Master's Thesis of Physics

Dynamical Chern-Simons gravity axion constraints with I-Love relation

dCS 중력 액시온의 I-Love 관계식을 통한 제약

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Abstract

We have studied how the presence of the dark matter field inside a neutron star modifies the I-Love relation and consequently the constraint on dynamical Chern-Simons(dCS) coupling. It is shown that DM inside of NS slightly changes dCS axion constraints and I-Love relation. Before considering DM existence, as a review, we have done I-Love test on dCS gravity theory without DM presence. As modified gravity theory, one of choices is an additional symmetry-violating Chern-Simons term in gravitational Lagrangian. Its dynamical scalar field version is dynamical Chern-Simons gravity theory and its coupling constant l_{dCS} is a key parameter. The previous GPB constraint of l_{dCS} is given by $l_{dCS} \leq O(10^8)$km order. Meanwhile, there is I-Love relation, a neutron star's quasi-universal relation between nondimensionalized moment of inertia and non-dimensionalized tidal Love number regardless of EOS. It can be used to examine gravity theory and GR passed it. Likewise, dCS gravity theory can be constrained by I-Love relation in a similar way. By this test, the upper bound for the coupling constant is reduced to O(10)km order. DM in NS can change NS configuration and shift I-Love curve. It causes reduced GR contribution of moment of inertia and tidal Love number and increased dCS correction to moment of inertia. However, the total effect of it does not change dCS constraints from I-Love relation significantly.

Keyword : dCS gravity, ALP, I-Love relation, NS, Dark matter
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국문 초록

중성자별 내부에 암흑물질이 존재하는 경우 I−Love 관계식과 그에 따른 dCS의 결합상수가 어떻게 바뀌는지에 대해 알아보았고, 그 결과 중성자별 내부의 암흑 물질은 I−Love 관계식과 dCS 액시온에 대한 제약을 아주 조금 바꾸는 것을 알 수 있었다. 암흑 물질의 존재를 고려하기 전 리뷰로써 암흑물질 없이 I−Love로 dCS 중력 이론에 대해 검증하였다. 수정 중력 이론 중 하나로 대칭성을 깨는 천−사이먼스 항을 중력에 대한 라그랑지안에 추가할 수 있고, 이 항이 역학적일 때의 이론을 dCS 중력 이론이라고 한다. 이 이론의 주요한 상수는 결합 상수 l_{dCS}이며 이에 대한 기존의 GPB실험으로부터의 상한은 $O(10^8)$km로 주어졌다. 한편 I−Love 관계식은 중성자별에 대해 성립하는 관성 모멘트와 조석 러브수 사이의 준 보편적인 관계로 중성자별 내부의 상태방정식과 무관하게 성립한다. 이를 중력이론을 검증하는데 사용할 수 있고, 일반 상대론 또한 이 검증을 통과했다. 이와 유사하게 dCS 중력이론에 대해서도 검증 방법을 적용 가능하고, 이를 통해 l_{dCS}의 상한을 $O(10^1)$km까지 줄일 수 있다. 중성자별 내부의 암흑물질은 중성자별 내부의 분포를 변화시킬 수 있고, 이에 따라 일반 상대론에 의한 관성 모멘트와 조석 러브 수는 줄고, dCS 이론에 의한 관성 모멘트의 보정은 커진다. 하지만 종합적인 변화는 적어 I−Love 곡선과 dCS에 대한 제약은 크게 변하지 않는다.

주요어: dCS 중력, ALP, I−Love, 중성자별, 암흑 물질

Contents

1 Introduction

Axion, originally introduced to deal with strong CP problem, appears in various parts of physics in similar form. From the original QCD axion, and axion like particle(axion) to condensed matter axion, all axion have characteristic interaction, Chern-Simons action $g\Theta G\widetilde{G}$. G is some gauge field and $\widetilde{G}$ is dual of the field. Given scalar field Θ interacts with the field G related to fundamental force in parity-violating way. By the way, general relativity (GR), the theory for one of the four fundamental forces in nature, has passed a series of examinations which have been getting subtle and precise. Still, GR is not the final theory for gravity, in view of quantum field theory, so that expecting higher order curvature term in the relevant lagrangian is natural.

One of such modified gravity theory is adding parity-violating term $g\Theta R\widetilde{R}$, which is called Chern-Simons gravity theory. In this theory, Θ is externally prescribed dynamical field which select certain direction in a flow of time [1]. For valid solutions of spacetime, its Pontryagin number $R\widetilde{R}$ of the theory has to be 0. [2]. To require a more realistic theory, one can change Θ to dynamical field by adding $\nabla_\mu\Theta\nabla^\mu\Theta+V(\Theta)$ and get more arbitrariness of Pontryagin density. It is dynamical Chern-Simons gravity theory (dCS gravity). In this theory, theory itself is parity conserved, however, some of the solutions can be parity-odd. In other words, in parity-odd configuration such as Kerr metric, Pontryagin number does not vanish. [3] Considering similarity of form, Θ in dCS gravity theory can be considered as ALP, or some ALP might have this kind of interaction with gravity.

Key parameter for ALPs is its coupling constant to the interacting field. Finding ALP largely depends on the strength of the interaction. For example, a lot of experiments searching QCD axion use conversion, axion to EM wave through this interaction, and provide a possible region of coupling strength and axion mass. [4], [5] Likewise, there were several tries to measure coupling constant for dCS gravity theory, and they provided relatively wide range of bounds due to high precision of GR accuracy, indicating extremely small effect of the modified theory. Gravity Probe B(GPB) is one of them. It measured frame dragging effect around Earth using a gyroscope of a satellite in a 650km-high orbit. The result coincided with GR expectation within 20% [6], which is converted to coupling constant bound, $l_{dCS} \leq O(10^8 km)$ bound. [7]

Briefly analyzing the result, frame dragging effect depends on $t-\pi$ component of metric, and using the result of [2], the bound is converted from inequalities for compactness and mass of the stellar object, $l_{dCS}^2 \leq M^2/C^3$. Compactness mainly determines the order of the upper bound, so that it can be inferred that experiments depending on more compact objects might generate stricter constraints on the coupling. However, applying a similar method of GPB to the desired stellar object is not an easy task due to its distance from the Earth. Another available quantity related to dCS gravity theory is correction to moment of inertia of rotating objects. But acquiring moment of inertia of stars is difficult and even if the information about some star is given, it is required to examine theoretical expectation that additional properties of the star, which are also hard to measure. At this point, it is helpful to utilize general relation between moment of inertia and another property of neutron star(NS), more compact object than Earth.

I-Love relation was originally discovered by Yagi and Yunes in 2013 [8] and it describes non-dimensionalized moment of inertia and tidal love number of NS, holding quasi-globally. Under the modified gravity theory, these two parameters can have additional contribution from the extra term and different form of the relation. With these facts, theoretical analysis expects that a certain kind of I-Love curve appears and depends on the coupling constant. Comparing this result to the observed one provided narrower range of possible value.

The majority of papers about dCS gravity theory's constraints don't consider additional information, for example, dark matter in NS. However, it seems to have no specific reason to ignore it. DM structure can change dCS correction of moment of inertia and tidal love number. Therefore, the form of I-Love relation and corresponding constraints for coupling constant could be different. In this thesis, the change is studied.

This thesis is organized as follows: in section 2, I-Love relation and Chern-Simons gravity theory are reviewed. To see the effect of DM presence, results without DM should be given, so that the results are reproduced in section 3 and 4. In section 3, theoretical expectations with and without dCS gravity are derived. In section 4, I-Love curve with some selected EOS are drawn, and the expectation with actual data is confirmed. In section 5, finally, DM field in NS is considered and how it can change I-Love curve is looked into. Also, massive scalar field constraints are considered in this context. In section 6, a summary is provided. Throughout this thesis, modified geometrized unit is adopted, $G = c = \mathrm{M}_\odot = 1$, therefore $[length] = [mass] = [time]$ unless specific unit is written. Also the MTW convention [10] is used.

2 I-Love relation and dCS gravity

2.1 I-Love relation

I-Love relation is quasi-global relation between moment of inertia and tidal love number. 'Quasi-global' means the relation holds for almost every realistic EOS of NS within $O(1)\%$. Moment of inertia I in GR is given by following [11], [25]

$$\frac{8\pi}{3}\frac{1}{\Omega}\int_0^{\mathrm{R}} \frac{e^{(-\nu+\lambda)/2}R^5(\rho+p)\widetilde{\omega}_1}{R-2M}dR \tag{2.1}$$

Information about each character is given in the next section. In general, moment of inertia is an ill-defined quantity, since angular momentum may not scale linearly with angular velocity. For slowly rotating approximation, however, angular momentum can be expanded by angular velocity and the definition can be extended to:

$$I = \lim_{\Omega\to 0}\frac{J}{\Omega} \tag{2.2}$$

For non-dimensionalization in geometrized units, dividing it by M^3 is enough. Tidal love number means how easily the object can be deformed. In Newtonian gravity, tidal love number λ is defined as

$$Q_{ij} = \lambda\epsilon_{ij} \tag{2.3}$$

where ϵ is tidal field from external gravitational force and Q is quadrupole moment. In GR, the quantity is calculated from metric using the definition above and following equation [12]:

$$\frac{(1-g_{tt})}{2} = \frac{-M}{r} + \frac{-3Q_{ij}}{2r^3}(n^in^j - \frac{1}{3}\delta^{ij}) + O(\frac{1}{r^3}) + \frac{1}{2}\epsilon_{ij}x^ix^j + O(r^3) \quad n^i = \frac{x^i}{r} \tag{2.4}$$

The dimension of tidal love number is $[length^5]$, $\frac{\lambda}{M^5}$ is non-dimensionalized one. [13] These two parameters seem to have dependence on inner properties of NS but it might not be true. Each quantity can be parametrized with compactness $\mathcal{C} = \frac{M}{\mathrm{R}}$, and quasi-general relation between them is derived from these parametrizations. Gravity theory determines form of the relation. For the Newtonian limit, I-Love relation is given by $\bar{I}_{\mathrm{N}} = \mathrm{C}\bar{\lambda}^{\frac{2}{5}}$. For GR, it is given by [14]

$$\bar{I}_{\mathrm{GR}} = \bar{\lambda}^{\frac{2}{5}}(c_0 + c_1\bar{\lambda}^{\frac{-1}{5}} + c_2\bar{\lambda}^{\frac{-2}{5}}) \tag{2.5}$$

Extra term can be added to RHS, for modified gravity theory. Using this result, gravity theory can be examined in following way. Measured I and Love number generate an error box in parameter space, and the measured point is on the space. If I-Love number curve derived by a certain gravity theory can't pass through the acquired region, then the theory is ruled out.

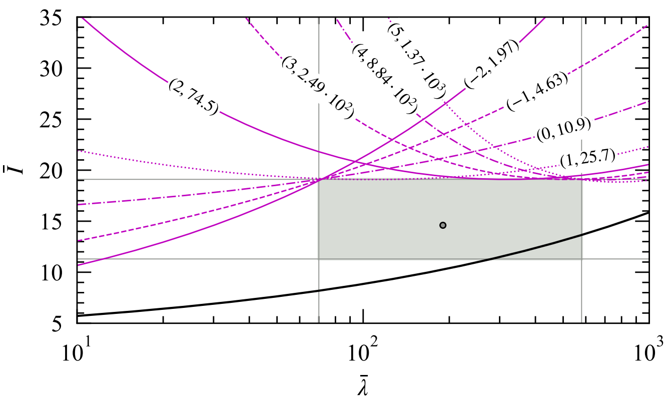

Figure 1. This figure is from [14] and an example of GR examination using I-Love relation. $\bar{I} = I/M^3$ means non-dimensionalized moment of inertia of NS and $\bar{\lambda} = \lambda/M^5$ means non-dimensionalized tidal Love number. As mentioned in the text, I-Love curve under GR passes the error box and GR is a valid theory.

In other words, this generality exists under modified gravity theory. Universality of this relation can be used in other kinds of test, extra dimension [15], dark matter detection [16] etc. Several reasoning for EOS-independence are suggested. One is that these quantities are mainly determined by contribution from the surface, so that physics inside NS, EOS, is concealed and similarity between different EOS appears. It can be supplemented by the fact that there is a non-ignorable deviation when applied EOS has radical change around the surface of NS. [8] Another available reason is compactness. As we will see in the next section, in the black hole limit, difference between EOSs gets smaller. Also, this I-Love relation is found in other compact stellar objects like polytropic star[17], white dwarf star [18] etc. and these phenomena might prove the effect of compactness on the generality.

2.2 Review of dCS gravity

Lagrangian for CS gravity theory is given by [3]

$$\begin{aligned} \mathcal{L} &= \sqrt{-g}(\kappa_g R + \frac{\alpha}{4}\Theta R\widetilde{R} - \frac{\beta}{2}(\nabla_\mu\Theta\nabla^\mu\Theta + 2V(\Theta)) + \mathcal{L}_{matter}), \\ \widetilde{R}^{\nu\mu\rho\delta} &= \frac{1}{2}\epsilon^{\rho\delta\alpha\beta}R^{\nu\mu}_{\alpha\beta} \quad \kappa_g = \frac{1}{16\pi} \end{aligned} \tag{2.6}$$

Many papers adopt α and β to denote coupling constant and separation of non-dynamic and dynamic Chern-Simons theory. If α has dimension of $[length^A]$, β has dimension of $[length^{2A-2}]$ and Θ has dimension of $[length^{-A}]$. For non-dynamical theory, $\beta = 0$ and α is non zero and both of them are non-zero in dynamical Chern-Simons theory. Dimension of each coupling parameter can be arbitrarily chosen, and one of common choices is $[\alpha] = -2$. By differentiating the Lagrangian by Riemann tensor, Einstein equation is derived with extra stress-energy tensor from theta field on its RHS:[3]

$$
\begin{aligned}
G_{\mu\nu} + \frac{\alpha}{\kappa_g} C_{\mu\nu} &= \frac{1}{2\kappa_g}(T^{mat}_{\mu\nu} + T^{\Theta}_{\mu\nu}) \\
C^{\mu\nu} &\equiv (\nabla_\sigma \Theta)\epsilon^{\sigma\delta\alpha(\mu}\nabla_\alpha R^{\nu)}_{\delta} + (\nabla_\sigma \nabla_\delta \Theta)\widetilde{R}^{\delta(\nu\mu)\sigma} \\
T^{\Theta}_{\mu\nu} &\equiv \beta(\nabla_\mu \Theta)(\nabla_\nu \Theta) - \frac{\beta}{2} g_{\mu\nu}\nabla_\delta \Theta \nabla^\delta \Theta - \beta g_{\mu\nu} V(\Theta)
\end{aligned}
\tag{2.7}
$$

C is 4-dimensional Cotton tensor. Considering dCS gravity, equation of motion for theta field is given by

$$
\Box\Theta = -\frac{\alpha}{4\beta} R_{\nu\mu\rho\sigma}\widetilde{R}^{\mu\nu\rho\sigma} + \frac{dV}{d\Theta} \tag{2.8}
$$

Freedom of dCS gravity theory is on scalar field potential, however, it is generally ignored by various reasons. The CS scalar field is moduli field in the context of string theory. The field has zero potential before stabilization and it is natural to ignore potential term which means massless axion in a classical or semi-classical scenario. Nonetheless, considering massive scalar field for dCS gravity theory has some interesting phenomena. For example, see Ref.[9]. By using Bianchi identity, it can be shown that divergence from normal matter stress-energy tensor is zero.[3] Dynamical theta field has a simple solution of $\Theta = C_\mu x^\mu$, [20] but this solution has severe constraints from cosmological observation [19]. Parity violation from this gravity theory differs on whether the theory is dynamical or not. For non-dynamical one, S invariant under spatial parity needs pseudo scalar field and Cotton tensor invariant under spatial parity needs pseudo scalar Θ or vector Θ. [3] For dynamical one, $\nabla_a v^a \propto R\widetilde{R}$ where v^a is covariant velocity of Θ, and forces the theta field to be pseudo scalar field so that theory itself is parity even. However, although theory itself is parity conserving, the solution of the dynamical theory might be parity odd. As an example, EM wave propagating into dielectric medium in Maxwell theory is the case. Likewise, dCS gravity shows similar phenomena. For example, Kerr metric has parity-odd spacetime, therefore solution in the metric shows parity-odd in the modified theory. This kind of solution is the desired non-trivial solution.

The application of dCS gravity theory is in various physical regions, such as canceling ABJ anomaly from gravity theory like QCD axion, string theory etc. Also, massive scalar field might be expected to have some properties of ALP.

3 Theoretical expression & expectation of dCS gravity

In following two sections, constraints on dCS coupling without DM is reproduced to compare it to the result with DM. Through the reproduction, it can be confirmed that GR passes the I-Love test and how dCS effect activates.

3.1 Assumption and Approximation

Several approximations are applied since solving equation of motion for theta field in general is not an easy problem. Slowly rotating approximation is used for order by order calculation and applying I-Love relation. Slowness is determined by comparing dimensionless spin parameter χ and mass of the object. Equivalently, this condition can be expressed with frequency, then the desired inequality is $\epsilon = \frac{f}{f_0} \ll 1, f_0 = \frac{(G/M^3)^{\frac{1}{2}}}{2\pi}$. Observed data with pulsar [21] are $f_0 = 205.53Hz$ and $\epsilon = 0.14$. Small coupling approximation is adopted for solving equations order by order and (2.6) becomes an effective theory. This smallness means dCS effect is quite smaller than the mass of the stellar object. α has $[length^4]$ dimension, and corresponding inequality is

$$\zeta \equiv \frac{\xi_{CS} M^2}{\mathrm{R}^6} \ll 1, \quad l_{dCS}^2 = \xi_{CS} = \frac{\alpha^2}{\beta \kappa_g}, \tag{3.1}$$

where M is the mass of the star and R is the radius of the star. For 1.4 $M_{\odot}$ NS, upper bound for Chern-Simons coupling l_{dCS} is given by $O(10^8 km)$. The previous constraint from GPB is compatible with this upper bound. As we will see in the following sections, results from I-love relation also match with the approximation. Using these approximations and following procedure of [20], all physical quantities can be expanded by χ and α', where α' is a book-keeping parameter of small coupling:

$$A = \sum_{m,n} \chi^m {\alpha'}^n A_{(n,m)} \tag{3.2}$$

where $A_{(n.m)}$ means quantity in $O(\chi^n {\alpha'}^m)$ order. Applying this expression to equation of motion, solution is acquired order by order.

As we discussed in section 2, parity-odd spacetime is required to observe the effect of dCS gravity. In here, an axially symmetrically rotating object is the spacetime. We will consider stationary and axially symmetric system and the expected metric is a function of r and Θ. Also, the inversion of time has the same result of the inversion of the rotating angle. In mathematical expression, this means $ds^2(-t, \vec{r}) = ds^2(t, r, \theta, -\phi)$. With this assumption, appropriate metric has the following form:[11]

$$\begin{aligned} ds^2 = & - e^{\bar{v}(r)}(1 + 2\bar{h}(r,\theta))dt^2 + e^{\bar{\lambda}(r)}(1 + \frac{2\overline{m}(r,\theta)}{r - 2\overline{M}(r)})dr^2 \\ & + r^2(1 + 2\bar{k}(r,\theta))(d\theta^2 + sin^2\theta(d\phi - \overline{\omega}(r,\theta)dt)^2) \end{aligned} \tag{3.3}$$

It is based on Schwarzschild-like metric and the extra contribution is come from slowly rotating approximation. $\overline{\lambda}, \overline{v}, \overline{M}$ refer to the original Schwarzschild-like contributions and functions of r only. With slowly rotating approximation, rotation can be treated as perturbation and $\overline{h}, \overline{m}, \overline{k}, \overline{\omega}$ refer to this perturbation. (t, r, θ, ϕ) are Hartle-Thorne coordinates and r, θ denote ordinary polar coordinates. In this coordinate system, parameters required for deriving the mass profile of some EOS would break perturbation assumption since distortion around the surface might exist due to rotation. By changing coordinates as following, $r(R, \theta') = R + \xi(R, \theta'), \theta = \theta'$, $\xi(R, \theta') = {\alpha^2}' \chi^2 \xi_{(2,2)}(R, \theta')$, the effect of rotation absorbed into the coordinates. After the coordinate transformation, corresponding metric is,

$$\begin{aligned}
ds^2 =& -((1+2h+\xi\frac{dv}{dR})e^v - R^2\omega^2 sin^2\theta')dt^2 - 2R^2\omega sin^2\theta' dt d\phi \\
&+ (R^2(1+2k)+2R\xi)sin^2\theta' d\phi^2 + e^\lambda(1+\frac{2m}{R-2M}+\xi\frac{d\lambda}{dR}+2\frac{\partial\xi}{\partial R})dR^2 \\
&+ 2e^\lambda\frac{\partial\xi}{\partial\theta'}dRd\theta' + (R^2(1+2k)+2R\xi)d\theta'^2 \\
&+ O(higherorder) \qquad M(R) \equiv \frac{(1-e^{-\lambda(R)})R}{2}
\end{aligned} \tag{3.4}$$

For the spherically symmetric circumstance, $R\widetilde{R} = 0$ and there is no source to the inhomogeneous equation of motion for theta field Θ if we ignore potential terms. As a result, Θ could not have terms which is χ^0 order. Also, even χ order of Θ terms vanish due to parity requiring, $P(\Theta) = -1$. Therefore, $\Theta = \alpha\chi\Theta_{(1,1)} + O(\alpha'\chi^3)$. To focus on dCS correction to GR, it is ignored that $O(\alpha^0\chi^2)$ like terms, which are higher order terms with respect to spin and have no contribution from small coupling. Considering parity, diagonal terms and the (R, θ') component is even power of χ and (t, ϕ) component has odd power of it only.[20] Deformation from rotation affects energy and pressure inside of the star only, so that the metric returns to the original one for the outside of the star by substituting R to r and θ' to θ. NS are treated as perfect fluid and its rotation as uniform one. With this assumption, stress energy tensor of normal matter has little contribution from other equations of motion. Then, we can derive equations of motion to solve order by order [20]. At zeroth order, energy density and pressure is calculated with certain EOS, and mass profile of NS is made. At first order in spin, the contribution from rotating, gravitomagnetic sector is calculated and using this result, theta field configuration at first order in spin and coupling is derived. Considering the theta field configuration as source term for the equation of motion of gravitomagnetic sector, dCS correction to moment of inertia is derived. Tidal love number is defined as (2.4), g_{tt} doesn't change radically under transformation to Cartesian frame, dCS correction to tidal Love number does not exist.

3.2 $O(\chi^0\alpha'^0)$ Order-Non Rotating Background

At $O(\chi^0\alpha'^0)$, there is no dCS correction due to its spherically symmetric property. At this order, the metric is reduced to non-rotating star's one and only diagonal terms remain. From the first two diagonal components,

$$\frac{dM}{dR} = 4\pi R^2\rho \tag{3.5}$$

$$\frac{dv}{dR} = 2\frac{4\pi R^3 p + M}{R(R-2M)} \tag{3.6}$$

It has four unknown functions to solve but only two differential equations are given. Instead of using the other two diagonal components, matter's stress-energy tensor's R component yields the following equation:

$$\frac{dv}{dR} = -\frac{2}{\rho+p}\frac{dp}{dR} \tag{3.7}$$

, which eliminates $\frac{dv}{dR}$ in coupled equation and becomes Tolman-Oppenheimer-Volkoff(TOV) equation.

$$\frac{dp}{dR} = -\frac{(4\pi R^3 p + M)(\rho + p)}{R(R - 2M)} \tag{3.8}$$

Still, another condition is required to solve equations completely and equation of states provides the one. EOS have form of $p = p(\rho)$, so that coupled differential equations (3.5), (3.8), EOS are closed.

For outside of NS, $p = \rho = 0$ so that

$$e^{-\lambda} = e^{v} = 1 - \frac{2M_{tot}}{R} \equiv f(R) \tag{3.9}$$

For inside of NS, (3.8) has a singularity at $R = 0$, therefore, initial point R_c should be non-zero and satisfy $R_c/R_e \ll 1$. Given equations are numerically solved by forth-order Runge-Kutta method with 10^{-3} step size. Other differential equations in this section are also solved by this method. ρ_c or p_c, value at initial point, is chosen arbitrarily and radius R_e is determined by the value at which $p(R_e) = 0$ after solving coupled differential equations. Non-rotating mass of NS is given by $M_{tot} = M(R_e)$. Varying ρ_c or p_c, mass-radius relation for each EOS can be obtained. Fig.2 is the relation for APR3[30].

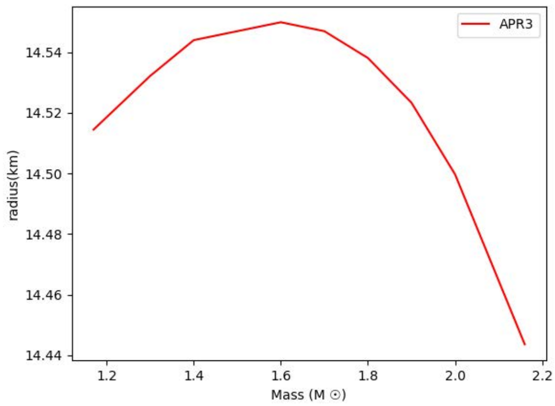

Figure 2. NS mass-NS radius relation for APR3 EOS

Substituting calculated $\rho(R), p(R)$ into (3.6), numerical solution for $v(R)$ is made. Initial condition is $\frac{4\pi}{3}(\rho_c + p_c)R_e^2$ and adding proper constant value to trial solution yields final solution satisfying the boundary condition $e^{v(R_e)} = f(R_e)$.

Piecewise polytropic EOSs are adopted for numerical method.[22], [23] In this method, every EOS is expressed in:

$$p_i = K_i \rho^{\Gamma_i} \quad \text{for} \quad \rho_{i-1} \leq \rho \leq \rho_i, \quad \rho_0 = 0 \tag{3.10}$$

K_i, Γ_i are chosen to have the least rms residual. For low rest mass density region, crust,($\rho \leq 10^{14} g/cm^3$) SLy EOS[24] is used with four different region. In $\rho \geq 10^{14} g/cm^3$ region, various EOS is applied with fixed division points, $10^{14.7}\mathrm{g/cm}^3, 10^{15}\mathrm{g/cm}^3$. At this order, expression for tidal love number is also derived. In [12], equation for g_{tt} component is following:

$$H'' = H'(\frac{2}{r} + e^{\lambda}[\frac{m(r)}{r^2} + 4\pi r(p-\rho)]) + H[\frac{-6e^{\lambda}}{r^2} + 4\pi e^{\lambda}(5\rho + 9p + \frac{\rho+p}{dp/d\rho}) - v'^2] = 0 \quad (3.11)$$

It can be derived from the equation given in previous subsection with setting $\omega = 0$. Initial condition for H is

$$H(r) = a_0 r^2(1 - O(r^2)) \quad r \to 0 \quad (3.12)$$

and a_0 is some arbitrary value. H means small perturbation to Schwarzschild metric. With this fact, tidal love number is given by,

$$\lambda_{tidal} = \frac{2}{3}R^5 \frac{8C^5}{5}(1-2C)^2[2 - 2C(y-1) - y](2C(6 - 3y + 3C(5y-8)) + 4C^3(13 - 11y + C(3y-2) + 2C^2(1+y)) + 3(1-2C)^2[2 - y + 2C(y-1)]\log(1-2C))^{-1} \quad (3.13)$$

where $y = R_e H'(R_e)/H(R_e)$ $C = M_{tot}/R_e$. It is not mandatory to solve the equation completely since the used value is the ratio between H and its derivative. Dividing H in both sides of (3.11) and considering F as $F = \log(H)$, then (3.11) become 1st order non-linear differential equation for F and its initial condition is given by an exact value not unknown. Then, tidal love number comes out.

3.3 $O(\chi^1\alpha'^0)$ Order-GR Rotating Contribution

At this order, only the rotational effect from GR exists and $\omega(R)$ contains the contribution. Non-vanishing component of the metric is $g_{t\phi}$ only and its differential equation is

$$\frac{\partial^2 \widetilde{\omega}_{(1,0)}}{\partial R^2} + 4\frac{1 - \pi R^2(\rho+p)e^{\lambda}}{R}\frac{\partial \widetilde{\omega}_{(1,0)}}{\partial R} + \frac{e^{\lambda}}{R^2}\left(\frac{\widetilde{\omega}_{(1,0)}}{\partial \theta'^2} + 3\cot\theta' \frac{\partial \widetilde{\omega}_{(1,0)}}{\partial \theta'}\right) - 16\pi(\rho+p)e^{\lambda}\widetilde{\omega}_{(1,0)} = 0 \qquad \widetilde{\omega}_{(1,0)} \equiv \Omega - \omega_{(1,0)} \quad (3.14)$$

where Ω is angular velocity of NS which is given by a priori condition. But Ω cannot be selected completely arbitrarily since I-Love relation disappears in fast rotating with fixed angular velocity. [36] Instead, with fixed small dimensionless spin parameter $\frac{\mathrm{J}}{\mathrm{M}^2}$, the relation remains. [37] [38]. For real calculation, $\Omega = 10^{-3}$ is used, since $f_{obs} = 205.53Hz$ is corresponding observed frequency from NICER data. [39], [40] For further simplification, $\widetilde{\omega}_{(1,0)}$ can be expanded by Legendre polynomial:

$$\widetilde{\omega}_{(1,0)}(R,\theta') = \sum_l \widetilde{\omega}_l(R)\left(-\frac{1}{\sin\theta'}\frac{\partial}{\partial \theta'}P_l(\cos\theta')\right) \quad (3.15)$$

Substituting expression above into (3.14) provides simpler equation.

$$\frac{d^2\widetilde{\omega}_l}{dR^2} + 4\frac{1 - \pi R^2(\rho+p)e^{\lambda}}{R}\frac{d\widetilde{\omega}_l}{dR} - \left(\frac{(l+2)(l-1)}{R^2} + 16\pi(\rho+p)\right)e^{\lambda}\widetilde{\omega}_l = 0 \quad (3.16)$$

By [11],

$$\widetilde{\omega} \to C_1 r^{S_+} + C_2 r^{S_-}, \quad S_{\pm} = -\frac{3}{2} \pm (\frac{9}{4} + \frac{l(l+1)-2}{j(0)})^{1/2}, \quad r \to 0 \quad (3.17)$$

For regularity at center, except for $l = 1$, C_2 should vanish. Also, outside of star

$$\widetilde{\omega} \to C_1 r^{-l-2} + C_2 r^{l-1}, \quad r \to \infty \tag{3.18}$$

For satisfying asymptotic flatness, the function should decrease faster than r^{-3}. Fixed inside solution and boundary conditions require two unknowns unless both of them vanish. Therefore, only $l = 1$ contribution can survive. Applying this fact to (3.16),

$$\frac{d^2\widetilde{\omega}_1}{dR^2} + 4\frac{1 - \pi R^2(\rho + p)e^{\lambda}}{R}\frac{d\widetilde{\omega}_1}{dR} - 16\pi(\rho + p)e^{\lambda}\widetilde{\omega}_1 = 0 \tag{3.19}$$

This is the differential equation for GR-rotating contribution. As mentioned in the previous subsection, $p = \rho = 0, \quad e^{-\lambda} = e^{v} = 1 - \frac{2M_{tot}}{R} \equiv f(R)$ for outside and $\widetilde{\omega}(R)$ can be solved analytically:

$$\widetilde{\omega}_{1out} = \Omega - \frac{2J}{R^3} \tag{3.20}$$

J is an integration constant which is determined by boundary condition and inside solution. This number indicates angular momentum of the star in GR. Interior solution can be obtained from (3.19) with boundary condition: $\widetilde{\omega}_1(R_e) = \Omega - \frac{2J}{R_e^3}$ and $\lim_{\varepsilon\to 0} \frac{d\widetilde{\omega}_1}{dR}|_{R_e-\epsilon}^{R_e+\epsilon} = 0$. Considering GR contribution only, moment of inertia has two different expressions;

$$I \equiv \frac{J}{\Omega}$$

and (2.1). It is the boundary condition that requiring same value for these two expression. Using the fact that (3.19) is linear and homogeneous, differential equation with boundary condition can be converted to initial condition one. Solving the equation with an arbitrary initial value ω_c at R_c, trial solution $\widetilde{\omega}_{1tr}$ is made. For some C_w, $C_w\widetilde{\omega}_{1tr}$ is solution too because of linearity and homogeneity so that true solution can be obtained by adjusting C_w to satisfy

$$C_w\widetilde{\omega}_{1tr}(R_e) = \Omega(1 - \frac{2C_w I_{tr}}{R^3}), \qquad I_{tr} = \frac{8\pi}{3}\frac{1}{\Omega}\int_0^{\mathrm{R_e}} \frac{e^{(-\nu+\lambda)/2}R^5(\rho + p)\widetilde{\omega}_{1tr}}{R - 2M}dR \tag{3.21}$$

then,

$$C_w = \frac{\Omega R_e^3}{\widetilde{\omega}_{1tr}(R_e)R_e^3 + 2I_{tr}\Omega} \tag{3.22}$$

Fig. 3 shows the results for some EOS.

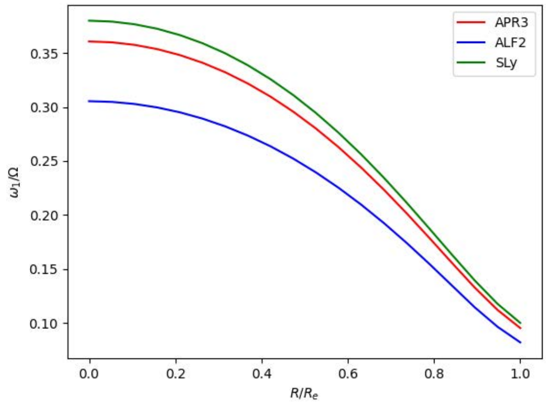

Figure 3. Normalized ω_1 and normalized NS radius relation for various EOS. Mass of neutron star is fixed as $1.4M_{\odot}$

3.4 $O(\chi^1\alpha'^1)$ Order-dCS Gravity Effect

GR-rotating effect activates dCS correction. In other words, the differential equation for Θ has non-zero source term if rotation exists and the equation is following:

$$\begin{aligned}&\frac{\partial^2\Theta_{(1,1)}}{\partial R^2}+\frac{1+e^{\lambda}(1-4\pi R^2(\rho-p))}{R}\frac{\partial\Theta_{(1,1)}}{\partial R}+\frac{e^{\lambda}}{R^2}\left(\frac{\partial^2\Theta_{(1,1)}}{\partial\theta'^2}+\cot\theta'\frac{\partial\Theta_{(1,1)}}{\partial\theta'}\right)\\&\qquad=8\pi\frac{\alpha}{\beta}(\rho-\frac{M}{(4/3)\pi R^3})e^{(\lambda-v)/2}\left(\sin\theta'\frac{\partial^2\widetilde{\omega}_{(1,0)}}{\partial R\partial\theta'}+2\cos\theta'\frac{\partial\widetilde{\omega}_{(1,0)}}{\partial R}\right)\end{aligned}\tag{3.23}$$

This differential equation ignores terms induced by Θ potential. If effect of potential is required, adding $V'(\Theta)$ to RHS is enough. As in $\widetilde{\omega}_{(1,0)}$ case, expanding $\Theta_{(1,1)}$ in Legendre polynomial and requiring regularity at center and asymptotic flatness, the equation above is simplified as

$$\begin{aligned}&\frac{d^2\theta_1}{dR^2}+\frac{1+e^{\lambda}(1-4\pi R^2(\rho-p))}{R}\frac{d\theta_1}{dR}-2\frac{e^{\lambda}}{R^2}\theta_1\\&\qquad=16\pi\frac{\alpha}{\beta}(\rho-\frac{M}{(4/3)\pi R^3})e^{(\lambda-v)/2}\frac{d\widetilde{\omega}_1}{dR}\end{aligned}\tag{3.24}$$

where θ_1 is $l=1$ component of $\Theta_{(1,1)}$. Exterior differential equation for θ_1 can be solved analytically due to $p=\rho=0,\quad e^{-\lambda}=e^{v}=f(R)$ and $\widetilde{\omega}_{out}=\Omega-\frac{2J}{R^3}$, if potential of θ_1 is negligible. The analytic solution is

$$\begin{aligned}\theta_1^{ex}=&\frac{5\alpha J}{8\beta M_{tot}^2R^2}(1+2\frac{M_{tot}}{R}+\frac{18M_{tot}^2}{5R^2}\\&+C_\theta\frac{R^2}{M_{tot}^2}[1+\frac{R}{2M_{tot}}(1-\frac{M_{tot}}{R})\ln f(r)])\end{aligned}\tag{3.25}$$

, where C_θ is an integration constant which is determined by boundary conditions. Another integration constant is set to 0 for asymptotic flatness. θ_1 differential equation does not have good properties like $\widetilde{\omega}$, interior solution satisfying boundary condition needs more process to be obtained. Asymptotic behavior at the center is given by[20],

$$\theta_1(R) = \theta'_{1c}R + O(R^3) \qquad R \to 0^+ \tag{3.26}$$

Setting initial condition as $\theta_1(R_c) = \theta'_{1c}R_c$, $\theta'_1(R_c) = \theta'_{1c}$ with some θ'_{1c} value and particular solution and homogeneous solution can be derived. True solution can be constructed as

$$\theta_1(R) = \theta_{1p}(R) + C_{\theta h}\theta_{1h}(R) \qquad R \leq R_e \tag{3.27}$$

$C_{\theta h}$ is some constant. Boundary conditions are

$$\begin{aligned} \frac{d\theta_1}{dR}|^{R_e+\epsilon}_{R_e-\epsilon} = 0 \\ \theta_1|^{R_e+\epsilon}_{R_e-\epsilon} = 0 \end{aligned} \tag{3.28}$$

There are two integration constants and two equations so that these equations are solvable. Fig.4 is θ_1-radius graph for selected EOS.

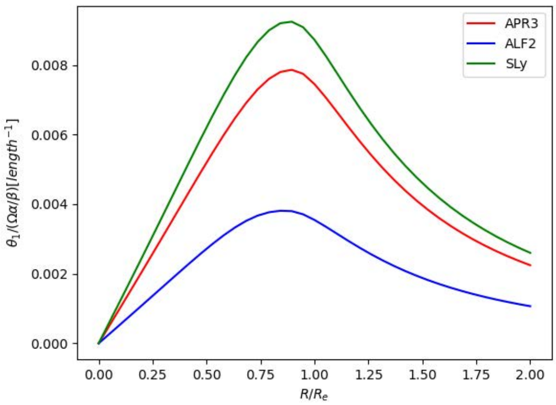

Figure 4. Normalized θ_1 is given as function of normarlized radius with selected EOS. Mass of neutron star is $1.4M_\odot$

3.5 $O(\chi^1\alpha'^2)$ Order-dCS Gravity Correction

Activated dCS correction is reflected in $O(\chi^1\alpha'^2)$ order of ω. Modified differential equation is come from $g_{t\phi}$ component:

$$\begin{aligned} \frac{\partial^2\omega_{(1,2)}}{\partial R^2} + 4\frac{1-\pi R^2(\rho+p)e^\lambda}{R}\frac{\partial\omega_{(1,2)}}{\partial R} + \frac{e^\lambda}{R^2}(\frac{\omega_{(1,2)}}{\partial\theta'^2} + 3\cot\theta'\frac{\partial\omega_{(1,2)}}{\partial\theta'}) \\ - 16\pi(\rho+p)e^\lambda\omega_{(1,2)} = \frac{128\pi^2\alpha e^{(v+\lambda)/2}}{R^3\sin\theta'}(\delta R\frac{\partial^2\Theta_{(1,1)}}{\partial R\partial\theta'} \\ + (R\frac{d\rho}{dR} - \delta)\frac{\Theta_{(1,1)}}{\partial\theta'}) \qquad \delta(R) = \rho(r) - \frac{M(R)}{(4/3)\pi R^3} \end{aligned} \tag{3.29}$$

There is no contribution from $T^{\Theta}_{\mu\nu}$ since Θ is $O(\chi^1\alpha'^1)$ order, thus there is no need to consider existence of potential term. It can be simplified with the same reason of the previous differential equations, and the equation above becomes

$$\begin{aligned}\frac{d^2\omega_2}{dR^2} + 4\frac{1-\pi R^2(\rho+p)e^{\lambda}}{R}\frac{d\omega_2}{dR} - 16\pi(\rho+p)e^{\lambda}\omega_2 \\ = -\frac{128\pi^2\alpha e^{(v+\lambda)/2}}{R^3}[\delta R\frac{d\theta_1}{dR} + (R\frac{d\rho}{dR} - \delta)\theta_1]\end{aligned} \tag{3.30}$$

This equation is also non-homogenous, true solution can be obtained by

$$\omega_2(R) = \omega_{2p} + C_{w2}\omega_{2h} \tag{3.31}$$

where C_{w2} is an integration constant. Exterior solution is analytic solution as previous subsection,

$$\begin{aligned}\omega_2^{out} =& \frac{2J_{CS}}{R^3} - \frac{5\xi_{CS}J}{8M_{tot}R^6}(1 + \frac{12M_{tot}}{7R} + 27M_{tot}^2 10R^2 - \frac{5C_\theta R^5}{32M_{tot}^5}[1 \\ &+ \frac{M_{tot}}{R} - \frac{54M_{tot}^3}{5R^3} + \frac{R}{2M_{tot}}(1 - \frac{64M_{tot}^3}{5R^3} + \frac{48M_{tot}^4}{5R^4})\ln f(R)])\end{aligned} \tag{3.32}$$

J_{CS} is an integration constant which is determined by boundary conditions and related to dCS correction to angular momentum. For setting initial conditions, asymptotic behavior of the interior solution is a good reference.

$$\omega_2(R) = \omega_c + \frac{8\pi}{5}[(\rho_c + p_c)\omega_c - 16\pi\alpha e^{v_c/2}\rho_c\theta'_{1c}]R^2 + O(R^3) \quad (R \to 0) \tag{3.33}$$

As in the previous subsection, two unknowns and two conditions are given so that they are solvable. Fig.5 is ω_2-radius graph for selected EOS.

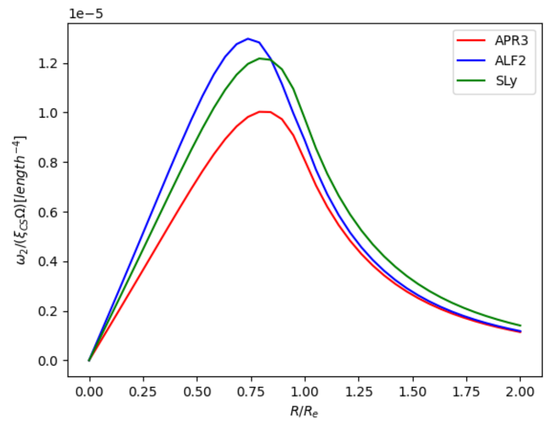

Figure 5. Normalized ω_2 at normalized radius for selected EOS. Mass of Neutron star is $1.4M_\odot$

Also, (2.1) is valid for dCS gravity theory [2] due to the form of the differential equation of ω_2. (3.29) can be expressed as

$$\frac{1}{R^4}(R^4 e^{-(\lambda+v)/2}\omega_2') + 16\pi(\rho+p)e^{(\lambda-v)/2}(\Omega - \omega_2) = -\frac{128\pi^2\alpha}{R^4}\frac{d}{dr}(\frac{4\pi}{3}R^2\delta\theta_1) \tag{3.34}$$

Multiplying R^4 on both sides and integrating 0 to R_e makes the same result of GR-only integration ,(2.1) because of the boundary condition for ω_2 and the form of ω_2^{out}. Using this formula, moment of inertia in dCS theory can be calculated in integral form, not a limit value. Finally, it is prepared that theoretical expression of moment of inertia in GR, tidal love number and moment of inertia correction in dCS gravity theory.

4 I-Love curve for selected EOS and Observed Data

Using expressions derived in a previous section, tidal love number, GR moment of inertia and dCS correction are calculated for fixed mass point. Maximum mass of NS is set as $2.16M_\odot$ [26], and minimum mass is set as $1.17M_\odot$[27]. For some EOS, extended mass range is used to check values in certain limits. Also, EOS are selected among ones which allow $M_{max} \geq 1.96M_\odot$, which is the least upper bound of NS by [28] and selected EOS are ALF2 [29], APR3[30], APR4[30], BCPM[31], BSR2[32], IOPB[33], ENG[34], FSUGarnet[35], SLy[24]. Results are Fig. 6, Fig. 7. As shown in the graphs, there is a positive dCS correction to moment of inertia for the same tidal love number since dCS doesn't change tidal love number. It can be interpreted as the following: Tidal love number is a property of how the object reacts to external deforming force so that the quantity does not rely on how dCS gravity consists, once the object is formed. Moment of inertia, on the contrary, depends on rotation and extra interaction with gravity makes 'dragging force' to rotation, which leads to increasing of the moment of inertia. Nonetheless, I-Love curves for each EOS are similar regardless of existence of a dCS correction. As mentioned in the previous section, 'errors' between different EOS are getting smaller as compactness of NS is approaching the black hole limit, C → 0.5. Also, in the non-slowly rotating and low mass region, differences are relatively larger.

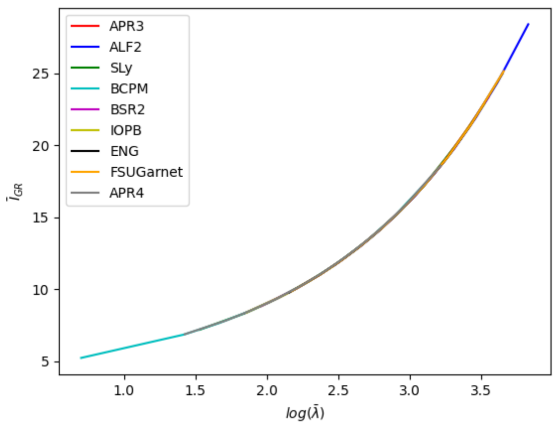

Figure 6. $\bar{I}_{GR}$-$\log_{10}\bar{\lambda}$ curve for selected EOS. $\bar{I}_{GR} = \frac{I}{M^3}$ means non-dimensionalized moment of inertia of NS and $\bar{\lambda} = \frac{\lambda}{M^5}$ means non-dimensionalized tidal Love number. Each EOS consists of certain part of total I_{GR}-Love curve.

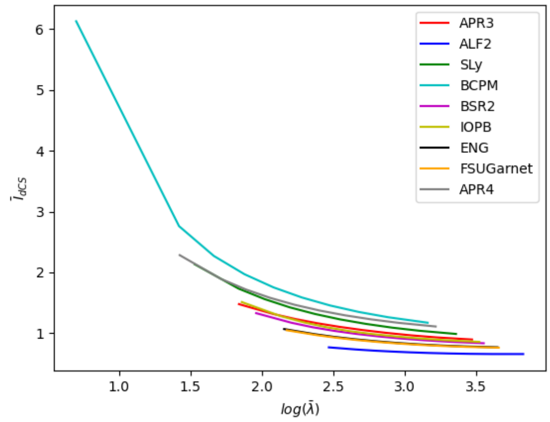

Figure 7. $\bar{I}_{dCS}$-$\log_{10}\bar{\lambda}$ curve for selected EOS with $\alpha = 10$, $\beta = 1$. $\bar{I}_{dCS} = I/M^3$ means dCS correction of non-dimensionalized moment of inertia of NS and $\bar{\lambda} = \lambda/M^5$ means non-dimensionalized tidal Love number.

Then, it becomes possible to apply this curve to observed data. Observed non-dimensionalized tidal love number is 190 for $1.4\mathrm{M}_\odot$ with 90% confidence from LIGO/Virgo data. [41] Direct observation of moment of inertia for $1.4\mathrm{M}_\odot$ NS is complicated problem, and inferred value $\bar{I} = 14.6$ is used with 90% confidence. [14] Inferring procedure is following: Apply two different analysis [39], [40] to NICER data [42] and get MCMC $M - R_e$ sample. Use these samples to construct a posterior distribution for compactness and get a posterior distribution for moment of inertia using quasi-universal relation between compactness and moment of inertia. [43], [44], [45] And check that the values from two different approach are almost same.

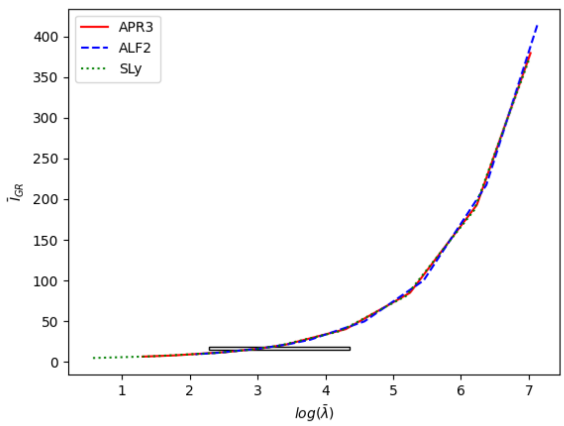

Figure 8. Examination of GR using I-Love relation with an extended range of NS mass using selected EOS. Grey bar is a 90% error box for observed data. $\bar{I}_{GR} = I/M^3$ means non-dimensionalized moment of inertia of NS and $\bar{\lambda} = \lambda/M^5$ means non-dimensionalized tidal Love number. The curve passes through the error box so that GR is valid with 90% confidence.

I-Love relation with fitting in GR is given by,

$$\bar{I}_{\rm GR} = \bar{\lambda}^{\frac{2}{5}}(c_0 + c_1\bar{\lambda}^{\frac{-1}{5}} + c_2\bar{\lambda}^{\frac{-2}{5}}) \tag{4.1}$$

with $c_0 = 0.52319258, c_1 = 1.57267013, c_2 = 1.66475892$ for APR3 EOS curve, which is similar to (2.5). Leading order contribution is come from the Newtonian limit so that it has high precision [14]. It can be found in Fig. 8 that I-Love curve in GR passes through the error box of observed/inferred data and GR is valid in 90% confidence for this test.

Varying dCS coupling constants l_{dCS}, possible space for it can be found in similar way. More precisely, fixing $\beta = 1$ and varying α is enough because α or $\frac{\alpha}{\beta}$ matters in theoretical expression. Then $\frac{\alpha}{\beta} = \alpha$, and define $l_{dCS} = \sqrt{\frac{16\pi\alpha^2}{\beta}}$ which has dimension of [length] in geometrized unit. In dCS gravity, with numerical fitting, I-Love relation is given by

$$\bar{I}_{\rm p} = \bar{I}_{\rm GR} + \gamma\bar{\lambda}^{-b/5} \tag{4.2}$$

with $\gamma = 6.2 * 10^{-2}\bar{\xi}$, $b_{dCS} = 4$. where $\bar{\xi} = \frac{l_{dCS}^2}{M^2}$. With this expression, bound for l_{dCS} is given by $l_{dCS} \leq O(10km)$.

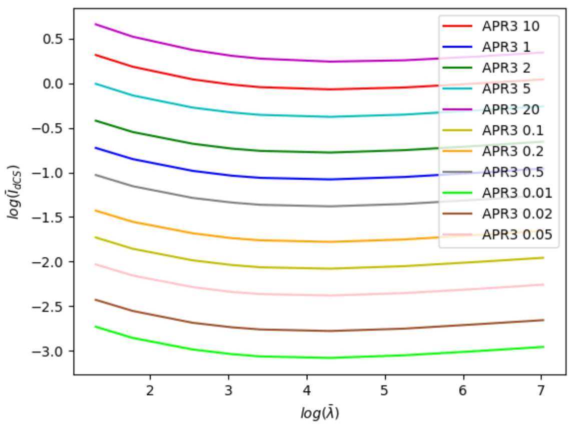

Figure 9. $\log_{10}\bar{I}_{dCS}$-$\log_{10}\bar{\lambda}$ curve with various α values with APR3 EOS. $\bar{I}_{dCS} = I/M^3$ means dCS correction of non-dimensionalized moment of inertia of NS and $\bar{\lambda} = \lambda/M^5$ means non-dimensionalized tidal Love number.

It is greatly reduced, compared to the former constraint $l_{dCS} \leq O(10^8 km)$ [46], [3], [7], using frame dragging effect around Earth. However, it might not be a remarkable point. The effect of frame dragging is promotional to $g_{t\phi}$ component, in other expression, $\omega = \omega_{GR}(1 + \frac{\omega_{dCS}}{\omega_{GR}})$ and $\frac{\omega_{dCS}}{\omega_{GR}}$ corresponds to dCS fraction of correction. If this method applied to NS with same 20% error, $l_{dCS}^2 \leq M^2/C^3$ would be converted to a similar $l_{dCS} \leq O(10km)$ bound. In other words, dCS correction might have a similar magnitude of effect for objects which have similar compactness, regardless of how the effect is realized. In addition to it, increasing accuracy of experiment might divide upper bound value by a $O(1)$ factor only. Therefore, if a stricter constraint is needed, more compact objects will be required.

5 NS with DM

Now we study how the presence of the dark matter field inside a neutron star modifies the I-Love relation and consequently the constraint on dynamical Chern-Simons(dCS) coupling.

So far, neutron stars were assumed to consist only of normal matter. With normal matter only, it might be hard to increase compactness of NS. However if dark matter exists inside of NS or around NS, 'overlapped' DM would raise compactness. Then, I-Love curve might show different behavior and harsher constraints would be expected. For simplicity, two fluid approach is used to deal with DM, in this work.

5.1 DM field in NS

In short, the presence of DM is important, for a wide variety of DM. In other words, once DM is captured, specified interaction or mechanism does not matter and only the mass profile of DM does. Therefore, change of dCS correction with DM inside of NS can be calculated in a similar way of previous sections without further manipulation. The detailed effect of DM inside of NS is the following: reduced maximum mass of NS, increased compactness and reduced tidal love number and GR contribution of moment of inertia, increased dCS correction to moment of inertia for given NS mass. Reduced maximum mass might rule out some EOS and recover some excluded EOS, for example H4 [50]. Numerical method is almost same with normal matter only NS. One of a few difference is how TOV is applied. Coupled TOV equation for two fluid approach is used [53]:

$$\begin{aligned}\frac{dp_{\rm DM}}{dR} &= -\frac{(4\pi R^3(p_{\rm DM}+p_{\rm NM})+M)(\rho_{\rm DM}+p_{\rm DM})}{R(R-2M)}\\ \frac{dp_{\rm NM}}{dR} &= -\frac{(4\pi R^3(p_{\rm DM}+p_{\rm NM})+M)(\rho_{\rm NM}+p_{\rm NM})}{R(R-2M)}\\ \frac{dM}{dR} &= 4\pi R^2(\rho_{\rm DM}+\rho_{\rm NM})\end{aligned} \tag{5.1}$$

There are several DM EOS which can be adopted. For bosonic DM, corresponding EOS is given by, [47]

$$P_{\rm DM} = \frac{\hbar^2\sqrt{\pi\sigma_{\rm D}}}{m_{\rm DM}^3}\rho_{\rm DM}^2 \tag{5.2}$$

where a certain kind of repulsive self-interaction is assumed and its cross section is given by $\sigma_{\rm D}$. For asymmetric fermionic DM without self-interaction which is replaced by Pauli exclusion principle, EOS is given by, [51], [52]

$$\begin{aligned}\rho_{\rm DM} &= \frac{m_{\rm DM}^4 c^6}{\hbar^3}\chi(x), \quad P_{\rm DM} = \frac{m_{\rm DM}^4 c^6}{\hbar^3}\phi(x)\\ \chi(x) &= \frac{x\sqrt{1+x^2}(1+2x^2)-\ln(x+\sqrt{1+x^2})}{8\pi^2},\\ \phi(x) &= \frac{x\sqrt{1+x^2}(2x^2/3-1)+\ln(x+\sqrt{1+x^2})}{8\pi^2}, \quad x = p_{\rm DM}/m_{\rm DM}\end{aligned} \tag{5.3}$$

where x denotes how relativistic the DM particles are and $p_{\rm D}$ denotes the Fermi momentum. If self-interaction is involved, additional terms are required for $\rho_{\rm D}$ and $P_{\rm D}$. Another difference is initial condition. DM part has a similar form of initial condition of normal matter as subsection 3.2. These two initial conditions are adjusted to have a certain value of total NS mass and fixed DM mass fraction. In this work, 5% mass fraction is set as default. Other steps are same. For simplicity, bosonic case only is considered here.

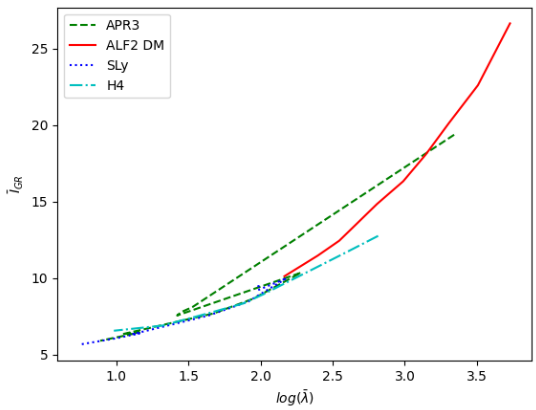

Figure 10. $\bar{I}_{GR}$-$\log_{10}\bar{\lambda}$ curve for selected EOS with 5% mass fraction of DM in NS. Coefficient of bosonic DM EOS, (5.2) is set to 10^5. $\bar{I}_{GR} = I/M^3$ means non-dimensionalized moment of inertia of NS and $\bar{\lambda} = \lambda/M^5$ means non-dimensionalized tidal Love number. This figure shows that the quasi-global relation exists for NS with DM.

With DM inside NS, the generality of I-Love curve does not disappear as Fig. 10. Fig.10 is acquired by solving (5.1) with mixed initial condition which is adjusted to meet 5% DM fraction. Coeffiecient of bosonic DM EOS, (5.2) is set to 10^5. Then, l_{dCS} constraint is slightly changed, but it might not change the order of magnitude of upper bound. Also, DM fraction affects how much I-Love curve changes, so that the possible DM fraction of NS is limited by actual observation of I-Love data. We can check that the tendency of NS configuration with a fixed DM mass fraction is relatively unstable so that the parameters easily bounce.

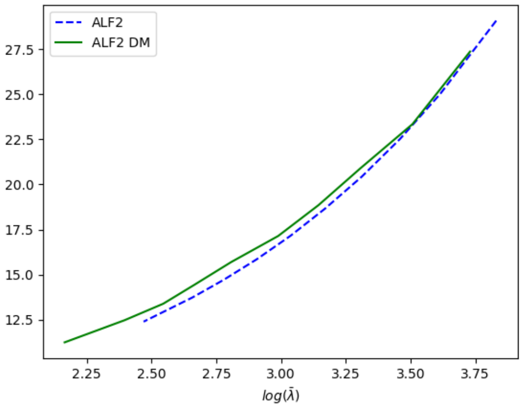

Figure 11. $\bar{I}$-$\log_{10}\bar{\lambda}$ curve for ALF2 EOS with/without DM. $\bar{I} = I/M^3$ means non-dimensionalized moment of inertia of NS and $\bar{\lambda} = \lambda/M^5$ means non-dimensionalized tidal Love number. Solid lines are for ALF2 EOS and 5% mass fraction of bosonic dark matter inside of NS and dashed lines are for without DM case. This figure shows DM inside of NS does not change I-Love curve significantly, however, each parameter has notable change; see Fig. 12

For checking details, ALF2 EOS case is focused on since it seems to be relatively stable. In Fig.11, the whole I-Love curve is shifted. Overall changes in I-Love curve do not seem significant, its underline values might not be the case. An interesting point is GR contribution to the moment of inertia is reduced about 10%, however, dCS contribution is increased as Fig.12.

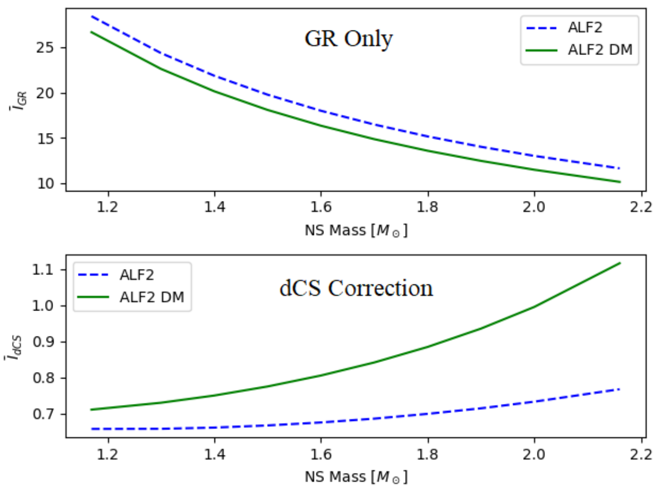

Figure 12. Non-dimensionalized moment of inertia curve as a function of NS mass with/without DM using ALF2 EOS. Solid lines are for NS with 5% mass fraction of bosonic dark matter inside of NS and dashed lines are for NS without DM. The upper panel is for GR contribution of moment of inertia and the lower panel is for dCS correction of moment of inertia. $\bar{I}_{GR}$ means GR contribution of non-dimensionalized moment of inertia of NS and $\bar{I}_{dCS}$ means dCS correction to moment of inertia of NS. DM makes GR contribution smaller and dCS correction larger.

Increased compactness is not hard to understand since 'overlap' of NM and DM generates a smaller radius for the given mass. Reduced tidal love number can be understood in a similar way if there is no additional interaction. Each part of NS is more tightly bounded by gravity due to density. Decreased GR moment of inertia might come from that effect of reduced radius wins over denser density. Increased dCS contribution can be perceived as a result of a more compact object. NS which captures DM inside becomes more compact with the same mass and angular velocity, which enlarges $g_{t\phi}$ component. As found in the interpretation of the constraint from GPB experiment, dCS effect turns on when the $g_{t\phi}$ component is non-zero, and therefore dCS contribution to moment of inertia gets larger when ω_1 becomes larger. It can be found in Fig.13. If DM is inside of NS, its $O(\chi^1, \alpha'^0)$ order of $g_{t\phi}$ component is larger than the function of NS without DM with same mass and then, corresponding θ_1 configuration with DM inside of NS has larger value at each point. This larger value means larger dCS effect. In short, observable and more compact objects can be realized with DM.

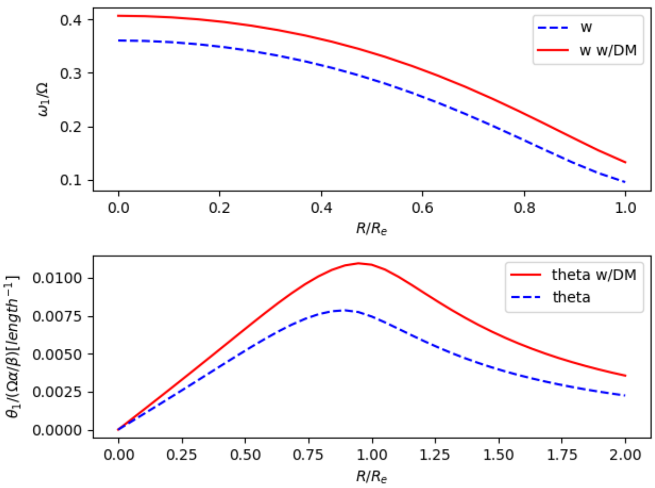

Figure 13. Supporting graphs of Fig.12. Solid lines are for ALF2 EOS and 5% mass fraction of bosonic dark matter inside of NS and dashed lines are for NS without DM. The upper panel shows normalized $g_{t\phi}$ component, ω_1 as a function of normalized radius setting NS mass as $1.4M_\odot$ at $O(\chi^1, \alpha'^0)$ order. The lower panel shows normalized θ_1 which is induced by ω_1, as a function of normalized radius setting NS mass as $1.4M_\odot$. This figure shows that larger ω_1 leads to larger dCS effect, θ_1.

As in [48], DM core structure inside of NS can be considered and its results depend on mass fraction directly, and $\sqrt{\sigma_\mathrm{D}}/\mathrm{m}_\mathrm{D}^3$ indirectly since this value determines whether DM structure's radius is smaller than NS radius. σ_D is self-interaction strength of bosonic DM. This structure shows the effect which is similar to results above. Also, DM halo effect is considered in [49] and it requires more parameters to determine desired observables.

5.2 dCS axion structure

Meanwhile, adding potential term $\frac{1}{2}m^2\theta_1$ to RHS of (3.24) would be enough to incoporrate the mass of dCS axion. As mentioned in 3.5, ω_2 doesn't have $T^{\Theta}_{\mu\nu}$ contributions. Also, ω_1 and other variables have $O(\alpha'^0)$ order and are not perturbed by dCS correction. However, there is a problem. If the mass of dCS axion is around 1meV, it is converted to about 1.5×10^{-69}km in geometrized units. The value is way too smaller than $2\frac{e^{\lambda}}{R^2}$, which is a coefficient of θ_1 in (3.24). The mass of axion might not cause any difference because of its smallness. If internal structure of dCS axion exists, however, the infinitesimal value might not matter.

It requires small enough self-interaction strength and σ_D to keep small coupling approximation and large enough coefficient of DM EOS to make EOS for dCS axion nonignorable, which means $\frac{\hbar\sqrt{\pi\sigma_{\mathrm{D}}}}{m^3_{\mathrm{dCS}}} \geq 1$. For example, $\mathrm{m_{dCS}}$ is around 1meV, then $\sigma_{\mathrm{D}} \geq \frac{10^{-18}\mathrm{eV}}{\hbar^4} \approx O(10^{-O(200)})\mathrm{km}^2$ and it satisfies conditions above. For simplicity, dCS axion core structure is briefly considered here. Formation of dCS axion core is given. Also, it is assumed that θ field configuration follows massless θ_1 configuration for the outside of dCS axion core due to its tiny mass. By using results of [48], total information required to get moment of inertia would be provided. These results can constrain α and then l_{dCS} as in the previous section. Also, it can constrain possible $\frac{\hbar\sqrt{\pi\sigma_{\mathrm{D}}}}{m^3_{\mathrm{dCS}}}$ values to keep DM radius is smaller than NS radius, so that dCS axion mass-interaction strength diagram could be drawn. dCS axion mixing into NS and dCS halo need different approach since EOS which is inserted by hand and induces such structure might not coincide with θ_1 configuration derived from order by order calculation. Still, structure of massive axion is an interesting topic and seems to need further study. Anyhow, relation between constraints and the mass of dCS axion would be indirect ways so far. Meanwhile, dCS axion mass and coupling constraints might be related to each other in a different way. Using [9]'s method and assuming dCS axion mass is small enough, possible mass range can be directly limited with l_{dCS} constraint given by massless I-Love relation or the result might constrain self-interaction strength. This can be another topic for dCS axion constraints.

6 Summary

In summary, we have added Chern-Simons term in gravitational Lagrangian as one of higher-order curvature terms and Θ is promoted to a dynamical field. Its coupling constant can be constrained by I-Love relation of NS. We have shown that the presence of DM in NS does not change I-Love curve and the resulting dCS constraints significantly, compared to the result of NS without DM. However, it is shown that DM in NS can induce notable change in the related observables; reduced GR contribution of moment of inertia and tidal Love number and increased dCS correction to moment of inertia.

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