit.

It is well known that two $(g+1)$-connective domains with non-degeneracy boundary continua are conformally equivalent if $3g-3$ real parameters agree for $g > 1$. If $g = 1$, there is only one real parameter (modulus) describing the conformal class and if $g = 0$, by Riemann mapping theorem, all such domains are conformally equivalent.

Moreover, a conformal map from D to chordal standard domain can be represented in terms of complex ER Poisson kernel, see e.g., [37, 42]. More precisely,

$$-ia\, H_D^{ER+}(\zeta, q) + b, \quad a, b \in \mathbb{R}_{\geq 0} \tag{4.1.4}$$

are conformal map from D to some chordal standard domain. Under these maps, $C_0 \mapsto \mathbb{R}$ and $q \mapsto \infty$.

In particular, if D is contained in the upper-half plane and $q = \infty$,

$$g(\zeta) = -2iH_D^{ER+}(\zeta, \infty) \tag{4.1.5}$$

is the unique conformal map satisfying

$$\lim_{\zeta \to \infty} (g(\zeta) - \zeta) = 0. \tag{4.1.6}$$

With this normalization, we call g the *canonical mapping* for D.

$\triangleright$ $g = 0$. Recall that

$$G_{\mathbb{D}}(\zeta, z) = \log \Big| \frac{1 - \zeta \bar{z}}{\zeta - z} \Big|. \tag{4.1.7}$$

Therefore

$$G^{+}_{\mathbb{D}}(\zeta, z) = \frac{1}{2} \log \Big(\frac{1 - \zeta \bar{z}}{\zeta - z} \Big). \tag{4.1.8}$$

Set $q = 1$. Then

$$H^{+}_{\mathbb{D}}(\zeta, 1) = -\frac{1}{2} \frac{1 + \zeta}{1 - \zeta}. \tag{4.1.9}$$

Therefore the function

$$f(z) := \frac{ia}{2} \frac{1 + z}{1 - z} + b.$$

is a conformal map from $\mathbb{D}$ onto $\mathbb{H}$.

$\triangleright$ $g = 1$. By (3.2.14), we have

$$G^{ER}_{\mathbb{A}_r}(\zeta, z) = \log \Big| \frac{\Theta(r, -i \log(\zeta \bar{z}))}{\Theta(r, -i \log(\zeta / z))} \Big|. \tag{4.1.10}$$

Therefore

$$G^{ER+}_{\mathbb{A}_r}(\zeta, z) = \frac{1}{2} \log \Big(\frac{\Theta(r, -i \log(\zeta \bar{z}))}{\Theta(r, -i \log(\zeta / z))} \Big). \tag{4.1.11}$$

Again set $q = 1$. Then

$$H^{ER+}_{\mathbb{A}_r}(\zeta, 1) = -\frac{i}{2} H(r, -i \log \zeta). \tag{4.1.12}$$

Notice that by (3.1.38), we have

$$\lim_{r \to \infty} H^{ER+}_{\mathbb{A}_r}(\zeta, 1) = \frac{1}{2} \frac{\sqrt{\zeta} + 1/\sqrt{\zeta}}{\sqrt{\zeta} - 1/\sqrt{\zeta}} = -\frac{1}{2} \frac{1 + \zeta}{1 - \zeta} = H^{+}_{\mathbb{D}}(\zeta, 1).$$

Therefore one can see that

$$f(z) := -\frac{a}{2} H(r, -i \log z) + b \tag{4.1.13}$$

is a conformal map from $\mathbb{A}_r$ onto a standard domain with one slit. Indeed,

since $\operatorname{Im} H(r, ir) = -1$, we have

$$\{f(z) : z \in \mathbb{T}_r\} \in \frac{a}{2}\, i + \mathbb{R}.$$

4.1.2 Loewner vector field and flow

Consider a chordal standard domain D. Let $\gamma : [0, \infty) \to D$ be a simple curve with $\gamma(0) \in \mathbb{R}$. Let g_t be the canonical mapping from $D \setminus \gamma[0, t]$ satisfying

$$g_t(z) = z + \frac{2t}{z} + o\Big(\frac{1}{|z|^2}\Big), \quad z \to \infty. \tag{4.1.14}$$

Let us denote

$$D_t = g_t(D \setminus \gamma[0, t]), \quad \xi_t = g_t(\gamma(t)), \quad C_j(t) = g_t(C_j), \tag{4.1.15}$$

where $j = 0, 1, \cdots, g$. Note that $C_0(t) = \mathbb{R}$. From now on, we add subscript t to the corresponding functions for D_t, e.g,

$$G_t^{ER}, \quad H_t^{ER}, \quad \boldsymbol{h}_t, \quad \boldsymbol{P}_t.$$

The *chordal Loewner equation* is given by

$$\partial_t g_t(z) = -4i\, H_t^{ER+}\Big(g_t(z), \xi(t)\Big), \quad g_0(z) = z. \tag{4.1.16}$$

see e.g., [23, Theorem 3.1].

$\triangleright$ $g = 0$. For $D(t) \equiv \mathbb{H}$, the Green's function is given by

$$G_{\mathbb{H}}(\zeta, z) = \log\Big|\frac{\zeta - \bar{z}}{\zeta - z}\Big|, \quad G_{\mathbb{H}}^{+}(\zeta, z) = \frac{1}{2}\log\frac{\zeta - \bar{z}}{\zeta - z}. \tag{4.1.17}$$

Also, its Poisson kernel is given by

$$H_{\mathbb{H}}^{+}(\zeta, x) = \frac{1}{2}\frac{\partial}{\partial y}\Big|_{y=0}\frac{1}{2}\log\frac{\zeta - x + iy}{\zeta - x - iy} = \frac{i}{2}\frac{1}{\zeta - x}. \tag{4.1.18}$$

Therefore in this case, (4.1.16) is written as

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \xi_t}, \quad g_0(z) = z, \tag{4.1.19}$$

which is well known chordal Loewner equation.

$\triangleright$ $g = 1$. Let D be a standard domain and denote by F the conformal map from chordal standard domain to the cylinder satisfying $H(r, F(z)) = -z$. Note that by (3.2.14), we have

$$G_D^{ER}(\zeta, z) = \log \Big| \frac{\Theta(r, F(\zeta) - \overline{F(z)})}{\Theta(r, F(\zeta) - F(z))} \Big|, \tag{4.1.20}$$

$$G^{ER+}(\zeta, z) = \frac{1}{2} \log \Big(\frac{\Theta(r, F(\zeta) - \overline{F(z)})}{\Theta(r, F(\zeta) - F(z))} \Big). \tag{4.1.21}$$

Thus Loewner vector field is given as

$$F'(x) H(r, F(z) - F(x))$$

and

$$\partial_t g_t(z) = F'(\xi_t) H(r, F(g_t(z)) - F(\xi_t)), \quad g_0(z) = z. \tag{4.1.22}$$

In this case $g = 1$, it is more convenient to take the reference domain as $\mathbb{A}_r$, where we follow Zhan's description [132]. In the annulus $\mathbb{A}_r$ the Loewner kernel $S(r, z)$ is given by

$$S(r, z) := \lim_{N \to \infty} \sum_{n=-N}^{N} \frac{e^{2nr} + z}{e^{2nr} - z} = H(r, -i \log z). \tag{4.1.23}$$

Thus we have

$$\partial_t g_t(z) = g_t(z) S(r - t, g_t(z)/e^{i\xi_t}), \quad g_0(z) = z. \tag{4.1.24}$$

On the other hand, the Loewner equation in the covering space $\mathbb{S}_r$ is given

by

$$\partial_t \tilde{g}_t(z) = H(r-t, \tilde{g}_t(z) - \xi_t), \quad \tilde{g}_0(z) = z. \tag{4.1.25}$$

4.2 Evolutionary description of SLE

4.2.1 SLE in a simply connected domain

For a simply connected domain (D, p, q) with two marked boundary points p, q, let us denote by $g_0 : (D, p, q) \to (\mathbb{H}, 0, \infty)$ given conformal map. The chordal SLE map g_t satisfies the Loewner's equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \xi_t}, \quad \xi_t = \sqrt{\kappa}\, B_t, \tag{4.2.1}$$

which is well defined up to the first time τ_z such that $g_t(z) = \xi_t$. For all $t < \tau_z$,

$$w_t(z) := g_t(z) - \xi_t$$

is a conformal map from the domain $D_t := \{z \in D : \tau_z > t\}$ onto $\mathbb{H}$. The chordal SLE trace γ is then defined by

$$\gamma_t := \lim_{z \to \xi_t} g_t^{-1}(z). \tag{4.2.2}$$

We now consider several force points $\boldsymbol{q} = \{q_1, \cdots, q_N\}$ with prescribed "charges" $\boldsymbol{\rho} = \{\rho_1, \cdots, \rho_N\}$. By definition, the chordal SLE$(\kappa, \boldsymbol{\rho})$ is the Loewner chain driven by

$$d\xi_t = \sqrt{\kappa}\, dB_t + \sum \frac{\rho_k\, dt}{\xi_t - q_k(t)}, \tag{4.2.3}$$

where q_k's follow Loewner's flow, i.e.,

$$dq_t(t) = \frac{2}{q_k(t) - \xi_t}\, dt. \tag{4.2.4}$$

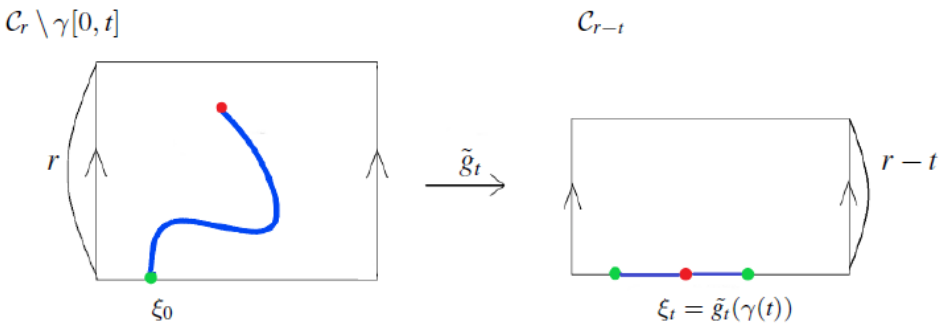


Figure 4.1: Loewner chain in the cylinder $\mathcal{C}_r$

Intuitively, each force point q_k repels (when $\rho_k > 0$) or attracts (when $\rho_k < 0$) the SLE$(\kappa, \boldsymbol{\rho})$ trace. Such generalization of chordal SLE is useful to describe statistical models with various boundary condition or the law of single interface curve under the presence of multiple interfaces, see e.g., [18, 66]. It is also shown by Miller and Sheffield [104] that the chordal SLE$(\kappa, \boldsymbol{\rho})$ trace is almost surely a continuous path.

4.2.2 SLE in a doubly connected domain

We now introduce the (covering) annulus SLE with N-force points $q_1, \cdots q_N$. Similarly as in a simply connected domain, the annulus SLE(κ, Λ) can be described by the associated Loewner's differential equation [133].

Let ξ be a (real-valued) continuous function defined on $[0, T)$ for some $0 < T < r$. For each $z \in \mathbb{A}_r$, let $g_t(z)$ be the solution (which exists up to the first time $\tau_z \in (0, T]$ that $g_t(z)$ hits ξ_t) of the annulus Loewner equation

$$\partial_t g_t(z) = g_t(z) S(r-t, g_t(z)/e^{i\xi_t}), \quad g_0(z) = z.$$

For each time $t \in [0, T)$, let $K_t := \{z \in \mathbb{A}_r : \tau_z \leq t\}$. Then K_t is a hull in $\mathbb{A}_r$ and has capacity t. Moreover, g_t is a conformal map from $(\mathbb{A}_r \setminus K_t, \mathbb{T}_r)$ onto $(\mathbb{A}_{r-t}, \mathbb{T}_{r-t})$.

It is convenient to describe annulus SLE in the covering space $\mathbb{S}_r$, where

the Loewner kernel H is expressed in terms of Θ as

$$H(r,z) := -i\, S(r,e^{iz}) = 2\,\partial_z \log \Theta(r,z).$$

For each $z \in \mathbb{S}_r$, we denote by $\tilde{g}_t(z)$ the solution of the equation

$$\partial_t \tilde{g}_t(z) = H(r-t, \tilde{g}_t(z) - \xi_t), \quad \tilde{g}_0(z) = z,$$

and set $\tilde{K}_t := \{z \in \mathbb{S}_r : \tau_z \le t\}$. Then $\tilde{g}_t$ is a conformal map from $(\mathbb{S}_r \setminus \tilde{K}_t; \mathbb{R}_r)$ onto $(\mathbb{S}_{r-t}; \mathbb{R}_{r-t})$. Moreover for each $z \in \mathbb{S}_r \setminus \tilde{K}_t$,

$$e^{i\tilde{g}_t(z)} = g_t(e^{iz}), \quad K_t = \{e^{iz} \in \mathbb{A}_r : z \in \tilde{K}_t\}.$$

In particular, $\tilde{K}_t$ is 2π-periodic and for each $n \in \mathbb{Z}$, we have $\tilde{g}_t(z+2n\pi) = \tilde{g}_t(z) + 2n\pi$. We call that ξ generates a trace γ if

$$\gamma_t = \gamma(t) := \lim_{z \to \xi_t} \tilde{g}_t^{-1}(z).$$

It was proved by Zhan in [133] that $\xi_t := \sqrt{\kappa}\, B_t$ $(t < r)$ a.s. generates continuous trace, where B_t is a standard one-dimensional Brownian motion.

Suppose Λ is a real-valued $C^{0,1}$ function on $(0,\infty) \times (\mathbb{C} \setminus \{z \in 2\pi\mathbb{Z} + 2ir\mathbb{Z}\})^N$. Let ξ_t be the solution, which exists up to the first time T such that $\xi_t - \tilde{g}_t(q_j) \in 2\pi\mathbb{Z}$, of the SDE

$$d\xi_t = \sqrt{\kappa}\, dB_t + \Lambda(r-t, \xi_t - \tilde{g}_t(q_1), \cdots, \xi_t - \tilde{g}_t(q_N))\, dt, \quad \xi_0 = p.$$

Applying the Girsanov theorem, it can be shown that up to time T, ξ_t a.s. generates continuous trace, see [139, Section 3.2] for further details. We call the trace generated by ξ_t the (covering) annulus SLE(κ, Λ) trace started from p with force points $q_1 . \cdots, q_N$. Furthermore if Λ is 2π-periodic in each space variable, ξ_t does not depend on the choice of points $p, q_1, \cdots q_N$ on the covering space, which allows us to define SLE(κ, Λ) only using the geometry of the base space, the annulus $\mathbb{A}_r$ with marked points.

Let us focus on the case there is only one force point q. It was shown by Zhan [139] that the associated SLE(κ, Λ) is reversible (if it connected p, q) if the translation invariant drift function $\Lambda(r, p-q)$ satisfies

$$\partial_r \Lambda = \frac{\kappa}{2}\Lambda'' + \Big(3 - \frac{\kappa}{2}\Big)H'' + \Lambda H' + H'\Lambda + \Lambda\Lambda' \tag{4.2.5}$$

for the chordal case and

$$\partial_r \Lambda = \frac{\kappa}{2}\Lambda'' + \Big(3 - \frac{\kappa}{2}\Big)H_I'' + \Lambda H_I' + H_I'\Lambda + \Lambda\Lambda'. \tag{4.2.6}$$

for the crossing case. In terms of the *annulus SLE partition function* $Z \equiv Z(r, \cdot)$ satisfying $\Lambda = \kappa\, Z'/Z$, the above equations are written as follows: for the chordal case,

$$\partial_r Z = \frac{\kappa}{2}Z'' + H\, Z' + \frac{6-\kappa}{2\kappa}H'\, Z + C(r)Z; \tag{4.2.7}$$

and for the crossing case,

$$\partial_r Z = \frac{\kappa}{2}Z'' + H_I\, Z' + \frac{6-\kappa}{2\kappa}H_I'\, Z + C(r)Z, \tag{4.2.8}$$

where $C(r)$ is a constant depending only on the modular parameter r.

Chapter 5

Conformal Fock space fields

In this chapter we implement a version of CFT of GFF in doubly connected domains. We start by reviewing some basic definitions and concepts in CFT. The conformal Fock space fields in this section are constructed from the GFFs and their derivatives by means of Wick's calculus [71,122]. Furthermore, they have well-defined correlation functions equipped with conformal structures, which are expressed in terms of elliptic functions.

5.1 Correlation functions of Fock space fields

In this section we introduce the one parameter family of GFFs in a doubly connected domain and recall some basic concepts such as Fock space fields and OPE products. As an expansion of the tensor product of two Fock space fields near diagonal, operator product expansion (OPE) gives rise to important binary operations, namely OPE multiplications or OPE products on Fock space fields.

5.1.1 One parameter family of Gaussian free fields

We denote by $\Phi \equiv \Phi_\chi$ the GFF with weighted boundary condition (3.2.5), see Figure 1.2.

By definition, the 2-point correlation function of GFF Φ in D_r is given by

$$\mathbf{E}[\Phi(\zeta)\Phi(z)] := 2G^{\chi}_{D_r}(\zeta, z) = 2\log\Big|\frac{\Theta_\chi(r, w(\zeta) - \overline{w(z)})}{\Theta_\chi(r, w(\zeta) - w(z))}\Big|, \tag{5.1.1}$$

where w is a conformal map from D_r onto $\mathcal{C}_r$. Moreover, the n-point correlation function of Φ is given by

$$\mathbf{E}[\Phi(z_1)\cdots\Phi(z_n)] = \sum\prod_k 2G^{\chi}_{D_r}(z_{i_k}, z_{j_k}), \tag{5.1.2}$$

where the sum is taken over all partitions of the set $\{1, \cdots, n\}$ into disjoint pairs $\{i_k, j_k\}$. Note that if n is odd, the n-point correlation function of GFF always vanishes.

The *current field* $J = \partial\Phi$, its conjugate $\bar{J} = \bar{\partial}\Phi$, and higher order derivatives of Φ are distributional fields (random generalized functions) and their correlators are computed by differentiating those of GFF. For example, for $\zeta \neq z$, in the identity chart of $\mathcal{C}_r$, we have

$$\mathbf{E}[J(\zeta)\Phi(z)] = \frac{1}{2}\Big(H_\chi(r, \zeta - \bar{z}) - H_\chi(r, \zeta - z)\Big), \tag{5.1.3}$$

$$\mathbf{E}[J(\zeta)J(z)] = \frac{1}{2}\,H'_\chi(r, \zeta - z). \tag{5.1.4}$$

5.1.2 Fock space fields

We treat the GFF as a Gaussian (correlation) functional-valued function and view its correlation function as the kernel representing the expectation of the value of GFF on test functions. We also treat the derivatives of GFF as centered (mean zero) Gaussian generalized functions and include Wick's product of derivatives of GFF in a collection of fields we consider.

By definition, a basic Fock space correlation functional is a (formal form of) Wick's product $\mathcal{X} = X_1(z_1) \odot \cdots \odot X_n(z_n)$ of derivatives X_j of the GFF, and it has zero correlation, i.e., $\mathbf{E}\mathcal{X} = 0$. Here, points $z_j \in D_r$ are not

necessarily distinct and called the *nodes* of $\mathcal{X}$. We write $S_{\mathcal{X}} \equiv S(\mathcal{X})$ for the set of nodes of $\mathcal{X}$.

For basic functionals $\mathcal{X}_j$ of the form $\mathcal{X}_j = X_{j1}(z_{j1}) \odot \cdots \odot X_{jn_j}(z_{jn_j})$ with pairwise disjoint $S(\mathcal{X}_j)$ we define the *tensor product* $\mathcal{X} = \mathcal{X}_1 \cdots \mathcal{X}_m$ by Wick's formula

$$\mathcal{X}_1 \cdots \mathcal{X}_m = \sum \prod_{\{v,v'\}} \mathbf{E}[X_v(z_v) X_{v'}(z_{v'})] \bigodot_{v''} X_{v''}(z_{v''}). \tag{5.1.5}$$

The sum in the right-hand side of (5.1.5) is over all graphs (Feynman diagrams) with vertices v labeled by functionals X_{jk} such that there are no edges (Wick's contractions) of vertices with the same j, and the Wick's products are over unpaired vertices v''. The tensor product of correlation functionals is commutative and associative, see [77, Proposition 1.1].

We identify two functionals $\mathcal{X}_1$ and $\mathcal{X}_2$ if $\mathbf{E}[\mathcal{X}_1\mathcal{Y}] = \mathbf{E}[\mathcal{X}_2\mathcal{Y}]$ holds for all functionals $\mathcal{Y}$ with $S_{\mathcal{Y}} \cap (S_{\mathcal{X}_1} \cup S_{\mathcal{X}_2}) = \emptyset$. The complex conjugation $\overline{\mathcal{X}}$ of $\mathcal{X}$ is the unique correlation functional (modulo trivial functionals) such that $\mathbf{E}[\overline{\mathcal{X}}\mathcal{Y}] = \overline{\mathbf{E}[\mathcal{X}\mathcal{Y}]}$ for all $\mathcal{Y}$'s of the form $\Phi(z_1) \odot \cdots \odot \Phi(z_n)$.

5.1.3 OPE calculations

The *operator product expansion* (OPE) of two Fock space fields X and Y is an asymptotic expansion of $X(\zeta)Y(z)$ near diagonal. In particular, if a field X is holomorphic (i.e., $\bar\partial X = 0$ within correlation), the OPE is defined as a formal Laurent series expansion

$$X(\zeta)Y(z) = \sum C_n(z)(\zeta - z)^n, \quad \text{as} \quad \zeta \to z.$$

More precisely, it means that

$$\mathbf{E}X(\zeta)Y(z)\mathcal{X} = \mathbf{E}\sum C_n(z)(\zeta - z)^n \mathcal{X}$$

for every $\mathcal{X}$ for $\zeta, z \notin S_{\mathcal{X}}$. In this case, $*_n$ product is defined as $X *_n Y := C_n$, the n-th OPE coefficient. In particular, we write $*$ for $*_0$ and call $X * Y$ the *OPE multiplication*, or the *OPE product* of X and Y. We remark that the $*_n$ product is neither associative nor commutative. On the other hand, it satisfies Leibniz's rule, $\partial(X *_n Y) = \partial X *_n Y + X *_n \partial Y$.

We now present some examples of OPEs. As $\zeta \to z, (\zeta \neq z)$,

$$\Phi(\zeta)\Phi(z) = \log \frac{1}{|\zeta - z|^2} + 2c(z) + \Phi^{\odot 2}(z) + o(1), \tag{5.1.6}$$

where $c(z) := u(z,z)$, $u(\zeta, z) := G^{\chi}_{D_r}(\zeta, z) + \log|\zeta - z|$. By definition, for $z \in D_r$,

$$\Phi * \Phi(z) = 2c(z) + \Phi^{\odot 2}(z), \tag{5.1.7}$$

where $c(z)$ is evaluated as

$$c(z) = \log \Big| \frac{\Theta_{\chi}(w(z) - \overline{w(z)})}{w'(z)\Theta'_{\chi}(0)} \Big|. \tag{5.1.8}$$

The OPE powers and exponentials follows from the following proposition. For this let us recall that Hermite polynomials H_n are given by

$$H_n(z) := (-1)^n e^{z^2} \frac{d^n}{dz^n} e^{-z^2}. \tag{5.1.9}$$

Then standard Wick's calculus together with 3-term recurrence relation and generating function of Hermite polynomials [124] leads to the following proposition.

Proposition 5.1.1 (Cf. Proposition 3.2 and 3.3 [77])**.** *We have*

$$\Phi^{\odot n} = (2c)^{n/2} H_n^*\Big(\frac{\Phi}{\sqrt{2}}\Big) \tag{5.1.10}$$

and

$$e^{*\alpha\Phi} = e^{\alpha^2 c} e^{\odot \alpha\Phi}. \tag{5.1.11}$$

Differentiating (5.1.6), we have

$$\begin{aligned} J(\zeta)\Phi(z) = &-\frac{1}{\zeta - z} + (J \odot \Phi)(z) \\ &+ \frac{w'(z)}{2} H_\chi(w(z) - \overline{w(z)}) - \frac{w''(z)}{2w'(z)} + o(1) \end{aligned} \tag{5.1.12}$$

and

$$\begin{aligned} J(\zeta)J(z) = &-\frac{1}{(\zeta - z)^2} + (J \odot J)(z) \\ &- \frac{1}{6} S_w(z) - w'(z)^2 \Big(\frac{\zeta_r(\pi)}{\pi} - \frac{1}{2(r+\chi)} \Big) + o(1). \end{aligned} \tag{5.1.13}$$

Here, S_h is the Schwarzian derivative S_h of a conformal map h, i.e.,

$$S_h := N_h' - \frac{1}{2} N_h^2, \quad N_h = \frac{h''}{h'}.$$

We sometimes use the notation $\sim$ for the singular part of the OPE. For instance, we have

$$J(\zeta)\Phi(z) \sim -\frac{1}{\zeta - z}, \quad J(\zeta)J(z) \sim -\frac{1}{(\zeta - z)^2}. \tag{5.1.14}$$

5.2 Conformal geometry of Fock space fields

We treat a stress tensor as a Lie derivative operator to state Ward's identities. For this purpose, it is needed to consider Fock space fields in a planar domain (e.g., a doubly connected domain) as defined on a Riemann surface. The correlation functions of conformal Fock space fields depend on the choice of local charts at their nodes.

By definition, a non-random conformal field f on a Riemann surface is a smooth function $(f\|\phi) : \phi U \to \mathbb{C}$ for each local chart ϕ. We define a general conformal Fock space field as $X = \sum_\alpha f_\alpha X_\alpha$, where f_α's are non-random conformal fields and basic fields $\mathcal{X}_\alpha$ are Wick's products of derivatives of GFF.

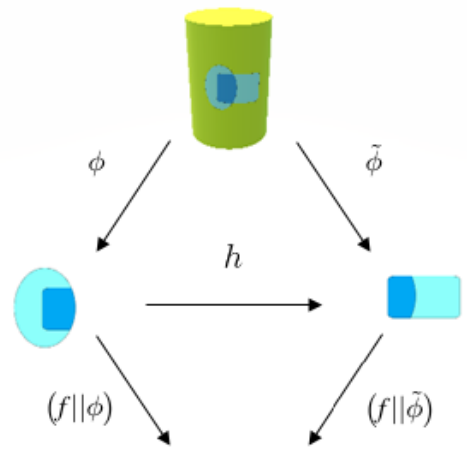


Figure 5.1: Transition map

We now list some basic non-random fields f with specific transformation laws between $f = (f\|\phi)$ and $\tilde{f} = (f\|\tilde{\phi})$ for any two overlapping charts ϕ and $\tilde{\phi}$. A non-random field f is called a *differential* of conformal dimension (λ, λ_*) if

$$f = (h')^{\lambda}(\overline{h'})^{\lambda_*}\,\tilde{f}\circ h.$$

Here h is the transition map between ϕ and $\tilde{\phi}$, i.e.,

$$h = \tilde{\phi}\circ\phi^{-1} : \phi(U\cap\tilde{U}) \to \tilde{\phi}(U\cap\tilde{U}),$$

see Figure 5.1.

By definition, pre-pre-Schwarzian forms (PPS-forms), pre-Schwarzian forms (PS-forms) and Schwazian forms (S-forms) of order μ are conformal Fock space fields with transformation laws

$$f = \tilde{f}\circ h + \mu\log h', \quad f = h'\tilde{f}\circ h + \mu N_h, \quad f = (h')^2\tilde{f}\circ h + \mu S_h$$

respectively.

5.2.1 Lie derivative of conformal Fock space field

Suppose a non-random smooth vector field v is holomorphic in some open set $U \subset M$. For a conformal Fock space field X, the Lie derivative $\mathcal{L}_v X$ in U is defined as

$$(\mathcal{L}_v X \| \phi) = \frac{d}{dt}\Big|_{t=0} (X \| \phi \circ \psi_{-t}),$$

where ψ_t is a local flow of v, and ϕ is a given chart. For example, we have the followings:

- $\mathcal{L}_v X = (v\partial + \bar{v}\bar{\partial} + \lambda v' + \lambda_* \overline{v'})X$ for a (λ, λ_*)-differential X;
- $\mathcal{L}_v X = (v\partial + v')X + \mu v''$ for a PS-form X of order μ;
- $\mathcal{L}_v X = (v\partial + 2v')X + \mu v'''$ for an S-form X of order μ,

see [77, Proposition 4.1]. For reader's convenience, we list some basic properties of the Lie derivatives:

- $\mathbf{E}[\mathcal{L}_v X] = \mathcal{L}_v \mathbf{E}[X]$;
- $\mathcal{L}_v(\bar{X}) = \overline{(\mathcal{L}_v X)}$;
- $\mathcal{L}_v(\partial X) = \partial(\mathcal{L}_v X)$;
- Leibniz's rule holds for tensor, Wick's, and OPE products.

We define the $\mathbb{C}$-linear part $\mathcal{L}_v^+$ and the anti-linear part $\mathcal{L}_v^-$ of $\mathcal{L}_v$ by

$$\mathcal{L}_v^+ := \frac{\mathcal{L}_v - i\mathcal{L}_{iv}}{2}, \quad \mathcal{L}_v^- := \frac{\mathcal{L}_v + i\mathcal{L}_{iv}}{2}$$

so that $\mathcal{L}_v$ is decomposed as $\mathcal{L}_v = \mathcal{L}_v^+ + \mathcal{L}_v^-$.

5.2.2 Stress energy tensor

In this subsection we recall the definition of a *stress tensor* and review its basic properties. See [77, Sections 5.2 5.3, Lecture 7, and Appendix 11] for more details.

A stress tensor for a family of conformal Fock space fields is defined as a pair of holomorphic and anti-holomorphic quadratic differentials which represent the Lie derivative operators within correlations of fields in this family. In the physics literature, special forms of formulas are known as (conformal) Ward's identities. The existence of a stress tensor is a remarkable property of certain families of conformal Fock space fields. For instance, the OPE family of GFFs has a common stress tensor.

A Fock space field X in D_r is said to have a stress tensor $(A, \bar{A})$ (or $X \in \mathcal{F}(A, \bar{A})$) if A is a holomorphic quadratic differential and the residue form of Ward's identity

$$\mathcal{L}_v^+ X(z) = \frac{1}{2\pi i} \oint_{(z)} vAX(z), \quad \mathcal{L}_v^- X(z) = -\frac{1}{2\pi i} \oint_{(z)} \bar{v}\bar{A}X(z)$$

holds on $D_{\mathrm{hol}}(v) \cap U$ for all smooth vector field v, where $D_{\mathrm{hol}}(v)$ is the the maximal open set where v is holomorphic. By [77, Proposition 5.8], $\mathcal{F}(A, \bar{A})$ is closed under $*_n$ product, in particular differentiations.

If X is a differential or a form, the residue form of Ward's identity holds for X if and only if Ward's OPE for X holds in some or every chart. For example, a (λ, λ_*)-differential X is in $\mathcal{F}(A, \bar{A})$ if and only if the following singular OPEs hold in every/some chart:

$$A(\zeta)X(z) \sim \frac{\lambda X(z)}{(\zeta - z)^2} + \frac{\partial X(z)}{\zeta - z}, \quad A(\zeta)\bar{X}(z) \sim \frac{\overline{\lambda}_* \bar{X}(z)}{(\zeta - z)^2} + \frac{\partial \bar{X}(z)}{\zeta - z}. \quad (5.2.1)$$

For any form X of order μ belongs to $\mathcal{F}(A, \bar{A})$ if and only if the following singular OPE holds in every/some chart:

$$A(\zeta)X(z) \sim \frac{\mu}{(\zeta - z)^2} + \frac{\partial X(z)}{\zeta - z} \qquad \text{for a PPS-form } X; \quad (5.2.2)$$

$$A(\zeta)X(z) \sim \frac{2\mu}{(\zeta - z)^3} + \frac{X(z)}{(\zeta - z)^2} + \frac{\partial X(z)}{\zeta - z} \qquad \text{for a PS-form } X; \quad (5.2.3)$$

$$A(\zeta)X(z) \sim \frac{6\mu}{(\zeta - z)^4} + \frac{2X(z)}{(\zeta - z)^2} + \frac{\partial X(z)}{\zeta - z} \qquad \text{for an S-form } X. \quad (5.2.4)$$

5.3 Central charge modification

In this subsection, we introduce the central charge modification $\Phi_{(b)}$ of GFF, where b is the real parameter related to the central charge c as $c = 1 - 12b^2$.

In the chordal/radial CFT in a simply connected domain with a marked boundary/interior point q, two neutrality conditions are imposed in the construction of OPE exponentials. They have Wick's part and the correlation part; Wick's part is Wick's exponential of the linear combination of bosonic fields and the correlation part can be expressed in terms of Coulomb gas correlation functions. We require that this linear combination of bosonic fields should be a well-defined Fock space field. This requirement is called a neutrality condition on the coefficients or charges of linear combination. At the same time, conformal invariance of Coulomb gas correlation functions in a simply connected domain gives rise to the other neutrality condition. Because the neutrality conditions are different, we need to place the background charge at q to reconcile these two neutrality conditions.

On the other hand, in a doubly connected domain (or on a compact Riemann surface of genus one), no background charges need to placed because background charges have zero sum due to a version of Gauss-Bonnet theorem and because two different background charge modification can be converted to each other by means of insertion of the corresponding OPE exponential.

We express a stress tensor and the Virasoro field of $\Phi_{(b)}$ in terms of current fields $J_{(b)} := \partial\Phi_{(b)}$. From now on, the subscript (0) is added to the notation of fields in the OPE family of the GFF, the algebra over $\mathbb{C}$ spanned by 1 and the derivatives of GFF under OPE multiplication. For instance, $\Phi_{(0)}$ is the new notation for the GFF and $J_{(0)} = \partial\Phi_{(0)}$. First, we recall the definition of Virasoro field.

Definition. A Fock space field T is said to be *Virasoro field* for $\mathcal{F}(A, \bar{A})$ if

- $T \in \mathcal{F}(A, \bar{A})$, and
- $T - A$ is a non-random meromorphic Schwarzian form.

The order of Schwarzian form $T - A$ is denoted by $\frac{1}{12}c$, where c is called the *central charge*.

For each real parameter b, we define

$$\Phi \equiv \Phi_{(b)} := \Phi_{(0)} - 2b \arg w', \quad J \equiv J_{(b)} := J_{(0)} + ib\frac{w''}{w'}, \tag{5.3.1}$$

where w is a conformal map from D_r onto $\mathcal{C}_r$.

Proposition 5.3.1. *The field $\Phi_{(b)}$ has a stress tensor*

$$A_{(b)} := -\frac{1}{2}J_{(0)} \odot J_{(0)} + (ib\partial - \jmath_{(b)})J_{(0)}, \quad \jmath_{(b)} := \mathbf{E}[J_{(b)}] = ib\frac{w''}{w'},$$

and its Virasoro field is given by

$$\begin{aligned} T_{(b)} &= -\frac{1}{2}J_{(b)} * J_{(b)} + ib\partial J_{(b)} \\ &= A_{(b)} + \frac{c}{12}S_w + (w')^2\Big(\frac{\zeta_r(\pi)}{2\pi} - \frac{1}{4(r+\chi)}\Big), \end{aligned}$$

where $c = 1 - 12b^2$.

Proof. First, we prove the case $b = 0$. It is obvious that $A_{(0)} := -\frac{1}{2}J_{(0)} \odot J_{(0)}$ is holomorphic quadratic differential. We claim that $W = (A_{(0)}, \bar{A}_{(0)})$ is a stress tensor for $\Phi_{(0)}$. Since $\Phi_{(0)}$ is a scalar field or a $(0,0)$-differential, by (5.2.1), all we need to verify is the following (Ward's) OPE in the identity chart of cylinder $\mathcal{C}_r$:

$$A_{(0)}(\zeta)\Phi_{(0)}(z) \sim \frac{J_{(0)}(z)}{\zeta - z}. \tag{5.3.2}$$

This is clear from (5.1.12) and Wick's calculus.

We now define the Virasoro field $T_{(0)}$ of $\Phi_{(0)}$ by $T_{(0)} := -\frac{1}{2}J_{(0)} * J_{(0)}$, equivalently by the following OPE:

$$J_{(0)}(\zeta)J_{(0)}(z) = -\frac{1}{(\zeta - z)^2} - 2T_{(0)}(z) + o(1). \tag{5.3.3}$$

Then it follows from (5.1.13) that

$$T_{(0)} = A_{(0)} + \frac{1}{12}S_w + (w')^2\Big(\frac{\zeta_r(\pi)}{2\pi} - \frac{1}{4(r+\chi)}\Big).$$

Next, let us prove the general case $b \neq 0$. It is easy to see that $ib\partial J_{(0)}$ and $\jmath J_{(0)}$ satisfy the following transformation rules:

$$\begin{aligned} ib\partial J_{(0)} &= ibh''\tilde{J}_{(0)} \circ h + ib(h')^2\partial\tilde{J}_{(0)} \circ h; \\ \jmath J_{(0)} &= ib\Big(\frac{h''}{h'}\Big)h'\tilde{J}_{(0)} \circ h + (h')^2(\tilde{\jmath}\tilde{J}_{(0)}) \circ h. \end{aligned}$$

Thus we readily see that $A_{(b)}$ is a holomorphic quadratic differential.

Again, we claim that $W = (A_{(b)}, \bar{A}_{(b)})$ is a stress tensor for $\Phi_{(b)}$. Since $\Phi_{(b)}$ is the real part of a PPS-form, by (5.2.2), it suffices to show the following (Ward's) OPE in the identity chart of cylinder $\mathcal{C}_r$:

$$A_{(b)}(\zeta)\Phi_{(b)}(z) = \Big(A_{(0)}(\zeta) + (ib\partial - \jmath_{(b)})J_{(0)}(\zeta)\Big)\Phi_{(b)}(z) \sim \frac{ib}{(\zeta - z)^2} + \frac{J_{(b)}(z)}{\zeta - z}.$$

The above OPE follows from (5.1.12), (5.1.13) and (5.3.2). Let

$$T_{(b)} = -\frac{1}{2}J_{(b)} * J_{(b)} + ib\partial J_{(b)}.$$

Then by (5.3.3), we obtain

$$\begin{aligned} T_{(b)} &= A_{(b)} + \frac{1}{12}S_w + (w')^2\Big(\frac{\zeta_r(\pi)}{2\pi} - \frac{1}{4(r+\chi)}\Big) - \frac{1}{2}\jmath_{(b)}^2 + ib\partial\jmath_{(b)} \\ &= A_{(b)} + \frac{1-12b^2}{12}S_w + (w')^2\Big(\frac{\zeta_r(\pi)}{2\pi} - \frac{1}{4(r+\chi)}\Big), \end{aligned}$$

which completes the proof. □

5.4 Formal bosonic fields and neutrality condition

In this section we introduce formal fields and their formal correlation functions. We use them to describe modified multi-vertex fields (OPE exponentials) in the next section.

We first define the *bi-variant* field $\Phi^+ \equiv \Phi^+_{(b)}$ by

$$\Phi^+_{(b)}(z,z_0) := \Big\{ \Phi^+_{(b)}(\gamma) = \int_\gamma J_{(b)}(\zeta)d\zeta \Big\} = \Phi^+_{(0)}(z,z_0) + ib\log\frac{w'(z)}{w'(z_0)},$$

where γ is a curve from z_0 to z and $w : D_r \to \mathcal{C}_r$ is a conformal map. Note that the values of Φ^+ are multivalued functionals. We define the complex Green's function in D_r as

$$G^{\chi+}_{D_r}(z,z_1) := \frac{1}{2}\log\Big(\frac{\Theta_\chi(r, w(z) - \overline{w(z_1)})}{\Theta_\chi(r, w(z) - w(z_1))}\Big).$$

Integrating (5.1.3) over a curve from z_0 to z, we obtain

$$\mathbf{E}[\Phi^+_{(0)}(z,z_0)\Phi_{(0)}(z_1)] = 2(G^{\chi+}_{D_r}(z,z_1) - G^{\chi+}_{D_r}(z_0,z_1)).$$

We remark that for the ER case when $\chi = \infty$, the complex Green's function plays an important role in the theory of conformal mappings between canonical multiply connected domains, see [83] and also a recent review [37].

Now we introduce *formal (1-point) fields* $\Phi^\pm_{(0)}$ in D_r as centered Gaussian formal fields with (formal) 2-point correlations:

$$\mathbf{E}[\Phi^+_{(0)}(z_1)\Phi^+_{(0)}(z_2)] = -\log\Theta_\chi(r, w(z_1) - w(z_2)); \tag{5.4.1}$$

$$\mathbf{E}[\Phi^+_{(0)}(z_1)\Phi^-_{(0)}(z_2)] = +\log\Theta_\chi(r, w(z_1) - \overline{w(z_2)}). \tag{5.4.2}$$

Formal fields $\Phi^\pm_{(0)}$ satisfy $\Phi_{(0)} = \Phi^+_{(0)} + \Phi^-_{(0)}$ within correlations, e.g.,

$$\mathbf{E}\,\Phi_{(0)}(z_1)\Phi_{(0)}(z_2) = \mathbf{E}(\Phi^+_{(0)}(z_1) + \Phi^-_{(0)}(z_1))(\Phi^+_{(0)}(z_2) + \Phi^-_{(0)}(z_2)),$$

where **E** in the left-hand side stands for correlation while we use formal correlations in the right-hand side. We define $\Phi^{\pm}_{(b)}$ by

$$\Phi^{+}_{(b)} = \Phi^{+}_{(0)} + ib\log w', \quad \Phi^{-}_{(b)} = \Phi^{-}_{(0)} - ib\log \overline{w'}. \tag{5.4.3}$$

By definition, the following formal rule holds:

$$\Phi^{+}_{(0)}(z, z_0) = \Phi^{+}_{(0)}(z) - \Phi^{+}_{(0)}(z_0).$$

The formal *dual boson* $\tilde{\Phi}_{(0)}$ is defined by

$$\tilde{\Phi}_{(0)} = -i\Big(\Phi^{+}_{(0)} - \Phi^{-}_{(0)}\Big), \quad \tilde{\Phi}_{(0)}(z, z_0) = \tilde{\Phi}_{(0)}(z) - \tilde{\Phi}_{(0)}(z_0).$$

Note that we have the following relations

$$2\Phi^{+}_{(0)} = \Phi_{(0)} + i\tilde{\Phi}_{(0)}, \quad 2\Phi^{-}_{(0)} = \Phi_{(0)} - i\tilde{\Phi}_{(0)}.$$

For a finite set of distinct points $\{z_j\}_{j=1}^{n}$ in $\bar{D}_r$, let us denote $\boldsymbol{\sigma} = \sum \sigma_j \cdot z_j$, where σ_j is a "charge" at each z_j. We also consider $\boldsymbol{\sigma}$ as a divisor, i.e., a function $\boldsymbol{\sigma} : \bar{D}_r \to \mathbb{R}$ which takes the value 0 at all points except for finitely many points, and $\boldsymbol{\sigma}(z_j) = \sigma_j$. Sometimes it is convenient to consider $\boldsymbol{\sigma}$ as an atomic measure $\boldsymbol{\sigma} = \sum \sigma_j \cdot \delta_{z_j}$. For example, $\int \boldsymbol{\sigma} = \sum \sigma_j$.

For a double divisor $(\boldsymbol{\sigma}, \boldsymbol{\sigma}_*)$, we define the formal bosonic field $\Phi\,[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]$ by

$$\Phi\,[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] := \sum \sigma_j \Phi^{+}(z_j) - \sigma_{*j}\Phi^{-}(z_j), \quad \Phi^{\pm} \equiv \Phi^{\pm}_{(0)}. \tag{5.4.4}$$

In general, it is not a well-defined Fock space field. The proposition below shows that the neutrality condition

$$\int \boldsymbol{\sigma} + \boldsymbol{\sigma}_* = 0 \tag{5.4.5}$$

guarantees well-definedness of $\Phi[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]$ as a Fock space field. For example, bi-variant fields satisfy the neutrality condition.

Proposition 5.4.1. *If a double divisor $(\boldsymbol{\sigma}, \boldsymbol{\sigma}_*)$ satisfies (5.4.5), then the formal bosonic field $\Phi[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]$ can be represented as a linear combination of well-defined Fock space fields.*

Proof. Let us choose any point $z_0 \in D_r$. Then

$$\Phi[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] = \Phi^+(z_0) \int \boldsymbol{\sigma} - \Phi^-(z_0) \int \boldsymbol{\sigma}_* + \sum \sigma_j \Phi^+(z_j, z_0) - \sigma_{*j} \Phi^-(z_j, z_0).$$

Under the neutrality condition, the first two terms in the right-hand side becomes the Fock space correlation functional:

$$\Phi^+(z_0) \int \boldsymbol{\sigma} - \Phi^-(z_0) \int \boldsymbol{\sigma}_* = \Phi(z_0) \int \boldsymbol{\sigma}.$$

□

5.5 Coulomb gas correlation function

In this section we construct modified multi-vertex fields and express their correlation functions. In particular, we explain the *Coulomb gas formalism*, which allows us to actually compute correlation functions of conformal fields.

Given divisors $\boldsymbol{\sigma} = \sum \sigma_j \cdot z_j, \boldsymbol{\tau} = \sum \tau_k \cdot \zeta_k$ for $z_j, \zeta_k \in D_r$, set

$$\Phi[\boldsymbol{\sigma}] := \sum \sigma_j \, \Phi(z_j), \quad \Phi[\boldsymbol{\tau}] := \sum \tau_k \, \Phi(\zeta_k),$$

where $\Phi \equiv \Phi_{(0)}$. Define

$$\mathbf{E}\Phi[\boldsymbol{\sigma}] * \Phi[\boldsymbol{\tau}] := \sum_{z_j \neq \zeta_k} \sigma_j \tau_k \, \mathbf{E}\Phi(z_j)\Phi(\zeta_k) + \sum_{z_j = \zeta_k} \sigma_j \tau_k \, \mathbf{E}\Phi * \Phi(z_j).$$

In particular, set

$$c[\boldsymbol{\sigma}] := \frac{1}{2} \, \mathbf{E}\Phi[\boldsymbol{\sigma}] * \Phi[\boldsymbol{\sigma}]$$

and define the (non-chiral) *Coulomb gas correlation function* $C_{(b)}$ as

$$C_{(b)}[\boldsymbol{\sigma}] := e^{-c[\boldsymbol{\sigma}]} \prod (w_j')^{-b\sigma_j} (\overline{w_j'})^{b\sigma_j},$$

where $w_j := w(z_j)$. By (5.1.1) and (5.1.8), if all z_j's are in the bulk, $C_{(b)}[\boldsymbol{\sigma}]$ is $(\sigma_j^2/2 - b\sigma_j, \sigma_j^2/2 + b\sigma_j)$-differential at z_j, and its evaluation in the identity chart of cylinder $\mathcal{C}_r$ is given as

$$C_{(b)}[\boldsymbol{\sigma}] = \Theta'_\chi(0)^{\sum \sigma_j^2} \prod \Theta_\chi(z_j - \bar{z}_j)^{-\sigma_j^2} \prod_{j<k} \left| \frac{\Theta_\chi(z_j - z_k)}{\Theta_\chi(z_j - \bar{z}_k)} \right|^{2\sigma_j\sigma_k}.$$

We remark that the terminology "Coulomb gas" comes from the resemblance of the two point function $\mathbf{E}\Phi(z_1)\Phi(z_2)$ with the electric potential energy between two unit charges in a planar domain (or on a compact Riemann surface). For the genus one case we refer to [56, 81] and references therein.

By definition, a non-chiral vertex field $\mathcal{V}_{(0)}[\boldsymbol{\sigma}]$ is given by the OPE exponential

$$\mathcal{V}_{(0)}[\boldsymbol{\sigma}] := e^{*i\Phi[\boldsymbol{\sigma}]} \equiv \sum_{n=0}^{\infty} \frac{i^n \Phi^{*n}[\boldsymbol{\sigma}]}{n!}.$$

Then it is easy to show that

$$\mathcal{V}_{(0)}[\boldsymbol{\sigma}] = C_{(0)}[\boldsymbol{\sigma}]\mathcal{V}^{\odot}[\boldsymbol{\sigma}], \quad \mathcal{V}^{\odot}[\boldsymbol{\sigma}] = e^{\odot i\Phi[\boldsymbol{\sigma}]}.$$

For further details, see [78, Section 5].

Let us denote by $(\boldsymbol{\sigma}, \boldsymbol{\sigma}_*) = (\sum \sigma_j \cdot z_j, \sum \sigma_{*j} \cdot z_j)$ a double divisor satisfying the neutrality condition (NC_0). We define (chiral) Coulomb gas correlation function $C_{(b)}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]$ as

$$\begin{aligned} C_{(b)}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] : = \Theta'_\chi(0)^{\sum \frac{1}{2}(\sigma_j^2 + \sigma_{*j}^2)} \prod \Theta_\chi(w_j - \bar{w}_j)^{\sigma_j\sigma_{*j}} \prod (w'_j)^{\lambda_j} (\overline{w'_j})^{\lambda_{*j}} \\ \times \prod_{j<k} \Theta_\chi(w_j - w_k)^{\sigma_j\sigma_k} \Theta_\chi(\bar{w}_j - w_k)^{\sigma_{*j}\sigma_k} \\ \times \prod_{j<k} \Theta_\chi(w_j - \bar{w}_k)^{\sigma_j\sigma_{*k}} \Theta_\chi(\bar{w}_j - \bar{w}_k)^{\sigma_{*j}\sigma_{*k}}, \end{aligned} \tag{5.5.1}$$

where the conformal dimensions are given as

$$(\lambda_j, \lambda_{*j}) = (\lambda(\sigma_j), \lambda(\sigma_{*j})), \quad \lambda(x) := x^2/2 - b\,x. \tag{5.5.2}$$

Notice here that the non-chiral Coulomb gas correlation function is related to chiral one as $C_{(b)}[\boldsymbol{\sigma}] = C_{(b)}[\boldsymbol{\sigma}, -\boldsymbol{\sigma}]$.

By definition, the *multi-vertex field* $\mathcal{O}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] \equiv \mathcal{O}_{(b)}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]$ is given by

$$\mathcal{O}_{(b)}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] := C_{(b)}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] V^{\odot}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*], \quad V^{\odot}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] = e^{\odot i\Phi_{(0)}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]}. \tag{5.5.3}$$

Then $\mathcal{O}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]$ is a differential and by (5.5.2), its conformal dimension (h_j, h_{*j}) at z_j is given as

$$(h_j, h_{*j}) = \Big(\frac{\sigma_j^2}{2} - b\sigma_j\,,\, \frac{\sigma_{*j}^2}{2} - b\sigma_{*j}\Big). \tag{5.5.4}$$

In particular, we have

$$\mathcal{O}_{(b)}[\boldsymbol{\sigma}] := \mathcal{O}_{(b)}[\boldsymbol{\sigma}, -\boldsymbol{\sigma}] = C_{(b)}[\boldsymbol{\sigma}] e^{\odot i\Phi_{(b)}[\boldsymbol{\sigma}]} = e^{*i\Phi_{(b)}[\boldsymbol{\sigma}]}. \tag{5.5.5}$$

Now we show that $\mathcal{O}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]$ satisfies the Ward's OPE.

Proposition 5.5.1. *Under the neutrality condition, $\mathcal{O}(\boldsymbol{z}) \equiv \mathcal{O}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*](\boldsymbol{z})$ satisfies Ward's OPE: as $\zeta \to z_j \in D_r$,*

$$T(\zeta)\mathcal{O}(\boldsymbol{z}) \sim h_j \frac{\mathcal{O}(\boldsymbol{z})}{(\zeta - z_j)^2} + \frac{\partial_j \mathcal{O}(\boldsymbol{z})}{\zeta - z_j}, \quad T(\zeta)\overline{\mathcal{O}}(\boldsymbol{z}) \sim \bar{h}_{*j} \frac{\overline{\mathcal{O}}(\boldsymbol{z})}{(\zeta - z_j)^2} + \frac{\partial_j \overline{\mathcal{O}}(\boldsymbol{z})}{\zeta - z_j}.$$

Proof. Since vertex fields are differentials, it is enough to prove the proposition in the identity chart. Recall that in the identity chart,

$$J_{(b)} = J_{(0)}, \quad T_{(b)} = -\frac{1}{2} J_{(0)} \odot J_{(0)} + ib\,\partial J_{(0)} + \Big(\frac{\zeta_r(\pi)}{2\pi} - \frac{1}{4(r+\chi)}\Big).$$

By Wick's calculus, we have

$$\partial_j \mathcal{O} = \frac{\partial_j C_{(b)}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]}{C_{(b)}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]} \mathcal{O} + i\sigma_j J_{(b)}(z_j) \odot \mathcal{O}.$$

First, let us show the Ward's OPE in the case $b = 0$. Let us denote

$$\begin{aligned} F(\zeta, \boldsymbol{z}) : &= \mathbf{E} J_{(0)}(\zeta) \sum \Big(i\sigma_j \Phi^+_{(0)}(z_j) - i\sigma_{*j} \Phi^-_{(0)}(z_j) \Big) \\ &= -\frac{i}{2} \sum \Big(\sigma_j H_\chi(r, \zeta - z_j) + \sigma_{*j} H_\chi(r, \zeta - \bar{z}_j) \Big). \end{aligned}$$

Then by (3.1.32),(5.5.1) and (5.5.4) as $\zeta \to z_j$, we have

$$F \sim -\frac{i\sigma_j}{\zeta - z_j}, \qquad F^2 \sim -\frac{2h_j}{(\zeta - z_j)^2} - \frac{\partial_j C_{(b)}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]}{C_{(b)}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]} \frac{2}{\zeta - z_j}.$$

On the other hand, by Wick's formula, we have

$$T_{(0)}(\zeta)\mathcal{O}(\boldsymbol{z}) = T_{(0)}(\zeta) \odot \mathcal{O}(\boldsymbol{z}) - F J_{(0)}(\zeta) \odot \mathcal{O}(\boldsymbol{z}) - \frac{1}{2} F^2 \mathcal{O}(\boldsymbol{z}).$$

Combining all above equations, our assertion in the case $b = 0$ follows. In the general case, we only need to check

$$ib\partial J_{(0)}(\zeta)\mathcal{O}(\boldsymbol{z}) \sim -\sigma_j b \frac{\mathcal{O}(\boldsymbol{z})}{(\zeta - z_j)^2}, \quad \text{as} \quad \zeta \to z_j.$$

Again, this follows from Wick's calculus, which completes the proof. We leave it to the reader to verify the proposition for $\overline{\mathcal{O}}$. □

5.6 Background charge modifications

Given marked points q_k, we call a divisor $\boldsymbol{\beta} = \sum_k \beta_k \cdot q_k$ satisfying the neutrality condition $\int \boldsymbol{\beta} = 0$ a *background charge*. By definition, a *background charge modification* $\Phi_{\boldsymbol{\beta}} \equiv \Phi_{\boldsymbol{\beta},(b)}$ of the GFF $\Phi \equiv \Phi_{(b)}$ is given as

$$\Phi_{\boldsymbol{\beta},(b)} := \Phi_{(0)} + \varphi_{\boldsymbol{\beta}}, \tag{5.6.1}$$

where the non-random harmonic function $\varphi_{\boldsymbol{\beta}}$ is given by

$$\varphi_{\boldsymbol{\beta}} := -2b \arg w'_z + \sum \beta_k \arg \Big\{ \Theta_\chi(w_k - w_z)\Theta_\chi(\bar{w}_k - w_z) \Big\}. \tag{5.6.2}$$

Here $w_z = w(z)$ and $w_k = w(q_k)$. Note that by Wick's calculus, we have

$$\mathbf{E}\Phi_{\boldsymbol{\beta}} = \mathbf{E}[e^{\odot i\Phi[\boldsymbol{\beta}/2,\boldsymbol{\beta}/2]}\Phi]. \tag{5.6.3}$$

The current field $J_{\boldsymbol{\beta}}$ is defined by

$$J_{\boldsymbol{\beta},(b)} \equiv J_{\boldsymbol{\beta}} := \partial\Phi_{\boldsymbol{\beta}} = J_{(0)} + \jmath_{\boldsymbol{\beta}}, \quad \jmath_{\boldsymbol{\beta}} := \partial\varphi_{\boldsymbol{\beta}}. \tag{5.6.4}$$

Note that

$$\jmath_{\boldsymbol{\beta}} := ib\frac{w''_z}{w'_z} + \frac{iw'_z}{4}\sum \beta_k \Big\{ H_\chi(w_k - w_z) + H_\chi(\bar{w}_k - w_z) \Big\}. \tag{5.6.5}$$

To clarify the notations, we remark that for the trivial divisor $\boldsymbol{\beta} = \mathbf{0}$,

$$\Phi_{\mathbf{0}} = \Phi_{(b)}, \quad J_{\mathbf{0}} = J_{(b)}. \tag{5.6.6}$$

We now generalize Proposition 5.3.1.

Proposition 5.6.1. *The field $\Phi_{\boldsymbol{\beta}}$ has a stress tensor*

$$A_{\boldsymbol{\beta}} := A_{(0)} + \Big(ib\partial - \jmath_{\boldsymbol{\beta}} \Big) J_{(0)},$$

and its Virasoro field is given as

$$T_{\boldsymbol{\beta}} := -\frac{1}{2} J_{\boldsymbol{\beta}} * J_{\boldsymbol{\beta}} + ib\,\partial J_{\boldsymbol{\beta}}.$$

In particular, $\mathbf{E}T_{\boldsymbol{\beta}}(z)$ *is evaluated in the identity chart of* $\mathcal{C}_r$ *as*

$$\begin{aligned}\mathbf{E}T_{\boldsymbol{\beta}}(z) = \frac{1}{32}\Big[\sum \beta_k\Big\{H_\chi(r, q_k - z) + H_\chi(r, \bar{q}_k - z)\Big\}\Big]^2 \\ + \frac{b}{4}\sum \beta_k\Big\{H'_\chi(r, q_k - z) + H'_\chi(r, \bar{q}_k - z)\Big\} + \frac{\zeta_r(\pi)}{2\pi} - \frac{1}{4(r+\chi)}.\end{aligned}$$

Proof. Throughout the proof, we denote by ϕ the local chart and $h = \tilde{\phi}\circ\phi^{-1}$ the transition map between ϕ and a chart $\tilde{\phi}$ overlapping with ϕ. We also use the tilde notation for conformal fields evaluated to the chart $\tilde{\phi}$, e.g., $\tilde{J}_{(0)} = (J_{(0)}\|\tilde{\phi})$.

We claim that $A_{\boldsymbol{\beta}}$ is a $(2,0)$-differential and the Ward's OPE

$$A_{\boldsymbol{\beta}}(\zeta)\Phi_{\boldsymbol{\beta}}(z) \sim \frac{ib}{(\zeta - z)^2} + \frac{J_{\boldsymbol{\beta}}(z)}{\zeta - z} \tag{5.6.7}$$

holds. Then the first part of the proposition immediately follow from (5.2.2).

Since $J_{(0)}$ is $(1,0)$-differential, we have

$$\partial J_{(0)} = h''\tilde{J}_{(0)}\circ h + (h')^2\partial\tilde{J}_{(0)}\circ h$$

and by (5.6.5),

$$\jmath_{\boldsymbol{\beta}} = h'\,\tilde{\jmath}_{\boldsymbol{\beta}}\circ h + ib\frac{h''}{h'}. \tag{5.6.8}$$

Combining above equations, one can observe that $(ib\partial - \jmath_{\boldsymbol{\beta}})J_{(0)}$ is a $(2,0)$-differential. On the other hand, note that

$$A_{\boldsymbol{\beta}}(\zeta)\Phi_{\boldsymbol{\beta}}(z) \sim A_{(0)}(\zeta)\Phi_{(0)}(z) + \Big(ib\partial - \jmath_{\boldsymbol{\beta}}\Big)J_{(0)}(\zeta)\Phi_{(0)}(z).$$

Therefore (5.6.7) follows from (5.1.14) and Proposition 5.3.1.

Next, we prove the second part of the proposition. Note that by Proposition 5.3.1, we have

$$T_{\boldsymbol{\beta}} = A_{\boldsymbol{\beta}} + \mathbf{E}\,T_{(0)} - \frac{1}{2}\jmath_{\boldsymbol{\beta}}^2 + ib\partial\jmath_{\boldsymbol{\beta}}, \tag{5.6.9}$$

and $\mathbf{E}\,T_{(0)}$ is an S-form of order 1/12. Moreover, by (5.6.8), the term

$$-\frac{1}{2}\jmath_{\boldsymbol{\beta}}^2 + ib\partial\jmath_{\boldsymbol{\beta}}$$

is an S-form of order $-b^2$, which completes the proof. □

We define holomorhpic (resp., anti-holomorphic) function $\varphi_{\boldsymbol{\beta}}^+$ (resp,$\varphi_{\boldsymbol{\beta}}^-$ as

$$\begin{aligned}\varphi_{\boldsymbol{\beta}}^+ &:= +ib\log w_z' - \frac{i}{2}\sum \beta_k \log\Big\{\Theta_\chi(w_k - w_z)\Theta_\chi(\bar{w}_k - w_z)\Big\},\\ \varphi_{\boldsymbol{\beta}}^- &:= -ib\log \overline{w_z'} + \frac{i}{2}\sum \beta_k \log\Big\{\Theta_\chi(\bar{w}_k - \bar{w}_z)\Theta_\chi(w_k - \bar{w}_z)\Big\}.\end{aligned} \tag{5.6.10}$$

The chiral and anti-chiral parts of $\Phi_{\boldsymbol{\beta}}$ are defined by

$$\Phi_{\boldsymbol{\beta}}^+ := \Phi_{(0)}^+ + \varphi_{\boldsymbol{\beta}}^+, \qquad \Phi_{\boldsymbol{\beta}}^- := \Phi_{(0)}^- + \varphi_{\boldsymbol{\beta}}^-. \tag{5.6.11}$$

Notice that by definition, $\Phi_{\boldsymbol{\beta}} = \Phi_{\boldsymbol{\beta}}^+ + \Phi_{\boldsymbol{\beta}}^-$.

For a double divisor $(\boldsymbol{\sigma}, \boldsymbol{\sigma}_*) = (\sum \sigma_j \cdot z_j, \sum \sigma_{*j} \cdot z_j)$ satisfying $\int(\boldsymbol{\sigma} + \boldsymbol{\sigma}_*) = 0$, we write

$$\varphi_{\boldsymbol{\beta}}\,[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] := \sum \sigma_j\, \varphi_{\boldsymbol{\beta}}^+ - \sigma_{*j}\, \varphi_{\boldsymbol{\beta}}^-. \tag{5.6.12}$$

We define the formal bosonic field $\Phi_{\boldsymbol{\beta}}\,[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]$ associated with the background charge $\boldsymbol{\beta}$ as

$$\begin{aligned}\Phi_{\boldsymbol{\beta}}\,[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] :&= \sum \sigma_j \Phi_{\boldsymbol{\beta}}^+(z_j) - \sigma_{*j}\Phi_{\boldsymbol{\beta}}^-(z_j)\\ &= \Phi_{(0)}\,[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] + \varphi_{\boldsymbol{\beta}}\,[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*].\end{aligned} \tag{5.6.13}$$

The multi-vertex field $\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]$ is defined by

$$\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] := \frac{C_{(b)}[\boldsymbol{\sigma} + \boldsymbol{\beta}/2, \boldsymbol{\sigma}_* + \boldsymbol{\beta}/2]}{C_{(b)}[\boldsymbol{\beta}/2, \boldsymbol{\beta}/2]} V^{\odot}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]. \tag{5.6.14}$$

In particular, if $\boldsymbol{\sigma}_* = -\boldsymbol{\sigma}$, we simply omit $\boldsymbol{\sigma}_*$ in the notations, e.g.,

$$\varphi_{\boldsymbol{\beta}}\,[\boldsymbol{\sigma}] := \sum \sigma_j \varphi_{\boldsymbol{\beta}}, \quad \Phi_{\boldsymbol{\beta}}\,[\boldsymbol{\sigma}] := \sum \sigma_j\, \Phi_{\boldsymbol{\beta}}(z_j). \tag{5.6.15}$$

The following proposition reveals the relation between vertex fields with OPE exponentials.

Proposition 5.6.2. *We have* $\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\sigma}] = e^{*i\Phi_{\boldsymbol{\beta}}[\boldsymbol{\sigma}]}$.

Proof. By [78, Lemma 5.2] and (5.5.5),

$$e^{*i\Phi_{\boldsymbol{\beta}}[\boldsymbol{\sigma}]} = e^{i\varphi_{\boldsymbol{\beta}}[\boldsymbol{\sigma}]} e^{*i\Phi_{(0)}[\boldsymbol{\sigma}]} = C_{(0)}[\boldsymbol{\sigma}] e^{i\varphi_{\boldsymbol{\beta}}[\boldsymbol{\sigma}]} e^{\odot i\Phi_{(0)}[\boldsymbol{\sigma}]}.$$

Therefore all we need to show is

$$C_{(0)}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] e^{i\varphi_{\boldsymbol{\beta}}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]} = \frac{C_{(b)}[\boldsymbol{\sigma} + \boldsymbol{\beta}/2, \boldsymbol{\sigma}_* + \boldsymbol{\beta}/2]}{C_{(b)}[\boldsymbol{\beta}/2, \boldsymbol{\beta}/2]}. \tag{5.6.16}$$

Observe that $e^{i\varphi_{\boldsymbol{\beta}}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]}$ is a scalar field with respect to each q_k and is a differential with dimension $(-b\sigma_j, -b\sigma_{*j})$ at each z_j. Therefore by (5.5.4), conformal dimensions of both sides of (5.6.16) are identical. Thus it is enough to show (5.6.16) in the identity chart of $\mathcal{C}_r$.

Note that by construction,

$$\begin{aligned} e^{i\varphi_{\boldsymbol{\beta}}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]} = &\prod_{j<k} \Big(\Theta_\chi(r, q_k - z_j) \Theta_\chi(r, \bar{q}_k - z_j) \Big)^{\frac{\sigma_j \beta_k}{2}} \\ &\times \prod_{j<k} \Big(\Theta_\chi(r, \bar{q}_k - \bar{z}_j) \Theta_\chi(r, q_k - \bar{z}_j) \Big)^{\frac{\sigma_{*j} \beta_k}{2}}. \end{aligned}$$

Therefore proposition follows from (5.5.1). □

We now generalize Proposition 5.5.1 for a non-trivial background charge $\boldsymbol{\beta}$.

Proposition 5.6.3. *The vertex field* $\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]$ *satisfies Ward's OPE*

$$T_{\boldsymbol{\beta}}(\zeta) \mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] \sim \lambda_j \frac{\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]}{(\zeta - z_j)^2} + \frac{\partial_{z_j} \mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]}{\zeta - z_j}, \quad \text{as } \zeta \to z_j,$$

and similar OPE holds for $\overline{\mathcal{O}}_{\boldsymbol{\beta}}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]$.

Proof. We write $\mathcal{O}_{\boldsymbol{\beta}} \equiv \mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]$ and $\mathcal{O}_{(b)} \equiv \mathcal{O}_{(b)}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]$ throughout the proof. Since $\mathcal{O}_{\boldsymbol{\beta}}$ is a differential, it suffices to show the proposition in the identity chart of $\mathcal{C}_r$. By combining Proposition 5.3.1 and 5.6.1, we have

$$T_{\boldsymbol{\beta}} = T_{(b)} - (\jmath_{\boldsymbol{\beta}} - \jmath_{(b)})J_{(0)} - \frac{1}{2}(\jmath_{\boldsymbol{\beta}}^2 - \jmath_{(b)}^2) + ib\partial(\jmath_{\boldsymbol{\beta}} - \jmath_{(b)}).$$

Then it follows from Proposition 5.5.1 and $\mathcal{O}_{\boldsymbol{\beta}} = (\mathbf{E}\mathcal{O}_{\boldsymbol{\beta}}/\mathbf{E}\mathcal{O}_{(b)})\mathcal{O}_{(b)}$ that

$$\begin{aligned} T_{\boldsymbol{\beta}}(\zeta)\mathcal{O}_{\boldsymbol{\beta}} &\sim \frac{\mathbf{E}\mathcal{O}_{\boldsymbol{\beta}}}{\mathbf{E}\mathcal{O}_{(b)}} T_{(b)}(\zeta)\mathcal{O}_{(b)} - (\jmath_{\boldsymbol{\beta}} - \jmath_{(b)})J_{(0)}(\zeta)\mathcal{O}_{\boldsymbol{\beta}} \\ &\sim \frac{\lambda_j\, \mathcal{O}_{\boldsymbol{\beta}}}{(\zeta - z_j)^2} + \frac{\mathbf{E}\mathcal{O}_{\boldsymbol{\beta}}}{\mathbf{E}\mathcal{O}_{(b)}} \frac{\partial_{z_j}\mathcal{O}_{(b)}}{\zeta - z_j} - (\jmath_{\boldsymbol{\beta}} - \jmath_{(b)})J_{(0)}(\zeta)\mathcal{O}_{\boldsymbol{\beta}} \\ &\sim \frac{\lambda_j\, \mathcal{O}_{\boldsymbol{\beta}}}{(\zeta - z_j)^2} + \frac{\partial_{z_j}\mathcal{O}_{\boldsymbol{\beta}}}{\zeta - z_j} - \frac{\partial_{z_j}(\mathbf{E}\mathcal{O}_{\boldsymbol{\beta}}/\mathbf{E}\mathcal{O}_{(b)})}{\mathbf{E}\mathcal{O}_{\boldsymbol{\beta}}/\mathbf{E}\mathcal{O}_{(b)}} \frac{\mathcal{O}_{\boldsymbol{\beta}}}{\zeta - z_j} - (\jmath_{\boldsymbol{\beta}} - \jmath_{(b)})J_{(0)}(\zeta)\mathcal{O}_{\boldsymbol{\beta}}. \end{aligned}$$

On the other hand, by Wick's calculus together with (5.4.1),(5.4.2) and (3.1.32), we have

$$J_{(0)}(\zeta)\mathcal{O}_{\boldsymbol{\beta}} \sim \mathcal{O}_{\boldsymbol{\beta}}\, \mathbf{E}J_{(0)}(\zeta) \sum (i\sigma_j \Phi_{(0)}^+(z_j) - i\sigma_{*j}\Phi_{(0)}^-(z_j)) \sim -i\sigma_j \frac{\mathcal{O}_{\boldsymbol{\beta}}}{\zeta - z_j}.$$

Moreover by (5.5.1) and (5.6.5),

$$\frac{\partial_{z_j}\mathbf{E}\mathcal{O}_{\boldsymbol{\beta}}}{\mathbf{E}\mathcal{O}_{\boldsymbol{\beta}}} - \frac{\partial_{z_j}\mathbf{E}\mathcal{O}_{(b)}}{\mathbf{E}\mathcal{O}_{(b)}} = i\sigma_j(\jmath_{\boldsymbol{\beta}} - \jmath_{(b)}),$$

which completes the proof. □

Chapter 6

Eguchi-Ooguri and Ward's equations

In [50], Eguchi and Ooguri stated Ward's equation for tensor product $\mathcal{X}$ of primary fields (differentials in the OPE family) on a complex torus of genus one in terms of linear partial differential operators of first order with respect to the nodes of fields and with respect to the modular parameter τ. They explained the so-called Teichmüller term $\partial_\tau \mathbf{E}\,\mathcal{X}$ by means of the Virasoro generator L_0, the zeroth mode of the Virasoro field. In this section, we derive Ward's equation in a doubly connected domain not from Eguchi and Ooguri's physical argument but from some basic identities in function theory together with Wick's calculus.

6.1 Ward's functional and Ward's identities

We first introduce global Ward's functional and present the relation between a stress tensor and Ward's functional.

Let $W = (A_{\boldsymbol{\beta}}, \bar{A}_{\boldsymbol{\beta}})$ be a stress tensor for $\mathcal{F}_{\boldsymbol{\beta}}$. By Proposition 5.6.1, it is easy to see that $A = A_{\boldsymbol{\beta}}$ is continuous and real on the boundary. For a given open set $U \subset D_r$ and a smooth vector field v in $\overline{U}$ which is continuous up to

the boundary, Ward's functionals $W^{\pm}(v;U)$ are defined by

$$W^{+}(v;U)=\frac{1}{2\pi i}\int_{\partial U}vA-\frac{1}{\pi}\iint_{U}(\bar{\partial}v)A,\qquad W^{-}(v;U)=\overline{W^{+}(v;U)}.$$

We also write $W(v;U):=2\mathrm{Re}\,W^{+}(v;U)$. Then for any Fock space functional $\mathcal{X}$ with nodes in D_r, $\mathbf{E}[W^{+}(v;U)\mathcal{X}]$ is a well-defined correlation function in $D_{\mathrm{hol}}(v)$.

We will frequently use the following proposition. See [77, Propositions 5.3 and 5.10] for more details. The last equations in the following proposition are known as Ward's identities.

Proposition 6.1.1. *The following statements are equivalent:*

- *Ward's OPE holds for a Fock space field X;*
- *The residue form of Ward's identity for X*

$$\mathcal{L}_vX(z)=\frac{1}{2\pi i}\oint_{(z)}vAX(z)-\frac{1}{2\pi i}\oint_{(z)}\overline{v}\bar{A}X(z)$$

  holds on $D_{hol}(v)\cap U$ for all smooth vector field v;

- *For all $z\in D_{hol}(v)$,*

$$\mathbf{E}[\mathcal{Y}\mathcal{L}(v,U)X(z)]=\mathbf{E}[\mathcal{Y}W(v,U)X(z)]$$

  holds for all Fock space functionals $\mathcal{Y}$ with nodes outside supp(v).

We remark that the definition of Ward's functionals can be extended to meromorphic vector fields. For a meromorphic vector field v with poles $\{p_k\}$ in D_r, we define

$$W^{+}(v)=W^{+}(v;D_r):=\lim_{\varepsilon\to 0}W^{+}(v;D_r^{\varepsilon}),$$

where $D_r^{\varepsilon}=D_r\setminus\bigcup_k B(p_k,\varepsilon)$. Thus, we have

$$W^{+}(v)=W^{+}(v;D_r)=\frac{1}{2\pi i}\int_{\partial D_r}vA-\frac{1}{\pi}\iint_{D_r}(\bar{\partial}v)A,$$

with the interpretation of $\bar{\partial} v$ in the sense of distributions.

Proposition 6.1.2. *In the identity chart* $\mathrm{id}_{\bar{\mathcal{C}}_r}$, *we have*

$$2A_{\boldsymbol{\beta}}(\zeta) = W^+(v_\zeta) + W^-(v_{\bar{\zeta}}) + \frac{1}{\pi}\int_{[-\pi+ir,\pi+ir]} A_{\boldsymbol{\beta}}(\xi)\, d\xi,$$

where $(v_\zeta \| \mathrm{id}_{\bar{\mathcal{C}}_r})(z) = H(r, \zeta - z)$.

Proof. Recall that a stress tensor

$$A_{\boldsymbol{\beta}} := A_{(0)} + \Big(ib\partial - \jmath_{\boldsymbol{\beta}}\Big) J_{(0)},$$

is a holomorphic quadratic differential in $\mathcal{C}_r$, and real on the boundary. For $\zeta \in \mathcal{C}_r$, the reflected vector field $v_\zeta^\sharp$ is defined by

$$v_\zeta^\sharp(z) := \overline{v_\zeta(\bar{z})} = v_{\bar{\zeta}}(z).$$

Since v_ζ is meromorphic on $\mathcal{C}_r$ with poles at $\{\zeta + 2\pi m + 2irn : m, n \in \mathbb{Z}\}$, the reflected vector field $v_\zeta^\sharp$ is holomorphic on $\mathcal{C}_r$. Then by definition,

$$W^+(v_\zeta) = -\frac{1}{\pi}\int_{\mathcal{C}_r} (\bar{\partial} v_\zeta) A_{\boldsymbol{\beta}} + \frac{1}{2\pi i}\int_{\partial \mathcal{C}_r} v_\zeta A_{\boldsymbol{\beta}}, \quad W^+(v_\zeta^\sharp) = \frac{1}{2\pi i}\int_{\partial \mathcal{C}_r} v_\zeta^\sharp A_{\boldsymbol{\beta}}.$$

Also, since $A_{\boldsymbol{\beta}} = \bar{A}_{\boldsymbol{\beta}}$ on $\partial \mathcal{C}_r$, we have

$$W^-(v_{\bar{\zeta}}) = \overline{W^+(v_\zeta^\sharp)} = \overline{\frac{1}{2\pi i}\int_{\partial \mathcal{C}_r} v_{\bar{\zeta}}(\xi) A_{\boldsymbol{\beta}}(\xi)\, d\xi} = -\frac{1}{2\pi i}\int_{\partial \mathcal{C}_r} v_\zeta(\bar{\xi}) A_{\boldsymbol{\beta}}(\xi)\, d\xi.$$

Then by the facts $\bar{\partial} v_\zeta = -2\pi \delta_\zeta$, and $H(r, z + 2ir) = H(r, z) - 2i$, we have

$$\begin{aligned} W^+(v_\zeta) + W^-(v_{\bar{\zeta}}) &= -\frac{1}{\pi}\int_{\mathcal{C}_r} (\bar{\partial} v_\zeta) A_{\boldsymbol{\beta}} - \frac{1}{\pi}\int_{[-\pi+ir,\pi+ir]} A_{\boldsymbol{\beta}}(\xi)\, d\xi \\ &= 2A_{\boldsymbol{\beta}}(\zeta) - \frac{1}{\pi}\int_{[-\pi+ir,\pi+ir]} A_{\boldsymbol{\beta}}(\xi)\, d\xi. \end{aligned}$$

Proposition now follows. □

6.2 Eguchi-Ooguri equations

We now state and prove a version of Eguchi-Ooguri equations.

First let us consider the case $\boldsymbol{\beta} = \mathbf{0}$.

Lemma 6.2.1. *For any $\mathcal{X} \in \mathcal{F}_{(b)}$, in the $\mathcal{C}_r$-uniformization,*

$$\frac{1}{\pi} \oint_{[-\pi+ir,\pi+ir]} \mathbf{E} A(\zeta) \mathcal{X} \, d\zeta = \partial_r \mathbf{E} \mathcal{X}. \tag{6.2.1}$$

Proof. Let us denote by $\mathcal{W}$ the family of Fock space fields satisfying (6.2.1). We show $\mathcal{F}_{(b)} \subset \mathcal{W}$ through the following steps.

Claim 1. Tensor product of two GFFs is contained in $\mathcal{W}$.

Claim 2. All tensor products of GFFs and their derivatives are contained in $\mathcal{W}$.

Claim 3. $\mathcal{W}$ is closed under linear combinations over $\mathbb{C}$ and OPE products $*_n$.

Proof of Claim 1. Let us denote $A \equiv A_{(b)}$, $\Phi \equiv \Phi_{(b)}$. Recall that in the identity chart $\mathrm{id}_{\mathcal{C}_r}$, $A = -\frac{1}{2} J_{(0)} \odot J_{(0)} + ib\partial J_{(0)}$, and $\Phi = \Phi_{(0)}$. Since ∂J is exact, by the fundamental theorem of calculus, and the fact that all lifted correlation functions are 2π periodic, it suffices to show the case $b = 0$.

First, we show the claim for the ER boundary condition (i.e., the case that $\chi = \infty$). Of course, there is nothing to prove for $\mathcal{X} = \Phi(z)$. For $\mathcal{X} = \Phi(z_1)\Phi(z_2)$, by (5.1.1) and the fact $\partial_r \log \Theta = H'/2 + H^2/4$ (which follows from (3.1.12) and (3.1.31)), we have

$$\partial_r \mathbf{E}[\Phi(z_1)\Phi(z_2)] = -\mathrm{I}(z_1, z_2) + \mathrm{I}(\bar{z}_1, z_2) + \mathrm{I}(z_1, \bar{z}_2) - \mathrm{I}(\bar{z}_1, \bar{z}_2),$$

where $\mathrm{I}(z, w) := H'(z - w)/2 + H^2(z - w)/4$. On the other hand, by Wick's calculus and (5.1.3), we have

$$\mathbf{E}[A(\zeta)\Phi(z_1)\Phi(z_2)] = -\mathrm{II}_{z_1,z_2}(\zeta) + \mathrm{II}_{\bar{z}_1,z_2}(\zeta) + \mathrm{II}_{z_1,\bar{z}_2}(\zeta) - \mathrm{II}_{\bar{z}_1,\bar{z}_2}(\zeta),$$

where $\mathrm{II}_{z,w}(\zeta) := H(\zeta - z)H(\zeta - w)/4$. Note that by (3.1.41), $\mathrm{II}_{z,w}(\zeta)$ can be rewritten as

$$\begin{aligned}\mathrm{II}_{z,w}(\zeta) = \frac{1}{2}\mathrm{I}(z,w) + \frac{1}{4}\mathrm{III}_{z,w}(\zeta) \\ + \frac{1}{8}H^2(\zeta - z) + \frac{1}{8}H^2(\zeta - w) + \frac{3\zeta_r(\pi)}{2\pi},\end{aligned} \tag{6.2.2}$$

where

$$\mathrm{III}_{z,w}(\zeta) = H'(\zeta - z) + H'(\zeta - w) + (H(\zeta - w) - H(\zeta - z))H(w - z).$$

Combining all above, we obtain

$$\begin{aligned}\mathbf{E}[A(\zeta)\Phi(z_1)\Phi(z_2)] = \frac{1}{2}\partial_r \mathbf{E}[\Phi(z_1)\Phi(z_2)] \\ - \frac{1}{4}\Big(\mathrm{III}_{z_1,z_2}(\zeta) - \mathrm{III}_{\bar{z}_1,z_2}(\zeta) - \mathrm{III}_{z_1,\bar{z}_2}(\zeta) + \mathrm{III}_{\bar{z}_1,\bar{z}_2}(\zeta)\Big).\end{aligned}$$

Finally, observe that by (3.1.31) and (3.1.11), we have

$$\oint_{[-\pi+ir,\pi+ir]} \mathrm{III}_{z,w}(\zeta)d\zeta \equiv 0,$$

which leads to (6.2.1) for $\mathcal{X} = \Phi(z_1)\Phi(z_2)$ with ER boundary condition.

We now show the claim for general boundary condition with generic values of $\chi \in [0, \infty)$. For notational convenience, we add the superscript "ER" for the field with ER boundary condition, e.g., Φ^{ER}, A^{ER} etc. Then by (3.2.15) and the periodicity of H, we have

$$\begin{aligned}&\frac{1}{\pi}\oint \mathbf{E}[A(\zeta)\Phi(z_1)\Phi(z_2)]\,d\zeta \\ &= \frac{1}{\pi}\oint \mathbf{E}[A^{ER}(\zeta)\Phi^{ER}(z_1)\Phi^{ER}(z_2)]\,d\zeta - \frac{(z_1 - \bar{z}_1)(z_2 - \bar{z}_2)}{2\,(r+\chi)^2}.\end{aligned}$$

On the other hand,

$$\partial_r\Big(\mathbf{E}\Phi(z_1)\Phi(z_2) - \mathbf{E}\Phi^{ER}(z_1)\Phi^{ER}(z_2)\Big) = \partial_r\Big(\frac{1}{r+\chi}\Big)\frac{(z_1 - \bar{z}_1)(z_2 - \bar{z}_2)}{2}.$$

Combining above identities, the claim follows.

Proof of Claim 2. Again, there is nothing to prove for $\mathcal{X} = \Phi(z_1)\cdots\Phi(z_{2n-1})$. Therefore it is enough to show (6.2.1) for $\mathcal{X} = \Phi(z_1)\cdots\Phi(z_{2n})$. For $j \neq k$, let us denote

$$\mathcal{X}_{jk} = \Phi(z_1)\cdots\Phi(z_{k-1})\Phi(z_{k+1})\cdots\Phi(z_{j-1})\Phi(z_{j+1})\cdots\Phi(z_{2n}).$$

Then by Wick's formula and an induction argument, we have

$$\begin{aligned}\frac{1}{\pi}\int \mathbf{E}[A(\zeta)\Phi(z_1)\cdots\Phi(z_{2n})]\,d\zeta &= \frac{1}{\pi}\int \sum_{j\neq k}\mathbf{E}[A(\zeta)\Phi(z_j)\Phi(z_k)]\mathbf{E}[\mathcal{X}_{jk}]\,d\zeta \\ &= \sum_{j\neq k}\partial_r\mathbf{E}[\Phi(z_j)\Phi(z_k)]\mathbf{E}[\mathcal{X}_{jk}].\end{aligned}$$

On the other hand, by (5.1.2) and Leibniz's rule, we express the left-hand side of (6.2.1) as

$$\begin{aligned}\partial_r\mathbf{E}[\Phi(z_1)\cdots\Phi(z_{2n})] &= \partial_r\sum\prod_k \mathbf{E}[\Phi(z_{i_k})\Phi(z_{j_k})] \\ &= \sum\Big(\prod_k \mathbf{E}[\Phi(z_{i_k})\Phi(z_{j_k})]\Big)\Big(\sum_k \frac{\partial_r\mathbf{E}[\Phi(z_{i_k})\Phi(z_{j_k})]}{\mathbf{E}[\Phi(z_{i_k})\Phi(z_{j_k})]}\Big),\end{aligned}$$

where the outer sum is taken over all partitions of the set $\{1,\cdots,2n\}$ into disjoint pairs $\{i_l, j_l\}$. Rearranging the terms, we rewrite the above identity as

$$\partial_r\mathbf{E}[\Phi(z_1)\cdots\Phi(z_{2n})] = \sum_{j\neq k}\Big(\partial_r\mathbf{E}[\Phi(z_j)\Phi(z_k)]\Big)\Big(\sum\prod_l \mathbf{E}[\Phi(z_{i_l})\Phi(z_{j_l})]\Big),$$

where the sum is over all partitions of the set $\{1,\cdots,2n\}\setminus\{j,k\}$ into disjoint pairs $\{i_k, j_k\}$. Therefore again, by (5.1.2), we have (6.2.1) for $\mathcal{X} = \Phi(z_1)\cdots\Phi(z_{2n})$. Differentiating term by term, we complete the proof of Claim 2.

Proof of Claim 3. The first part of this claim is obvious. For the second part,

we consider the case that the field X is holomorphic. The other cases can be treated in a similar way. Suppose that we have the following OPE

$$X(\zeta)Y(z) = \sum C_n(z)(\zeta - z)^n, \quad \zeta \to z.$$

If both X and Y are in $\mathcal{W}$, then the term by term integration gives

$$\begin{aligned} &\frac{1}{\pi}\int_{[-\pi+ir,\pi+ir]} \mathbf{E}[A(\xi)X(\zeta)Y(z)]d\xi \\ &= \sum_n \Big(\frac{1}{\pi}\int_{[-\pi+ir,\pi+ir]} \mathbf{E}[A(\xi)C_n(z)]d\xi\Big)(\zeta - z)^n \end{aligned}$$

as $\zeta \to z$. On the other hand, differentiating term by term, we have

$$\partial_r \mathbf{E}[X(\zeta)Y(z)] = \sum \partial_r \mathbf{E}[C_n(z)](\zeta - z)^n.$$

Therefore, by comparing the coefficients of the two expansions above, we conclude $C_n \in \mathcal{W}$. □

Remark. One may apply similar arguments to the OPE family generated by a GFF Φ whose 2-point correlation function is given by the probabilistic interpolation (via Bernoulli distribution with probability $p \equiv p(r)$) of Green's functions with ER and Dirichlet boundary conditions, i.e.,

$$\begin{aligned} \mathbf{E}\Phi(\zeta)\Phi(z) &= 2\Big(p(r)G_r^{Diri}(\zeta, z) + (1 - p(r))G_r^{ER}(\zeta, z)\Big) \\ &= 2\log\Big|\frac{\Theta(r, \zeta - \bar{z})}{\Theta(r, \zeta - z)}\Big| - \frac{2\,p(r)}{r}\operatorname{Im}\zeta \operatorname{Im} z. \end{aligned}$$

Then the conclusion of Lemma 6.2.1 can be drawn if and only if $\partial_r(r/p(r)) = 1$. In other words, to derive Eguchi-Ooguri type equation (6.2.1), it is required that

$$p(r) = \frac{r}{r + \chi} \quad (\chi \geq 0), \tag{6.2.3}$$

which corresponds to the rational interpolation (3.2.15).

Next, we generalize Lemma 6.2.1. We write $(\boldsymbol{\sigma}, \boldsymbol{\sigma_*}) = (\sum \sigma_j \cdot z_j, \sum \sigma_{*j} \cdot z_j)$

for a double divisor satisfying (NC_0) and denote by $\mathcal{O} \equiv \mathcal{O}_{(b)}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]$ the associated multi-vertex field.

Proposition 6.2.2. *Let $\gamma = [-\pi + ir, \pi + ir]$. Then for any $\mathcal{X} \in \mathcal{F}_{(b)}$,*

$$\frac{1}{\pi} \oint_\gamma \mathbf{E} A(\zeta) \mathcal{O}\mathcal{X}\, d\zeta = \partial_r \mathbf{E}\, \mathcal{O}\mathcal{X}, \tag{6.2.4}$$

where all fields are evaluated in the identity chart of $\mathcal{C}_r$.

Proof. We present the proof for ER boundary conditions and leave to the reader the verification of the proposition for the general case.

Again, since $A_{(b)} = -\frac{1}{2} J_{(0)} \odot J_{(0)} + ib\partial J_{(0)}$, and ∂J is exact, it suffices to show the case $b = 0$. For notational convenience, we omit the subscript (0), e.g., $A \equiv A_{(0)}, C \equiv C_{(b)}$. Throughout the proof, we use symbols $\mathrm{I}(z, w), \mathrm{II}_{z,w}(\zeta)$ and $\mathrm{III}_{z,w}(\zeta)$ given in the proof of Lemma 6.2.1.

By definition, (6.2.4) is equivalent to

$$\begin{aligned} \frac{1}{\pi} \oint_\gamma \mathbf{E} A(\zeta) V^{\odot}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] \mathcal{X} d\zeta = \frac{\partial_r C[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]}{C[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]} \mathbf{E} V^{\odot}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] \mathcal{X} \\ + \partial_r \mathbf{E} V^{\odot}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] \mathcal{X}. \end{aligned} \tag{6.2.5}$$

Using similar arguments in the proof of Lemma 6.2.1, it suffices to show the identity (6.2.5) for tensor products of GFFs.

Let us first show that the identity (6.2.5) holds for trivial string $\mathcal{X} = 1$. By (5.5.1), the evaluation of $C[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]$ in the identity chart of $\mathcal{C}_r$ is given by

$$\begin{aligned} &\Theta'(0)^{\sum_j \frac{1}{2}(\sigma_j^2 + \sigma_{*j}^2)} \prod_j \Theta(z_j - \bar{z}_j)^{\sigma_j \sigma_{*j}} \\ &\times \prod_{j<k} \Theta(z_j - z_k)^{\sigma_j \sigma_k} \Theta(\bar{z}_j - z_k)^{\sigma_{*j} \sigma_k} \Theta(z_j - \bar{z}_k)^{\sigma_j \sigma_{*k}} \Theta(\bar{z}_j - \bar{z}_k)^{\sigma_{*j} \sigma_{*k}}. \end{aligned} \tag{6.2.6}$$

Combining (3.1.12) with (3.1.31) and (3.1.25), $\partial_r \log C[\boldsymbol{\sigma}, \boldsymbol{\sigma_*}]$ is given as

$$\begin{aligned}
&-\frac{3\zeta_r(\pi)}{2\pi}\sum(\sigma_j^2+\sigma_{*j}^2)+\sum\sigma_j\sigma_{*j}\mathrm{I}(z_j,\bar{z}_j)\\
&+\sum_{j<k}\sigma_j\sigma_k\mathrm{I}(z_j,z_k)+\sigma_{*j}\sigma_k\mathrm{I}(\bar{z}_j,z_k)+\sigma_j\sigma_{*k}\mathrm{I}(z_j,\bar{z}_k)+\sigma_{*j}\sigma_{*k}\mathrm{I}(\bar{z}_j,\bar{z}_k).
\end{aligned} \tag{6.2.7}$$

Differentiating (5.4.1) and (5.4.2), we have

$$\mathbf{E}J(\zeta)\Phi\left[\boldsymbol{\sigma},\boldsymbol{\sigma_*}\right]=-\frac{1}{2}\sum\sigma_j H(\zeta-z_j)+\sigma_{*j}H(\zeta-\bar{z}_j). \tag{6.2.8}$$

By Wick's formula and (6.2.8),

$$\begin{aligned}
&\mathbf{E}A(\zeta)V^{\odot}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]=\frac{1}{2}\Big(\mathbf{E}J(\zeta)\Phi\left[\boldsymbol{\sigma},\boldsymbol{\sigma_*}\right]\Big)^2\\
&=\frac{1}{8}\sum\sigma_j^2H^2(\zeta-z_j)+\sigma_{*j}^2H^2(\zeta-\bar{z}_j)+\sum\sigma_j\sigma_{*j}\mathrm{II}_{z_j,\bar{z}_j}(\zeta)\\
&+\sum_{j<k}\sigma_j\sigma_k\mathrm{II}_{z_j,z_k}(\zeta)+\sigma_{*j}\sigma_k\mathrm{II}_{\bar{z}_j,z_k}(\zeta)+\sigma_j\sigma_{*k}\mathrm{II}_{z_j,\bar{z}_k}(\zeta)+\sigma_{*j}\sigma_{*k}\mathrm{II}_{\bar{z}_j,\bar{z}_k}(\zeta).
\end{aligned} \tag{6.2.9}$$

Using (6.2.2), is easy to observe that by the neutrality condition (5.4.5), all terms of the form $H^2(\zeta-z_j)$ and $H^2(\zeta-\bar{z}_j)$ vanish. For instance, the coefficient of $H^2(\zeta-z_j)$ for each j is given as

$$\frac{1}{8}\Big(\sigma_j^2+\sigma_j\sigma_{*j}+\sum_{k\neq j}\sigma_j\sigma_k+\sigma_j\sigma_{*k}\Big)=\frac{\sigma_j}{8}\sum_k(\sigma_k+\sigma_{*k})=0.$$

The neutrality condition (5.4.5) also implies that the coefficient of the $3\zeta_r(\pi)/2\pi$ is given as

$$\sum_j\sigma_j\sigma_{*j}+\sum_{j<k}\sigma_j\sigma_k+\sigma_{*j}\sigma_k+\sigma_j\sigma_{*k}+\sigma_{*j}\sigma_{*k}=-\frac{1}{2}\sum_j(\sigma_j^2+\sigma_{*j}^2).$$

Combining all above, we obtain

$$\begin{aligned}&\mathbf{E}A(\zeta)V^{\odot}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]-\frac{1}{2}\partial_r\log C[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]\\&=\frac{1}{4}\sum_{j<k}\sigma_j\sigma_k\mathrm{III}_{z_j,z_k}(\zeta)+\sigma_{*j}\sigma_k\mathrm{III}_{\bar{z}_j,z_k}(\zeta)+\sigma_j\sigma_{*k}\mathrm{III}_{z_j,\bar{z}_k}(\zeta)+\sigma_{*j}\sigma_{*k}\mathrm{III}_{\bar{z}_j,\bar{z}_k}(\zeta).\end{aligned}$$

Now claim follows from the fact that $\oint_\gamma \mathrm{III}_{z,w}(\zeta)d\zeta\equiv 0$.

Next, we show that the identity (6.2.5) holds for the case $\mathcal{X}=\Phi$. Combining Wick's formula with (5.4.1) and (5.4.2), we have

$$\begin{aligned}\mathbf{E}V^{\odot}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]\Phi(z)&=i\,\mathbf{E}\Phi\,[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]\Phi(z)\\&=i\sum\sigma_j\log\frac{\Theta(z_j-\bar{z})}{\Theta(z_j-z)}+\sigma_{*j}\log\frac{\Theta(\bar{z}_j-\bar{z})}{\Theta(\bar{z}_j-z)}.\end{aligned}\tag{6.2.10}$$

Therefore by (3.1.12),

$$\partial_r\mathbf{E}V^{\odot}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]\Phi(z)=i\sum\sigma_j\Big(\mathrm{I}(z_j,\bar{z})-\mathrm{I}(z_j,z)\Big)+\sigma_{*j}\Big(\mathrm{I}(\bar{z}_j,\bar{z})-\mathrm{I}(\bar{z}_j,z)\Big).$$

On the other hand, by (6.2.9) and (6.2.10),

$$\mathbf{E}A(\zeta)V^{\odot}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]\Phi(z)=\Phi\,[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]+F(\zeta,z,\boldsymbol{z})-iG(\zeta,z,\boldsymbol{z}),\tag{6.2.11}$$

where

$$\begin{aligned}F(\zeta,z,\boldsymbol{z})&=\mathbf{E}A(\zeta)V^{\odot}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]\mathbf{E}V^{\odot}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]\Phi(z),\\G(\zeta,z,\boldsymbol{z})&=\mathbf{E}J(\zeta)\Phi(z)\mathbf{E}J(\zeta)\Phi\,[\boldsymbol{\sigma},\boldsymbol{\sigma}_*].\end{aligned}$$

Note that

$$\begin{aligned}\frac{1}{\pi}\oint_\gamma F(\zeta,z,\boldsymbol{z})\,d\zeta&=\frac{1}{\pi}\oint_\gamma\mathbf{E}A(\zeta)V^{\odot}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]d\zeta\,\mathbf{E}V^{\odot}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]\Phi(z)\\&=\frac{\partial_r C[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]}{C[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]}\mathbf{E}V^{\odot}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]\Phi(z).\end{aligned}\tag{6.2.12}$$

Thus it follows from the neutrality condition (5.4.5) and (6.2.8) that

$$\begin{aligned} G(\zeta, z, \boldsymbol{z}) &= \sum \sigma_j \Big(\mathrm{II}_{z_j,z}(\zeta) - \mathrm{II}_{z_j,\bar{z}}(\zeta) \Big) + \sigma_{*j} \Big(\mathrm{II}_{\bar{z}_j,z}(\zeta) - \mathrm{II}_{\bar{z}_j,\bar{z}}(\zeta) \Big) \\ &= \frac{1}{2} \sum \sigma_j \Big(\mathrm{I}(z_j, z) - \mathrm{I}(z_j, \bar{z}) \Big) + \sigma_{*j} \Big(\mathrm{I}(\bar{z}_j, z) - \mathrm{I}(\bar{z}_j, \bar{z}) \Big) \\ &+ \frac{1}{4} \sum \sigma_j \Big(\mathrm{III}_{z_j,z}(\zeta) - \mathrm{III}_{z_j,\bar{z}}(\zeta) \Big) + \sigma_{*j} \Big(\mathrm{III}_{\bar{z}_j,z}(\zeta) - \mathrm{III}_{\bar{z}_j,\bar{z}}(\zeta) \Big). \end{aligned}$$

Therefore we obtain

$$-\frac{i}{\pi} \oint_\gamma G(\zeta, z, \boldsymbol{z}) d\zeta = \partial_r \mathbf{E} V^{\odot}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] \Phi(z), \tag{6.2.13}$$

which shows (6.2.5) for $\mathcal{X} = \Phi$.

For $m \geq 2$, set $\mathcal{X} = \Phi(w_1) \cdots \Phi(w_m)$. We write

$$\begin{aligned} \mathcal{X}_k &= \Phi(w_1) \cdots \Phi(w_{k-1}) \Phi(w_{k+1}) \cdots \Phi(w_m), \\ \mathcal{X}_{jk} &= \Phi(w_1) \cdots \Phi(w_{k-1}) \Phi(w_{k+1}) \cdots \Phi(w_{j-1}) \Phi(w_{j+1}) \cdots \Phi(w_m). \end{aligned}$$

By Wick's formula,

$$\mathbf{E} A(\zeta) V^{\odot}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] \mathcal{X} = \mathbf{E} A(\zeta) \Big(\sum_{m=0}^{\infty} \frac{i^m}{m!} \Phi[\boldsymbol{\sigma}, \boldsymbol{\sigma_*}]^{\odot m} \Big) \mathcal{X} = \mathrm{I}(\zeta) + \mathrm{II}(\zeta) + \mathrm{III}(\zeta),$$

where

$$\begin{aligned} \mathrm{I}(\zeta) &= \frac{1}{2} \Big(\mathbf{E} J(\zeta) \Phi \left[\boldsymbol{\sigma}, \boldsymbol{\sigma_*} \right] \Big)^2 \mathbf{E} V^{\odot}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] \mathcal{X}, \\ \mathrm{II}(\zeta) &= -i \, \mathbf{E} J(\zeta) \Phi \left[\boldsymbol{\sigma}, \boldsymbol{\sigma_*} \right] \sum_{k=1}^{m} \mathbf{E} J(\zeta) \Phi(w_k) \mathbf{E} V^{\odot}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] \mathcal{X}_k, \\ \mathrm{III}(\zeta) &= \sum_{j \neq k} \mathbf{E} A(\zeta) \Phi(w_j) \Phi(w_k) \mathbf{E} V^{\odot}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] \mathcal{X}_{jk}. \end{aligned}$$

By (6.2.9) and the identity (6.2.5) for $\mathcal{X} = 1$,

$$\begin{aligned}\frac{1}{\pi}\oint_\gamma \mathrm{I}(\zeta)\,d\zeta &= \frac{1}{\pi}\oint_\gamma \mathbf{E}A(\zeta)V^{\odot}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]\,d\zeta\,\mathbf{E}V^{\odot}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]\mathcal{X}\\ &= \frac{\partial_r C[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]}{C[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]}\mathbf{E}V^{\odot}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]\mathcal{X}.\end{aligned}$$

Therefore to complete the proof, we need to show

$$\frac{1}{\pi}\oint_\gamma (\mathrm{II}+\mathrm{III})(\zeta)\,d\zeta = \partial_r \mathbf{E}V^{\odot}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]\mathcal{X}. \tag{6.2.14}$$

Note that by (6.2.13),

$$\begin{aligned}\frac{1}{\pi}\oint_\gamma \mathrm{II}(\zeta)\,d\zeta &= -\frac{i}{\pi}\sum_{k=1}^{m}\oint_\gamma G(\zeta,w_k,\boldsymbol{z})\,d\zeta\,\mathbf{E}V^{\odot}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]\mathcal{X}_k\\ &= \sum_{k=1}^{m}\partial_r \mathbf{E}V^{\odot}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]\Phi(w_k)\mathbf{E}V^{\odot}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]\mathcal{X}_k.\end{aligned}$$

Also by Claim 1 in the proof of Lemma 6.2.1,

$$\begin{aligned}\frac{1}{\pi}\oint_\gamma \mathrm{III}(\zeta)\,d\zeta &= \frac{1}{\pi}\int\sum_{j\neq k}\mathbf{E}[A(\zeta)\Phi(w_j)\Phi(w_k)]\mathbf{E}V^{\odot}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]\mathcal{X}_{jk}\,d\zeta\\ &= \sum_{j\neq k}\partial_r\mathbf{E}[\Phi(w_j)\Phi(w_k)]\mathbf{E}V^{\odot}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]\mathcal{X}_{jk}.\end{aligned}$$

Now (6.2.14) follows from a similar argument of Claim 2 in the proof of Lemma 6.2.1. □

Let us define *effective multi-vertex field* $\mathcal{O}^{\mathrm{eff}} \equiv \mathcal{O}^{\mathrm{eff}}_{(b)}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]$ as

$$\mathcal{O}^{\mathrm{eff}}_{(b)}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*] := \mathcal{O}_{(b)}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]Z_r, \tag{6.2.15}$$

where

$$Z_r = \Theta'(r,0)^{-1/3}(r+\chi)^{-1/2}. \tag{6.2.16}$$

In the physics literature, Z_r is called *CFT partition function*, see e.g., [63].

By Proposition 6.2.2, we obtain following corollary.

Corollary 6.2.3. *Let $\gamma = [-\pi + ir, \pi + ir]$. Then for any $\mathcal{X} \in \mathcal{F}_{(b)}$,*

$$\frac{1}{\pi}\oint_\gamma \mathbf{E}T(\zeta)\mathcal{O}^{\mathrm{eff}}\mathcal{X}\,d\zeta = \partial_r \mathbf{E}\mathcal{O}^{\mathrm{eff}}\mathcal{X}, \tag{6.2.17}$$

where all fields are evaluated in the identity chart of $\mathcal{C}_r$.

Proof. By Proposition 6.2.2, we have

$$\begin{aligned}\frac{1}{\pi}\oint_\gamma \mathbf{E}T(\zeta)\mathcal{O}^{\mathrm{eff}}\mathcal{X}\,d\zeta =& \Big(\frac{1}{\pi}\oint_\gamma \mathbf{E}T(\zeta)\mathcal{O}\mathcal{X}\,d\zeta\Big)Z_r \\ =& \Big(\frac{1}{\pi}\oint_\gamma \mathbf{E}A(\zeta)\mathcal{O}\mathcal{X}\,d\zeta\Big)Z_r + \Big(\frac{1}{\pi}\oint_\gamma \mathbf{E}T(\zeta)\,d\zeta\Big)\mathbf{E}\mathcal{O}\mathcal{X}Z_r \\ =& \Big(\partial_r \mathbf{E}\mathcal{O}\mathcal{X}\Big)Z_r + 2\mathbf{E}T\,\mathbf{E}\mathcal{O}\mathcal{X}Z_r.\end{aligned}$$

Note that by Proposition 5.3.1 and (3.1.25), we have

$$\mathbf{E}T = -\frac{1}{6}\frac{\Theta'''(r,0)}{\Theta'(r,0)} - \frac{1}{4(r+\chi)}. \tag{6.2.18}$$

Thus it follows from (6.2.16) that $2\mathbf{E}T = \partial_r \log Z_r$. Therefore we conclude that

$$\frac{1}{\pi}\oint_\gamma \mathbf{E}T(\zeta)\mathcal{O}^{\mathrm{eff}}\mathcal{X}\,d\zeta = \Big(\partial_r \mathbf{E}\mathcal{O}\mathcal{X}\Big)Z_r + \mathbf{E}\mathcal{O}\mathcal{X}\partial_r Z_r = \partial_r \mathbf{E}\mathcal{O}^{\mathrm{eff}}\mathcal{X},$$

which completes the proof. □

We now extend Lemma 6.2.1 to the OPE family $\mathcal{F}_{\boldsymbol{\beta}}$ with non-trivial background charges $\boldsymbol{\beta}$.

Proposition 6.2.4. *For any $\mathcal{X} \in \mathcal{F}_{\boldsymbol{\beta}}$, in the $\mathcal{C}_r$-uniformization,*

$$\frac{1}{\pi}\oint_\gamma \mathbf{E}A_{\boldsymbol{\beta}}(\zeta)\mathcal{X}\,d\zeta = \partial_r \mathbf{E}\mathcal{X}, \tag{6.2.19}$$

where $\gamma := [-\pi + ir, \pi + ir]$.

Proof. Again, it suffices to show that the GFF $\Phi_{\boldsymbol{\beta}}$ and tensor products of GFFs satisfy (6.2.19). Here we present proof of (6.2.19) for ER boundary conditions. The verification for generic values of χ follows from the same argument with slight modifications.

Recall that by Proposition 5.6.1,

$$A_{\boldsymbol{\beta}} := A_{(0)} + \Big(ib\partial - \jmath_{\boldsymbol{\beta}}\Big)J_{(0)}, \quad A_{(0)} := -\frac{1}{2}J_{(0)} \odot J_{(0)}, \quad \jmath_{\boldsymbol{\beta}} = \mathbf{E}J_{\boldsymbol{\beta}}.$$

Note that since $\oint \mathbf{E}\, \partial J_{(0)}\mathcal{X}$ always vanishes, it suffices to show the case $b = 0$.

We first prove (6.2.19) for $\mathcal{X} = \Phi_{\boldsymbol{\beta}}$. Applying the Wick's formula, we have

$$\mathbf{E}A_{\boldsymbol{\beta}}(\zeta)\Phi_{\boldsymbol{\beta}}(z) = -\jmath_{\boldsymbol{\beta}}(\zeta)\, \mathbf{E}J_{(0)}(\zeta)\Phi_{(0)}(z).$$

Therefore all we need to show is

$$-\frac{1}{\pi}\oint_{\gamma} \jmath_{\boldsymbol{\beta}}(\zeta)\, \mathbf{E}J_{(0)}(\zeta)\Phi_{(0)}(z)d\zeta = \partial_r\varphi_{\boldsymbol{\beta}}(z). \tag{6.2.20}$$

By (5.6.5) and (5.1.3), we have

$$\jmath_{\boldsymbol{\beta}}(\zeta)\, \mathbf{E}J_{(0)}(\zeta)\Phi_{(0)}(z) = \frac{i}{2}\sum \beta_k\Big(\mathrm{II}_{z,q_k}(\zeta) + \mathrm{II}_{z,\bar{q}_k}(\zeta) - \mathrm{II}_{\bar{z},q_k}(\zeta) - \mathrm{II}_{\bar{z},\bar{q}_k}(\zeta)\Big).$$

On the other hand, by (5.6.2) and (3.1.12),

$$\partial_r\mathbf{E}\Phi_{\boldsymbol{\beta}}(z) = \partial_r\varphi_{\boldsymbol{\beta}}(z) = -\frac{i}{2}\sum \beta_k\Big(\mathrm{I}(z,q_k) + \mathrm{I}(z,\bar{q}_k) - \mathrm{I}(\bar{z},q_k) - \mathrm{I}(\bar{z},\bar{q}_k)\Big).$$

Combining (6.2.2) with the neutrality condition $\sum \beta_k = 0$, the desired identity (6.2.20) follows.

We now prove (6.2.19) for $\mathcal{X} = \Phi_{\boldsymbol{\beta}}(z_1)\cdots\Phi_{\boldsymbol{\beta}}(z_n)$. Note that for $n = 2$, it immediately follows from Lemma 6.2.1 and (6.2.20) since

$$\partial_r\mathbf{E}\Phi_{\boldsymbol{\beta}}(z_1)\Phi_{\boldsymbol{\beta}}(z_2) = \partial_r\mathbf{E}\Phi_{(0)}(z_1)\Phi_{(0)}(z_2) + \varphi_{\boldsymbol{\beta}}(z_1)\partial_r\varphi_{\boldsymbol{\beta}}(z_2) + \varphi_{\boldsymbol{\beta}}(z_2)\partial_r\varphi_{\boldsymbol{\beta}}(z_1).$$

and

$$\begin{aligned}\oint_\gamma \mathbf{E}A_{\boldsymbol{\beta}}(\zeta)\Phi_{\boldsymbol{\beta}}(z_1)\Phi_{\boldsymbol{\beta}}(z_2)\,d\zeta &= \oint_\gamma \mathbf{E}A_{(0)}(\zeta)\Phi_{(0)}(z_1)\Phi_{(0)}(z_2)\,d\zeta \\ &\quad - \varphi_{\boldsymbol{\beta}}(z_2)\oint_\gamma \jmath_{\boldsymbol{\beta}}(\zeta)\,\mathbf{E}J_{(0)}(\zeta)\Phi_{(0)}(z_1)\,d\zeta \\ &\quad - \varphi_{\boldsymbol{\beta}}(z_1)\oint_\gamma \jmath_{\boldsymbol{\beta}}(\zeta)\,\mathbf{E}J_{(0)}(\zeta)\Phi_{(0)}(z_2)\,d\zeta.\end{aligned}$$

To handle the general case $n > 2$, for each $k \le n$, let us denote by

$$I_k := \sum \Big(\prod_{l=1}^{k} \varphi_{\boldsymbol{\beta}}(z_{i_l})\Big)\Big(\prod_{l=1}^{n-k} \Phi_{(0)}(z_{j_l})\Big).$$

Then we have

$$\Phi_{\boldsymbol{\beta}}(z_1)\cdots\Phi_{\boldsymbol{\beta}}(z_n) = I_0 + \cdots + I_n.$$

We write

$$\begin{aligned}A_k &:= \sum \prod_{l=1}^{k} \varphi_{\boldsymbol{\beta}}(z_{i_l})\,\partial_r \mathbf{E}\prod_{l=1}^{n-k}\Phi_{(0)}(z_{j_l}), \\ B_k &:= \sum_{j=1}^{n} \partial_r\varphi_{\boldsymbol{\beta}}(z_j)\sum\prod_{l=1}^{k-1}\varphi_{\boldsymbol{\beta}}(z_{i_l})\mathbf{E}\prod_{l=1}^{n-k}\Phi_{(0)}(z_{j_l}).\end{aligned}$$

By Leibniz's rule, we have $\partial_r \mathbf{E} I_k = A_k + B_k$.

On the other hand,

$$\frac{1}{\pi}\oint_\gamma \mathbf{E}A_{\boldsymbol{\beta}}(\zeta)I_k\,d\zeta = C_k + D_k,$$

where

$$C_k := \frac{1}{\pi}\oint_\gamma \mathbf{E}A_{(0)}(\zeta)I_k\,d\zeta, \quad D_k := -\frac{1}{\pi}\oint_\gamma \jmath_{\boldsymbol{\beta}}(\zeta)\mathbf{E}J_{(0)}(\zeta)I_k\,d\zeta.$$

Note that by Lemma 6.2.1, $A_k = C_k$. Moreover it follows from (6.2.20) that $B_{k+1} = D_k$. This completes the proof. □

6.3 Ward's equations

In this section, we derive Ward's equations (Theorem A) in terms of a stress tensor, Lie derivative operator and the modular parameter. Later, we combine these equations with the level two degeneracy equations for the one-leg operator to derive BPZ-Cardy equations.

Proof of Theorem A. Let us choose $v(z) = v_\zeta(z) = H(r, \zeta - z)$ with $\zeta \in \mathcal{C}_r$. By Proposition 6.1.2, Eguchi-Ooguri equations (Proposition 6.2.4), and Ward's identities (see Proposition 6.1.1), we obtain

$$\begin{aligned} 2\mathbf{E}A_{\boldsymbol{\beta}}(\zeta)\mathcal{X} &= \mathbf{E}W^+(v_\zeta)\mathcal{X} + \mathbf{E}W^-(v_{\bar{\zeta}})\mathcal{X} + \partial_r \mathbf{E}\mathcal{X} \\ &= \Big(\mathcal{L}^+_{v_\zeta} + \mathcal{L}^-_{v_{\bar{\zeta}}}\Big)\mathbf{E}\mathcal{X} + \partial_r \mathbf{E}\mathcal{X}, \end{aligned}$$

which completes the proof. □

As a corollary of Theorem A, we derive the following form of Ward's equations.

Corollary 6.3.1. *For a holomorphic differential $Y \in \mathcal{F}_{\boldsymbol{\beta}}$ with conformal dimension h, and $X = X_1 \cdots X_n$, $(X_j \in \mathcal{F}_{\boldsymbol{\beta}})$, we have*

$$2\mathbf{E}A_{\boldsymbol{\beta}} * Y(z)X = \mathbf{E}Y(z)\mathcal{L}^+_{v_z}X + \mathcal{L}^-_{v_{\bar{z}}}\mathbf{E}Y(z)X + \Big(2h\frac{\zeta_r(\pi)}{\pi} + \partial_r\Big)\mathbf{E}Y(z)X,$$

where all fields are evaluated in the identity chart of $\mathcal{C}_r$.

Proof. By Ward's OPE, we have the following singular part of the operator product expansion:

$$\mathrm{Sing}_{\zeta\to z}\, A_{\boldsymbol{\beta}}(\zeta)Y(z)X = \frac{h}{(\zeta - z)^2}Y(z)X + \frac{\partial_z}{\zeta - z}Y(z)X.$$

On the other hand, by Theorem A, we have

$$\begin{aligned} 2\mathbf{E}A_{\boldsymbol{\beta}}(\zeta)Y(z)X = {} & (hv'_\zeta(z) + v_\zeta(z)\partial_z + \partial_r)\mathbf{E}Y(z)X \\ & + \mathbf{E}Y(z)\mathcal{L}^+_{v_\zeta}X + \mathcal{L}^-_{v_{\bar{\zeta}}}\mathbf{E}Y(z)X. \end{aligned}$$

Subtracting the singular part from the both sides of the above and then taking the limit as $\zeta \to z$, we obtain

$$2\mathbf{E}A_{\boldsymbol{\beta}} * Y(z)X = \mathbf{E}Y(z)\mathcal{L}^{+}_{v_z}X + \mathcal{L}^{-}_{v_{\bar{z}}}\mathbf{E}Y(z)X + \partial_r \mathbf{E}Y(z)X + \lim_{\zeta\to z}\Big\{hv'_\zeta(z) - \frac{2h}{(\zeta-z)^2} + \Big(v_\zeta(z) - \frac{2}{\zeta-z}\Big)\partial_z\Big\}\mathbf{E}Y(z)X.$$

Corollary now follows from the asymptotic behavior (3.1.32) of the Loewner vector field $v_\zeta(z) = H(r, \zeta - z)$. □

6.4 Generalized Eguchi-Ooguri equations in a multiply connected domain

In this section we discuss the generalization of Eguchi-Ooguri and Ward's equations, key ingredients to construct CFT and its precise relation with SLE theory in a general multiply connected domain.

Let us consider OPE family $\mathcal{F}$ of GFF Φ in a multiply connected domain D. Here we only focus on the case that the central charge $c = 1$. Let us first show the following.

Proposition 6.4.1. *The GFF has a stress tensor*

$$A := -\frac{1}{2}J \odot J.$$

Proof. Since J is holomorphic 1-differential, it is clear that A is holomorphic quadratic differential. Note that

$$A(\zeta)\Phi(z) = -\mathbf{E}[J(\zeta)\Phi(z)]\, J(\zeta).$$

It follows from the definition of Greens' function that as $\zeta \to z$,

$$\Phi(\zeta)\Phi(z) \sim \log\frac{1}{|\zeta - z|^2}, \quad J(\zeta)\Phi(z) \sim -\frac{1}{\zeta - z}.$$

Thus we obtain the (Ward's) OPE

$$A(\zeta)\Phi(z) \sim \frac{J(z)}{\zeta - z}, \tag{6.4.1}$$

which completes the proof. □

Next, we show the following representation of a stress tensor in terms of Ward's functional. Recall that we call the upper-half plane minus a finite number of horizontal slits $[z_k^l, z_k^r]$ $(k = 1, \cdots, g)$ a chordal standard domain.

Proposition 6.4.2. *In the identity chart* id_D *of a chordal standard domain* D*, we have*

$$2A(\zeta) = W^+(v_\zeta) + W^-(v_{\bar{\zeta}}) - \frac{1}{2\pi i}\int_{\partial D}\Big(v_\zeta(\xi) - v_\zeta(\bar{\xi})\Big)A(\xi)\,d\xi,$$

where v_ζ *is the Loewner vector field.*

Proof. For $\zeta \in D$, let us define the reflected vector field $v_\zeta^\sharp$ by

$$v_\zeta^\sharp(z) := \overline{v_\zeta(\bar{z})} = v_{\bar{\zeta}}(z).$$

Note that since v_ζ is meromorphic on D with pole at ζ, the reflected vector field $v_\zeta^\sharp$ is holomorphic on D, which leads to

$$W^+(v_\zeta) = -\frac{1}{\pi}\int_D (\bar{\partial} v_\zeta)A + \frac{1}{2\pi i}\int_{\partial D} v_\zeta A, \quad W^+(v_\zeta^\sharp) = \frac{1}{2\pi i}\int_{\partial D} v_\zeta^\sharp A.$$

Since $A = \bar{A}$ on ∂D, we have

$$W^-(v_{\bar{\zeta}}) = \overline{W^+(v_\zeta^\sharp)} = \overline{\frac{1}{2\pi i}\int_{\partial D} v_{\bar{\zeta}}(\xi)A(\xi)\,d\xi} = -\frac{1}{2\pi i}\int_{\partial D} v_\zeta(\bar{\xi})A(\xi)\,d\xi.$$

Due to $\bar{\partial} v_\zeta = -2\pi\delta_\zeta$, we have

$$\begin{aligned} W^+(v_\zeta) + W^-(v_{\bar{\zeta}}) &= 2A(\zeta) + \frac{1}{2\pi i}\int_{\partial D} v_\zeta A + \overline{\frac{1}{2\pi i}\int_{\partial D} v_{\bar{\zeta}} A}. \\ &= 2A(\zeta) + \frac{1}{2\pi i}\int_{\partial D} \Big(v_\zeta(\xi) - v_\zeta(\bar{\xi})\Big) A(\xi)\, d\xi. \end{aligned}$$

□

Let us define the differential operator

$$\nabla_M := \sum_k v_p(z_k^l)\partial_{z_k^l} + v_p(\bar{z}_k^l)\bar{\partial}_{z_k^l} + v_p(z_k^r)\partial_{z_k^r} + v_p(\bar{z}_k^r)\bar{\partial}_{z_k^r}. \tag{6.4.2}$$

We remark that Dubédat's uses such differential operator to describe commuting SLEs in a multiply connected domains, see [46, Section 9].

We now show Eguchi-Ooguri type equation in a multiply connected domain, which we call the generalized Eguchi-Ooguri equation.

Proposition 6.4.3. *The generalized Eguchi-Ooguri equation holds:*

$$\frac{1}{2\pi i}\int_{\partial D} v_\zeta A + \overline{\frac{1}{2\pi i}\int_{\partial D} v_{\bar{\zeta}} A} + \nabla_M = 0, \tag{6.4.3}$$

i.e.,

$$\frac{1}{2\pi i}\int_{\partial D} \Big(v_\zeta(\xi) - v_\zeta(\bar{\xi})\Big) A(\xi)\, d\xi + \nabla_M = 0. \tag{6.4.4}$$

Proof. For $p \in \mathbb{R}$, let

$$P(z) := \frac{\partial}{\partial n}\Big|_{\zeta=p} G(\zeta, z)$$

be the Poisson kernel. It was shown by Izyurov and Kytölä [69] that the Hadamard's formula for Loewner chains is given by

$$\frac{d}{dt} G_{\Omega_t}(g_t(z_1), g_t(z_2))\Big|_{t=0} = -P(z_1)P(z_2). \tag{6.4.5}$$

It follows from the Loewner's equation

$$\frac{d}{dt} g_t(z) = -v_{\xi_t}(g_t(z)) \tag{6.4.6}$$

that the Hadamard's formula is equivalent to

$$\Big(\mathcal{L}_{v_p} + \nabla_M\Big)G(z_1, z_2) + 4\partial_\zeta G(\zeta, z_1)\partial_\zeta G(\zeta, z_2)\Big|_{\zeta=p} = 0. \tag{6.4.7}$$

Recall here that

$$\mathcal{L}_{v_p}G(z_1, z_2) = \Big(v_p(z_1)\partial_{z_1} + v_p(\bar{z}_1)\bar{\partial}_{z_1} + v_p(z_2)\partial_{z_2} + v_p(\bar{z}_2)\bar{\partial}_{z_2}\Big)G(z_1, z_2) \tag{6.4.8}$$

In general, for $\zeta \in D$,

$$\Big(\mathcal{L}^+_{v_\zeta} + \mathcal{L}^-_{v_{\bar{\zeta}}} + \nabla_M\Big)G(z_1, z_2) + 4\partial_\zeta G(\zeta, z_1)\partial_\zeta G(\zeta, z_2) = 0. \tag{6.4.9}$$

Therefore, we have

$$\nabla_M G(z_1, z_2) = -\Big(\mathcal{L}^+_{v_\zeta} + \mathcal{L}^-_{v_{\bar{\zeta}}}\Big)G(z_1, z_2) - 4\partial_\zeta G(\zeta, z_1)\partial_\zeta G(\zeta, z_2). \tag{6.4.10}$$

Using the argument presented in Lemma 6.2.4, it suffices to show that

$$\frac{1}{2\pi i}\int_{\partial D} v_\zeta \mathbf{E}A\mathcal{X} + \overline{\frac{1}{2\pi i}\int_{\partial D} v_{\bar{\zeta}}\mathbf{E}A\mathcal{X}} + \nabla_M \mathbf{E}\mathcal{X} = 0 \tag{6.4.11}$$

for $\mathcal{X} = \Phi(z_1)\Phi(z_2)$.

Note that as $\zeta \to z$,

$$v_\zeta(z) = \frac{2}{\zeta - z} + O(1), \quad \partial_\zeta G(\zeta, z) = -\frac{1}{2}\frac{1}{\zeta - z} + O(1). \tag{6.4.12}$$

Thus we have

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\partial D} v_\zeta(\xi)\mathbf{E}A(\xi)\Phi(z_1)\Phi(z_2)\, d\xi \\
&= -\frac{2}{\pi i}\int_{\partial D} v_\zeta(\xi)\partial_\xi G(\xi, z_1)\partial_\xi G(\xi, z_2)\, d\xi \\
&= 8\partial_\zeta G(\zeta, z_1)\partial_\zeta G(\zeta, z_2) + 2\Big(v_\zeta(z_1)\partial_{z_1} + v_\zeta(z_2)\partial_{z_2}\Big)G(z_1, z_2).
\end{aligned}$$

On the other hand,

$$\begin{aligned}\overline{\frac{1}{2\pi i}\int_{\partial D} v_{\bar\zeta}(\xi)\mathbf{E}A(\xi)\Phi(z_1)\Phi(z_2)\,d\xi} &= -\overline{\frac{2}{\pi i}\int_{\partial D} v_{\bar\zeta}(\xi)\partial_\xi G(\xi,z_1)\partial_\xi G(\xi,z_2)\,d\xi}\\ &= 2\overline{\Big(v_{\bar\zeta}(z_1)\partial_{z_1}+v_{\bar\zeta}(z_2)\partial_{z_2}\Big)}G(z_1,z_2)\\ &= 2\Big(v_{\zeta}(\bar z_1)\bar\partial_{z_1}+v_{\zeta}(\bar z_2)\bar\partial_{z_2}\Big)G(z_1,z_2).\end{aligned}$$

Therefore Eguchi-Ooguri equation (6.4.11) follows from (6.4.10) together with residue calculus.

□

Chapter 7

Annulus SLE martingale-observables

In this chapter, we present the precise relation between CFT and SLE in a doubly connected domain. In the first subsection, we define the one-leg operator $\Psi_{\boldsymbol{\beta}}[\boldsymbol{\tau}]$ and explain its implementation as a boundary condition changing operator. In Section 7.3, we show a version of BPZ-Cardy type equation to prove Theorem C. Further generalizations are discussed in the remaining subsections.

7.1 One-leg operator and BPZ equations

In this section we introduce the *one-leg operator* $\Psi_{\boldsymbol{\beta}}$, a specific form of the OPE exponential set to satisfy level two degeneracy equation. Throughout this section, we denote by $\boldsymbol{\beta} = \sum_k \beta_k \cdot q_k$ a given background charge.

By definition, for $n \in \mathbb{Z}$, Virasoro and current generators L_n, J_n are given as

$$L_n(z) := \frac{1}{2\pi i} \oint_{(z)} (\zeta - z)^{n+1} T_{\boldsymbol{\beta}}(\zeta)\, d\zeta, \tag{7.1.1}$$

$$J_n(z) := \frac{1}{2\pi i} \oint_{(z)} (\zeta - z)^{n} J_{\boldsymbol{\beta}}(\zeta)\, d\zeta, \tag{7.1.2}$$

respectively. First, let us recall the following proposition.

Proposition 7.1.1 (Cf. Proposition 7.5 [77]). *Let X be a Fock space field. Then any two of the following assertions imply the third one:*

- *X belongs to $\mathcal{F}(A, \bar{A})$;*
- *X is a $[\lambda, \lambda_*]$-differential;*
- *$L_n X = 0$ for all $n \geq 1$, $L_0 X = \lambda X, L_{-1} X = \partial X$, and similar equations hold for $\bar{X}$.*

Any field satisfying all conditions in the previous proposition is called a *(Virasoro) primary* field in $\mathcal{F}(A, \bar{A})$. By definition, a Virasoro primary field X is called *current primary* if $J_n X = J_n \bar{X} = 0$ for all $n \geq 1$ and $J_0 X = -iqX$, $J_0 \bar{X} = i\overline{q}_* \bar{X}$ for some numbers q and q_*, which are called charges. One of the main ingredients to connect CFT with SLE theory is the following *level two degeneracy equation.*

Proposition 7.1.2 (Cf. Proposition 11.2 [77]). *For a current primary field V with charges q,q_* in $\mathcal{F}_{\boldsymbol{\beta}}$, we have*

$$L_{-2} V = \frac{1}{2q^2} L_{-1}^2 V \tag{7.1.3}$$

provided $2q(b+q) = 1$.

Suppose that the real parameters a and b are related to the SLE parameter κ as

$$a = \sqrt{2/\kappa}, \qquad b = a(\kappa/4 - 1). \tag{7.1.4}$$

In the sequel, we sometimes use the following Kac labeling of the conformal weights or dimensions:

$$h_{r,s} = \frac{(r\kappa - 4s)^2 - (\kappa - 4)^2}{16\kappa}.$$

For a divisor $\boldsymbol{\tau} = \sum_{j=1}^{N} \tau_j \cdot z_j$ satisfying

$$a + \int \boldsymbol{\tau} = 0, \tag{7.1.5}$$

we define the one-leg operator $\Psi_{\boldsymbol{\beta}}[\boldsymbol{\tau}]$ as

$$\Psi_{\boldsymbol{\beta}}[\boldsymbol{\tau}] := \mathcal{O}_{\boldsymbol{\beta}}\Big[a \cdot z + \frac{\boldsymbol{\tau}}{2}, \frac{\boldsymbol{\tau}}{2}\Big]. \tag{7.1.6}$$

By definition, $\Psi_{\boldsymbol{\beta}}[\boldsymbol{\tau}]$ is a scalar field with respect to the each node q_j of the background charge $\boldsymbol{\beta}$. On the other hand, by (5.5.4), conformal dimensions of $\Psi_{\boldsymbol{\beta}}[\boldsymbol{\tau}]$ with respect to z and z_j are given by

$$h_z = \frac{6-\kappa}{2\kappa} = h_{1,2}, \qquad h_j = h_{*j} = \frac{\tau_j^2}{8} - \frac{\tau_j b}{2}. \tag{7.1.7}$$

It follows from (5.6.14) that

$$\mathbf{E}\Psi_{\boldsymbol{\beta}}[\boldsymbol{\tau}] = \frac{C_{(b)}[a \cdot z + \boldsymbol{\tau}/2 + \boldsymbol{\beta}/2, \boldsymbol{\tau}/2 + \boldsymbol{\beta}/2]}{C_{(b)}[\boldsymbol{\beta}/2, \boldsymbol{\beta}/2]}. \tag{7.1.8}$$

Therefore by (5.5.1), the evaluation of $\mathbf{E}\Psi_{\boldsymbol{\beta}}[\boldsymbol{\tau}]$ in the identity chart of the cylinder $\mathcal{C}_r$ is given as

$$\begin{aligned}
\mathbf{E}\Psi_{\boldsymbol{\beta}}[\boldsymbol{\tau}] = \Theta'_\chi(0)^{\frac{a^2}{2}+\sum \frac{\tau_j^2}{4}} & \prod_j \Big(\Theta_\chi(z-q_j)\Theta_\chi(z-\bar{q}_j)\Big)^{\frac{a\beta_j}{2}} \\
& \times \prod_j \Big(\Theta_\chi(z-z_j)\Theta_\chi(z-\bar{z}_j)\Big)^{\frac{a\tau_j}{2}} \Theta_\chi(z_j-\bar{z}_j)^{\frac{\tau_j^2}{4}} \\
& \times \prod_{j<k} \Big(\Theta_\chi(z_j-z_k)\Theta_\chi(\bar{z}_j-z_k)\Theta_\chi(z_j-\bar{z}_k)\Theta_\chi(\bar{z}_j-\bar{z}_k)\Big)^{\frac{\tau_j\tau_k}{4}} \\
& \times \prod_{j,k} \Big(\Theta_\chi(z_j-q_k)\Theta_\chi(\bar{z}_j-q_k)\Theta_\chi(z_j-\bar{q}_k)\Theta_\chi(\bar{z}_j-\bar{q}_k)\Big)^{\frac{\tau_j\beta_k}{4}}.
\end{aligned} \tag{7.1.9}$$

From now on, we call

$$Z_{\boldsymbol{\beta}} \equiv Z_{\boldsymbol{\beta}}(z, \boldsymbol{z}) := \mathbf{E}\Psi_{\boldsymbol{\beta}}[\boldsymbol{\tau}]$$

the *partition function* and set

$$\Lambda_{\boldsymbol{\beta}} \equiv \Lambda_{\boldsymbol{\beta}}(z, \boldsymbol{z}) := \kappa\, \partial_z \log Z_{\boldsymbol{\beta}}.$$

By (7.1.8) and (5.5.1), $\Lambda_{\boldsymbol{\beta}}$ is evaluated in the identity chart of $\mathcal{C}_r$ as

$$\begin{aligned} \Lambda_{\boldsymbol{\beta}} = &\sqrt{\frac{\kappa}{2}} \sum \frac{\beta_j}{2} \Big(H_\chi(r, z - z_j) + H_\chi(r, z - \bar{z}_j) \Big) \\ &+ \sqrt{\frac{\kappa}{2}} \sum \frac{\tau_j}{2} \Big(H_\chi(r, z - q_j) + H_\chi(r, z - \bar{q}_j) \Big). \end{aligned} \tag{7.1.10}$$

Now we derive the level two degeneracy equation for $\Psi_{\boldsymbol{\beta}}[\boldsymbol{\tau}]$.

Lemma 7.1.3. *The one-leg operator $\Psi \equiv \Psi_{\boldsymbol{\beta}}[\boldsymbol{\tau}]$ is a current primary field with charges $q = a, q_* = 0$ at z, i.e.,*

$$J_0\Psi = -ia\Psi, \quad J_0\overline{\Psi} = 0, \quad J_n\Psi = J_n\overline{\Psi} = 0 \quad (n \geq 1).$$

Proof. To show above equations, it is enough to show that as $\zeta \to z$,

$$J_{(0)}(\zeta)\Psi \sim -ia\frac{\Psi}{\zeta - z}, \quad J_{(0)}(\zeta)\overline{\Psi} \sim 0$$

hold in the identity chart. By Wick's formula, we have

$$\begin{aligned} J_{(0)}(\zeta)\Psi = &J_{(0)}(\zeta) \odot \Psi + ia\mathbf{E}[J_{(0)}(\zeta)\Phi^+_{(0)}(z)]\Psi \\ &- \sum \frac{\tau_j}{2}\mathbf{E}[J_{(0)}(\zeta)\tilde{\Phi}_{(0)}(z_j)]\Psi - \sum \frac{\beta_j}{2}\mathbf{E}[J_{(0)}(\zeta)\tilde{\Phi}_{(0)}(q_j)]\Psi \end{aligned}$$

and a similar expression for $\overline{\Psi}$. Now, lemma follows from (5.4.1) and (3.1.32). □

By Proposition 7.1.2, we immediately obtain the following level two de-

generacy equation of $\Psi \equiv \Psi_{\boldsymbol{\beta}}[\boldsymbol{\tau}]$.

Proposition 7.1.4. *If* $2a(a+b)=1$, *then*

$$T_{\boldsymbol{\beta}} *_z \Psi_{\boldsymbol{\beta}}[\boldsymbol{\tau}] = \frac{1}{2a^2}\partial_z^2 \Psi_{\boldsymbol{\beta}}[\boldsymbol{\tau}]. \tag{7.1.11}$$

Now we show BPZ equation, Theorem B.

Proof of Theorem B. By Proposition 7.1.4, we have

$$\frac{1}{a^2}\partial_z^2 \mathbf{E}\Psi_{\boldsymbol{\beta}}[\boldsymbol{\tau}]\mathcal{X} = 2\mathbf{E}(T_{\boldsymbol{\beta}} *_z \Psi_{\boldsymbol{\beta}}[\boldsymbol{\tau}])\mathcal{X}.$$

Then it follows from Corollary 6.3.1 that

$$2\mathbf{E}(T_{\boldsymbol{\beta}} *_z \Psi_{\boldsymbol{\beta}}[\boldsymbol{\tau}])\mathcal{X} = 2\mathbf{E}(A_{\boldsymbol{\beta}} *_z \Psi_{\boldsymbol{\beta}}[\boldsymbol{\tau}])\mathcal{X} + 2\mathbf{E}T_{\boldsymbol{\beta}}\,\mathbf{E}\Psi_{\boldsymbol{\beta}}[\boldsymbol{\tau}]\mathcal{X},$$

which completes the proof. □

We now discuss how the one-leg operator $\Psi_{\boldsymbol{\beta}}[\boldsymbol{\tau}]$ acts on Fock space fields in D_r as a boundary condition changing operator. Given marked boundary points $p \in \mathbb{R}$ and $\{q_j\}, \{\xi_j\} \in \partial\mathcal{C}_r$, set $\boldsymbol{\beta} = \sum \beta_j \cdot q_j$ and $\boldsymbol{\tau} = \sum \tau_j \cdot \xi_j$. By abuse of notation, for $z_j \in \mathcal{C}_r$, we sometimes use the same symbol $\boldsymbol{\tau}$ for $\sum \tau_j \cdot z_j$. From now on, we write $w_z = w(z)$, $w_p := w(p)$, $w_j := w(\xi_j)$ and $\tilde{w}_j := w(q_j)$ for a conformal transformation $w : D_r \to \mathcal{C}_r$. The insertion of field $\Psi_{\boldsymbol{\beta}}[\boldsymbol{\tau}]/\mathbf{E}\Psi_{\boldsymbol{\beta}}[\boldsymbol{\tau}]$ produces an operator $\mathcal{X} \mapsto \widehat{\mathcal{X}}$ on Fock space fields. It is given by

$$\partial\mathcal{X} \mapsto \partial\widehat{\mathcal{X}}, \quad \bar{\partial}\mathcal{X} \mapsto \bar{\partial}\widehat{\mathcal{X}}, \quad \alpha\mathcal{X} + \beta\mathcal{Y} \mapsto \alpha\widehat{\mathcal{X}} + \beta\widehat{\mathcal{Y}}, \quad \mathcal{X} \odot \mathcal{Y} \mapsto \widehat{\mathcal{X}} \odot \widehat{\mathcal{Y}}$$

and by the formula:

$$\begin{aligned}\widehat{\Phi}_{\boldsymbol{\beta}}(z) = \Phi_{\boldsymbol{\beta}}(z) &+ 2a \arg \Theta_\chi(w_p - w_z) \\ &+ \sum \tau_j \arg\Big(\Theta_\chi(w_j - w_z)\Theta_\chi(\bar{w}_j - w_z)\Big).\end{aligned} \tag{7.1.12}$$

When $\kappa = 4$, this insertion procedure produces specific height gap of GFF, which gives rise to the coupling with SLE$(4, \Lambda)$. For instance, in [63], Hagendorf, Bernard and Bauer studied coupling of GFF with Dirichlet boundary condition and SLE$(4, \Lambda)$ with one force point $q = \xi_1$ using the one-leg operator $\Psi \equiv \mathcal{O}[a \cdot p - a \cdot q, \mathbf{0}]$. In the case of multiple force points, the (multivalued) harmonic function $\widehat{\varphi} := \mathbf{E}\widehat{\Phi}$ in $\bar{\mathcal{C}}_r \setminus \{p, \xi_1, \cdots, \xi_N\}$ has piecewise Dirichlet boundary conditions having additional jump of $2\pi a$ at p and $2\pi\tau_j$ at ξ_j's. Due to the neutrality condition, we require that all jumps should add up to 0. This type of boundary condition of GFF was studied by Izyurov and Kytölä, see [69].

As in Section 2.3, we let

$$\widehat{\mathbf{E}}[\mathcal{X}] = \mathbf{E}\Big[e^{\odot ia\Phi^+_{(0)}(p) - \sum \frac{\tau_j}{2}\tilde{\Phi}_{(0)}(\xi_j)}\mathcal{X}\Big]. \tag{7.1.13}$$

Then it is easy to check that, for all $\mathcal{X} \in \mathcal{F}_{\boldsymbol{\beta}}$, we have

$$\widehat{\mathbf{E}}[\mathcal{X}] = \mathbf{E}[\widehat{\mathcal{X}}].$$

Examples. Let us consider the case with trivial background charge $\boldsymbol{\beta} = \mathbf{0}$. Then as an operator acting on $\mathcal{F}_{(b)}$, $\mathcal{X} \mapsto \widehat{\mathcal{X}}$ plays a similar role to imposing background charge $a \cdot z + \boldsymbol{\tau}$, see Section 5.6.

• The current field $\widehat{J}$ is a PS-form of order ib. We have

$$\widehat{J}(z) = J(z) + \frac{iaw'_z}{2}H_\chi(w_p - w_z) + \frac{iw'_z}{4}\sum \tau_j\Big(H_\chi(w_j - w_z) + H_\chi(\bar{w}_j - w_z)\Big).$$

• The Virasoro field $\widehat{T}$ is an S-form of order $c/12$. We have

$$\begin{aligned}\widehat{T}(z) &= -\frac{1}{2}\widehat{J} * \widehat{J} + ib\partial\widehat{J} \\ &= T - (\widehat{\jmath} - \jmath)J_{(0)} - \frac{1}{2}(\widehat{\jmath}^2 - \jmath^2) + ib\partial(\widehat{\jmath} - \jmath) \\ &= A_{(0)} - \widehat{\jmath}J_{(0)} + ib\partial J_{(0)} + \frac{c}{12}S_w + (w'_z)^2\mathbf{E}T - \frac{1}{2}(\widehat{\jmath}^2 - \jmath^2) + ib\partial(\widehat{\jmath} - \jmath),\end{aligned}$$

where $\jmath := \mathbf{E}J$ and $\widehat{\jmath} := \mathbf{E}\widehat{J}$. By Proposition 5.3.1, its evaluation in the identity chart of $\mathcal{C}_r$ is given as

$$\begin{aligned}\mathbf{E}\widehat{T}(z) = \frac{\zeta_r(\pi)}{2\pi} - \frac{1}{4(r+\chi)} \\ + \frac{1}{8}\Big[aH_\chi(p-z) + \frac{1}{2}\sum \tau_j\Big(H_\chi(\xi_j - z) + H_\chi(\bar{\xi}_j - z)\Big)\Big]^2 \\ + \frac{b}{2}\Big[aH'_\chi(p-z) + \frac{1}{2}\sum \tau_j\Big(H'_\chi(\xi_j - z) + H'_\chi(\bar{\xi}_j - z)\Big)\Big].\end{aligned}$$

• The multi-vertex field $\widehat{\mathcal{O}}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]$ whose nodes do not intersect with $\mathcal{S}_\Psi$ is a differential with conformal dimension (h_j, h_{*j}). We have

$$\begin{aligned}\mathbf{E}\widehat{\mathcal{O}}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] &= C_{(b)}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]\mathbf{E}V^{\odot}[a\cdot p + \boldsymbol{\beta}/2, \boldsymbol{\beta}/2]V^{\odot}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] \\ &= C_{(b)}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]e^{-\mathbf{E}\Phi_{(0)}[a\cdot p+\boldsymbol{\beta}/2, \boldsymbol{\beta}/2]\mathbf{E}\Phi_{(0)}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]}\end{aligned}$$

and its evaluation in the identity chart of $\mathcal{C}_r$ is given as (up to multiplicative constant)

$$\begin{aligned}\mathbf{E}\widehat{\mathcal{O}} = \Theta'_\chi(0)^{\sum \frac{1}{2}(\sigma_j^2+\sigma_{*j}^2)} \prod \Theta_\chi(p-z_j)^{a\sigma_j}\Theta_\chi(p-\bar{z}_j)^{a\sigma_{*j}}\Theta_\chi(z_j-\bar{z}_j)^{\sigma_j\sigma_{*j}} \\ \times \prod_{j,k}(\Theta_\chi(q_j-z_k)\Theta_\chi(\bar{q}_j-z_k))^{\beta_j\sigma_k/2}(\Theta_\chi(q_j-\bar{z}_k)\Theta_\chi(\bar{q}_j-\bar{z}_k))^{\beta_j\sigma_{*k}/2} \\ \times \prod_{j<k}\Theta_\chi(z_j-z_k)^{\sigma_j\sigma_k}\Theta_\chi(\bar{z}_j-z_k)^{\sigma_{*j}\sigma_k}\Theta_\chi(z_j-\bar{z}_k)^{\sigma_j\sigma_{*k}}\Theta_\chi(\bar{z}_j-\bar{z}_k)^{\sigma_{*j}\sigma_{*k}}.\end{aligned}$$

7.2 Null-vector equations

Throughout this section until further notice, we mainly consider the case that all marked points are on the outer boundary component. In what follows, we simply consider the one-leg operator of the form

$$\Psi_{\boldsymbol{\beta}}[\boldsymbol{\tau}] := \mathcal{O}_{\boldsymbol{\beta}}[a\cdot p + \boldsymbol{\tau},\ \mathbf{0}], \quad \boldsymbol{\beta} = \sum \beta_j \cdot q_j, \quad \boldsymbol{\tau} = \sum \tau_j \cdot \xi_j. \tag{7.2.1}$$

Here we require that

$$\sum \beta_j = a + \sum \tau_j = 0. \tag{7.2.2}$$

The associated partition function $Z_{\boldsymbol{\beta}}[\boldsymbol{\tau}] := \mathbf{E}\Psi_{\boldsymbol{\beta}}[\boldsymbol{\tau}]$ is evaluated in the $\mathcal{C}_r$ as

$$\begin{aligned} Z_{\boldsymbol{\beta}}[\boldsymbol{\tau}] = C_{\boldsymbol{\tau}} \prod_j \Theta_\chi(r, p-\xi_j)^{a\tau_j} \prod_j \Theta_\chi(r, p-q_j)^{a\beta_j} \\ \times \prod_{j<k} \Theta_\chi(r, \xi_j-\xi_k)^{\tau_j\tau_k} \prod_{j,k} \Theta_\chi(r, \xi_j-q_k)^{\tau_j\beta_k}, \end{aligned} \tag{7.2.3}$$

where $C_{\boldsymbol{\tau}} = \Theta'(r,0)^{\frac{a^2}{2}+\sum\frac{\tau_j^2}{2}}$. It follows from neutrality conditions that

$$\begin{aligned} Z_{\boldsymbol{\beta}}[\boldsymbol{\tau}] = C_{\boldsymbol{\tau}} \exp\Big(-\frac{(ap+\int z\, d(\boldsymbol{\tau}+\boldsymbol{\beta}))^2 - (\int z\, d\boldsymbol{\beta})^2}{4(r+\chi)}\Big) \\ \times \prod_j \Theta(r, p-\xi_j)^{a\tau_j} \prod_j \Theta(r, p-q_j)^{a\beta_j} \\ \times \prod_{j<k} \Theta(r, \xi_j-\xi_k)^{\tau_j\tau_k} \prod_{j,k} \Theta(r, \xi_j-q_k)^{\tau_j\beta_k}. \end{aligned} \tag{7.2.4}$$

We write λ_p, λ_j the (holomorphic) conformal dimension of $\Psi_{\boldsymbol{\beta}}$ with respect to p and ξ_j's. Then by (5.5.4), we have

$$\lambda_p = \frac{6-\kappa}{2\kappa}, \quad \lambda_j = \frac{\beta_j^2}{2} - \beta_j b. \tag{7.2.5}$$

It is obvious but noteworthy that in the expression (7.2.4), the sum of all the exponents vanishes due to the neutrality condition, i.e.,

$$\begin{aligned} &\frac{a^2}{2} + \sum_j \frac{\tau_j^2}{2} + \sum_j a\tau_j + \sum_k a\beta_k + \sum_{j<k} \tau_j\tau_k + \sum_{j,k} \tau_j\beta_k \\ &= \frac{1}{2}\Big(a + \sum_j \tau_j + \sum_k \beta_k\Big)^2 - \Big(\sum_k \beta_k\Big)^2 = 0. \end{aligned}$$

Therefore it follows from (3.1.8) that the partition function $Z_{\boldsymbol{\beta}}$ has a non-trivial limit when $r \to \infty$. Indeed, given $\chi \in [0, \infty]$, the partition functions

degenerate to (up to a multiplicative constant)

$$\begin{aligned} Z_\infty := \lim_{r\to\infty} Z_{\boldsymbol{\beta}}[\boldsymbol{\tau}] = \prod_j \sin^{a\tau_j}\Big(\frac{p-\xi_j}{2}\Big)\prod_j \sin^{a\beta_j}\Big(\frac{p-q_j}{2}\Big) \\ \times \prod_{j<k} \sin^{\tau_j\tau_k}\Big(\frac{\xi_j-\xi_k}{2}\Big)\prod_{j,k} \sin^{\tau_j\beta_k}\Big(\frac{\xi_j-q_k}{2}\Big). \end{aligned} \tag{7.2.6}$$

As a consequence of BPZ equation (Theorem B), we obtain a general form of null-vector equation for the partition function $Z_{\boldsymbol{\beta}}[\boldsymbol{\tau}]$. Here and subsequently, ∂ and $'$ stand for the differentiation with respect to p.

Corollary 7.2.1. *The partition function $Z_{\boldsymbol{\beta}}[\boldsymbol{\tau}]$ satisfies following partial differential equations:*

$$\begin{aligned} \partial_r Z_{\boldsymbol{\beta}}[\boldsymbol{\tau}] &= \frac{\kappa}{2}\partial^2 Z_{\boldsymbol{\beta}}[\boldsymbol{\tau}] \\ &\quad + \sum_j \Big(\lambda_j H'(r,p-\xi_j) - H(r,p-\xi_j)\partial_{\xi_j}\Big) Z_{\boldsymbol{\beta}}[\boldsymbol{\tau}] \\ &\quad - \sum_j H(r,p-q_j)\partial_{q_j} Z_{\boldsymbol{\beta}}[\boldsymbol{\tau}] + C(r,\boldsymbol{q}) Z_{\boldsymbol{\beta}}[\boldsymbol{\tau}], \end{aligned} \tag{7.2.7}$$

where

$$\begin{aligned} C(r,\boldsymbol{q}) &= -\frac{6}{\kappa}\frac{\zeta_r(\pi)}{\pi} + \frac{1}{2(r+\chi)} \\ &\quad - \frac{1}{16}\Big[\sum \beta_k \Big\{ H_\chi(r,q_k-p) + H_\chi(r,\bar{q}_k-p)\Big\}\Big]^2 \\ &\quad - \frac{b}{2}\sum \beta_k \Big\{ H'_\chi(r,q_k-p) + H'_\chi(r,\bar{q}_k-p)\Big\}. \end{aligned} \tag{7.2.8}$$

Proof. By applying trivial string $\mathcal{X}\equiv 1$ to Theorem B, we have

$$\partial_r Z_{\boldsymbol{\beta}}[\boldsymbol{\tau}] = \frac{\kappa}{2}\partial^2 Z_{\boldsymbol{\beta}}[\boldsymbol{\tau}] - \mathcal{L}^+_{v_p} Z_{\boldsymbol{\beta}}[\boldsymbol{\tau}] + C(r,\boldsymbol{q}) Z_{\boldsymbol{\beta}}[\boldsymbol{\tau}],$$

where

$$C(r,\boldsymbol{q}) = -2h_{1,2}\frac{\zeta_r(\pi)}{\pi} - 2\mathbf{E}T_{\boldsymbol{\beta}}.$$

Now lemma follows from Proposition 5.6.1 and the fact that

$$\mathcal{L}_{v_p}^{+} Z_{\boldsymbol{\beta}}[\boldsymbol{\tau}] = \sum \Big(v_p(\xi_j)\partial_{\xi_j} + \lambda_j v_p'(\xi_j) \Big) Z_{\boldsymbol{\beta}}[\boldsymbol{\tau}] + \sum v_p(q_j)\partial_{q_j} Z_{\boldsymbol{\beta}}[\boldsymbol{\tau}].$$

□

Example. Let us consider the simplest case that

$$\boldsymbol{\beta} = \mathbf{0}, \quad \boldsymbol{\tau} = -a \cdot q, \quad (q \in \mathbb{R})$$

and set

$$\Psi_* := \mathcal{O}[a \cdot p - a \cdot q, \mathbf{0}].$$

By (7.2.4), we have

$$Z_* := \mathbf{E}\Psi_* = \Theta_\chi'(r,0)^{\frac{2}{\kappa}} \Theta_\chi(r, p-q)^{-\frac{2}{\kappa}}. \tag{7.2.9}$$

By translation invariance, we have $\partial_q Z_* = -Z_*'$. Also by (7.2.5), the conformal dimension λ_q of Ψ_* with respect to q is given as

$$\lambda_q = \frac{a^2}{2} + ab = \frac{1}{2} - \frac{1}{\kappa}.$$

Therefore due to Corollary 7.2.1, Z_* satisfies

$$\partial_r Z_* = \frac{\kappa}{2} Z_*'' + H Z_*' + \Big(\frac{1}{2} - \frac{1}{\kappa} \Big) H' Z_* + C(r) Z_*,$$

where

$$C(r) = -\frac{6}{\kappa} \frac{\zeta_r(\pi)}{\pi} + \frac{1}{2(r+\chi)}.$$

7.3 BPZ-Cardy equations

In this section, we prove BPZ-Cardy type equation, a key ingredient to connect CFT with SLE theory and present the proof of Theorem C. See [31] for Cardy's version of BPZ-type equations.

Fix a marked point $p = \xi$ on the outer boundary. Recall the definitions

$$Z_{\boldsymbol{\beta}}(z, \boldsymbol{z}) := \mathbf{E}\Psi_{\boldsymbol{\beta}}(z, \boldsymbol{z}), \quad \Lambda_{\boldsymbol{\beta}}(z, \boldsymbol{z}) := \kappa\partial_z \log Z_{\boldsymbol{\beta}}(z, \boldsymbol{z}).$$

For a Fock space functional $\mathcal{X}$ whose nodes do not intersect marked boundary points, set

$$R_z := \widehat{\mathbf{E}}_z[\mathcal{X}] := \frac{\mathbf{E}[\Psi_{\boldsymbol{\beta}}(z, \boldsymbol{z})\mathcal{X}]}{Z_{\boldsymbol{\beta}}(z, \boldsymbol{z})},$$

and

$$R_\xi := \widehat{\mathbf{E}}_\xi[\mathcal{X}] = \mathbf{E}\Big[e^{\odot ia\Phi^+_{(0)}(\xi) - \sum \frac{\tau_j}{2}\tilde{\Phi}_{(0)}(\xi_j)}\mathcal{X}\Big].$$

Let us denote $\check{\mathcal{L}}_{v_z} := \mathcal{L}_{v_z}(\bar{D}_r \setminus (\{z\} \cup \mathcal{S}_\Psi))$, i.e., $\check{\mathcal{L}}_{v_z}\Psi Y(z)X := \Psi Y(z)\check{\mathcal{L}}_{v_z}X$ if the nodes of $\mathcal{X}$ do not intersect z and $\mathcal{S}_\Psi$.

Now we prove the following BPZ-Cardy equation.

Proposition 7.3.1. *For any $\mathcal{X} \in \mathcal{F}_{\boldsymbol{\beta}}$, in the $\mathcal{C}_r$-uniformization, we have*

$$\frac{1}{a^2}\partial_\xi^2 R_\xi + \Lambda_{\boldsymbol{\beta}}(\xi, \boldsymbol{\xi})\partial_\xi R_\xi = \check{\mathcal{L}}_{v_\xi}R_\xi + \Big(\sum v_\xi(\xi_j)\partial_{\xi_j} + \bar{v}_\xi(\xi_j)\bar{\partial}_{\xi_j}\Big)R_\xi + \partial_r R_\xi.$$

Here $\partial_\xi = \partial + \bar{\partial}$ is the operator of differentiation with respect to the real variable ξ whereas ∂_{q_j} denotes the one with respect to the complex variable ξ_j.

Proof. Let us write $\boldsymbol{\zeta} = S_{\mathcal{X}}$. Since $\Psi_{\boldsymbol{\beta}}(z, \boldsymbol{z})$ is a holomorphic differential with respect to z, we rewrite Theorem B as

$$\begin{aligned}\frac{1}{a^2}\partial_z^2\Big(R_z Z_{\boldsymbol{\beta}}(z, \boldsymbol{z})\Big) &= \Big(\mathcal{L}^+_{v_z} + \mathcal{L}^-_{v_{\bar{z}}}\Big)(\boldsymbol{\zeta}, \boldsymbol{z}, \boldsymbol{q})R_z Z_{\boldsymbol{\beta}}(z, \boldsymbol{z}) \\ &\quad + \Big(\partial_r + 2h_{1,2}\frac{\zeta_r(\pi)}{\pi} + 2\mathbf{E}T_{\boldsymbol{\beta}}\Big)R_z Z_{\boldsymbol{\beta}}(z, \boldsymbol{z}).\end{aligned}$$

By applying the trivial string $\mathcal{X} \equiv 1$ to above, we have

$$\frac{1}{a^2}\partial_z^2 Z_{\boldsymbol{\beta}}(z,\boldsymbol{z}) = \Big(\mathcal{L}^+_{v_z} + \mathcal{L}^-_{v_{\bar{z}}}\Big)(\boldsymbol{\zeta},\boldsymbol{z},\boldsymbol{q}) Z_{\boldsymbol{\beta}}(z,\boldsymbol{z}) + \Big(\partial_r + 2h_{1,2}\frac{\zeta_r(\pi)}{\pi} + 2\mathbf{E}T_{\boldsymbol{\beta}}\Big) Z_{\boldsymbol{\beta}}(z,\boldsymbol{z}).$$

Subtracting above two equations, we obtain

$$\frac{1}{a^2}\partial_z^2 R_z + \Lambda_{\boldsymbol{\beta}}(z,\boldsymbol{z})\partial_z R_z = \Big(\mathcal{L}^+_{v_z} + \mathcal{L}^-_{v_{\bar{z}}}\Big)(\boldsymbol{\zeta},\boldsymbol{z},\boldsymbol{q}) R_z + \partial_r R_z.$$

We now take the limit $z \to \xi$ and $z_j \to \xi_j$. Then, due to $\partial_\xi = \partial + \bar{\partial}$, we obtain

$$\frac{1}{a^2}\partial_\xi^2 R_\xi + \Lambda_{\boldsymbol{\beta}}(\xi,\boldsymbol{\xi})\partial_\xi R_\xi = \mathcal{L}_{v_\xi}(\boldsymbol{\zeta},\boldsymbol{q}) R_\xi + \mathcal{L}_{v_\xi}(\boldsymbol{\xi}) R_\xi + \partial_r R_\xi.$$

Now proposition follows from the fact R_ξ is a scalar field with respect to each ξ_j. □

Now we prove one of our main theorems.

Proof of Theorem C. Let us denote $R_\xi = \widehat{\mathbf{E}}_\xi[\mathcal{X}]$. Then

$$M_t = m\Big(r-t, \xi_t, \tilde{g}_t(\boldsymbol{\xi})\Big), \quad m(r,\xi,\boldsymbol{\xi},t) = \Big(R(r,\xi,\boldsymbol{\xi}) \| \tilde{g}_t^{-1}\Big).$$

Hence, by Itô's formula, we have

$$dM_t = -\partial_r m\, dt + \partial_\xi m\, d\xi_t + \partial_\xi^2 m \frac{d\langle \xi\rangle_t}{2} + (L_t R \| \tilde{g}_t{}^{-1})\, dt + \sum_{j=1}^N \Big[\partial_{\xi_j} m\, d\xi_j(t) + \bar{\partial}_{\xi_j} m\, d\overline{\xi_j(t)}\Big],$$

where

$$L_t := \frac{d}{ds}\Big|_{s=0}\Big(R(r-t,\xi_t,\tilde{g}_t(\boldsymbol{\xi})) \| \tilde{g}_{t+s}^{-1}\Big).$$

Let $f_{s,t} = \tilde{g}_{t+s}^{-1} \circ \tilde{g}_t^{-1}$. Then since the time-dependent flow $f_{s,t}$ satisfies the

ODE

$$\frac{d}{ds} f_{s,t} = -v_{\xi_{t+s}} f_{s,t}, \quad v_\xi(z) = H(r, \xi - z),$$

we obtain

$$f_{s,t} = \mathrm{id} - s\, v_{\xi_t} + o(s).$$

From the fact that the fields in $\mathcal{F}_{\boldsymbol{\beta}}$ depends smoothly on local charts, it follows

$$L_t = -\Big(\check{\mathcal{L}}_{v_{\xi_t}} R_\xi \| \tilde{g}_t^{-1}\Big).$$

By the Loewner's equation $d\xi_j(t) = -v_{\xi_t}(\xi_j(t))$, it follows from Proposition 7.3.1 that the drift term of dM_t vanishes. This completes the proof of theorem. □

7.4 Examples

In this section, we present several examples of annulus SLE martingale-observables.

7.4.1 Bosonic observable

The simplest examples of SLE martingale-observables are the 1-point function $M(z)$ and the 2-point function $M(z_1, z_2)$ of the bosonic field $\widehat{\Phi}_{(b)}$ when $\boldsymbol{\beta} = \mathbf{0}$. For instance, when $\chi = \infty$, for all κ,

$$dM_t(z) = -\sqrt{2}\,\mathrm{Im}\, H(w_t(z))\, dB_t + \Big(a + \sum \tau_j\Big) \mathrm{Im}\,\Big(\frac{H^2}{2} + H'\Big)(w_t(z))\, dt.$$

Due to the neutrality condition (7.1.5), M_t is driftless. For the 2-point function, one can easily show that $dM_t(z_1, z_2)$ is expressed as

$$\begin{aligned}
&-\sqrt{2}\,\Big(M_t(z_1)\,\mathrm{Im}\, H(w_t(z_2)) + M_t(z_2)\,\mathrm{Im}\, H(w_t(z_1))\Big)\, dB_t \\
&+ \Big(a + \sum \tau_j\Big)\Big\{M_t(z_1)\Big(\frac{H^2}{2} + H'\Big)(w_t(z_2)) + M_t(z_2)\Big(\frac{H^2}{2} + H'\Big)(w_t(z_1))\Big\}\, dt,
\end{aligned}$$

so the drifts disappear due to the neutrality condition again. In [69], Izyurov and Kytölä studied these types of martingale-observables with Dirichlet boundary condition.

Let us show the above equations by direct computations.

Proof. Recall that ξ_t satisfies

$$d\xi_t = \sqrt{\kappa}\, dB_t + \sqrt{\frac{\kappa}{2}} \sum \tau_k \operatorname{Re} H(r-t, \xi_t - \tilde{g}_t(\xi_k))\, dt.$$

Set $w_t(z) := \tilde{g}_t(z) - \xi_t$. Then by Loewner equation, we have

$$dw_t(z) = \Big[H(r-t, w_t(z)) - \sqrt{\frac{\kappa}{2}} \sum \tau_k \operatorname{Re} H(r-t, \xi_t - \tilde{g}_t(\xi_k)) \Big]\, dt - \sqrt{\kappa}\, dB_t.$$

Then we have $\operatorname{Im} N_t = M_t$, where

$$\begin{aligned} N_t = &-2b \log w_t'(z) + 2a \log \Theta(r-t, w_t(z)) \\ &+ \sum \tau_k \Big\{ \log \Theta(r-t, w_t(\xi_k) - w_t(z)) + \log \Theta(r-t, \overline{w_t(\xi_k)} - w_t(z)) \Big\}. \end{aligned}$$

On the other hand, by Itô's formula, we have

$$d \log w_t'(z) = H'(r-t, w_t(z))\, dt$$

and

$$\begin{aligned} d \log \Theta(r-t, w_t(z)) = &-\frac{1}{\sqrt{2a}} H(r-t, w_t(z))\, dB_t \\ &- \frac{1}{2a} H(r-t, w_t(z)) \sum \tau_k \operatorname{Re} H(r-t, w_t(\xi_k))\, dt \\ &+ \frac{1}{4} H^2(r-t, w_t(z))\, dt + \Big(\frac{\kappa}{4} - \frac{1}{2} \Big) H'(r-t, w_t(z))\, dt. \end{aligned}$$

Furthermore, by pseudo addition formula (3.1.41),

$$\begin{aligned} & d\log\Theta(w_t(z_1)-w_t(z_2)) \\ &= \frac{1}{4}\Big(H(w_t(z_1))-H(w_t(z_2))\Big)^2\,dt \\ &+ \frac{1}{2}H'(w_t(z_1))\,dt + \frac{1}{2}H'(w_t(z_2))\,dt + \frac{3}{\pi}\zeta_r(\pi)\,dt, \end{aligned} \tag{7.4.1}$$

which leads to

$$\begin{aligned} & d\sum\tau_k\Big(\log\Theta(w_t(\xi_k)-w_t(z)) + \log\Theta(r-t,\overline{w_t(\xi_k)}-w_t(z))\Big) \\ &= \Big(\sum\tau_k\Big)\Big[\Big(\frac{H^2}{2}+H'\Big)(w_t(z)) + \frac{6}{\pi}\zeta_r(\pi)\Big]\,dt \\ &+ \sum\frac{\tau_k}{2}\Big[\mathrm{Re}\,(H^2+2H')(w_t(\xi_k)) - 2H(w_t(z))\mathrm{Re}\,H(w_t(\xi_k))\Big]\,dt. \end{aligned}$$

Combining all of the above equations,

$$\begin{aligned} dN_t &= -\sqrt{2}H(w_t(z))\,dB_t + \sum\tau_k\,\mathrm{Re}\Big(\frac{H^2}{2}+H'\Big)(w_t(\xi_k))\,dt \\ &+ \Big(\sum\tau_k\Big)\frac{6}{\pi}\zeta_r(\pi)\,dt + \Big(a+\sum\tau_k\Big)\Big(\frac{H^2}{2}+H'\Big)(w_t(z))\,dt. \end{aligned}$$

Then since $\zeta_r(\pi)$ is real, we obtain dM_t.

Next, let us compute $M(z_1,z_2)$. By definition,

$$M(z_1,z_2) = 2G^{ER}_{D_r}(z_1,z_2) + M(z_1)M(z_2), \tag{7.4.2}$$

where $M(z)$ is 1-point function of bosonic field. Therefore, we have

$$M_t(z_1,z_2) = 2\log\Big|\frac{\Theta(r-t,w_t(z_1)-\overline{w_t(z_2)})}{\Theta(r-t,w_t(z_1)-w_t(z_2))}\Big| + M_t(z_1)M_t(z_2).$$

Then by (7.4.1),

$$d\log\Big|\frac{\Theta(r-t,w_t(z_1)-\overline{w_t(z_2)})}{\Theta(r-t,w_t(z_1)-w_t(z_2))}\Big| = -\mathrm{Im}\,H(w_t(z_1))\mathrm{Im}\,H(w_t(z_2))\,dt.$$

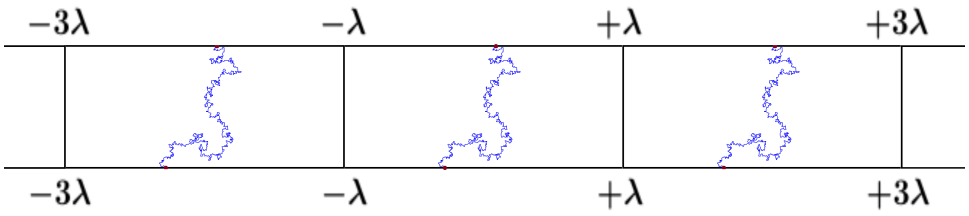


Figure 7.1: Level lines of compactified GFF

Therefore we have

$$dM_t(z_1, z_2) = M_t(z_1)\, dM_t(z_2) + M_t(z_2)\, dM_t(z_1),$$

which completes the proof. □

7.4.2 Schramm's observable

Let us consider the case that there is only one force point q on $\mathbb{R}_r$. Here due to neutrality condition, we impose the symmetric charge $(-a/2, -a/2)$ at q.

By (7.1.10), the drift function Λ is given by

$$\Lambda(p, q) = H_I(r, p - \operatorname{Re} q) + \frac{p - \operatorname{Re} q}{r + \chi}. \tag{7.4.3}$$

Thus the driving process ξ_t is given by

$$d\xi_t = \sqrt{\kappa}\, dB_t + \left(H_I(r-t, \xi_t - \operatorname{Re} \tilde{g}_t(q)) + \frac{\xi_t - \operatorname{Re} \tilde{g}_t(q)}{r + \chi - t} \right) dt, \quad \xi_0 = p. \tag{7.4.4}$$

Note that in the degenerate limit $r \to \infty$, the associated SLE(κ, Λ) converges to the law of radial SLE(κ). We remark that when $\kappa = 4$, the SLE$(4, \Lambda)$ corresponds to the scaling limit of level lines of *compactified* GFF, see Figure 7.1.

Proposition 7.4.1. *For $\chi = 0$ let η be the trace of SLE(κ, Λ), where Λ is given by* (7.4.3)*. Then η a.s. ends at the force point q. Furthermore, when*

$\kappa = 4$, the probability $\mathbf{P}(z)$ that $z \in \mathbb{S}_r$ is left to η is given by

$$\mathbf{P}(z) = \frac{1}{\pi}\Big(\arg \tilde{\Theta}(r, z-p) - \arg \tilde{\Theta}(r, z-q)\Big). \tag{7.4.5}$$

Proof. By Loewner's equation, the angle-difference process $X_t := \xi_t - \operatorname{Re} \tilde{g}_t(q)$ satisfies

$$dX_t = \sqrt{\kappa}\, dB_t + \frac{X_t\, dt}{r + \chi - t}, \quad X_0 = p - \operatorname{Re} q. \tag{7.4.6}$$

Therefore the process X_t is realized as a Brownian bridge starting from X_0, which reaches 0 at time $t = r + \chi$, see e.g., [107]. Thus when $\chi = 0$, we have $\lim_{t\uparrow r} X_t = 0$, which means η a.s. ends at the force point q.

For $\kappa = 4$, by Theorem C, the bosonic observable

$$M_t(z) := \frac{1}{\pi}\Big(\arg \tilde{\Theta}(r_t, w_t(z)) - \arg \tilde{\Theta}(r, w_t(z) - w_t(q))\Big)$$

is a (bounded) martingale. On the other hand, note that $M = M_0$ is a harmonic function satisfying the boundary condition

$$M(z) = \begin{cases} 0 & \text{if} \quad z \in (0,p) \cup (ir, q), \\ 1 & \text{if} \quad z \in (p, 2\pi) \cup (q, 2\pi + ir). \end{cases}$$

Let us write E the event that z is left to η. Then we have

$$\lim_{t\uparrow r} M_t(z) = \begin{cases} 0 & \text{on } E, \\ 1 & \text{otherwise.} \end{cases}$$

Then proposition follows from optional stopping theorem. □

For the graph of $\mathbf{P}(z)$, see Figure 7.2.

7.4.3 Restriction observable

First we shall present a basic example of one-leg operator which guarantees the positivity of $\widehat{\mathbf{E}}T$. This value is closely related to Friedrich-Werner's

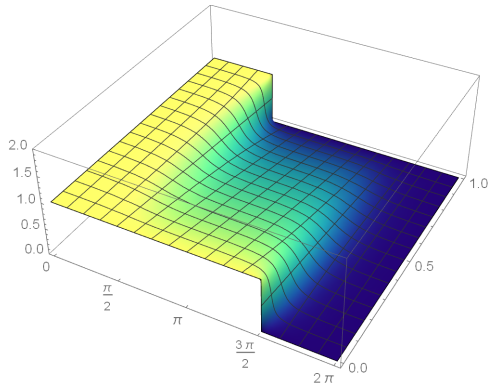


Figure 7.2: The plot displays the left passage probability $\mathbf{P}(z)$ given by (7.4.5), where $p = \frac{3}{2}\pi$, $q = \frac{1}{2}\pi + ir$ and $r = 1$

formula of SLE(8/3) [58, 59]. See also [79] for its dipolar SLE version.

Proposition 7.4.2. *Suppose $\kappa \leq 4$ and let $\Psi = \mathcal{O}[a \cdot \xi + \boldsymbol{\tau}, \mathbf{0}]$ for Dirichlet boundary conditions (i.e., $\chi = 0$). Moreover assume that $\tau_k \geq 0$ for every $\xi_k \in \mathbb{R}$, and $\xi_k \leq 0$ for every $\xi_k \in \mathbb{R}_r$. Then for any $x \in \mathbb{R}$,*

$$\widehat{\mathbf{E}}T(x) = \mathbf{E}\Psi T(x)/\mathbf{E}\Psi > 0.$$

Proof. By the neutrality condition, we have

$$\widehat{\mathbf{E}}T = \frac{1}{8}\Big(a\tilde{H}(x) - \sum \tau_k \tilde{H}(\xi_k - x)\Big)^2 + \frac{b}{2}\Big(aH'(x) + \sum \tau_k H'(\xi_k - x)\Big).$$

Note that by the assumption $\kappa \leq 4$, we have $b \leq 0$. Therefore it suffices to show

$$aH'(x) + \sum \tau_k H'(\tau_k - x) < 0. \tag{7.4.7}$$

By (3.1.31) and (3.1.22), we have

$$H'(x) - H'(q - x) = 2\zeta_r'(x) - 2\zeta_r'(q - x) = 2\wp_r(q - x) - 2\wp_r(x),$$

where $\wp_r$ denotes Weierstrass elliptic function with basic periods $(2\pi, 2ir)$.

Therefore (7.4.7) is rewritten as

$$a\wp_r(x) + \sum \tau_k \wp_r(\xi_k - x) > 0.$$

Set $e_1 := \wp_r(\pi)$, $e_2 := \wp_r(\pi + ir)$ and $e_3 := \wp_r(ir)$.

Recall that in the cylinder $\mathcal{C}_r$, $\wp_r$ is meromorphic function having double pole only at the origin and its derivative vanishes only at π, ir and $\pi + ir$. Therefore by periodicity of $\wp_r$, it is clear that

$$\min\{\wp(x) : x \in \mathbb{R}\} = \wp(\pi) = e_1$$

and

$$\max\{\wp(y + ir) : y \in \mathbb{R}\} = \wp(\pi + ir) = e_2.$$

By assumptions on the divisor and (3.1.24), we have

$$\begin{aligned} a\wp(x) + \sum \tau_k \wp(\xi_k - x) &= a\wp(x) + \sum_{\xi_k \in \mathbb{R}} \tau_k \wp(\xi_k - x) + \sum_{\xi_k \in \mathbb{R}_r} \tau_k \wp(\xi_k - x) \\ &\geq \Big(a + \sum_{\xi_k \in \mathbb{R}} \tau_k\Big) e_1 + \Big(\sum_{\xi_k \in \mathbb{R}_r} \tau_k\Big) e_2 \\ &= \Big(a + \sum_{\xi_k \in \mathbb{R}} \tau_k\Big)(e_1 - e_2) > 0, \end{aligned}$$

which completes the proof. □

The restriction property of chordal SLE(8/3) was obtained by Lawler, Schramm and Werner. In [90], they showed that for a compact hull K, there is a local martingale N_t such that an SLE in $\mathbb{H} \setminus K$ can be thought of an SLE in $\mathbb{H}$ weighted by N_t. In [93], N_t is called the *fundamental martingale* of SLE. Also Lawler and Werner presented the geometric representation of fundamental martingale in terms of loop measure and central charge, see [96]. In this manner, the restriction property of random curves can be considered in a general domain where the concept of loop measure is naturally defined. See also [21, 44, 46, 138] for studies on restriction properties of SLE.

We now explain how the fundamental martingale is expressed in terms of correlation function. Without loss of generality, we assume that SLE trace grows from the origin. For a given compact hull $K \subset \mathcal{C}_r$, let

$$\psi_K : (\mathcal{C}_r \setminus K, 0) \to (\mathcal{C}_{r_K}, 0)$$

be a conformal map, where r_K is the modulus of $\mathcal{C}_r \setminus K$. For an SLE$(\kappa, \Lambda)$ trace $\gamma[0,t]$, let us denote by D_t the Loewner domain and $r(t)$ the modulus of D_t. On the event $\gamma[0,r] \cap K = \emptyset$, we define a conformal map h_t as

$$h_t : \Omega_t = \tilde{g}_t(\mathcal{C}_r \setminus K) \to \mathcal{C}_{r_K(t)}, \quad h_t := \tilde{g}_{K,t} \circ \psi_K \circ \tilde{g}_t^{-1},$$

where $\tilde{g}_{K,t} : \mathcal{C}_{r_K} \setminus \psi_K(\gamma[0,t]) \to \mathcal{C}_{r_K(t)}$ is a conformal map.

Now let us define

$$M_t := \frac{(\mathbf{E}\Psi^{\mathrm{eff}}(\gamma_t; D_t \setminus K) \| \tilde{g}_t)}{(\mathbf{E}\Psi^{\mathrm{eff}}(\gamma_t; D_t) \| \tilde{g}_t)} = \frac{(\mathbf{E}\Psi(\gamma_t; D_t \setminus K) \| \tilde{g}_t) Z_{r_K(t)}}{(\mathbf{E}\Psi(\gamma_t; D_t) \| \tilde{g}_t) Z_{r(t)}} \tag{7.4.8}$$

and

$$N_t = M_t \exp\Big(-\frac{c}{6} \int_0^t Sh_s(\xi_s) ds \Big). \tag{7.4.9}$$

We now present a conformal field theoretic proof of martingale property of N_t. In the proof, we explain why the central charge and the Schwarzian derivative appear in the expression of fundamental martingale.

Proposition 7.4.3. *The process N_t is a local martingale, i.e.,*

$$\textit{drift term of } \quad \frac{dM_t}{M_t} = \frac{c}{6} S_{h_t}(\xi_t)\, dt. \tag{7.4.10}$$

Moreover, the following holds:

$$\frac{dN_t}{N_t} = \Big[h_{1,2} \frac{h_t''(\xi_t)}{h_t'(\xi_t)} + \frac{h_t'(\xi_t)}{\kappa} \Lambda(r_K(t), h_t(\xi_t), h_t(\boldsymbol{\xi})) - \frac{1}{\kappa} \Lambda(r-t, \xi_t, \tilde{g}_t(\boldsymbol{q})) \Big] \sqrt{\kappa}\, dB_t.$$

Before presenting the proof of Proposition 7.4.3, let us consider simplest

case that there exists only one force point q. By the neutrality condition,

$$\Psi^{\mathrm{eff}} := \mathcal{O}^{\mathrm{eff}}[a \cdot z_0 - a \cdot q, \ \mathbf{0}], \quad q \in \mathbb{R}.$$

Then for ER boundary condition, by (7.4.8) and (6.2.15)

$$\begin{aligned} M_t &= h_t'(\xi_t)^{\frac{3}{\kappa}-\frac{1}{2}} h_t'(q_t)^{\frac{1}{2}-\frac{1}{\kappa}} \\ &\quad \times \Big(\frac{\Theta'(r_K(t),0)}{\Theta'(r(t),0)}\Big)^{\frac{6-\kappa}{3\kappa}} \Big(\frac{\Theta(r_K(t), h_t(\xi_t)-h_t(q_t))}{\Theta(r(t),\xi_t-q_t)}\Big)^{-\frac{2}{\kappa}}, \end{aligned} \tag{7.4.11}$$

where $q_t := \tilde{g}_t(q)$. We present a proof of Proposition 7.4.3 for this example by direct Itô's stochastic calculus.

Proof. Recall that by (7.1.10), the driving process ξ_t is given as

$$d\xi_t = \sqrt{\kappa}\, dB_t - H(r(t), \xi_t - q_t)\, dt.$$

We denote by N_h (resp., S_h) pre-Schwarzian (resp., Schwarzian) derivative, i.e.,

$$S_h := \frac{h'''}{h'} - \frac{3}{2}N_h^2, \quad N_h = \frac{h''}{h'}.$$

By Proposition 5.3.1, the central charge c is given as

$$c = \frac{(6-\kappa)(3\kappa-8)}{2\kappa}.$$

Set

$$X_t := h_t'(\xi_t)^{\frac{3}{\kappa}-\frac{1}{2}}.$$

It follow from Itô's formula that

$$dh_t'(\xi_t) = h_t''(\xi_t)\, d\xi_t + \frac{\kappa}{2} h_t'''(\xi_t)\, dt + \partial_t h_t'(\xi_t)\, dt$$

and

$$\frac{\partial_t h_t'(\xi_t)}{h_t'(\xi_t)} = \frac{1}{2}N_{h_t}(\xi_t)^2 - \frac{4}{3}\frac{h_t'''(\xi_t)}{h_t(\xi_t)} + \frac{2}{3}h_t'(\xi_t)^2 \frac{\Theta'''}{\Theta'}(r_K(t),0) - \frac{2}{3}\frac{\Theta'''}{\Theta'}(r(t),0).$$

Thus we have

$$\begin{aligned}\frac{dX_t}{X_t} &= \Big(\frac{3}{\kappa}-\frac{1}{2}\Big)\frac{dh_t'(\xi_t)}{h_t'(\xi_t)} + \frac{3(6-\kappa)(2-\kappa)}{8\kappa}N_{h_t}^2(\xi_t)\\ &= \Big(\frac{3}{\kappa}-\frac{1}{2}\Big)N_{h_t}(\xi_t)\,d\xi_t + \frac{c}{6}S_{h_t}(\xi_t)\,dt\\ &\quad + \frac{6-\kappa}{3\kappa}\Big(h_t'(\xi_t)^2\frac{\Theta'''}{\Theta'}(r_K(t),0) - \frac{\Theta'''}{\Theta'}(r(t),0)\Big)\,dt.\end{aligned}$$

Let

$$Y_t := h_t'(q_t)^{\frac{1}{2}-\frac{1}{\kappa}}.$$

Recall that by Loewner's differential equation,

$$dq_t = H(r(t), q_t-\xi_t)\,dt, \quad dh_t(q_t) = h_t'(\xi_t)^2 H(r_K(t), h_t(q_t)-h_t(\xi_t))\,dt$$

and

$$\frac{dh_t'(q_t)}{h_t'(q_t)} = h_t'(\xi_t)^2 H'(r_K(t), h_t(q_t)-h_t(\xi_t))\,dt - H'(r(t), q_t-\xi_t)\,dt.$$

Therefore

$$\frac{dY_t}{Y_t} = \Big(\frac{1}{2}-\frac{1}{\kappa}\Big)\Big[h_t'(\xi_t)^2 H'(r_K(t), h_t(q_t)-h_t(\xi_t)) - H'(r(t), q_t-\xi_t)\Big]\,dt.$$

Set

$$Z_t := \Big(\frac{\Theta'(r_K(t),0)}{\Theta'(r(t),0)}\Big)^{\frac{6-\kappa}{3\kappa}}.$$

It is straightforward to show that (see [138])

$$\partial_t r(t) = -1, \quad \partial_t r_K(t) = -h_t'(\xi_t)^2 \tag{7.4.12}$$

Then by (7.4.12),

$$\frac{dZ_t}{Z_t} = \Big(\frac{1}{3}-\frac{2}{\kappa}\Big)\Big(h_t'(\xi_t)^2\frac{\Theta'''}{\Theta'}(r_K(t),0) - \frac{\Theta'''}{\Theta'}(r(t),0)\Big)\,dt.$$

Finally, let

$$W_t := \Theta(r_K(t), h_t(\xi_t) - h_t(q_t))^{-\frac{2}{\kappa}} \Theta(r(t), \xi_t - q_t)^{\frac{2}{\kappa}}.$$

Note that by definition,

$$dh_t(\xi_t) = h_t'(\xi_t)\, d\xi_t + \Big(\frac{\kappa}{2} - 3\Big) h_t''(\xi_t)\, dt.$$

Therefore we have

$$\begin{aligned} d\Big(h_t(\xi_t) - h_t(q_t)\Big) &= \sqrt{\kappa}\, h_t'(\xi_t)\, dB_t - h_t'(\xi_t) H(r(t), \xi_t - q_t)\, dt \\ &\quad + \Big[\Big(\frac{\kappa}{2} - 3\Big) h_t''(\xi_t) + h_t'(\xi_t)^2 H(r_K(t), h_t(\xi_t) - h_t(q_t))\Big]\, dt. \end{aligned}$$

Combining all above, we have

$$\begin{aligned} d\Theta(r_K(t), h_t(\xi_t) - h_t(q_t)) &= \Theta'(r_K(t), h_t(\xi_t) - h_t(q_t))\, d(h_t(\xi_t) - h_t(q_t)) \\ &\quad + \Big(\frac{\kappa}{2} - 1\Big) h_t'(\xi_t)^2 \Theta''(r_K(t), h_t(\xi_t) - h_t(q_t))\, dt. \end{aligned}$$

On the other hand, since $d(\xi_t - q_t) = \sqrt{\kappa}\, dB_t$,

$$d\Theta(r(t), \xi_t - q_t) = \sqrt{\kappa}\Theta'(r(t), \xi_t - q_t)\, dB_t + \Big(\frac{\kappa}{2} - 1\Big)\Theta''(r(t), \xi_t - q_t)\, dt.$$

Therefore we have

$$\begin{aligned} \frac{d\Theta(r_K(t), h_t(\xi_t) - h_t(q_t))^{-2/\kappa}}{\Theta(r_K(t), h_t(\xi_t) - h_t(q_t))^{-2/\kappa}} &= -\frac{1}{\kappa} H(r_K(t), h_t(\xi_t) - h_t(q_t))\, d(h_t(\xi_t) - h_t(q_t)) \\ &\quad + \Big(\frac{1}{\kappa} - \frac{1}{2}\Big) h_t'(\xi_t)^2 H'(r_K(t), h_t(\xi_t) - h_t(q_t))\, dt \\ &\quad + \frac{1}{\kappa} h_t'(\xi_t)^2 H^2(r_K(t), h_t(\xi_t) - h_t(q_t))\, dt \end{aligned}$$

and

$$\frac{d\Theta(r(t), \xi_t - q_t)^{2/\kappa}}{\Theta(r(t), \xi_t - q_t)^{2/\kappa}} = \frac{1}{\sqrt{\kappa}} H(r(t), \xi_t - q_t)\, dB_t + \Big(\frac{1}{2} - \frac{1}{\kappa}\Big) H'(r(t), \xi_t - q_t)\, dt.$$

This leads to

$$\frac{dW_t}{W_t} = \frac{d\Theta(r_K(t), h_t(\xi_t) - h_t(q_t))^{-2/\kappa}}{\Theta(r_K(t), h_t(\xi_t) - h_t(q_t))^{-2/\kappa}} + \frac{d\Theta(r(t), \xi_t - q_t)^{2/\kappa}}{\Theta(r(t), \xi_t - q_t)^{2/\kappa}} - \frac{1}{\kappa} h_t'(\xi_t) H(r_K(t), h_t(\xi_t) - h_t(q_t)) H(r(t), \xi_t - q_t)\, dt.$$

Combining above computations, we have

$$\frac{dM_t}{M_t} = \frac{dX_t}{X_t} + \frac{dY_t}{Y_t} + \frac{dZ_t}{Z_t} + \frac{dW_t}{W_t} + \frac{\langle dX, dW\rangle_t}{X_t W_t}.$$

Note that

$$\frac{\langle dX, dW\rangle_t}{X_t W_t} = \frac{6-\kappa}{2\kappa} h_t''(\xi_t) H(r_K(t), h_t(\xi_t)\, dt - \frac{6-\kappa}{2\kappa} N_{h_t}(\xi_t) H(r(t), \xi_t - q_t)\, dt$$

Therefore we conclude (7.4.10). Moreover, if $c = 0$, we have

$$\frac{dM_t}{M_t} = \frac{6-\kappa}{2\sqrt{\kappa}} \frac{h_t''(\xi_t)}{h_t'(\xi_t)}\, dB_t + \frac{1}{\sqrt{\kappa}} \Big[h_t'(\xi_t) H(r_K(t), h_t(q_t) - h_t(\xi_t)) - H(r(t), q_t - \xi_t) \Big]\, dB_t$$

□

Now we prove (7.4.10) in general by means of conformal field theoretic method.

Proof of Proposition 7.4.3. For notational convenience, we write

$$dX_t \overset{d}{=} b(X_t), \quad \text{for} \quad X_t = a(X_t)\, dB_t + b(X_t)\, dt.$$

Let us denote

$$F(z, \boldsymbol{\xi}, t) := \Big(\mathbf{E}\Psi_{\Omega_t}(z, \boldsymbol{\xi}) \Big\| \mathrm{id}_{\Omega_t} \Big), \quad G(z, \boldsymbol{\xi}, t) := \Big(\mathbf{E}\Psi_{\mathcal{C}_{r(t)}}(z, \boldsymbol{\xi}) \Big\| \mathrm{id}_{\mathcal{C}_{r(t)}} \Big).$$

During the proof, we write the subscript for differentiation, e.g., $F_\xi = \partial_\xi F(\xi, \cdot)$, $G_{\xi\xi} = \partial_\xi^2 G(\xi, \cdot)$.

By (7.4.8), we have

$$M_t = \frac{F(\xi_t, \boldsymbol{\xi}(t), t)}{G(\xi_t, \boldsymbol{\xi}(t), t)} \frac{Z_{r_K(t)}}{Z_{r(t)}}, \tag{7.4.13}$$

where $\boldsymbol{\xi}(t) := \{\tilde{g}_t(\xi_1), \cdots, \tilde{g}_t(\xi_N)\}$. By (6.2.16),

$$\frac{Z_{r_K(t)}}{Z_{r(t)}} = \Big(\frac{\Theta'(r_K(t), 0)}{\Theta'(r(t), 0)}\Big)^{-1/3} \Big(\frac{r_K(t) + \chi}{r(t) + \chi}\Big)^{-1/2}. \tag{7.4.14}$$

By (7.4.12), we have

$$\frac{d(\Theta'(r_K(t), 0)/\Theta'(r(t), 0))^{-1/3}}{(\Theta'(r_K(t), 0)/\Theta'(r(t), 0))^{-1/3}} \overset{d}{=} \frac{1}{3}\Big(h_t'(\xi_t)^2 \frac{\Theta'''}{\Theta'}(r_K(t), 0) - \frac{\Theta'''}{\Theta'}(r(t), 0)\Big)$$

and

$$\frac{d((r_K(t) + \chi)/(r(t) + \chi))^{-1/2}}{((r_K(t) + \chi)/(r(t) + \chi))^{-1/2}} \overset{d}{=} \frac{1}{2}\Big(h_t'(\xi_t)^2 \frac{1}{r_K(t) + \chi} - \frac{1}{r(t) + \chi}\Big).$$

On the other hand, by Loewner's equation,

$$d\xi_t = \sqrt{\kappa}\, dB_t + \kappa \frac{G_\xi(\xi_t, \boldsymbol{\xi}(t))}{G(\xi_t, \boldsymbol{\xi}(t))}\, dt, \quad \dot{\xi}_k(t) = H(r(t), \xi_k(t) - \xi_t).$$

Also it is easy to see that for $z \in \Omega_t$ (see [138])

$$\partial_t h_t(z) = h_t'(\xi_t)^2 H(r_K(t), h_t(z) - \tilde{\xi}_t) - h_t'(z) H(r(t), z - \xi_t)$$

and

$$[\partial_t h_t](\xi_t) = \lim_{z \to \xi_t} \partial_t h_t(z) = -3h_t''(\xi_t),$$

where $\tilde{\xi}_t = h_t(\xi_t)$ Then it follows from Itô's formula that

$$\frac{d(F/G)}{F/G} \overset{d}{=} \frac{\dot{F}}{F} - \frac{\dot{G}}{G} + \frac{\kappa}{2}\Big(\frac{F_{\xi\xi}}{F} - \frac{G_{\xi\xi}}{G}\Big) - \Big(\frac{\mathcal{L}_{v_{\xi_t}} F}{F} - \frac{\mathcal{L}_{v_{\xi_t}} G}{G}\Big),$$

where Lie derivatives do not apply to ξ_t. Also we represent $\dot{F}$ in terms of the Lie derivatives as

$$\begin{aligned}\dot{F}(z,\boldsymbol{\xi}(t),t) &= \frac{d}{ds}\Big|_{s=0}\mathbf{E}[\Psi_{\mathcal{C}_{r_K(t)}}\|h_{t+s}^{-1}](z,\boldsymbol{\xi}(t)) + \frac{dr_K(t)}{dt}\partial_r F(z,\boldsymbol{\xi}(t),t)\\ &= \frac{d}{ds}\Big|_{s=0}\mathbf{E}[\Psi_{\mathcal{C}_{r_K(t)}}\|h_t^{-1}\circ f_{s,t}^{-1}](z,\boldsymbol{\xi}(t)) + \frac{dr_K(t)}{dt}\partial_r F(z,\boldsymbol{\xi}(t),t)\\ &= (\mathbf{E}\mathcal{L}_v\Psi_{\mathcal{C}_{r_K(t)}}\|h_t^{-1})(z,\boldsymbol{\xi}(t)) + \frac{dr_K(t)}{dt}\partial_r F(z,\boldsymbol{\xi}(t),t),\end{aligned}$$

where $f_{s,t} = h_{t+s}\circ h_t^{-1}$ and

$$\begin{aligned}(v\|\mathrm{id}_{\mathcal{C}_{r_K(t)}})(\eta) &= \frac{d}{ds}\Big|_{s=0} f_{s,t}\\ &= -h_t'(\xi_t)^2 H(r_K(t),\tilde{\xi}_t-\eta) + h_t'(h_t^{-1}(\eta))H(r(t),\xi_t-h_t^{-1}(\eta)).\end{aligned}$$

Then by the definition of Lie derivative and Proposition 6.1.2, we obtain

$$\begin{aligned}&(\mathbf{E}\mathcal{L}_v\Psi_{\mathcal{C}_{r_K(t)}}\|h_t^{-1})(z,\boldsymbol{\xi}(t))\\ &= -h_t'(\xi_t)^2 h_t'(z)^{h_{1,2}}\prod h_t'(\xi_k)^{h_l}(\mathbf{E}\mathcal{L}_{v_{\tilde{\xi}_t}}\Psi_{\mathcal{C}_{r_K(t)}}\|\mathrm{id}_{\mathcal{C}_{r_K(t)}})(h_t(z),h_t(\boldsymbol{\xi}(t)))\\ &\quad + (\mathbf{E}\mathcal{L}_{v_{\xi_t}}\Psi_{\Omega_t}\|\mathrm{id}_{\Omega_t})(z,\boldsymbol{\xi}(t))\\ &= -2(\mathbf{E}A_{\Omega_t}(\xi_t)\Psi_{\Omega_t}(z,\boldsymbol{\xi}(t))\|\mathrm{id}_{\Omega_t}) - \frac{dr_K(t)}{dt}\partial_r F(z,\boldsymbol{\xi}(t),t)\\ &\quad + (\mathbf{E}\mathcal{L}_{v_{\xi_t}}\Psi_{\Omega_t}\|\mathrm{id}_{\Omega_t})(z,\boldsymbol{\xi}(t)).\end{aligned}$$

Therefore we obtain

$$\dot{F}(z,\boldsymbol{\xi}(t),t) = -2(\mathbf{E}A_{\Omega_t}(\xi_t)\Psi_{\Omega_t}(z,\boldsymbol{\xi}(t))\|\mathrm{id}_{\Omega_t}) + (\mathbf{E}\mathcal{L}_{v_{\xi_t}}\Psi_{\Omega_t}\|\mathrm{id}_{\Omega_t})(z,\boldsymbol{\xi}(t)).$$

As $z\to\xi_t$, by the similar argument in the proof of Corollary 6.3.1, it is easy to show that

$$\begin{aligned}\dot{F}(\xi_t,\boldsymbol{\xi}(t),t) = &-\frac{2h_{1,2}}{3}\frac{\Theta'''}{\Theta'}(r(t),0)F - 2(\mathbf{E}A_{\Omega_t}*_\xi\Psi_{\Omega_t}(\xi_t,\boldsymbol{\xi}(t))\|\mathrm{id}_{\Omega_t})\\ &+ (\mathbf{E}\mathcal{L}_{v_{\xi_t}}\Psi_{\Omega_t}\|\mathrm{id}_{\Omega_t})(\xi_t,\boldsymbol{\xi}(t)).\end{aligned}$$

Moreover, by the level two degeneracy equation (Proposition 7.1.4), we have

$$\frac{\kappa}{2}F_{\xi\xi} = 2\Big(\mathbf{E}T_{\Omega_t} *_\xi \Psi_{\Omega_t}(\xi_t, \boldsymbol{\xi}(t))\|\mathrm{id}_{\Omega_t}\Big).$$

Combining all of the above equations with Proposition 5.3.1, we obtain

$$\frac{\dot{F}}{F} + \frac{\kappa}{2}\frac{F_{\xi\xi}}{F} - \frac{\mathcal{L}_{v_{\xi_t}}F}{F} = 2\Big(T_{\Omega_t} - A_{\Omega_t}\Big)(\xi_t) - \frac{2h_{1,2}}{3}\frac{\Theta'''}{\Theta'}(r(t),0).$$

Note that by Proposition 5.3.1 and (3.1.25),

$$2\Big(T_{\Omega_t} - A_{\Omega_t}\Big)(\xi_t) = \frac{c}{6}S_{h_t}(\xi_t) - h_t'(\xi_t)^2\Big(\frac{1}{3}\frac{\Theta'''}{\Theta'}(r_K(t),0) - \frac{1}{2(r_K(t)+\chi)}\Big).$$

On the other hand,

$$\dot{G} = \frac{dr(t)}{dt}\partial_r G = -\partial_r G.$$

By Proposition 7.1.4 and Theorem B, we obtain

$$\begin{aligned}\frac{\kappa}{2}G_{\xi\xi} &= 2\Big(\mathbf{E}T_{\mathcal{C}_{r(t)}} *_\xi \Psi_{\mathcal{C}_{r(t)}}(\xi_t, \boldsymbol{\xi}(t))\|\mathrm{id}_{\Omega_t}\Big)\\ &= \mathcal{L}_{v_{\xi_t}}G - \Big(\frac{2h_{1,2}+1}{3}\frac{\Theta'''}{\Theta'}(r(t),0) + \frac{1}{2(r(t)+\chi)}\Big)G + \partial_r G,\end{aligned}$$

which leads to

$$\frac{\dot{G}}{G} + \frac{\kappa}{2}\frac{G_{\xi\xi}}{G} - \frac{\mathcal{L}_{v_{\xi_t}}G}{G} = -\frac{2h_{1,2}+1}{3}\frac{\Theta'''}{\Theta'}(r(t),0) - \frac{1}{2(r(t)+\chi)}$$

Combining all of the above, the drift term of $d(F/G)/(F/G)$ is given by

$$\begin{aligned}&\frac{c}{6}S_{h_t}(\xi_t) - \frac{1}{3}\Big(h_t'(\xi_t)^2\frac{\Theta'''}{\Theta'}(r_K(t),0) - \frac{\Theta'''}{\Theta'}(r(t),0)\Big)\\ &\qquad - \frac{1}{2}\Big(h_t'(\xi_t)^2\frac{1}{r_K(t)+\chi} - \frac{1}{r(t)+\chi}\Big).\end{aligned}$$

Now proposition follows from (7.4.13) and (7.4.14). □

7.5 Non-atomic background charges

In this section, we implement general background charges beyond linear combination of atomic masses.

Let us denote $\check{\boldsymbol{\beta}}$ a (signed) measure on $\mathbb{R}_r$ with total mass $-2b$ and write

$$\boldsymbol{\beta} := -2b \cdot q + \check{\boldsymbol{\beta}}, \quad q \in \mathbb{R}. \tag{7.5.1}$$

Its discrete counter part $\boldsymbol{\beta}_n$ is given by

$$\boldsymbol{\beta}_n := 2b \cdot q + \sum_{k=1}^{n} \frac{\beta_k}{n} \cdot q_k, \quad (\sum \beta_k = -2b). \tag{7.5.2}$$

Let us define

$$\Psi_{\boldsymbol{\beta}}[a \cdot p - a \cdot q] := \lim_{n\to\infty} \Psi_{\boldsymbol{\beta}_n}[a \cdot p - a \cdot q] \tag{7.5.3}$$

and write

$$Z_{\boldsymbol{\beta}} = \mathbf{E}\Psi_{\boldsymbol{\beta}}[a \cdot p - a \cdot q], \quad Z_{\boldsymbol{\beta}_n} = \mathbf{E}\Psi_{\boldsymbol{\beta}_n}[a \cdot p - a \cdot q]. \tag{7.5.4}$$

Denoting $q_k = s_k + ir$, in the identity chart of $\mathcal{C}_r$, we have (up to multiplicative constant)

$$\begin{aligned} Z_{\boldsymbol{\beta}_n} = \Theta(p-q)^{1-\frac{6}{\kappa}} \prod_{k=1}^{n} \Big(\frac{\Theta_I(p-s_k)}{\Theta_I(q-s_k)}\Big)^{\frac{a\beta_k}{n}} \\ \times \exp\Big[\Big(1-\frac{6}{k}\Big)\frac{(p-q)^2}{4(r+\chi)} + a\frac{p-q}{4(r+\chi)}\sum_{k=1}^{n}\frac{\beta_k}{n}(p+q-2s_k)\Big]. \end{aligned} \tag{7.5.5}$$

Thus in the continuum limit, we have

$$\begin{aligned} Z_{\boldsymbol{\beta}} = \Theta'(0)^{\frac{1}{2}}\Theta(p-q)^{1-\frac{6}{\kappa}} \exp\Big(a\int \log\frac{\Theta_I(p-s)}{\Theta_I(q-s)}\, d\check{\boldsymbol{\beta}}(s)\Big) \\ \times \exp\Big[\Big(1-\frac{6}{k}\Big)\frac{(p-q)^2}{4(r+\chi)} + a\frac{p-q}{4(r+\chi)}\int (p+q-2s)\; d\check{\boldsymbol{\beta}}(s)\Big]. \end{aligned} \tag{7.5.6}$$

Without loss of generality, let $p+q=0$.

We now derive the null-vector equation for $Z_{\boldsymbol{\beta}}$.

Proposition 7.5.1. *The partition function $Z_{\boldsymbol{\beta}}$ satisfies*

$$\partial_r Z_{\boldsymbol{\beta}} = \frac{\kappa}{2}\partial_p^2 Z_{\boldsymbol{\beta}} + H(p-q)\partial_p Z_{\boldsymbol{\beta}} + h_{1,2}H'(p-q)Z_{\boldsymbol{\beta}} + F_{\boldsymbol{\beta}} Z_{\boldsymbol{\beta}}, \tag{7.5.7}$$

where

$$\begin{aligned} F_{\boldsymbol{\beta}} &= -2\Big(\mathbf{E}T_{\boldsymbol{\beta}}(p) + h_{1,2}\frac{\zeta_r(\pi)}{\pi}\Big) \\ &\quad - \frac{a}{2}\int \Big(H(p-q) - H_I(p-s)\Big)\Big(H_I(p-s) - H_I(q-s) + \frac{p-q}{r+\chi}\Big)\, d\check{\boldsymbol{\beta}}(s). \end{aligned}$$

Here

$$\begin{aligned} \mathbf{E}T_{\boldsymbol{\beta}}(p) &= \frac{\zeta_r(\pi)}{2\pi} - \frac{1}{4(r+\chi)} + b^2\Big[H'(p-q) + \frac{1}{2}\Big(H(p-q) + \frac{p-q}{r+\chi}\Big)^2\Big] \\ &\quad + \frac{1}{8}\Big[\int \Big(H_I(p-s) + \frac{p-s}{r+\chi}\Big)\, d\check{\boldsymbol{\beta}}(s)\Big]^2 + \frac{b}{2}\int H_I'(p-s)\, d\check{\boldsymbol{\beta}}(s). \end{aligned}$$

Proof. By Theorem B, we have

$$\begin{aligned} \partial_r Z_{\boldsymbol{\beta}_n} &= \frac{\kappa}{2}\partial_p^2 Z_{\boldsymbol{\beta}_n} - H(p-q)\partial_q Z_{\boldsymbol{\beta}_n} + h_{1,2}H'(p-q)Z_{\boldsymbol{\beta}_n} \\ &\quad - 2\Big(\mathbf{E}T_{\boldsymbol{\beta}_n}(p) + h_{1,2}\frac{\zeta_r(\pi)}{\pi}\Big)Z_{\boldsymbol{\beta}_n} \\ &\quad - \sum H_I(p-s_k)\partial_{s_k} Z_{\boldsymbol{\beta}_n} + i(\partial_{q_k} - \bar{\partial}_{q_k})Z_{\boldsymbol{\beta}_n}. \end{aligned}$$

It follows from the translation invariance

$$\Big(\partial_p + \partial_q + \sum \partial_{s_k}\Big)Z_{\boldsymbol{\beta}_n} = 0$$

that

$$\begin{aligned}\partial_r Z_{\boldsymbol{\beta}_n} &= \frac{\kappa}{2}\partial_p^2 Z_{\boldsymbol{\beta}_n} + H(p-q)\partial_p Z_{\boldsymbol{\beta}_n} + h_{1,2}H'(p-q)Z_{\boldsymbol{\beta}_n} \\ &\quad - 2\Big(\mathbf{E}T_{\boldsymbol{\beta}_n}(p) + h_{1,2}\frac{\zeta_r(\pi)}{\pi}\Big)Z_{\boldsymbol{\beta}_n} \\ &\quad + \sum\Big(H(p-q) - H_I(p-s_k)\Big)\partial_{s_k} Z_{\boldsymbol{\beta}_n} + i(\partial_{q_k} - \bar{\partial}_{q_k})Z_{\boldsymbol{\beta}_n}.\end{aligned}$$

Note that KZ equation gives rise to

$$\frac{\partial_{q_k} Z_{\boldsymbol{\beta}_n}}{Z_{\boldsymbol{\beta}_n}} = -\frac{a}{2}\frac{\beta_k}{n}\Big(H_I(p-s_k) - H_I(q-s_k) + \frac{p-q}{r+\chi}\Big).$$

Therefore we have

$$\partial_r Z_{\boldsymbol{\beta}_n} = \frac{\kappa}{2}\partial_p^2 Z_{\boldsymbol{\beta}_n} + H(p-q)\partial_p Z_{\boldsymbol{\beta}_n} + h_{1,2}H'(p-q)Z_{\boldsymbol{\beta}_n} + F_{\boldsymbol{\beta}_n}Z_{\boldsymbol{\beta}_n},$$

where

$$\begin{aligned}F_{\boldsymbol{\beta}_n} &= -2\Big(\mathbf{E}T_{\boldsymbol{\beta}_n}(p) + h_{1,2}\frac{\zeta_r(\pi)}{\pi}\Big) \\ &\quad - \frac{a}{2}\sum\frac{\beta_k}{n}\Big(H(p-q) - H_I(p-s_k)\Big)\Big(H_I(p-s_k) - H_I(q-s_k) + \frac{p-q}{r+\chi}\Big).\end{aligned}$$

Furthermore, by Proposition 5.6.1,

$$\begin{aligned}\mathbf{E}T_{\boldsymbol{\beta}_n}(p) &= \frac{\zeta_r(\pi)}{2\pi} - \frac{1}{4(r+\chi)} + b^2\Big[H'(p-q) + \frac{1}{2}\Big(H(p-q) + \frac{p-q}{r+\chi}\Big)^2\Big] \\ &\quad + \frac{1}{8}\Big[\sum\frac{\beta_k}{n}\Big(H_I(p-s_k) + \frac{p-s_k}{r+\chi}\Big)\Big]^2 + \frac{b}{2}\sum\frac{\beta_k}{n}H_I'(p-s_k).\end{aligned}$$

Now proposition follows by letting $n \to \infty$. □

Example. The simplest example is the case that $\check{\boldsymbol{\beta}}$ is the uniform (signed) measure on $[-\pi, \pi]$ with total mass $-2b$. In this case, by periodicity of $\Theta_I(r, \cdot)$, we have

$$Z_{\boldsymbol{\beta}} = \Theta'(0)^{\frac{1}{2}}\Theta(p-q)^{1-\frac{6}{\kappa}}\exp\Big[\Big(1-\frac{6}{k}\Big)\frac{(p-q)^2}{4(r+\chi)}\Big]. \tag{7.5.8}$$

Note also that we have

$$\begin{aligned}\mathbf{E}T_{\boldsymbol{\beta}}(p) = \frac{\zeta_r(\pi)}{2\pi} - \frac{1}{4(r+\chi)} + b^2\Big(H' + \frac{H^2}{2}\Big)(p-q)\\ + b^2\Big(\frac{p-q}{r+\chi}H(p-q) + \frac{5}{8}\frac{(p-q)^2}{(r+\chi)^2}\Big).\end{aligned}$$

On the other hand, it follows from (3.1.41) that

$$\begin{aligned}&H_I(p-s)\Big(H_I(p-s) - H_I(q-s)\Big)\\ &= -\Big(H' + \frac{H^2}{2}\Big)(p-q) - \frac{6}{\pi}\zeta_r(\pi) + \frac{1}{2}\Big(H_I^2(p-s) - H_I^2(q-s)\Big)\\ &+ H(p-q)\Big(H_I(p-s) - H_I(q-s)\Big) - H_I'(p-s) - H_I'(q-s),\end{aligned}$$

which leads to

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(H(p-q) - H_I(p-s)\Big)\Big(H_I(p-s) - H_I(q-s) + \frac{p-q}{r+\chi}\Big)\,ds\\ &= \frac{p-q}{r+\chi}\,H(p-q) - \frac{1}{2\pi}\int_{-\pi}^{\pi}H_I(p-s)\Big(H_I(p-s) - H_I(q-s)\Big)\,ds\\ &= \frac{p-q}{r+\chi}\,H(p-q) + \Big(H' + \frac{H^2}{2}\Big)(p-q) + \frac{6}{\pi}\zeta_r(\pi).\end{aligned}$$

Therefore we obtain (up to additive constant)

$$\begin{aligned}F_{\boldsymbol{\beta}} = &\Big(2 - \frac{\kappa}{2}\Big)\Big(1 - \frac{6}{\kappa}\Big)\\ &\times\Big[\Big(\frac{H'}{2} + \frac{H^2}{4}\Big)(p-q) + \frac{p-q}{2(r+\chi)}\,H(p-q) + \frac{(p-q)^2}{4(r+\chi)^2}\Big].\end{aligned} \tag{7.5.9}$$

7.6 Periodization

In this section, we present a way of extending our CFT framework to generalize the family of SLE(κ, Λ) under consideration. This extension is achieved through the weighted summation of the chiral conformal fields. We refer to [126] for a similar idea (under the name of *chiral bosonization*) on compact Riemann surfaces.

For a suitable weight function $\omega : \mathbb{Z} \to \mathbb{R}_+$, let us define

$$\Psi^{\omega}_{\boldsymbol{\beta}}(p, \boldsymbol{\xi}) := \sum_{n \in \mathbb{Z}} \omega(n) \Psi_{\boldsymbol{\beta}}(p + 2n\pi, \boldsymbol{\xi})$$

and write

$$Z^{\omega}_{\boldsymbol{\beta}}(p, \boldsymbol{\xi}) := \mathbf{E}\Psi^{\omega}_{\boldsymbol{\beta}}(p, \boldsymbol{\xi}), \quad \Lambda^{\omega}(r, p, \boldsymbol{\xi}) := \kappa\, \partial_p \log Z^{\omega}_{\boldsymbol{\beta}}(p, \boldsymbol{\xi}).$$

We remark that in [139], Zhan used the weight function

$$\omega(n) := e^{-\frac{2\pi s_0}{\kappa} n}, \quad (s_0 \in \mathbb{R})$$

to construct partition functions having rotational periodicity.

For instance, let us consider the case that all ξ_j's and q_k's are on $\mathbb{R}_r$. Then it follows from (7.1.9) that up to multiplicative constant,

$$\begin{aligned} Z_{\boldsymbol{\beta}}(p, \boldsymbol{\xi}) = \exp\Big(- \frac{(ap + \sum_j \tau_j \operatorname{Re} \xi_j + \sum_k \beta_k \operatorname{Re} q_k)^2}{4(r + \chi)} \Big) \\ \times \prod_j \Theta_I(r, p - \operatorname{Re} \xi_j)^{a\tau_j} \prod_j \Theta_I(r, p - \operatorname{Re} q_j)^{a\beta_j} \\ \times \prod_{j<k} \Theta(r, \xi_j - \xi_k)^{\tau_j \tau_k} \prod_{j,k} \Theta(r, \xi_j - q_k)^{\tau_j \beta_k}. \end{aligned} \tag{7.6.1}$$

Note that for $\chi < \infty$, the partition function $Z_{\boldsymbol{\beta}}(p, \boldsymbol{\xi})$ is not 2π-periodic with respect to the space variable p. Thus the associated drift function $\Lambda_{\boldsymbol{\beta}}(p, \boldsymbol{\xi})$ is defined only on the covering space $\mathbb{S}_r$.

Set $\omega(n) \equiv 1$. By (3.1.14), we have

$$\begin{aligned} Z^{\omega}_{\boldsymbol{\beta}}(p, \boldsymbol{\xi}) = \Theta_I\Big(\frac{\kappa}{2}(r + \chi), p + \sum_j \frac{\tau_j}{a} \operatorname{Re} \xi_j + \sum_k \frac{\beta_k}{a} \operatorname{Re} q_k + \pi\Big) \\ \times \prod_j \Theta_I(r, p - \operatorname{Re} \xi_j)^{a\tau_j} \prod_k \Theta_I(r, p - \operatorname{Re} q_k)^{a\beta_k} \\ \times \prod_{j<k} \Theta(r, \xi_j - \xi_k)^{\tau_j \tau_k} \prod_{j,k} \Theta(r, \xi_j - q_k)^{\tau_j \beta_k} \end{aligned} \tag{7.6.2}$$

and

$$\begin{aligned}\Lambda^{\omega}_{\boldsymbol{\beta}}(p,\boldsymbol{\xi}) = \frac{\kappa}{2}H_I\Big(\frac{\kappa}{2}(r+\chi), p + \sum_j \frac{\tau_j}{a}\operatorname{Re}\xi_j + \sum_k \frac{\beta_k}{a}\operatorname{Re}q_k + \pi\Big) \\ + \sqrt{\frac{\kappa}{2}}\Big[\sum_j \tau_j H_I(r, p - \operatorname{Re}\xi_j) + \sum_k \beta_k H_I(r, p - \operatorname{Re}q_k)\Big].\end{aligned} \tag{7.6.3}$$

Then it follows from the periodicity of H_I (3.1.33) that $\Lambda^{\omega}_{\boldsymbol{\beta}}$ is well-defined in the cylinder $\mathcal{C}_r$.

We now prove Theorem D.

Proof of Theorem D. Let $\mathcal{X}$ be a string of fields in the OPE family $\mathcal{F}_{\boldsymbol{\beta}}$ and set

$$R_\xi = \widehat{\mathbf{E}}\mathcal{X} := \frac{\mathbf{E}\Psi^{\omega}_{\boldsymbol{\beta}}(\xi,\boldsymbol{\xi})\mathcal{X}}{Z^{\omega}_{\boldsymbol{\beta}}[\boldsymbol{\tau}]}.$$

It follows from the argument presented in the proof of Theorem C, it suffices to show the following form of BPZ-Cardy equation

$$\frac{1}{a^2}\partial_\xi^2 R_\xi + \Lambda^{\omega}_{\boldsymbol{\beta}}\partial_\xi R_\xi = \mathcal{L}_{v_\xi}R_\xi + \partial_r R_\xi. \tag{7.6.4}$$

By Theorem B, we have

$$\frac{1}{a^2}\partial_z^2\mathbf{E}\Psi_{\boldsymbol{\beta}}(\xi,\boldsymbol{\xi})\mathcal{X} = \Big(\mathcal{L}_{v_\xi} + \partial_r + 2h_{1,2}\frac{\zeta_r(\pi)}{\pi} + 2\mathbf{E}T_{\boldsymbol{\beta}}(\xi)\Big)\mathbf{E}\Psi_{\boldsymbol{\beta}}(\xi,\boldsymbol{\xi})\mathcal{X}.$$

Therefore we obtain

$$\begin{aligned}\frac{1}{a^2}\partial_z^2\mathbf{E}\Psi^{\omega}_{\boldsymbol{\beta}}[\boldsymbol{\tau}]\mathcal{X} = \Big(\mathcal{L}_{v_\xi} + \partial_r + 2h_{1,2}\frac{\zeta_r(\pi)}{\pi}\Big)\mathbf{E}\Psi^{\omega}_{\boldsymbol{\beta}}(\xi,\boldsymbol{\xi})\mathcal{X} \\ + 2\sum_{n\in\mathbb{Z}}\mathbf{E}T_{\boldsymbol{\beta}}(\xi+2n\pi)\mathbf{E}\Psi_{\boldsymbol{\beta}}(\xi+2n\pi,\boldsymbol{\xi})\mathcal{X}.\end{aligned} \tag{7.6.5}$$

By Proposition 5.6.1 and the neutrality condition $\int \boldsymbol{\beta} = 0$,

$$\begin{aligned}\mathbf{E}T_{\boldsymbol{\beta}}(\xi) = \frac{1}{32}\Big[\sum \beta_k\Big\{H(r, q_k - \xi) + H(r, \bar{q}_k - \xi) + 2\frac{\operatorname{Re} q_k}{r+\chi}\Big\}\Big]^2 \\ + \frac{b}{4}\sum \beta_k\Big\{H'(r, q_k - \xi) + H'(r, \bar{q}_k - \xi)\Big\} \\ + \frac{\zeta_r(\pi)}{2\pi} - \frac{1}{4(r+\chi)}.\end{aligned} \tag{7.6.6}$$

Thus we have for any $n \in \mathbb{Z}$,

$$\mathbf{E}T_{\boldsymbol{\beta}}(\xi + 2n\pi) = \mathbf{E}T_{\boldsymbol{\beta}}(\xi), \tag{7.6.7}$$

which leads to

$$\frac{1}{a^2}\partial_z^2 R_\xi Z^\omega_{\boldsymbol{\beta}}(\xi, \boldsymbol{\xi}) = \Big(\mathcal{L}_{v_\xi} + \partial_r + 2h_{1,2}\frac{\zeta_r(\pi)}{\pi} + 2\mathbf{E}T_{\boldsymbol{\beta}}(\xi)\Big) R_\xi Z^\omega_{\boldsymbol{\beta}}(\xi, \boldsymbol{\xi}). \tag{7.6.8}$$

By applying the trivial string $\mathcal{X} \equiv 1$ to above, we have the null-vector equation

$$\frac{1}{a^2}\partial_z^2 Z^\omega_{\boldsymbol{\beta}}(\xi, \boldsymbol{\xi}) = \Big(\mathcal{L}_{v_\xi} + \partial_r + 2h_{1,2}\frac{\zeta_r(\pi)}{\pi} + 2\mathbf{E}T_{\boldsymbol{\beta}}(\xi)\Big) Z^\omega_{\boldsymbol{\beta}}(\xi, \boldsymbol{\xi}). \tag{7.6.9}$$

Subtracting above two equations, we obtain (7.6.4), which completes the proof.

□

7.7 Screening

In this section, we discuss the screening method to construct family of solutions to the null-vector equations

$$\partial_r Z = \frac{\kappa}{2}Z'' + HZ' + \Big(\frac{3}{\kappa} - \frac{1}{2}\Big)H'Z + C(r)Z, \tag{7.7.1}$$

$$\partial_r Z = \frac{\kappa}{2}Z'' + H_I Z' + \Big(\frac{3}{\kappa} - \frac{1}{2}\Big)H_I'Z + C(r)Z \tag{7.7.2}$$

and prove Theorem E. From now on, we thus focus on an SLE(κ, Λ) starting from p with one force point q. In particular, we present family of explicit solutions to the null-vector equations (7.7.1),(7.7.2) for general $\kappa > 0$. The main idea is so-called screening, a well-known method in CFT to find solutions of the Knizhnik-Zamolodchikov type equations, see e.g., [51, 125].

In general, it has been turned out that the screening method is useful in the theory of multiple SLEs. For instance, in [45], Dubédat found Euler integral representations for solutions to a system of PDEs characterizing commuting SLEs in $\mathbb{H}$ by means of screening. The full solution space was discovered by Flores and Kleban in the series of works [52–55]. Furthermore, we refer the reader to [70, 87, 97, 98, 112] among others for recent studies on analysis of solutions to such PDE system and the geometric properties of associated multiple SLEs.

For $p, q \in \mathbb{R}$ and $q_1, q_2, \zeta \in \mathbb{C}$, we set

$$\boldsymbol{\tau} = a \cdot p + a \cdot q - 2a \cdot \zeta, \quad \boldsymbol{\beta} = \beta \cdot q_1 - \beta \cdot q_2. \tag{7.7.3}$$

Here we assume that q_1, q_2 are two different fibers of a marked point in annulus satisfying $q_2 = q_1 + 2\pi$. Write

$$\Psi^{\sharp}_{\beta}(p, q, \zeta) := \mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau}, \mathbf{0}], \quad Z^{\sharp}_{\beta} = \mathbf{E}\Psi^{\sharp}_{\beta}. \tag{7.7.4}$$

Then $\Psi^{\sharp}$ has the same conformal dimensions both at p, q and it is a $(1, 0)$-differential with respect to the screening variable ζ, i.e.,

$$\lambda_p = \lambda_q = h_{1,2} = \frac{6 - \kappa}{2\kappa}, \quad \lambda_\zeta = 1. \tag{7.7.5}$$

By (7.2.4), in the identity chart of $\mathcal{C}_r$, we have

$$\begin{aligned} Z^{\sharp}_{\beta}(p, q, \zeta) = {} & \Theta'(0)^{\frac{6}{\kappa}} \Theta_\chi(p - q)^{\frac{2}{\kappa}} \Theta_\chi(p - \zeta)^{-\frac{4}{\kappa}} \Theta_\chi(q - \zeta)^{-\frac{4}{\kappa}} \\ & \times \left| \frac{\Theta_\chi(p - z_0)}{\Theta_\chi(p - z_1)} \frac{\Theta_\chi(q - z_0)}{\Theta_\chi(q - z_1)} \frac{\Theta_\chi(\zeta - z_1)^2}{\Theta_\chi(\zeta - z_0)^2} \right|^{a\beta}. \end{aligned} \tag{7.7.6}$$

In particular due to the fact $q_2 - q_1 = 2\pi$ and (3.1.10), this expression reduces to (up to multiplicative constant)

$$Z_\beta^\sharp(p,q,\zeta) = \Theta(p-q)^{\frac{2}{\kappa}}\Theta(p-\zeta)^{-\frac{4}{\kappa}}\Theta(\zeta-q)^{-\frac{4}{\kappa}}\exp\Big(-\frac{(\Sigma-2\beta\pi)^2}{4(r+\chi)}\Big), \tag{7.7.7}$$

where $\Sigma = a(p+q-2\zeta)$. For generic $\kappa > 0$, we need to specify a principal branch (as a function of ζ) in the expression (7.7.7). In what follows, we use the branch cut $[-\infty, q] \cup [p, \infty]$.

For a closed contour γ, set

$$\Psi_\beta \equiv \Psi_\beta(p,q) := C(\kappa)\oint_\gamma \Psi_\beta^\sharp(p,q,\zeta)\,d\zeta, \tag{7.7.8}$$

where the normalization constant $C(\kappa)$ is given by

$$C(\kappa) = \sin^{-2}\Big(\frac{4\pi}{\kappa}\Big)\frac{1}{4\Gamma(1-4/\kappa)}. \tag{7.7.9}$$

Here Γ is the Gamma function. It is easy to see that $C(\kappa)$ has a simple pole at κ if and only if $4/\kappa$ is a positive integer. By (7.7.7), the partition function

$$Z_\beta \equiv Z_\beta(r, p-q) := \mathbf{E}\Psi_\beta = C(\kappa)\oint_\gamma Z_\beta^\sharp(p,q,\zeta)\,d\zeta \tag{7.7.10}$$

is evaluated as (up to multiplicative constant)

$$Z_\beta = \Theta(p-q)^{\frac{2}{\kappa}}\oint_\gamma \Theta(p-\zeta)^{-\frac{4}{\kappa}}\Theta(q-\zeta)^{-\frac{4}{\kappa}}\exp\Big(-\frac{(\Sigma-2\beta\pi)^2}{4(r+\chi)}\Big)\,d\zeta. \tag{7.7.11}$$

Remark. In general, partition functions $Z_\beta(r,\cdot)$ do not have periodic property with respect to 2π except the case $\chi = \infty$ (i.e., ER b.c.). On the other hand, for $\chi = 0$ (i.e., Dirichlet b.c.) Z_β has following rotational periodicity with

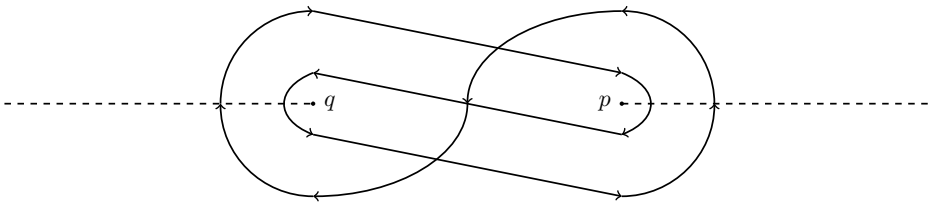


Figure 7.3: Pochhammer contour

respect to $2ir$.

$$Z_\beta(r, x + 2ir) = \exp\Big(a(2\beta - a)\pi i\Big) Z_\beta(r, x). \tag{7.7.12}$$

We denote by $\Pi_1 \equiv \Pi_1(\mathbb{C} \setminus \{p, q\}, \zeta_0)$ the fundamental group of $\mathbb{C} \setminus \{p, q\}$ with base point ζ_0. If $4/\kappa$ is not a positive integer, in order to make the partition function Z_β single-valued, the homotopy class $[\gamma]$ should reside in the commutator subgroup of the Π_1. Moreover, this specific choice of $C(\kappa)$ makes the expression (7.7.10) non-trivial for all values of $\kappa > 0$ after removing possible singularities. We will explain this in Proposition 7.7.1 below.

The simplest example of such γ is the Pochhammer contour $\mathcal{P}(p, q)$ entwining p and q. See Figure 7.3 below for the description of $\mathcal{P}(p, q)$, where we indicate the branch cut as a dotted line. It is an elementary but remarkable property that the winding number of $\mathcal{P}(p, q)$ with respect to each point p, q is zero. This property allows us to define $Z^\sharp(p, q, \cdot)$ as a single-valued analytic function on the contour.

Indeed, the Pochhammer contour is the typical example of path contained in the class of *first twisted homology* or *loaded cycle*, see [125, 130]. In other words, for any (single-valued) function $G(\zeta)$ defined on $\mathcal{P}(p, q)$, we have

$$\oint_{\mathcal{P}(p,q)} \partial_\zeta G(\zeta)\, d\zeta = 0. \tag{7.7.13}$$

We now present the following version of null-vector equations, which is precisely the first assertion of Theorem E.

Proposition 7.7.1. *For any $\kappa > 0$, $Z_\beta(r,\cdot)$ is a (non-trivial) real-valued solution to*

$$\partial_r Z = \frac{\kappa}{2}Z'' + HZ' + \Big(\frac{3}{\kappa} - \frac{1}{2}\Big)H'Z + C(r)Z, \tag{7.7.14}$$

where $C(r)$ is given as

$$C(r) = -\frac{6}{\kappa}\frac{\zeta_r(\pi)}{\pi} + \frac{1}{2(r+\chi)} - \Big(\frac{\beta\pi}{r+\chi}\Big)^2. \tag{7.7.15}$$

Proof. Recall that $Z^\sharp_\beta(p,q,\zeta) := \mathbf{E}\Psi^\sharp_\beta[p,q,\zeta]$. Combining Corollary 7.2.1 with (7.7.5),

$$\begin{aligned}\partial_r Z^\sharp_\beta = \frac{\kappa}{2}\partial^2 Z^\sharp_\beta + \Big[\Big(\frac{3}{\kappa} - \frac{1}{2}\Big)H' + H'(p-\zeta)\Big]Z^\sharp_\beta \\ - \Big(H\partial_q Z^\sharp_\beta + H(p-\zeta)\partial_\zeta Z^\sharp_\beta\Big) + C(r)Z^\sharp_\beta.\end{aligned}$$

Here we simplify the term $C(r,\boldsymbol{q})$ in (7.2.8) by the (2π-) periodicity of $H(r,\cdot)$ and the fact $q_2 - q_1 = 2\pi$.

On the other hand, observe that by the evaluation (7.7.7),

$$(\partial + \partial_q + \partial_\zeta)Z^\sharp(p,q,\zeta) = 0.$$

Therefore the above equation is rewritten as

$$\partial_r Z^\sharp_\beta = \frac{\kappa}{2}\partial^2 Z^\sharp_\beta + H\partial Z^\sharp_\beta + \Big(\frac{3}{\kappa} - \frac{1}{2}\Big)H'Z^\sharp_\beta + C(r)Z^\sharp_\beta + \partial_\zeta F(\zeta), \tag{7.7.16}$$

where

$$F(\zeta) := H(p-q)\,Z^\sharp_\beta - H(p-\zeta)Z^\sharp_\beta.$$

By integrating (7.7.16) with respect to screening variable ζ along $\mathcal{P}(p,q)$, the desired null-vector equation (7.7.14) follows from (7.7.13).

It remains to verify the real-valuedness and non-triviality of $Z_\beta(r,\cdot)$. For

Figure 7.4: A contour deformation

$\kappa > 4$, the expression (7.7.10) can be decomposed as

$$Z_\beta = C(\kappa) \sum_{j=1}^{4} \int_{\gamma_j} Z_\beta^\sharp(p, q, \zeta)\, d\zeta, \tag{7.7.17}$$

where the path γ_j's are given as in Figure 7.4.

We denote by $f(\cdot)$ the principal branch of $Z_\beta^\sharp(p, q, \cdot)$ defined on $\mathbb{C} \setminus ([-\infty, q] \cup [p, \infty])$. Then one can express $Z_\beta^\sharp(p, q, \cdot)$ on γ_j as

$$Z_\beta^\sharp(p, q, \zeta) = \begin{cases} f(\zeta) & \text{if} \quad \zeta \in \gamma_1, \\ f(\zeta) e^{-2\pi i \frac{4}{\kappa}\pi} & \text{if} \quad \zeta \in \gamma_2, \\ f(\zeta) & \text{if} \quad \zeta \in \gamma_3, \\ f(\zeta) e^{2\pi i \frac{4}{\kappa}\pi} & \text{if} \quad \zeta \in \gamma_4. \end{cases}$$

Therefore the Pochhammer contour integration (7.7.17) coincides with the following Euler type integral

$$Z_\beta(r, p - q) = \frac{1}{\Gamma(1 - 4/\kappa)} \int_q^p Z_\beta^\sharp(p, q, t)\, dt. \tag{7.7.18}$$

The integrability condition is satisfied both at p and q if and only if $\kappa > 4$. Note that the integrand $Z_\beta^\sharp(p, q, \cdot)$ is a positive function since $\Theta_\chi(r, \cdot)$ is a non-negative function in the interval $[0, 2\pi]$ vanishing only at 0,2π. Therefore the positivity of $Z_\beta(r, \cdot)$ for $\kappa > 4$ also follows.

We remark that as a function of κ, Z_β can be interpreted as an ana-

lytic continuation of the right-hand side of (7.7.18), which implies the real-valuedness of Z_β for $\kappa \leq 4$. Now it remains to check the non-triviality of Z_β. By (7.2.6), we have (up to a multiplicative constant)

$$\begin{aligned} Z_\infty(p-q) &:= \lim_{r\to\infty} Z_\beta(r, p-q) \\ &= C(\kappa) \sin^{\frac{2}{\kappa}}\left(\frac{p-q}{2}\right) \oint_{\mathcal{P}(p,q)} \sin^{-\frac{4}{\kappa}}\left(\frac{p-\zeta}{2}\right) \sin^{-\frac{4}{\kappa}}\left(\frac{\zeta-q}{2}\right) d\zeta. \end{aligned}$$

Therefore for $\kappa > 4$,

$$Z_\infty(p-q) = \frac{1}{\Gamma(1-4/\kappa)} \sin^{\frac{2}{\kappa}}\left(\frac{p-q}{2}\right) \int_q^p \sin^{-\frac{4}{\kappa}}\left(\frac{p-t}{2}\right) \sin^{-\frac{4}{\kappa}}\left(\frac{t-q}{2}\right) dt.$$

Recall that the regularized hypergeometric function

$${}_2\boldsymbol{F}_1(a,b;c;z) := \frac{{}_2F_1(a,b;c;z)}{\Gamma(c)}$$

is an entire function of a, b, c. Then by [108, Eq.(15.6.1)], one can express Z_∞ in terms of ${}_2\boldsymbol{F}_1$ as follows:

$$Z_\infty(x) = \cos^{\frac{2}{\kappa}-1}\left(\frac{x}{4}\right) \sin^{1-\frac{6}{\kappa}}\left(\frac{x}{4}\right) {}_2\boldsymbol{F}_1\left(\frac{1}{2}, 1-\frac{4}{\kappa}; \frac{3}{2}-\frac{4}{\kappa}, -\tan^2\left(\frac{x}{4}\right)\right). \quad (7.7.19)$$

Therefore by the identity theorem, (7.7.19) holds not only for $\kappa > 4$ but also for $\kappa \leq 4$. Thus we conclude that $Z_\beta(r,\cdot)$ is a non-trivial function for all $\kappa > 0$. □

Remark. We emphasize here that not only partition functions $Z_\beta(r,x)$, but also their derivatives with respect to β or χ, i.e.,

$$\partial_\beta^m \partial_\chi^n Z_\beta(r,x), \quad m,n \in \mathbb{Z}_+.$$

also satisfy the null-vector equations (7.7.1).

Example. As a consequence of Proposition 7.7.1, it is easy to observe that

Table 7.1: Partition functions Z_∞

κ	$Z_\infty(\cdot)$
4	$\sin_2^{-1/2}$
2	$\sin_2^{-1}\cot_2$
4/3	$\sin_2^{-3/2}\left(3\cot_2^2 - 2\cot_2' + \frac{1}{3}\right)$
1	$\sin_2^{-2}\left(4\cot_2^3 - 6\cot_2\cot_2' + \cot_2'' + \cot_2\right)$

Z_∞ solves the ordinary differential equation

$$0 = \frac{\kappa}{2}Z_\infty'' + \cot\left(\frac{x}{2}\right)Z_\infty' + \left(\frac{3}{\kappa} - \frac{1}{2}\right)\cot\left(\frac{x}{2}\right)' Z_\infty - \frac{1}{2\kappa}Z_\infty. \tag{7.7.20}$$

We remark that the ordinary differential equation (7.7.20) was also introduced by Zhan as a commutation relation for the radial SLE(κ, Λ) process, see [139, Section 4.5]. In particular for the values of κ such that $4/\kappa$ is a positive integer, the expression (7.7.19) is simplified and expressed in terms of trigonometric functions, see e.g., [108, Section 15.4]. See for instance Table 7.1, where we write $\sin_2(x) := \sin(x/2)$ and $\cot_2(x) = \cot(x/2)$ to lighten notations.

Example: Some particular solutions. Observe that if $4/\kappa$ is a positive integer, $Z_\beta^\sharp(p, q, \cdot)$ is a well-defined meromorphic function in $\mathbb{C}$ having poles only at p and q. Therefore one can choose sufficiently small circle around p as an integration contour in (7.7.10). In this case, it is straightforward to compute $Z \equiv Z_\beta$ by residue calculus. We present some particular solutions (up to multiplicative constant) in Table 7.2 for ER boundary condition with $\beta = 0$.

Table 7.2: Partition functions Z

κ	$Z(r,\cdot)$
4	$\Theta^{-1/2}$
2	$\Theta^{-1}H$
4/3	$\Theta^{-3/2}\Big(3H^2-2H'+4\frac{\zeta_r(\pi)}{\pi}\Big)$
1	$\Theta^{-2}\Big(4H^3-6HH'+H''+12\frac{\zeta_r(\pi)}{\pi}H\Big)$

Now we present the proof of Theorem E. Recall that

$$\Psi_\beta \equiv \Psi_\beta(\xi,q) := C(\kappa)\oint_{\mathcal{P}(\xi,q)} \Psi_\beta^\sharp(\xi,q,\zeta)\,d\zeta.$$

By definition, $\mathbf{E}\Psi = Z$ and $\Lambda = \kappa(\log Z)'$.

Proof of Theorem E. By Proposition 7.7.1, it remains to prove the second assertion. For any string $\mathcal{X}$ of fields in the OPE family $\mathcal{F}_{\boldsymbol{\beta}}$, let

$$R_\xi = \widehat{\mathbf{E}}\mathcal{X} := \frac{\mathbf{E}\Psi_\beta(\xi,q)\mathcal{X}}{\mathbf{E}\Psi_\beta(\xi,q)}.$$

Note that all we need to show is the following version of BPZ-Cardy equation

$$\frac{1}{a^2}\partial_\xi^2 R_\xi + \Lambda\partial_\xi R_\xi = \mathcal{L}_{v_\xi}R_\xi + \partial_r R_\xi. \tag{7.7.21}$$

By Theorem B, we have

$$\begin{aligned}\frac{1}{a^2}\partial_\xi^2\mathbf{E}[\Psi_\beta^\sharp(\xi,q,\zeta)\mathcal{X}] &= \mathcal{L}_{v_\xi}(q,S_\mathcal{X})\mathbf{E}[\Psi_\beta^\sharp(\xi,q,\zeta)\mathcal{X}]\\ &\quad+\mathcal{L}_{v_\xi}(\zeta)\mathbf{E}[\Psi_\beta^\sharp(\xi,q,\zeta)\mathcal{X}]\\ &\quad+(\partial_r - C(r))\mathbf{E}[\Psi_\beta^\sharp(\xi,q,\zeta)\mathcal{X}],\end{aligned} \tag{7.7.22}$$

where $C(r)$ is given as (7.7.15). Note that since $\Psi_\beta^\sharp$ is a $(1,0)$-differential with respect to ζ, we have

$$\mathcal{L}_{v_\xi}(\zeta)\mathbf{E}\Big[\Psi_\beta^\sharp(\xi,q,\zeta)\mathcal{X}\Big] = \partial_\zeta\Big[v_\xi(\zeta)\mathbf{E}[\Psi_\beta^\sharp(\xi,q,\zeta)\mathcal{X}]\Big].$$

Therefore by (7.7.13),

$$\oint_{\mathcal{P}(p,q)} \mathcal{L}_{v_p}(\zeta)\mathbf{E}[\Psi_\beta^\sharp(p,q,\zeta)\mathcal{X}]d\zeta = 0.$$

Integrating (7.7.22) with respect to ζ along $\mathcal{P}(p,q)$, we obtain

$$\frac{1}{a^2}\partial_\xi^2\Big(Z_\beta R_\xi\Big) = \mathcal{L}_{v_\xi}\Big(Z_\beta R_\xi\Big) + (\partial_r - C(r))\Big(Z_\beta R_\xi\Big). \tag{7.7.23}$$

Combining (7.7.23) with Proposition 7.7.1 we conclude (7.7.21), which completes the proof. □

Remark. For $\kappa = 4$, let γ be a contour around q not encircling p and write

$$\Psi = \mathcal{O}[a\cdot p - a\cdot q], \quad \Psi_\gamma = \int_\gamma \mathcal{O}[a\cdot p + a\cdot q - 2a\cdot\zeta]\,d\zeta.$$

Then one can easily see that $\mathbf{E}\Psi = \mathbf{E}\Psi_\gamma$. Thus the insertion of such one-leg operators produces martingale-observables for same SLE$(4,\Lambda)$. However, we emphasize here that these are not same in general as operators $\mathcal{X} \to \widehat{\mathcal{X}}$. For instance let us assume that γ encircles the node z. Then for ER boundary condition, we have

$$\frac{\mathbf{E}J(z)\Psi(p,q)}{\mathbf{E}\Psi(p,q)} = X(z) := w_z'\Big(H(w_p - w_z) - H(w_q - w_z)\Big),$$

where as

$$\frac{\mathbf{E}J(z)\Psi_\gamma(p,q)}{\mathbf{E}\Psi_\gamma(p,q)} = X(z) + Y(z), \quad Y(z) := \frac{w_z'\,\Theta'(0)\Theta(w_p - w_q)}{\Theta(w_p - w_z)\Theta(w_q - w_z)}.$$

Moreover, by combining Theorem C and E, we also obtain $Y_t(z)$ is a martingale-

observable for SLE$(4, \Lambda)$ where $\Lambda(r, x) = -H(r, x)$.

7.7.1 Partition functions for two commuting SLEs

We briefly discuss double SLEs with force points from the viewpoint of CFT. Although considerable research has been devoted to multiple SLEs in simply connected domains, rather less achievement has been obtained for those in multiply connected domains. Nevertheless, Dubédat presented null-vector equations for chordal type multiple SLEs in the chordal standard domain, see [46, Section 9]. We now describe such equations in the cylinder, whose specific case when $\kappa = 6$ is presented in [43]. Following the setup in [46], we consider an SLE(κ, Λ_1) starting from $x \in \mathbb{R}$ and an SLE(κ, Λ_2) starting from $y \in \mathbb{R}$, whose drift functions are given as

$$\Lambda_1 := \kappa\, \partial_x \log Z, \quad \Lambda_2 := \kappa\, \partial_y \log Z$$

for some partition function $Z \equiv Z(r, x, y, \boldsymbol{z})$. For simplicity, we assume that all force points are on $\mathbb{R}$ and there is no force point between x and y. Let us consider the following relation

$$\mathcal{M}_1\, Z = \mathcal{M}_2\, Z = 0, \tag{7.7.24}$$

where the operators $\mathcal{M}_1$ and $\mathcal{M}_2$ are given by

$$\begin{aligned} \mathcal{M}_1 :=& -\partial_r + \frac{\kappa}{2}\partial_x^2 + H(y-x)\partial_y + \sum_k H(z_k - x)\partial_k \\ &+ \Big(\frac{3}{\kappa} - \frac{1}{2}\Big) H'(y-x) + h_1 \end{aligned} \tag{7.7.25}$$

and

$$\begin{aligned} \mathcal{M}_2 :=& -\partial_r + \frac{\kappa}{2}\partial_x^2 + H(x-y)\partial_x + \sum_k H(z_k - y)\partial_k \\ &+ \Big(\frac{3}{\kappa} - \frac{1}{2}\Big) H'(x-y) + h_2. \end{aligned} \tag{7.7.26}$$

Here, $h_1 \equiv h_1(r, x, \boldsymbol{z})$ and $h_2 \equiv h_2(r, y, \boldsymbol{z})$ are some real-valued functions and ∂_k denotes the operator of differentiation with respect to the real variable z_k. We now present an explicit formula for the partition function Z satisfying (7.7.24). Moreover we explain a field theoretical interpretation of such functions h_1, h_2 in terms of Virasoro field with background charges.

Let $\boldsymbol{\beta} = \sum_k \beta_k \cdot z_k$ be an arbitrary divisor satisfying the neutrality condition $\sum_k \beta_k = 0$ and set

$$\begin{aligned} Z^\sharp(x, y, \zeta, \boldsymbol{z}) :&= \mathbf{E}\, \mathcal{O}_{\boldsymbol{\beta}}[a \cdot x + a \cdot y - 2a \cdot \zeta, \mathbf{0}] \\ &= \frac{\mathbf{E}\, \mathcal{O}[a \cdot x + a \cdot y - 2a \cdot \zeta + \boldsymbol{\beta}, \mathbf{0}]}{\mathbf{E}\, \mathcal{O}[\boldsymbol{\beta}, \mathbf{0}]}. \end{aligned}$$

Then, by (5.5.1), the evaluation of $Z^\sharp$ in the identity chart is given as

$$\begin{aligned} Z^\sharp(x, y, \zeta, \boldsymbol{z}) = \Theta_\chi'(0)^{\frac{6}{\kappa}} \Theta_\chi(y - x)^{\frac{2}{\kappa}} \prod \Theta_\chi(x - z_k)^{a\beta_k} \Theta_\chi(y - z_k)^{a\beta_k} \\ \times \Theta_\chi(\zeta - x)^{-\frac{4}{\kappa}} \Theta_\chi(y - \zeta)^{-\frac{4}{\kappa}} \prod \Theta_\chi(\zeta - z_k)^{-2a\beta_k}. \end{aligned}$$

The following proposition holds for each $\chi \in [0, \infty]$.

Proposition 7.7.2. *Let $\mathcal{P}(x, y)$ be a Pochhammer contour entwining x and y, which does not encircle z_k for every $k = 1, \cdots, N$. Then the partition function*

$$Z(x, y, \boldsymbol{z}) := \oint_{\mathcal{P}(x,y)} Z^\sharp(x, y, \zeta, \boldsymbol{z})\, d\zeta$$

is real-valued and satisfies (7.7.24)*. Moreover the functions $h_1(r, x, \boldsymbol{z})$ and $h_2(r, y, \boldsymbol{z})$ are given as*

$$h_1(r, \cdot, \boldsymbol{z}) = h_2(r, \cdot, \boldsymbol{z}) = \Big(1 - \frac{6}{\kappa}\Big) \frac{\zeta_r(\pi)}{\pi} - 2\mathbf{E}\, T_{\boldsymbol{\beta}}(\cdot). \tag{7.7.27}$$

Proof. Since the partition function Z is symmetric for the variables x and y, it suffices to show $\mathcal{M}_1 Z = 0$. Throughout the proof, we write $Z_{\boldsymbol{\beta}} = \mathbf{E}\mathcal{O}[\boldsymbol{\beta}, \mathbf{0}]$ to lighten notations.

Let $\lambda_k := \beta_k^2/2 - b\beta_k$ be the conformal dimension of

$$\mathbf{E}\mathcal{O}[a\cdot x + a\cdot y - 2a\cdot\zeta + \boldsymbol{\beta}, \mathbf{0}]$$

at z_k. By applying Corollary 7.2.1 to $\mathbf{E}\mathcal{O}[a\cdot x+a\cdot y-2a\cdot\zeta+\boldsymbol{\beta},\mathbf{0}]$, we obtain

$$\begin{aligned}0 =&\Big[-\partial_r + \frac{\kappa}{2}\partial_x^2 + H(y-x)\partial_y + \sum H(z_k-x)\partial_k + \Big(\frac{3}{\kappa}-\frac{1}{2}\Big)H'(y-x)\Big]Z^\sharp\\ &+\Big[\sum\Big(\lambda_k H'(z_k-x) + H(z_k-x)\frac{\partial_k Z_{\boldsymbol{\beta}}}{Z_{\boldsymbol{\beta}}}\Big) - \frac{\partial_r Z_{\boldsymbol{\beta}}}{Z_{\boldsymbol{\beta}}} + C(r)\Big]Z^\sharp\\ &+\partial_\zeta\Big[H(\zeta-x)Z^\sharp\Big].\end{aligned}$$

By integrating above equation with respect to the variable ζ along the contour $\mathcal{P}(x,y)$, we conclude that $\mathcal{M}_1 Z = 0$, where

$$\begin{aligned}h_1(r,x,\boldsymbol{z}) =& \Big(1-\frac{6}{\kappa}\Big)\frac{\zeta_r(\pi)}{\pi} - 2\mathbf{E}\,T_{(b)}(x)\\ &+\sum_k\Big[\Big(\frac{\beta_k^2}{2}-b\beta_k\Big)H'(z_k-x) + H(z_k-x)\frac{\partial_k Z_{\boldsymbol{\beta}}}{Z_{\boldsymbol{\beta}}}\Big] - \frac{\partial_r Z_{\boldsymbol{\beta}}}{Z_{\boldsymbol{\beta}}}.\end{aligned}$$

Here we have used the fact the constant $C(r)$ is related to $\mathbf{E}\,T_{(b)}$ as

$$C(r) = -2\mathbf{E}\,T_{(b)} - 2h_{1,2}\frac{\zeta_r(\pi)}{\pi}.$$

The real-valuedness of Z can be similarly obtained following the proof of Proposition 7.7.1, where the requirement that there is no force point between x and y is used.

We now prove the second assertion of the proposition for ER boundary condition. The verification for generic values of χ follows from the same argument with slight modifications. By direct calculation, we have

$$\mathbf{E}\,T_{\boldsymbol{\beta}}(x) = \mathbf{E}\,T_{(b)}(x) + \frac{1}{8}\Big(\sum\beta_k H(z_k-x)\Big)^2 + \frac{b}{2}\Big(\sum\beta_k H'(z_k-x)\Big).$$

Therefore to prove (7.7.27), it suffices to show

$$\sum_k \Big(\frac{\beta_k^2}{2}H'(z_k-x)+H(z_k-x)\frac{\partial_k Z_{\boldsymbol{\beta}}}{Z_{\boldsymbol{\beta}}}\Big)=\frac{\partial_r Z_{\boldsymbol{\beta}}}{Z_{\boldsymbol{\beta}}}-\frac{1}{4}\Big(\sum \beta_k H(z_k-x)\Big)^2.$$

Note that by (5.5.1),

$$Z_{\boldsymbol{\beta}}=\Theta'(0)^{\sum\frac{\beta_k^2}{2}}\prod_{k<l}\Theta(z_k-z_l)^{\beta_k\beta_l}.$$

Therefore by (3.1.12) and (3.1.25),

$$\frac{\partial_r Z_{\boldsymbol{\beta}}}{Z_{\boldsymbol{\beta}}}=-\frac{3}{2}\Big(\sum\beta_k^2\Big)\frac{\zeta_r(\pi)}{\pi}+\sum_{k<l}\beta_k\beta_l\Big(\frac{1}{2}H'+\frac{1}{4}H^2\Big)(z_k-z_l).$$

On the other hand, by the pseudo-addition formula (3.1.41),

$$\begin{aligned}&\sum_k H(z_k-x)\frac{\partial_k Z_{\boldsymbol{\beta}}}{Z_{\boldsymbol{\beta}}}=\frac{1}{2}\sum_{k<l}\beta_k\beta_l\,H(z_k-z_l)\Big(H(z_k-x)-H(z_l-x)\Big)\\&=\frac{1}{4}\sum_{k<l}\beta_k\beta_l\Big[H^2(z_k-z_l)+\Big(H(z_k-x)-H(z_l-x)\Big)^2\Big]\\&+\frac{1}{2}\sum_{k<l}\beta_k\beta_l\Big[H'(z_k-z_l)+H'(z_k-x)+H'(z_l-x)\Big]+\frac{3}{2}\Big(\sum_{k<l}2\beta_k\beta_l\Big)\frac{\zeta_r(\pi)}{\pi}.\end{aligned}$$

Combining above equations with the neutrality condition $\sum\beta_k=0$, the desired identity follows. □

7.7.2 Crossing annulus SLE partition function

Let $p\in\mathbb{R}$ and $q\in\mathbb{R}_r$. Without loss of generality write $p=x$, $q=ir$. Set

$$\Psi:=\oint_\gamma\Psi^\sharp\,d\zeta,\quad \Psi^\sharp:=\mathcal{O}\Big[a\cdot p+\frac{a}{2}\cdot q-2a\cdot\zeta\,,\,\frac{a}{2}\cdot q\Big],\tag{7.7.28}$$

where γ is the Pocchhammer contour entwining $q,\bar{q}$. In the sequel, we sometimes use the convention that two functions are equal if they differ by a multiplicative constant.

By definition, the pre-screened partition function $Z^\sharp$ evaluates as

$$\begin{aligned} Z^\sharp &= |\Theta_\chi(x-ir)|^{\frac{2}{\kappa}}\Theta_\chi(x-\zeta)^{-\frac{4}{\kappa}}\Theta_\chi(\zeta-ir)^{-\frac{2}{\kappa}}\Theta_\chi(\zeta+ir)^{-\frac{2}{\kappa}} \\ &= |\Theta(x-ir)|^{\frac{2}{\kappa}}\Theta(x-\zeta)^{-\frac{4}{\kappa}}\Theta(\zeta-ir)^{-\frac{2}{\kappa}}\Theta(\zeta+ir)^{-\frac{2}{\kappa}}\exp\Big(-\frac{(x-2\zeta)^2}{2\kappa(r+\chi)}\Big). \end{aligned}$$

Note that it follows from the relations

$$\Theta(z-ir) = -i\,\Theta_I(z)\exp\Big(\frac{r}{4}+\frac{iz}{2}\Big), \quad \Theta(z+ir) = i\,\Theta_I(z)\exp\Big(\frac{r}{4}-\frac{iz}{2}\Big).$$

that

$$\Theta(z-ir)\Theta(z+ir) = \Theta_I(z)^2\exp\Big(\frac{r}{2}\Big). \tag{7.7.29}$$

Therefore $Z^\sharp$ is rewritten as

$$Z^\sharp(x,\zeta) = \Theta_I(x)^{\frac{2}{\kappa}}\Theta(x-\zeta)^{-\frac{4}{\kappa}}\Theta_I(\zeta)^{-\frac{4}{\kappa}}\exp\Big(-\frac{(x-2\zeta)^2}{2\kappa(r+\chi)}\Big) \tag{7.7.30}$$

and we have

$$Z(x) = \Theta_I(x)^{\frac{2}{\kappa}}\oint_\gamma \Theta(x-\zeta)^{-\frac{4}{\kappa}}\Theta_I(\zeta)^{-\frac{4}{\kappa}}\exp\Big(-\frac{(x-2\zeta)^2}{2\kappa(r+\chi)}\Big)\,d\zeta. \tag{7.7.31}$$

On the other hand, due to BPZ equation (Theorem B), we have

$$\partial_r Z^\sharp = \frac{\kappa}{2}\partial_x^2 Z^\sharp + H_I\partial_x Z^\sharp + \Big(\frac{3}{\kappa}-\frac{1}{2}\Big)H_I' Z^\sharp + C(r)Z^\sharp + \partial_\zeta F(\zeta), \tag{7.7.32}$$

where

$$F(\zeta) := H_I(x)\,Z^\sharp - H(x-\zeta)Z^\sharp.$$

Therefore it follows from

$$\oint_\gamma \partial_\zeta F(\zeta)d\zeta \equiv 0$$

that Z satisfies the null-vector equation (7.7.2).

Let us also present direct computations for (7.7.32).

Proof. For simplicity, let us consider the ER case and set

$$Z^\sharp = \Theta_I(x)^{\frac{2}{\kappa}}\Theta(x-\zeta)^{-\frac{4}{\kappa}}\Theta_I(\zeta)^{-\frac{4}{\kappa}}. \tag{7.7.33}$$

It suffices to check that

$$\begin{aligned}\frac{\partial_r Z^\sharp}{Z^\sharp} &= \frac{\kappa}{2}\Big[\partial_x\Big(\frac{\partial_x Z^\sharp}{Z^\sharp}\Big)+\Big(\frac{\partial_x Z^\sharp}{Z^\sharp}\Big)^2\Big]+H_I(x)\frac{\partial_x Z^\sharp}{Z^\sharp}\\ &\quad+\Big(\frac{3}{\kappa}-\frac{1}{2}\Big)H_I'(x)+H'(x-\zeta)+\Big(H_I(x)-H(x-\zeta)\Big)\frac{\partial_\zeta Z^\sharp}{Z^\sharp}\end{aligned} \tag{7.7.34}$$

up to additive constant.

Since

$$\frac{\partial_x Z^\sharp}{Z^\sharp}=\frac{1}{\kappa}H_I(x)-\frac{2}{\kappa}H(x-\zeta),\quad \frac{\partial_\zeta Z^\sharp}{Z^\sharp}=\frac{2}{\kappa}H(x-\zeta)-\frac{2}{\kappa}H_I(\zeta),$$

we have

$$\begin{aligned}&\frac{\kappa}{2}\Big[\partial_x\Big(\frac{\partial_x Z^\sharp}{Z^\sharp}\Big)+\Big(\frac{\partial_x Z^\sharp}{Z^\sharp}\Big)^2\Big]\\ &=\frac{1}{2}H_I'(x)-H'(x-\zeta)+\frac{1}{2\kappa}H_I^2(x)+\frac{2}{\kappa}H^2(x-\zeta)-\frac{2}{\kappa}H_I(x)H(x-\zeta)\end{aligned}$$

and

$$H_I(x)\frac{\partial_x Z^\sharp}{Z^\sharp}=\frac{1}{\kappa}H_I^2(x)-\frac{2}{\kappa}H_I(x)H(x-\zeta).$$

Therefore

$$\begin{aligned}&\frac{\kappa}{2}\Big[\partial_x\Big(\frac{\partial_x Z^\sharp}{Z^\sharp}\Big)+\Big(\frac{\partial_x Z^\sharp}{Z^\sharp}\Big)^2\Big]+H_I(x)\frac{\partial_x Z^\sharp}{Z^\sharp}+\Big(\frac{3}{\kappa}-\frac{1}{2}\Big)H_I'(x)+H'(x-\zeta)\\ &=\frac{3}{\kappa}H_I'(x)+\frac{3}{2\kappa}H_I^2(x)+\frac{2}{\kappa}H^2(x-\zeta)-\frac{4}{\kappa}H_I(x)H(x-\zeta).\end{aligned} \tag{7.7.35}$$

Note that

$$\Big(H_I(x)-H(x-\zeta)\Big)\frac{\partial_\zeta Z^\sharp}{Z^\sharp}=\frac{2}{\kappa}H_I(x)H(x-\zeta)\\ -\frac{2}{\kappa}H(x-\zeta)^2-\frac{2}{\kappa}H_I(\zeta)\Big(H_I(x)-H(x-\zeta)\Big).$$

Then it follows from (3.1.41) that

$$\begin{aligned}\Big(H_I(x)-H(x-\zeta)\Big)\frac{\partial_\zeta Z^\sharp}{Z^\sharp}&=\frac{4}{\kappa}H_I(x)H(x-\zeta)-\frac{3}{\kappa}H^2(x-\zeta)\\ &\quad-\frac{1}{\kappa}H_I^2(\zeta)-\frac{1}{\kappa}H_I^2(x)\\ &\quad-\frac{2}{\kappa}H_I'(\zeta)-\frac{2}{\kappa}H_I'(x)-\frac{2}{\kappa}H'(x-\zeta).\end{aligned}\tag{7.7.36}$$

Combining (7.7.35) and (7.7.36), we conclude that the right-hand side of (7.7.34) is given by

$$\frac{2}{\kappa}\Big(\frac{1}{2}H_I'+\frac{1}{4}H_I^2\Big)(x)-\frac{4}{\kappa}\Big(\frac{1}{2}H'+\frac{1}{4}H^2\Big)(x-\zeta)-\frac{4}{\kappa}\Big(\frac{1}{2}H_I'+\frac{1}{4}H_I^2\Big)(\zeta).$$

Now (7.7.34) follows from the heat equation (3.1.12) of theta functions. □

We now verify the real-valuedness and the non-triviality. As before, it suffices to consider the case $\kappa>4$. Then the case $\kappa\le 4$ follows from argument using the analytic continuation.

Suppose that $\kappa>4$ and let $\gamma=[\bar{q},q]$. Then up to multiplicative constant we have

$$Z=\Theta_I(x)^{\frac{2}{\kappa}}\int_{-r}^{r}\Theta(x-is)^{-\frac{4}{\kappa}}\Theta_I(is)^{-\frac{4}{\kappa}}\exp\Big(-\frac{(x-2is)^2}{2\kappa(r+\chi)}\Big)\,ds.$$

Then the real-valuedness easily follows from

$$\begin{aligned}Z&=\Theta_I(x)^{\frac{2}{\kappa}}\int_{-r}^{r}\Theta(x+is)^{-\frac{4}{\kappa}}\Theta_I(-is)^{-\frac{4}{\kappa}}\exp\Big(-\frac{(x+2is)^2}{2\kappa(r+\chi)}\Big)\,ds\\ &=\overline{\Theta_I(x)^{\frac{2}{\kappa}}\int_{-r}^{r}\Theta(x-is)^{-\frac{4}{\kappa}}\Theta_I(is)^{-\frac{4}{\kappa}}\exp\Big(-\frac{(x-2is)^2}{2\kappa(r+\chi)}\Big)\,ds}=\bar{Z}.\end{aligned}$$

On the other hand, using (3.1.9), we have

$$\begin{aligned} Z_\infty(x) :&= \lim_{r\to\infty} Z(r,x) = \int_{\mathbb{R}} \sin^{-\frac{4}{\kappa}}\Big(\frac{x-is}{2}\Big)\, ds \\ &= \int_{\mathbb{R}} \sin^{-\frac{4}{\kappa}}\Big(\frac{x+is}{2}\Big)\, ds = 2\int_{\mathbb{R}} \sin^{-\frac{4}{\kappa}}\Big(\frac{x}{2}+is\Big)\, ds \\ &\equiv \frac{2}{\sqrt{\pi}}\cos\Big(\frac{2\pi}{\kappa}\Big)\Gamma\Big(\frac{1}{2}-\frac{2}{\kappa}\Big)\Gamma\Big(\frac{2}{\kappa}\Big), \end{aligned}$$

which leads to the non-triviality of Z.

7.7.3 A classical limit $\kappa \to 0$

In this subsection, we discuss a classical limit $\kappa \to 0$ of the null-vector equation and its solutions. In a simply connected domain, such differential equations for multiple SLE(0) (equivalently, multichordal geodesic) were studied by Peltola and Wang [111].

Recall that the null-vector equation (7.7.1) is equivalent to the fact that the drift function $\Lambda := \kappa(\log Z)'$ satisfies

$$\partial_r \Lambda = \frac{\kappa}{2}\Lambda'' + \Big(3-\frac{\kappa}{2}\Big)H'' + \Lambda H' + H'\Lambda + \Lambda\Lambda'. \tag{7.7.37}$$

For a solution Z to the null-vector equation (7.7.1), let us define

$$Z_0 := \lim_{\kappa\to 0} Z^\kappa, \quad \Lambda_0 := (\log Z_0)', \tag{7.7.38}$$

if the limit exists. Then the functions Z_0, Λ_0 satisfy following non-linear partial differential equations

$$\partial_r Z_0 = HZ_0' + 3H'Z_0 + \frac{(Z_0')^2}{2Z_0} + C_0(r)Z \tag{7.7.39}$$

$$\partial_r \Lambda = 3H'' + \Lambda H' + H'\Lambda + \Lambda\Lambda'. \tag{7.7.40}$$

On the other hand, by means of screening, we have general solutions

$$Z(r,x) = \Theta(r,x)^{\frac{2}{\kappa}} \oint \Theta(r,x-\zeta)^{-\frac{4}{\kappa}} \Theta(r,\zeta)^{-\frac{4}{\kappa}} \exp\Big(-\frac{(x-2\zeta)^2}{2\kappa(r+\chi)}\Big)\, d\zeta \quad (7.7.41)$$

to the null-vector equation (7.7.1). Thus by the method of stationary phase, we formally obtain

$$Z_0(r,x) = \lim_{\kappa\to 0} \max_{0\le\zeta\le x} \Big[\Theta(r,x)^2 \Theta(r,x-\zeta)^{-4} \Theta(r,\zeta)^{-4} \exp\Big(-\frac{(x-2\zeta)^2}{2(r+\chi)}\Big)\Big].$$

Note that the critical point ζ of the integrand satisfies

$$H(r,x-\zeta) - H(r,\zeta) + \frac{x-2\zeta}{r+\chi} = 0. \quad (7.7.42)$$

Therefore we have $\zeta = x/2$. Thus we obtain that

$$Z_0(r,x) = \Theta(r,x)^2 \Theta(r,x/2)^{-8} \quad (7.7.43)$$

$$\Lambda_0(r,x) = H(r,x) - 2H(r,x/2), \quad (7.7.44)$$

which corresponds Zhan's special solution found in [139, Section 8.4].

Chapter 8

Martingale-observables for continuum interface curves

In this chapter we discuss a way to produce martingale-observables associated with scaling limits of lattice models and present several examples.

8.1 GFF observables

Since Schramm and Sheffield discovered the relation between level lines of discrete GFF and SLE(4) type curves (see [118]), it has played an important role in the study of random conformal geometries, see e.g., [48]. One way of observing such relations is to obtain the coupling of GFF and SLE in the continumm level by means of associated bosonic observables. In this spirit, Izyurov and Kytölä proposed a general framework to develop Schramm-Sheffield's coupling relations in various conformal setups including doubly connected domains with Dirichlet or Dirichlet-Neumann mixed boundary conditions with several force points.

For $\kappa = 4$, let us consider the one-leg operator Ψ_β of the form

$$\Psi_\beta(p, q) := \mathcal{O}_{\boldsymbol{\beta}}[a \cdot p - a \cdot q], \tag{8.1.1}$$

where $\boldsymbol{\beta}$ is given as (7.7.3). For the chordal case when $p, q \in \mathbb{R}$, we have

$$Z_\beta(r,x) = \Theta(r,x)^{-\frac{1}{2}} \exp\Big(-\frac{(x-\mu)^2}{8(r+\chi)}\Big), \quad \mu = 2\beta\pi/a. \tag{8.1.2}$$

Here we write $p = x$ and $q = 0$.

For the Dirichlet boundary condition (i.e., $\chi = 0$), such GFFs correspond to those studied in [63, 69]. On the other hand, the associated partition functions are presented in [139, Section 8.1] as a specific solutions for the null-vector equation with $\kappa = 4$. Here one may find a simple geometric interpretation of $\beta \in \mathbb{R}$ since the bosonic field $\widehat{\Phi}_{\boldsymbol{\beta}}$ has a piecewise Dirichlet boundary condition whose value is given as (up to additive constant)

$$\mathbf{E}\widehat{\Phi}_{\boldsymbol{\beta}}(z) = \begin{cases} -\lambda & \text{if} \quad z \in (0,p) \cup (q, 2\pi), \\ +\lambda & \text{if} \quad z \in (p,q), \\ (1-\mu/\pi)\lambda & \text{if} \quad z \in \mathbb{R}_r, \end{cases} \tag{8.1.3}$$

where $\lambda = a\pi = \pi/\sqrt{2}$.

For the weight function $\omega(n) := (-1)^{-\frac{n}{2}}$, one can easily observe from (3.1.14) that (up to multiplicative constant)

$$\begin{aligned} Z_\beta^\omega(r,x) :&= \sum_{n\in\mathbb{Z}} Z_\beta(r, x+2\pi n) \\ &= \Theta(r,x)^{-\frac{1}{2}} \sum_{n\in\mathbb{Z}} \exp\Big(-\frac{(x-\mu)^2}{8(r+\chi)}\Big) \\ &= \Theta(r,x)^{-\frac{1}{2}} \Theta_I(2r+2\chi, x-\mu+\pi). \end{aligned}$$

In particular, for $\chi = 0$ and $\beta = a/2$, we have

$$Z_\beta^\omega(r,x) = \Theta(r,x)^{-\frac{1}{2}} \Theta_I(2r,x)$$

and the SLE$(4,\Lambda)$ associated with this partition function is coupled with GFF having specific height gap on the outer boundary component which

obeys Neumann condition on the inner one, see [63, 69].

8.1.1 Hitting probabilities of the inner boundary component

For SLE$(4,\Lambda)$ processes associated with partition functions of the form (8.1.2), we obtain their inner circle hitting probabilities by basic properties of Brownian motions.

Proposition 8.1.1. *Let η be the trace of* SLE$(4,\Lambda)$ *whose partition function is given as* (8.1.2)*. Then the probability $\mathbf{P}_\chi^\mu(x)$ that η hits the inner boundary component is given by*

$$\begin{aligned}\mathbf{P}_\chi^\mu(x) &= \frac{1}{4\pi}\sqrt{\frac{r+\chi}{\chi}}e^{\frac{(x-\mu)^2}{8(r+\chi)}} \\ &\times \int_0^{2\pi} e^{-\frac{(s-\mu)^2}{\sqrt{8}\chi}}\left[\Theta_I\Big(\frac{r}{2},\frac{x-s}{2}+\pi\Big)-\Theta_I\Big(\frac{r}{2},\frac{x+s}{2}+\pi\Big)\right]ds.\end{aligned} \tag{8.1.4}$$

Proof. By definition and (8.1.2), the driving process ξ_t of SLE$(4,\Lambda)$ is given by

$$d\xi_t = 2\,dB_t - \left(H(r-t,\xi_t-q_t)+\frac{\xi_t-q_t-\mu}{r+\chi-t}\right)dt, \quad q_t := \tilde{g}_t(q).$$

We denote by $X_t := \xi_t - q_t$ the angle difference process. Then $\mathbf{P}_\chi^\mu(x)$ is same with the probability that X_t stays in the interval $[0,2\pi]$ up to time r.

By Loewner's equation, X_t satisfies following SDE:

$$dX_t := 2\,dB_t - \frac{X_t-\mu}{r+\chi-t}\,dt, \quad X_0 = x.$$

Therefore the process X_t is identified to a Brownian bridge starting from x, which reaches μ at time $t = r+\chi$, see e.g., [107]. Thus we obtain

$$\mathbf{P}_\chi^\mu(x) = \Pr\left\{0 < \min_{0\le s\le r} B(s) < \max_{0\le s\le r} B(s) < \pi \,\middle|\, B(0)=\frac{x}{2}, B(r+\chi)=\frac{\mu}{2}\right\}.$$

Thus it follows from the well-known density of Brownian motion that

$$\begin{aligned}\mathbf{P}^{\mu}_{\chi}(x) &= \sqrt{2\pi(r+\chi)}\exp\Big(\frac{(\mu/2-x/2)^2}{2(r+\chi)}\Big)\\ &\quad\times\int_0^{\pi}\frac{1}{\sqrt{2\pi\chi}}\exp\Big(-\frac{(z-\mu/2)^2}{2\chi}\Big)\,d\sigma(z),\end{aligned}\tag{8.1.5}$$

where $d\sigma$ is given as

$$\begin{aligned}d\sigma(z) &= \Pr\Big\{0<\min_{0\le s\le r}B(s)<\max_{0\le s\le r}B(s)<\pi, B(r)\in dz\Big|B(0)=\frac{x}{2}\Big\}\\ &= \frac{dz}{\sqrt{2\pi r}}\sum_{n\in\mathbb{Z}}\Big[\exp\Big(-\frac{(x-2z+4n\pi)^2}{2r}\Big)-\exp\Big(-\frac{(x+2z+4n\pi)^2}{2r}\Big)\Big].\end{aligned}$$

Now proposition follows from (3.1.14). □

Note that for ER boundary condition ($\chi=\infty$), the angle difference process X_t is simply a Brownian motion with speed 2. In this case (8.1.4) is given as

$$\mathbf{P}^{\mu}_{\infty}(x)=\frac{1}{4\pi}\int_0^{2\pi}\Big[\Theta_I\Big(\frac{r}{2},\frac{x-s}{2}+\pi\Big)-\Theta_I\Big(\frac{r}{2},\frac{x+s}{2}+\pi\Big)\Big]\,ds,\tag{8.1.6}$$

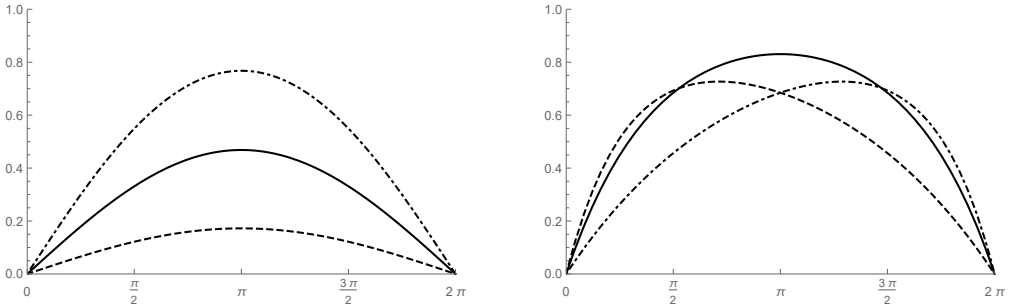
which corresponds to the formula given in [106, Theorem 7.45]. On the other hand, for Dirichlet boundary condition ($\chi=0$), due to the Gaussian approximation of Dirac delta measure

$$\frac{1}{\sqrt{8\chi\pi}}\exp\Big(-\frac{(s-\mu)^2}{\sqrt{8\chi}}\Big)\to\delta_{\mu}(s),$$

we have

$$\begin{aligned}\mathbf{P}^{\mu}_{0}(x) &= \mathbf{1}_{(0,2\pi)}(\mu)\cdot\sqrt{\frac{r}{2\pi}}\exp\Big(\frac{(x-\mu)^2}{8r}\Big)\\ &\quad\times\Big[\Theta_I\Big(\frac{r}{2},\frac{x-\mu}{2}+\pi\Big)-\Theta_I\Big(\frac{r}{2},\frac{x+\mu}{2}+\pi\Big)\Big].\end{aligned}\tag{8.1.7}$$

By (3.1.14), this probability agrees with the one presented in [63]. Observe here that the requirement $\mu\in(0,2\pi)$ for the positive probability is same as



(a) $\mathbf{P}^{\mu}_{\infty}(x)$ (b) $\mathbf{P}^{\mu}_{0}(x)$

Figure 8.1: (a): The plot displays $\mathbf{P}^{\mu}_{\infty}(x)$ with modulus $r = 1$ (dot-dashed line), $r = 2$ (full line) and $r = 4$ (dashed line). (b): The plot displays $\mathbf{P}^{\mu}_{0}(x)$ with $r = 2$ for $\mu = \pi/2$ (dot-dashed line), $\mu = \pi$ (full line) and $\mu = 3\pi/2$ (dashed line).

the one that the boundary value of $\widehat{\Phi}_{\boldsymbol{\beta}}$ on the inner circle is between two heights on the outer circle, see (8.1.3).

On the other hand, by the asymptotic behavior of Θ_I when $r \to \infty$

$$\Theta_I(r, z) = 1 - e^{-r}\cos(z) + O(e^{-2r}), \tag{8.1.8}$$

we obtain that as $r \to \infty$,

$$\mathbf{P}^{\mu}_{0}(x) = (1 + o(1))\sqrt{\frac{8r}{\pi}}\sin\left(\frac{\mu}{2}\right)e^{-r/2}\sin\left(\frac{x}{2}\right). \tag{8.1.9}$$

This asymptotic is closely related to Green's function for SLE(4), see e.g., [7, 98, 114]. In particular, the exponent $r/2$ is related to Hausdorff dimension $d = 1 + \kappa/8$ as $r/2 = (2 - d)r$.

We now discuss the observables for $\mathbf{P}^{\mu}_{0}(x)$ using CFT correlation functions. For $\chi = 0$, set

$$\mathcal{O}^{1,n} := \mathcal{O}[a \cdot q - a \cdot q_n], \quad \mathcal{O}^{2,n} := \mathcal{O}[a \cdot q - a \cdot q_n - 2\boldsymbol{\beta}],$$

where $q_n := q - 4n\pi$. Then one can check that after renormalization, the

vertex fields $\widehat{\mathcal{O}}^{1,n}$, $\widehat{\mathcal{O}}^{2,n}$ have conformal dimension 0 and evaluates as

$$\begin{aligned} M^{1,n} &:= \mathbf{E}\widehat{\mathcal{O}}^{1,n} = \exp\Big(\frac{(x-\mu)^2}{8r}\Big)\exp\Big(-\frac{(x-4\pi n-\mu)^2}{8r}\Big), \\ M^{2,n} &:= \mathbf{E}\widehat{\mathcal{O}}^{2,n} = \exp\Big(\frac{(x-\mu)^2}{8r}\Big)\exp\Big(-\frac{(x-4\pi n+\mu)^2}{8r}\Big). \end{aligned} \tag{8.1.10}$$

Here notice that by (3.1.14), we have

$$\mathbf{P}_0^{\mu}(x) = \sum_{n\in\mathbb{Z}}(M^{1,n} - M^{2,n}).$$

Such martingale-observables (8.1.10) are utilized in [63] to calculate $\mathbf{P}_0^{\mu}(x)$.

8.1.2 Crossing case

Let us consider the case that $p \in \mathbb{R}$, $q+ir \in \mathbb{R}_r$. Then the partition function associated to the one-leg operator (8.1.1) is given as

$$Z_\beta(r,x) = \Theta_I(r,x)^{-\frac{1}{2}}\exp\Big(-\frac{(x-\mu)^2}{8(r+\chi)}\Big), \quad \mu = 2\beta\pi/a.$$

In particular, if $\chi = \beta = 0$, each partition function $Z_\beta^n(x) := Z_\beta(x+2n\pi)$ corresponds to the crossing type SLE$(4,\Lambda)$ connecting two marked points with prescribed winding number, see [139, Section 6.3]. Moreover the 2π-periodic partition function

$$Z_\beta^\omega(r,x) := \Theta_I(r,x)^{-1/2}\Theta_I(2r,x+\pi), \quad \omega(n)\equiv 1 \tag{8.1.11}$$

agrees with Zhan's Feynman-Kac expression of the crossing type annulus partition function, see [139, Section 8.1].

8.1.3 GFF with general boundary data

In this subsection, we consider GFF with general Dirichlet boundary condition on the inner circle. We refer to [113] for this type of GFF and its

coupling with SLE in a simply connected domain.

Here we focus on the Dirichlet boundary condition, i.e., $\chi = 0$. For given marked points $p, q \in \mathbb{R}$, set

$$\Psi(p,q) := \mathcal{O}_{\boldsymbol{\beta}}[a \cdot p - a \cdot q], \tag{8.1.12}$$

where $\boldsymbol{\beta}$ is a measure defined on the upper boundary component of $\mathcal{C}_r$ satisfying

$$\int \boldsymbol{\beta} = 0. \tag{8.1.13}$$

The associated bosonic field $\mathbf{E}\widehat{\Phi}_{\boldsymbol{\beta}}$ satisfies the boundary condition

$$\mathbf{E}\widehat{\Phi}_{\boldsymbol{\beta}}(z) = \begin{cases} -\lambda & \text{if} \quad z \in (0,p) \cup (q, 2\pi), \\ +\lambda & \text{if} \quad z \in (p,q), \\ \Big(1 - \dfrac{2}{a}\displaystyle\int_0^{\operatorname{Re} z} d\boldsymbol{\beta}(s)\Big)\lambda & \text{if} \quad z \in \mathbb{R}_r. \end{cases} \tag{8.1.14}$$

Thus the neutrality condition (8.1.13) is equivalent to the fact that such harmonic function is well-defined in the cylinder.

To see (8.1.14), note that if $\boldsymbol{\beta}$ is a linear combination of atomic measure of the form

$$\boldsymbol{\beta} = \sum_{k=1}^{n} \frac{\beta_k}{n} \cdot \delta_{q_k}, \quad q_k \in \mathbb{R}_r, \tag{8.1.15}$$

the correlation function of the bosonic observable has piecewise Dirichlet boundary condition having additional jump of $2\pi a$ at p, $-2\pi a$ at q and $2\pi\beta_k/n$ at q_k's.

The associated partition function $Z(p,q) \equiv Z_{\boldsymbol{\beta}}(p,q)$ is evaluated as

$$\begin{aligned} Z(p,q) &= \Theta'(0)^{\frac{1}{2}} \tilde{\Theta}(p-q)^{-\frac{1}{2}} \prod_{k=1}^{n} \Big(\frac{\tilde{\Theta}(p-q_k)}{\tilde{\Theta}(q-q_k)}\Big)^{\frac{a\beta_k}{n}} \\ &= \Theta'(0)^{\frac{1}{2}} \tilde{\Theta}(p-q)^{-\frac{1}{2}} \exp\Big(-\frac{a}{2r}(p-q)\sum_{k=1}^{n}\frac{\beta_k}{n}q_k\Big) \prod_{k=1}^{n} \Big(\frac{\Theta(p-q_k)}{\Theta(q-q_k)}\Big)^{\frac{a\beta_k}{n}}. \end{aligned}$$

Therefore for a general background charge $\boldsymbol{\beta}$, one can write $Z_{\boldsymbol{\beta}}$ as (up to multiplicative constant)

$$\begin{aligned} Z_{\boldsymbol{\beta}}(p,q) = \Theta(x)^{-\frac{1}{2}} \exp\Big(- \frac{(x+\sqrt{2}\int z\, d\boldsymbol{\beta}(z))^2}{8r}\Big) \\ \times \exp\Big(\frac{1}{\sqrt{2}}\int \log\Big(\frac{\Theta(p-z)}{\Theta(q-z)}\Big) d\boldsymbol{\beta}(z)\Big). \end{aligned} \tag{8.1.16}$$

We remark that in general, the partition function (8.1.16) is not translation invariant. In other words, for generic $\boldsymbol{\beta}$, the associated $Z(p,q)$ is not a function of the variable $x = p - q$.

Example. Note that if $\boldsymbol{\beta} = \beta \cdot q_1 - \beta \cdot q_2$ for some $q_1, q_2 \in \mathbb{R}_r$ satisfying $q_2 = q_1 + 2\pi$, the boundary condition (8.1.14) reduces to (8.1.3). On the other hand, the partition function (8.1.16) corresponds to the partition function (8.1.2) for GFF with constant boundary value on $\mathbb{R}_r$.

By Proposition 7.5.1, the null-vector equation for $Z_{\boldsymbol{\beta}}$ is given by

$$\partial_r Z_{\boldsymbol{\beta}} = 2\partial_p^2 Z_{\boldsymbol{\beta}} + H(p-q)\partial_p Z_{\boldsymbol{\beta}} + \frac{1}{4}H'(p-q)Z_{\boldsymbol{\beta}} + F_{\boldsymbol{\beta}} Z_{\boldsymbol{\beta}}, \tag{8.1.17}$$

where

$$\begin{aligned} F_{\boldsymbol{\beta}} = -\frac{3}{2}\frac{\zeta_r(\pi)}{\pi} + \frac{1}{2r} - \frac{1}{4}\Big[\int \Big(H_I(p-s) + \frac{p-s}{r}\Big)\, d\boldsymbol{\beta}(s)\Big]^2 \\ + \frac{1}{2\sqrt{2}}\int H_I(p-s)\Big(H_I(p-s) - H_I(q-s) + \frac{p-q}{r}\Big)\, d\boldsymbol{\beta}(s). \end{aligned}$$

Moreover, by (3.1.46), we have

$$\begin{aligned} & H_I(p-s)\Big(H_I(p-s) - H_I(q-s)\Big) \\ &= H_I(p-q)\Big(H_I(p-s) - H_I(q-s)\Big) - \Big(H_I' + \frac{H_I^2}{2}\Big)(p-q) \\ &- \Big(H_I' - \frac{H_I^2}{2}\Big)(p-s) - \Big(H_I' + \frac{H_I^2}{2}\Big)(q-s) - \frac{6}{\pi}\zeta_r(\pi). \end{aligned}$$

Thus we further simplify $F_{\boldsymbol{\beta}}$ as

$$\begin{aligned} F_{\boldsymbol{\beta}} = &-\frac{3}{2}\frac{\zeta_r(\pi)}{\pi} + \frac{1}{2r} - \frac{1}{4}\Big[\int \Big(H_I(p-s) + \frac{p-s}{r}\Big)\, d\boldsymbol{\beta}(s)\Big]^2 \\ &- \frac{1}{2\sqrt{2}}\int \Big(H_I' - \frac{H_I^2}{2}\Big)(p-s) + \Big(H_I' + \frac{H_I^2}{2}\Big)(q-s)\, d\boldsymbol{\beta}(s) \\ &+ \frac{1}{2\sqrt{2}}\frac{p-q}{r}\int H_I(p-s)\, d\boldsymbol{\beta}(s). \end{aligned}$$

Remark. Let $\boldsymbol{\beta} = \beta\cdot q_1 - \beta\cdot q_2$ for some $q_1, q_2 \in \mathbb{R}_r$ satisfying $q_2 = q_1 + 2\pi$. Then by the periodicity of $H(r,\cdot)$, we have

$$F_{\boldsymbol{\beta}}(p,q) = -\frac{3}{2}\frac{\zeta_r(\pi)}{\pi} + \frac{1}{2r} - \Big(\frac{\beta\pi}{r}\Big)^2.$$

One can check directly that the partition function

$$Z(p,q) = \Theta'(0)^{\frac{1}{2}}\Theta(r,x)^{-\frac{1}{2}}\exp\Big(-\frac{x^2-2\mu x}{8r}\Big), \quad \mu = 2\sqrt{2}\beta\pi$$

satisfy the null-vector equation.

8.1.4 Vertex observable

For $x_1, x_2, x_3, x_4 \in \mathbb{R}$, let us consider the one-leg operator

$$\Psi = \mathcal{O}[a\cdot x_1 - a\cdot x_2 + a\cdot x_3 - a\cdot x_4]. \tag{8.1.18}$$

Then the associated partition function $Z \equiv Z(r,x_1,x_2,x_3,x_4) := \mathbf{E}\Psi$ is evaluated in the identity chart of $\mathcal{C}_r$ as

$$\begin{aligned} Z = &\,\Theta_\chi(r,x_1-x_2)^{-\frac{1}{2}}\Theta_\chi(r,x_1-x_3)^{\frac{1}{2}}\Theta_\chi(r,x_1-x_4)^{-\frac{1}{2}} \\ &\times \Theta_\chi(r,x_2-x_3)^{-\frac{1}{2}}\Theta_\chi(r,x_2-x_4)^{\frac{1}{2}}\Theta_\chi(r,x_3-x_4)^{-\frac{1}{2}}. \end{aligned} \tag{8.1.19}$$

Therefore the drift function $\Lambda \equiv \Lambda(r,x_1,x_2,x_3,x_4) := 4\partial_{x_1}\log Z$ is given as

$$\Lambda = -H_\chi(r,x_1-x_2) + H_\chi(r,x_1-x_3) - H_\chi(r,x_1-x_4). \tag{8.1.20}$$

Let us define

$$\mathcal{O} := \mathcal{O}[-2a \cdot \zeta + 2a \cdot \eta], \quad \zeta, \eta \in \mathcal{C}_r. \tag{8.1.21}$$

Then $\widehat{\mathcal{O}}$ is a differential with conformal dimension 1 at ζ, η and scalar field with respect to x_j's. Its evaluation in the identity chart of $\mathcal{C}_r$ is given as

$$\mathbf{E}\widehat{\mathcal{O}} = \frac{\Theta_\chi'(0)^2}{\Theta_\chi(\zeta-\eta)^2} \frac{\Theta_\chi(x_1-\eta)}{\Theta_\chi(x_1-\zeta)} \frac{\Theta_\chi(x_2-\zeta)}{\Theta_\chi(x_2-\eta)} \frac{\Theta_\chi(x_3-\eta)}{\Theta_\chi(x_3-\zeta)} \frac{\Theta_\chi(x_4-\zeta)}{\Theta_\chi(x_4-\eta)}.$$

Now let us take the limit $\zeta \to x_3$, $\eta \to x_4$. Then by the rooting procedure (OPE)

$$\Theta_\chi(w(x_3)-w(\zeta)) \to \Theta_\chi'(0)w'(x_3), \quad \Theta_\chi(w(x_4)-w(\eta)) \to \Theta_\chi'(0)w'(x_4),$$

we obtain

$$\mathbf{E}\widehat{\mathcal{O}} = \frac{\Theta_\chi(w(x_1)-w(x_4))}{\Theta_\chi(w(x_1)-w(x_3))} \frac{\Theta_\chi(w(x_2)-w(x_3))}{\Theta_\chi(w(x_2)-w(x_4))}. \tag{8.1.22}$$

By Theorem C, the vertex observable $\mathbf{E}\widehat{\mathcal{O}}$ is a local martingale. In the degenerate limit $r \to \infty$, such martingale-observable was utilized by Peltola and Wu [112] to derive the connection probabilities for $\mathrm{SLE}_4(-2, 2, -2)$ process starting from x_1 with force points x_2, x_3, x_4.

For $\chi = \infty$, let us show the martingale property of (8.1.22) by direct computations.

Proof. Note that the driving process is given by

$$d\xi_t = 2\, dB_t + \Big[H(r-t, w_t(x_2)) - H(r-t, w_t(x_3)) + H(r-t, w_t(x_4))\Big]\, dt,$$

where we write $w_t := \tilde{g}_t - \xi_t$.

Set

$$M_t := \frac{\Theta(r-t, \tilde{g}_t(x_3)-\tilde{g}_t(x_2))\Theta(r-t, \tilde{g}_t(x_4)-\xi_t)}{\Theta(r-t, \tilde{g}_t(x_4)-\tilde{g}_t(x_2))\Theta(r-t, \tilde{g}_t(x_3)-\xi_t)}.$$

Let us write

$$X_t := \frac{\Theta(r-t, \tilde{g}_t(x_4) - \xi_t)}{\Theta(r-t, \tilde{g}_t(x_4) - \tilde{g}_t(x_2))}, \quad Y_t := \frac{\Theta(r-t, \tilde{g}_t(x_3) - \tilde{g}_t(x_2))}{\Theta(r-t, \tilde{g}_t(x_3) - \xi_t)}$$

so that

$$\frac{dM_t}{M_t} = \frac{dX_t}{X_t} + \frac{dY_t}{Y_t} + \frac{d\langle X, Y\rangle_t}{X_t Y_t}.$$

We first compute dX_t/X_t. By Itô's formula, we have

$$\begin{aligned} d\Theta(w_t(x_4)) &= \Theta'(w_t(x_4))\, d(g_t(x_4) - \xi_t) \\ &\quad + \frac{1}{2}\Theta''(w_t(x_4))\, d\langle g_t(x_4) - \xi_t\rangle - \partial_r \Theta(w_t(x_4))\, dt. \end{aligned}$$

Note that by Loewner equation,

$$d(g_t(x_4) - \xi_t) = -2\, dB_t + \Big[H(w_t(x_3)) - H(w_t(x_2))\Big]\, dt.$$

Thus we have

$$\begin{aligned} d\Theta(w_t(x_4)) &= -2\Theta'(w_t(x_4))\, dB_t \\ &\quad - \Theta'(w_t(x_4))\Big[H(w_t(x_2)) - H(w_t(x_3))\Big]\, dt + \Theta''(w_t(x_4))\, dt, \end{aligned}$$

thus leading to

$$\begin{aligned} \frac{d\Theta(w_t(x_4))}{\Theta(w_t(x_4))} &= -H(w_t(x_4))\, dB_t - \frac{1}{2}H(w_t(x_4))\Big[H(w_t(x_2)) - H(w_t(x_3))\Big]\, dt \\ &\quad + \Big(\frac{1}{2}H' + \frac{1}{4}H^2\Big)(w_t(x_4))\, dt. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} d\Theta\big(\tilde{g}_t(x_4) - \tilde{g}_t(x_2)\big) &= \Theta'(w_t(x_4) - w_t(x_2))\Big[H(w_t(x_4)) - H(w_t(x_2))\Big]\, dt \\ &\quad - \Theta''(w_t(x_4) - w_t(x_2))\, dt. \end{aligned}$$

Thus

$$\frac{d\Theta(w_t(x_4)-w_t(x_2))}{\Theta(w_t(x_4)-w_t(x_2))} = \frac{1}{2}H(w_t(x_4)-w_t(x_2))\Big[H(w_t(x_4))-H(w_t(x_2))\Big]\,dt \\ -\Big(\frac{1}{2}H'+\frac{1}{4}H^2\Big)(w_t(x_4)-w_t(x_2))\,dt.$$

Therefore we obtain

$$\frac{dX_t}{X_t} = -H(w_t(x_4))\,dB_t + \frac{1}{2}H(w_t(x_4))\Big[H(w_t(x_3))-H(w_t(x_2))\Big]\,dt \\ +\Big(\frac{1}{2}H'+\frac{1}{4}H^2\Big)(w_t(x_4))\,dt + \Big(\frac{1}{2}H'+\frac{1}{4}H^2\Big)(w_t(x_4)-w_t(x_2))\,dt \\ -\frac{1}{2}H(w_t(x_4)-w_t(x_2))\Big[H(w_t(x_4))-H(w_t(x_2))\Big]\,dt.$$

Next, we compute dY_t/Y_t. First note that

$$d\Theta(w_t(x_3)-w_t(x_2)) = \Theta'(w_t(x_3)-w_t(x_2))\Big[H(w_t(x_3))-H(w_t(x_2))\Big]\,dt \\ -\Theta''(w_t(x_3)-w_t(x_2))\,dt.$$

Thus,

$$\frac{d\Theta(w_t(x_3)-w_t(x_2))}{\Theta(w_t(x_3)-w_t(x_2))} = \frac{1}{2}H(w_t(x_3)-w_t(x_2))\Big[H(w_t(x_3))-H(w_t(x_2))\Big]\,dt \\ -\Big(\frac{1}{2}H'+\frac{1}{4}H^2\Big)(w_t(x_3)-w_t(x_2))\,dt.$$

Note that

$$d(g_t(x_3)-\xi_t) = -2\,dB_t + \Big[2H(w_t(x_3))-H(w_t(x_2))-H(w_t(x_4))\Big]\,dt.$$

Therefore we have

$$d\Theta(w_t(x_3)) = -2\Theta'(w_t(x_3))\,dB_t + \Theta''(w_t(x_3))\,dt. \\ +\Theta'(w_t(x_3))\Big[2H(w_t(x_3))-H(w_t(x_2))-H(w_t(x_4))\Big]\,dt.$$

Combining all of the above,

$$\frac{d\Theta(w_t(x_3))}{\Theta(w_t(x_3))} = -H(w_t(x_3))\,dB_t + \Big(\frac{1}{2}H' + \frac{1}{4}H^2\Big)(w_t(x_3))\,dt + \frac{1}{2}H(w_t(x_3))\Big[2H(w_t(x_3)) - H(w_t(x_2)) - H(w_t(x_4))\Big]\,dt.$$

Therefore

$$\begin{aligned}\frac{dY_t}{Y_t} &= H(w_t(x_3))\,dB_t + \frac{1}{2}H(w_t(x_3) - w_t(x_2))\Big[H(w_t(x_3)) - H(w_t(x_2))\Big]\,dt \\ &\quad - \Big(\frac{1}{2}H' + \frac{1}{4}H^2\Big)(w_t(x_3) - w_t(x_2))\,dt - \Big(\frac{1}{2}H' + \frac{1}{4}H^2\Big)(w_t(x_3))\,dt \\ &\quad + \frac{1}{2}H(w_t(x_3))\Big[H(w_t(x_2)) + H(w_t(x_4))\Big]\,dt \\ &\quad + \frac{1}{2}H(w_t(x_3) - w_t(x_2))\Big[H(w_t(x_3)) - H(w_t(x_2))\Big]\,dt.\end{aligned}$$

On the other hand, by pseudo addition formula (3.1.41),

$$\begin{aligned}&- \frac{1}{2}H(w_t(x_4) - w_t(x_2))\Big[H(w_t(x_4)) - H(w_t(x_2))\Big] \\ &+ \frac{1}{2}H(w_t(x_3) - w_t(x_2))\Big[H(w_t(x_3)) - H(w_t(x_2))\Big] \\ &= -\Big(\frac{1}{2}H' + \frac{1}{4}H^2\Big)(w_t(x_4) - w_t(x_2)) - \Big(\frac{1}{2}H' + \frac{1}{4}H^2\Big)(w_t(x_4)) \\ &+ \Big(\frac{1}{2}H' + \frac{1}{4}H^2\Big)(w_t(x_3) - w_t(x_2)) + \Big(\frac{1}{2}H' + \frac{1}{4}H^2\Big)(w_t(x_3)) \\ &+ \frac{1}{2}H(w_t(x_4))H(w_t(x_2)) - \frac{1}{2}H(w_t(x_3))H(w_t(x_2)).\end{aligned}$$

Combining all above equations, we obtain

$$\frac{dX_t}{X_t} + \frac{dY_t}{Y_t} \overset{d}{=} H(w_t(x_4))H(w_t(x_3)).$$

Therefore we conclude that

$$\frac{dM_t}{M_t} = \frac{dX_t}{X_t} + \frac{dY_t}{Y_t} + \frac{d\langle X, Y\rangle_t}{X_t Y_t} \overset{d}{=} 0,$$

i.e., M_t is a local martingale. □

8.2 LERW observables

In this section, we present a way of constructing annulus CFT, which produces martingale-observables for variants of loop-erased random walk (LERW). In the physics literature, a version of CFT based on (sympletic) fermions has been proposed for LERW, see e.g., [20, 30, 62, 101] and references therein.

In doubly connected domains, one may consider variants of LERWs by imposing boundary conditions on the inner boundary components. For instance, let us consider a simple random walk on the lattice approximation of the annulus started from an interior vertex near the initial boundary point conditioned to hit a boundary vertex near the target point. We assume that the random walk continues when it hits the inner circle, and stops when it visit the outer one. Then after the loop-erasing procedure, the resulting curve is the LERW connecting two marked points on the outer boundary component, obeying Neumann condition on the inner one.

For generic LERW with certain boundary condition, it is well known that the associated partition function Z is expressed as

$$Z(p,q) = \frac{\partial^2}{\partial n_\zeta \partial n_z}\Big|_{\zeta=p,z=q} G(\zeta,z), \tag{8.2.1}$$

see e.g., [132, Section 4.5], [46, Section 9] and [102]. Here the Green's function G satisfies same boundary condition with LERW.

We now present a way of constructing LERW partition functions using the CFT framework developed above. Then by Theorem D, the construction of associated martingale-observables also follow.

Recall that for $4/\kappa \in \mathbb{Z}_+$, one can choose the integration contour γ as a circle encircles p or/and q in the expression (7.7.11). With such choice of γ, by residue calculus, we obtain following two parameter family of solutions to

null-vector equation (2.3.1) for $\kappa = 2$:

$$Z_\beta(r,x) = \frac{H(r,x) + \dfrac{x-2\beta\pi}{r+\chi}}{\Theta(r,x)\exp\Big(\dfrac{(x-2\beta\pi)^2}{4(r+\chi)}\Big)}. \tag{8.2.2}$$

Note that by (3.1.10),

$$\begin{aligned} Z_\beta(r,x+2n\pi) = {} & \frac{H(r,x)}{\Theta(r,x)}(-1)^n \exp\Big(-\frac{(x+2n\pi-2\beta\pi)^2}{4(r+\chi)}\Big) \\ & + \frac{1}{\Theta(r,x)}(-1)^n \frac{x+2n\pi-2\beta\pi}{4(r+\chi)} \exp\Big(-\frac{(x+2n\pi-2\beta\pi)^2}{4(r+\chi)}\Big). \end{aligned}$$

Therefore using the weight functions $\omega(n) \equiv 1$ (or $\omega(n) \equiv (-1)^n$), we obtain following family of solutions having 2π-(anti) periodicity:

$$Z_\beta^\omega = \begin{cases} \dfrac{\Theta(r+\chi, x+\pi-2\beta\pi)}{\Theta(r,x)}\Big(H(r,x) - H(r+\chi, x+\pi-2\beta\pi)\Big) \\ \dfrac{\Theta_I(r+\chi, x+\pi-2\beta\pi)}{\Theta(r,x)}\Big(H(r,x) - H_I(r+\chi, x+\pi-2\beta\pi)\Big) \end{cases} \tag{8.2.3}$$

We emphasize here that the 2π-periodic partition functions (i.e., $\omega(n) \equiv 1$) Z_β^ω are positive for all β. Let us also point out that such partition functions for $\beta \in \{0, b\}$ with $\chi = 0$ correspond to those found by Zhan in [139, Section 8.2].

8.2.1 LERW partition functions

Let us now focus on the case with $\chi = 0$. Note that due to (7.7.12), such 2π-periodic partition functions further satisfies rotational periodicities with respect to $2ir$, i.e.,

$$Z_\beta^\omega(r, x+2ir) = \exp\Big((2\beta-1)\pi i\Big) Z_\beta^\omega(r,x). \tag{8.2.4}$$

- *LERW with Dirichlet/Riemann-Hilbert mixed boundary conditions.* Let

$\omega(n) = 1$. Then by (8.2.3), we have

$$Z^{\omega}_{\beta}(r,x) = \frac{\Theta(r, x+\pi-2\beta\pi)}{\Theta(r,x)}\Big(H(r,x) - H(r, x+\pi-2\beta\pi)\Big). \tag{8.2.5}$$

In particular, for $\beta = 0$, we have an alternative expression

$$Z(r,x) = H'(2r,x) - H'_I(2r,x). \tag{8.2.6}$$

On the other hand, when $\beta \to b$, after normalization, we have

$$Z(r,x) = H'(r,x). \tag{8.2.7}$$

The partition functions (8.2.5) corresponds to the continuous LERW with Riemann-Hilbert boundary condition (cf. [27]), i.e., the requirement that the derivative along the obliquely direction (indexed by β) vanishes. For instance, the case $\beta = 0$ (resp., $\beta \to b$) corresponds to the LERW with Neumann (resp., ER) b.c. This follows from (8.2.1) and the representation of Green's function (3.2.41) satisfying Riemann-Hilbert boundary condition on the inner boundary component.

• *LERW with Dirichlet/Dirichlet boundary condition.* Set

$$\Psi^{\omega}_{\beta} := \sum_{n\in\mathbb{Z}} n\Psi_b(x+2n\pi). \tag{8.2.8}$$

In other words, we set $\beta = b$ and $\omega(n) = n$. Then we have

$$Z^{\omega}_{\beta}(r,x) = H'(x) + \frac{1}{r}. \tag{8.2.9}$$

This corresponds to the LERW partition function with Dirichlet boundary condition, which appears in [137].

We shall present a short computation for (8.2.9).

Proof. Note that

$$\mathbf{E}\Psi_b(r,x) = \Theta(r,x)^{-1}\exp\Big(-\frac{(x+\pi)^2}{4r}\Big)\Big(H(r,x)+\frac{x+\pi}{r}\Big).$$

Therefore we have

$$\begin{aligned} Z_\beta^\omega = {} & \frac{H(r,x)}{\Theta(r,x)}\sum_{n\in\mathbb{Z}}(-1)^n\, n\, \exp\Big(-\frac{(x+\pi+2n\pi)^2}{4r}\Big) \\ & + \frac{1}{\Theta(r,x)}\sum_{n\in\mathbb{Z}}(-1)^n\, n\Big(\frac{x+\pi+2n\pi}{r}\Big)\exp\Big(-\frac{(x+\pi+2n\pi)^2}{4r}\Big). \end{aligned}$$

It follows from (3.1.14) that

$$\sum_{n\in\mathbb{Z}}(-1)^n\, n\, \exp\Big(-\frac{(x+\pi+2n\pi)^2}{4r}\Big) = \frac{r}{\pi}\sqrt{\frac{r}{\pi}}\Big(\Theta'(r,x)+\frac{x+\pi}{2r}\Theta(r,x)\Big).$$

Thus we obtain (up to multiplicative constant)

$$Z_\beta^\omega = H(r,x)\Big(\frac{\Theta'(r,x)}{\Theta(r,x)}+\frac{x+\pi}{2r}\Big) - \Big(2\frac{\Theta''(r,x)}{\Theta(r,x)}+\frac{1}{r}+\frac{x+\pi}{r}\frac{\Theta'(r,x)}{\Theta(r,x)}\Big).$$

Now the expression (8.2.9) follow from (3.1.12) and (3.1.31). □

Remark. In parallel, one can construct such partition functions for the crossing case that $p \in \mathbb{R}$ and $q \in \mathbb{R}_r$. In particular, for $\beta = b, \chi = 0$ with the choice of weight function $\omega(n) = n$, we obtain

$$Z_\beta^\omega = H_I'(r,x) + \frac{1}{r}, \tag{8.2.10}$$

which corresponds to Zhan's Feynmann-Kac solution, see [139, Section 8.2].

• *LERW aiming side arc.* Recall that the screening variable has the two choice of charges

$$-2a \quad \text{or} \quad 2(a+b) = 1/a.$$

When $\kappa = 2$, due to a special feature $a = 1/a$, if we set charge a to the "target point" q as usual, the node q can also be utilized as a screening variable. This

fact allows us to describe the one-leg operator for LERW aiming not only a target point but also given side arc.

For a subset I of boundary components ($I \cap p = \emptyset$), set

$$\Psi(p, I) = \frac{1}{|I|} \int_I \Psi_b^{\omega}(p, q)\, dq, \quad \omega(n) = n. \tag{8.2.11}$$

Then the associated partition function $Z(p, I) = \mathbf{E}\Psi(p, \gamma)$ is given by

$$Z(p, I) := \frac{1}{|I|} \int_I H'(r, p - q)\, dq + \frac{1}{r}. \tag{8.2.12}$$

This corresponds to the partition function of continuous LERW whose target set is I, see [137].

In particular, when $I = \mathbb{T}_r$, it describes LERW starting at p, which stops whenever it hits the inner boundary component. Then by the periodicity (3.1.33), the associated partition function is given by

$$Z(p, \mathbb{T}_r) = \frac{1}{r}, \tag{8.2.13}$$

which can be realized as a partition function for standard SLE_2, see [133,137].

In the sequel, let Ψ be the one-leg operator associated with variants of LERWs described above. We present some import martingale-observables.

8.2.2 Generalized Poisson kernel

For simplicity we only consider the case that target set is a boundary point q. Set

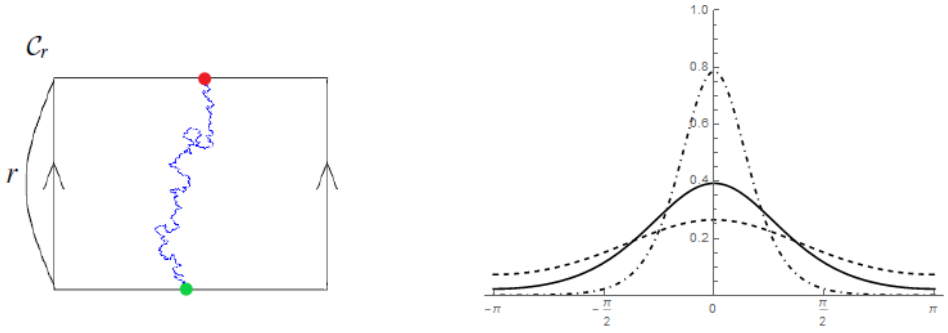
$$\mathcal{O} := \mathcal{O}[a \cdot z - a \cdot q].$$

Then we have

$$\widehat{\mathbf{E}}\mathcal{O}(z) = \frac{w'(z) Z(r, w(z) - w(p))}{w'(q) Z(r, w(q) - w(p))}.$$

Let us define

$$M(z) := \frac{1}{2i} \int_{\bar{z}}^{z} \widehat{\mathbf{E}}\mathcal{O}(\zeta)\, d\zeta. \tag{8.2.14}$$



(a) continuum LERW (b) ending point distribution

Figure 8.2: (a): An illustration of continuum LERW in the cylinder $\mathcal{C}_r$. (b): The plot displays ending point distribution with modulus $r = 1$ (dot-dashed line), $r = 2$ (full line) and $r = 3$ (dashed line)

The scalar field M is called generalized Poisson kernel.

For the Dirichlet/Dirichlet boundary condition, Zhan utilized the bounded martingale

$$M_t(z) = \frac{1}{w_t'(q)} \frac{\operatorname{Im} \tilde{H}(r_t, w_t(z))}{\tilde{H}'(r_t, w_t(q))} \tag{8.2.15}$$

to show the convergence of LERW, see [132, 137].

8.2.3 Ending point distribution of standard SLE_2

Set $I = [q_1 + ir, q_2 + ir] \subset \mathbb{T}_r$. Let us write $\Psi(p, \mathbb{T}_r)$ for the one-leg operator for the standard SLE_2. Set

$$\mathcal{O} = \frac{1}{2\pi} \int_I \mathcal{O}[a \cdot z - a \cdot q] \, dq.$$

Then the martingale-observable

$$M(z) = \widehat{\mathbf{E}}\mathcal{O} = \frac{\mathbf{E}\Psi(p, \mathbb{T}_r)\mathcal{O}}{\mathbf{E}\Psi(p, \mathbb{T}_r)} = \frac{1}{2\pi} \int_{q_1}^{q_2} \Big(r H_I'(r, p - q) + 1 \Big) \, dq$$

yields the probability that standard SLE_2 ends at I, see [134].

8.2.4 Bosonic observables for LERW

Recall that the partition function Z for LERW with marked points $p, q \in \mathbb{R}$ under ER boundary condition is given as $Z(r,x) = H'(r,x)$. Therefore we have

$$dw_t(z) = -\sqrt{2}\, dB_t + \Big[H(r_t, w_t(z)) + \frac{2H''}{H'}(r_t, w_t(q))\Big]\, dt, \quad r_t = r - t.$$

By choosing integration contour γ as a simple curve entwining both p and q, we have

$$\mathbf{E}\widehat{\Phi}(z) = -2b \arg w_z' - \frac{a}{2}\frac{H}{H'}(w_p - w_q)\mathrm{Im}\Big[H(w_p - w_z) - H(w_q - w_z)\Big].$$

Let us define

$$N_t(z) = \log w_t'(z) - \frac{1}{2}\frac{H}{H'}(w_t(q))\Big(H(w_t(z)) + H(w_t(q) - w_t(z))\Big)$$

so that

$$M_t(z) = \mathrm{Im}\, N_t(z).$$

By (3.1.40), we have

$$\begin{aligned} dH(w_t(q)) &= -(H'' + H'H)(w_t(q)) + H'(w_t(q))\, dw_t(q) + H''(w_t(q))\, dt \\ &= -\sqrt{2}H'(w_t(q))dB_t + 2H''(w_t(q))\, dt. \end{aligned}$$

Similarly,

$$\begin{aligned} dH'(w_t(q)) &= -(H''H + (H')^2)(w_t(q))\, dt + H''(w_t(q))\, dw_t(q) \\ &= -\sqrt{2}H''(w_t(q))\, dB_t + \Big[-(H')^2 + 2\frac{(H'')^2}{H'}\Big](w_t(q))\, dt. \end{aligned}$$

Combining above two equations, we obtain that

$$d\Big(\frac{H}{H'}(w_t(q))\Big) = -\sqrt{2}\Big(1 - \frac{HH''}{(H')^2}(w_t(q))\Big)\, dB_t + H(w_t(q))\, dt.$$

On the other hand, we have

$$dH(w_t(z)) = -\sqrt{2}H'(w_t(z))\,dB_t + 2H'(w_t(z))\frac{H''}{H'}(w_t(q))\,dt.$$

Differentiating psuedo-addition formula (3.1.41) with respect to the variable z, we have

$$\begin{aligned} H'(z-w)\Big(H(z)-H(w)\Big) &= (H''+HH')(z-w) + (H''+HH')(z) \\ &\quad - \Big(H(z-w)+H(w)\Big)H'(z). \end{aligned}$$

Using this identity, we obtain that

$$\begin{aligned} dH(w_t(q)-w_t(z)) &= -\,(H''+H'H)(w_t(q)-w_t(z))\,dt \\ &\quad + H'(w_t(q)-w_t(z))\Big(H(w_t(q))-H(w_t(z))\Big)\,dt \\ &= +\,(H''+H'H)(w_t(q))\,dt \\ &\quad - \Big(H(w_t(q)-w_t(z))+H(w_t(z))\Big)H'(w_t(q))\,dt. \end{aligned}$$

Using above equations, we obtain that

$$\begin{aligned} & d\Big[\frac{H}{H'}(w_t(q))\Big(H(w_t(z))+H(w_t(q)-w_t(z))\Big)\Big] \\ \stackrel{d}{=} & +\Big[2H'(w_t(z)) + \Big(\frac{HH''}{H'}+H^2\Big)(w_t(q))\Big]\,dt. \end{aligned}$$

Since

$$d\log w_t'(z) = H'(w_t(z))\,dt,$$

we have

$$dN_t(z) \stackrel{d}{=} -\frac{1}{2}\Big(\frac{HH''}{H'}+H^2\Big)(w_t(q))\,dt.$$

Then it follows from $q \in \mathbb{R}$ that $M_t(z)$ is a local martingale.

8.3 Critical Ising observables

In this section, we focus on the case that $\kappa = 3$. As in Section 7.5, let us denote by $\check{\boldsymbol{\beta}}$ the (signed) uniform measure on $[ir - \pi, ir + \pi]$ with total mass $-2b$ and let $\boldsymbol{\beta} = 2b \cdot q + \check{\boldsymbol{\beta}}$. Then we have

$$Z_{\boldsymbol{\beta}}(r,x) = \mathbf{E}\Psi_{\boldsymbol{\beta}}[a \cdot p - a \cdot q] = \Theta(r,x)^{-1} \exp\Big(-\frac{x^2}{4r}\Big), \quad x = p - q. \tag{8.3.1}$$

The null-vector equation for $Z_{\boldsymbol{\beta}}$ is given as

$$\partial_r Z_{\boldsymbol{\beta}} = \frac{3}{2} Z''_{\boldsymbol{\beta}} + H Z'_{\boldsymbol{\beta}} + \frac{1}{2} H' Z_{\boldsymbol{\beta}} + F_{\boldsymbol{\beta}} Z_{\boldsymbol{\beta}}, \tag{8.3.2}$$

where

$$F_{\boldsymbol{\beta}}(r,x) = -\Big(\frac{H'}{2} + \frac{H^2}{4}\Big)(r,x) - \frac{x}{2r} H(r,x) - \frac{x^2}{4r^2}. \tag{8.3.3}$$

Set

$$\Psi := \sum_{n \in \mathbb{Z}} (-1)^n \Psi_{\boldsymbol{\beta}}[a \cdot (p + 2n\pi) - a \cdot q]. \tag{8.3.4}$$

Then by (3.1.14), we have

$$Z(r,x) := \mathbf{E}\Psi = \sum_{n \in \mathbb{Z}} (-1)^n Z_{\boldsymbol{\beta}}(r, x + 2n\pi) = \sqrt{\frac{r}{\pi}} \frac{\Theta_I(r, x + \pi)}{\Theta(r,x)}. \tag{8.3.5}$$

This corresponds to the partition function for critical Ising interface studied by Izyurov [67]. Therefore insertion of the field Ψ provides martingale-observables for the continuum limit of the interface curve.

Since the interface curve satisfies reversibility, it follows from Izyurov's convergence result [67] and Zhan's theory on commuting annulus SLE that the partition function (8.3.5) satisfies the null-vector equation (2.3.1) with $\kappa = 3$. This fact also follows from the martingale property of parafermionic observable in the continuum limit. See [68] for a verification of BPZ type equations of the partition function in this manner.

Here we present a field theoretical way to show that Z satisfies (2.3.1)

based on BPZ equation derived in Section 7.5.

Proof. By (8.3.2), all we need to show is

$$Z(r,x)^{-1}\sum_{n\in\mathbb{Z}}(-1)^n F_{\boldsymbol{\beta}}(x+2n\pi)Z_{\boldsymbol{\beta}}(r,x+2n\pi) \tag{8.3.6}$$

is a constant depending only on the modular parameter r. By differentiating (3.1.14), we have

$$\sum_{n\in\mathbb{Z}}\frac{x+2n\pi}{2r}\exp\Big(-\frac{(x+2n\pi)^2}{4r}\Big)=-\sqrt{\frac{r}{\pi}}\Theta_I'(r,x+\pi)$$

and

$$\sum_{n\in\mathbb{Z}}\frac{(x+2n\pi)^2}{4r^2}\exp\Big(-\frac{(x+2n\pi)^2}{4r}\Big)=\sqrt{\frac{r}{\pi}}\Big(\Theta_I''(r,x+\pi)+\frac{1}{2r}\Theta_I(r,x+\pi)\Big).$$

Using these identities, we obtain that (8.3.6) is simplified to

$$\frac{1}{2}H(r,x)H_I(r,x+\pi)-\Big(\frac{H_I'}{2}+\frac{H_I^2}{4}\Big)(r,x+\pi)-\Big(\frac{H_I'}{2}+\frac{H_I^2}{4}\Big)(r,x+\pi)-\frac{1}{2r}.$$

Now the claim follows from (3.1.47). □

Example. Let

$$\mathcal{O}_{\boldsymbol{\beta}}=\mathcal{O}_{\boldsymbol{\beta}}[(2b-a)\cdot z-(2b-a)\cdot q].$$

Then $M(z):=\widehat{\mathbf{E}}\mathcal{O}_{\boldsymbol{\beta}}$ provides the local martingale

$$M_t(z)=\frac{w_t'(z)^{\frac{1}{2}}Z(r_t,w_t(z))}{w_t'(q)^{\frac{1}{2}}Z(r_t,w_t(q))}. \tag{8.3.7}$$

This process is the parafermionic observable utilized in [67].

Using a similar idea in previous section, one can construct further partition functions Z of the form

$$\frac{\Theta(r,x+\pi-2\beta\pi)}{\Theta(r,x)},\quad \frac{\Theta_I(r,x+\pi-2\beta\pi)}{\Theta(r,x)}, \tag{8.3.8}$$

which satisfies (2.3.1) and have 2π-(anti) periodicity as well as $2ir$-rotational periodicity. Moreover, the construction in this section also works for the crossing case.

8.4 Annulus SLE with one marked point

For $k \in \{1, \dots, n\}$, let us denote

$$q_k := ir + 2\pi \frac{k}{n}.$$

Set

$$\Psi_n(p) = \mathcal{O}\Big[a \cdot p - \sum_{k=1}^{n} \frac{a}{2n} \cdot q_k, -\sum_{k=1}^{n} \frac{a}{2n} \cdot q_k\Big]. \tag{8.4.1}$$

By (7.1.10),

$$\begin{aligned} \Lambda_n(p) &= -\frac{1}{2n} \sum_{k=1}^{n} H_\chi(r, p - q_k) + H_\chi(p - \bar{q}_k) \\ &= -\frac{1}{n} \sum_{k=1}^{n} H_I(r, p - \operatorname{Re} q_k) + \frac{p - \operatorname{Re} q_k}{r + \chi}. \end{aligned} \tag{8.4.2}$$

We denote by

$$\Psi(p) := \lim_{n \to \infty} \Psi_n(p) = \mathcal{O}\Big[a \cdot p - \frac{a}{2} \cdot \boldsymbol{q}, -\frac{a}{2} \cdot \boldsymbol{q}\Big], \tag{8.4.3}$$

where $\boldsymbol{q}$ is the uniform measure on $[q - \pi, q + \pi]$ for $q \in \mathbb{R}_r$. Let us write $\Lambda := \lim_{n \to \infty} \Lambda_n$. Note that Ψ is a differential with conformal dimension $h_{1,2}$ at p. On the other hand, due to 2π-periodicity of $H_I(r, \cdot)$, we have

$$\Lambda(p) = \frac{\operatorname{Re} q - p}{r + \chi}. \tag{8.4.4}$$

By Theorem C, the insertion of the one-leg operator $\Psi(p)$ produces martingale-

observables for SLE with driving process

$$d\xi_t = \sqrt{\kappa}\, dB_t + \frac{\mathrm{Re}\, q - \xi_t}{r + \chi - t}\, dt, \quad \xi_0 = p. \tag{8.4.5}$$

Notice that this corresponds to the law of Brownian bridge.

In particular, for the ER b.c. case when $\chi = \infty$ (thus $\Lambda \equiv 0$), it corresponds to the standard SLE(κ). On the other hand for the Dirichlet b.c. case when $\chi = 0$, then the SLE trace ends at the point q.

8.4.1 Bosonic observables

In the identity chart of $\mathcal{C}_r$, the bosonic observable $M(z) := \mathbf{E}\widehat{\Phi}(z)$ is evaluated as

$$M(z) = 2a \arg \Theta_\chi(r, z-p) - a \int \arg \Theta_\chi(r, z-\boldsymbol{q})\Theta_\chi(r, z-\bar{\boldsymbol{q}})\, d\boldsymbol{q}. \tag{8.4.6}$$

For the Dirichlet b.c., the boundary value of M has discontinuity $2a\pi$ at $p+2n\pi$ and has linear growth with speed a on $\mathbb{R}_r$. To be precise, $M(z+2\pi) = M(z) + 2\lambda$ and

$$M(z) = \begin{cases} -\lambda & \text{if} \quad z \in (0, p) \\ +\lambda & \text{if} \quad z \in (p, 2\pi)\,, \\ \lambda \dfrac{\mathrm{Re}\,(z-q)}{\pi} & \text{if} \quad z \in \mathbb{R}_r \end{cases} \tag{8.4.7}$$

where $\lambda = a\pi = \pi/\sqrt{2}$.

8.4.2 Vertex observables for standard SLE

For the ER b.c., let us denote by Z_0 the partition function associated with the standard $SLE(\kappa)$. From the expression (7.1.8), it is not straightforward to obtain a simple evaluation of Z_0 since it contains certain non-trivial integrations of theta functions. Nevertheless, we claim that the partition function

Z_0 is given by

$$Z_0 = (w_p')^{h_{1,2}}\,\Theta'(r,0)^{a^2} \tag{8.4.8}$$

up to an universal (i.e., independent of the modular parameter r) multiplicative constant.

For a divisor $\boldsymbol{\tau} = \sum \tau_k \cdot z_k$ satisfying $\int \boldsymbol{\tau} = -a$, write

$$Z(p,\boldsymbol{\tau}) = \mathcal{O}[a\cdot p + \boldsymbol{\tau}, \mathbf{0}].$$

The evaluation of $Z(p,\boldsymbol{\tau})$ is given by

$$\begin{aligned} Z(p,\boldsymbol{\tau}) = \Theta'(r,0)^{\frac{a^2}{2}+\sum\frac{\tau_k^2}{2}}(w_p')^{h_{1,2}}\prod_k (w_k')^{\lambda_k} \\ \times \prod_k \Theta(r,p-z_k)^{a\tau_k}\prod_{j>k}\Theta(r,z_j-z_k)^{\tau_j\tau_k}. \end{aligned} \tag{8.4.9}$$

where $w_k' = w'(z_k)$ and $\lambda_k = \lambda(z_k)$.

Set

$$\mathcal{O} := \mathcal{O}\Big[\boldsymbol{\tau} + \frac{a}{2}\cdot\boldsymbol{q}, \frac{a}{2}\cdot\boldsymbol{q}\Big].$$

Then the vertex observable $\widehat{\mathbf{E}}\mathcal{O}$ is given by

$$\widehat{\mathbf{E}}\mathcal{O} = \frac{\mathbf{E}\Psi(p)\mathcal{O}}{\mathbf{E}\Psi(p)} = \frac{Z(p,\boldsymbol{\tau})}{Z_0}. \tag{8.4.10}$$

Such observables defined by the ratio of two partition functions are used when one applies Girsanov's theorem to different SLEs, see e.g., [139].

By Theorem C, $\widehat{\mathbf{E}}\mathcal{O}$ is martingale-observable for SLE(κ). Therefore the evaluation of the constant term $\Theta'(r,0)^{\frac{2}{\kappa}}$ in (8.4.8) also follows from the following proposition.

Proposition 8.4.1. *Let $w_t(z) := \tilde{g}_t(z) - \sqrt{\kappa}\, B_t$. Then*

$$\begin{aligned} M_t := \Theta'(r_t,0)^{-\frac{a^2}{2}+\sum\frac{\tau_k^2}{2}} \prod_k w_t'(z_k)^{\lambda_k} \\ \times \prod_k \Theta(r_t, w_t(z_k))^{a\tau_k} \prod_{j>k} \Theta(r_t, w_t(z_j) - w_t(z_k))^{\tau_j\tau_k} \end{aligned} \tag{8.4.11}$$

is a local martingale.

Proof. By (3.1.25), we have

$$d\log \Theta'(r_t,0)^{-\frac{a^2}{2}+\sum\frac{\tau_k^2}{2}} = -\frac{1}{2}\Big(a^2 - \sum \tau_k^2\Big)\frac{3}{\pi}\zeta_r(\pi). \tag{8.4.12}$$

By Loewner's equation, we have

$$dw_t(z) = -\sqrt{\kappa}\, dB_t + H(r_t, w_t(z))\, dt.$$

Therefore we have $d\log w_t'(z) = H'(r_t, w_t(z))\, dt$, which leads to

$$d\log \prod w_t'(z_k)^{\lambda_k} = \sum \lambda_k H'(r_t, w_t(z))\, dt. \tag{8.4.13}$$

Let

$$X_t := \prod_k \Theta(r_t, w_t(z_k))^{a\tau_k}, \quad Y_t := \prod_{j>k} \Theta(r_t, w_t(z_j) - w_t(z_k))^{\tau_j\tau_k}.$$

Let us first compute dX_t/X_t. Combining Itô's formula with (3.1.12), (3.1.31), we have

$$d\Theta(r_t, w_t(z)) = -\sqrt{\kappa}\, \Theta'(r_t, w_t(z))\, dB_t + \Big(\Theta' H + \frac{\kappa-2}{2}\Theta''\Big)(r_t, w_t(z))\, dt.$$

and

$$\frac{d\Theta(r_t, w_t(z))}{\Theta(r_t, w_t(z))} = -\frac{\sqrt{\kappa}}{2} H(r_t, w_t(z))\, dB_t + \Big(\frac{\kappa-2}{2}H' + \frac{\kappa+2}{8}H^2\Big)(r_t, w_t(z))\, dt.$$

This leads to

$$\frac{d\Theta(r_t, w_t(z_k))^{a\tau_k}}{\Theta(r_t, w_t(z_k))^{a\tau_k}} = -\frac{\tau_k}{\sqrt{2}} H(r_t, w_t(z_k))\, dB_t + a\tau_k \Big(\frac{\kappa - 2}{4} H' + \frac{2 + a\tau_k \kappa}{8} H^2 \Big)(r_t, w_t(z_k))\, dt.$$

Therefore we obtain

$$\begin{aligned} \frac{dX_t}{X_t} &= -\frac{1}{\sqrt{2}} \sum_k \tau_k H(r_t, w_t(z_k))\, dB_t \\ &\quad + \sum_k \tau_k \Big(a \frac{\kappa - 2}{4} H' + \frac{a + \tau_k}{4} H^2 \Big)(r_t, w_t(z_k))\, dt \\ &\quad + \sum_{j>k} \frac{\tau_j \tau_k}{2} H(r_t, w_t(z_j)) H(r_t, w_t(z_k))\, dt. \end{aligned} \tag{8.4.14}$$

Next, we compute dY_t/Y_t. Note that

$$\begin{aligned} &\frac{d\Theta(r_t, w_t(z_j) - w_t(z_k))}{\Theta(r_t, w_t(z_j) - w_t(z_k))} \\ &= \frac{1}{2} H(r_t, w_t(z_j) - w_t(z_k)) \Big(H(r_t, w_t(z_j)) - H(r_t, w_t(z_k)) \Big)\, dt \\ &- \Big(\frac{1}{2} H' + \frac{1}{4} H^2 \Big)(r_t, w_t(z_j) - w_t(z_k))\, dt. \end{aligned}$$

Therefore by pseudo-addition formula (3.1.41), we obtain

$$\begin{aligned} \frac{dY_t}{Y_t} &= \sum_{j>k} \frac{\tau_j \tau_k}{4} \Big(H(r_t, w_t(z_j)) - H(r_t, w_t(z_k)) \Big)^2\, dt \\ &\quad + \sum_{j>k} \tau_j \tau_k \Big[\frac{1}{2} H'(r_t, w_t(z_j)) + \frac{1}{2} H'(r_t, w_t(z_k)) + \frac{3}{\pi} \zeta_r(\pi) \Big]\, dt. \end{aligned} \tag{8.4.15}$$

Combining all of the above with the neutrality condition $\int \boldsymbol{\tau} = -a$, one can observe that dM_t/M_t is driftless. For instance, the coefficient of $H'(r_t, w_t(z_k)\, dt$

is computed as

$$\lambda_k + a\tau_k\frac{\kappa-2}{4} + \tau_k\sum_{j(\neq k)}\frac{\tau_j}{2} = \frac{\tau_k^2}{2} - \tau_k b + a\tau_k\frac{\kappa-2}{4} - \tau_k\frac{a+\tau_k}{2} = 0$$

and the coefficient of $\frac{3}{\pi}\zeta_r(\pi)\,dt$ is computed as

$$-\frac{1}{2}\Big(a^2 - \sum\tau_k^2\Big) + \sum_{j>k}\tau_k\tau_k = -\frac{1}{2}\Big(a^2 - \sum\tau_k^2\Big) + \frac{1}{2}\Big((\sum\tau_k)^2 - \sum\tau_k^2\Big) = 0.$$

Therefore we conclude

$$\frac{dM_t}{M_t} = -\frac{1}{\sqrt{2}}\sum_k \tau_k H(r_t, w_t(z_k))\,dB_t, \tag{8.4.16}$$

which completes the proof. □

Chapter 9

Ward's identities in random matrix theory

Particle systems that follow Boltzmann-Gibbs law enjoy an intimate relationship with CFT. We give a brief exposition on the use of Ward's identities to study eigenvalue statistics in random matrix theory or 2D Coulomb gases in general.

9.1 Global Ward's identities and their applications

Let us review Ward's identities for finite Boltzmann-Gibbs ensembles. We consider N interacting particle system $\boldsymbol{\zeta} := \{\zeta_j\}_{j=1}^N$ whose joint probability function $\mathbf{P}_N$ is given by

$$d\mathbf{P}_N(\boldsymbol{\zeta}) := \frac{1}{Z_N} e^{H_N(\boldsymbol{\zeta})} \prod_{j=1}^N dA(\zeta_j), \quad Z_N := \int_{\mathbb{C}^N} e^{H_N(\boldsymbol{\zeta})} \prod_{j=1}^N dA(\zeta_j), \quad (9.1.1)$$

where dA is the area measure (i.e., 2D Lebesgue measure divided by π) and the normalization constant Z_N, which turns $\mathbf{P}_N$ into a probability measure is called the partition function. Here H_N is a given Hamiltonian. For instance,

2D Coulomb gas corresponds to the system with Hamiltonian

$$H_N(\boldsymbol{\zeta}) := \beta \sum_{j>k} \log|\zeta_j - \zeta_k| - m \sum_{j=1}^{N} Q(\zeta_j), \tag{9.1.2}$$

where $\beta > 0$ is a inverse temperature, $m > 0$ is a scaling factor and $Q : \mathbb{C} \to \mathbb{R}$ is an external potential, which satisfies proper growth condition so that $Z_N < \infty$. Some particular case of 2D Coulomb gases can be realized as eigenvalue statistics of certain random matrix models and we refer to [1, 57] for a comprehensive account on Coulomb gases and random matrix theory. We also denote by

$$\mathbf{R}_{N,k}(\zeta_1, \cdots, \zeta_k) := \frac{1}{Z_N} \frac{N!}{(N-k)!} \int_{\mathbb{C}^{N-k}} e^{-H_N(\boldsymbol{\zeta})} \prod_{j=k+1}^{N} dA(\zeta_j) \tag{9.1.3}$$

the k-point correlation function, which corresponds to the normalized probability that particle exists near a prescribed points $\zeta_1, \cdots, \zeta_k$.

For a smooth vector field v in $\mathbb{C}$ with compact support, let us write ψ_t for its flow. We also denote by Ψ_t the flow of particle system $\{\zeta_j\} \mapsto \{\psi_t \zeta_j\}$ in $\mathbb{C}^N$. Then it follows from the fact that Z_N is invariant with respect to the reparametrization that

$$\int e^{H(\boldsymbol{\zeta})} \prod_{j=1}^{N} dA(\zeta_j) = \int e^{H \diamond \Psi_t(\boldsymbol{\eta})} J_t(\boldsymbol{\eta}) \prod_{j=1}^{N} dA(\eta_j), \tag{9.1.4}$$

where $J_t(\boldsymbol{\eta})$ is the Jacobian. Note here that as $t \to 0$,

$$J_t(\boldsymbol{\eta}) = 1 + t \sum [\partial v(\eta_j) + \bar{\partial} v(\eta_j)].$$

We now define the Ward's functional W_v as

$$\begin{aligned} W_v :&= \frac{d}{dt}\Big|_{t=0} [H \diamond \Psi_t + J_t] \\ &= \sum [v(\eta_j)\partial_j H + \overline{v(\eta_j)} \bar{\partial}_j H + \partial v(\eta_j) + \bar{\partial} v(\eta_j)], \end{aligned} \tag{9.1.5}$$

where $\partial_j = \partial_{\eta_j}$. The $\mathbb{C}$-linear part W_v^+ is then given by

$$W_v^+ := \sum_j [v(\eta_j)\partial_j H + \partial v(\eta_j)]. \tag{9.1.6}$$

For a random variable $F = F(\boldsymbol{\eta})$ on $\mathbb{C}^N$, let us define

$$\nabla_v F := \frac{d}{dt} F \diamond \Psi_t = \sum [v(\eta_j)\partial_j F + \overline{v(\eta_j)}\bar{\partial}_j F] \tag{9.1.7}$$

and

$$\nabla_v^+ F := \sum v(\eta_j)\partial_j F. \tag{9.1.8}$$

Then we have the following form of Ward's identities.

Proposition 9.1.1 (Cf. Proposition 6.1 [77])**.** *We have*

$$\mathbf{E}[W_v F] + \mathbf{E}[\nabla_v F]. \tag{9.1.9}$$

Note that for 2D Coulomb gases, we have

$$W_v^+ = \frac{\beta}{2} \sum_{j<k} \frac{v(\zeta_j) - v(\zeta_k)}{\zeta_j - \zeta_k} + \sum [\partial v(\zeta_j) - mv\partial Q(\zeta_j)]. \tag{9.1.10}$$

In this case the Ward's identity (9.1.9) is also well known under the name of *loop equation* or *genus expansion* [84, 131]. In particular, this gives rise to the following relations between 1- and 2-point correlation functions

$$\frac{\beta}{2} \int_{\mathbb{C}} \frac{\mathbf{R}_{N,2}(\zeta, \eta)}{\zeta - \eta}\, dA(\eta) = m(\partial_\zeta Q(\zeta))\mathbf{R}_{N,1}(\zeta) + \partial_\zeta \mathbf{R}_{N,1}(\zeta), \tag{9.1.11}$$

see e.g., [2]. Such Ward's identities were utilized to study the macroscopic behavior of 2D Coulomb gases [2, 131] as well as the fluctuations of random matrix ensembles (when $\beta = 2$) [10, 11, 72].

Let us turn to the large-N behaviour of (9.1.1) in the high temperature regime $\beta \to 0$. It is clear that this regime implies weaker correlations among particles. In the extreme case $\beta \equiv 0$, the particles become independent from

each other, their k-point correlation functions trivially factorise and become proportional to $\prod_{j=1}^{k} e^{-Q(\zeta_j)}$, normalised with respect to the area measure. In [2], the crossover phenomenon between fixed and vanishing β was investigated. For instance, in the case of a Gaussian potential $Q(\zeta) = |\zeta|^2$, we study the smooth interpolation between Ginibre's circular law and the Gaussian distribution. The precise scaling we have to impose in (9.1.1) is to set

$$m = 1, \quad \beta = \frac{2c}{N} \quad \text{for fixed} \quad c \in (-2, \infty). \tag{9.1.12}$$

Here, c is kept fixed when $N \to \infty$, and we can allow for a weakly attracting interaction with negative c as well. The same scaling (9.1.12) was found on the real line in one dimension [8], with fixed $c \in (-1, \infty)$.

On the other hand, for the more standard scaling

$$m = \frac{\beta}{2} N, \tag{9.1.13}$$

In [32], Chafaï, Hardy and Maïda showed on a non optimal scale, using concentration of measure in Wasserstein distance, that if β is bigger than $\beta_0 \log N/N$ for some constant β_0, there is no such crossover phenomenon, and the limiting global density ρ follows from the Laplacian growth

$$0 = \Delta Q(\zeta) - \rho(\zeta). \tag{9.1.14}$$

In the high temperature regime (9.1.12) the *free energy functional* $F[\rho]$ in terms of the probability density function ρ, associated with the Gibbs measure (9.1.1) is given by

$$\begin{aligned} F \equiv F_c[\rho] = &\int_{\mathbb{C}} Q(\zeta)\rho(\zeta) dA(\zeta) \\ &- c \int_{\mathbb{C}^2} \log|\zeta - \eta| \rho(\zeta)\rho(\eta) dA(\zeta) dA(\eta) \\ &+ \int_{\mathbb{C}} \rho(\zeta) \log \rho(\zeta) dA(\zeta). \end{aligned} \tag{9.1.15}$$

While the first and the second lines can be easily seen to follow from the energy in (9.1.1), the third line is the so-called entropy contribution. The saddle point condition

$$0 = \frac{\delta F_c[\rho]}{\delta \rho(\zeta)} = Q(\zeta) - 2c \int_{\mathbb{C}} \log|\zeta - \eta| \rho(\eta)\, dA(\eta) + \log \rho(\zeta) + 1 \qquad (9.1.16)$$

is imposed in order to extremise the free energy. Whether it is a minimum or a maximum is determined by a second functional derivative of $F_c[\rho]$. Equation (9.1.16) has the limiting density

$$\rho_c(\zeta) = \lim_{N\to\infty} \frac{1}{N} R_{N,1}(\zeta)$$

as its solution. Applying the Laplace operator $\Delta = \partial\bar{\partial}$ to (9.1.16) and using that its Green's function is the logarithm, we obtain that the crossover density ρ_c satisfies (9.1.17) below. Namely, the limiting density function ρ_c minimises (resp., maximises) the free energy functional $F_c[\rho]$ given in (9.1.15) for $c > 0$ (resp., < 0), and solves the following *mean field equation*:

$$0 = \Delta Q(\zeta) - c\, \rho_c(\zeta) + \Delta \log \rho_c(\zeta)\ . \qquad (9.1.17)$$

We turn to the discussion of features of the solution of the mean field equation (9.1.17). Defining $\psi_c := \log \rho_c$, it is rewritten as follows

$$c\, e^{\psi_c(\zeta)} = \Delta Q(\zeta) + \Delta \psi_c(\zeta), \qquad (9.1.18)$$

which is a differential equation of *generalised Liouville type*. In case that $\Delta Q \equiv 0$ would hold, the equation (9.1.18) reduces to the standard Liouville equation whose explicit solutions are well known, see e.g., [39]. However, also in view of the result (9.1.14) in the standard scaling limit, we cannot assume that ΔQ is small or even negligible in any sense. For that reason it is not possible to provide an explicit solution for (9.1.17), or equivalently (9.1.18), even in the Gaussian case. For further examples, we refer the reader to [2].

9.2 Rescaled Ward's identities and their applications

The rescaled version of Ward's identities (called Ward's equation) was introduced in [12] as a tool to study local universality conjectures in random matrix theory. Intuitively, it says that the local eigenvalue statistics do not depend on the exact probability distribution of the ensemble, but only on some general characteristics. An important feature of Ward's equation is that in the large-N limit, it does not depend on the choice of the potential Q if the base point we look at the local statistics is regular in a proper sense. Due to this property, Ward's equation has been utilized to show the universality problems in various situations, see e.g., [4, 9, 13, 14] and references therein.

As an example, we shall present the implementation Ward's equation for the study of chiral ensembles. For given non-negative integers N, ν, let P and Q be $N \times (N+\nu)$ random matrices with independent complex Gaussian entries of mean 0 and variance $1/(4N)$. These are the building blocks of the two correlated random matrices

$$X_1 = \sqrt{1+\tau}\, P + \sqrt{1-\tau}\, Q, \quad X_2 = \sqrt{1+\tau}\, P - \sqrt{1-\tau}\, Q. \tag{9.2.1}$$

Here, $\tau \in [0,1]$ is a *non-Hermiticity* parameter. Only for $\tau = 0$, X_1 and X_2 are again uncorrelated Gaussian random matrices. In the other extremal case $\tau = 1$, the two matrices become perfectly correlated, $X_1 = X_2$. In the following we consider the complex eigenvalues of the non-Hermitian Wishart matrix X, given by the product of the two:

$$X := X_1 X_2^*. \tag{9.2.2}$$

Throughout this article, we shall use the notation

$$\alpha_N := \frac{\nu}{N}, \quad \lim_{N\to\infty} \alpha_N = \alpha \in [0, \infty). \tag{9.2.3}$$

The *chiral* version of X is given by $(2N+\nu)\times(2N+\nu)$ random *Dirac matrix*

$$\mathcal{D} := \begin{pmatrix} 0 & X_1 \\ X_2^* & 0 \end{pmatrix}. \tag{9.2.4}$$

In particular, for the extremal case $\tau = 1$, this non-Hermitian two-matrix ensemble $\mathcal{D}$ reduces to the standard Hermitian chiral Gaussian unitary ensemble with $X_1 = X_2$.

We investigate the local statistics of the ensemble $\mathcal{D}$ when the droplet splits into two connected components. Since the case $\alpha = 0$ does not reveal such a multi-critical behaviour, let us assume $\alpha > 0$ in the sequel and define the rescaled point process $\boldsymbol{z} = \{z_j\}_{j=1}^N$ at the origin as

$$z_j = (N\delta)^{1/4} \cdot \zeta_j, \quad \delta = \frac{1}{(1-\tau^2)^2\alpha}. \tag{9.2.5}$$

Here, the rescaling order is chosen so that the mean eigenvalue spacing of the rescaled process $\boldsymbol{z}$ is of order $O(1)$. Recall that the k-point correlation function $R_{N,k}(z_1, \cdots, z_k)$ of the system $\boldsymbol{z}$ is given by

$$R_{N,k}(z_1, \cdots, z_k) := \lim_{\epsilon \downarrow 0} \frac{\mathbb{P}(\exists \text{ at least one particle in } \mathbb{D}(z_j, \epsilon), j = 1, \cdots, k)}{\epsilon^{2k}},$$

where $\mathbb{D}(z_j, \epsilon)$ is a disc with centre z_j and radius ϵ.

Then it was shown in [3, Theorem 3] that for each $k \in \mathbb{N}$, we have

$$\lim_{N\to\infty} R_{N,k}(z_1, \cdots, z_k) = R_k(z_1, \cdots, z_k) = \det\Big[K(z_j, z_l)\Big]_{1\le j,l\le k} \tag{9.2.6}$$

uniformly for $z_1, \cdots, z_k$ in compact subsets of $\mathbb{C}$, where

$$K(z,w) = |zw|\, e^{z^2\bar{w}^2 - |z|^4/2 - |w|^4/2} \mathrm{erfc}\Big(-\frac{z^2+\bar{w}^2}{\sqrt{2}}\Big). \tag{9.2.7}$$

This provides the first explicit result for the local statistics at the singular boundary of lemniscate type, see [29] for a recent work on a non-chiral version. In [3, Section 4.1], the chiral version of Ward's equation plays the role in

heuristically guessing the explicit formula (9.2.7). For this, let us define the *Berezin kernel*

$$B(z,w) := \frac{|K(z,w)|^2}{R(z)}. \tag{9.2.8}$$

Then one can show that the following form of Ward's equation hold:

$$\bar{\partial}\widetilde{C}(z) = 2R(z) - \Delta V_0(z) - \Delta \log R(z), \tag{9.2.9}$$

where

$$\widetilde{C}(z) := 2z \int_{\mathbb{C}} \frac{B(z,w)}{z^2 - w^2}\, dA(w) \tag{9.2.10}$$

and $V_0(z) := |z|^4 - 2\log|z|$. Then the explicit form (9.2.7) can be obtained as a special solution to such an integral differential equation.

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국문초록

본 학위논문의 목적은 이중연결영역에서의 가우스 자유장으로부터 건설된 등각장론과 이의 환 슈람-뢰브너 전개 (SLE) 이론에의 구현을 정립하는 것이다. 특히, 리 (Lie) 미분 연산자 및 타이히뮐러 (Teichmüller) 매개 변수를 통해 비라소로 (Virasoro) 장을 기술하는 에구치-오구리 (Eguchi-Ooguri) 버전의 와드 (Ward) 방정식들을 기반으로 특정 유형의 BPZ 방정식들을 도출한다. 이에 대한 응용으로서, 환 SLE 분배함수에 대한 영-벡터 방정식의 쿨롱 (Coulomb) 기체 해들을 찾고, 여러가지 환 SLE 들에 대한 마팅게일 관측들을 건설한다.

주요어휘: 등각장론, 슈람-뢰브러 전개, 가우스 자유장, 이중연결영역, 분배함수, 마팅게일 관측
학번: 2015-20264

감사의 글

우선 박사학위 과정을 무사히 마칠수 있도록 지도해주신 강남규 교수님께 깊은 감사를 드립니다. 교수님의 아낌없는 지원과 가르침 덕분에 학위 과정 중에 좋은 환경에서 많은 것을 경험하며 전문 수학자로서 성장할수 있었다고 생각합니다. 무엇보다 항상 즐겁게 연구할수 있는 연구자로서의 자세를 배울수 있었습니다. 앞으로도 교수님의 귀중한 가르침을 잊지 않으며 더욱 성장하도록 하겠습니다. 진심으로 감사드립니다.

박사 학위 심사를 비롯하여 좋은 연구를 할수 있도록 소중한 도움과 조언을 주신 김판기 교수님께 감사를 드립니다. 대학원 생활 중에 교수님의 아낌없는 지원 덕분에 아무런 불편함없이 연구에만 몰두 할수 있었습니다. 특히 IRTG 프로그램을 통해 독일에서 생활하고 연구할수 있는 기회를 주신 덕분에 제 연구분야도 넓히고 더 넓은 시야를 얻을수 있었습니다.

독일 빌레펠트 대학교에서 교환학생으로 있을때 멘토셨던 Gernot Akemann 교수님께 이 자리를 빌어 감사를 드립니다. 저를 지도학생처럼 아껴주시며 여러 공동연구를 하며 많은 가르침을 주신 덕분에 지금의 제가 있다고 생각합니다. 또한 학회 및 워크샵을 개최하실 때마다 저를 초청해주신 덕분에 동 연구분야 사람들과 교류하며 학자로서 성장할수 있었습니다.

대학원 기간 중에 공동연구를 해온 Yacin Ameur 교수님, 이지운 교수님, 이승엽 교수님, Dapeng Zhan 교수님께도 깊이 감사를 드립니다. 교수님들의 전문 분야지식을 아낌없이 가르쳐주신 덕분에 제 연구의 지평을 넓힐 수 있었습니다. 또한 저를 수차례 초청해주신 덕분에 대학원 생활의 일부를 해외에서 즐겁게 연구하며 보낼수 있었고, 교수님들께서 저를 학생보다는 한명의 연구자로서 생각해주신 덕분에 책임감 있는 주체적인 공동연구자로서의 자세를 배울수 있었습니다. 앞으로도 더욱 활발히 연구하는 연구자가 되도록 하겠습니다.

바쁘신 와중에도 박사 학위 심사를 맡아주신 서인석 교수님, 서의린 교수님, Tom Alberts 교수님께도 감사의 인사를 전합니다. 또한 항상 진심어린 격려와 소중한 조언을 주셨던 서울대학교 수리과학부의 여러 교수님들 및 대한수학회 확률론 분과 교수님들께도 감사의 인사를 전합니다. 특히 저를 초청해주시고 강연의 기회를 주셨던 Paul Jung 교수님, 김건우 교수님께 감사를 드립니다.

BIBLIOGRAPHY

처음 연구를 시작할때부터 많은 것을 가르쳐주시고 이끌어주신 서성미 박사님, 탁희준 박사님께도 감사를 드립니다. 학부때부터 같이 공부하고 대학원에서도 같은 연구실을 쓰며 항상 서로 응원하며 고민을 나누던 민기, 현준에게도 감사를 전합니다. 대학원 생활 내내 많은 수학 이야기를 나누며 함께 생활했던 서울대학교 수리과학부 학생들과 고등과학원 박사님들께도 모두 감사를 전합니다. IRTG 프로그램에서 함께 활동했던 한국, 독일의 학생들에게도 감사를 드립니다. 많은 시간들을 함께하며 쌓았던 추억들 덕분에 대학원 생활을 더욱 즐겁게 할수 있었습니다. 특히 독일에서 교환학생으로서 같이 생활하며 서로 의지했던 김민현 박사님, 이재훈 박사님께도 이 기회를 빌어 다시 한번 감사를 전합니다. 또한 오랜 시간을 함께 하며 응원해줬던 여러 친구들에게도 고맙다는 말을 전합니다. 지면이 부족해 모든 분들을 언급할수는 없지만, 제가 깊이 감사한다는 것을 알고 계시리라 믿습니다.

항상 저를 믿고 응원하며 무조건적인 지원을 해준 엄마, 큰누나, 작은누나에게 이자리를 빌어 감사를 드립니다. 대학원 생활 내내 함께 하며 큰 도움을 주었던 화현이에게도 고맙다는 말을 전합니다. 마지막으로 이 순간을 자랑스러워하실 아버지께 이 논문을 바칩니다.