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Abstract

Coefficients of $e_{n-1,1}$ of chromatic quasisymmetric functions

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Coefficients of chromatic quasisymmetric functions for the class of unit interval graphs with bipartite complement in the elementary basis can be characterized as certain q-hit numbers. We introduce a bounce q-hit number which is a refined notion of a q-hit number. We characterize coefficients of $e_{n-1,1}$ of chromatic quasisymmetric functions for the class of unit interval graphs on the elementary basis and show that it is positive. This is partial proof of the Stanley-Stembridge conjecture on chromatic quasisymmetric functions.

Keywords : chromatic quasisymmetric function, rook placement, e-positivity
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Contents

1 Introduction

In 1995, Stanley [12] introduced a *chromatic symmetric function* $X_G(x)$, which is a symmetric function generalization of a chromatic polynomial of a simple graph G. Given a simple graph $G = (V, E)$ it is defined as

$$X_G(x) := \sum_k \prod_{v \in V} x_{k(v)}$$

where the sum runs through all proper colorings of the vertices $k : V(G) \to \mathbb{N}$. A coloring k is proper if $k(u) \neq k(v)$ whenever $\{u, v\} \in E$. It is straightforward to confirm that $X_G(x)$ lies in Λ which is the algebra of symmetric functions. There has been plenty of researche about the chromatic symmetric function in diverse areas. [11, 2, 9, 3, 8, 1, 4]

Recall that the *incomparability graph* inc(P) of a poset P is a graph with vertex P, edges of which are pairs of incomparable elements. Also, a poset P is called *(r+s)-free* if P does not contain an induced subposet isomorphic to the direct sum of an r element chain and s element chain. We say that symmetric function f is *e-positive* if coefficients of f with respect to the elementary basis $(e_\lambda)_{\lambda \in \mathrm{Par}}$ of Λ are all nonnegative. The following is one of the famous conjectures on chromatic symmetric function presented by Stanley and Stembridge.

Conjecture 1.1. (Stanley-Stembridge Conjecture [13, Conjecture 5.5]). *Let G = (V,E) be the incomparability graph of a (3+1)-free poset. Then $X_G(x)$ is e-positive.*

In 2016, Shareshian and Wachs [11] introduced a *chromatic quasisymmetric function* which is a quasisymmetric refinement of the chromatic symmetric function. For a simple graph $G = (V, E)$, the refinement is given as follows.

$$X_G(x, q) := \sum_k \prod_{v \in V} q^{asc(k)} x_{k(v)}$$

where $\mathrm{asc}(k) := |\{\{i, j\} \in E : i < j \text{ and } k(i) < k(j)\}|$. Then $X_G(x, 1) = X_G(x)$ and $X_G(x, q) \in \mathrm{QSym}[q]$. Moreover it is well known that $X_G(x, q) \in$

$\Lambda[q]$ when G is incomparability graph of (3+1)-free poset [11, Theorem 4.5]. This lets us refine the Stanley-Stembridge conjecture.

Conjecture 1.2. ([11, Conjecture 1.3]). *Let G be the incomparability graph of a (3+1)-free poset. Then $X_G(x,q)$ is e-positive. That is, if $X_G(x,q) = \sum_{j=1}^{m} a_j(x)q^j$ then $a_j(x)$ is e-positive for all j.*

It has been proved in various ways that $X_G(x,q)$ is e-positive for a unit interval graph $G = (V,E)$ which has a bipartite complement, i.e., $\exists$ nonzero distinct sets A, B such that $A \cup B = V$ and $\{i,j\} \notin E,\ \forall i \in A \text{ and } j \in B$. The goal of this paper is to find a positive formula of the coefficient of $e_{n-1,1}$ of $X_G(x,q)$ for an arbitrary unit interval graph G. For a partition λ contained in a $n \times n$ board, let $H_j^n(\lambda)$ denotes the q-hit number as in [4]. When $q = 1$, $H_j^n(\lambda)$ counts the number of rook placements with n rooks on a $n \times n$ board such that precisely j rooks are in λ. The goal can be accomplished by proving the following main theorem.

Theorem 1.3. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition with $\lambda_1 \leq n-1$ and $k \leq n-1$.*

$$X_\lambda(x,q) \mid_{e_{n-1,1}} = \begin{cases} q[n-2]_q H_1^{n-2}(\lambda) + q[n-1]_q H_{[0,1],n}^{n-2}(\lambda) & \text{if } \lambda_1, k \leq n-2, \\ [n-1]_q H_0^{n-2}((\lambda_2, \ldots, \lambda_k)) & \text{if } \lambda_1 = n-1, \\ [n-1]_q H_0^{n-2}((\lambda_1 - 1, \ldots, \lambda_k - 1)) & \text{if } k = n-1. \end{cases}$$

We can easily find out that it is a q-polynomial, every coefficient of which is nonnegative. This is very partial proof of Conjecture 1.2.

The paper is organized as follows. In Section 2, we lay out some known results required to prove Theorem 1.3. In Section 3, we review Abreu and Nigro's results about rook placements and q-hit numbers. Also, we introduce a generalized definition of q-hit numbers with rook placements and some of their properties. In Section 4, we prove Theorem 1.3.

2 Preliminaries

2.1 Natural unit interval orders

A chromatic quasisymmetric function is a function from a finite simple graph to $\Lambda[q]$ and we want to focus on the case that the finite simple graph is an incomparability graph of some (3+1)-free poset. But it is quite complicated to deal with such a graph. In this section, we introduce the way to substitute our object from graph to partition. It starts from the following result, which Guay-Paquet introduced in 2013.

Theorem 2.1. ([7, Theorem 5.1]). *If Conjecture 1.1 holds for every incomparability graph of (3+1)-free and (2+2)-free poset, then it also holds for every incomparability graph of (3+1)-free poset.*

So, it is enough to consider the set of (3+1)-free and (2+2)-free posets which is a subset of the set of (3+1)-free posets. *Natural unit interval order* describes our reduced set.

Definition 2.2. ([11]). Let P be a poset on finite subset of $\mathbb{N}$. We say P is a *natural unit interval order* if it satisfies both conditions:

- $x <_P y$ implies $x < y$ in the natural order on $\mathbb{N}$, and
- if the direct sum $\{x <_P z\} + \{y\}$ is an induced subposet of $\mathbb{N}$ then $x < y < z$ in the natural order in $\mathbb{N}$.

Theorem 2.3. ([11, Proposition 4.1]). *Let P be a poset on $[n]$. The following conditions on P are equivalent.*

1. *P is a natural unit interval order.*
2. *There exist n real numbers $y_1 < \ldots < y_n$ such that, for i,j $\in [n]$, $y_i + 1 < y_j$ if and only if $i <_P j$.*
3. *There exist weakly increasing sequence $\mathbf{m} = (m_1, \ldots, m_{n-1})$ such that $i \leq m_i \leq n$ for all i and $i <_P j$ if only if $m_i < j$.*

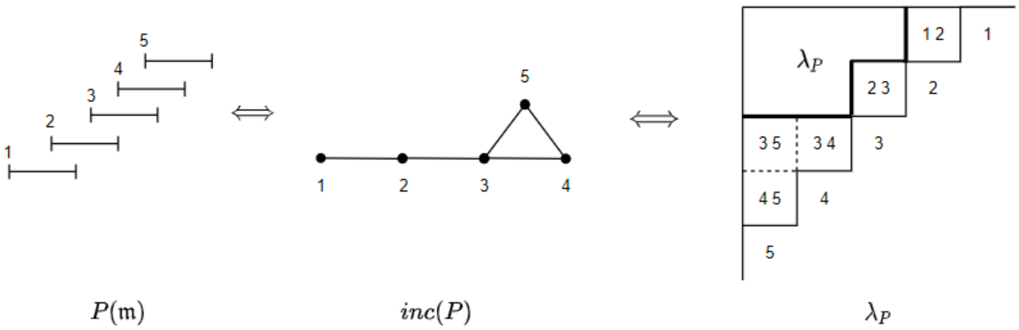

Figure 1: $P(\mathbf{m})$, $inc(P)$ and λ_P when $\mathbf{m}$ = (2,3,5,5)

It is easy to check that if P is a (3+1)-free and (2+2)-free poset on $[n]$, then P is a natural unit interval order. Furthermore, if poset P on $[n]$ satisfies condition *2* above, then it is (3+1)-free and (2+2)-free (See [10]). Therefore we can conclude that the set of (3+1)-free and (2+2)-free posets on $[n]$ is equal to the set of posets on $[n]$ satisfying condition *3*.

For a given sequence in *3*, let λ be a partition defined by $\lambda_i = n - m_i$ for $1 \leq i \leq n-1$. Note that there is a one-to-one correspondence between $\mathbf{m}$ and a partition λ contained in $\delta_n = (n-1, n-2, \ldots, 1)$. Consequently, for given positive integer n there is one-to-one correspondence between the partitions contained in $\delta_n = (n-1, n-2, \ldots, 1)$ and incomparability graphs of (3+1)-free and (2+2)-free posets. For a partition λ contained in δ_n, denote P_λ to be a corresponding poset. (See Figure 1)

2.2 Linear relation

Denote $[n] = \{1, 2, \ldots, n\}$, $[n]_q = \frac{q^n-1}{q-1}$ and $[n]_q! = \prod_{k=1}^{n}[k]_q$ as usual. Also, let Λ_q be the algebra of symmetric functions with coefficients in $\mathbb{Q}(q)$.

Definition 2.4. We say that a function $f : \{\text{Partitions of } \mathbb{N}\} \to \Lambda_q$ satisfies the *linear relation* if

$$(1+q)f(\lambda^1) = f(\lambda^0) + qf(\lambda^2) \tag{2.1}$$

whenever one of the following conditions hold

1. There exists $i \geq 2$ such that $\lambda_i + 2 \leq \lambda_{i-1}$ and $\lambda_{n-\lambda_i-1} = \lambda_{n-\lambda_i}$. Moreover, λ^1 and λ^2 satisfy $\lambda_j^a = \lambda_j^0$ if $j \neq i$ and $\lambda_i^a = \lambda_i^0 + a$ for a $=$ $1, 2$.

2. μ^0, μ^1 and μ^2 satisfies the condition 1 when μ^k is a transposition of λ^k for k = 0,1,2.

Let us call condition 1 and 2 be a *row condition* and *column condition* respectively. Then we say f satisfies the *row linear relation*[*column linear relation*] if the equation (2.1) holds for λ^i's which satisfies the row condition [column condition].

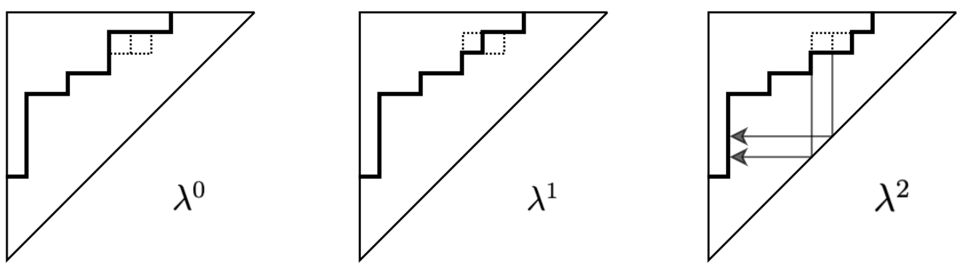

Figure 2: λ^0, λ^1 and λ^2 satisfying condition 1.

Theorem 2.5. ([9, Theorem 3.4]). *Chromatic quasisymmetric function satisfies the linear relation.*

Theorem 2.6. ([1, Theorem 1.1]). *Chromatic quasisymmetric function* $X : \{\text{unit interval graphs}\} \to \Lambda_q$ *is the unique function that has the following three properties.*

- *It satisfies the linear relation.*
- *It is multiplicative,* $X_{G_1 \sqcup G_2} = X_{G_1} X_{G_2}$.
- *It has values at complete graphs given by* $X_{K_n} = [n]_q! e_n$.

Theorem 2.7. ([1, Theorem 1.2]). *Let* A *be a* $\mathbb{Q}(q)$*-algebra and let* $f : \{\text{unit interval graphs}\} \to A$ *be a function that satisfies the linear relation. Then* f *is determined by its values* $f(K_{n_1} \sqcup K_{n_2} \sqcup \cdots \sqcup K_{n_m})$ *at the disjoint ordered.*

3 Rook placements and q-hit numbers

Rook placements that count the number of placing non-attacking rooks on a given board are a generalization of permutation diagrams. Garsia and Remmel give q-analogs of rook placements by counting inversions [6]. It has been revealed that there is a strong connection between rook placements and chromatic quasisymmetric functions [1, 2, 4]. In this section, we present a generalized definition of q-hit numbers with rook placements and some of their properties. Our new definition will appear in section 4 to illustrate an explicit formula of the coefficient of $e_{n-1,1}$. First, we review the known results.

Let λ be a partition contained in a $n \times n$ board. Define $B_j^n(\lambda)$ to be the set of placements of n rooks on a $n \times n$ board such that precisely j rooks are in λ. Each rook placement has a λ-weight defined as in [5]. The weight is the number of cells c in a $n \times n$ board such that

1. there is no rook in c,
2. there is no rook to the left of c,
3. if c is in λ then the rook on the same column of c is in λ and below c,
4. if c is not in λ then the rook on same column of c is either in λ or below c.

We define a *q-hit number*

$$H_j^n(\lambda) := \sum_{\sigma \in B_j^n(\lambda)} q^{\mathrm{wt}_\lambda(\sigma)}.$$

where $\mathrm{wt}_\lambda(\sigma)$ is the λ-weight of σ.

Theorem 3.1. ([1, Theorem 4.3])

Remark. For partition λ, if $\lambda_1 \leq j$ or $l(\lambda) \leq j$ then $H_j^m(\lambda) = 0$. If not, $q^j[m-j-1]_q H_j^m(\lambda)$ is the coefficient of $e_{m-j+1,j}$ of $X_\lambda(x,q)$ by Theorem 3.1. Recall that chromatic quasisymmetric function is transpose invariant i.e., $X_\lambda(x,q) = X_{\lambda^t}(x,q)$ ([1, Theorem 1.2.]). Hence $q^j[m-j-1]_q H_j^m(\lambda) =$

$q^j[m-j-1]_q H_j^m(\lambda)^t$. In conclusion,$H_j^m(\lambda) = H_j^m(\lambda^t)$ namely q-hit numbers are transpose invariant.

Definition 3.2. Let λ be a partition contained in a $n \times n$ board. Define $B^n_{[0,1],m}(\lambda)$ to be the set of placements of n rooks on a $n \times n$ board such that precisely two rooks satisfying the following condition are in λ.

- Let c_1 and c_2 be cells in λ containing rook and c_2 locates lower than c_1. Then $a + b = m + 1$ when c_1 and c_2 are cells with $a - th$ column and $b - th$ row respectively.

For each rook placement define *bounce λ-weight* with refined statistic. The bounce weight is the number of cells c in a $n \times n$ board such that

1. there is no rook in c,
2. there is no rook to the left of c,
3. if c is in λ then the c satisfies both of the followings
   (a) the rook on the same column of c is in λ and below c
   (b) if the rook on the same column is not the rightmost rook in λ, the rook on the same row is in λ.
4. if c is not in λ then the rook on same column of c is either in λ or below c.

See Figure 3 for an example, where the black circles are the rooks, while the c corresponds to the cells c satisfying the above condition and gray cells are partition λ.

We define *bounce q-hit number*

$$H^n_{[0,1],m}(\lambda) := \sum_{\sigma \in B^n_{[0,1],m}(\lambda)} q^{\mathrm{bwt}_\lambda(\sigma)}.$$

where $\mathrm{bwt}_\lambda(\sigma)$ is the bounce λ-weight of σ.

Proposition 3.3. *Bounce q-hit number satisfies the linear relation.*

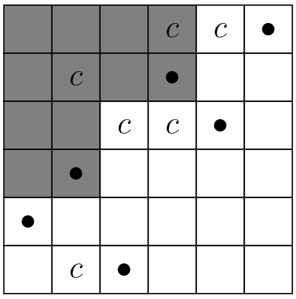

Figure 3: A rook placement in $B^6_{[0,1],7}(\lambda)$ with bounce λ-weight 5 with $\lambda =$ (4,4,2,2)

Proof. We need to show that (2.1) holds for $H^n_{[0,1],m}$ with a partition satisfying row or column condition of Definition 2.4..

Let us starts from the row linear relation. Let λ^0, λ^1 and λ^2 be partitions satisfying row condition with i-th row. Then for $k = 0, 1, 2$ $B^n_{[0,1],m}(\lambda^k)$ is the union of four disjoint subsets:

- $A_k = \{\sigma \in B^n_{[0,1],m}(\lambda^k) \mid \text{there is a rook on } (i, r) \text{ and } r > \lambda_i + 2\}$
- $B_k = \{\sigma \in B^n_{[0,1],m}(\lambda^k) \mid \text{there is a rook on } (i, r) \text{ and } r \le \lambda_i\}$
- $C_k = \{\sigma \in B^n_{[0,1],m}(\lambda^k) \mid \text{there is a rook on } (i, r) \text{ and } r = \lambda_i + 1\}$
- $D_k = \{\sigma \in B^n_{[0,1],m}(\lambda^k) \mid \text{there is a rook on } (i, r) \text{ and } r = \lambda_i + 2\}$

Let σ_k, τ_k be placements of rooks on a $n \times n$ board, location of which are exactly same for all k. If $\sigma_k \in A_k$, $\mathrm{btw}_{\lambda^0}(\sigma_0) = \mathrm{btw}_{\lambda^1}(\sigma_1) + 1 = \mathrm{btw}_{\lambda^2}(\sigma_2) + 2$. So we can induce

$$(1+q)\sum_{\sigma\in A_1} q^{\mathrm{bwt}_{\lambda^1}(\sigma)} = \sum_{\sigma\in A_0} q^{\mathrm{bwt}_{\lambda^0}(\sigma)} + q\sum_{\sigma\in A_2} q^{\mathrm{bwt}_{\lambda^2}(\sigma)}. \tag{3.1}$$

If $\sigma_k \in B_k$, $\mathrm{btw}_{\lambda^0}(\sigma_0) = \mathrm{btw}_{\lambda^1}(\sigma_1) = \mathrm{btw}_{\lambda^2}(\sigma_2)$. And this gives

$$(1+q)\sum_{\sigma\in B_1} q^{\mathrm{bwt}_{\lambda^1}(\sigma)} = \sum_{\sigma\in B_0} q^{\mathrm{bwt}_{\lambda^0}(\sigma)} + q\sum_{\sigma\in B_2} q^{\mathrm{bwt}_{\lambda^2}(\sigma)}. \tag{3.2}$$

Define a *transition map* $\mathcal{S}_i^{col}$ $[\mathcal{S}_i^{row}]$ to be a function that maps rook placements on a $n \times n$ board to themselves by switching the location of rooks on the $i-th$ and $i+1-th$ column [row]. (See Figure 4)

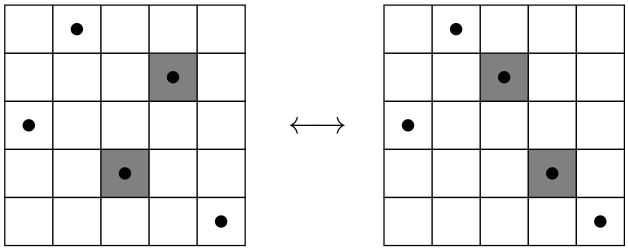

Figure 4: $\mathcal{S}_3^{col}$

Now assume $\sigma_k \in C_k$ and $\tau_k \in D_k$. Then $\mathrm{btw}_{\lambda^1}(\sigma_1) = \mathrm{btw}_{\lambda^2}(\sigma_2)$ and $\mathrm{btw}_{\lambda^1}(\tau_1) + 1 = \mathrm{btw}_{\lambda^0}(\tau_0)$. This gives

$$(3.3) \qquad q \sum_{\sigma \in C_1} q^{\mathrm{bwt}_{\lambda^1}(\sigma)} = q \sum_{\sigma \in C_2} q^{\mathrm{bwt}_{\lambda^2}(\sigma)}, \quad q \sum_{\tau \in D_1} q^{\mathrm{bwt}_{\lambda^1}(\tau)} = \sum_{\tau \in D_0} q^{\mathrm{bwt}_{\lambda^0}(\tau)}.$$

claim: $\sum_{\tau \in D_1} q^{\mathrm{bwt}_{\lambda^1}(\tau)} = \sum_{\sigma \in C_0} q^{\mathrm{bwt}_{\lambda^0}(\sigma)}$.

(i) Assume that the rook on the $\lambda_i + 2 - th$ column is not in the partition λ. Then $\mathcal{S}_{\lambda_i+1}^{col}$ is a bijection from $C_0^{(1)} = \{\sigma \in C_0 \mid$ the rook on the $\lambda_i + 2$ - th column is not in the partition $\lambda\}$ to $D_1^{(1)} = \{\sigma \in D_1 \mid$ the rook on the $\lambda_i + 1$ - th column is not in the partition $\lambda\}$ and $\mathrm{btw}_{\lambda^1}((\tau_1)) = \mathrm{btw}_{\lambda^1}(\mathcal{S}_{\lambda_i+1}^{col}(\sigma_0)) = \mathrm{btw}_{\lambda^0}(\sigma_0)$.

(ii) Assume contrary that the rook on the $\lambda_i + 2 - th$ column is in the partition λ. We can number rooks in the partition from right to left. Since there are only two rooks in the partition, we call the rightmost rook the 1st rook and the leftmost rook the 2nd rook.

If the rook on the $\lambda_i + 2 - th$ column is the 2nd rook, $\mathcal{S}_{\lambda_i+1}^{col}$ is a bijection from $C_0^{(2)} = \{\sigma \in C_0 \mid$ the rook on the $\lambda_i + 2$ - th column is the 2nd rook in the partition $\lambda\}$ to $D_1^{(2)} = \{\sigma \in D_1 \mid$ the rook on the $\lambda_i + 1$ - th column is the 2nd rook in the partition $\lambda\}$ and bounce q-hit weights are invariant under this map.

If the rook on the $\lambda_i + 2 - th$ column is the 1st rook, $\mathcal{S}_{m-\lambda_i}^{row}\mathcal{S}_{\lambda_i+1}^{col}$ is a bijection from $C_0^{(3)} = \{\sigma \in C_0 \mid$ the rook on the $\lambda_i + 2$ - th column is the 1st rook in the partition $\lambda\}$ to $D_1^{(3)} = \{\sigma \in D_1 \mid$ the rook on the $\lambda_i + 1$ - th column is the 1st rook in the partition $\lambda\}$ and bounce q-hit weights are invariant under this map.

Note that C_0 and D_0 are the disjoint unions of $C_0^{(i)}$'s and $D_0^{(i)}$'s. So the multiset of bounce q-hit weights of C_0 is equal to that of D_0. This proves the claim.

claim: $\sum_{\sigma\in C_1} q^{\mathrm{bwt}_{\lambda^1}(\sigma)} = q\sum_{\tau\in D_0} q^{\mathrm{bwt}_{\lambda^0}(\tau)}$.

Similar to the previous claim, it can be proved with the following function. Let

$$f(\sigma) := \begin{cases} \mathcal{S}^{row}_{m-\lambda_i}\mathcal{S}^{col}_{\lambda_i+1}(\sigma) & \text{if the rook on the } \lambda_i\text{+1-th column is the 1st rook,} \\ \mathcal{S}^{col}_{\lambda_i+1}(\sigma) & \text{if the rook on the } \lambda_i\text{+1-th column is the 2nd rook.} \end{cases}$$

Then f is a bijection from C_1 to D_0 such that $\mathrm{bwt}_{\lambda^1}(\sigma)+1 = \mathrm{bwt}_{\lambda^0}(f(\sigma))$. So the claim is proved.

Combining (3.1) - (3.3) and two claims,

$$(1+q)B^n_{[0,1],m}(\lambda^1) = B^n_{[0,1],m}(\lambda^0) + qB^n_{[0,1],m}(\lambda^2).$$

Therefore $H^n_{[0,1],m}$ satisfies the row linear relation.

Now let us move on to the column linear relation. The technique of proof is exactly the same as the row case. Let λ^0, λ^1 and λ^2 be partitions satisfying column condition in definition 2.4. with $\lambda^0_j + 1 = \lambda^1_j = \lambda^2_j$ and $\lambda^1_{j+1}+1 = \lambda^2_{j+1} = \lambda^2_j$. Then for $k = 0,1,2$, $B^n_{[0,1],m}(\lambda^k)$ is the union of four disjoint subsets:

- $A_k = \{\sigma \in B^n_{[0,1],m}(\lambda^k) \mid$ rooks on the j-th and $j+1$-th row are not in λ^2 $\}$
- $B_k = \{\sigma \in B^n_{[0,1],m}(\lambda^k) \mid$ rooks on the j-th and $j+1$-th row are in λ^0 $\}$
- $C_k = \{\sigma \in B^n_{[0,1],m}(\lambda^k) \mid$ the rook on the j-th row is not in λ^2 and the rook on the $j+1$-th row is in λ^0 $\}$
- $D_k = \{\sigma \in B^n_{[0,1],m}(\lambda^k) \mid$ the rook on the j-th row is in λ^0 and the rook on the $j+1$-th row is not in λ^2 $\}$
- $E_k = \{\sigma \in B^n_{[0,1],m}(\lambda^k) \mid$ the rook on the j-th row is on λ^2_j-th column and the rook on the $j+1$-th row is in λ^0 $\}$
- $F_k = \{\sigma \in B^n_{[0,1],m}(\lambda^k) \mid$ the rook on the j-th row is on in λ^0 and the rook on the $j+1$-th row is on λ^2_j-th column $\}$

- $G_k = \{\sigma \in B^n_{[0,1],m}(\lambda^k) \mid$ the rook on the j-th row is on λ^2_j-th column and the rook on the $j+1$-th row is not in λ^2 $\}$
- $H_k = \{\sigma \in B^n_{[0,1],m}(\lambda^k) \mid$ the rook on the j-th row is on not in λ^2 and the rook on the $j+1$-th row is on λ^2_j-th column $\}$

First, notice that E_1, E_2 and F_2 are empty sets.
With an identity map, we can obtain the following.

$$\sum_{\sigma\in A_1} q^{\mathrm{bwt}_{\lambda^1}(\sigma)} = q\sum_{\sigma\in A_2} q^{\mathrm{bwt}_{\lambda^2}(\sigma)}, \quad q\sum_{\sigma\in A_1} q^{\mathrm{bwt}_{\lambda^1}(\sigma)} = \sum_{\sigma\in A_0} q^{\mathrm{bwt}_{\lambda^0}(\sigma)},$$

$$\sum_{\sigma\in B_1} q^{\mathrm{bwt}_{\lambda^1}(\sigma)} = \sum_{\sigma\in B_0} q^{\mathrm{bwt}_{\lambda^0}(\sigma)}, \quad q\sum_{\sigma\in B_1} q^{\mathrm{bwt}_{\lambda^1}(\sigma)} = q\sum_{\sigma\in B_2} q^{\mathrm{bwt}_{\lambda^2}(\sigma)},$$

$$q\sum_{\sigma\in C_1} q^{\mathrm{bwt}_{\lambda^1}(\sigma)} = q\sum_{\sigma\in C_2} q^{\mathrm{bwt}_{\lambda^2}(\sigma)}, \quad q\sum_{\sigma\in D_1} q^{\mathrm{bwt}_{\lambda^1}(\sigma)} = \sum_{\sigma\in D_0} q^{\mathrm{bwt}_{\lambda^0}(\sigma)},$$

$$q\sum_{\sigma\in F_1} q^{\mathrm{bwt}_{\lambda^1}(\sigma)} = \sum_{\sigma\in F_0} q^{\mathrm{bwt}_{\lambda^0}(\sigma)}, \quad \sum_{\sigma\in G_1} q^{\mathrm{bwt}_{\lambda^1}(\sigma)} = q\sum_{\sigma\in G_2} q^{\mathrm{bwt}_{\lambda^2}(\sigma)},$$

$$q\sum_{\sigma\in H_1} q^{\mathrm{bwt}_{\lambda^1}(\sigma)} = \sum_{\sigma\in H_0} q^{\mathrm{bwt}_{\lambda^0}(\sigma)}.$$

Let

$$g(\sigma) := \begin{cases} \mathcal{S}^{row}_j & \text{if the rook on the } j\text{-th or } j{+}1\text{-th row is the 1st rook,} \\ \mathcal{S}^{col}_{m+1-j}\mathcal{S}^{row}_j(\sigma) & \text{if the rook on the } j\text{-th or } j{+}1\text{-th row is the 1st rook.} \end{cases}$$

By g we can obtain

$$\sum_{\sigma\in C_1} q^{\mathrm{bwt}_{\lambda^1}(\sigma)} = q\sum_{\sigma\in D_2} q^{\mathrm{bwt}_{\lambda^2}(\sigma)}, \quad q\sum_{\sigma\in D_1} q^{\mathrm{bwt}_{\lambda^1}(\sigma)} = \sum_{\sigma\in C_0} q^{\mathrm{bwt}_{\lambda^0}(\sigma)},$$

$$\sum_{\sigma\in F_1} q^{\mathrm{bwt}_{\lambda^1}(\sigma)} = \sum_{\sigma\in E_0} q^{\mathrm{bwt}_{\lambda^0}(\sigma)}, \quad q\sum_{\sigma\in G_1} q^{\mathrm{bwt}_{\lambda^1}(\sigma)} = q\sum_{\sigma\in H_2} q^{\mathrm{bwt}_{\lambda^2}(\sigma)}.$$

Lastly, the map $\mathcal{S}^{row}_j$ draws

$$\sum_{\sigma\in H_1} q^{\mathrm{bwt}_{\lambda^1}(\sigma)} = \sum_{\sigma\in G_0} q^{\mathrm{bwt}_{\lambda^0}(\sigma)}.$$

Add whole equations. Then we can see that the column linear relation also holds. Both row and column linear relations hold. Finally, proposition 3.3. is proved. $\square$

4 Coefficients of $e_{n-1,1}$

In this section, we find the coefficient of $e_{n-1,1}$ of chromatic quasisymmetric function with the expansion of elementary basis e.

For partition λ and μ let $X_\lambda(x,q)\mid_{e_\mu}$ be the coefficient of e_μ with the expansion of elementary basis e.

Theorem 4.1 (Theorem 1.3.). *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition with $\lambda_1 \le n-1$ and $k \le n-1$.*

$$X_\lambda(x,q)\mid_{e_{n-1,1}} = \begin{cases} q[n-2]_q H_1^{n-2}(\lambda) + q[n-1]_q H_{[0,1],n}^{n-2}(\lambda) & \text{if } \lambda_1, k \le n-2, \\ [n-1]_q H_0^{n-2}((\lambda_2,\ldots,\lambda_k)) & \text{if } \lambda_1 = n-1, \\ [n-1]_q H_0^{n-2}((\lambda_1 - 1, \ldots, \lambda_k - 1)) & \text{if } k = n-1. \end{cases}$$

If $\lambda_1, k = n-1$, $H_0^{n-2}((\lambda_2,\ldots,\lambda_k))$ and $H_0^{n-2}((\lambda_1-1,\ldots,\lambda_k-1))$ are both zeros. So $X_\lambda(x,q)\mid_{e_{n-1,1}}$ is well defined.

Lemma 4.2. *For $m \in \mathbb{N}$ and a partition λ in a $(m-1)\times(m-1)$ board,*

$$H_1^m((m,\lambda)) = H_0^m(\lambda).$$

Proof. Let $f = \mathcal{S}_{m-1}^{row} \cdots \mathcal{S}_2^{row}\mathcal{S}_1^{row} : B_1^m((m,\lambda)) \to B_0^m(\lambda)$. Since every rook in partition of the $\sigma \in B_1^m((m,\lambda))$ is on the first row, it is a bijection. For any $\sigma \in B_1^m((m,\lambda))$, $\mathrm{wt}_{(m,\lambda)}(\sigma) = \mathrm{wt}_\lambda(f(\sigma))$. So,

$$H_1^m((m,\lambda)) = \sum_\sigma q^{\mathrm{wt}_{(m,\lambda)}(\sigma)} = \sum_\sigma q^{\mathrm{wt}_\lambda(f(\sigma))} = H_0^m(\lambda).$$

□

Lemma 4.3. *For $m \in \mathbb{N}$ and partition λ,*

$$H_0^m(\lambda) = [m]_q H_0^{m-1}(\lambda) + H_1^{m-1}(\lambda).$$

Proof. To show that $H_0^m(\lambda)$ the is addition of two different q-hit numbers, I will divide $B_0^m(\lambda)$ into two disjoint sets and map each of them to $B_0^{m-1}(\lambda)$ and $B_1^{m-1}(\lambda)$ respectively.

Board sizes of $B_0^m(\lambda)$ and $B_k^{m-1}(\lambda)$ are different and it makes it hard to compare them. So from now on let us identify an element in $B_k^{m-1}(\lambda)$ with an element in $B_k^m(\lambda)$ by locating all rooks as before and adding one more rook in the bottom right corner. Note that λ-weight of them are equal.

First, let us think of $\sigma \in B_0^m(\lambda)$ bottom rook on which is on the k-th column and there is no rook on $(i, \lambda_i + 1)$ if $\lambda_i \leq k$ (first and second coordinates imply row and column respectively). Let $A \subset B_0^m(\lambda)$ be a set of such elements.f

Let $f_i = \mathcal{S}_{m-i}^{col} \cdots \mathcal{S}_{m-2}^{col}\mathcal{S}_{m-1}^{col} : B_0^{m-1}(\lambda) \to B_0^m(\lambda)$ for $i = 1, \ldots, m-1$ and $f_0 : B_0^{m-1}(\lambda) \to B_0^m(\lambda)$ be the identity map. Then f_i's are injective and

$$A = \bigsqcup_{i=0}^{m-1} f_i(B_0^{m-1}(\lambda)).$$

Moreover, for $\sigma \in B_0^{m-1}(\lambda)$, $i + \mathrm{wt}_\lambda(\sigma) = \mathrm{wt}_\lambda(f_i(\sigma))$. Therefore,

$$\sum_{\sigma \in A} q^{\mathrm{wt}_\lambda(\sigma)} = \sum_{i=0}^{m-1} \sum_{\sigma \in B_0^{m-1}(\lambda)} q^{\mathrm{wt}_\lambda(f_i(\sigma))} = [m]_q \sum_{\sigma \in B_0^{m-1}(\lambda)} q^{\mathrm{wt}_\lambda(\sigma)}. \tag{4.1}$$

Now let $\tau \in B_0^m(\lambda) \backslash A$ which has a rook on (m, l). Then we can correspond τ to an element in $B_1^{m-1}(\lambda)$ using the following algorithm;

1. Choose the rightmost cell containing a rook among $(i, \lambda_i + 1)$ for $i = 1, 2, \ldots, m$ (there exist a cell containing rook since $\tau \in B_0^m(\lambda) \setminus A$) and let the cell be $(k, \lambda_k + 1)$.
2. Put τ in $\mathcal{S}_m^{col} \cdots \mathcal{S}_{\lambda_k+2}^{col}\mathcal{S}_{\lambda_k+1}^{col}$.
3. Remove rooks on (m, l) and (k, m) and put rooks on (k, l) and (m, m) to replace them.

Let $g : B_0^m(\lambda) \setminus A \to B_1^{m-1}(\lambda)$ be the function that represents the algorithm. Since we can exactly reverse the given algorithm, g is a bijection. And it is not that hard to check that $\mathrm{wt}_\lambda(\tau) = \mathrm{wt}_\lambda(g(\tau))$. Therefore,

$$\sum_{\tau \in B_0^m(\lambda) \backslash A} q^{\mathrm{wt}_\lambda(\tau)} = \sum_{\tau \in B_1^{m-1}(\lambda)} q^{\mathrm{wt}_\lambda(g(\tau))} = \sum_{\tau \in B_1^{m-1}(\lambda)} q^{\mathrm{wt}_\lambda(\tau)}. \tag{4.2}$$

Combine (4.1) and (4.2). Then

$$\begin{aligned}
H_0^m(\lambda) &= \sum_{\sigma \in A} q^{\mathrm{wt}_\lambda(\sigma)} + \sum_{\tau \in B_0^m(\lambda)\setminus A} q^{\mathrm{wt}_\lambda(\tau)} \\
&= [m]_q \sum_{\sigma \in B_0^{m-1}(\lambda)} q^{\mathrm{wt}_\lambda(\sigma)} + \sum_{\tau \in B_1^{m-1}(\lambda)} q^{\mathrm{wt}_\lambda(\tau)} = [m]_q H_0^{m-1}(\lambda) + H_1^{m-1}(\lambda).
\end{aligned}$$

□

Now we are ready to prove our main theorem.

proof of Theorem 4.1. Let

$$h(\lambda) = \begin{cases} q[n-2]_q H_1^{n-2}(\lambda) + q[n-1]_q H_{[0,1],n}^{n-2}(\lambda) & \text{if } \lambda_1, k \leq n-2, \\ [n-1]_q H_0^{n-2}((\lambda_2, \dots, \lambda_k)) & \text{if } \lambda_1 = n-1, \\ [n-1]_q H_0^{n-2}((\lambda_1 - 1, \dots, \lambda_k - 1)) & \text{if } k = n-1. \end{cases}$$

for convenience. By Theorem 2.7, it is enough to prove the following two statements.

1. $X_{K_{n_1} \sqcup K_{n_2} \sqcup \cdots \sqcup K_{n_m}}(x, q) \mid_{e_{n-1,1}} = h(K_{n_1} \sqcup K_{n_2} \sqcup \cdots \sqcup K_{n_m})$ for arbitrary disjoint complete graph K_{n_i}'s.

2. h satisfies the linear relation.

If $n = 1$, $X_\phi(x, q) = \sum_{k=1}^{\infty} x_k = e_1$ and $h(\phi) = 0$. So from now on assume that $n \geq 2$.

Let us begin with 1. By the second property of Theorem 2.6,

$$X_{K_{n_1} \sqcup K_{n_2} \sqcup \cdots \sqcup K_{n_m}} = X_{K_{n_1}} X_{K_{n_2}} \cdots X_{K_{n_m}} = [n_1]_q! \cdots [n_m]_q! e_{n_1} \cdots e_{n_m}.$$

So the coefficient of $e_{n-1,1}$ of a chromatic quasisymmetric function is nonzero only if $K_1 \sqcup K_{n-1}$ and it is $[n-1]_q!$. The corresponding partition of $K_1 \sqcup K_{n-1}$ is $(n-1)$ or $(1, 1, \dots, 1)$ ($n-1$ many 1's). Moreover,

$$h((n-1)) = [n-1]_q H_0^{n-2}(\phi) = [n-1]_q!,$$

$$h((1, 1, \dots, 1)) = [n-1]_q H_0^{n-2}(\phi) = [n-1]_q!$$

and

$$h(\lambda) = 0$$

for the other partition λ because $H_1^{n-2}(\lambda) = H_{[0,1],n}^{n-2}(\lambda) = 0$ (since λ touches diagonal line from $(n,1)$ to $(1,n)$ there should be more rooks in λ). So, we proved what we want.

Now we need to show that h satisfies the linear relation.

First, assume that λ is inside the $(n-2)\times(n-2)$ board i.e., $\lambda_1, k \leq n-2$. By [1, Lemma 4.2.] and Proposition 3.4., both q-hit numbers and bounce q-hit numbers satisfy the linear relation. So $h(\lambda) = q[n-2]_q H_1^{n-2}(\lambda) + q[n-1]_q H_{[0,1],n}^{n-2}(\lambda)$ also satisfies the linear relation.

Second, assume that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n-1})$ with $\lambda_1 = n-1$ and $\lambda_{n-1} = 1$. To show that h satisfies the linear relation with such λ, we have to show that it satisfies both row and column linear relations. Let us see the row linear relation first. Let $\lambda^0 = (\lambda_1, \ldots, \lambda_i - 2, \ldots, \lambda_{n-1})$, $\lambda^1 = (\lambda_1, \ldots, \lambda_i - 1, \ldots, \lambda_{n-1})$ and $\lambda^2 = \lambda$ be partitions satisfy the row condition. If $i \neq 1$, $h(\lambda^0) = h(\lambda^1) = h(\lambda^2) = 0$. So the equation (2.1) holds. So, now we suppose that $i = 1$. Let μ^j be the partition that can be obtained by subtracting the first column of λ^j i.e., $\mu^j = (n-4+j, \lambda_2 - 1, \ldots, \lambda_{n-1} - 1)$. Then,

$$h(\lambda^0) = [n-1]_q H_0^{n-2}(\mu^0),$$

$$h(\lambda^1) = [n-1]_q H_0^{n-2}(\mu^1),$$

$$h(\lambda^2) = 0.$$

Let $A = \{\sigma \in B_0^{n-2}(\mu^0) \mid \text{there is a rook on top right corner}\}$. With the identity map and $\mathcal{S}_{n-3}^{col}$, it can be easily shown that

$$\sum_{\sigma \in A} q^{\mathrm{wt}_{\mu^0}(\sigma)} = qH_0^{n-2}(\mu^1), \quad \sum_{\sigma \in B_0^{n-2} \setminus A} q^{\mathrm{wt}_{\mu^0}(\sigma)} = H_0^{n-2}(\mu^1).$$

Therefore,

$$\begin{aligned}(1+q)h(\lambda^1) &= [n-1]_q (1+q) H_0^{n-2}(\mu^1) \\ &= [n-1]_q H_0^{n-2}(\mu^0) = h(\lambda^0) = h(\lambda^0) + h(\lambda^2).\end{aligned}$$

So, the row linear relation holds for such λ. Since the q-hit number is transpose invariant, it follows from row linearity that λ also satisfies the column linear relation.

Third, assume that $\lambda = (n-1, \lambda_2, \ldots, \lambda_k)$ with $k \leq n-2$ and let $\lambda' = (\lambda_2, \ldots, \lambda_k)$. The column linear relation and the row linear relation with regard to the m-th row for $(m \geq 2)$ follow from the linearity of H_0^{n-2}. So it is enough to show the following equation.

$$
\begin{aligned}
& (1+q)(q[n-2]_q H_1^{n-2}(n-2, \lambda') + q[n-1]_q H_{[0,1],n}^{n-2}(n-2, \lambda')) \\
= \; & q[n-2]_q H_1^{n-2}(n-3, \lambda') + q[n-1]_q H_{[0,1],n}^{n-2}(n-3, \lambda') \\
& + [n-1]_q H_0^{n-2}(\lambda')
\end{aligned} \tag{4.3}
$$

Let

$$
\begin{aligned}
(1) = \; & [n-2]_q H_1^{n-2}(n-2, \lambda') + [n-1]_q H_{[0,1],n}^{n-2}(n-2, \lambda') \\
& -([n-2]_q H_1^{n-2}(n-3, \lambda') + [n-1]_q H_{[0,1],n}^{n-2}(n-3, \lambda'))
\end{aligned}
$$

and

$$
\begin{aligned}
(2) = \; & q[n-2]_q H_1^{n-2}(n-2, \lambda') + q[n-1]_q H_{[0,1],n}^{n-2}(n-2, \lambda') \\
& -[n-1]_q H_0^{n-2}(\lambda').
\end{aligned}
$$

Then equation (4.3) holds iff $(1) + (2) = 0$. Let $B = \{\sigma \in B_{[0,1],n}^{n-2}(\lambda^2) \mid$ there is a rook on top right corner$\}$ and $C = \{\sigma \in B_{[0,1],n}^{n-2}(\lambda^1) \mid$ there is a rook on top right corner$\}$. Then,

$$
(1) = [n-2]_q (H_0^{n-3}(\lambda') - qH_1^{n-3}(\lambda')) + [n-1]_q (\sum_{\sigma \in B} q^{\mathrm{bwt}(\sigma)} - \sum_{\sigma \in C} q^{\mathrm{bwt}(\sigma)}).
$$

By lemma 4.2.,

$$
(2) = q[n-1]_q H_{[0,1],n}^{n-2}(n-2, \lambda') - H_0^{n-2}(\lambda').
$$

Adding (1) and (2) and applying Lemma 4.2. gives

$$(1) + (2) = [n-1]_q\Big(\sum_{\sigma \in B} q^{\mathrm{bwt}(\sigma)} - \sum_{\sigma \in C} q^{\mathrm{bwt}(\sigma)} + qH^{n-2}_{[0,1],n}(n-2, \lambda') - H^{n-3}_1(\lambda')\Big).$$

So, I want to show that

$$\sum_{\sigma \in B} q^{\mathrm{bwt}(\sigma)} + qH^{n-2}_{[0,1],n}(n-2, \lambda') = \sum_{\sigma \in C} q^{\mathrm{bwt}(\sigma)} + H^{n-3}_1(\lambda').$$

With maps we used to prove Lemma 4.3, the equation similarly can be proven. So we omit this part.

In the last case when $\lambda = (\lambda_1, \ldots, \lambda_{n-2}, 1)$ with $\lambda_1 \leq n-2$, it can be proved as the same as the third case. Hence we finally proved that h satisfies the linear relation. □

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국문초록

보완 그래프가 이분되는 단위 간격 그래프들에 대하여 채색 준대칭 함수의 기본 기저에 대한 계수를 생각해보자. 계수들을 특정 q-hit number들로 표현할 수 있다는 사실이 알려져 있다. 우리는 q-hit number를 개선한 bounce q-hit number를 정의하고, 임의의 단위 간격 그래프들에 대하여 채색 준대칭 함수의 $e_{n-1,1}$의 계수를 이로서 표현했다. $e_{n-1,1}$의 계수가 양부호가 된다는 것 또한 보였는데, 이는 채색 준대칭 함수에 대한 Stanley-Stembridge 가설의 부분적인 증명이다.

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