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Abstract

Conserved Quantities in Double Field Theory

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Double Field Theory (DFT) is an extended gravitational theory that geometrises T-duality, which is one of the fundamental and unique symmetries in String Theory. Einstein gravity has a manifest covariance over Poincaré symmetry and diffeomorphism of Riemannian geometry. Double Field Theory has a manifest covariance over the T-duality group $\mathbf{O}(D,D)$ and the generalised diffeomorphism, which includes diffeomorphism and gauge transform of other massless sector of String Theory.

This thesis studies the standard way to produce conserved charge in Double Field Theory. To study this, first, the thesis discusses the difficulties of Noether theorem to be applied to gravitational theories and reviews ADT formalism and Wald's formalism, which gives conserved charges corresponding to energy and momentum in the way equivalent to Hamiltonian formalism. Then, the construction of Double Field Theory itself and its covariant derivative is reviewed.

Double Field Theory is written in a completely new geometry called Double geometry instead of Riemannian geometry. This study modifies ADT formalism to be developed in Double geometry and finds new quantities corresponding to essential quantities in Einstein gravity. The Noether current of DFT is found to be written in Noether potential form, and the potential is in well-known Komar-like form. Also, in the process to obtain the Noether current, the Einstein tensor, a curvature tensor with zero divergence, is found, which is not exactly the same with the equations of motion in DFT but is a certain combination of them. Lastly, the boundary action of DFT is also studied to construct the charge formula. Using all those ingredients we found above, the off-shell ADT conserved charge of DFT is constructed.

The new DFT charges consist of Riemannian ADM charges and the winding charges in String Theory. This is expected in string theory sense, but this study confirms that it is also explicitly confirmed in field theory sense, by applying the DFT charge formula to the known string backgrounds. Before the application, this thesis argues that proper orthonormal vectors for time, space, and winding directions should be fixed for the DFT charges to represent the proper energy, momenta, and winding charges. Then, the formula is applied to pure-Einstein background, 1-branes, 5-branes, and an asymptotically non-flat background. The application to the pure-Einstein background confirms that our charge formula is equivalent to ADM formula in pure-Einstein region. The application to the fundamental string and its

T-duals (1-branes) confirms the definition of T-duality: momentum and winding charge interchanges under the T-duality. Moreover, it is confirmed that the application to a non-Riemannian background is also well-defined, where any Riemannian description is mal-defined. The application to DFT monopoles (5-branes) also gives the known brane tension formula as ADM energy. Lastly, the application to the asymptotically non-flat background confirms that our formula successfully suppress its divergence from the extrinsic curvature by the compensation term.

This study concludes that the standard way to construct the ADM/ADT charge formula is successfully established in a bizarre Double geometry and its gravitational theory, Double Field Theory. The study may also extended to Exceptional Field Theory, which is a gravitational theory geometrising all the symmetry in String Theory.

keywords : Double Field Theory, String Theory, Gravity, T-duality, Noether Theorem, Conserved Charge

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Chapter 1 Introduction

String Theory is one of the basic tools for the theoretical particle physicists, as it has the largest symmetry so can be reduced to most field theories, possibly any possible field theories. T-duality symmetry is one of the most important symmetry in String Theory. *Double Field Theory* (DFT) [56, 58, 60–62][1] is a new approach to construct a gravitational field theory which holds the ($\mathbb{R}$-valued extension of) $\mathbf{O}(D, D)$ T-duality symmetries manifestly as its geometrical structure and which thus unifies and geometrises all gauge invariance of the massless field (NSNS sector) in String Theory. Based on the study [1] from our group, this thesis systematically formulates the conserved charge formula associated with the DFT geometric symmetry, which corresponds to ADM mass, momentum, or winding charges in conventional theories.

To understand the novelty of Double Field Theory, we first explain T-duality and its group $\mathbf{O}(D, D)$, which and other symmetries would be explained in detail in Section 1.1. T-duality is one of the essential pieces of the second superstring revolution that saved String Theory and connects all the

[1] The best summaries are [56, 57].

different kinds of string theories into one larger symmetry group. T-duality is an equivalence between a string theory in a torus with radius R and a string theory with an inverse radius and strings with the momentum and the winding number exchanged. Each mode of a string means a particle in the field theory, so the field contents are also exchanged under the T-duality.

The massless (NSNS) sector, which we concern in gravitational theories, and which consists of the metric $G_{\mu\nu}$, the anti-symmetric Kalb-Ramond gauge field $B_{\mu\nu}$, and the dilaton Φ, also mixes itself under the duality. (The mixing rule is called Buscher rules.[17, 18]) Those mixing combined with already existing geometric and gauge symmetries compose the T-duality group $\mathbf{O}(D,D;\mathbb{Z})$. By extending $\mathbf{O}(D,D;\mathbb{Z})$ to $\mathbf{O}(D,D;\mathbb{R})$, and by taking it as the basic tensor structure by doubling the spacetime dimension, we can construct Double Field Theory, which is a stringy gravitational theory and gives us physical insights into the intricacies of String Theory.

There are special examples showing that DFT is not simple rephrasing of the Riemannian NSNS gravity. *A non-Riemannian background* is a background that has diverging Riemannian description. This background is induced by performing T-dualities to the conventional branes, so this background has a well-defined DFT description. Another example is *a non-geometric background*. Originally in supergravity (SUGRA), a background having monodromy has exist. In that background, the metric is defined patch by patch which are connected by geometric transformations. However, after performing T-duality to the background, we get a special background where the patches are connected by T-dualities. Also, as $G_{\mu\nu}$ and $B_{\mu\nu}$ are mixed each other, the charge of $G_{\mu\nu}$ and $B_{\mu\nu}$ for this background is hard to define, and the conserved charge of this background would be very exotic. [2] [102, 104, 105, 108] Even for these backgrounds, DFT gives a good description, geometrising all those non-geometric properties.

However, Double Field Theory has too many degrees of freedom to define the connections and the curvature tensors uniquely[56]. J.-H. Park and his colleagues Lee and Jeon developed “semi-covariant formulation” of Double Field Theory which compromise on the tensority of the undetermined components but fixes all the components and then extract only covariant components by “projections” [69, 71, 74, 75]. Under this formulation, we can more easily define all the math theorems corresponding to that all we have used conveniently in the Riemannian-geometric theories. Our conserved charge formula is written under this formulation.

Our DFT ADT formula is constructed analogously to the original ADT formalism in the Einstein gravity. Originally, before the Gibbon-Hawking-York boundary term had been developed, the ADM formalism, which is the Hamiltonian formalism with certain timeslice and dividing the vector space into 3+1 dimensions, was developed to obtain the energy formula of the Einstein gravity theory. ADT formalism formulates the ADM energy and also the momentum of the gravitational background without braking the spacetime symmetry and maintaining the tensor indices. What we need to obtain as the

[2] Those backgrounds are also called *T-folds* in the sense that the patches are mended only by including T-duality, or called *exotic branes* in the sense that the brane charge is exotic.

ADT charge value of a gravitational background should be equivalent to the Noether charge of the matter fields living on the background.

Remind that the Noether charge of matter fields ϕ's associated with a spacetime symmetry $x^\mu \to x^\mu + \xi^\mu$ is

$$Q[\xi] = \int_{\mathcal{M}} \mathrm{d}^{D-1}x \, \sqrt{|G|}\, T^\mu{}_\nu[\phi \cdots]\xi^\nu, \tag{1.1}$$

where $\mathcal{M}$ is the timeslice, and $G_{\mu\nu}$ is the metric of the spacetime, and $T^\mu{}_\nu$ is the energy-momentum tensor that appears in the Noether theorem:

$$\sqrt{|G|}T^\mu{}_\mu \simeq \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\nu \phi - \delta^\mu_\nu \mathcal{L}, \tag{1.2}$$

where $\mathcal{L}$ is the Lagrangian density including the volume element. Also remind that the energy-momentum tensor can be symmetrised by adding antisymmetric potential terms, [36]

$$T^{\mu\nu} \to T^{\mu\nu} + \partial_\sigma f^{\mu\nu\sigma}, \tag{1.3}$$

without braking the conservation law, where $f^{\mu\nu\sigma}$ satisfies both

$$f^{\mu\nu\sigma} = -f^{\mu\sigma\nu}, \tag{1.4}$$

$$\partial_\nu \partial_\sigma f^{\mu\nu\sigma} = 0. \tag{1.5}$$

The symmetrised energy-momentum tensor couples to the gravity, and the relation is represented by the Einstein equation:

$$\mathcal{G}_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R\, G_{\mu\nu} = \kappa^2 T_{\mu\nu}, \tag{1.6}$$

where $\mathcal{G}_{\mu\nu}$ is the Einstein curvature tensor defined as above, and $\kappa^2 := 8\pi G_N$ is the gravitational constant. Thus, the energy or momentum $Q[\xi^\mu]$ of the matter fields associated with the spacetime symmetry ξ^μ can equivalently be constructed using only gravitational background fields by substituting the energy-momentum tensor of the matter $T^{\mu\nu}$ by the Einstein tensor $\mathcal{G}^{\mu\nu}$,

$$Q[\xi] = \frac{1}{\kappa^2} \int\limits_{\mathcal{M}} \mathrm{d}^{D-1}x\, \sqrt{|G|} \mathcal{G}^\mu{}_\nu \xi^\nu. \tag{1.7}$$

Also, we have to note that for the conserved charge to be formulated properly, the associated spacetime transformation $\xi^\mu(x)$ must be a global symmetry[3] of the gravitational background. The symmetry of the background[4] is defined by the *Killing equations*,

$$\widehat{\mathcal{L}}_\xi G_{\mu\nu} = 0, \tag{1.8}$$

where $\widehat{\mathcal{L}}_\xi$ is the Lie derivative with respect to the vector field ξ^μ *i.e.* the difference of the metric $G_{\mu\nu}$ after the transformation $x^\mu \to x^\mu + \xi^\mu$.

[3]such as global translation or global rotation. It does not matter to have the coordinate dependence, but here I mean that there is no proper local currents, but there is only global charges.

[4]The symmetry of the gravity theory itself is the diffeomorphism, but here we mean that the symmetry of the system. Remind the Noether theorem of the matter field. The spacetime symmetry of the matter fields casts the symmetry of the background $G_{\mu\nu}$ such as global translation or global rotation, and the symmetry of the background can be obtained by the Killing equations.

Using the basic identites of Riemannian geometry like the Bianchi identity, we can show that the charge (1.7) can be written as a integral of an off-shell current. Furthermore, we can show that the ADT charge formula can be written as a Noether charge of the action that consists of Einstein-Hilbert action and a Gibbons-Hawking action. By introducing the boundary term that can act as the Gibbons-Hawking, and obtaining the Noether charge of the total action, we can obtain the ADT formula under these conditions: (i) the integral domain is defined *i.e.* there is no local currents but at least quasi-local charges[33, 35, 41]; (ii) the associated symmetry is a linear translation or a rotation but no highlier coordinate-dependent transformation. Also, the charge formula takes the form of an integral of an anti-symmetric Noether potential $K^{[\mu\nu]}$ *s.t.* $J^{\mu} = \partial_{\nu} K^{\mu\nu}$ on the boundary $\partial\mathcal{M}$ of the timeslice $\mathcal{M}$.

In our study, we formulate this argument analogously in Double Field Theory, establishing one of the backbones of Double Field Theory, the Noether theorem and the ADT formalism.

In Section 1.1 later in the introduction, I explain T-duality and other String symmetries in detail. In Section 1.2, I declare and clarify the notation manner in this thesis to avoid confusions between the objects with similar notations in other literatures and to introduce new notations for those objects. In Chapter 2, I explain the ADT formalism in the conventional (Einstein) gravity and Riemannian geometry we have discussed in detail.

In Chapter 3, the formulation of Double Field Theory is explained. Double geometry is not only the doubling dimension of the Riemannian geometry, but all the fundamental definitions of the its geometrical structure is different from the Riemannian one to inscribe the $\mathbf{O}(D, D)$ structure into the geometry. Furthermore, we need "semi-covariant formulation" of Double Field Theory to define the covariant derivative explicitly[69, 71, 74, 75]. All the geometric theorems written in that notation is used in the charge formulation.

In Chapter 4, we develop the ADT charge formulation process with the DFT action and Double Geometry apparati. First, we define the Killing equations in DFT to define available symmetries for each DFT background. Then, we formulate the conserved charges in Double Field Theory analagously to the ADT formalism in the Einstein gravity with the semi-covarint tools. During the process, we also discover other quantities that has similar structure in the Einstein gravity. We discover the Einstein curvature tensor, which is different from the equations of motion in DFT, and the off-shell Noether current of bulk DFT action, which also has the similar form to the Einstein version so-called 'Komar form'. In conclusion, we sum up all the contributions from each part of the calculation and display the total ADT global charge of Double Field Theory with respect to the local translation X^A,

$$\begin{aligned} Q[X] = \frac{1}{2\kappa} \oint\limits_{\partial\mathcal{M}} \mathrm{d}^{D-2}x_{AB}\ e^{-2d} \Big[K^{AB} + 2X^{[A}B^{B]} + 4K_0 X^{[A}\tilde{N}^{B]} \Big] \\ + \frac{1}{g_{\mathrm{YM}}^2} \oint\limits_{\partial\mathcal{M}} \mathrm{d}^{D-2}x_{AB}\, e^{-2d}\, \mathrm{Tr}\Big\{ 12(P\mathcal{F}\bar{P})^{[AB}V^{C]}X_C \Big\}, \end{aligned} \tag{1.9}$$

which is an integral of the Noether potential on the boundary of the timeslice, and which shows the similar form to the Einstein ADT formula, but the definition of each symbol is DFT version, different from the Einstein gravity.

Furthermore, in the last part of the chapter, I make a note to other charges in Double Field Theory discovered by other than our group.

In Chapter 5, we discuss the application of our formulation to various known geometric and non-geometric string backgrounds. Before we go into specific cases, we first discuss the right way to fix the Killing vector basis to obtain the appropriate notion of energy and momentum. Then, we discuss specific cases, especially for 1-branes and 5-branes. 1-brane examples which we consider are interesting because they are T-duals of the fundamental string, and, in other words, they are null waves of Double Field Theory[94]. 5-branes with 2 torus transverse directions are interesting, for they have monodromies as they are codimension-2 objects, and and of them have monopole charges. Additionally, we push applicability of our formula to the background with extrinsic curvature and check the function of the extrinsic curvature term in the charge formula. [5]

In Chapter 6, I summarise all the arguments, theories, theorems, formulations, and discussions and conclude my dissertation.

This thesis is written based on our group's study in 2015 [1] with additional my own study. Also, similar study was done by Blair [2] in parallel.

[5]The extrinsic curvature term in the Einstein gravity is implemented to remove the divergence in the various quantities when the background has the non-zero background curvature. A typical example with the constant extrinsic curvature is a (anti-)de Sitter space.

1.1 Symmetries of String Theory

As I specified in the introduction, symmetries of String Theory is the inspiration and the basic structure of the novel gravitational theories. The most important symmetries in String Theory are S- and T-dualities, and these are also what we want to implement as new geometric structures for new theories. The T-duality group $\mathbf{O}(D,D)$ can be easily represented within the $\mathbf{GL}(2D)$ representation, thus simply doubling the vector indices or further extending the spacetime dimension double and then reducing the physical dimension by a certain condition can handle the $\mathbf{O}(D,D)$ structure in the geometrical sense. Manifestly T-duality-covariant geometric theory especially for the NSNS sector is called Double Field Theory.

On the other hand, S-duality is another symmetry of String Theory. S-duality is one of the classical duality relations since the gauge theory. In the Yang-Mills theory, we are familiar that there is the electric-magnetic duality symmetry. In String Theory, there is also a S-duality that the string coupling goes inverted $g_s' = 1/g_s$ and that the field strength of the p-form field goes to its hodge dual $F' = *F$.

Combining the S- and T-duality induce the bigger symmetry: U-duality. U-duality is necessarily induced by S- and T-duality, but the U-duality group has a lot more dimensions (actually infinite dimention) than those of the S- and T-duality combined. As all kinds of string theories may be described as a descendant of the M-theory, all kinds of the U-duality group is a subgroup of the exceptional group $E_{11(11)}$, and the gravitational theory that geometrise that group is also formulated and called *Exceptional Field Theory (ExFT)* [119].

In this section, we discuss what is the S-duality and T-duality in String Theory sense and the field theory sense and how the T-duality group and the U-duality group may be constructed.

1.1.1 T-duality

T-duality is one of the most fundamental and most important duality symmetries in String Theory, which connects different kinds of string theories and has no similar counterparts in field theories and thus is one of the most stringy symmetries. The T-duality arises from the toroidal compactification of string theories. The mass spectrum of string in a torus with radius R and in a torus with radius α'/R match [Chap. 8 in 7, 10]. The correspondence also switch the quantum numbers, the momentum p in the torus direction and the winding charge w, of each string mode, as depicted in Figure 1.1.

In the field theory sense, we can obtain the correspondence of every p-forms in the theory. In this thesis, we are interested in NSNS sector, which is a massless sector of the string, and also which couples the string geometrically. In this subsection, firstly we consider the sigma model action and its geometric background, the NSNS sector, and compactify one direction in the spacetime. Then, we produce the T-duality procedure so that we formulate how the NSNS fields transforms under the T-duality. Finally, we show that the T-duality forms $\mathbf{O}(D,D)$ T-duality group and the NSNS fields are in the $\mathbf{O}(D,D)$ representations, using the T-duality formula in the NSNS sector.

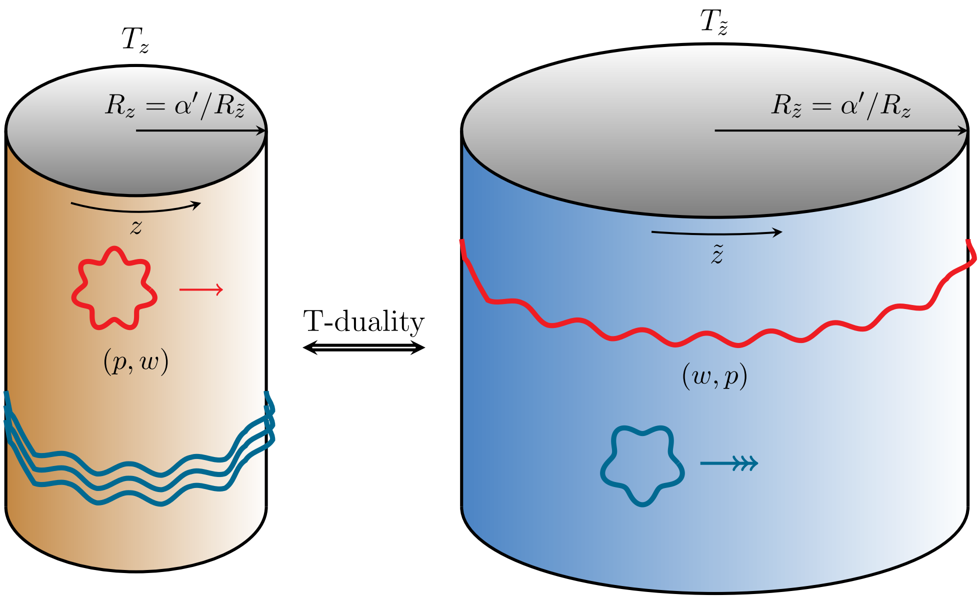

Figure 1.1: T-duality in String Theory in diagram. String Theory in torus with radius R and the thoery with torus radius α'/R have the correspondence in their mass spectra. Under this correspondence *i.e.* T-duality, a string with quantum numbers (momentum p, winding number w) corresponds to a string on the other theory with quantum numbers exchanged (momentum w, winding number p).

T-duality and Buscher rules

We show how the metric $G_{\mu\nu}$ and Kalb-Ramond field $B_{\mu\nu}$ under the T-duality.

First, we start from the string action:

$$S[X^\mu(\sigma)] = \frac{-T}{2}\int_{\Sigma_2} \mathrm{d}^2\sigma \left(\eta^{ab}G_{\mu\nu}[X] + \epsilon^{ab}B_{\mu\nu}[X]\right) \partial_a X^\mu(\sigma)\, \partial_b X^\nu(\sigma), \quad (1.10)$$

where $X^\mu[\sigma]$ is the worldsheet of the string, and μ and ν are the spacetime indices, and a and b are the worldsheet indices.

To perform T-duality, we first compactify the spacetime in the direction we want to perform the duality. In Kaluza-Klein compactification process, we normally kill the coordinate dependence along the compactified directions and make the spacetime symmetries in the compact directions internal symmetries. T-duality also requires Killing symmetry in its performing direction. From the total 10 dimensions (or D-dimensions) which is noted by the small Greek indices, we note the T-duality direction d, and the rest directions m or n, in this topic, and we write the sigma model action into the divided form:

$$\begin{aligned} &S[X^m, X^d] \\ &= \frac{-T}{2}\int_{\Sigma_2} \mathrm{d}^2\sigma \Big[G_{mn}\eta^{ab}\partial_a X^m \partial_b X^n + 2G_{md}\eta^{ab}\partial_a X^m \partial_b X^d + G_{dd}\eta^{ab}\partial_a X^d \partial_b X^d \\ &\qquad + B_{mn}\epsilon^{ab}\partial_a X^m \partial_b X^n + 2B_{md}\epsilon^{ab}\partial_a X^m \partial_b X^d \Big]. \end{aligned} \tag{1.11}$$

Assuming the Killing equations

$$\frac{\partial G_{\mu\nu}}{\partial X^d} = 0, \quad \frac{\partial B_{\mu\nu}}{\partial X^d} = 0 \tag{1.12}$$

are satisfied along the torus direction, the equation of motion F^a and the Bianchi equation $\tilde{F}^a$ in the torus direction are

$$F^a = \frac{\partial \mathcal{L}}{\partial(\partial_a X^d)}, \quad \widetilde{F}^a = \epsilon^{ab}\partial_b X^d, \tag{1.13}$$

both of which must be zero in the end. The Bianchi equation must be zero, so we may convert this action by converting all X^d's to $\tilde{F}^a$ and by inserting the Lagrangian multiplier scalar $\tilde{X}^d$ that gives the Bianchi equation as the constraint:

$$
\begin{aligned}
&S[X^m, \widetilde{F}^a, \widetilde{X}^d] \\
&= \frac{-T}{2}\int_{\Sigma_2} \mathrm{d}^2\sigma \Big[G_{mn}\eta^{ab}\partial_a X^m \partial_b X^n + 2G_{md}\epsilon^{ab}\partial_a X^m \widetilde{F}_b + G_{dd}\eta^{ab}\widetilde{F}_a\widetilde{F}_b \\
&\qquad + B_{mn}\epsilon^{ab}\partial_a X^m \partial_b X^n + 2B_{md}\partial_a X^m \widetilde{F}^a - 2\widetilde{F}^a\partial_a\widetilde{X}^d \Big]. \qquad (1.14)
\end{aligned}
$$

By integrating the Lagrangian multiplier, we get the original action (1.11).

To perform T-duality, we solve the action (1.14) by taking variation of $\widetilde{F}^a$ first without integrating out the multiplier $\widetilde{X}^d$, and then we can the equivalent action with X^d and $\widetilde{X}^d$ exchanged, where $\widetilde{X}^d$ in the new action will act as a new internal coordinate, called *winding coordinate*[11, 15]. The equation of motion of $\widetilde{F}^a$ is

$$
\frac{\delta S}{\delta \widetilde{F}^a} = 2G_{dd}\widetilde{F}_a + 2G_{md}\epsilon_{ba}\partial^b X^m + 2B_{md}\partial_a X^m - 2\partial_a\widetilde{X}^d = 0, \qquad (1.15)
$$

which should be satisfied. The solution of this equation is

$$
\widetilde{F}_a = \frac{1}{G_{dd}}\left(\partial_a\widetilde{X}^d - G_{md}\epsilon_{ba}\partial^b X^m - B_{md}\partial_a X^m\right) \qquad (1.16)
$$

Putting this solution to the action (1.14) gives the new sigma model action

$$
\begin{aligned}
&S[X^m, \widetilde{X}^d] \\
&= \frac{-T}{2}\int_{\Sigma_2} \mathrm{d}^2\sigma \Big[\widetilde{G}_{mn}\eta^{ab}\partial_a X^m \partial_b X^n + 2\widetilde{G}_{md}\eta^{ab}\partial_a X^m \partial_b \widetilde{X}^d + \widetilde{G}_{dd}\eta^{ab}\partial_a\tilde{X}^d\partial_b\widetilde{X}^d \\
&\qquad + \widetilde{B}_{mn}\epsilon^{ab}\partial_a X^m \partial_b X^n + 2\widetilde{B}_{md}\epsilon^{ab}\partial_a X^m \partial_b\widetilde{X}^d \Big], \qquad (1.17)
\end{aligned}
$$

with the new coefficients $\widetilde{G}_{\mu\nu}$ and $\widetilde{B}_{\mu\nu}$, other than $G_{\mu\nu}$ and $B_{\mu\nu}$. Those new coefficients are

Buscher Rules:

$$\begin{aligned}\widetilde{G}_{dd} = \frac{1}{G_{dd}}, \quad G_{md} = \frac{B_{md}}{G_{dd}}, \quad \widetilde{G}_{mn} - \frac{G_{md}G_{nd} - B_{md}B_{nd}}{G_{dd}}, \\ \widetilde{B}_{md} = \frac{G_{md}}{G_{dd}}, \quad \widetilde{B}_{mn} = B_{mn} - \frac{B_{md}G_{nd} - G_{md}B_{nd}}{G_{dd}},\end{aligned} \tag{1.18}$$

which is called *Buscher rules*, the rules how the fields transforms under the T-duality along the direction noted as d. [11, 15–18]

As you can see, $\widetilde{G}_{dd} = \frac{1}{G_{dd}}$, so the scale of the original X^d and the winding direction $\widetilde{X}^d$ are in the inverse relation, in the natural units with the unit length of the string length l_s. This coincides with the previous explanation that the T-duality connects two torus space with radius R and $1/R$.

The dilaton Φ also couples the sigma model[7] and we may derive the Buscher rule, and in the 10-dimension,

$$\sqrt{|G|}e^{-2\Phi} = \text{invariant under T-duality.} \tag{1.19}$$

$\mathbf{O}(D,D)$ T-duality group

If we suppose that we can perform T-duality in all D directions, and that there is also $\mathbf{GL}(D)$ coordinate transform symmetry in that D-dimensions, we can show that the $\mathbf{O}(D,D)$ T-duality group is constructable. In T^D, there is $\mathbf{GL}(D;\mathbb{Z})$, and we can perform T-dualities in all directions, the T-duality group is $\mathbf{O}(D,D;\mathbb{Z})$.

In Double Field Theory, we extend the group into the continuous region. Note that the Buscher rules (1.18) has no radius R information, so we can use that as a geometric symmetry for the general situation. Retaining all geometric symmetries (coordinate transform symmetries) and gauge symmetries with additionally accepting that T-duality is applicable without torus, we can extend the T-duality group to the continuous $\mathbf{O}(D,D;\mathbb{R})$.

We define metric in the $\mathbf{O}(D,D)$ representation, *the generalised metric* $\mathcal{H}_{MN}$ $2D\times 2D$ matrix:

$$\mathcal{H}_{MN} = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G-BG^{-1}B \end{pmatrix}. \tag{1.20}$$

We can show that this matrix is in $\mathbf{O}(D,D)$ representation by introducing the $\mathbf{O}(D,D)$ metric and showing its orthogonality:

$$\mathcal{H}_{MP}\mathcal{J}^{PQ}\mathcal{H}_{QN} = \mathcal{J}_{MN} \tag{1.21}$$

where the constant $\mathbf{O}(D,D)$ metric is defined by

$$\mathcal{J}_{PQ} = \mathcal{J}^{PQ} = \begin{pmatrix} 0 & 1_{D\times D} \\ 1_{D\times D} & 0 \end{pmatrix}. \tag{1.22}$$

Be careful that this is not diagonal but off-diagonal. The capital Roman index is the $2D$-dimensional matrix index.

To demonstrate the role of each component of $\mathbf{O}(D,D)$ transformation, let us see how the generalised metric transforms under the general $\mathbf{O}(D,D)$

transformation ${h_M}^P$:

$$\mathcal{H}'_{MN} = {h_M}^P {h_N}^Q \mathcal{H}_{PQ}. \tag{1.23}$$

The $\mathbf{O}(D,D)$ transformation ${h_M}^P$ consists of

$$\textbf{Coordinate transform } \mathbf{GL}(D)\textbf{:} \quad {h_M}^P = \begin{pmatrix} \frac{\partial x'^\mu}{\partial x^\sigma} & 0 \\ 0 & \frac{\partial x^\sigma}{\partial x'^\mu} \end{pmatrix}, \tag{1.24}$$

$$B_{\mu\nu}\textbf{-shift:} \quad {h_M}^P = \begin{pmatrix} 0 & 0 \\ \Delta B_{\mu\sigma} & 0 \end{pmatrix}, \tag{1.25}$$

$$\begin{array}{l}\textbf{T-dual of } B_{\mu\nu}\textbf{-shift:} \\ \textbf{(in all } D\textbf{-directions)}\end{array} \quad {h_M}^P = \begin{pmatrix} 0 & \Delta\beta^{\mu\sigma} \\ 0 & 0 \end{pmatrix}, \tag{1.26}$$

$$\textbf{T-duality in ``}d\textbf{''-direction:} \quad {h_M}^P = \begin{pmatrix} \delta^\mu_\sigma - \delta^\mu_d \delta^d_\sigma & \delta^\mu_d \delta^\sigma_d \\ \delta^d_\mu \delta^d_\sigma & \delta^\sigma_\mu - \delta^d_\mu \delta^\sigma_d \end{pmatrix}. \tag{1.27}$$

Note that the T-duality acts as a parity transformation in $\mathbf{O}(D,D)$, and it can be reproduced by T-duality in another direction and other gauge transforms.

1.1.2 U-duality

U-duality is introduced in [§17.3 in 13, 21] as the union of the S-duality and the T-duality. S-duality is a electric-magnetic duality, which exists in most of gauge theories.[23] Just as the T-duality connects between type IIA and IIB, the U-duality connects all five kinds of string theories and is the one big symmetry group of the unified string theory, *M-theory*. Understanding the U-duality group is essential to understand the M-theory, and $\mathbb{R}$-generaralised U-duality group may act as a gauge group of the M-theory and its low energy field theory, *Exceptional Field Theory* (ExFT). Exceptional Field Theory is the gravitaitional field theory which manifestly contains the U-duality group as a diffeomorphism group, which is similar to the formulation of DFT regarding to the T-duality group.

In the D-dimensional supergravity with n-dimensional torus T^n, T-dualities can be performed in those n-directions, and the T-duality group, which is the smallest group containing the T-dualities and the geometric transformation $\mathbf{GL}(n)$, must be $\mathbf{O}(n, n, \mathbb{Z})$. Thus, brane charges are in $\mathbf{O}(n, n, \mathbb{Z})$ representations, and all the fields in the theory can be written in $\mathbf{O}(n, n, \mathbb{R})$ representations. We can generalise the T-duality without assuming torus, and we can construct DFT as a $O(D, D)$ gauge theory.

In the same way, first we look at the U-duality group in each topological configuration of M-theory, and then we may generalise it and geometrise/gauge it to construct the ExFT. The U-duality group is stated in [21] and also in Table 1.1 in this thesis. The first torus that compactifies the 11D M-theory into 10D supergravity theories gives the $\mathbf{SL}(2)$ S-duality. Additional compactification gives T-dualities. The maximal U-duality group can be seen at the T^{11} M-theory and is $E_{11(11)}$ exceptional group. Any de-compactification restores the noncompact physical dimensions and breaks the symmetry group down into the geometric diffeomorphism group and the rest gauge group, which is the consequences of the section condition in the perspective of

Theories and Compactified directions	T-duality	Corresponding M-theory	U-duality
		$\mathbb{R}^{11}$ M-theory	
$\mathbb{R}^{10}$ IIB SUGRA		$\mathbb{R}^{10} \times T$ M-theory	$\mathbf{SL}(2,\mathbb{Z})$ S-duality
$\mathbb{R}^{9} \times T$ SUGRA	$\mathbb{Z}_2$	$\mathbb{R}^{9} \times T^2$ M-theory	$\mathbf{SL}(2,\mathbb{Z}) \times \mathbb{Z}_2$
$\mathbb{R}^{8} \times T^2$ SUGRA	$\mathbf{O}(2,2;\mathbb{Z})$	$\mathbb{R}^{8} \times T^3$ M-theory	$\mathbf{SL}(3,\mathbb{Z}) \times \mathbf{SL}(2,\mathbb{Z})$
$\mathbb{R}^{7} \times T^3$ SUGRA	$\mathbf{O}(3,3;\mathbb{Z})$	$\mathbb{R}^{7} \times T^4$ M-theory	$\mathbf{SL}(5,\mathbb{Z})$
$\mathbb{R}^{6} \times T^4$ SUGRA	$\mathbf{O}(4,4;\mathbb{Z})$	$\mathbb{R}^{6} \times T^5$ M-theory	$\mathbf{O}(5,5;\mathbb{Z})$
$\mathbb{R}^{5} \times T^5$ SUGRA	$\mathbf{O}(5,5;\mathbb{Z})$	$\mathbb{R}^{5} \times T^6$ M-theory	$\mathbf{E}_{6(6)}(\mathbb{Z})$
$\mathbb{R}^{4} \times T^6$ SUGRA	$\mathbf{O}(6,6;\mathbb{Z})$	$\mathbb{R}^{4} \times T^7$ M-theory	$\mathbf{E}_{7(7)}(\mathbb{Z})$
$\mathbb{R}^{3} \times T^7$ SUGRA	$\mathbf{O}(7,7;\mathbb{Z})$	$\mathbb{R}^{3} \times T^8$ M-theory	$\mathbf{E}_{8(8)}(\mathbb{Z})$
$\mathbb{R}^{2} \times T^8$ SUGRA	$\mathbf{O}(8,8;\mathbb{Z})$	$\mathbb{R}^{2} \times T^9$ M-theory	$\mathbf{E}_{9(9)}(\mathbb{Z})$
$\mathbb{R} \times T^9$ SUGRA	$\mathbf{O}(9,9;\mathbb{Z})$	$\mathbb{R} \times T^{10}$ M-theory	$\mathbf{E}_{10(10)}(\mathbb{Z})$
T^{10} SUGRA	$\mathbf{O}(10,10;\mathbb{Z})$	T^{11} M-theory	$\mathbf{E}_{11(11)}(\mathbb{Z})$

Table 1.1: The U-duality and T-duality groups for various compactification settings. $\mathbb{Z}$ is generalised to $\mathbb{R}$ to construct T/U-duality groups and generalised gravitational theory with the bigger symmetry group.

ExFT. Assuming the implementation of the section condition, ExFT can be constructed geometrising the maximal symmetry group that unifies every fields existing in the theoretical physics including R-R sector. [121, 126]

1.1.3 Branes in String Theory

Branes, including the strings as 1-branes, are the fundamental objects in String Theory. Every object in String Theory comes from the combination of branes, and thus to study any DFT background (*i.e.* DFT solution), we need to know properties and explicit expressions of the branes. In this subsection, we look into how we derive, sort, and name all the branes in String Theory.

Branes possess charges from the supersymmetry in String Theory. This equivalently means that the branes are in the U-duality group representation, as the branes are all related each other by U-duality, and being said in bigger picture, M-theory is eventually U-duality gauge theory. Traditionally well-known branes have the charges which can be described in Riemannian representation. By the U-duality, we can derive further unfamiliar "exotic" branes, which had not been derived naturally in the conventional String Theory or the low-energy Supergravity, and which has a monodromy so has to be defined by patches related to each other by T-dualities or U-dualities not only by the geometric transforms. Those branes are called *T-folds* or *U-folds* compared to 'manifolds', and they are also one of the big motivation to develop the T-duality-covariant or U-duality-covariant gravity theory so that we can finally patch all kinds of branes or backgrounds by "geometric" sense.

To use those exotic branes as application examples in our study, here I state the naming of the branes and the relation between the branes.

Brane Notation

We usually deal with D-branes in the 10-dimensional string theories and M-branes in the M-theory. Furthermore, we see NS-branes, F1-strings, and null waves in String theory. However, we can also derive branes other than represented by letters by performing T-duality to the conventional branes. One of the important quantum charges of branes is the tension, or equivalently said, energy in the field theory sense. The tension is composed of the string coupling, the string scale, and the torus radius of the background topology. We can name the brane with the power numbers of these constants. [106]

A p_n^q-*brane* is defined as a brane with the tension proportional to

$$T[p_n^q] = \frac{1}{g_s^n l_s} \left(\frac{R_{i_1} \cdots R_{i_p}}{l_s^p} \right) \left(\frac{R_{j_1} \cdots R_{j_q}}{l_s^q} \right)^2, \tag{1.28}$$

where $l_s = \sqrt{\alpha'}$ is the string scale, and g_s is the string coupling constant. Note that $\left(R_{i_1} \cdots R_{i_p}\right)$ is the string volume (not including time direction) so you can extend this definition to the noncompact case without torus. The q directions are the smeared transverse directions along which the T-dualities are performed on the conventional p-branes.

Smearing means to remove the coordinate dependence in a certain direction[14]. Smearing a brane along the x^j-direction means to position infinitely many branes parallel to the original brane by every R_j distance, or equivalently, to position the brane in the topology with torus in the x^j-direction with the radius R_j, and then to assume that R_j is small enough so that we can ignore x^j-dependence in the field theory.

By performing further T-dualities ans S-dualities, we may get higher powers of the torus in the tension formula, so we may extend the notation as

$$T[p_n^{(q_3,q_2)}] = \frac{1}{g_s^n l_s} \left(\frac{R_{i_1} \cdots R_{i_p}}{l_s^p} \right) \left(\frac{R_{j_1} \cdots R_{j_{q_2}}}{l_s^{q_2}} \right)^2 \left(\frac{R_{k_1} \cdots R_{k_{q_3}}}{l_s^{q_3}} \right)^3, \tag{1.29}$$

and so on. Furthermore, if a brane tension has no terms with higher power of the torus radius than the quadratic, we can omit the superscript.

The usual p-branes may be written equivalently in this notation[12, 106]:

- F1-brane = 1_0-brane (fundamental string),

- Dp-brane = p_1-brane,
- NSp-brane = p_3-brane,
- KK5-brane = 5^1_2-brane (Kaluza-Klein monopole in $T^7 \times \mathbb{R}^{2,1}$),

which are the branes predicted by the supersymmetry algebras in string theories. There are other branes arising by performing the dualities to the conventional branes, called *exotic brane*[97, 105]. Those exotic branes are arisen by T-duality which mixes $G_{\mu\nu}$ and $B_{\mu\nu}$, so they possess exotic charges which are neither geometric nor $B_{\mu\nu}$ gauge charge, but those charges may be geometrised under DFT, still are the monodromy charges either.

Brane duality

S- and T-duality transforms the constants as [106]

$$\textbf{T-duality:}\quad R'_y = \frac{l_s^2}{R_y}, \quad g'_s = \frac{l_s}{R_y} g_s, \quad l'_s = l_s, \tag{1.30}$$

$$\textbf{S-duality:}\quad g'_s = \frac{1}{g_s}, \quad l'_s = g_s^{1/2} l_s, \tag{1.31}$$

where R_y is the radius of the torus in the T-duality direction. The transformation of the constants also transforms the mass of the brane, and the brane is named after its mass formula, so we know which brane transforms into which brane:

T-duality along one of q_i directions:

$$p_n^{(\cdots q_i \cdots q_{n-i} \cdots)} \to p_n^{(\cdots q_i - 1 \cdots q_{n-i}+1 \cdots)} \quad \text{if } n - i \geq 2, \tag{1.32}$$

$$p_n^{(\cdots q_i \cdots)} \to (p+1)_n^{(\cdots q_i - 1 \cdots)} \quad \text{if } n - i = 1, \tag{1.33}$$

$$p_n^{(\cdots q_i \cdots)} \to p_n^{(\cdots q_i - 1 \cdots)} \quad \text{if } n - i = 0, \tag{1.34}$$

$$\text{undefined} \quad \text{if } n - i \leq 0; \tag{1.35}$$

T-duality along one of p directions:

$$p_n^{(\cdots q_{n-1} \cdots)} \to (p-1)_n^{(\cdots q_{n-1}+1 \cdots)} \quad \text{if } n - 1 \geq 2, \tag{1.36}$$

$$p_n^{(\cdots)} \to p_n^{(\cdots)} \quad \text{if } n - 1 = 1, \tag{1.37}$$

$$p_n^{(\cdots)} \to (p-1)_n^{(\cdots)} \quad \text{if } n - 1 = 0; \tag{1.38}$$

T-duality along neither direction:

$$p_n^{(\cdots q_n \cdots)} \to p_n^{(\cdots q_n + 1 \cdots)} \quad \text{if } n \geq 2, \tag{1.39}$$

$$p_n^{(\cdots)} \to (p+1)_n^{(\cdots)} \quad \text{if } n = 1; \tag{1.40}$$

S-duality:

$$p_n^{(\cdots q_j \cdots)} \leftrightarrow p^{(\cdots q_j \cdots)}_{(1+p+\sum_{j=2} j q_j)/2 - n} \quad \text{if } \frac{1 + p + \sum_{j=2} j q_j}{2} - n \geq 0, \tag{1.41}$$

$$\text{undefined} \quad \text{otherwise.} \tag{1.42}$$

If you carefully see these relations, they are still duality relations; you can always apply the same duality transformation along the same direction to undo it and go back to the original brane.

Duality web

Starting from one specific brane and using the duality relations between branes stated above, we may construct a web of branes called *duality web of branes*[105, 106]. Be aware that the T-duality only can be taken along the torus direction in the physical sense while DFT removed that restriction. The most interested example is the D7-brane. The D7-brane has (2 + 1) dimensions in the transverse direction, which is similar to the point particle in the (2 + 1)-dimension. The branes we get from performing S-dualities and T-dualities to the D7-brane along its spanning directions are figured in Figure 1.2. Since the S-duality is analogous to the electric-magnetic duality and the T-duality mixed $G_{\mu\nu}$ and $B_{\mu\nu}$, you can easily guess that those branes with 2 spatial noncompact directions may have monopole charges, monodromies, and exotic charges which is neither geometric nor Kalb-Ramond.

Those new branes which are not predicted by the spacetime super-symmetry algebras of the string theories are called *exotic branes*, and they possess exotic charges.[97] Those *monodrofolds* that have monodromies in geometry must only describe its geometry patch by patch, and if those patches are connected by T-duality, they are called *T-folds*[98–100], and if those patches are connected by U-duality, they are called *U-folds*[99, 110–114], compared to the manifolds where the patches are connected by geometric relations.

In Figure 1.2, there are some interesting branes we would like to use as applications in our study. The first objects we want to use are in the bottom purple box in Figure 1.2. Not clear with the brane duality relations (1.32–1.42), the T-duality can be performed on the macroscopic fundamental string along its spanning direction, and the result is a null wave, which has a momentum in its spanning direction, as the winding charge is converted into the momentum under the T-duality. Also, we can think of time-T-duality theoretically, as it means only to exchange the Dirichlet condition and the Neumann condition[22]. If we further perfome the T-duality in time direction to the null wave assuming the Euclidean time circle, and adjust the gauge parameter spot on, we get a *non-Riemannian background* which is noted as 'n-R' in Figure 1.2, and where all the Riemannian description diverges. In this background, it is hard to discuss geometric charges like energy and momentum. We are coming back to this topic in Section 5.3.

Another interesting part of the duality web is the other purple box above. Those 5-branes have monopole charges and are interested as monopoles in DFT. They are also a bridge to the exotic part of the web. Thus, the first exotic brane we encounter, 5^2_2, is presented quite often in many literatures. The monopole charge in DFT is also handled many times[4, 106]. In this thesis, we mainly focus on reproducing the correct energy formula to those exotic branes.

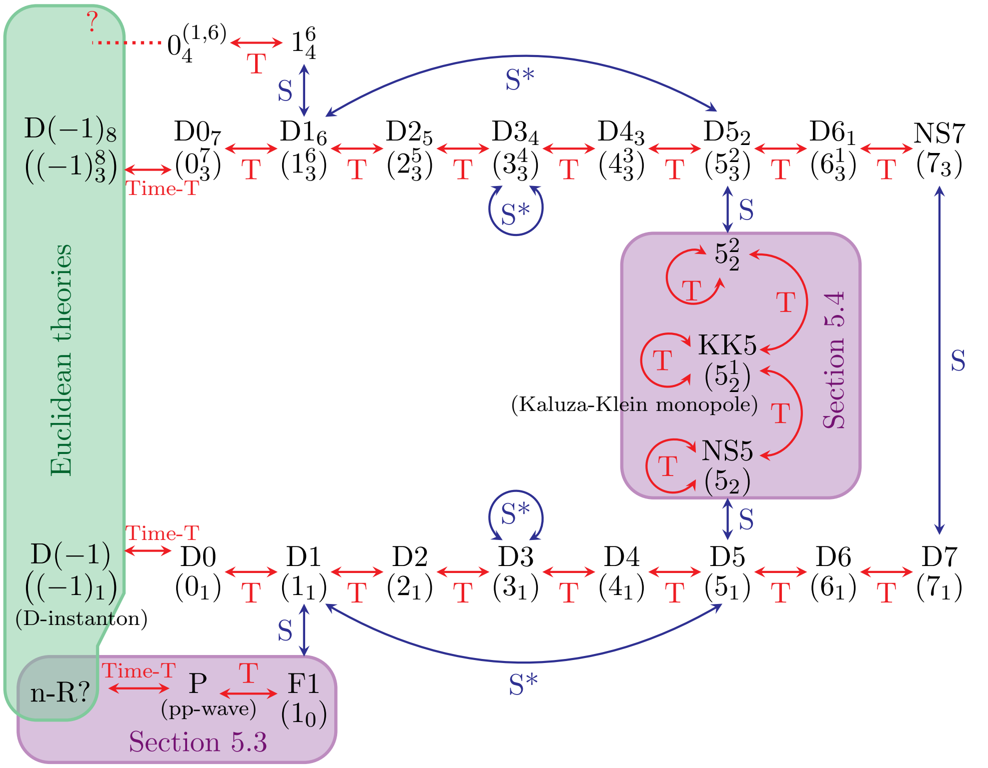

Figure 1.2: The U-duality web originated from the D7-brane in the torii along all its spanning dimensions: $\mathbb{R}^{2,1} \times T^7_{\text{transverse}}$. It means that all the p and q's directions should be in the torus, $p + \sum q_i \leq 7$, except the time-T-duality and time-circles are considered. The time-T-duality is also considered by implementing Euclidean time circle[22], which produces instantons[106], as you can see in the green region.

Exceptionally, "n-R" in the bottom left corner contains a non-Riemannian background, which is not only hard to interprete as a brane but has no Riemannian description at all. Including this, objects in the bottom purple region are the T-duals of the fundamental string (F1) and also null waves in T-fold or DFT sense.Theses examples are considered in Section 5.3.

Also, we consider a set of 5-branes including an exotic 5_2^2-brane, which are derived by performing T-dualities to a black brane called NS5-brane[90, 91], and which are utilised in the application section 5.4 either.

The S-duality between 1- and 5-branes (noted as S*) is possible in the electric-magnetic duality sense. In the electric-magnetic duality, the 2-form field and the 6-form (hodge dual or its 7-form field strength is 3-form field strength) may be mixed under the S-duality relation in 10-dimensional String Theory.

1.2 Notation

The notation in this thesis basically follows our paper [1]. The Riemannian fields, the metric $G_{\mu\nu}$ and the dilaton Φ, are basically noted with capital letters to make parallel to $B_{\mu\nu}$ because the small letters are assigned for the asymptotic values in the paper [1]. Most other papers writes the Riemannian metric as $g_{\mu\nu}$, so this is the most distinct notation in this thesis. Meanwhile, the DFT fields are written in the following way: the generalised metric $\mathcal{H}_{MN}$ and the $\mathbf{O}(D,D)$ dilaton d. Note that the physical spacetime dimensions is noted as D.

Furthermore, as most papers follows, in this thesis, I also follows the rule that the large Roman alphabet index means the $2D$-dimensional vector index in the $\mathbf{O}(D,D)$ representation, and that the small Greek letter index is the Riemannian D-dimensional index.

The small Greek letters also serve for another role. In DFT, we need to apply a section condition which fixes D physical dimensions and eliminates coordinate dependence to the unphysical dimensions. The standard way to select the physical directions is to select the lower part of the matrix in the current $\mathbf{O}(D,D)$ representation:

$$x^A = \begin{pmatrix} 0 \\ x^\mu \end{pmatrix}. \tag{1.43}$$

Under this section, we name each D-dimensional blocks of the DFT quantities with the quantities with small Greek letter indices, which is called

Riemannian parametrisation. Each DFT quantity has different Riemannian parametrisation to achieve succesfully reducing the DFT action to the Einstein action.

Most of the DFT quantities have B-twisted way of Riemannian parametrisation. For example, a Double Yang-Mills vector gauge field is normally Riemannian-parametrised by

$$V_A = \begin{pmatrix} \varphi^\lambda \\ A_\mu + B_{\mu\nu}\phi^\nu \end{pmatrix}, \tag{1.44}$$

where A_μ is the Riemannian Yang-Mills field, and the φ^λ is an auxiliary field for dual directions, and the $B_{\mu\nu}$ is the Kalb-Ramond field, one of the components of the generalised metric. This parametrisation is also widely be seen for other Double vector quantities to correctly parametrise Double Field Theory into the conventional Riemannian theory [1, 70, 77]. Even the Riemannian parametrisation of the generalised metric,

$$\mathcal{H}_{MN} = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix}, \tag{1.45}$$

is a B-twisted parametrisation from the diagonal metric

$$\begin{pmatrix} G^{-1} & 0 \\ 0 & G \end{pmatrix}. \tag{1.46}$$

The reason of this B-twisted parametrisation is studied solely in this thesis after the publication of our study [1] and explained in Section 1.2.4 and 5.1 in this thesis.

We also develop some convenient notation to indicate components of DFT fields throughout the thesis. For some quantities which are defined only in DFT and of which the representing letter is uniquely assigned, the lower or upper index directly means the lower or upper block component of the quantity. For example, the curly letter $\mathcal{H}$ is uniquely assigned to the generalised metric $\mathcal{H}_{MN}$, and we may conveniently indicate each $D \times D$ blocks of the metric by

$$\mathcal{H}_{MN} = \begin{pmatrix} \mathcal{H}^{\mu\nu} & \mathcal{H}^{\mu}{}_{\nu} \\ \mathcal{H}_{\mu}{}^{\nu} & \mathcal{H}_{\mu\nu} \end{pmatrix}, \tag{1.47}$$

or by raising the $\mathbf{O}(D,D)$ index by the $\mathbf{O}(D,D)$ metric $\mathcal{J}^{MN}$ (1.22):

$$\mathcal{H}^{MN} = \begin{pmatrix} \mathcal{H}_{\mu\nu} & \mathcal{H}_{\mu}{}^{\nu} \\ \mathcal{H}^{\mu}{}_{\nu} & \mathcal{H}^{\mu\nu} \end{pmatrix}. \tag{1.48}$$

The beauty of this indication is that this rule does not change: the right lower matrix (physical-dimensional) component has the index with the same position as the $\mathbf{O}(D,D)$ representation.

There are some quantity that we cannot parametrise. For example, the parametrisation of the normal vector to some surface is hard to be discussed. To discuss a parametrisation, we first fix the section. After fix the section and the physical directions, any surface may live on the section, and the sole parametrisation should be trivial,

$$N_A = \begin{pmatrix} 0 \\ n_\mu \end{pmatrix} \tag{1.49}$$

Other than parametrisation, you have to be careful that there are pairs of different quantities that share the notation because they have similar roles in Riemannian theory and DFT. The Christoffel connection is written as $\Gamma^\sigma{}_{\mu\nu}$ in Riemannian geometry and Γ_{CAB} in Double Field Theory; they have no direct connection by calculation; $\Gamma^\sigma{}_{\mu\nu}$ is not the Riemannian parametrisation of Γ_{CAB}, but we use the same Γ because they both serve as the Christoffel connection in each of their theories.

Also be careful that there are some quantities that share the symbol but has no similarity at all. In this thesis, we find the DFT Einstein tensor and will be noted as G_{AB}, which has no connection to the Riemannian metric $G_{\mu\nu}$. This came from the convention where the metric is noted as $g_{\mu\nu}$ and the Einstein curvature tensor is noted as $G_{\mu\nu}$, and which is not used in this thesis.

In this section, I list up all conventions used in this thesis from the index conventions to the symbols, ahead of the main context. Necessarily, I need to write some expressions without detailed backgrounds or definitions as this is the spoiler before the explanations in the main text, so it may be hard to understand by itself. You should look at the text afterwards to understand what the notations here mean, and conversely you should watch here to understand what each letter means.

String constants and gravitational constants.

Before we start to discuss dynamic variables, let me briefly mention about the constants in String Theory, related to the gravitational constant in Double Field Theory, for the gravitational constant is often not mentioned in DFT studies, so it is hard to check whether our result of charge formula exactly coincide with the tension in String Theory for the branes introduced in §1.1.3.

In String Theory, the gravitational constant is defined as

$$G_N := g_s^2 l_s^{D-2}, \tag{1.50}$$

where D is the dimension of the domain, and g_s is the string coupling constant, and l_s is the string scale length. The gravitational constant also can be written in different notation:

$$\kappa^2 := 8\pi G_N, \tag{1.51}$$

or possibly with other coefficient but always defined $\kappa^2 \propto G_N$ [pg. 150 in 8] so that κ^2 simplifies the equation of motion as $\mathcal{G}_{\mu\nu} = \kappa^2 T_{\mu\nu}$ where $\mathcal{G}_{\mu\nu}$ is the Einstein curvature tensor and $T_{\mu\nu}$ is the energy-momentum tensor. The string scale also can be written in α':

$$l_s = \sqrt{\alpha'}; \tag{1.52}$$

α' appears in the sigma-model action, and the string tension is mostly defined as $1/(2\pi\alpha')$.

Remind that the Einstein-Hilbert action is written as

$$I = \frac{1}{16\pi G_N} \int \mathrm{d}^D x \, \sqrt{|G|} R, \tag{1.53}$$

or in the string frame[Chap. 3 in 7],

$$I = \frac{1}{16\pi G_N} \int \mathrm{d}^D x \, e^{-2\Phi} \sqrt{|G|} R, \tag{1.54}$$

where $\sqrt{|G|}$ is the volume element and Φ is the dilaton. In most DFT literature, the Lagrangian is defined as

$$\mathcal{L}_{\mathrm{DFT}} \sim e^{-2\Phi} \sqrt{|G|} R \quad \text{(in the sense of overall coefficient)} \tag{1.55}$$

without the gravitational constant including our own paper [1]. In this thesis, we keep the Lagrangian without the constant as (1.55) to maintain consistency with other literature, and we define the DFT action as

$$I_{DFT} = \frac{1}{2\kappa^2} \int_{\Sigma_D} \mathrm{d}^D x \, \mathcal{L}_{DFT}. \tag{1.56}$$

Note that the overall coefficient is proportional to g_s^{-2}:

$$\frac{1}{2\kappa^2} \propto \frac{1}{g_s^2 l_s^{D-2}}, \tag{1.57}$$

which is natural because in the NSNS sector action, the graviton is coupled with the dilaton in the form of $e^{-2\Phi}$; originally the expectation value of the dilaton is the string coupling: [pg. 150 in 8, 9]

$$g_s := e^{\phi_0} := e^{\langle\phi\rangle}, \quad \Phi = \phi - \phi_0 \tag{1.58}$$

if the gravitational constant does not contain g_s^{-2}. In this literature, I put the g_s dependence into the gravitational constant, and the expectation value of the dilaton is set to zero.

(Anti-)Symmetrised (Pseudo-)Tensors. I also want to clarify the overall coefficient of the tensors with symmetrised indices. Some undergraduate students may confuse because the bracket on the operators and the bracket on the indices have different overall coefficients in the most used convention. While the brackets on operators are defined as

$$[A, B] := AB - BA \tag{1.59}$$

$$\{A, B\} := AB + BA \tag{1.60}$$

in most literature and also in this thesis, the index (anti-)symmetrisation is defined by

$$T_{(ij)} = \frac{1}{2}\left(T_{ij} + T_{ji}\right), \tag{1.61}$$

$$T_{(a_1 \cdots a_n)} = \frac{1}{n!}\sum_{\sigma\in S_n} T_{a_{\sigma 1} \cdots a_{\sigma n}}, \tag{1.62}$$

$$T_{A[B}{}^{C(DE}S^{FG)HI}{}_{JKL}Z^{MN}{}_{O]PQ}$$

Figure 1.3: Symmetrisation of multiple multilinear objects. The red indices are symmetrised, and the blue indices are anti-symmetrised.

$$T_{[ij]} = \frac{1}{2}\left(T_{ij} - T_{ji}\right), \tag{1.63}$$

$$T_{(a_1\cdots a_n)} = \frac{1}{n!}\sum_{\sigma\in S_n}(-1)^{\operatorname{sgn}\sigma}T_{a_{\sigma_1}\cdots a_{\sigma_n}}, \tag{1.64}$$

where S_n is the set of all permutations of $\{a_1\cdots a_n\}$ and the $\operatorname{sgn}\sigma$ is the parity ±1 of the permutation σ, for both Riemannian quantities and Double quantities in this thesis and most of the other literature.

The symmetrisation is also applicable for the combined tensor, and the bracket can be split with some conditions: (i) both the beginning and the end of a bracket must be in the same upper or lower indices; (ii) all the indices between the two end of the bracket and in the same upper or lower side with the bracket is subject to the symmetrisation. This explanation is described in Figure 1.3.

From here, I state notations of all the fields in Riemannian geometry and Double geometry.

1.2.1 Fundamental Fields in Riemannian Geometry

The fundamental field in Riemannian geometry is only the metric $G_{\mu\nu}$. In this thesis, we are dealing with the NSNS sector string frame action:

$$I = \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, e^{-2\Phi}\left(R[G] - \frac{1}{12}H[B]^2 + 4\left(\partial\Phi\right)^2\right), \tag{1.65}$$

so the fundamental fields in the Riemannian NSNS theory are

- D-dimensional (symmetric) metric $G_{\mu\nu} \in \mathbf{GL}(D-1,1)/\mathbf{O}(D-1,1)$,
- Anti-symmetric Kalb-Ramond gauge field $B_{\mu\nu}$,
- Dilaton $e^{-2\Phi}$.

Be aware that the metric is written as $G_{\mu\nu}$ not $g_{\mu\nu}$ as in [1, 60].

Note that the total degrees of freedom for $G_{\mu\nu}$, $B_{\mu\nu}$, and Φ are respectively $D(D+1)/2$, $D(D-1)/2$, and 1 for the spacetime dimension D. You can notice that the degree of freedom (DOF) of $G_{\mu\nu}$ and $B_{\mu\nu}$ combined is D^2, which is the target DOF of the generalised metric in DFT.

Gauge symmetries. Component-wise, this argument is correct, but as both $G_{\mu\nu}$ and $B_{\mu\nu}$ have the gauge symmetries, we have redundencies called gauge orbit to reduce to obtain the true degrees of freedom. The gauge orbit is defined by the gauge transform with the gauge parameter with one rank down.

The gauge symmetry for the metric (and the spacetime) is known as *diffeomorphism*, and it is defined by the *Lie derivative*:

$$\delta_\xi G_{\mu\nu} = -\nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu, \tag{1.66}$$

which is the derivative for the infinitesimal spacetime transformation $x^\mu \to x^\mu + \xi^\mu$. The *gauge transformation* of Kalb-Ramond field follows the law for the standard 2-form gauge field:

$$\delta_\lambda B_{\mu\nu} = \partial_\mu \lambda_\nu - \partial_\nu \lambda_\mu, \tag{1.67}$$

where the $\lambda_\mu(x)$ is the gauge parameter. Also here, the gauge parameter is a vector.

Double Field Theory takes aim to unify the momentum and winding charge into the single $2D$-dimensional charge and to combine the local translation and the local gauge transform into a $2D$-dimensional local translation:

$$\text{New gauge parameter } X^A = (\lambda_\mu, \xi^\mu). \tag{1.68}$$

Thus, except the dilaton's 1 DOF, we can expect that DFT defines a generalised metric with $2D \times 2D$ size but somehow having D^2 components by some restrictions, and we may also have the gauge transformation for the generalised metric with a $2D$-dimensional vector generalised gauge parameter.

1.2.2 Fundamental Fields in Double Field Theory

In Double Field Theory, all fields have $\mathbf{O}(D,D)$ indices explained in Subsection 1.1.1. Thus, we expect that all the matrices may have $2D$-dimensional indices but also have additional conditions. The fundamental fields in Double Field Theory are

- Generalised metric $\mathcal{H}_{MN}$
$$\begin{aligned} &\text{(Riemannian Parametrisation)} \\ &= \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix} \\ &\in \frac{\mathbf{O}(D,D)}{\mathbf{O}(D-1,1)\times\mathbf{O}(D-1,1)} \end{aligned} \tag{1.69}$$

- $\mathbf{O}(D,D)$-scalar dilaton e^{-2d}
$$\begin{aligned} &= e^{-2\Phi}\sqrt{|G|} \\ &\text{(act as volume element)} \end{aligned} \tag{1.70}$$

and also please keep in mind that the constant $\mathbf{O}(D,D)$ metric $\mathcal{J}_{MN}$ exists all around Double Field Theory. In fact, we define to use $\underline{\mathcal{J}_{MN}}$ not the metric $\mathcal{H}_{MN}$ to *lower or raise the Double indices*. You may notice that the $\mathcal{J}_{MN}$ also has a property like metric:

$$\mathcal{J}^{MN} = \mathcal{J}^{MP}\mathcal{J}_{PQ}\mathcal{J}^{QN} = \mathcal{J}_{MN} = \begin{pmatrix} 1_{D\times D} & 0 \\ 0 & 1_{D\times D} \end{pmatrix}, \tag{1.71}$$

not only the generalised metric $\mathcal{H}_{MN}$:

$$\mathcal{H}^{MN} := \mathcal{J}^{MP}\mathcal{H}_{PQ}\mathcal{J}^{QN} = \mathcal{H}^{MP}\mathcal{H}_{PQ}\mathcal{H}^{QN} = (\mathcal{H}_{MN})^{-1}\,. \tag{1.72}$$

As you can see above, both the Riemannian parametrisation and the group $\mathbf{O}(D,D)/\mathbf{O}(D)\times\mathbf{O}(D)$ suggests that the degree of freedom of $\mathcal{H}_{MN}$ is D^2, which equals to $G_{\mu\nu}$ and $B_{\mu\nu}$ combined, as we expected in the Riemannian theory (section 1.2.1). The same expression may be equivalent written that the generalised metric $\mathcal{H}_{MN}$ is a symmetric $2D\times 2D$ matrix with these conditions:

$$\mathcal{H}_{MN} \in \mathbf{O}(D,D), \quad \mathcal{H}^{MN} = \mathcal{J}^{MP}\mathcal{H}_{PQ}\mathcal{J}^{QP}, \quad \mathcal{H}_{MP}\mathcal{H}^{PN} = \delta_M^N. \tag{1.73}$$

as stated at (3.23) in [57].

Projection matrices. There is another expression of $\mathcal{H}_{MN}$ which is *projection matrices*: we define two redundant matrices:

$$P_{MN} = \frac{1}{2}\left(\mathcal{J}_{MN} + \mathcal{H}_{MN}\right), \tag{1.74}$$

$$\bar{P}_{MN} = \frac{1}{2}\left(\mathcal{J}_{MN} - \mathcal{H}_{MN}\right), \tag{1.75}$$

which satisfy the conditions for projections for each D-dimension space that quotient (codimension) each other:

$$P_A{}^B P_B{}^C = P_A{}^C, \tag{1.76}$$

$$\bar{P}_A{}^B \bar{P}_B{}^C = \bar{P}_A{}^C, \tag{1.77}$$

$$P_A{}^B \bar{P}_B{}^C = 0, \tag{1.78}$$

where the raising and the lowering of the index is done by (1.22). The projection matrices are convenient to extract specific components from a tensor covariantly, so in the main text of this thesis, projections are more used than the generalised metric.

Also, we may introduce the notation to omit the indices of the projections, in the sense that

$$(PV)^A := P^A{}_B V^B, \tag{1.79}$$

in this thesis.

Gauge Symmetries. The gauge symmetry of the metric is defined by the *Lie derivative* as in 1.2.1. In the same sense, we define *the generalised Lie derivative*, which defines "Double" local translations for the generalised metric, the $\mathbf{O}(D, D)$ dilaton, and all other Double-geometric quantities. The gauge parameter X^A is defined by simply parallelly combining the gauge parameters of $B_{\mu\nu}$ and $G_{\mu\nu}$:

$$X^A := \begin{pmatrix} \lambda_\mu \\ \xi^\mu \end{pmatrix}, \tag{1.80}$$

where the X is used as the Double version of ξ, and the *generalised Lie derivative* is defined in the way that it can define both the Riemannian diffeomorphism and 2-form gauge transform simultaneously with the X^A:

$$\begin{aligned} \widehat{\mathcal{L}}_X T_{MN\cdots} = X^P \partial_P T_{MN\cdots} + \omega(\partial_P X^P) T_{MN\cdots} \\ + (\partial_M X^P - \partial^P X_M) T_{PN\cdots} + (\partial_N X^P - \partial^P X_N) T_{MP\cdots} + \cdots \end{aligned} \tag{1.81}$$

where T is the general tensor of Double geometry, and ω is the scaling weight of T *i.e.* the order of e^{-2d} in T. The generalised Lie derivative of tensors with upper indices is also easily defined by switching the upper/lower positions of contracted indices, for the index-raising/lowering tensor $\mathcal{J}_{MN}$ is a constant, not a variable as in Riemannian geometry.

We will not differentiate the notation of the Lie derivative and the generalised Lie derivative and use $\widehat{\mathcal{L}}$ for both case. We will differentiate only by the subscript, the parameter notation. Riemannian diffeomorphism parameters will only be noted by small Greek letters like ξ while the generalised diffemorphism parameters will be noted by large Roman alphabets like X. By the notation of the parameter and the context of the literature, you should determine whether $\widehat{\mathcal{L}}$ means the diffeomorphism or the generalised diffeomorphism defined by the generalised Lie derivative.

Details will be explained in Chapter 3.

Yang-Mills field in DFT

Yang-Mills theory also can be written in Double geometry [70, 77]. Like in those papers, we note the Double Yang-Mills field as V_A and the Riemannian parametrisation is given as

$$V_A = \begin{pmatrix} \varphi^\lambda \\ A_\mu + B_{\mu\nu}\varphi^\nu \end{pmatrix}, \tag{1.82}$$

where A_μ is the Riemannian Yang-Mills field and φ^ν is an auxiliary field in the winding directions, so that the Double Yang-Mills action is parametrised into Riemannian Yang-Mills action. This $B_{\mu\nu}$-twisted parametrisation is necessary to obtain the pure Riemannian directional components, and you can find this kind of parametrisation in many Double vectors including the Double Killing vector appearing in this study[1]. For more details, see Section 1.2.4 and 5.1.

Furthermore, following the notations in [1, 70, 77], the semi-covariant Double Yang-Mills field strength is defined by

$$\mathcal{F}_{AB} := \nabla_A V_B - \nabla_B V_A - i[V_A, V_B] \tag{1.83}$$

with semi-covariant derivatives. The covariant components of this semi-covariant field strength is also defined by

$$P_A{}^C \mathcal{F}_{CD} \bar{P}_B{}^D, \tag{1.84}$$

which is not anti-symmetric at all. The Yang-Mills action is defined by

$$I_{\mathrm{YM}} = g_{\mathrm{YM}}^{-2} \int\limits_{\Sigma_D} \mathrm{d}^D x \, e^{-2d} \, \mathrm{Tr} \left[P^{AC} \bar{P}^{BD} \mathcal{F}_{AB} \mathcal{F}_{CD} \right], \tag{1.85}$$

which is reduced to the Riemannian Yang-Mills

$$I_{\mathrm{YM}} = g_{\mathrm{YM}}^{-2} \int\limits_{\Sigma_D} \mathrm{d}^D x \, e^{-2\Phi} \sqrt{|G|} \, \mathrm{Tr} \left[F^2 \right] \tag{1.86}$$

with the Riemannian Yang-Mills field strength definition

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \tag{1.87}$$

if the auxiliary field is turned off: $\varphi_\mu = 0$, according to [70].

1.2.3 Double Coordinates

In the sense of Double index of $\mathcal{H}_{MN}$ and the definition of X^A, we also define the 2D-dimensional spacetime by combining the physical coordinates x and the *dual coordinates* or the "*winding directions*" $\tilde{x}$ that we invent here for the winding charges:

$$x^A = \begin{pmatrix} \tilde{x}_\mu \\ x^\mu \end{pmatrix}. \tag{1.88}$$

Be aware that both the physical coordinates and the Double coordinates are noted as x while the "winding" coordinates is noted as $\tilde{x}$.

As the whole $2D$ dimensions are not all physical, we need to reduce the dimension by half but in $\mathbf{O}(D,D)$-covariant way because DFT's objective is to make $\mathbf{O}(D,D)$-covariant theory. "Section condition" or "strong constraint" is the $\mathbf{O}(D,D)$-covariant equation, a solution of which is a D-dimensional section that eliminates coordinate dependence on the co-space of the section from the theory and that breaks the $\mathbf{O}(D,D)$ symmetry in the theory. Obviously, when we deal with a solution of DFT so-called "background", the section should also be fixed, as the section is a part of the solution of DFT. We will note a given section in the text as Σ_D.

The most standard section is simply eliminating $\tilde{x}$ and makes

$$x^A = \begin{pmatrix} 0 \\ x^\mu \end{pmatrix}. \tag{1.89}$$

We will call this section *the usual section*, *the conventioal section*, *the canonical section*, or *Riemannian section*. The parametrisation of DFT fields to the D-dimensional blocks under this section is called *the Riemannian parametrisation*, which is explained at the beginning of this section 1.2.

1.2.4 Line element

As double coordinates defined, we may also write the line element expression in Double Field Theory.

Remind that in Riemannian geometry, the metric also can be written in *line element* form:

$$\mathrm{d}s^2 = G_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu, \tag{1.90}$$

where s states the distance in Riemannian geometry.

In the same manner, we may define the *line element* expression of the generalised metric in Double geometry using the double coordinates that we have defined in the previous subsection 1.2.3:

$$\begin{aligned}\mathrm{d}\mathsf{s}^2 &= \mathcal{H}_{MN}\,\mathrm{d}x^M\,\mathrm{d}x^N & (1.91)\\ &= G^{\mu\nu}\Big(\mathrm{d}\tilde{x}_\mu - B_{\mu\sigma}\,\mathrm{d}x^\sigma\Big)\Big(\mathrm{d}\tilde{x}_\nu - B_{\nu\lambda}\,\mathrm{d}x^\lambda\Big) + G_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu, & (1.92)\end{aligned}$$

where we assign the Sans Serif–styled small s for the distance in Double geometry.

The expression (1.92) gives us a hint how we can diagonalise the generalised metric with $\mathbf{O}(D, D)$ transformation.

If a background in Riemannian geometry has an asymptotic value that is not trivial $\eta_{\mu\nu} = \mathrm{diag}\{-1, 1, \cdots\}$ and the background has translational symmetries in time and space, still a Killing vector ∂_t may not give the correct notion of energy, and a Killing vector ∂_μ would not provide the correct momentum. If possible, you have to diagonalise the asymptotic behaviour by $\mathbf{GL}(D)$ transformation to get the right notion of momentum. In Double Field Theory, we have similar argument, which will be discussed in Section 5.1 in detail.

To get the right notion of conserved charges, for asymptotically flat case, you should diagonalise the asymptotic behaviour by $\mathbf{O}(D,D)$ rotation.[6] First, apply a transformation which can eliminate $B_{\mu\nu}$ component, or equivalently, the block-off-diagonal terms from the generalised metric,

$$\mathrm{d}\tilde{x}_\mu \to \mathrm{d}\tilde{x}_\mu + b_{\mu\sigma}\,\mathrm{d}x^\sigma, \tag{1.93}$$

where $b_{\mu\nu}$ is the asymptotic value of $B_{\mu\nu}$, and which can be obtained easily from the line element form in (1.92). In the next step, you should apply the $\mathbf{GL}(D)$ transformation, which can diagonalise $g_{\mu\nu} = G_{\mu\nu}(r\to\infty)$, and which is explained in Riemannian Geometry. Then, we get the background with asymptotic value

$$\mathcal{H}_{MN} \to \begin{pmatrix} \eta^{\mu\nu} & 0 \\ 0 & \eta_{\mu\nu} \end{pmatrix}, \tag{1.94}$$

and in this coordinate system and vector basis, we get the right notion of spatial directions associated with the energy and momentum and of winding directions associated with the winding charges.

[6] $\mathbf{GL}(D)$ transformation in the Riemannian geometry corresponds to $\mathbf{O}(D,D)$ transformation in Double geometry.

1.2.5 Normal Vectors

In Riemannian geometry, especially during manipulating the action and obtaing the Noether current, by using Stokes' and Divergence theorem, we often convert a bulk integral into a surface integral and vice versa. To define the surface integral domain and converting the bulk integrand into the surface integrand, we need a normal vector to take a divergence and make a projection of the integrand. (i) The normal vector n_μ is obviously defined perpendicular to the surface to which we want to define the normal vector with respect:

$$\text{surface } \{f(x) = 0\} \quad \Rightarrow \quad \text{normal vector } n_\mu \propto \partial_\mu f, \tag{1.95}$$

(ii) and the sign of the normal vector is defined in the way that n^μ points outwards. (iii) Furthermore, the norm of the normal vector must be normalised to be ± 1 depending on its timelikeness:

$$G_{\mu\nu} n^\mu n^\nu = \begin{cases} +1 & \text{if the normal vector is spacelike,} \\ -1 & \text{if the normal vector is timelike,} \end{cases} \tag{1.96}$$

if we have the convention that the timelike distance has the negative sign.

In the same way, we also define the normal vectors in Double spacetime. There are some people that consider $2D$-dimensional Double integral in the action and then apply the section condition as a part of a DFT solution. In this thesis, we follow Park's notation[69, 71, 74, 75], which we already state on the integral that we will apply some kind of section condition and the integral will be done with respect to only physical D-dimensions. The D-dimensional *section* is noted by Σ_D in the integral in this thesis.

Especially, in this thesis, we will only consider *Riemannian section* in the application to define proper time- and spatial direction and thus to define right notion of energy, momentum and winding charges. Thus, we will use the similar notation of timelikeness in (1.96) to define the surfaces and their normal vectors in Double geometry.

In Double geometry, we define the surface and normal vectors analogously to the Riemannian geometry version stated as (i), (ii), and (iii) around (1.95) and (1.96).

For those who consider DFT as a $2D$-dimensional theory, $(2D-1)$-dimensional hyperspace may be defined as $\{f(x)=0\}$ where x is the Double coordinates and the normal vector is defined by

$$N_A \propto \partial_A f. \tag{1.97}$$

Then the Riemannian section is applied, so the hypersurface we consider will be reduced to $(D-1)$ dimensional intersection between the section and the original surface. The normal vector also lose its dual dimensions and is

Riemannian-parametrised to

$$N_A = \begin{pmatrix} 0 \\ n_\mu \end{pmatrix}. \tag{1.98}$$

In this thesis, we apply the section Σ_D in the first place, so the hypersurface in DFT in our description only keeps $(D-1)$ physical dimensions and throws D unphysical dimensions. In this case, the hypersurface is defined only within the Riemmanian space and the definition is the same as (1.95). The $2D$-dimensional normal vector is constructed like (1.98).

Now, to discuss (ii) and (iii), we need a metric to raise the index and make a norm of the normal vector. In DFT, usually we use $\mathbf{O}(D,D)$ metric $\mathcal{J}_{MN}$ to lower/raise indices. However, please keep in mind that still in DFT, the generalised metric $\mathcal{H}_{MN}$ is the distance-measuring matrix as we discussed in 1.2.4. Thus, the norm should be defined by

$$\mathcal{H}_{MN} N_A N_B = \begin{cases} +1 & \text{if the normal vector is spacelike,} \\ -1 & \text{if the normal vector is timelike,} \end{cases} \tag{1.99}$$

and the upper-indexed normal vector to discuss (ii) also should be defined by

$$\tilde{N}^A = \mathcal{H}^{AB} N_B. \tag{1.100}$$

In this thesis, we define $\tilde{N}$ notation uniquely to the normal vector to specially mark the $\mathcal{H}$ index raising. In some literature [68], they define $N^A := \mathcal{H}^{AB} N_B$ only for the normal vector, but this notation is too confusing, so we will not use this notation.

You can see that this definition of the norm and the index raising of the normal vector is very natural if you apply the Riemannian parametrisation to the definition. By applying the parametrisation (1.98) and (1.69), the norm becomes

$$\mathcal{H}_{MN} N_A N_B = G^{\mu\nu} n_\mu n_\nu, \tag{1.101}$$

and the upper-indexed normal vector becomes

$$\tilde{N}^A = \mathcal{H}^{AB} N_B = G^{\mu\nu} n_\nu. \tag{1.102}$$

Thus, we may say that this kind of definition perfectly describes surfaces in Double geometry and is coherent to the Riemannian definition connected by the Riemannian parametrisation.

1.2.6 Connections and Curvatures

Connections and covariant derivatives

In this subsection, we discuss the notations of connections and curvatures in Riemannian geometry and Double geometry, the essentials of each differential geometry. The connection is the derivative of the metric, which corresponds to the field strength in the gauge theory. There are several kinds of connections: *the Weitzenböck connection* is the field strength in the vielbein description;

the Christoffel connection or the Christoffel symbols is the most common connection notation we learn in the Einstein gravity textbook, and in the torsionless basis, the Christoffel symbols in the Riemannian geometry is defined by

$$\Gamma^{\sigma}{}_{\mu\nu} = \frac{1}{2} G^{\sigma\lambda} \left(\partial_{\mu} G_{\nu\lambda} + \partial_{\nu} G_{\lambda\mu} - \partial_{\lambda} G_{\mu\nu} \right), \tag{1.103}$$

where we use the large Greek Γ with Riemannian (small Greek) indices for the Riemannian Christoffel symbols as in most other texts. The Christoffel connection without the torsion terms is also called *the Levi-Civita connection*.

In Double Field Theory, we may also define all kinds of connections: we may formulate Double Field Theory in vielbein and/or with torsion. However, in this text, we mainly consider the generalised metric formulation without torsion. *The Christoffel or Levi-Civita connection in DFT* is defined by the definition of *the covariant derivative of DFT*:

$$\begin{aligned} \nabla_C T_{A_1 A_2 \cdots A_n} := {} & \partial_C T_{A_1 A_2 \cdots A_n} - \omega_T \Gamma^{B}{}_{BC} T_{A_1 A_2 \cdots A_n} \\ & + \sum_{i=1}^{N} \Gamma_{C A_i}{}^{B} T_{A_1 \cdots A_{i-1} B A_{i+1} \cdots A_n}, \end{aligned} \tag{1.104}$$

where T is a general DFT tensor, and ω_T is the dilatation weight of T, as explained in (1.81). Note that not only the DFT Christoffel connection and the Riemannian Christoffel connection have the same symbol Γ while they have different indices and different definitions, but also the covariant derivatives of Riemannian geometry and Double geometry both are stated with the same symbol ∇ but only different index notation. The DFT covariant derivative is, of course, covariant in the sense of generalised Lie derivative.

The torsionless Christoffel connection in DFT may be obtained by solving the metricity condition and the torsionless conditions, as in Riemannian geometry. However, as DFT has too many components, we cannot fix all components of the connection in DFT. In most literature like Hohm, Hull, Zweibach, *et al.* [56, 57], they do not obtain the explicit form of the connection because they cannot, but they only obtain other solvable covariant quantities. Park *et al.*[69, 71, 74, 75] developed *the semi-covariant formulation of DFT* where they define the projection tensors that can divide the solvable components and the unsolvable ones, and where they make the connection formulation that fix the unsolvable components to zero by breaking some part of the covariance, but that can still lead to final covariant curvature tensor. In this thesis, we will expand our arguments mainly on the semi-covariant formulation and write everything in the explicit but also manifestly "semi-covariant" expressions.

Curvatures

In Einstein theory, as we defined the metric $G_{\mu\nu}$ in this thesis, we may assign another character for the Einstein curvature tensor:

$$\mathcal{G}_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}RG_{\mu\nu} \tag{1.105}$$

In DFT, we find that the definition of the curvatures are different from the Riemannian theories in the first place. However, we may find the (semi-)covariant curvature tensors with 4 indices and 2 indices, which we may call corresponding to the Riemann tensor and the Ricci tensor. Because of the difference between the Riemannian Riemann tensor and the DFT Riemann tensor, we may assign another character for the DFT Riemann tensor S_{ABCD}, and then the DFT Ricci tensor is assigned to be $\mathcal{S}_{AB}$, and the DFT scalar curvature is denoted by $\mathcal{S}$.

Also, in our study, we discovered a DFT tensor which appears in the place of the Einstein tensor in the ADT formalism, and which is not identicla to the DFT equations of motion. We note this as G_{AB}, and the symmetrised one as $\mathcal{G}_{AB}$. [7]

[7] The Einstein tensor was not trivially symmetric.

Chapter 2 Conserved Quantities in Gravitational Theory

DFT is a gravitational theory, so it is good to know how the conserved charges in gravitational theories in Riemannian geometry have been formulated before we go into the formulation of the charges in DFT. In gravitational theories, it is difficult to construct the Noether currents in the same way for the matter field, as the action of gravitational theories involves 2nd-order derivative terms, curvatures. To make the Dirichlet condition available, we need to apply some adequate boundary term.

There are several ways to acquire such boundary term or the total conserved charge formula. The very first attempt to obtain the conserved charge from the gravitational background which corresponds to the matter field spacetime symmetry charge is done by Arnowitt, Deser, and Misner[29, 30], so it is called *ADM* formalism. ADM formalism is the realisation of the Hamiltonian in the gravitational theory. It first defines the timeslices and reduces all the indices into the spatial space. Under this completely time-and-space-divided formalism, they obtain the Hamiltonian which indicates the energy of the system. This method is indeed canonical but breaking the spacetime symmetry explicitly.

Another attempt were made by Komar[31]. He attempted to obtain the Noether current of the Einstein-Hilbert action, and he found that we can construct off-shell currents without assuming the equation of motion because of the differential Bianchi identity of Riemannian geometry. Thus, the current can be written by potential, and the conserved charge is written by the integral of the current on the timeslice, equivalent to the integral of the Noether potential on the boundary of the timeslice. The Noether currents and the Noether potentials are indeed manifestly covariant, and it gave the conserved charge for the rotational symmetries and the translational symmetries, but it did not give the consistent notion of energy: you can easily check that the Komar mass is negative in some cases. This current is a Noether current associated with the general diffeomorphism, but not the charge with canonical notion of energy-momentum associated with the symmetry of the given solution background.

Thus, Abbott-Deser-Tekin (ADT) charge is developed, which is formulated in the covariant formalism but gives the standard notion of energy and charges equivalent to the ADM formalism. ADT charge is, roughly said, equivalent to a Noether charge of the total action that is composed of the Einstein-Hilbert action and a Gibbons-Hawking-York boundary term. However, as the boundary must be defined, we cannot define the local conserved current which correspond to the energy-momentum. We can use a trick defining a small integral domain, so we call the conserved charge in General Relativity "quasi-local" in that sense. [33, 41, 42, 45, 52–54] Furthermore, the ADT charge consists of the diffeomorphism Noether current and the boundary terms, and using the off-shell Noether current gives us the *off-shell* ADT conserved charge.

Note that Wald *et al.* [46–48] studied the presymplectic forms in the gravity and relations between the Noether current and the Hamiltonian, and they also argued that the black hole entropy can also be represented as a Noether charge. This gives us a hint that our charge formula in DFT in the later chapter may also represent entropy.

In this chapter, I review the ADT charge formalism in Einstein gravity, to which we develop the conserved charge formulation in DFT later analogously. In §2.2, I review the Noether theorem in the gravitational theory and the construction of the off-shell Noether current. In Einstein gravity, an off-shell current is possible to be constructed, which also has to be proven in DFT in the later chapter. In §2.3, based on [41], I review the ADT formalism of constructing the conserved charge from the gravitational background corresponding to the matter energy-momentum. Using the off-shell current constructed in the previous section, we construct an off-shell global (quasi-local) charge.

2.1 Riemannian geometry and Einstein gravity

2.1.1 Local symmetry of the Einstein gravity

The Einstein-Hilbert action without any boundary terms is very important because it is the simplest action of gravity with the manifest covariance. It has, obviously, the local symmetry of diffeomorphism. Diffeomorphism is a general coordinate transformation with differentiability. In the infinitesimal form we may write:

$$x^\mu \to x^\mu + \epsilon \xi^\mu(x^\mu), \tag{2.1}$$

where ϵ is the infinitesimal parameter.

Lie Derivative

The *Lie derivative* is a derivative for certain direction ξ^μ of the diffeomorphism:

$$\widehat{\mathcal{L}}_\xi \phi(x) := \frac{\phi(x + \epsilon\xi^\mu(x^\mu)) - \phi(x)}{\epsilon}, \tag{2.2}$$

and the Lie derivative of the general (Riemannian) tensor associated with the parameter ξ^μ is thus defined by

$$\widehat{\mathcal{L}}_\xi T_{\mu_1\cdots}{}^{\nu_1\cdots} = \xi^\sigma \partial_\sigma T_{\mu_1\cdots}{}^{\nu_1\cdots} + (\partial_{\mu_1}\xi^\sigma) T_{\sigma\cdots}{}^{\nu_1\cdots} - (\partial_{\nu_1}\xi^\sigma) T_{\mu_1\cdots}{}^{\sigma\cdots}. \tag{2.3}$$

in Riemannian geometry. This definition indeed satisfies the distributive law for constant coefficients, so it can be handled as a “derivative”.

Despite its form, the Lie derivative is covariant. [24, 27] Thus, under torsion-free condition[1], the Lie derivative can be also written by

$$\widehat{\mathcal{L}}_\xi T_{\mu_1\cdots}{}^{\nu_1\cdots} = \xi^\sigma \nabla_\sigma T_{\mu_1\cdots}{}^{\nu_1\cdots} + (\nabla_{\mu_1}\xi^\sigma) T_{\sigma\cdots}{}^{\nu_1\cdots} - (\nabla_{\nu_1}\xi^\sigma) T_{\mu_1\cdots}{}^{\sigma\cdots}, \tag{2.4}$$

where ∇_μ is the covariant derivative

$$\nabla_\sigma T_{\mu_1\cdots}{}^{\nu_1\cdots} = \partial_\sigma T_{\mu_1\cdots}{}^{\nu_1\cdots} - \Gamma^\lambda{}_{\sigma\mu_1} T_{\lambda\cdots}{}^{\nu_1\cdots} + \Gamma^{\nu_1}{}_{\sigma\lambda} T_{\mu_1\cdots}{}^{\lambda\cdots}, \tag{2.5}$$

and under the torsion-free condition, the Christoffel connection $\Gamma^\sigma{}_{\mu\nu}$ is defined by the *Levi-Civita connection*:

$$\Gamma^\sigma{}_{\mu\nu} = \frac{1}{2} G^{\sigma\lambda} \left(\partial_\mu G_{\nu\lambda} + \partial_\nu G_{\lambda\mu} - \partial_\lambda G_{\mu\nu} \right), \tag{2.6}$$

where $G_{\mu\nu}$ is the metric. This definition of connection can be fixed by the conditions:

$$\nabla_\sigma G_{\mu\nu} = 0 \qquad \textbf{(metricity)}, \tag{2.7}$$

$$\Gamma^\sigma{}_{\mu\nu} = \Gamma^\sigma{}_{\nu\mu} \qquad \textbf{(torsion-free condition)}. \tag{2.8}$$

The torsion-free condition can also be given by

$$\widehat{\mathcal{L}}_\xi[\nabla] = \widehat{\mathcal{L}}_\xi[\partial] \tag{2.9}$$

Also, the Lie derivative has a closed algebra. We may define the bracket of

[1] If we use the coordinate basis for the vector space, the torsion in the connection is zero

vectors by

$$[X, Y]^\sigma := X^\mu \partial_\mu Y^\sigma - Y^\mu \partial_\mu X^\sigma = \widehat{\mathcal{L}}_X Y^\sigma, \tag{2.10}$$

and then we may show the Lie derivative has a closed algebra.

$$[\widehat{\mathcal{L}}_X, \widehat{\mathcal{L}}_Y] = \widehat{\mathcal{L}}_{[X,Y]} \tag{2.11}$$

This is a part of what is called *Cartan calculus*.[2]

Local symmetry of the Einstein-Hilbert action

The Lie derivative of the metric is thus defined by

$$\widehat{\mathcal{L}}_\xi G_{\mu\nu}(x) = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu, \tag{2.12}$$

which is the derivative under the infinitesimal transformation $x^\mu \to x^\mu + \delta\xi^\mu$. The principle of relativity established by Galilei and modified by Einstein demands that any physical action must have symmetry of diffeomorphism, *i.e.* the action must be invariant under the diffeomorphism, and the Lagrangian density must transform as a product of a volume element and a scalar 'density' under the diffeomorphism.

[2]In DFT, there is DFT version of Cartan calculus. However, similar algebra cannot be constructed with the Lie bracket, so we define *C-bracket* in the *generalised Cartan calculus* of DFT. There are several other brackets in the *generalised Cartan calculus*, but anyway they compose a closed algebra.

Einstein-Hilbert action

The scalar curvature in Riemannian geometry is defined by

$$
\begin{aligned}
R &:= R^{\mu}{}_{\mu} && \textbf{(scalar curvature)}, && (2.13)\\
R_{\mu\nu} &:= R^{\sigma}{}_{\mu\sigma\nu} && \textbf{(Ricci curvature tensor)}, && (2.14)\\
R^{\rho}{}_{\sigma\mu\nu} &:= \mathbf{e}^{\rho}\cdot[\nabla_{\mu},\nabla_{\nu}]\mathbf{e}_{\sigma} && \textbf{(Riemann curvature tensor)} && \\
&= \partial_{\mu}\Gamma^{\rho}{}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\sigma}, && && (2.15)
\end{aligned}
$$

where the raising and the lowering of indices are, of course, done by the metric $G_{\mu\nu}$. Indeed, the scalar curvature transforms as a scalar under the Lie derivative: [3]

$$\widehat{\mathcal{L}}_{\xi}R = \xi^{\mu}\partial_{\mu}R. \tag{2.16}$$

Then, the scalar integrated by the standard volume element is invariant under the coordinate transform:

$$I = \int \mathrm{d}^{D}x\,\sqrt{|G|}R[G]. \tag{2.17}$$

This property is manifest and undoubtedly accepted by most people who studied general relativity. In the next section, I discuss how the scalar integrated by the volume element is invariant under the diffeomorphism, and I review the Noether analysis of general relativity under the diffeomorphism.

[3] We can also prove this by brute-force calculation.

2.2 Off-shell Noether current associated with diffeomorphism

In this section, I explicitly write the symmetry *i.e.* diffeomorphism, the action, and how the action changes under the diffeomorphism. Then, I review the Noether analysis associated with the spacetime transformation that is explained in any regular textbook [28], in equivalent but different footing.

The analysis is following: simply altering the integrated coordinates in the action does not change the value of action:

$$\int_{\mathcal{D}} \mathrm{d}^D x \, \sqrt{|G|(x)} R[G(x)] = \int_{\mathcal{D}'} \mathrm{d}^D x' \, \sqrt{|G|(x')} R[G(x')], \tag{2.18}$$

where $\mathcal{D}$'s are the domain in each coordinate system. To simplify, here I mention the Lagrangian as

$$\mathcal{L}_{\text{grav}} = \sqrt{|G|} R. \tag{2.19}$$

The identity transformation (2.18) can be divided into two parts: (i) the active transformation of fields, which means that the fields shift by the transformation and (ii) the passive transformation of the integral measure and domain, which follows the shift to make the total difference to be zero.

$$
\begin{aligned}
&\int_{\mathcal{D}'} \mathrm{d}^D x' \, \mathcal{L}_{\text{grav}}(x') - \int_{\mathcal{D}} \mathrm{d}^D x \, \mathcal{L}_{\text{grav}}(x) = 0 \\
&= \underbrace{\int_{\mathcal{D}} \mathrm{d}^D x \, \big(\mathcal{L}_{\text{grav}}(x') - \mathcal{L}_{\text{grav}}(x)\big)}_{\text{active transform}} + \underbrace{\left(\int_{\mathcal{D}'} \mathrm{d}^D x' \, \mathcal{L}_{\text{grav}}(x') - \int_{\mathcal{D}} \mathrm{d}^D x \, \mathcal{L}_{\text{grav}}(x') \right)}_{\text{passive transform}}
\end{aligned}
\tag{2.20}
$$

The goal here is to obtain the expression of the active field variation and the passive domain variation, and by assuming the identity relation that the active and passive transforms are equal so cancel each other, we may obtain the conserved current.

2.2.1 Active transform of the Lagrangian density

The deviation of the Lagrangian density in the Einstein-Hilbert action (2.17) under the infinitesimal diffeomorphism $x^\mu \to x'^\mu = x^\mu + \epsilon\xi^\mu$ is defined by the Lie derivative as we defined previously:

$$
\delta_\xi \mathcal{L} = \widehat{\mathcal{L}}_\xi \left(\sqrt{|G|} R \right) . \tag{2.21}
$$

Here, the variation of fields will be called *active*; under a coordinate transformation, fields transforms *actively*, and coordinate transforms *passively* [28].

By the way, the volume element $\sqrt{|G|}$ transforms as

$$\widehat{\mathcal{L}}_\xi \sqrt{|G|} = \frac{1}{2}\sqrt{|G|}G^{\mu\nu}\widehat{\mathcal{L}}_\xi G_{\mu\nu} = \sqrt{|G|}\nabla_\mu \xi^\mu = \partial_\mu\left(\sqrt{|G|}\xi^\mu\right), \tag{2.22}$$

which can be obtained using the derivative formula of the determinant of the matrix.

Thus, the active transformation of the Lagrangian density is

$$\delta_\xi \mathcal{L} = \widehat{\mathcal{L}}_\xi\left(\sqrt{|G|}R\right) = \partial_\mu\left(\xi^\mu\sqrt{|G|}R\right). \tag{2.23}$$

2.2.2 Passive transform of the integral measure

Under the diffeomorphism (or coordinate transform), not only the active transform of fields occurs, but also the integral measure and the integral domain also transforms as the coordinate itself transforms *passively.* The integral measure $\mathrm{d}^D x$ transforms as

$$\mathrm{d}^D x \to \mathrm{d}^D x \left|\delta^\nu_\mu - \partial_\mu \xi^\nu\right| = \mathrm{d}^D x\left(1 - \partial_\mu\xi^\mu\right), \tag{2.24}$$

where the minus sign is because it is passive.

This only counts the scaling factor linear transformation-wise, but the translation is not counted here. By the translation (even for local translation), the boundary is changed, and the action flowed out and in because of the variation of the boundary is

$$-\int \mathrm{d}^D x\, \xi^\mu \partial_\mu\left(\sqrt{|G|}R\right), \tag{2.25}$$

where the minus sign is because it is passive.

The total passive transform of the integral by the domain movement is

$$\begin{aligned}
\delta_\xi^{(\mathrm{domain})} I &= -\int \mathrm{d}^D x \, (\partial_\mu \xi^\mu) \left(\sqrt{|G|} R\right) - \int \mathrm{d}^D x \, \xi^\mu \partial_\mu \left(\sqrt{|G|} R\right) \\
&= -\int \mathrm{d}^D x \, \partial_\mu \left(\xi^\mu \sqrt{|G|} R\right) . \qquad (2.26)
\end{aligned}$$

2.2.3 Principle of Relativity and Lagrangian density

Thus, the total variation of the Einstein-Hilbert action under the diffeomorphism is

$$\delta_\xi I = \delta_\xi^{(\mathrm{fields})} I + \delta_\xi^{(\mathrm{domain})} I = 0, \qquad (2.27)$$

$$\delta_\xi^{(\mathrm{fields})} I = \int \mathrm{d}^D x \, \delta_\xi \mathcal{L} = \int \mathrm{d}^D x \, \partial_\mu \left(\xi^\mu \sqrt{|G|} R\right), \qquad (2.28)$$

$$\delta_\xi^{(\mathrm{domain})} I = -\int \mathrm{d}^D x \, \partial_\mu \left(\xi^\mu \sqrt{|G|} R\right). \qquad (2.29)$$

It is proven that the Einstein-Hilbert action is invariant under the diffeomorphism. Also, in other words, if you review the proof, you can also say that to guarantee the diffeomorphism invariance of the action, the Langrangian density should act as "Lagrangian density" or "volume element" under the diffeomorphism:

$$\delta_\xi \mathcal{L} = \partial_\mu \left(\xi^\mu \mathcal{L}\right), \qquad (2.30)$$

or equivalently said, the Lagrangian density should take the form of (volume element $\sqrt{|G|}$) $\times$ (scalar).

As the principle of relativity established by Galilei says that the physical law should be invariant regardless of the coordinate choice, any action physically making sense should be diffeomorphism-invariant.[4] It means that the Lagrangian density must satisfy (2.30).[5]

2.2.4 Noether Analysis in textbook

The identity for the Lagrangian density (2.30) gives the trivial identity:

$$\delta_\xi \mathcal{L} - \partial_\mu \left(\xi^\mu \mathcal{L}\right) = 0, \tag{2.31}$$

which is the start point of the Noether analysis.

This analysis was put differently in other textbooks[28]. In other textbooks, the logics are following: If we do not assume the Lagrangian density as (scalar) $\times \sqrt{|G|}$ (like we don't know what is the action but we expand the Noether analysis for the general action), the total action variation under the diffeomorphism shall be

$$\delta I = \int \mathrm{d}^D x \, \left(\delta_\xi \mathcal{L} - \partial_\mu(\xi^\mu \mathcal{L})\right). \tag{2.32}$$

To guarantee the diffeomorphism invariance, this quantity should be zero.

Now, in typical field theories without curvature, the variation is defined by

$$\delta \mathcal{L} = \frac{\delta I}{\delta \phi_i}\delta\phi_i + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\delta\phi_i\right), \tag{2.33}$$

[4] unless you choose not to do so.
[5] even for DFT.

so the action variation under the diffeomorphism (2.32) becomes

$$\delta_\xi I = \int \mathrm{d}^D x \left[\cancel{\underbrace{\frac{\delta I}{\delta \phi_i}}_{\substack{\text{Equation}\\ \text{of motion}}} \widehat{\mathcal{L}}_\xi \phi_i} + \partial_\mu \underbrace{\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \widehat{\mathcal{L}}_\xi \phi_i - \xi^\mu \mathcal{L} \right)}_{\text{Noether current } J^\mu} \right], \tag{2.34}$$

where ϕ_i's are the variables (fields) of the action.

However, in gravitational theory, where the curvature exists in the action, the integral by part will not separate the equation of motion bulk part and the boundary terms that clearly. Instead, we have much more strong condition, actually identity, to eliminate the bulk part from (2.32).

2.2.5 Off-shell Noether analysis

Instead of the equation of motion, the differential Bianchi identity in Riemannian geometry eliminates the bulk part from (2.32) in a different way.

First, let us write the variation of the Einstein-Hilbert Lagrangian density:

$$\mathcal{L}_{EH} := \sqrt{|G|} R, \tag{2.35}$$

$$\delta \mathcal{L}_{EH} = -\sqrt{|G|} \mathcal{G}^{\mu\nu} \delta G_{\mu\nu} + \partial_\mu \left(\sqrt{|G|} \Theta^\mu (G_{\alpha\beta}, \delta G_{\alpha\beta}) \right), \tag{2.36}$$

$$\mathcal{G}_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R G_{\mu\nu} \qquad \textbf{(equation of motion)}, \tag{2.37}$$

$$\Theta^\mu := (G^{\sigma\nu} \delta \Gamma^\mu{}_{\nu\sigma} - G^{\sigma\mu} \delta \Gamma^\nu{}_{\nu\sigma}) \quad \textbf{(pseudovector boundary)}, \tag{2.38}$$

where $\Theta^\mu(G_{\alpha\beta}, \delta G_{\alpha\beta})$ is the boundary term of the general variation of the action. We can put $\widehat{\mathcal{L}}_\xi G_{\alpha\beta}$ instead of $\delta G_{\alpha\beta}$ for the diffeomorphism case. The action variation under the diffeomorphism (2.32) may be written with the expression we just defined:

$$\delta_\xi I = \int \mathrm{d}^D x \left[-\sqrt{|G|}\, \mathcal{G}^{\mu\nu} \widehat{\mathcal{L}}_\xi G_{\mu\nu} + \partial_\mu \left(\sqrt{|G|} \left(\Theta^\mu (G_{\alpha\beta}, \widehat{\mathcal{L}}_\xi G_{\alpha\beta}) - \xi^\mu R \right) \right) \right]. \tag{2.39}$$

In regular Noether analysis, the bulk term, which is the equation of motion of the theory $\mathcal{G}^{\mu\nu}$, becomes zero assuming the on-shell condition. However, in this subsection §2.2.5, we want to make Noether analysis about the diffeomorphism symmetry of the theory without applying the on-shell condition. To apply the equations of motion, we need the Gibbons-Hawkings boundary term, which makes the Dirichlet condition of the general variation of the action definable and makes Hamiton's principle working in the Einstein theory so that the equations of motion are derived.

In Einstein gravity, the bulk term is removable by the property of Riemannian geometry itself. Let us expand the bulk term:

$$\begin{aligned} -\int \mathrm{d}^D x\, \sqrt{|G|} \mathcal{G}^{\mu\nu} \widehat{\mathcal{L}}_\xi G_{\mu\nu} &= -2 \int \mathrm{d}^D x\, \sqrt{|G|} \mathcal{G}^{\mu\nu} \nabla_\mu \xi_\nu \\ &= -2 \int \mathrm{d}^D x \left[\partial_\mu \left(\sqrt{|G|} \mathcal{G}^{\mu\nu} \xi_\nu \right) - \cancel{\sqrt{|G|} \underbrace{(\nabla_\mu \mathcal{G}^{\mu\nu})}_{\text{Bianchi identity}} \xi_\nu} \right]. \end{aligned} \tag{2.40}$$

The bulk term expands to the boundary term using the differential Bianchi identity $\nabla_\mu \mathcal{G}^{\mu\nu} = 0$ of Riemannian geometry rather than the equations of motion. Thus, the Noether current we are obtaining in this subsection is the off-shell Noether current.

Using (2.40), the diffeomorphism-variation of the action (2.39) becomes [41]

$$\delta_\xi I = \int \mathrm{d}^D x\, \partial_\mu \left(\sqrt{|G|}\, \underbrace{\left(-2\mathcal{G}^{\mu\nu}\xi_\nu + \Theta^\mu(G_{\alpha\beta}, \widehat{\mathcal{L}}_\xi G_{\alpha\beta}) - \xi^\mu R\right)}_{\text{Off-shell Noether current } J^\mu_{\text{off}}} \right), \tag{2.41}$$

which must be zero because of the diffeomorphism invariance of gravity. Thus, the off-shell Noether current of the Einstein gravity is

$$J^\mu_{\text{off}} = -2\mathcal{G}^{\mu\nu}\xi_\nu + \Theta^\mu(G_{\alpha\beta}, \widehat{\mathcal{L}}_\xi G_{\alpha\beta}) - \xi^\mu R. \tag{2.42}$$

Noether potential

A great feature of the off-shell conserved current is that the Noether potential can be defined. The off-shell Noether current is trivially conserved by the Bianchi identity, so the Noether potential $K^{\mu\nu}$ is defined as an anti-symmetric tensor *s.t.*

$$\sqrt{|G|}J^\mu =: \partial_\nu(\sqrt{|G|}K^{\mu\nu}), \tag{2.43}$$

or by the theorem that for an anti-symmetric tensor $F^{\mu\nu}$,

$$\partial_\mu(\sqrt{|G|}F^{\mu\nu}) = \sqrt{|G|}\nabla_\mu F^{\mu\nu} \tag{2.44}$$

in Riemannian geometry, the definition of the Noether potential can also be written by

$$J^\mu = \nabla_\nu K^{\mu\nu}. \tag{2.45}$$

You can easily show that the existence of the Noether potential means that the Noether current is trivially conserved: the conservation law $\partial_\mu(\sqrt{|G|}J^\mu) = 0$ is trivially satisfied by the index symmetric structure:

$$\partial_\mu(\sqrt{|G|}J^\mu = \partial_\mu\partial_\nu(\sqrt{|G|}K^{\mu\nu}) = \partial_{(\mu}\partial_{\nu)}(\sqrt{|G|}K^{[\mu\nu]}) = 0. \tag{2.46}$$

Let us find an off-shell Noether potential form by expanding and simplifying the Noether current (2.42). The boundary term in the general variation, $\Theta^\mu(G_{\alpha\beta}, \delta G_{\alpha\beta})$, can be explicitly written in $\delta G_{\alpha\beta}$ using the famous theorem in Riemannian geometry,

$$\delta\Gamma^\sigma{}_{\mu\nu} = \frac{1}{2}G^{\sigma\lambda}\left(\nabla_\mu\delta G_{\nu\lambda} + \nabla_\nu\delta G_{\lambda\mu} - \nabla_\lambda\delta G_{\mu\nu}\right), \tag{2.47}$$

and be simplified to

$$\Theta^\mu(G_{\alpha\beta}, \delta G_{\alpha\beta}) = G^{\mu\nu}\nabla^\rho\delta G_{\nu\rho} - G^{\rho\sigma}\nabla^\mu\delta G_{\rho\sigma}. \tag{2.48}$$

Using the explicit expression of Θ, the off-shell current (2.42) expands:

$$J^\mu_{\text{off}} = -2\mathcal{G}^{\mu\nu}\xi_\nu + \left(G^{\mu\nu}\nabla^\rho\widehat{\mathcal{L}}_\xi G_{\nu\rho} - G^{\rho\sigma}\nabla^\mu\widehat{\mathcal{L}}_\xi G_{\rho\sigma}\right) - \xi^\mu R \tag{2.49}$$

$$
\begin{aligned}
&= -2\mathcal{G}^{\mu\nu}\xi_\nu + \left(\nabla^\rho\nabla^\mu\xi_\rho + \nabla^\rho\nabla_\rho\xi^\mu - 2\nabla^\mu\nabla_\rho\xi^\rho\right) - \xi^\mu R \\
&= -2\mathcal{G}^{\mu\nu}\xi_\nu + \Big(2\nabla_\rho\nabla^{[\rho}\xi^{\mu]} + 2[\nabla^\rho, \nabla^\mu]\xi_\rho\Big) - \xi^\mu R \\
&= -2\mathcal{G}^{\mu\nu}\xi_\nu + 2\nabla_\rho\nabla^{[\rho}\xi^{\mu]} + 2\mathcal{G}^{\lambda\mu}\xi_\lambda \\
&= -2\nabla_\rho\nabla^{[\mu}\xi^{\rho]},
\end{aligned}
\tag{2.50}
$$

so the Noether potential of the Einstein-Hilbert action is defined by

$$
K^{\mu\nu} := -2\nabla^{[\mu}\xi^{\nu]}. \tag{2.51}
$$

This is called Komar form [31, pg 289 in 25], as Komar first found this form and claimed that the integral of this potential becomes the mass (conserved charge) of the gravitational background. The Noether charge is defined by the integration of the Noether current over the timeslice, which may be substituted by the integral of the Noether potential over the boundary of the timeslice:

$$
\begin{aligned}
\int_{\mathcal{M}} \mathrm{d}^{D-1}x_\mu \sqrt{|G|}J^\mu &= \int_{\mathcal{M}} \mathrm{d}^{D-1}x_\mu\, \partial_\nu(\sqrt{|G|}K^{\mu\nu}) \\
&= \oint_{\partial\mathcal{M}} \mathrm{d}^{D-2}x_{\mu\nu} \sqrt{|G|}K^{\mu\nu},
\end{aligned}
\tag{2.52}
$$

where $\mathcal{M}$ is the timeslice. This charge is called *Komar mass* and had been believed to be the energy formula of the gravitational background, but this conserved charge does not give the right coeffiecient to be the energy and occasionally gives the negative value[32]. To get the right notion of energy, we have to consider additional boundary term of the action, which is discussed in te next subsection §2.3.

Freedom of choice of Noether current and potential

You should be aware that there are freedom of choice of the off-shell conserved current:

$$\partial_\mu \left(\sqrt{|G|} J^\mu \right) = 0 \Rightarrow \partial_\mu \left(\sqrt{|G|} J^\mu + \alpha \mathcal{G}^{\mu\nu} \xi_\nu \right) = 0 \tag{2.53}$$

for any divergence-free $\mathcal{G}^{\mu\nu}\xi_\nu$ *s.t.* $\nabla_\mu(\mathcal{G}^{\mu\nu}\xi_\nu) = 0$, where α is an arbitrary coefficient. For $\mathcal{G}^{\mu\nu}$ *s.t.* $\nabla_\mu \mathcal{G}^{\mu\nu} = 0$ such as Einstein tensor, ξ^μ is required to be a Killing vector for the expression $\nabla_\mu(\mathcal{G}^{\mu\nu}\xi_\nu) = 0$ to be satisfied. Also, without the divergence-free property $\nabla_\mu(\mathcal{G}^{\mu\nu}\xi_\nu) = 0$, in the sense that $\mathcal{G}^{\mu\nu}$ is the equation of motion, the new current is an on-shell conserved current.[48]

Also, I may argue that there are additional freedom of choice of the Noether potential that gives the same Noether current.

$$\sqrt{|G|} J^\mu = \partial_\nu \left(\sqrt{|G|} K^{\mu\nu} \right) = \partial_\nu \left(\sqrt{|G|} K^{\mu\nu} + \alpha \mathcal{G}^{\mu\nu} \right) \tag{2.54}$$

for any divergence-free $\mathcal{G}^{\mu\nu}$ *s.t.* $\partial_\nu \mathcal{G}^{\mu\nu} = 0$. Apparently this new potential is not guaranteed to be anti-symmetric. Thus, here I have shown that the Noether potential is not necessarily anti-symmetric. But I will only refer to the canonical anti-symmetric definition for any future mention of the Noether potential in this thesis.

2.3 ADT charges

Up to previous section, we have discussed the pure Einstein theory without matter. In reality, matter fields exists and makes additional curvatures from the pure-Einstein metric background. To get the physical energy and momentum from the gravitational background, we have to derive the conserved charges that is equivalent to the conserved charges from the energy-momentum tensor of the matter part:

$$Q[\xi] \simeq \int_{\mathcal{M}} \mathrm{d}^{D-1}x_\mu \sqrt{|G|}\, T^\mu{}_\nu \xi^\nu, \tag{2.55}$$

where $\mathcal{M}$ is the timeslice.

The coupling between the gravitational field and the matter fields is realised by adding the matter part of the action and introducing the coupling constant κ^2 and the boundary action which well-defines the equation of motion:

$$I = \frac{1}{2\kappa^2} \int \mathrm{d}^D x \left(\sqrt{|G|}R - \partial_\mu(\sqrt{|G|}B^\mu)\right) + \int \mathrm{d}^D x\, \mathcal{L}_{\text{matter}}, \tag{2.56}$$

where the boundary term is defined by

$$\delta(\sqrt{|G|}B^\mu[G_{\alpha\beta}]) = \Theta(G_{\alpha\beta}, \delta G_{\alpha\beta}) \tag{2.57}$$

for the general variation. This coupled action derives the equations of motion:

$$\mathcal{G}^{\mu\nu} = \kappa^2 T^{\mu\nu}, \tag{2.58}$$

where the coupling constant is also called *the gravitational constant* $8\pi G_N = \kappa^2$, and the energy-momentum tensor of the matter is defined by

$$\begin{aligned} T_{\mu\nu} &:= -\frac{2}{\sqrt{|G|}}\frac{\delta\left(\int \mathrm{d}^D x\, \mathcal{L}_{\text{matter}}\right)}{\delta G^{\mu\nu}} \\ &= G_{\mu\alpha}G_{\nu\beta}\frac{2}{\sqrt{|G|}}\frac{\delta\left(\int \mathrm{d}^D x\, \mathcal{L}_{\text{matter}}\right)}{\delta G_{\alpha\beta}}. \end{aligned} \tag{2.59}$$

Thus, the gravitational conserverd charge equivalent to the energy-momentum may be defined by

$$Q[\xi] \simeq \frac{1}{\kappa^2}\int\limits_{\mathcal{M}} \mathrm{d}^{D-1}x_\mu\, \sqrt{|G|}\,\mathcal{G}^\mu{}_\nu \xi^\nu. \tag{2.60}$$

More precisely, considered the possibility of other terms that gives the extrinsic structure to the gravitational background, such as the cosmological constant, the ADT charge, which is the canonical charge of the gravitational theory equivalent to the energy-momentum, is defined by [43]

$$Q[\xi] \simeq \frac{1}{\kappa^2}\int\limits_{\mathcal{M}} \mathrm{d}^{D-1}x_\mu\, \delta(\sqrt{|G|}\mathcal{G}^\mu{}_\nu)\xi^\nu, \tag{2.61}$$

where $\delta\mathcal{G}^\mu{}_\nu$ here means the deviation of the Einstein tensor by the local concentration of matter from the extrinsic gravitational background.[6]

The ADT current is conserved only if the background has the symmetry

[6]For a background that has an extrinsic curvature, the ADT formula for the flat case would diverge, so we add a counter-term to cancel out the divergence, as discussed in §4.2.3.

to the associated transformation. The conservation law of the ADT current is written as

$$\begin{aligned}
\partial_\mu\left(\sqrt{|G|}J^\mu_{\mathrm{ADT}}\right) &= \partial_\mu\left(\delta\left(\sqrt{|G|}\mathcal{G}^\mu{}_\nu\right)\xi^\nu\right)\\
&= \partial_\mu\left(\left(\sqrt{|G'|}\,\mathcal{G}'^\mu{}_\nu - \sqrt{|\overline{G}|}\,\overline{\mathcal{G}}^\mu{}_\nu\right)\xi^\nu\right)\\
&= \sqrt{|G'|}\left((\nabla'_\mu\mathcal{G}'^\mu{}_\nu)\xi^\nu + \mathcal{G}'^\mu{}_\nu\nabla'_\mu\xi^\nu\right)\\
&\qquad - \sqrt{|\overline{G}|}\left((\overline{\nabla}_\mu\overline{\mathcal{G}}^\mu{}_\nu)\xi^\nu + \overline{\mathcal{G}}^\mu{}_\nu\overline{\nabla}_\mu\xi^\nu\right),
\end{aligned} \tag{2.62}$$

which should be zero, where $G'_{\alpha\beta}$ is the metric with the matter contribution while $\overline{G}_{\alpha\beta}$ is the background metric without the matter but only pure Einstein, and the Einstein tensor notations are also simplified as $\mathcal{G}'^\mu{}_\nu := \mathcal{G}^\mu{}_\nu[G'_{\alpha\beta}]$ and $\overline{\mathcal{G}}^\mu{}_\nu := \mathcal{G}^\mu{}_\nu[\overline{G}_{\alpha\beta}]$, and also the covariant derivative notations are also $\nabla'_\mu := \nabla_\mu[G'_{\alpha\beta}]$ and $\overline{\nabla}_\mu := \nabla_\mu[\overline{G}_{\alpha\beta}]$. The differential bianchi identity is satisfied in both the pure-Einstein background and the gravitational field with the matter case:

$$\overline{\nabla}_\mu\overline{\mathcal{G}}^{\mu\nu} = 0, \tag{2.63}$$

$$\nabla'_\mu\mathcal{G}'^{\mu\nu} = 0. \tag{2.64}$$

Thus the conservation law (2.62) is simplified as

$$\partial_\mu\left(\sqrt{|G|}J^\mu_{\mathrm{ADT}}\right) = \sqrt{|G'|}\mathcal{G}'^{\mu\nu}\,\nabla'_\mu(G'_{\nu\sigma}\xi_\sigma) - \sqrt{|\overline{G}|}\,\overline{\mathcal{G}}^{\mu\nu}\overline{\nabla}_\mu(\overline{G}_{\nu\sigma}\xi_\sigma) \tag{2.65}$$

and requires only *the Killing equations* to be satisfy in the $\overline{G}_{\alpha\beta}$ and $G'_{\alpha\beta}$:

$$\textbf{Killing equations:} \quad \widehat{\mathcal{L}}_\xi G_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)} = 0, \tag{2.66}$$

for the ADT current to be conserved. Satisfying the Killing equations means that the metric is invariant under the given transformation $x^\mu \to x^\mu + \xi^\mu$, and the infinitesimal transformation ξ^μ that satisfies the Killing equations is called a *Killing vector*.

2.3.1 Off-shell ADT charge

Normally conserved charges are defined to be conserved on-shell. However, here we want to construct the conserved charge formula using the Off-shell Noether potential formula. [43, 44] To accomplish this, we first want to derive the relation between the Noether current and the ADT current. From the explicit expression of off-shell Noether current (2.42), we take the variation of it:

$$\delta\left(\sqrt{|G|}J^\mu_{\text{off}}\right) = -2\delta\left(\mathcal{G}^{\mu\nu}G_{\nu\sigma}\xi^\sigma\right) + \delta\left(\sqrt{|G|}\Theta^\mu(\widehat{\mathcal{L}}_\xi)\right) - \xi^\mu\delta\left(\sqrt{|G|}R\right), \tag{2.67}$$

where we use the simplified notation $\Theta^\mu(\delta) := \Theta^\mu(G_{\alpha\beta}, \delta G_{\alpha\beta})$.

Using the variation of the Einstein-Hilbert action and the off-shell Noether current (2.42) with the potential form, the equation above is re-written into [7]

$$\delta\left(\sqrt{|G|}\mathcal{G}^{\mu\rho}G_{\rho\nu}\xi^{\nu}\right)-\frac{1}{2}(\delta G_{\rho\sigma})\mathcal{G}^{\rho\sigma}\xi^{\mu}$$
$$=\frac{1}{2}\partial_{\nu}\left[\delta\left(\sqrt{|G|}K^{\mu\nu}\right)+2\sqrt{|G|}\xi^{[\mu}\Theta^{\nu]}(\delta)\right]+\frac{1}{2}\sqrt{|G|}\Omega^{\mu}(\delta,\widehat{\mathcal{L}}_{\xi}), \qquad (2.68)$$

where the presymplectic form is define by

$$\sqrt{|G|}\,\Omega^{\mu}(\delta,\widehat{\mathcal{L}}_{\xi}):=\sqrt{|G|}\,\Omega^{\mu}(G_{\alpha\beta},\delta G_{\alpha\beta},\widehat{\mathcal{L}}_{\xi}G_{\alpha\beta})$$
$$:=\delta\left(\sqrt{|G|}\Theta^{\mu}(\widehat{\mathcal{L}}_{\xi})\right)-\widehat{\mathcal{L}}_{\xi}\left(\sqrt{|G|}\Theta^{\mu}(\delta)\right), \qquad (2.69)$$

which is zero for the Killing vector ξ^{μ}. [50] The left hand side of (2.68) is the off-shell ADT current defined in [41, 43, 44], which is equivalent to the on-shell ADT current (2.62) only under the on-shell condition.

Thus, the off-shell ADT charge may be defined by the integration of the off-shell ADT current:

$$Q_{\text{off-shell-ADT}}[\xi]:=\frac{1}{\kappa^2}\int_{\mathcal{M}}\mathrm{d}^{D-1}x_{\mu}\,\sqrt{|G|}J^{\mu}_{\text{off-shell-ADT}}[\xi]$$
$$=\frac{1}{\kappa^2}\int_{\mathcal{M}}\mathrm{d}^{D-1}x_{\mu}\left[\delta\left(\sqrt{|G|}\mathcal{G}^{\mu\rho}G_{\rho\nu}\xi^{\nu}\right)-\frac{1}{2}(\delta G_{\rho\sigma})\mathcal{G}^{\rho\sigma}\xi^{\mu}\right]$$
$$=\frac{1}{2\kappa^2}\oint_{\partial\mathcal{M}}\mathrm{d}^{D-2}x_{\mu\nu}\left[\delta\left(\sqrt{|G|}K^{\mu\nu}\right)+2\sqrt{|G|}\xi^{[\mu}\Theta^{\nu]}(\delta)\right], \qquad (2.70)$$

which is also written in the integration on the boundary of the timeslice

[7] See (2.42) and compare with [41, 43]. The sign of the Noether current and the equation of motion is opposite.

domain.

Note that not only the Noether current but also the ADT current also can be written in potential form, so the conserved charge is written in the boundary integral and only depends on the boundary (or asymptotic) behaviour of the metric.

Now, we can integrate the δ explicitly, given that the background is asymptotically flat. In the flat background, most things are trivially zero, so we can simply say

$$Q_{\text{off-shell-ADT}}[\xi] = \frac{1}{2\kappa^2} \oint_{\partial\mathcal{M}} \mathrm{d}^{D-2}x_{\mu\nu} \sqrt{|G|} \left[K^{\mu\nu} + 2\xi^{[\mu} B^{\nu]} \right], \tag{2.71}$$

where B^μ is the pseudo-vector boundary term *s.t.*

$$\delta B^\mu = \Theta^\mu(\delta) \tag{2.72}$$

for the general variation. It means that B^μ acts as a Gibbons-Hawking term *s.t.* the total action

$$\int \mathrm{d}^D x \left[\sqrt{|G|} R - \partial_\mu \left(\sqrt{|G|} B^\mu \right) \right] \tag{2.73}$$

has a well-defined variation under the Dirichlet condition. Thus, we can say that the Noether charge of total action[8] is equivalent to the off-shell ADT charge.

[8]the sum of the Einstein-Hilbert and the Gibbons-Hawking, or we could say the Γ^2 action.

For backgrounds having asymptotic behaviours other than the flat, the integration of the variations $\delta K^{\mu\nu}$ and $\Theta^{\mu}(\delta)$ must start from non-zero value. Especially, B^{μ} diverges with the non-zero extrinsic curvature. Thus, in this case, we may add compensation value to the charge formula (2.71).

One of the interesting findings in the ADT charge is that the boundary term only contributes to the linear momenta but has no contribution to the rotational charge[41]. Also, it is commonly known that the higher order symmetries other than linear translation and rotation is not considered for the conserved charge in the gravitational theory [33, 35]. [9] Also, this charge is called *quasi-local*: it must have the boundary as it has the boundary term, so we cannot define the local current alone, but we have to define the integral domain; we can define a small integral domain for certain local confinement to mimic the locality. [33, 41, 42, 45, 52–54]

[9] Considering that the ADT charge comes from the energy-momentum tensor, I think we can make the rotational charge $x^{[\mu}T^{\nu]\rho}$ but for no higher order of coordinate dependence or higher spin spherical harmonics. Also, Prof. Seok Kim argues that this is also relevant to the indivisual Virasoro operators not having its conserved charges.

2.4 Wald's formalism

2.4.1 Hamiltonian in presymplectic form

Wald *et al.* [46–48] showed that the ADT formula is also derivable in Hamiltonian formulation, using the presymplectic form (2.69). The "Hamiltonian" is defined in symplectic form [46, 50]:

$$\delta H[\xi] := \int_{\mathcal{M}} \mathrm{d}^{D-1}x_\mu \sqrt{|G|}\Omega^\mu(\delta, \widehat{\mathcal{L}}_\xi) \tag{2.74}$$

for general ξ^μ, where $\Omega^\mu(\delta, \widehat{\mathcal{L}}_\xi)$ is not trivially zero because the Killing equation is not demanded here.

We want to write the Hamiltonian using the Noether current. The standard on-shell Noether current is written by

$$\begin{aligned}\delta\left(\sqrt{|G|}J^\mu_{\mathrm{on}}[\xi]\right) &= \delta\left(\sqrt{|G|}\left(\Theta^\mu(\widehat{\mathcal{L}}_\xi) - \xi^\mu R\right)\right) \\ &= \sqrt{|G|}\Omega^\mu(\delta, \widehat{\mathcal{L}}_\xi) - 2\partial_\nu\left(\sqrt{|G|}\xi^{[\mu}\Theta^{\nu]}(\delta)\right) + \xi^\mu \mathcal{G}^{\sigma\lambda}\delta G_{\sigma\lambda},\end{aligned} \tag{2.75}$$

so the Hamiltonian is re-written in

$$\delta H[\xi] = \int_{\mathcal{M}} \mathrm{d}^{D-1}x_\mu \left(\sqrt{|G|}J^\mu_{\mathrm{on}}[\xi] + 2\partial_\nu\left(\sqrt{|G|}\xi^{[\mu}\Theta^{\nu]}(\delta)\right)\right) \tag{2.76}$$

under the on-shell condition $\mathcal{G}^{\sigma\lambda} = 0$. [10]

[10] As this expression transforms to the boundary integral later, the on-shell condition is satisfied as far as the boundary is kept away from the matter.

By the way, the on-shell Noether current and the off-shell Noether current is equivalent. The off-shell current (2.42) and the on-shell current differs by

$$J^{\mu}_{\text{off}} = J^{\mu}_{\text{on}} - 2\mathcal{G}^{\mu\nu}\xi_{\nu}. \tag{2.77}$$

As far as the Killing equation is satisfied:

$$\nabla_{\mu}\xi_{\nu} + \nabla_{\nu}\xi_{\mu} = 0 \quad \Rightarrow \quad \nabla_{\mu}(\mathcal{G}^{\mu\nu}\xi_{\nu}) = 0, \tag{2.78}$$

The on-shell current and the off-shell current is equivalent off-shell level as we discussed in (2.53).

Thus, the Hamiltonian (2.76) can be re-written in the off-shell current, or potential:

$$\delta H[\xi] = \oint_{\partial\mathcal{M}} \mathrm{d}^{D-2}x_{\mu\nu} \left[\delta\left(\sqrt{|G|}K^{\mu\nu}\right) + 2\sqrt{|G|}\xi^{[\mu}\Theta^{\nu]}(\delta)\right], \tag{2.79}$$

which is equivalent to the ADT charge for Killing vector ξ^{μ}.

2.4.2 Black Hole entropy

Wald *et al.* [46–48] applied this Hamiltonian formalism to the black hole background, implementing the boundary term also to the horizon as an internal boundary. The charge formula gives the relation between charges like energy and angular momenta, the laws of the black hole mechanics [49]. In that study, they showed that the entropy of the black hole is written in form of Noether charge.

Chapter 3 Double Field Theory

The goal of our study is to derive the $\mathbf{O}(D,D)$-covariant conserved charge formula in Double Field Theory basis. Double Field Theory is the T-duality-covariant gravitational field theory with curvatures defined in the new geometry called *Double geometry*[59, 67, 69, 71, 74, 75, 79, 80]. For the study similar to the discussion in Chapter 2 to be developed in Double Field Theory, we should understand what is Double Field Theory, how DFT develops, what is the Double geometry, and how different it is from the Riemannian geometry.

In this chapter, we review the formulation of Double Field Theory [56, 60–63]. Especially, in Double Field Theory, the Christoffel connection cannot be fully fixed because of its abundance of degrees of freedom. In this chapter, we review the semi-covariant formulation of Double Field Theory[69, 71, 74, 75], which separates the determinable and undeterminable components of connections in $\mathbf{O}(D,D)$-covariant way, and fixes the undeterminable components to zero.

For detailed review of Double Field Theory, see the summary paper by Olaf Hohm and Barton Zwiebach [65] or the paper by Gerardo Aldazabal, Diego Marqués, and Carmen Núñez [57]. Our publication [1] and most other papers by Jeong-Hyuck Park *et al.* also review semi-covariant formulation of Double Field Theory.[69, 71, 74, 75]

3.1 Double-yet-gauged Spacetime

We begin with self-contained review of the semi-covariant formulation of the bosonic DFT for the NS-NS sector [69, 71] and also the Yang-Mills sector [70, 77]. They constitute the massless modes of string theory at leading order in string coupling perturbation theory. For further extensions beyond the leading order, we refer readers to [82] for fermions, [84] for the R-R sector, and [72, 73, 76] for the (gauged) maximal and half-maximal supersymmetric completions.[1]

Doubled Spacetime and Invariant Metric

The DFT is defined over the doubled, $(D+D)$-dimensional spacetime. Denote the $\mathbf{O}(D,D)$ vector indices by capital Latin letters, $A, B, C, \cdots = 1, 2, \cdots, D{+}D$. There exists a unique $\mathbf{O}(D,D)$ invariant constant metric,

$$\mathcal{J}_{AB} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} . \tag{3.1}$$

Using this invariant metric, we freely raise and lower the $\mathbf{O}(D,D)$ tensor indices.

Gauge Equivalence

The actual physics is realized in D-dimensional subspace. As the DFT starts with doubled $(D+D)$-dimensional spacetime, this doubled spacetime must be

[1] In particular, thanks to the twofold spin groups, *i.e.* $\mathbf{Spin}(1,9) \times \mathbf{Spin}(9,1)$, the distinction between IIA and IIB disappears [73], and the maximal $D=10$ supersymmetric double field theory *unifies* IIA and IIB supergravities.

projected appropriately. We do this by imposing the property that the doubled coordinate system satisfies *local equivalence relations* [74, 75],

$$x^A \;\simeq\; x^A + \phi(x)\partial^A\varphi(x)\,, \tag{3.2}$$

which was termed as 'coordinate gauge symmetry'. In (3.2), $\phi(x)$ and $\varphi(x)$ are arbitrary smooth functions in DFT. Each equivalence class or each gauge orbit defined by the equivalence relation (3.2) represents a single physical point, and diffeomorphism invariance refers to a symmetry under arbitrary reparametrizations of the gauge orbits.

The equivalence relation (3.2) is realized in DFT by enforcing that arbitrary functions and their arbitrary derivatives, denoted here collectively by Φ, are invariant under the coordinate gauge transformations *shift*,

$$\Phi(x+\Delta) = \Phi(x)\,, \qquad \Delta^A = \phi\partial^A\varphi\,. \tag{3.3}$$

The coordinate gauge symmetry can be also realized as a local Noether symmetry on a string worldsheet [75].

Section Condition

The symmetry under the coordinate gauge transformation (3.3) is equivalent (*i.e.* necessary [74] and sufficient [75]) to the *section condition* [56],

$$\partial_A\partial^A = 0\,. \tag{3.4}$$

Acting on arbitrary functions, Φ, Φ', and their products, the section condition leads to

$$\partial_A \partial^A \Phi = 0 \quad \text{(weak constraint)}\,, \tag{3.5}$$

$$\partial_A \Phi \partial^A \Phi' = 0 \quad \text{(strong constraint)}\,. \tag{3.6}$$

Generalized Lie Derivatives

Diffeomorphism transformation in the doubled-yet-gauged coordinate system is generated by a *generalised Lie derivative* [58, 66, 101]. Acting on n-indexed field, it is defined by

$$\begin{aligned}\widehat{\mathcal{L}}_X T_{A_1\cdots A_n} := {}& X^B \partial_B T_{A_1\cdots A_n} + \omega_T\, \partial_B X^B T_{A_1\cdots A_n} \\ & + \sum_{i=1}^{n} (\partial_{A_i} X_B - \partial_B X_{A_i}) T_{A_1\cdots A_{i-1}}{}^{B}{}_{A_{i+1}\cdots A_n}\,.\end{aligned} \tag{3.7}$$

Here, ω_T denotes the weight of the T field. In particular, the generalized Lie derivative of the $\mathbf{O}(D,D)$ invariant metric is trivial,

$$\widehat{\mathcal{L}}_X \mathcal{J}_{AB} = 0\,. \tag{3.8}$$

The commutator of the generalized Lie derivatives is closed by the C-bracket [58, 61],

$$\left[\widehat{\mathcal{L}}_X, \widehat{\mathcal{L}}_Y\right] = \widehat{\mathcal{L}}_{[X,Y]_{\mathrm{C}}}\,, \tag{3.9}$$

$$[X,Y]^A_{\mathrm{C}} = X^B \partial_B Y^A - Y^B \partial_B X^A + \frac{1}{2} Y^B \partial^A X_B - \frac{1}{2} X^B \partial^A Y_B\,. \tag{3.10}$$

In the NS-NS sector, dynamical contents of the DFT consist of the dilaton, $d(x)$, and a pair of the projection fields $P_{AB}, \bar{P}_{AB}$, obeying the properties

$$P_{AB} = P_{BA}\,, \quad \bar{P}_{AB} = \bar{P}_{BA}\,, \quad P_A{}^B P_B{}^C = P_A{}^C\,, \quad \bar{P}_A{}^B \bar{P}_B{}^C = \bar{P}_A{}^C\,. \quad (3.11)$$

Further, the projection fields are orthogonal and complementary:

$$P_A{}^B \bar{P}_B{}^C = 0\,, \quad \bar{P}_A{}^B P_B{}^C = 0\,, \quad P_{AB} + \bar{P}_{AB} = \mathcal{J}_{AB}\,. \qquad (3.12)$$

The two projection fields are not independent, since

$$P_A{}^B + \bar{P}_A{}^B = \mathcal{J}_A{}^B. \qquad (3.13)$$

The dynamical contents are contained (in addition to the dilaton) in the difference of the projection fields

$$P_{AB} - \bar{P}_{AB} = \mathcal{H}_{AB}. \qquad (3.14)$$

This is the well-known *generalized metric* [56], which can be also independently defined as a symmetric $\mathbf{O}(D,D)$ element having the properties

$$\mathcal{H}_{AB} = \mathcal{H}_{BA} \quad \text{and} \quad \mathcal{H}_A{}^B \mathcal{H}_B{}^C = \delta_A^{\ C}. \qquad (3.15)$$

The projection fields and dilaton are naturally in the string frame. To facilitate the $\mathbf{O}(D,D)$ invariant integral calculus, we assign the scaling weight of these fields as

$$\omega[P] = \omega[\bar{P}] = 0\,, \qquad \omega[e^{-2d}] = 1\,. \tag{3.16}$$

3.2 Semi-covariant Formulation of Double Field Theory

Semi-covariant derivative

The central construction of the DFT starts with the semi-covariant derivative, defined by [69, 71]

$$\begin{aligned}\nabla_C T_{A_1 A_2 \cdots A_n} := \partial_C T_{A_1 A_2 \cdots A_n} - \omega_T \Gamma^B{}_{BC} T_{A_1 A_2 \cdots A_n} \\ + \sum_{i=1}^{n} \Gamma_{CA_i}{}^B T_{A_1 \cdots A_{i-1} B A_{i+1} \cdots A_n}\,,\end{aligned} \tag{3.17}$$

The connection is defined by [71]:[2]

$$\begin{aligned}\Gamma_{CAB} = 2\left(P\partial_C P\bar{P}\right)_{[AB]} + 2\left(\bar{P}_{[A}{}^D \bar{P}_{B]}{}^E - P_{[A}{}^D P_{B]}{}^E\right)\partial_D P_{EC} \\ - \tfrac{4}{D-1}\left(\bar{P}_{C[A}\bar{P}_{B]}{}^D + P_{C[A}P_{B]}{}^D\right)\left(\partial_D d + (P\partial^E P\bar{P})_{[ED]}\right)\,,\end{aligned} \tag{3.18}$$

Below, we shall elaborate the uniqueness of this connection. The semi-covariant derivative obeys the Leibniz rule and annihilates the $\mathbf{O}(D, D)$ invariant constant metric:

$$\nabla_A \mathcal{J}_{BC} = 0\,. \tag{3.19}$$

Unlike the Levi-Civita connection in the Riemannian Einstein gravity, the diffeomorphism transformation (3.7) cannot put the connection (3.18) to

[2] In this review of the *bosonic* DFT, we focus on the above 'torsionless' connection (3.18). Yet, in *supersymmetric* DFT, it is necessary to include torsions in order to ensure the '1.5 formalism' [72, 73, 84].

vanish pointwise in doubled spacetime. One may view this as failure of the equivalence principle. This is not surprising since it is know that in string theory the equivalence principle no longer holds due to the Kalb-Ramond field and the dilaton field.

Semi-Covariant Curvatures

The semi-covariant Riemann curvature calculates the field strength of the connection (3.18):

$$S_{ABCD} := \frac{1}{2}\left(R_{ABCD} + R_{CDAB} - \Gamma^{E}{}_{AB}\Gamma_{ECD}\right) . \tag{3.20}$$

Here, R_{ABCD} denotes the ordinary Riemann curvature associated with the connection:

$$R_{CDAB} = \partial_A \Gamma_{BCD} - \partial_B \Gamma_{ACD} + \Gamma_{AC}{}^{E}\Gamma_{BED} - \Gamma_{BC}{}^{E}\Gamma_{AED} . \tag{3.21}$$

A crucial defining property of the semi-covariant Riemann curvature is that, under arbitrary variation of the connection (3.18), its variation takes the form *total derivative* [71],

$$\delta S_{ABCD} = \nabla_{[A}\delta\Gamma_{B]CD} + \nabla_{[C}\delta\Gamma_{D]AB} . \tag{3.22}$$

Further, the semi-covariant Riemann curvature satisfies precisely the same symmetric properties as the ordinary Riemann curvature, including the

Bianchi identity,

$$S_{ABCD} = S_{[AB][CD]} = S_{CDAB}\,, \qquad S_{[ABC]D} = 0\,. \tag{3.23}$$

In addition, when projected by $P, \bar{P}$, the semi-covariant Riemann curvature obeys identities [71],

$$P_I{}^A P_J{}^B \bar{P}_K{}^C \bar{P}_L{}^D S_{ABCD} = 0\,, \qquad P_I{}^A \bar{P}_J{}^B P_K{}^C \bar{P}_L{}^D S_{ABCD} = 0\,, \tag{3.24}$$

$$(P^{AB}P^{CD} + \bar{P}^{AB}\bar{P}^{CD})S_{ACBD} = 0\,, \tag{3.25}$$

$$P_I{}^A \bar{P}_J{}^C P^{BD} S_{ABCD} = P_I{}^A \bar{P}_J{}^C \bar{P}^{BD} S_{ABCD} = \frac{1}{2} P_I{}^A \bar{P}_J{}^C S_{AC}\,. \tag{3.26}$$

As in the Riemannian case, we also define the semi-covariant Ricci curvature as the trace part of the semi-covariant Riemann curvature,

$$S_{AC} := S_{ABCD}\mathcal{J}^{BD} = S_{ABC}{}^B, \tag{3.27}$$

However, unlike the Riemannian case, the above Bianchi identities imply that the Ricci curvature is traceless

$$S_A{}^A = S^{AB}{}_{AB} = 0\,. \tag{3.28}$$

Uniqueness of Connection

The alluded connection (3.18) turns out to be *the unique solution* to the following five requirements [71]:

$$\nabla_A P_{BC} = 0\,, \qquad \nabla_A \bar{P}_{BC} = 0\,, \tag{3.29}$$

$$\nabla_A d = -\frac{1}{2}e^{2d}\nabla_A(e^{-2d}) = \partial_A d + \frac{1}{2}\Gamma^B{}_{BA} = 0\,, \tag{3.30}$$

$$\Gamma_{ABC} + \Gamma_{ACB} = 0\,, \tag{3.31}$$

$$\Gamma_{ABC} + \Gamma_{BCA} + \Gamma_{CAB} = 0\,, \tag{3.32}$$

$$\mathcal{P}_{ABC}{}^{DEF}\Gamma_{DEF} = 0\,, \qquad \bar{\mathcal{P}}_{ABC}{}^{DEF}\Gamma_{DEF} = 0\,. \tag{3.33}$$

The first two relations, (3.29), (3.30), are the compatibility conditions with all the geometric objects –or the NS-NS sector– in DFT. The third constraint (3.31) is the compatibility condition with the $\mathbf{O}(D,D)$ invariant constant metric, (3.19), which is also consistent with (3.12) and (3.29). The next cyclic property, (3.32), makes the semi-covariant derivative compatible with the generalized Lie derivative as well as with the C-bracket, [61, 64, 81]

$$\widehat{\mathcal{L}}_X(\partial) = \widehat{\mathcal{L}}_X(\nabla)\,, \qquad [X,Y]_{\mathrm{C}}(\partial) = [X,Y]_{\mathrm{C}}(\nabla)\,. \tag{3.34}$$

The last conditions (3.33) assert that the connection belongs to the kernel of the triple-projection fields $\mathcal{P}_{ABC}{}^{DEF}, \bar{\mathcal{P}}_{ABC}{}^{DEF}$. They are properties of the connection (3.18) which completes to ensure the uniqueness.

Triple-Projection Fields

The triple-projection fields carrying six indices,$\mathcal{P}_{ABC}{}^{DEF}$, $\bar{\mathcal{P}}_{ABC}{}^{DEF}$, used in (3.33), are explicitly given by

$$\mathcal{P}_{CAB}{}^{DEF} := P_C{}^D P_{[A}{}^{[E} P_{B]}{}^{F]} + \tfrac{2}{D-1} P_{C[A} P_{B]}{}^{[E} P^{F]D}\,, \tag{3.35}$$

$$\bar{\mathcal{P}}_{CAB}{}^{DEF} := \bar{P}_C{}^D \bar{P}_{[A}{}^{[E} \bar{P}_{B]}{}^{F]} + \tfrac{2}{D-1} \bar{P}_{C[A} \bar{P}_{B]}{}^{[E} \bar{P}^{F]D}\,, \tag{3.36}$$

which satisfy the 'projection' properties,

$$\mathcal{P}_{ABC}{}^{DEF} \mathcal{P}_{DEF}{}^{GHI} = \mathcal{P}_{ABC}{}^{GHI}\,, \qquad \bar{\mathcal{P}}_{ABC}{}^{DEF} \bar{\mathcal{P}}_{DEF}{}^{GHI} = \bar{\mathcal{P}}_{ABC}{}^{GHI}\,. \tag{3.37}$$

They are symmetric and traceless,

$$\mathcal{P}_{ABCDEF} = \mathcal{P}_{DEFABC}\,, \qquad \bar{\mathcal{P}}_{ABCDEF} = \bar{\mathcal{P}}_{DEFABC}\,, \tag{3.38}$$

$$\mathcal{P}_{ABCDEF} = \mathcal{P}_{A[BC]D[EF]}\,, \qquad \bar{\mathcal{P}}_{ABCDEF} = \bar{\mathcal{P}}_{A[BC]D[EF]}\,, \tag{3.39}$$

$$P^{AB} \mathcal{P}_{ABCDEF} = 0\,, \qquad \bar{P}^{AB} \bar{\mathcal{P}}_{ABCDEF} = 0\,. \tag{3.40}$$

The triple-projection fields describe anomalous part of the semi-covariant derivative and the semi-covariant Riemann curvature under the generalized diffeomorphism transformations. From

$$(\delta_X - \hat{\mathcal{L}}_X)\Gamma_{CAB} = 2\left[(\mathcal{P} + \bar{\mathcal{P}})_{CAB}{}^{FDE} - \delta_C^F \delta_A^D \delta_B^E\right] \partial_F \partial_{[D} X_{E]}\,, \tag{3.41}$$

it is straightforward to see that the generalized diffeomorphism anomalies are all given by the triple-projection fields,

$$(\delta_X-\widehat{\mathcal{L}}_X)\nabla_C T_{A_1\cdots A_n}=\sum_{i=1}^{n}2(\mathcal{P}+\bar{\mathcal{P}})_{CA_i}{}^{BDEF}\partial_D\partial_E X_F\,T_{A_1\cdots A_{i-1}BA_{i+1}\cdots A_n}\,, \tag{3.42}$$

$$\begin{aligned}(\delta_X-\widehat{\mathcal{L}}_X)S_{ABCD}=&\,2\nabla_{[A}\Big((\mathcal{P}+\bar{\mathcal{P}})_{B][CD]}{}^{EFG}\partial_E\partial_F X_G\Big)\\&+2\nabla_{[C}\Big((\mathcal{P}+\bar{\mathcal{P}})_{D][AB]}{}^{EFG}\partial_E\partial_F X_G\Big).\end{aligned} \tag{3.43}$$

Hence, one can easily project out the anomalies through appropriate contractions with the triple-projection fields.

Full Covariance

The fully covariant derivatives are obtainable by further projections,

$$\begin{array}{rl}\textbf{Gradients:} & P_C{}^D\bar{P}_{A_1}{}^{B_1}\cdots\bar{P}_{A_n}{}^{B_n}\nabla_D T_{B_1\cdots B_n}\,,\\ & \bar{P}_C{}^D P_{A_1}{}^{B_1}\cdots P_{A_n}{}^{B_n}\nabla_D T_{B_1\cdots B_n}\,,\end{array} \tag{3.44}$$

$$\begin{array}{rl}\textbf{Divergences:} & P^{AB}\bar{P}_{C_1}{}^{D_1}\cdots\bar{P}_{C_n}{}^{D_n}\nabla_A T_{BD_1\cdots D_n}\,,\\ & \bar{P}^{AB}P_{C_1}{}^{D_1}\cdots P_{C_n}{}^{D_n}\nabla_A T_{BD_1\cdots D_n}\,,\end{array} \tag{3.45}$$

$$\begin{array}{rl}\textbf{Laplacians:} & P^{AB}\bar{P}_{C_1}{}^{D_1}\cdots\bar{P}_{C_n}{}^{D_n}\nabla_A\nabla_B T_{D_1\cdots D_n}\,,\\ & \bar{P}^{AB}P_{C_1}{}^{D_1}\cdots P_{C_n}{}^{D_n}\nabla_A\nabla_B T_{D_1\cdots D_n}\,.\end{array} \tag{3.46}$$

Correspondingly, fully covariant Ricci curvature and fully covariant curvature scalar are [3]

$$\textbf{Ricci curvature: } \mathcal{S}_{AB} := P_A{}^C \bar{P}_B{}^D S_{CED}{}^E = P_A{}^C \bar{P}_B{}^D S_{CD} = (PS\bar{P})_{AB}\,, \tag{3.47}$$

$$\textbf{scalar curvature:} \quad \mathcal{S} := (P^{AC}P^{BD} - \bar{P}^{AC}\bar{P}^{BD})S_{ABCD}\,. \tag{3.48}$$

Because of the triviality of the semi-covariant scalar curvature, $S_A{}^A = S_{AB}{}^{AB} = 0$ (3.28), hereafter we use the calligraphic font to denote the nontrivial Ricci curvature and scalar curvature, '$\mathcal{S}_{AB},\ \mathcal{S}$'.

A remark is in order at this point. As an alternative to the semi-covariant approach described above, from the outset, one may wish to postulate a "perfectly well-behaving" connection, say $\widetilde{\Gamma}_{CAB}$, such that it would transform as $(\delta_X - \widehat{\mathcal{L}}_X)\widetilde{\Gamma}_{CAB} = -2\partial_C\partial_{[A}X_{B]}$ instead of (3.41) and hence there would be no diffeomorphism anomalies like (3.43). Yet, this is most likely not true in generic DFT, since such a "perfect" connection cannot always be constructed solely out of the NS-NS sector. It requires extra unphysical degrees of freedom or "undetermined" part [65]. After projecting them out, the final results would be reduced to the semi-covariant formalism.

[3] For the torsionless connection (3.18), the following identities hold as for the completely covariant scalar curvature,

$$P^{AB}P^{CD}S_{ACBD} = P^{AB}S_{ACB}{}^C = -\bar{P}^{AB}\bar{P}^{CD}S_{ACBD} = -\bar{P}^{AB}S_{ACB}{}^C\,.$$

However, it is the expression in (3.48) that ensures the '1.5 formalism' in the supersymmetric DFTs with torsions [72, 73].

3.3 DFT other than NSNS sector: Yang-Mills

We now extend the semi-covariant DFT formulation to YM sector. For a given Lie algebra-valued vector potential, V_A, the semi-covariant YM field strength is defined by [70, 77]

$$\mathcal{F}_{AB} := \nabla_A V_B - \nabla_B V_A - i[V_A, V_B] \,. \tag{3.49}$$

Unlike the Riemannian torsionless case, the connections above are not cancelled but yields non-derivative and non-commutator contribution.

As seen from the generic formula (3.43), the semi-covariant YM field strength is not completely covariant but rather semi-covariant under diffeomorphisms. Again, the anomalous part is parametrized by the triple-projection fields,

$$(\delta_X - \widehat{\mathcal{L}}_X)\mathcal{F}_{AB} = -2(\mathcal{P} + \bar{\mathcal{P}})^C{}_{AB}{}^{DEF}\partial_D\partial_{[E}X_{F]}V_C \,. \tag{3.50}$$

Thus, following the general prescription (3.46), the completely covariant YM field strength is

$$(P\mathcal{F}\bar{P})_{AB} = -(\bar{P}\mathcal{F}P)_{BA} = P_A{}^C\bar{P}_B{}^D\mathcal{F}_{CD} \,. \tag{3.51}$$

The YM gauge transformation is realized by the action

$$V_A \longrightarrow \mathbf{g}V_A\mathbf{g}^{-1} - i(\partial_A\mathbf{g})\mathbf{g}^{-1}\,, \tag{3.52}$$

$$\mathcal{F}_{AB} \longrightarrow \mathbf{g}\mathcal{F}_{AB}\mathbf{g}^{-1} + i\Gamma^{C}{}_{AB}(\partial_C\mathbf{g})\mathbf{g}^{-1}\,, \tag{3.53}$$

$$(P\mathcal{F}\bar{P})_{AB} \longrightarrow \mathbf{g}(P\mathcal{F}\bar{P})_{AB}\mathbf{g}^{-1}\,. \tag{3.54}$$

One finds that a two-derivative scalar fully invariant with respect to both the diffeomorphism and the YM gauge transformations is

$$\mathrm{Tr}\left[(P\mathcal{F}\bar{P})_{AB}(P\mathcal{F}\bar{P})^{AB}\right] = \mathrm{Tr}\left[P^{AC}\bar{P}^{BD}\mathcal{F}_{AB}\mathcal{F}_{CD}\right]\,. \tag{3.55}$$

Clearly, there appear doubled off-shell degrees of freedom in the $(D{+}D)$-component gauge potential. In order to halve them, if wanted, we may impose the ‘gauged section condition’ [77]:

$$(\partial_A - iV_A)(\partial^A - iV^A) = 0\,, \tag{3.56}$$

which, along with the original section condition (3.4), implies $V_A\partial^A{=}\,0$, $\partial_A V^A{=}\,0$, $V_A V^A{=}\,0$. Here, it is implicit that the connections are in an irreducible representation of the fields that the covariant derivative acts on.

For consistency, the condition (3.56) is preserved under all the symmetry transformations: $\mathbf{O}(D,D)$ rotations, diffeomorphisms (3.7) and the Yang-Mills gauge symmetry (3.54).

3.4 DFT Action

We can construct the DFT action I_{DFT} for the NS-NS sector coupled to YM sector and cosmological constant as

$$I_{\mathrm{DFT}} = \int_{\Sigma_D} \mathcal{L}_{\mathrm{DFT}}\,, \qquad \mathcal{L}_{\mathrm{DFT}} = \mathcal{L}_{\mathrm{NSNS}} + \mathcal{L}_{\mathrm{YM}} - 2\Lambda e^{-2d}\,, \tag{3.57}$$

where the integral is taken over a D-dimensional section or their 'manifold-like' patch, Σ_D. Here, Λ denotes the DFT-cosmological constant [71]. The fully invariant Lagrangian densities are given for each sector by

$$\mathcal{L}_{\mathrm{NSNS}} = e^{-2d}(P^{AC}P^{BD} - \bar{P}^{AC}\bar{P}^{BD})S_{ABCD} = e^{-2d}\mathcal{S}\,, \tag{3.58}$$

$$\mathcal{L}_{\mathrm{YM}} = g_{\mathrm{YM}}^{-2} e^{-2d}\mathrm{Tr}\left[P^{AC}\bar{P}^{BD}\mathcal{F}_{AB}\mathcal{F}_{CD}\right]\,. \tag{3.59}$$

Section

At the outset, one needs to impose appropriate section condition. Up to $\mathbf{O}(D,D)$ duality rotations, the solution to the section condition is locally unique. It is a D-dimensional *section*, Σ_D, characterized by the independence of the dual "winding" coordinates,

$$\frac{\partial}{\partial \tilde{x}_\mu} \equiv 0\,. \tag{3.60}$$

Here, the Greek letters are D-dimensional indices on the section Σ_D. In this foliation, the whole doubled coordinates are given by

$$x^A = (\tilde{x}_\mu, x^\nu)\,. \tag{3.61}$$

To perform the 'Riemannian' reduction onto the D-dimensional section, Σ_D, one only needs to parametrize the projection fields and the dilaton in terms of a D-dimensional Riemannian metric, $G_{\mu\nu}$, an ordinary dilaton, Φ, and a Kalb-Ramond two-form potential, $B_{\mu\nu}$ [56],

$$\mathcal{H}_{AB} := P_{AB} - \bar{P}_{AB} = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix}\,, \qquad e^{-2d} = \sqrt{|G|}e^{-2\Phi}\,. \tag{3.62}$$

The DFT scalar curvature (3.48) reduces upon the section condition to

$$\mathcal{S}|_{\Sigma_D} = R_G + 4\Delta\Phi - 4\partial_\mu\Phi\partial^\mu\Phi - \tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu}\,, \tag{3.63}$$

where *the Kalb-Ramond field strength* $H_{\lambda\mu\nu} := 3\partial_{[\lambda}B_{\mu\nu]}$.

Up to field redefinitions, Eq.(3.62) is the most general parametrization of the "generalized metric", $\mathcal{H}_{AB} = P_{AB} - \bar{P}_{AB}$, of which the upper left $D \times D$ block is non-degenerate.

Yang-Mills

The parametrization of the doubled YM vector potential reads from [70],

$$V_A = \begin{pmatrix} \varphi^\lambda \\ A_\mu + B_{\mu\nu}\varphi^\nu \end{pmatrix}, \tag{3.64}$$

of which the D-dimensional vector, φ^λ, which is in the YM adjoint representation can be put trivial upon the 'gauged section condition' (3.56) [77]. For the consequent expression of the completely covariant YM field strength in terms of φ^λ and A_μ, we refer readers to (3.19) and (3.21) of [70].

Non-Riemannian Background

When the upper left $(D \times D)$ block of the generalized metric is degenerate –where G^{-1} is positioned in (3.62)– the Riemannian metric ceases to exist upon the section, Σ_D (3.60). Nevertheless, the $\mathbf{O}(D,D)$ DFT and a doubled sigma model [75] have no problem with describing such a non-Riemannian background, as long as the generalized metric is a symmetric $\mathbf{O}(D,D)$ element, satisfying $\mathcal{H}_{AB} = \mathcal{H}_{BA}$ and $\mathcal{H}_A{}^B\mathcal{H}_B{}^C = \delta_A^{\ C}$. We refer readers to [75] for a concrete example (see also a math literature [78]).

Chapter 4 Canonical Formulation of Conserved Quantities in Double Field Theory

In the previous chapters, we have reviewed the formulation of DFT itself and the ADT charge formulation in the conventional gravity theory. Using those background knowledge, we formulate the ADT charge of DFT in this chapter.

Remind the ADT formalism briefly here. Remind that in the Noether theorem, the Noether charge of matter fields associated with spacetime symmetry is written as

$$Q[\xi] = \int_{\mathcal{M}} \mathrm{d}^{D-1}x_{\mu}\sqrt{-G}\, T^{\mu}{}_{\nu}\, \xi^{\nu}, \tag{4.1}$$

where $\mathcal{M}$ is the timeslice, and $\sqrt{-G}$ is the volume element of the D-dimensional domain, and $T_{\mu\nu}$ is the energy-momenum tensor. The energy-momentum tensor that appears in the Noether theorem can be symmetrised so can be connected to the energy-momentum tensor in the gravity theory. Then, by using the equations of motion of the Einstein theory

$\mathcal{G}_{\mu\nu} = 8\pi G_N T_{\mu\nu}$, the Noether charge equivalently can be written as:

$$Q[\xi] = \frac{1}{\kappa^2} \int_{\mathcal{M}} \mathrm{d}^{D-1}x_\mu \, \sqrt{-G}\, \mathcal{G}^\mu{}_\nu \, \xi^\nu, \tag{4.2}$$

where $\mathcal{G}_{\mu\nu} := R_{\mu\nu}[G] - \frac{1}{2}R[G]G_{\mu\nu}$ is the traceless Einstein curvature tensor, and $G_{\mu\nu}$ is the bulk metric, and $\kappa^2 := 8\pi G_N$ is the gravitational constant.

Now, we show that this charge is the same as the charge integrated from the off-shell Noether current of the action consisting of the Einstein-Hilbert action and a Gibbons-Hawking-like action [30, 39–41]. Here, "Gibbons-Hawking-like" means that the variation of the total action must become

$$\delta I = \frac{1}{2\kappa^2} \int_{\mathcal{D}} \mathrm{d}^D x \, \sqrt{-G}\, \mathcal{G}_{\mu\nu} \, \delta G^{\mu\nu}, \tag{4.3}$$

where $\mathcal{D}$ is the D-dimensional spacetime domain, and $G^{\mu\nu}$ is the inverse of the D-dimensional metric. $\delta G_{\mu\nu}$ under a spacetime tranformation $x^\mu \to x^\mu + \xi^\mu$ is defined by the Lie derivative:

$$\delta G_{\mu\nu} := 2\nabla_{(\mu}\xi_{\nu)} := \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \tag{4.4}$$

Thus, the variation of the action under the spacetime tranformation becomes:

$$\delta_\xi I = \frac{1}{\kappa^2} \int_{\mathcal{D}} \mathrm{d}^D x \, \sqrt{-G}\, \mathcal{G}_{\mu\nu} \nabla^\mu \xi^\nu \tag{4.5}$$

$$= \frac{1}{\kappa^2} \int_{\mathcal{D}} \mathrm{d}^D x \left\{ \partial_\mu \underbrace{\left(\sqrt{-G}\, \mathcal{G}^\mu{}_\nu \xi^\nu \right)}_{\sqrt{-G} J^\mu_{\mathrm{ADT(off\text{-}shell)}}} - \cancelto{0}{\sqrt{-G}\, \xi^\nu \nabla_\mu \mathcal{G}^\mu{}_\nu} \right\}, \tag{4.6}$$

where the last line is derived by integral by part. The latter term of the last line becomes zero by the differential Bianchi identity. As we can write the varation into a total derivative, we can derive an off-shell current $J^{\mu}_{\text{ADT(off-shell)}}$, and the charges from this current is the same with (4.2), so we can say that the charges from the off-shell current of the gravity action combined with its Gibbons-Hawking term applied to a gravitational background is equivalent to the momenta of the matter living on that gravitational background.[1]

In this chapter, we construct the DFT ADT formalism by obtaining off-shell charges of the DFT action and its Gibbons-Hawking-like boundary term [1].

In §4.1, we formulate the off-shell current of the Einstein-Hilbert-like DFT bulk action *i.e.* the integral of the scalar curvature. We show that the off-shell Noether current can be expressed with the Komar-like Noether potential,[2] and also additionally we find a divergence-free curvature tensor like the Einstein tensor, not equal to equations of motion, during developing the formulation.

In §4.2, we find the proper boundary term of the action for the ADT formalism, and find the DFT ADT charge associated with the given killing vector. We show that our formalism is only applied to DFT version of linear and angular momenta, just as in the Einstein gravity.[3] Also, we consider other

[1]Also important to note that the conserved charge can only be obtained for simpler spacetime symmetries upto rotational symmetries. It is famous that we can only discuss linear and angular momentum but no more in the gravitational conserved charge formulation [33, 41, 42, 45, 52–54]. In Double Field Theory, this is the same; DFT ADT formalism can only be applied to symmetries upto $X^A \sim O(x)$ but not to $X^A \sim O(x^2)$. This is discussed in this chapter. (or see our paper[1])

[2]Similar to the Einstein gravity. For more information about Komar form in the Einstein gravity, see [31, pp. 285–297 in 25]

[3]See the previous paragraphs and footnote 1 that explains this, or see the main text of the given section 4.2.

boundary terms of DFT action. I compare a DFT boundary term from another paper to our boundary term and discuss its physical meaning. Also, I discuss DFT version of extrinsic curvature term to cancel divergence of our boundary term when the DFT background is not asymptotically flat.

In further sections, we discuss additional background fields that can be added to the Doubled space. Following rank-2 field and rank-0 field, we can think of a Double-vector field background that can be added to the DFT action. 1-form gauge field is also a part of the massless section of the string[pp. 30 in 7], and it may also be considered as a background of the matter, so in §4.3, we may also consider to formulate the conserved charge contribution of the doubled Yang-Mills action formulated in [70]. In §4.4, we also discuss the charge contribution of the cosmological constant.

In conclusion, in §4.5, I gather all the contributions from the background fields above and write the total ADT conserved charge in the complete form, which is the Noether charge of the total action, including the boundary terms constructed in the text.

Additionally, there are charges we cannot capture using our formulation. Not only D-brane charges, we also cannot obtain the monopole charges in the NSNS background. In §4.6, we discuss other charges including DFT monopole charges, and cite some references.

4.1 Off-shell Noether analysis on DFT of NS-NS sector

Recall that dynamical field contents of the NS-NS sector DFT are the projection field P_{AB} and the dilaton field d. Under variation of these fields, variation of the DFT Lagrangian (3.59) takes the structure [4]

$$\delta\mathcal{L}_{\rm NSNS} = -2\delta d\, e^{-2d}\mathcal{S} + 4e^{-2d}(P\delta P\bar{P})^{AB}(PS\bar{P})_{AB} + \partial_A\left[e^{-2d}\Theta^A\right]\,, \quad (4.7)$$

where Θ^A in the last surface term denotes

$$\Theta^A := 2(P^{AC}P^{BD} - \bar{P}^{AC}\bar{P}^{BD})\delta\Gamma_{BCD}\,. \quad (4.8)$$

For infinitesimal variations, (4.7) yields the equations of motion for the projection field, respectively, the dilaton field: [5]

$$(PS\bar{P})_{(AB)} = P_{(A}{}^C\bar{P}_{B)}{}^D S_{CD} \simeq 0\,, \quad (4.9)$$

$$\mathcal{S} = (P^{AC}P^{BD} - \bar{P}^{AC}\bar{P}^{BD})S_{ABCD} \simeq 0\,. \quad (4.10)$$

So, on the shell, the Lagrangian vanishes: the on-shell action would be given entirely by a surface term one may add to it.

[4] Note that $\delta P_{AB} = (P\delta P\bar{P})_{AB} + (\bar{P}\delta PP)_{AB}$.

[5] Note the equivalence, $P_A{}^C\bar{P}_B{}^D S_{CD} \simeq 0 \;\Leftrightarrow\; P_{(A}{}^C\bar{P}_{B)}{}^D S_{CD} \simeq 0$.

The variation of the connection in (4.8) is given by [71],

$$\begin{aligned}\delta\Gamma_{CAB} = {} & 2P_{[A}{}^{D}\bar{P}_{B]}{}^{E}\nabla_C\delta P_{DE} + 2(\bar{P}_{[A}{}^{D}\bar{P}_{B]}{}^{E} - P_{[A}{}^{D}P_{B]}{}^{E})\nabla_D\delta P_{EC} \\ & - \tfrac{4}{D-1}(\bar{P}_{C[A}\bar{P}_{B]}{}^{D} + P_{C[A}P_{B]}{}^{D})(\partial_D\delta d + P_{E[G}\nabla^G\delta P^E{}_{D]}) \\ & - \Gamma_{FDE}\,\delta(\mathcal{P}+\bar{\mathcal{P}})_{CAB}{}^{FDE}\,. \end{aligned} \tag{4.11}$$

The last line does not contribute to (4.8) as, from (3.33) and (3.40), the following projection properties follow:

$$P^{CB}\Gamma_{FDE}\,\delta(\mathcal{P}+\bar{\mathcal{P}})_{CAB}{}^{FDE} = -P^{CB}(\mathcal{P}+\bar{\mathcal{P}})_{CAB}{}^{FDE}\delta\Gamma_{FDE} = 0\,, \tag{4.12}$$

$$\bar{P}^{CB}\Gamma_{FDE}\,\delta(\mathcal{P}+\bar{\mathcal{P}})_{CAB}{}^{FDE} = -\bar{P}^{CB}(\mathcal{P}+\bar{\mathcal{P}})_{CAB}{}^{FDE}\delta\Gamma_{FDE} = 0\,. \tag{4.13}$$

Using also the relations

$$P^{BD}\delta\Gamma_{BCD} = 2P_C{}^B\partial_B\delta d - \nabla^B\delta P_{BC}\,, \tag{4.14}$$

$$\bar{P}^{BD}\delta\Gamma_{BCD} = 2\bar{P}_C{}^B\partial_B\delta d + \nabla^B\delta P_{BC}\,, \tag{4.15}$$

$$\delta\Gamma_{CAB}P^{CF}\bar{P}^{BG} = (P\nabla)^F(P\delta P\bar{P})_A{}^G + (\bar{P}\nabla)_A(P\delta P\bar{P})^{FG} - (\bar{P}\nabla)^G(P\delta P\bar{P})^F{}_A\,, \tag{4.16}$$

$$\delta\Gamma_{CAB}\bar{P}^{CF}P^{BG} = (P\nabla)^G(P\delta P\bar{P})_A{}^F - (\bar{P}\nabla)^F(P\delta P\bar{P})^G{}_A - (P\nabla)_A(P\delta P\bar{P})^{GF}\,, \tag{4.17}$$

we can simplify Θ^A into the form

$$\Theta^A(d,P,\delta d,\delta P) = 4(P-\bar{P})^{AB}\partial_B\delta d - 2\nabla_B\delta P^{AB}\,. \tag{4.18}$$

Consider now arbitrary generalised diffeomorphism gauge transformations. They are generated by *the generalised Lie derivatives*, so

$$
\begin{aligned}
\delta_X d \;=\; \widehat{\mathcal{L}}_X d \quad &= X^A\partial_A d - \tfrac{1}{2}\partial_A X^A = -\tfrac{1}{2}\nabla_A X^A\,,\\
\delta_X P_{AB} = \widehat{\mathcal{L}}_X P_{AB} \quad &= X^C\partial_C P_{AB} + (\partial_A X^C - \partial^C X_A)P_{CB} + (\partial_B X^C - \partial^C X_B)P_{AC}\\
&= (\nabla_A X^C - \nabla^C X_A)P_{CB} + (\nabla_B X^C - \nabla^C X_B)P_{AC}\\
&= 2(\bar{P}\nabla)_{(A}(PX)_{B)} - 2(P\nabla)_{(A}(\bar{P}X)_{B)}\,.
\end{aligned}
\tag{4.19}
$$

These equations give the $\mathbf{O}(D,D)$-*covariant Killing equations in DFT*:

$$
\nabla_A X^A = 0 \qquad \text{and} \qquad (P\nabla)_A(\bar{P}X)_B - (\bar{P}\nabla)_B(PX)_A = 0\,, \tag{4.20}
$$

which are equivalent to

$$
\widehat{\mathcal{L}}_X d = 0 \qquad \text{and} \qquad \widehat{\mathcal{L}}_X \mathcal{H}_{AB} = 0. \tag{4.21}
$$

From (4.19), it also follows that the connection transforms as [71]

$$
\delta_X \Gamma_{CAB} = \widehat{\mathcal{L}}_X \Gamma_{CAB} + 2\left[(\mathcal{P}+\bar{\mathcal{P}})_{CAB}{}^{FDE} - \delta_C^{~F}\delta_A^{~D}\delta_B^{~E}\right]\partial_F\partial_{[D}X_{E]}\,. \tag{4.22}
$$

This is in fact the generic variation (4.11) when restricted to the generalised diffeomorphism gauge transformations.

To derive off-shell conserved Noether current, we start with the covariance of the weight-one Lagrangian under the general diffeomorphism gauge transformation:

$$\delta_X \mathcal{L}_{\rm NSNS} = \partial_A \left(X^A \mathcal{L}_{\rm NSNS} \right) . \tag{4.23}$$

It gives the identity:

$$\begin{aligned}\partial_A \left(X^A e^{-2d} \mathcal{S} \right) = {} & e^{-2d} X_B \nabla_A \left[4(P^{AC}\bar{P}^{BD} - \bar{P}^{AC}P^{BD})S_{CD} - \mathcal{J}^{AB}\mathcal{S} \right] \\ & + \partial_A \Big[4e^{-2d}(\bar{P}^{AC}P^{DE} - P^{AC}\bar{P}^{DE})S_{CD}X_E \\ & \qquad + 2e^{-2d}(P^{AC}P^{BD} - \bar{P}^{AC}\bar{P}^{BD})\delta_X\Gamma_{BCD} \Big] \\ & + \partial_A \left(X^A e^{-2d} \mathcal{S} \right) ,\end{aligned} \tag{4.24}$$

implying that the sum of the first line and the second line should vanish identically. Actually, the above identity (4.24) holds not just for generalised gauge transformations but also for arbitrary local transformations generated by the vector field, X^A. Therefore, the first line and the second line of (4.24) ought to vanish independently. [6] Consequently, we obtain an off-shell, covariantly conserved two-index curvature field G_{AB}, which we propose as the DFT counterpart of the *Einstein curvature tensor*:

$$G^{AB} := 2(P^{AC}\bar{P}^{BD} - \bar{P}^{AC}P^{BD})S_{CD} - \tfrac{1}{2}\mathcal{J}^{AB}\mathcal{S} \quad \text{obeying} \quad \nabla_A G^{AB} = 0 \,, \tag{4.25}$$

[6] For those readers not convinced, consider the case where the vector field X^A is a distribution on an arbitrary point as a Dirac delta function and integrate (4.24) over a section. This will confirm that the first line vanishes by itself.

and an *off-shell, covariantly conserved Noether current*,

$$\begin{array}{l} J^{A} := 4(\bar{P}^{AC}P^{D}{}_{E} - P^{AC}\bar{P}^{D}{}_{E})X^{E}S_{CD} + 2(P^{AC}P^{BD} - \bar{P}^{AC}\bar{P}^{BD})\delta_{X}\Gamma_{BCD}\,, \\ \text{obeying} \quad \nabla_{A}J^{A} = 0\,, \qquad \partial_{A}(e^{-2d}J^{A}) = 0\,. \end{array} \tag{4.26}$$

We can further decompose the conservation law of the generalised Einstein curvature G_{AB} by projecting $\nabla_A G^{AB} = 0$ with P or $\bar{P}$. We obtain a pair of conservation relations:

$$\nabla_A \left(4P^{AC}\bar{P}^{BD}S_{CD} - \bar{P}^{AB}\mathcal{S}\right) = 0\,, \qquad \nabla_A \left(4\bar{P}^{AC}P^{BD}S_{CD} + P^{AB}\mathcal{S}\right) = 0\,, \tag{4.27}$$

which can be re-expressed as

$$4(P\nabla)_A(PS\bar{P})^{AB} - (\bar{P}\nabla)^B\mathcal{S} = 0\,, \qquad 4(\bar{P}\nabla)_A(\bar{P}SP)^{AB} + (P\nabla)^B\mathcal{S} = 0\,. \tag{4.28}$$

We see that these conservation relations are precisely the 'differential Bianchi identities' obtained previously in [65]. While the difference of the two projected curvatures in (4.27) yields back the generalised Einstein tensor (4.25), their sum leads to nothing new. Rather, it yields the conservation relation of 'symmetric' Einstein curvature tensor:

$$\nabla_A \left[\mathcal{G}^{(AB)}\right] = 0 \quad \text{where} \quad \mathcal{G}^{(AB)} := G^{AC}\mathcal{H}_C{}^B = -4(PS\bar{P})^{(AB)} - \tfrac{1}{2}\mathcal{H}^{AB}\mathcal{S}\,. \tag{4.29}$$

Note also that the equations of motion (4.9) and (4.10) are equivalent to the vanishing of the generalised Einstein curvature,

$$G^{AB} \simeq 0\,. \tag{4.30}$$

Note that the off-shell, conserved Noether current is expressible as

$$\begin{aligned} J^A &= 8(\bar{P}SP)^{[AB]}X_B \\ &\quad + 4\nabla_B\left[(P\nabla)^{(A}(\bar{P}X)^{B)} - (\bar{P}\nabla)^{(A}(PX)^{B)} - \frac{1}{2}\mathcal{H}^{AB}\nabla_C X^C\right]\,, \end{aligned} \tag{4.31}$$

and, also in terms of the Einstein curvature tensor and the Θ-term (4.18), as

$$J^A = -2G^{AB}X_B + \Theta^A(d, P, \delta_X d, \delta_X P) - \mathcal{S}X^A\,. \tag{4.32}$$

This then leads to the on-shell Noether current:

$$H^A := \Theta^A(d, P, \delta_X d, \delta_X P) - \mathcal{S}X^A\,. \tag{4.33}$$

Taking the divergence, we obtain

$$\nabla_A H^A = 2G^{AB}\nabla_A X_B = 2\mathcal{S}\delta_X d - 4(PS\bar{P})^{AB}\delta_X P_{AB} \simeq 0\,. \tag{4.34}$$

Indeed, the right-hand side vanishes either on-shell or, alternatively, for a Killing vector, X^A, satisfying (4.20). We would like to further re-express the Noether current such that conservation relation is manifest. We do so by searching for a skew-symmetric *Noether potential*, K^{AB}, in terms of which the off-shell conserved Noether current is given by

$$e^{-2d}J^{A}=\partial_{B}(e^{-2d}K^{AB})+\Phi\partial^{A}\Phi'\,,\qquad K^{AB}=-K^{BA}\,. \tag{4.35}$$

In this form, the conservation relation is manifest up to the section condition. Note that $\Phi\partial^{A}\Phi'$ takes the generic form of a 'derivative-index-valued vector' [74] which generates the coordinate gauge symmetry (3.3). As such, upon imposing the section condition, it is automatically conserved. Moreover, it will not contribute to the global charge in the next subsection, which is defined on a given choice of the section by the spatial integral of the conserved charge density. Hence, we may freely drop off such derivative-index-valued vectors and take the Noether current up to the derivative-index-valued-vectors as

$$e^{-2d}J^{A}\equiv\partial_{B}(e^{-2d}K^{AB})\,. \tag{4.36}$$

To find explicit expression of the Noether potential, we utilize the projection field identities (3.26) and commutator relations:

$$\big[(P\nabla)_{B},(\bar{P}\nabla)_{A}\big]\,X^{B}=(\bar{P}SP)_{AB}X^{B}\,, \tag{4.37}$$

$$\big[(\bar{P}\nabla)_{B},(P\nabla)_{A}\big]\,X^{B}=(PS\bar{P})_{AB}X^{B}\,, \tag{4.38}$$

and rewrite the off-shell conserved current J^A in the form

$$J^A = 4\nabla_B \left[(\bar{P}\nabla)^{[A}(PX)^{B]} - (P\nabla)^{[A}(\bar{P}X)^{B]}\right] + 2\partial^A \left[(\bar{P} - P)^{BC}\nabla_B X_C\right] . \tag{4.39}$$

This expression naturally suggests to define the skew-symmetric Noether potential as

$$K^{AB} := 4(\bar{P}\nabla)^{[A}(PX)^{B]} - 4(P\nabla)^{[A}(\bar{P}X)^{B]} . \tag{4.40}$$

With this definition, we finally get

$$\begin{aligned} e^{-2d}J^A = {}& \partial_B(e^{-2d}K^{[AB]}) + 2e^{-2d}\partial^A \left[(\bar{P} - P)^{BC}\nabla_B X_C\right] \\ & + 2e^{-2d}(P\partial^A P\bar{P})^{BC}\left[(\bar{P}\nabla)_C(PX)_B + (P\nabla)_B(\bar{P}X)_C\right] . \end{aligned} \tag{4.41}$$

In going from (4.39) to (4.41), one needs to take care of the semi-covariant derivative connections. It turns out that they merely yield derivative-index-valued vectors, as in the second line of (4.41). For this, it is worth to note that $\Gamma^A{}_{BC}P^B{}_D\bar{P}^C{}_E = (P\partial^A P\bar{P})_{DE}$ is a derivative-index-valued vector as well.

4.2 Foundation of Boundary Term of DFT Action and Its Contribution to Charge Formula

We now proceed to construct conserved global charges, which will constitute generators of asymptotic symmetry algebra. The starting point is the two-form Komar function. While our Noether potential (4.40) can correctly reproduce the two-form Komar integrand, it is well known that the Komar integrand itself needs to be further corrected [46–48] (see also [41] and references therein). Here, we derive such a correction and define a corresponding conserved global charge.

We start by rewriting the semi-covariant four-index Riemann curvature in terms of the semi-covariant derivative acting on the connection:

$$
\begin{aligned}
&S_{ABCD}\\
&= \frac{1}{2}\Big(\Gamma^{E}{}_{AB}\Gamma_{ECD} + \Gamma_{CA}{}^{E}\Gamma_{DBE} - \Gamma_{CB}{}^{E}\Gamma_{DAE} + \Gamma_{AC}{}^{E}\Gamma_{BDE} - \Gamma_{AD}{}^{E}\Gamma_{BCE}\Big)\\
&\quad + \nabla_{[A}\Gamma_{B]CD} + \nabla_{[C}\Gamma_{D]AB}\,,
\end{aligned}
\tag{4.42}
$$

This enables us to isolate the two-derivative terms ('accelerations') from the one-derivative terms ('velocities') in the DFT Lagrangian:

$$
\begin{aligned}
&e^{-2d}(P^{AC}P^{BD}-\bar{P}^{AC}\bar{P}^{BD})S_{ABCD}\\
&=e^{-2d}(P^{AC}P^{BD}-\bar{P}^{AC}\bar{P}^{BD})(\Gamma_{AC}{}^{E}\Gamma_{BDE}-\Gamma_{AB}{}^{E}\Gamma_{DCE}+\frac{1}{2}\Gamma^{E}{}_{AB}\Gamma_{ECD})\\
&\qquad+2\partial_A\left[e^{-2d}(P^{AC}P^{BD}-\bar{P}^{AC}\bar{P}^{BD})\Gamma_{BCD}\right] \qquad (4.43)\\
&=e^{-2d}(P^{AC}P^{BD}-\bar{P}^{AC}\bar{P}^{BD})(\Gamma_{AC}{}^{E}\Gamma_{BDE}-\Gamma_{AB}{}^{E}\Gamma_{DCE}+\frac{1}{2}\Gamma^{E}{}_{AB}\Gamma_{ECD})\\
&\qquad+\partial_A\left[e^{-2d}\Big\{4(P-\bar{P})^{AB}\partial_B d-2\partial_B P^{AB}\Big\}\right]. \qquad (4.44)
\end{aligned}
$$

Motivated by this observation, we define a composite vector field:

$$
\begin{aligned}
B^A\ &:=2(P^{AC}P^{BD}-\bar{P}^{AC}\bar{P}^{BD})\Gamma_{BCD}\\
&=4(P-\bar{P})^{AB}\partial_B d-2\partial_B P^{AB}\,.
\end{aligned}
\qquad (4.45)
$$

Note that this is not quite a diffeomorphism covariant vector: because of (3.41), B^A transforms anomalously,

$$
\delta_X B^A=\widehat{\mathcal{L}}_X B^A+4(\bar{P}^{AC}\bar{P}^{BD}-P^{AC}P^{BD})\partial_B\partial_{[C}X_{D]}\,. \qquad (4.46)
$$

Only if the vector field X^A can be restricted to satisfy $\partial_B\partial_{[C}X_{D]}=0$, the composite vector field B^A transforms covariantly under the generalised diffeomorphism gauge transformations. We will see momentarily that this condition can be arranged by modifying the Noether potential in a specific way.

The idea is that we would like to remove the two-derivative terms. To do so, we consider modifying the DFT Lagrangian with a specific surface term:

$$\begin{aligned}\widehat{\mathcal{L}}_{\rm NSNS} &= \mathcal{L}_{\rm NSNS} - \partial_A(e^{-2d}B^A)\\ &= e^{-2d}(P^{AC}P^{BD} - \bar{P}^{AC}\bar{P}^{BD})(\Gamma_{AC}{}^{E}\Gamma_{BDE} - \Gamma_{AB}{}^{E}\Gamma_{DCE} + \frac{1}{2}\Gamma^{E}{}_{AB}\Gamma_{ECD}).\end{aligned} \tag{4.47}$$

The idea is analogous to the modification of the Einstein-Hilbert action to the Schrödinger action *a la* Dirac [26] that is free of two-derivative terms (Γ^2 action). While the equations of motion remain intact, the theta term in the variation of the Lagrangian, (4.7), gets modified to

$$e^{-2d}\widehat{\Theta}^A(d,P,\delta d,\delta P) = e^{-2d}\Theta^A(d,P,\delta d,\delta P) - \delta(e^{-2d}B^A)\,, \tag{4.48}$$

such that the new theta term, $\widehat{\Theta}^A(d,P,\delta d,\delta P)$, no longer contains the derivative of the variations. In particular, for the generalised diffeomorphism gauge transformations, we have from (4.46) that

$$\begin{aligned}&e^{-2d}\widehat{\Theta}^A(d,P,\delta_X d,\delta_X P)\\ &= e^{-2d}\Theta^A(d,P,\delta_X d,\delta_X P) + 2\partial_B(e^{-2d}X^{[A}B^{B]}) - e^{-2d}B_B\partial^A X^B\\ &\quad - X^A\partial_B(e^{-2d}B^B) + 4e^{-2d}(P^{AC}P^{BD} - \bar{P}^{AC}\bar{P}^{BD})\partial_B\partial_{[C}X_{D]}\,.\end{aligned} \tag{4.49}$$

The off-shell Noether current (4.32) now receives extra contributions

$$
\begin{aligned}
e^{-2d}\widehat{J}^A &= e^{-2d}\left[-2G^{AB}X_B + \widehat{\Theta}^A(d,P,\delta_X d,\delta_X P) - (\mathcal{S} - \nabla_B B^B)X^A\right] \\
&= \underbrace{e^{-2d}J^A + 2\partial_B(e^{-2d}X^{[A}B^{B]})}_{\text{ADT current}} - \underbrace{e^{-2d}B_B\partial^A X^B}_{\substack{\text{trivially conserved} \\ \text{by the strong constraint}}} \\
&\quad + \underbrace{4e^{-2d}(P^{AC}P^{BD} - \bar{P}^{AC}\bar{P}^{BD})\partial_B\partial_{[C}X_{D]}}_{\text{additional condition}} \,.
\end{aligned}
\tag{4.50}
$$

Correspondingly, we have the modified Noether potential

$$
\widehat{K}^{AB} = K^{AB} + 2X^{[A}B^{B]} \,. \tag{4.51}
$$

We need to ensure that the modified Noether current is conserved. Taking the divergence, one finds that

$$
\partial_A(e^{-2d}\widehat{J}^A) = 4e^{-2d}(P^{AC}P^{BD} - \bar{P}^{AC}\bar{P}^{BD})\nabla_{[A}(\partial_{B]}\partial_{[C}X_{D]}) \,. \tag{4.52}
$$

Thus, the modified off-shell Noether current is not always conserved. However, we can ensure the conservation relation provided we impose the diffeomorphism vector field to obey the condition

$$
\nabla_{[A}(\partial_{B]}\partial_{[C}X_{D]}) = 0 \,, \tag{4.53}
$$

or more strongly the condition

$$
\partial_B\partial_{[C}X_{D]} = 0 \,. \tag{4.54}
$$

Wonderfully, the latter condition is precisely the condition we needed in order to ensure the composite vector field B^A transform covariantly (4.46). The simplest example of such restricted vector field is when X^A is a constant vector, corresponding to a rigid translation in doubled spacetime. Also, this result coincides with the fact that the conserved charge of the Einstein gravity also can only be discussed with rigid translations.

With the conserved modified Noether current at hand, we finally obtain the conserved global charge as surface integral:

$$\boxed{Q[X] := \int_{\mathcal{M}} \mathrm{d}^{D-1} x_A \; e^{-2d} \widehat{J}^A = \oint_{\partial\mathcal{M}} \mathrm{d}^{D-2} x_{AB} \; e^{-2d} \left(K^{AB} + 2X^{[A} B^{B]} \right)} \,. \tag{4.55}$$

Here, $\mathcal{M}$ denotes a timelike hypersurface inside the section, $\Sigma_D = \mathbb{R}_t \times \mathcal{M}$, while $\partial\mathcal{M}$ corresponds to its asymptotic boundary.

Intuitively, we can also motivate the conserved global charge proposed above from the method adopted by Wald [46–48]. Modulo the equations of motion, the variation of the on-shell conserved Noether current (4.33) reads

$$\delta(e^{-2d} H^A) \simeq e^{-2d} \Omega^A(\delta, \delta_X) - 2\partial_B \left(e^{-2d} X^{[A} \Theta^{B]} \right) + e^{-2d} \Theta_B \partial^A X^B \,, \tag{4.56}$$

where Ω^A denotes the (Hamiltonian) symplectic structure defined by

$$e^{-2d} \Omega^A(\delta_1, \delta_2) := \delta_1 \left[e^{-2d} \Theta^A(d, P, \delta_2 d, \delta_2 P) \right] - \delta_2 \left[e^{-2d} \Theta^A(d, P, \delta_1 d, \delta_1 P) \right] . \tag{4.57}$$

The above variation (4.56) then reveals an on-shell relation

$$e^{-2d}\Omega^A(\delta,\delta_X) \simeq \partial_B\left[\delta\left(e^{-2d}K^{AB}\right) + 2\left(e^{-2d}X^{[A}\Theta^{B]}\right)\right] - e^{-2d}\Theta_B\partial^A X^B\,. \tag{4.58}$$

Again, the last term is a derivative-index-valued vector and can be dropped off when integrated over a section. Finally, by assuming proper asymptotic fall-off behaviour at infinity, the left-hand side of (4.48) can be made to vanish at infinity [7]. This facilitates to approximate

$$e^{-2d}\Omega^A(\delta,\delta_X) \approx \partial_B\delta\left[e^{-2d}\left(K^{AB} + 2X^{[A}B^{B]}\right)\right]\,. \tag{4.59}$$

The final expression then supports the validity of our proposed expression for the conserved global charge (4.55).

4.2.1 Integral-by-part Boundary Term in $\mathcal{H}$–expression by Hohm, Hull, and Zwiebach

Early Double Field Theory studies by Hohm, Hull, and Zwiebach [56] is done without semi-covariant formalism but only with the generalised metric $\mathcal{H}_{MN}$ and the $\mathbf{O}(D,D)$ dilaton e^{-2d}. The Double Field Theory action in those fields and its generalised-diffeomorphism invariance were also well-studied in that formalism. In this subsection, I briefly show their work and compare and check correspondence to our work.

Most Double Field Theory studies use Lagrangians equivalent to each other

[7] See §5.1.3 for explicit demonstration of this for the asymptotically flat hypersurface.

up to total derivatives. The action with only kinetic terms [8] is defined as (4.11) in [56]:

$$\mathcal{L}_{\text{kinetic}} = e^{-2d}\left[\frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_N\mathcal{H}^{KL}\partial_L\mathcal{H}_{MK}\right.$$
$$\left. - 2\partial_M d\,\partial_N\mathcal{H}^{MN} + 4\mathcal{H}^{MN}\partial_M d\,\partial_N d\right]. \tag{4.60}$$

For example, Berman, Musaev, and Perry's boundary term study [68] explained in Subsection 4.2.2 also used this action.

However, Hohm, Hull, and Zwiebach already studied the covariant scalar curvature in Double Field Theory in $\mathcal{H}_{MN}$ and e^{-2d}–expression. In equation (4.24) and afterwards in Hohm, Hull, and Zwiebach's paper[56], they proved that the covariant scalar curvature must be only defined by

$$\begin{aligned}\mathcal{S} = \ & \frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_K\mathcal{H}_{NL} \\ & - 4\mathcal{H}^{MN}\partial_M d\,\partial_N d + 4\partial_M\mathcal{H}^{MN}\partial_N d \\ & + 4\mathcal{H}^{MN}\partial_M\partial_N d - \partial_M\partial_N\mathcal{H}^{MN},\end{aligned} \tag{4.61}$$

and the boundary term difference between the lagrangian defined by the curvature and the lagrangian with only kinetic terms is naturally derived,

$$\mathcal{L} = e^{-2d}\mathcal{S} + \partial_M\left(e^{-2d}\left[\partial_N\mathcal{H}^{MN} - 4\mathcal{H}^{MN}\partial_N d\right]\right), \tag{4.62}$$

which is also written in (4.27) in [56].

[8] Here, the kinetic term is defined to be in the form of $A^{MN}\partial_M\phi\,\partial_N\varphi$ where A^{MN} is an arbitrary rank-2 tensor field, and where ϕ and φ are arbitrary fields with arbitrary tensor ranks.

Remind that the Gibbons–Hawking-like boundary term (4.45) obtained before this subsection, and re-write the term in the $\mathcal{H}$–expression:

$$B^A = 4(P - \bar{P})^{AB}\partial_B d - 2\partial_B P^{AB} = 4\mathcal{H}^{AB}\partial_B d - \partial_B \mathcal{H}^{AB}, \tag{4.63}$$

which is already appeared in (4.62).

4.2.2 Comparing to Berman–Musaev–Perry Boundary Term

Berman, Musaev, and Perry also developed the boundary term for Double Field Theory which is equivalent to the Gibbons-Hawking term in the Riemannian parametrisation[68]. In this subsection 4.2.2, I compare the Berman–Musaev–Perry boundary term to our[1] or Hohm, Hull, and Zwiebach's[56] boundary term, explained in the previous subsections.

To compare our boundary term to the Gibbons-Hawking term written with the normal vector, I may convert the boundary term (4.45) into the boundary quantity with the normal vector. In manner of the Stokes' theorem in DFT explained in (4.74), I may try to simply substitute

$$\partial_A \to \epsilon N_A, \tag{4.64}$$

where ϵ is the spacelikeness of the normal vector and is $+1$ if the normal vector is spacelike while $\epsilon = -1$ if the normal vector is timelike.

This substitution changes the total action (from the total Lagrangian

(4.47)) to

$$2\kappa^2 I_{\text{total}} = \int_{\Sigma_D} \mathrm{d}^D x \left[e^{-2d}\mathcal{S} - \partial_A \left[e^{-2d} B^A \right] \right] \tag{4.65}$$

$$= \int_{\Sigma_D} \mathrm{d}^D x \left[e^{-2d}\mathcal{S} - \partial_A \left[e^{-2d} \left(4\mathcal{H}^{AB}\partial_B d - \partial_B \mathcal{H}^{AB} \right) \right] \right] \tag{4.66}$$

$$= \int_{\Sigma_D} \mathrm{d}^D x \, e^{-2d}\mathcal{S} + \oint_{\partial\Sigma_D} \mathrm{d}^{D-1} x \; \epsilon \, e^{-2d} \left(2\mathcal{H}^{AB}\partial_A N_B + N_A \partial_B \mathcal{H}^{AB} \right)$$
$$+ \underbrace{\cancel{\oint_{\partial\Sigma_D} \mathrm{d}^{D-1} x \, \partial_A \left(-2\epsilon \, e^{-2d} \mathcal{H}^{AB} N_B \right)}}_{\text{boundary of boundary}} . \tag{4.67}$$

Meanwhile, Berman, Musaev, and Perry [68] constructed the DFT action equivalent to the total action (Einstein-Hilbert action + Gibbons-Hawking term) in Einstein gravity, assuming Euclidean spacetime:

$$\int \mathrm{d}^D x \, \sqrt{G} \, e^{-2\Phi} \left(R + 4(\partial\Phi)^2 - \frac{1}{12} H^2 \right) + 2 \oint \mathrm{d}^{D-1} x \, \sqrt{g} \, e^{-2\Phi} K$$
$$\simeq \int \mathrm{d}^D x \, \mathcal{L}_{\text{kinetic}} + \oint \mathrm{d}^{D-1} x \, e^{-2d} \left(2\mathcal{H}^{AB}\partial_A N_B + N_A \partial_B \mathcal{H}^{AB} \right) . \tag{4.68}$$

Comparing the purple parts in (4.67) and (4.68) makes us suspect that the Berman–Musaev–Perry boundary term [68] and our boundary term (4.45) might be related. However, Berman *et al.* used $\mathcal{L}_{kinetic}$ instead of the fully covariant Lagrangian $e^{-2d}\mathcal{S}$ in their paper (green part in (4.68)). It is very difficult to follow and mix up all the DFT studies exactly including the boundary term because most of DFT papers expand their studies only up to boundary term. Thus, I cannot strongly argue here any correlations between the Berman–Musaev–Perry term and our boundary term (4.45).

Hopefully, in the future study, I may prove the relations between those boundary term, and furthermore I may write more precise DFT version of Gibbons-Hawking term which written only in boundary quantities rather than bulk quantities on boundary as in this study.

4.2.3 Extrinsic Curvature of Boundary

Extrinsic boundary curvature term from the conventional theory. All the previous boundary term discussion only can be guaranteed to work in asymptotically flat backgrounds. In an asymptotically flat case *i.e.* a locally deviated background around the p-brane(s) from the flat spacetime $\mathbb{R}^{D-s-1,1}$ with s-dimensional transverse smeared directions, the boundary $S^{D-p-s-2} \times \mathbb{R}^{p+s}$ of the integral domain at the spatial direction has a vanishing curvature as the domain radius goes infinity $r_{S^{D-p-s-2}} \to \infty$. In a background with a distinct infinity behaviour where the hypersurface does not have a vanishing curvature, the boundary action, which is an integral of the boundary curvature over the boundary area, may have a divergent value as the boundary area goes infinity. In classical gravity, we substract the non-dynamic[9] extrinsic curvature of boundary (convergent value of the curvature of the boundary) from the Gibbons–Hawking boundary term to cancel out the divergence of the boundary term when the extrinsic curvature of the boundary of the domain does not

[9]The "non-dymamic" term means the term that does not depend on fields but can depend on coordinates so is constant in variational sense.

converges to zero.

$$2\kappa^2 I = \int_{\mathcal{D}} \mathrm{d}^D x \sqrt{|G|}\, R[G] + 2 \oint_{\partial\mathcal{D}} \mathrm{d}^{D-1} x\, \epsilon \sqrt{|h|}\, K[h] \tag{4.69}$$

$$\to \int_{\mathcal{D}} \mathrm{d}^D x \sqrt{|G|}\, R[G] + 2 \oint_{\partial\mathcal{D}} \mathrm{d}^{D-1} x\, \epsilon \sqrt{|h|}\, (K[h] - K_0) \tag{4.70}$$

where $G_{\mu\nu}$ is the metric of the D-dimensional domain $\mathcal{D}$ in the D-dimensional spacetime, and h is the $(D-1)$-dimensional induced metric of the hypersurface $\partial\mathcal{D}$, and the curvature of boundary $K[h]$ can be defined by normal vector n_μ of the boundary:

$$K := G^{\mu\nu} \nabla_\mu n_\nu, \tag{4.71}$$

and ϵ is the space-like-ness or time-like-ness of the boundary:

$$\epsilon = G^{\mu\nu} n_\mu n_\nu = \begin{cases} -1 & \text{for timelike normal vector,} \\ & \text{(boundary in the time direction)} \\ +1 & \text{for spacelike normal vector.} \\ & \text{(boundary in the spatial direction)} \end{cases} \tag{4.72}$$

The extrinsic curvature term in (4.70) can be converted into a bulk integral,

$$\begin{aligned} -2 \int_{\partial\mathcal{D}} \mathrm{d}^{D-1} x\, \epsilon \sqrt{|h|}\, K_0 &= -2 \int_{\partial\mathcal{D}} \mathrm{d}^{D-1} x\, n_\mu G^{\mu\nu} n_\nu \sqrt{|h|}\, K_0 \\ &= -2 \int_{\mathcal{D}} \mathrm{d}^D x\, \partial_\mu \left[\epsilon \sqrt{|G|}\, K_0\, G^{\mu\nu}\, n_\nu \right], \end{aligned} \tag{4.73}$$

where K_0 and n_ν in the last line is arbitrary continuations into $\mathcal{D}$ from the surface quantities K_0 and n_ν. The last line calculation, which is the conversion from the surface integral into the bulk integral, can be derived by *the generalised Stokes' theorem* (or the divergence theorem):

$$\int_{\mathcal{D}} \mathrm{d}^D x\, \partial_\mu \left(\sqrt{|G|} V^\mu\right) = \oint_{\partial\mathcal{D}} \mathrm{d}^{D-1} x \sqrt{|h|}\, \epsilon\, n_\mu V^\mu. \tag{4.74}$$

To guarantee the variation with Dirichlet condition works and our charge formula converges with asymptotically non-flat backgrounds also in Double Field Theory, we need the extrinsic curvature term of boundary in Double Field Theory that cancels out the divergence of our Gibbons–Hawking-like boundary term $\int \partial_A(e^{-2d} B^A)$, corresponding to the term discussed above. To write the non-dynamic extrinsic curvature term into the DFT version, I needed to write the surface integral into the bulk integral (4.73). From here, we discuss how to write (4.73) into a $2D$-dimensional version.

Introduction of the extrinsic curvature correction term to the DFT boundary action. In Double Field Theory, the extrinsic curvature term I_{ext}

is constructed analogical to (4.73) and be added to the DFT action $\widehat{I}_{\text{NSNS}}$,

$$I = \widehat{I}_{\text{NSNS}} \rightarrow \widehat{I}_{\text{NSNS}} + I_{\text{ext}}, \tag{4.75}$$

$$\begin{aligned} 2\kappa^2\,\widehat{I}_{\text{NSNS}} &:= \int\limits_{\Sigma_D} \mathrm{d}^D x\,\widehat{\mathcal{L}}_{\text{NSNS}} = \int\limits_{\Sigma_D} \mathrm{d}^D x\,\left[e^{-2d}\mathcal{S} - \partial_A\left(e^{-2d}B^A\right)\right] \\ &\simeq \int\limits_{\Sigma_D} \mathrm{d}^D x\, e^{-2d}\mathcal{S}, \end{aligned} \tag{4.76}$$

$$2\kappa^2\, I_{\text{ext}} := -2\int\limits_{\Sigma_D} \mathrm{d}^D x\,\partial_A\left[e^{-2d}\,\epsilon\, K_0\,\mathcal{H}^{AB}N_B\right], \tag{4.77}$$

where, of course, analogically ϵ in DFT can be defined as:

$$\epsilon := \mathcal{H}^{AB}N_A N_B = \pm 1, \tag{4.78}$$

and N_A is the normal vector of the boundary of the domain. In other words, the normal vector N_A is defined as a unit vector, and the definition of the unit vector must be (4.78). Here, for the consistency of this thesis, we do integral over the D-dimensional section Σ_D, but on different formalism, you can also integral over $2D$-dimensional space with the section condition implemented manually. Anyway, the extrinsic curvature of boundary term can be written in covariant bulk action as you can see. Also, as the geometry-defining metric of DFT is the generalised metric $\mathcal{H}_{MN}$ but not the constant index-raising/lowering $O(D,D)$ metric $\mathcal{J}_{MN}$, the metric $G^{\mu\nu}$ in the Einstein gravity should correspond to the generalised metric $\mathcal{H}^{MN}$ but not $\mathcal{J}^{MN}$. Thus, you should be careful to corresponds n_μ to N_M not $n^\mu = G^{\mu\nu}n_\nu$ to N^M because n_μ comes first than n^μ in the Stokes' theorem (or divergence theorem) (See Section 1.2.5). DFT normal vector N_A of the DFT surface is parametrised as:

$$N_A = \begin{pmatrix} 0 \\ n_\mu \end{pmatrix} \tag{4.79}$$

with the usual Riemannian section. Here, n_μ is the normal vector of the boundary surface $\partial\Sigma_D$ of the domain in the Riemannian section.

The analogical construction I explain here does not only mean that (4.77) is constructed in the same format of expression with (4.73), but also (4.77) is exactly the same with (4.73) under the Riemannian section condition. The Riemannian section forces $\tilde{\partial}^\mu = 0$ so

$$\partial_A = \begin{pmatrix} 0 \\ \partial_\mu \end{pmatrix} . \tag{4.80}$$

Putting (4.80) and (4.79) into (4.77) gives:

$$\begin{aligned} 2\kappa^2 I_{\text{ext}} &= -2 \int\limits_{\Sigma_D} \mathrm{d}^D x \, \partial_\mu \left[e^{-2d} \, \epsilon \, K_0 \, \mathcal{H}^{\mu\nu} N_\nu \right] \\ &= -2 \int\limits_{\Sigma_D} \mathrm{d}^D x \, \partial_\mu \left[e^{-2\Phi} \sqrt{-G} \, \epsilon \, K_0 \, G^{\mu\nu} n_\nu \right] , \end{aligned} \tag{4.81}$$

which is exactly the extrinsic curvature term of Einstein gravity with additional dilaton field, compared to (4.73).

Interpretation of the normal vector parametrisation and the new upper-indexed normal vector. The same normal vector convention also appears in Berman–Musaev–Perry's paper[68]. Even though they established their formalism bottom-up (from the SUGRA action to the DFT level), they have the same normal vector convention as my top-down formalism. Also, in their paper, they define the normal vector with upper index as:

$$\tilde{N}^A := \mathcal{H}^{AB} N_B, \tag{4.82}$$

which corresponds to n^μ in the Riemannian version. In their paper, they use the N^A notation without tilde for this, but in the common notation, DFT index lowering/raising is defined to use $O(D, D)$ metric $\mathcal{J}_{MN}$, so here I define $\tilde{N}$ to avoid misunderstandings. Anyway, Berman *et al.* and I stress that you should use $\mathcal{H}_{MN}$ to raise/lower the index of the normal vector, as it is the distance-defining metric in Double Field Theory.

Another reasoning of this notation of normal vector is also possible. Upper-indexed normal vector in Riemannian section can be parametrised as:

$$\tilde{N}^A = \begin{pmatrix} b_{\mu\nu} g^{\nu\rho} n_\rho \\ g^{\mu\nu} n_\nu \end{pmatrix}, \tag{4.83}$$

where $g_{\mu\nu}$ and $b_{\mu\nu}$ is the value of $G_{\mu\nu}$ and $B_{\mu\nu}$ at the boundary. This parametrisation is similar to the Riemannian parametrisation of other DFT vectors.[10][1, 70] $b_{\mu\nu}$ in (4.83) unwinds the $O(D, D)$ rotation by

[10] See Section 5.1 of this thesis for the parametrisation of the Killing vector and equation 3.17 in [70] for the parametrisation of the DFT Yang-Mills field.

$B_{\mu\nu}$ in the generalised metric $\mathcal{H}_{MN}$ and gives us parametrisation of Riemannian-geometrical directions. See Section 1.2.5 for more explanation in the notation and the parametrisation of the normal vector.

Implementing this extrinsic curvature of boundary term is motivated by an asymptotically non-flat example of linear dilaton background in Section 4.7 of our paper[1]. In this thesis, I organise its formulation in this section and the application in Section 5.5 in the application chapter. Obtaining the extrinsic boundary curvature in an actual example is well-described in that section.

Contribution of Extrinsic Boundary Curvature to the Charge Formula. In summary, the total action should be

$$2\kappa^2 I_{\mathrm{NSNSTotal}} = \left[\int_{\Sigma_D} \mathrm{d}^D x\, \widehat{\mathcal{L}}_{\mathrm{NSNS}}\right] + 2\kappa^2 I_{\mathrm{ext}} \tag{4.84}$$

$$= \int_{\Sigma_D} \mathrm{d}^D x \left[e^{-2d}\mathcal{S} - \partial_A\left(e^{-2d}B^A\right) - 2\partial_A\left(e^{-2d}\,\epsilon\, K_0 \tilde{N}^A\right)\right] \tag{4.85}$$

and equivalently I can say that the boundary term of the original action has transformed as:

$$B^A \to B^A + 2\epsilon\, K_0\, \tilde{N}^A. \tag{4.86}$$

Using this total boundary term, I can easily get the modified ADT charge formula, but before that I can fix the sign constant ϵ. Usually when we obtain conserved charges, we integrate conserved current over a timeslice, and the boundary of the domain (timeslice) should be in the spatial direction. Thus,

$$\epsilon = 1 \tag{4.87}$$

in a usual charge obtaining process. Thus, the charge formula (4.55) with divergence corrected from the extrinsic boundary curvature is written:

$$\boxed{Q[X] = \frac{1}{2\kappa^2} \oint_{\partial\mathcal{M}} \mathrm{d}x_{AB}\, e^{-2d} \left[K^{[AB]} + 2X^{[A}B^{B]} + 4K_0 X^{[A}\tilde{N}^{B]} \right].} \tag{4.88}$$

Remind that usually timeslices are used to obtain conserved charges, so here the normal vectors are for the spatial boundary of the domain.

4.3 Extension to Yang-Mills Sector

A proper account of low-energy string theory requires inclusion of the Yang-Mills sector and the cosmological constant in addition to the NS-NS sector. Here, we consider the DFT in which the NS-NS sector is coupled to Yang-Mills sector:

$$\begin{aligned} I = {} & \frac{1}{2\kappa^2} \int_{\Sigma_D} \mathrm{d}^D x \, e^{-2d} \left[\mathcal{S} + (\text{boundary terms}) \right] \\ & + \frac{1}{g_{\mathrm{YM}}^2} \int_{\Sigma_D} \mathrm{d}^D x \, e^{-2d} \, \mathrm{Tr}\Big\{ P^{AC} \bar{P}^{BD} \mathcal{F}_{AB} \mathcal{F}_{CD} \Big\} \end{aligned} \tag{4.89}$$

with the Yang-Mills coupling constant g_{YM}, and we construct corresponding extension of the conserved global charges. [1, 70]

Consider arbitrary variations of the projection field and the vector potential,

$$\begin{aligned} & \delta(\mathcal{F}_{AB}) = 2\mathcal{D}_{[A}\delta V_{B]} - \delta\Gamma^{C}{}_{AB} V_C \,, \qquad \mathcal{D}_A := \nabla_A - i\,[V_A \,, \quad] \,, \\ & P_A{}^C \bar{P}_B{}^D \delta(\mathcal{F}_{CD}) = 2 P_A{}^{[C} \bar{P}_B{}^{D]} \mathcal{D}_C \delta V_D - \nabla^C (P\delta P \bar{P})_{AB} V_C \,. \end{aligned} \tag{4.90}$$

This induces variation of the YM part in the DFT Lagrangian as

$$
\begin{aligned}
&\delta\mathrm{Tr}\left(P^{AC}\bar{P}^{BD}\mathcal{F}_{AB}\mathcal{F}_{CD}\right)\\
&\qquad\qquad\qquad = -4\mathrm{Tr}\left[\delta V_B\mathcal{D}_A\left(P\mathcal{F}\bar{P}\right)^{[AB]}\right]\\
&\qquad\qquad\qquad\quad + 2(P\delta P\bar{P})^{AB}\mathrm{Tr}\left[(P\mathcal{F}\mathcal{H}\mathcal{F}\bar{P})_{AB}+\nabla^{C}\left\{(P\mathcal{F}\bar{P})_{AB}V_C\right\}\right]\\
&\qquad\qquad\qquad\quad + \nabla_A\mathrm{Tr}\left[4(P\mathcal{F}\bar{P})^{[AB]}\delta V_B - 2V^{A}(P\mathcal{F}\bar{P})_{CD}\delta P^{CD}\right].
\end{aligned}
\tag{4.91}
$$

For local variations, from the first line, we find the YM equation of motion: [11]

$$
\mathcal{D}_A\left(P\mathcal{F}\bar{P}\right)^{[AB]}\simeq 0\,. \tag{4.92}
$$

From the second line in (4.91), we find that the equation of motion of the NSNS sector DFT has modified: the equation of motion of the projection field P_{AB}, $(PS\bar{P})_{AB}\simeq 0$, is modified to

$$
\frac{1}{2\kappa^2}(PS\bar{P})_{AB}+\frac{1}{2g_{\mathrm{YM}}^2}\left[(P\mathcal{F}\mathcal{H}\mathcal{F}\bar{P})_{AB}+\nabla_C\left\{(P\mathcal{F}\bar{P})_{AB}V^{C}\right\}\right]\simeq 0\,; \tag{4.93}
$$

respectively, the equation of motion of the dilaton e^{-2d}, $\mathcal{S}\simeq 0$, is now modified to

$$
\frac{1}{2\kappa^2}\mathcal{S}+\frac{1}{g_{\mathrm{YM}}^2}\mathrm{Tr}\left(P^{AC}\bar{P}^{BD}\mathcal{F}_{AB}\mathcal{F}_{CD}\right)\simeq 0\,. \tag{4.94}
$$

[11] It is useful to note

$$
(P\mathcal{F}\bar{P})_{[AB]}=(\bar{P}\mathcal{F}P)_{[AB]}\,.
$$

Once again, the DFT Lagrangian vanishes on-shell. For consistency, it is straightforward to check that

$$(P\mathcal{F}\mathcal{H}\mathcal{F}\bar{P})_{AB} + \nabla_C \left[(P\mathcal{F}\bar{P})_{AB} V^C \right] . \tag{4.95}$$

is indeed fully covariant under both the generalised diffeomorphisms and the YM gauge transformations given in Section 3.3.

The last line in (4.91) is the YM contribution to Θ^A. Then, in steps completely parallel to the analysis of the NS-NS sector DFT as carried out in Section 4.1, we can straightforwardly obtain the off-shell conserved Noether current associated with the diffeomorphism transformation to the total DFT. Modulo the part identifiable with derivative-index-valued-vectors, we get

$$\begin{aligned} \tfrac{1}{2\kappa^2} e^{-2d} \widehat{J}^A_{\text{total}} &\equiv \tfrac{1}{2\kappa^2} e^{-2d} \widehat{J}^A + 12 g_{\text{YM}}^{-2} \partial_B \text{Tr} \left(e^{-2d} (P\mathcal{F}\bar{P})^{[AB} V^{C]} X_C \right) \\ &= \partial_B \left[\tfrac{1}{2\kappa^2} e^{-2d} K^{[AB]} + 12 g_{\text{YM}}^{-2} e^{-2d} \text{Tr} \left\{ (P\mathcal{F}\bar{P})^{[AB} V^{C]} X_C \right\} \right] . \end{aligned} \tag{4.96}$$

Thus, the ADT charge contribution from the Yang-Mills action associated with the Killing vector X^A is

$$\boxed{Q_{\text{YM}}[X] = \oint_{\partial\mathcal{M}} \mathrm{d}x_{AB}\ e^{-2d} \left[\frac{1}{g_{\text{YM}}^2} \text{Tr} \left\{ 12 (P\mathcal{F}\bar{P})^{[AB} V^{C]} X_C \right\} \right] .} \tag{4.97}$$

4.4 Extension to Cosmological Constant

A proper account of low-energy string theory also requires inclusion of the cosmological constant in addition to the NS-NS sector:

$$2\kappa^2 I = \int_{\Sigma_D} \mathrm{d}^D x\, \mathcal{S} \to \int_{\Sigma_D} \mathrm{d}^D x\, e^{-2d}\, [\mathcal{S} - 2\Lambda]\,. \tag{4.98}$$

Here, we discuss the contribution of the additional action

$$2\kappa^2 I_\Lambda = \int_{\Sigma_D} \mathrm{d}^D x\, e^{-2d}(-2\Lambda) \tag{4.99}$$

to the conserved charge formula (4.55).

Remind that the off-shell Noether current for the generalised diffeomorphism $x^A \to x^A + X^A$ is:

$$\partial_A(e^{-2d} J^A_{\text{off-shell}}) = \delta_X \mathcal{L} - \partial_A(X^A \mathcal{L}). \tag{4.100}$$

The Lagrangian density we consider is in (4.99):

$$\mathcal{L}_\Lambda = -2\Lambda e^{-2d}. \tag{4.101}$$

The only dependence of this Lagrangian is the DFT dilaton e^{-2d}, so the variation of the dilation (the generalised Lie derivative of the DFT dilaton previously defined) is only taken into account:

$$\hat{\mathcal{L}}_X d = -\frac{1}{2}\nabla_A X^A. \tag{4.102}$$

Thus, the Noether current is:

$$\begin{aligned}
\delta_X \mathcal{L} - \partial_A(X^A \mathcal{L}) &= -2(\delta_X d)\, e^{-2d}(-2\Lambda) - \partial_A \left(X^A e^{-2d}(-2\Lambda)\right) && (4.103)\\
&= -2(\hat{\mathcal{L}}_X d)\, e^{-2d}(-2\Lambda) - \partial_A \left(X^A e^{-2d}(-2\Lambda)\right) && (4.104)\\
&= -2\left(-\frac{1}{2}\nabla_A X^A\right) e^{-2d}(-2\Lambda) - \partial_A \left(X^A e^{-2d}(-2\Lambda)\right) && (4.105)\\
&= \partial_A \left(X^A\, e^{-2d}(-2\Lambda)\right) - \partial_A \left(X^A e^{-2d}(-2\Lambda)\right) && (4.106)\\
&= 0. && (4.107)
\end{aligned}$$

We can see that the Noether current is trivially zero, so we conclude that the cosmological constant does not contribute to the conserved charge formula.

This result is briefly mentioned in Section 3.3 of our paper[1].

4.5 The Total ADT Charge Formula

In this section, we sum up all the contributions we have considered in previous sections.

We have considered the conserved charges of NSNS sector DFT, the Double Yang-Mills field, and the cosmological constant term.

$$I = \int_{\Sigma_D} \mathrm{d}^D x \left[e^{-2d} \left(\frac{1}{2\kappa^2} (\mathcal{S} - 2\Lambda) + g_{\mathrm{YM}}^{-2} \mathrm{Tr} P^{AC} \bar{P}^{BD} \mathcal{F}_{AB} \mathcal{F}_{CD} \right) \right] \tag{4.108}$$

During developing the conserved charge of DFT, we developed the boundary term, which is required in ADT formalism, and which acts as a Gibbons-Hawking term in DFT so can well-define the variation in the Dirichlet condition.

$$\begin{aligned} 2\kappa^2\, I = \int_{\Sigma_D} \mathrm{d}^D x \Big[& e^{-2d} \left(\mathcal{S} - 2\Lambda + 2\kappa^2 g_{\mathrm{YM}}^{-2} \mathrm{Tr} P^{AC} \bar{P}^{BD} \mathcal{F}_{AB} \mathcal{F}_{CD} \right) \\ & - \partial_A \left(e^{-2d} B^A + 2K_0 \tilde{N}^A \right) \Big] \end{aligned} \tag{4.109}$$

We have formulate the total DFT charge using this total action, and we have found that Noether potentials can be defined for all the terms we considered, so the conserved charge is defined as a surface integral of the potential on the boundary of a timeslice.

$$\boxed{\begin{aligned} Q_{\text{total}}[X] = \frac{1}{2\kappa^2} \oint\limits_{\partial\mathcal{M}} \mathrm{d}x_{AB}\ e^{-2d} \Bigg[& K^{[AB]} + 2X^{[A}B^{B]} + 4K_0 X^{[A}\tilde{N}^{B]} \\ & + \frac{2\kappa^2}{g_{\text{YM}}^2} \mathrm{Tr}\left\{ 12(P\mathcal{F}\bar{P})^{[AB}V^{C]}X_C \right\} \Bigg] \,. \end{aligned}} \tag{4.110}$$

Also, note that during the formulation of the Noether potential of the boundary term, we have found that the total conserved charge could only formulated under lower-order spacetime symmetries such as linear translation or rotation, just as the ADT charge of Einstein Gravity can only be defined for linear momenta and angular momenta. Furthermore, as we needed non-covariant boundary term for the charge, we need to define a closed boundary, which means that we need to define certain integration domain and we can define quasi-local charge but we cannot define local conserved current, as in the Einstein gravity[35, 40, 42].

4.5.1 Charge in Explicit Expression in $\mathcal{H}_{MN}$ and e^{-2d}

To use in practical applications, here we expand the semi-covariant charge formula (4.110) into $\mathcal{H}_{MN}$ and e^{-2d}.

The Komar form (4.40) expands:

$$K^{AB}[X] = 4\left(\bar{P}^{C[A}P^{B]D} - P^{C[A}\bar{P}^{B]D}\right)\left(\partial_C X_D + \Gamma_{CDE}X^E\right). \tag{4.111}$$

Here the connection with the given projection combination can be simplified by the properties of the projection matrices, and can be re-written as

$$\begin{aligned}
&4\left(\bar{P}^{C[A}P^{B]D} - P^{C[A}\bar{P}^{B]D}\right)\Gamma_{CDE} \\
&= 8\left(\bar{P}^{AC}\,P^{BD} - P^{AC}\,\bar{P}^{BD}\right)\Big[\left(P\,\partial_C P\bar{P}\right)_{[DE]} \\
&\qquad\qquad + \left(\bar{P}_{[D}{}^{F}\,\bar{P}_{E]}{}^{G} - P_{[D}{}^{F}\,P_{E]}{}^{G}\right)\partial_F P_{GC}\Big] \\
&= 4\Big[2\,\bar{P}^{C[A}\left(P\,\partial_C P\bar{P}\right)^{B]}{}_{E} + 2\,P^{C[A}\left(P\,\partial_C P\bar{P}\right)_{E}{}^{B]} \\
&\qquad\qquad - \mathcal{H}_E{}^C\left(P\,\partial_C P\bar{P}\right)^{[AB]} + \left(P\,\partial_E P\bar{P}\right)^{(AB)}\Big] \\
&= -2\,\mathcal{H}^{C[A}\,\mathcal{H}^{B]D}\,\partial_C\mathcal{H}_{DE} + 2\,\partial^{[A}\mathcal{H}^{B]}{}_E - \mathcal{H}_E{}^C\,\mathcal{H}^{[A}{}_D\,\partial_C\mathcal{H}^{B]D} + \partial_E\mathcal{H}^{AB}.
\end{aligned} \tag{4.112}$$

Using the identity above (4.112), the Komar form (4.111) can be written in $\mathcal{H}_{MN}$:

$$\begin{aligned}
K^{AB}[X] &= -2\mathcal{H}^{C[A}\left(\partial_C X^{B]} + \partial^{B]}X_C + \mathcal{H}^{B]D}(\partial_C\mathcal{H}_{DE})X^E\right) \\
&\quad + \left(2\partial^{[A}\mathcal{H}^{B]}{}_E - \mathcal{H}_E{}^C\mathcal{H}^{[A}{}_D\partial_C\mathcal{H}^{B]D}\right)X^E.
\end{aligned} \tag{4.113}$$

Meanwhile, the boundary term B^A (4.45) also can be rewritten by e^{-2d}

and $\mathcal{H}_{AB}$:

$$B^A = -2\mathcal{H}^{AB}\partial_B \ln e^{-2d} - \partial_B \mathcal{H}^{AB}. \tag{4.114}$$

Using those explicit forms, we can apply to various backgrounds to calculate their charges. You may also be interested to compare our Noether currents to the other's Noether currents [2]. Blair [2] also developed Noether currents in DFT in many different ways and obtained many different forms of currents.

We may further parametrise the DFT fields to Riemannian fields to check the correspondence between our charge to the ADT charge in Einstein gravity and to make easier to apply the formula. Actually, under Riemannian section, we only need the Riemannian part $K^{\mu\nu}$ of the DFT Komar form K^{AB}, as we only integrate on physical dimensions. The Riemannian parametrisation of $K^{\mu\nu}$ is discussed in Section 5.1.

4.6 Discussion on Other Charges

In this thesis, we discuss DFT charge corresponding ADT energy and momentum which is a shadow of the matter field Noether charge on the gravitational background. We do not capture other various conserved charges existing in String Theory.

p-brane charges

First, there are Dp-brane charges, which appears on the $(p+1)$-form field. We can also obtain the T-duality transformation of the p-form by starting from considering the compactification of the p-form [19]; The p-form fields $C^{(p)}$ mix with $B_{\mu\nu}$ under the T-duality[20], which makes sense because $B_{\mu\nu}$ is a part of the geometric background in the T-duality covariant gravitational theory. To obtain Dp-brane charges, we may express the p-form in Double geometry and perform Noether procedure on it to obtain the Dp charge formula in field theory side.

Monopole charges

Furthermore, there are monopole charges of conventional fields, or the DFT fields. The NS5-brane has a magnetic monopole as it is a magnetic dual of the D5-brane. Its T-dual, the 5^2_2-brane, has monopole charge on T-folds: you can only describe this charge properly on the T-duality-covariant formalism. Integrating "field-strength" (*i.e.* connections in gravitational sense) on a cycle may show the monopole charge of the DFT background [4, 106, 109].

Black hole charges

Wald and other people showed that the Noether charge in the gravitational theory deeply related to the black entropy[46–48]. Our DFT formula may also be used to calculate the black hole charges. Actually, there are already some studies about black hole thermodynamics in DFT[5]. It would also be interesting to push this direction in the future.

Chapter 5 Applications: $2D$-momenta

In this chapter, we apply the generalised ADT formula of the conserved global charge (4.55) to various well-known DFT backgrounds with constant Killing vectors to obtain their $2D$-momenta (Double Field Theory version of linear momenta). According to our understandings of Double Field Theory, these new momenta should be a set of ADM momenta and winding charges, but we need a specific parametrisation of Killing vector to see the right form of the Riemannian ADM momenta. In the first section §5.1, we discuss the right Riemannian parametrisation of Killing vectors and the right way to choose the basis of the Killing vector space in backgrounds with various asymptotic topologies.

Using this parametrisation, we apply our formula to backgrounds the most well studied in Double Field Theory, waves and monopoles of DFT.[1] Among those backgrounds with different constant parameters and different positions in the $O(D,D)$ orbit, there are a "non-geometric" solution and a "non-Riemannian" solution, which are interesting solutions because they do not have a good description in Riemannian geometry, but have a clearer description in DFT. In the same section, we also present the common form and

[1] Those backgrounds, *i.e.* plane waves and monopoles of DFT, are well-reviewed by Rudolph's thesis.[86]

topologies of our examples and discuss the Killing vector space in (doubled) spacetime of such form. In the remaining sections, we calculate and discuss energy, momentum and winding charges of each example.[2]

In the next section §5.2, we briefly check our results with classical Einstein gravity backgrounds. First, we prove that our formula (4.55) is the same with ADM momentum formula eliminating the Kalb-Ramond field $B_{\mu\nu}$ and the dilaton Φ. Also, we check our result with some well-known pure-Einstein-gravity backgrounds giving correct ADM momenta.

In §5.3, we consider a macroscopic infinite fundamental string stretched in one direction in the asymptotically flat spacetime, which is a null wave in DFT point of view. Considering conserved charge in Doubled spacetime shows a good picture of how charges change under $O(D,D)$ T-dualities. Especially, there is a non-Riemannian solution in the $O(D,D)$ orbit of the null wave where the Riemannian description diverges, and we can handle this solution in DFT without any problem. Hence, we discuss its conserved charges afterwards.

In §5.4, we consider monopoles of DFT. A Kaluza-Klein monopole in 10-dimension with $\mathbb{R}^2 \times T^2$ topology in its transverse directions is a good start point, as it is obviously a monopole in Riemannian geometry and thus in Double geometry. Its T-dual objects are the NS5-brane and an "exotic" 5^2_2-brane.[86, 90, 91] We obtain linear momenta and winding charges for these objects and compare them to the known charges in String Theory. [3] Comparing

[2] Other kinds of charge and further examples are also considered in papers other than ours[1]. Other kinds of charge such as black hole entropy is considered in [5]. More exotic examples such as negative strings are considered in [6].

[3] In our application, there is no charge to compare other than the energy. The magnetic charge (*i.e.* monopole charge) can be obtained by considering "field strength (flux)" but not by a global charge formula. The monopole charge in DFT is studied by Musaev *et al.*[4]

known quantum numbers to the field theory charge formula may strengthen the legitimacy of the formula. Also, as a NS5-brane is an extremal case of the black 5-brane, we also study the general case, the black brane case.

In the last section §5.5, we discuss an asymptotically non-flat example. A "linear dilaton" background gives diverging conserved charge term, which should be compensated by extrinsic curvature term. We show that the extrinsic curvature surface term introduced in the previous chapter can successfully compensate for the diverging term, perfectly analogous to how diverging charges are compensated in Einstein Gravity.

5.1 Killing Vector Normalisation and Riemannian Parametrisation

In this section, we show the exact settings for the section and the integral domain boundary, and discuss the right notion to obtain the ADM momenta and the winding charges. In Riemannian geometry, we have to use the right notion of time and space that $G_{\mu\nu}(r \to \infty) \to \eta_{\mu\nu}$: using $\mathbf{GL}(D)$-twisted notion of time and space gives conserved charges that do not match the known energy and momentum. In DFT, we also found that we have to untwist the off-diagonal components ($B_{\mu\nu}$) of the generalised metric $\mathcal{H}_{MN}$ at the infinity to get the right notion of the ADM momenta and the winding charges[1]. In this section, we discuss how we find this process, and what settings we have to apply on the charge formula (4.110) to calculate the right energy of the given background so that we can compare to the given string quantum number.

5.1.1 Riemannian Parametrisation of the Killing Vector for Right ADM Charges

To apply our formula to a specific background, we need to fix the section, which is normally the Riemannian section. After fixing the section and defining the physical and winding directions, we may perform Riemannian parametrisation to the DFT fields and our formula to confirm that our formula indeed gives us the right ADM momenta and winding charges.

We want to reduce the terms we have to calculate from the charge formula

(4.55):

$$Q[X] = \int_{\partial\mathcal{M}} \mathrm{d}^{D-1}x_{AB}\, e^{-2d} \left[K^{AB} + 2X^{[A}B^{B]}\right]. \tag{5.1}$$

Supposing the Riemannian section means that the integral domain is within the section, so we only need Riemannian components of the DFT Komar form K^{AB} and the boundary term B^A:

$$Q[X] = \int_{\partial\mathcal{M}} \mathrm{d}^{D-1}x_{\mu\nu}\, e^{-2d} \left[K^{\mu\nu} + 2X^{[\mu}B^{\nu]}\right], \tag{5.2}$$

where $K^{\mu\nu}$ means the Riemannian components of K^{AB} in the sense of (1.48), not the Komar form in the Riemannian geometry.

From the explicit form of the DFT Komar form (4.113), its Riemannian part $K^{\mu\nu}$ may be further simplified by the section condition $\tilde{\partial} = 0$:

$$K^{\mu\nu}[X] = -2\mathcal{H}^{\rho[\mu}\left(\partial_\rho X^{\nu]} + \mathcal{H}^{\nu]D}(\partial_\rho \mathcal{H}_{DE})X^E\right) - \mathcal{H}_E{}^\rho \mathcal{H}^{[\mu}{}_D(\partial_\rho \mathcal{H}^{\nu]D})X^E, \tag{5.3}$$

while the Riemannian part of the boundary term (4.114) is written by

$$B^\mu[X] = -2\mathcal{H}^{\mu\nu}\partial_\nu \ln e^{-2d} - \partial_\nu \mathcal{H}^{\mu\nu}. \tag{5.4}$$

We further parametrise the generalised metric by the Riemannian parametrisation given by (3.62). First, the boundary term trivially parametrised as

$$B^{\mu} = -2G^{\mu\nu}\partial_{\nu} \ln(e^{-2\Phi}\sqrt{|G|}) - \partial_{\nu}G^{\mu\nu}, \tag{5.5}$$

which does not involve the $B_{\mu\nu}$-field and is purely geometric except the dilaton term, so this corresponds to the boundary term in the Einstein theory. On the other hand, we found that the Komar form is not parametrised into the Noether potentials of the $G_{\mu\nu}$ and $B_{\mu\nu}$ with the simple Killing vector parametrisation $X^A = \begin{pmatrix} \zeta_\mu \\ \xi^\mu \end{pmatrix}$ [1]. By only introducing the twisted parametrisation

$$X^A = \begin{pmatrix} \zeta_\mu + B_{\mu\nu}\xi^\nu \\ \xi^\mu \end{pmatrix}, \tag{5.6}$$

which is motivated by [70], the DFT Komar form (5.3) is parametrised into

$$K^{\mu\nu}[X] = -2\nabla^{[\mu}\xi^{\nu]} - H^{\mu\nu\rho}\zeta_{\rho}, \tag{5.7}$$

where ∇^{μ} is the covariant derivative in Riemannian geometry, and $H^{\mu\nu\rho}$ is the field strengh of $B_{\mu\nu}$. This is the form we want as the Noether potential.

Thus, here we could see that the parametrisation (5.6) is the uniquely correct way to parametrise the DFT charge into the geometric charges (momenta) and the winding charges.

Further Direction to Yang-Mills

Furthermore, obviously, this Killing vector parametrisation also parametrise the Yang-Mills term $V^C X_C$ into

$$V^C X_C = A_\mu \xi^\mu + \varphi^\mu \zeta_\mu, \tag{5.8}$$

where

$$V_A = \begin{pmatrix} \varphi^\lambda \\ A_\mu + B_{\mu\nu}\varphi^\nu \end{pmatrix}. \tag{5.9}$$

Using this, we may also try to parametrise the Noether potential of the Yang-Mills field:

$$(P\mathcal{F}\bar{P})^{[AB}V^{C]}X_C, \tag{5.10}$$

in the future.

5.1.2 Reasoning for the $B_{\mu\nu}$-twisted Parametrisation of the Killing vector

In our original study [1], the reasoning for this parametrisation of the DFT Killing vector was not discussed. During preparation of my thesis, I got the idea of the reasoning from the discussion with my current instructor.

Even in the conventional gravitational theory, we have to look whether we have the right notion of time and space to get the right energy and momentum.

If the gravitational background does not converge to the trivial $\eta_{\mu\nu}$ but goes to another asymptotic values, which may be related with $\eta_{\mu\nu}$ by scaling or linear transformations, we may get the time-translation charge scaled of tilted from the right energy. To calculate energy in such backgrounds, we have to rotate the background itself, or we can rotate the Killing vector basis at the boundary.

Also in DFT, the Killing vector parametrisation is the basis rotation to get the pure geometric charge from the DFT charge formula. The parametrisation (5.6) is the equivalent to $\mathbf{O}(D,D)$-rotating of $\mathcal{H}_{MN}$ to eliminate $B_{\mu\nu}$ and to make $\mathcal{H}_{MN} \rightarrow \begin{pmatrix} G^{-1} & \\ & G \end{pmatrix}$. Going further, as we discussed for the conventional theory, we also have to diagonalise $G_{\mu\nu}$ to get the proper time and space unit vectors, at the boundary. (5.6) is a part of the diagonalisation of the generalised metric:

$$X'^A \mathcal{H}_{AB} X^B = \zeta'_\mu G^{\mu\nu} \zeta_\nu + \xi'^\mu G_{\mu\nu} \xi^\nu. \tag{5.11}$$

$B_{\mu\nu}$-tilting (5.6) is also important when we think the definition of ADT charge. ADT charge is constructed as a charge of the gravitational background equivalent to the spacetime charge (energy-momentum tensor) of the matter field. Thus, the translational Noether charge of the total action $\int (R - \frac{1}{12} H^2) \cdots$ gives the combination of the ADT charge from the gravitational background and the energy-momentum tensor from $B_{\mu\nu}$:

$$Q[\xi^\mu] = Q_{ADM}[\xi^\mu] + \int \mathrm{d}^{D-1}x_\nu \, \xi^\mu t^\nu_\mu[B_{\alpha\beta}], \tag{5.12}$$

where $J^\nu[B_{\alpha\beta};\xi^\mu] = \xi^\mu t^\nu_\mu[B_{\alpha\beta}]$ is the Noether current of Kalb-Ramond field action $I_B = \int \mathrm{d}^D x \, e^{-2\Phi}\sqrt{-G}H_{\mu\nu\rho}H^{\mu\nu\rho}$. This is double counting of the energy-momentum of $B_{\mu\nu}$. The $B_{\mu\nu}$-tilting (or twisting) in (5.6) may eliminate this double counting.

5.1.3 Settings for Practical Applications

Obviously, we do not want to $B_{\mu\nu}$-tilt(twist) throughout the domain, which makes $\mathcal{H}_{MN} = \eta_{MN} = \begin{pmatrix} \eta^{\mu\nu} & \\ & \eta_{\mu\nu} \end{pmatrix}$ everywhere and makes the theory vielbein formalism. We allow local deviations and control only asymptotic behaviours, by performing global $\mathbf{O}(D,D)$ transformations.

Topological structure of general solutions

To define the boundary, we have to look the topological structure (and the isometries) of the background. A p-brane with n compactified directions has $9-p-n$ noncompact transverse directions. The background has no coordinate dependence in the spanning dimensions and the compact dimensions of the brane. This is similar to a point particle in spatial $\mathbb{R}^{9-p-n}$. The radius r is defined as the distance from this point particle in $\mathbb{R}^{9-p-n}$, and the boundary is defined as $S^{8-p-n} \times$ (volume of the compact dimensions).

In this chapter, in most of cases, we follow this notation:

$$x^{\mu} = (t, x^1, x^2, \cdots, x^9) \tag{5.13}$$

$$\Downarrow$$

$$x^{\mu} = (\underbrace{t, z^1, z^2, \cdots, z^p}_{\text{spanning}}, \underbrace{x^{p+1}, \cdots, x^{p+n}}_{\text{compact transverse}}, \underbrace{y^1, \cdots, y^{9-p-n}}_{\text{noncompact transverse}}), \tag{5.14}$$

where the spatial spanning p-dimensions of p-brane is denoted by z^i's, and the noncompact transverse directions is denoted by y^i's. In the y-space, we use the integral techinique which we are used to exercise in $\mathbb{R}^{9-p-n}$. We can convert $\mathbb{R}^{9-p-n}$ into the spherical coordinate and we define r:

$$r^2 = \sum_i \left(y^i\right)^2. \tag{5.15}$$

The integral domain would be the sphere S^{8-p-n} and $r \to \infty$.

Application of our formula to these solutions: only remaining terms

In this setting, the charge formula may be written by

$$Q[X] \sim V[T_{n+p}] \int_{r\to\infty} \mathrm{d}\Omega_{8-p-n}\, r^{8-p-n} \left[K^{tr} + 2X^{[t}B^{r]}\right] \tag{5.16}$$

where Ω_{8-p-n} is the total surface of the unit sphere S^{8-p-n} and $V[T_{n+p}]$ is the volume of the spatial compact dimensions.

In this background, the possible candidates of the Killing vector space are the torus dimensions and the spanning dimensions including the time. As r is not a Killing direction, we do not want to put any X^M with $X^r \neq 0$.[4] Thus, the formula simplifies:

$$Q[X] \sim \int \mathrm{d}\Omega\, r^{8-p-n} \left[K^{tr} + X^t B^r\right], \tag{5.17}$$

so we show that the only components we have to know in normal situations are K^{tr} and B^r.

Suppose we write the generalised metric background divided into the asymptotic part and the deviation part:

$$\mathcal{H}_{AB} = \mathcal{H}^{(0)}_{AB} + \Delta_{AB}, \quad \mathcal{H}^{(0)}_{AB} = \begin{pmatrix} g^{\mu\nu} & -g^{\mu\rho} b_{\rho\nu} \\ b_{\mu\rho} g^{\rho\nu} & g_{\mu\mu} - b_{\mu\rho} g^{\rho\lambda} b_{\lambda\nu} \end{pmatrix}, \tag{5.18}$$

where $g_{\mu\nu}$ and $b_{\mu\nu}$ are the asymptotic value of $G_{\mu\nu}$ and $B_{\mu\nu}$. Also suppose

$$\Delta_{AB} \sim r^{-\alpha}, \tag{5.19}$$

which is where the charge comes from. As the Komar form consists of connections and field strength, which $\sim r^{-\alpha-1}$, the behaviour of the total charge (5.17) goes to

$$Q[X] \sim r^{8-p-n} \cdot r^{-\alpha-1}, \tag{5.20}$$

unless y^i's are also compact *e.g.* AdS or dS.[5] Thus, the only term that gives

[4] Cartesian y-directions may be asymptotic Killing directions. But anyway, we only want to deal with object not moving in y-directions.

[5] We did not consider such case from the start of the formulation first place. We may also

the converging contribution to the charge has the order of [6]

$$\alpha = 7 - p - n \Rightarrow \Delta_{AB} \sim r^{p+n-7}, e^{-2d} = 1 + \mathcal{O}(r^{p+n-7}). \tag{5.21}$$

Any terms with larger order should be mended by the extrinsic curvature term (§4.2.3), and any terms with smaller order vanish.

The O(D, D) global rotation of the asymptotic background discussed in §5.1.2, and the normalization of the Killing vector basis

Now, the background field $\mathcal{H}^{(0)}_{AB}$ also should be fully diagonalised as we discussed in previous subsections. After the full orthonormalization of the basis, we denote all the normal $2D$ unit vectors by

$$E_A := \begin{pmatrix} (e^{-1})_\mu{}^a \tilde{\partial}^\mu \\ e^\mu{}_a(\partial_\mu - b_{\mu\nu}\tilde{\partial}^\nu) \end{pmatrix}, \tag{5.22}$$

where each line of the matrix is each unit vector, and the charge associated with those orthonormal unit vector by

$$P_A := Q[E_A]. \tag{5.23}$$

Be aware that only here, "$_A$" index is not a $2D$-vector index but an index counting $2D$ entities of vectors or charges. This is the right way to obtain the linear DFT charges, which will correctly correspond to energy, momentum,

further extend our study to such area where the boundaries are not well-defined.

[6]Defined $\left\langle e^{-2d} \right\rangle = 1$ in (1.58).

and winding charges; hereby, we define P_A the $2D$-momenta.

The importance of choosing the right basis is more immersively studied by the examples in §5.3. These examples have Killing vector space along the spanning dimensions and freedom to choose the basis by Galilean boost.[7] In §5.3, the basis choosing process may be seen in concrete examples.

Remark. DFT is manifestly covariant under $\mathbf{O}(D, D)$ rotations which act on both the tensor indices and the arguments of the tensor, *i.e.* coordinates. In this case, the whole configuration including the section itself is rotated, and there should be no change in physics. However, if a specific given background admits an isometry, there is ambiguity of choosing the section. Without rotating the section, it is possible to rotate the tensor indices only and this can generate a physically different configuration. Thus, while our global charge is manifestly $\mathbf{O}(D, D)$ covariant, it may not transform covariantly if we keep the section fixed and rotate only the tensor indices.

Future study. For more complexity, we may try to obtain rotational charge of DFT: *e.g.* T-dual of a Kerr black hole may be an interesting example. However, Killing vector space of the rotational symmetry is much more rigid: (3+1)-dimensional black holes have only one kind of rotational charge while they can have linear momentum because the asymptotic Killing equation is satisfied. I suspect that we do not have to discuss the normalisation of the Killing vector space that much in the rotational symmetry area. But,

[7] This idea came from the discussion with Prof. Seok Kim in 2021.

on the other hand, it is quite interesting to watch rotating object in higher dimensions. To study the rotating T-folds might be very interesting topic for future study.[8]

[8] Charges of the rotating black hole in the Einstein gravity is discussed in [51].

5.2 Application to the pure-Einstein backgrounds

Here, considering the pure Einstein gravity, *i.e.*

$$\Phi = 0\,, \qquad\qquad B_{\mu\nu} = 0\,, \tag{5.24}$$

we show that the ADM mass, P_t, defined on (5.23), evaluated for a background with $\mathcal{M} = \mathbb{R}^{D-1}$, correctly reproduces the well-known ADM mass formula [29],

$$E_{\text{ADM}} = \frac{1}{2\kappa_D^2} \int_{S_\infty^{D-2}} \mathrm{d}\Omega_{D-2}\, r^{D-2}\, \hat{n}^k\, G^{ij} \left(\partial_i G_{kj} - \partial_k G_{ij}\right)\,, \tag{5.25}$$

where $\hat{n}^k\partial_k = \partial_r$ is a radial vector that becomes a unit vector at infinity. Note that the gravitational constant $2\kappa_D^2$ will be omit in this discusssion and so forth time to time.

From $B_{\mu\nu} = 0$, the DFT-Noether potential (5.7) is reduced to the standard Komar potential:

$$K^{\mu\nu}[X] = 2\,\xi^{[\mu;\nu]}\,. \tag{5.26}$$

On the other hand, B^μ, in the Cartesian coordinates, simply becomes

$$B^\mu = -2\,\mathcal{H}^{\mu\nu}\,\partial_\nu \ln\sqrt{|G|} - \partial_\nu\mathcal{H}^{\mu\nu} = G^{\mu\nu}\,G^{\rho\sigma}\left(\partial_\rho G_{\nu\sigma} - \partial_\nu G_{\rho\sigma}\right). \tag{5.27}$$

In particular, the identically conserved current, $\partial_\nu(\delta^{[\mu}{}_\lambda B^{\nu]})$, corresponds to the Einstein pseudo-tensor *a la* Dirac [26].

Our definition of the ADM mass is now

$$\begin{aligned} P_t = Q[\partial_t] &= \frac{1}{2\kappa_D^2} \oint_{\partial\mathcal{M}} \mathrm{d}^{D-2}x_{\mu\nu}\,\sqrt{|G|}\,\left(K^{\mu\nu}[\partial_t] + 2X^{[\mu}\,B^{\nu]}\right) \\ &= \frac{1}{2\kappa_D^2}\cdot 2\int_{S_\infty^{D-2}} \mathrm{d}^{D-2}x_{tr}\,\sqrt{|G|}\,\left(K^{tr}[\partial_t] + B^r\right), \end{aligned} \tag{5.28}$$

and we have as r goes to infinity,

$$K^{tr}[\partial_t] = -2\,G^{t\mu}\,G^{r\nu}\,\partial_{[\mu}G_{\nu]t} \approx -\hat{n}^k\left(\partial_k G_{tt} - \partial_t G_{kt}\right), \tag{5.29}$$

$$B^r \approx \hat{n}^k\left(\partial_k G_{tt} - \partial_t G_{kt}\right) + \hat{n}^k\,G^{ij}\left(\partial_i G_{kj} - \partial_k G_{ij}\right). \tag{5.30}$$

These precisely coincide with the known results of [47]. Finally, summing them up and using the expression for the integral measure at the spatial infinity, we have

$$2\int_{S_\infty^{D-2}} \mathrm{d}^{D-2}x_{tr}\sqrt{|G|}\quad\cdots\quad = \int_{S_\infty^{D-2}} \mathrm{d}\Omega_{D-2}\,r^{D-2}\quad\cdots\,. \tag{5.31}$$

Using this result and expressions (5.29) and (5.30), we obtain that (5.28) equals (5.25).

5.2.1 Reissner-Nordström Black Hole

As a simplest example of pure-Einstein background, we consider the Schwarzschild solution. In the same time, we want to see a bit more generalisation into the Yang-Mills coupling. As a simplest example of pure-Einstein and YM-coupled DFT, let us consider the Reissner-Nordström black hole,

$$
\begin{aligned}
\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + f^{-1}(r)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2_{D-2}\,, \quad \Phi = 0\,,\\
A = -\frac{2g_{\mathrm{YM}}Q}{r^{D-3}}\,\mathrm{d}t\,, \quad B_{\mu\nu} = 0\,,\\
f(r) := 1 - \frac{2\mu}{r^{D-3}} + \frac{q^2}{r^{2(D-3)}}\,, \quad \mu := \frac{M}{2(D-2)\,\Omega_{D-2}}\,, \quad q^2 := \frac{2(D-3)\,Q^2}{(D-2)}\,,
\end{aligned}
\tag{5.32}
$$

The doubled vector potential satisfies (3.56) and is parametrized by $V_A = (0,\, A_\mu)$. Consequently, we have $(P\mathcal{F}\bar{P})^{[\mu\nu]} = -f^{\mu\nu} = -G^{\mu\rho}G^{\nu\sigma}(\partial_\rho A_\sigma - \partial_\sigma A_\rho)$, see (3.18) of [70].

In this background, the global charge (4.110) becomes

$$
Q[X] = 2\int_{\partial\mathcal{M}} \mathrm{d}^{D-2}x_{tr}\,\sqrt{|G|}\,\big(K^{[tr]} + 2\,\xi^{[t}B^{r]} - 4\,g_{\mathrm{YM}}^{-2}\,f^{tr}A_t\,\xi^t\big)\,, \tag{5.33}
$$

with the natural units $2\kappa^2 = 1$ in this example. From the asymptotic behaviour, $f^{tr}A_t \propto r^{-(2D-5)}$, the last term does not contribute to the surface integral. The nontrivial contributions come from

$$
K^{tr}[\partial_t] \approx 2(D-3)\,\mu\,r^{-(D-2)}\,, \quad B^r \approx 2\,\mu\,r^{-(D-2)}\,, \tag{5.34}
$$

such that we can recover the correct ADM mass,[9]

$$
P_t = Q[\partial_t] = 2(D-2)\,\Omega_{D-2}\,\mu = M\,. \tag{5.35}
$$

[9] If we make a constant shift in the gauge field, $A_\mu \to A_\mu + a_\mu$, the $f^{tr}A_t$ term also gives a contribution to the ADM mass; $Q[\partial_t] \to Q[\partial_t] - 8(D-3)\,\Omega_{D-2}\,g_{\mathrm{YM}}^{-1}\,Q\,a_t$, which depends on the free parameter, a_t. Like the definition of the ADM momentum given in (5.23), we can define a gauge invariant combination, $\widehat{P}_t := P_t + 8(D-3)\,\Omega_{D-2}\,g_{\mathrm{YM}}^{-1}\,Q\,A_t$.

5.3 A Fundamental String, a Null Wave, and a Non-Riemannian Background

In this section, we discuss backgrounds of null wave in Double Field Theory propagating in various directions. The fundamental string is one of the fundamental element in String Theory. The background of the massless mode of the macroscopic stable fundamental string object (F1 brane) described in [87] has a winding charge that has the same value of its energy. In Double Field Theory, this can be interpreted as a 1-brane having light-like $2D$-momentum *i.e.* a null wave. Indeed, performing a T-duality along its spanning direction in which the winding charge is gives a null wave solution in Riemannian spacetime.[6, 93] In our study, we apply our formula (4.55) and show that we get the expected $2D$-momenta for those backgrounds. Additionally, we find that the gauge transform of those backgrounds does not affect the momenta even after the T-duality transform while the T-dual of the gauge transform seems nontrivial in Riemannian sense.

Moreover, performing a T-duality along time direction gives a DFT solution that has no Riemannian description.[22, 67, 107] This kind of solutions is called the *non-Riemannian background*, and seems not physical, but is clearly a solution of String Theory.[10] We can well-describe this object in DFT fields, and we shall have no problem to obtain the $2D$-momentum of this

[10]C. Hull is the pioneer in studying timelike T-duality even before DFT. "T-duality on a timelike circle does not interchange IIA and IIB string theories," but transforms a type II theory into type II* theory. There are many exotic properties in type II* theories.[22] In Double Field Theory, T-duality is a simple double-spacetime rotation, so we can explore objects in type II* theories with DFT descriptions.

object. We show that our formula (4.110) gives non-diverging $2D$-momentum from the given non-Riemannian background.

5.3.1 A Fundamental String (F1 string)

The macroscopic string solution has been studied in [87]. However, the result in this paper is in Einstein frame, we need to convert it to the string frame. In the string frame, a macroscopic F1 string stretched in straight line in D-dimensional spacetime is given by [11]

$$\begin{aligned}
\mathrm{d}s^2 &= f(r)^{\frac{2-4\sqrt{2/(D-2)}}{D-2}} \left(\big(H(r)\big)^{-1} \left(-\,\mathrm{d}t^2 + \mathrm{d}z^2\right) + \delta_{ij}\,\mathrm{d}y^i\,\mathrm{d}y^j \right) \\
B^{(2)} &= \left(c - \big(H(r)\big)^{-1} \right) dt \wedge dz \\
e^{-2\Phi} &= \big(H(r)\big)^{\sqrt{\frac{8}{D-2}}}\,,
\end{aligned} \tag{5.36}$$

which is simplified in 10-dimension[1, 75] [12] as

$$\begin{aligned}
\mathrm{d}s^2 &= H^{-1}(r)\left(-\,\mathrm{d}t^2 + \mathrm{d}z^2\right) + \left((\mathrm{d}y^1)^2 + \cdots + (\mathrm{d}y^{D-2})^2 \right), \\
B^{(2)} &= \left(c - H^{-1}(r) \right) dt \wedge dz, \\
e^{-2\Phi} &= H(r),
\end{aligned} \tag{5.37}$$

[11] The overall constant of the dilaton $e^{-2\Phi}$ also can be a free parameter, but we already defined as $\langle \Phi \rangle = 0$ in (1.58).

[12] The solution in [87] is in Einstein frame. Also, (5.17) in [75] and (4.32) in [1] directly mention this example, but different $B_{\mu\nu}$ sign from [87]. We follow the original sign of [87].

where c is the constant gauge shift parameter of $B_{\mu\nu}$ that we can freely adjust, and $H(r)$ is the harmonic function which satisfies

$$\nabla_\mu \nabla^\mu H(r) = -16\pi G_N \mu\, \delta^{(D-2)}(x), \tag{5.38}$$

where μ is the mass density of the string (up to overall constant). Also, $H^{-1}(r)$ means $1/H(r)$. The harmonic function in this thesis and in most other literature is defined to satisfy $\lim_{r\to\infty} H(r) = 1$ to make the metric go to the trivial identity matrix at the infinity:

$$H(r) = 1 - \int_r^\infty \mathrm{d}r' \frac{-16\pi G_N \mu}{\Omega_{D-3} r'^{D-3}} = \begin{cases} 1 + \dfrac{16\pi G_N \mu}{(D-4)\Omega_{D-3} r^{D-4}} & \text{if } D > 4, \\ 1 - 8 G_N \mu \ln \left| \dfrac{r}{r_c} \right| & \text{if } D = 4, \end{cases} \tag{5.39}$$

where $\Omega_{D-3} = \frac{2\pi^{D/2-1}}{\Gamma(D/2-1)}$ is the surface area of the unit sphere S^{D-3}, and r_c is the cutoff radius.[13] [87]

The background in 10-dimension may be converted into the DFT expression:

$$\begin{aligned} \mathrm{d}s^2 &= (\mathrm{d}y^i)^2 + (\mathrm{d}\tilde{y}_i)^2 + H^{-1}\left(-\,\mathrm{d}t^2 + \mathrm{d}z^2\right) \\ &\quad + H\left(-\left(\mathrm{d}\tilde{t} - (c - H^{-1})\,\mathrm{d}z\right)^2 + \left(\mathrm{d}\tilde{z} + (c - H^{-1})\,\mathrm{d}t\right)^2\right), \\ e^{-2d} &= 1, \end{aligned} \tag{5.40}$$

[13] As the log function diverges at infinity, we define the cutoff radius where the boundary sits.

using (1.92). We may also express the generalised metric with the flat part and additional parts:

$$
\begin{aligned}
\mathsf{ds}^2 &= \eta^{\mu\nu}\,\mathrm{d}\tilde{x}_\mu\,\mathrm{d}\tilde{x}_\nu + \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + (H-1)(-\,\mathrm{d}\tilde{t}^2 + \mathrm{d}\tilde{z}^2) \\
&\qquad -(c^2H - 2c + 1)(-\,\mathrm{d}t^2 + \mathrm{d}z^2) + 2(cH-1)\,\mathrm{d}\tilde{t}\;\mathrm{d}z + 2(cH-1)\,\mathrm{d}\tilde{z}\;\mathrm{d}t, \\
e^{-2d} &= 1,
\end{aligned}
\tag{5.41}
$$

where μ's are covering whole 10 dimensions. Here, as you can see in (5.37), c is simply a B-gauge shift parameter, which is also a constant $\mathbf{O}(D,D)$ transformation parameter, and we can simply fix the gauge $c=1$ to remove nonzero $B_{\mu\nu}(r\to\infty)$ as we argued in Section 5.1. Then, in this gauge, the background simplifies: [1, 93]

$$
\begin{aligned}
\mathsf{ds}^2 &= \eta^{\mu\nu}\,\mathrm{d}\tilde{x}_\mu\,\mathrm{d}\tilde{x}_\nu + \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + (H-1)\Big((\mathrm{d}t + \mathrm{d}\tilde{z})^2 - (\mathrm{d}\tilde{t} - \mathrm{d}z)^2\Big)\,, \\
e^{-2d} &= 1.
\end{aligned}
\tag{5.42}
$$

Conserved charge

Now, I apply the ADT formula to the string background (5.41). In this background, $\partial_\mu d = 0$. Also, we are considering the $2D$ linear momenta here, so $\partial_\rho X^\nu = 0$. The Komar form and the boundary term (5.3) and (5.4) are calculated as:[14]

$$K^{tr} = \mathcal{H}^{tD}\partial_r\mathcal{H}_{DE}X^E = -cH'(r)X^t - H'(r)X_z, \tag{5.43}$$

$$B^r = 0. \tag{5.44}$$

Note that there is no boundary contribution in this background.

In the right gauge $c = 1$ that makes asymptotic $b_{\mu\nu} = 0$,

$$K^{tr}[\partial_t] = -\partial_r H(r)\,, \quad K^{tr}[\tilde{\partial}^z] = -\partial_r H(r)\,, \tag{5.45}$$

and the ADT momenta are

$$P_t = Q[\partial_t] = \frac{1}{16\pi G_N}\int \mathrm{d}z \int_{r\to\infty} \mathrm{d}\Omega_{D-3}\, r^{D-3}(-\partial_r H) = \int \mathrm{d}z\,\mu, \tag{5.46}$$

$$\tilde{P}^z = Q[\tilde{\partial}^z] = \frac{1}{16\pi G_N}\int \mathrm{d}z \int_{r\to\infty} \mathrm{d}\Omega_{D-3}\, r^{D-3}(-\partial_r H) = \int \mathrm{d}z\,\mu, \tag{5.47}$$

where μ is the energy density of the string. As these are the only non-zero components, the ADM momentum in the string background becomes

$$P_A^{(\text{string})} = n_D\,(0, +1, 0 \ldots, 0\,;\, +1, 0, \ldots, 0) \tag{5.48}$$

with the ADM energy n_D. If the z-direction is compact and its radius is R_z,

[14]The $2D$-dimension is constructed in Cartesian sense here. The radius r is not a spherical coordinate but simply an alias of coordinates and a unit vector direction that is not a coordinate basis. So, $\mathcal{H}^{rr}$ is not $\sim r^{-2}$ in spherical coordinate sense, but $\hat{r}_{y_i}\hat{r}_{y_j}\mathcal{H}^{y_i y_j} = 1$.

the energy must be

$$n_D = \mu\,(2\pi R_z) \simeq \frac{R_z}{l_s^2} \simeq T[\mathrm{F1}] \tag{5.49}$$

which coincides with the string tension formula (1.28), as the mass density has the dimension of $\mu \sim l_s^{-2}$.

We discuss each charge, the string winding charge and the physical ADM momentum, as following:

String winding charge. We can identify the well-known string winding charge (along z direction) given by

$$Q_{\mathrm{F1}} \propto \int_{S_\infty^{D-3}} e^{-2\Phi} \;*_D H\,, \tag{5.50}$$

as the global charge, $Q[\tilde{\partial}^z]$, for the winding direction $\tilde{z}$, *i.e.* ADM momentum along the dual direction. Here, let us focus on a spacetime of toroidal topology, $\mathcal{M} = S_z \times \mathbb{R}^{D-2}$, such that a string is winding along the compactified z direction. The momentum in the dual $\tilde{z}$ direction reads

$$\tilde{P}^z = Q[\tilde{\partial}^z] = 2\int_{\partial\mathcal{M}} \mathrm{d}^{D-2}x_{tr}\, e^{-2\Phi}\,\sqrt{|G|}\,K^{tr}[\tilde{\partial}^z]\,, \tag{5.51}$$

with the natural units $2\kappa^2 = 1$. By using the formula (5.7), the DFT-Noether potential becomes, restricted on the section,

$$K^{\mu\nu}[\tilde{\partial}^z] = -H^{\mu\nu z}\,. \tag{5.52}$$

Therefore, the ADM momentum becomes

$$\begin{aligned}\tilde{P}^z = Q[\tilde{\partial}^z] &= -\int \mathrm{d}z \int_{S^{D-3}_\infty} \mathrm{d}^{D-3}\theta\, \varepsilon_{trz\theta_1\cdots\theta_{D-3}}\, e^{-2\Phi}\, \sqrt{|G|}\, G^{t\rho}\, G^{r\sigma}\, G^{z\delta}\, H_{\rho\sigma\delta} \\ &= 2\pi R_z \int_{S^{D-3}_\infty} \mathrm{d}^{D-3}\theta\, e^{-2\Phi}\, \sqrt{|G|}\, \varepsilon_{tzr\theta_1\cdots\theta_{D-3}}\, G^{t\mu}\, G^{z\nu}\, G^{r\rho}\, H_{\mu\nu\rho} \\ &= 2\pi R_z \int_{S^{D-3}_\infty} \mathrm{d}\Omega_{D-3}\, e^{-2\Phi}\, *_D\, H\,,\end{aligned} \tag{5.53}$$

where θ_a $(a = 1, \ldots, D-3)$ are angular coordinates. Namely, the well-known flux integral for the string winding charge (in the z direction) precisely matches with the quasi-local ADM momentum in the $\tilde{z}$-direction. This supports the idea [93] that strings are waves in doubled spacetimes.

Also, the gauge parameter c in (5.37) never changes the field strength, the winding charge $Q[\tilde{\partial}^z] \sim \int e^{-2\Phi} *_D H$ also never changes. This interpretation coincides with (5.43) where the coefficient of X_z has no c dependence.

Physical ADM momentum. The constant B-field shift c affects the ADM energy of the string. From (5.37), the boundary value of B-field in expression of c is

$$b_{tz} = c - 1 + \mathcal{O}\left(r^{-(D-4)}\right). \tag{5.54}$$

Note that the gauge shift is also a $\mathbf{O}(D,D)$ rotation. Given (5.43), the conserved global charges for various gauge settings are

$$Q[\partial_A] = n_D\,(\,\underbrace{0,1,0\ldots,0}_{\text{winding charges}}\,;\;\;\underbrace{c,0,\ldots,0}_{\text{ADM momenta}}\,)\,. \tag{5.55}$$

You can see that the ADM momentum in time direction is c and can be negative. As we discussed in §5.1, We must $\mathbf{O}(D,D)$-rotate the background to the proper spatial and winding-direction basis so that $b_{tz}\to 0$. That $\mathbf{O}(D,D)$ rotation is equivalent to the process that assigns $c=0$. Regardless of the value of c, we can always find the proper $2D$ unit vector basis which is also unique in some sense where the $2D$ ADM momenta is fixed as

$$P_A^{(\text{string})} = n_D\,(\underbrace{0,+1,0\ldots,0}_{\tilde{P}^\mu};\,\underbrace{+1,0,\ldots,0}_{P_\mu})\,. \tag{5.56}$$

In this proper basis, the ADM energy $P_t^{(\text{string})}$, is always positive.

5.3.2 A Null Wave

The string background is a null wave in the dual direction $\tilde{z}$ in DFT [93] and we may obtain a Riemannian null wave (1-brane with null momentum) by performing T-duality along the z-direction to (5.40):

$$
\begin{aligned}
\mathsf{d}s^2 &= (\mathrm{d}y^i)^2 + (\mathrm{d}\tilde{y}_i)^2 + H^{-1}\Big(-\,\mathrm{d}t^2 + \mathrm{d}\tilde{z}^2\Big) \\
&\quad + H\left(-\Big(\mathrm{d}\tilde{t} - (c - H^{-1})d\tilde{z}\Big)^2 + \Big(\mathrm{d}z + (c - H^{-1})\,\mathrm{d}t\Big)^2\right) \\
&= \eta_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B + (c^2H - 2c + 1)\,\mathrm{d}t^2 + 2(cH-1)\,\mathrm{d}t\,\mathrm{d}z + (H-1)\,\mathrm{d}z^2 \\
&\quad - (H-1)\,\mathrm{d}\tilde{t}^2 + 2(cH-1)\,\mathrm{d}\tilde{t}\,\mathrm{d}\tilde{z} - (c^2H - 2c + 1)\,\mathrm{d}\tilde{z}^2, \\
e^{-2d} &= 1.
\end{aligned}
\tag{5.57}
$$

where η_{AB} is the flat generalised metric *s.t.* $\eta_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + \eta^{\mu\nu}\,\mathrm{d}\tilde{x}_\mu\,\mathrm{d}\tilde{x}_\nu$. This background is laid on the circle in z-direction with the radius $\tilde{R}_z$ inverse from the string background radius R_z in §5.3.1, as the T-duality is performed:

$$
\tilde{R}_z = \frac{l_s^2}{R_z}. \tag{5.58}
$$

The constants in the harmonic function should also be re-written in the null wave constants which is T-dual from the string background constants: the gravitational constant contains the string coupling constant, and the string coupling changes under T-duality, so the string coupling in the F1-string example and that in this null wave example is different. Following the T-duality relation of constants (1.30), the gravitational constant transforms $G_N \to (l_s^2/\tilde{R}_z^2)\tilde{G}_N$. Thus, the harmonic function is re-written in the new constants:

$$
H(r) \sim 1 + \frac{16\pi\tilde{G}_N(l_s^2/\tilde{R}_z^2)\mu}{(D-4)\Omega_{D-3}r^{D-4}}. \tag{5.59}
$$

Fixing the gauge $c = 1$ may give the proper directions for time, space, and winding directions as we discussed in §5.3.1, and it gives the simple generalised metric form given in [93]:[15]

$$\mathsf{ds}^2 = \underbrace{\eta_{AB}\, \mathrm{d}x^A\, \mathrm{d}x^B}_{\text{flat background}} + \underbrace{(H-1)\left[(\mathrm{d}t + \mathrm{d}z)^2 - (\mathrm{d}\tilde{t} - \mathrm{d}\tilde{z})^2\right]}_{\text{local fluctuation}} . \tag{5.60}$$

These (5.57) and (5.60) solutions are purely gravitational solution in D-dimensions, as they have no off-block-diagonal components in the generalised metric, and the solution (5.60) can, of course, be written as the Riemannian expression:

$$\begin{aligned} \mathrm{d}s^2 &= (H-2)\ \mathrm{d}t^2 + 2\,(H-1)\ \mathrm{d}t\ \mathrm{d}z + H\ \mathrm{d}z^2 + \delta_{mn}\ \mathrm{d}y^m\ \mathrm{d}y^n\,, \\ \Phi &= 0\,, \qquad B_{\mu\nu} = 0\,. \end{aligned} \tag{5.61}$$

Here, we compactified the z-direction with a radius R_z, in order to make the value of the global charge finite. The remaining y^m directions are treated as non-compact.

Charge for the proper basis background

In this background (5.61), using the simplifications for the pure-Einstein solutions $\partial_\mu d = 0$ and $K^{\mu\nu} = -2\, G^{\rho[\mu}\, G^{\nu]\delta}\, \partial_\rho \xi_\delta$ as stated in §5.2 , we have

[15] but the sign (noted red in (5.60)) is different because we picked the notation of [87] rather than [1, 93].

at infinity,

$$K^{tr}[\partial_t] \approx -\partial_r H(r)\,, \quad K^{tr}[\partial_z] \approx -\partial_r H(r)\,, \quad B^r \approx 0\,. \tag{5.62}$$

Same with the string case, the boundary term does not contribute to the conserved charge here. Then, it is straightforward to show that the ADM energy and the momentum in the z-direction becomes

$$\begin{aligned} P_t = Q[\partial_t] &= \frac{1}{2\tilde{\kappa}^2}\cdot 2\oint_{\partial\mathcal{M}} \mathrm{d}^{D-2}x_{tr}\, K^{tr}[\partial_t] \\ &= -\frac{1}{2\tilde{\kappa}^2}\int \mathrm{d}z \int_{S^{D-3}_\infty} \mathrm{d}\Omega_{D-3}\, R^{D-3}\,\partial_r H = (2\pi\tilde{R}_z)\left(\frac{l_s^2}{\tilde{R}_z^2}\mu\right)\,, \end{aligned} \tag{5.63}$$

$$\begin{aligned} P_z = Q[\partial_z] &= \frac{1}{2\tilde{\kappa}^2}\cdot 2\oint_{\partial\mathcal{M}} \mathrm{d}^{D-2}x_{tr}\, K^{tr}[\partial_t] \\ &= -\frac{1}{2\tilde{\kappa}^2}\int \mathrm{d}z \int_{S^{D-3}_\infty} \mathrm{d}\Omega_{D-3}\, R^{D-3}\,\partial_r H = (2\pi\tilde{R}_z)\left(\frac{l_s^2}{\tilde{R}_z^2}\mu\right)\,, \end{aligned} \tag{5.64}$$

where $2\tilde{\kappa}^2 = 16\pi\tilde{G}_N$. Here, $\partial\mathcal{M}$ is the surface of constant t and $r = R$ in the $R\to\infty$ limit, and Ω_{D-3} is a surface area of a $(D-3)$-sphere with a unit radius; $\Omega_{D-3} = 2\pi^{(D-2)/2}/\Gamma((D-2)/2)$. As the momenta in other directions are trivial, the $2D$-momentum becomes

$$P_A^{(\mathrm{wave})} = (\tilde{P}^t,\, \tilde{P}^z,\, \tilde{P}^m\,;\, P_t,\, P_z,\, P_m) = n_D\,(0,\ldots,0\,;\, +1,+1,0,\ldots,0)\,, \tag{5.65}$$

$$n_D := (2\pi\tilde{R}_z)\frac{l_s^2}{\tilde{R}_z^2}\mu \simeq \frac{1}{\tilde{R}_z} \quad (\text{mass density } \mu\sim l_s^{-2})\,, \tag{5.66}$$

which is indeed a null vector at the flat spatial infinity:

$$\big(\mathcal{H}^{(0)}\big)^{AB}\, P_A^{(\text{wave})}\, P_B^{(\text{wave})} = 0\,. \tag{5.67}$$

Also, we can confirm that the ADM energy is positive definite and proportional to $\frac{1}{R_z}$ which has negative order of R_z, not common for branes.

The gauge parameter c in the null wave

After our original study [1], during the preparation of this thesis, I obtained the conserved charges for the general gauge as in (5.57), and found that the charges depends on the gauge parameter c just like (5.55). The problem we did not discussed in our paper [1] and other papers is that even we are already in the gauge that $B_{\mu\nu} = 0$, we could not find the proper energy notion that always give the definite positive. (Other gauges are not considered in the first place.)

In the discussion with my current instruction, Prof. Seok Kim, we realised that the B-gauge transformation in the string background is transformed to the Galilean transformation in this pure-Einstein null wave background under the T-duality, and the gauge parameter $c - 1 \sim b_{tz}$ in the string background transforms into the linear transformation parameter galilean speed in the Riemannian null wave background.

In the asymptotic region, $H \to 1$, so the null wave (5.57) converges to

$$\mathsf{d}s^2 \to (\mathrm{d}y^i)^2 + (\mathrm{d}\tilde{y}_i)^2 - \mathrm{d}t^2 + \underbrace{\Big(\mathrm{d}z + \overbrace{(c-1)}^{\text{Galilean speed}}\mathrm{d}t\Big)^2}_{\text{Galilean-boosted } z} - \Big(\mathrm{d}\tilde{t} - (c-1)\,\mathrm{d}\tilde{z}\Big)^2 + \mathrm{d}\tilde{z}^2. \tag{5.68}$$

Here, you can confirm that the gauge parameter $c - 1 = b_{tz}$ in the string background (5.37) is transformed into the Galilean boost speed $c - 1$ in the null wave background by the T-duality along z-direction. You can clearly see that ∂_t is not perpendicular to ∂_z in general case, so conserved charges $Q[\partial_t]$ and $Q[\partial_z]$ does not gives the proper notion of energy and momentum.

The Double Komar form for (5.57) is

$$\begin{aligned} K^{tr} &= \mathcal{H}^{tt}(\partial_r\mathcal{H}_{tt}X^t + \partial_r\mathcal{H}_{tz}X^z) + \mathcal{H}^{tz}(\partial_r\mathcal{H}_{zt}X^t + \partial_r\mathcal{H}_{zz}X^z) \\ &= -H(c^2H'X^t + cH'X^z) + (cH-1)(cH'X^t + H'X^z) \\ &= (-cX^t - X^z)H'\big(1 + \mathcal{O}(H-1)\big) \end{aligned} \tag{5.69}$$

from (5.3), so compared to the previous examples, the conserved charges are easily found to be

$$Q[\partial_t] = c \cdot n_D, \qquad Q[\partial_z] = n_D. \tag{5.70}$$

Here, you can see that $Q[\partial_t]$ can have negative value. By applying a $\mathbf{GL}(D)$ Riemannian transformation which diagonalise and normalise the asymptotic

Riemannian metric

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \underset{r\to\infty}{\to} -\mathrm{d}t^2 + \big(\mathrm{d}z + (c-1)\,\mathrm{d}t\big)^2 + (\mathrm{d}y^i)^2 \tag{5.71}$$

from the asymptotic generalised metric (5.68), the generic null wave solution (5.57) transforms into the $c = 1$ background (5.60), and the proper $2D$ ADM momenta become

$$P_A = n_D\,(0,\dots,0\,;\,+1,+1,0,\dots,0)\,. \tag{5.72}$$

5.3.3 Comment on $\mathbf{O}(D,D)$ rotation relation between a null wave and a string

The $2D$-momentum of the fundamental string in §5.3.1 is given by

$$P_A = n_D\,(0,+1,\dots,0\,;\,+1,0,0,\dots,0), \tag{5.73}$$

while the $2D$-momentum of the null wave in §5.3.2 is

$$P_A = n_D\,(0,0,\dots,0\,;\,+1,+1,0,\dots,0), \tag{5.74}$$

where n_D is the string tension for each case of the fundamental string and the null wave solution.

This is the expected result: our global charge formula is covariant under global $\mathbf{O}(D,D)$ transformations, so the ADM momentum in the string background should be related to that in the null-wave background by an

$\mathbf{O}(D,D)$ rotation, $\Lambda_A{}^B$, which corresponds to the T-duality along the z direction,

$$P_A^{(\text{string})} = \Lambda_A{}^B \, P_B^{(\text{wave})} \,, \quad \tilde{R}_z = \frac{l_s^2}{R_z} \,. \tag{5.75}$$

Furthermore, the energy n_D is written in F1 constants and null wave constants respectively,

$$n_D \simeq \frac{R_z}{l_s^2} = \frac{1}{\tilde{R}_z} \tag{5.76}$$

where R_z is the z-circle radius in the F1 example and $\tilde{R}_z$ is the z-circle radius in the null wave example.

T-duality interpretation in aspect of DFT topology

As T-duality in DFT is interpreted as a coordinate swap between a physical coordinate and its dual coordinate. To make this coordinate transform automatically give the radius inverting relation of T-duality, I may suppose the topology $S^1_z[R_z] \times S^1_{\tilde{z}}[l_s^2/R_z]$ in the doubled space or further strongly argue that any circle in the physical direction always brings the circle in its dual direction with the inverse radius.

Gauge parameter component under T-duality

Interestingly, I could also see that the $B_{\mu\nu}$-gauge parameter in the string background transforms into the $\mathbf{GL}(D)$ spacetime linear transform parameter in the null wave background. This is naturally interpreted when we see the definition of the generalised diffeomorphism. The generalised diffeomorphism along the dual directions is, by definition, the gauge transform of $B_{\mu\nu}$, while the generalised diffeomorphism along the physical directions is Riemannian diffeomorphism. The generalised diffeomorphism gauge parameter is also a DFT vector and transforms under T-duality. T-duality exchanges the physical component and the dual component in the DFT vector, so the B-gauge parameter and the Riemannian diffeomorphism parameter exchanges under the T-duality. This feature is explicitly portraited through the gauge parameter of this DFT null wave example.

5.3.4 A Non-Riemannian background

Now, in order to obtain a non-Riemannian background, we further perform T-duality along time direction on the null wave example or perform double T-duality transformations along the isometric t- and z-directions (see (5.23) in [75] for the explicit form) on the F1 example. Through the $\mathbf{O}(D,D)$ rotations [16], the $(t,z,\tilde{t},\tilde{z})$ part of the generalized metric, $\mathcal{H}_{AB}$, has the following form: [17]

$$\mathcal{H}_{AB} = \begin{array}{c} \tilde{t} \\ \tilde{z} \\ t \\ z \end{array} \begin{pmatrix} c\,(cH-2) & 0 & 0 & cH-1 \\ 0 & -c\,(cH-2) & cH-1 & 0 \\ 0 & cH-1 & -H & 0 \\ cH-1 & 0 & 0 & H \end{pmatrix}, \tag{5.77}$$

[16] Note that the $\mathbf{O}(D,D)$ rotation here may not correspond to the traditional T-duality rotation. In backgrounds with isometries, we can choose the coordinates, $x^A = (\tilde{x}_a, x^a, \tilde{x}_i, x^i)$ $(a = 1, \ldots, D-n,\ i = D-n+1, \ldots, D)$, such that the background fields are independent of $\tilde{x}_a$ and $x^I = (\tilde{x}_i, x^i)$. In such backgrounds, a global $\mathbf{O}(D,D)$ rotation, $\mathcal{H}_{AB} \to O_A{}^C\, O_B{}^D\, \mathcal{H}_{CD}$ with $O_A{}^B = \left(\begin{smallmatrix} 1 & 0 \\ 0 & O_I{}^J \end{smallmatrix}\right) \in \mathbf{O}(D,D)$ (keeping the coordinates fixed), transforms the equation of motion of DFT covariantly. We used this rotation as a solution generating method. For discussions of related subtle issues, see [67, 107].

[17] Be aware that we set the definition of $B_{\mu\nu}$ in F1 string differently from [75]. The sign of B-field is different, and the relative sign of the gauge parameter$c$ is different.

where H is the harmonic function. Then, in the $c \to 0$ limit, the upper-left (2×2) block vanishes which would correspond to the inverse of the Riemannian metric. Namely, this background becomes singular in the conventional Riemannian sense, and accordingly is called a *non-Riemannian background* [75].

In such background, which is still a T-dual of a background with Riemannian description so has a well-defined DFT description, $B_{\mu\nu}$ and Φ also diverges so no Riemannian description is available. While the block-off-diagonal of the generalised metric $G^{-1}B$ is finite, $|G|^{-1} \to 0$, so B is ill-defined. While the $\mathbf{O}(D,D)$ dilaton $\sqrt{|G|}e^{-2\Phi}$ is finite, $\sqrt{|G|}$ diverges, so Φ also diverges. This is natural because $B_{\mu\nu}$ and Φ is defined on Riemannian geometry, and the Riemannian geometry itself is globally ill-defined in this case.

Even in this background, the DFT charge is well-defined, which corresponds to the energy, momentum, and winding charge for the Riemannian description–available background. In this background, the DFT Komar form (5.3) is written by

$$
\begin{aligned}
K^{tr} &= \mathcal{H}^{tD}\partial_r\mathcal{H}_{DE}X^E &(5.78)\\
&= \mathcal{H}^{tt}(\partial_r\mathcal{H}_t{}^zX_z + \partial_r\mathcal{H}_{tt}X^t) + \mathcal{H}^t{}_z(\partial_r\mathcal{H}^{zz}X_z + \partial_r\mathcal{H}^z{}_tX^t) &(5.79)\\
&= cH'X^t - c^2H'X_z + \mathcal{O}(1-H), &(5.80)
\end{aligned}
$$

and thus the asymptotic forms are

$$K^{tr}[\partial_t] \approx c\,\partial_r H(r)\,, \quad K^{tr}[\tilde{\partial}^z] \approx -c^2\,\partial_r H(r)\,, \quad B^r \approx 0\,, \tag{5.81}$$

where the boundary contribution is still zero, same with other 1-brane examples. From this, we obtain the DFT charges for the non-Riemannian (n-R) background,[18]

$$Q^{(\text{n-R})}[\partial_A] = n_D\,(0, c^2, 0\ldots, 0\,; -c, 0, 0, \ldots, 0)\,, \tag{5.82}$$

where n_D is the same constant as in other 1-brane examples but written in the constants time-T-dual to the null wave constants.

Diagonalisation of charges

As we put constant Killing vectors in the charge formula, which is also composed of only the Komar bulk form, (5.3) or (5.80), without boundary terms in the 1-brane examples, the set of DFT charges is covariant under the global constant $\mathbf{O}(D, D)$ rotation:

$$Q'[\partial_A] = \Lambda_A{}^B Q[\partial_B]. \tag{5.83}$$

As we have obtained the DFT charges (5.82) approaching to absolute zero as $c \to 0$, even after the proper diagonalisation of the basis vector, the ADM

[18] For the non-Riemannian case, in order to calculate the ADM momentum explicitly, we need to use a general formula (5.3), instead of (5.7). Further, since we cannot define $b_{\mu\nu}$ for the non-Riemannian case, the ADM momentum should be defined using a different parametrization the generalized metric, see *e.g.* [106].

$2D$-momentum is zero in all components.

Still, let us try explicit diagonalisation of the charges. The asymptotic generalised metric is

$$\mathcal{H}_{AB} \to \begin{pmatrix} c\,(c-2) & 0 & 0 & c-1 \\ 0 & -c\,(c-2) & c-1 & 0 \\ 0 & c-1 & -1 & 0 \\ c-1 & 0 & 0 & 1 \end{pmatrix} . \tag{5.84}$$

The Riemannian parametrisation of this asymptotic generalised metric is

$$g_{\mu\nu} = \begin{array}{c} t \\ z \end{array} \begin{pmatrix} \frac{1}{c(c-2)} & 0 \\ 0 & -\frac{1}{c(c-2)} \end{pmatrix}, \qquad b_{tz} = -\frac{c-1}{c(c-2)}. \tag{5.85}$$

Our original paper [1] only considers the block diagonalisation of the generalised metric (only mending $B_{\mu\nu}$ twist) like (5.6), so it claimes that the ADM $2D$-momentum is

$$P_A^{(\text{n-R})'} = n_D \left(0, c^2, 0\ldots, 0\,;\, \frac{c}{c-2}, 0, 0, \ldots, 0\right), \tag{5.86}$$

where signs of c and 2 differ from the original paper, considering the difference in relative signs of c and H in the background.

However, the basis vector set used here

$$E'_A = \begin{pmatrix} \tilde{\partial}^\mu \\ \partial_\mu - b_{\mu\nu}\tilde{\partial}^\nu \end{pmatrix} \tag{5.87}$$

equivalent to (5.6) only block-diagonalise the generalised metric:

$$\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}^T \mathcal{H}_{AB} \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \underset{r\to\infty}{\longrightarrow} \begin{pmatrix} c(c-2) & 0 & 0 & 0 \\ 0 & -c(c-2) & 0 & 0 \\ 0 & 0 & \frac{1}{c(c-2)} & 0 \\ 0 & 0 & 0 & -\frac{1}{c(c-2)} \end{pmatrix} = \begin{pmatrix} g^{-1} & 0 \\ 0 & g \end{pmatrix}. \tag{5.88}$$

To get the proper notion of energy and momentum, I should also diagonalise and normalise the Riemannian metric $g_{\mu\nu}$. Fortunately, the asymptotic Riemannian metric is already diagonal as in (5.85). Only scaling each coordinate is required to fully orthonormalise the basis like (5.22). The scaled canonical ADM $2D$-momentum is

$$P_A^{(\text{n-R})} = n_D \left(0, \sqrt{\frac{c^3}{2-c}}, 0\ldots, 0\,; -\sqrt{\frac{c^3}{2-c}}, 0, 0, \ldots, 0\right). \tag{5.89}$$

You should carefully consider the scaling factor in each condition of $c < 0$, $0 < c < 2$, and $c > 2$ to get the proper signature of metric $g_{\mu\nu} \to \operatorname{diag}\{-1, 1\}$. As we expected, even after the orthonormalisation, all the ADM $2D$-linear-momentum goes to zero in the non-Riemannian limit $c \to 0$.

I may consider another interesting limit $c \to 2$ where the ADM energy diverges. $c \to 2$ is not a non-Riemannian background but has a Riemannian description if you look at the full background expression (5.77). $c \to 2$ only makes the asymptotic Riemannian metric diverges, but in any finite region $r < \infty$, the Riemannian description is well-defined. It may be interesting to study this background further in the future, which has no well-defined boundary at the asymptotic region and has the diverging $2D$-momentum.

5.4 Black NS5-brane and Exotic 5^2_2-brane

5-brane are very interesting topics in DFT. As depicted in Figure 1.2 in Section 1.1.3, 5-branes (NS5, KK5, and 5^2_2) connect the D-brane area and the exotic brane area and also behave as magnetic charges as they are generated by performing S-duality (electric-magnetic duality) on the D5-brane.

In this section, we consider NS5-brane and 5^2_2-brane. First we consider a NS5-brane as a black brane in $\mathbb{R}^10$, which is also one of the generalisation of black hole solutions into general dimensions in String Theory. In this background, the NS5 charge is not captured by our formula. We mainly focus on checking the mass formula of the NS5-brane. The NS5 charge, which is a magnetic charge of $B_{\mu\nu}$ should be handled by integrating the field strength.

The 5^2_2-brane also has its own 5^2_2 exotic charge, which only appears with the 2 transverse compactified directions T^2 along which the 2 T-dualities are performed on the NS5-brane, similar to the Kaluza-Klein monopole (KK5) which only appears by compactifying one transverse direction of the NS5-brane. In this case, its charge is a magnetic charge of DFT fields not Riemannian fields. This charge is discussed in [106]. In our study, we focus on reproducing the mass formula of 5^2_2 in the field theory side.

We show that in both cases, the energy given by our formula succesfully reproduces the known energy from the each of the reference literature. Hereby, we argue that our ADT formula is the legitimate charge formula that reproduces quantum charges in String Theory in all cases.

5.4.1 Black 5-brane (NS5-brane)

Here, we consider the black 5-brane in $\mathbb{R}^10$, whose background reads [88]:

$$\mathrm{d}s^2 = -\frac{\left(1 - \frac{r_+^2}{r^2}\right)}{\left(1 - \frac{r_-^2}{r^2}\right)}\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\left(1 - \frac{r_+^2}{r^2}\right)\left(1 - \frac{r_-^2}{r^2}\right)} + r^2\,\mathrm{d}\Omega_3^2 + \sum_{s=1}^{5}(\mathrm{d}z^s)^2\,, \tag{5.90}$$

$$e^{-2\Phi} = 1 - \frac{r_-^2}{r^2}\,, \quad \mathrm{d}B = Q\,\boldsymbol{\epsilon}_3\,, \quad Q := r_+ r_-\,, \tag{5.91}$$

where $\boldsymbol{\epsilon}_3$ is the volume element on the unit 3-sphere, satisfying $\int_{S^3}\boldsymbol{\epsilon}_3 = 2\pi^2$, and $r_\pm$ is the radius of the outer and the inner horizons. In order to make the conserved charges finite, we assumed that the z^s-directions to be a five-torus with the volume, V_{T^5}. In the extremal limit, $r_+ \to r_-$, this background approaches that of the NS5-brane.

In the Cartesian coordinates, we obtain the asymptotic form,

$$K^{tr}[\partial_t] \approx \frac{2\,(r_+^2 - r_-^2)}{r^3}\,, \quad B^r \approx \frac{r_+^2 + r_-^2}{r^3}\,. \tag{5.92}$$

As it is expected, the only non-vanishing component of the ADM momentum is the ADM energy,

$$P_t = Q[\partial_t] = \frac{1}{2\kappa^2}\int \mathrm{d}^5 z \int_{S^3_\infty} \mathrm{d}\Omega_3\left(3r_+^2 - r_-^2\right) = \frac{1}{2\kappa^2} V_{T^5}\,2\pi^2\left(3r_+^2 - r_-^2\right), \tag{5.93}$$

which reproduces the known result, (3.14) in [89], especially the mass of the NS5-brane as the extremal limit, $r_\pm \to N^{1/2} l_s$. Note that the ADM momentum is timelike in this background,

$$(\mathcal{H}^{(0)})^{AB}\, P_A\, P_B < 0\,. \tag{5.94}$$

As expected, unlike the case of the fundamental string, the charge of the NS5-brane does not appear as the ADM momentum. The charge of the NS5-brane will appear as the NUT charge in doubled spacetime since it is T-dual to the Kaluza-Klein monopole, which has the NUT charge. Another possibility is that, as discussed in [94, 115], since monopoles are simultaneously interpreted as null waves in the 'Exceptional Field Theory', it may be possible to describe the charge of the NS5-brane as an ADM momentum in the extended spacetime.

5.4.2 An exotic brane (5^2_2-brane)

A 5^2_2-brane is a brane that is realised by performing T-dualities along the two transverse directions on a NS5-brane in $\mathbb{R}^{2,1}\times T^7$ as noted in Figure 1.2. This is the first exotic brane we meet in the duality web of branes. The 5^2_2 background may be described as [103–105]

$$\begin{aligned}
\mathrm{d}s^2 = H(r)\left(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2\right) + H(r)K^{-1}(r,\theta)\ \mathrm{d}x^2_{89} + \mathrm{d}x^2_{034567},\\
e^{2\Phi} = H(r)K^{-1}(r,\theta), \qquad B_{89} = -\sigma\theta K^{-1}(r,\theta),\\
H(r) := \sigma\ln(r_c/r), \qquad K(r,\theta) := H^2(r) + \sigma^2\theta^2,\\
\sigma := R_8R_9/(2\pi l_s^2),
\end{aligned} \tag{5.95}$$

where $\mathrm{d}x^2_{89} := (\mathrm{d}x^8)^2 + (\mathrm{d}x^9)^2$, and $\mathrm{d}x^2_{034567} := -(\mathrm{d}t)^2 + (\mathrm{d}x^3)^2 + (\mathrm{d}x^4)^2 + (\mathrm{d}x^5)^2 + (\mathrm{d}x^6)^2 + (\mathrm{d}x^7)^2$. We are using the polar coodinates (r, θ) for the $(x^1 = r\cos\theta,\, x^2 = r\sin\theta)$ directions here. Also be aware that the cutoff radius $\sim r_c$ must be set to get the converging expression in this $\mathbb{R}^{2,1} \times T^7$ background[1, 103–105].

We may also write the background in DFT. In Cartesian coordinates,

$$\begin{aligned} \mathrm{d}s^2 &= H^{-1}\,\mathrm{d}\tilde{x}^2_{12} + \mathrm{d}\tilde{x}^2_{034567} \\ &\quad + H^{-1}K\left[\Big(\mathrm{d}\tilde{x}_8 + \sigma\theta K^{-1}(r,\theta)\,\mathrm{d}x^9\Big)^2 + \Big(\mathrm{d}\tilde{x}_9 - \sigma\theta K^{-1}\,\mathrm{d}x^8\Big)^2\right] \\ &\quad + H\,\mathrm{d}x^2_{12} + HK^{-1}\,\mathrm{d}x^2_{89} + \mathrm{d}x^2_{034567} \\ e^{-2d} &= H, \end{aligned} \tag{5.96}$$

where $H = H(r)$, and $K = K(r,\theta)$, and $\mathrm{d}\tilde{x}^2_{034567} := -(\mathrm{d}\tilde{t})^2 + (\mathrm{d}\tilde{x}_3)^2 + (\mathrm{d}\tilde{x}_4)^2 + (\mathrm{d}\tilde{x}_5)^2 + (\mathrm{d}\tilde{x}_6)^2 + (\mathrm{d}\tilde{x}_7)^2$, and $\mathrm{d}x^2_{12} = (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2$, and $\mathrm{d}\tilde{x}^2_{12} = (\mathrm{d}x_1)^2 + (\mathrm{d}x_2)^2$.

As you can see, the metric depends on $H(r)$ which diverges in both limits $r \to 0$ and $r \to \infty$. In this thesis, we want to set the metric goes to flat $G_{\mu\nu} \to \eta_{\mu\nu}$, $\mathcal{H}_{MN} = \mathrm{diag}\{\eta^{\mu\nu}, \eta_{\mu\nu}\}$ for the noncompact directions at the boundary of the timeslice in the sense that discussed in Section 5.1. We can scale r or set the r_c to the appropriate value to achieve this, but only by assuming the existence of the cutoff radius that limits the maximum value of r where the boundary sits on rather than letting r goes to infinity. You can show that $H(r) = 1$ at $r = r_c\, e^{-1/\sigma}$ where the boundary is.

Noether potential and conserved charge.

Interestingly enough, in this background, we obtain

$$K^{tr} = \mathcal{H}^{rr}\mathcal{H}^{tt}\cancel{\partial_r \mathcal{H}_{tt}}X^t = 0, \qquad B^r = -\frac{\partial_r H(r)}{H^2(r)} = \partial_r H^{-1}(r), \tag{5.97}$$

so there is no contribution from the Komar potential to the conserved charge associated with any Double-space symmetry, which is also discussed in [4] after our work [1]. Without time dependence in the background, $B^t = 0$ also trivially. Thus, the only Noether charge in this background is the charge associated with time translation: ADM energy; other charges only comes from K^{tr}. The time direction charge is

$$\begin{aligned}
Q[\partial_t] &= \frac{1}{2\kappa^2}\oint_{\partial\mathcal{M}} \mathrm{d}^{D-2}x_{tr}\; e^{-2d}B^r & (5.98)\\
&= \frac{1}{2\kappa^2}\int_{T_{3456789}} \mathrm{d}^7 z \int d\theta\, rH(r)B^r(r)\Big|_{r=r_c\,e^{-1/\sigma}} & (5.99)\\
&= \frac{2\pi}{2\kappa^2}V_{T_{3456789}}\left[-rH^{-1}(r)\partial_r H(r)\right]_{r=r_c\,e^{-1/\sigma}} & (5.100)\\
&= \frac{2\pi\sigma}{2\kappa^2}V_{T_{3456789}}\left[H^{-1}(r)\right]_{r=r_c\,e^{-1/\sigma}\Leftrightarrow H(r)=1} & (5.101)\\
&= \frac{2\pi\sigma}{2\kappa^2}V_{T_{3456789}} \propto \frac{1}{g_s^2 l_s}\left(\frac{R_3R_4R_5R_6R_7}{l_s^5}\right)\left(\frac{R_8R_9}{l_s^2}\right)^2 = T[5^2_2], & (5.102)
\end{aligned}$$

where R's are the radius in each direction of the torus, and $V_{T_{3456789}}$ means the volume of the torus T^7. I show that this charge is equivalent to the tension of the brane given in (1.28) up to the overall numeral constant.

This charge is indeed interpreted as the ADM energy even if you see the mixture of $\mathrm{d}x^8$ and $\mathrm{d}x^9$ and their dual directions in the generalised metric.

Only non-compact directions (t, x^1, and x^2) are the isometry directions. Compactified directions are interpreted as internal symmetry groups, so there is no problem to accept t as the properly normalised time notion which gives the proper energy notion.

Another charge (exotic charge) which is not captured by our formula. Each brane is characterised by its charge. A 5^2_2-brane also has its own 5^2_2 exotic charge, which appears as a magnetic charge of Q-flux, which is the T-dual of the Kalb-Ramond B-field[106]. This charge is not captured by our formula because it is not electric.

There are monopoles in DFT, which are called *T-folds*. They have monodromies and must be patched by T-duality-covariant geometry because the relation between the patches always involves the T-duality. Monodromies in Riemannian geometry do not disappear in DFT but are well patched.

Those monodromies, or magnetic charges should be obtained by integrating the field strength in the field theory sense, as in the Maxwell theory. In this 5^2_2 background, β^{89}, the T-dual of B_{89} along the x^8- and x^9-directions, is what we have to look into. The magnetic charge is also briefly discussed in Section 4.6.

5.5 Linear Dilaton Background

Finally, we demonstrate that our general formula (4.55) is also applicable for a non-asymptotically flat background, and that the extrinsic curvature term in 4.2.3 adds a suitable counterterm to remove the divergence of the charge formula.

Definition of Linear Dilaton Background

As an example of a non-asymptotically flat background, consider the asymptotically linear dilaton background, which can be obtained by taking a decoupling limit and performing a coordinate transformation from a near-extremal 5-brane solution in 10-dimension without torus $(\mathbb{R}_t \times (\mathbb{R}_r \times S^3)_{\text{transverse}} \times \mathbb{R}^5_{\text{span}})$ [92]:

$$\begin{aligned}
\mathrm{d}s^2 &= -f(r)\ \mathrm{d}t^2 + \frac{Nl_s^2}{r^2 f(r)}\ \mathrm{d}r^2 + Nl_s^2\ \mathrm{d}\Omega_3^2 + \sum_{s=1}^{5}(\mathrm{d}z^s)^2\,, \\
e^{2\Phi} &= \frac{Nl_s^2}{r^2}\,, \quad \mathrm{d}B = Q\,\boldsymbol{\epsilon}_3\,, \quad f(r) := 1 - \frac{r_0^2}{r^2}\,,
\end{aligned} \tag{5.103}$$

where r_0 is the corresponds to the outer horizon and N corresponds to the number of the five-branes. z-directions are the spanning directions of the brane. Also, if we take convention of $d\Omega_3^2 = \mathrm{d}\alpha_1^2 + \sin^2\alpha_1\,\mathrm{d}\alpha_2^2 + \sin^2\alpha_1 \sin^2\alpha_2\,\mathrm{d}\alpha_3^2$ where α's are the angles in S^3, the definition of epsilon is $\boldsymbol{\epsilon}_3 = \sin^2\alpha_1 \sin\alpha_2 \cdot \mathrm{d}\alpha_1 \wedge \mathrm{d}\alpha_2 \wedge \mathrm{d}\alpha_3$. Furthermore, z's are the coordinate in brane-spanning directions. [88]

The expression above sufficiently well-defines the background we want to see, and we can put those Riemannian fields into the Riemannian-parametrised formula to obtain the charges, but let me construct DFT expressions for the background. To construct the generalised metric, first I have to obtain the gauge field $B_{\mu\nu}$ from the field strength $(dB)_{[\mu\nu\rho]} = H_{\mu\nu\rho}$ by solving the given equation:

$$\partial_{\alpha_1} B_{\alpha_2\alpha_3} + \partial_{\alpha_2} B_{\alpha_3\alpha_1} + \partial_{\alpha_3} B_{\alpha_1\alpha_2} = Q \sin^2 \alpha_1 \sin \alpha_2. \tag{5.104}$$

The general solution to the equation is

$$B_{\alpha_2\alpha_3} = \beta_1 \int \mathrm{d}\alpha_1 \, Q \sin^2 \alpha_1 \sin \alpha_2 = \frac{\beta_1 Q}{2} (\alpha_1 - \sin \alpha_1 \cos \alpha_1 \sin \alpha_2) \tag{5.105}$$

$$B_{\alpha_3\alpha_1} = \beta_2 \int \mathrm{d}\alpha_2 \, Q \sin^2 \alpha_1 \sin \alpha_2 = -\beta_2 Q \sin^2 \alpha_1 \cos \alpha_2 \tag{5.106}$$

$$B_{\alpha_1\alpha_2} = \beta_3 \int \mathrm{d}\alpha_3 \, Q \sin^2 \alpha_1 \sin \alpha_2 = \beta_3 Q \, \alpha_3 \sin^2 \alpha_1 \sin \alpha_2 \tag{5.107}$$

$$1 = \beta_1 + \beta_2 + \beta_3 \qquad \text{(Coefficient Constraints)} \tag{5.108}$$

up to integral constants.

Using this result, I can write DFT fields in the line element form $\mathrm{ds}^2 = \mathcal{H}_{MN} dx^M dx^N$. Among all possible solution with various choice of β's and constants of integrations, I show one of the simplest case $\beta_1 = \beta_3 = 0$ with zero integration constant:

$$B_{\alpha_3\alpha_1} = -Q \sin^2 \alpha_1 \cos \alpha_2. \tag{5.109}$$

In this case, the DFT fields are written by

$$
\begin{aligned}
e^{-2d} &= rNl_s^2\epsilon_3, && (5.110)\\
\mathsf{ds}^2 &= -\frac{1}{f(r)}\,\mathrm{d}\tilde{t}^2 + \frac{r^2 f(r)}{Nl_s^2}\,\mathrm{d}\tilde{r}^2 + \frac{1}{Nl_s^2}\,\mathrm{d}\widetilde{\Omega}_3^2 + \mathrm{d}\tilde{z}_5^2 \\
&\quad + 2Q\sin^2\alpha_1\cos\alpha_2\,\mathrm{d}\alpha_3\,\mathrm{d}\tilde{\alpha}_1 - 2Q\cot\alpha_2\csc\alpha_2\,\mathrm{d}\alpha_1\,\mathrm{d}\tilde{\alpha}_3 \\
&\quad - f(r)\,\mathrm{d}t^2 + \frac{Nl_s^2}{r^2 f(r)}\,\mathrm{d}r^2 + \mathrm{d}z_5^2 \\
&\quad + \Big(1 + Q^2\sin^2\alpha_1\cot^2\alpha_2\Big)\,\mathrm{d}\alpha_1^2 + \sin^2\alpha_1\,\mathrm{d}\alpha_2^2 \\
&\quad + \Big(\sin^2\alpha_1\sin^2\alpha_2 + Q^2\sin^4\alpha_1\cos^2\alpha_2\Big)\,\mathrm{d}\alpha_3^2, && (5.111)
\end{aligned}
$$

where

$$
\begin{aligned}
\epsilon_3 = \sin^2\alpha_1\sin\alpha_2, \qquad \mathrm{d}\tilde{\Omega}_3^2 &= \mathrm{d}\tilde{\alpha}_1^2 + \frac{\mathrm{d}\tilde{\alpha}_2^2}{\sin^2\alpha_1} + \frac{\mathrm{d}\tilde{\alpha}_3^2}{\sin^2\alpha_1\sin^2\alpha_2}, \\
\mathrm{d}z_5^2 = \sum_{s=1}^{5}\left(\mathrm{d}z^s\right)^2, &\qquad \mathrm{d}\tilde{z}_5^2 = \sum_{s=1}^{5}\left(\mathrm{d}\tilde{z}^s\right)^2. && (5.112)
\end{aligned}
$$

Charge Formulation

In this case, we have

$$
K^{tr}[\partial_t] = \frac{r^2}{Nl_s^2}\,\partial_r f(r)\,, \quad B^r = -\frac{4r}{Nl_s^2}\,f(r) - \frac{r^2}{Nl_s^2}\,\partial_r f(r)\,. \tag{5.113}
$$

Since the constant term in B^r gives a divergent value to the conserved charge, we add the following boundary term to the action:[19]

$$S_0 = -\int_{\mathbb{R}_t \times \partial\mathcal{M}} \sqrt{h}\, e^{-2\Phi}\, b_0 \,. \tag{5.114}$$

Here, h is the induced metric on the boundary, $\mathbb{R}_t \times \partial\mathcal{M}$, which is in our case a constant r surface, and b_0 is a function of h. This corresponds to the shift in B^A by

$$B^A \to \bar{B}^A := B^A + b_0\, \hat{n}^A \,, \tag{5.115}$$

where $\hat{n}^A := \mathcal{H}^{AB}\, n_A$ and n_A denote the unit normal vector at the boundary, $\mathcal{H}^{AB} n_A n_B = 1$.[20] In the present case, its asymptotic form becomes $\hat{n}^A \partial_A \approx \partial_r$. Thus, the shift simply changes B^r component as

$$B^r \to \bar{B}^r = B^r + b_0 \,. \tag{5.116}$$

Here, we simply choose b_0 as the minus of the leading term in B^r, evaluated on the extremal background, $r_0 = 0$; $b_0 = 4/(N^{1/2} l_s)$. We then obtain

$$K^{tr}[\partial_t] + \bar{B}^r \approx \frac{2r_0^2}{N l_s^2}\, \frac{1}{r} \,. \tag{5.117}$$

[19] I explain this process in general in Section 4.2.3.

[20] Note that n_A is defined through Stokes' theorem for an arbitrary vector, K^A,

$$\int \mathrm{d}^D x\, \partial_A \big(e^{-2d} K^A \big) = \oint \mathrm{d}^{D-1} x\, e^{-2d} n_A\, K^A \,.$$

We note that $n^A = \mathcal{J}^{AB}\, n_B$ is different from $\hat{n}^A$ appearing in (5.115). Similarly, the normal vector, $\hat{n}^k \partial_k$, appearing in (5.25) should be also understood as the D-dimensional components of $\hat{n}^A$.

Therefore, the ADM mass, which is the only non-vanishing component of the ADM momentum, becomes

$$Q[\partial_t] = \int \mathrm{d}^5 z \int_{S^3_\infty} \mathrm{d}\Omega_3 \, e^{-2d} \big(K^{tr}[\partial_t] + \bar{B}^r\big) = 2r_0^2 \, \Omega_3 \, V_{T^5} \, . \tag{5.118}$$

This result matches with (138) in [95], where the mass was obtained from an approach of Brown and York [40] as well as of Hawking and Horowitz [38]. See also (6.17) of [96], where another approach was used.

Chapter 6 Conclusion

In this thesis, we have formulated conserved charge in DFT by developing standard gravitational conserved charge formulation (ADT) in the new Double geometry:

$$\boxed{\begin{aligned}Q_{\text{total}}[X] = \frac{1}{2\kappa^2}\oint_{\partial\mathcal{M}} \mathrm{d}x_{AB}\ e^{-2d}\Bigg[&K^{[AB]} + 2X^{[A}B^{B]} + 4K_0X^{[A}\tilde{N}^{B]}\\ &+ \frac{2\kappa^2}{g_{\text{YM}}^2}\mathrm{Tr}\left\{12(P\mathcal{F}\bar{P})^{[AB}V^{C]}X_C\right\}\Bigg].\end{aligned}} \tag{6.1}$$

To develop the ADT charge formula, first we have developed off-shell Noether current associated with the generalised diffeomorphism. The current have been able to be expressed as a divergence of an antisymmetric Noether potential with some section condition terms:

$$\partial_A(e^{-2d}J^A) \simeq \partial_A\partial_B(e^{-2d}K^{[AB]}) \simeq 0 \quad \text{up to section condition } (\partial_A\partial^A = 0). \tag{6.2}$$

Also we have found that the Noether potential can be written in the form similar to Komar form [1, 31]:

$$K^{AB}[X] = K^{[AB]}[X] = 4(\bar{P}\nabla)^{[A}(PX)^{B]} - 4(P\nabla)^{[A}(\bar{P}X)^{B]}. \tag{6.3}$$

Furthermore, during the formulation of the Noether current, we have found a divergence-free DFT tensor equivalent to the Einstein tensor, which is essential to the existence of the off-shell current:

$$G^{AB} := 2(P^{AC}\bar{P}^{BD} - \bar{P}^{AC}P^{BD})S_{CD} - \frac{1}{2}\mathcal{J}AB\mathcal{S}; \tag{6.4}$$

and the symmetrised one:

$$\mathcal{G}^{(AB)} := G^{AC}\mathcal{H}_C{}^B = -4(PS\bar{P})^{(AB)} - \frac{1}{2}\mathcal{H}^{AB}\mathcal{S}. \tag{6.5}$$

In classical gravity, it is stated that the Noether charge of the total action of gravity, including the Gibbons-Hawking boundary term, gives the conserved charge equivalent to the Noether charge of the matter, with some additional conditions that the local Noether current of the gravity becomes ill-defined and the integral domain of the current is required[41]. In our study, we have found the boundary term that can act as a Gibbons-Hawking term:

$$I_{GHY} = -\frac{1}{2\kappa^2}\int\limits_{\Sigma_D} \mathrm{d}^D x\, e^{-2d}\partial_A(e^{-2d}B^A) \tag{6.6}$$

$$\begin{aligned} B^A &:= 2(P^{AC}P^{BD} - \bar{P}^{AC}\bar{P}^{BD})\Gamma_{BCD} \\ &= 4(P-\bar{P})^{AB}\partial_B d - 2\partial_B P^{AB}, \end{aligned} \tag{6.7}$$

where B^A is a pseudo-DFT-vector. The total ADT charge becomes the timeslice integration of the total Noether current including the boundary term:

$$Q_{\text{total}}[X] = \frac{1}{2\kappa^2} \oint_{\partial\mathcal{M}} \mathrm{d}x_{AB}\; e^{-2d} \left(K^{[AB]} + 2X^{[A}B^{B]} \right), \tag{6.8}$$

where $\mathcal{M}$ is the timeslice. This quantity is conserved only if

(i) the integral domain is defined *i.e.* the boundary is defined where B^A is defined, and which means that the local current cannot be defined;

(ii) the symmetry and its Killing vector field X^A satisfies the condition:

$$\begin{aligned} &\int_{\Sigma_D} \mathrm{d}^D x\, \partial_A (e^{-2d} J^A_{\text{total}}) = \int_{\Sigma_D} \mathrm{d}^D x\, \partial_A \partial_B \left(e^{-2d} (K^{AB} + 2X^{[A}B^{B]}) \right) = \\ &\qquad \int_{\partial\Sigma_D} \mathrm{d}^{D-1} x_A\; e^{-2d} (P^{C[A}P^{B]D} - \bar{P}^{C[A}\bar{P}^{B]D}) \partial_B \partial_{[C} X_{D]} = 0, \end{aligned} \tag{6.9}$$

which is mostly satisfied for translational and rotational symmetries, but hardly satisfied for the higher coordinate dependent symmetries.[1]

Finally, I also have introduced the extrinsic curvature term to the DFT action and the conserved global (quasi-local) charge, which can cancel out the diverging terms when the DFT background is asymptotically non-flat and has the extrinsic curvature.

Including other fields that can affect the massless sector background such as

[1] Seok Kim suggested that this may be related to the Virasoro algebra. Virasoro operators have higher coordinate dependence and we cannot obtain the charge for each operator.

the cosmological constant and the Yang-Mills sector, The total charge formula have been formulated as (6.1).

We also have applied our result to various string backgrounds, which well-display the role of each term of the charge formula explicitly, and have confirmed that the applications show the expected mass, momentum, and winding charges that we expect in String Theory as quantum numbers. Before the application, we have learned that to obtain the correct energy, momentum, and winding charge values for each explicit background from the $\mathbf{O}(D,D)$-covariant charge formula, we have to normalise the Killing vector space well to fix the correct time, space, and winding direction notions. Using all the techniques and formulae we had developed, we first have applied to the purely Einstein backgrounds and confirmed that the result matches to the classical results.

Then, we have applied the formula to 1-branes and 5-branes.

For the 1-brane (string) examples, we have confirmed that our formula correctly reproduces the energy, momentum, and winding charges of the strings which expected as quantum numbers, and that the winding charge and the momentum is exchanged under the T-duality in field theory perspective. Furthermore, we have been able to apply our formula to even a non-Riemannian background, which has no Riemannian description but a DFT description, and we have been able to calculate the conserved $2D$-momenta of it, which is hard to be interpreted in the classical sense.

We have also checked the $2D$-momenta of the 5-branes with monopole charges such as a NS5-brane and a 5^2_2-brane. They are objects that are expected to have energy and the NSNS monopole charge. We have confirmed that those objects gives time-translational charge only, which is exactly the , and no other $2D$ linear momenta with our formula.

6.1 Further Directions

Ramond-Ramond sector, Heterotic strings, and others

There are further directions our result can be extended. One would like to include the R-R sector fields and also to substantiate fermionic Noether currents and fermionic global charges as odd-grading part of supersymmetric asymptotic symmetries in superstring theories. Fermionic Double Field Theory has been studied in [82–84] including the spinor representation and the gamma matrices in Double notation. Also, Heterotic strings are studied in the Double notation[85]. Thus, most of the components in String Theory or M-theory are studied in DFT notation. We can make the Noether process for those matter fields.

Exceptional Field Theory

All those studies to present M-theory in representations in larger groups brought *Exceptional Field Theory (ExFT)* [116–126], which is the gravitational theory with manifest U-duality covariance, including all fields into one large generalised metric. The full 11-dimensional ExFT has $E_{11(11)}$ symmetry, and the section condition breaks the symmetry into the geometric $\mathbf{GL}(D)$ and the rest. Or, you can construct D-(internal-)dimensional theory with the exceptional symmetry group E_D. (See Section 1.1.2.) In ExFT, we may also try to develop ADM charge formula in the future.

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Appendices

Appendix A The Covariant Divergence Theorem in Double Geometry

The utility of the covariant derivative is not only manifestly guaranteeing the covariance but also opens the use of variety of geometric theorems, which simplifies complicated computations and also gives geometric meanings. The Riemannian-geometric theorem,

$$\partial_\mu(\sqrt{|G|}V^\mu) = \sqrt{|G|}\nabla_\mu V^\mu \tag{A.1}$$

for an vector V^μ, must be true for Riemannian geometry to be consistent and guarantees us consistent generalisation of the divergence theorem in curved spaces.

In Double Field Theory, similarly

$$\nabla_\mu(e^{-2d}V^\mu) = \partial_\mu(e^{-2d}V^\mu) = e^{-2d}\nabla_\mu V^\mu \tag{A.2}$$

is satisfied, which is necessary to well-define the geometry. ∂d term in the DFT-connection is added to satisfy this relation in the first place (*see* (13) *in* [71]).

Another important theorem about divergence in Riemannian geometry is

$$\partial_\mu\left(\sqrt{|G|}F^{\mu\nu}\right)=\sqrt{|G|}\nabla_\mu F^{\mu\nu} \tag{A.3}$$

for any anti-symmetric tensor $F^{\mu\nu}=F^{[\mu\nu]}$, which is very important property to construct the Maxwell equation from the Yang-Mills action, and which is also crucial for the existence of the Noether potential 'tensor':

$$\sqrt{|G|}J^\mu=\partial_\nu(\sqrt{|G|}K^{\mu\nu})=\sqrt{|G|}\nabla_\nu K^{\mu\nu}. \tag{A.4}$$

In our study, we could not find such theorem for general anti-symmetric DFT-tensors, but still, we could define the (semi-)covariant Noether potential tensor because this kind of relation is satisfied for the Noether potential up to the section condition (*see* (4.39) to (4.41)).

Appendix B The Einstein Frame Parametrisation of Double Field Theory

The DFT fields, the generalised metric and the $\mathbf{O}(D,D)$ dilaton is defined in the string frame quantities in the main text:

$$\begin{aligned}\mathcal{H}_{MN} &= \begin{pmatrix} G^{-1}_{(\mathrm{string})} & -G^{-1}_{(\mathrm{string})}B \\ BG^{-1}_{(\mathrm{string})} & G_{(\mathrm{string})} - BG^{-1}_{(\mathrm{string})}B \end{pmatrix}, \\ e^{-2d} &= e^{-2\Phi}\sqrt{\left|G_{(\mathrm{string})}\right|}. \end{aligned} \tag{B.1}$$

However, the vacuum expectation value of the dilaton Φ is generally not zero, so we may redefine the dilaton field as the deviation from the expectation value

$$\Phi^{(\mathrm{E})} = \Phi^{(\mathrm{string})} - \left\langle \Phi^{(\mathrm{string})} \right\rangle, \tag{B.2}$$

and the graviton field may also be redefined by a Weyl transformation

$$G^{(\mathrm{string})}_{\mu\nu} = e^{\frac{4}{D-2}\Phi^{(\mathrm{E})}} G^{(\mathrm{E})}_{\mu\nu}\,. \tag{B.3}$$

to avoid the coupling between the dilaton and the graviton and to maintain the Lagrangian in kinetic form

$$I \simeq \frac{1}{2\kappa^2_{(\text{string})}} \int \mathrm{d}^D x \left[\sqrt{|G|} e^{-2\Phi} \left(R - \frac{1}{12} H^2 + 4(\partial\Phi)^2 \right) \right]^{(\text{string})} \tag{B.4}$$

$$\simeq \frac{1}{2\kappa^2_{(\text{E})}} \int \mathrm{d}^D x \left[\sqrt{|G|} \left(R - \frac{1}{12} e^{-\frac{8}{D-2}\Phi} H^2 - \frac{4}{D-2} (\partial\Phi)^2 \right) \right]^{(\text{E})} \tag{B.5}$$

This redefinition of dilaton and graviton is called *Einstein frame* [pg. 114 in 7, 9, 55], and the Einstein frame quantities will be denoted by (E) in this appendix.

The raw gravitational constant $\kappa_{(\text{string})} \sim l_s^{(D-2)/2}$ only depends on the string scale $\alpha' \sim l_s^2$. The expectation value of $e^{-2\Phi^{(\text{string})}}$ in (B.4) comes out from the integral and added to the gravitational constant, so the gravitational constant in Einstein frame becomes

$$\kappa_{(\text{E})} = e^{\langle \Phi^{(\text{string})} \rangle} \kappa_{(\text{string})} \sim g_s l_s^{(D-2)/2}, \tag{B.6}$$

which is also mentioned in §1.2.

The raw gravitational constant $\kappa_{(\text{string})}$ has no physical significance because we can always redefine (add constant to) Φ, not changing the string frame action form but changing the gravitational constant. The only physically significant gravitational constant is determined by defining the dilaton with its expectation value zero, pushing the expectation value factor to the overall constant, the gravitational constant. This is discussed in §1.2, and the physical gravitational constant is

$$\frac{1}{16\pi G_N} = \frac{1}{2\kappa^2} = \frac{1}{2\kappa^2_{(\mathrm{E})}} \sim \frac{1}{g_s^2 l_s^{D-2}}. \tag{B.7}$$

Double Field Theory

In this appendix, we try to describe DFT itself and our charge formula in Einstein frame. DFT fields are parametrised in Einstein frame as

$$\begin{aligned}
\mathcal{H}_{MN} &= e^{-\frac{4}{D-2}\Phi} \begin{pmatrix} G^{-1}_{(\mathrm{E})} & -G^{-1}_{(\mathrm{E})}B \\ BG^{-1}_{(\mathrm{E})} & e^{\frac{8}{D-2}\Phi}G_{(\mathrm{E})} - BG^{-1}_{(\mathrm{E})}B \end{pmatrix}, \\
e^{-2d} &= e^{\frac{4}{D-2}\Phi}\sqrt{\left|G_{(\mathrm{E})}\right|}.
\end{aligned} \tag{B.8}$$

Charge Formula

The Riemannian parametrisation of the charge formula,

$$Q[X] = \frac{1}{2\kappa^2}\int_{\partial\mathcal{M}} \mathrm{d}^{D-2}x_{\mu\nu}\, e^{-2d}\left(K^{\mu\nu}[X] + 2X^{[\mu}B^{\nu]}\right), \tag{B.9}$$

should also parametrised in Einstein frame, where the Killing vector is Riemannian-parametrised in both frame as

$$X^M = \begin{pmatrix} \zeta_\mu + B_{\mu\nu}\xi^\nu \\ \xi^\mu \end{pmatrix}. \tag{B.10}$$

The Komar form (5.7) is parametrised in Einstein frame as following: [1]

$$
\begin{aligned}
K^{\mu\nu}[X] &= \Big[-2\nabla^{[\mu}\xi^{\nu]} - H^{\mu\nu\rho}\zeta_\rho\Big]^{(\text{string})} \\
&= \left[e^{-\frac{4}{D-2}\Phi}\left(-2\nabla^{[\mu}\xi^{\nu]} - e^{-\frac{8}{D-2}\Phi}H^{\mu\nu\rho}\zeta_\rho + \frac{8}{D-2}\xi^{[\mu}\partial^{\nu]}\Phi\right)\right]^{(\text{E})}, \qquad \text{(B.11)}
\end{aligned}
$$

and the boundary term is parametrised as

$$
\begin{aligned}
B^\mu &= \Big[2G^{\mu\nu}(2\partial_\nu\Phi - \partial_\nu \ln\sqrt{|G|}) - \partial_\nu G^{\mu\nu}\Big]^{(\text{string})} \\
&= \left[e^{-\frac{4}{D-2}\Phi}\left(-2G^{\mu\nu}\partial_\nu \ln\sqrt{|G|} - \partial_\nu G^{\mu\nu}\right)\right]^{(\text{E})}. \qquad \text{(B.12)}
\end{aligned}
$$

Interestingly, the Einstein frame parametrisation of both Komar form and the boundary term have the $e^{-\frac{4}{D-2}\Phi}$ factor in front, which cancels out the dilaton factor in the volume element e^{-2d}.

Thus, the total charge formula is parametrised in the Einstein frame as

$$
Q[X] = \frac{1}{2\kappa^2}\int_{\partial\mathcal{M}} \mathrm{d}^{D-2}x_{\mu\nu}\sqrt{|G_{(\text{E})}|}\,\Big[\widetilde{K}^{\mu\nu} + 2\xi^{[\mu}\widetilde{B}^{\nu]}\Big], \qquad \text{(B.13)}
$$

$$
\widetilde{K}^{\mu\nu} := \left[-2\nabla^{[\mu}\xi^{\nu]} - e^{-\frac{8}{D-2}\Phi}H^{\mu\nu\rho}\zeta_\rho + \frac{8}{D-2}\xi^{[\mu}\partial^{\nu]}\Phi\right]^{(\text{E})}, \qquad \text{(B.14)}
$$

$$
\widetilde{B}^\mu := \Big[-2G^{\mu\nu}\partial_\nu \ln\sqrt{|G|} - \partial_\nu G^{\mu\nu}\Big]^{(\text{E})}. \qquad \text{(B.15)}
$$

[1] Interestingly enough, the Komar form in Einstein frame shows the canonical momentum form $\sim \partial^\mu\Phi$.

Appendix C Komar Form Calculation in Linear $2D$-momentum of Black Strings/Branes

In this appendix, I talk about how the $2D$-momentum formula is simplified for the backgrounds like 1-branes examples in §5.3. Specifically, the conditions we demand in this appendix are

$$\partial_N X^M = 0, \qquad \text{($2D$ Linear momentum)} \tag{C.1}$$

$$\begin{aligned} &\mathcal{H}^{i\mu} = \delta^{i\mu},\ \mathcal{H}^{i}{}_{\mu} = 0, \\ &\mathcal{H}_{i}{}^{\mu} = 0,\ \mathcal{H}_{i\mu} = \delta_{i\mu}, \end{aligned} \qquad \begin{pmatrix} \text{the non-smeared transverse direction} \\ \text{has trivial (generalised) metric and} \\ \text{does not mix with other coordinates.} \end{pmatrix} \tag{C.2}$$

$$\mathcal{H}^{i}{}_{A} = \mathcal{H}^{j}{}_{A},\ \mathcal{H}_{iA} = \mathcal{H}^{jA}, \quad \text{(isometry of y's)} \tag{C.3}$$

$$\mathcal{H}_{MN} \text{ only depends on } r, \tag{C.4}$$

where the index i or j stands for the non-smeared transverse direction y^i, and r stands for the asymptotic direction $r^2 = \sum_i (y^i)^2$.

$$\partial_\lambda \xi^\mu = 0, \tag{C.5}$$

$$\partial_\lambda \zeta_\mu = 0 \text{ (assuming } \partial_\lambda b_{\mu\nu} \to 0 + \mathcal{O}(r^{p+n-8})), \tag{C.6}$$

$$G^{i\mu} = \delta^{i\mu}, \tag{C.7}$$

$$B_{i\mu} = 0, \tag{C.8}$$

$$\text{Harmonic function } H = H(r), \tag{C.9}$$

where $b_{\mu\nu}$ is the asymptotic value of the Kalb-Ramond field, and $\mathcal{O}(r^{p+n-8})$ means the order that does not contribute to the charge as discussed in (5.21).

Under this condition, we simplify the Komar form (5.3):

$$K^{\mu\nu}[X] = -2\mathcal{H}^{\rho[\mu}\left(\partial_\rho X^{\nu]} + \mathcal{H}^{\nu]D}(\partial_\rho \mathcal{H}_{DE})X^E\right) - \mathcal{H}_E{}^\rho \mathcal{H}^{[\mu}{}_D(\partial_\rho \mathcal{H}^{\nu]D})X^E. \tag{C.10}$$

As we assumed the linear momentum case (C.1) and the r-only-dependence (C.9), the Riemannnian component of the DFT Komar form simplifies as

$$K^{\mu\nu}[X] = -2\mathcal{H}^{i[\mu}\mathcal{H}^{\nu]D}\left(\frac{\partial r}{\partial y^i}\cdot \partial_r \mathcal{H}_{DE}\right)X^E - 2\mathcal{H}_E{}^i\mathcal{H}^{[\mu}{}_D\left(\frac{\partial r}{\partial y^i}\cdot \partial_r \mathcal{H}^{\nu]D}\right)X^E. \tag{C.11}$$

Among Riemannian components of the Komar form, the only require component to obtain the conserved charge is K^{tr} as discussed in Chapter

5, and K^{tr} is defined by

$$K^{tr} := \frac{\partial r}{\partial y^i} K^{ti} \tag{C.12}$$

in the Cartesian coordinates, which is used throughtout this thesis. Using the condition (C.2), the tr-component of (C.11) is specified as

$$K^{tr}[X] = \frac{\partial r}{\partial y^i}\frac{\partial r}{\partial y^j}\left[\delta^{ij}\mathcal{H}^{tD}\left(\partial_r \mathcal{H}_{DE}\right)X^E + (\partial_r \mathcal{H}^t{}_j)X_i\right]. \tag{C.13}$$

Be careful that $\delta^{ij}\mathcal{H}_{Aj} = \mathcal{H}_{Ai} \neq \mathcal{H}_A{}^i$. As $\frac{\partial r}{\partial y^i}\frac{\partial r}{\partial y^j} = 1$, this expression is simplified as

$$K^{tr}[X] = \mathcal{H}^{tD}\left(\partial_r \mathcal{H}_{DE}\right)X^E + \frac{\partial r}{\partial y^j}(\partial_r \mathcal{H}^t{}_j)X_r. \tag{C.14}$$

Because of (C.3), we can define

$$\mathcal{H}^t{}_r := \mathcal{H}^t{}_i(\text{equal for all } i) = \frac{\partial r}{\partial y^j}\mathcal{H}^t{}_j, \tag{C.15}$$

so we can write (C.14) into

$$K^{tr}[X] = \mathcal{H}^{tD}\left(\partial_r \mathcal{H}_{DE}\right)X^E + (\partial_r \mathcal{H}^t{}_r)X_r. \tag{C.16}$$

In Chapter 5, we denoted the r-index of $\mathcal{H}$ as just an alias of the Cartesian

y's and considered as

$$\mathcal{H}_{rA} := \mathcal{H}_{iA}. \tag{C.17}$$

In this appendix, I have explained the exact theories behind this explicitly.

Index

초 록

이중장론(DFT)은 끈이론의 중요한 대칭 중 하나인 T-이중성 대칭성을 기하학적인 대칭으로 포함하는 새로운 중력이론이다. 아인슈타인의 일반상대성이론은 푸앙카레 대칭과 일반좌표변환 대칭을 가지고 있다. 이중장론은 T-이중성 군 $\mathbf{O}(D,D)$에 대한 대칭과 끈이론의 다른 무질량 입자들의 게이지 대칭을 포함하는 확장미분동형사상에 대한 대칭을 가진다.

이 논문에서는 이중장론에서 보존량을 구하는 표준적인 공식을 구축하는 일을 하였다. 이 일에 앞서 우선 뇌터정리를 중력이론에서 그대로 적용하기는 어려움을 보이고, ADT 방법이나 Wald의 방법 등 해밀토니언에 해당하는 보존량 공식 유도의 표준적인 방법을 살펴보았다. 그 후, 이중장론의 구축 및 그 공변미분의 구축에 대해서 살펴보았다.

이중장론은 리만기하학이 아닌 완전히 새로운 기하인 이중기하로 쓰여진 중력이론이다. 이 연구에서는 이중기하에서 ADT 방법론을 어떻게 쓸 수 있을지 연구하였고, 그 과정에서 아인슈타인 중력이론에서 중요한 양들에 해당하는 새로운 양들을 찾을 수 있었다. 이중장론의 뇌터흐름은 뇌터퍼텐셜의 형태로 쓰여질 수 있음을 알아내었고, 뇌터퍼텐셜의 형태는 잘 알려진 Komar 형태를 띄었다. 또한, 뇌터흐름을 구하는 과정에서 아인슈타인 텐서에 해당하는 양을 찾을 수 있었는데, 다이버전스가 0이면서 운동방정식과 정확히 일치하지는 않고 그 조합인 텐서이다. 마지막으로 이중장론의 경계 작용 또한 연구되었다. 이러한 요소들을 활용하여 이중장론에서 운동방정식을 가정하지 않고도 보존되는 해밀토니언에 해당하는 보존량 공식을 개발할 수 있었다.

이 새로운 이중장론의 보존량은 기존의 ADM 전하와 끈이론의 감음수로

구성되어 있다. 이 사실은 끈이론에서 예측되어 왔지만, 이 연구는 장론의 관점에서 이 사실을 확인하였다. 이 사실을 확인하기 위해 우리는 끈이론의 여러가지 해에 우리의 공식을 적용해보았다. 적용하기 전에, 이 논문에서는 올바른 에너지, 운동량, 감음수를 얻기 위해서는 서로 직교하는 올바른 시간, 공간, 감음공간의 방향의 단위벡터를 얻어야 한다는 사실을 논한다. 그리고 우리는 보존량 공식을 순수한 아인슈타인 중력의 예, 끈, 5-막, 그리고 점근 행태가 평평하지 않은 예에 대하여 적용해보았다. 끈에 대한 공식의 적용은 T-이중성은 운동량과 감음수를 교환하는 이중성임을 다시금 확인시켜 주었다. 또한, 리만기하학적 표현이 존재하지 않 비리만배경에 대한 공식의 적용도 문제없이 가능함을 확인할 수 있었다. 이중장론 홀극 (5-막) 에 대한 적용 또한 잘 알려진 막의 장력을 에너지로써 나타내주었다. 마지막으로, 점근 행태가 평평하지 않은 예에서는 우리의 공식의 발산이 보정항에 의해 잘 제어됨을 확인할 수 있었다.

이 연구는 ADM/ADT 방법에 해당하는 표준적인 이중장론에서의 보존량 공식 구성 방법을 확립하였다고 결론지을 수 있다. 이 연구는 앞으로 예외장론이라고 하는 끈이론의 모든 대칭을 기하학적인 구조로써 포함하는 더 큰 중력이론에 대해서도 적용할 수 있을 것으로 기대한다.

주요어 : 이중장론, 끈이론, 중력, T-이중성, 뇌터정리, 보존량

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