Ph.D. Dissertation of Natural Sciences

Multicomponent and Spinor-Dipolar Bose-Einstein Condensates as Laboratories for Fundamental Physics and Metrology

다성분 및 스핀-쌍극자 보즈 아인슈타인 응집체의 기초 물리 및 계측학 연구로의 활용

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Abstract

This thesis is focused on theoretical studies of the possibility of applying Bose-Einstein condensates (BEC) as laboratories for fundamental physics and metrology.

First topic is how one can get the value of the damping parameter in the modified mean-field theory of the spinor BEC. Mean-field theory of BEC [1, 2] is widely used to study behaviors and characteristics of BEC, but they have one major problem: it cannot explain the collective damping of BEC. To remedy this problem, Pitaevskiĭ introduced dimensionless phenomenological damping parameter γ_p [3] and Choi, Morgan, and Burnett estimated $\gamma_p \simeq 0.03$ [4] from the date of scalar ^{23}Na BEC experiment [5]. Later, people tried to derive this phenomenological equations for scalar BEC [6] but so far it has been done with introducing correction factor to match the value of γ_p to be 0.03 [7, 8]. In other words, no complete microscopic derivation for γ_p is done yet. Moreover, we find out that the damping parameter for spinor BEC is commonly set to be 0.03 without any justifications, e.g. [9, 10], although there is a possiblity that the damping parameter might be different on different systems and it may depend on spin indices.

Based on our Physical Review A paper [11], we show that one may get the value of the damping parameter by measuring the switching time of the direction of the spin of the spinor-dipolar BEC if its local spin orientation is homogeneous. By assuming that the damping paramter for spinor BEC does not depend on spin indices [9, 10], we were able to derive the Landau-Lifshitz-Gilbert equation which is phenomenological equation to describe the behavior of the ferromagnets under external magnetic field. We also obtain Stoner-Wohlfarth Hamiltonian if there is no dissipation in the spinor-dipolar BEC and if local spin direction of the spinor-dipolar BEC is same everywhere. It has been verified experimentally that spinor-dipolar BEC with homogeneous local spin orientation can be made [12], so our suggestion to get damping parameter from the switching time of the direction of the spin of the spinor-dipolar BEC is not just a theoretical toy model.

Second topic is the possilibity of estimating the magnitude of the external perturbation by measuring the number of BEC molecules created by ultracold chemical reaction. There are proposals that BEC can act as sensors for measuring the acceleration [13], for measuring the detection of gravitational waves (GW) [14, 15, 16, 17], for measuring the gravitational field gradient on a millimetre scale [18], and for the detection of dark matter [19], but they do not calculate classical Fisher information and hence the lower bound of the variance of the estimation could be bigger. Moreover, those proposed sensors are based on measuring number of phonons in BEC but single phonon detection in condensates has been achieved experimentally so far only in the superfluid helium II [20] and there is no report of achieving single phonon detection in BECs yet (it is difficult to measure the number of phonons in BEC, for example, see [21]).

In our to-be-submitted paper, we study scalar BEC system under ultracold chemical reaction with homogeneous but time-dependent density perturbation being applied to that system at some time $t = 0$. By calculating quantum Fisher information (QFI) and the lower bound of the classical Fisher information (CFI) when estimating the maximum magnitude of that perturbation by measuring the number of BEC molecules created by ultracold chemical reaction, we found out that the sensitivity of this method can be close to the ultimate possible limit. In addition, since number of BEC molecules created by ultracold chemical reaction can be measured (for example, see [22, 23, 24, 25]), our scheme implies that there could be BEC sensors more easy to implement than previous BEC sensors based on phonons.

keywords: Bose-Einstein condensates, Damping parameter, Stoner-Wohlfarth switching, Ultracold chemical reaction, Fisher information

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Introduction

Ancient Greek philosopher Democritus (460 BC - 370 BC) thought that there are fundamental particles (atoms) which consist everything in our world. John Dalton (1766 - 1844) proposed the first atomic model that atoms cannot be cut and different atoms have differnt size and mass. In 1897, Joseph John Thomson (1856 - 1940) discovered that there are more smaller particles (electrons) in the atom and the existence of the nucleus is discovered by Ernest Rutherford (1871 - 1937) in 1909. People continued to find fundamental particles which cannot be cut and which compose our universe, and developed the Standard Model (see Fig. 1) to explain what consist ordinary matter and energy.

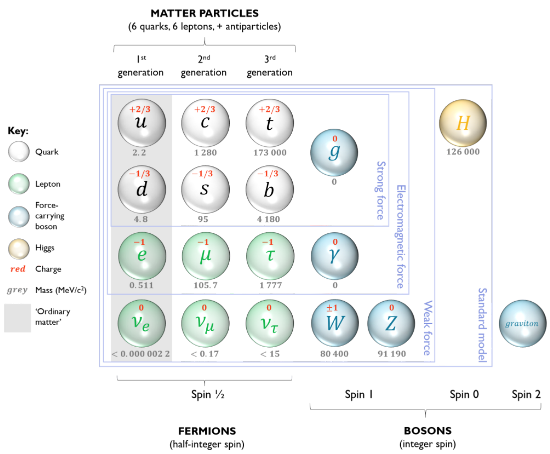


Figure 1: Standard model. This figure is from Science by degrees.

In this model, the Higgs boson gives mass to particles. To verify its existence, the Large Hadron Collider (LHC) is made and ATLAS and CMS experiments at the LHC discovered Higgs bosons in 2012.

However, only about 5 percent of the matter in the universe is ordinary matter and energy we know. About 27 percent is thought to be dark matter and 68 percent is dark energy which we do not know yet. Weakly interacting massive particles (WIMPs) is one of the candidate for dark matter and the European Organization for Nuclear Reserach (CERN) plans to make more larger collider, Future Circular Collider (FCC) in Fig. 2, which is expected to verify some of models of WIMPs.

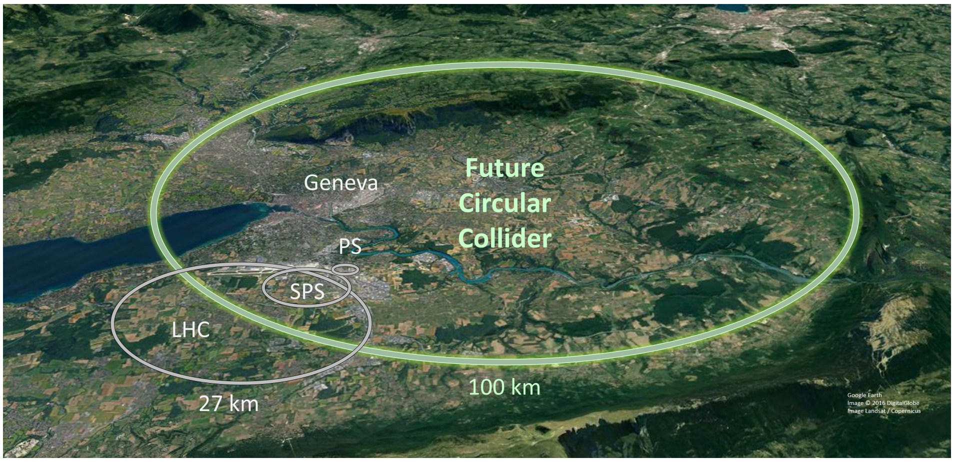


Figure 2: Schematic map of Future Circular Collider (FCC). Large Hadron Collider (LHC) is also shown to illustrate the size of FCC. This figure is from CERN (Conseil Européen pour la Recherche Nucléaire).

However, finding fundamental particles using collider requires a lot of energy and space since one has to accelerate particles with enough speed. This can be roughly explained if we think of LEGO blocks. Suppose that there are spheres made of several LEGO blocks like Fig. 3. If directly disassembling this LEGO sphere by hand is not possible but we want to get LEGO blocks which consist this sphere, we may get them by preparing two LEGO spheres and throwing them to each other so that two spheres are broken into pieces. If each blocks are tightly joined together, one would need more speed when throwing those spheres else they will just bounce.

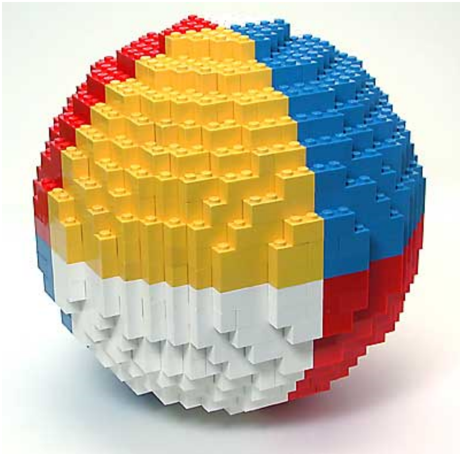


Figure 3: A figure of a LEGO sphere from philohome.com.

One may ask whether using collider is the only way to study fundamental particles, and the answer is no. For example, [26] showed that quark confinement in quantum chromodynamics can be simulated by using half-quantum vortices in BEC. Fig. 4 shows that one may simulate $\bar{u}u \to \bar{u}\bar{d} + du \to \bar{d}d$, $\bar{u}u \to \bar{u}\bar{d} + du$, and $\bar{u}u \to \bar{u}\bar{d} + d\bar{d} + du$ using half-quantum vortices in BEC where u is the up quark, d is the down quark, $\bar{u}$ is the antiparticle of u, and $\bar{d}$ is the antiparticle of d.

Fig. 5 shows that BEC can be also used to simulate early universe [27]. There are also theoretical proposals using BEC to estimate the magnitude of the gravitational wave [14, 15, 16, 17] and to detect dark matter [19], so there are many possibilities of using BEC to study fundamental physics.

One of the benefit of using BEC to study fundamental physics is that the size of BEC is typically less than 1mm, which means it takes less money and space than building a new collider. For example, half-quantum vortices is observed in spin-1 ^{23}Na BEC where the radius of the BEC is about 200μm [28] (see Fig. 6).

In this thesis we show that (1) one can use BEC to measure the damping parameter

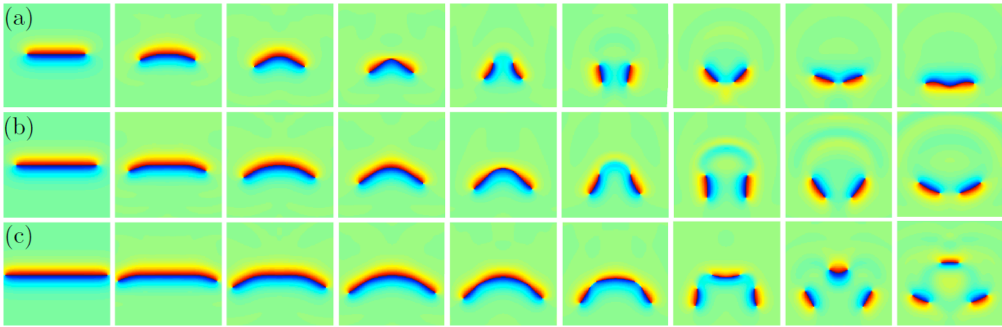


Figure 4: Simulating quark confinement using half-quantum vortices in BEC. (a) corresponds to $\bar{u}u \to \bar{u}\bar{d} + du \to \bar{d}d$, (b) corresponds to $\bar{u}u \to \bar{u}\bar{d} + du$, and (c) corresponds to $\bar{u}u \to \bar{u}\bar{d} + d\bar{d} + du$ where u is the up quark, d is the down quark, $\bar{u}$ is the antiparticle of u, and $\bar{d}$ is the antiparticle of d. This figure is from [26].

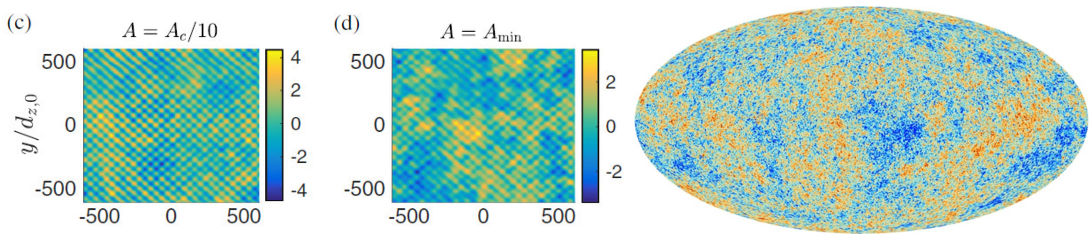


Figure 5: Simulating cosmic microwave background using BEC. (c) and (d) are from [27], and the right figure is the cosmic microwave background from the European Space Agency.

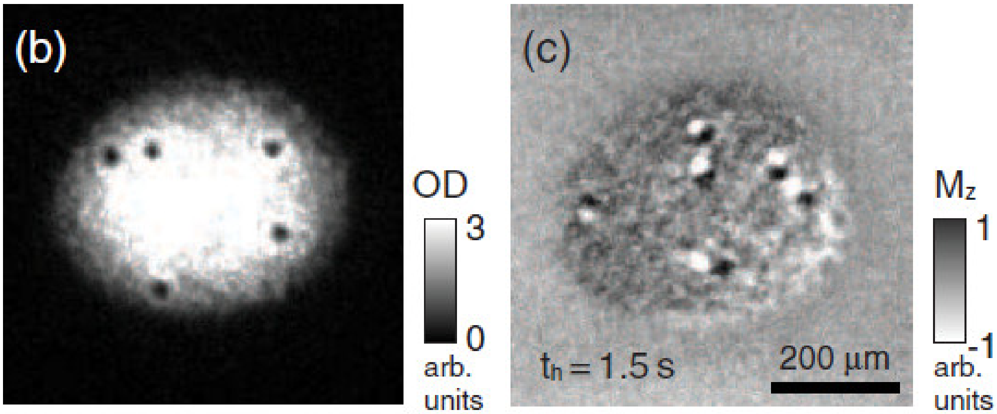


Figure 6: Observation of the dissociation of single-quantum vortices into half-quantum vortices in spin 1 ^{23}Na BEC. (b) Single-quantum vortices (black dots) when the BEC is in polar phase state. (c) Observation of half-quantum vortices (black and white dots) when the BEC is in antiferromagnetic phase state by applying external microwave field to BEC. Figures are from [28].

of the collective oscillation of BEC which has been estimated for only one specific system so far, and (2) one can estimate the magnitude of the external time-dependent homogeneous perturbation by measuring number of BEC molecules created by ultra-cold chemical reaction. These two applications show that BEC can act as laboratories for studying fundamental physics and metrology.

0.1 Structure of the Thesis and How to Read

This thesis is organized as three chapters and appendicies. Chapter 2 and 3 are independent, so one may choose to read either of those chapters or both if interested.

First chapter is to introduce necessary theories to understand contents of this thesis. In section 1.1, we introduce general theories on multicomponent and spinor-dipolar BEC including Hamiltonian 1.1.1, mean-field theory and its limitations 1.1.2, and Bogoliubov theory 1.1.3. Time evolution operator is introduced in section 1.2 with Dyson series expansion and the symplectic formalism which enables one to calculate time evolution operator with desired accuracy. Final section 1.3 is focused on the Cramér-Rao theorem and Fisher information, which tells us how well one can estimate the value of the physical quantity A by measuring other quantities which depend on A and are more easy to measure than A.

For those who want to understand contents in the second chapter should check section 1.1.2 and see if they are familiar with theories in there. For those who want to understand contents in the third chapter, it is advised to check sections 1.1.1, 1.1.3, 1.2, and 1.3.

Second chapter is about measuring the damping parameter by measuring the switching time of the spin of the spinor-dipolar BEC with homogeneous local spin orientation. We first introduce Stoner-Wohlfarth (SW) model and Landau-Lifshitz-Gilbert (LLG) equation, which are basic theories on ferromagnets. Motivated by these theories, we consider a spinor-dipolar BEC where local spin direction is same everywhere, like a single-domain. The difference to the single domain is that the magnitude of the magnetic dipole moment may be inhomogeneous. We consider two cases: box trap which makes our system homogeneous, and harmonic trap where the number density and the magnitude of the magnetic dipole moment may depend on the position.

Third chapter is how well one can estimate the magnitude of the external perturbation by measuring the number of molecules created by ultracold chemical reaction in BEC. We first discuss the difference between usual chemical reaction and ultracold chemical reaction. Then we show that there is a quasiparticle, reacton, which describes the reaction rate of the ultracold chemical reaction. By writing our system in terms of reactons, we calculated Fisher information when estimating the magnitude of the external perturbation by measuring number of BEC molecules created by this perturbation and show that the accuracy of this estimation can reach 60% or more of the ultimate limit given by the quantum Fisher information. Lastly, we discuss differences between our work and previous papers on the theoretical possibility of using BEC as sensors [13, 14, 15, 16, 17, 18, 19].

Appendices are to show how we obtained formula we used to derive our results. We tried to write those derivations in detail so that those who are determined to read and follow lengthy equations can get same conclusions as we obtained. Those who are not interested in checking every technical details may skip appendices. We tried our best so that one may understand our work even if one does not read appendices.

We summarize symbols and abbreviations we used in Table 1 and 2.

Table 1: Table of symbols

Symbol	Definition
$\mathbb{R}$	Set of real numbers
s	Spin of the particle
$m_z, m_1, m_2, \cdots$	Magnetic quantum number
m	Mass of the particle
$\hat{A}$, A	Quantum operator $\hat{A}$ and its expectation value
$\boldsymbol{v}$	Vector
$\hbar\hat{\boldsymbol{f}}$	Spin operator ($\hbar$ is the reduced Planck constant)
$\boldsymbol{B}$	External magnetic field
$\tilde{A}$	Fourier transform of A
$\hat{H}$	Hamiltonian
$\hat{U}$	Time evolution operator
I	Identity matrix or identity operator
Subscript S	Operator or state in the Schrödinger picture
Subscript I	Operator or state in the interaction picture
δ_{n_1,n_2}	Kronecker delta
$\delta(\boldsymbol{r})$	Dirac delta function
$\bar{V}_a$	Scaled magnitude of the external perturbation V_a
$I_{Q,\text{ex}}$	Quantum Fisher information using symplectic formalism
$I_{C,\text{ex}}$	The lower bound of the classical Fisher information using symplectic formalism
I_Q	Quantum Fisher information expanded up to 0th order in $\bar{V}_a$
I_C	The lower bound of the classical Fisher information expanded up to 0th order in $\bar{V}_a$

Table 2: Table of abbreviations

Abbreviation	Definition
BEC	Bose-Einstein condensates
h.c.	Hermitian conjugate
QFI	Quantum Fisher information
CFI	Classical Fisher information
SW	Stoner-Wohlfarth
LLG	Landau-Lifshitz-Gilbert

Chapter 1

Theoretical Framework

This chapter is to introduce necessary theories to understand subjects of this thesis. Readers who already know contents in following three sections may skip this chapter and directly go to either chapter 2 (measuring the damping parameter of the collective oscillation of the spinor Bose-Einstein condensates from Stoner-Wohlfarth switching) or chapter 3 (estimating the magnitude of the external perturbation from ultracold chemical reaction in Bose-Einstein condensates). If not, we recommend readers to go to necessary sections and then continue to either chapter 2 or chapter 3.

First section 1.1 is about general theory on multicomponent and spinor-dipolar Bose-Einstein condensates (BEC) including Hamiltonian 1.1.1, mean-field theory and its limitations 1.1.2, and Bogoliubov theory 1.1.3. Section 1.1.2 is needed to understand contents in chapter 2. Sections 1.1.1 and 1.1.3 are needed for chapter 3.

Second and third sections are needed to understand contents in chapter 3. Second section 1.2 is about the time evolution operator, and in that section, Dyson series expansion (section 1.2.1) and symplectic formalism of the time evolution operator (section 1.2.2) will be introduced.

Third section 1.3 is about estimation theory focusing on the Cramér-Rao theorem and the Fisher information. We will introduce classical Fisher information in 1.3.1 and quantum Fisher information in 1.3.2.

1.1 General Theory on Multicomponent and Spinor-Dipolar Bose-Einstein Condensates

This section is to introduce theoretical description of Bose-Einstein condensates (BEC). We will first consider a system of identical spin s bosons with mass m and then consider multicomponent BEC. After presenting Hamiltonians in section 1.1.1, mean-field description and its limitations will be discussed in section 1.1.2. Bogoliubov theory will be discussed at the end of this section, 1.1.3, including a specific example in scalar atomic and molecular BEC under ultracold chemical reaction to illustrate how one can apply Bogoliubov theory.

Most of the contents of this section are already thoroughly reviewed by others. [2] is the standard reference for spinor BEC. There are textbooks for scalar BEC, e.g., [1, 31, 32].

1.1.1 Hamiltonian

Let $\hat{\psi}(\boldsymbol{r})$ be the bosonic field operator at position $\boldsymbol{r}$ and m_z be the magnetic quantum number. For spin s bosons, $\hat{\psi}(\boldsymbol{r})$ has $2s+1$ components with

$$\hat{\psi}(\boldsymbol{r}) = \left[\begin{array}{ccc} \hat{\psi}_{m_z=s}(\boldsymbol{r}), & \cdots, & \hat{\psi}_{m_z=-s}(\boldsymbol{r}) \end{array}\right]^T. \tag{1.1}$$

For convenience, we will denote $\hat{\psi}_{m_z=m_1}(\boldsymbol{r})$ as $\hat{\psi}_{m_1}(\boldsymbol{r})$ where $-s \leq m_1 \leq s$. Following canonical commutation relations are satisfied:

$$\left[\hat{\psi}_{m_1}(\boldsymbol{r}_1), \hat{\psi}^{\dagger}_{m_2}(\boldsymbol{r}_2)\right] = \delta_{m_1,m_2}\delta(\boldsymbol{r}_1 - \boldsymbol{r}_2), \quad \left[\hat{\psi}_{m_1}(\boldsymbol{r}_1), \hat{\psi}_{m_2}(\boldsymbol{r}_2)\right] = 0, \tag{1.2}$$

where δ_{m_1,m_2} is Kronecker delta and $\delta(\boldsymbol{r}_1 - \boldsymbol{r}_2)$ is Dirac delta function.

Let $\hbar\hat{\boldsymbol{f}} = \hbar \sum_{\nu=x,y,z} \hat{f}_\nu \boldsymbol{e}_\nu$ be the spin s operator where $\boldsymbol{e}_\nu$ is unit vector along $+\nu$ axis. By denoting orthonormal spinor basis with spin s and magnetic quantum number $m_z = m_1$ as $|s, m_1\rangle$, we get $\hat{f}_z|s, m_1\rangle = m_1|s, m_1\rangle$ and $\hat{f}_\pm := \hat{f}_x \pm i\hat{f}_y$ where $\hat{f}_\pm|s, m_1\rangle = \sqrt{s(s+1) - m_1(m_1 \pm 1)}\,|s, m_1 \pm 1\rangle$.

Under an external magnetic field $\boldsymbol{B}$ and a microwave or a light field, quadratic Zeeman effect occurs. Let q_B be the quadratic Zeeman effect due to $\boldsymbol{B}$, and q_{MW} be the quadratic Zeeman effect due to a microwave or a light field. Since applying a linearly polarized microwave field can change q_{MW} without affecting q_B [33, 34], we will assume that q_{MW} is set to be the value which enables one to neglect quadratic Zeeman effect. Then the Hamiltonian of spinor-dipolar BEC in three-dimensional space can be written as

$$
\begin{aligned}
\hat{H} = & \int d^3r\, \hat{\psi}^\dagger(\boldsymbol{r}) \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\mathrm{tr}}(\boldsymbol{r}) - g_F \mu_B \boldsymbol{B}\cdot\hat{\boldsymbol{f}} \right] \hat{\psi}(\boldsymbol{r}) \\
& + \frac{1}{2}\int d^3r \int d^3r' \sum_{\mathcal{S}=0,2,\cdots,2s} V_{\mathrm{int}}^{(\mathcal{S})}(\boldsymbol{r},\boldsymbol{r}') \sum_{m_1,m_2,m_1',m_2'=-s}^{s} \sum_{\mathcal{M}=-\mathcal{S}}^{\mathcal{S}} \langle s, m_2'; s, m_1' | \mathcal{S}, \mathcal{M} \rangle \langle \mathcal{S}, \mathcal{M} | s, m_1; s, m_2 \rangle \\
& \times \hat{\psi}^\dagger_{m_2'}(\boldsymbol{r}') \hat{\psi}^\dagger_{m_1'}(\boldsymbol{r}) \hat{\psi}_{m_1}(\boldsymbol{r}) \hat{\psi}_{m_2}(\boldsymbol{r}') \\
& + \frac{c_{dd}}{2}\int d^3r \int d^3r' \sum_{\nu,\nu'=x,y,z} \sum_{m_1,m_2,m_1',m_2'=-s}^{s} \left(\hat{f}_\nu\right)_{m_1,m_2} Q_{\nu,\nu'}(\boldsymbol{r}-\boldsymbol{r}') \left(\hat{f}_{\nu'}\right)_{m_1',m_2'} \\
& \times \hat{\psi}^\dagger_{m_1}(\boldsymbol{r}) \hat{\psi}^\dagger_{m_1'}(\boldsymbol{r}') \hat{\psi}_{m_2'}(\boldsymbol{r}') \hat{\psi}_{m_2}(\boldsymbol{r}),
\end{aligned} \tag{1.3}
$$

where $\hbar$ is the reduced Planck constant, $V_{\mathrm{tr}}(\boldsymbol{r})$ is the trap potential, g_F is the Landé g-factor, μ_B is the Bohr magneton, $V_{\mathrm{int}}^{(\mathcal{S})}(\boldsymbol{r},\boldsymbol{r}')$ is a scalar function describing interactions between particles, $Q_{\nu,\nu'}(\boldsymbol{r}) := \left(r^2\delta_{\nu,\nu'} - 3r_\nu r_{\nu'}\right)/r^5$, $c_{dd} := \mu_0 \left(g_F\mu_B\right)^2/(4\pi)$ is the magnetic dipole-dipole interaction coefficient, and μ_0 is the magnetic permeability of free space.

Typical number density at the center of BEC is about 10^{18} to $10^{20}\mathrm{m}^{-3}$ [35, 36, 1], whereas number density of air at atmospheric pressure (1013hPa) and room temperature (27°C) is $2.4\times10^{25}\mathrm{m}^{-3}$ from the ideal gas law. It is usually assumed that BEC is dilute, and in that case we may use s-wave scattering limit which gives $V_{\mathrm{int}}^{(\mathcal{S})}(\boldsymbol{r},\boldsymbol{r}') = 4\pi\hbar^2 a_S \delta(\boldsymbol{r}-\boldsymbol{r}')/m$ where a_S is the s-wave scattering length of the total spin S channel. In this approximation, Eq. (1.3) can be written as

$$
\begin{aligned}
\hat{H} = &\int d^3r\, \hat{\psi}^\dagger(\boldsymbol{r}) \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{tr}}(\boldsymbol{r}) - g_F\mu_B \boldsymbol{B}\cdot\hat{\boldsymbol{f}} + \frac{c_0}{2}\hat{n}(\boldsymbol{r})\right]\hat{\psi}(\boldsymbol{r}) \\
&+\sum_{j=1}^{s}\frac{c_{2j}}{2}\int d^3r \sum_{\nu_1,\cdots,\nu_j=x,y,z}\ \sum_{m_1,m_2=-s}^{s} \hat{\psi}^\dagger_{m_1}(\boldsymbol{r})\,\hat{F}_{\nu_1,\cdots,\nu_j}(\boldsymbol{r})\left(\hat{f}_{\nu_1}\cdots\hat{f}_{\nu_j}\right)_{m_1,m_2}\hat{\psi}_{m_2}(\boldsymbol{r}) \\
&+\frac{c_{dd}}{2}\int d^3r\int d^3r' \sum_{\nu,\nu'=x,y,z}\ \sum_{m_1,m_2=-s}^{s}\hat{\psi}^\dagger_{m_1}(\boldsymbol{r})\left(\hat{f}_\nu\right)_{m_1,m_2} Q_{\nu,\nu'}\left(\boldsymbol{r}-\boldsymbol{r}'\right)\hat{F}_{\nu'}\left(\boldsymbol{r}'\right)\hat{\psi}_{m_2}(\boldsymbol{r}),
\end{aligned}
\tag{1.4}
$$

where $\hat{n}(\boldsymbol{r}) := \hat{\psi}^\dagger(\boldsymbol{r})\,\hat{\psi}(\boldsymbol{r})$ and $\hat{F}_{\nu_1,\cdots,\nu_j}(\boldsymbol{r}) := \hat{\psi}^\dagger(\boldsymbol{r})\,\hat{f}_{\nu_1}\cdots\hat{f}_{\nu_j}\hat{\psi}(\boldsymbol{r})$.

Multicomponent BEC is a BEC with its bosonic field operator $\hat{\psi}(\boldsymbol{r})$ having multiple components but particles may not be identical. ^{47}K - ^{87}Rb BEC [37] or a system of ^{87}Rb BEC atoms and $^{87}Rb_2$ BEC molecules (created by ultracold chemical reaction) [38] are examples of multicomponent BEC. By writing $\hat{\psi}_j(\boldsymbol{r})$ as a bosonic field operator with mass m_j and $\hat{H}_j$ as a Hamiltonian of identical bosons with mass m_j, total Hamiltonian $\hat{H}$ can be written as $\sum_j \hat{H}_j + \hat{H}_{\text{int}}$ where $\hat{H}_{\text{int}}$ is the interaction between different particles in BEC.

In chapter 3, we will consider ultracold chemical reaction. Let us consider a system of scalar BEC atoms A and scalar BEC molecules A_2 with $A + A \leftrightarrow A_2$. By denoting $\hat{\psi}_a(\boldsymbol{r})$ as a field operator for A and $\hat{\psi}_m(\boldsymbol{r})$ as a field operator for A_2, total Hamiltonian $\hat{H}$ can be written as [30, 39, 40]

$$
\begin{aligned}
\hat{H} \;=\; &\int d^3r\, \hat{\psi}^\dagger_a(\boldsymbol{r})\left\{-\frac{\hbar^2}{2m_a}\nabla^2 + V_a(\boldsymbol{r}) + \frac{g_a}{2}\hat{\psi}^\dagger_a(\boldsymbol{r})\,\hat{\psi}_a(\boldsymbol{r})\right\}\hat{\psi}_a(\boldsymbol{r}) \\
&+g_{am}\int d^3r\, \hat{\psi}^\dagger_a(\boldsymbol{r})\,\hat{\psi}_a(\boldsymbol{r})\,\hat{\psi}^\dagger_m(\boldsymbol{r})\,\hat{\psi}_m(\boldsymbol{r}) + \frac{\alpha}{\sqrt{2}}\int d^3r\, \left\{\hat{\psi}^\dagger_m(\boldsymbol{r})\,\hat{\psi}_a(\boldsymbol{r})\,\hat{\psi}_a(\boldsymbol{r}) + h.c.\right\} \\
&+\int d^3r\, \hat{\psi}^\dagger_m(\boldsymbol{r})\left\{-\frac{\hbar^2}{2m_m}\nabla^2 + V_m(\boldsymbol{r}) + \epsilon + \frac{g_m}{2}\hat{\psi}^\dagger_m(\boldsymbol{r})\,\hat{\psi}_m(\boldsymbol{r})\right\}\hat{\psi}_m(\boldsymbol{r}),
\end{aligned}
\tag{1.5}
$$

where m_a is the mass of the atom, $m_m\,(\simeq 2m_a)$ is the mass of the molecule, $V_a(\boldsymbol{r})$ is the trap potential applied to atoms, $V_m(\boldsymbol{r})$ is the trap potential applied to molecule, g_a is the contact-interaction coupling between two atoms, g_m is the contact-interaction coupling between two molecules, g_{am} is the contact-interaction couplings between atom and molecule, ϵ is the energy difference between two atoms and a molecule, α is the coupling coefficient that determines the coherent conversion rate of the ultracold chemical reaction, and $h.c.$ is the hermitian conjugate.

1.1.2 Mean-field Description and Its Limitations

In Schrödinger picture we used so far, operators are constant in time and the state $|\Psi_S(t)\rangle$ follows Schrödinger equation $i\hbar\partial|\Psi_S(t)\rangle/\partial t = \hat{H}|\Psi_S(t)\rangle$. However, to show how mean-field theory can be obtained, it is better to start from Heisenberg picture where the state is constant in time and the observable operator $\hat{A}$ follows Heisenberg equation of motion with $i\hbar\partial\hat{A}/\partial t = \left[\hat{A}, \hat{H}\right]$. One can get mean-field theory by replacing the field operator $\hat{\psi}(\boldsymbol{r}, t)$ with its expectation value $\psi(\boldsymbol{r}, t) := \left\langle\hat{\psi}(\boldsymbol{r}, t)\right\rangle$. As an example, from Eq. (1.4), one can get

$$\begin{aligned} i\hbar\frac{\partial\hat{\psi}(\boldsymbol{r},t)}{\partial t} &= \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{tr}}(\boldsymbol{r}) + c_0\hat{n}(\boldsymbol{r},t) + \sum_{j=1}^{s} c_{2j} \sum_{\nu_1,\cdots,\nu_j=x,y,z} \hat{F}_{\nu_1,\cdots,\nu_j}(\boldsymbol{r},t)\left(\hat{f}_{\nu_1}\cdots\hat{f}_{\nu_j}\right)\right]\hat{\psi}(\boldsymbol{r},t) \\ &\quad - \sum_{\nu=x,y,z}\left[g_F\mu_B B_\nu - c_{dd}\sum_{\nu'=x,y,z}\left\{\int d^3r'\, Q_{\nu,\nu'}\left(\boldsymbol{r}-\boldsymbol{r}'\right)\hat{F}_{\nu'}\left(\boldsymbol{r}',t\right)\right\}\right]\hat{f}_\nu\hat{\psi}(\boldsymbol{r},t), \end{aligned} \tag{1.6}$$

and thus mean-field equation for spinor-dipolar BEC is

$$\begin{aligned} i\hbar\frac{\partial\psi(\boldsymbol{r},t)}{\partial t} &= \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{tr}}(\boldsymbol{r}) + c_0 n(\boldsymbol{r},t) + \sum_{j=1}^{s} c_{2j} \sum_{\nu_1,\cdots,\nu_j=x,y,z} F_{\nu_1,\cdots,\nu_j}(\boldsymbol{r},t)\left(\hat{f}_{\nu_1}\cdots\hat{f}_{\nu_j}\right)\right]\psi(\boldsymbol{r},t) \\ &\quad - \sum_{\nu=x,y,z}\left[g_F\mu_B B_\nu - c_{dd}\sum_{\nu'=x,y,z}\left\{\int d^3r'\, Q_{\nu,\nu'}\left(\boldsymbol{r}-\boldsymbol{r}'\right)F_{\nu'}\left(\boldsymbol{r}',t\right)\right\}\right]\hat{f}_\nu\psi(\boldsymbol{r},t), \end{aligned} \tag{1.7}$$

where $n(\boldsymbol{r},t) := \psi^\dagger(\boldsymbol{r},t)\psi(\boldsymbol{r},t)$ is the number density of the mean-field and $F_{\nu_1,\cdots,\nu_j}(\boldsymbol{r},t) := \psi^\dagger(\boldsymbol{r},t)\left(\hat{f}_{\nu_1}\cdots\hat{f}_{\nu_j}\right)\psi(\boldsymbol{r},t)$.

In general, mean-field equation is written as

$$i\hbar\frac{\partial\psi\left(\boldsymbol{r},t\right)}{\partial t}=\left\{-\frac{\hbar^{2}}{2m}\nabla^{2}+H_{\text{other}}\left(\boldsymbol{r},t\right)\right\}\psi\left(\boldsymbol{r},t\right), \tag{1.8}$$

where $H_{\text{other}}\left(\boldsymbol{r},t\right)$ is Hermitian matrix which may contain $\psi\left(\boldsymbol{r},t\right)$. From Eq. (1.8), one can get following continuity equation:

$$\frac{\partial n\left(\boldsymbol{r},t\right)}{\partial t}+\nabla\cdot\left\{n\left(\boldsymbol{r},t\right)\boldsymbol{v}_{s}\left(\boldsymbol{r},t\right)\right\}=\frac{2}{\hbar}\text{Im}\left\{\psi^{\dagger}\left(\boldsymbol{r},t\right)H_{\text{other}}\left(\boldsymbol{r},t\right)\psi\left(\boldsymbol{r},t\right)\right\}, \tag{1.9}$$

where $n\left(\boldsymbol{r},t\right)\boldsymbol{v}_{s}\left(\boldsymbol{r},t\right):=\left(\hbar/m\right)\text{Im}\left\{\psi^{\dagger}\left(\boldsymbol{r},t\right)\nabla\psi\left(\boldsymbol{r},t\right)\right\}$. Therefore, as long as $H_{\text{other}}\left(\boldsymbol{r},t\right)$ is Hermitian, collective oscillation of BEC stays forever which is contradictory to the finite lifetime of the collective oscillation of BEC (order of 10 seconds, e.g., [35]).

To solve this discrepancy, Pitaevskiĭ introduced dimensionless phenomenological damping parameter γ_{p} [3] where

$$i\hbar\frac{\partial\psi\left(\boldsymbol{r},t\right)}{\partial t}=\left(1-i\gamma_{p}\right)\left\{-\frac{\hbar^{2}}{2m}\nabla^{2}+H_{\text{other}}\left(\boldsymbol{r},t\right)\right\}\psi\left(\boldsymbol{r},t\right), \tag{1.10}$$

which gives

$$\begin{aligned}&\frac{\partial n\left(\boldsymbol{r},t\right)}{\partial t}+\nabla\cdot\left\{n\left(\boldsymbol{r},t\right)\boldsymbol{v}_{s}\left(\boldsymbol{r},t\right)\right\}\\&=-\frac{2\gamma_{p}}{\hbar}\left[\left\{P_{Q}\left(\boldsymbol{r},t\right)+\frac{m}{2}v_{s}^{2}\left(\boldsymbol{r},t\right)\right\}n\left(\boldsymbol{r},t\right)+\text{Re}\left\{\psi^{\dagger}\left(\boldsymbol{r},t\right)H_{\text{other}}\left(\boldsymbol{r},t\right)\psi\left(\boldsymbol{r},t\right)\right\}\right],\end{aligned} \tag{1.11}$$

where $P_{Q}\left(\boldsymbol{r},t\right):=-\left\{\hbar^{2}/\left(2m\right)\right\}\left\{\nabla^{2}\sqrt{n\left(\boldsymbol{r},t\right)}\right\}/\sqrt{n\left(\boldsymbol{r},t\right)}$ is quantum pressure. Based on the rate equation for the $n\left(\boldsymbol{r},t\right)$ in [41], Choi, Morgan, and Burnett estimated $\gamma_{p}\simeq 0.03$ [4] from the date of scalar ^{23}Na BEC experiment [5].

Later, for scalar BEC, Zaremba, Nikuni, and Griffin expanded $\hat{\psi}\left(\boldsymbol{r},t\right)$ as $\psi\left(\boldsymbol{r},t\right)+\delta\hat{\psi}\left(\boldsymbol{r},t\right)$ where $\left\langle\delta\hat{\psi}\left(\boldsymbol{r},t\right)\right\rangle=0$ and obtained [6]

$$\begin{aligned}&\frac{\partial n\left(\boldsymbol{r},t\right)}{\partial t}+\nabla\cdot\left\{n\left(\boldsymbol{r},t\right)\boldsymbol{v}_{s}\left(\boldsymbol{r},t\right)\right\}\\&=\frac{2c_{0}}{\hbar}\text{Im}\left[\left\{\psi^{*}\left(\boldsymbol{r},t\right)\right\}^{2}\left\langle\hat{\psi}\left(\boldsymbol{r},t\right)\hat{\psi}\left(\boldsymbol{r},t\right)\right\rangle+\psi^{*}\left(\boldsymbol{r},t\right)\left\langle\hat{\psi}^{\dagger}\left(\boldsymbol{r},t\right)\hat{\psi}\left(\boldsymbol{r},t\right)\hat{\psi}\left(\boldsymbol{r},t\right)\right\rangle\right].\end{aligned} \tag{1.12}$$

Gardiner, Anglin, and Fudge managed to get [7]

$$(i-\gamma)\hbar\frac{\partial\psi(\boldsymbol{r},t)}{\partial t}=\left\{-\frac{\hbar^2}{2m}\nabla^2+V_{\rm tr}(\boldsymbol{r})+c_0 n(\boldsymbol{r},t)+i\gamma\mu_{\rm nc}\right\}\psi(\boldsymbol{r},t), \quad (1.13)$$

with $\gamma := 4mg_{\rm corr}a^2k_BT/\left(\pi\hbar^2\right)$ where $\mu_{\rm nc}$ is chemical potential of noncondensed particles, $g_{\rm corr}=3$ is correction term to make γ to be order of 0.01, a is the scattering length of the density-density interaction, k_B is the Boltzmann constant, and T is the temperature. From [6] and [7], Kasamatsu, Tsubota, and Ueda obtained [8]

$$(i-\gamma)\hbar\frac{\partial\psi(\boldsymbol{r},t)}{\partial t}=\left\{-\frac{\hbar^2}{2m}\nabla^2+V_{\rm tr}(\boldsymbol{r})+c_0 n(\boldsymbol{r},t)-\mu\right\}\psi(\boldsymbol{r},t), \quad (1.14)$$

when $\mu_{\rm nc}$ is independent of space and time. Here, μ is chemical potential of BEC. Note that Eq. (1.14) can be also written as

$$\begin{aligned} i\hbar\frac{\partial\psi(\boldsymbol{r},t)}{\partial t} &= \frac{i}{i-\gamma}\left\{-\frac{\hbar^2}{2m}\nabla^2+V_{\rm tr}(\boldsymbol{r})+c_0 n(\boldsymbol{r},t)-\mu\right\}\psi(\boldsymbol{r},t) \\ &= \frac{1-i\gamma}{1+\gamma^2}\left\{-\frac{\hbar^2}{2m}\nabla^2+V_{\rm tr}(\boldsymbol{r})+c_0 n(\boldsymbol{r},t)-\mu\right\}\psi(\boldsymbol{r},t). \quad (1.15) \end{aligned}$$

By comparing with Eq. (1.10), $\gamma_p \simeq \gamma$ if $|\gamma_p| \ll 1$.

So far, there is no complete microscopic derivations for the damping parameter since $g_{\rm corr}$ is an arbitrary term introduced to match the value of the damping parameter to the estimation in [4]. And even though above derivations of γ is for scalar BEC, it is commonly used for spinor BEC by replacing $i\hbar\partial\psi(\boldsymbol{r},t)/\partial t$ with $(i-\gamma)\hbar\partial\psi(\boldsymbol{r},t)/\partial t$ without proper justification, e.g., [9] and [10]. In chapter 2, we show that γ might be measured by Stoner-Wohlfarth switching if replacing $i\hbar\partial\psi(\boldsymbol{r},t)/\partial t$ with $(i-\gamma)\hbar\partial\psi(\boldsymbol{r},t)/\partial t$ is valid for spinor BEC. Measuring γ will give us a way to check theories on γ and increase the accuracy of the numerical calculations on the dynamics of BEC.

1.1.3 Bogoliubov Theory

To study effects beyond mean-field theory, one writes $\hat{\psi}(\boldsymbol{r})=\psi(\boldsymbol{r})+\delta\hat{\psi}(\boldsymbol{r})$ where $\psi(\boldsymbol{r}) := \left\langle\hat{\psi}(\boldsymbol{r})\right\rangle$ is the mean-field of the field operator $\hat{\psi}(\boldsymbol{r})$ and expand the Hamiltonian up to the second order in $\delta\hat{\psi}(\boldsymbol{r})$ by assuming that $\delta\hat{\psi}(\boldsymbol{r}) \ll \psi(\boldsymbol{r})$. Then the expanded Hamiltonian can be diagonalized by Bogoliubov transformation [42], and hence this approach is called as Bogoliubov theory. We will use Bogoliubov theory in chapter 3 with homogeneous mean-field background. Thus we start from the Hamiltonian in Eq. (1.5) with $V_j(\boldsymbol{r}) = 0$ for $j = a, m$.

In Heisenberg picture, let

$$\hat{N}_a(t) := \int d^3r\ \hat{\psi}_a^\dagger(\boldsymbol{r},t)\,\hat{\psi}_a(\boldsymbol{r},t), \quad \hat{N}_m(t) := \int d^3r\ \hat{\psi}_m^\dagger(\boldsymbol{r},t)\,\hat{\psi}_m(\boldsymbol{r},t), \tag{1.16}$$

$N_a(t) := \int d^3r\ |\psi_a(\boldsymbol{r},t)|^2$, and $N_m(t) := \int d^3r\ |\psi_m(\boldsymbol{r},t)|^2$. Then one can get

$$\begin{aligned} i\hbar\frac{\partial\hat{N}_a(t)}{\partial t} &= \int d^3r\ \left[\hat{\psi}_a^\dagger(\boldsymbol{r},t)\left\{i\hbar\frac{\partial\hat{\psi}_a(\boldsymbol{r},t)}{\partial t}\right\} - \left\{i\hbar\frac{\partial\hat{\psi}_a(\boldsymbol{r},t)}{\partial t}\right\}^\dagger\hat{\psi}_a(\boldsymbol{r},t)\right] \\ &= \alpha\sqrt{2}\int d^3r\ \left\{\hat{\psi}_a^\dagger(\boldsymbol{r},t)\,\hat{\psi}_a^\dagger(\boldsymbol{r},t)\,\hat{\psi}_m(\boldsymbol{r},t) - h.c.\right\} \\ &= -2i\hbar\frac{\partial\hat{N}_m(t)}{\partial t}, \end{aligned} \tag{1.17}$$

which shows that $\hat{N} := \hat{N}_a(t) + 2\hat{N}_m(t)$ is constant in time t. Note that one A_2 molecule is composed of two A atoms and thus $\hat{N}$ represents total number of A atoms in the system.

Since we consider box trap case where $V_j(\boldsymbol{r}) = 0$, we will write $\hat{\psi}_j(\boldsymbol{r},t) = \psi_j(t) + \delta\hat{\psi}_j(\boldsymbol{r},t)$ (homogeneous mean field). According to mean-field equations, we get

$$\begin{aligned} i\hbar\frac{\partial\psi_a(t)}{\partial t} &= \left\{g_a|\psi_a(t)|^2 + g_{am}|\psi_m(t)|^2\right\}\psi_a(t) + \alpha\sqrt{2}\psi_m(t)\,\psi_a^*(t), \\ i\hbar\frac{\partial\psi_m(t)}{\partial t} &= \left\{g_m|\psi_m(t)|^2 + g_{am}|\psi_a(t)|^2 + \epsilon\right\}\psi_m(t) + \frac{\alpha}{\sqrt{2}}\psi_a^2(t), \end{aligned} \tag{1.18}$$

which gives $\hbar\partial N_a(t)/\partial t = 2V\alpha\sqrt{2}\mathrm{Im}\left\{\psi_m(t)\,\psi_a^{*2}(t)\right\} = -(1/2)\,\hbar\partial N_m(t)/\partial t$ where V is the volume of the system and thus $N := N_a(t) + 2N_m(t)$ is constant in time t.

Let $n \coloneqq N/V$, $\psi_a(t) = x(t)\sqrt{n}e^{i\{\varphi_a(t)-\mu t/\hbar\}}$, and $\psi_m(t) = y(t)\sqrt{n}e^{i\{\varphi_m(t)-2\mu t/\hbar\}}$ where μ is the chemical potential, $x(t) \coloneqq |\psi_a(t)|/\sqrt{n} \geq 0$, $y(t) \coloneqq |\psi_m(t)|/\sqrt{n} \geq 0$, $\varphi_a(t) \in \mathbb{R}$ is the mean-field phase of the atom, and $\varphi_m(t) \in \mathbb{R}$ is the mean-field phase of the molecule. Note that $x^2 + 2y^2 = 1$ from the definition of $N_a(t)$, $N_m(t)$, N, and n. From Eqs. (1.18),

$$i\frac{\partial y(t)}{\partial \tilde{t}} - \frac{\partial \varphi_m(t)}{\partial \tilde{t}} y(t) = \left\{\tilde{g}_m y^2(t) + \tilde{g}_{am} x^2(t) + \tilde{\epsilon} - 2\tilde{\mu}\right\} y(t) + \frac{\tilde{\alpha}}{2} x^2(t) e^{-i\varphi_{am}(t)},$$
$$i\frac{\partial x(t)}{\partial \tilde{t}} - \frac{\partial \varphi_a(t)}{\partial \tilde{t}} x(t) = \left\{x^2(t) + \tilde{g}_{am} y^2(t) - \tilde{\mu} + \tilde{\alpha} y(t) e^{i\varphi_{am}(t)}\right\} x(t), \qquad (1.19)$$

$$\frac{\partial x(t)}{\partial \tilde{t}} = \tilde{\alpha} x(t) y(t) \sin\varphi_{am}(t), \quad \frac{\partial y(t)}{\partial \tilde{t}} = -\frac{\tilde{\alpha}}{2} x^2(t) \sin\varphi_{am}(t), \qquad (1.20)$$

and

$$\frac{\partial \varphi_{am}(t)}{\partial \tilde{t}} = 2x^2(t) + \tilde{g}_{am}\left\{2y^2(t) - x^2(t)\right\} - \tilde{g}_m y^2(t) - \tilde{\epsilon} + \frac{\tilde{\alpha}}{2}\left\{4y(t) - \frac{x^2(t)}{y(t)}\right\}\cos\varphi_{am}(t), \qquad (1.21)$$

where $\tilde{t} \coloneqq g_a n t/\hbar$, $\tilde{g}_{am} \coloneqq g_{am}/g_a$, $\tilde{g}_m \coloneqq g_m/g_a$, $\tilde{\alpha} \coloneqq \alpha\sqrt{2n}/(g_a n)$, $\tilde{\mu} \coloneqq \mu/(g_a n)$, $\tilde{\epsilon} \coloneqq \epsilon/(g_a n)$, and $\varphi_{am}(t) \coloneqq \varphi_m(t) - 2\varphi_a(t)$.

By replacing $\hat{\psi}_j(\boldsymbol{r}, t)$ with $\psi_j(t)$ in the Hamiltonian, the mean-field Hamiltonian $H_0(t)$ is obtained, which is

$$\frac{H_0(t)}{N g_a n} = \frac{1}{2}x^4(t) + \frac{\tilde{g}_m}{2}y^4(t) + \tilde{\epsilon} y^2(t) + \tilde{g}_{am} x^2(t) y^2(t) + \tilde{\alpha} x^2(t) y(t) \cos\varphi_{am}(t), \qquad (1.22)$$

and if $g_a > 0$, $H_0(t)$ is minimized when $\alpha\cos\varphi_{am}(t) = -|\alpha|$. Note that $\sin\varphi_{am} = 0$ in this case. Then $x(t)$ and $y(t)$ are constant according to Eqs. (1.20). Therefore we will write $x(t)$ as x, $y(t)$ as y, and $\varphi_{am}(t)$ as φ_{am} from now on. Eq. (1.21) shows that $\alpha\cos\varphi_{am}(t) = -|\alpha|$ can be achieved when

$$\tilde{\epsilon} = 2x^2 + \tilde{g}_{am}\left(2y^2 - x^2\right) - \tilde{g}_m y^2 - \frac{|\tilde{\alpha}|}{2}\left(4y - \frac{x^2}{y}\right), \qquad (1.23)$$

which gives

$$\frac{H_0}{Ng_a n} = \frac{1}{2}x^4 - \frac{1}{2}\left(\tilde{g}_m - 4\tilde{g}_{am}\right) y^4 + 2x^2y^2 - 2\left|\tilde{\alpha}\right| y^3 - \frac{\left|\tilde{\alpha}\right|}{2}x^2 y, \tag{1.24}$$

from Eqs. (1.22) and (1.23).

Since we set x, y, and φ_{am} to be constant in time, we may set $\varphi_a(t)$ and $\varphi_m(t)$ to be also constant in time. These conditions can be achieved when (see Eqs. (1.19))

$$\tilde{\mu} = x^2 + \tilde{g}_{am}y^2 - \left|\tilde{\alpha}\right| y. \tag{1.25}$$

From Eqs. (1.24) and (1.25),

$$\frac{H_0 - \mu N}{Ng_a n} = -\frac{1}{2} - \left\{2 + \frac{1}{2}\left(\tilde{g}_m - 4\tilde{g}_{am}\right)\right\} y^4 + \left(2 - \tilde{g}_{am}\right) y^2 - \left|\tilde{\alpha}\right| y^3 + \frac{\left|\tilde{\alpha}\right|}{2} y, \tag{1.26}$$

which is minimized when

$$\left(8 + 2\tilde{g}_m - 8\tilde{g}_{am}\right) y^3 + 3\left|\tilde{\alpha}\right| y^2 - 2\left(2 - \tilde{g}_{am}\right) y - \frac{\left|\tilde{\alpha}\right|}{2} = 0, \tag{1.27}$$

with $y \geq 0$ from the definition of y.

From above settings, we write $\hat{\psi}_j(\boldsymbol{r}, t) = \psi_j(t) + \delta\hat{\psi}_j(\boldsymbol{r}, t)$ where $\psi_a(t) = x\sqrt{n}e^{i(\varphi_a - \mu t/\hbar)}$, $\psi_m(t) = y\sqrt{n}e^{i(\varphi_m - 2\mu t/\hbar)}$,

$$\delta\hat{\psi}_a(\boldsymbol{r}, t) = e^{-i\mu t/\hbar}\frac{1}{\sqrt{V}}\sum_{\boldsymbol{k}\neq 0} e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\delta\hat{\Psi}_a(\boldsymbol{k}, t), \quad \delta\hat{\psi}_m(\boldsymbol{r}, t) = e^{-2i\mu t/\hbar}\frac{1}{\sqrt{V}}\sum_{\boldsymbol{k}\neq 0} e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\delta\hat{\Psi}_m(\boldsymbol{k}, t). \tag{1.28}$$

Note that $\psi_j(t)$ corresponds to $\boldsymbol{k} = 0$ component of the Fourier transform of $\hat{\psi}_j(\boldsymbol{r}, t)$.

From canonical bosonic commutation relations $\left[\hat{\psi}_j(\boldsymbol{r}_1, t), \hat{\psi}_l^\dagger(\boldsymbol{r}_2, t)\right] = \delta_{j,l}\delta(\boldsymbol{r}_1 - \boldsymbol{r}_2)$ and $\left[\hat{\psi}_j(\boldsymbol{r}_1, t), \hat{\psi}_l(\boldsymbol{r}_2, t)\right] = 0$ where $j, l = a, m$, $\left[\delta\hat{\Psi}_j(\boldsymbol{k}_1, t), \delta\hat{\Psi}_l^\dagger(\boldsymbol{k}_2, t)\right] = \delta_{j,l}\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}$ and $\left[\delta\hat{\Psi}_j(\boldsymbol{k}_1, t), \delta\hat{\Psi}_l(\boldsymbol{k}_2, t)\right] = 0$.

Let $\delta\Psi(\boldsymbol{k}, t) := \begin{bmatrix} \delta\hat{\Psi}_a(\boldsymbol{k}, t) & \delta\hat{\Psi}_m(\boldsymbol{k}, t) & \delta\hat{\Psi}_a^\dagger(-\boldsymbol{k}, t) & \delta\hat{\Psi}_m^\dagger(-\boldsymbol{k}, t) \end{bmatrix}^T$ and $\delta\Psi^\dagger(\boldsymbol{k}, t) := \begin{bmatrix} \delta\hat{\Psi}_a^\dagger(\boldsymbol{k}, t) & \delta\hat{\Psi}_m^\dagger(\boldsymbol{k}, t) & \delta\hat{\Psi}_a(-\boldsymbol{k}, t) & \delta\hat{\Psi}_m(-\boldsymbol{k}, t) \end{bmatrix}$. Expanding the Hamiltonian up to the second order in $\delta\hat{\Psi}_j$ gives

$$\hat{H}(t) - \mu\hat{N} = H_0 - \mu N - \frac{g_a n}{2}\sum_{\boldsymbol{k}\neq 0}\{M_{11}(k) + M_{22}(k)\} + O\left(\delta\hat{\Psi}_j^3\right)$$

$$+\frac{g_a n}{2}\sum_{\boldsymbol{k}\neq 0}\delta\Psi^\dagger(\boldsymbol{k},t)\begin{bmatrix} M_{11}(k) & M_{12} & M_{13} & M_{14} \\ M_{12}^* & M_{22}(k) & M_{14} & M_{24} \\ M_{13}^* & M_{14}^* & M_{11}(k) & M_{12}^* \\ M_{14}^* & M_{24}^* & M_{12} & M_{22}(k) \end{bmatrix}\delta\Psi(\boldsymbol{k},t), \quad (1.29)$$

where μ is in Eq. (1.25), $H_0 - \mu N$ in Eq. (1.26),

$$M_{11}(k) := \frac{\hbar^2 k^2}{2m_a g_a n} + 2x^2 + \tilde{g}_{am}y^2 - \tilde{\mu} = k^2\xi_a^2 + 1 - 2y^2 + |\tilde{\alpha}|\,y, \quad (1.30)$$

$$M_{22}(k) := \frac{\hbar^2 k^2}{2m_m g_a n} + \tilde{\epsilon} + 2\tilde{g}_m y^2 + \tilde{g}_{am}x^2 - 2\tilde{\mu} = \frac{1}{2}k^2\xi_a^2 + \tilde{g}_m y^2 - |\tilde{\alpha}|\left(y - \frac{1}{2y}\right), \quad (1.31)$$

$$M_{12} := \{\tilde{\alpha} - \mathrm{sign}(\tilde{\alpha})\,\tilde{g}_{am}y\}\sqrt{1-2y^2}, \quad M_{13} := 1 - 2y^2 - |\tilde{\alpha}|\,y, \quad (1.32)$$

and

$$M_{14} := -\mathrm{sign}(\tilde{\alpha})\,\tilde{g}_{am}y\sqrt{1-2y^2}, \quad M_{24} := \tilde{g}_m y^2. \quad (1.33)$$

Here, for convenience, we set $\varphi_a = 0$ and define $\mathrm{sign}(\alpha) = 1$ if $\alpha \geq 0$ and $\mathrm{sign}(\alpha) = -1$ if $\alpha < 0$.

For notational convenience, let us define 4×4 matrix $M(k)$ as

$$M(k) := \begin{bmatrix} M_{11}(k) & M_{12} & M_{13} & M_{14} \\ M_{12}^* & M_{22}(k) & M_{14} & M_{24} \\ M_{13}^* & M_{14}^* & M_{11}(k) & M_{12}^* \\ M_{14}^* & M_{24}^* & M_{12} & M_{22}(k) \end{bmatrix} = \begin{bmatrix} M_{B,1}(k) & M_{B,2} \\ M_{B,2}^* & M_{B,1}^*(k) \end{bmatrix}, \quad (1.34)$$

with $M_{B,\zeta}$ being 2×2 matrices ($\zeta = 1, 2$).

From Bogoliubov expansion [1, 2], let

$$\begin{bmatrix} \delta\hat{\Psi}_a(\boldsymbol{k},t) \\ \delta\hat{\Psi}_m(\boldsymbol{k},t) \\ \delta\hat{\Psi}_a^\dagger(-\boldsymbol{k},t) \\ \delta\hat{\Psi}_m^\dagger(-\boldsymbol{k},t) \end{bmatrix} = \begin{bmatrix} U(k) & V^*(k) \\ V(k) & U^*(k) \end{bmatrix} \begin{bmatrix} \hat{b}_1(\boldsymbol{k},t) \\ \hat{b}_2(\boldsymbol{k},t) \\ \hat{b}_1^\dagger(-\boldsymbol{k},t) \\ \hat{b}_2^\dagger(-\boldsymbol{k},t) \end{bmatrix}, \tag{1.35}$$

where $U(k)$ and $V(k)$ are 2×2 matrices and $\hat{b}_p(\boldsymbol{k},t)$ are bosonic annihilation operators ($p = 1, 2$). Then they should follow bosonic commutation relations, i.e.,

$$\left[\hat{b}_p(\boldsymbol{k}_1,t), \hat{b}_q^\dagger(\boldsymbol{k}_2,t)\right] = \delta_{p,q}\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}, \quad \left[\hat{b}_p(\boldsymbol{k}_1,t), \hat{b}_q(\boldsymbol{k}_2,t)\right] = 0, \tag{1.36}$$

for $p, q = 1, 2$.

From [2], by writing

$$U(k) := \begin{bmatrix} u_1(k) & u_2(k) \end{bmatrix}, \quad V(k) := \begin{bmatrix} v_1(k) & v_2(k) \end{bmatrix}, \tag{1.37}$$

where $u_p(k) = \begin{bmatrix} u_{p1}(k) & u_{p2}(k) \end{bmatrix}^T$ and $v_p(k) = \begin{bmatrix} v_{p1}(k) & v_{p2}(k) \end{bmatrix}^T$ are 2×1 matrices ($p = 1, 2$), Eq. (1.35) can also be written as

$$\begin{aligned} \delta\hat{\Psi}_\zeta(\boldsymbol{k},t) &= \sum_{q=1,2} \left\{u_{q\zeta}(k)\,\hat{b}_q(\boldsymbol{k},t) + v_{q\zeta}^*(k)\,\hat{b}_q^\dagger(-\boldsymbol{k},t)\right\}, \\ \delta\hat{\Psi}_\zeta^\dagger(\boldsymbol{k},t) &= \sum_{q=1,2} \left\{u_{q\zeta}^*(k)\,\hat{b}_q^\dagger(\boldsymbol{k},t) + v_{q\zeta}(k)\,\hat{b}_q(-\boldsymbol{k},t)\right\}, \end{aligned} \tag{1.38}$$

where $\zeta = a, m$.

Using Eqs. (1.36) leads us to

$$\begin{aligned} \left[\delta\hat{\Psi}_{\zeta_1}(\boldsymbol{k}_1,t), \delta\hat{\Psi}_{\zeta_2}^\dagger(\boldsymbol{k}_2,t)\right] &= \sum_{q=1,2} \left\{u_{q\zeta_1}(k_1)\,u_{q\zeta_2}^*(k_2) - v_{q\zeta_1}^*(k_1)\,v_{q\zeta_2}(k_2)\right\}\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2} \\ &= \left\{U^*(k_1)\,U^T(k_1) - V(k_1)\,V^\dagger(k_1)\right\}_{\zeta_2,\zeta_1}\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}, \end{aligned} \tag{1.39}$$

and

$$\begin{aligned}\left[\delta\hat{\Psi}_{\zeta_1}(\boldsymbol{k}_1,t),\delta\hat{\Psi}_{\zeta_2}(\boldsymbol{k}_2,t)\right] &= \sum_{q=1,2}\left\{u_{q\zeta_1}(k_1)\,v^*_{q\zeta_2}(k_2)-v^*_{q\zeta_1}(k_1)\,u_{q\zeta_2}(k_2)\right\}\delta_{\boldsymbol{k}_1,-\boldsymbol{k}_2}\\ &= \left\{V^*(k_1)\,U^T(k_1)-U(k_1)\,V^\dagger(k_1)\right\}_{\zeta_2,\zeta_1}\delta_{\boldsymbol{k}_1,-\boldsymbol{k}_2},\end{aligned} \tag{1.40}$$

which give

$$\begin{aligned}&\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\left\{U(k_1)\,U^\dagger(k_1)-V^*(k_1)\,V^T(k_1)\right\}=\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}I,\\ &\delta_{\boldsymbol{k}_1,-\boldsymbol{k}_2}\left\{U(k_1)\,V^\dagger(k_1)-V^*(k_1)\,U^T(k_1)\right\}=0,\end{aligned} \tag{1.41}$$

where I is the identity operator. By following conventions, we set

$$U(k)\,U^\dagger(k)-V^*(k)\,V^T(k):=I,\quad U(k)\,V^\dagger(k)-V^*(k)\,U^T(k):=0, \tag{1.42}$$

which is the first constraints of $U(k)$ and $V(k)$.

Then Eq. (1.35) can be written as

$$\begin{bmatrix}\hat{b}_1(\boldsymbol{k},t)\\ \hat{b}_2(\boldsymbol{k},t)\\ \hat{b}_1^\dagger(-\boldsymbol{k},t)\\ \hat{b}_2^\dagger(-\boldsymbol{k},t)\end{bmatrix}=\begin{bmatrix}U^\dagger(k) & -V^\dagger(k)\\ -V^T(k) & U^T(k)\end{bmatrix}\begin{bmatrix}\delta\hat{\Psi}_a(\boldsymbol{k},t)\\ \delta\hat{\Psi}_m(\boldsymbol{k},t)\\ \delta\hat{\Psi}_a^\dagger(-\boldsymbol{k},t)\\ \delta\hat{\Psi}_m^\dagger(-\boldsymbol{k},t)\end{bmatrix}, \tag{1.43}$$

which gives

$$\begin{aligned}\hat{b}_q(\boldsymbol{k},t) &= \sum_{\zeta=a,m}\left\{u^*_{q\zeta}(k)\,\delta\hat{\Psi}_\zeta(\boldsymbol{k},t)-v^*_{q\zeta}(k)\,\delta\hat{\Psi}^\dagger_\zeta(-\boldsymbol{k},t)\right\},\\ \hat{b}^\dagger_q(\boldsymbol{k},t) &= \sum_{\zeta=a,m}\left\{u_{q\zeta}(k)\,\delta\hat{\Psi}^\dagger_\zeta(\boldsymbol{k},t)-v_{q\zeta}(k)\,\delta\hat{\Psi}_\zeta(-\boldsymbol{k},t)\right\},\end{aligned} \tag{1.44}$$

$$\begin{aligned}\left[\hat{b}_{q_1}(\boldsymbol{k}_1,t),\hat{b}^\dagger_{q_2}(\boldsymbol{k}_2,t)\right] &= \sum_{\zeta=a,m}\left\{u^*_{q_1\zeta}(k_1)\,u_{q_2\zeta}(k_2)-v^*_{q_1\zeta}(k_1)\,v_{q_2\zeta}(k_2)\right\}\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\\ &= \left\{U^\dagger(k_1)\,U(k_1)-V^\dagger(k_1)\,V(k_1)\right\}_{q_1,q_2}\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2},\end{aligned} \tag{1.45}$$

and

$$
\begin{aligned}
\left[\hat{b}_{q_1}(\boldsymbol{k}_1,t),\hat{b}_{q_2}(\boldsymbol{k}_2,t)\right] &= -\sum_{\zeta=a,m}\left\{u^*_{q_1\zeta}(k_1)\,v^*_{q_2\zeta}(k_2)-v^*_{q_1\zeta}(k_1)\,u^*_{q_2\zeta}(k_2)\right\}\delta_{\boldsymbol{k}_1,-\boldsymbol{k}_2}\\
&= -\left\{U^\dagger(k_1)\,V^*(k_1)-V^\dagger(k_1)\,U^*(k_1)\right\}_{q_1,q_2}\delta_{\boldsymbol{k}_1,-\boldsymbol{k}_2}. \qquad (1.46)
\end{aligned}
$$

Therefore, we get the second constraints of $U(k)$ and $V(k)$,

$$
U^\dagger(k)\,U(k)-V^\dagger(k)\,V(k)=I, \quad U^T(k)\,V(k)-V^T(k)\,U(k)=0. \qquad (1.47)
$$

By following the approach in [2], suppose that $M_{B,p}$, $U(k)$, and $V(k)$ satisfy

$$
g_a n\begin{bmatrix} M_{B,1}(k) & M_{B,2} \\ -M^*_{B,2} & -M^*_{B,1}(k)\end{bmatrix}\begin{bmatrix} U(k) \\ V(k)\end{bmatrix}=\begin{bmatrix} U(k) \\ V(k)\end{bmatrix}\mathcal{E}(k),
$$

$$
\text{where } \mathcal{E}(k) := \hbar\begin{bmatrix} \omega_1(k) & 0 \\ 0 & \omega_2(k)\end{bmatrix}. \qquad (1.48)
$$

Let $\sigma_z := \begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix}$. Then, from Eq. (1.48), $\mathcal{V}(k) := \begin{bmatrix} U(k) \\ V(k)\end{bmatrix}$ is the 4×2 eigenmatrix of $\sigma_z M(k)$ and Eq. (1.47) can be written as

$$
\mathcal{V}^\dagger(k)\begin{bmatrix} I & 0 \\ 0 & -I\end{bmatrix}\mathcal{V}(k)=I. \qquad (1.49)
$$

Note that, if Eq. (1.48) is satisfied,

$$
g_a n\begin{bmatrix} M_{B,1}(k) & M_{B,2} \\ -M^*_{B,2} & -M^*_{B,1}(k)\end{bmatrix}\begin{bmatrix} V^*(k) \\ U^*(k)\end{bmatrix}=-\begin{bmatrix} V^*(k) \\ U^*(k)\end{bmatrix}\mathcal{E}^*(k), \qquad (1.50)
$$

so $\begin{bmatrix} V^\dagger(k) & U^\dagger(k)\end{bmatrix}^T$ is also eigenmatrix of $\sigma_z M(k)$.

However,

$$
\begin{bmatrix} V^*(k) \\ U^*(k)\end{bmatrix}^\dagger\begin{bmatrix} I & 0 \\ 0 & -I\end{bmatrix}\begin{bmatrix} V^*(k) \\ U^*(k)\end{bmatrix}=-\left\{U^\dagger(k)\,U(k)-V^\dagger(k)\,V(k)\right\}^*=-I, \qquad (1.51)
$$

and thus it does not satisfy Eq. (1.49). Therefore, half of eigenvalues are unphysical values (in the sense that their eigenmatrices $\mathcal{V}(k)$ do not satisfy Eq. (1.49) and hence bosonic commutation relations are not satisfied).

After neglecting those unphysical eigenvalues, Eq. (1.29) can be written as

$$\hat{H}(t)-\mu\hat{N}=H_0-\mu N+\frac{1}{2}\sum_{\boldsymbol{k}\neq 0}\left[\hbar\left\{\omega_1(k)+\omega_2(k)\right\}^*-g_a n\left\{M_{11}(k)+M_{22}(k)\right\}\right]$$
$$+\hbar\sum_{\boldsymbol{k}\neq 0}\left[\mathrm{Re}\left\{\omega_1(k)\right\}\hat{b}_1^\dagger(\boldsymbol{k},t)\,\hat{b}_1(\boldsymbol{k},t)+\mathrm{Re}\left\{\omega_2(k)\right\}\hat{b}_2^\dagger(\boldsymbol{k},t)\,\hat{b}_2(\boldsymbol{k},t)\right]+O\left(\delta\hat{\Psi}_j^3\right). \quad (1.52)$$

Here, by defining $\tilde{\omega}_p(k):=\hbar\omega_p(k)/(g_a n)$, $\tilde{\omega}_p(k)$ satisfy

$$\begin{bmatrix} M_{B,1}(k) & M_{B,2} \\ -M_{B,2}^* & -M_{B,1}^*(k) \end{bmatrix}\begin{bmatrix} u_p(k) \\ v_p(k) \end{bmatrix}=\tilde{\omega}_p(k)\begin{bmatrix} u_p(k) \\ v_p(k) \end{bmatrix}, \quad (1.53)$$

according to Eq. (1.48). Note that, if $M_{B,1}(k)$, $M_{B,2}$, and $\tilde{\omega}_p(k)$ are all real, then $u_p(k)$ and $v_p(k)$ should be also real from Eq. (1.53). In terms of the above eigenvectors, the 4×4 eigenproblem reads

$$\begin{bmatrix} M_{11}(k) & M_{12} & M_{13} & M_{14} \\ M_{12}^* & M_{22}(k) & M_{14} & M_{24} \\ -M_{13}^* & -M_{14}^* & -M_{11}(k) & -M_{12}^* \\ -M_{14}^* & -M_{24}^* & -M_{12} & -M_{22}(k) \end{bmatrix}\begin{bmatrix} u_{p1}(k) \\ u_{p2}(k) \\ v_{p1}(k) \\ v_{p2}(k) \end{bmatrix}=\tilde{\omega}_p(k)\begin{bmatrix} u_{p1}(k) \\ u_{p2}(k) \\ v_{p1}(k) \\ v_{p2}(k) \end{bmatrix}, \quad (1.54)$$

where $M_{\zeta\eta}$ is defined from Eqs. (1.30) to (1.33) ($\zeta,\eta=1,2$). Based on the approach of [43], we show analytic expressions of $\tilde{\omega}_p(k)$, $u_{pq}(k)$, and $v_{pq}(k)$ in Appendix D.

1.2 Time Evolution Operator

In Schrödinger picture, operators are constant in time and the state $|\Psi_S(t)\rangle$ follow Schrödinger equation $i\hbar\partial|\Psi_S(t)\rangle/\partial t=\hat{H}|\Psi_S(t)\rangle$. It can be also written as $|\Psi_S(t)\rangle=\hat{U}_S(t)|\Psi_S(0)\rangle$ where $\hat{U}_S(t)$ is the time evolution operator in Schrödinger picture which satisfy $i\hbar\partial\hat{U}_S(t)/\partial t = \hat{H}\hat{U}_S(t)$, $\hat{U}_S^\dagger(t)\hat{U}_S(t) = \hat{U}_S(t)\hat{U}_S^\dagger(t) = I$, and $\hat{U}_S(0) = I$ where I is the identity operator.

In Heisenberg picture, the state $|\Psi\rangle := |\Psi_S(0)\rangle$ is constant in time and operators $\hat{A}(t)$ change in time as $\hat{A}(t) = \hat{U}_S^\dagger(t)\hat{A}_S\hat{U}_S(t)$ where $\hat{A}_S$ is the operator $\hat{A}$ in Schrödinger picture. Note that $\langle\Psi_S(t)|\hat{A}_S|\Psi_S(t)\rangle = \langle\Psi|\hat{A}(t)|\Psi\rangle$ and hence observable values are same in both pictures.

If one adds time dependent perturbation ($\hat{V}_S(t)$ in Schrödinger picture) to the original Hamiltonian (time independent operator $\hat{H}_0$ in Schrödinger picture), it is convenient to use interaction picture where the state is $|\Psi_I(t)\rangle := \hat{U}_0(t)|\Psi_S(t)\rangle$ with $\hat{U}_0(t) := e^{i\hat{H}_0 t/\hbar}$. Let us define the time evolution operator $\hat{U}_I(t)$ in the interaction picture where $|\Psi_I(t)\rangle := \hat{U}_I(t)|\Psi_S(0)\rangle$, $\hat{U}_I^\dagger(t)\hat{U}_I(t) = \hat{U}_I(t)\hat{U}_I^\dagger(t) = I$, and $\hat{U}_I(0) = I$. Then we get

$$
\begin{aligned}
i\hbar\frac{\partial|\Psi_I(t)\rangle}{\partial t} &= i\hbar\frac{\partial\hat{U}_I(t)}{\partial t}|\Psi_S(0)\rangle \\
&= -\hat{H}_0(t)\hat{U}_0(t)|\Psi_S(t)\rangle + \hat{U}_0(t)\left\{\hat{H}_0 + \hat{V}_S(t)\right\}|\Psi_S(t)\rangle \\
&= \left\{\hat{U}_0(t)\hat{V}_S(t)\hat{U}_0^\dagger(t)\right\}\hat{U}_I(t)|\Psi_S(0)\rangle, \qquad (1.55)
\end{aligned}
$$

which gives $i\hbar\partial\hat{U}_I(t)/\partial t = \hat{V}_I(t)\hat{U}_I(t)$ where the operator $\hat{A}_S$ in Schrödinger picture is written as $\hat{A}_I(t) := \hat{U}_0(t)\hat{A}_S\hat{U}_0^\dagger(t)$ in the interaction picture. Note that $\langle\Psi_I(t)|\hat{A}_I(t)|\Psi_I(t)\rangle = \langle\Psi_S(t)|\hat{A}_S|\Psi_S(t)\rangle$ and hence observable values are also same in the interaction picture.

To get the time evolution operator to calculate dynamics of the observable values, one has to solve the differential equation of the form $i\hbar\partial\hat{U}(t)/\partial t = \hat{H}(t)\hat{U}(t)$ with $\hat{U}^\dagger(t)\hat{U}(t) = \hat{U}(t)\hat{U}^\dagger(t) = I$ and $\hat{U}(0) = I$. We will introduce two methods to solve this equation.

1.2.1 Dyson Series Expansion

Dyson showed that

$$\hat{U}(t) = I - \frac{i}{\hbar}\int_0^t dt_1\ \hat{H}(t_1) + \sum_{j=2}^{\infty}\left(-\frac{i}{\hbar}\right)^j \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{j-1}} dt_j\ \hat{H}(t_1)\hat{H}(t_2)\cdots\hat{H}(t_j), \tag{1.56}$$

is the solution of $i\hbar\partial\hat{U}(t)/\partial t = \hat{H}(t)\hat{U}(t)$ with $\hat{U}^\dagger(t)\hat{U}(t) = \hat{U}(t)\hat{U}^\dagger(t) = I$ and $\hat{U}(0) = I$ [44]. If the magnitude of the perturbation $\hat{V}_S(t)$ is small, using interaction picture is advisable since higher order terms in $\hat{V}_I(t)$ would be negligible (Note that in the interaction picture, $\hat{H}(t) \to \hat{V}_I(t)$). However, this method requires one to perform infinite summation and hence it is mainly used to get the approximate expression of the time evolution operator (such as first order or second order expansion of $\hat{U}(t)$ in $\hat{V}_S(t)$) . See Appendix F.1 for more detailed calculations.

1.2.2 Symplectic Formalism

Wei and Norman showed that if $\hat{H}(t)$ can be written as $\hat{H}(t) = \sum_{j=1}^{n_1} c_j(t)\hat{G}_j$ where $c_j(t)$ are a set of linearly independent functions, there are n_2 linearly independent operators $\hat{G}_j$ with $n_2 \geq n_1$ which enables us to write $\hat{U}(t) = \hat{U}_1(t)\hat{U}_2(t)\cdots\hat{U}_{n_2}(t)$ with $\hat{U}_l(t) := \exp\left\{-iF_l(t)\hat{G}_l\right\}$ [45]. Since our time evolution operator $\hat{U}(t)$ should satisfy $\hat{U}^\dagger(t)\hat{U}(t) = \hat{U}(t)\hat{U}^\dagger(t) = I$, we set $\hat{G}_l$ to be Hermitian which makes $F_l(t) \in \mathbb{R}$. $F_l(t)$ satisfy

$$\begin{aligned}\hbar\left[\frac{\partial F_1(t)}{\partial t}\hat{G}_1 + \frac{\partial F_2(t)}{\partial t}\hat{U}_1(t)\hat{G}_2\hat{U}_1^\dagger(t) + \cdots + \frac{\partial F_{n_2}(t)}{\partial t}\hat{U}_1(t)\hat{U}_2(t)\cdots\hat{U}_{n_2}(t)\hat{G}_{n_2}\hat{U}_{n_2}^\dagger(t)\cdots\hat{U}_2^\dagger(t)\hat{U}_1^\dagger(t)\right] \\ = \sum_{j=1}^{n_1} c_j(t)\hat{G}_j,\end{aligned} \tag{1.57}$$

with $F_l(0) = 0$ from $\hat{U}(0) = I$.

From Baker-Campbell-Hausdorff (BCH) theorem [46, 47, 48, 49, 50, 51],

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \cdots + \frac{1}{n!}[\underbrace{A, [A, \cdots, [A}_{n \text{ times}}, B]]] + \cdots, \tag{1.58}$$

and thus

$$\hat{U}_j(t)\,\hat{G}_l\hat{U}_j^\dagger(t) = \hat{G}_l - iF_j(t)\left[\hat{G}_j,\hat{G}_l\right] + \frac{1}{2!}\{-iF_j(t)\}^2\left[\hat{G}_j,\left[\hat{G}_j,\hat{G}_l\right]\right] + \frac{1}{n!}\{-iF_j(t)\}^n \underbrace{[\hat{G}_j,[\hat{G}_j,\cdots,[\hat{G}_j}_{n\text{ times}},\hat{G}_l]]] + \cdots . \tag{1.59}$$

For more detailed calculations, see Appendix G.1.

In section 3.3, by comparing with results from symplectic formalism, we show in Fig. 3.5 that Dyson series expansion up to second order in $\hat{H}(t)$ is enough for small perturbations such as the density modulation due to the gravitational wave.

1.3 Parameter Estimation Theory

Suppose that we want to estimate the temperature of the system by measuring speed of particles in the system. We get v_1 at the first measurement, v_2 at the second measurement, and so on where v_j can be same as v_l for some $j \neq l$. Let $P(v)$ be the probability that the particle with speed v is measured. If $P(v)$ follows Maxwell-Boltzmann distribution as Fig. 1.1,

$$P(v) = 4\pi\left(\frac{m}{2\pi k_B T}\right)^{3/2} v^2 e^{-mv^2/(2k_BT)}, \tag{1.60}$$

one may estimate the temperature T by calculating the mean speed of particles from N measurements and then compare with $\int_0^\infty dv\;\, vP(v) = \sqrt{8k_BT/(\pi m)}$. These estimations on T will be more accurate as N increases.

There are textbooks on this parameter estimation theory, for example, [52, 53, 54, 55, 56]. We will focus on Cramér-Rao theorem and Fisher information which are used in section 3.3.

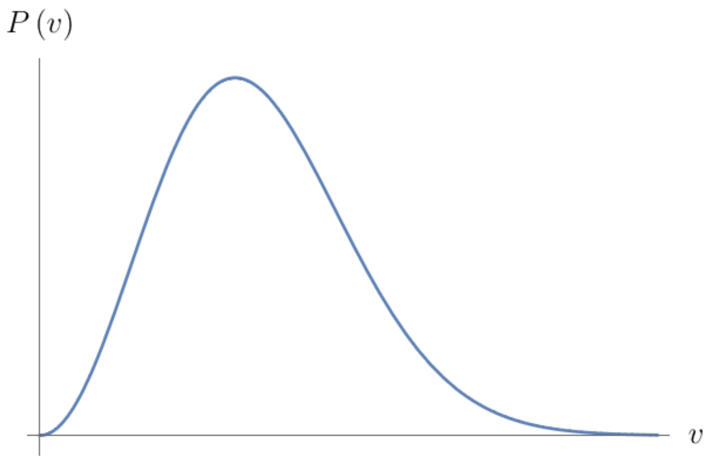


Figure 1.1: Schematic figure of the probability distribution $P(v)$ as a function of the speed v of particles.

1.3.1 Cramér-Rao Theorem and Classical Fisher Information

Suppose that we want to estimate the value of θ by measuring x whose probability distribution $P(x)$ depends on θ. Let $\hat{\theta}$ be the estimator which estimates θ from x. We assume that the mean value of $\hat{\theta}(x)$ is equal to θ, i.e., we assume that our estimator is not biased. Then the variance $\mathrm{Var}\left(\hat{\theta}\right)$ is bounded by

$$\mathrm{Var}\left(\hat{\theta}\right) \geq \frac{1}{N I_C(\theta)}, \quad I_C(\theta) := E\left[\left\{\frac{\partial \ln P(x)}{\partial \theta}\right\}^2\right], \tag{1.61}$$

where $E(y)$ is the mean value of y, N is the number of measurements, and $I_C(\theta)$ is the classical Fisher information (CFI) [57, 58]. As Eq. (1.61) shows, CFI depends on the way of measurements: If one measures y whose probability distribution is $P(y)$ and then tries to estimate θ, CFI will be changed in general. Also, as expected, our estimation becomes more accurate as N increases.

[59] showed that there is a lower bound in CFI,

$$I_C(\theta) \geq \frac{1}{\mathrm{Var}(x)}\left\{\frac{\partial E(x)}{\partial \theta}\right\}^2, \tag{1.62}$$

which is more easy to calculate when one does not know $P(x)$. From now on, we will use this lower bound of CFI as we know that $E(x) = \langle\Psi|\,\hat{x}\,|\Psi\rangle$ and $\mathrm{Var}(x) = \langle\Psi|\,\hat{x}^2\,|\Psi\rangle - E^2(x)$ for the quantum state $|\Psi\rangle$ but it is in general difficult to express the probability distribution of $\hat{x}$ in that state.

1.3.2 Quantum Cramér-Rao Theorem and Quantum Fisher Information

Quantum Cramér-Rao theorem is similar to the Cramér-Rao theorem:

$$\mathrm{Var}\left(\hat{\theta}\right) \geq \frac{1}{N I_Q(\theta)}, \quad I_Q(\theta) := \mathrm{Tr}\left(\hat{L}^2\hat{\rho}\right), \quad \frac{\hat{L}\hat{\rho}+\hat{\rho}\hat{L}}{2} := \frac{\partial\hat{\rho}}{\partial\theta}, \tag{1.63}$$

where $\hat{\rho}$ is the density operator and $I_Q(\theta)$ is the quantum Fisher information (QFI) [60]. The difference to the Cramér-Rao theorem is that QFI does not depend on the way of measurements. It depends on the density operator and thus QFI gives ideal limit of our estimation for the given system.

By writing the density operator as $\hat{\rho} = \sum_j p_j\,|\psi_j\rangle\,\langle\psi_j|$ where p_j is the probability that the state is in $|\psi_j\rangle$, QFI can be written as [61]

$$I_Q(\theta) = \sum_j \frac{1}{p_j}\left(\frac{\partial p_j}{\partial\theta}\right)^2 + 4\sum_j p_j\left\langle\frac{\partial\psi_j}{\partial\theta}\middle|\frac{\partial\psi_j}{\partial\theta}\right\rangle - 8\sum_{j,l}\frac{p_j p_l}{p_j+p_l}\left|\left\langle\psi_j\middle|\frac{\partial\psi_l}{\partial\theta}\right\rangle\right|^2. \tag{1.64}$$

For a pure Gaussian state $|\psi_{\mathrm{pure,G}}\rangle$, there is a simple expression for QFI [62]. A pure Gaussian state is defined as the state which makes the Wigner function as Gaussian, and the Wigner function is defined as follows: In D-dimensional space with position $\boldsymbol{r}$ and momentum $\boldsymbol{p}$, for a pure state $|\psi_{\mathrm{pure}}\rangle$, the Wigner function $W(\boldsymbol{r},\boldsymbol{p})$ is defined as

$$W(\boldsymbol{r},\boldsymbol{p}) := \frac{1}{(2\pi)^D}\int d^D r_1\ \langle\boldsymbol{r}-\boldsymbol{r}_1|\,\psi_{\mathrm{pure}}\rangle\,\langle\psi_{\mathrm{pure}}|\,\boldsymbol{r}+\boldsymbol{r}_1\rangle\,e^{i\boldsymbol{p}\cdot\boldsymbol{r}_1}. \tag{1.65}$$

For $\hat{\mathbb{X}} := (\hat{\boldsymbol{r}},\hat{\boldsymbol{p}})^T$ and the symmetrized covariance matrix Γ with $\Gamma_{j,l} := \left\langle\left\{\hat{\mathbb{X}}_j,\hat{\mathbb{X}}_l\right\}\right\rangle - 2\left\langle\hat{\mathbb{X}}_j\right\rangle\left\langle\hat{\mathbb{X}}_l\right\rangle$, QFI can be written as

$$I_Q(\theta) = \left(\frac{\partial\left\langle\hat{\mathbb{X}}\right\rangle}{\partial\theta}\right)^T \Gamma^{-1}\left(\frac{\partial\left\langle\hat{\mathbb{X}}\right\rangle}{\partial\theta}\right) + \frac{1}{4}\mathrm{Tr}\left\{\left(\frac{\partial\Gamma}{\partial\theta}\Gamma^{-1}\right)^2\right\}. \tag{1.66}$$

In section 3.3, we will use Eqs. (1.62) and (1.66) together with symplectic formalism 1.2.2 to calculate exact values of QFI and the lower bound of CFI. We will also use Eqs. (1.62) and (1.64) together with Dyson series expansion 1.2.1 up to the second order and show that they are consistent for small external perturbations. For those who want to see detailed calculations how QFI and CFI can be calculated by using Dyson series expansion, see Appendix F. For those who want to see detailed calculations how QFI and CFI can be calculated by using symplectic formalism, see Appendix G.

This ends the introduction to the theoretical framework of this thesis. Now we will begin our first topic, measuring damping parameter γ using Stoner-Wohlfarth switching in spinor-dipolar BEC where its spin always points to the same direction.

Chapter 2

Stoner-Wohlfarth Switching and Damping Parameter Measurement

As we discussed in 1.1.2, damping parameter γ is introduced to solve the problem that the mean-field theory of BEC cannot explain the damping of the collective oscillation. We could not find complete microscopic derivations for this γ, especially for spinor BEC, and there is only one estimation of γ for scalar ^{23}Na BEC. In this chapter, based on our paper [11], we propose a way to measure γ by Stoner-Wohlfarth (SW) switching in the spinor-dipolar BEC whose spin orientation is homogeneous.

If the conventional assumption (the mean-field correction for the scalar BEC also holds for the spinor BEC) is true, one may measure the value of γ by measuring SW switching time. If this assumption does not hold, then SW switching time would be different from what we obtain here. Thus our method not only enables one to measure γ but also gives us a way to check the validity of the conventional mean-field corrections for the spinor BEC.

We will first introduce SW model and Landau-Lifshitz-Gilbert (LLG) equation, which are basic theories on single-domain ferromagnets. Then we will show how one can measure γ by measuring SW switching time. Readers who already know what SW model is and what LLG equation is may skip the first section and directly go to the section 2.2.

2.1 The Stoner-Wohlfarth Model and the Landau-Lifshitz-Gilbert Equation

Stoner-Wohlfarth Model

In magnetism, single-domain state means that the magnetization is homogeneous: not only its orientation but also its magnitude is same everywhere. Stoner and Wohlfarth studied the single-domain ferromagnet and obtained hysteresis curve which shows that irreversible change of magnetization can occur by changing external magnetic field [29].

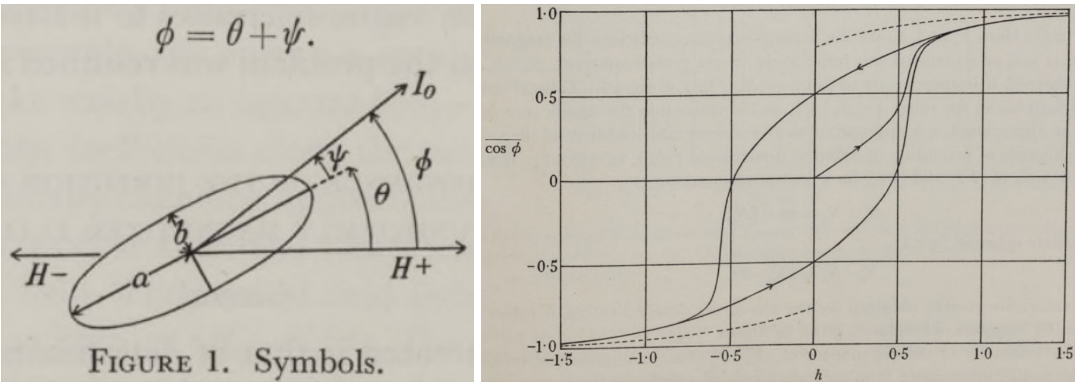


Figure 2.1: (Left) Schematic figure of the single-domain ferromagnet under external magnetic field $\boldsymbol{H}$ on z axis. $\boldsymbol{I}_0$ is the magnetization. (Right) Hysteresis curve obtained from SW model. $\overline{\cos\phi}$ is the mean value of $\cos\phi$ over solid angle of the ferromagnet, i.e., $\overline{\cos\phi} := \left(2\pi\int d\theta\ \sin\theta\cos\phi\right) / \left(2\pi\int d\theta\ \sin\theta\right)$, and $h := |\boldsymbol{H}| / \{(N_b - N_a)\, I_0\}$ where N_a and N_b are demagnetization coefficients along the polar axis (a axis in the left figure) and equatorial axis (b axis in the left figure). Figures are from [29].

The Hamiltonian of the SW model is $\hat{H} = K\sin^2\psi - \boldsymbol{H}\cdot\boldsymbol{I}_0$ by following notations in Fig. 2.1, and it is widely used, for example, in hard disk storage.

Landau-Lifshitz-Gilbert Equation

Landau, Lifshitz [63], and Gilbert [64] (based on his unpublished Ph.D thesis in 1956) introduced the following Landau-Lifshitz-Gilbert (LLG) equation:

$$\frac{\partial \boldsymbol{M}}{\partial t} = \gamma_e \boldsymbol{M} \times \boldsymbol{H} - \beta \boldsymbol{M} \times \frac{\partial \boldsymbol{M}}{\partial t}, \tag{2.1}$$

where γ_e is the electron gyromagnetic ratio, $\boldsymbol{M}$ is the magnetization, $\boldsymbol{H}$ is the external magnetic field, and β is the damping parameter. It is phenomenological equation which is not derived from microscopic theories.

In the next section, we will show that one can derive both SW model and LLG equation from the mean-field theory of the spinor-dipolar BEC provided that corrections to the mean-field equation of the scalar BEC (replacing $i\hbar\partial/\partial t \to (i - \gamma)\, \hbar\partial/\partial t$) is also valid for the mean-field equation of the spinor-dipolar BEC. From these results, we propose a way to measure γ by measuring the SW switching time (introduced in the next section).

2.2 Spinor-Dipolar Bose-Einstein Condensates as Detector

We suggest that those who are not familiar with the mean-field equation of the spinor-dipolar BEC read section 1.1 first (at least until section 1.1.2) and continue to read this section.

First two parts (section 2.2.1 and 2.2.2) of this section is deriving LLG equation and the Hamiltonian of the SW model from conventional mean-field equation of spinor-dipolar BEC with damping parameter. This shows the possibility that the phenomenological LLG equation might be derivable from the miscroscopic theories such as the Heisenberg equations of motion.

Third part (section 2.2.3) is how in principle one can measure the damping parameter γ by using spinor-dipolar BEC with homogeneous local spin orientations. One can not only measure the value of γ but also check the validity of the conventional mean-field equation of the spinor-dipolar BEC with damping parameter since our results are based on this conventional mean-field equation. Measuring γ would help us to make theoretical simulations of the dynamics of BEC more accurate and broaden our understandings on the finite lifetime of the collective oscillation of BEC.

2.2.1 Derivations on Landau-Lifshitz-Gilbert equation

Let $\psi_{m_1}(\boldsymbol{r})$ be the mean field of the spinor-dipolar BEC with the spin s and magnetic quantum number $m_z = m_1$. From Eq. (1.7) with replacing $i\hbar\partial/\partial t \to (i-\gamma)\hbar\partial/\partial t$ (see section 1.1.2), we get conventional mean-field equation of the spinor-dipolar BEC with damping:

$$\begin{aligned}(i-\gamma)\hbar\frac{\partial\psi(\boldsymbol{r},t)}{\partial t} &= \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\mathrm{tr}}(\boldsymbol{r}) + c_0 n(\boldsymbol{r},t) + \sum_{j=1}^{s} c_{2j}\sum_{\nu_1,\cdots,\nu_j=x,y,z} F_{\nu_1,\cdots,\nu_j}(\boldsymbol{r},t)\left(\hat{f}_{\nu_1}\cdots\hat{f}_{\nu_j}\right)\right]\psi(\boldsymbol{r},t)\\ &\quad -\sum_{\nu=x,y,z}\left[g_F\mu_B B_\nu - c_{dd}\sum_{\nu'=x,y,z}\left\{\int d^3r'\,Q_{\nu,\nu'}\left(\boldsymbol{r}-\boldsymbol{r}'\right)F_{\nu'}\left(\boldsymbol{r}',t\right)\right\}\right]\hat{f}_\nu\psi(\boldsymbol{r},t),\end{aligned} \tag{2.2}$$

where $\psi(\boldsymbol{r},t) = \left[\begin{array}{ccc}\psi_{-s}(\boldsymbol{r},t), & \cdots, & \psi_s(\boldsymbol{r},t)\end{array}\right]^T$, $\hbar$ is the reduced Planck constant, m is the mass, $V_{\mathrm{tr}}(\boldsymbol{r})$ is the trap potential, c_0 is the density-density interaction coefficient, $n(\boldsymbol{r},t) := \psi^\dagger(\boldsymbol{r},t)\psi(\boldsymbol{r})$ is the number density of the mean-field, $\hbar\hat{\boldsymbol{f}}$ is the spin operator, $F_{\nu_1,\cdots,\nu_j}(\boldsymbol{r},t) := \psi^\dagger(\boldsymbol{r},t)\left(\hat{f}_{\nu_1}\cdots\hat{f}_{\nu_j}\right)\psi(\boldsymbol{r})$, $\boldsymbol{B}$ is the external magnetic field, g_F is the Landé g-factor, μ_B is the Bohr magneton, $Q_{\nu,\nu'}(\boldsymbol{r}) := \left(r^2\delta_{\nu,\nu'} - 3r_\nu r_{\nu'}\right)/r^5$, $c_{dd} := \mu_0\left(g_F\mu_B\right)^2/(4\pi)$ is the magnetic dipole-dipole interaction coefficient, and μ_0 is the magnetic permeability of free space. For the unit vector $\boldsymbol{e}_\nu$ in $+\nu$ axis, we define $A_\nu := \boldsymbol{A}\cdot\boldsymbol{e}_\nu$ for any vector $\boldsymbol{A}$.

From now on, we will consider a quasi-1D spinor-dipolar BEC on z axis where $V_{\mathrm{tr}}(\boldsymbol{r}) = (m/2)\,\omega_\perp^2\left(x^2+y^2\right) + V(z)$ and the magnetic dipole moment $\boldsymbol{d}(z,t) = g_F\mu_B\boldsymbol{F}(\boldsymbol{r},t)$ points to the same direction everywhere, i.e., $\boldsymbol{M}(t) := \boldsymbol{d}(z,t)/|\boldsymbol{d}(z,t)| = (\sin\theta(t)\cos\phi(t), \sin\theta(t)\sin\phi(t), \cos\theta(t))$ (see Fig. 2.2).

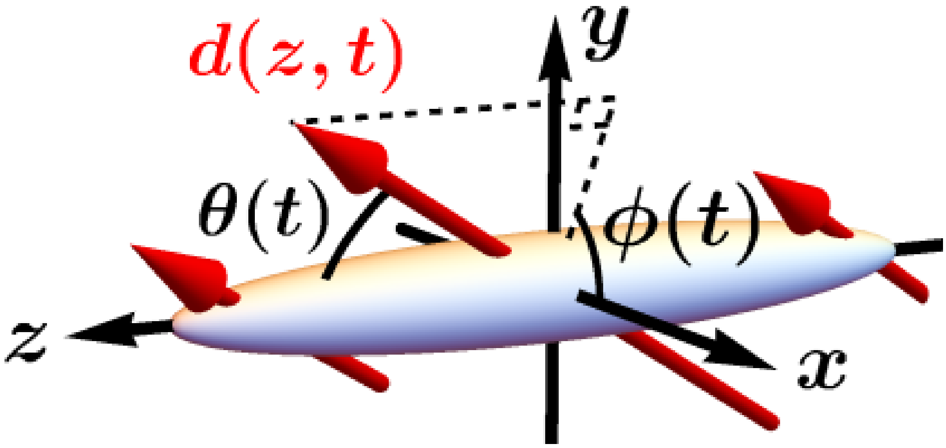


Figure 2.2: Schematic figure of the quasi-1D spinor-dipolar BEC we used in [11], represented as the ellipsoid. Red arrows represent the magnetic dipole moment $\boldsymbol{d}(z,t) = g_F \mu_B \boldsymbol{F}(\boldsymbol{r},t)$ with their length being proportional to $|\boldsymbol{d}(z,t)|$.

Using the ansatz

$$\psi_{m_1}(\boldsymbol{r},t) = \frac{e^{-\rho^2/\left(2l_\perp^2\right)}}{l_\perp\sqrt{\pi}}\Psi(z,t)\,\zeta_{m_1}(t)\,e^{-\frac{i+\gamma}{1+\gamma^2}\omega_\perp t}, \quad \zeta_{m_1}(t) := \langle m_1|\,e^{-i\hat{f}_z\phi(t)}e^{-i\hat{f}_y\theta(t)}\,|s\rangle\,, \tag{2.3}$$

where $\rho = \sqrt{x^2+y^2}$, $l_\perp := \sqrt{\hbar/(m\omega_\perp)}$ is the harmonic oscillator length in xy plane, and $\zeta^\dagger(t)\,\zeta(t) = 1$, we get

$$\begin{aligned}
\hbar F_x(\boldsymbol{r},t) &= \hbar s\frac{e^{-\rho^2/l_\perp^2}}{\pi l_\perp^2}|\Psi(z,t)|^2\,e^{-2\gamma\omega_\perp t/\left(1+\gamma^2\right)}\sin\theta(t)\cos\phi(t)\,,\\
\hbar F_y(\boldsymbol{r},t) &= \hbar s\frac{e^{-\rho^2/l_\perp^2}}{\pi l_\perp^2}|\Psi(z,t)|^2\,e^{-2\gamma\omega_\perp t/\left(1+\gamma^2\right)}\sin\theta(t)\sin\phi(t)\,,\\
\hbar F_z(\boldsymbol{r},t) &= \hbar s\frac{e^{-\rho^2/l_\perp^2}}{\pi l_\perp^2}|\Psi(z,t)|^2\,e^{-2\gamma\omega_\perp t/\left(1+\gamma^2\right)}\cos\theta(t)\,,
\end{aligned} \tag{2.4}$$

which is consistent with our definition of $\boldsymbol{M}(t)$.

By integrating out the x and y directions, the mean-field equation of our quasi-1D spin-S BEC can be written as (for a detailed derivation, see Appendix A)

$$(i-\gamma)\hbar\frac{\partial\left\{\Psi(z,t)\zeta_{m_1}(t)\right\}}{\partial t}=\left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}+V(z)+\frac{c_0}{2\pi l_\perp^2}n(z,t)\right\}\Psi(z,t)\zeta_{m_1}(t)$$
$$+\hbar\left[-\boldsymbol{b}+s\left\{\boldsymbol{M}(t)-3M_z(t)\boldsymbol{e}_z\right\}P_{dd}(z,t)\right]\cdot\left\{\sum_{m_2=-s}^{s}\left(\hat{\boldsymbol{f}}\right)_{m_1,m_2}\Psi(z,t)\zeta_{m_2}(t)\right\}$$
$$+\sum_{j=1}^{s}\frac{c_{2j}}{2\pi l_\perp^2}n(z,t)\sum_{\nu_1,\nu_2,\cdots,\nu_j=x,y,z}sM_{\nu_1,\nu_2,\cdots,\nu_j}(t)\left\{\sum_{m_2=-s}^{s}\left(\hat{f}_{\nu_1}\hat{f}_{\nu_2}\cdots\hat{f}_{\nu_j}\right)_{m_1,m_2}\Psi(z,t)\zeta_{m_2}(t)\right\}, \quad (2.5)$$

where $\boldsymbol{b}\coloneqq g_F\mu_B\boldsymbol{B}/\hbar$ is the Larmor frequency vector and we defined the two functions

$$M_{\nu_1,\nu_2,\cdots,\nu_j}(t)\coloneqq\frac{1}{s}\sum_{m_1,m_2=-s}^{s}\zeta_{m_1}^{\dagger}(t)\left(\hat{f}_{\nu_1}\hat{f}_{\nu_2}\cdots\hat{f}_{\nu_j}\right)_{m_1,m_2}\zeta_{m_2}(t), \quad (2.6)$$

$$P_{dd}(z,t)\coloneqq\frac{c_{dd}}{2\hbar l_\perp^3}\int_{-\infty}^{\infty}dz'\ n(z',t)\left\{G\left(\frac{|z-z'|}{l_\perp}\right)-\frac{4}{3}\delta\left(\frac{z-z'}{l_\perp}\right)\right\}, \quad (2.7)$$

with the axial density $n(z,t)\coloneqq\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\ |\psi(\boldsymbol{r},t)|^2=|\Psi(z,t)|^2e^{-2\gamma\omega_\perp t/\left(1+\gamma^2\right)}$, and

$$G(\lambda)\coloneqq\sqrt{\frac{\pi}{2}}\left(\lambda^2+1\right)e^{\lambda^2/2}\mathrm{Erfc}\left(\frac{\lambda}{\sqrt{2}}\right)-\lambda. \quad (2.8)$$

By applying normalization condition on $\zeta(t)$, we get (see Appendix B for details)

$$\frac{\partial\boldsymbol{M}(t)}{\partial t}=\boldsymbol{M}(t)\times\left\{\boldsymbol{b}+s\Lambda'_{dd}(t)M_z(t)\boldsymbol{e}_z\right\}-\gamma\boldsymbol{M}(t)\times\frac{\partial\boldsymbol{M}(t)}{\partial t}, \quad (2.9)$$

where $\Lambda'_{dd}(t)=\{3/N(t)\}\int_{-\infty}^{\infty}dz\ n(z,t)P_{dd}(z,t)$ and $N(t)\coloneqq\int d^3r\ |\psi(\boldsymbol{r},t)|^2=\int_{-\infty}^{\infty}dz\ n(z,t)$. Note that $\Lambda'_{dd}(t)$ is connected to the dipole-dipole interaction contribution $V_{dd}(t)$ by $V_{dd}(t)=(\hbar/2)s^2N(t)\Lambda'_{dd}(t)\left\{(1/3)-\cos^2\theta(t)\right\}$.

Eq. (2.9) has the same form as Eq. (2.1) with $\gamma_e\to g_F\mu_B/\hbar$, $\boldsymbol{H}\to\boldsymbol{B}+\hbar s\Lambda'_{dd}(t)M_z(t)\boldsymbol{e}_z/(g_F\mu_B)$, and $\beta\to\gamma$. Thus we managed to derive LLG equation if conventional mean-field equation of the spinor-dipolar BEC with damping is valid.

2.2.2 Derivations on the Hamiltonian of the Stoner-Wohlfarth Model

When there is no damping in collective oscillation, i.e., $\gamma = 0$, and if we set $\boldsymbol{B} = B_x \boldsymbol{e}_x + B_z \boldsymbol{e}_z$, Eq. (2.9) can be written as

$$\frac{d\theta(t)}{dt} = b_x \sin\phi(t), \quad \frac{d\phi(t)}{dt} = b_x \cot\theta(t)\cos\phi(t) - b_z - s\Lambda'_{dd}(t)\cos\theta(t). \tag{2.10}$$

By using the Lagrangian formalism introduced in [65], the Lagrangian L of this system can be written as

$$L(t) = \hbar\left[\frac{\partial\phi(t)}{\partial t}\cos\theta(t) + b_x \sin\theta(t)\cos\phi(t) + b_z\cos\theta(t) + \frac{s}{4}\Lambda'_{dd}(t)\cos\{2\theta(t)\}\right], \tag{2.11}$$

and the equations of motion are

$$\begin{aligned}
&\frac{\partial L(t)}{\partial\theta(t)} = \hbar\left[-\dot{\phi}(t)\sin\theta(t) + b_x\cos\theta(t)\cos\phi(t) - b_z\sin\theta(t) - \frac{s}{2}\Lambda'_{dd}(t)\sin\{2\theta(t)\}\right], \\
&\frac{\partial L(t)}{\partial\dot{\theta}(t)} = 0, \quad \frac{\partial L(t)}{\partial\phi(t)} = -\hbar b_x\sin\theta(t)\sin\phi(t), \quad \frac{\partial L(t)}{\partial\dot{\phi}(t)} = \hbar\cos\theta(t).
\end{aligned} \tag{2.12}$$

where $\dot{\phi}(t) := \partial\phi(t)/\partial t$ and $\dot{\theta}(t) := \partial\theta(t)/\partial t$. One can verify that Eq. (2.11) is indeed the Lagrangian which gives Eqs. (2.10). Let $p_\xi(t)$ be the conjugate momentum of the coordinate ξ. Since $p_\theta(t) = 0$ and $p_\phi(t) = \hbar\cos\theta(t)$ ($\hbar$ times the z component of $\boldsymbol{M}$), the Hamiltonian H can be written as

$$H(t) = -b_x\sqrt{\hbar^2 - p_\phi^2(t)}\cos\phi(t) - b_z p_\phi(t) + \frac{\hbar^2 - 2p_\phi^2(t)}{4\hbar}s\Lambda'_{dd}(t). \tag{2.13}$$

Since $\Lambda'_{dd}(t)$ does not explicitly depend on $\theta(t)$ and $\phi(t)$, we may express the Hamiltonian as $\tilde{H}(t) := H(t) + \hbar s\Lambda'_{dd}(t)/4$, which gives

$$\tilde{H}(t) = \hbar\left\{-\boldsymbol{b}\cdot\boldsymbol{M}(t) + \frac{s}{2}\Lambda'_{dd}(t)\sin^2\theta(t)\right\}. \tag{2.14}$$

This has the same form as the SW hamiltonian with $K \to s\Lambda'_{dd}(t)/2$, and thus we also managed to derive the SW hamiltonian.

2.2.3 Measuring the Damping Parameter

For simplicity, we will assume that $\boldsymbol{B}$ is on the z axis. Then Eq. (2.9) becomes

$$\frac{\partial \boldsymbol{M}(t)}{\partial t} = s\Lambda'_{dd}\boldsymbol{M}(t) \times \boldsymbol{e}_z \left\{M_z(t) - (M_z)_{\rm cr}(t)\right\} - \gamma \boldsymbol{M}(t) \times \frac{\partial \boldsymbol{M}(t)}{\partial t}, \qquad (2.15)$$

where $(M_z)_{\rm cr}(t) := -b_z / \left\{s\Lambda'_{dd}(t)\right\}$.

Since $|\boldsymbol{M}| := \sqrt{\boldsymbol{M}(t) \cdot \boldsymbol{M}(t)} = 1$, by taking the cross product with $\boldsymbol{M}(t)$ on both sides of Eq. (2.15), we get

$$\frac{\partial M_z(t)}{\partial t} = -\frac{\gamma s \Lambda'_{dd}(t)}{1+\gamma^2} \left\{M_z(t) - (M_z)_{\rm cr}(t)\right\} \left\{M_z^2(t) - 1\right\}, \qquad (2.16)$$

which shows that there are two stationary stable solutions $M_z(t) = \pm 1$. If $(M_z)_{\rm cr}(t)$ does not depend on t, $M_z(t) = (M_z)_{\rm cr}$ would be a stationary unstable solution. Excluding these unstable solutions, $M_z(t)$ will switch to either 1 or -1 and we will name this switching as the Stoner-Wohlfarth (SW) switching.

If the damping parameter γ is small, we may assume that $\Lambda'_{dd}(t)$ (and thus $(M_z)_{\rm cr}(t)$) does not change until SW switching occurs. Since the time dependence of $\Lambda'_{dd}(t)$ is due to the time dependence in the axial density $n(z,t)$ (see section 2.2.1), this assumption is equivalent to assuming that the number of BEC does not change until SW switching occurs. We will show later that our SW switching time is below 10 percent of the typical lifetime of the collective oscillation of BEC for the typical value of $\gamma \simeq 0.03$, which justifies our assumptions.

When $\Lambda'_{dd}(t)$ does not depend on t, the solution of Eq. (2.16) is

$$t = \frac{1+\gamma^2}{\gamma s \Lambda'_{dd}} \left[\begin{array}{l} \frac{1}{\left\{(M_z)_{\rm cr}\right\}^2 - 1} \ln \left\{\frac{(M_z)_{\rm in} - (M_z)_{\rm cr}}{M_z(t) - (M_z)_{\rm cr}}\right\} - \frac{1}{2\left\{1-(M_z)_{\rm cr}\right\}} \ln \left\{\frac{1-M_z(t)}{1-(M_z)_{\rm in}}\right\} \\ + \frac{1}{2\left\{1+(M_z)_{\rm cr}\right\}} \ln \left\{\frac{1+(M_z)_{\rm in}}{1+M_z(t)}\right\} \end{array} \right], \qquad (2.17)$$

where $(M_z)_{\rm in} := M_z(0)$ is the initial value of M_z. To avoid the divergence in our solution (Eq. (2.17)), we define a critical switching time $t_{\rm cr}$ to be the time when

$M_z(t_{\text{cr}}) = \pm 0.99$. Note that even though we chose the spatial dimension of our system to be quasi-1D for convenience, Eq. (2.17) is still valid for other dimensions by suitably changing $\Lambda'_{dd}(t)$ (see Appendix A and B).

If $\gamma \ll 1$ and the magnetic dipole-dipole interaction is large so that $|(M_z)_{\text{cr}}| \ll 1$, we get

$$t_{\text{cr}} \simeq \frac{1}{\gamma s \Lambda'_{dd}} \ln \left[\frac{5\sqrt{2(1-(M_z)_{\text{in}}^2)}}{|(M_z)_{\text{in}} - (M_z)_{\text{cr}}|} \right]. \tag{2.18}$$

Eq. (2.18) clearly shows that the magnetic dipole-dipole interaction accelerates the decay of $M_z(t)$. Hence, by using a dipolar spinor BEC with large magnetic dipole moment such as produced from ^{164}Dy or ^{166}Er, one may observe the relaxation of $M_z(t)$ to the stable state before the collective oscillation of BEC breaks down, enabling the measurement of γ in BEC experiments.

The homogeneous-local-spin-orientation approximation is valid when the system size is on the order of the spin healing length ξ_s or less, which has been experimentally verified in in [12]. Using $c_0 \simeq 100\ |c_2|$ [2, 12], $\xi_s \simeq 10\,\xi_d$ where $\xi_d = \sqrt{\hbar^2/(2mc_0\bar{n})}$ is the density healing length and $\xi_s = \sqrt{\hbar^2/(2m|c_2|\bar{n})}$ is the spin healing length where $\bar{n}$ is the mean number density. Thus, if the size $2L_z$ of our quasi-1D spinor-dipolar BEC is on the order of $10\,\xi_d$, the homogeneous-local-spin-orientation approximation is justified.

Using the $s = 6$ element ^{166}Er, we can provide numerical values which satisfy both the quasi-1D and homogeneous-local-spin-orientation limits, as well as they enable us to explicitly show how t_{cr} depends on $(M_z)_{\text{in}}$ in a realizable setup. We consider below two cases: (A) box trap along z and (B) harmonic trap along z.

Box traps

We set $V(z) = 0$ for $|z| < L_z$ and ∞ otherwise. For small γ, we may write $n(z,t) = N/(2L_z)$ and hence we estimate $\mu \simeq \left(c_0 + 2\pi s^2 c_{dd}/3\right) N/\left(4\pi l_\perp^2 L_z\right)$ (see Appendix C).

In this case, we get $\Lambda'_{dd} = \Lambda_{dd}(L_z/l_\perp)$ with

$$\Lambda_{dd}(\lambda) = \frac{3Nc_{dd}}{2\hbar l_\perp^3}\frac{1}{\lambda}\left\{\int_0^{2\lambda} dv\ \left(1-\frac{v}{2\lambda}\right)G(v)-\frac{2}{3}\right\} \simeq \frac{Nc_{dd}}{2\hbar l_\perp^3}\frac{1}{\lambda} \quad \text{for } \lambda = L_z/l_\perp \gg 1. \tag{2.19}$$

Fixing $B_z = -0.03\,\mathrm{mG}$ and $N = 100$, we consider the following two cases: (1) $\omega_\perp/(2\pi) = 2.4\times10^4\,\mathrm{Hz}$ and $L_z = 3.125\,\mu\mathrm{m}$. Then $L_z/l_\perp = 62.03$, $\mu/(\hbar\omega_\perp) = 0.1692$, and $L_z/\xi_d = 29.55$. Thus, the system is in both the quasi-1D and homogeneous-local-spin-orientation limit. $s\Lambda_{dd}(L_z/l_\perp) = 4.074\times10^3\,\mathrm{Hz}$, $\hbar s\Lambda_{dd}(L_z/l_\perp)/(g_F\mu_B) = 0.3969\,\mathrm{mG}$, and $\theta_{\mathrm{cr}} := \cos^{-1}(M_z)_{\mathrm{cr}}$ is $85.67\,^\circ$.

(2) $\omega_\perp/(2\pi) = 1.2\times10^4\,\mathrm{Hz}$ and $L_z = 6.250\,\mu\mathrm{m}$. Then $L_z/l_\perp = 87.72$, $\mu/(\hbar\omega_\perp) = 0.0846$, and $L_z/\xi_d = 29.55$. Thus, again the system is in both the quasi-1D and homogeneous-local-spin-orientation limits. Specifically, $s\Lambda_{dd}(L_z/l_\perp) = 1.028\times10^3\,\mathrm{Hz}$, $\hbar s\Lambda_{dd}(L_z/l_\perp)/(g_F\mu_B) = 0.1002\,\mathrm{mG}$, and $\theta_{\mathrm{cr}} = 72.57\,^\circ$. Fig. 2.3 shows the relation between t_{cr} and $(M_z)_{\mathrm{in}}$.

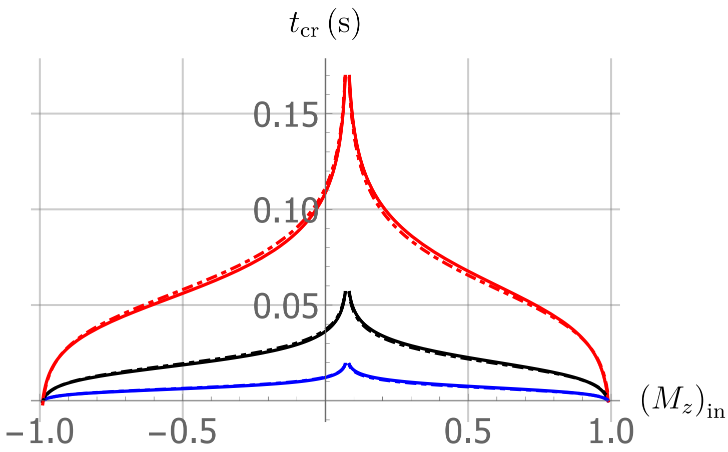
Harmonic traps

We set $V(z) = m\omega_z^2 z^2/2$. Using the Thomas-Fermi approximation, $\mu = m\omega_z^2 L_z^2/2$ where (see Appendix C)

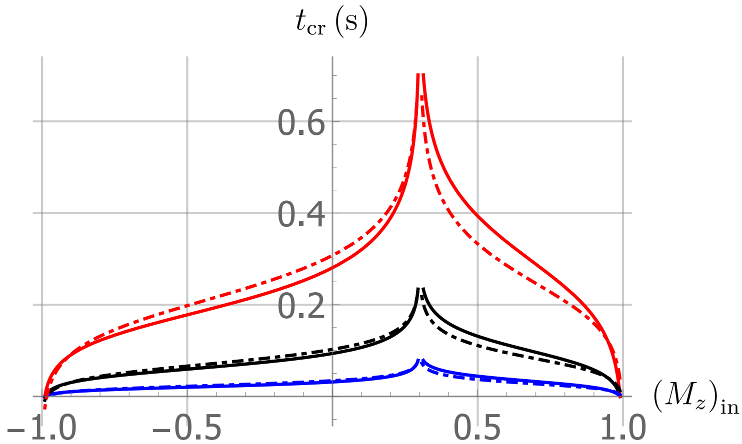
$$L_z = \left\{\frac{3\left(c_0 + 2\pi s^2 c_{dd}/3\right)N}{4\pi m\omega_z^2 l_\perp^2}\right\}^{1/3}, \tag{2.20}$$

and for small γ, we may write $\left(c_0 + 2\pi s^2 c_{dd}/3\right)n(z,t)/\left(\pi l_\perp^2\right) = m\omega_z^2\left(L_z^2 - z^2\right)$ for $|z| \leq L_z$ and $n(z,t) = 0$ for $|z| > L_z$. From this $n(z,t)$, we performed a numerical integration to calculate Λ'_{dd}. Fixing $B_z = -0.03\,\mathrm{mG}$, we consider the following two cases:

(1) $N = 240$, $\omega_\perp/(2\pi) = 2000\,\mathrm{Hz}$, and $\omega_z/(2\pi) = 50\,\mathrm{Hz}$, for which $L_z = 5.703\,\mu\mathrm{m}$ and $L_z/l_\perp = 32.68$. We obtain again the quasi-1D and homogeneous-local-spin-orientation limits since $\mu/(\hbar\omega_\perp) = 0.3337$ and $L_z/\xi_d = 17.85$. Furthermore, $s\Lambda'_{dd} = 1.644\times10^3\,\mathrm{Hz}$, $\hbar s\Lambda'_{dd}/(g_F\mu_B) = 1.602\times10^{-1}\,\mathrm{mG}$, and $\theta_{\mathrm{cr}} = 79.21\,^\circ$.



$\omega_\perp / (2\pi) = 2.4 \times 10^4$ Hz, $L_z = 3.125\,\mu$m, and $l_\perp = 0.0504\,\mu$m where
$N/\left(4\pi L_z l_\perp^2\right) = 10.03 \times 10^{20}\,\mathrm{m}^{-3}$ ($(M_z)_{\rm cr} = 0.0756$).



$\omega_\perp / (2\pi) = 1.2 \times 10^4$ Hz, $L_z = 6.250\,\mu$m, and $l_\perp = 0.0712\,\mu$m where
$N/\left(4\pi L_z l_\perp^2\right) = 2.508 \times 10^{20}\,\mathrm{m}^{-3}$ ($(M_z)_{\rm cr} = 0.2995$).

Figure 2.3: $t_{\rm cr}$ as a function of $(M_z)_{\rm in}$ when $\boldsymbol{B} = B_z \boldsymbol{e}_z$ where $B_z = -0.03\,\mathrm{mG}$ and particle number $N = 100$. $\gamma = 0.01$, 0.03, and 0.09 for red, black, and blue lines respectively. Lines are calculated from exact analytic formula in Eq. (2.17), and dot-dashed are from asymptotic expression in Eq. (2.18). These figures are from our paper [11].

(2) $N = 340, \omega_\perp / (2\pi) = 1000\,\text{Hz}$, and $\omega_z / (2\pi) = 25\,\text{Hz}$, where $L_z = 8.070\,\mu\text{m}$ and $L_z/l_\perp = 32.70$. Again, we have the quasi-1D and with homogeneous-local-spin-orientation limits fulfilled due to $\mu / (\hbar\omega_\perp) = 0.3341$ and $L_z/\xi_d = 17.87$. In addition, $s\Lambda'_{dd} = 8.230 \times 10^2\,\text{Hz}$, $\hbar s \Lambda'_{dd} / (g_F \mu_B) = 8.019 \times 10^{-2}\,\text{mG}$, and $\theta_{\text{cr}} = 68.03\,°$.

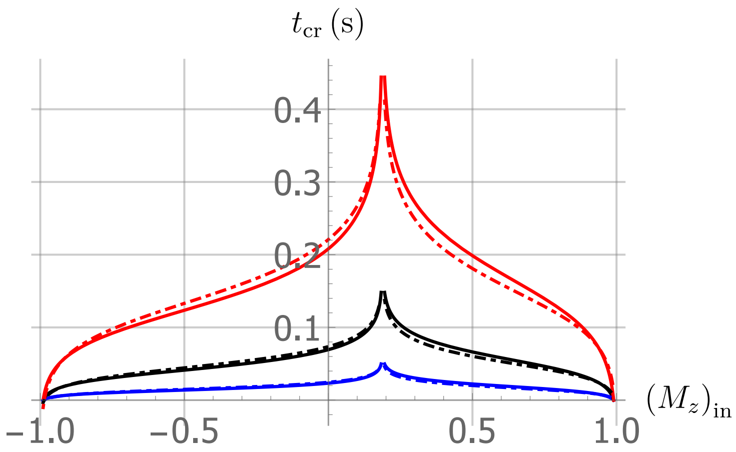
Fig. 2.4 shows for the harmonic traps the relation between t_{cr} and $(M_z)_{\text{in}}$.

Measurability of critical switching time

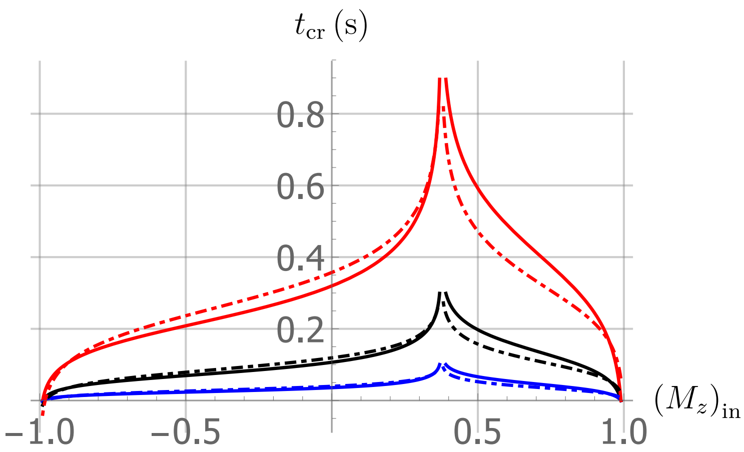
Figs. 2.3 and 2.4 demonstrate that the critical switching time t_{cr} is much smaller than the lifetime of the collective oscillation of BEC (several seconds [66]) and thus, by measuring t_{cr} by varying $(M_z)_{\text{in}}$, one will be able to obtain the value of γ provided that γ indeed does not depend on spin indices as [9, 10] have assumed. Conversely, if one obtains from the measurements a different functional relation which does not follow Eq. (2.17), it implies that γ may depend on spin indices.

Note that both figures, Figs. 2.3 and 2.4, show that t_{cr} is *inversely* proportional to the mean number density $N / \left(4\pi L_z l_\perp^2\right)$. Eq. (2.18) states that t_{cr} is inversely proportional to Λ'_{dd}, but except for the box trap case, in which one can analytically calculate $\Lambda'_{dd} = \Lambda_{dd}(L_z/l_\perp)$, the dependence of Λ'_{dd} and the mean number density $N / \left(4\pi L_z l_\perp^2\right)$ is not immediately apparent. Thus, at least for harmonic traps, and in the Thomas-Fermi approximation, one may use the box trap results to provide an approximate estimate of the behavior of t_{cr}.

This ends our first topic. In the next chapter, we will show our to-be-submitted work on the theoretical possibilities of BEC as sensors using its ultracold chemical reaction.



$N = 240$, $\omega_\perp/(2\pi) = 2000\,\mathrm{Hz}$, and $\omega_z/(2\pi) = 50\,\mathrm{Hz}$. $L_z = 5.703\,\mu\mathrm{m}$ and $l_\perp = 0.1745\,\mu\mathrm{m}$ where $N/\left(4\pi L_z l_\perp^2\right) = 1.010 \times 10^{20}\,\mathrm{m}^{-3}$ ($(M_z)_{\rm cr} = 0.1873$).



$N = 340$, $\omega_\perp/(2\pi) = 1000\,\mathrm{Hz}$, and $\omega_z/(2\pi) = 25\,\mathrm{Hz}$. $L_z = 8.070\,\mu\mathrm{m}$ and $l_\perp = 0.2468\,\mu\mathrm{m}$ where $N/\left(4\pi L_z l_\perp^2\right) = 0.550 \times 10^{20}\,\mathrm{m}^{-3}$ ($(M_z)_{\rm cr} = 0.3741$).

Figure 2.4: $t_{\rm cr}$ as a function of $(M_z)_{\rm in}$ when $\boldsymbol{B} = B_z\boldsymbol{e}_z$ where $B_z = -0.03\,\mathrm{mG}$, for two particle numbers N as shown. $\gamma = 0.01$, 0.03, and 0.09 for red, black, and blue lines respectively. Lines are calculated from exact analytic formula in Eq. (2.17), and dot-dashed are from asymptotic expression in Eq. (2.18). These figures are from our paper [11].

Chapter 3

BEC as Detector – Ultracold Chemical Reaction

Now we will show how accurately one can estimate the magnitude of the external perturbation by measuring number of BEC molecules created by ultracold chemical reaction.

Previous work on the possibility of using BEC as sensors to measure various physical quantities [13, 14, 15, 16, 17, 18, 19] rely on measuring number of phonons in BEC. However, it is difficult to measure number of phonons in BEC [21]. So far, it has been only achieved in the superfluid helium II experiment [20]. Measuring number of BEC molecules under ultracold chemical reaction can be done with spectroscopic techniques [22, 23] and already achieved in, for example, Cs - Cs_2 BEC under ultracold chemical reaction [25]. Thus we focus on the theoretical limit of estimating the magnitude of the external homogeneous density perturbation by measuring number of BEC molecules.

This chapter is composed of three sections. First section, 3.1, is to introduce the difference between ordinary chemical reaction and ultracold chemical reaction. Starting from the Hamiltonian in Eq. (1.5), we define the reaction rate operator $\hat{R}(t) := \{\hbar/(g_a n)\}\,\partial\hat{N}_a(t)/\partial t$ which describes the change of the number of BEC atoms as a function of time t and show that $\hat{R}(t)$ can be written as pair creation and annihilation of quasiparticles which we will name as reactons. In the final section 3.3 we set our system so that mean-field contributions to the ultracold chemical reaction is zero to see characteristics of reactons. Then we calculate Fisher information for estimating the magnitude of the external perturbations to see its theoretical accuracy. At the end of this chapter, we briefly discuss benefits of our approach by comparing previous theoretical works on BEC as sensors by measuring number of phonons in BEC.

3.1 Short Introduction on Ultracold Chemical Reaction

Ordinary chemical reaction has activation energy which should be overcome to make chemical reaction occur as is shown in Fig. 3.1. Note that due to thermal energy in the background, forward and backward reactions occur at the same time and they eventually reach the dynamical equilibrium where the ratio of atoms does not change.

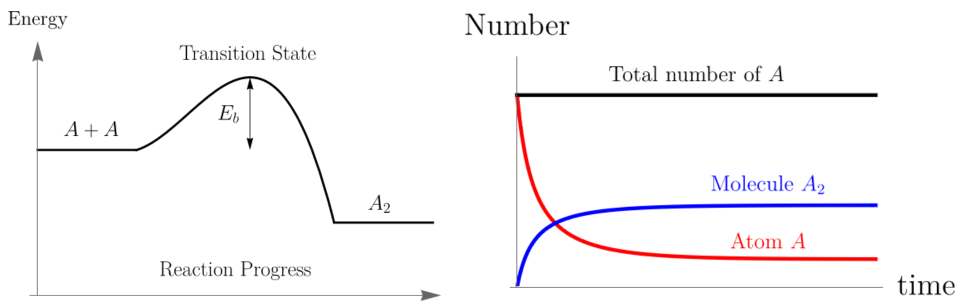


Figure 3.1: Schematics of ordinary chemical reaction $A + A \rightarrow A_2$ (left) and the change of ratio as a function of time (right). In the right panel, red line represents the number of atoms (excluding atoms in molecules), blue line represents the number of molecules, and black line represents the total number of A which is conserved (one molecule has two A).

In the BEC ultracold chemical reaction, bound state (BEC molecules) and free state (BEC atoms) can be in resonance by changing external magnetic field. We call this as Feshbach resonance, and in this state, number of atoms and number of molecules oscillate as time goes on (see Fig. 3.2). [25] observed that Cs_2 BEC molecules are created by ultracold chemical reaction in Cs BEC.

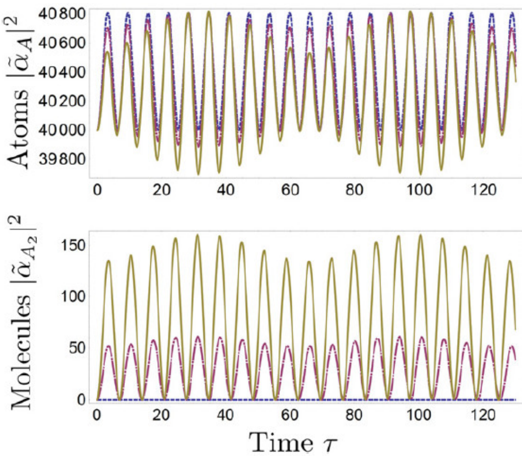


Figure 3.2: Number of atoms and molecules in ultracold chemical reaction as a function of time. This figure is from [30].

3.2 Reaction Rate Operator and Reactons

In Heisenberg picture, let $\hat{N}_j(t) := \int d^3r\ \hat{\psi}_j^\dagger(\boldsymbol{r},t)\,\hat{\psi}_j(\boldsymbol{r},t)$ where $j = a, m$. From Eq. (1.5),

$$\begin{aligned} i\hbar\frac{\partial\hat{\psi}_a(\boldsymbol{r},t)}{\partial t} &= \left\{-\frac{\hbar^2}{2m_a}\nabla^2 + V_a(\boldsymbol{r},t) + g_a\hat{\psi}_a^\dagger(\boldsymbol{r},t)\,\hat{\psi}_a(\boldsymbol{r},t) + g_{am}\hat{\psi}_m^\dagger(\boldsymbol{r},t)\,\hat{\psi}_m(\boldsymbol{r},t)\right\}\hat{\psi}_a(\boldsymbol{r},t) \\ &\quad +\alpha\sqrt{2}\hat{\psi}_a^\dagger(\boldsymbol{r},t)\,\hat{\psi}_m(\boldsymbol{r},t)\,, \\ i\hbar\frac{\partial\hat{\psi}_m(\boldsymbol{r},t)}{\partial t} &= \left\{-\frac{\hbar^2}{2m_m}\nabla^2 + V_m(\boldsymbol{r},t) + \epsilon + g_m\hat{\psi}_m^\dagger(\boldsymbol{r},t)\,\hat{\psi}_m(\boldsymbol{r},t) + g_{am}\hat{\psi}_a^\dagger(\boldsymbol{r},t)\,\hat{\psi}_a(\boldsymbol{r},t)\right\}\hat{\psi}_m(\boldsymbol{r},t) \\ &\quad +\frac{\alpha}{\sqrt{2}}\left\{\hat{\psi}_a(\boldsymbol{r},t)\right\}^2, \end{aligned} \tag{3.1}$$

and we get

$$
\begin{aligned}
i\hbar\frac{\partial\hat{N}_a(t)}{\partial t} &= \alpha\sqrt{2}\int d^3r\ \left\{\hat{\psi}_a^\dagger(\boldsymbol{r},t)\,\hat{\psi}_a^\dagger(\boldsymbol{r},t)\,\hat{\psi}_m(\boldsymbol{r},t) - h.c.\right\} \\
&= -2i\hbar\frac{\partial\hat{N}_m(t)}{\partial t}.
\end{aligned}
\tag{3.2}
$$

We define the dimensionless *reaction rate operator* $\hat{R}(t)$ by

$$
\hat{R}(t) \coloneqq \frac{\hbar}{g_a n}\frac{\partial\hat{N}_a(t)}{\partial t} = \frac{\partial\hat{N}_a(t)}{\partial\tilde{t}} = i\frac{\alpha\sqrt{2}}{g_a n}\int d^3r\ \left\{\hat{\psi}_a(\boldsymbol{r},t)\,\hat{\psi}_a(\boldsymbol{r},t)\,\hat{\psi}_m^\dagger(\boldsymbol{r},t) - h.c.\right\},
\tag{3.3}
$$

where $\tilde{t} \coloneqq g_a n t/\hbar$ is dimensionless time. This $\hat{R}(t)$ describes how many atoms are converted into molecules per time.

By using expansions in Eqs. (1.28),

$$
\begin{aligned}
\hat{R}(t) &= 2N\tilde{\alpha}x^2 y\sin\varphi_{am} \\
&\quad -2i\tilde{\alpha}x\sum_{\boldsymbol{k}\neq 0}\left\{e^{-i\varphi_a}\delta\hat{\Psi}_a^\dagger(\boldsymbol{k},t)\,\delta\hat{\Psi}_m(\boldsymbol{k},t) - e^{i\varphi_a}\delta\hat{\Psi}_a(\boldsymbol{k},t)\,\delta\hat{\Psi}_m^\dagger(\boldsymbol{k},t)\right\} \\
&\quad -i\tilde{\alpha}y\sum_{\boldsymbol{k}\neq 0}\left\{e^{i\varphi_m}\delta\hat{\Psi}_a^\dagger(\boldsymbol{k},t)\,\delta\hat{\Psi}_a^\dagger(-\boldsymbol{k},t) - e^{-i\varphi_m}\delta\hat{\Psi}_a(\boldsymbol{k},t)\,\delta\hat{\Psi}_a(-\boldsymbol{k},t)\right\} \\
&\quad -i\frac{\tilde{\alpha}}{\sqrt{N}}\sum_{\boldsymbol{k}_1\neq 0}\sum_{\boldsymbol{k}_2\neq 0}\left\{\delta\hat{\Psi}_a^\dagger(\boldsymbol{k}_1,t)\,\delta\hat{\Psi}_a^\dagger(\boldsymbol{k}_2,t)\,\delta\hat{\Psi}_m(\boldsymbol{k}_1+\boldsymbol{k}_2,t) - h.c.\right\},
\end{aligned}
\tag{3.4}
$$

and the mean-field contribution $2N\tilde{\alpha}x^2 y\sin\varphi_{am}$ can be set to be zero by minimizing the mean-field Hamiltonian (see Eq. (1.22)).

Let

$$
\begin{aligned}
\hat{R}_2(t) &\coloneqq -2i\tilde{\alpha}x\sum_{\boldsymbol{k}\neq 0}\left\{e^{-i\varphi_a}\delta\hat{\Psi}_a^\dagger(\boldsymbol{k},t)\,\delta\hat{\Psi}_m(\boldsymbol{k},t) - e^{i\varphi_a}\delta\hat{\Psi}_a(\boldsymbol{k},t)\,\delta\hat{\Psi}_m^\dagger(\boldsymbol{k},t)\right\} \\
&\quad -i\tilde{\alpha}y\sum_{\boldsymbol{k}\neq 0}\left\{e^{i\varphi_m}\delta\hat{\Psi}_a^\dagger(\boldsymbol{k},t)\,\delta\hat{\Psi}_a^\dagger(-\boldsymbol{k},t) - e^{-i\varphi_m}\delta\hat{\Psi}_a(\boldsymbol{k},t)\,\delta\hat{\Psi}_a(-\boldsymbol{k},t)\right\}.
\end{aligned}
\tag{3.5}
$$

Then for $N \gg 1$ and neglecting mean-field contribution to $\hat{R}(t)$, $\hat{R}(t) \simeq \hat{R}_2(t)$.

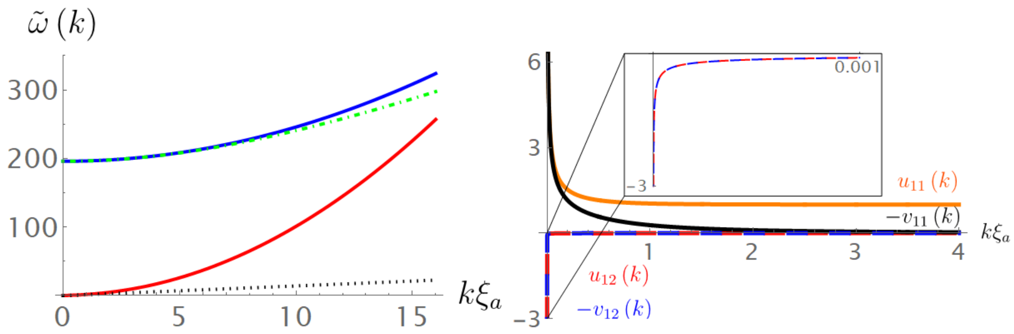


Figure 3.3: (Left) $\tilde{\omega}_p(k)$ as a function of $k\xi_a$ where ξ_a is the density healing length of the BEC atom. Red line shows $\tilde{\omega}_1(k)$ and blue line shows $\tilde{\omega}_2(k)$. The expansion of $\tilde{\omega}_1(k)$ up to $O(k)$ (black dotted) and $\tilde{\omega}_2(k)$ up to $O(k^2)$ (green dot-dashed) are also shown. (Right) Bogoliubov mode amplitudes in Eqs. (1.53) as a function of $k\xi_a$.

Imposing Bogoliubov transformation in Eq. (1.35) leads us to

$$\begin{aligned}\hat{R}_2(t) &= \tilde{\alpha}\sum_{\boldsymbol{k}\neq 0}\mathrm{Im}\{\Xi_{11}(k)\}\left\{2\hat{b}_1^\dagger(\boldsymbol{k},t)\hat{b}_1(\boldsymbol{k},t)+1\right\}+\tilde{\alpha}\sum_{\boldsymbol{k}\neq 0}\mathrm{Im}\{\Xi_{22}(k)\}\left\{2\hat{b}_2^\dagger(\boldsymbol{k},t)\hat{b}_2(\boldsymbol{k},t)+1\right\}\\ &-2i\tilde{\alpha}\sum_{\boldsymbol{k}\neq 0}\left\{\Xi_{12}(k)\hat{b}_1^\dagger(\boldsymbol{k},t)\hat{b}_2(\boldsymbol{k},t)-h.c.\right\}-i\tilde{\alpha}\sum_{\boldsymbol{k}\neq 0}\left\{\Xi_{13}(k)\hat{b}_1^\dagger(\boldsymbol{k},t)\hat{b}_1^\dagger(-\boldsymbol{k},t)-h.c.\right\}\\ &-2i\tilde{\alpha}\sum_{\boldsymbol{k}\neq 0}\left\{\Xi_{14}(k)\hat{b}_1^\dagger(\boldsymbol{k},t)\hat{b}_2^\dagger(-\boldsymbol{k},t)-h.c.\right\}-i\tilde{\alpha}\sum_{\boldsymbol{k}\neq 0}\left\{\Xi_{24}(k)\hat{b}_2^\dagger(\boldsymbol{k},t)\hat{b}_2^\dagger(-\boldsymbol{k},t)-h.c.\right\},\end{aligned} \tag{3.6}$$

where we define $\Xi_{pq}(k)$ in Appendix E.

To simplify the expression in Eq. (3.6), we choose our parameter to be $\tilde{g}_{am}=100$, $\tilde{g}_m=0$, and $\tilde{\alpha}=1$ which gives $y=0.0026$ and $\omega_1(k)\ll\omega_2(0)$ within our k range (see Fig. 3.3) such that effects of $\hat{b}_2(\boldsymbol{k},t)$ can be neglected. Then we have

$$\hat{R}_2(t)=\tilde{\alpha}\sum_{\boldsymbol{k}\neq 0}\mathrm{Im}\{\Xi_{11}(k)\}\left\{2\hat{b}_1^\dagger(\boldsymbol{k},t)\hat{b}_1(\boldsymbol{k},t)+1\right\}-i\tilde{\alpha}\sum_{\boldsymbol{k}\neq 0}\left\{\Xi_{13}(k)\hat{b}_1^\dagger(\boldsymbol{k},t)\hat{b}_1^\dagger(-\boldsymbol{k},t)-h.c.\right\}. \tag{3.7}$$

With our parameters where Bogoliubov excitation energies are all real, $\varphi_a=0$, and $\cos\varphi_m=-\mathrm{sign}(\alpha)$, $\Xi_{11}(k)=0$ and $\Xi_{13}(k)$ is real (see Appendix E.1. More specifically, Eqs. (E.19) and (E.29)). We plot $\alpha\Xi_{13}(k)$ as a function of $\tilde{g}_{am}$ and $\tilde{\alpha}$ in Fig. 3.4. Note that reaction rate is symmetric in α, which implies that the sign of α does not change the direction of ultracold chemical reaction and only the magnitude of α is important. This effect cannot be seen if mean-field contribution (which is proportional to α) is not zero.

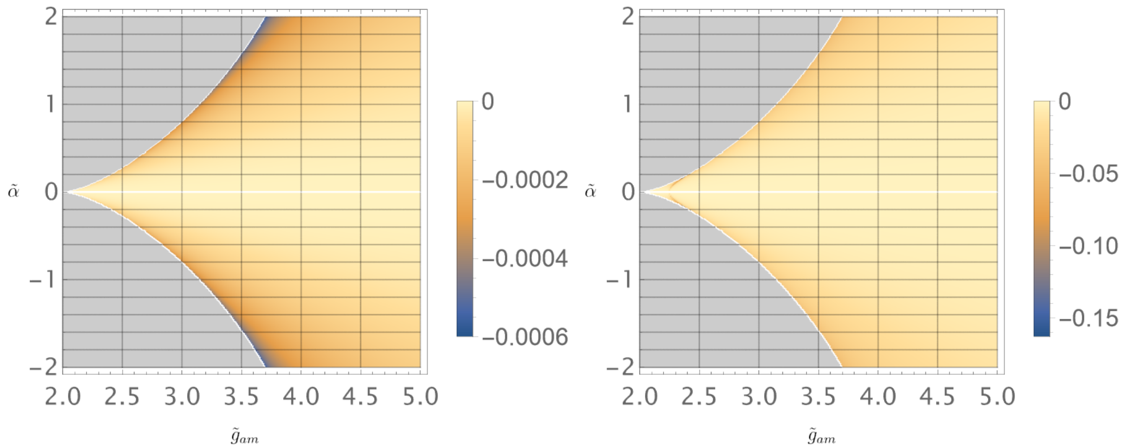


Figure 3.4: $\alpha\Xi_{13}(k)$ as a function of $\tilde{g}_{am}$ and $\tilde{\alpha}$ for $k = 2\pi/L$ (left) and $k = 20\pi/L$ (right) where L is the length of one-dimensional BEC system with $L/\xi_a = 161.3$. System is not stable in grey regions (number of BEC atoms or molecules are negative or Bogoliubov excitation energies are imaginary in those regions).

3.3 BEC as Detector - Estimating External Perturbation

After applying Bogoliubov transformation and neglecting effects of $\hat{b}_2(\boldsymbol{k}, t)$ as previous section, in Schrödinger picture, our Hamiltonian $\hat{H}$ is (see Eq. (1.52))

$$\hat{H}-\mu\hat{N} = H_0-\mu N-\frac{g_a n}{2}\sum_{\boldsymbol{k}\neq 0}\left\{M_{11}(k)+M_{22}(k)\right\}$$
$$+\hbar\sum_{\boldsymbol{k}\neq 0}\omega_1(k)\frac{\hat{b}_1^\dagger(\boldsymbol{k})\hat{b}_1(\boldsymbol{k})+\hat{b}_1(\boldsymbol{k})\hat{b}_1^\dagger(\boldsymbol{k})}{2}+O\left(\delta\hat{\Psi}_j^3\right), \quad (3.8)$$

which is constant in time t.

In Schrödinger picture, if we apply external perturbation $\hat{V}_S(t)$ from $t=0$ where

$$\hat{V}_S(t)=V_a f(t)\sum_{\boldsymbol{k}\neq 0}\left[\mathbb{V}_1(k)+\mathbb{V}_2(k)\hat{b}_1^\dagger(\boldsymbol{k})\hat{b}_1(\boldsymbol{k})+\left\{\mathbb{V}_3(k)\hat{b}_1^\dagger(\boldsymbol{k})\hat{b}_1^\dagger(-\boldsymbol{k})+h.c.\right\}\right], \quad (3.9)$$

it would affect the ultracold chemical reaction and hence the number of molecules created by ultracold chemical reaction will change.

As an example, we consider density perturbation

$$\hat{V}_S(t)=V_a f(t)\int d^3r\ \left\{\hat{\psi}_a^\dagger(\boldsymbol{r})\hat{\psi}_a(\boldsymbol{r})+2\hat{\psi}_m^\dagger(\boldsymbol{r})\hat{\psi}_m(\boldsymbol{r})\right\}, \quad (3.10)$$

and define $\bar{V}_a := V_a/(g_a n)$. Then we calculate quantum Fisher information $I_Q(t)$ and the lower bound of the classical Fisher information $I_C(t)$ when estimating the value of $\bar{V}_a = 10^{-6}$ by measuring number of BEC molecules created by ultracold chemical reaction.

In the Appendix F.5.3 (specifically, in Eqs. (F.68) and (F.69)), we show that if the perturbation is sinusoidal in time, k component which satisfy $\omega_a = 2\omega_1(k)$ dominates in the calculation of $I_Q(t)$ or $I_C(t)$. And Fig. 3.3 shows that magnitudes of Bogoliubov $U(k)$ and $V(k)$ have maximum value as $k\to 0$. Thus, for $f(t)=\sin(\omega_a t)$, we set $\omega_a = 2\omega_1(k=2\pi/L) = 2.7\times 10^3$Hz to maximize both $I_Q(t)$ and $I_C(t)$. Then our calculation shows that the Fisher information calculated by using Dyson series expansion of the time evolution operator up to $O\left(\bar{V}_a^2\right)$ is within 0.06% of the Fisher information calculated by using symplectic formalism of the time evolution operator (see Fig. 3.5), and as t increases, Fisher information calculated by using Dyson series expansion and Fisher information calculated by using symplectic formalism gives

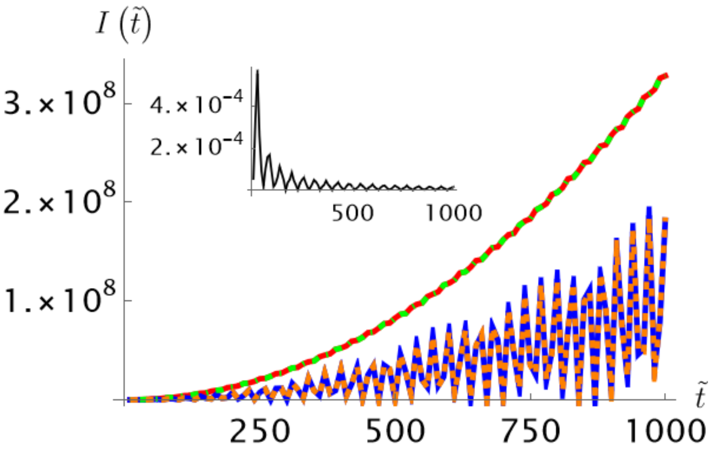


Figure 3.5: QFI (red line) and the lower bound of the CFI (blue line) calculated by using Dyson series expansion up to $O\left(\bar{V}_a^2\right)$, and QFI (green dashed) and the lower bound of the CFI (orange dot-dashed) calculated by using symplectic formalism as functions of dimensionless time $\tilde{t} := g_a n t/\hbar$ with perturbation profiles $f(t) = \sin(\omega_a t)$. (Inset) (Fisher information from Dyson series expansion up to $O\left(\bar{V}_a^2\right)$ - Fisher information from symplectic formalism) / (Fisher information from symplectic formalism) as a function of dimensionless time $\tilde{t}$.

same value. Hence one may use Dyson series expansion to see long-time-behavior of Fisher information (For example, Eqs. (F.68) and (F.69) imply that $I_Q(t) \propto t^2$ as $t \to \infty$ if the perturbation is sinusoidal in time t).

In Fig. 3.6, we calculated $I_Q(t)$ and $I_C(t)$ for various $f(t)$. It shows that $I_C\left(\tilde{t}\right)/I_Q\left(\tilde{t}\right)$ oscillates and its maximum value is about 0.6 for every $f(t)$ we considered. As we only calculated the lower bound of the CFI, CFI can be within 0.6 QFI $\leq$ CFI $\leq$ QFI. From Cramér-Rao theorem, for single measurement, the variance of estimating $\bar{V}_a$ by measuring number of BEC molecules created by ultracold chemical reaction is equal to or greater than $1/$CFI, and the ideal variance of this estimation is equal to or greater than $1/$QFI. Thus our result implies that measuring number of BEC molecules created by ultracold chemical reaction might be an ideal method to estimate the magnitude of

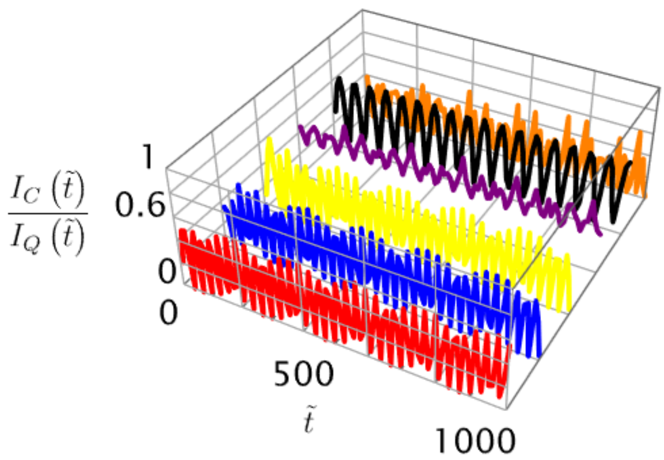


Figure 3.6: The lower bound of the CFI ($I_C(\tilde{t})$) over QFI ($I_Q(\tilde{t})$) for density perturbation as a function of dimensionless time $\tilde{t} := g_a n t/\hbar$ with perturbation profiles $f(t) = \sin(\omega_a t)$ for $t \geq 0$ (Red line), $f(t) = \cos(\omega_a t)$ for $t \geq 0$ (Blue line), $f(t) = \{1 - \cos(\omega_a t)\}/2$ for $t \geq 0$ (Yellow line), $f(t) = \{1 + \cos(\omega_a t)\}/2$ for $t \geq 0$ (Purple line), $f(t) = \theta(t)$ for $t \geq 0$ (Black line), and $f(t) = \delta(t)$ (Orange line). Here, we set $\omega_a = 2\omega_1 (k = 2\pi/L) = 2.7 \times 10^3$Hz to maximize both $I_Q(\tilde{t})$ and $I_C(\tilde{t})$.

the homogeneous density perturbation.

Conclusion

In this thesis, the theoretical possibilities that Bose-Einstein condensates (BEC) can act as a sensor for measuring damping parameter and estimating the magnitude of the external homogeneous density perturbation are discussed.

In chapter 2, we have studied the limitation of the conventional mean-field theory in BEC with dissipation and proposed a new method to measure the damping parameter by using Stoner-Wohlfarth swithing in spinor-dipolar BEC with homogeneous local spin orientation [11]. By starting from the conventional mean-field theory and assuming that it is valid for spinor-dipolar BEC with homogeneous local spin orientation, we derived Landau-Lifshitz-Gilbert (LLG) equation and Stoner-Wohlfarth Hamiltonian and show that the direction of the magnetization is switched to $\pm z$ axis when the external magnetic field is applied on z axis. By solving LLG equation, we obtained the formula for the switching time as a function of the damping parameter γ and it suggests that (1) γ can be measured by measuring the switching time and (2) the validity of the conventional mean-field theory on spinor-dipolar BEC can be checked by comparing the measured swtiching time and our formula for the switching time. This will help us to see whether γ is the universal parameter as is usually assumed ($\gamma \simeq 0.03$ and does not depend on spin) and it will provide useful information on studies with BEC simulations.

We show in chapter 3 that the magnitude of external perturbation can be estimated by measuring number of BEC molecules created by ultracold chemical reaction. Starting from the Hamiltonian for ultracold chemical reaction suggested in [30], we obtain reaction rate operator and show that it can be expressed as pair creation and annihilation of quasiparticles, which we named as reactons. To show its characteristics, we set our system such that mean-field effects are zero and then calculate Fisher information for estimating the magnitude of external homogeneous density perturbation by measuring number of BEC molecules created by ultracold chemical reaction. As it is difficult to measure number of phonons in BEC experiment [21], our proposal is expected to be more easy to apply than previous suggestions which rely on measuring number of phonons in BEC [13, 14, 15, 16, 17, 18, 19].

As we mentioned in the introduction, BEC can be used to study quark confinement simulation [26], simulate early universe [27],and measure various physical quantities [13, 14, 15, 16, 17, 18, 19]. So there would be many other possibilities to apply BEC as laboratories for fundamental physics. We hope that our thesis contribute to spread this application.

Chapter A

Quasi-1D Gross-Pitaevskiĭ equation with dissipation

We start from our conventional mean-field equation for the spinor-dipolar BEC with dissipation in Eq. (2.2):

$$
\begin{aligned}
(i-\gamma)\hbar\frac{\partial\psi(\boldsymbol{r},t)}{\partial t} &= \left[-\frac{\hbar^2}{2m}\nabla^2+V_{\mathrm{tr}}(\boldsymbol{r})+c_0 n(\boldsymbol{r},t)+\sum_{j=1}^{s}c_{2j}\sum_{\nu_1,\cdots,\nu_j=x,y,z}F_{\nu_1,\cdots,\nu_j}(\boldsymbol{r},t)\left(\hat{f}_{\nu_1}\cdots\hat{f}_{\nu_j}\right)\right]\psi(\boldsymbol{r},t) \\
&\quad -\sum_{\nu=x,y,z}\left[g_F\mu_B B_\nu - c_{dd}\sum_{\nu'=x,y,z}\left\{\int d^3r'\; Q_{\nu,\nu'}\left(\boldsymbol{r}-\boldsymbol{r}'\right)F_{\nu'}\left(\boldsymbol{r}',t\right)\right\}\right]\hat{f}_\nu\psi(\boldsymbol{r},t). \qquad \text{(A.1)}
\end{aligned}
$$

We will use the following trap potential

$$
V_{\mathrm{tr}}(\boldsymbol{r})=\frac{1}{2}m\omega_\perp^2\left(x^2+y^2\right)+V(z), \qquad \text{(A.2)}
$$

and set our ansatz for the wavefunction $\psi(\boldsymbol{r},t)$ as (see Eq. (2.3))

$$
\psi(\boldsymbol{r},t)=\frac{e^{-\rho^2/\left(2l_\perp^2\right)}}{l_\perp\sqrt{\pi}}\Psi(z,t)\,\zeta(t)\,e^{-(i+\gamma)\omega_\perp t/\left(1+\gamma^2\right)}, \qquad \text{(A.3)}
$$

where $l_\perp := \sqrt{\hbar/(m\omega_\perp)}$ is the harmonic oscillator length in xy plane. Therefore, to get quasi-1D equations, we have to multiply $e^{-\rho^2/\left(2l_\perp^2\right)}/\left(l_\perp\sqrt{\pi}\right)$ (xy plane component of the wavefunction) before integrating out contributions of xy components.

From Eq. (2.4),

$$
\boldsymbol{F}(\boldsymbol{r},t)=s\frac{e^{-\rho^2/l_\perp^2}}{\pi l_\perp^2}\left|\Psi(z,t)\right|^2 e^{-2\gamma\omega_\perp t/\left(1+\gamma^2\right)}\boldsymbol{M}(t)=s\boldsymbol{M}(t)\,n(\boldsymbol{r},t). \qquad \text{(A.4)}
$$

A.1 Step 1. Dipole-Dipole Interaction Term

From the definition of $b_{dd}(\boldsymbol{r},t)$, one can get

$$\begin{aligned}
&\int d\boldsymbol{\rho}\,\frac{1}{l_\perp\sqrt{\pi}}e^{-\rho^2/\left(2l_\perp^2\right)}\left[c_{dd}\sum_{\nu'=x,y,z}\left\{\int d^3r'\,Q_{\nu,\nu'}\left(\boldsymbol{r}-\boldsymbol{r}'\right)F_{\nu'}\left(\boldsymbol{r}',t\right)\right\}\hat{f}_\nu\psi\left(\boldsymbol{r},t\right)\right]\\
&=\int d\boldsymbol{\rho}\,\frac{1}{l_\perp\sqrt{\pi}}e^{-\rho^2/\left(2l_\perp^2\right)}\left\{\hbar\boldsymbol{b}_{dd}\left(\boldsymbol{r}\right)\cdot\hat{\boldsymbol{f}}\right\}\psi\left(\boldsymbol{r},t\right)\\
&=\frac{c_{dd}}{\pi l_\perp^2}\int dz'\int d\boldsymbol{\rho}\int d\boldsymbol{\rho'}\sum_{\nu,\nu'=x,y,z}Q_{\nu,\nu'}\left(\boldsymbol{r}-\boldsymbol{r'}\right)s\frac{e^{-\rho'^2/l_\perp^2}}{\pi l_\perp^2}\left|\Psi\left(z',t\right)\right|^2e^{-2\gamma\omega_\perp t/\left(1+\gamma^2\right)}\\
&\quad\times M_{\nu'}\left(t\right)f_\nu e^{-\rho^2/l_\perp^2}\Psi\left(z,t\right)\zeta\left(t\right)e^{-(i+\gamma)\omega_\perp t/\left(1+\gamma^2\right)}\\
&=\frac{c_{dd}}{\pi^2 l_\perp^4}\int dz'\,n\left(z',t\right)\int d\boldsymbol{\rho}\int d\boldsymbol{\rho'}\sum_{\nu,\nu'=x,y,z}Q_{\nu,\nu'}\left(\boldsymbol{r}-\boldsymbol{r'}\right)M_{\nu'}\left(t\right)\\
&\quad\times e^{-\rho'^2/l_\perp^2}e^{-\rho^2/l_\perp^2}\Psi\left(z,t\right)e^{-\frac{i+\gamma}{1+\gamma^2}\omega_\perp t}sf_\nu\zeta\left(t\right),
\end{aligned}\tag{A.5}$$

where $\boldsymbol{\rho}\coloneqq(x,y)$ with $\rho\coloneqq\sqrt{x^2+y^2}$, $\tan\phi_\rho\coloneqq y/x$, and $\int d\boldsymbol{\rho}\coloneqq\int d\phi_\rho\int d\rho\;\rho=\int dx\int dy$.

Due to the symmetry, after performing angular part of $\int d\boldsymbol{\rho}\int d\boldsymbol{\rho'}$, nonzero term of $\sum_{\nu'=x,y,z}Q_{\nu,\nu'}\left(\boldsymbol{r}-\boldsymbol{r'}\right)M_{\nu'}\left(t\right)$ is

$$\begin{aligned}
&\left\{\sqrt{\frac{2\pi}{15}}\frac{1}{\eta^3}\sqrt{6}\sin\theta\left(t\right)\cos\phi\left(t\right)\delta_{\nu,x}+\sqrt{\frac{2\pi}{15}}\frac{1}{\eta^3}\sqrt{6}\sin\theta\left(t\right)\sin\phi\left(t\right)\delta_{\nu,y}-4\sqrt{\frac{\pi}{5}}\frac{1}{\eta^3}\cos\theta\left(t\right)\delta_{\nu,z}\right\}Y_2^0\left(\boldsymbol{e_\eta}\right)\\
&=\left\{\sqrt{\frac{2\pi}{15}}\frac{1}{\eta^3}\sqrt{6}\sin\theta\left(t\right)\cos\phi\left(t\right)\delta_{\nu,x}+\sqrt{\frac{2\pi}{15}}\frac{1}{\eta^3}\sqrt{6}\sin\theta\left(t\right)\sin\phi\left(t\right)\delta_{\nu,y}-4\sqrt{\frac{\pi}{5}}\frac{1}{\eta^3}\cos\theta\left(t\right)\delta_{\nu,z}\right\}\\
&\qquad\times\frac{1}{4}\sqrt{\frac{5}{\pi}}\left(3\frac{\eta_z^2}{\eta^2}-1\right)\\
&=\left\{\frac{1}{2}\sin\theta\left(t\right)\cos\phi\left(t\right)\delta_{\nu,x}+\frac{1}{2}\sin\theta\left(t\right)\sin\phi\left(t\right)\delta_{\nu,y}-\cos\theta\left(t\right)\delta_{\nu,z}\right\}\frac{1}{\eta^3}\left(3\frac{\eta_z^2}{\eta^2}-1\right)\\
&=\left\{M_\nu\left(t\right)-3M_z\left(t\right)\delta_{\nu,z}\right\}\frac{1}{2\eta^3}\left(3\frac{\eta_z^2}{\eta^2}-1\right),
\end{aligned}\tag{A.6}$$

where $\boldsymbol{\eta}\coloneqq\boldsymbol{r}-\boldsymbol{r'}$ and $Y_l^m\left(\boldsymbol{e_\eta}\right)$ are the usual spherical harmonics.

Therefore,

$$\int d\boldsymbol{\rho}\frac{1}{l_\perp\sqrt{\pi}}e^{-\rho^2/\left(2l_\perp^2\right)}\left\{\hbar\boldsymbol{b}_{dd}\left(\boldsymbol{r}\right)\cdot\hat{\boldsymbol{f}}\right\}\psi\left(\boldsymbol{r},t\right)$$
$$=\frac{c_{dd}}{2\pi^2 l_\perp^4}\int dz'\,n\left(z',t\right)\int d\boldsymbol{\rho}\int d\boldsymbol{\rho'}\frac{e^{-\left(\rho^2+\rho'^2\right)/l_\perp^2}}{\left(\left|\boldsymbol{\rho}-\boldsymbol{\rho'}\right|^2+\left|z-z'\right|^2\right)^{3/2}}\left(3\frac{\left|z-z'\right|^2}{\left|\boldsymbol{\rho}-\boldsymbol{\rho'}\right|^2+\left|z-z'\right|^2}-1\right)$$
$$\times\Psi\left(z,t\right)e^{-\frac{i+\gamma}{1+\gamma^2}\omega_\perp t}s\left\{\boldsymbol{M}\left(t\right)-3M_z\left(t\right)\boldsymbol{e}_z\right\}\cdot\hat{\boldsymbol{f}}\zeta\left(t\right). \tag{A.7}$$

It is not easy to directly calculate double integration with $\int d\boldsymbol{\rho}\int d\boldsymbol{\rho'}$ in Eq. (A.7), so we will make a detour. Final result is in section A.1.2.

A.1.1 Fourier Space Representation

The magnetic dipole-dipole interaction $V_{dd}\left(t\right)$ can be written as

$$\begin{aligned} V_{dd}\left(t\right) &= \frac{c_{dd}}{2}\int d^3r\int d^3r'\sum_{\nu,\nu'=x,y,z}F_\nu\left(\boldsymbol{r},t\right)Q_{\nu,\nu'}\left(\boldsymbol{r}-\boldsymbol{r'}\right)F_{\nu'}\left(\boldsymbol{r'},t\right) \\ &= s^2\frac{c_{dd}}{2\pi^2 l_\perp^4}\int dz\int dz'\,n\left(z,t\right)n\left(z',t\right) \\ &\quad\times\int d\boldsymbol{\rho}\int d\boldsymbol{\rho'}\,e^{-\left(\rho^2+\rho'^2\right)/l_\perp^2}\sum_{\nu'=x,y,z}M_\nu\left(t\right)Q_{\nu,\nu'}\left(\boldsymbol{r}-\boldsymbol{r'}\right)M_{\nu'}\left(t\right), \end{aligned} \tag{A.8}$$

and from Eq. (A.6), nonzero term of $\int d\boldsymbol{\rho}\int d\boldsymbol{\rho'}\,e^{-\left(\rho^2+\rho'^2\right)/l_\perp^2}\sum_{\nu'=x,y,z}M_\nu\left(t\right)Q_{\nu,\nu'}\left(\boldsymbol{r}-\boldsymbol{r'}\right)M_{\nu'}\left(t\right)$ is

$$\frac{1}{2}\left\{1-3\cos^2\theta\left(t\right)\right\}\int d\boldsymbol{\rho}\int d\boldsymbol{\rho'}\frac{e^{-\left(\rho^2+\rho'^2\right)/l_\perp^2}}{\left(\left|\boldsymbol{\rho}-\boldsymbol{\rho'}\right|^2+\left|z-z'\right|^2\right)^{3/2}}\left(3\frac{\left|z-z'\right|^2}{\left|\boldsymbol{\rho}-\boldsymbol{\rho'}\right|^2+\left|z-z'\right|^2}-1\right). \tag{A.9}$$

Therefore, $V_{dd}\left(t\right)$ can be written as

$$V_{dd}\left(t\right)=\frac{c_{dd}}{2}\int dz\int dz'\,n\left(z,t\right)n\left(z',t\right)V_{\text{eff}}\left(z-z',t\right), \tag{A.10}$$

with

$$V_{\text{eff}}\left(z,t\right):=\left\{1-3\cos^2\theta\left(t\right)\right\}\frac{s^2}{2\pi^2 l_\perp^4}\int d\boldsymbol{\rho}\int d\boldsymbol{\rho'}\frac{e^{-\left(\rho^2+\rho'^2\right)/l_\perp^2}}{\left(\left|\boldsymbol{\rho}-\boldsymbol{\rho'}\right|^2+z^2\right)^{3/2}}\left(3\frac{z^2}{\left|\boldsymbol{\rho}-\boldsymbol{\rho'}\right|^2+z^2}-1\right). \tag{A.11}$$

Thus, calculating $V_{\text{eff}}(z,t)$ will make us to calculate Eq. (A.7). We will first calculate $V_{dd}(t)$ in Fourier space and then perform the inverse Fourier transform to get $V_{\text{eff}}(z,t)$.

Note that $V_{dd}(t)$ can also be written as

$$\begin{aligned} V_{dd}(t) &= \frac{c_{dd}}{2}\int d^3r \int d^3r'\; n(\boldsymbol{r},t)\, n(\boldsymbol{r}',t)\, U_{dd}(\boldsymbol{r}-\boldsymbol{r}',t) \\ &= \frac{c_{dd}}{2}(2\pi)^{3/2}\int d^3k\; \tilde{n}(\boldsymbol{k},t)\,\tilde{n}(-\boldsymbol{k},t)\,\tilde{U}_{dd}(\boldsymbol{k},t), \end{aligned} \tag{A.12}$$

where

$$\begin{aligned} U_{dd}(\boldsymbol{r},t) &:= s^2\sum_{\nu'=x,y,z} M_\nu(t)\, Q_{\nu,\nu'}(\boldsymbol{r})\, M_{\nu'}(t) \\ &= -\frac{s^2}{r^3}\sqrt{\frac{6\pi}{5}}\left[\begin{array}{l} \left\{Y_2^2(\boldsymbol{e_r})\,e^{-2i\phi(t)} + Y_2^{-2}(\boldsymbol{e_r})\,e^{2i\phi(t)}\right\}\sin^2\theta(t) \\ -\left\{Y_2^1(\boldsymbol{e_r})\,e^{-i\phi(t)} - Y_2^{-1}(\boldsymbol{e_r})\,e^{i\phi(t)}\right\}\sin\{2\theta(t)\} \end{array}\right] \\ &\quad + \frac{s^2}{r^3}\sqrt{\frac{6\pi}{5}}Y_2^0(\boldsymbol{e_r})\sqrt{\frac{2}{3}}\left\{3\sin^2\theta(t) - 2\right\}, \end{aligned} \tag{A.13}$$

and $\tilde{g}(\boldsymbol{k}) = (2\pi)^{-3/2}\int d^3r\; e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} g(\boldsymbol{r})$ is the Fourier transform of the function $g(\boldsymbol{r})$.

Fourier Transform of $n(\boldsymbol{r},t)$

Since $\boldsymbol{\rho}\cdot\boldsymbol{k}_\rho = \rho k_\rho \cos(\phi_\rho - \phi_{k_\rho})$ where $\boldsymbol{k}_\rho := (k_x, k_y)$ with $k_\rho := \sqrt{k_x^2 + k_y^2}$, $\tan\phi_{k_\rho} := k_y/k_x$, and $\int d^2\boldsymbol{k}_\rho := \int d\phi_{k_\rho}\int dk_\rho\, k_\rho$,

$$\begin{aligned} \tilde{n}(\boldsymbol{k},t) &= \frac{1}{(2\pi)^{3/2}}\int d\boldsymbol{\rho}\int_{-\infty}^{\infty} dz\; \frac{1}{\pi l_\perp^2} e^{-(\rho/l_\perp)^2} n(z,t)\, e^{-i\boldsymbol{\rho}\cdot\boldsymbol{k}_\rho} e^{-ik_z z} \\ &= \frac{\tilde{n}(k_z,t)}{2\pi}\frac{1}{\pi l_\perp^2}\int_0^\infty d\rho\;\rho\int_0^{2\pi} d\phi_\rho\; e^{-(\rho/l_\perp)^2} e^{-i\rho k_\rho\cos(\phi_\rho-\phi_{k_\rho})}. \end{aligned} \tag{A.14}$$

We know that

$$e^{-i\rho k_\rho\cos(\phi_\rho - \phi_{k_\rho})} = \sum_{m=-\infty}^{\infty}(-i)^m J_m(\rho k_\rho)\, e^{-im(\phi_\rho - \phi_{k_\rho})}, \tag{A.15}$$

which is Jacobi-Anger expansion ($J_m(x)$ is Bessel function of the first kind). Since $\int_0^{2\pi} d\phi_\rho\ e^{-im(\phi_\rho-\phi_{k_\rho})} = 2\pi\delta_{m,0}$ for any integer m where $\delta_{m,m'}$ is Kronecker delta function, Eq. (A.14) can be written as

$$\tilde{n}(\boldsymbol{k},t) = \tilde{n}(k_z,t)\frac{1}{\pi l_\perp^2}\int_0^\infty d\rho\ \rho\ e^{-(\rho/l_\perp)^2} J_0(\rho k_\rho) = \frac{1}{2\pi}\tilde{n}(k_z,t)\,e^{-(k_\rho l_\perp)^2/4}. \qquad \text{(A.16)}$$

Fourier Transform of $U_{dd}(\boldsymbol{r})$

It is known that

$$\int d\Omega_{\boldsymbol{r}}\ e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} Y_l^m(\boldsymbol{e_r}) = 4\pi(-i)^l j_l(kr)\,Y_l^m(\boldsymbol{e_k}), \qquad \text{(A.17)}$$

where $\int d\Omega_{\boldsymbol{r}}$ is the integration of the angular part of $\boldsymbol{r}$ and $j_m(x)$ is the m-th spherical Bessel function. Hence, the Fourier transform of $U_{dd}(\boldsymbol{r})$ (see Eq. (A.13)) is

$$\begin{aligned}\tilde{U}_{dd}(\boldsymbol{k}) &= -\frac{s^2}{(2\pi)^{3/2}}\sqrt{\frac{6\pi}{5}}\int_0^\infty dr\ \frac{1}{r}\int d\Omega_{\boldsymbol{r}}\ \begin{bmatrix}\sin^2\theta(t)\left\{Y_2^2(\boldsymbol{e_r})\,e^{-2i\phi}+Y_2^{-2}(\boldsymbol{e_r})\,e^{2i\phi}\right\}\\ -\sin\{2\theta(t)\}\left\{Y_2^1(\boldsymbol{e_r})\,e^{-i\phi}-Y_2^{-1}(\boldsymbol{e_r})\,e^{i\phi}\right\}\end{bmatrix} e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\\ &\quad+\frac{s^2}{(2\pi)^{3/2}}\sqrt{\frac{6\pi}{5}}\sqrt{\frac{2}{3}}\int_0^\infty dr\ \frac{1}{r}\int d\Omega_{\boldsymbol{r}}\ \left\{3\sin^2\theta(t)-2\right\}Y_2^0(\boldsymbol{e_r})\,e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\\ &= \frac{4\pi}{(2\pi)^{3/2}}\sqrt{\frac{6\pi}{5}}s^2\begin{bmatrix}\sin^2\theta(t)\left\{Y_2^2(\boldsymbol{e_k})\,e^{-2i\phi}+Y_2^{-2}(\boldsymbol{e_k})\,e^{2i\phi}\right\}\\ -\sin\{2\theta(t)\}\left\{Y_2^1(\boldsymbol{e_k})\,e^{-i\phi}-Y_2^{-1}(\boldsymbol{e_k})\,e^{i\phi}\right\}\end{bmatrix}\int_0^\infty dr\ \frac{j_2(kr)}{r}\\ &\quad-\frac{4\pi}{(2\pi)^{3/2}}\sqrt{\frac{6\pi}{5}}\sqrt{\frac{2}{3}}s^2\left\{3\sin^2\theta(t)-2\right\}Y_2^0(\boldsymbol{e_k})\int_0^\infty dr\ \frac{j_2(kr)}{r}. \qquad \text{(A.18)}\end{aligned}$$

Since $\int_0^\infty dr\ j_2(kr)/r = 1/3$, we get

$$\begin{aligned}\tilde{U}_{dd}(\boldsymbol{k}) &= \frac{1}{\sqrt{2\pi}}s^2\left[\sin^2\theta_k\sin^2\theta(t)\cos\{2(\phi_k-\phi)\}+\sin(2\theta_k)\sin\{2\theta(t)\}\cos(\phi_k-\phi)\right]\\ &\quad+\frac{1}{(2\pi)^{3/2}}\frac{4\pi}{3}s^2\left\{1-\frac{3}{2}\sin^2\theta(t)\right\}\left(3\cos^2\theta_k-1\right). \qquad \text{(A.19)}\end{aligned}$$

Fourier Transform of $V_{\text{eff}}(z,t)$ and Its Inverse Fourier Transform

From Eqs. (A.10), (A.12), (A.16), and (A.19), we get

$$
\begin{aligned}
V_{dd}(t) &= \frac{c_{dd}}{2}\int dz \int dz'\; n(z,t)\, n(z',t)\, V_{\text{eff}}(z-z',t) \\
&= \frac{c_{dd}}{2}(2\pi)^{3/2}\int d^3k\; \tilde{n}(\boldsymbol{k},t)\,\tilde{n}(-\boldsymbol{k},t)\,\tilde{U}_{dd}(\boldsymbol{k},t) \\
&= \frac{c_{dd}}{2}(2\pi)^{3/2}\int_{-\infty}^{\infty} dk_z \int_0^{\infty} dk_\rho\; k_\rho\; 2\pi \left(\frac{1}{2\pi}\right)^2 \tilde{n}(k_z,t)\,\tilde{n}(-k_z,t)\, e^{-k_\rho^2 l_\perp^2/2} \\
&\quad \times \frac{1}{(2\pi)^{3/2}}\frac{4\pi}{3}s^2\left\{1-\frac{3}{2}\sin^2\theta(t)\right\}\left(3\frac{k_z^2}{k_\rho^2+k_z^2}-1\right) \\
&= \frac{c_{dd}}{2}\sqrt{2\pi}\int_{-\infty}^{\infty} dk_z\; \tilde{n}(k_z,t)\,\tilde{n}(-k_z,t)\,\frac{s^2}{\sqrt{2\pi}}\frac{2}{3}\left\{1-\frac{3}{2}\sin^2\theta(t)\right\} \\
&\quad \times \int_0^{\infty} dk_\rho\; k_\rho\; e^{-k_\rho^2 l_\perp^2/2}\left(3\frac{k_z^2}{k_\rho^2+k_z^2}-1\right). \qquad \text{(A.20)}
\end{aligned}
$$

Hence the Fourier transform of $V_{\text{eff}}(z,t)$ is

$$
\tilde{V}_{\text{eff}}(k_z,t) = \frac{s^2}{\sqrt{2\pi}}\frac{2}{3}\left\{1-\frac{3}{2}\sin^2\theta(t)\right\}\int_0^{\infty} dk_\rho\; k_\rho\; e^{-k_\rho^2 l_\perp^2/2}\left(3\frac{k_z^2}{k_\rho^2+k_z^2}-1\right). \qquad \text{(A.21)}
$$

Let $t=\left(k_\rho^2+k_z^2\right) l_\perp^2/2$. Then $dt = l_\perp^2 k_\rho dk_\rho$ and

$$
\begin{aligned}
\tilde{V}_{\text{eff}}(k_z,t) &= \frac{s^2}{\sqrt{2\pi}}\frac{2}{3}\left\{1-\frac{3}{2}\sin^2\theta(t)\right\}\int_{k_z^2 l_\perp^2/2}^{\infty} dt\; \frac{1}{l_\perp^2}e^{-t}e^{k_z^2 l_\perp^2/2}\left(3\frac{k_z^2 l_\perp^2}{2t}-1\right) \\
&= \frac{2s^2}{l_\perp^2\sqrt{2\pi}}\left\{1-\frac{3}{2}\sin^2\theta(t)\right\} e^{k_z^2 l_\perp^2/2}\int_{k_z^2 l_\perp^2/2}^{\infty} dt\; e^{-t}\left(\frac{k_z^2 l_\perp^2}{2t}-\frac{1}{3}\right) \\
&= \frac{2s^2}{l_\perp^2\sqrt{2\pi}}\left\{1-\frac{3}{2}\sin^2\theta(t)\right\}\left\{\left(k_z^2 l_\perp^2/2\right) e^{k_z^2 l_\perp^2/2}E_1\left(k_z^2 l_\perp^2/2\right)-\frac{1}{3}\right\}, \qquad \text{(A.22)}
\end{aligned}
$$

where $E_1(x)=\int_x^\infty du\; e^{-u}/u$ is exponential integral. Then we have

$$
V_{\text{eff}}(z,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk_z\; \frac{2s^2}{l_\perp^2\sqrt{2\pi}}\left\{1-\frac{3}{2}\sin^2\theta(t)\right\}\left\{\left(k_z^2 l_\perp^2/2\right) e^{k_z^2 l_\perp^2/2}E_1\left(k_z^2 l_\perp^2/2\right)-\frac{1}{3}\right\} e^{-ik_z z}. \qquad \text{(A.23)}
$$

From the definition of the exponential integral $E_1(x)$,

$$
\int_{-\infty}^{\infty} dx\; e^{x^2}E_1\left(x^2\right)e^{-ikx} = \int_{-\infty}^{\infty} dx\; e^{-ikx}\int_{x^2}^{\infty} dt\; \frac{1}{t}e^{-\left(t-x^2\right)}. \qquad \text{(A.24)}
$$

Let $s^2 = t - x^2$. Then $2sds = dt$ and Eq. (A.24) can be written as

$$\int_{-\infty}^{\infty} dx\; e^{x^2} E_1\left(x^2\right) e^{-ikx} = \int_{-\infty}^{\infty} dx\; e^{-ikx} \int_0^{\infty} ds\; \frac{2s}{x^2+s^2} e^{-s^2} = 2\int_0^{\infty} ds\; se^{-s^2} \int_{-\infty}^{\infty} dx\; \frac{1}{x^2+s^2} e^{-ikx}. \tag{A.25}$$

Using contour integral, $\int_{-\infty}^{\infty} dx\; e^{-ikx}/\left(x^2+s^2\right) = \left(\pi/\left|s\right|\right) e^{-|ks|}$ and thus Eq. (A.24) becomes

$$\begin{aligned}\int_{-\infty}^{\infty} dx\; e^{x^2} E_1\left(x^2\right) e^{-ikx} &= 2\pi \int_0^{\infty} ds\; e^{-s^2} e^{-|k|s} = 2\pi \int_0^{\infty} ds\; e^{-\{s+(|k|/2)\}^2} e^{k^2/4} \\ &= (\pi)^{3/2} e^{k^2/4} \mathrm{Erfc}\left(|k|/2\right).\end{aligned} \tag{A.26}$$

By differentiating Eq. (A.25) with respect to k two times,

$$\begin{aligned}\int_{-\infty}^{\infty} dx\; x^2 e^{x^2} E_1\left(x^2\right) e^{-ikx} &= 2\int_0^{\infty} ds\; se^{-s^2} \int_{-\infty}^{\infty} dx\; \frac{x^2}{x^2+s^2} e^{-ikx} \\ &= 2\int_0^{\infty} ds\; se^{-s^2} \int_{-\infty}^{\infty} dx\; \left(1 - \frac{s^2}{x^2+s^2}\right) e^{-ikx} \\ &= 4\pi\delta\left(k\right) \int_0^{\infty} ds\; se^{-s^2} - 2\pi \int_0^{\infty} ds\; s^2 e^{-s^2} e^{-|k|s} \\ &= 2\pi\delta\left(k\right) - 2\pi \int_0^{\infty} ds\; s^2 e^{-s^2} e^{-|k|s}.\end{aligned} \tag{A.27}$$

Note that

$$\begin{aligned}2\pi \int_0^{\infty} ds\; s^2 e^{-s^2} e^{-|k|s} &= \frac{\partial^2}{\partial\left|k\right|^2} 2\pi \int_0^{\infty} ds\; e^{-s^2} e^{-|k|s} = \frac{\partial^2}{\partial\left|k\right|^2} \int_{-\infty}^{\infty} dx\; e^{x^2} E_1\left(x^2\right) e^{-ikx} \\ &= (\pi)^{3/2} \frac{\partial^2}{\partial\left|k\right|^2} \left\{ e^{k^2/4} \mathrm{Erfc}\left(|k|/2\right) \right\},\end{aligned} \tag{A.28}$$

$$\begin{aligned}\frac{\partial}{\partial x} \left\{ e^{x^2/4} \mathrm{Erfc}\left(x/2\right) \right\} &= \frac{x}{2} e^{x^2/4} \mathrm{Erfc}\left(x/2\right) - \frac{1}{2} e^{x^2/4} \frac{2}{\sqrt{\pi}} e^{-x^2/4} \\ &= \frac{x}{2} e^{x^2/4} \mathrm{Erfc}\left(x/2\right) - \frac{1}{\sqrt{\pi}},\end{aligned} \tag{A.29}$$

and

$$\frac{\partial^2}{\partial x^2}\left\{e^{x^2/4}\mathrm{Erfc}\left(x/2\right)\right\} = \frac{1}{2}e^{x^2/4}\mathrm{Erfc}\left(x/2\right)+\frac{x}{2}\left\{\frac{x}{2}e^{x^2/4}\mathrm{Erfc}\left(x/2\right)-\frac{1}{\sqrt{\pi}}\right\}$$
$$= \frac{1}{2}\left(\frac{x^2}{2}+1\right)e^{x^2/4}\mathrm{Erfc}\left(x/2\right)-\frac{x}{2\sqrt{\pi}}. \quad \text{(A.30)}$$

Therefore, from Eqs. (A.27) and (A.28), one can get

$$\int_{-\infty}^{\infty} dx\; x^2 e^{x^2} E_1\left(x^2\right)e^{-ikx} = 2\pi\delta\left(k\right)-\left(\pi\right)^{3/2}\left\{\frac{1}{2}\left(\frac{k^2}{2}+1\right)e^{k^2/4}\mathrm{Erfc}\left(\left|k\right|/2\right)-\frac{\left|k\right|}{2\sqrt{\pi}}\right\}$$
$$= -\left(\pi\right)^{3/2}\left\{\frac{1}{2}\left(\frac{k^2}{2}+1\right)e^{k^2/4}\mathrm{Erfc}\left(\left|k\right|/2\right)-\frac{\left|k\right|}{2\sqrt{\pi}}-\frac{2}{\sqrt{\pi}}\delta\left(k\right)\right\}, \quad \text{(A.31)}$$

and

$$V_{\mathrm{eff}}\left(z,t\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk_z\;\frac{2s^2}{l_\perp^2\sqrt{2\pi}}\left\{1-\frac{3}{2}\sin^2\theta\left(t\right)\right\}\left\{\left(k_z^2l_\perp^2/2\right)e^{k_z^2l_\perp^2/2}E_1\left(k_z^2l_\perp^2/2\right)-\frac{1}{3}\right\}e^{-ik_zz}$$
$$= \frac{s^2}{l_\perp^3}\left\{\frac{3}{2}\sin^2\theta\left(t\right)-1\right\}\left\{G\left(\left|z\right|/l_\perp\right)-\frac{4}{3}\delta\left(z/l_\perp\right)\right\}, \quad \text{(A.32)}$$

where $G\left(\lambda\right) := \sqrt{\pi/2}\left(\lambda^2+1\right)e^{\lambda^2/2}\mathrm{Erfc}\left(\lambda/\sqrt{2}\right)-\lambda$ and $\delta\left(x\right)$ is the Dirac delta function.

A.1.2 Final Result

Since we get $V_{\mathrm{eff}}\left(z,t\right)$ in Eq. (A.32), one can get

$$\frac{1}{\pi^2 l_\perp^4}\int d\boldsymbol{\rho}\int d\boldsymbol{\rho}'\;\frac{e^{-\left(\rho^2+\rho'^2\right)/l_\perp^2}}{\left(\left|\boldsymbol{\rho}-\boldsymbol{\rho}'\right|^2+\left|z-z'\right|^2\right)^{3/2}}\left(3\frac{\left|z-z'\right|^2}{\left|\boldsymbol{\rho}-\boldsymbol{\rho}'\right|^2+\left|z-z'\right|^2}-1\right)$$
$$= \frac{1}{l_\perp^3}\left\{G\left(\left|z-z'\right|/l_\perp\right)-\frac{4}{3}\delta\left(\frac{z-z'}{l_\perp}\right)\right\}, \quad \text{(A.33)}$$

and

$$\int d\boldsymbol{\rho}\;\frac{1}{l_\perp\sqrt{\pi}}e^{-\rho^2/\left(2l_\perp^2\right)}\left\{\hbar\boldsymbol{b}_{dd}\left(\boldsymbol{r}\right)\cdot\hat{\boldsymbol{f}}\right\}\psi\left(\boldsymbol{r},t\right)$$
$$= \frac{c_{dd}}{2l_\perp^3}\int dz'\;n\left(z',t\right)\left\{G\left(\frac{\left|z-z'\right|}{l_\perp}\right)-\frac{4}{3}\delta\left(\frac{z-z'}{l_\perp}\right)\right\}\Psi\left(z,t\right)e^{-\frac{i+\gamma}{1+\gamma^2}\omega_\perp t}s\left\{\boldsymbol{M}\left(t\right)-3M_z\left(t\right)\boldsymbol{e}_z\right\}\cdot\hat{\boldsymbol{f}}\zeta\left(t\right). \quad \text{(A.34)}$$

A.2 Step 2. Final Result

We have

$$
\begin{aligned}
(i-\gamma)\,\hbar\frac{\partial\psi(\boldsymbol{r},t)}{\partial t} &= \frac{e^{-\rho^2/\left(2l_\perp^2\right)}}{l_\perp\sqrt{\pi}}\,(i-\gamma)\,\hbar\frac{\partial}{\partial t}\left\{\Psi(z,t)\,\zeta(t)\,e^{-\frac{i+\gamma}{1+\gamma^2}\omega_\perp t}\right\} \\
&= \frac{e^{-\rho^2/\left(2l_\perp^2\right)}}{l_\perp\sqrt{\pi}}e^{-\frac{i+\gamma}{1+\gamma^2}\omega_\perp t}\,(i-\gamma)\,\hbar\frac{\partial\left\{\Psi(z,t)\,\zeta(t)\right\}}{\partial t} + \frac{e^{-\rho^2/\left(2l_\perp^2\right)}}{l_\perp\sqrt{\pi}}\Psi(z,t)\,\zeta(t)\,(i-\gamma)\,\hbar\frac{\partial}{\partial t}\left\{e^{-\frac{i+\gamma}{1+\gamma^2}\omega_\perp t}\right\} \\
&= \frac{e^{-\rho^2/\left(2l_\perp^2\right)}}{l_\perp\sqrt{\pi}}e^{-\frac{i+\gamma}{1+\gamma^2}\omega_\perp t}\,(i-\gamma)\,\hbar\frac{\partial\left\{\Psi(z,t)\,\zeta(t)\right\}}{\partial t} + \frac{e^{-\rho^2/\left(2l_\perp^2\right)}}{l_\perp\sqrt{\pi}}\Psi(z,t)\,\zeta(t)\,\hbar\omega_\perp e^{-\frac{i+\gamma}{1+\gamma^2}\omega_\perp t} \\
&= \frac{e^{-\rho^2/\left(2l_\perp^2\right)}}{l_\perp\sqrt{\pi}}e^{-\frac{i+\gamma}{1+\gamma^2}\omega_\perp t}\,(i-\gamma)\,\hbar\frac{\partial\left\{\Psi(z,t)\,\zeta(t)\right\}}{\partial t} + \hbar\omega_\perp\psi(\boldsymbol{r},t)\,, \qquad \text{(A.35)}
\end{aligned}
$$

$$
\begin{aligned}
-\frac{\hbar^2}{2m}\nabla^2\psi(\boldsymbol{r},t) &= -\frac{\hbar^2}{2m}\frac{1}{l_\perp\sqrt{\pi}}\zeta(t)\,e^{-\frac{i+\gamma}{1+\gamma^2}\omega_\perp t}\nabla^2\left\{e^{-\rho^2/\left(2l_\perp^2\right)}\Psi(z,t)\right\} \\
&= -\frac{\hbar^2}{2m}\frac{1}{l_\perp\sqrt{\pi}}\zeta(t)\,e^{-\frac{i+\gamma}{1+\gamma^2}\omega_\perp t}\left[\Psi(z,t)\,\frac{1}{\rho}\frac{\partial}{\partial\rho}\left\{\rho\frac{\partial e^{-\rho^2/\left(2l_\perp^2\right)}}{\partial\rho}\right\} + e^{-\rho^2/\left(2l_\perp^2\right)}\frac{\partial^2\Psi(z,t)}{\partial z^2}\right] \\
&= -\frac{\hbar^2}{2m}\frac{1}{l_\perp\sqrt{\pi}}\zeta(t)\,e^{-\frac{i+\gamma}{1+\gamma^2}\omega_\perp t}\left[\Psi(z,t)\,\frac{1}{\rho}\frac{\partial}{\partial\rho}\left\{-\frac{\rho^2}{l_\perp^2}e^{-\rho^2/\left(2l_\perp^2\right)}\right\} + e^{-\rho^2/\left(2l_\perp^2\right)}\frac{\partial^2\Psi(z,t)}{\partial z^2}\right] \\
&= -\frac{\hbar^2}{2m}\frac{1}{l_\perp\sqrt{\pi}}\zeta(t)\,e^{-\frac{i+\gamma}{1+\gamma^2}\omega_\perp t}\left[\Psi(z,t)\,\frac{1}{l_\perp^2}\left(\frac{\rho^2}{l_\perp^2}-2\right)e^{-\rho^2/\left(2l_\perp^2\right)} + e^{-\rho^2/\left(2l_\perp^2\right)}\frac{\partial^2\Psi(z,t)}{\partial z^2}\right] \\
&= -\frac{\hbar^2}{2ml_\perp^2}\left(\frac{\rho^2}{l_\perp^2}-2\right)\psi(\boldsymbol{r},t) - \frac{e^{-\rho^2/\left(2l_\perp^2\right)}}{l_\perp\sqrt{\pi}}e^{-\frac{i+\gamma}{1+\gamma^2}\omega_\perp t}\frac{\hbar^2}{2m}\frac{\partial^2\left\{\Psi(z,t)\,\zeta(t)\right\}}{\partial z^2} \\
&= \left(-\frac{m}{2}\omega_\perp^2\rho^2 + \hbar\omega_\perp\right)\psi(\boldsymbol{r},t) - \frac{e^{-\rho^2/\left(2l_\perp^2\right)}}{l_\perp\sqrt{\pi}}e^{-\frac{i+\gamma}{1+\gamma^2}\omega_\perp t}\frac{\hbar^2}{2m}\frac{\partial^2\left\{\Psi(z,t)\,\zeta(t)\right\}}{\partial z^2}. \qquad \text{(A.36)}
\end{aligned}
$$

Therefore, Eq. (A.1) can be written as

$$
\begin{aligned}
&\frac{e^{-\rho^2/\left(2l_\perp^2\right)}}{l_\perp\sqrt{\pi}} e^{-\frac{i+\gamma}{1+\gamma^2}\omega_\perp t}\left(i-\gamma\right)\hbar\frac{\partial\left\{\Psi\left(z,t\right)\zeta\left(t\right)\right\}}{\partial t} \\
&\quad= \frac{e^{-\rho^2/\left(2l_\perp^2\right)}}{l_\perp\sqrt{\pi}} e^{-\frac{i+\gamma}{1+\gamma^2}\omega_\perp t}\left[-\frac{\hbar^2}{2m}\frac{\partial^2\left\{\Psi\left(z,t\right)\zeta\left(t\right)\right\}}{\partial z^2}\right] \\
&\qquad+\frac{e^{-\rho^2/\left(2l_\perp^2\right)}}{l_\perp\sqrt{\pi}} e^{-\frac{i+\gamma}{1+\gamma^2}\omega_\perp t}\left\{V\left(z\right)+c_0\frac{e^{-\rho^2/l_\perp^2}}{\pi l_\perp^2}n\left(z,t\right)\right\}\Psi\left(z,t\right)\zeta\left(t\right) \\
&\qquad+\frac{e^{-\rho^2/\left(2l_\perp^2\right)}}{l_\perp\sqrt{\pi}} e^{-\frac{i+\gamma}{1+\gamma^2}\omega_\perp t}\sum_{j=1}^{s}c_{2j}\sum_{\nu_1,\nu_2,\cdots,\nu_j=x,y,z} s\frac{e^{-\rho^2/l_\perp^2}}{\pi l_\perp^2}n\left(z,t\right)M_{\nu_1,\nu_2,\cdots,\nu_j}\left(t\right)\hat{f}_{\nu_1}\hat{f}_{\nu_2}\cdots\hat{f}_{\nu_j}\Psi\left(z,t\right)\zeta\left(t\right) \\
&\qquad+\frac{e^{-\rho^2/\left(2l_\perp^2\right)}}{l_\perp\sqrt{\pi}} e^{-\frac{i+\gamma}{1+\gamma^2}\omega_\perp t}\left(-\hbar\boldsymbol{b}\cdot\hat{\boldsymbol{f}}\right)\Psi\left(z,t\right)\zeta\left(t\right)+\hbar\boldsymbol{b}_{dd}\left(\boldsymbol{r},t\right)\cdot\hat{\boldsymbol{f}}\psi\left(\boldsymbol{r},t\right),
\end{aligned}
\tag{A.37}
$$

where

$$
M_{\nu_1,\nu_2,\cdots,\nu_j}\left(t\right) \coloneqq \frac{1}{s}\sum_{m_1,m_2=-s}^{s}\zeta_{m_1}^{\dagger}\left(t\right)\left(\hat{f}_{\nu_1}\hat{f}_{\nu_2}\cdots\hat{f}_{\nu_j}\right)_{m_1,m_2}\zeta_{m_2}\left(t\right). \tag{A.38}
$$

Since $n\left(\boldsymbol{r},t\right)=\left(1/\pi l_\perp^2\right)e^{-\rho^2/l_\perp^2}n\left(z,t\right)$, to integrate out x and y directions, we have to multiply $\left(1/l_\perp\sqrt{\pi}\right)e^{-\rho^2/\left(2l_\perp^2\right)}$ to Eq. (A.37) and integrate in x and y axis. The result is

$$
\begin{aligned}
&(i-\gamma)\hbar\frac{\partial\left\{\Psi(z,t)\zeta(t)\right\}}{\partial t} \\
&=\left(-\frac{\hbar^2}{2m}\frac{\partial^2\left\{\Psi(z,t)\zeta(t)\right\}}{\partial z^2}\right)+\left[V(z)-\hbar\boldsymbol{b}\cdot\hat{\boldsymbol{f}}+c_0\int d\boldsymbol{\rho}\,\frac{e^{-2\rho^2/l_\perp^2}}{\pi^2 l_\perp^4}n(z,t)\right]\Psi(z,t)\zeta(t) \\
&\quad+e^{\frac{i+\gamma}{1+\gamma^2}\omega_\perp t}\int d\boldsymbol{\rho}\,\frac{1}{l_\perp\sqrt{\pi}}e^{-\rho^2/\left(2l_\perp^2\right)}\left\{\hbar\boldsymbol{b}_{dd}(\boldsymbol{r})\cdot\hat{\boldsymbol{f}}\right\}\psi(\boldsymbol{r},t) \\
&\quad+\sum_{j=1}^{s}c_{2j}\sum_{\nu_1,\nu_2,\cdots,\nu_j=x,y,z}s\int d\boldsymbol{\rho}\,\frac{e^{-2\rho^2/l_\perp^2}}{\pi^2 l_\perp^4}n(z,t)\,M_{\nu_1,\nu_2,\cdots,\nu_j}(t)\,\hat{f}_{\nu_1}\hat{f}_{\nu_2}\cdots\hat{f}_{\nu_j}\Psi(z,t)\zeta(t) \\
&=\left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}+V(z)-\hbar\boldsymbol{b}\cdot\hat{\boldsymbol{f}}+c_0\frac{1}{\pi^2 l_\perp^4}\frac{\pi l_\perp^2}{2}n(z,t)\right\}\Psi(z,t)\zeta(t) \\
&\quad+\frac{c_{dd}}{2l_\perp^3}\int dz'\,n(z',t)\left\{G\left(\frac{|z-z'|}{l_\perp}\right)-\frac{4}{3}\delta\left(\frac{z-z'}{l_\perp}\right)\right\}\Psi(z,t)\,s\left\{\boldsymbol{M}(t)-3M_z(t)\,\boldsymbol{e}_z\right\}\cdot\hat{\boldsymbol{f}}\zeta(t) \\
&\quad+\sum_{j=1}^{s}c_{2j}\sum_{\nu_1,\nu_2,\cdots,\nu_j=x,y,z}s\frac{1}{\pi^2 l_\perp^4}\frac{\pi l_\perp^2}{2}n(z,t)\,M_{\nu_1,\nu_2,\cdots,\nu_j}(t)\,\hat{f}_{\nu_1}\hat{f}_{\nu_2}\cdots\hat{f}_{\nu_j}\Psi(z,t)\zeta(t) \\
&=\left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}+V(z)-\hbar\boldsymbol{b}\cdot\hat{\boldsymbol{f}}+\frac{c_0}{2\pi l_\perp^2}n(z,t)\right\}\Psi(z,t)\zeta(t) \\
&\quad+\hbar P_{dd}(z,t)\,\Psi(z,t)\,s\left\{\boldsymbol{M}(t)-3M_z(t)\,\boldsymbol{e}_z\right\}\cdot\hat{\boldsymbol{f}}\zeta(t) \\
&\quad+\sum_{j=1}^{s}\frac{c_{2j}}{2\pi l_\perp^2}n(z,t)\sum_{\nu_1,\nu_2,\cdots,\nu_j=x,y,z}sM_{\nu_1,\nu_2,\cdots,\nu_j}(t)\,\hat{f}_{\nu_1}\hat{f}_{\nu_2}\cdots\hat{f}_{\nu_j}\Psi(z,t)\zeta(t) \\
&=\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}+V(z)+\hbar\left[-\boldsymbol{b}+s\left\{\boldsymbol{M}(t)-3M_z(t)\,\boldsymbol{e}_z\right\}P_{dd}(z,t)\right]\cdot\hat{\boldsymbol{f}}+\frac{c_0}{2\pi l_\perp^2}n(z,t)\right)\Psi(z,t)\zeta(t) \\
&\quad+\sum_{j=1}^{s}\frac{c_{2j}}{2\pi l_\perp^2}n(z,t)\sum_{\nu_1,\nu_2,\cdots,\nu_j=x,y,z}sM_{\nu_1,\nu_2,\cdots,\nu_j}(t)\,\hat{f}_{\nu_1}\hat{f}_{\nu_2}\cdots\hat{f}_{\nu_j}\Psi(z,t)\zeta(t),
\end{aligned}
\tag{A.39}
$$

where

$$
\begin{aligned}
P_{dd}(z,t) &:= \frac{c_{dd}}{2\hbar l_\perp^3}\int_{-\infty}^{\infty}dz'\,n(z',t)\left\{G\left(\frac{|z-z'|}{l_\perp}\right)-\frac{4}{3}\delta\left(\frac{z-z'}{l_\perp}\right)\right\} \\
&= \frac{c_{dd}}{\hbar S^2\left\{3\sin^2\theta(t)-2\right\}}\int_{-\infty}^{\infty}dz'\,n(z',t)\,V_{\text{eff}}(z-z',t).
\end{aligned}
\tag{A.40}
$$

Chapter B

Derivation of the Landau-Lifshitz-Gilbert Equation in Spinor-Dipolar BEC with homogeneous local spin orientations

From now on, if there is no ambiguity, and for brevity, we drop the arguments such as x, y, z, t from the functions. From Eq. (A.39), we get

$$\begin{aligned}\hbar\frac{\partial\zeta_{m_1}}{\partial t} &= -\frac{\hbar}{\Psi}\frac{\partial\Psi}{\partial t}\zeta_{m_1} - \frac{\gamma+i}{1+\gamma^2}\left(-\frac{\hbar^2}{2m}\frac{1}{\Psi}\frac{\partial^2\Psi}{\partial z^2} + V + \frac{c_0}{2\pi l_\perp^2}n\right)\zeta_{m_1} \\ &+\frac{\gamma+i}{1+\gamma^2}\left\{\hbar\boldsymbol{b} - s\left(\boldsymbol{M} - 3M_z\boldsymbol{e}_z\right)\hbar P_{dd}\right\}\cdot\left\{\sum_{m_2=-s}^{s}\left(\hat{\boldsymbol{f}}\right)_{m_1,m_2}\zeta_{m_2}\right\} \\ &-\frac{\gamma+i}{1+\gamma^2}\sum_{j=1}^{s}\frac{c_{2j}}{2\pi l_\perp^2}n\sum_{\nu_1,\nu_2,\cdots,\nu_j=x,y,z} sM_{\nu_1,\nu_2,\cdots,\nu_j}\left\{\sum_{m_2=-s}^{s}\left(\hat{f}_{\nu_1}\hat{f}_{\nu_2}\cdots\hat{f}_{\nu_j}\right)_{m_1,m_2}\zeta_{m_2}\right\}. \end{aligned} \tag{B.1}$$

Since $\partial\left|\zeta\right|^2/\partial t = 0$ due to the normalization $\left|\zeta\right|^2 = 1$, we have

$$\begin{aligned}0 = 2\mathrm{Re}&\left\{-\frac{\hbar}{\Psi}\frac{\partial\Psi}{\partial t} - \frac{\gamma}{1+\gamma^2}\left(-\frac{\hbar^2}{2m}\frac{1}{\Psi}\frac{\partial^2\Psi}{\partial z^2} + V + \frac{c_0}{2\pi l_\perp^2}n\right)\right\} + \frac{i}{1+\gamma^2}\frac{\hbar^2}{2m}\left(\frac{1}{\Psi}\frac{\partial^2\Psi}{\partial z^2} - \frac{1}{\Psi^*}\frac{\partial^2\Psi^*}{\partial z^2}\right) \\ &+\frac{2\gamma}{1+\gamma^2}s\left\{\hbar\boldsymbol{b} - s\left(\boldsymbol{M} - 3M_z\boldsymbol{e}_z\right)\hbar P_{dd}\right\}\cdot\boldsymbol{M} - \frac{2\gamma}{1+\gamma^2}\sum_{j=1}^{s}\frac{c_{2j}}{2\pi l_\perp^2}n\sum_{\nu_1,\nu_2,\cdots,\nu_j=x,y,z}s^2M^2_{\nu_1,\nu_2,\cdots,\nu_j}. \end{aligned} \tag{B.2}$$

Hence the dynamics of the magnetization direction follows the equation

$$
\begin{aligned}
\hbar s\frac{\partial M_{\nu}}{\partial t} &= 2\mathrm{Re}\left\{\sum_{m_1,m_2=-s}^{s}\zeta_{m_1}^{\dagger}\left(\hat{f}_{\nu}\right)_{m_1,m_2}\left(\hbar\frac{\partial\zeta_{m_2}}{\partial t}\right)\right\} \\
&= -\frac{2\gamma}{1+\gamma^2}s^2M_{\nu}\left\{\hbar\boldsymbol{b}-s\left(\boldsymbol{M}-3M_z\boldsymbol{e}_z\right)\hbar P_{dd}\right\}\cdot\boldsymbol{M} \\
&\quad +\frac{2\gamma}{1+\gamma^2}M_{\nu}\sum_{j=1}^{s}\frac{c_{2j}}{2\pi l_{\perp}^2}n\sum_{\nu_1,\nu_2,\cdots,\nu_j=x,y,z}s^3M_{\nu_1,\nu_2,\cdots,\nu_j}^2 \\
&\quad +\frac{\gamma}{1+\gamma^2}\sum_{\mu=x,y,z}\left\{\hbar b_{\mu}-s\left(M_{\mu}-3M_z\delta_{\mu,z}\right)\hbar P_{dd}\right\}s\left\{\delta_{\mu,\nu}+(2s-1)\,M_{\mu}M_{\nu}\right\} \\
&\quad -2\mathrm{Re}\left\{\frac{\gamma+i}{1+\gamma^2}\sum_{j=1}^{s}\frac{c_{2j}}{2\pi l_{\perp}^2}n\sum_{\nu_1,\nu_2,\cdots,\nu_j=x,y,z}sM_{\nu_1,\nu_2,\cdots,\nu_j}\sum_{m_1,m_2=-s}^{s}\zeta_{m_1}^{\dagger}\left(\hat{f}_{\nu}\hat{f}_{\nu_1}\hat{f}_{\nu_2}\cdots\hat{f}_{\nu_j}\right)_{m_1,m_2}\zeta_{m_2}\right\} \\
&\quad -\frac{1}{1+\gamma^2}\sum_{\mu,\kappa=x,y,z}\left\{\hbar b_{\mu}-s\left(M_{\mu}-3M_z\delta_{\mu,z}\right)\hbar P_{dd}\right\}\epsilon_{\nu,\mu,\kappa}sM_{\kappa},
\end{aligned}
\tag{B.3}
$$

since the scalar product $\zeta^{\dagger}\left(\hat{f}_{\alpha}\hat{f}_{\beta}+\hat{f}_{\beta}\hat{f}_{\alpha}\right)\zeta = s\left\{\delta_{\alpha,\beta}+(2s-1)\,M_{\alpha}M_{\beta}\right\}$ [10].

By direct comparison, we can identify Eq. (B.4) below as being identical to Eq. (B21) in [10], the only difference consisting in the definition of $M_{\nu_1,\nu_2,\cdots,\nu_k}$: We employ a scaled version of $M_{\nu_1,\nu_2,\cdots,\nu_k}$, which is normalized to s in [10]. From Eq. (2.3) in the main text,

$$
\begin{aligned}
&\sum_{\nu_1,\nu_2,\cdots,\nu_j=x,y,z}M_{\nu_1,\nu_2,\cdots,\nu_j}\sum_{m_1,m_2=-s}^{s}\zeta_{m_1}^{\dagger}\left(\hat{f}_{\nu}\hat{f}_{\nu_1}\hat{f}_{\nu_2}\cdots\hat{f}_{\nu_j}\right)_{m_1,m_2}\zeta_{m_2} \\
&\qquad = \sum_{\nu_1,\nu_2,\cdots,\nu_j=x,y,z}M_{\nu_1,\nu_2,\cdots,\nu_j}^2 s^2M_{\nu},
\end{aligned}
\tag{B.4}
$$

which is real. Therefore, Eq. (B.3) can be written in the following form

$$
\begin{aligned}
\frac{\partial\boldsymbol{M}}{\partial t} &= -\frac{\gamma}{1+\gamma^2}\boldsymbol{M}\times\left[\boldsymbol{M}\times\left\{\boldsymbol{b}-s\left(\boldsymbol{M}-3M_z\boldsymbol{e}_z\right)P_{dd}\right\}\right]+\frac{1}{1+\gamma^2}\boldsymbol{M}\times\left\{\boldsymbol{b}-s\left(\boldsymbol{M}-3M_z\boldsymbol{e}_z\right)P_{dd}\right\} \\
&= \frac{1}{1+\gamma^2}\boldsymbol{M}\times\left(\boldsymbol{b}+3sP_{dd}M_z\boldsymbol{e}_z\right)-\frac{\gamma}{1+\gamma^2}\boldsymbol{M}\times\left[\boldsymbol{M}\times\left(\boldsymbol{b}+3sP_{dd}M_z\boldsymbol{e}_z\right)\right] \\
&= \boldsymbol{M}\times\left(\boldsymbol{b}+3sP_{dd}M_z\boldsymbol{e}_z\right)-\gamma\boldsymbol{M}\times\frac{\partial\boldsymbol{M}}{\partial t},
\end{aligned}
\tag{B.5}
$$

since $\boldsymbol{M} \cdot \frac{\partial \boldsymbol{M}}{\partial t} = 0$ holds.

As P is a function of z and t, but $\boldsymbol{M}$ is independent of z [$\boldsymbol{M}$ is the scaled local magnetization and our aim is to study a dipolar spinor BEC with unidirectional local magnetization (the homogeneous-local-spin-orientation limit)], by multiplying with $n\,(z,t)$ both sides of Eq. (B.5) and integrating along z, we finally get the LLG equation

$$\frac{\partial \boldsymbol{M}}{\partial t} = \boldsymbol{M} \times \left(\boldsymbol{b} + s\Lambda'_{dd} M_z \boldsymbol{e}_z\right) - \gamma \boldsymbol{M} \times \frac{\partial \boldsymbol{M}}{\partial t}, \tag{B.6}$$

where $\Lambda'_{dd}\,(t) = \{3/N\,(t)\} \int_{-\infty}^{\infty} dz\; n\,(z,t)\, P_{dd}\,(z,t)$.

Chapter C

Description of magnetostriction

For a dipolar spinor BEC without quadratic Zeeman term, when there is no dissipation ($\gamma = 0$), the mean-field equation in Eq. (1.7) can be written as

$$\begin{aligned}\mu_{m_1}(t)\,\psi_{m_1}(\boldsymbol{r},t) = &\left\{-\frac{\hbar^2}{2m}\nabla^2 + V_{\rm tr}(\boldsymbol{r}) + c_0\left|\psi(\boldsymbol{r},t)\right|^2\right\}\psi_{m_1}(\boldsymbol{r},t)\\ &+\sum_{j=1}^{s} c_{2j}\sum_{\nu_1,\nu_2,\cdots,\nu_j=x,y,z}\ \sum_{m_2=-s}^{s} F_{\nu_1,\nu_2,\cdots,\nu_j}(\boldsymbol{r},t)\left(\hat{f}_{\nu_1}\hat{f}_{\nu_2}\cdots\hat{f}_{\nu_j}\right)_{m_1,m_2}\psi_{m_2}(\boldsymbol{r},t)\\ &-\hbar\left\{\boldsymbol{b}-\boldsymbol{b}_{dd}(\boldsymbol{r},t)\right\}\cdot\sum_{m_2=-s}^{s}\left(\hat{\boldsymbol{f}}\right)_{m_1,m_2}\psi_{m_2}(\boldsymbol{r},t),\end{aligned} \tag{C.1}$$

where we have substituted $i\hbar\partial\psi_{m_1}(\boldsymbol{r},t)/\partial t = \mu_{m_1}(t)\,\psi_{m_1}(\boldsymbol{r},t)$.

Since we consider the homogeneous-local-spin-orientation limit, we may write $\psi_{m_1}(\boldsymbol{r},t) = \Psi_{\rm uni}(\boldsymbol{r},t)\,\zeta_{m_1}(t)$. In this limit, we have

$$\left|\psi(\boldsymbol{r},t)\right|^2 := \psi^\dagger(\boldsymbol{r},t)\,\psi(\boldsymbol{r},t) = \sum_{m_1=-s}^{s}\psi^\dagger_{m_1}(\boldsymbol{r},t)\,\psi_{m_1}(\boldsymbol{r},t) = \left|\Psi_{\rm uni}(\boldsymbol{r},t)\right|^2, \tag{C.2}$$

since $\sum_{m_1=-s}^{s}\left|\zeta_{m_1}(t)\right|^2 = 1$ from the definition of $\zeta_{m_1}(t)$ in Eq. (1.7). Thus $\left|\Psi_{\rm uni}(\boldsymbol{r},t)\right|^2$ is equal to the number density.

By introducing the chemical potential $\mu(t)$ as $\mu(t) := \sum_{m_1=-s}^{s}\mu_{m_1}(t)\left|\zeta_{m_1}(t)\right|^2$, one obtains

$$\mu(t)\,\Psi_{\rm uni}(\boldsymbol{r},t) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\rm tr}(\boldsymbol{r}) + \left[\Phi_{dd}(\boldsymbol{r},t) - s\hbar\left\{\boldsymbol{b}\cdot\boldsymbol{M}(t)\right\}\right]\right)\Psi_{\rm uni}(\boldsymbol{r},t)$$
$$+\left\{c_0 + s^2\sum_{j=1}^{s} c_{2j}\sum_{\nu_1,\nu_2,\cdots,\nu_j=x,y,z} M^2_{\nu_1,\nu_2,\cdots,\nu_j}(t)\right\}\left|\Psi_{\rm uni}(\boldsymbol{r},t)\right|^2\Psi_{\rm uni}(\boldsymbol{r},t), \quad \text{(C.3)}$$

where

$$\Phi_{dd}(\boldsymbol{r},t) := s^2 c_{dd}\left[\int d^3r' \left\{\sum_{\nu,\nu'=x,y,z} M_\nu(t)\,Q_{\nu,\nu'}\left(\boldsymbol{r}-\boldsymbol{r'}\right)M_{\nu'}(t)\right\}\left|\Psi_{\rm uni}\left(\boldsymbol{r'},t\right)\right|^2\right] \quad \text{(C.4)}$$

is the dipole-dipole mean-field potential [67].

Since we set our system so that $M_x(t) = \sin\theta(t)\cos\phi(t)$, $M_y(t) = \sin\theta(t)\sin\phi(t)$, and $M_z(t) = \cos\theta(t)$, from Eq. (A.19), $\Phi_{dd}(\boldsymbol{r},t)$ can be written as

$$\Phi_{dd}(\boldsymbol{r},t) = s^2 c_{dd}\left[\int d^3r'\,\frac{|\boldsymbol{r}-\boldsymbol{r'}|^2 - 3\left\{(\boldsymbol{r}-\boldsymbol{r'})\cdot\boldsymbol{M}(t)\right\}^2}{|\boldsymbol{r}-\boldsymbol{r'}|^5}\left|\Psi_{\rm uni}\left(\boldsymbol{r'},t\right)\right|^2\right]$$
$$= s^2 c_{dd}\left[\int d^3\bar{r}'\,\frac{|\bar{\boldsymbol{r}}-\bar{\boldsymbol{r}}'|^2 - 3\left\{(\bar{\boldsymbol{r}}-\bar{\boldsymbol{r}}')\cdot\boldsymbol{M}(t)\right\}^2}{|\bar{\boldsymbol{r}}-\bar{\boldsymbol{r}}'|^5}\left|\Psi_{\rm uni}\left(\bar{\boldsymbol{r}}',t\right)\right|^2\right]$$
$$= -\frac{3}{2}s^2 c_{dd}\sin^2\theta(t)\int d^3\bar{\eta}\,\left|\Psi_{\rm uni}(\bar{\boldsymbol{\eta}}+\bar{\boldsymbol{r}},t)\right|^2\frac{1}{\bar{\eta}^5}\left[\bar{\eta}^2 - \bar{\eta}_z^2 - 2\left\{\bar{\eta}_x\sin\phi(t) - \bar{\eta}_y\cos\phi(t)\right\}^2\right]$$
$$-3s^2 c_{dd}\sin\left\{2\theta(t)\right\}\int d^3\bar{\eta}\,\left|\Psi_{\rm uni}(\bar{\boldsymbol{\eta}}+\bar{\boldsymbol{r}},t)\right|^2\frac{\bar{\eta}_z}{\bar{\eta}^5}\left\{\bar{\eta}_x\cos\phi(t) + \bar{\eta}_y\sin\phi(t)\right\}$$
$$+\frac{1}{2}s^2 c_{dd}\left\{1 - 3\cos^2\theta(t)\right\}\int d^3\bar{\eta}\,\left|\Psi_{\rm uni}(\bar{\boldsymbol{\eta}}+\bar{\boldsymbol{r}},t)\right|^2\frac{1}{\bar{\eta}^5}\left(3\bar{\eta}_z^2 - \bar{\eta}^2\right). \quad \text{(C.5)}$$

where $\bar{\boldsymbol{r}} := \boldsymbol{r}/L$ with L being some length which scales r (so that $\bar{\boldsymbol{r}}$ is a dimensionless vector). For example, in quasi-1D with trap potential (Eq. (A.2)), $L = l_\perp$. Note that, in the special case where $\boldsymbol{M}(t) = M_z(t)\,\boldsymbol{e}_z$, the form of Eq. (C.5) becomes identical to Eq.(6) in Ref. [68].

Since we concentrate on quasi-1D gases with trap potential in Eq. (A.2), we will explicitly compute the form of $\Phi_{dd}(\boldsymbol{r},t)$ for the quasi-1D setup. By writing

$$\left|\Psi_{\rm uni}(\boldsymbol{r},t)\right|^2 = \frac{e^{-\rho^2/l_\perp^2}}{\pi l_\perp^2}\left|\Psi(z,t)\right|^2, \quad \text{(C.6)}$$

and integrating out x and y directions, one can get the quasi-1D dipole-dipole-interaction mean-field potential $\Phi_{dd}(z,t)$ as

$$\Phi_{dd}(z,t) = \frac{c_{dd}}{2l_\perp^2} s^2 \left\{1 - 3M_z^2(t)\right\} \left\{ \int_{-\infty}^{\infty} d\bar{z}\ \left|\Psi(z + \bar{z}l_\perp, t)\right|^2 G(|\bar{z}|) - \frac{4}{3} \left|\Psi(z,t)\right|^2 \right\}. \tag{C.7}$$

Now, let us consider box trap in quasi-1D case, i.e. $V(z) = 0$ for $|z| \leq L_z$ and $V(z) = \infty$ for $|z| > L_z$. Then we may write

$$\left|\Psi(z,t)\right|^2 = \left[\begin{array}{cl} N/(2L_z) & \text{for } |z| \leq L_z, \\ 0 & \text{for } |z| > L_z, \end{array} \right. \tag{C.8}$$

since $V(z) = 0$ for $-L_z \leq z \leq L_z$. Thus, $\Phi_{dd}(z,t)$ can be written as

$$\Phi_{dd}(z,t) = \left[\begin{array}{ll} \bar{\Phi}_{dd}(t) \left\{ \int_{-(L_z+z)/l_\perp}^{(L_z-z)/l_\perp} d\bar{z}\ G(|\bar{z}|) - \frac{4}{3} \right\} & \text{for } |z| \leq L_z, \\ \\ \bar{\Phi}_{dd}(t) \int_{-(L_z+z)/l_\perp}^{(L_z-z)/l_\perp} d\bar{z}\ G(|\bar{z}|) & \text{for } |z| > L_z, \end{array} \right. \tag{C.9}$$

where $\bar{\Phi}_{dd}(t) := N c_{dd} S^2 \left\{1 - 3M_z^2(t)\right\} / \left(2L_z l_\perp^2\right)$. $\Phi_{dd}(z,t)$ is discontinuous at $z = \pm L_z$ because of the sudden change of the density at the boundary ($z = \pm L_z$) due to box trap potential.

Defining the scaled density-density mean-field potential $\bar{\Phi}_{dd}(z) := \Phi_{dd}(z,t)/\bar{\Phi}_{dd}(t)$, we obtain Fig. C.1, for two different axial extensions, $L_z/l_\perp = 10$ and 30. As Fig. C.1 clearly illustrates, in a box-trapped quasi-1D gas, $\Phi_{dd}(z,t)$ becomes approximately constant for $|z| < L_c$ and $L_c \to L_z$ for $L_z/l_\perp \gg 1$. Depending on the value of $\boldsymbol{M}(t)$, $\Phi_{dd}(\boldsymbol{r},t)$ will introduce either a repulsive or an attractive force. This force will however exist only near the boundary for a box trap, where it can lead to a slight modification of the density of atomes. Its relative influence decreases with increasing extension of the trapped gas along the z axis, and can therefore be consistently neglected in the approximation of constant particle-density.

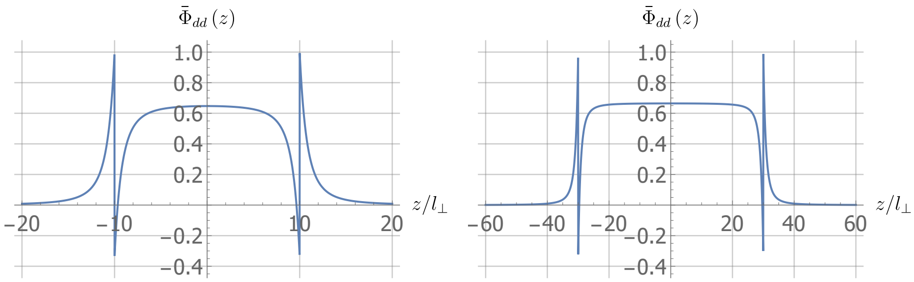


Figure C.1: Scaled dipole-dipole mean-field potential $\bar{\Phi}_{dd}(z)$ as a function of z for a quasi-1D box trap. (Left) $L_z/l_\perp = 10$. (Right) $L_z/l_\perp = 30$. These figures are from our paper [11].

However, to assess whether significant magnetostriction occurs, one has to consider, in addition to Φ_{dd}, the trap potential V_{tr} and the 'quasi' density-density interaction mean field potential Φ_0 defined as

$$\Phi_0(\boldsymbol{r},t) := \left\{ c_0 + s^2 \sum_{j=1}^{s} c_{2j} \sum_{\nu_1,\nu_2,\cdots,\nu_j=x,y,z} M^2_{\nu_1,\nu_2,\cdots,\nu_j}(t) \right\} |\Psi_{\text{uni}}(\boldsymbol{r},t)|^2 . \tag{C.10}$$

We can coin $\Phi_0(\boldsymbol{r},t)$ a 'quasi' density-density interaction mean field potential because only c_0 is a density-density interaction coefficient (c_{2j} are interaction coefficients parametrizing the spin-spin interactions for spin-s gas where j is an integer with $1 \leq j \leq s$. For example, c_2 is the spin-spin interaction coefficient of a spin-one gas). In our quasi-1D case, this $\Phi_0(\boldsymbol{r},t)$ potential is $\Phi_0(z,t)$ where

$$\Phi_0(z,t) := \left\{ \frac{c_0}{2\pi l_\perp^2} + s^2 \sum_{j=1}^{s} \frac{c_{2j}}{2\pi l_\perp^2} \sum_{\nu_1,\nu_2,\cdots,\nu_j=x,y,z} M^2_{\nu_1,\nu_2,\cdots,\nu_j}(t) \right\} |\Psi(z,t)|^2 . \tag{C.11}$$

In the main text, we assume that $c_0 \gg s^2 \sum_{j=1}^{s} c_{2j} \sum_{\nu_1,\nu_2,\cdots,\nu_j=x,y,z} M^2_{\nu_1,\nu_2,\cdots,\nu_j}(t)$. For spin-1 ^{23}Na or ^{87}Rb, $s = 1$ and $c_0 \simeq 100\,|c_2|$ [2, 12], so this is an appropriate assumption (note that $|\boldsymbol{M}(t)| = 1$). The values of the c_{2j} are not yet established for ^{166}Er. We therefore tacitly assume in the main text, when calculating concrete numerical examples for ^{166}Er, that the above condition also still holds, despite the prefactor s^2 enhancing the importance of spin-spin interactions in $\Phi_0(z,t)$. When this assumption is not applicable, one is required to take into account the time dependence of $\Phi_0(z,t)$ due to $\boldsymbol{M}(t)$ together with magnetostriction due to $\Phi_{dd}(z,t)$, which will change the system size L_z as a function of t. This will in turn change the integration domain and quasi-1D density $n(z,t) = |\Psi(z,t)|^2$, and incur also a changed time dependence of $\Lambda'_{dd}(t)$, and the solution of the coupled system of equations (B.6) and (C.3) needs to be found self-consistently.

For a harmonic trap, due to the resulting inhomogeneity of $|\Psi(z,t)|^2$, $\Phi_{dd}(z,t)$ will have more significant spatial dependence than its box trap counterpart shown in Fig. C.1. Here, we note that Ref. [68] has already shown (for a spin-polarized gas, c_{2k} couplings not included), that magnetostriction occurs in a harmonic trap. The effect of magnetostriction is generally expected to be larger in a harmonic trap when compared to a box trap with similar geometrical and dynamical parameters for large relative system size $L_z/l_\perp \gg 1$, at least under the above condition that the $s^2 c_{2k}/c_0$ are sufficiently small.

Chapter D

Analytic Expressions of $\tilde{\omega}_p(k)$, $u_{pq}(k)$, and $v_{pq}(k)$

Based on the approach of [43], we show analytic expressions of $\tilde{\omega}_p(k)$, $u_{pq}(k)$, and $v_{pq}(k)$ in Eqs. (1.54) where $p, q = 1, 2$. The basic idea is rotating $U(k)$ and $V(k)$ in Eqs. (1.37) to make eigenvalue problem have more simple form. We define

$$U_R(k) := \begin{bmatrix} u_{R,11}(k) & u_{R,21}(k) \\ u_{R,12}(k) & u_{R,22}(k) \end{bmatrix}, \quad V_R(k) := \begin{bmatrix} v_{R,11}(k) & v_{R,21}(k) \\ v_{R,12}(k) & v_{R,22}(k) \end{bmatrix}, \tag{D.1}$$

and set $U(k) = \Theta(k) U_R(k)$ and $V(k) = \Theta(k) V_R(k)$ where

$$\Theta(k) := \begin{bmatrix} \cos\phi(k) & -\sin\phi(k) \\ \sin\phi(k) & \cos\phi(k) \end{bmatrix}. \tag{D.2}$$

Then Eq. (1.48) can be written as

$$\begin{bmatrix} M_{B,1}(k) & M_{B,2} \\ -M_{B,2}^* & -M_{B,1}^*(k) \end{bmatrix} \begin{bmatrix} U(k) \\ V(k) \end{bmatrix} = \begin{bmatrix} U(k) \\ V(k) \end{bmatrix} \tilde{\mathcal{E}}(k), \qquad \tilde{\mathcal{E}}(k) := \begin{bmatrix} \tilde{\omega}_1(k) & 0 \\ 0 & \tilde{\omega}_2(k) \end{bmatrix}.$$

$$\begin{aligned} \Rightarrow \quad & \begin{bmatrix} \Theta(k) & 0 \\ 0 & \Theta(k) \end{bmatrix}^{-1} \begin{bmatrix} M_{B,1}(k) & M_{B,2} \\ -M_{B,2}^* & -M_{B,1}^*(k) \end{bmatrix} \begin{bmatrix} \Theta(k) & 0 \\ 0 & \Theta(k) \end{bmatrix} \begin{bmatrix} U_R(k) \\ V_R(k) \end{bmatrix} \\ & = \begin{bmatrix} U_R(k) \\ V_R(k) \end{bmatrix} \tilde{\mathcal{E}}(k). \end{aligned} \tag{D.3}$$

Let

$$M_R(k) := \begin{bmatrix} \Theta(k) & 0 \\ 0 & \Theta(k) \end{bmatrix}^{-1} \begin{bmatrix} M_{B,1}(k) & M_{B,2} \\ -M_{B,2}^* & -M_{B,1}^*(k) \end{bmatrix} \begin{bmatrix} \Theta(k) & 0 \\ 0 & \Theta(k) \end{bmatrix}. \tag{D.4}$$

Since $\Theta^{-1}(k) = \Theta^T(k)$, we need to calculate $\Theta^T(k) M_{B,1}(k) \Theta(k)$ and $\Theta^T(k) M_{B,2}\Theta(k)$ to get components of $M_R(k)$.

D.0.1 Part 1 - $\Theta^T(k) M_{B,1}(k) \Theta(k)$

$$\begin{aligned} \Theta^T(k) M_{B,1}(k) \Theta(k) &= \begin{bmatrix} \cos\phi(k) & \sin\phi(k) \\ -\sin\phi(k) & \cos\phi(k) \end{bmatrix} \begin{bmatrix} M_{11}(k) & M_{12} \\ M_{12}^* & M_{22}(k) \end{bmatrix} \begin{bmatrix} \cos\phi(k) & -\sin\phi(k) \\ \sin\phi(k) & \cos\phi(k) \end{bmatrix} \\ &= \begin{bmatrix} A_{11}(k) & A_{12}(k) \\ A_{12}^*(k) & A_{22}(k) \end{bmatrix}, \end{aligned} \tag{D.5}$$

where

$$\begin{aligned} A_{11}(k) &:= M_{11}(k)\cos^2\phi(k) + M_{22}(k)\sin^2\phi(k) + \mathrm{Re}(M_{12})\sin\{2\phi(k)\} \\ &= \frac{M_{11}(k)+M_{22}(k)}{2} + \frac{M_{11}(k)-M_{22}(k)}{2}\cos\{2\phi(k)\} + \mathrm{Re}(M_{12})\sin\{2\phi(k)\}, \\ A_{22}(k) &:= M_{22}(k)\cos^2\phi(k) + M_{11}(k)\sin^2\phi(k) - \mathrm{Re}(M_{12})\sin\{2\phi(k)\} \\ &= \frac{M_{11}(k)+M_{22}(k)}{2} - \frac{M_{11}(k)-M_{22}(k)}{2}\cos\{2\phi(k)\} - \mathrm{Re}(M_{12})\sin\{2\phi(k)\}, \\ A_{12}(k) &:= M_{12}\cos^2\phi(k) - M_{12}^*\sin^2\phi(k) - \frac{1}{2}\{M_{11}(k) - M_{22}(k)\}\sin\{2\phi(k)\} \\ &= i\mathrm{Im}(M_{12}) + \mathrm{Re}(M_{12})\cos\{2\phi(k)\} - \frac{1}{2}\{M_{11}(k) - M_{22}(k)\}\sin\{2\phi(k)\}. \end{aligned} \tag{D.6}$$

D.0.2 Part 2 - $\Theta^T(k) M_{B,2}\Theta(k)$

$$\begin{aligned} \Theta^T(k) M_{B,2}(k) \Theta(k) &= \begin{bmatrix} \cos\phi(k) & \sin\phi(k) \\ -\sin\phi(k) & \cos\phi(k) \end{bmatrix} \begin{bmatrix} M_{13} & M_{14} \\ M_{14} & M_{24} \end{bmatrix} \begin{bmatrix} \cos\phi(k) & -\sin\phi(k) \\ \sin\phi(k) & \cos\phi(k) \end{bmatrix} \\ &= \begin{bmatrix} A_{13}(k) & A_{14}(k) \\ A_{14}(k) & A_{24}(k) \end{bmatrix}, \end{aligned} \tag{D.7}$$

where

$$
\begin{aligned}
A_{13}(k) &:= \frac{M_{13}+M_{24}}{2}+\frac{M_{13}-M_{24}}{2}\cos\{2\phi(k)\}+M_{14}\sin\{2\phi(k)\}, \\
A_{24}(k) &:= \frac{M_{13}+M_{24}}{2}-\frac{M_{13}-M_{24}}{2}\cos\{2\phi(k)\}-M_{14}\sin\{2\phi(k)\}, \\
A_{14}(k) &:= M_{14}\cos\{2\phi(k)\}-\frac{1}{2}(M_{13}-M_{24})\sin\{2\phi(k)\}. \qquad \text{(D.8)}
\end{aligned}
$$

D.0.3 Determining Rotation Angle $\phi(k)$

Note that, from Eqs. (D.6) and (D.8), we get

$$
\begin{aligned}
A_{11}(k) &= \frac{\{M_{11}(k)-M_{13}\}+\{M_{22}(k)-M_{24}\}}{2} \\
&\quad +\frac{\{M_{11}(k)-M_{13}\}-\{M_{22}(k)-M_{24}\}}{2}\cos\{2\phi(k)\} \\
&\quad +\{\mathrm{Re}(M_{12})-M_{14}\}\sin\{2\phi(k)\}+A_{13}(k), \\
A_{22}(k) &= \frac{\{M_{11}(k)-M_{13}\}+\{M_{22}(k)-M_{24}\}}{2} \\
&\quad -\frac{\{M_{11}(k)-M_{13}\}-\{M_{22}(k)-M_{24}\}}{2}\cos\{2\phi(k)\} \\
&\quad -\{\mathrm{Re}(M_{12})-M_{14}\}\sin\{2\phi(k)\}+A_{24}(k), \\
A_{12}(k) &= i\mathrm{Im}(M_{12})+\{\mathrm{Re}(M_{12})-M_{14}\}\cos\{2\phi(k)\} \\
&\quad -\frac{1}{2}[\{M_{11}(k)-M_{13}\}-\{M_{22}(k)-M_{24}\}]\sin\{2\phi(k)\}+A_{14}(k). \qquad \text{(D.9)}
\end{aligned}
$$

When the system is stable, Bogoliubov excitation energies $\tilde{\omega}_p(k)$ are real. These conditions are satisfied if every $M_{\zeta\eta}$ are real (see Eqs. (1.54)). In that case, we can make $A_{12}(k)=A_{14}(k)$ by setting

$$
\tan\{2\phi(k)\}=\frac{2(M_{12}-M_{14})}{\{M_{11}(k)-M_{13}\}-\{M_{22}(k)-M_{24}\}}. \qquad \text{(D.10)}
$$

Then Eqs. (D.3) (or equivalently Eqs. (1.48)) become

$$
\begin{bmatrix} \mathcal{A}_1(k) & \mathcal{A}_2(k) \\ -\mathcal{A}_2(k) & -\mathcal{A}_1(k) \end{bmatrix}
\begin{bmatrix} u_{R,p}(k) \\ v_{R,p}(k) \end{bmatrix}
=\tilde{\omega}_p(k)\begin{bmatrix} u_{R,p}(k) \\ v_{R,p}(k) \end{bmatrix}, \qquad \text{(D.11)}
$$

where $u_{R,p}(k) := \begin{bmatrix} u_{R,p1}(k) & u_{R,p2}(k) \end{bmatrix}^T$, $v_{R,p}(k) := \begin{bmatrix} v_{R,p1}(k) & v_{R,p2}(k) \end{bmatrix}^T$,

$$\mathcal{A}_2(k) := \begin{bmatrix} A_{13}(k) & A_{14}(k) \\ A_{14}(k) & A_{24}(k) \end{bmatrix}, \quad \mathcal{A}_1(k) := \mathcal{A}_2(k) + \begin{bmatrix} \tilde{\omega}_{a1}(k) & 0 \\ 0 & \tilde{\omega}_{a2}(k) \end{bmatrix}, \tag{D.12}$$

$$\begin{aligned} \tilde{\omega}_{a1}(k) &:= \frac{\{M_{11}(k) - M_{13}\} + \{M_{22}(k) - M_{24}\}}{2} \\ &\quad + \frac{\{M_{11}(k) - M_{13}\} - \{M_{22}(k) - M_{24}\}}{2} \cos\{2\phi(k)\} + (M_{12} - M_{14}) \sin\{2\phi(k)\} \\ &= \frac{\tilde{\omega}_c(k) + \tilde{\omega}_d(k) + \sqrt{\{\tilde{\omega}_c(k) - \tilde{\omega}_d(k)\}^2 + 4(M_{12} - M_{14})^2}}{2}, \\ \tilde{\omega}_{a2}(k) &:= \frac{\{M_{11}(k) - M_{13}\} + \{M_{22}(k) - M_{24}\}}{2} \\ &\quad - \frac{\{M_{11}(k) - M_{13}\} - \{M_{22}(k) - M_{24}\}}{2} \cos\{2\phi(k)\} - (M_{12} - M_{14}) \sin\{2\phi(k)\} \\ &= \frac{\tilde{\omega}_c(k) + \tilde{\omega}_d(k) - \sqrt{\{\tilde{\omega}_c(k) - \tilde{\omega}_d(k)\}^2 + 4(M_{12} - M_{14})^2}}{2}, \end{aligned} \tag{D.13}$$

where we defined $\tilde{\omega}_c(k) := M_{11}(k) - M_{13}$ and $\tilde{\omega}_d(k) := M_{22}(k) - M_{24}$.

By plugging Eqs. (1.30), (1.31), (1.32), and (1.33), we get $\tilde{\omega}_c(k) = k^2\xi_a^2 + 2|\tilde{\alpha}|y$, $\tilde{\omega}_d(k) = \left(k^2\xi_a^2/2\right) + |\tilde{\alpha}|\left(1 - 2y^2\right)/(2y)$, and $M_{12} - M_{14} = \tilde{\alpha}\sqrt{1 - 2y^2}$.

D.1 Analytic Expressions of $\tilde{\omega}_p(k)$

From Eqs. (D.11), (D.12), and (D.13), one can get

$$\tilde{\omega}_p(k) = \sqrt{\frac{\tilde{\omega}_{a1}'^2(k) + \tilde{\omega}_{a2}'^2(k) + (2p-3)\sqrt{\left\{\tilde{\omega}_{a1}'^2(k) - \tilde{\omega}_{a2}'^2(k)\right\}^2 + 16A_{14}^2(k)\,\tilde{\omega}_{a1}(k)\,\tilde{\omega}_{a2}(k)}}{2}}, \tag{D.14}$$

which is same as [43] where $\tilde{\omega}_{a1}'^2(k) := \tilde{\omega}_{a1}(k)\{\tilde{\omega}_{a1}(k) + 2A_{13}(k)\}$ and $\tilde{\omega}_{a2}'^2(k) := \tilde{\omega}_{a2}(k)\{\tilde{\omega}_{a2}(k) + 2A_{24}(k)\}$.

D.2 Analytic Expressions of $u_{pq}(k)$, and $v_{pq}(k)$

From Eqs. (1.42) and (1.47), as $U(k) = \Theta(k) U_R(k)$ and $V(k) = \Theta(k) V_R(k)$, $U_R(k)$ and $V_R(k)$ satisfy Bogoliubov constraints where

$$\begin{aligned} &U_R(k) U_R^\dagger(k) - V_R^*(k) V_R^T(k) = I, \quad U_R(k) V_R^\dagger(k) - V_R^*(k) U_R^T(k) = 0, \\ &U_R^\dagger(k) U_R(k) - V_R^\dagger(k) V_R(k) = I, \quad U_R^T(k) V_R(k) - V_R^T(k) U_R(k) = 0. \end{aligned} \tag{D.15}$$

Since we consider stable solution where $\tilde{\omega}_p(k)$ are all real, from Eqs. (D.11), (D.12), and (D.13), $u_{R,p}(k)$ and $v_{R,p}(k)$ are also real where

$$u_{R,p1}(k) = \frac{\{\tilde{\omega}_{a1}(k) + \tilde{\omega}_p(k)\} |A_{14}(k)| \sqrt{\tilde{\omega}_{a2}(k)}}{\sqrt{\tilde{\omega}_p(k) \left[4\tilde{\omega}_{a1}(k)\tilde{\omega}_{a2}(k) A_{14}^2(k) + \left\{\tilde{\omega}_{a1}'^2(k) - \tilde{\omega}_p^2(k)\right\}^2\right]}}, \tag{D.16}$$

$$u_{R,p2}(k) = -\text{sign}\{A_{14}(k)\} \frac{\{\tilde{\omega}_{a2}(k) + \tilde{\omega}_p(k)\}\left\{\tilde{\omega}_{a1}'^2(k) - \tilde{\omega}_p^2(k)\right\}}{2\sqrt{\tilde{\omega}_{a2}(k)\tilde{\omega}_p(k) \left[4\tilde{\omega}_{a1}(k)\tilde{\omega}_{a2}(k) A_{14}^2(k) + \left\{\tilde{\omega}_{a1}'^2(k) - \tilde{\omega}_p^2(k)\right\}^2\right]}}, \tag{D.17}$$

$$v_{R,p1}(k) = \frac{\{\tilde{\omega}_{a1}(k) - \tilde{\omega}_p(k)\} |A_{14}(k)| \sqrt{\tilde{\omega}_{a2}(k)}}{\sqrt{\tilde{\omega}_p(k) \left[4\tilde{\omega}_{a1}(k)\tilde{\omega}_{a2}(k) A_{14}^2(k) + \left\{\tilde{\omega}_{a1}'^2(k) - \tilde{\omega}_p^2(k)\right\}^2\right]}}, \tag{D.18}$$

and

$$v_{R,p2}(k) = -\text{sign}\{A_{14}(k)\} \frac{\{\tilde{\omega}_{a2}(k) - \tilde{\omega}_p(k)\}\left\{\tilde{\omega}_{a1}'^2(k) - \tilde{\omega}_p^2(k)\right\}}{2\sqrt{\tilde{\omega}_{a2}(k)\tilde{\omega}_p(k) \left[4\tilde{\omega}_{a1}(k)\tilde{\omega}_{a2}(k) A_{14}^2(k) + \left\{\tilde{\omega}_{a1}'^2(k) - \tilde{\omega}_p^2(k)\right\}^2\right]}}. \tag{D.19}$$

Chapter E

Definitions of $\Xi_{pq}(k)$

By applying Eqs. (1.28) to $\hat{R}_2(t)$ in Eq. (3.5), we get

$$\hat{R}_2(t) = -i\tilde{\alpha}\sum_{\boldsymbol{k}\neq 0}\delta\Psi^\dagger(\boldsymbol{k},t)\begin{bmatrix} M_{R,1} & M_{R,2} \\ -M_{R,2}^* & -M_{R,1}^* \end{bmatrix}\delta\Psi(\boldsymbol{k},t), \tag{E.1}$$

where

$$M_{R,1} := \begin{bmatrix} 0 & xe^{-i\varphi_a} \\ -xe^{i\varphi_a} & 0 \end{bmatrix}, \quad M_{R,2} := \begin{bmatrix} ye^{i\varphi_m} & 0 \\ 0 & 0 \end{bmatrix}. \tag{E.2}$$

Imposing Bogoliubov transformation in Eq. (1.35) leads us to

$$\hat{R}_2(t) = -i\tilde{\alpha}\sum_{\boldsymbol{k}\neq 0}\begin{bmatrix} \hat{b}_1^\dagger(\boldsymbol{k},t) & \hat{b}_2^\dagger(\boldsymbol{k},t) & \hat{b}_1(-\boldsymbol{k},t) & \hat{b}_2(-\boldsymbol{k},t) \end{bmatrix} M_{R,B}(k)\begin{bmatrix} \hat{b}_1(\boldsymbol{k},t) \\ \hat{b}_2(\boldsymbol{k},t) \\ \hat{b}_1^\dagger(-\boldsymbol{k},t) \\ \hat{b}_2^\dagger(-\boldsymbol{k},t) \end{bmatrix}, \tag{E.3}$$

where

$$
\begin{aligned}
M_{R,B}(k) &:= \begin{bmatrix} U^{\dagger}(k) & V^{\dagger}(k) \\ V^{T}(k) & U^{T}(k) \end{bmatrix} \begin{bmatrix} M_{R,1} & M_{R,2} \\ -M_{R,2}^{*} & -M_{R,1}^{*} \end{bmatrix} \begin{bmatrix} U(k) & V^{*}(k) \\ V(k) & U^{*}(k) \end{bmatrix} \\
&= \begin{bmatrix} M_{R,B;1}(k) & M_{R,B;2}(k) \\ -M_{R,B;2}^{*}(k) & -M_{R,B;1}^{*}(k) \end{bmatrix},
\end{aligned} \tag{E.4}
$$

$$
M_{R,B;1}(k) := U^{\dagger}(k) M_{R,1} U(k) - V^{\dagger}(k) M_{R,1}^{*} V(k) + U^{\dagger}(k) M_{R,2} V(k) - V^{\dagger}(k) M_{R,2}^{*} U(k), \tag{E.5}
$$

and

$$
M_{R,B;2}(k) := U^{\dagger}(k) M_{R,1} V^{*}(k) - V^{\dagger}(k) M_{R,1}^{*} U^{*}(k) + U^{\dagger}(k) M_{R,2} U^{*}(k) - V^{\dagger}(k) M_{R,2}^{*} V^{*}(k). \tag{E.6}
$$

Since $M_{R,1}^{\dagger} = -M_{R,1}$ and $M_{R,2}^{\dagger} = M_{R,2}^{*}$, $M_{R,B}^{\dagger} = -M_{R,B}$. Thus, we may write

$$
M_{R,B;1}(k) = \begin{bmatrix} \Xi_{11}(k) & \Xi_{12}(k) \\ -\Xi_{12}^{*}(k) & \Xi_{22}(k) \end{bmatrix}, \quad M_{R,B;2}(k) = \begin{bmatrix} \Xi_{13}(k) & \Xi_{14}(k) \\ \Xi_{14}(k) & \Xi_{24}(k) \end{bmatrix}, \tag{E.7}
$$

with $\Xi_{\zeta\zeta}(k)$ being pure imaginary ($\zeta = 1, 2$) as $M_{R,B;1}^{\dagger}(k) = -M_{R,B;1}(k)$ and $M_{R,B;2}^{\dagger}(k) = M_{R,B;2}^{*}(k)$.

E.1 Analytic Expressions

From D, by using $M_{R,1}^{T} = -M_{R,1}^{*}$ and $M_{R,2}^{\dagger} = M_{R,2}^{*}$, Eqs. (E.5) and (E.6) can be written as

$$
\begin{aligned}
M_{R,B;1}(k) &= U_R^{\dagger}(k)\,\Theta^T(k)\,M_{R,1}\Theta(k)\,U_R(k) - V_R^{\dagger}(k)\,\Theta^T(k)\,M_{R,1}^{*}\Theta(k)\,V_R(k) \\
&\quad + U_R^{\dagger}(k)\,\Theta^T(k)\,M_{R,2}\Theta(k)\,V_R(k) - \left\{U_R^{\dagger}(k)\,\Theta^T(k)\,M_{R,2}\Theta(k)\,V_R(k)\right\}^{\dagger}, \\
M_{R,B;2}(k) &= U_R^{\dagger}(k)\,\Theta^T(k)\,M_{R,1}\Theta(k)\,V_R^{*}(k) + \left\{U_R^{\dagger}(k)\,\Theta^T(k)\,M_{R,1}\Theta(k)\,V_R^{*}(k)\right\}^{T} \\
&\quad + U_R^{\dagger}(k)\,\Theta^T(k)\,M_{R,2}\Theta(k)\,U_R^{*}(k) - V_R^{\dagger}(k)\,\Theta^T(k)\,M_{R,2}^{*}\Theta(k)\,V_R^{*}(k)\,.
\end{aligned}
\tag{E.8}
$$

E.1.1 Part 1 - $\Theta^T(k)\,M_{R,1}\Theta(k)$

$$
\begin{aligned}
\Theta^T(k)\,M_{R,1}\Theta(k) &= \begin{bmatrix} \cos\phi(k) & \sin\phi(k) \\ -\sin\phi(k) & \cos\phi(k) \end{bmatrix} \begin{bmatrix} xe^{-i\varphi_a}\sin\phi(k) & xe^{-i\varphi_a}\cos\phi(k) \\ -xe^{i\varphi_a}\cos\phi(k) & xe^{i\varphi_a}\sin\phi(k) \end{bmatrix} \\
&= \begin{bmatrix} -ix\sin\varphi_a\sin\{2\phi(k)\} & x\left[\cos\varphi_a - i\sin\varphi_a\cos\{2\phi(k)\}\right] \\ -x\left[\cos\varphi_a + i\sin\varphi_a\cos\{2\phi(k)\}\right] & ix\sin\varphi_a\sin\{2\phi(k)\} \end{bmatrix}.
\end{aligned}
\tag{E.9}
$$

E.1.2 Part 2 - $\Theta^T(k)\,M_{R,2}\Theta(k)$

$$
\begin{aligned}
\Theta^T(k)\,M_{R,2}\Theta(k) &= ye^{i\varphi_m} \begin{bmatrix} \cos\phi(k) & \sin\phi(k) \\ -\sin\phi(k) & \cos\phi(k) \end{bmatrix} \begin{bmatrix} \cos\phi(k) & -\sin\phi(k) \\ 0 & 0 \end{bmatrix} \\
&= ye^{i\varphi_m} \begin{bmatrix} \frac{1}{2} + \frac{1}{2}\cos\{2\phi(k)\} & -\frac{1}{2}\sin\{2\phi(k)\} \\ -\frac{1}{2}\sin\{2\phi(k)\} & \frac{1}{2} - \frac{1}{2}\cos\{2\phi(k)\} \end{bmatrix}.
\end{aligned}
\tag{E.10}
$$

E.1.3 Final Results - $\Xi_{pp}(k)$ and $\Xi_{12}(k)$

From Eqs. (E.8), (E.9), and (E.10), we get

$$
U_R^{\dagger}(k)\,\Theta^T(k)\,M_{R,1}\Theta(k)\,U_R(k) = \begin{bmatrix} \mathcal{M}_{R1,11}(k) & \mathcal{M}_{R1,12}(k) \\ \mathcal{M}_{R1,21}(k) & \mathcal{M}_{R1,22}(k) \end{bmatrix}, \tag{E.11}
$$

where

$$\begin{aligned}
\mathcal{M}_{R1,pp}(k) &:= -ix\sin\varphi_a\sin\{2\phi(k)\}\left|u_{R,p1}(k)\right|^2 + x\left[\cos\varphi_a - i\sin\varphi_a\cos\{2\phi(k)\}\right]u^*_{R,p1}(k)\,u_{R,p2}(k) \\
&\quad -x\left[\cos\varphi_a + i\sin\varphi_a\cos\{2\phi(k)\}\right]u_{R,p1}(k)\,u^*_{R,p2}(k) + ix\sin\varphi_a\sin\{2\phi(k)\}\left|u_{R,p2}(k)\right|^2 \\
&= -ix\sin\varphi_a\sin\{2\phi(k)\}\left\{\left|u_{R,p1}(k)\right|^2 - \left|u_{R,p2}(k)\right|^2\right\} + 2ix\cos\varphi_a\mathrm{Im}\left\{u^*_{R,p1}(k)\,u_{R,p2}(k)\right\} \\
&\quad -2ix\sin\varphi_a\cos\{2\phi(k)\}\,\mathrm{Re}\left\{u^*_{R,p1}(k)\,u_{R,p2}(k)\right\},
\end{aligned} \tag{E.12}$$

$$\begin{aligned}
\mathcal{M}_{R1,12}(k) &:= -ix\sin\varphi_a\sin\{2\phi(k)\}\,u^*_{R,11}(k)\,u_{R,21}(k) + x\left[\cos\varphi_a - i\sin\varphi_a\cos\{2\phi(k)\}\right]u^*_{R,11}(k)\,u_{R,22}(k) \\
&\quad -x\left[\cos\varphi_a + i\sin\varphi_a\cos\{2\phi(k)\}\right]u^*_{R,12}(k)\,u_{R,21}(k) + ix\sin\varphi_a\sin\{2\phi(k)\}\,u^*_{R,12}(k)\,u_{R,22}(k) \\
&= -ix\sin\varphi_a\sin\{2\phi(k)\}\left\{u^*_{R,11}(k)\,u_{R,21}(k) - u^*_{R,12}(k)\,u_{R,22}(k)\right\} \\
&\quad +x\cos\varphi_a\left\{u^*_{R,11}(k)\,u_{R,22}(k) - u^*_{R,12}(k)\,u_{R,21}(k)\right\} \\
&\quad -ix\sin\varphi_a\cos\{2\phi(k)\}\left\{u^*_{R,11}(k)\,u_{R,22}(k) + u^*_{R,12}(k)\,u_{R,21}(k)\right\},
\end{aligned} \tag{E.13}$$

and

$$\begin{aligned}
\mathcal{M}_{R1,21}(k) &:= -ix\sin\varphi_a\sin\{2\phi(k)\}\,u^*_{R,21}(k)\,u_{R,11}(k) + x\left[\cos\varphi_a - i\sin\varphi_a\cos\{2\phi(k)\}\right]u^*_{R,21}(k)\,u_{R,12}(k) \\
&\quad -x\left[\cos\varphi_a + i\sin\varphi_a\cos\{2\phi(k)\}\right]u^*_{R,22}(k)\,u_{R,11}(k) + ix\sin\varphi_a\sin\{2\phi(k)\}\,u^*_{R,22}(k)\,u_{R,12}(k) \\
&= -ix\sin\varphi_a\sin\{2\phi(k)\}\left\{u^*_{R,11}(k)\,u_{R,21}(k) - u^*_{R,12}(k)\,u_{R,22}(k)\right\}^* \\
&\quad -x\cos\varphi_a\left\{u^*_{R,11}(k)\,u_{R,22}(k) - u^*_{R,12}(k)\,u_{R,21}(k)\right\}^* \\
&\quad -ix\sin\varphi_a\cos\{2\phi(k)\}\left\{u^*_{R,11}(k)\,u_{R,22}(k) + u^*_{R,12}(k)\,u_{R,21}(k)\right\}^* = -\mathcal{M}^*_{R1,12}(k).
\end{aligned} \tag{E.14}$$

Also,

$$U_R^\dagger(k)\,\Theta^T(k)\,M_{R,2}\Theta(k)\,V_R(k) = \begin{bmatrix} \mathcal{M}_{R2,11}(k) & \mathcal{M}_{R2,12}(k) \\ \mathcal{M}_{R2,21}(k) & \mathcal{M}_{R2,22}(k) \end{bmatrix}, \tag{E.15}$$

where

$$
\begin{aligned}
\mathcal{M}_{R2,pp}(k) &:= \frac{y}{2}e^{i\varphi_m}\left[1+\cos\{2\phi(k)\}\right]u^*_{R,p1}(k)\,v_{R,p1}(k)-\frac{y}{2}e^{i\varphi_m}\sin\{2\phi(k)\}\,u^*_{R,p1}(k)\,v_{R,p2}(k)\\
&\quad -\frac{y}{2}e^{i\varphi_m}\sin\{2\phi(k)\}\,u^*_{R,p2}(k)\,v_{R,p1}(k)+\frac{y}{2}e^{i\varphi_m}\left[1-\cos\{2\phi(k)\}\right]u^*_{R,p2}(k)\,v_{R,p2}(k)\\
&= \frac{y}{2}e^{i\varphi_m}\left\{u^*_{R,p1}(k)\,v_{R,p1}(k)+u^*_{R,p2}(k)\,v_{R,p2}(k)\right\}\\
&\quad +\frac{y}{2}e^{i\varphi_m}\cos\{2\phi(k)\}\left\{u^*_{R,p1}(k)\,v_{R,p1}(k)-u^*_{R,p2}(k)\,v_{R,p2}(k)\right\}\\
&\quad -\frac{y}{2}e^{i\varphi_m}\sin\{2\phi(k)\}\left\{u^*_{R,p1}(k)\,v_{R,p2}(k)+u^*_{R,p2}(k)\,v_{R,p1}(k)\right\}, \qquad \text{(E.16)}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{M}_{R2,12}(k) &:= \frac{y}{2}e^{i\varphi_m}\left[1+\cos\{2\phi(k)\}\right]u^*_{R,11}(k)\,v_{R,21}(k)-\frac{y}{2}e^{i\varphi_m}\sin\{2\phi(k)\}\,u^*_{R,11}(k)\,v_{R,22}(k)\\
&\quad -\frac{y}{2}e^{i\varphi_m}\sin\{2\phi(k)\}\,u^*_{R,12}(k)\,v_{R,21}(k)+\frac{y}{2}e^{i\varphi_m}\left[1-\cos\{2\phi(k)\}\right]u^*_{R,12}(k)\,v_{R,22}(k)\\
&= \frac{y}{2}e^{i\varphi_m}\left\{u^*_{R,11}(k)\,v_{R,21}(k)+u^*_{R,12}(k)\,v_{R,22}(k)\right\}\\
&\quad +\frac{y}{2}e^{i\varphi_m}\cos\{2\phi(k)\}\left\{u^*_{R,11}(k)\,v_{R,21}(k)-u^*_{R,12}(k)\,v_{R,22}(k)\right\}\\
&\quad -\frac{y}{2}e^{i\varphi_m}\sin\{2\phi(k)\}\left\{u^*_{R,11}(k)\,v_{R,22}(k)+u^*_{R,12}(k)\,v_{R,21}(k)\right\}, \qquad \text{(E.17)}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{M}_{R2,21}(k) &:= \frac{y}{2}e^{i\varphi_m}\left[1+\cos\{2\phi(k)\}\right]u^*_{R,21}(k)\,v_{R,11}(k)-\frac{y}{2}e^{i\varphi_m}\sin\{2\phi(k)\}\,u^*_{R,21}(k)\,v_{R,12}(k)\\
&\quad -\frac{y}{2}e^{i\varphi_m}\sin\{2\phi(k)\}\,u^*_{R,22}(k)\,v_{R,11}(k)+\frac{y}{2}e^{i\varphi_m}\left[1-\cos\{2\phi(k)\}\right]u^*_{R,22}(k)\,v_{R,12}(k)\\
&= \frac{y}{2}e^{i\varphi_m}\left\{u_{R,21}(k)\,v^*_{R,11}(k)+u_{R,22}(k)\,v^*_{R,12}(k)\right\}^*\\
&\quad +\frac{y}{2}e^{i\varphi_m}\cos\{2\phi(k)\}\left\{u_{R,21}(k)\,v^*_{R,11}(k)-u_{R,22}(k)\,v^*_{R,12}(k)\right\}^*\\
&\quad -\frac{y}{2}e^{i\varphi_m}\sin\{2\phi(k)\}\left\{u_{R,21}(k)\,v^*_{R,12}(k)+u_{R,22}(k)\,v^*_{R,11}(k)\right\}^*. \qquad \text{(E.18)}
\end{aligned}
$$

Therefore, from Eqs. (E.7) and (E.8), we get

$$
\begin{aligned}
\Xi_{pp}(k) &= -ix\sin\varphi_a \sin\{2\phi(k)\}\left\{|u_{R,p1}(k)|^2 + |v_{R,p1}(k)|^2 - |u_{R,p2}(k)|^2 - |v_{R,p2}(k)|^2\right\} \\
&\quad +2ix\cos\varphi_a \mathrm{Im}\left\{u^*_{R,p1}(k)\,u_{R,p2}(k) - v^*_{R,p1}(k)\,v_{R,p2}(k)\right\} \\
&\quad -2ix\sin\varphi_a\cos\{2\phi(k)\}\,\mathrm{Re}\left\{u^*_{R,p1}(k)\,u_{R,p2}(k) + v^*_{R,p1}(k)\,v_{R,p2}(k)\right\} \\
&\quad +iy\mathrm{Im}\left[e^{i\varphi_m}\left\{u^*_{R,p1}(k)\,v_{R,p1}(k) + u^*_{R,p2}(k)\,v_{R,p2}(k)\right\}\right] \\
&\quad +iy\cos\{2\phi(k)\}\,\mathrm{Im}\left[e^{i\varphi_m}\left\{u^*_{R,p1}(k)\,v_{R,p1}(k) - u^*_{R,p2}(k)\,v_{R,p2}(k)\right\}\right] \\
&\quad -iy\sin\{2\phi(k)\}\,\mathrm{Im}\left[e^{i\varphi_m}\left\{u^*_{R,p1}(k)\,v_{R,p2}(k) + u^*_{R,p2}(k)\,v_{R,p1}(k)\right\}\right] \\
&= -ix\sin\varphi_a \sin\{2\phi(k)\}\left\{|u_{R,p1}(k)|^2 + |v_{R,p1}(k)|^2 - |u_{R,p2}(k)|^2 - |v_{R,p2}(k)|^2\right\} \\
&\quad -2ix\sin\varphi_a\cos\{2\phi(k)\}\,\mathrm{Re}\left\{u^*_{R,p1}(k)\,u_{R,p2}(k) + v^*_{R,p1}(k)\,v_{R,p2}(k)\right\} \\
&\quad +2ix\cos\varphi_a \mathrm{Im}\left\{u^*_{R,p1}(k)\,u_{R,p2}(k) - v^*_{R,p1}(k)\,v_{R,p2}(k)\right\} \\
&\quad +iy\mathrm{Im}\left[e^{i\varphi_m}\left\{u^*_{R,p1}(k)\,v_{R,p1}(k) + u^*_{R,p2}(k)\,v_{R,p2}(k)\right\}\right] \\
&\quad +iy\cos\{2\phi(k)\}\,\mathrm{Im}\left[e^{i\varphi_m}\left\{u^*_{R,p1}(k)\,v_{R,p1}(k) - u^*_{R,p2}(k)\,v_{R,p2}(k)\right\}\right] \\
&\quad -iy\sin\{2\phi(k)\}\,\mathrm{Im}\left[e^{i\varphi_m}\left\{u^*_{R,p1}(k)\,v_{R,p2}(k) + u^*_{R,p2}(k)\,v_{R,p1}(k)\right\}\right], \qquad \text{(E.19)}
\end{aligned}
$$

which is always imaginary and

$$
\begin{aligned}
\Xi_{12}(k) &= -ix\sin\varphi_a \sin\{2\phi(k)\}\left\{\begin{array}{c} u^*_{R,11}(k)\,u_{R,21}(k)+v^*_{R,11}(k)\,v_{R,21}(k) \\ -u^*_{R,12}(k)\,u_{R,22}(k)-v^*_{R,12}(k)\,v_{R,22}(k) \end{array}\right\} \\
&\quad -ix\sin\varphi_a \cos\{2\phi(k)\}\left\{\begin{array}{c} u^*_{R,11}(k)\,u_{R,22}(k)+v^*_{R,11}(k)\,v_{R,22}(k) \\ +u^*_{R,12}(k)\,u_{R,21}(k)+v^*_{R,12}(k)\,v_{R,21}(k) \end{array}\right\} \\
&\quad +x\cos\varphi_a\left\{u^*_{R,11}(k)\,u_{R,22}(k)-v^*_{R,11}(k)\,v_{R,22}(k)-u^*_{R,12}(k)\,u_{R,21}(k)+v^*_{R,12}(k)\,v_{R,21}(k)\right\} \\
&\quad +\frac{y}{2}e^{i\varphi_m}\left\{u^*_{R,11}(k)\,v_{R,21}(k)+u^*_{R,12}(k)\,v_{R,22}(k)\right\} \\
&\quad +\frac{y}{2}e^{i\varphi_m}\cos\{2\phi(k)\}\left\{u^*_{R,11}(k)\,v_{R,21}(k)-u^*_{R,12}(k)\,v_{R,22}(k)\right\} \\
&\quad -\frac{y}{2}e^{i\varphi_m}\sin\{2\phi(k)\}\left\{u^*_{R,11}(k)\,v_{R,22}(k)+u^*_{R,12}(k)\,v_{R,21}(k)\right\} \\
&\quad -\frac{y}{2}e^{-i\varphi_m}\left\{u_{R,21}(k)\,v^*_{R,11}(k)+u_{R,22}(k)\,v^*_{R,12}(k)\right\} \\
&\quad -\frac{y}{2}e^{-i\varphi_m}\cos\{2\phi(k)\}\left\{u_{R,21}(k)\,v^*_{R,11}(k)-u_{R,22}(k)\,v^*_{R,12}(k)\right\} \\
&\quad +\frac{y}{2}e^{-i\varphi_m}\sin\{2\phi(k)\}\left\{u_{R,21}(k)\,v^*_{R,12}(k)+u_{R,22}(k)\,v^*_{R,11}(k)\right\} \\
&= -ix\sin\varphi_a \sin\{2\phi(k)\}\left\{\begin{array}{c} u^*_{R,11}(k)\,u_{R,21}(k)+v^*_{R,11}(k)\,v_{R,21}(k) \\ -u^*_{R,12}(k)\,u_{R,22}(k)-v^*_{R,12}(k)\,v_{R,22}(k) \end{array}\right\} \\
&\quad -ix\sin\varphi_a \cos\{2\phi(k)\}\left\{\begin{array}{c} u^*_{R,11}(k)\,u_{R,22}(k)+v^*_{R,11}(k)\,v_{R,22}(k) \\ +u^*_{R,12}(k)\,u_{R,21}(k)+v^*_{R,12}(k)\,v_{R,21}(k) \end{array}\right\} \\
&\quad +i\frac{y}{2}\sin\varphi_m\left\{u^*_{R,11}(k)\,v_{R,21}(k)+u_{R,21}(k)\,v^*_{R,11}(k)+u^*_{R,12}(k)\,v_{R,22}(k)+u_{R,22}(k)\,v^*_{R,12}(k)\right\} \\
&\quad +i\frac{y}{2}\sin\varphi_m\cos\{2\phi(k)\}\left\{\begin{array}{c} u^*_{R,11}(k)\,v_{R,21}(k)+u_{R,21}(k)\,v^*_{R,11}(k) \\ -u^*_{R,12}(k)\,v_{R,22}(k)-u_{R,22}(k)\,v^*_{R,12}(k) \end{array}\right\} \\
&\quad -i\frac{y}{2}\sin\varphi_m\sin\{2\phi(k)\}\left\{\begin{array}{c} u^*_{R,11}(k)\,v_{R,22}(k)+u^*_{R,12}(k)\,v_{R,21}(k) \\ +u_{R,21}(k)\,v^*_{R,12}(k)+u_{R,22}(k)\,v^*_{R,11}(k) \end{array}\right\} \\
&\quad +x\cos\varphi_a\left\{u^*_{R,11}(k)\,u_{R,22}(k)-v^*_{R,11}(k)\,v_{R,22}(k)-u^*_{R,12}(k)\,u_{R,21}(k)+v^*_{R,12}(k)\,v_{R,21}(k)\right\} \\
&\quad +\frac{y}{2}\cos\varphi_m\left\{u^*_{R,11}(k)\,v_{R,21}(k)-u_{R,21}(k)\,v^*_{R,11}(k)+u^*_{R,12}(k)\,v_{R,22}(k)-u_{R,22}(k)\,v^*_{R,12}(k)\right\} \\
&\quad +\frac{y}{2}\cos\varphi_m\cos\{2\phi(k)\}\left\{\begin{array}{c} u^*_{R,11}(k)\,v_{R,21}(k)-u_{R,21}(k)\,v^*_{R,11}(k) \\ -u^*_{R,12}(k)\,v_{R,22}(k)+u_{R,22}(k)\,v^*_{R,12}(k) \end{array}\right\} \\
&\quad -\frac{y}{2}\cos\varphi_m\sin\{2\phi(k)\}\left\{\begin{array}{c} u^*_{R,11}(k)\,v_{R,22}(k)+u^*_{R,12}(k)\,v_{R,21}(k) \\ -u_{R,21}(k)\,v^*_{R,12}(k)-u_{R,22}(k)\,v^*_{R,11}(k) \end{array}\right\}. \qquad \text{(E.20)}
\end{aligned}
$$

E.1.4 Final Results - $\Xi_{13}(k)$, $\Xi_{14}(k)$, and $\Xi_{24}(k)$

From Eqs. (E.8), (E.9), and (E.10), we get

$$U_R^{\dagger}(k)\,\Theta^T(k)\,M_{R,1}\Theta(k)\,V_R^*(k) = \begin{bmatrix} \mathcal{M}_{R3,11}(k) & \mathcal{M}_{R3,12}(k) \\ \mathcal{M}_{R3,21}(k) & \mathcal{M}_{R3,22}(k) \end{bmatrix}, \tag{E.21}$$

where

$$\begin{aligned}
\mathcal{M}_{R3,pp}(k) &\coloneqq -ix\sin\varphi_a\sin\{2\phi(k)\}\,u_{R,p1}^*(k)\,v_{R,p1}^*(k) + x\left[\cos\varphi_a - i\sin\varphi_a\cos\{2\phi(k)\}\right]u_{R,p1}^*(k)\,v_{R,p2}^*(k)\\
&\quad -x\left[\cos\varphi_a + i\sin\varphi_a\cos\{2\phi(k)\}\right]u_{R,p2}^*(k)\,v_{R,p1}^*(k) + ix\sin\varphi_a\sin\{2\phi(k)\}\,u_{R,p2}^*(k)\,v_{R,p2}^*(k)\\
&= -ix\sin\varphi_a\sin\{2\phi(k)\}\left\{u_{R,p1}(k)\,v_{R,p1}(k) - u_{R,p2}(k)\,v_{R,p2}(k)\right\}^*\\
&\quad -ix\sin\varphi_a\cos\{2\phi(k)\}\left\{u_{R,p1}(k)\,v_{R,p2}(k) + u_{R,p2}(k)\,v_{R,p1}(k)\right\}^*\\
&\quad +x\cos\varphi_a\left\{u_{R,p1}(k)\,v_{R,p2}(k) - u_{R,p2}(k)\,v_{R,p1}(k)\right\}^*,
\end{aligned} \tag{E.22}$$

$$\begin{aligned}
\mathcal{M}_{R3,12}(k) &\coloneqq -ix\sin\varphi_a\sin\{2\phi(k)\}\,u_{R,11}^*(k)\,v_{R,21}^*(k) + x\left[\cos\varphi_a - i\sin\varphi_a\cos\{2\phi(k)\}\right]u_{R,11}^*(k)\,v_{R,22}^*(k)\\
&\quad -x\left[\cos\varphi_a + i\sin\varphi_a\cos\{2\phi(k)\}\right]u_{R,12}^*(k)\,v_{R,21}^*(k) + ix\sin\varphi_a\sin\{2\phi(k)\}\,u_{R,12}^*(k)\,v_{R,22}^*(k)\\
&= -ix\sin\varphi_a\sin\{2\phi(k)\}\left\{u_{R,11}(k)\,v_{R,21}(k) - u_{R,12}(k)\,v_{R,22}(k)\right\}^*\\
&\quad -ix\sin\varphi_a\cos\{2\phi(k)\}\left\{u_{R,11}(k)\,v_{R,22}(k) + u_{R,12}(k)\,v_{R,21}(k)\right\}^*\\
&\quad +x\cos\varphi_a\left\{u_{R,11}(k)\,v_{R,22}(k) - u_{R,12}(k)\,v_{R,21}(k)\right\}^*,
\end{aligned} \tag{E.23}$$

and

$$\begin{aligned}
\mathcal{M}_{R3,21}(k) &\coloneqq -ix\sin\varphi_a\sin\{2\phi(k)\}\,u_{R,21}^*(k)\,v_{R,11}^*(k) + x\left[\cos\varphi_a - i\sin\varphi_a\cos\{2\phi(k)\}\right]u_{R,21}^*(k)\,v_{R,12}^*(k)\\
&\quad -x\left[\cos\varphi_a + i\sin\varphi_a\cos\{2\phi(k)\}\right]u_{R,22}^*(k)\,v_{R,11}^*(k) + ix\sin\varphi_a\sin\{2\phi(k)\}\,u_{R,22}^*(k)\,v_{R,12}^*(k)\\
&= -ix\sin\varphi_a\sin\{2\phi(k)\}\left\{u_{R,21}(k)\,v_{R,11}(k) - u_{R,22}(k)\,v_{R,12}(k)\right\}^*\\
&\quad -ix\sin\varphi_a\cos\{2\phi(k)\}\left\{u_{R,21}(k)\,v_{R,12}(k) + u_{R,22}(k)\,v_{R,11}(k)\right\}^*\\
&\quad +x\cos\varphi_a\left\{u_{R,21}(k)\,v_{R,12}(k) - u_{R,22}(k)\,v_{R,11}(k)\right\}^*.
\end{aligned} \tag{E.24}$$

Also,

$$U_R^{\dagger}(k)\,\Theta^T(k)\,M_{R,2}\Theta(k)\,U_R^{*}(k) = \begin{bmatrix} \mathcal{M}_{R4,11}(k) & \mathcal{M}_{R4,12}(k) \\ \mathcal{M}_{R4,21}(k) & \mathcal{M}_{R4,22}(k) \end{bmatrix}, \tag{E.25}$$

where

$$\begin{aligned} \mathcal{M}_{R4,pp}(k) &:= \frac{y}{2}e^{i\varphi_m}\left[1+\cos\{2\phi(k)\}\right]u^*_{R,p1}(k)\,u^*_{R,p1}(k) - \frac{y}{2}e^{i\varphi_m}\sin\{2\phi(k)\}\,u^*_{R,p1}(k)\,u^*_{R,p2}(k) \\ &\quad -\frac{y}{2}e^{i\varphi_m}\sin\{2\phi(k)\}\,u^*_{R,p2}(k)\,u^*_{R,p1}(k) + \frac{y}{2}e^{i\varphi_m}\left[1-\cos\{2\phi(k)\}\right]u^*_{R,p2}(k)\,u^*_{R,p2}(k) \\ &= \frac{y}{2}e^{i\varphi_m}\left\{u^2_{R,p1}(k)+u^2_{R,p2}(k)\right\}^* + \frac{y}{2}e^{i\varphi_m}\cos\{2\phi(k)\}\left\{u^2_{R,p1}(k)-u^2_{R,p2}(k)\right\}^* \\ &\quad -ye^{i\varphi_m}\sin\{2\phi(k)\}\left\{u_{R,p1}(k)\,u_{R,p2}(k)\right\}^*, \end{aligned} \tag{E.26}$$

$$\begin{aligned} \mathcal{M}_{R4,12}(k) &:= \frac{y}{2}e^{i\varphi_m}\left[1+\cos\{2\phi(k)\}\right]u^*_{R,11}(k)\,u^*_{R,21}(k) - \frac{y}{2}e^{i\varphi_m}\sin\{2\phi(k)\}\,u^*_{R,11}(k)\,u^*_{R,22}(k) \\ &\quad -\frac{y}{2}e^{i\varphi_m}\sin\{2\phi(k)\}\,u^*_{R,12}(k)\,u^*_{R,21}(k) + \frac{y}{2}e^{i\varphi_m}\left[1-\cos\{2\phi(k)\}\right]u^*_{R,12}(k)\,u^*_{R,22}(k) \\ &= \frac{y}{2}e^{i\varphi_m}\left\{u_{R,11}(k)\,u_{R,21}(k)+u_{R,12}(k)\,u_{R,22}(k)\right\}^* \\ &\quad +\frac{y}{2}e^{i\varphi_m}\cos\{2\phi(k)\}\left\{u_{R,11}(k)\,u_{R,21}(k)-u_{R,12}(k)\,u_{R,22}(k)\right\}^* \\ &\quad -\frac{y}{2}e^{i\varphi_m}\sin\{2\phi(k)\}\left\{u_{R,11}(k)\,u_{R,22}(k)+u_{R,12}(k)\,u_{R,21}(k)\right\}^*, \end{aligned} \tag{E.27}$$

and

$$\begin{aligned} \mathcal{M}_{R4,21}(k) &:= \frac{y}{2}e^{i\varphi_m}\left[1+\cos\{2\phi(k)\}\right]u^*_{R,21}(k)\,u^*_{R,11}(k) - \frac{y}{2}e^{i\varphi_m}\sin\{2\phi(k)\}\,u^*_{R,21}(k)\,u^*_{R,12}(k) \\ &\quad -\frac{y}{2}e^{i\varphi_m}\sin\{2\phi(k)\}\,u^*_{R,22}(k)\,u^*_{R,11}(k) + \frac{y}{2}e^{i\varphi_m}\left[1-\cos\{2\phi(k)\}\right]u^*_{R,22}(k)\,u^*_{R,12}(k) \\ &= \frac{y}{2}e^{i\varphi_m}\left\{u_{R,11}(k)\,u_{R,21}(k)+u_{R,12}(k)\,u_{R,22}(k)\right\}^* \\ &\quad +\frac{y}{2}e^{i\varphi_m}\cos\{2\phi(k)\}\left\{u_{R,11}(k)\,u_{R,21}(k)-u_{R,12}(k)\,u_{R,22}(k)\right\}^* \\ &\quad -\frac{y}{2}e^{i\varphi_m}\sin\{2\phi(k)\}\left\{u_{R,11}(k)\,u_{R,22}(k)+u_{R,12}(k)\,u_{R,21}(k)\right\}^* = \mathcal{M}_{R4,12}(k). \end{aligned} \tag{E.28}$$

Therefore, from Eqs. (E.7) and (E.8), we get

$$
\begin{aligned}
\Xi_{13}(k) &= -2ix\sin\varphi_a \sin\{2\phi(k)\}\{u_{R,11}(k)\,v_{R,11}(k) - u_{R,12}(k)\,v_{R,12}(k)\}^* \\
&\quad -2ix\sin\varphi_a \cos\{2\phi(k)\}\{u_{R,11}(k)\,v_{R,12}(k) + u_{R,12}(k)\,v_{R,11}(k)\}^* \\
&\quad +2x\cos\varphi_a \{u_{R,11}(k)\,v_{R,12}(k) - u_{R,12}(k)\,v_{R,11}(k)\}^* \\
&\quad +\frac{y}{2}e^{i\varphi_m}\left\{u_{R,11}^2(k) + u_{R,12}^2(k)\right\}^* + \frac{y}{2}e^{i\varphi_m}\cos\{2\phi(k)\}\left\{u_{R,11}^2(k) - u_{R,12}^2(k)\right\}^* \\
&\quad -ye^{i\varphi_m}\sin\{2\phi(k)\}\{u_{R,11}(k)\,u_{R,12}(k)\}^* \\
&\quad -\frac{y}{2}e^{-i\varphi_m}\left\{v_{R,11}^2(k) + v_{R,12}^2(k)\right\}^* - \frac{y}{2}e^{-i\varphi_m}\cos\{2\phi(k)\}\left\{v_{R,11}^2(k) - v_{R,12}^2(k)\right\}^* \\
&\quad +ye^{-i\varphi_m}\sin\{2\phi(k)\}\{v_{R,11}(k)\,v_{R,12}(k)\}^* \\
&= -2ix\sin\varphi_a \sin\{2\phi(k)\}\{u_{R,11}(k)\,v_{R,11}(k) - u_{R,12}(k)\,v_{R,12}(k)\}^* \\
&\quad -2ix\sin\varphi_a \cos\{2\phi(k)\}\{u_{R,11}(k)\,v_{R,12}(k) + u_{R,12}(k)\,v_{R,11}(k)\}^* \\
&\quad +i\frac{y}{2}\sin\varphi_m\left\{u_{R,11}^2(k) + v_{R,11}^2(k) + u_{R,12}^2(k) + v_{R,12}^2(k)\right\}^* \\
&\quad +i\frac{y}{2}\sin\varphi_m\cos\{2\phi(k)\}\left\{u_{R,11}^2(k) + v_{R,11}^2(k) - u_{R,12}^2(k) - v_{R,12}^2(k)\right\}^* \\
&\quad -iy\sin\varphi_m\sin\{2\phi(k)\}\{u_{R,11}(k)\,u_{R,12}(k) + v_{R,11}(k)\,v_{R,12}(k)\}^* \\
&\quad +2x\cos\varphi_a \{u_{R,11}(k)\,v_{R,12}(k) - u_{R,12}(k)\,v_{R,11}(k)\}^* \\
&\quad +\frac{y}{2}\cos\varphi_m\left\{u_{R,11}^2(k) - v_{R,11}^2(k) + u_{R,12}^2(k) - v_{R,12}^2(k)\right\}^* \\
&\quad +\frac{y}{2}\cos\varphi_m\cos\{2\phi(k)\}\left\{u_{R,11}^2(k) - v_{R,11}^2(k) - u_{R,12}^2(k) + v_{R,12}^2(k)\right\}^* \\
&\quad -y\cos\varphi_m\sin\{2\phi(k)\}\{u_{R,11}(k)\,u_{R,12}(k) - v_{R,11}(k)\,v_{R,12}(k)\}^*, \qquad \text{(E.29)}
\end{aligned}
$$

$$\begin{aligned}
\Xi_{24}(k) = & -2ix\sin\varphi_a \sin\{2\phi(k)\}\{u_{R,21}(k)\,v_{R,21}(k) - u_{R,22}(k)\,v_{R,22}(k)\}^* \\
& -2ix\sin\varphi_a \cos\{2\phi(k)\}\{u_{R,21}(k)\,v_{R,22}(k) + u_{R,22}(k)\,v_{R,21}(k)\}^* \\
& +i\frac{y}{2}\sin\varphi_m \left\{u_{R,21}^2(k) + v_{R,21}^2(k) + u_{R,22}^2(k) + v_{R,22}^2(k)\right\}^* \\
& +i\frac{y}{2}\sin\varphi_m \cos\{2\phi(k)\}\left\{u_{R,21}^2(k) + v_{R,21}^2(k) - u_{R,22}^2(k) - v_{R,22}^2(k)\right\}^* \\
& -iy\sin\varphi_m \sin\{2\phi(k)\}\{u_{R,21}(k)\,u_{R,22}(k) + v_{R,21}(k)\,v_{R,22}(k)\}^* \\
& +2x\cos\varphi_a\{u_{R,21}(k)\,v_{R,22}(k) - u_{R,22}(k)\,v_{R,21}(k)\}^* \\
& +\frac{y}{2}\cos\varphi_m \left\{u_{R,21}^2(k) - v_{R,21}^2(k) + u_{R,22}^2(k) - v_{R,22}^2(k)\right\}^* \\
& +\frac{y}{2}\cos\varphi_m \cos\{2\phi(k)\}\left\{u_{R,21}^2(k) - v_{R,21}^2(k) - u_{R,22}^2(k) + v_{R,22}^2(k)\right\}^* \\
& -y\cos\varphi_m \sin\{2\phi(k)\}\{u_{R,21}(k)\,u_{R,22}(k) - v_{R,21}(k)\,v_{R,22}(k)\}^*, \qquad \text{(E.30)}
\end{aligned}$$

and

$$
\begin{aligned}
\Xi_{14}(k) &= -ix\sin\varphi_a \sin\{2\phi(k)\}\{u_{R,11}(k)v_{R,21}(k) - u_{R,12}(k)v_{R,22}(k)\}^* \\
&\quad -ix\sin\varphi_a\cos\{2\phi(k)\}\{u_{R,11}(k)v_{R,22}(k) + u_{R,12}(k)v_{R,21}(k)\}^* \\
&\quad +x\cos\varphi_a\{u_{R,11}(k)v_{R,22}(k) - u_{R,12}(k)v_{R,21}(k) + u_{R,21}(k)v_{R,12}(k) - u_{R,22}(k)v_{R,11}(k)\}^* \\
&\quad -ix\sin\varphi_a\sin\{2\phi(k)\}\{u_{R,21}(k)v_{R,11}(k) - u_{R,22}(k)v_{R,12}(k)\}^* \\
&\quad -ix\sin\varphi_a\cos\{2\phi(k)\}\{u_{R,21}(k)v_{R,12}(k) + u_{R,22}(k)v_{R,11}(k)\}^* \\
&\quad +\frac{y}{2}\left[e^{i\varphi_m}\{u_{R,11}(k)u_{R,21}(k) + u_{R,12}(k)u_{R,22}(k)\} - e^{-i\varphi_m}\{v_{R,11}(k)v_{R,21}(k) + v_{R,12}(k)v_{R,22}(k)\}\right]^* \\
&\quad +\frac{y}{2}e^{i\varphi_m}\cos\{2\phi(k)\}\{u_{R,11}(k)u_{R,21}(k) - u_{R,12}(k)u_{R,22}(k)\}^* \\
&\quad -\frac{y}{2}e^{i\varphi_m}\sin\{2\phi(k)\}\{u_{R,11}(k)u_{R,22}(k) + u_{R,12}(k)u_{R,21}(k)\}^* \\
&\quad -\frac{y}{2}e^{-i\varphi_m}\cos\{2\phi(k)\}\{v_{R,11}(k)v_{R,21}(k) - v_{R,12}(k)v_{R,22}(k)\}^* \\
&\quad +\frac{y}{2}e^{-i\varphi_m}\sin\{2\phi(k)\}\{v_{R,11}(k)v_{R,22}(k) + v_{R,12}(k)v_{R,21}(k)\}^* \\
&= -ix\sin\varphi_a\sin\{2\phi(k)\}\left\{\begin{array}{c} u_{R,11}(k)v_{R,21}(k) - u_{R,12}(k)v_{R,22}(k) \\ +u_{R,21}(k)v_{R,11}(k) - u_{R,22}(k)v_{R,12}(k) \end{array}\right\}^* \\
&\quad -ix\sin\varphi_a\cos\{2\phi(k)\}\left\{\begin{array}{c} u_{R,11}(k)v_{R,22}(k) + u_{R,12}(k)v_{R,21}(k) \\ +u_{R,21}(k)v_{R,12}(k) + u_{R,22}(k)v_{R,11}(k) \end{array}\right\}^* \\
&\quad +i\frac{y}{2}\sin\varphi_m\{u_{R,11}(k)u_{R,21}(k) + v_{R,11}(k)v_{R,21}(k) + u_{R,12}(k)u_{R,22}(k) + v_{R,12}(k)v_{R,22}(k)\}^* \\
&\quad +i\frac{y}{2}\sin\varphi_m\cos\{2\phi(k)\}\left\{\begin{array}{c} u_{R,11}(k)u_{R,21}(k) + v_{R,11}(k)v_{R,21}(k) \\ -u_{R,12}(k)u_{R,22}(k) - v_{R,12}(k)v_{R,22}(k) \end{array}\right\}^* \\
&\quad -i\frac{y}{2}\sin\varphi_m\sin\{2\phi(k)\}\left\{\begin{array}{c} u_{R,11}(k)u_{R,22}(k) + v_{R,11}(k)v_{R,22}(k) \\ +u_{R,12}(k)u_{R,21}(k) + v_{R,12}(k)v_{R,21}(k) \end{array}\right\}^* \\
&\quad +x\cos\varphi_a\{u_{R,11}(k)v_{R,22}(k) - u_{R,12}(k)v_{R,21}(k) + u_{R,21}(k)v_{R,12}(k) - u_{R,22}(k)v_{R,11}(k)\}^* \\
&\quad +\frac{y}{2}\cos\varphi_m\{u_{R,11}(k)u_{R,21}(k) - v_{R,11}(k)v_{R,21}(k) + u_{R,12}(k)u_{R,22}(k) - v_{R,12}(k)v_{R,22}(k)\}^* \\
&\quad +\frac{y}{2}\cos\varphi_m\cos\{2\phi(k)\}\left\{\begin{array}{c} u_{R,11}(k)u_{R,21}(k) - v_{R,11}(k)v_{R,21}(k) \\ -u_{R,12}(k)u_{R,22}(k) + v_{R,12}(k)v_{R,22}(k) \end{array}\right\}^* \\
&\quad -\frac{y}{2}\cos\varphi_m\sin\{2\phi(k)\}\left\{\begin{array}{c} u_{R,11}(k)u_{R,22}(k) - v_{R,11}(k)v_{R,22}(k) \\ +u_{R,12}(k)u_{R,21}(k) - v_{R,12}(k)v_{R,21}(k) \end{array}\right\}^* .
\end{aligned}
\tag{E.31}
$$

Chapter F

Fisher Information Calculated by Using Dyson Series Expansion

In the Schrödinger picture, we start from the time dependent bilinear Hamiltonian $\hat{H}_S(t)$ where

$$\hat{H}_S(t) = \hbar \sum_{\boldsymbol{k} \neq 0} \omega_1(k) \hat{b}_1^\dagger(\boldsymbol{k}) \hat{b}_1(\boldsymbol{k}) + \hat{V}_S(t), \tag{F.1}$$

and

$$\hat{V}_S(t) = V_a f(t) \sum_{\boldsymbol{k} \neq 0} \left\{ \begin{array}{l} \mathbb{V}_1(k) + \mathbb{V}_2(k) \hat{b}_1^\dagger(\boldsymbol{k}) \hat{b}_1(\boldsymbol{k}) + \mathbb{V}_3(k) \hat{b}_1^\dagger(\boldsymbol{k}) \hat{b}_1^\dagger(-\boldsymbol{k}) \\ +\mathbb{V}_3^*(k) \hat{b}_1(\boldsymbol{k}) \hat{b}_1(-\boldsymbol{k}) \end{array} \right\}. \tag{F.2}$$

Here, V_a, $f(t)$, $\mathbb{V}_1(k)$, and $\mathbb{V}_2(k)$ are real and $f(t) = 0$ for $t < 0$ so that our Hamiltonian is time independent operator at $t < 0$. We assume small perturbation, i.e. $0 \leq V_a \ll \hbar\omega_1(k)$ for $k \neq 0$, and also assume that Bogoliubov excitation energy $\omega_1(k)$ is real and positive so that the system is in stable state.

F.1 Time Evolution Operator

To solve this perturbation problem with time dependent perturbation $\hat{V}_S(t)$, we will use interaction picture. Let $\hat{H}_{0,S} := \hbar \sum_{\boldsymbol{k}\neq 0} \omega_1(k)\, \hat{b}_1^\dagger(\boldsymbol{k})\, \hat{b}_1(\boldsymbol{k})$. By denoting states in Schrödinger picture as $|\psi_S(t)\rangle$, states in interaction picture as $|\psi_I(t)\rangle := \hat{U}_0(t)\,|\psi_S(t)\rangle$ where $\hat{U}_0(t) := \exp\left(i\hat{H}_{0,S}t/\hbar\right)$, and the time evolution operator as $\hat{U}_I(t)$ where $|\psi_I(t)\rangle = \hat{U}_I(t)\,|\psi_I(0)\rangle$, we get

$$\hat{U}_I(t) = 1 - \frac{i}{\hbar}\int_0^t dt_1\ \hat{V}_I(t_1) - \frac{1}{\hbar^2}\int_0^t dt_1\ \hat{V}_I(t_1)\int_0^{t_1} dt_2\ \hat{V}_I(t_2) + O\left(\hat{V}_I^3(t)\right), \tag{F.3}$$

where $\hat{V}_I(t) := \hat{U}_0(t)\,\hat{V}_S(t)\,\hat{U}_0^\dagger(t)$.

With our $\hat{H}_{0,S}$,

$$\left[it\sum_{\boldsymbol{k}_1\neq 0}\omega_1(k_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1(\boldsymbol{k}_1)\,,\hat{b}_1(\boldsymbol{k})\right] = -i\omega_1(k)\,\hat{b}_1(\boldsymbol{k})\,, \tag{F.4}$$

and from Baker-Campbell-Hausdorff formula,

$$\hat{U}_0(t)\,\hat{b}_1(\boldsymbol{k})\,\hat{U}_0^\dagger(t) = e^{-i\omega_1(k)t}\hat{b}_1(\boldsymbol{k})\,. \tag{F.5}$$

From Eqs. (F.2) and (F.5),

$$\hat{V}_I(t) = V_a f(t)\sum_{\boldsymbol{k}\neq 0}\left\{\begin{array}{l} \mathbb{V}_1(k) + \mathbb{V}_2(k)\,\hat{b}_1^\dagger(\boldsymbol{k})\,\hat{b}_1(\boldsymbol{k}) + e^{2i\tilde{\omega}_1(k)\tilde{t}}\mathbb{V}_3(k)\,\hat{b}_1^\dagger(\boldsymbol{k})\,\hat{b}_1^\dagger(-\boldsymbol{k}) \\ +e^{-2i\tilde{\omega}_1(k)\tilde{t}}\mathbb{V}_3^*(k)\,\hat{b}_1(\boldsymbol{k})\,\hat{b}_1(-\boldsymbol{k}) \end{array}\right\}, \tag{F.6}$$

as $\tilde{\omega}_1(k) := \hbar\omega_1(k)/(g_a n)$ and $\tilde{t} := g_a n t/\hbar$. This leads us to

$$
\begin{aligned}
&\frac{i}{\hbar}\int_0^t dt_1\ \hat{V}_I\left(t_1\right)=\frac{i}{g_a n}\int_0^{\tilde{t}} d\tilde{t}_1\ \hat{V}_I\left(\tilde{t}_1\right)\\
&=i\bar{V}_a\left\{\int_0^{\tilde{t}} d\tilde{t}_1\ f\left(\tilde{t}_1\right)\right\}\sum_{\boldsymbol{k}\neq 0}\left\{\mathbb{V}_1\left(k\right)+\mathbb{V}_2\left(k\right)\hat{b}_1^{\dagger}\left(\boldsymbol{k}\right)\hat{b}_1\left(\boldsymbol{k}\right)\right\}\\
&\quad+i\bar{V}_a\sum_{\boldsymbol{k}\neq 0}\left[\left\{\int_0^{\tilde{t}} d\tilde{t}_1\ e^{2i\tilde{\omega}_1(k)\tilde{t}_1}f\left(\tilde{t}_1\right)\right\}\mathbb{V}_3\left(k\right)\hat{b}_1^{\dagger}\left(\boldsymbol{k}\right)\hat{b}_1^{\dagger}\left(-\boldsymbol{k}\right)+h.c.\right],
\end{aligned}
\tag{F.7}
$$

where $\bar{V}_a \coloneqq V_a/\left(g_a n\right)$ and $h.c.$ represents Hermitian conjugate.

For convenience, we define

$$
T_{1,0}\left(\tilde{t}\right)\coloneqq\int_0^{\tilde{t}} d\tilde{t}_1\ f\left(\tilde{t}_1\right),\quad T_{1,\pm}\left(k_1,\tilde{t}\right)\coloneqq\int_0^{\tilde{t}} d\tilde{t}_1\ e^{\pm 2i\tilde{\omega}_1(k_1)\tilde{t}_1}f\left(\tilde{t}_1\right). \tag{F.8}
$$

Then we get

$$
\begin{aligned}
\frac{i}{\hbar}\int_0^t dt_1\ \hat{V}_I\left(t_1\right)&=i\bar{V}_a T_{1,0}\left(\tilde{t}\right)\sum_{\boldsymbol{k}\neq 0}\left\{\mathbb{V}_1\left(k\right)+\mathbb{V}_2\left(k\right)\hat{b}_1^{\dagger}\left(\boldsymbol{k}\right)\hat{b}_1\left(\boldsymbol{k}\right)\right\}\\
&\quad+i\bar{V}_a\sum_{\boldsymbol{k}\neq 0}\left\{T_{1,+}\left(k,\tilde{t}\right)\mathbb{V}_3\left(k\right)\hat{b}_1^{\dagger}\left(\boldsymbol{k}\right)\hat{b}_1^{\dagger}\left(-\boldsymbol{k}\right)+T_{1,-}\left(k,\tilde{t}\right)\mathbb{V}_3^{*}\left(k\right)\hat{b}_1\left(\boldsymbol{k}\right)\hat{b}_1\left(-\boldsymbol{k}\right)\right\}.
\end{aligned}
\tag{F.9}
$$

F.1.1 Second Order Terms

$$\begin{aligned}
\hat{V}_I(t_1)\hat{V}_I(t_2) &= V_a^2 f(t_1)\sum_{\boldsymbol{k}_1\neq 0}\left[\mathbb{V}_1(k_1)+\mathbb{V}_2(k_1)\hat{b}_1^\dagger(\boldsymbol{k}_1)\hat{b}_1(\boldsymbol{k}_1)+\left\{e^{2i\omega_1(k_1)t_1}\mathbb{V}_3(k_1)\hat{b}_1^\dagger(\boldsymbol{k}_1)\hat{b}_1^\dagger(-\boldsymbol{k}_1)+h.c.\right\}\right]\\
&\quad\times f(t_2)\sum_{\boldsymbol{k}_2\neq 0}\left[\mathbb{V}_1(k_2)+\mathbb{V}_2(k_2)\hat{b}_1^\dagger(\boldsymbol{k}_2)\hat{b}_1(\boldsymbol{k}_2)+\left\{e^{2i\omega_1(k_2)t_2}\mathbb{V}_3(k_2)\hat{b}_1^\dagger(\boldsymbol{k}_2)\hat{b}_1^\dagger(-\boldsymbol{k}_2)+h.c.\right\}\right]\\
&= V_a^2 f(\tilde{t}_1)f(\tilde{t}_2)\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}\mathbb{V}_1(k_1)\left\{\begin{array}{l}\mathbb{V}_1(k_2)+\mathbb{V}_2(k_2)\hat{b}_1^\dagger(\boldsymbol{k}_2)\hat{b}_1(\boldsymbol{k}_2)+e^{2i\omega_1(k_2)t_2}\mathbb{V}_3(k_2)\hat{b}_1^\dagger(\boldsymbol{k}_2)\hat{b}_1^\dagger(-\boldsymbol{k}_2)\\ \quad +e^{-2i\omega_1(k_2)t_2}\mathbb{V}_3^*(k_2)\hat{b}_1(\boldsymbol{k}_2)\hat{b}_1(-\boldsymbol{k}_2)\end{array}\right\}\\
&\quad+V_a^2 f(\tilde{t}_1)f(\tilde{t}_2)\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}\mathbb{V}_2(k_1)\hat{b}_1^\dagger(\boldsymbol{k}_1)\hat{b}_1(\boldsymbol{k}_1)\left[\begin{array}{l}\mathbb{V}_1(k_2)+\mathbb{V}_2(k_2)\hat{b}_1^\dagger(\boldsymbol{k}_2)\hat{b}_1(\boldsymbol{k}_2)\\ \quad +\left\{e^{2i\omega_1(k_2)t_2}\mathbb{V}_3(k_2)\hat{b}_1^\dagger(\boldsymbol{k}_2)\hat{b}_1^\dagger(-\boldsymbol{k}_2)+h.c.\right\}\end{array}\right]\\
&\quad+V_a^2 f(\tilde{t}_1)f(\tilde{t}_2)\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}e^{2i\omega_1(k_1)t_1}\mathbb{V}_3(k_1)\hat{b}_1^\dagger(\boldsymbol{k}_1)\hat{b}_1^\dagger(-\boldsymbol{k}_1)\left[\begin{array}{l}\mathbb{V}_1(k_2)+\mathbb{V}_2(k_2)\hat{b}_1^\dagger(\boldsymbol{k}_2)\hat{b}_1(\boldsymbol{k}_2)\\ +\left\{e^{2i\omega_1(k_2)t_2}\mathbb{V}_3(k_2)\hat{b}_1^\dagger(\boldsymbol{k}_2)\hat{b}_1^\dagger(-\boldsymbol{k}_2)+h.c.\right\}\end{array}\right]\\
&\quad+V_a^2 f(\tilde{t}_1)f(\tilde{t}_2)\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}e^{-2i\omega_1(k_1)t_1}\mathbb{V}_3^*(k_1)\hat{b}_1(\boldsymbol{k}_1)\hat{b}_1(-\boldsymbol{k}_1)\left[\begin{array}{l}\mathbb{V}_1(k_2)+\mathbb{V}_2(k_2)\hat{b}_1^\dagger(\boldsymbol{k}_2)\hat{b}_1(\boldsymbol{k}_2)\\ +\left\{e^{2i\omega_1(k_2)t_2}\mathbb{V}_3(k_2)\hat{b}_1^\dagger(\boldsymbol{k}_2)\hat{b}_1^\dagger(-\boldsymbol{k}_2)+h.c.\right\}\end{array}\right]
\end{aligned} \tag{F.10}$$

Simplifying above equations gives

$$V_a^2 f\left(\tilde{t}_1\right) f\left(\tilde{t}_2\right) \left(\left\{\sum_{\boldsymbol{k}\neq 0} \mathbb{V}_1(k)\right\}^2 + \sum_{\boldsymbol{k}\neq 0}\left\{\sum_{\boldsymbol{k}_1\neq 0} \mathbb{V}_1(k_1)\right\}\left[\begin{array}{l} 2\mathbb{V}_2(k)\,\hat{b}_1^\dagger(\boldsymbol{k})\,\hat{b}_1(\boldsymbol{k}) \\ +\left\{e^{2i\omega_1(k)t_1} + e^{2i\omega_1(k)t_2}\right\}\mathbb{V}_3(k)\,\hat{b}_1^\dagger(\boldsymbol{k})\,\hat{b}_1^\dagger(-\boldsymbol{k}) \end{array}\right]\right)$$

$$+V_a^2 f\left(\tilde{t}_1\right) f\left(\tilde{t}_2\right) \sum_{\boldsymbol{k}\neq 0}\left\{\sum_{\boldsymbol{k}_1\neq 0} \mathbb{V}_1(k_1)\right\}\left\{e^{-2i\omega_1(k)t_1} + e^{-2i\omega_1(k)t_2}\right\}\mathbb{V}_3^*(k)\,\hat{b}_1(\boldsymbol{k})\,\hat{b}_1(-\boldsymbol{k})$$

$$+V_a^2 f\left(\tilde{t}_1\right) f\left(\tilde{t}_2\right) \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0} \mathbb{V}_2(k_1)\,\mathbb{V}_2(k_2)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1(\boldsymbol{k}_2)$$

$$+V_a^2 f\left(\tilde{t}_1\right) f\left(\tilde{t}_2\right) \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0} e^{2i\omega_1(k_2)t_2}\mathbb{V}_2(k_1)\,\mathbb{V}_3(k_2)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2)$$

$$+V_a^2 f\left(\tilde{t}_1\right) f\left(\tilde{t}_2\right) \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0} e^{-2i\omega_1(k_2)t_2}\mathbb{V}_2(k_1)\,\mathbb{V}_3^*(k_2)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1(\boldsymbol{k}_1)\,\hat{b}_1(\boldsymbol{k}_2)\,\hat{b}_1(-\boldsymbol{k}_2)$$

$$+V_a^2 f\left(\tilde{t}_1\right) f\left(\tilde{t}_2\right) \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0} e^{2i\omega_1(k_1)t_1}\mathbb{V}_3(k_1)\,\mathbb{V}_2(k_2)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1(\boldsymbol{k}_2)$$

$$+V_a^2 f\left(\tilde{t}_1\right) f\left(\tilde{t}_2\right) \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0} e^{2i\omega_1(k_1)t_1}e^{2i\omega_1(k_2)t_2}\mathbb{V}_3(k_1)\,\mathbb{V}_3(k_2)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2)$$

$$+V_a^2 f\left(\tilde{t}_1\right) f\left(\tilde{t}_2\right) \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0} e^{2i\omega_1(k_1)t_1}e^{-2i\omega_1(k_2)t_2}\mathbb{V}_3(k_1)\,\mathbb{V}_3^*(k_2)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1)\,\hat{b}_1(\boldsymbol{k}_2)\,\hat{b}_1(-\boldsymbol{k}_2)$$

$$+V_a^2 f\left(\tilde{t}_1\right) f\left(\tilde{t}_2\right) \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0} e^{-2i\omega_1(k_1)t_1}\mathbb{V}_3^*(k_1)\,\mathbb{V}_2(k_2)\,\hat{b}_1(\boldsymbol{k}_1)\,\hat{b}_1(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1(\boldsymbol{k}_2)$$

$$+V_a^2 f\left(\tilde{t}_1\right) f\left(\tilde{t}_2\right) \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0} e^{-2i\omega_1(k_1)t_1}e^{2i\omega_1(k_2)t_2}\mathbb{V}_3^*(k_1)\,\mathbb{V}_3(k_2)\,\hat{b}_1(\boldsymbol{k}_1)\,\hat{b}_1(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2)$$

$$+V_a^2 f\left(\tilde{t}_1\right) f\left(\tilde{t}_2\right) \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0} e^{-2i\omega_1(k_1)t_1}e^{-2i\omega_1(k_2)t_2}\mathbb{V}_3^*(k_1)\,\mathbb{V}_3^*(k_2)\,\hat{b}_1(\boldsymbol{k}_1)\,\hat{b}_1(-\boldsymbol{k}_1)\,\hat{b}_1(\boldsymbol{k}_2)\,\hat{b}_1(-\boldsymbol{k}_2)\,. \tag{F.11}$$

Now, let

$$T_{2,0,0}\left(\tilde{t}\right) \coloneqq \int_0^{\tilde{t}} d\tilde{t}_1\; f\left(\tilde{t}_1\right) \int_0^{\tilde{t}_1} d\tilde{t}_2\; f\left(\tilde{t}_2\right), \quad T_{2,\pm,0}\left(k,\tilde{t}\right) \coloneqq \int_0^{\tilde{t}} d\tilde{t}_1\; e^{\pm 2i\tilde{\omega}_1(k)\tilde{t}_1} f\left(\tilde{t}_1\right) \int_0^{\tilde{t}_1} d\tilde{t}_2\; f\left(\tilde{t}_2\right), \tag{F.12}$$

$$T_{2,0,\pm}\left(k,\tilde{t}\right) \coloneqq \int_0^{\tilde{t}} d\tilde{t}_1\; f\left(\tilde{t}_1\right) \int_0^{\tilde{t}_1} d\tilde{t}_2\; e^{\pm 2i\tilde{\omega}_1(k)\tilde{t}_2} f\left(\tilde{t}_2\right), \tag{F.13}$$

$$T_{2,\pm,+}\left(k_1,k_2,\tilde{t}\right) \coloneqq \int_0^{\tilde{t}} d\tilde{t}_1\; e^{\pm 2i\tilde{\omega}_1(k_1)\tilde{t}_1} f\left(\tilde{t}_1\right) \int_0^{\tilde{t}_1} d\tilde{t}_2\; e^{2i\tilde{\omega}_1(k_2)\tilde{t}_2} f\left(\tilde{t}_2\right), \tag{F.14}$$

and

$$T_{2,\pm,-}\left(k_1,k_2,\tilde{t}\right) \coloneqq \int_0^{\tilde{t}} d\tilde{t}_1\ e^{\pm 2i\tilde{\omega}_1(k_1)\tilde{t}_1} f\left(\tilde{t}_1\right) \int_0^{\tilde{t}_1} d\tilde{t}_2\ e^{-2i\tilde{\omega}_1(k_2)\tilde{t}_2} f\left(\tilde{t}_2\right). \quad \text{(F.15)}$$

Then we get

$$\begin{aligned}
&-\frac{1}{\hbar^2}\int_0^t dt_1\ \hat{V}_I(t_1)\int_0^{t_1} dt_2\ \hat{V}_I(t_2) = \frac{i}{\hbar}\int_0^t dt_1\ \hat{V}_I(t_1)\,\frac{i}{\hbar}\int_0^{t_1} dt_2\ \hat{V}_I(t_2) \\
&= -\bar{V}_a^2\left[T_{2,0,0}\left(\tilde{t}\right)\left\{\sum_{\boldsymbol{k}\neq 0}\mathbb{V}_1(k)\right\}^2 + 2T_{2,0,0}\left(\tilde{t}\right)\sum_{\boldsymbol{k}\neq 0}\left\{\sum_{\boldsymbol{k}_1\neq 0}\mathbb{V}_1(k_1)\right\}\mathbb{V}_2(k)\,\hat{b}_1^\dagger(\boldsymbol{k})\,\hat{b}_1(\boldsymbol{k})\right] \\
&\quad -\bar{V}_a^2\sum_{\boldsymbol{k}\neq 0}\left\{\sum_{\boldsymbol{k}_1\neq 0}\mathbb{V}_1(k_1)\right\}\left\{T_{2,+,0}\left(k,\tilde{t}\right)+T_{2,0,+}\left(k,\tilde{t}\right)\right\}\mathbb{V}_3(k)\,\hat{b}_1^\dagger(\boldsymbol{k})\,\hat{b}_1^\dagger(-\boldsymbol{k}) \\
&\quad -\bar{V}_a^2\sum_{\boldsymbol{k}\neq 0}\left\{\sum_{\boldsymbol{k}_1\neq 0}\mathbb{V}_1(k_1)\right\}\left\{T_{2,-,0}\left(k,\tilde{t}\right)+T_{2,0,-}\left(k,\tilde{t}\right)\right\}\mathbb{V}_3^*(k)\,\hat{b}_1(\boldsymbol{k})\,\hat{b}_1(-\boldsymbol{k}) \\
&\quad -\bar{V}_a^2 T_{2,0,0}\left(\tilde{t}\right)\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}\mathbb{V}_2(k_1)\,\mathbb{V}_2(k_2)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1(\boldsymbol{k}_2) \\
&\quad -\bar{V}_a^2\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}T_{2,0,+}\left(k_2,\tilde{t}\right)\mathbb{V}_2(k_1)\,\mathbb{V}_3(k_2)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2) \\
&\quad -\bar{V}_a^2\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}T_{2,0,-}\left(k_2,\tilde{t}\right)\mathbb{V}_2(k_1)\,\mathbb{V}_3^*(k_2)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1(\boldsymbol{k}_1)\,\hat{b}_1(\boldsymbol{k}_2)\,\hat{b}_1(-\boldsymbol{k}_2) \\
&\quad -\bar{V}_a^2\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}T_{2,+,0}\left(k_1,\tilde{t}\right)\mathbb{V}_3(k_1)\,\mathbb{V}_2(k_2)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1(\boldsymbol{k}_2) \\
&\quad -\bar{V}_a^2\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}T_{2,+,+}\left(k_1,k_2,\tilde{t}\right)\mathbb{V}_3(k_1)\,\mathbb{V}_3(k_2)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2) \\
&\quad -\bar{V}_a^2\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}T_{2,+,-}\left(k_1,k_2,\tilde{t}\right)\mathbb{V}_3(k_1)\,\mathbb{V}_3^*(k_2)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1)\,\hat{b}_1(\boldsymbol{k}_2)\,\hat{b}_1(-\boldsymbol{k}_2) \\
&\quad -\bar{V}_a^2\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}T_{2,-,0}\left(k_1,\tilde{t}\right)\mathbb{V}_3^*(k_1)\,\mathbb{V}_2(k_2)\,\hat{b}_1(\boldsymbol{k}_1)\,\hat{b}_1(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1(\boldsymbol{k}_2) \\
&\quad -\bar{V}_a^2\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}T_{2,-,+}\left(k_1,k_2,\tilde{t}\right)\mathbb{V}_3^*(k_1)\,\mathbb{V}_3(k_2)\,\hat{b}_1(\boldsymbol{k}_1)\,\hat{b}_1(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2) \\
&\quad -\bar{V}_a^2\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}T_{2,-,-}\left(k_1,k_2,\tilde{t}\right)\mathbb{V}_3^*(k_1)\,\mathbb{V}_3^*(k_2)\,\hat{b}_1(\boldsymbol{k}_1)\,\hat{b}_1(-\boldsymbol{k}_1)\,\hat{b}_1(\boldsymbol{k}_2)\,\hat{b}_1(-\boldsymbol{k}_2). \quad \text{(F.16)}
\end{aligned}$$

Now, we order operators so that annihilation operators position right hand side, whereas creation operators position left hand side (normal-ordered).

1. 4th line in Eq. (F.16)

$$\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1(\boldsymbol{k}_2) = \delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1(\boldsymbol{k}_2) + \hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1(\boldsymbol{k}_1)\,\hat{b}_1(\boldsymbol{k}_2)$$

$$\begin{aligned}\Rightarrow &\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}\mathbb{V}_2(k_1)\,\mathbb{V}_2(k_2)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1(\boldsymbol{k}_2)\\ &= \sum_{\boldsymbol{k}\neq 0}\mathbb{V}_2^2(k)\,\hat{b}_1^\dagger(\boldsymbol{k})\,\hat{b}_1(\boldsymbol{k}) + \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}\mathbb{V}_2(k_1)\,\mathbb{V}_2(k_2)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1(\boldsymbol{k}_1)\,\hat{b}_1(\boldsymbol{k}_2)\,,\end{aligned} \tag{F.17}$$

2. 5th line in Eq. (F.16)

$$\begin{aligned}\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2) &= \delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2) + \delta_{\boldsymbol{k}_1,-\boldsymbol{k}_2}\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\\ &\quad+\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2)\,\hat{b}_1(\boldsymbol{k}_1)\end{aligned}$$

$$\begin{aligned}\Rightarrow &\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}T_{2,0,+}\left(k_2,\tilde{t}\right)\mathbb{V}_2(k_1)\,\mathbb{V}_3(k_2)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2)\\ &= 2\sum_{\boldsymbol{k}\neq 0}T_{2,0,+}\left(k,\tilde{t}\right)\mathbb{V}_2(k)\,\mathbb{V}_3(k)\,\hat{b}_1^\dagger(\boldsymbol{k})\,\hat{b}_1^\dagger(-\boldsymbol{k})\\ &\quad+\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}T_{2,0,+}\left(k_2,\tilde{t}\right)\mathbb{V}_2(k_1)\,\mathbb{V}_3(k_2)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2)\,\hat{b}_1(\boldsymbol{k}_1)\,,\end{aligned} \tag{F.18}$$

3. 10th line in Eq. (F.16)

$$\begin{aligned}\hat{b}_1(\boldsymbol{k}_1)\,\hat{b}_1(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1(\boldsymbol{k}_2) &= \delta_{\boldsymbol{k}_1,-\boldsymbol{k}_2}\hat{b}_1(\boldsymbol{k}_1)\,\hat{b}_1(\boldsymbol{k}_2) + \delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\hat{b}_1(-\boldsymbol{k}_1)\,\hat{b}_1(\boldsymbol{k}_2)\\ &\quad+\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1(\boldsymbol{k}_1)\,\hat{b}_1(-\boldsymbol{k}_1)\,\hat{b}_1(\boldsymbol{k}_2)\end{aligned}$$

$$\begin{aligned}\Rightarrow &\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}T_{2,-,0}\left(k_1,\tilde{t}\right)\mathbb{V}_3^*(k_1)\,\mathbb{V}_2(k_2)\,\hat{b}_1(\boldsymbol{k}_1)\,\hat{b}_1(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1(\boldsymbol{k}_2)\\ &= 2\sum_{\boldsymbol{k}\neq 0}T_{2,-,0}\left(k,\tilde{t}\right)\mathbb{V}_3^*(k)\,\mathbb{V}_2(k)\,\hat{b}_1(\boldsymbol{k})\,\hat{b}_1(-\boldsymbol{k})\\ &\quad+\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}T_{2,-,0}\left(k_2,\tilde{t}\right)\mathbb{V}_3^*(k_2)\,\mathbb{V}_2(k_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1(\boldsymbol{k}_1)\,\hat{b}_1(\boldsymbol{k}_2)\,\hat{b}_1(-\boldsymbol{k}_2)\,,\end{aligned} \tag{F.19}$$

4. 11th line in Eq. (F.16)

$$
\begin{aligned}
&\hat{b}_1(\boldsymbol{k}_1)\,\hat{b}_1(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2) \\
&\quad = \delta_{\boldsymbol{k}_1,-\boldsymbol{k}_2}\hat{b}_1(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2) + \delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\hat{b}_1(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2) + \delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1(\boldsymbol{k}_1) + \delta_{\boldsymbol{k}_1,-\boldsymbol{k}_2}\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1(-\boldsymbol{k}_1) \\
&\qquad + \hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2)\,\hat{b}_1(\boldsymbol{k}_1)\,\hat{b}_1(-\boldsymbol{k}_1) \\
&\quad = (\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2} + \delta_{\boldsymbol{k}_1,-\boldsymbol{k}_2})\left\{1 + \hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1(\boldsymbol{k}_1) + \hat{b}_1^\dagger(-\boldsymbol{k}_1)\,\hat{b}_1(-\boldsymbol{k}_1)\right\} + \hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2)\,\hat{b}_1(\boldsymbol{k}_1)\,\hat{b}_1(-\boldsymbol{k}_1) \\
&\Rightarrow \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0} T_{2,-,+}\left(k_1,k_2,\tilde{t}\right)\mathbb{V}_3^*(k_1)\,\mathbb{V}_3(k_2)\,\hat{b}_1(\boldsymbol{k}_1)\,\hat{b}_1(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2) \\
&\qquad = 2\sum_{\boldsymbol{k}\neq 0} T_{2,-,+}\left(k,k,\tilde{t}\right)|\mathbb{V}_3(k)|^2\left\{1 + 2\hat{b}_1^\dagger(\boldsymbol{k})\,\hat{b}_1(\boldsymbol{k})\right\} \\
&\qquad\quad + \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0} T_{2,-,+}\left(k_2,k_1,\tilde{t}\right)\mathbb{V}_3^*(k_2)\,\mathbb{V}_3(k_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1)\,\hat{b}_1(\boldsymbol{k}_2)\,\hat{b}_1(-\boldsymbol{k}_2)\,. \qquad \text{(F.20)}
\end{aligned}
$$

Now we get

$$
\begin{aligned}
&-\frac{1}{\hbar^2}\int_0^t dt_1\,\hat{V}_I(t_1)\int_0^{t_1} dt_2\,\hat{V}_I(t_2) = \frac{i}{\hbar}\int_0^t dt_1\,\hat{V}_I(t_1)\,\frac{i}{\hbar}\int_0^{t_1} dt_2\,\hat{V}_I(t_2) \\
&= -\bar{V}_a^2\left[T_{2,0,0}\left(\tilde{t}\right)\left\{\sum_{\boldsymbol{k}\neq 0}\mathbb{V}_1(k)\right\}^2 + 2T_{2,0,0}\left(\tilde{t}\right)\sum_{\boldsymbol{k}\neq 0}\left\{\sum_{\boldsymbol{k}_1\neq 0}\mathbb{V}_1(k_1)\right\}\mathbb{V}_2(k)\,\hat{b}_1^\dagger(\boldsymbol{k})\,\hat{b}_1(\boldsymbol{k})\right] \\
&\quad -\bar{V}_a^2\sum_{\boldsymbol{k}\neq 0}\left\{\sum_{\boldsymbol{k}_1\neq 0}\mathbb{V}_1(k_1)\right\}\left\{T_{2,+,0}\left(k,\tilde{t}\right)+T_{2,0,+}\left(k,\tilde{t}\right)\right\}\mathbb{V}_3(k)\,\hat{b}_1^\dagger(\boldsymbol{k})\,\hat{b}_1^\dagger(-\boldsymbol{k}) \\
&\quad -\bar{V}_a^2\sum_{\boldsymbol{k}\neq 0}\left\{\sum_{\boldsymbol{k}_1\neq 0}\mathbb{V}_1(k_1)\right\}\left\{T_{2,-,0}\left(k,\tilde{t}\right)+T_{2,0,-}\left(k,\tilde{t}\right)\right\}\mathbb{V}_3^*(k)\,\hat{b}_1(\boldsymbol{k})\,\hat{b}_1(-\boldsymbol{k}) \\
&\quad -\bar{V}_a^2 T_{2,0,0}\left(\tilde{t}\right)\sum_{\boldsymbol{k}\neq 0}\mathbb{V}_2^2(k)\,\hat{b}_1^\dagger(\boldsymbol{k})\,\hat{b}_1(\boldsymbol{k}) - \tilde{V}_a^2 T_{2,0,0}\left(\tilde{t}\right)\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}\mathbb{V}_2(k_1)\,\mathbb{V}_2(k_2)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1(\boldsymbol{k}_1)\,\hat{b}_1(\boldsymbol{k}_2) \\
&\quad -2\bar{V}_a^2\sum_{\boldsymbol{k}\neq 0}T_{2,0,+}\left(k,\tilde{t}\right)\mathbb{V}_2(k)\,\mathbb{V}_3(k)\,\hat{b}_1^\dagger(\boldsymbol{k})\,\hat{b}_1^\dagger(-\boldsymbol{k}) \\
&\quad -\tilde{V}_a^2\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}T_{2,0,+}\left(k_2,\tilde{t}\right)\mathbb{V}_2(k_1)\,\mathbb{V}_3(k_2)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2)\,\hat{b}_1(\boldsymbol{k}_1) \\
&\quad -\bar{V}_a^2\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}T_{2,0,-}\left(k_2,\tilde{t}\right)\mathbb{V}_2(k_1)\,\mathbb{V}_3^*(k_2)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1(\boldsymbol{k}_1)\,\hat{b}_1(\boldsymbol{k}_2)\,\hat{b}_1(-\boldsymbol{k}_2) \\
&\quad -\bar{V}_a^2\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}T_{2,+,0}\left(k_1,\tilde{t}\right)\mathbb{V}_3(k_1)\,\mathbb{V}_2(k_2)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1(\boldsymbol{k}_2) \\
&\quad -\bar{V}_a^2\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}T_{2,+,+}\left(k_1,k_2,\tilde{t}\right)\mathbb{V}_3(k_1)\,\mathbb{V}_3(k_2)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2) \\
&\quad -\bar{V}_a^2\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}T_{2,+,-}\left(k_1,k_2,\tilde{t}\right)\mathbb{V}_3(k_1)\,\mathbb{V}_3^*(k_2)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1)\,\hat{b}_1(\boldsymbol{k}_2)\,\hat{b}_1(-\boldsymbol{k}_2) \\
&\quad -2\bar{V}_a^2\sum_{\boldsymbol{k}\neq 0}T_{2,-,0}\left(k,\tilde{t}\right)\mathbb{V}_3^*(k)\,\mathbb{V}_2(k)\,\hat{b}_1(\boldsymbol{k})\,\hat{b}_1(-\boldsymbol{k}) \\
&\quad -\tilde{V}_a^2\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}T_{2,-,0}\left(k_2,\tilde{t}\right)\mathbb{V}_3^*(k_2)\,\mathbb{V}_2(k_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1(\boldsymbol{k}_1)\,\hat{b}_1(\boldsymbol{k}_2)\,\hat{b}_1(-\boldsymbol{k}_2) \\
&\quad -2\bar{V}_a^2\sum_{\boldsymbol{k}\neq 0}T_{2,-,+}\left(k,k,\tilde{t}\right)\left|\mathbb{V}_3(k)\right|^2\left\{1+2\hat{b}_1^\dagger(\boldsymbol{k})\,\hat{b}_1(\boldsymbol{k})\right\} \\
&\quad -\bar{V}_a^2\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}T_{2,-,+}\left(k_2,k_1,\tilde{t}\right)\mathbb{V}_3^*(k_2)\,\mathbb{V}_3(k_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1)\,\hat{b}_1(\boldsymbol{k}_2)\,\hat{b}_1(-\boldsymbol{k}_2) \\
&\quad -\bar{V}_a^2\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}T_{2,-,-}\left(k_1,k_2,\tilde{t}\right)\mathbb{V}_3^*(k_1)\,\mathbb{V}_3^*(k_2)\,\hat{b}_1(\boldsymbol{k}_1)\,\hat{b}_1(-\boldsymbol{k}_1)\,\hat{b}_1(\boldsymbol{k}_2)\,\hat{b}_1(-\boldsymbol{k}_2)\,. \qquad \text{(F.21)}
\end{aligned}
$$

We re-order terms from 4th to 15th lines in Eqs. (F.21):

$$
\begin{aligned}
&-2\tilde{V}_a^2 \sum_{\boldsymbol{k}\neq 0} T_{2,-,+}\left(k,k,\tilde{t}\right)\left|\mathbb{V}_3\left(k\right)\right|^2 - \tilde{V}_a^2 \sum_{\boldsymbol{k}\neq 0}\left\{T_{2,0,0}\left(\tilde{t}\right)\mathbb{V}_2^2\left(k\right) + 4T_{2,-,+}\left(k,k,\tilde{t}\right)\left|\mathbb{V}_3\left(k\right)\right|^2\right\}\hat{b}_1^\dagger\left(\boldsymbol{k}\right)\hat{b}_1\left(\boldsymbol{k}\right) \\
&-2\tilde{V}_a^2 \sum_{\boldsymbol{k}\neq 0} T_{2,0,+}\left(k,\tilde{t}\right)\mathbb{V}_2\left(k\right)\mathbb{V}_3\left(k\right)\hat{b}_1^\dagger\left(\boldsymbol{k}\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}\right) - 2\tilde{V}_a^2 \sum_{\boldsymbol{k}\neq 0} T_{2,-,0}\left(k,\tilde{t}\right)\mathbb{V}_2\left(k\right)\mathbb{V}_3^*\left(k\right)\hat{b}_1\left(\boldsymbol{k}\right)\hat{b}_1\left(-\boldsymbol{k}\right) \\
&-\tilde{V}_a^2 T_{2,0,0}\left(\tilde{t}\right) \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0} \mathbb{V}_2\left(k_1\right)\mathbb{V}_2\left(k_2\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_2\right)\hat{b}_1\left(\boldsymbol{k}_1\right)\hat{b}_1\left(\boldsymbol{k}_2\right) \\
&-\tilde{V}_a^2 \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}\left\{T_{2,+,-}\left(k_1,k_2,\tilde{t}\right) + T_{2,-,+}\left(k_2,k_1,\tilde{t}\right)\right\}\mathbb{V}_3\left(k_1\right)\mathbb{V}_3^*\left(k_2\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_1\right)\hat{b}_1\left(\boldsymbol{k}_2\right)\hat{b}_1\left(-\boldsymbol{k}_2\right) \\
&-\tilde{V}_a^2 \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0} T_{2,0,+}\left(k_2,\tilde{t}\right)\mathbb{V}_3\left(k_2\right)\mathbb{V}_2\left(k_1\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_2\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_2\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_1\right)\hat{b}_1\left(\boldsymbol{k}_1\right) \\
&-\tilde{V}_a^2 \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0} T_{2,+,0}\left(k_1,\tilde{t}\right)\mathbb{V}_3\left(k_1\right)\mathbb{V}_2\left(k_2\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_2\right)\hat{b}_1\left(\boldsymbol{k}_2\right) \\
&-\tilde{V}_a^2 \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}\left\{T_{2,-,0}\left(k_2,\tilde{t}\right) + T_{2,0,-}\left(k_2,\tilde{t}\right)\right\}\mathbb{V}_3^*\left(k_2\right)\mathbb{V}_2\left(k_1\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_1\right)\hat{b}_1\left(\boldsymbol{k}_1\right)\hat{b}_1\left(\boldsymbol{k}_2\right)\hat{b}_1\left(-\boldsymbol{k}_2\right) \\
&-\tilde{V}_a^2 \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0} T_{2,+,+}\left(k_1,k_2,\tilde{t}\right)\mathbb{V}_3\left(k_1\right)\mathbb{V}_3\left(k_2\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_2\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_2\right) \\
&-\tilde{V}_a^2 \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0} T_{2,-,-}\left(k_1,k_2,\tilde{t}\right)\mathbb{V}_3^*\left(k_1\right)\mathbb{V}_3^*\left(k_2\right)\hat{b}_1\left(\boldsymbol{k}_1\right)\hat{b}_1\left(-\boldsymbol{k}_1\right)\hat{b}_1\left(\boldsymbol{k}_2\right)\hat{b}_1\left(-\boldsymbol{k}_2\right).
\end{aligned}
\tag{F.22}
$$

From (F.22), Eqs. (F.21) can be written as

$$
\begin{aligned}
&-\frac{1}{\hbar^2}\int_0^t dt_1\;\hat{V}_I\left(t_1\right)\int_0^{t_1} dt_2\;\hat{V}_I\left(t_2\right)=\frac{i}{\hbar}\int_0^t dt_1\;\hat{V}_I\left(t_1\right)\frac{i}{\hbar}\int_0^{t_1} dt_2\;\hat{V}_I\left(t_2\right)\\
&\quad=-\bar{V}_a^2\left[T_{2,0,0}\left(\tilde{t}\right)\left\{\sum_{\boldsymbol{k}\neq 0}\mathbb{V}_1\left(k\right)\right\}^2+2\sum_{\boldsymbol{k}\neq 0}T_{2,-,+}\left(k,k,\tilde{t}\right)\left|\mathbb{V}_3\left(k\right)\right|^2\right]\\
&\qquad-\bar{V}_a^2\sum_{\boldsymbol{k}\neq 0}\left\{T_{2,0,0}\left(\tilde{t}\right)\mathbb{V}_2^2\left(k\right)+2T_{2,0,0}\left(\tilde{t}\right)\mathbb{V}_2\left(k\right)\sum_{\boldsymbol{k}_1\neq 0}\mathbb{V}_1\left(k_1\right)+4T_{2,-,+}\left(k,k,\tilde{t}\right)\left|\mathbb{V}_3\left(k\right)\right|^2\right\}\hat{b}_1^\dagger\left(\boldsymbol{k}\right)\hat{b}_1\left(\boldsymbol{k}\right)\\
&\qquad-\bar{V}_a^2\sum_{\boldsymbol{k}\neq 0}\left[\left\{T_{2,+,0}\left(k,\tilde{t}\right)+T_{2,0,+}\left(k,\tilde{t}\right)\right\}\sum_{\boldsymbol{k}_1\neq 0}\mathbb{V}_1\left(k_1\right)+2T_{2,0,+}\left(k,\tilde{t}\right)\mathbb{V}_2\left(k\right)\right]\mathbb{V}_3\left(k\right)\hat{b}_1^\dagger\left(\boldsymbol{k}\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}\right)\\
&\qquad-\bar{V}_a^2\sum_{\boldsymbol{k}\neq 0}\left[\left\{T_{2,-,0}\left(k,\tilde{t}\right)+T_{2,0,-}\left(k,\tilde{t}\right)\right\}\sum_{\boldsymbol{k}_1\neq 0}\mathbb{V}_1\left(k_1\right)+2T_{2,-,0}\left(k,\tilde{t}\right)\mathbb{V}_2\left(k\right)\right]\mathbb{V}_3^*\left(k\right)\hat{b}_1\left(\boldsymbol{k}\right)\hat{b}_1\left(-\boldsymbol{k}\right)\\
&\qquad-\bar{V}_a^2T_{2,0,0}\left(\tilde{t}\right)\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}\mathbb{V}_2\left(k_1\right)\mathbb{V}_2\left(k_2\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_2\right)\hat{b}_1\left(\boldsymbol{k}_1\right)\hat{b}_1\left(\boldsymbol{k}_2\right)\\
&\qquad-\bar{V}_a^2\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}\left\{T_{2,+,-}\left(k_1,k_2,\tilde{t}\right)+T_{2,-,+}\left(k_2,k_1,\tilde{t}\right)\right\}\mathbb{V}_3\left(k_1\right)\mathbb{V}_3^*\left(k_2\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_1\right)\hat{b}_1\left(\boldsymbol{k}_2\right)\hat{b}_1\left(-\boldsymbol{k}_2\right)\\
&\qquad-\bar{V}_a^2\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}\left\{T_{2,+,0}\left(k_1,\tilde{t}\right)+T_{2,0,+}\left(k_1,\tilde{t}\right)\right\}\mathbb{V}_3\left(k_1\right)\mathbb{V}_2\left(k_2\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_2\right)\hat{b}_1\left(\boldsymbol{k}_2\right)\\
&\qquad-\bar{V}_a^2\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}\left\{T_{2,-,0}\left(k_1,\tilde{t}\right)+T_{2,0,-}\left(k_1,\tilde{t}\right)\right\}\mathbb{V}_3^*\left(k_1\right)\mathbb{V}_2\left(k_2\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_2\right)\hat{b}_1\left(\boldsymbol{k}_2\right)\hat{b}_1\left(\boldsymbol{k}_1\right)\hat{b}_1\left(-\boldsymbol{k}_1\right)\\
&\qquad-\bar{V}_a^2\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}T_{2,+,+}\left(k_1,k_2,\tilde{t}\right)\mathbb{V}_3\left(k_1\right)\mathbb{V}_3\left(k_2\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_2\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_2\right)\\
&\qquad-\bar{V}_a^2\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}T_{2,-,-}\left(k_1,k_2,\tilde{t}\right)\mathbb{V}_3^*\left(k_1\right)\mathbb{V}_3^*\left(k_2\right)\hat{b}_1\left(\boldsymbol{k}_1\right)\hat{b}_1\left(-\boldsymbol{k}_1\right)\hat{b}_1\left(\boldsymbol{k}_2\right)\hat{b}_1\left(-\boldsymbol{k}_2\right).
\end{aligned}
\tag{F.23}
$$

Combining Eqs. (F.9) and (F.23) gives our time evolution operator

$$
\hat{U}_I\left(\tilde{t}\right)=1-i\bar{V}_a T_{1,0}\left(\tilde{t}\right)\sum_{\boldsymbol{k}\neq 0}\mathbb{V}_1\left(k\right)-\bar{V}_a^2\left[T_{2,0,0}\left(\tilde{t}\right)\left\{\sum_{\boldsymbol{k}\neq 0}\mathbb{V}_1\left(k\right)\right\}^2+2\sum_{\boldsymbol{k}\neq 0}T_{2,-,+}\left(k,k,\tilde{t}\right)\left|\mathbb{V}_3\left(k\right)\right|^2\right]
$$

$$
-i\bar{V}_a\sum_{\boldsymbol{k}\neq 0}\left(T_{1,0}\left(\tilde{t}\right)\mathbb{V}_2\left(k\right)-i\bar{V}_a\left[\begin{array}{l}T_{2,0,0}\left(\tilde{t}\right)\mathbb{V}_2\left(k\right)\left\{\mathbb{V}_2\left(k\right)+2\sum_{\boldsymbol{k}_1\neq 0}\mathbb{V}_1\left(k_1\right)\right\}\\ +4T_{2,-,+}\left(k,k,\tilde{t}\right)\left|\mathbb{V}_3\left(k\right)\right|^2\end{array}\right]\right)\hat{b}_1^\dagger\left(\boldsymbol{k}\right)\hat{b}_1\left(\boldsymbol{k}\right)
$$

$$
-i\bar{V}_a\sum_{\boldsymbol{k}\neq 0}\left(T_{1,+}\left(k,\tilde{t}\right)-i\bar{V}_a\left[\begin{array}{l}\left\{T_{2,+,0}\left(k,\tilde{t}\right)+T_{2,0,+}\left(k,\tilde{t}\right)\right\}\sum_{\boldsymbol{k}_1\neq 0}\mathbb{V}_1\left(k_1\right)\\ +2T_{2,0,+}\left(k,\tilde{t}\right)\mathbb{V}_2\left(k\right)\end{array}\right]\right)\mathbb{V}_3\left(k\right)\hat{b}_1^\dagger\left(\boldsymbol{k}\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}\right)
$$

$$
-i\bar{V}_a\sum_{\boldsymbol{k}\neq 0}\left(T_{1,-}\left(k,\tilde{t}\right)-i\bar{V}_a\left[\begin{array}{l}\left\{T_{2,-,0}\left(k,\tilde{t}\right)+T_{2,0,-}\left(k,\tilde{t}\right)\right\}\sum_{\boldsymbol{k}_1\neq 0}\mathbb{V}_1\left(k_1\right)\\ +2T_{2,-,0}\left(k,\tilde{t}\right)\mathbb{V}_2\left(k\right)\end{array}\right]\right)\mathbb{V}_3^*\left(k\right)\hat{b}_1\left(\boldsymbol{k}\right)\hat{b}_1\left(-\boldsymbol{k}\right)
$$

$$
-\bar{V}_a^2 T_{2,0,0}\left(\tilde{t}\right)\sum_{\boldsymbol{k}_1\neq 0}\sum_{\boldsymbol{k}_2\neq 0}\mathbb{V}_2\left(k_1\right)\mathbb{V}_2\left(k_2\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_2\right)\hat{b}_1\left(\boldsymbol{k}_2\right)\hat{b}_1\left(\boldsymbol{k}_1\right)
$$

$$
-\bar{V}_a^2\sum_{\boldsymbol{k}_1\neq 0}\sum_{\boldsymbol{k}_2\neq 0}\left\{T_{2,+,-}\left(k_1,k_2,\tilde{t}\right)+T_{2,-,+}\left(k_2,k_1,\tilde{t}\right)\right\}\mathbb{V}_3\left(k_1\right)\mathbb{V}_3^*\left(k_2\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_1\right)\hat{b}_1\left(\boldsymbol{k}_2\right)\hat{b}_1\left(-\boldsymbol{k}_2\right)
$$

$$
-\bar{V}_a^2\sum_{\boldsymbol{k}_1\neq 0}\sum_{\boldsymbol{k}_2\neq 0}\left\{T_{2,+,0}\left(k_1,\tilde{t}\right)+T_{2,0,+}\left(k_1,\tilde{t}\right)\right\}\mathbb{V}_3\left(k_1\right)\mathbb{V}_2\left(k_2\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_2\right)\hat{b}_1\left(\boldsymbol{k}_2\right)
$$

$$
-\bar{V}_a^2\sum_{\boldsymbol{k}_1\neq 0}\sum_{\boldsymbol{k}_2\neq 0}\left\{T_{2,-,0}\left(k_1,\tilde{t}\right)+T_{2,0,-}\left(k_1,\tilde{t}\right)\right\}\mathbb{V}_3^*\left(k_1\right)\mathbb{V}_2\left(k_2\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_2\right)\hat{b}_1\left(\boldsymbol{k}_2\right)\hat{b}_1\left(\boldsymbol{k}_1\right)\hat{b}_1\left(-\boldsymbol{k}_1\right)
$$

$$
-\bar{V}_a^2\sum_{\boldsymbol{k}_1\neq 0}\sum_{\boldsymbol{k}_2\neq 0}T_{2,+,+}\left(k_1,k_2,\tilde{t}\right)\mathbb{V}_3\left(k_1\right)\mathbb{V}_3\left(k_2\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_2\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_2\right)
$$

$$
-\bar{V}_a^2\sum_{\boldsymbol{k}_1\neq 0}\sum_{\boldsymbol{k}_2\neq 0}T_{2,-,-}\left(k_1,k_2,\tilde{t}\right)\mathbb{V}_3^*\left(k_1\right)\mathbb{V}_3^*\left(k_2\right)\hat{b}_1\left(\boldsymbol{k}_1\right)\hat{b}_1\left(-\boldsymbol{k}_1\right)\hat{b}_1\left(\boldsymbol{k}_2\right)\hat{b}_1\left(-\boldsymbol{k}_2\right)+O\left(\bar{V}_a^3\right). \quad \text{(F.24)}
$$

F.2 Reaction Rate - Initially in Bogoliubov Vacuum State

Since $\hat{b}_1\left(\boldsymbol{k}\right)\left|\text{vac}\right\rangle=0$, from Eq. (F.24), $\left|\Psi_0\left(t\right)\right\rangle:=\hat{U}_I\left(t\right)\left|\text{vac}\right\rangle$ is

$$|\Psi_0(t)\rangle = \left(1 - i\bar{V}_a T_{1,0}\left(\tilde{t}\right)\sum_{\boldsymbol{k}\neq 0}\mathbb{V}_1(k) - \bar{V}_a^2\left[\begin{array}{c} T_{2,0,0}\left(\tilde{t}\right)\left\{\sum_{\boldsymbol{k}\neq 0}\mathbb{V}_1(k)\right\}^2 \\ +2\sum_{\boldsymbol{k}\neq 0}T_{2,-,+}\left(k,k,\tilde{t}\right)|\mathbb{V}_3(k)|^2 \end{array}\right]\right)|\mathrm{vac}\rangle$$
$$-i\bar{V}_a\sum_{\boldsymbol{k}\neq 0}\left(T_{1,+}\left(k,\tilde{t}\right) - i\bar{V}_a\left[\begin{array}{c}\left\{T_{2,+,0}\left(k,\tilde{t}\right)+T_{2,0,+}\left(k,\tilde{t}\right)\right\}\sum_{\boldsymbol{k}_1\neq 0}\mathbb{V}_1(k_1) \\ +2T_{2,0,+}\left(k,\tilde{t}\right)\mathbb{V}_2(k)\end{array}\right]\right)\mathbb{V}_3(k)\,\hat{b}_1^\dagger(\boldsymbol{k})\,\hat{b}_1^\dagger(-\boldsymbol{k})|\mathrm{vac}\rangle$$
$$-\bar{V}_a^2\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2\neq 0}T_{2,+,+}\left(k_1,k_2,\tilde{t}\right)\mathbb{V}_3(k_1)\,\mathbb{V}_3(k_2)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2)|\mathrm{vac}\rangle + O\left(\bar{V}_a^3\right). \tag{F.25}$$

For convenience, we will define

$$\psi_{0,1}\left(\tilde{t}\right) = T_{1,0}\left(\tilde{t}\right)\sum_{\boldsymbol{k}\neq 0}\mathbb{V}_1(k), \quad \psi_{0,2}\left(\tilde{t}\right) = T_{2,0,0}\left(\tilde{t}\right)\left\{\sum_{\boldsymbol{k}\neq 0}\mathbb{V}_1(k)\right\}^2 + 2\sum_{\boldsymbol{k}\neq 0}T_{2,-,+}\left(k,k,\tilde{t}\right)|\mathbb{V}_3(k)|^2,$$
$$\psi_{2,1}\left(k,\tilde{t}\right) = T_{1,+}\left(k,\tilde{t}\right), \quad \psi_{2,2}\left(k,\tilde{t}\right) = \left\{T_{2,+,0}\left(k,\tilde{t}\right)+T_{2,0,+}\left(k,\tilde{t}\right)\right\}\sum_{\boldsymbol{k}_1\neq 0}\mathbb{V}_1(k_1) + 2T_{2,0,+}\left(k,\tilde{t}\right)\mathbb{V}_2(k),$$
$$\psi_{4,2}\left(k_1,k_2,\tilde{t}\right) = T_{2,+,+}\left(k_1,k_2,\tilde{t}\right), \tag{F.26}$$

$|\boldsymbol{k},-\boldsymbol{k}\rangle := \hat{b}_1^\dagger(\boldsymbol{k})\,\hat{b}_1^\dagger(-\boldsymbol{k})|\mathrm{vac}\rangle$, $|\boldsymbol{k}_1,-\boldsymbol{k}_1,\boldsymbol{k}_2,-\boldsymbol{k}_2\rangle := \hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2)|\mathrm{vac}\rangle$, and $|\boldsymbol{k}_1,-\boldsymbol{k}_1,\boldsymbol{k}_2,-\boldsymbol{k}_2,\boldsymbol{k}_3,-\boldsymbol{k}_3\rangle := \hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2)\,\hat{b}_1^\dagger(\boldsymbol{k}_3)\,\hat{b}_1^\dagger(-\boldsymbol{k}_3)|\mathrm{vac}\rangle$.

With this state,

$$\hat{b}_1(\boldsymbol{k})\,\hat{b}_1(-\boldsymbol{k})|\Psi_0(t)\rangle = -i\bar{V}_a\left\{\hat{b}_1(\boldsymbol{k})\,\hat{b}_1(-\boldsymbol{k})\sum_{\boldsymbol{k}_1\neq 0}\psi_{2,1}\left(k_1,\tilde{t}\right)\mathbb{V}_3(k_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1)|\mathrm{vac}\rangle + O\left(\bar{V}_a\right)\right\}$$
$$= -i\bar{V}_a\left[\sum_{\boldsymbol{k}_1\neq 0}\psi_{2,1}\left(k_1,\tilde{t}\right)\mathbb{V}_3(k_1)\left\{\delta_{\boldsymbol{k}_1,-\boldsymbol{k}}\hat{b}_1(\boldsymbol{k})\,\hat{b}_1^\dagger(-\boldsymbol{k}_1) + \delta_{\boldsymbol{k}_1,\boldsymbol{k}}\hat{b}_1(-\boldsymbol{k})\,\hat{b}_1^\dagger(-\boldsymbol{k}_1)\right\}|\mathrm{vac}\rangle + O\left(\bar{V}_a\right)\right]$$
$$= -2i\bar{V}_a\left[\psi_{2,1}\left(k,\tilde{t}\right)\mathbb{V}_3(k)|\mathrm{vac}\rangle + O\left(\bar{V}_a\right)\right]. \tag{F.27}$$

In interaction picture, using Eq. (F.5), reaction operator $\hat{R}_2$ becomes

$$
\hat{U}_0(t)\left[i\tilde{\alpha}\sum_{\boldsymbol{k}\neq 0}\Xi_{13}(k)\left\{\hat{b}_1(\boldsymbol{k})\hat{b}_1(-\boldsymbol{k})-\hat{b}_1^\dagger(\boldsymbol{k})\hat{b}_1^\dagger(-\boldsymbol{k})\right\}\right]\hat{U}_0^\dagger(t)
$$

$$
= i\tilde{\alpha}\sum_{\boldsymbol{k}\neq 0}\Xi_{13}(k)\left\{e^{-2i\tilde{\omega}_1(k)\tilde{t}}\hat{b}_1(\boldsymbol{k})\hat{b}_1(-\boldsymbol{k})-e^{2i\tilde{\omega}_1(k)\tilde{t}}\hat{b}_1^\dagger(\boldsymbol{k})\hat{b}_1^\dagger(-\boldsymbol{k})\right\}. \quad \text{(F.28)}
$$

Therefore,

$$
\begin{aligned}
\left\langle \hat{R}_2(t)\right\rangle &= -2\tilde{\alpha}\sum_{\boldsymbol{k}\neq 0}\Xi_{13}(k)\,\mathrm{Im}\left\{e^{-2i\tilde{\omega}_1(k)\tilde{t}}\langle\Psi_0(t)|\,\hat{b}_1(\boldsymbol{k})\hat{b}_1(-\boldsymbol{k})\,|\Psi_0(t)\rangle\right\}\\
&= 4\tilde{\alpha}\bar{V}_a\sum_{\boldsymbol{k}\neq 0}\Xi_{13}(k)\,\mathrm{Im}\left\{e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,1}\left(k,\tilde{t}\right)\mathbb{V}_3(k)\right\}+O\left(\bar{V}_a^2\right). \quad \text{(F.29)}
\end{aligned}
$$

F.3 Quantum Fisher information

Our state in Eq. (F.25) can be written as

$$
\begin{aligned}
|\Psi_0(t)\rangle &:= \hat{U}_I(t)\,|\mathrm{vac}\rangle\\
&= \left\{1-i\bar{V}_a\psi_{0,1}\left(\tilde{t}\right)-\bar{V}_a^2\psi_{0,2}\left(\tilde{t}\right)\right\}|\mathrm{vac}\rangle-2i\bar{V}_a\sum_{\boldsymbol{k}>0}\left\{\psi_{2,1}\left(k,\tilde{t}\right)-i\bar{V}_a\psi_{2,2}\left(k,\tilde{t}\right)\right\}\mathbb{V}_3(k)\,|\boldsymbol{k},-\boldsymbol{k}\rangle\\
&\quad -4\bar{V}_a^2\sum_{\boldsymbol{k}_1>0}\sum_{\boldsymbol{k}_2>0}\psi_{4,2}\left(k_1,k_2,\tilde{t}\right)\mathbb{V}_3(k_1)\mathbb{V}_3(k_2)\,|\boldsymbol{k}_1,-\boldsymbol{k}_1,\boldsymbol{k}_2,-\boldsymbol{k}_2\rangle+O\left(\bar{V}_a^3\right), \quad \text{(F.30)}
\end{aligned}
$$

since $|\boldsymbol{k},-\boldsymbol{k}\rangle := \hat{b}_1^\dagger(\boldsymbol{k})\,\hat{b}_1^\dagger(-\boldsymbol{k})\,|\mathrm{vac}\rangle = |-\boldsymbol{k},\boldsymbol{k}\rangle$ and

$$
\begin{aligned}
|\boldsymbol{k}_1,-\boldsymbol{k}_1,\boldsymbol{k}_2,-\boldsymbol{k}_2\rangle &:= \hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2)\,|\mathrm{vac}\rangle\\
&= |\boldsymbol{k}_1,-\boldsymbol{k}_1,\pm\boldsymbol{k}_2,\mp\boldsymbol{k}_2\rangle = |-\boldsymbol{k}_1,\boldsymbol{k}_1,\pm\boldsymbol{k}_2,\mp\boldsymbol{k}_2\rangle. \quad \text{(F.31)}
\end{aligned}
$$

. Hence, each state in Eq. (F.30) are orthogonal.

Let $P_0(t) := |\langle\Psi_0(t)|\mathrm{vac}\rangle|^2$, $P_2(\boldsymbol{k},t) := |\langle\Psi_0(t)|\,\boldsymbol{k},-\boldsymbol{k}\rangle|^2$ for $\boldsymbol{k}>0$, and $P_4(\boldsymbol{k}_1,\boldsymbol{k}_2,t) := |\langle\Psi_0(t)|\,\boldsymbol{k}_1,-\boldsymbol{k}_1,\boldsymbol{k}_2,-\boldsymbol{k}_2\rangle|^2$ for $\boldsymbol{k}_1>0$ and $\boldsymbol{k}_2>0$. Then we get

$$
\begin{aligned}
P_0(t) &= \left\{1+i\bar{V}_a\psi_{0,1}^*\left(\tilde{t}\right)-\bar{V}_a^2\psi_{0,2}^*\left(\tilde{t}\right)+O\left(\bar{V}_a^3\right)\right\}\left\{1-i\bar{V}_a\psi_{0,1}\left(\tilde{t}\right)-\bar{V}_a^2\psi_{0,2}\left(\tilde{t}\right)+O\left(\bar{V}_a^3\right)\right\}\\
&= 1-\bar{V}_a^2\left[2\mathrm{Re}\left\{\psi_{0,2}\left(\tilde{t}\right)\right\}-\left|\psi_{0,1}\left(\tilde{t}\right)\right|^2\right]+O\left(\bar{V}_a^3\right), \quad \text{(F.32)}
\end{aligned}
$$

$$
\begin{aligned}
P_2(\boldsymbol{k},t) = 4\bar{V}_a^2 &\left| \sum_{\boldsymbol{k}_1>0} \left\{ \psi_{2,1}\left(k_1,\tilde{t}\right) - i\bar{V}_a\psi_{2,2}\left(k_1,\tilde{t}\right) \right\} \mathbb{V}_3(k_1) \langle \mathrm{vac}| \hat{b}_1(\boldsymbol{k})\, \hat{b}_1(-\boldsymbol{k})\, \hat{b}_1^\dagger(\boldsymbol{k}_1)\, \hat{b}_1^\dagger(-\boldsymbol{k}_1) |\mathrm{vac}\rangle \right|^2 \\
&+ O\left(\bar{V}_a^3\right) \\
= 4\bar{V}_a^2 &\left| \sum_{\boldsymbol{k}_1>0} \left\{ \psi_{2,1}\left(k_1,\tilde{t}\right) - i\bar{V}_a\psi_{2,2}\left(k_1,\tilde{t}\right) \right\} \mathbb{V}_3(k_1) \langle \mathrm{vac}| \delta_{\boldsymbol{k}_1,-\boldsymbol{k}} \hat{b}_1(\boldsymbol{k})\, \hat{b}_1^\dagger(-\boldsymbol{k}_1) + \delta_{\boldsymbol{k}_1,\boldsymbol{k}} \hat{b}_1(\boldsymbol{k})\, \hat{b}_1^\dagger(\boldsymbol{k}_1) |\mathrm{vac}\rangle \right|^2 \\
&+ O\left(\bar{V}_a^3\right) \\
= 4\bar{V}_a^2 &\left| \psi_{2,1}\left(k,\tilde{t}\right) \mathbb{V}_3(k) \right|^2 + O\left(\bar{V}_a^3\right), \qquad \text{(F.33)}
\end{aligned}
$$

since we defined $P_2(\boldsymbol{k},t)$ for $\boldsymbol{k} > 0$, and $P_4(\boldsymbol{k}_1,\boldsymbol{k}_2,t) = O\left(\bar{V}_a^4\right)$.

When measuring $\bar{V}_a$, using formula in [61], quantum Fisher information $I_Q\left(\bar{V}_a,\tilde{t}\right)$ is

$$
\begin{aligned}
I_Q\left(\bar{V}_a,\tilde{t}\right) =& \frac{1}{P_0\left(t\right)}\left(\frac{\partial P_0\left(t\right)}{\partial \bar{V}_a}\right)^2 + \sum_{\boldsymbol{k}>0}\frac{1}{P_2\left(\boldsymbol{k},t\right)}\left(\frac{\partial P_2\left(\boldsymbol{k},t\right)}{\partial \bar{V}_a}\right)^2 + O\left(\bar{V}_a^2\right) \\
&+4P_0\left(t\right)\left\langle \text{vac}\left|\frac{\partial\left\{1+i\bar{V}_a\psi_{0,1}^*\left(\tilde{t}\right)-\bar{V}_a^2\psi_{0,2}^*\left(\tilde{t}\right)+O\left(\bar{V}_a^3\right)\right\}}{\partial\bar{V}_a}\frac{\partial\left\{1-i\bar{V}_a\psi_{0,1}\left(\tilde{t}\right)-\bar{V}_a^2\psi_{0,2}\left(\tilde{t}\right)+O\left(\bar{V}_a^3\right)\right\}}{\partial\bar{V}_a}\right|\text{vac}\right\rangle \\
&+4\sum_{\boldsymbol{k}>0}P_2\left(t\right)\left\langle \boldsymbol{k},-\boldsymbol{k}\left|\frac{\partial\left[2i\bar{V}_a\left\{\psi_{2,1}^*\left(k,\tilde{t}\right)+i\bar{V}_a\psi_{2,2}^*\left(k,\tilde{t}\right)+O\left(\bar{V}_a^2\right)\right\}\mathbb{V}_3^*\left(k\right)\right]}{\partial\bar{V}_a}\right.\right. \\
&\qquad\times\left.\left.\frac{\partial\left[-2i\bar{V}_a\left\{\psi_{2,1}\left(k,\tilde{t}\right)-i\bar{V}_a\psi_{2,2}\left(k,\tilde{t}\right)+O\left(\bar{V}_a^2\right)\right\}\mathbb{V}_3\left(k\right)\right]}{\partial\bar{V}_a}\right|\boldsymbol{k},-\boldsymbol{k}\right\rangle \\
&-4P_0\left(t\right)\left|\left\langle \text{vac}\left|\left\{1+i\bar{V}_a\psi_{0,1}^*\left(\tilde{t}\right)-\bar{V}_a^2\psi_{0,2}^*\left(\tilde{t}\right)+O\left(\bar{V}_a^3\right)\right\}\frac{\partial\left\{1-i\bar{V}_a\psi_{0,1}\left(\tilde{t}\right)-\bar{V}_a^2\psi_{0,2}\left(\tilde{t}\right)+O\left(\bar{V}_a^3\right)\right\}}{\partial\bar{V}_a}\right|\text{vac}\right\rangle\right|^2 \\
&-4\sum_{\boldsymbol{k}>0}P_2\left(t\right)\left|\left\langle \boldsymbol{k},-\boldsymbol{k}\left|\left[2i\bar{V}_a\left\{\psi_{2,1}^*\left(k,\tilde{t}\right)+i\bar{V}_a\psi_{2,2}^*\left(k,\tilde{t}\right)+O\left(\bar{V}_a^2\right)\right\}\mathbb{V}_3^*\left(k\right)\right]\right.\right.\right. \\
&\qquad\times\left.\left.\left.\frac{\partial\left[-2i\bar{V}_a\left\{\psi_{2,1}\left(k,\tilde{t}\right)-i\bar{V}_a\psi_{2,2}\left(k,\tilde{t}\right)+O\left(\bar{V}_a^2\right)\right\}\mathbb{V}_3\left(k\right)\right]}{\partial\bar{V}_a}\right|\boldsymbol{k},-\boldsymbol{k}\right\rangle\right|^2 \\
=& \frac{1}{1-\bar{V}_a^2\left[2\mathrm{Re}\left\{\psi_{0,2}\left(\tilde{t}\right)\right\}-\left|\psi_{0,1}\left(\tilde{t}\right)\right|^2\right]+O\left(\bar{V}_a^3\right)}\left(\frac{\partial\left(1-\bar{V}_a^2\left[2\mathrm{Re}\left\{\psi_{0,2}\left(\tilde{t}\right)\right\}-\left|\psi_{0,1}\left(\tilde{t}\right)\right|^2\right]+O\left(\bar{V}_a^3\right)\right)}{\partial\tilde{V}_a}\right)^2 \\
&+\sum_{\boldsymbol{k}>0}\frac{1}{4\bar{V}_a^2\left\{\left|\psi_{2,1}\left(k,\tilde{t}\right)\mathbb{V}_3\left(k\right)\right|^2+O\left(\bar{V}_a\right)\right\}}\left(\frac{\partial\left[4\bar{V}_a^2\left\{\left|\psi_{2,1}\left(k,\tilde{t}\right)\mathbb{V}_3\left(k\right)\right|^2+O\left(\bar{V}_a\right)\right\}\right]}{\partial\bar{V}_a}\right)^2+O\left(\bar{V}_a^2\right) \\
&+4\left\langle\text{vac}\left|\left\{i\psi_{0,1}^*\left(\tilde{t}\right)-2\bar{V}_a\psi_{0,2}^*\left(\tilde{t}\right)+O\left(\bar{V}_a^2\right)\right\}\left\{-i\psi_{0,1}\left(\tilde{t}\right)-2\bar{V}_a\psi_{0,2}\left(\tilde{t}\right)+O\left(\bar{V}_a^2\right)\right\}\right|\text{vac}\right\rangle \\
&-4\left|\left\langle\text{vac}\left|\left\{1+i\bar{V}_a\psi_{0,1}^*\left(\tilde{t}\right)-\bar{V}_a^2\psi_{0,2}^*\left(\tilde{t}\right)+O\left(\bar{V}_a^3\right)\right\}\left\{-i\psi_{0,1}\left(\tilde{t}\right)-2\bar{V}_a\psi_{0,2}\left(\tilde{t}\right)+O\left(\bar{V}_a^2\right)\right\}\right|\text{vac}\right\rangle\right|^2 \\
=& \left(1+\bar{V}_a^2\left[2\mathrm{Re}\left\{\psi_{0,2}\left(\tilde{t}\right)\right\}-\left|\psi_{0,1}\left(\tilde{t}\right)\right|^2\right]+O\left(\bar{V}_a^3\right)\right)\left(-2\bar{V}_a\left[2\mathrm{Re}\left\{\psi_{0,2}\left(\tilde{t}\right)\right\}-\left|\psi_{0,1}\left(\tilde{t}\right)\right|^2+O\left(\bar{V}_a\right)\right]\right)^2 \\
&+\sum_{\boldsymbol{k}>0}\frac{1}{4\bar{V}_a^2\left|\psi_{2,1}\left(k,\tilde{t}\right)\mathbb{V}_3\left(k\right)\right|^2}\left\{1-O\left(\bar{V}_a\right)\right\}\left[8\bar{V}_a\left\{\left|\psi_{2,1}\left(k,\tilde{t}\right)\mathbb{V}_3\left(k\right)\right|^2+O\left(\bar{V}_a\right)\right\}\right]^2+O\left(\bar{V}_a^2\right) \\
&+4\left[\left|\psi_{0,1}\left(\tilde{t}\right)\right|^2+2\bar{V}_a\mathrm{Im}\left\{\psi_{0,1}^*\left(\tilde{t}\right)\psi_{0,2}\left(\tilde{t}\right)\right\}-\left|-i\psi_{0,1}\left(\tilde{t}\right)-\bar{V}_a\left\{2\psi_{0,2}\left(\tilde{t}\right)-\left|\psi_{0,1}\left(\tilde{t}\right)\right|^2\right\}+O\left(\bar{V}_a^2\right)\right|^2\right] \\
=& \sum_{\boldsymbol{k}>0}16\left|\psi_{2,1}\left(k,\tilde{t}\right)\mathbb{V}_3\left(k\right)\right|^2+O\left(\bar{V}_a\right).
\end{aligned}
\tag{F.34}
$$

F.4 Lower bound of Classical Fisher Information

Let $\hat{M}_S = \sum_{\boldsymbol{k}\neq 0} \left\{ \mathbb{M}_1(k) + \mathbb{M}_2(k)\, \hat{b}_1^\dagger(\boldsymbol{k})\, \hat{b}_1(\boldsymbol{k}) + \mathbb{M}_3(k)\, \hat{b}_1^\dagger(\boldsymbol{k})\, \hat{b}_1^\dagger(-\boldsymbol{k}) + \mathbb{M}_3^*(k)\, \hat{b}_1(\boldsymbol{k})\, \hat{b}_1(-\boldsymbol{k}) \right\}$ be the physical operator which we will measure in order to estimate the value of $\bar{V}_a$.

From Eq. (F.5), $\hat{M}_I(t) \coloneqq \hat{U}_0(t)\, \hat{M}_S \hat{U}_0^\dagger(t)$ is

$$
\begin{aligned}
\hat{M}_I(t) &= \sum_{\boldsymbol{k}\neq 0} \left\{ \mathbb{M}_1(k) + \mathbb{M}_2(k)\, \hat{b}_1^\dagger(\boldsymbol{k})\, \hat{b}_1(\boldsymbol{k}) + e^{2i\tilde{\omega}_1(k)\tilde{t}} \mathbb{M}_3(k)\, \hat{b}_1^\dagger(\boldsymbol{k})\, \hat{b}_1^\dagger(-\boldsymbol{k}) + e^{-2i\tilde{\omega}_1(k)\tilde{t}} \mathbb{M}_3^*(k)\, \hat{b}_1(\boldsymbol{k})\, \hat{b}_1(-\boldsymbol{k}) \right\} \\
&= 2 \sum_{\boldsymbol{k}>0} \left\{ \mathbb{M}_1(k) + \mathbb{M}_2(k)\, \hat{b}_1^\dagger(\boldsymbol{k})\, \hat{b}_1(\boldsymbol{k}) + e^{2i\tilde{\omega}_1(k)\tilde{t}} \mathbb{M}_3(k)\, \hat{b}_1^\dagger(\boldsymbol{k})\, \hat{b}_1^\dagger(-\boldsymbol{k}) + e^{-2i\tilde{\omega}_1(k)\tilde{t}} \mathbb{M}_3^*(k)\, \hat{b}_1(\boldsymbol{k})\, \hat{b}_1(-\boldsymbol{k}) \right\}.
\end{aligned}
\tag{F.35}
$$

Using our state in Eq. (F.30), we get

$$
\begin{aligned}
|\Psi_0(t)\rangle = &\left\{ 1 - i\bar{V}_a \psi_{0,1}(\tilde{t}) - \bar{V}_a^2 \psi_{0,2}(\tilde{t}) \right\} |\mathrm{vac}\rangle - 2i\bar{V}_a \sum_{\boldsymbol{k}>0} \left\{ \psi_{2,1}(k,\tilde{t}) - i\bar{V}_a \psi_{2,2}(k,\tilde{t}) \right\} \mathbb{V}_3(k)\, |\boldsymbol{k}, -\boldsymbol{k}\rangle \\
&-4\bar{V}_a^2 \sum_{\boldsymbol{k}_1>0} \sum_{\boldsymbol{k}_2>0} \psi_{4,2}(k_1,k_2,\tilde{t})\, \mathbb{V}_3(k_1)\, \mathbb{V}_3(k_2)\, |\boldsymbol{k}_1, -\boldsymbol{k}_1, \boldsymbol{k}_2, -\boldsymbol{k}_2\rangle + O\left(\bar{V}_a^3\right),
\end{aligned}
\tag{F.36}
$$

$$
\begin{aligned}
&\hat{M}_I\left(\tilde{t}\right)\left|\hat{\Psi}_0\left(\tilde{t}\right)\right\rangle \\
&= 2\left\{1 - i\bar{V}_a\psi_{0,1}\left(\tilde{t}\right) - \bar{V}_a^2\psi_{0,2}\left(\tilde{t}\right)\right\}\sum_{\boldsymbol{k}>0}\left\{\mathbb{M}_1\left(k\right) + e^{2i\tilde{\omega}_1(k)\tilde{t}}\mathbb{M}_3\left(k\right)\hat{b}_1^\dagger\left(\boldsymbol{k}\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}\right)\right\}\left|\mathrm{vac}\right\rangle + O\left(\bar{V}_a^3\right) \\
&\quad -4i\bar{V}_a\sum_{\boldsymbol{k}>0}\left\{\psi_{2,1}\left(k,\tilde{t}\right) - i\bar{V}_a\psi_{2,2}\left(k,\tilde{t}\right)\right\}\mathbb{V}_3\left(k\right)\sum_{\boldsymbol{k}_1>0}\left\{\mathbb{M}_1\left(k_1\right) + \mathbb{M}_2\left(k_1\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_1\right)\hat{b}_1\left(\boldsymbol{k}_1\right)\right\}\hat{b}_1^\dagger\left(\boldsymbol{k}\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}\right)\left|\mathrm{vac}\right\rangle \\
&\quad -4i\bar{V}_a\sum_{\boldsymbol{k}_1>0}e^{2i\tilde{\omega}_1(k_1)\tilde{t}}\mathbb{M}_3\left(k_1\right)\sum_{\boldsymbol{k}>0}\left\{\psi_{2,1}\left(k,\tilde{t}\right) - i\bar{V}_a\psi_{2,2}\left(k,\tilde{t}\right)\right\}\mathbb{V}_3\left(k\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(\boldsymbol{k}\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}\right)\left|\mathrm{vac}\right\rangle \\
&\quad -4i\bar{V}_a\sum_{\boldsymbol{k}_1>0}e^{-2i\tilde{\omega}_1(k_1)\tilde{t}}\mathbb{M}_3^*\left(k_1\right)\sum_{\boldsymbol{k}>0}\left\{\psi_{2,1}\left(k,\tilde{t}\right) - i\bar{V}_a\psi_{2,2}\left(k,\tilde{t}\right)\right\}\mathbb{V}_3\left(k\right)\hat{b}_1\left(\boldsymbol{k}_1\right)\hat{b}_1\left(-\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(\boldsymbol{k}\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}\right)\left|\mathrm{vac}\right\rangle \\
&\quad -8\bar{V}_a^2\sum_{\boldsymbol{k}_3>0}\mathbb{M}_1\left(k_3\right)\sum_{\boldsymbol{k}_1>0}\sum_{\boldsymbol{k}_2>0}\psi_{4,2}\left(k_1,k_2,\tilde{t}\right)\mathbb{V}_3\left(k_1\right)\mathbb{V}_3\left(k_2\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_2\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_2\right)\left|\mathrm{vac}\right\rangle \\
&\quad -8\bar{V}_a^2\sum_{\boldsymbol{k}_3>0}\mathbb{M}_2\left(k_3\right) \\
&\qquad\qquad \times\sum_{\boldsymbol{k}_1>0}\sum_{\boldsymbol{k}_2>0}\psi_{4,2}\left(k_1,k_2,\tilde{t}\right)\mathbb{V}_3\left(k_1\right)\mathbb{V}_3\left(k_2\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_3\right)\hat{b}_1\left(\boldsymbol{k}_3\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_2\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_2\right)\left|\mathrm{vac}\right\rangle \\
&\quad -8\bar{V}_a^2\sum_{\boldsymbol{k}_3>0}e^{-2i\tilde{\omega}_1(k_3)\tilde{t}}\mathbb{M}_3^*\left(k_3\right) \\
&\qquad\qquad \times\sum_{\boldsymbol{k}_1>0}\sum_{\boldsymbol{k}_2>0}\psi_{4,2}\left(k_1,k_2,\tilde{t}\right)\mathbb{V}_3\left(k_1\right)\mathbb{V}_3\left(k_2\right)\hat{b}_1\left(\boldsymbol{k}_3\right)\hat{b}_1\left(-\boldsymbol{k}_3\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_2\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_2\right)\left|\mathrm{vac}\right\rangle \\
&\quad -8\bar{V}_a^2\sum_{\boldsymbol{k}_3>0}e^{2i\tilde{\omega}_1(k_3)\tilde{t}}\mathbb{M}_3\left(k_3\right) \\
&\qquad\qquad \times\sum_{\boldsymbol{k}_1>0}\sum_{\boldsymbol{k}_2>0}\psi_{4,2}\left(k_1,k_2,\tilde{t}\right)\mathbb{V}_3\left(k_1\right)\mathbb{V}_3\left(k_2\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_3\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_3\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_2\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_2\right)\left|\mathrm{vac}\right\rangle .
\end{aligned}
\tag{F.37}
$$

1. 2nd line in Eq. (F.37)

$$
\begin{aligned}
\hat{b}_1^\dagger\left(\boldsymbol{k}_1\right)\hat{b}_1\left(\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(\boldsymbol{k}\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}\right)\left|\mathrm{vac}\right\rangle &= \delta_{\boldsymbol{k}_1,\boldsymbol{k}}\hat{b}_1^\dagger\left(\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}\right)\left|\mathrm{vac}\right\rangle + \delta_{\boldsymbol{k}_1,-\boldsymbol{k}}\hat{b}_1^\dagger\left(\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(\boldsymbol{k}\right)\left|\mathrm{vac}\right\rangle \\
&= \left(\delta_{\boldsymbol{k}_1,\boldsymbol{k}} + \delta_{\boldsymbol{k}_1,-\boldsymbol{k}}\right)\hat{b}_1^\dagger\left(\boldsymbol{k}\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}\right)\left|\mathrm{vac}\right\rangle .
\end{aligned}
\tag{F.38}
$$

2. 4th line in Eq. (F.37)

$$\begin{aligned}\hat{b}_1(\boldsymbol{k}_1)\,\hat{b}_1(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k})\,\hat{b}_1^\dagger(-\boldsymbol{k})\,|\mathrm{vac}\rangle &= \delta_{\boldsymbol{k}_1,-\boldsymbol{k}}\hat{b}_1(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k})\,|\mathrm{vac}\rangle + \delta_{\boldsymbol{k}_1,\boldsymbol{k}}\hat{b}_1(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k})\,|\mathrm{vac}\rangle \\ &= \left(\delta_{\boldsymbol{k}_1,\boldsymbol{k}} + \delta_{\boldsymbol{k}_1,-\boldsymbol{k}}\right)|\mathrm{vac}\rangle\,. \end{aligned} \tag{F.39}$$

3. 6th line in Eq. (F.37)

$$\begin{aligned}&\hat{b}_1^\dagger(\boldsymbol{k}_3)\,\hat{b}_1(\boldsymbol{k}_3)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2)\,|\mathrm{vac}\rangle \\ &= \delta_{\boldsymbol{k}_3,\boldsymbol{k}_1}\hat{b}_1^\dagger(\boldsymbol{k}_3)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2)\,|\mathrm{vac}\rangle + \delta_{\boldsymbol{k}_3,-\boldsymbol{k}_1}\hat{b}_1^\dagger(\boldsymbol{k}_3)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2)\,|\mathrm{vac}\rangle \\ &\quad +\delta_{\boldsymbol{k}_3,\boldsymbol{k}_2}\hat{b}_1^\dagger(\boldsymbol{k}_3)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2)\,|\mathrm{vac}\rangle + \delta_{\boldsymbol{k}_3,-\boldsymbol{k}_2}\hat{b}_1^\dagger(\boldsymbol{k}_3)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,|\mathrm{vac}\rangle \\ &= \left(\delta_{\boldsymbol{k}_3,\boldsymbol{k}_1} + \delta_{\boldsymbol{k}_3,-\boldsymbol{k}_1} + \delta_{\boldsymbol{k}_3,\boldsymbol{k}_2} + \delta_{\boldsymbol{k}_3,-\boldsymbol{k}_2}\right)\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2)\,|\mathrm{vac}\rangle\,. \end{aligned} \tag{F.40}$$

4. 7th line in Eq. (F.37)

$$
\begin{aligned}
&\hat{b}_1(\boldsymbol{k}_3)\,\hat{b}_1(-\boldsymbol{k}_3)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2)\,|\mathrm{vac}\rangle \\
&\quad = \delta_{\boldsymbol{k}_3,-\boldsymbol{k}_1}\hat{b}_1(\boldsymbol{k}_3)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2)\,|\mathrm{vac}\rangle + \delta_{\boldsymbol{k}_3,\boldsymbol{k}_1}\hat{b}_1(\boldsymbol{k}_3)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2)\,|\mathrm{vac}\rangle \\
&\qquad +\delta_{\boldsymbol{k}_3,-\boldsymbol{k}_2}\hat{b}_1(\boldsymbol{k}_3)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2)\,|\mathrm{vac}\rangle + \delta_{\boldsymbol{k}_3,\boldsymbol{k}_2}\hat{b}_1(\boldsymbol{k}_3)\,\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\,|\mathrm{vac}\rangle \\
&\quad = \delta_{\boldsymbol{k}_3,-\boldsymbol{k}_1}\left\{\delta_{\boldsymbol{k}_3,-\boldsymbol{k}_1}\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2) + \delta_{\boldsymbol{k}_3,\boldsymbol{k}_2}\hat{b}_1^\dagger(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2) + \delta_{\boldsymbol{k}_3,-\boldsymbol{k}_2}\hat{b}_1^\dagger(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\right\}|\mathrm{vac}\rangle \\
&\qquad +\delta_{\boldsymbol{k}_3,\boldsymbol{k}_1}\left\{\delta_{\boldsymbol{k}_3,\boldsymbol{k}_1}\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2) + \delta_{\boldsymbol{k}_3,\boldsymbol{k}_2}\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2) + \delta_{\boldsymbol{k}_3,-\boldsymbol{k}_2}\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\right\}|\mathrm{vac}\rangle \\
&\qquad +\delta_{\boldsymbol{k}_3,-\boldsymbol{k}_2}\left\{\delta_{\boldsymbol{k}_3,\boldsymbol{k}_1}\hat{b}_1^\dagger(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2) + \delta_{\boldsymbol{k}_3,-\boldsymbol{k}_1}\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2) + \delta_{\boldsymbol{k}_3,-\boldsymbol{k}_2}\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1)\right\}|\mathrm{vac}\rangle \\
&\qquad +\delta_{\boldsymbol{k}_3,\boldsymbol{k}_2}\left\{\delta_{\boldsymbol{k}_3,\boldsymbol{k}_1}\hat{b}_1^\dagger(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2) + \delta_{\boldsymbol{k}_3,-\boldsymbol{k}_1}\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2) + \delta_{\boldsymbol{k}_3,\boldsymbol{k}_2}\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1)\right\}|\mathrm{vac}\rangle \\
&\quad = \delta_{\boldsymbol{k}_3,-\boldsymbol{k}_1}\left\{\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2) + \delta_{\boldsymbol{k}_2,-\boldsymbol{k}_1}\hat{b}_1^\dagger(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2) + \delta_{\boldsymbol{k}_2,\boldsymbol{k}_1}\hat{b}_1^\dagger(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\right\}|\mathrm{vac}\rangle \\
&\qquad +\delta_{\boldsymbol{k}_3,\boldsymbol{k}_1}\left\{\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2) + \delta_{\boldsymbol{k}_2,\boldsymbol{k}_1}\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2) + \delta_{\boldsymbol{k}_2,-\boldsymbol{k}_1}\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2)\right\}|\mathrm{vac}\rangle \\
&\qquad +\delta_{\boldsymbol{k}_3,-\boldsymbol{k}_2}\left\{\delta_{\boldsymbol{k}_2,-\boldsymbol{k}_1}\hat{b}_1^\dagger(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2) + \delta_{\boldsymbol{k}_2,\boldsymbol{k}_1}\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2) + \hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1)\right\}|\mathrm{vac}\rangle \\
&\qquad +\delta_{\boldsymbol{k}_3,\boldsymbol{k}_2}\left\{\delta_{\boldsymbol{k}_2,\boldsymbol{k}_1}\hat{b}_1^\dagger(-\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2) + \delta_{\boldsymbol{k}_2,-\boldsymbol{k}_1}\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(\boldsymbol{k}_2) + \hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1)\right\}|\mathrm{vac}\rangle \\
&\quad = \delta_{\boldsymbol{k}_3,-\boldsymbol{k}_1}\left\{\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2) + \left(\delta_{\boldsymbol{k}_2,\boldsymbol{k}_1} + \delta_{\boldsymbol{k}_2,-\boldsymbol{k}_1}\right)\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1)\right\}|\mathrm{vac}\rangle \\
&\qquad +\delta_{\boldsymbol{k}_3,\boldsymbol{k}_1}\left\{\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2) + \left(\delta_{\boldsymbol{k}_2,\boldsymbol{k}_1} + \delta_{\boldsymbol{k}_2,-\boldsymbol{k}_1}\right)\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1)\right\}|\mathrm{vac}\rangle \\
&\qquad +\delta_{\boldsymbol{k}_3,-\boldsymbol{k}_2}\left\{\left(\delta_{\boldsymbol{k}_2,\boldsymbol{k}_1} + \delta_{\boldsymbol{k}_2,-\boldsymbol{k}_1}\right)\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1) + \hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1)\right\}|\mathrm{vac}\rangle \\
&\qquad +\delta_{\boldsymbol{k}_3,\boldsymbol{k}_2}\left\{\left(\delta_{\boldsymbol{k}_2,\boldsymbol{k}_1} + \delta_{\boldsymbol{k}_2,-\boldsymbol{k}_1}\right)\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1) + \hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1)\right\}|\mathrm{vac}\rangle \\
&\quad = \left\{\left(\delta_{\boldsymbol{k}_3,\boldsymbol{k}_2} + \delta_{\boldsymbol{k}_3,-\boldsymbol{k}_2}\right) + \left(\delta_{\boldsymbol{k}_3,\boldsymbol{k}_1} + \delta_{\boldsymbol{k}_3,-\boldsymbol{k}_1} + \delta_{\boldsymbol{k}_3,\boldsymbol{k}_2} + \delta_{\boldsymbol{k}_3,-\boldsymbol{k}_2}\right)\left(\delta_{\boldsymbol{k}_2,\boldsymbol{k}_1} + \delta_{\boldsymbol{k}_2,-\boldsymbol{k}_1}\right)\right\}\hat{b}_1^\dagger(\boldsymbol{k}_1)\,\hat{b}_1^\dagger(-\boldsymbol{k}_1)\,|\mathrm{vac}\rangle \\
&\qquad + \left(\delta_{\boldsymbol{k}_3,\boldsymbol{k}_1} + \delta_{\boldsymbol{k}_3,-\boldsymbol{k}_1}\right)\hat{b}_1^\dagger(\boldsymbol{k}_2)\,\hat{b}_1^\dagger(-\boldsymbol{k}_2)\,|\mathrm{vac}\rangle\,. \qquad \text{(F.41)}
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
&\hat{M}_I\left(\tilde{t}\right)\left|\hat{\Psi}_0\left(\tilde{t}\right)\right\rangle \\
&=2\left\{1-i\bar{V}_a\psi_{0,1}\left(\tilde{t}\right)-\bar{V}_a^2\psi_{0,2}\left(\tilde{t}\right)\right\}\sum_{\boldsymbol{k}>0}\mathbb{M}_1\left(k\right)|\text{vac}\rangle+O\left(\bar{V}_a^3\right) \\
&\quad+2\left\{1-i\bar{V}_a\psi_{0,1}\left(\tilde{t}\right)-\bar{V}_a^2\psi_{0,2}\left(\tilde{t}\right)\right\}\sum_{\boldsymbol{k}>0}e^{2i\tilde{\omega}_1(k)\tilde{t}}\mathbb{M}_3\left(k\right)\hat{b}_1^\dagger\left(\boldsymbol{k}\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}\right)|\text{vac}\rangle \\
&\quad-4i\bar{V}_a\sum_{\boldsymbol{k}>0}\left\{\psi_{2,1}\left(k,\tilde{t}\right)-i\bar{V}_a\psi_{2,2}\left(k,\tilde{t}\right)\right\}\mathbb{V}_3\left(k\right)\sum_{\boldsymbol{k}_1>0}\mathbb{M}_1\left(k_1\right)\hat{b}_1^\dagger\left(\boldsymbol{k}\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}\right)|\text{vac}\rangle \\
&\quad-4i\bar{V}_a\sum_{\boldsymbol{k}>0}\left\{\psi_{2,1}\left(k,\tilde{t}\right)-i\bar{V}_a\psi_{2,2}\left(k,\tilde{t}\right)\right\}\mathbb{V}_3\left(k\right)\mathbb{M}_2\left(k\right)\hat{b}_1^\dagger\left(\boldsymbol{k}\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}\right)|\text{vac}\rangle \\
&\quad-4i\bar{V}_a\sum_{\boldsymbol{k}_1>0}e^{2i\tilde{\omega}_1(k_1)\tilde{t}}\mathbb{M}_3\left(k_1\right)\sum_{\boldsymbol{k}>0}\left\{\psi_{2,1}\left(k,\tilde{t}\right)-i\bar{V}_a\psi_{2,2}\left(k,\tilde{t}\right)\right\}\mathbb{V}_3\left(k\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(\boldsymbol{k}\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}\right)|\text{vac}\rangle \\
&\quad-4i\bar{V}_a\sum_{\boldsymbol{k}>0}\left\{\psi_{2,1}\left(k,\tilde{t}\right)-i\bar{V}_a\psi_{2,2}\left(k,\tilde{t}\right)\right\}\mathbb{V}_3\left(k\right)e^{-2i\tilde{\omega}_1(k)\tilde{t}}\mathbb{M}_3^*\left(k\right)|\text{vac}\rangle \\
&\quad-8\bar{V}_a^2\sum_{\boldsymbol{k}_1>0}\sum_{\boldsymbol{k}_2>0}\psi_{4,2}\left(k_1,k_2,\tilde{t}\right)\left\{\sum_{\boldsymbol{k}_3>0}\mathbb{M}_1\left(k_3\right)\right\}\mathbb{V}_3\left(k_1\right)\mathbb{V}_3\left(k_2\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_2\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_2\right)|\text{vac}\rangle \\
&\quad-8\bar{V}_a^2\sum_{\boldsymbol{k}_1>0}\sum_{\boldsymbol{k}_2>0}\psi_{4,2}\left(k_1,k_2,\tilde{t}\right)\mathbb{V}_3\left(k_1\right)\mathbb{V}_3\left(k_2\right)\left\{\mathbb{M}_2\left(k_1\right)+\mathbb{M}_2\left(k_2\right)\right\}\hat{b}_1^\dagger\left(\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_2\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_2\right)|\text{vac}\rangle \\
&\quad-8\bar{V}_a^2\sum_{\boldsymbol{k}_1>0}\sum_{\boldsymbol{k}_2>0}\psi_{4,2}\left(k_1,k_2,\tilde{t}\right)\mathbb{V}_3\left(k_1\right)\mathbb{V}_3\left(k_2\right)e^{-2i\tilde{\omega}_1(k_2)\tilde{t}}\mathbb{M}_3^*\left(k_2\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_1\right)|\text{vac}\rangle \\
&\quad-16\bar{V}_a^2\sum_{\boldsymbol{k}_1>0}\psi_{4,2}\left(k_1,k_1,\tilde{t}\right)\mathbb{V}_3\left(k_1\right)\mathbb{V}_3\left(k_1\right)e^{-2i\tilde{\omega}_1(k_1)\tilde{t}}\mathbb{M}_3^*\left(k_1\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_1\right)|\text{vac}\rangle \\
&\quad-8\bar{V}_a^2\sum_{\boldsymbol{k}_1>0}\sum_{\boldsymbol{k}_2>0}\psi_{4,2}\left(k_1,k_2,\tilde{t}\right)\mathbb{V}_3\left(k_1\right)\mathbb{V}_3\left(k_2\right)e^{-2i\tilde{\omega}_1(k_1)\tilde{t}}\mathbb{M}_3^*\left(k_1\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_2\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_2\right)|\text{vac}\rangle \\
&\quad-8\bar{V}_a^2\sum_{\boldsymbol{k}_3>0}e^{2i\tilde{\omega}_1(k_3)\tilde{t}}\mathbb{M}_3\left(k_3\right) \\
&\qquad\times\sum_{\boldsymbol{k}_1>0}\sum_{\boldsymbol{k}_2>0}\psi_{4,2}\left(k_1,k_2,\tilde{t}\right)\mathbb{V}_3\left(k_1\right)\mathbb{V}_3\left(k_2\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_3\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_3\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_2\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_2\right)|\text{vac}\rangle\,.
\end{aligned}
\tag{F.42}
$$

Ordering Eq. (F.42) gives

$$
\begin{aligned}
&\hat{M}_I\left(\tilde{t}\right)\left|\hat{\Psi}_0\left(\tilde{t}\right)\right\rangle \\
&=2\sum_{\boldsymbol{k}>0}\left[\begin{array}{c}\mathbb{M}_1\left(k\right)-i\bar{V}_a\left\{\psi_{0,1}\left(\tilde{t}\right)\mathbb{M}_1\left(k\right)+2e^{-2i\tilde{\omega}_1\left(k\right)\tilde{t}}\psi_{2,1}\left(k,\tilde{t}\right)\mathbb{M}_3^*\left(k\right)\mathbb{V}_3\left(k\right)\right\}\\-\bar{V}_a^2\left\{\psi_{0,2}\left(\tilde{t}\right)\mathbb{M}_1\left(k\right)+2e^{-2i\tilde{\omega}_1\left(k\right)\tilde{t}}\psi_{2,2}\left(k,\tilde{t}\right)\mathbb{M}_3^*\left(k\right)\mathbb{V}_3\left(k\right)\right\}\end{array}\right]\left|\text{vac}\right\rangle+O\left(\bar{V}_a^3\right)\\
&\quad+2\sum_{\boldsymbol{k}>0}\left\{e^{2i\tilde{\omega}_1\left(k\right)\tilde{t}}\mathbb{M}_3\left(k\right)-i\bar{V}_ae^{2i\tilde{\omega}_1\left(k\right)\tilde{t}}\psi_{0,1}\left(\tilde{t}\right)\mathbb{M}_3\left(k\right)-\bar{V}_a^2e^{2i\tilde{\omega}_1\left(k\right)\tilde{t}}\psi_{0,2}\left(\tilde{t}\right)\mathbb{M}_3\left(k\right)\right\}\hat{b}_1^\dagger\left(\boldsymbol{k}\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}\right)\left|\text{vac}\right\rangle\\
&\quad+2\sum_{\boldsymbol{k}>0}\left\{-2i\bar{V}_a\psi_{2,1}\left(k,\tilde{t}\right)-2\bar{V}_a^2\psi_{2,2}\left(k,\tilde{t}\right)\right\}\left\{\mathbb{M}_2\left(k\right)+\sum_{\boldsymbol{k}_1>0}\mathbb{M}_1\left(k_1\right)\right\}\mathbb{V}_3\left(k\right)\hat{b}_1^\dagger\left(\boldsymbol{k}\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}\right)\left|\text{vac}\right\rangle\\
&\quad-2\bar{V}_a^2\sum_{\boldsymbol{k}>0}8e^{-2i\tilde{\omega}_1\left(k\right)\tilde{t}}\psi_{4,2}\left(k,k,\tilde{t}\right)\mathbb{M}_3^*\left(k\right)\mathbb{V}_3^2\left(k\right)\hat{b}_1^\dagger\left(\boldsymbol{k}\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}\right)\left|\text{vac}\right\rangle\\
&\quad-2\bar{V}_a^2\sum_{\boldsymbol{k}>0}4\sum_{\boldsymbol{k}_1>0}e^{-2i\tilde{\omega}_1\left(k_1\right)\tilde{t}}\left\{\psi_{4,2}\left(k_1,k,\tilde{t}\right)+\psi_{4,2}\left(k,k_1,\tilde{t}\right)\right\}\mathbb{M}_3^*\left(k_1\right)\mathbb{V}_3\left(k_1\right)\mathbb{V}_3\left(k\right)\hat{b}_1^\dagger\left(\boldsymbol{k}\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}\right)\left|\text{vac}\right\rangle\\
&\quad-4i\bar{V}_a\sum_{\boldsymbol{k}_1>0}\sum_{\boldsymbol{k}_2>0}e^{2i\tilde{\omega}_1\left(k_1\right)\tilde{t}}\left\{\psi_{2,1}\left(k_2,\tilde{t}\right)-i\bar{V}_a\psi_{2,2}\left(k_2,\tilde{t}\right)\right\}\mathbb{M}_3\left(k_1\right)\mathbb{V}_3\left(k_2\right)\\
&\qquad\qquad\times\hat{b}_1^\dagger\left(\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_2\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_2\right)\left|\text{vac}\right\rangle\\
&\quad-4\bar{V}_a^2\sum_{\boldsymbol{k}_1>0}\sum_{\boldsymbol{k}_2>0}2\psi_{4,2}\left(k_1,k_2,\tilde{t}\right)\left\{\sum_{\boldsymbol{k}_3>0}\mathbb{M}_1\left(k_3\right)\right\}\mathbb{V}_3\left(k_1\right)\mathbb{V}_3\left(k_2\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_2\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_2\right)\left|\text{vac}\right\rangle\\
&\quad-4\bar{V}_a^2\sum_{\boldsymbol{k}_1>0}\sum_{\boldsymbol{k}_2>0}2\psi_{4,2}\left(k_1,k_2,\tilde{t}\right)\left\{\mathbb{M}_2\left(k_1\right)+\mathbb{M}_2\left(k_2\right)\right\}\mathbb{V}_3\left(k_1\right)\mathbb{V}_3\left(k_2\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_2\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_2\right)\left|\text{vac}\right\rangle\\
&\quad-8\bar{V}_a^2\sum_{\boldsymbol{k}_1>0}\sum_{\boldsymbol{k}_2>0}\sum_{\boldsymbol{k}_3>0}e^{2i\tilde{\omega}_1\left(k_1\right)\tilde{t}}\psi_{4,2}\left(k_2,k_3,\tilde{t}\right)\mathbb{M}_3\left(k_1\right)\mathbb{V}_3\left(k_2\right)\mathbb{V}_3\left(k_3\right)\\
&\qquad\qquad\times\hat{b}_1^\dagger\left(\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_2\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_2\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_3\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_3\right)\left|\text{vac}\right\rangle. \qquad \text{(F.43)}
\end{aligned}
$$

From Eq. (F.39), $\langle\boldsymbol{k},-\boldsymbol{k}|\,\boldsymbol{k}_1,-\boldsymbol{k}_1\rangle=\langle\text{vac}|\,\hat{b}_1\left(\boldsymbol{k}\right)\hat{b}_1\left(-\boldsymbol{k}\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_1\right)|\text{vac}\rangle=\left(\delta_{\boldsymbol{k}_1,\boldsymbol{k}}+\delta_{\boldsymbol{k}_1,-\boldsymbol{k}}\right)$. Hence, from Eq. (F.43) and our state $\left|\hat{\Psi}_0\left(\tilde{t}\right)\right\rangle$ in Eq. (F.30),

$$
\begin{aligned}
&\left\langle \hat{\Psi}_0\left(\tilde{t}\right)\right| \hat{M}_I\left(\tilde{t}\right)\left|\hat{\Psi}_0\left(\tilde{t}\right)\right\rangle \\
&= 2\sum_{\boldsymbol{k}>0}\left(\begin{array}{l}
\mathbb{M}_1(k) - i\bar{V}_a\left[2i\mathrm{Im}\left\{\psi_{0,1}\left(\tilde{t}\right)\right\}\mathbb{M}_1(k) + 2e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,1}\left(k,\tilde{t}\right)\mathbb{M}_3^*(k)\,\mathbb{V}_3(k)\right] \\
+\bar{V}_a^2\left[\left|\psi_{0,1}\left(\tilde{t}\right)\right|^2\mathbb{M}_1(k) - 2\mathrm{Re}\left\{\psi_{0,2}\left(\tilde{t}\right)\right\}\mathbb{M}_1(k)\right] \\
+\bar{V}_a^2\left[2e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{0,1}^*\left(\tilde{t}\right)\psi_{2,1}\left(k,\tilde{t}\right)\mathbb{M}_3^*(k)\,\mathbb{V}_3(k) - 2e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,2}\left(k,\tilde{t}\right)\mathbb{M}_3^*(k)\,\mathbb{V}_3(k)\right]
\end{array}\right) \\
&+2\sum_{\boldsymbol{k}>0}\left[\begin{array}{l}
2i\bar{V}_a e^{2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,1}^*\left(k,\tilde{t}\right)\mathbb{M}_3(k)\,\mathbb{V}_3^*(k) \\
+\bar{V}_a^2\left\{2e^{2i\tilde{\omega}_1(k)\tilde{t}}\psi_{0,1}\left(\tilde{t}\right)\psi_{2,1}^*\left(k,\tilde{t}\right)\mathbb{M}_3(k)\,\mathbb{V}_3^*(k) - 2e^{2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,2}^*\left(k,\tilde{t}\right)\mathbb{M}_3(k)\,\mathbb{V}_3^*(k)\right\} \\
+4\bar{V}_a^2\left|\psi_{2,1}\left(k,\tilde{t}\right)\right|^2\left\{\mathbb{M}_2(k) + \sum_{\boldsymbol{k}_2>0}\mathbb{M}_1(k_2)\right\}\left|\mathbb{V}_3(k)\right|^2
\end{array}\right] + O\left(\bar{V}_a^3\right) \\
&= 2\sum_{\boldsymbol{k}>0}\left(\begin{array}{l}
\mathbb{M}_1(k) + 4\bar{V}_a\mathrm{Im}\left\{e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,1}\left(k,\tilde{t}\right)\mathbb{M}_3^*(k)\,\mathbb{V}_3(k)\right\} \\
+\bar{V}_a^2\left[\left|\psi_{0,1}\left(\tilde{t}\right)\right|^2\mathbb{M}_1(k) - 2\mathrm{Re}\left\{\psi_{0,2}\left(\tilde{t}\right)\right\}\mathbb{M}_1(k) + 4\left|\psi_{2,1}\left(k,\tilde{t}\right)\right|^2\left\{\mathbb{M}_2(k) + \sum_{\boldsymbol{k}_2>0}\mathbb{M}_1(k_2)\right\}\left|\mathbb{V}_3(k)\right|^2\right] \\
+4\bar{V}_a^2\left[\psi_{0,1}\left(\tilde{t}\right)\mathrm{Re}\left\{e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,1}\left(k,\tilde{t}\right)\mathbb{M}_3^*(k)\,\mathbb{V}_3(k)\right\} - \mathrm{Re}\left\{e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,2}\left(k,\tilde{t}\right)\mathbb{M}_3^*(k)\,\mathbb{V}_3(k)\right\}\right]
\end{array}\right) \\
&\quad +O\left(\bar{V}_a^3\right),
\end{aligned}
\tag{F.44}
$$

since $\psi_{0,1}\left(\tilde{t}\right) \coloneqq T_{1,0}\left(\tilde{t}\right)\sum_{\boldsymbol{k}\neq 0}\mathbb{V}_1(k)$ is real (see Eq. (F.8)).

From Eq. (F.43),

$$
\begin{aligned}
&\left\langle \hat{\Psi}_0\left(\tilde{t}\right)\right| \hat{M}_I^2\left(\tilde{t}\right)\left|\hat{\Psi}_0\left(\tilde{t}\right)\right\rangle \\
&= 4\left\langle \mathrm{vac}\right| \sum_{\boldsymbol{k}>0}\left[\begin{array}{l} \mathbb{M}_1\left(k\right) + i\bar{V}_a\left\{\psi_{0,1}\left(\tilde{t}\right)\mathbb{M}_1\left(k\right) + 2e^{2i\tilde{\omega}_1(k)\tilde{t}}\psi^*_{2,1}\left(k,\tilde{t}\right)\mathbb{M}_3\left(k\right)\mathbb{V}_3^*\left(k\right)\right\} \\ -\bar{V}_a^2\left\{\psi^*_{0,2}\left(\tilde{t}\right)\mathbb{M}_1\left(k\right) + 2e^{2i\tilde{\omega}_1(k)\tilde{t}}\psi^*_{2,2}\left(k,\tilde{t}\right)\mathbb{M}_3\left(k\right)\mathbb{V}_3^*\left(k\right)\right\} \end{array}\right] \\
&\qquad \times \sum_{\boldsymbol{k}_1>0}\left[\begin{array}{l} \mathbb{M}_1\left(k_1\right) - i\bar{V}_a\left\{\psi_{0,1}\left(\tilde{t}\right)\mathbb{M}_1\left(k_1\right) + 2e^{-2i\tilde{\omega}_1(k_1)\tilde{t}}\psi_{2,1}\left(k_1,\tilde{t}\right)\mathbb{M}_3^*\left(k_1\right)\mathbb{V}_3\left(k_1\right)\right\} \\ -\bar{V}_a^2\left\{\psi_{0,2}\left(\tilde{t}\right)\mathbb{M}_1\left(k_1\right) + 2e^{-2i\tilde{\omega}_1(k_1)\tilde{t}}\psi_{2,2}\left(k_1,\tilde{t}\right)\mathbb{M}_3^*\left(k_1\right)\mathbb{V}_3\left(k_1\right)\right\} \end{array}\right]\left|\mathrm{vac}\right\rangle \\
&\quad + 4\sum_{\boldsymbol{k}>0}\left\langle \boldsymbol{k},-\boldsymbol{k}\right| \left(\begin{array}{l} e^{-2i\tilde{\omega}_1(k)\tilde{t}}\mathbb{M}_3^*\left(k\right) \\ +i\bar{V}_a\left[e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{0,1}\left(\tilde{t}\right)\mathbb{M}_3^*\left(k\right) + 2\psi^*_{2,1}\left(k,\tilde{t}\right)\left\{\mathbb{M}_2\left(k\right) + \sum_{\boldsymbol{k}_1>0}\mathbb{M}_1\left(k_1\right)\right\}\mathbb{V}_3^*\left(k\right)\right] \\ -\bar{V}_a^2\left[\begin{array}{l} e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi^*_{0,2}\left(\tilde{t}\right)\mathbb{M}_3^*\left(k\right) + 2\psi^*_{2,2}\left(k,\tilde{t}\right)\left\{\mathbb{M}_2\left(k\right) + \sum_{\boldsymbol{k}_1>0}\mathbb{M}_1\left(k_1\right)\right\}\mathbb{V}_3^*\left(k\right) \\ +4\sum_{\boldsymbol{k}_1>0} e^{2i\tilde{\omega}_1(k_1)\tilde{t}}\left\{\psi^*_{4,2}\left(k_1,k,\tilde{t}\right) + \psi^*_{4,2}\left(k,k_1,\tilde{t}\right)\right\}\mathbb{M}_3\left(k_1\right)\mathbb{V}_3^*\left(k_1\right)\mathbb{V}_3^*\left(k\right) \\ +8e^{2i\tilde{\omega}_1(k)\tilde{t}}\psi^*_{4,2}\left(k,k,\tilde{t}\right)\mathbb{M}_3\left(k\right)\left\{\mathbb{V}_3^*\left(k\right)\right\}^2 \end{array}\right] \end{array}\right) \\
&\quad \times \sum_{\boldsymbol{k}'>0}\left(\begin{array}{l} e^{2i\tilde{\omega}_1(k')\tilde{t}}\mathbb{M}_3\left(k'\right) \\ -i\bar{V}_a\left[e^{2i\tilde{\omega}_1(k')\tilde{t}}\psi_{0,1}\left(\tilde{t}\right)\mathbb{M}_3\left(k'\right) + 2\psi_{2,1}\left(k',\tilde{t}\right)\left\{\mathbb{M}_2\left(k'\right) + \sum_{\boldsymbol{k}'_1>0}\mathbb{M}_1\left(k'_1\right)\right\}\mathbb{V}_3\left(k'\right)\right] \\ -\bar{V}_a^2\left[\begin{array}{l} e^{2i\tilde{\omega}_1(k')\tilde{t}}\psi_{0,2}\left(\tilde{t}\right)\mathbb{M}_3\left(k'\right) + 2\psi_{2,2}\left(k',\tilde{t}\right)\left\{\mathbb{M}_2\left(k'\right) + \sum_{\boldsymbol{k}'_1>0}\mathbb{M}_1\left(k'_1\right)\right\}\mathbb{V}_3\left(k'\right) \\ +4\sum_{\boldsymbol{k}_1>0} e^{-2i\tilde{\omega}_1(k'_1)\tilde{t}}\left\{\psi_{4,2}\left(k'_1,k',\tilde{t}\right) + \psi_{4,2}\left(k',k'_1,\tilde{t}\right)\right\}\mathbb{M}_3^*\left(k'_1\right)\mathbb{V}_3\left(k'_1\right)\mathbb{V}_3\left(k'\right) \\ +8e^{-2i\tilde{\omega}_1(k')\tilde{t}}\psi_{4,2}\left(k',k',\tilde{t}\right)\mathbb{M}_3^*\left(k'\right)\mathbb{V}_3^2\left(k'\right) \end{array}\right] \end{array}\right)\left|\boldsymbol{k}',-\boldsymbol{k}'\right\rangle \\
&\quad + 16\bar{V}_a^2\sum_{\boldsymbol{k}_1>0}\sum_{\boldsymbol{k}_2>0}\left\langle \boldsymbol{k}_1,-\boldsymbol{k}_1,\boldsymbol{k}_2,-\boldsymbol{k}_2\right| e^{-2i\tilde{\omega}_1(k_1)\tilde{t}}\psi^*_{2,1}\left(k_2,\tilde{t}\right)\mathbb{M}_3^*\left(k_1\right)\mathbb{V}_3^*\left(k_2\right) \\
&\qquad \times \sum_{\boldsymbol{k}'_1>0}\sum_{\boldsymbol{k}'_2>0} e^{2i\tilde{\omega}_1\left(k'_1\right)\tilde{t}}\psi_{2,1}\left(k'_2,\tilde{t}\right)\mathbb{M}_3\left(k'_1\right)\mathbb{V}_3\left(k'_2\right)\left|\boldsymbol{k}'_1,-\boldsymbol{k}'_1,\boldsymbol{k}'_2,-\boldsymbol{k}'_2\right\rangle + O\left(\bar{V}_a^3\right). \qquad \text{(F.45)}
\end{aligned}
$$

Now, we divide Eq. (F.45) into 3 parts:

Eq. (F.45) **- Part 1**

$$
\begin{aligned}
&4\sum_{\boldsymbol{k}>0}\left[\begin{array}{c}\mathbb{M}_1(k)+i\bar{V}_a\left\{\psi_{0,1}\left(\tilde{t}\right)\mathbb{M}_1(k)+2e^{2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,1}^{*}\left(k,\tilde{t}\right)\mathbb{M}_3(k)\,\mathbb{V}_3^{*}(k)\right\}\\-\bar{V}_a^2\left\{\psi_{0,2}^{*}\left(\tilde{t}\right)\mathbb{M}_1(k)+2e^{2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,2}^{*}\left(k,\tilde{t}\right)\mathbb{M}_3(k)\,\mathbb{V}_3^{*}(k)\right\}\end{array}\right]\\
&\quad\times\sum_{\boldsymbol{k}_1>0}\left[\begin{array}{c}\mathbb{M}_1(k_1)-i\bar{V}_a\left\{\psi_{0,1}\left(\tilde{t}\right)\mathbb{M}_1(k_1)+2e^{-2i\tilde{\omega}_1(k_1)\tilde{t}}\psi_{2,1}\left(k_1,\tilde{t}\right)\mathbb{M}_3^{*}(k_1)\,\mathbb{V}_3(k_1)\right\}\\-\bar{V}_a^2\left\{\psi_{0,2}\left(\tilde{t}\right)\mathbb{M}_1(k_1)+2e^{-2i\tilde{\omega}_1(k_1)\tilde{t}}\psi_{2,2}\left(k_1,\tilde{t}\right)\mathbb{M}_3^{*}(k_1)\,\mathbb{V}_3(k_1)\right\}\end{array}\right]\\
&=4\left[\begin{array}{l}\left\{\sum_{\boldsymbol{k}>0}\mathbb{M}_1(k)\right\}^2+4\bar{V}_a\sum_{\boldsymbol{k}>0}\mathrm{Im}\left\{e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,1}\left(k,\tilde{t}\right)\mathbb{M}_3^{*}(k)\,\mathbb{V}_3(k)\right\}\sum_{\boldsymbol{k}_1>0}\mathbb{M}_1(k_1)\\-2\bar{V}_a^2\mathrm{Re}\left\{\psi_{0,2}\left(\tilde{t}\right)\right\}\left\{\sum_{\boldsymbol{k}>0}\mathbb{M}_1(k)\right\}^2-4\bar{V}_a^2\sum_{\boldsymbol{k}>0}\mathrm{Re}\left\{e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,2}\left(k,\tilde{t}\right)\mathbb{M}_3^{*}(k)\,\mathbb{V}_3(k)\right\}\sum_{\boldsymbol{k}_1>0}\mathbb{M}_1(k_1)\\+\bar{V}_a^2\psi_{0,1}^2\left(\tilde{t}\right)\left\{\sum_{\boldsymbol{k}>0}\mathbb{M}_1(k)\right\}^2+4\bar{V}_a^2\psi_{0,1}\left(\tilde{t}\right)\mathrm{Re}\left\{\sum_{\boldsymbol{k}>0}e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,1}\left(k,\tilde{t}\right)\mathbb{M}_3^{*}(k)\,\mathbb{V}_3(k)\right\}\sum_{\boldsymbol{k}_1>0}\mathbb{M}_1(k_1)\\+4\bar{V}_a^2\left|\sum_{\boldsymbol{k}>0}e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,1}\left(k,\tilde{t}\right)\mathbb{M}_3^{*}(k)\,\mathbb{V}_3(k)\right|^2\end{array}\right]\\
&\quad+O\left(\bar{V}_a^3\right)\\
&=4\left(\begin{array}{l}\left\{\sum_{\boldsymbol{k}>0}\mathbb{M}_1(k)\right\}^2+4\bar{V}_a\sum_{\boldsymbol{k}>0}\mathrm{Im}\left\{e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,1}\left(k,\tilde{t}\right)\mathbb{M}_3^{*}(k)\,\mathbb{V}_3(k)\right\}\sum_{\boldsymbol{k}_1>0}\mathbb{M}_1(k_1)\\+\bar{V}_a^2\left[\begin{array}{l}\psi_{0,1}^2\left(\tilde{t}\right)\left\{\sum_{\boldsymbol{k}>0}\mathbb{M}_1(k)\right\}^2+4\psi_{0,1}\left(\tilde{t}\right)\mathrm{Re}\left\{\sum_{\boldsymbol{k}>0}e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,1}\left(k,\tilde{t}\right)\mathbb{M}_3^{*}(k)\,\mathbb{V}_3(k)\right\}\sum_{\boldsymbol{k}_1>0}\mathbb{M}_1(k_1)\\+4\left|\sum_{\boldsymbol{k}>0}e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,1}\left(k,\tilde{t}\right)\mathbb{M}_3^{*}(k)\,\mathbb{V}_3(k)\right|^2-2\mathrm{Re}\left\{\psi_{0,2}\left(\tilde{t}\right)\right\}\left\{\sum_{\boldsymbol{k}>0}\mathbb{M}_1(k)\right\}^2\\-4\sum_{\boldsymbol{k}>0}\mathrm{Re}\left\{e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,2}\left(k,\tilde{t}\right)\mathbb{M}_3^{*}(k)\,\mathbb{V}_3(k)\right\}\sum_{\boldsymbol{k}_1>0}\mathbb{M}_1(k_1)\end{array}\right]\end{array}\right)\\
&\quad+O\left(\bar{V}_a^3\right).
\end{aligned}
\tag{F.46}
$$

Eq. (F.45) **- Part 2**

Below Eq. (F.43), we show that $\langle\boldsymbol{k},-\boldsymbol{k}|\,\boldsymbol{k}_1,-\boldsymbol{k}_1\rangle=(\delta_{\boldsymbol{k}_1,\boldsymbol{k}}+\delta_{\boldsymbol{k}_1,-\boldsymbol{k}})$. Hence, we get

$$
4\sum_{\boldsymbol{k}>0}\langle \boldsymbol{k},-\boldsymbol{k}|\left(\begin{array}{l}
e^{-2i\tilde{\omega}_1(k)\tilde{t}}\mathbb{M}_3^*(k) \\
\quad +i\bar{V}_a\left[e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{0,1}\left(\tilde{t}\right)\mathbb{M}_3^*(k)+2\psi_{2,1}^*\left(k,\tilde{t}\right)\left\{\mathbb{M}_2(k)+\sum_{\boldsymbol{k}_1>0}\mathbb{M}_1(k_1)\right\}\mathbb{V}_3^*(k)\right] \\
\quad -\bar{V}_a^2\left[\begin{array}{l}
e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{0,2}^*\left(\tilde{t}\right)\mathbb{M}_3^*(k)+2\psi_{2,2}^*\left(k,\tilde{t}\right)\left\{\mathbb{M}_2(k)+\sum_{\boldsymbol{k}_1>0}\mathbb{M}_1(k_1)\right\}\mathbb{V}_3^*(k) \\
+4\sum_{\boldsymbol{k}_1>0}e^{2i\tilde{\omega}_1(k_1)\tilde{t}}\left\{\psi_{4,2}^*\left(k_1,k,\tilde{t}\right)+\psi_{4,2}^*\left(k,k_1,\tilde{t}\right)\right\}\mathbb{M}_3(k_1)\mathbb{V}_3^*(k_1)\mathbb{V}_3^*(k) \\
+8e^{2i\tilde{\omega}_1(k)\tilde{t}}\psi_{4,2}^*\left(k,k,\tilde{t}\right)\mathbb{M}_3(k)\left\{\mathbb{V}_3^*(k)\right\}^2
\end{array}\right]
\end{array}\right)
$$

$$
\times\sum_{\boldsymbol{k}'>0}\left(\begin{array}{l}
e^{2i\tilde{\omega}_1(k')\tilde{t}}\mathbb{M}_3(k') \\
\quad -i\bar{V}_a\left[e^{2i\tilde{\omega}_1(k')\tilde{t}}\psi_{0,1}\left(\tilde{t}\right)\mathbb{M}_3\left(k'\right)+2\psi_{2,1}\left(k',\tilde{t}\right)\left\{\mathbb{M}_2\left(k'\right)+\sum_{\boldsymbol{k}_1'>0}\mathbb{M}_1\left(k_1'\right)\right\}\mathbb{V}_3\left(k'\right)\right] \\
-\bar{V}_a^2\left[\begin{array}{l}
e^{2i\tilde{\omega}_1(k')\tilde{t}}\psi_{0,2}\left(\tilde{t}\right)\mathbb{M}_3\left(k'\right)+2\psi_{2,2}\left(k',\tilde{t}\right)\left\{\mathbb{M}_2\left(k'\right)+\sum_{\boldsymbol{k}_1'>0}\mathbb{M}_1\left(k_1'\right)\right\}\mathbb{V}_3\left(k'\right) \\
+4\sum_{\boldsymbol{k}_1'>0}e^{-2i\tilde{\omega}_1\left(k_1'\right)\tilde{t}}\left\{\psi_{4,2}\left(k_1',k',\tilde{t}\right)+\psi_{4,2}\left(k',k_1',\tilde{t}\right)\right\}\mathbb{M}_3^*\left(k_1'\right)\mathbb{V}_3\left(k_1'\right)\mathbb{V}_3\left(k'\right) \\
+8e^{-2i\tilde{\omega}_1(k')\tilde{t}}\psi_{4,2}\left(k',k',\tilde{t}\right)\mathbb{M}_3^*\left(k'\right)\mathbb{V}_3^2\left(k'\right)
\end{array}\right]
\end{array}\right)\left|\boldsymbol{k}',-\boldsymbol{k}'\right\rangle
$$

$$
=4\sum_{\boldsymbol{k}>0}\left(\begin{array}{l}
\left|\mathbb{M}_3(k)\right|^2+4\bar{V}_a\left\{\mathbb{M}_2(k)+\sum_{\boldsymbol{k}_1>0}\mathbb{M}_1(k_1)\right\}\mathrm{Im}\left\{e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,1}\left(k,\tilde{t}\right)\mathbb{M}_3^*(k)\mathbb{V}_3(k)\right\} \\
+\bar{V}_a^2\left[\begin{array}{l}
-2\mathrm{Re}\left\{\psi_{0,2}\left(\tilde{t}\right)\right\}\left|\mathbb{M}_3(k)\right|^2-4\left\{\mathbb{M}_2(k)+\sum_{\boldsymbol{k}_1>0}\mathbb{M}_1(k_1)\right\}\mathrm{Re}\left\{e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,2}\left(k,\tilde{t}\right)\mathbb{M}_3^*(k)\mathbb{V}_3(k)\right\} \\
-8\sum_{\boldsymbol{k}_1>0}\mathrm{Re}\left[e^{-2i\left\{\tilde{\omega}_1(k)+\tilde{\omega}_1(k_1)\right\}\tilde{t}}\left\{\psi_{4,2}\left(k_1,k,\tilde{t}\right)+\psi_{4,2}\left(k,k_1,\tilde{t}\right)\right\}\mathbb{M}_3^*(k)\mathbb{M}_3^*(k_1)\mathbb{V}_3(k)\mathbb{V}_3(k_1)\right] \\
-16\mathrm{Re}\left[e^{-4i\tilde{\omega}_1(k)\tilde{t}}\psi_{4,2}\left(k,k,\tilde{t}\right)\left\{\mathbb{M}_3^*(k)\right\}^2\mathbb{V}_3^2(k)\right] \\
+\psi_{0,1}^2\left(\tilde{t}\right)\left|\mathbb{M}_3(k)\right|^2+4\psi_{0,1}\left(\tilde{t}\right)\left\{\mathbb{M}_2(k)+\sum_{\boldsymbol{k}_1>0}\mathbb{M}_1(k_1)\right\}\mathrm{Re}\left[e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,1}\left(k,\tilde{t}\right)\mathbb{M}_3^*(k)\mathbb{V}_3(k)\right] \\
+4\left\{\mathbb{M}_2(k)+\sum_{\boldsymbol{k}_1>0}\mathbb{M}_1(k_1)\right\}^2\left|\psi_{2,1}\left(k,\tilde{t}\right)\mathbb{V}_3(k)\right|^2
\end{array}\right]
\end{array}\right)
$$

$$
+O\left(\bar{V}_a^3\right), \tag{F.47}
$$

Eq. (F.45) **- Part 3**

From Eq. (F.39), we have

$$\begin{aligned}
&\langle\text{vac}|\,\hat{b}_1\left(\boldsymbol{k}_1\right)\hat{b}_1\left(-\boldsymbol{k}_1\right)\hat{b}_1\left(\boldsymbol{k}_2\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_1'\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_2'\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_2'\right)|\text{vac}\rangle \\
&\quad=\delta_{\boldsymbol{k}_1',-\boldsymbol{k}_2}\langle\text{vac}|\,\hat{b}_1\left(\boldsymbol{k}_1\right)\hat{b}_1\left(-\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_2'\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_2'\right)|\text{vac}\rangle+\delta_{\boldsymbol{k}_2',\boldsymbol{k}_2}\langle\text{vac}|\,\hat{b}_1\left(\boldsymbol{k}_1\right)\hat{b}_1\left(-\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_1'\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_2'\right)|\text{vac}\rangle \\
&\qquad+\delta_{\boldsymbol{k}_2',-\boldsymbol{k}_2}\langle\text{vac}|\,\hat{b}_1\left(\boldsymbol{k}_1\right)\hat{b}_1\left(-\boldsymbol{k}_1\right)\hat{b}_1^\dagger\left(-\boldsymbol{k}_1'\right)\hat{b}_1^\dagger\left(\boldsymbol{k}_2'\right)|\text{vac}\rangle \\
&\quad=\delta_{\boldsymbol{k}_1',-\boldsymbol{k}_2}\left(\delta_{\boldsymbol{k}_2',\boldsymbol{k}_1}+\delta_{\boldsymbol{k}_2',-\boldsymbol{k}_1}\right)+\delta_{\boldsymbol{k}_2',\boldsymbol{k}_2}\left(\delta_{\boldsymbol{k}_1',\boldsymbol{k}_1}+\delta_{\boldsymbol{k}_2',\boldsymbol{k}_1}\right)+\delta_{\boldsymbol{k}_2',-\boldsymbol{k}_2}\left(\delta_{\boldsymbol{k}_1',\boldsymbol{k}_1}+\delta_{\boldsymbol{k}_2',-\boldsymbol{k}_1}\right) \\
&\quad=\delta_{\boldsymbol{k}_1',-\boldsymbol{k}_2}\left(\delta_{\boldsymbol{k}_2',\boldsymbol{k}_1}+\delta_{\boldsymbol{k}_2',-\boldsymbol{k}_1}\right)+\delta_{\boldsymbol{k}_2',\boldsymbol{k}_2}\left(\delta_{\boldsymbol{k}_1',\boldsymbol{k}_1}+\delta_{\boldsymbol{k}_2,\boldsymbol{k}_1}\right)+\delta_{\boldsymbol{k}_2',-\boldsymbol{k}_2}\left(\delta_{\boldsymbol{k}_1',\boldsymbol{k}_1}+\delta_{\boldsymbol{k}_2,\boldsymbol{k}_1}\right) \\
&\quad=\delta_{\boldsymbol{k}_1',\boldsymbol{k}_1}\left(\delta_{\boldsymbol{k}_2',\boldsymbol{k}_2}+\delta_{\boldsymbol{k}_2',-\boldsymbol{k}_2}\right)+\delta_{\boldsymbol{k}_1',-\boldsymbol{k}_2}\left(\delta_{\boldsymbol{k}_2',\boldsymbol{k}_1}+\delta_{\boldsymbol{k}_2',-\boldsymbol{k}_1}\right)+\delta_{\boldsymbol{k}_2,\boldsymbol{k}_1}\left(\delta_{\boldsymbol{k}_2',\boldsymbol{k}_2}+\delta_{\boldsymbol{k}_2',-\boldsymbol{k}_2}\right).
\end{aligned} \tag{F.48}$$

Therefore,

$$
\begin{aligned}
&\langle \boldsymbol{k}_1, -\boldsymbol{k}_1, \boldsymbol{k}_2, -\boldsymbol{k}_2 | \boldsymbol{k}'_1, -\boldsymbol{k}'_1, \boldsymbol{k}'_2, -\boldsymbol{k}'_2 \rangle \\
&\quad = \delta_{\boldsymbol{k}'_1,-\boldsymbol{k}_2} \langle \mathrm{vac} | \hat{b}_1(\boldsymbol{k}_1) \hat{b}_1(-\boldsymbol{k}_1) \hat{b}_1(\boldsymbol{k}_2) \hat{b}_1^\dagger(-\boldsymbol{k}'_1) \hat{b}_1^\dagger(\boldsymbol{k}'_2) \hat{b}_1^\dagger(-\boldsymbol{k}'_2) | \mathrm{vac} \rangle \\
&\qquad + \delta_{\boldsymbol{k}'_1,\boldsymbol{k}_2} \langle \mathrm{vac} | \hat{b}_1(\boldsymbol{k}_1) \hat{b}_1(-\boldsymbol{k}_1) \hat{b}_1(\boldsymbol{k}_2) \hat{b}_1^\dagger(\boldsymbol{k}'_1) \hat{b}_1^\dagger(\boldsymbol{k}'_2) \hat{b}_1^\dagger(-\boldsymbol{k}'_2) | \mathrm{vac} \rangle \\
&\qquad + \delta_{\boldsymbol{k}'_2,-\boldsymbol{k}_2} \langle \mathrm{vac} | \hat{b}_1(\boldsymbol{k}_1) \hat{b}_1(-\boldsymbol{k}_1) \hat{b}_1(\boldsymbol{k}_2) \hat{b}_1^\dagger(\boldsymbol{k}'_1) \hat{b}_1^\dagger(-\boldsymbol{k}'_1) \hat{b}_1^\dagger(-\boldsymbol{k}'_2) | \mathrm{vac} \rangle \\
&\qquad + \delta_{\boldsymbol{k}'_2,\boldsymbol{k}_2} \langle \mathrm{vac} | \hat{b}_1(\boldsymbol{k}_1) \hat{b}_1(-\boldsymbol{k}_1) \hat{b}_1(\boldsymbol{k}_2) \hat{b}_1^\dagger(\boldsymbol{k}'_1) \hat{b}_1^\dagger(-\boldsymbol{k}'_1) \hat{b}_1^\dagger(\boldsymbol{k}'_2) | \mathrm{vac} \rangle \\
&\quad = \delta_{\boldsymbol{k}'_1,-\boldsymbol{k}_2} \left\{ \delta_{\boldsymbol{k}'_1,\boldsymbol{k}_1} \left(\delta_{\boldsymbol{k}'_2,\boldsymbol{k}_2} + \delta_{\boldsymbol{k}'_2,-\boldsymbol{k}_2} \right) + \delta_{\boldsymbol{k}'_1,-\boldsymbol{k}_2} \left(\delta_{\boldsymbol{k}'_2,\boldsymbol{k}_1} + \delta_{\boldsymbol{k}'_2,-\boldsymbol{k}_1} \right) + \delta_{\boldsymbol{k}_2,\boldsymbol{k}_1} \left(\delta_{\boldsymbol{k}'_2,\boldsymbol{k}_2} + \delta_{\boldsymbol{k}'_2,-\boldsymbol{k}_2} \right) \right\} \\
&\qquad + \delta_{\boldsymbol{k}'_1,\boldsymbol{k}_2} \left\{ \delta_{\boldsymbol{k}'_1,-\boldsymbol{k}_1} \left(\delta_{\boldsymbol{k}'_2,\boldsymbol{k}_2} + \delta_{\boldsymbol{k}'_2,-\boldsymbol{k}_2} \right) + \delta_{\boldsymbol{k}'_1,\boldsymbol{k}_2} \left(\delta_{\boldsymbol{k}'_2,\boldsymbol{k}_1} + \delta_{\boldsymbol{k}'_2,-\boldsymbol{k}_1} \right) + \delta_{\boldsymbol{k}_2,\boldsymbol{k}_1} \left(\delta_{\boldsymbol{k}'_2,\boldsymbol{k}_2} + \delta_{\boldsymbol{k}'_2,-\boldsymbol{k}_2} \right) \right\} \\
&\qquad + \delta_{\boldsymbol{k}'_2,-\boldsymbol{k}_2} \left\{ \delta_{\boldsymbol{k}'_2,\boldsymbol{k}_1} \left(\delta_{\boldsymbol{k}'_1,\boldsymbol{k}_2} + \delta_{\boldsymbol{k}'_1,-\boldsymbol{k}_2} \right) + \delta_{\boldsymbol{k}'_2,-\boldsymbol{k}_2} \left(\delta_{\boldsymbol{k}'_1,\boldsymbol{k}_1} + \delta_{\boldsymbol{k}'_1,-\boldsymbol{k}_1} \right) + \delta_{\boldsymbol{k}_2,\boldsymbol{k}_1} \left(\delta_{\boldsymbol{k}'_1,\boldsymbol{k}_2} + \delta_{\boldsymbol{k}'_1,-\boldsymbol{k}_2} \right) \right\} \\
&\qquad + \delta_{\boldsymbol{k}'_2,\boldsymbol{k}_2} \left\{ \delta_{\boldsymbol{k}'_2,-\boldsymbol{k}_1} \left(\delta_{\boldsymbol{k}'_1,\boldsymbol{k}_2} + \delta_{\boldsymbol{k}'_1,-\boldsymbol{k}_2} \right) + \delta_{\boldsymbol{k}'_2,\boldsymbol{k}_2} \left(\delta_{\boldsymbol{k}'_1,\boldsymbol{k}_1} + \delta_{\boldsymbol{k}'_1,-\boldsymbol{k}_1} \right) + \delta_{\boldsymbol{k}_2,\boldsymbol{k}_1} \left(\delta_{\boldsymbol{k}'_1,\boldsymbol{k}_2} + \delta_{\boldsymbol{k}'_1,-\boldsymbol{k}_2} \right) \right\} \\
&\qquad = \delta_{\boldsymbol{k}'_1,-\boldsymbol{k}_2} \left\{ \delta_{\boldsymbol{k}_2,-\boldsymbol{k}_1} \left(\delta_{\boldsymbol{k}'_2,\boldsymbol{k}_2} + \delta_{\boldsymbol{k}'_2,-\boldsymbol{k}_2} \right) + \left(\delta_{\boldsymbol{k}'_2,\boldsymbol{k}_1} + \delta_{\boldsymbol{k}'_2,-\boldsymbol{k}_1} \right) + \delta_{\boldsymbol{k}_2,\boldsymbol{k}_1} \left(\delta_{\boldsymbol{k}'_2,\boldsymbol{k}_2} + \delta_{\boldsymbol{k}'_2,-\boldsymbol{k}_2} \right) \right\} \\
&\qquad + \delta_{\boldsymbol{k}'_1,\boldsymbol{k}_2} \left\{ \delta_{\boldsymbol{k}_2,-\boldsymbol{k}_1} \left(\delta_{\boldsymbol{k}'_2,\boldsymbol{k}_2} + \delta_{\boldsymbol{k}'_2,-\boldsymbol{k}_2} \right) + \left(\delta_{\boldsymbol{k}'_2,\boldsymbol{k}_1} + \delta_{\boldsymbol{k}'_2,-\boldsymbol{k}_1} \right) + \delta_{\boldsymbol{k}_2,\boldsymbol{k}_1} \left(\delta_{\boldsymbol{k}'_2,\boldsymbol{k}_2} + \delta_{\boldsymbol{k}'_2,-\boldsymbol{k}_2} \right) \right\} \\
&\qquad + \delta_{\boldsymbol{k}'_2,-\boldsymbol{k}_2} \left\{ \delta_{\boldsymbol{k}_2,-\boldsymbol{k}_1} \left(\delta_{\boldsymbol{k}'_1,\boldsymbol{k}_2} + \delta_{\boldsymbol{k}'_1,-\boldsymbol{k}_2} \right) + \left(\delta_{\boldsymbol{k}'_1,\boldsymbol{k}_1} + \delta_{\boldsymbol{k}'_1,-\boldsymbol{k}_1} \right) + \delta_{\boldsymbol{k}_2,\boldsymbol{k}_1} \left(\delta_{\boldsymbol{k}'_1,\boldsymbol{k}_2} + \delta_{\boldsymbol{k}'_1,-\boldsymbol{k}_2} \right) \right\} \\
&\qquad + \delta_{\boldsymbol{k}'_2,\boldsymbol{k}_2} \left\{ \delta_{\boldsymbol{k}_2,-\boldsymbol{k}_1} \left(\delta_{\boldsymbol{k}'_1,\boldsymbol{k}_2} + \delta_{\boldsymbol{k}'_1,-\boldsymbol{k}_2} \right) + \left(\delta_{\boldsymbol{k}'_1,\boldsymbol{k}_1} + \delta_{\boldsymbol{k}'_1,-\boldsymbol{k}_1} \right) + \delta_{\boldsymbol{k}_2,\boldsymbol{k}_1} \left(\delta_{\boldsymbol{k}'_1,\boldsymbol{k}_2} + \delta_{\boldsymbol{k}'_1,-\boldsymbol{k}_2} \right) \right\} \\
&\quad = \delta_{\boldsymbol{k}'_1,-\boldsymbol{k}_2} \left\{ (\delta_{\boldsymbol{k}_2,\boldsymbol{k}_1} + \delta_{\boldsymbol{k}_2,-\boldsymbol{k}_1}) \left(\delta_{\boldsymbol{k}'_2,\boldsymbol{k}_2} + \delta_{\boldsymbol{k}'_2,-\boldsymbol{k}_2} \right) + \left(\delta_{\boldsymbol{k}'_2,\boldsymbol{k}_1} + \delta_{\boldsymbol{k}'_2,-\boldsymbol{k}_1} \right) \right\} \\
&\qquad + \delta_{\boldsymbol{k}'_1,\boldsymbol{k}_2} \left\{ (\delta_{\boldsymbol{k}_2,\boldsymbol{k}_1} + \delta_{\boldsymbol{k}_2,-\boldsymbol{k}_1}) \left(\delta_{\boldsymbol{k}'_2,\boldsymbol{k}_2} + \delta_{\boldsymbol{k}'_2,-\boldsymbol{k}_2} \right) + \left(\delta_{\boldsymbol{k}'_2,\boldsymbol{k}_1} + \delta_{\boldsymbol{k}'_2,-\boldsymbol{k}_1} \right) \right\} \\
&\qquad + \delta_{\boldsymbol{k}'_2,-\boldsymbol{k}_2} \left\{ (\delta_{\boldsymbol{k}_2,\boldsymbol{k}_1} + \delta_{\boldsymbol{k}_2,-\boldsymbol{k}_1}) \left(\delta_{\boldsymbol{k}'_1,\boldsymbol{k}_2} + \delta_{\boldsymbol{k}'_1,-\boldsymbol{k}_2} \right) + \left(\delta_{\boldsymbol{k}'_1,\boldsymbol{k}_1} + \delta_{\boldsymbol{k}'_1,-\boldsymbol{k}_1} \right) \right\} \\
&\qquad + \delta_{\boldsymbol{k}'_2,\boldsymbol{k}_2} \left\{ (\delta_{\boldsymbol{k}_2,\boldsymbol{k}_1} + \delta_{\boldsymbol{k}_2,-\boldsymbol{k}_1}) \left(\delta_{\boldsymbol{k}'_1,\boldsymbol{k}_2} + \delta_{\boldsymbol{k}'_1,-\boldsymbol{k}_2} \right) + \left(\delta_{\boldsymbol{k}'_1,\boldsymbol{k}_1} + \delta_{\boldsymbol{k}'_1,-\boldsymbol{k}_1} \right) \right\} \\
&\quad = \left(\delta_{\boldsymbol{k}'_1,\boldsymbol{k}_2} + \delta_{\boldsymbol{k}'_1,-\boldsymbol{k}_2} \right) \left\{ (\delta_{\boldsymbol{k}_2,\boldsymbol{k}_1} + \delta_{\boldsymbol{k}_2,-\boldsymbol{k}_1}) \left(\delta_{\boldsymbol{k}'_2,\boldsymbol{k}_2} + \delta_{\boldsymbol{k}'_2,-\boldsymbol{k}_2} \right) + \left(\delta_{\boldsymbol{k}'_2,\boldsymbol{k}_1} + \delta_{\boldsymbol{k}'_2,-\boldsymbol{k}_1} \right) \right\} \\
&\qquad + \left(\delta_{\boldsymbol{k}'_2,\boldsymbol{k}_2} + \delta_{\boldsymbol{k}'_2,-\boldsymbol{k}_2} \right) \left\{ (\delta_{\boldsymbol{k}_2,\boldsymbol{k}_1} + \delta_{\boldsymbol{k}_2,-\boldsymbol{k}_1}) \left(\delta_{\boldsymbol{k}'_1,\boldsymbol{k}_2} + \delta_{\boldsymbol{k}'_1,-\boldsymbol{k}_2} \right) + \left(\delta_{\boldsymbol{k}'_1,\boldsymbol{k}_1} + \delta_{\boldsymbol{k}'_1,-\boldsymbol{k}_1} \right) \right\},
\end{aligned}
\tag{F.49}
$$

and we get

$$
\begin{aligned}
&16\bar{V}_a^2 \sum_{\boldsymbol{k}_1>0}\sum_{\boldsymbol{k}_2>0} \langle \boldsymbol{k}_1,-\boldsymbol{k}_1,\boldsymbol{k}_2,-\boldsymbol{k}_2| \, e^{-2i\tilde{\omega}_1(k_1)\tilde{t}}\psi_{2,1}^*\left(k_2,\tilde{t}\right)\mathbb{M}_3^*\left(k_1\right)\mathbb{V}_3^*\left(k_2\right)\\
&\qquad\times\sum_{\boldsymbol{k}_1'>0}\sum_{\boldsymbol{k}_2'>0} e^{2i\tilde{\omega}_1\left(k_1'\right)\tilde{t}}\psi_{2,1}\left(k_2',\tilde{t}\right)\mathbb{M}_3\left(k_1'\right)\mathbb{V}_3\left(k_2'\right)\left|\boldsymbol{k}_1',-\boldsymbol{k}_1',\boldsymbol{k}_2',-\boldsymbol{k}_2'\right\rangle\\
&=16\bar{V}_a^2 \sum_{\boldsymbol{k}_1>0}\sum_{\boldsymbol{k}_2>0}\sum_{\boldsymbol{k}_1'>0}\sum_{\boldsymbol{k}_2'>0} e^{-2i\tilde{\omega}_1(k_1)\tilde{t}}e^{2i\tilde{\omega}_1\left(k_1'\right)\tilde{t}}\psi_{2,1}^*\left(k_2,\tilde{t}\right)\psi_{2,1}\left(k_2',\tilde{t}\right)\mathbb{M}_3^*\left(k_1\right)\mathbb{M}_3\left(k_1'\right)\mathbb{V}_3^*\left(k_2\right)\mathbb{V}_3\left(k_2'\right)\\
&\qquad\times\left(\delta_{\boldsymbol{k}_1',\boldsymbol{k}_2}+\delta_{\boldsymbol{k}_1',-\boldsymbol{k}_2}\right)\left\{\left(\delta_{\boldsymbol{k}_2,\boldsymbol{k}_1}+\delta_{\boldsymbol{k}_2,-\boldsymbol{k}_1}\right)\left(\delta_{\boldsymbol{k}_2',\boldsymbol{k}_2}+\delta_{\boldsymbol{k}_2',-\boldsymbol{k}_2}\right)+\left(\delta_{\boldsymbol{k}_2',\boldsymbol{k}_1}+\delta_{\boldsymbol{k}_2',-\boldsymbol{k}_1}\right)\right\}\\
&\quad+16\bar{V}_a^2 \sum_{\boldsymbol{k}_1>0}\sum_{\boldsymbol{k}_2>0}\sum_{\boldsymbol{k}_1'>0}\sum_{\boldsymbol{k}_2'>0} e^{-2i\tilde{\omega}_1(k_1)\tilde{t}}e^{2i\tilde{\omega}_1\left(k_1'\right)\tilde{t}}\psi_{2,1}^*\left(k_2,\tilde{t}\right)\psi_{2,1}\left(k_2',\tilde{t}\right)\mathbb{M}_3^*\left(k_1\right)\mathbb{M}_3\left(k_1'\right)\mathbb{V}_3^*\left(k_2\right)\mathbb{V}_3\left(k_2'\right)\\
&\qquad\times\left(\delta_{\boldsymbol{k}_2',\boldsymbol{k}_2}+\delta_{\boldsymbol{k}_2',-\boldsymbol{k}_2}\right)\left\{\left(\delta_{\boldsymbol{k}_2,\boldsymbol{k}_1}+\delta_{\boldsymbol{k}_2,-\boldsymbol{k}_1}\right)\left(\delta_{\boldsymbol{k}_1',\boldsymbol{k}_2}+\delta_{\boldsymbol{k}_1',-\boldsymbol{k}_2}\right)+\left(\delta_{\boldsymbol{k}_1',\boldsymbol{k}_1}+\delta_{\boldsymbol{k}_1',-\boldsymbol{k}_1}\right)\right\}\\
&=16\bar{V}_a^2 \sum_{\boldsymbol{k}_1>0}\sum_{\boldsymbol{k}_2>0}\sum_{\boldsymbol{k}_1'>0}\sum_{\boldsymbol{k}_2'>0} e^{-2i\tilde{\omega}_1(k_1)\tilde{t}}e^{2i\tilde{\omega}_1\left(k_1'\right)\tilde{t}}\psi_{2,1}^*\left(k_2,\tilde{t}\right)\psi_{2,1}\left(k_2',\tilde{t}\right)\mathbb{M}_3^*\left(k_1\right)\mathbb{M}_3\left(k_1'\right)\mathbb{V}_3^*\left(k_2\right)\mathbb{V}_3\left(k_2'\right)\\
&\qquad\times\delta_{\boldsymbol{k}_1',\boldsymbol{k}_2}\left(\delta_{\boldsymbol{k}_2',\boldsymbol{k}_2}\delta_{\boldsymbol{k}_2,\boldsymbol{k}_1}+\delta_{\boldsymbol{k}_2',\boldsymbol{k}_1}\right)\\
&\quad+16\bar{V}_a^2 \sum_{\boldsymbol{k}_1>0}\sum_{\boldsymbol{k}_2>0}\sum_{\boldsymbol{k}_1'>0}\sum_{\boldsymbol{k}_2'>0} e^{-2i\tilde{\omega}_1(k_1)\tilde{t}}e^{2i\tilde{\omega}_1\left(k_1'\right)\tilde{t}}\psi_{2,1}^*\left(k_2,\tilde{t}\right)\psi_{2,1}\left(k_2',\tilde{t}\right)\mathbb{M}_3^*\left(k_1\right)\mathbb{M}_3\left(k_1'\right)\mathbb{V}_3^*\left(k_2\right)\mathbb{V}_3\left(k_2'\right)\\
&\qquad\times\delta_{\boldsymbol{k}_2',\boldsymbol{k}_2}\left(\delta_{\boldsymbol{k}_1',\boldsymbol{k}_2}\delta_{\boldsymbol{k}_2,\boldsymbol{k}_1}+\delta_{\boldsymbol{k}_1',\boldsymbol{k}_1}\right)\\
&=16\bar{V}_a^2\sum_{\boldsymbol{k}_1>0}\left|\psi_{2,1}\left(k_1,\tilde{t}\right)\mathbb{M}_3\left(k_1\right)\mathbb{V}_3\left(k_1\right)\right|^2\\
&\quad+16\bar{V}_a^2\sum_{\boldsymbol{k}_1>0}e^{-2i\tilde{\omega}_1(k_1)\tilde{t}}\psi_{2,1}\left(k_1,\tilde{t}\right)\mathbb{M}_3^*\left(k_1\right)\mathbb{V}_3\left(k_1\right)\sum_{\boldsymbol{k}_2>0}e^{2i\tilde{\omega}_1(k_2)\tilde{t}}\psi_{2,1}^*\left(k_2,\tilde{t}\right)\mathbb{M}_3\left(k_2\right)\mathbb{V}_3^*\left(k_2\right)\\
&\quad+16\bar{V}_a^2\sum_{\boldsymbol{k}_1>0}\left|\psi_{2,1}\left(k_1,\tilde{t}\right)\mathbb{M}_3\left(k_1\right)\mathbb{V}_3\left(k_2\right)\right|^2+16\bar{V}_a^2\sum_{\boldsymbol{k}_1>0}\left|\mathbb{M}_3\left(k_1\right)\right|^2\sum_{\boldsymbol{k}_2>0}\left|\psi_{2,1}\left(k_2,\tilde{t}\right)\mathbb{V}_3\left(k_2\right)\right|^2\\
&=\bar{V}_a^2\left\{\begin{array}{l}16\left|\displaystyle\sum_{\boldsymbol{k}>0}e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,1}\left(k,\tilde{t}\right)\mathbb{M}_3^*\left(k\right)\mathbb{V}_3\left(k\right)\right|^2+32\displaystyle\sum_{\boldsymbol{k}>0}\left|\psi_{2,1}\left(k,\tilde{t}\right)\mathbb{M}_3\left(k\right)\mathbb{V}_3\left(k\right)\right|^2\\+16\displaystyle\sum_{\boldsymbol{k}_1>0}\left|\psi_{2,1}\left(k_1,\tilde{t}\right)\mathbb{V}_3\left(k_1\right)\right|^2\sum_{\boldsymbol{k}_2>0}\left|\mathbb{M}_3\left(k_2\right)\right|^2\end{array}\right\}.
\end{aligned}
\tag{F.50}
$$

Final Result of Eq. (F.45)

By combining Eqs. (F.46), (F.47), and (F.50), we get

$$
\begin{aligned}
&\left\langle \hat{\Psi}_0\left(\tilde{t}\right)\right| \hat{M}_I^2\left(\tilde{t}\right)\left|\hat{\Psi}_0\left(\tilde{t}\right)\right\rangle \\
&= 4\left[\begin{array}{l}
\left\{\sum_{\boldsymbol{k}>0}\mathbb{M}_1\left(k\right)\right\}^2+\sum_{\boldsymbol{k}>0}\left|\mathbb{M}_3\left(k\right)\right|^2 \\
+4\bar{V}_a\sum_{\boldsymbol{k}>0}\left\{\mathbb{M}_2\left(k\right)+2\sum_{\boldsymbol{k}_1>0}\mathbb{M}_1\left(k_1\right)\right\}\mathrm{Im}\left\{e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,1}\left(k,\tilde{t}\right)\mathbb{M}_3^*\left(k\right)\mathbb{V}_3\left(k\right)\right\}
\end{array}\right] \\
&\quad +4\bar{V}_a^2\left\{\begin{array}{l}
\psi_{0,1}^2\left(\tilde{t}\right)\left[\left\{\sum_{\boldsymbol{k}>0}\mathbb{M}_1\left(k\right)\right\}^2+\sum_{\boldsymbol{k}>0}\left|\mathbb{M}_3\left(k\right)\right|^2\right] \\
+4\psi_{0,1}\left(\tilde{t}\right)\sum_{\boldsymbol{k}>0}\left\{\mathbb{M}_2\left(k\right)+2\sum_{\boldsymbol{k}_1>0}\mathbb{M}_1\left(k_1\right)\right\}\mathrm{Re}\left\{e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,1}\left(k,\tilde{t}\right)\mathbb{M}_3^*\left(k\right)\mathbb{V}_3\left(k\right)\right\} \\
+8\left|\sum_{\boldsymbol{k}>0}e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,1}\left(k,\tilde{t}\right)\mathbb{M}_3^*\left(k\right)\mathbb{V}_3\left(k\right)\right|^2+4\sum_{\boldsymbol{k}>0}\left\{\mathbb{M}_2\left(k\right)+\sum_{\boldsymbol{k}_1>0}\mathbb{M}_1\left(k_1\right)\right\}^2\left|\psi_{2,1}\left(k,\tilde{t}\right)\mathbb{V}_3\left(k\right)\right|^2 \\
+8\sum_{\boldsymbol{k}>0}\left|\psi_{2,1}\left(k,\tilde{t}\right)\mathbb{M}_3\left(k\right)\mathbb{V}_3\left(k\right)\right|^2+4\sum_{\boldsymbol{k}_1>0}\left|\psi_{2,1}\left(k_1,\tilde{t}\right)\mathbb{V}_3\left(k_1\right)\right|^2\sum_{\boldsymbol{k}_2>0}\left|\mathbb{M}_3\left(k_2\right)\right|^2 \\
-2\mathrm{Re}\left\{\psi_{0,2}\left(\tilde{t}\right)\right\}\left[\left\{\sum_{\boldsymbol{k}>0}\mathbb{M}_1\left(k\right)\right\}^2+\sum_{\boldsymbol{k}>0}\left|\mathbb{M}_3\left(k\right)\right|^2\right] \\
-4\sum_{\boldsymbol{k}>0}\left\{\mathbb{M}_2\left(k\right)+2\sum_{\boldsymbol{k}_1>0}\mathbb{M}_1\left(k_1\right)\right\}\mathrm{Re}\left\{e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,2}\left(k,\tilde{t}\right)\mathbb{M}_3^*\left(k\right)\mathbb{V}_3\left(k\right)\right\} \\
-8\sum_{\boldsymbol{k}>0}\sum_{\boldsymbol{k}_1>0}\mathrm{Re}\left[e^{-2i\{\tilde{\omega}_1(k)+\tilde{\omega}_1(k_1)\}\tilde{t}}\left\{\psi_{4,2}\left(k_1,k,\tilde{t}\right)+\psi_{4,2}\left(k,k_1,\tilde{t}\right)\right\}\mathbb{M}_3^*\left(k\right)\mathbb{M}_3^*\left(k_1\right)\mathbb{V}_3\left(k\right)\mathbb{V}_3\left(k_1\right)\right] \\
-16\sum_{\boldsymbol{k}>0}\mathrm{Re}\left[e^{-4i\tilde{\omega}_1(k)\tilde{t}}\psi_{4,2}\left(k,k,\tilde{t}\right)\left\{\mathbb{M}_3^*\left(k\right)\right\}^2\mathbb{V}_3^2\left(k\right)\right]
\end{array}\right\} \\
&\quad +O\left(\bar{V}_a^3\right).
\end{aligned} \tag{F.51}
$$

F.4.1 Final Results

From Eq. (F.44), we can express

$$
\left\langle \hat{\Psi}_0\left(\tilde{t}\right)\right| \hat{M}_I\left(\tilde{t}\right)\left|\hat{\Psi}_0\left(\tilde{t}\right)\right\rangle = 2\sum_{\boldsymbol{k}>0}\left\{\mathbb{M}_1\left(k\right)+4\tilde{V}_a\left\langle \hat{M}\left(k,\tilde{t}\right)\right\rangle'+\tilde{V}_a^2\left\langle \hat{M}\left(k,\tilde{t}\right)\right\rangle''\right\}+O\left(\tilde{V}_a^3\right), \tag{F.52}
$$

where

$$\left\langle \hat{M}\left(k,\tilde{t}\right)\right\rangle' := \mathrm{Im}\left\{e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,1}\left(k,\tilde{t}\right)\mathbb{M}_3^*\left(k\right)\mathbb{V}_3\left(k\right)\right\},$$

$$\left\langle \hat{M}\left(k,\tilde{t}\right)\right\rangle'' := \left|\psi_{0,1}\left(\tilde{t}\right)\right|^2\mathbb{M}_1\left(k\right) - 2\mathrm{Re}\left\{\psi_{0,2}\left(\tilde{t}\right)\right\}\mathbb{M}_1\left(k\right) + 4\left|\psi_{2,1}\left(k,\tilde{t}\right)\right|^2\left\{\mathbb{M}_2\left(k\right) + \sum_{\boldsymbol{k}_2>0}\mathbb{M}_1\left(k_2\right)\right\}\left|\mathbb{V}_3\left(k\right)\right|^2$$
$$+4\psi_{0,1}\left(\tilde{t}\right)\mathrm{Re}\left\{e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,1}\left(k,\tilde{t}\right)\mathbb{M}_3^*\left(k\right)\mathbb{V}_3\left(k\right)\right\} - 4\mathrm{Re}\left\{e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,2}\left(k,\tilde{t}\right)\mathbb{M}_3^*\left(k\right)\mathbb{V}_3\left(k\right)\right\}. \tag{F.53}$$

The expectation value of the variance of the $\hat{M}$ in interaction picture is

$$\begin{aligned}
&\left\langle \hat{\Psi}_0\left(\tilde{t}\right)\middle|\hat{M}_I^2\left(\tilde{t}\right)\middle|\hat{\Psi}_0\left(\tilde{t}\right)\right\rangle - \left\langle \hat{\Psi}_0\left(\tilde{t}\right)\middle|\hat{M}_I\left(\tilde{t}\right)\middle|\hat{\Psi}_0\left(\tilde{t}\right)\right\rangle^2 \\
&= 4\left[\begin{array}{l}\left\{\sum_{\boldsymbol{k}>0}\mathbb{M}_1\left(k\right)\right\}^2 + \sum_{\boldsymbol{k}>0}\left|\mathbb{M}_3\left(k\right)\right|^2 \\ +4\bar{V}_a\sum_{\boldsymbol{k}>0}\left\{\mathbb{M}_2\left(k\right) + 2\sum_{\boldsymbol{k}_1>0}\mathbb{M}_1\left(k_1\right)\right\}\mathrm{Im}\left\{e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,1}\left(k,\tilde{t}\right)\mathbb{M}_3^*\left(k\right)\mathbb{V}_3\left(k\right)\right\}\end{array}\right] \\
&\quad -4\sum_{\boldsymbol{k}>0}\sum_{\boldsymbol{k}'>0}\left\{\mathbb{M}_1\left(k\right) + 4\bar{V}_a\left\langle \hat{M}\left(k,\tilde{t}\right)\right\rangle' + \bar{V}_a^2\left\langle \hat{M}\left(k,\tilde{t}\right)\right\rangle''\right\} \\
&\qquad\qquad \times\left\{\mathbb{M}_1\left(k'\right) + 4\bar{V}_a\left\langle \hat{M}\left(k',\tilde{t}\right)\right\rangle' + \bar{V}_a^2\left\langle \hat{M}\left(k',\tilde{t}\right)\right\rangle''\right\} \\
&\quad +O\left(\bar{V}_a^2\right) \\
&= 4\left[\left\{\sum_{\boldsymbol{k}>0}\mathbb{M}_1\left(k\right)\right\}^2 + \sum_{\boldsymbol{k}>0}\left|\mathbb{M}_3\left(k\right)\right|^2 + 4\bar{V}_a\sum_{\boldsymbol{k}>0}\left\{\mathbb{M}_2\left(k\right) + 2\sum_{\boldsymbol{k}_1>0}\mathbb{M}_1\left(k_1\right)\right\}\left\langle \hat{M}\left(k,\tilde{t}\right)\right\rangle'\right] \\
&\quad -4\sum_{\boldsymbol{k}>0}\sum_{\boldsymbol{k}'>0}\left\{\mathbb{M}_1\left(k\right)\mathbb{M}_1\left(k'\right) + 4\bar{V}_a\mathbb{M}_1\left(k\right)\left\langle \hat{M}\left(k',\tilde{t}\right)\right\rangle' + 4\bar{V}_a\left\langle \hat{M}\left(k,\tilde{t}\right)\right\rangle'\mathbb{M}_1\left(k'\right)\right\} + O\left(\bar{V}_a^2\right) \\
&= 4\left\{\sum_{\boldsymbol{k}>0}\left|\mathbb{M}_3\left(k\right)\right|^2 + 4\bar{V}_a\sum_{\boldsymbol{k}>0}\mathbb{M}_2\left(k\right)\left\langle \hat{M}\left(k,\tilde{t}\right)\right\rangle'\right\} + O\left(\bar{V}_a^2\right).
\end{aligned} \tag{F.54}$$

Then lower bound of Fisher information $I_C\left(\bar{V}_a,\tilde{t}\right)$ to measure $\bar{V}_a$ is [59]

$$I_C\left(\bar{V}_a,\tilde{t}\right)=\frac{1}{\left\langle\hat{\Psi}_0\left(\tilde{t}\right)\middle|\hat{M}_I^2\left(\tilde{t}\right)\middle|\hat{\Psi}_0\left(\tilde{t}\right)\right\rangle-\left\langle\hat{\Psi}_0\left(\tilde{t}\right)\middle|\hat{M}_I\left(\tilde{t}\right)\middle|\hat{\Psi}_0\left(\tilde{t}\right)\right\rangle^2}\left\{\frac{\partial}{\partial\bar{V}_a}\left\langle\hat{\Psi}_0\left(\tilde{t}\right)\middle|\hat{M}_I\left(\tilde{t}\right)\middle|\hat{\Psi}_0\left(\tilde{t}\right)\right\rangle\right\}^2$$

$$=\frac{1}{4\sum_{\boldsymbol{k}>0}\left|\mathbb{M}_3\left(k\right)\right|^2}\left\{1-4\bar{V}_a\frac{\sum_{\boldsymbol{k}>0}\mathbb{M}_2\left(k\right)\left\langle\hat{M}\left(k,\tilde{t}\right)\right\rangle'}{4\sum_{\boldsymbol{k}>0}\left|\mathbb{M}_3\left(k\right)\right|^2}+O\left(\bar{V}_a^2\right)\right\}$$

$$\times\left[2\sum_{\boldsymbol{k}>0}\left\{4\left\langle\hat{M}\left(k,\tilde{t}\right)\right\rangle'+2\bar{V}_a\left\langle\hat{M}\left(k,\tilde{t}\right)\right\rangle''\right\}+O\left(\bar{V}_a^2\right)\right]^2$$

$$=\frac{1}{\sum_{\boldsymbol{k}>0}\left|\mathbb{M}_3\left(k\right)\right|^2}\left\{1-\bar{V}_a\frac{\sum_{\boldsymbol{k}>0}\mathbb{M}_2\left(k\right)\left\langle\hat{M}\left(k,\tilde{t}\right)\right\rangle'}{\sum_{\boldsymbol{k}>0}\left|\mathbb{M}_3\left(k\right)\right|^2}+O\left(\bar{V}_a^2\right)\right\}16\left\{\sum_{\boldsymbol{k}>0}\left\langle\hat{M}\left(k,\tilde{t}\right)\right\rangle'\right\}^2$$

$$\times\left\{1+\bar{V}_a\frac{\sum_{\boldsymbol{k}>0}\left\langle\hat{M}\left(k,\tilde{t}\right)\right\rangle''}{\sum_{\boldsymbol{k}>0}\left\langle\hat{M}\left(k,\tilde{t}\right)\right\rangle'}+O\left(\bar{V}_a^2\right)\right\}$$

$$=16\frac{\left\{\sum_{\boldsymbol{k}>0}\left\langle\hat{M}\left(k,\tilde{t}\right)\right\rangle'\right\}^2}{\sum_{\boldsymbol{k}>0}\left|\mathbb{M}_3\left(k\right)\right|^2}\left[1+\bar{V}_a\left\{\frac{\sum_{\boldsymbol{k}>0}\left\langle\hat{M}\left(k,\tilde{t}\right)\right\rangle''}{\sum_{\boldsymbol{k}>0}\left\langle\hat{M}\left(k,\tilde{t}\right)\right\rangle'}-\frac{\sum_{\boldsymbol{k}>0}\mathbb{M}_2\left(k\right)\left\langle\hat{M}\left(k,\tilde{t}\right)\right\rangle'}{\sum_{\boldsymbol{k}>0}\left|\mathbb{M}_3\left(k\right)\right|^2}\right\}+O\left(\bar{V}_a^2\right)\right]$$

$$=16\frac{\left[\sum_{\boldsymbol{k}>0}\mathrm{Im}\left\{e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,1}\left(k,\tilde{t}\right)\mathbb{M}_3^*\left(k\right)\mathbb{V}_3\left(k\right)\right\}\right]^2}{\sum_{\boldsymbol{k}>0}\left|\mathbb{M}_3\left(k\right)\right|^2}+O\left(\bar{V}_a\right). \tag{F.55}$$

F.5 Summary and Specific Examples

From Eq. (F.26), $\psi_{2,1}\left(k,\tilde{t}\right):=T_{1,+}\left(k,\tilde{t}\right)$ where $T_{1,+}\left(k,\tilde{t}\right)$ is defined in Eq. (F.8).

Since

$$T_{1,+}\left(k,\tilde{t}\right):=\int_0^{\tilde{t}}d\tilde{t}_1\;e^{2i\tilde{\omega}_1(k)\tilde{t}_1}f\left(\tilde{t}_1\right), \tag{F.56}$$

Eq. (F.34) gives

$$
\begin{aligned}
I_Q\left(\bar{V}_a,\tilde{t}\right) &= 16\sum_{\boldsymbol{k}>0}\left|\psi_{2,1}\left(k,\tilde{t}\right)\mathbb{V}_3\left(k\right)\right|^2+O\left(\bar{V}_a\right)=16\sum_{\boldsymbol{k}>0}\left|e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,1}\left(k,\tilde{t}\right)\right|^2\left|\mathbb{V}_3\left(k\right)\right|^2+O\left(\bar{V}_a\right)\\
&= 16\sum_{\boldsymbol{k}>0}\left|e^{-2i\tilde{\omega}_1(k)\tilde{t}}\int_0^{\tilde{t}}d\tilde{t}_1\ e^{2i\tilde{\omega}_1(k)\tilde{t}_1}f\left(\tilde{t}_1\right)\right|^2\left|\mathbb{V}_3\left(k\right)\right|^2+O\left(\bar{V}_a\right), \qquad \text{(F.57)}
\end{aligned}
$$

and Eq. (F.55) gives

$$
\begin{aligned}
I_C\left(\bar{V}_a,\tilde{t}\right) &= 16\frac{\left[\sum_{\boldsymbol{k}>0}\text{Im}\left\{e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,1}\left(k,\tilde{t}\right)\mathbb{M}_3^*\left(k\right)\mathbb{V}_3\left(k\right)\right\}\right]^2}{\sum_{\boldsymbol{k}>0}\left|\mathbb{M}_3\left(k\right)\right|^2}+O\left(\bar{V}_a\right)\\
&= 16\frac{\left[\sum_{\boldsymbol{k}>0}\text{Im}\left\{e^{-2i\tilde{\omega}_1(k)\tilde{t}}\int_0^{\tilde{t}}d\tilde{t}_1\ e^{2i\tilde{\omega}_1(k)\tilde{t}_1}f\left(\tilde{t}_1\right)\mathbb{M}_3^*\left(k\right)\mathbb{V}_3\left(k\right)\right\}\right]^2}{\sum_{\boldsymbol{k}>0}\left|\mathbb{M}_3\left(k\right)\right|^2}+O\left(\bar{V}_a\right). \quad \text{(F.58)}
\end{aligned}
$$

Note that

$$
\begin{aligned}
&e^{-2i\tilde{\omega}_1(k)\tilde{t}}\int_0^{\tilde{t}}d\tilde{t}_1\ e^{2i\tilde{\omega}_1(k)\tilde{t}_1}f\left(\tilde{t}_1\right)\\
&=\left[\cos\left\{2\tilde{\omega}_1\left(k\right)\tilde{t}\right\}-i\sin\left\{2\tilde{\omega}_1\left(k\right)\tilde{t}\right\}\right]\left[\text{Re}\left\{\int_0^{\tilde{t}}d\tilde{t}_1\ e^{2i\tilde{\omega}_1(k)\tilde{t}_1}f\left(\tilde{t}_1\right)\right\}+i\text{Im}\left\{\int_0^{\tilde{t}}d\tilde{t}_1\ e^{2i\tilde{\omega}_1(k)\tilde{t}_1}f\left(\tilde{t}_1\right)\right\}\right]\\
&=\text{Re}\left\{\int_0^{\tilde{t}}d\tilde{t}_1\ e^{2i\tilde{\omega}_1(k)\tilde{t}_1}f\left(\tilde{t}_1\right)\right\}\cos\left\{2\tilde{\omega}_1\left(k\right)\tilde{t}\right\}+\text{Im}\left\{\int_0^{\tilde{t}}d\tilde{t}_1\ e^{2i\tilde{\omega}_1(k)\tilde{t}_1}f\left(\tilde{t}_1\right)\right\}\sin\left\{2\tilde{\omega}_1\left(k\right)\tilde{t}\right\}\\
&\quad+i\left[\text{Im}\left\{\int_0^{\tilde{t}}d\tilde{t}_1\ e^{2i\tilde{\omega}_1(k)\tilde{t}_1}f\left(\tilde{t}_1\right)\right\}\cos\left\{2\tilde{\omega}_1\left(k\right)\tilde{t}\right\}-\text{Re}\left\{\int_0^{\tilde{t}}d\tilde{t}_1\ e^{2i\tilde{\omega}_1(k)\tilde{t}_1}f\left(\tilde{t}_1\right)\right\}\sin\left\{2\tilde{\omega}_1\left(k\right)\tilde{t}\right\}\right]. \quad \text{(F.59)}
\end{aligned}
$$

F.5.1 Delta Function Perturbation

Suppose that $f\left(t\right)=\delta\left(t-t_p\right)$ for $t\geq 0$ (assuming $t_p\geq 0$). Then

$$
\int_0^{\tilde{t}}d\tilde{t}_1\ e^{2i\tilde{\omega}_1(k)\tilde{t}_1}\delta\left(\tilde{t}_1-\tilde{t}_p\right)=\theta\left(\tilde{t}-\tilde{t}_p\right)e^{2i\tilde{\omega}_1(k)\tilde{t}_p}, \qquad \text{(F.60)}
$$

where $\theta(t) = 1$ if $t \geq 0$ and 0 otherwise. Thus

$$I_Q\left(\bar{V}_a, \tilde{t}\right) = 16\theta\left(\tilde{t} - \tilde{t}_p\right) \sum_{k>0} \left|\mathbb{V}_3(k)\right|^2 + O\left(\bar{V}_a\right),$$
$$I_C\left(\bar{V}_a, \tilde{t}\right) = 16\theta\left(\tilde{t} - \tilde{t}_p\right)$$
$$\times \frac{\left(\sum_{k>0} \left[\text{Re}\left\{\mathbb{M}_3^*(k)\,\mathbb{V}_3(k)\right\} \sin\left\{2\tilde{\omega}_1(k)\left(\tilde{t} - \tilde{t}_p\right)\right\} - \text{Im}\left\{\mathbb{M}_3^*(k)\,\mathbb{V}_3(k)\right\} \cos\left\{2\tilde{\omega}_1(k)\left(\tilde{t} - \tilde{t}_p\right)\right\}\right]\right)^2}{\sum_{k>0} \left|\mathbb{M}_3(k)\right|^2}$$
$$+O\left(\bar{V}_a\right). \tag{F.61}$$

Note that $I_Q\left(\bar{V}_a, \tilde{t}\right)$ is constant in scaled time $\tilde{t}$ for $\tilde{t} \geq \tilde{t}_p$.

F.5.2 Step Function Perturbation

If $f(t) = \theta(t - t_p)$ for $t \geq 0$ (assuming $t_p \geq 0$).

$$\int_0^{\tilde{t}} d\tilde{t}_1\; e^{2i\tilde{\omega}_1(k)\tilde{t}_1} \theta\left(\tilde{t}_1 - \tilde{t}_p\right) = \theta\left(\tilde{t} - \tilde{t}_p\right) \int_{\tilde{t}_p}^{\tilde{t}} d\tilde{t}_1\; e^{2i\tilde{\omega}_1(k)\tilde{t}_1} = \theta\left(\tilde{t} - \tilde{t}_p\right) \frac{e^{2i\tilde{\omega}_1(k)\tilde{t}} - e^{2i\tilde{\omega}_1(k)\tilde{t}_p}}{2i\tilde{\omega}_1(k)}$$
$$= e^{i\tilde{\omega}_1(k)\left(\tilde{t}+\tilde{t}_p\right)} \theta\left(\tilde{t} - \tilde{t}_p\right) \frac{e^{i\tilde{\omega}_1(k)\left(\tilde{t}-\tilde{t}_p\right)} - e^{-i\tilde{\omega}_1(k)\left(\tilde{t}-\tilde{t}_p\right)}}{2i\tilde{\omega}_1(k)} = e^{i\tilde{\omega}_1(k)\left(\tilde{t}+\tilde{t}_p\right)} \theta\left(\tilde{t} - \tilde{t}_p\right) \frac{\sin\left\{\tilde{\omega}_1(k)\left(\tilde{t} - \tilde{t}_p\right)\right\}}{\tilde{\omega}_1(k)}. \tag{F.62}$$

Hence,

$$I_Q\left(\bar{V}_a, \tilde{t}\right) = 16\theta\left(\tilde{t} - \tilde{t}_p\right) \sum_{k>0} \frac{\sin^2\left\{\tilde{\omega}_1(k)\left(\tilde{t} - \tilde{t}_p\right)\right\}}{\tilde{\omega}_1^2(k)} \left|\mathbb{V}_3(k)\right|^2 + O\left(\bar{V}_a\right),$$
$$I_C\left(\bar{V}_a, \tilde{t}\right) = 16\theta\left(\tilde{t} - \tilde{t}_p\right) \times$$
$$\frac{\left(\sum_{k>0} \frac{\sin\left\{\tilde{\omega}_1(k)\left(\tilde{t} - \tilde{t}_p\right)\right\}}{\tilde{\omega}_1(k)} \left[\text{Re}\left\{\mathbb{M}_3^*(k)\,\mathbb{V}_3(k)\right\} \sin\left\{\tilde{\omega}_1(k)\left(\tilde{t} - \tilde{t}_p\right)\right\} - \text{Im}\left\{\mathbb{M}_3^*(k)\,\mathbb{V}_3(k)\right\} \cos\left\{\tilde{\omega}_1(k)\left(\tilde{t} - \tilde{t}_p\right)\right\}\right]\right)^2}{\sum_{k>0} \left|\mathbb{M}_3(k)\right|^2}$$
$$+O\left(\bar{V}_a\right). \tag{F.63}$$

F.5.3 Sinusoidal Perturbation

If $f(t) = \cos(\omega_a t + \delta_a)$ for $t \geq 0$, we have to calculate

$$e^{-2i\tilde{\omega}_1(k)\tilde{t}} \int_0^{\tilde{t}} d\tilde{t}_1 \; e^{2i\tilde{\omega}_1(k)\tilde{t}_1} f\left(\tilde{t}_1\right) = e^{-2i\tilde{\omega}_1(k)\tilde{t}} \int_0^{\tilde{t}} d\tilde{t}_1 \; e^{2i\tilde{\omega}_1(k)\tilde{t}_1} \cos\left(\tilde{\omega}_a \tilde{t}_1 + \delta_a\right), \tag{F.64}$$

where $\tilde{\omega}_a := \omega_a / (g_a n)$.

After some calculation,

$$\begin{aligned}
&e^{-2i\tilde{\omega}_1(k)\tilde{t}} \int_0^{\tilde{t}} d\tilde{t}_1 \; e^{2i\tilde{\omega}_1(k)\tilde{t}_1} \cos\left(\tilde{\omega}_a \tilde{t}_1 + \delta_a\right) \\
&= \frac{\tilde{\omega}_a \sin\left(\tilde{\omega}_a \tilde{t} + \delta_a\right) + 2i\tilde{\omega}_1(k) \cos\left(\tilde{\omega}_a \tilde{t} + \delta_a\right) - e^{-2i\tilde{\omega}_1(k)\tilde{t}} \left\{\tilde{\omega}_a \sin(\delta_a) + 2i\tilde{\omega}_1(k)\cos(\delta_a)\right\}}{\tilde{\omega}_a^2 - 4\tilde{\omega}_1^2(k)} \\
&= \frac{\tilde{\omega}_a \sin\left(\tilde{\omega}_a \tilde{t} + \delta_a\right) + 2i\tilde{\omega}_1(k) \cos\left(\tilde{\omega}_a \tilde{t} + \delta_a\right)}{\tilde{\omega}_a^2 - 4\tilde{\omega}_1^2(k)} \\
&\quad - \frac{\tilde{\omega}_a \cos\left\{2\tilde{\omega}_1(k)\tilde{t}\right\} \sin(\delta_a) + 2i\tilde{\omega}_1(k) \cos\left\{2\tilde{\omega}_1(k)\tilde{t}\right\} \cos(\delta_a)}{\tilde{\omega}_a^2 - 4\tilde{\omega}_1^2(k)} \\
&\quad - \frac{2\tilde{\omega}_1(k) \sin\left\{2\tilde{\omega}_1(k)\tilde{t}\right\} \cos(\delta_a) - i\tilde{\omega}_a \sin\left\{2\tilde{\omega}_1(k)\tilde{t}\right\} \sin(\delta_a)}{\tilde{\omega}_a^2 - 4\tilde{\omega}_1^2(k)} \\
&= \frac{\tilde{\omega}_a \sin\left(\tilde{\omega}_a \tilde{t}\right) \cos(\delta_a) + \tilde{\omega}_a \cos\left(\tilde{\omega}_a \tilde{t}\right) \sin(\delta_a) + 2i\tilde{\omega}_1(k) \cos\left(\tilde{\omega}_a \tilde{t}\right) \cos(\delta_a) - 2i\tilde{\omega}_1(k) \sin\left(\tilde{\omega}_a \tilde{t}\right) \sin(\delta_a)}{\tilde{\omega}_a^2 - 4\tilde{\omega}_1^2(k)} \\
&\quad - \frac{\tilde{\omega}_a \cos\left\{2\tilde{\omega}_1(k)\tilde{t}\right\} \sin(\delta_a) + 2\tilde{\omega}_1(k) \sin\left\{2\tilde{\omega}_1(k)\tilde{t}\right\} \cos(\delta_a)}{\tilde{\omega}_a^2 - 4\tilde{\omega}_1^2(k)} \\
&\quad - i\frac{2\tilde{\omega}_1(k) \cos\left\{2\tilde{\omega}_1(k)\tilde{t}\right\} \cos(\delta_a) - \tilde{\omega}_a \sin\left\{2\tilde{\omega}_1(k)\tilde{t}\right\} \sin(\delta_a)}{\tilde{\omega}_a^2 - 4\tilde{\omega}_1^2(k)}.
\end{aligned} \tag{F.65}$$

Now, let $\epsilon(k) := 2\tilde{\omega}_1(k) - \tilde{\omega}_a$. Then we get

$$
\begin{aligned}
&\tilde{\omega}_a \sin\left(\tilde{\omega}_a \tilde{t}\right) \cos(\delta_a) + \tilde{\omega}_a \cos\left(\tilde{\omega}_a \tilde{t}\right) \sin(\delta_a) + 2i\tilde{\omega}_1(k) \cos\left(\tilde{\omega}_a \tilde{t}\right) \cos(\delta_a) - 2i\tilde{\omega}_1(k) \sin\left(\tilde{\omega}_a \tilde{t}\right) \sin(\delta_a) \\
&\quad -\tilde{\omega}_a \cos\left\{2\tilde{\omega}_1(k)\tilde{t}\right\} \sin(\delta_a) - 2\tilde{\omega}_1(k) \sin\left\{2\tilde{\omega}_1(k)\tilde{t}\right\} \cos(\delta_a) - 2i\tilde{\omega}_1(k) \cos\left\{2\tilde{\omega}_1(k)\tilde{t}\right\} \cos(\delta_a) \\
&\quad +i\tilde{\omega}_a \sin\left\{2\tilde{\omega}_1(k)\tilde{t}\right\} \sin(\delta_a) \\
&= \tilde{\omega}_a \sin\left(\tilde{\omega}_a \tilde{t}\right) \cos(\delta_a) + \tilde{\omega}_a \cos\left(\tilde{\omega}_a \tilde{t}\right) \sin(\delta_a) + i\epsilon(k) \cos\left(\tilde{\omega}_a \tilde{t}\right) \cos(\delta_a) + i\tilde{\omega}_a \cos\left(\tilde{\omega}_a \tilde{t}\right) \cos(\delta_a) \\
&\quad -i\epsilon(k) \sin\left(\tilde{\omega}_a \tilde{t}\right) \sin(\delta_a) - i\tilde{\omega}_a \sin\left(\tilde{\omega}_a \tilde{t}\right) \sin(\delta_a) - \tilde{\omega}_a \cos\left[\left\{\epsilon(k) + \tilde{\omega}_a\right\}\tilde{t}\right] \sin(\delta_a) \\
&\quad -\epsilon(k) \sin\left[\left\{\epsilon(k) + \tilde{\omega}_a\right\}\tilde{t}\right] \cos(\delta_a) - \tilde{\omega}_a \sin\left[\left\{\epsilon(k) + \tilde{\omega}_a\right\}\tilde{t}\right] \cos(\delta_a) - i\epsilon(k) \cos\left[\left\{\epsilon(k) + \tilde{\omega}_a\right\}\tilde{t}\right] \cos(\delta_a) \\
&\quad -i\tilde{\omega}_a \cos\left[\left\{\epsilon(k) + \tilde{\omega}_a\right\}\tilde{t}\right] \cos(\delta_a) + i\tilde{\omega}_a \sin\left[\left\{\epsilon(k) + \tilde{\omega}_a\right\}\tilde{t}\right] \sin(\delta_a) \\
&= \tilde{\omega}_a \left(\sin\left(\tilde{\omega}_a \tilde{t}\right) - \sin\left[\left\{\epsilon(k) + \tilde{\omega}_a\right\}\tilde{t}\right]\right) \cos(\delta_a) + \tilde{\omega}_a \left(\cos\left(\tilde{\omega}_a \tilde{t}\right) - \cos\left[\left\{\epsilon(k) + \tilde{\omega}_a\right\}\tilde{t}\right]\right) \sin(\delta_a) \\
&\quad -\epsilon(k) \sin\left[\left\{\epsilon(k) + \tilde{\omega}_a\right\}\tilde{t}\right] \cos(\delta_a) + i\tilde{\omega}_a \left(\cos\left(\tilde{\omega}_a \tilde{t}\right) - \cos\left[\left\{\epsilon(k) + \tilde{\omega}_a\right\}\tilde{t}\right]\right) \cos(\delta_a) \\
&\quad -i\tilde{\omega}_a \left(\sin\left(\tilde{\omega}_a \tilde{t}\right) - \sin\left[\left\{\epsilon(k) + \tilde{\omega}_a\right\}\tilde{t}\right]\right) \sin(\delta_a) - i\epsilon(k) \cos\left[\left\{\epsilon(k) + \tilde{\omega}_a\right\}\tilde{t}\right] \cos(\delta_a) + i\epsilon(k) \cos\left(\tilde{\omega}_a \tilde{t} + \delta_a\right) \\
&= -2\tilde{\omega}_a \cos\left[\left\{\frac{\epsilon(k)}{2} + \tilde{\omega}_a\right\}\tilde{t}\right] \sin\left\{\frac{\epsilon(k)}{2}\tilde{t}\right\} \cos(\delta_a) + 2\tilde{\omega}_a \sin\left[\left\{\frac{\epsilon(k)}{2} + \tilde{\omega}_a\right\}\tilde{t}\right] \sin\left\{\frac{\epsilon(k)}{2}\tilde{t}\right\} \sin(\delta_a) \\
&\quad -\epsilon(k) \sin\left[\left\{\epsilon(k) + \tilde{\omega}_a\right\}\tilde{t}\right] \cos(\delta_a) + 2i\tilde{\omega}_a \sin\left[\left\{\frac{\epsilon(k)}{2} + \tilde{\omega}_a\right\}\tilde{t}\right] \sin\left\{\frac{\epsilon(k)}{2}\tilde{t}\right\} \cos(\delta_a) \\
&\quad +2i\tilde{\omega}_a \cos\left[\left\{\frac{\epsilon(k)}{2} + \tilde{\omega}_a\right\}\tilde{t}\right] \sin\left\{\frac{\epsilon(k)}{2}\tilde{t}\right\} \sin(\delta_a) - i\epsilon(k) \left(\cos\left[\left\{\epsilon(k) + \tilde{\omega}_a\right\}\tilde{t}\right] \cos(\delta_a) - \cos\left(\tilde{\omega}_a \tilde{t} + \delta_a\right)\right) \\
&= -2\tilde{\omega}_a \cos\left[\left\{\frac{\epsilon(k)}{2} + \tilde{\omega}_a\right\}\tilde{t} + \delta_a\right] \sin\left\{\frac{\epsilon(k)}{2}\tilde{t}\right\} - \epsilon(k) \sin\left[\left\{\epsilon(k) + \tilde{\omega}_a\right\}\tilde{t}\right] \cos(\delta_a) \\
&\quad +2i\tilde{\omega}_a \sin\left[\left\{\frac{\epsilon(k)}{2} + \tilde{\omega}_a\right\}\tilde{t} + \delta_a\right] \sin\left\{\frac{\epsilon(k)}{2}\tilde{t}\right\} - i\epsilon(k) \left(\cos\left[\left\{\epsilon(k) + \tilde{\omega}_a\right\}\tilde{t}\right] \cos(\delta_a) - \cos\left(\tilde{\omega}_a \tilde{t} + \delta_a\right)\right).
\end{aligned}
\tag{F.66}
$$

Since $\tilde{\omega}_a^2 - 4\tilde{\omega}_1^2(k) = -\epsilon(k)\left\{\epsilon(k) + 2\tilde{\omega}_a\right\}$, we get

$$
\begin{aligned}
&e^{-2i\tilde{\omega}_1(k)\tilde{t}} \int_0^{\tilde{t}} d\tilde{t}_1\, e^{2i\tilde{\omega}_1(k)\tilde{t}_1} \cos\left(\tilde{\omega}_a \tilde{t}_1 + \delta_a\right) \\
&= \frac{2\tilde{\omega}_a \cos\left[\left\{\frac{\epsilon(k)}{2} + \tilde{\omega}_a\right\}\tilde{t} + \delta_a\right] \sin\left\{\frac{\epsilon(k)}{2}\tilde{t}\right\}}{\epsilon(k)\left\{\epsilon(k) + 2\tilde{\omega}_a\right\}} + \frac{\sin\left[\left\{\epsilon(k) + \tilde{\omega}_a\right\}\tilde{t}\right] \cos(\delta_a)}{\epsilon(k) + 2\tilde{\omega}_a} \\
&\quad + i\frac{\cos\left[\left\{\epsilon(k) + \tilde{\omega}_a\right\}\tilde{t}\right] \cos(\delta_a) - \cos\left(\tilde{\omega}_a \tilde{t} + \delta_a\right)}{\epsilon(k) + 2\tilde{\omega}_a} - i\frac{2\tilde{\omega}_a \sin\left[\left\{\frac{\epsilon(k)}{2} + \tilde{\omega}_a\right\}\tilde{t} + \delta_a\right] \sin\left\{\frac{\epsilon(k)}{2}\tilde{t}\right\}}{\epsilon(k)\left\{\epsilon(k) + 2\tilde{\omega}_a\right\}},
\end{aligned}
\tag{F.67}
$$

and

$$I_Q\left(\bar{V}_a,\tilde{t}\right)=16\sum_{k>0}\left|e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,1}\left(k,\tilde{t}\right)\right|^2\left|\mathbb{V}_3\left(k\right)\right|^2+O\left(\bar{V}_a\right),$$

$$I_C\left(\bar{V}_a,\tilde{t}\right)=$$

$$16\frac{\left[\sum_{k>0}\mathrm{Im}\left\{e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,1}\left(k,\tilde{t}\right)\right\}\mathrm{Re}\left\{\mathbb{M}_3^*\left(k\right)\mathbb{V}_3\left(k\right)\right\}+\mathrm{Re}\left\{e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,1}\left(k,\tilde{t}\right)\right\}\mathrm{Im}\left\{\mathbb{M}_3^*\left(k\right)\mathbb{V}_3\left(k\right)\right\}\right]^2}{\sum_{k>0}\left|\mathbb{M}_3\left(k\right)\right|^2}$$

$$+O\left(\bar{V}_a\right), \qquad \text{(F.68)}$$

where

$$\mathrm{Re}\left\{e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,1}\left(k,\tilde{t}\right)\right\}=\frac{\cos\delta_a\sin\left[\left\{\tilde{\omega}_a+\epsilon\left(k\right)\right\}\tilde{t}\right]}{2\tilde{\omega}_a+\epsilon\left(k\right)}+\frac{\tilde{\omega}_a\cos\left[\left\{\tilde{\omega}_a+\frac{\epsilon(k)}{2}\right\}\tilde{t}+\delta_a\right]}{2\tilde{\omega}_a+\epsilon\left(k\right)}\frac{\sin\left\{\epsilon\left(k\right)\tilde{t}/2\right\}}{\epsilon\left(k\right)/2},$$

$$\mathrm{Im}\left\{e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,1}\left(k,\tilde{t}\right)\right\}=\frac{\cos\delta_a\cos\left[\left\{\tilde{\omega}_a+\epsilon\left(k\right)\right\}\tilde{t}\right]-\cos\left(\tilde{\omega}_a\tilde{t}+\delta_a\right)}{2\tilde{\omega}_a+\epsilon\left(k\right)}$$

$$-\frac{\tilde{\omega}_a\sin\left[\left\{\tilde{\omega}_a+\frac{\epsilon(k)}{2}\right\}\tilde{t}+\delta_a\right]}{2\tilde{\omega}_a+\epsilon\left(k\right)}\frac{\sin\left\{\epsilon\left(k\right)\tilde{t}/2\right\}}{\epsilon\left(k\right)/2}. \qquad \text{(F.69)}$$

Note that both $\mathrm{Im}\left\{e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,1}\left(k,\tilde{t}\right)\right\}$ and $\left|e^{-2i\tilde{\omega}_1(k)\tilde{t}}\psi_{2,1}\left(k,\tilde{t}\right)\right|$ have maximum value at $2\tilde{\omega}_1\left(k\right)=\tilde{\omega}_a$ (where $\epsilon\left(k\right)=0$).

Chapter G

Fisher Information Calculated by Using Symplectic Formalism

Suppose that the Hamiltonian $\hat{H}_S(t)$ in Schrödinger picture is given as $\hat{H}_S(t) = \sum_{\boldsymbol{k}>0} \hat{H}_c(\boldsymbol{k}, t)$. Time evolution operator $\hat{U}_S(t)$ in Schrödinger picture satisfy $i\hbar \frac{\partial \hat{U}_S(t)}{\partial t} = \hat{H}_S(t)\hat{U}_S(t)$.

G.1 Time Evolution Operator

From Appendix A of [69], let $\hat{U}_S(t) = \hat{U}_1(t)\hat{U}_2(t)\hat{U}_3(t)$ with

$$\hat{U}_j(t) = \exp\left\{-i\sum_{\boldsymbol{k}>0} F_j(\boldsymbol{k}, t)\hat{G}_j(\boldsymbol{k})\right\}, \tag{G.1}$$

for $j = 1, 2, 3$. Then we get

$$\begin{aligned}\hat{H}_S(t) &= \hbar\sum_{\boldsymbol{k}>0}\left\{\frac{\partial F_1(\boldsymbol{k}, t)}{\partial t}\hat{G}_1(\boldsymbol{k}) + \frac{\partial F_2(\boldsymbol{k}, t)}{\partial t}\hat{U}_1(t)\hat{G}_2(\boldsymbol{k})\hat{U}_1^\dagger(t)\right\} \\ &\quad + \hbar\sum_{\boldsymbol{k}>0}\frac{\partial F_3(\boldsymbol{k}, t)}{\partial t}\hat{U}_1(t)\hat{U}_2(t)\hat{G}_3(\boldsymbol{k})\hat{U}_2^\dagger(t)\hat{U}_1^\dagger(t).\end{aligned} \tag{G.2}$$

Let $n_b(\boldsymbol{k})$ be the number of particles with momentum $\boldsymbol{k} > 0$. From now on, we will order positive momentum as $0 < \boldsymbol{k}_1 < \boldsymbol{k}_2 < \cdots < \boldsymbol{k}_c$ where $\boldsymbol{k}_c$ is the cutoff. For the state $|n_b(\boldsymbol{k}_1), n_b(\boldsymbol{k}_2), \cdots n_b(\boldsymbol{k}_c)\rangle$ where the number of particles with momentum $\boldsymbol{k}_j$ (which is $n_b(\boldsymbol{k}_j)$) is same as the number of particles with momentum $-\boldsymbol{k}_j$ for $\boldsymbol{k}_1 \le \boldsymbol{k}_j \le \boldsymbol{k}_c$, by defining

$$\hat{K}_z(\boldsymbol{k}) \coloneqq \frac{1}{2}\left\{\hat{b}_1^\dagger(\boldsymbol{k})\,\hat{b}_1(\boldsymbol{k}) + \hat{b}_1(-\boldsymbol{k})\,\hat{b}_1^\dagger(-\boldsymbol{k})\right\}, \quad \hat{K}_x(\boldsymbol{k}) \coloneqq \frac{1}{2}\left\{\hat{b}_1^\dagger(\boldsymbol{k})\,\hat{b}_1^\dagger(-\boldsymbol{k}) + \hat{b}_1(\boldsymbol{k})\,\hat{b}_1(-\boldsymbol{k})\right\},$$
$$\hat{K}_y(\boldsymbol{k}) \coloneqq \frac{1}{2i}\left\{\hat{b}_1^\dagger(\boldsymbol{k})\,\hat{b}_1^\dagger(-\boldsymbol{k}) - \hat{b}_1(\boldsymbol{k})\,\hat{b}_1(-\boldsymbol{k})\right\}, \tag{G.3}$$

and $\hat{K}_\pm(\boldsymbol{k}) \coloneqq \hat{K}_x(\boldsymbol{k}) \pm i\hat{K}_y(\boldsymbol{k})$, we get

$$\begin{aligned}
\hat{K}_+(\boldsymbol{k}_j)\,|n_b(\boldsymbol{k}_1), \cdots n_b(\boldsymbol{k}_c)\rangle &= \hat{b}_1^\dagger(\boldsymbol{k}_j)\,\hat{b}_1^\dagger(-\boldsymbol{k}_j)\,|n_b(\boldsymbol{k}_1), \cdots n_b(\boldsymbol{k}_c)\rangle \\
&= \underbrace{\sqrt{n_b(\boldsymbol{k}_j)+1}}_{\text{with } \hat{b}_1^\dagger(\boldsymbol{k}_j)}\underbrace{\sqrt{n_b(\boldsymbol{k}_j)+1}}_{\text{with } \hat{b}_1^\dagger(-\boldsymbol{k}_j)}\,|n_b(\boldsymbol{k}_1), \cdots n_b(\boldsymbol{k}_j)+1, \cdots n_b(\boldsymbol{k}_c)\rangle \\
&= \{n_b(\boldsymbol{k}_j)+1\}\,|n_b(\boldsymbol{k}_1), \cdots n_b(\boldsymbol{k}_j)+1, \cdots n_b(\boldsymbol{k}_c)\rangle,
\end{aligned} \tag{G.4}$$

$$\begin{aligned}
\hat{K}_-(\boldsymbol{k}_j)\,|n_b(\boldsymbol{k}_1), \cdots n_b(\boldsymbol{k}_c)\rangle &= \hat{b}_1(\boldsymbol{k}_j)\,\hat{b}_1(-\boldsymbol{k}_j)\,|n_b(\boldsymbol{k}_1), \cdots n_b(\boldsymbol{k}_c)\rangle \\
&= \underbrace{\sqrt{n_b(\boldsymbol{k}_j)}}_{\text{with } \hat{b}_1(\boldsymbol{k}_j)}\underbrace{\sqrt{n_b(\boldsymbol{k}_j)}}_{\text{with } \hat{b}_1(-\boldsymbol{k}_j)}\,|n_b(\boldsymbol{k}_1), \cdots n_b(\boldsymbol{k}_j)-1, \cdots n_b(\boldsymbol{k}_c)\rangle \\
&= n_b(\boldsymbol{k}_j)\,|n_b(\boldsymbol{k}_1), \cdots n_b(\boldsymbol{k}_j)-1, \cdots n_b(\boldsymbol{k}_c)\rangle,
\end{aligned} \tag{G.5}$$

and

$$\begin{aligned}
\hat{K}_z(\boldsymbol{k}_j)\,|n_b(\boldsymbol{k}_1), \cdots n_b(\boldsymbol{k}_c)\rangle &= \frac{1}{2}\left\{\hat{b}_1^\dagger(\boldsymbol{k}_j)\,\hat{b}_1(\boldsymbol{k}_j) + \hat{b}_1(-\boldsymbol{k}_j)\,\hat{b}_1^\dagger(-\boldsymbol{k}_j)\right\}|n_b(\boldsymbol{k}_1), \cdots n_b(\boldsymbol{k}_c)\rangle \\
&= \frac{1}{2}\{\underbrace{n_b(\boldsymbol{k}_j)}_{\text{with } \hat{b}_1^\dagger(\boldsymbol{k}_j)\hat{b}_1(\boldsymbol{k}_j)} + \underbrace{n_b(\boldsymbol{k}_j)+1}_{\text{with } \hat{b}_1(-\boldsymbol{k}_j)\hat{b}_1^\dagger(-\boldsymbol{k}_j)}\}\,|n_b(\boldsymbol{k}_1), \cdots n_b(\boldsymbol{k}_j), \cdots n_b(\boldsymbol{k}_c)\rangle \\
&= \left\{n_b(\boldsymbol{k}_j) + \frac{1}{2}\right\}|n_b(\boldsymbol{k}_1), \cdots n_b(\boldsymbol{k}_c)\rangle.
\end{aligned} \tag{G.6}$$

Therefore, we get

$$|n_b(\boldsymbol{k}_1), \cdots n_b(\boldsymbol{k}_c)\rangle = \prod_{j=1} \frac{1}{n_b(\boldsymbol{k}_j)!} \left\{\hat{K}_+(\boldsymbol{k}_j)\right\}^{n_b(\boldsymbol{k}_j)} |\text{vac}\rangle, \qquad \text{(G.7)}$$

where $|\text{vac}\rangle$ is Bogoliubov vacuum state.

For $\boldsymbol{k}_1 > 0$ and $\boldsymbol{k}_2 > 0$, we have

$$\begin{aligned}
\left[\hat{K}_z(\boldsymbol{k}_1), \hat{K}_x(\boldsymbol{k}_2)\right] &= i\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\hat{K}_y(\boldsymbol{k}_1), \quad \left[\hat{K}_z(\boldsymbol{k}_1), \hat{K}_y(\boldsymbol{k}_2)\right] = -i\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\hat{K}_x(\boldsymbol{k}_1), \\
\left[\hat{K}_x(\boldsymbol{k}_1), \hat{K}_y(\boldsymbol{k}_2)\right] &= -i\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\hat{K}_z(\boldsymbol{k}_1), \\
\left[\hat{K}_z(\boldsymbol{k}_1), \hat{K}_\pm(\boldsymbol{k}_2)\right] &= i\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\hat{K}_y(\boldsymbol{k}_1) \pm \delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\hat{K}_x(\boldsymbol{k}_1) = \pm\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\hat{K}_\pm(\boldsymbol{k}_1), \qquad \text{(G.8)}
\end{aligned}$$

which gives

$$\begin{aligned}
& e^{-iF_1(\boldsymbol{k}_1,t)\hat{K}_z(\boldsymbol{k}_1)}\hat{K}_\pm(\boldsymbol{k})\, e^{iF_1(\boldsymbol{k}_1,t)\hat{K}_z(\boldsymbol{k}_1)} \\
&= \hat{K}_\pm(\boldsymbol{k}) + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} F_1^n(\boldsymbol{k}_1, t) \, [\underbrace{\hat{K}_z(\boldsymbol{k}_1), [\hat{K}_z(\boldsymbol{k}_1), \cdots, [\hat{K}_z(\boldsymbol{k}_1)}_{n \text{ times}}, \hat{K}_\pm(\boldsymbol{k})] \cdots]] \\
&= \hat{K}_\pm(\boldsymbol{k}) + \delta_{\boldsymbol{k},\boldsymbol{k}_1} \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} F_1^n(\boldsymbol{k}_1, t) (\pm 1)^n \hat{K}_\pm(\boldsymbol{k}) = \left[(1-\delta_{\boldsymbol{k},\boldsymbol{k}_1}) + \delta_{\boldsymbol{k},\boldsymbol{k}_1} \exp\{\mp iF_1(\boldsymbol{k}, t)\}\right] \hat{K}_\pm(\boldsymbol{k}). \qquad \text{(G.9)}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\ & e^{-iF_1(\boldsymbol{k}_1,t)\hat{K}_z(\boldsymbol{k}_1)}\hat{K}_x(\boldsymbol{k})\, e^{iF_1(\boldsymbol{k}_1,t)\hat{K}_z(\boldsymbol{k}_1)} \\
&= (1-\delta_{\boldsymbol{k},\boldsymbol{k}_1})\hat{K}_x(\boldsymbol{k}) + \frac{1}{2}\delta_{\boldsymbol{k},\boldsymbol{k}_1}\left[\exp\{-iF_1(\boldsymbol{k},t)\}\hat{K}_+(\boldsymbol{k}) + \exp\{iF_1(\boldsymbol{k},t)\}\hat{K}_-(\boldsymbol{k})\right] \\
&= (1-\delta_{\boldsymbol{k},\boldsymbol{k}_1})\hat{K}_x(\boldsymbol{k}) + \delta_{\boldsymbol{k},\boldsymbol{k}_1}\left[\cos\{F_1(\boldsymbol{k},t)\}\hat{K}_x(\boldsymbol{k}) + \sin\{F_1(\boldsymbol{k},t)\}\hat{K}_y(\boldsymbol{k})\right], \\
& e^{-iF_1(\boldsymbol{k}_1,t)\hat{K}_z(\boldsymbol{k}_1)}\hat{K}_y(\boldsymbol{k})\, e^{iF_1(\boldsymbol{k}_1,t)\hat{K}_z(\boldsymbol{k}_1)} \\
&= (1-\delta_{\boldsymbol{k},\boldsymbol{k}_1})\hat{K}_y(\boldsymbol{k}) + \frac{1}{2i}\delta_{\boldsymbol{k},\boldsymbol{k}_1}\left[\exp\{-iF_1(\boldsymbol{k},t)\}\hat{K}_+(\boldsymbol{k}) - \exp\{iF_1(\boldsymbol{k},t)\}\hat{K}_-(\boldsymbol{k})\right] \\
&= (1-\delta_{\boldsymbol{k},\boldsymbol{k}_1})\hat{K}_y(\boldsymbol{k}) + \delta_{\boldsymbol{k},\boldsymbol{k}_1}\left[-\sin\{F_1(\boldsymbol{k},t)\}\hat{K}_x(\boldsymbol{k}) + \cos\{F_1(\boldsymbol{k},t)\}\hat{K}_y(\boldsymbol{k})\right], \qquad \text{(G.10)}
\end{aligned}$$

and

$$
\begin{aligned}
& e^{-iF_2(\boldsymbol{k}_1,t)\hat{K}_x(\boldsymbol{k}_1)}\hat{K}_y(\boldsymbol{k})\,e^{iF_2(\boldsymbol{k}_1,t)\hat{K}_x(\boldsymbol{k}_1)} \\
&= \hat{K}_y(\boldsymbol{k}) + \sum_{n=1}^{\infty}\frac{(-i)^n}{n!}F_2^n(\boldsymbol{k}_1,t)\,[\underbrace{\hat{K}_x(\boldsymbol{k}_1),[\hat{K}_x(\boldsymbol{k}_1),\cdots,[\hat{K}_x(\boldsymbol{k}_1)}_{n\text{ times}},\hat{K}_y(\boldsymbol{k})]\cdots]] \\
&= \hat{K}_y(\boldsymbol{k}) + \delta_{\boldsymbol{k},\boldsymbol{k}_1}\sum_{j=0}^{\infty}\frac{(-i)^{2j+1}}{(2j+1)!}F_2^{2j+1}(\boldsymbol{k}_1,t)(-i)^{2j+1}\hat{K}_z(\boldsymbol{k}) + \delta_{\boldsymbol{k},\boldsymbol{k}_1}\sum_{j=1}^{\infty}\frac{(-i)^{2j}}{(2j)!}F_2^{2j}(\boldsymbol{k}_1,t)(-i)^{2j}\hat{K}_y(\boldsymbol{k}) \\
&= (1-\delta_{\boldsymbol{k},\boldsymbol{k}_1})\hat{K}_y(\boldsymbol{k}) - \delta_{\boldsymbol{k},\boldsymbol{k}_1}\sum_{j=0}^{\infty}\frac{1}{(2j+1)!}F_2^{2j+1}(\boldsymbol{k},t)\hat{K}_z(\boldsymbol{k}) + \delta_{\boldsymbol{k},\boldsymbol{k}_1}\sum_{j=0}^{\infty}\frac{1}{(2j)!}F_2^{2j}(\boldsymbol{k},t)\hat{K}_y(\boldsymbol{k}) \\
&= (1-\delta_{\boldsymbol{k},\boldsymbol{k}_1})\hat{K}_y(\boldsymbol{k}) + \delta_{\boldsymbol{k},\boldsymbol{k}_1}\Big[\cosh\{F_2(\boldsymbol{k},t)\}\hat{K}_y(\boldsymbol{k}) - \sinh\{F_2(\boldsymbol{k},t)\}\hat{K}_z(\boldsymbol{k})\Big]. \qquad \text{(G.11)}
\end{aligned}
$$

Here, we used

$$
[\underbrace{\hat{K}_x(\boldsymbol{k}_1),[\hat{K}_x(\boldsymbol{k}_1),\cdots,[\hat{K}_x(\boldsymbol{k}_1)}_{2n\text{ times}},\hat{K}_y(\boldsymbol{k}_2)]\cdots]] = \delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}(-i)^{2n}\hat{K}_y(\boldsymbol{k}_2), \qquad \text{(G.12)}
$$

and

$$
[\underbrace{\hat{K}_x(\boldsymbol{k}_1),[\hat{K}_x(\boldsymbol{k}_1),\cdots,[\hat{K}_x(\boldsymbol{k}_1)}_{2n+1\text{ times}},\hat{K}_y(\boldsymbol{k}_2)]\cdots]] = \delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}(-i)^{2n+1}\hat{K}_z(\boldsymbol{k}_2). \qquad \text{(G.13)}
$$

Since $\Big[\hat{K}_z(\boldsymbol{k}_1),\hat{K}_z(\boldsymbol{k}_2)\Big]=0$, $\Big[\hat{K}_x(\boldsymbol{k}_1),\hat{K}_x(\boldsymbol{k}_2)\Big]=0$, and $\Big[\hat{K}_y(\boldsymbol{k}_1),\hat{K}_y(\boldsymbol{k}_2)\Big]=0$, we have

$$
\begin{aligned}
&\exp\left\{-i\sum_{\boldsymbol{k}>0}F_1(\boldsymbol{k},t)\hat{K}_z(\boldsymbol{k})\right\} = \prod_{\boldsymbol{k}>0}e^{-iF_1(\boldsymbol{k},t)\hat{K}_z(\boldsymbol{k})}, \quad \exp\left\{-i\sum_{\boldsymbol{k}>0}F_2(\boldsymbol{k},t)\hat{K}_x(\boldsymbol{k})\right\} = \prod_{\boldsymbol{k}>0}e^{-iF_2(\boldsymbol{k},t)\hat{K}_x(\boldsymbol{k})}, \\
&\exp\left\{-i\sum_{\boldsymbol{k}>0}F_3(\boldsymbol{k},t)\hat{K}_y(\boldsymbol{k})\right\} = \prod_{\boldsymbol{k}>0}e^{-iF_3(\boldsymbol{k},t)\hat{K}_y(\boldsymbol{k})}, \qquad \text{(G.14)}
\end{aligned}
$$

where order of product of each $\boldsymbol{k}$ component does not matter.

Therefore, by choosing $\hat{G}_1(\boldsymbol{k}) = \hat{K}_z(\boldsymbol{k})$, $\hat{G}_2(\boldsymbol{k}) = \hat{K}_x(\boldsymbol{k})$, and $\hat{G}_3(\boldsymbol{k}) =$

$\hat{K}_y(\boldsymbol{k})$, from Eq.(A.3) in [69], we get

$$
\begin{aligned}
\frac{1}{\hbar}\hat{H}_c(\boldsymbol{k},t) &= \dot{F}_1(\boldsymbol{k}.t)\,\hat{K}_z(\boldsymbol{k}) + \dot{F}_2(\boldsymbol{k},t)\left[\cos\{F_1(\boldsymbol{k},t)\}\,\hat{K}_x(\boldsymbol{k}) + \sin\{F_1(\boldsymbol{k},t)\}\,\hat{K}_y(\boldsymbol{k})\right] \\
&\quad + \dot{F}_3(\boldsymbol{k},t)\,\hat{U}_1(\boldsymbol{k},t)\left\{\hat{U}_2(\boldsymbol{k},t)\,\hat{K}_y(\boldsymbol{k})\,\hat{U}_2^{\dagger}(\boldsymbol{k},t)\right\}\hat{U}_1^{\dagger}(\boldsymbol{k},t) \\
&= \dot{F}_1(\boldsymbol{k},t)\,\hat{K}_z(\boldsymbol{k}) + \dot{F}_2(\boldsymbol{k},t)\cos\{F_1(\boldsymbol{k},t)\}\,\hat{K}_x(\boldsymbol{k}) + \dot{F}_2(\boldsymbol{k},t)\sin\{F_1(\boldsymbol{k},t)\}\,\hat{K}_y(\boldsymbol{k}) \\
&\quad + \dot{F}_3(\boldsymbol{k},t)\cosh\{F_2(\boldsymbol{k},t)\}\left[-\sin\{F_1(\boldsymbol{k},t)\}\,\hat{K}_x(\boldsymbol{k}) + \cos\{F_1(\boldsymbol{k},t)\}\,\hat{K}_y(\boldsymbol{k})\right] \\
&\quad - \dot{F}_3(\boldsymbol{k},t)\sinh\{F_2(\boldsymbol{k},t)\}\,\hat{K}_z(\boldsymbol{k}) \\
&= \left[\dot{F}_1(\boldsymbol{k},t) - \dot{F}_3(\boldsymbol{k},t)\sinh\{F_2(\boldsymbol{k},t)\}\right]\hat{K}_z(\boldsymbol{k}) \\
&\quad + \left[\dot{F}_2(\boldsymbol{k},t)\cos\{F_1(\boldsymbol{k},t)\} - \dot{F}_3(\boldsymbol{k},t)\sin\{F_1(\boldsymbol{k},t)\}\cosh\{F_2(\boldsymbol{k},t)\}\right]\hat{K}_x(\boldsymbol{k}) \\
&\quad + \left[\dot{F}_2(\boldsymbol{k},t)\sin\{F_1(\boldsymbol{k},t)\} + \dot{F}_3(\boldsymbol{k},t)\cos\{F_1(\boldsymbol{k},t)\}\cosh\{F_2(\boldsymbol{k},t)\}\right]\hat{K}_y(\boldsymbol{k}) \\
&= \left[\dot{F}_1(\boldsymbol{k},t) - \dot{F}_3(\boldsymbol{k},t)\sinh\{F_2(\boldsymbol{k},t)\}\right]\hat{K}_z(\boldsymbol{k}) \\
&\quad + \frac{1}{2}\left[\dot{F}_2(\boldsymbol{k},t)\cos\{F_1(\boldsymbol{k},t)\} - \dot{F}_3(\boldsymbol{k},t)\sin\{F_1(\boldsymbol{k},t)\}\cosh\{F_2(\boldsymbol{k},t)\}\right]\left\{\hat{K}_+(\boldsymbol{k}) + \hat{K}_-(\boldsymbol{k})\right\} \\
&\quad - \frac{i}{2}\left[\dot{F}_2(\boldsymbol{k},t)\sin\{F_1(\boldsymbol{k},t)\} + \dot{F}_3(\boldsymbol{k},t)\cos\{F_1(\boldsymbol{k},t)\}\cosh\{F_2(\boldsymbol{k},t)\}\right]\left\{\hat{K}_+(\boldsymbol{k}) - \hat{K}_-(\boldsymbol{k})\right\} \\
&= \left[\dot{F}_1(\boldsymbol{k},t) - \dot{F}_3(\boldsymbol{k},t)\sinh\{F_2(\boldsymbol{k},t)\}\right]\hat{K}_z(\boldsymbol{k}) \\
&\quad + \frac{1}{2}\left[\dot{F}_2(\boldsymbol{k},t)\exp\{-iF_1(\boldsymbol{k},t)\} - i\dot{F}_3(\boldsymbol{k},t)\exp\{-iF_1(\boldsymbol{k},t)\}\cosh\{F_2(\boldsymbol{k},t)\}\right]\hat{K}_+(\boldsymbol{k}) \\
&\quad + \frac{1}{2}\left[\dot{F}_2(\boldsymbol{k},t)\exp\{iF_1(\boldsymbol{k},t)\} + i\dot{F}_3(\boldsymbol{k},t)\exp\{iF_1(\boldsymbol{k},t)\}\cosh\{F_2(\boldsymbol{k},t)\}\right]\hat{K}_-(\boldsymbol{k}) . \qquad \text{(G.15)}
\end{aligned}
$$

Since we have

$$
\begin{aligned}
\hat{H}_c(\boldsymbol{k},t) &= \left[\hbar\omega_1(k) + V_a f(t)\left\{u_{11}^2(k) + v_{11}^2(k)\right\}\right]\left\{\hat{b}_1^{\dagger}(\boldsymbol{k})\,\hat{b}_1(\boldsymbol{k}) + \hat{b}_1^{\dagger}(-\boldsymbol{k})\,\hat{b}_1(-\boldsymbol{k})\right\} \\
&\quad + 2V_a f(t)\,u_{11}(k)\,v_{11}(k)\left\{\hat{b}_1^{\dagger}(\boldsymbol{k})\,\hat{b}_1^{\dagger}(-\boldsymbol{k}) + \hat{b}_1(\boldsymbol{k})\,\hat{b}_1(-\boldsymbol{k})\right\} \\
&= 2\left[\hbar\omega_1(k) + V_a f(t)\left\{u_{11}^2(k) + v_{11}^2(k)\right\}\right]\hat{K}_z(\boldsymbol{k}) + 2V_a f(t)\,u_{11}(k)\,v_{11}(k)\left\{\hat{K}_+(\boldsymbol{k}) + \hat{K}_-(\boldsymbol{k})\right\}, \qquad \text{(G.16)}
\end{aligned}
$$

we have

$$
\begin{aligned}
&\dot{F}_1(\boldsymbol{k},t) - \dot{F}_3(\boldsymbol{k},t)\sinh\{F_2(\boldsymbol{k},t)\} = g_1(k,t)\,, \\
&\dot{F}_2(\boldsymbol{k},t)\exp\{-iF_1(\boldsymbol{k},t)\} - i\dot{F}_3(\boldsymbol{k},t)\exp\{-iF_1(\boldsymbol{k},t)\}\cosh\{F_2(\boldsymbol{k},t)\} = 2g_2(k,t)\,, \\
&\dot{F}_2(\boldsymbol{k},t)\exp\{iF_1(\boldsymbol{k},t)\} + i\dot{F}_3(\boldsymbol{k},t)\exp\{iF_1(\boldsymbol{k},t)\}\cosh\{F_2(\boldsymbol{k},t)\} = 2g_2(k,t)\,, \qquad \text{(G.17)}
\end{aligned}
$$

where

$$g_1(k,t) \coloneqq 2\left[\omega_1(k) + (V_a/\hbar) f(t)\left\{u_{11}^2(k) + v_{11}^2(k)\right\}\right], \quad g_2(k,t) \coloneqq 2(V_a/\hbar) f(t)\, u_{11}(k)\, v_{11}(k). \tag{G.18}$$

By introducing dimensionless time $\tilde{t} \coloneqq g_a n t/\hbar$, dimensionless values $\tilde{\omega}_1(k) \coloneqq \hbar\omega_1(k)/(g_a n)$, and $\bar{V}_a \coloneqq V_a/(g_a n)$, we get

$$\begin{aligned}
&\frac{\partial F_1(\boldsymbol{k},\tilde{t})}{\partial \tilde{t}} - \frac{\partial F_3(\boldsymbol{k},\tilde{t})}{\partial \tilde{t}} \sinh\left\{F_2(\boldsymbol{k},\tilde{t})\right\} = \tilde{g}_1(k,\tilde{t}),\\
&\frac{\partial F_2(\boldsymbol{k},\tilde{t})}{\partial \tilde{t}} \cos\left\{F_1(\boldsymbol{k},\tilde{t})\right\} - \frac{\partial F_3(\boldsymbol{k},\tilde{t})}{\partial \tilde{t}} \sin\left\{F_1(\boldsymbol{k},\tilde{t})\right\}\cosh\left\{F_2(\boldsymbol{k},\tilde{t})\right\} = 2\tilde{g}_2(k,\tilde{t}),\\
&\frac{\partial F_2(\boldsymbol{k},\tilde{t})}{\partial \tilde{t}} \sin\left\{F_1(\boldsymbol{k},\tilde{t})\right\} + \frac{\partial F_3(\boldsymbol{k},\tilde{t})}{\partial \tilde{t}} \cos\left\{F_1(\boldsymbol{k},\tilde{t})\right\}\cosh\left\{F_2(\boldsymbol{k},\tilde{t})\right\} = 0,
\end{aligned} \tag{G.19}$$

$$\begin{aligned}
\frac{\partial \tilde{g}_1(k,\tilde{t})}{\partial \bar{V}_a} &= \left\{\frac{\partial}{\partial \tilde{t}}\frac{\partial F_1(\boldsymbol{k},\tilde{t})}{\partial \bar{V}_a}\right\} - \left\{\frac{\partial}{\partial \tilde{t}}\frac{\partial F_3(\boldsymbol{k},\tilde{t})}{\partial \bar{V}_a}\right\}\sinh\left\{F_2(\boldsymbol{k},\tilde{t})\right\} - \frac{\partial F_3(\boldsymbol{k},\tilde{t})}{\partial \tilde{t}}\frac{\partial F_2(\boldsymbol{k},\tilde{t})}{\partial \bar{V}_a}\cosh\left\{F_2(\boldsymbol{k},\tilde{t})\right\},\\
2\frac{\partial \tilde{g}_2(k,\tilde{t})}{\partial \bar{V}_a} &= \left\{\frac{\partial}{\partial \tilde{t}}\frac{\partial F_2(\boldsymbol{k},\tilde{t})}{\partial \bar{V}_a}\right\}\cos\left\{F_1(\boldsymbol{k},\tilde{t})\right\} - \frac{\partial F_2(\boldsymbol{k},\tilde{t})}{\partial \tilde{t}}\frac{\partial F_1(\boldsymbol{k},\tilde{t})}{\partial \bar{V}_a}\sin\left\{F_1(\boldsymbol{k},\tilde{t})\right\}\\
&\quad - \left[\left\{\frac{\partial}{\partial \tilde{t}}\frac{\partial F_3(\boldsymbol{k},\tilde{t})}{\partial \bar{V}_a}\right\}\sin\left\{F_1(\boldsymbol{k},\tilde{t})\right\} + \frac{\partial F_3(\boldsymbol{k},\tilde{t})}{\partial \tilde{t}}\frac{\partial F_1(\boldsymbol{k},\tilde{t})}{\partial \bar{V}_a}\cos\left\{F_1(\boldsymbol{k},\tilde{t})\right\}\right]\cosh\left\{F_2(\boldsymbol{k},\tilde{t})\right\}\\
&\quad - \frac{\partial F_3(\boldsymbol{k},\tilde{t})}{\partial \tilde{t}}\frac{\partial F_2(\boldsymbol{k},\tilde{t})}{\partial \bar{V}_a}\sin\left\{F_1(\boldsymbol{k},\tilde{t})\right\}\sinh\left\{F_2(\boldsymbol{k},\tilde{t})\right\},\\
0 &= \left\{\frac{\partial}{\partial \tilde{t}}\frac{\partial F_2(\boldsymbol{k},\tilde{t})}{\partial \bar{V}_a}\right\}\sin\left\{F_1(\boldsymbol{k},\tilde{t})\right\} + \frac{\partial F_2(\boldsymbol{k},\tilde{t})}{\partial \tilde{t}}\frac{\partial F_1(\boldsymbol{k},\tilde{t})}{\partial \bar{V}_a}\cos\left\{F_1(\boldsymbol{k},\tilde{t})\right\}\\
&\quad + \left[\left\{\frac{\partial}{\partial \tilde{t}}\frac{\partial F_3(\boldsymbol{k},\tilde{t})}{\partial \bar{V}_a}\right\}\cos\left\{F_1(\boldsymbol{k},\tilde{t})\right\} - \frac{\partial F_3(\boldsymbol{k},\tilde{t})}{\partial \tilde{t}}\frac{\partial F_1(\boldsymbol{k},\tilde{t})}{\partial \bar{V}_a}\sin\left\{F_1(\boldsymbol{k},\tilde{t})\right\}\right]\cosh\left\{F_2(\boldsymbol{k},\tilde{t})\right\}\\
&\quad + \frac{\partial F_3(\boldsymbol{k},\tilde{t})}{\partial \tilde{t}}\frac{\partial F_2(\boldsymbol{k},\tilde{t})}{\partial \bar{V}_a}\cos\left\{F_1(\boldsymbol{k},\tilde{t})\right\}\sinh\left\{F_2(\boldsymbol{k},\tilde{t})\right\},
\end{aligned} \tag{G.20}$$

where $\tilde{g}_j(k,\tilde{t}) \coloneqq \hbar g_j(k,\tilde{t})/(g_a n)$ are dimensionless ($j = 1, 2$).

From Eqs. (G.19), we can get

$$\begin{aligned}
&\frac{\partial F_1(\boldsymbol{k},\tilde{t})}{\partial \tilde{t}} = \tilde{g}_1(k,\tilde{t}) - 2\tilde{g}_2(k,\tilde{t})\sin\left\{F_1(\boldsymbol{k},\tilde{t})\right\}\tanh\left\{F_2(\boldsymbol{k},\tilde{t})\right\}, \frac{\partial F_2(\boldsymbol{k},\tilde{t})}{\partial \tilde{t}} = 2\tilde{g}_2(k,\tilde{t})\cos\left\{F_1(\boldsymbol{k},\tilde{t})\right\},\\
&\frac{\partial F_3(\boldsymbol{k},\tilde{t})}{\partial \tilde{t}} = -2\tilde{g}_2(k,\tilde{t})\sin\left\{F_1(\boldsymbol{k},\tilde{t})\right\}/\cosh\left\{F_2(\boldsymbol{k},\tilde{t})\right\},
\end{aligned} \tag{G.21}$$

and thus Eqs. (G.20) can be written as

$$\frac{\partial \tilde{g}_1\left(k,\tilde{t}\right)}{\partial \bar{V}_a} = \left\{\frac{\partial}{\partial \tilde{t}}\frac{\partial F_1\left(\boldsymbol{k},\tilde{t}\right)}{\partial \bar{V}_a}\right\} - \left\{\frac{\partial}{\partial \tilde{t}}\frac{\partial F_3\left(\boldsymbol{k},\tilde{t}\right)}{\partial \bar{V}_a}\right\}\sinh\left\{F_2\left(\boldsymbol{k},\tilde{t}\right)\right\} + 2\tilde{g}_2\left(k,\tilde{t}\right)\sin\left\{F_1\left(\boldsymbol{k},\tilde{t}\right)\right\}\frac{\partial F_2\left(\boldsymbol{k},\tilde{t}\right)}{\partial \bar{V}_a},$$

$$2\frac{\partial \tilde{g}_2\left(k,\tilde{t}\right)}{\partial \bar{V}_a} = \left\{\frac{\partial}{\partial \tilde{t}}\frac{\partial F_2\left(\boldsymbol{k},\tilde{t}\right)}{\partial \bar{V}_a}\right\}\cos\left\{F_1\left(\boldsymbol{k},\tilde{t}\right)\right\} - \left\{\frac{\partial}{\partial \tilde{t}}\frac{\partial F_3\left(\boldsymbol{k},\tilde{t}\right)}{\partial \bar{V}_a}\right\}\sin\left\{F_1\left(\boldsymbol{k},\tilde{t}\right)\right\}\cosh\left\{F_2\left(\boldsymbol{k},\tilde{t}\right)\right\}$$
$$+2\tilde{g}_2\left(k,\tilde{t}\right)\sin^2\left\{F_1\left(\boldsymbol{k},\tilde{t}\right)\right\}\tanh\left\{F_2\left(\boldsymbol{k},\tilde{t}\right)\right\}\frac{\partial F_2\left(\boldsymbol{k},\tilde{t}\right)}{\partial \bar{V}_a},$$

$$0 = \left\{\frac{\partial}{\partial \tilde{t}}\frac{\partial F_2\left(\boldsymbol{k},\tilde{t}\right)}{\partial \bar{V}_a}\right\}\sin\left\{F_1\left(\boldsymbol{k},\tilde{t}\right)\right\} + \left\{\frac{\partial}{\partial \tilde{t}}\frac{\partial F_3\left(\boldsymbol{k},\tilde{t}\right)}{\partial \bar{V}_a}\right\}\cos\left\{F_1\left(\boldsymbol{k},\tilde{t}\right)\right\}\cosh\left\{F_2\left(\boldsymbol{k},\tilde{t}\right)\right\}$$
$$+2\tilde{g}_2\left(k,\tilde{t}\right)\frac{\partial F_1\left(\boldsymbol{k},\tilde{t}\right)}{\partial \bar{V}_a} - 2\tilde{g}_2\left(k,\tilde{t}\right)\sin\left\{F_1\left(\boldsymbol{k},\tilde{t}\right)\right\}\cos\left\{F_1\left(\boldsymbol{k},\tilde{t}\right)\right\}\tanh\left\{F_2\left(\boldsymbol{k},\tilde{t}\right)\right\}\frac{\partial F_2\left(\boldsymbol{k},\tilde{t}\right)}{\partial \bar{V}_a}, \tag{G.22}$$

which gives

$$\left\{\frac{\partial}{\partial \tilde{t}}\frac{\partial F_1\left(\boldsymbol{k},\tilde{t}\right)}{\partial \bar{V}_a}\right\} = \frac{\partial \tilde{g}_1\left(k,\tilde{t}\right)}{\partial \bar{V}_a} - 2\frac{\partial \tilde{g}_2\left(k,\tilde{t}\right)}{\partial \bar{V}_a}\sin\left\{F_1\left(\boldsymbol{k},\tilde{t}\right)\right\}\tanh\left\{F_2\left(\boldsymbol{k},\tilde{t}\right)\right\}$$
$$-2\tilde{g}_2\left(k,\tilde{t}\right)\frac{\partial F_1\left(\boldsymbol{k},\tilde{t}\right)}{\partial \bar{V}_a}\cos\left\{F_1\left(\boldsymbol{k},\tilde{t}\right)\right\}\tanh\left\{F_2\left(\boldsymbol{k},\tilde{t}\right)\right\}$$
$$-2\tilde{g}_2\left(k,\tilde{t}\right)\frac{\sin\left\{F_1\left(\boldsymbol{k},\tilde{t}\right)\right\}}{\cosh^2\left\{F_2\left(\boldsymbol{k},\tilde{t}\right)\right\}}\frac{\partial F_2\left(\boldsymbol{k},\tilde{t}\right)}{\partial \bar{V}_a},$$

$$\left\{\frac{\partial}{\partial \tilde{t}}\frac{\partial F_2\left(\boldsymbol{k},\tilde{t}\right)}{\partial \bar{V}_a}\right\} = 2\frac{\partial \tilde{g}_2\left(k,\tilde{t}\right)}{\partial \bar{V}_a}\cos\left\{F_1\left(\boldsymbol{k},\tilde{t}\right)\right\} - 2\tilde{g}_2\left(k,\tilde{t}\right)\frac{\partial F_1\left(\boldsymbol{k},\tilde{t}\right)}{\partial \bar{V}_a}\sin\left\{F_1\left(\boldsymbol{k},\tilde{t}\right)\right\},$$

$$\left\{\frac{\partial}{\partial \tilde{t}}\frac{\partial F_3\left(\boldsymbol{k},\tilde{t}\right)}{\partial \bar{V}_a}\right\} = -2\frac{\partial \tilde{g}_2\left(k,\tilde{t}\right)}{\partial \bar{V}_a}\frac{\sin\left\{F_1\left(\boldsymbol{k},\tilde{t}\right)\right\}}{\cosh\left\{F_2\left(\boldsymbol{k},\tilde{t}\right)\right\}} - 2\tilde{g}_2\left(k,\tilde{t}\right)\frac{\partial F_1\left(\boldsymbol{k},\tilde{t}\right)}{\partial \bar{V}_a}\frac{\cos\left\{F_1\left(\boldsymbol{k},\tilde{t}\right)\right\}}{\cosh\left\{F_2\left(\boldsymbol{k},\tilde{t}\right)\right\}}$$
$$+2\tilde{g}_2\left(k,\tilde{t}\right)\sin\left\{F_1\left(\boldsymbol{k},\tilde{t}\right)\right\}\frac{\tanh\left\{F_2\left(\boldsymbol{k},\tilde{t}\right)\right\}}{\cosh\left\{F_2\left(\boldsymbol{k},\tilde{t}\right)\right\}}\frac{\partial F_2\left(\boldsymbol{k},\tilde{t}\right)}{\partial \bar{V}_a}. \tag{G.23}$$

Solving coupled differential Eqs. (G.21) and Eqs. (G.23) gives time evolution operator via Eqs. (G.1).

G.2 Quantum Fisher Information

For positive $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$, we have

$$\begin{aligned}
\left[\hat{K}_z(\boldsymbol{k}_1), \hat{b}_1(\pm\boldsymbol{k}_2)\right] &= \frac{1}{2}\left[\hat{b}_1^\dagger(\boldsymbol{k}_1)\hat{b}_1(\boldsymbol{k}_1) + \hat{b}_1(-\boldsymbol{k}_1)\hat{b}_1^\dagger(-\boldsymbol{k}_1), \hat{b}_1(\pm\boldsymbol{k}_2)\right] = -\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\frac{1}{2}\hat{b}_1(\pm\boldsymbol{k}_2), \\
\left[\hat{K}_x(\boldsymbol{k}_1), \hat{b}_1(\pm\boldsymbol{k}_2)\right] &= \frac{1}{2}\left[\hat{b}_1^\dagger(\boldsymbol{k}_1)\hat{b}_1^\dagger(-\boldsymbol{k}_1) + \hat{b}_1(\boldsymbol{k}_1)\hat{b}_1(-\boldsymbol{k}_1), \hat{b}_1(\pm\boldsymbol{k}_2)\right] = -\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\frac{1}{2}\hat{b}_1^\dagger(\mp\boldsymbol{k}_2), \\
\left[\hat{K}_y(\boldsymbol{k}_1), \hat{b}_1(\pm\boldsymbol{k}_2)\right] &= -\frac{i}{2}\left[\hat{b}_1^\dagger(\boldsymbol{k}_1)\hat{b}_1^\dagger(-\boldsymbol{k}_1) - \hat{b}_1(\boldsymbol{k}_1)\hat{b}_1(-\boldsymbol{k}_1), \hat{b}_1(\pm\boldsymbol{k}_2)\right] = \delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\frac{i}{2}\hat{b}_1^\dagger(\mp\boldsymbol{k}_2).
\end{aligned} \tag{G.24}$$

Let $\hat{q}(\pm\boldsymbol{k}) := \frac{1}{\sqrt{2}}\left\{\hat{b}_1^\dagger(\pm\boldsymbol{k}) + \hat{b}_1(\pm\boldsymbol{k})\right\}$ and $\hat{p}(\pm\boldsymbol{k}) := \frac{i}{\sqrt{2}}\left\{\hat{b}_1^\dagger(\pm\boldsymbol{k}) - \hat{b}_1(\pm\boldsymbol{k})\right\}$.
Then we have $[\hat{q}(\boldsymbol{k}_1), \hat{p}(\boldsymbol{k}_2)] = i\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}$,

$$\begin{aligned}
\left[\hat{K}_z(\boldsymbol{k}_1), \hat{q}(\pm\boldsymbol{k}_2)\right] &= \delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\frac{1}{2\sqrt{2}}\left\{\hat{b}_1^\dagger(\pm\boldsymbol{k}_2) - \hat{b}_1(\pm\boldsymbol{k}_2)\right\} = -\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\frac{i}{2}\hat{p}(\pm\boldsymbol{k}_2), \\
\left[\hat{K}_z(\boldsymbol{k}_1), \hat{p}(\pm\boldsymbol{k}_2)\right] &= \delta_{\boldsymbol{k}_1,\boldsymbol{k}_2} i\frac{1}{2\sqrt{2}}\left\{\hat{b}_1^\dagger(\pm\boldsymbol{k}_2) + \hat{b}_1(\pm\boldsymbol{k}_2)\right\} = \delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\frac{i}{2}\hat{q}(\pm\boldsymbol{k}_2), \\
\left[\hat{K}_x(\boldsymbol{k}_1), \hat{q}(\pm\boldsymbol{k}_2)\right] &= \delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\frac{1}{2\sqrt{2}}\left\{\hat{b}_1(\mp\boldsymbol{k}_2) - \hat{b}_1^\dagger(\mp\boldsymbol{k}_2)\right\} = \delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\frac{i}{2}\hat{p}(\mp\boldsymbol{k}_2), \\
\left[\hat{K}_x(\boldsymbol{k}_1), \hat{p}(\pm\boldsymbol{k}_2)\right] &= \delta_{\boldsymbol{k}_1,\boldsymbol{k}_2} i\frac{1}{2\sqrt{2}}\left\{\hat{b}_1(\mp\boldsymbol{k}_2) + \hat{b}_1^\dagger(\mp\boldsymbol{k}_2)\right\} = \delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\frac{i}{2}\hat{q}(\mp\boldsymbol{k}_2), \\
\left[\hat{K}_y(\boldsymbol{k}_1), \hat{q}(\pm\boldsymbol{k}_2)\right] &= \delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\frac{i}{2\sqrt{2}}\left\{\hat{b}_1(\mp\boldsymbol{k}_2) + \hat{b}_1^\dagger(\mp\boldsymbol{k}_2)\right\} = \delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\frac{i}{2}\hat{q}(\mp\boldsymbol{k}_2), \\
\left[\hat{K}_y(\boldsymbol{k}_1), \hat{p}(\pm\boldsymbol{k}_2)\right] &= -\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\frac{1}{2\sqrt{2}}\left\{\hat{b}_1(\mp\boldsymbol{k}_2) - \hat{b}_1^\dagger(\mp\boldsymbol{k}_2)\right\} = -\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\frac{i}{2}\hat{p}(\mp\boldsymbol{k}_2).
\end{aligned} \tag{G.25}$$

So we get

$$\begin{aligned}
&e^{iF_1(\boldsymbol{k}_1,t)\hat{K}_z(\boldsymbol{k}_1)}\hat{q}(\pm\boldsymbol{k}_2)e^{-iF_1(\boldsymbol{k}_1,t)\hat{K}_z(\boldsymbol{k}_1)} \\
&\quad= \hat{q}(\pm\boldsymbol{k}_2) + \delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\left[\sum_{n=1}^{\infty}\frac{i^{2n}}{(2n)!}\left\{\frac{1}{2}F_1(\boldsymbol{k}_1,t)\right\}^{2n}\hat{q}(\pm\boldsymbol{k}_2) - i\sum_{n=0}^{\infty}\frac{i^{2n+1}}{(2n+1)!}\left\{\frac{1}{2}F_1(\boldsymbol{k}_1,t)\right\}^{2n+1}\hat{p}(\pm\boldsymbol{k}_2)\right] \\
&\quad= (1-\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2})\hat{q}(\pm\boldsymbol{k}_2) + \delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\left[\cos\left\{\frac{1}{2}F_1(\boldsymbol{k}_2,t)\right\}\hat{q}(\pm\boldsymbol{k}_2) + \sin\left\{\frac{1}{2}F_1(\boldsymbol{k}_2,t)\right\}\hat{p}(\pm\boldsymbol{k}_2)\right],
\end{aligned} \tag{G.26}$$

$$
\begin{aligned}
&e^{iF_1(\boldsymbol{k}_1,t)\hat{K}_z(\boldsymbol{k}_1)}\hat{p}\left(\pm\boldsymbol{k}_2\right)e^{-iF_1(\boldsymbol{k}_1,t)\hat{K}_z(\boldsymbol{k}_1)} \\
&\quad = \hat{p}\left(\pm\boldsymbol{k}_2\right) + \delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\left[\sum_{n=1}^{\infty}\frac{i^{2n}}{(2n)!}\left\{\frac{1}{2}F_1\left(\boldsymbol{k}_1,t\right)\right\}^{2n}\hat{p}\left(\pm\boldsymbol{k}_2\right) + i\sum_{n=0}^{\infty}\frac{i^{2n+1}}{(2n+1)!}\left\{\frac{1}{2}F_1\left(\boldsymbol{k}_1,t\right)\right\}^{2n+1}\hat{q}\left(\pm\boldsymbol{k}_2\right)\right] \\
&\quad = \left(1-\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\right)\hat{p}\left(\pm\boldsymbol{k}_2\right) + \delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\left[-\sin\left\{\frac{1}{2}F_1\left(\boldsymbol{k}_2,t\right)\right\}\hat{q}\left(\pm\boldsymbol{k}_2\right) + \cos\left\{\frac{1}{2}F_1\left(\boldsymbol{k}_2,t\right)\right\}\hat{p}\left(\pm\boldsymbol{k}_2\right)\right],
\end{aligned}
\tag{G.27}
$$

$$
\begin{aligned}
&e^{iF_2(\boldsymbol{k}_1,t)\hat{K}_x(\boldsymbol{k}_1)}\hat{q}\left(\pm\boldsymbol{k}_2\right)e^{-iF_2(\boldsymbol{k}_1,t)\hat{K}_x(\boldsymbol{k}_1)} \\
&\quad = \hat{q}\left(\pm\boldsymbol{k}_2\right) + \delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\left[\begin{array}{l}\displaystyle\sum_{n=1}^{\infty}\frac{i^{2n}}{(2n)!}\left(-1\right)^n\left\{\frac{1}{2}F_2\left(\boldsymbol{k}_1,t\right)\right\}^{2n}\hat{q}\left(\pm\boldsymbol{k}_2\right) \\ \displaystyle +i\sum_{n=0}^{\infty}\frac{i^{2n+1}}{(2n+1)!}\left(-1\right)^n\left\{\frac{1}{2}F_2\left(\boldsymbol{k}_1,t\right)\right\}^{2n+1}\hat{p}\left(\mp\boldsymbol{k}_2\right)\end{array}\right] \\
&\quad = \left(1-\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\right)\hat{q}\left(\pm\boldsymbol{k}_2\right) + \delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\left[\cosh\left\{\frac{1}{2}F_2\left(\boldsymbol{k}_2,t\right)\right\}\hat{q}\left(\pm\boldsymbol{k}_2\right) - \sinh\left\{\frac{1}{2}F_2\left(\boldsymbol{k}_2,t\right)\right\}\hat{p}\left(\mp\boldsymbol{k}_2\right)\right],
\end{aligned}
\tag{G.28}
$$

$$
\begin{aligned}
&e^{iF_2(\boldsymbol{k}_1,t)\hat{K}_x(\boldsymbol{k}_1)}\hat{p}\left(\pm\boldsymbol{k}_2\right)e^{-iF_2(\boldsymbol{k}_1,t)\hat{K}_x(\boldsymbol{k}_1)} \\
&\quad = \hat{p}\left(\pm\boldsymbol{k}_2\right) + \delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\left[\begin{array}{l}\displaystyle\sum_{n=1}^{\infty}\frac{i^{2n}}{(2n)!}\left(-1\right)^n\left\{\frac{1}{2}F_2\left(\boldsymbol{k}_1,t\right)\right\}^{2n}\hat{p}\left(\pm\boldsymbol{k}_2\right) \\ \displaystyle +i\sum_{n=0}^{\infty}\frac{i^{2n+1}}{(2n+1)!}\left(-1\right)^n\left\{\frac{1}{2}F_2\left(\boldsymbol{k}_1,t\right)\right\}^{2n+1}\hat{q}\left(\mp\boldsymbol{k}_2\right)\end{array}\right] \\
&\quad = \left(1-\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\right)\hat{p}\left(\pm\boldsymbol{k}_2\right) + \delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\left[-\sinh\left\{\frac{1}{2}F_2\left(\boldsymbol{k}_2,t\right)\right\}\hat{q}\left(\mp\boldsymbol{k}_2\right) + \cosh\left\{\frac{1}{2}F_2\left(\boldsymbol{k}_2,t\right)\right\}\hat{p}\left(\pm\boldsymbol{k}_2\right)\right],
\end{aligned}
\tag{G.29}
$$

$$
\begin{aligned}
&e^{iF_3(\boldsymbol{k}_1,t)\hat{K}_y(\boldsymbol{k}_1)}\hat{q}\left(\pm\boldsymbol{k}_2\right)e^{-iF_3(\boldsymbol{k}_1,t)\hat{K}_y(\boldsymbol{k}_1)} \\
&\quad = \hat{q}\left(\pm\boldsymbol{k}_2\right) + \delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\left[\begin{array}{l}\displaystyle\sum_{n=1}^{\infty}\frac{i^{2n}}{(2n)!}\left(-1\right)^n\left\{\frac{1}{2}F_3\left(\boldsymbol{k}_1,t\right)\right\}^{2n}\hat{q}\left(\pm\boldsymbol{k}_2\right) \\ \displaystyle +i\sum_{n=0}^{\infty}\frac{i^{2n+1}}{(2n+1)!}\left(-1\right)^n\left\{\frac{1}{2}F_3\left(\boldsymbol{k}_1,t\right)\right\}^{2n+1}\hat{q}\left(\mp\boldsymbol{k}_2\right)\end{array}\right] \\
&\quad = \left(1-\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\right)\hat{q}\left(\pm\boldsymbol{k}_2\right) + \delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\left[\cosh\left\{\frac{1}{2}F_3\left(\boldsymbol{k}_2,t\right)\right\}\hat{q}\left(\pm\boldsymbol{k}_2\right) - \sinh\left\{\frac{1}{2}F_3\left(\boldsymbol{k}_2,t\right)\right\}\hat{q}\left(\mp\boldsymbol{k}_2\right)\right],
\end{aligned}
\tag{G.30}
$$

and

$$
\begin{aligned}
&e^{iF_3(\boldsymbol{k}_1,t)\hat{K}_y(\boldsymbol{k}_1)}\hat{p}\left(\pm\boldsymbol{k}_2\right)e^{-iF_3(\boldsymbol{k}_1,t)\hat{K}_y(\boldsymbol{k}_1)}\\
&=\hat{p}\left(\pm\boldsymbol{k}_2\right)+\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\left[\begin{array}{l}\displaystyle\sum_{n=1}^{\infty}\frac{i^{2n}}{(2n)!}\left(-1\right)^n\left\{\frac{1}{2}F_3\left(\boldsymbol{k}_1,t\right)\right\}^{2n}\hat{p}\left(\pm\boldsymbol{k}_2\right)\\ \displaystyle -i\sum_{n=0}^{\infty}\frac{i^{2n+1}}{(2n+1)!}\left(-1\right)^n\left\{\frac{1}{2}F_3\left(\boldsymbol{k}_1,t\right)\right\}^{2n+1}\hat{p}\left(\mp\boldsymbol{k}_2\right)\end{array}\right]\\
&=\left(1-\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\right)\hat{p}\left(\pm\boldsymbol{k}_2\right)+\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\left[\cosh\left\{\frac{1}{2}F_3\left(\boldsymbol{k}_2,t\right)\right\}\hat{p}\left(\pm\boldsymbol{k}_2\right)+\sinh\left\{\frac{1}{2}F_3\left(\boldsymbol{k}_2,t\right)\right\}\hat{p}\left(\mp\boldsymbol{k}_2\right)\right].
\end{aligned}
\tag{G.31}
$$

From above equations (Eqs. (G.26) to Eqs. (G.31)),

$$
\begin{aligned}
&e^{iF_3(\boldsymbol{k}_1,t)\hat{K}_y(\boldsymbol{k}_1)}e^{iF_2(\boldsymbol{k}_1,t)\hat{K}_x(\boldsymbol{k}_1)}e^{iF_1(\boldsymbol{k}_1,t)\hat{K}_z(\boldsymbol{k}_1)}\hat{q}\left(\pm\boldsymbol{k}_2\right)e^{-iF_1(\boldsymbol{k}_1,t)\hat{K}_z(\boldsymbol{k}_1)}e^{-iF_2(\boldsymbol{k}_1,t)\hat{K}_x(\boldsymbol{k}_1)}e^{-iF_3(\boldsymbol{k}_1,t)\hat{K}_y(\boldsymbol{k}_1)}\\
&=\left(1-\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\right)\hat{q}\left(\pm\boldsymbol{k}_2\right)\\
&\quad+\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\cos\left\{F_1\left(\boldsymbol{k}_2,t\right)/2\right\}e^{iF_3(\boldsymbol{k}_1,t)\hat{K}_y(\boldsymbol{k}_1)}\\
&\qquad\times\left[\cosh\left\{F_2\left(\boldsymbol{k}_2,t\right)/2\right\}\hat{q}\left(\pm\boldsymbol{k}_2\right)-\sinh\left\{F_2\left(\boldsymbol{k}_2,t\right)/2\right\}\hat{p}\left(\mp\boldsymbol{k}_2\right)\right]e^{-iF_3(\boldsymbol{k}_1,t)\hat{K}_y(\boldsymbol{k}_1)}\\
&\quad+\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\sin\left\{F_1\left(\boldsymbol{k}_2,t\right)/2\right\}e^{iF_3(\boldsymbol{k}_1,t)\hat{K}_y(\boldsymbol{k}_1)}\\
&\qquad\times\left[-\sinh\left\{F_2\left(\boldsymbol{k}_2,t\right)/2\right\}\hat{q}\left(\mp\boldsymbol{k}_2\right)+\cosh\left\{F_2\left(\boldsymbol{k}_2,t\right)/2\right\}\hat{p}\left(\pm\boldsymbol{k}_2\right)\right]e^{-iF_3(\boldsymbol{k}_1,t)\hat{K}_y(\boldsymbol{k}_1)}\\
&=\left(1-\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\right)\hat{q}\left(\pm\boldsymbol{k}_2\right)\\
&\quad+\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\left[\begin{array}{l}\cos\left\{F_1\left(\boldsymbol{k}_2,t\right)/2\right\}\cosh\left\{F_2\left(\boldsymbol{k}_2,t\right)/2\right\}\cosh\left\{F_3\left(\boldsymbol{k}_2,t\right)/2\right\}\\ +\sin\left\{F_1\left(\boldsymbol{k}_2,t\right)/2\right\}\sinh\left\{F_2\left(\boldsymbol{k}_2,t\right)/2\right\}\sinh\left\{F_3\left(\boldsymbol{k}_2,t\right)/2\right\}\end{array}\right]\hat{q}\left(\pm\boldsymbol{k}_2\right)\\
&\quad+\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\left[\begin{array}{l}\sin\left\{F_1\left(\boldsymbol{k}_2,t\right)/2\right\}\cosh\left\{F_2\left(\boldsymbol{k}_2,t\right)/2\right\}\cosh\left\{F_3\left(\boldsymbol{k}_2,t\right)/2\right\}\\ -\cos\left\{F_1\left(\boldsymbol{k}_2,t\right)/2\right\}\sinh\left\{F_2\left(\boldsymbol{k}_2,t\right)/2\right\}\sinh\left\{F_3\left(\boldsymbol{k}_2,t\right)/2\right\}\end{array}\right]\hat{p}\left(\pm\boldsymbol{k}_2\right)\\
&\quad-\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\left[\begin{array}{l}\cos\left\{F_1\left(\boldsymbol{k}_2,t\right)/2\right\}\cosh\left\{F_2\left(\boldsymbol{k}_2,t\right)/2\right\}\sinh\left\{F_3\left(\boldsymbol{k}_2,t\right)/2\right\}\\ +\sin\left\{F_1\left(\boldsymbol{k}_2,t\right)/2\right\}\sinh\left\{F_2\left(\boldsymbol{k}_2,t\right)/2\right\}\cosh\left\{F_3\left(\boldsymbol{k}_2,t\right)/2\right\}\end{array}\right]\hat{q}\left(\mp\boldsymbol{k}_2\right)\\
&\quad+\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\left[\begin{array}{l}\sin\left\{F_1\left(\boldsymbol{k}_2,t\right)/2\right\}\cosh\left\{F_2\left(\boldsymbol{k}_2,t\right)/2\right\}\sinh\left\{F_3\left(\boldsymbol{k}_2,t\right)/2\right\}\\ -\cos\left\{F_1\left(\boldsymbol{k}_2,t\right)/2\right\}\sinh\left\{F_2\left(\boldsymbol{k}_2,t\right)/2\right\}\cosh\left\{F_3\left(\boldsymbol{k}_2,t\right)/2\right\}\end{array}\right]\hat{p}\left(\mp\boldsymbol{k}_2\right),
\end{aligned}
\tag{G.32}
$$

and

$$
\begin{aligned}
&e^{iF_3(\boldsymbol{k}_1,t)\hat{K}_y(\boldsymbol{k}_1)}e^{iF_2(\boldsymbol{k}_1,t)\hat{K}_x(\boldsymbol{k}_1)}e^{iF_1(\boldsymbol{k}_1,t)\hat{K}_z(\boldsymbol{k}_1)}\hat{p}\left(\pm\boldsymbol{k}_2\right)e^{-iF_1(\boldsymbol{k}_1,t)\hat{K}_z(\boldsymbol{k}_1)}e^{-iF_2(\boldsymbol{k}_1,t)\hat{K}_x(\boldsymbol{k}_1)}e^{-iF_3(\boldsymbol{k}_1,t)\hat{K}_y(\boldsymbol{k}_1)}\\
&=\left(1-\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\right)\hat{p}\left(\pm\boldsymbol{k}_2\right)\\
&\quad-\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\sin\left\{F_1\left(\boldsymbol{k}_2,t\right)/2\right\}e^{iF_3(\boldsymbol{k}_1,t)\hat{K}_y(\boldsymbol{k}_1)}\\
&\qquad\times\left[\cosh\left\{F_2\left(\boldsymbol{k}_2,t\right)/2\right\}\hat{q}\left(\pm\boldsymbol{k}_2\right)-\sinh\left\{F_2\left(\boldsymbol{k}_2,t\right)/2\right\}\hat{p}\left(\mp\boldsymbol{k}_2\right)\right]e^{-iF_3(\boldsymbol{k}_1,t)\hat{K}_y(\boldsymbol{k}_1)}\\
&\quad+\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\cos\left\{F_1\left(\boldsymbol{k}_2,t\right)/2\right\}e^{iF_3(\boldsymbol{k}_1,t)\hat{K}_y(\boldsymbol{k}_1)}\\
&\qquad\times\left[-\sinh\left\{F_2\left(\boldsymbol{k}_2,t\right)/2\right\}\hat{q}\left(\mp\boldsymbol{k}_2\right)+\cosh\left\{F_2\left(\boldsymbol{k}_2,t\right)/2\right\}\hat{p}\left(\pm\boldsymbol{k}_2\right)\right]e^{-iF_3(\boldsymbol{k}_1,t)\hat{K}_y(\boldsymbol{k}_1)}\\
&=\left(1-\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\right)\hat{p}\left(\pm\boldsymbol{k}_2\right)\\
&\quad-\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\left[\begin{array}{l}\sin\left\{F_1\left(\boldsymbol{k}_2,t\right)/2\right\}\cosh\left\{F_2\left(\boldsymbol{k}_2,t\right)/2\right\}\cosh\left\{F_3\left(\boldsymbol{k}_2,t\right)/2\right\}\\-\cos\left\{F_1\left(\boldsymbol{k}_2,t\right)/2\right\}\sinh\left\{F_2\left(\boldsymbol{k}_2,t\right)/2\right\}\sinh\left\{F_3\left(\boldsymbol{k}_2,t\right)/2\right\}\end{array}\right]\hat{q}\left(\pm\boldsymbol{k}_2\right)\\
&\quad+\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\left[\begin{array}{l}\cos\left\{F_1\left(\boldsymbol{k}_2,t\right)/2\right\}\cosh\left\{F_2\left(\boldsymbol{k}_2,t\right)/2\right\}\cosh\left\{F_3\left(\boldsymbol{k}_2,t\right)/2\right\}\\+\sin\left\{F_1\left(\boldsymbol{k}_2,t\right)/2\right\}\sinh\left\{F_2\left(\boldsymbol{k}_2,t\right)/2\right\}\sinh\left\{F_3\left(\boldsymbol{k}_2,t\right)/2\right\}\end{array}\right]\hat{p}\left(\pm\boldsymbol{k}_2\right)\\
&\quad+\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\left[\begin{array}{l}\sin\left\{F_1\left(\boldsymbol{k}_2,t\right)/2\right\}\cosh\left\{F_2\left(\boldsymbol{k}_2,t\right)/2\right\}\sinh\left\{F_3\left(\boldsymbol{k}_2,t\right)/2\right\}\\-\cos\left\{F_1\left(\boldsymbol{k}_2,t\right)/2\right\}\sinh\left\{F_2\left(\boldsymbol{k}_2,t\right)/2\right\}\cosh\left\{F_3\left(\boldsymbol{k}_2,t\right)/2\right\}\end{array}\right]\hat{q}\left(\mp\boldsymbol{k}_2\right)\\
&\quad+\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\left[\begin{array}{l}\cos\left\{F_1\left(\boldsymbol{k}_2,t\right)/2\right\}\cosh\left\{F_2\left(\boldsymbol{k}_2,t\right)/2\right\}\sinh\left\{F_3\left(\boldsymbol{k}_2,t\right)/2\right\}\\+\sin\left\{F_1\left(\boldsymbol{k}_2,t\right)/2\right\}\sinh\left\{F_2\left(\boldsymbol{k}_2,t\right)/2\right\}\cosh\left\{F_3\left(\boldsymbol{k}_2,t\right)/2\right\}\end{array}\right]\hat{p}\left(\mp\boldsymbol{k}_2\right).
\end{aligned}
\tag{G.33}
$$

By defining $\hat{\mathbb{X}}\left(\boldsymbol{k}\right) \coloneqq \left[\begin{array}{cccc}\hat{q}\left(\boldsymbol{k}\right), & \hat{p}\left(\boldsymbol{k}\right), & \hat{q}\left(-\boldsymbol{k}\right), & \hat{p}\left(-\boldsymbol{k}\right)\end{array}\right]^T$, we get

$$
\begin{aligned}
&\left\{\prod_{\boldsymbol{k}_1>0}e^{iF_3(\boldsymbol{k}_1,t)\hat{K}_y(\boldsymbol{k}_1)}e^{iF_2(\boldsymbol{k}_1,t)\hat{K}_x(\boldsymbol{k}_1)}e^{iF_1(\boldsymbol{k}_1,t)\hat{K}_z(\boldsymbol{k}_1)}\right\}\hat{\mathbb{X}}\left(\boldsymbol{k}\right)\left\{\prod_{\boldsymbol{k}_2>0}e^{-iF_1(\boldsymbol{k}_2,t)\hat{K}_z(\boldsymbol{k}_2)}e^{-iF_2(\boldsymbol{k}_2,t)\hat{K}_x(\boldsymbol{k}_2)}e^{-iF_3(\boldsymbol{k}_2,t)\hat{K}_y(\boldsymbol{k}_2)}\right\}\\
&\quad=\Lambda\left(\boldsymbol{k},t\right)\hat{\mathbb{X}}\left(\boldsymbol{k}\right),
\end{aligned}
\tag{G.34}
$$

where

$$\Lambda(\boldsymbol{k},t) \coloneqq \begin{bmatrix} \Lambda_{1,1}(\boldsymbol{k},t) & \Lambda_{1,2}(\boldsymbol{k},t) & \Lambda_{1,3}(\boldsymbol{k},t) & \Lambda_{1,4}(\boldsymbol{k},t) \\ -\Lambda_{1,2}(\boldsymbol{k},t) & \Lambda_{1,1}(\boldsymbol{k},t) & \Lambda_{1,4}(\boldsymbol{k},t) & -\Lambda_{1,3}(\boldsymbol{k},t) \\ \Lambda_{1,3}(\boldsymbol{k},t) & \Lambda_{1,4}(\boldsymbol{k},t) & \Lambda_{1,1}(\boldsymbol{k},t) & \Lambda_{1,2}(\boldsymbol{k},t) \\ \Lambda_{1,4}(\boldsymbol{k},t) & -\Lambda_{1,3}(\boldsymbol{k},t) & -\Lambda_{1,2}(\boldsymbol{k},t) & \Lambda_{1,1}(\boldsymbol{k},t) \end{bmatrix}, \quad \text{(G.35)}$$

with

$$\begin{aligned}
\Lambda_{1,1}(\boldsymbol{k},t) &= \cos\{F_1(\boldsymbol{k},t)/2\}\cosh\{F_2(\boldsymbol{k},t)/2\}\cosh\{F_3(\boldsymbol{k},t)/2\} \\
&\quad + \sin\{F_1(\boldsymbol{k},t)/2\}\sinh\{F_2(\boldsymbol{k},t)/2\}\sinh\{F_3(\boldsymbol{k},t)/2\}, \\
\Lambda_{1,2}(\boldsymbol{k},t) &= \sin\{F_1(\boldsymbol{k},t)/2\}\cosh\{F_2(\boldsymbol{k},t)/2\}\cosh\{F_3(\boldsymbol{k},t)/2\} \\
&\quad - \cos\{F_1(\boldsymbol{k},t)/2\}\sinh\{F_2(\boldsymbol{k},t)/2\}\sinh\{F_3(\boldsymbol{k},t)/2\}, \\
\Lambda_{1,3}(\boldsymbol{k},t) &= -\cos\{F_1(\boldsymbol{k},t)/2\}\cosh\{F_2(\boldsymbol{k},t)/2\}\sinh\{F_3(\boldsymbol{k},t)/2\} \\
&\quad - \sin\{F_1(\boldsymbol{k},t)/2\}\sinh\{F_2(\boldsymbol{k},t)/2\}\cosh\{F_3(\boldsymbol{k},t)/2\}, \\
\Lambda_{1,4}(\boldsymbol{k},t) &= \sin\{F_1(\boldsymbol{k},t)/2\}\cosh\{F_2(\boldsymbol{k},t)/2\}\sinh\{F_3(\boldsymbol{k},t)/2\} \\
&\quad - \cos\{F_1(\boldsymbol{k},t)/2\}\sinh\{F_2(\boldsymbol{k},t)/2\}\cosh\{F_3(\boldsymbol{k},t)/2\}. \quad \text{(G.36)}
\end{aligned}$$

For nonzero $\boldsymbol{k}$,

$$\begin{aligned}
\hat{q}(\boldsymbol{k})\hat{q}(\pm\boldsymbol{k}) &= \frac{1}{2}\left\{\hat{b}_1^\dagger(\boldsymbol{k})\hat{b}_1^\dagger(\pm\boldsymbol{k}) + \hat{b}_1^\dagger(\boldsymbol{k})\hat{b}_1(\pm\boldsymbol{k}) + \hat{b}_1(\boldsymbol{k})\hat{b}_1^\dagger(\pm\boldsymbol{k}) + \hat{b}_1(\boldsymbol{k})\hat{b}_1(\pm\boldsymbol{k})\right\}, \\
\hat{q}(\boldsymbol{k})\hat{p}(\pm\boldsymbol{k}) &= \frac{i}{2}\left\{\hat{b}_1^\dagger(\boldsymbol{k})\hat{b}_1^\dagger(\pm\boldsymbol{k}) - \hat{b}_1^\dagger(\boldsymbol{k})\hat{b}_1(\pm\boldsymbol{k}) + \hat{b}_1(\boldsymbol{k})\hat{b}_1^\dagger(\pm\boldsymbol{k}) - \hat{b}_1(\boldsymbol{k})\hat{b}_1(\pm\boldsymbol{k})\right\}, \\
\hat{p}(\boldsymbol{k})\hat{q}(\pm\boldsymbol{k}) &= \frac{i}{2}\left\{\hat{b}_1^\dagger(\boldsymbol{k})\hat{b}_1^\dagger(\pm\boldsymbol{k}) + \hat{b}_1^\dagger(\boldsymbol{k})\hat{b}_1(\pm\boldsymbol{k}) - \hat{b}_1(\boldsymbol{k})\hat{b}_1^\dagger(\pm\boldsymbol{k}) - \hat{b}_1(\boldsymbol{k})\hat{b}_1(\pm\boldsymbol{k})\right\}, \\
\hat{p}(\boldsymbol{k})\hat{p}(\pm\boldsymbol{k}) &= -\frac{1}{2}\left\{\hat{b}_1^\dagger(\boldsymbol{k})\hat{b}_1^\dagger(\pm\boldsymbol{k}) - \hat{b}_1^\dagger(\boldsymbol{k})\hat{b}_1(\pm\boldsymbol{k}) - \hat{b}_1(\boldsymbol{k})\hat{b}_1^\dagger(\pm\boldsymbol{k}) + \hat{b}_1(\boldsymbol{k})\hat{b}_1(\pm\boldsymbol{k})\right\}, \quad \text{(G.37)}
\end{aligned}$$

and hence, for $j = 1, 2, 3, 4$,

$$\begin{aligned}
\langle\mathrm{vac}|\hat{\mathbb{X}}_1(\boldsymbol{k})\hat{\mathbb{X}}_j(\boldsymbol{k})|\mathrm{vac}\rangle &= \frac{1}{2}(\delta_{j,1} + i\delta_{j,2}), \quad \langle\mathrm{vac}|\hat{\mathbb{X}}_2(\boldsymbol{k})\hat{\mathbb{X}}_j(\boldsymbol{k})|\mathrm{vac}\rangle = -\frac{i}{2}(\delta_{j,1} + i\delta_{j,2}), \\
\langle\mathrm{vac}|\hat{\mathbb{X}}_3(\boldsymbol{k})\hat{\mathbb{X}}_j(\boldsymbol{k})|\mathrm{vac}\rangle &= \frac{1}{2}(\delta_{j,3} + i\delta_{j,4}), \quad \langle\mathrm{vac}|\hat{\mathbb{X}}_4(\boldsymbol{k})\hat{\mathbb{X}}_j(\boldsymbol{k})|\mathrm{vac}\rangle = -\frac{i}{2}(\delta_{j,3} + i\delta_{j,4}). \quad \text{(G.38)}
\end{aligned}$$

Let $\Gamma_{j,l}(\boldsymbol{k},t) := \left\langle \left\{ \hat{\mathbb{X}}_j(\boldsymbol{k}), \hat{\mathbb{X}}_l(\boldsymbol{k}) \right\} \right\rangle - 2\left\langle \hat{\mathbb{X}}_j(\boldsymbol{k}) \right\rangle \left\langle \hat{\mathbb{X}}_l(\boldsymbol{k}) \right\rangle$. From Eq. (G.34), with our state $|\Psi_S(t)\rangle = \prod_{\boldsymbol{k}>0} e^{-iF_1(\boldsymbol{k},t)\hat{K}_z(\boldsymbol{k})} e^{-iF_2(\boldsymbol{k},t)\hat{K}_x(\boldsymbol{k})} e^{-iF_3(\boldsymbol{k},t)\hat{K}_y(\boldsymbol{k})} |\text{vac}\rangle$, $\left\langle \hat{\mathbb{X}}_j(\boldsymbol{k}) \right\rangle = 0$ and we get

$$\begin{aligned}
\Gamma_{j,l}(\boldsymbol{k},t) &= \sum_{j_1,j_2=1}^{4} \left\{ \Lambda_{j,j_1}(\boldsymbol{k},t)\Lambda_{l,j_2}(\boldsymbol{k},t) + \Lambda_{j,j_2}(\boldsymbol{k},t)\Lambda_{l,j_1}(\boldsymbol{k},t) \right\} \langle \text{vac}| \hat{\mathbb{X}}_{j_1}(\boldsymbol{k}) \hat{\mathbb{X}}_{j_2}(\boldsymbol{k}) |\text{vac}\rangle \\
&= \sum_{j_2=1}^{4} \left\{ \Lambda_{j,1}(\boldsymbol{k},t)\Lambda_{l,j_2}(\boldsymbol{k},t) + \Lambda_{j,j_2}(\boldsymbol{k},t)\Lambda_{l,1}(\boldsymbol{k},t) \right\} \langle \text{vac}| \hat{\mathbb{X}}_{1}(\boldsymbol{k}) \hat{\mathbb{X}}_{j_2}(\boldsymbol{k}) |\text{vac}\rangle \\
&\quad + \sum_{j_2=1}^{4} \left\{ \Lambda_{j,2}(\boldsymbol{k},t)\Lambda_{l,j_2}(\boldsymbol{k},t) + \Lambda_{j,j_2}(\boldsymbol{k},t)\Lambda_{l,2}(\boldsymbol{k},t) \right\} \langle \text{vac}| \hat{\mathbb{X}}_{2}(\boldsymbol{k}) \hat{\mathbb{X}}_{j_2}(\boldsymbol{k}) |\text{vac}\rangle \\
&\quad + \sum_{j_2=1}^{4} \left\{ \Lambda_{j,3}(\boldsymbol{k},t)\Lambda_{l,j_2}(\boldsymbol{k},t) + \Lambda_{j,j_2}(\boldsymbol{k},t)\Lambda_{l,3}(\boldsymbol{k},t) \right\} \langle \text{vac}| \hat{\mathbb{X}}_{3}(\boldsymbol{k}) \hat{\mathbb{X}}_{j_2}(\boldsymbol{k}) |\text{vac}\rangle \\
&\quad + \sum_{j_2=1}^{4} \left\{ \Lambda_{j,4}(\boldsymbol{k},t)\Lambda_{l,j_2}(\boldsymbol{k},t) + \Lambda_{j,j_2}(\boldsymbol{k},t)\Lambda_{l,4}(\boldsymbol{k},t) \right\} \langle \text{vac}| \hat{\mathbb{X}}_{4}(\boldsymbol{k}) \hat{\mathbb{X}}_{j_2}(\boldsymbol{k}) |\text{vac}\rangle \\
&= \frac{1}{2} \sum_{j_2=1}^{4} \left\{ \Lambda_{j,1}(\boldsymbol{k},t)\Lambda_{l,j_2}(\boldsymbol{k},t) + \Lambda_{j,j_2}(\boldsymbol{k},t)\Lambda_{l,1}(\boldsymbol{k},t) \right\} (\delta_{j_2,1} + i\delta_{j_2,2}) \\
&\quad - \frac{i}{2} \sum_{j_2=1}^{4} \left\{ \Lambda_{j,2}(\boldsymbol{k},t)\Lambda_{l,j_2}(\boldsymbol{k},t) + \Lambda_{j,j_2}(\boldsymbol{k},t)\Lambda_{l,2}(\boldsymbol{k},t) \right\} (\delta_{j_2,1} + i\delta_{j_2,2}) \\
&\quad + \frac{1}{2} \sum_{j_2=1}^{4} \left\{ \Lambda_{j,3}(\boldsymbol{k},t)\Lambda_{l,j_2}(\boldsymbol{k},t) + \Lambda_{j,j_2}(\boldsymbol{k},t)\Lambda_{l,3}(\boldsymbol{k},t) \right\} (\delta_{j_2,3} + i\delta_{j_2,4}) \\
&\quad - \frac{i}{2} \sum_{j_2=1}^{4} \left\{ \Lambda_{j,4}(\boldsymbol{k},t)\Lambda_{l,j_2}(\boldsymbol{k},t) + \Lambda_{j,j_2}(\boldsymbol{k},t)\Lambda_{l,4}(\boldsymbol{k},t) \right\} (\delta_{j_2,3} + i\delta_{j_2,4}) \\
&= \sum_{j_1=1}^{4} \Lambda_{j,j_1}(\boldsymbol{k},t)\Lambda_{l,j_1}(\boldsymbol{k},t) = \left\{ \Lambda(\boldsymbol{k},t)\Lambda^T(\boldsymbol{k},t) \right\}_{j,l}. \qquad \text{(G.39)}
\end{aligned}$$

From [62], quantum Fisher information (QFI) is

$$I_Q(\bar{V}_a, t) = \frac{1}{4} \sum_{\boldsymbol{k}>0} \text{Tr}\left[\left\{ \frac{\partial \Gamma(\boldsymbol{k},t)}{\partial \bar{V}_a} \Gamma^{-1}(\boldsymbol{k},t) \right\}^2 \right], \qquad \text{(G.40)}$$

where $\Gamma(\boldsymbol{k},t) = \Lambda(\boldsymbol{k},t)\Lambda^T(\boldsymbol{k},t)$ as is shown in Eq. (G.39).

As we can get $F_j(\boldsymbol{k},t)$ from Eqs. (G.21), $\Gamma(\boldsymbol{k},t)$ and $\Gamma^{-1}(\boldsymbol{k},t)$ can be calculated from Eqs. (G.35) and (G.36). Also, $\partial\Gamma(\boldsymbol{k},t)/\partial\bar{V}_a$ can also be calculated by using Eqs. (G.23). Specifically, from Eqs. (G.36),

$$
\begin{aligned}
\frac{\partial\Lambda_{1,1}(\boldsymbol{k},t)}{\partial\bar{V}_a} &= -\frac{1}{2}\frac{\partial F_1(\boldsymbol{k},t)}{\partial\bar{V}_a}\left[\begin{array}{l}\sin\{F_1(\boldsymbol{k},t)/2\}\cosh\{F_2(\boldsymbol{k},t)/2\}\cosh\{F_3(\boldsymbol{k},t)/2\}\\ -\cos\{F_1(\boldsymbol{k},t)/2\}\sinh\{F_2(\boldsymbol{k},t)/2\}\sinh\{F_3(\boldsymbol{k},t)/2\}\end{array}\right]\\
&\quad+\frac{1}{2}\frac{\partial F_2(\boldsymbol{k},t)}{\partial\bar{V}_a}\left[\begin{array}{l}\sin\{F_1(\boldsymbol{k},t)/2\}\cosh\{F_2(\boldsymbol{k},t)/2\}\sinh\{F_3(\boldsymbol{k},t)/2\}\\ +\cos\{F_1(\boldsymbol{k},t)/2\}\sinh\{F_2(\boldsymbol{k},t)/2\}\cosh\{F_3(\boldsymbol{k},t)/2\}\end{array}\right]\\
&\quad+\frac{1}{2}\frac{\partial F_3(\boldsymbol{k},t)}{\partial\bar{V}_a}\left[\begin{array}{l}\cos\{F_1(\boldsymbol{k},t)/2\}\cosh\{F_2(\boldsymbol{k},t)/2\}\sinh\{F_3(\boldsymbol{k},t)/2\}\\ +\sin\{F_1(\boldsymbol{k},t)/2\}\sinh\{F_2(\boldsymbol{k},t)/2\}\cosh\{F_3(\boldsymbol{k},t)/2\}\end{array}\right]\\
&= \frac{1}{2}\frac{\partial F_2(\boldsymbol{k},t)}{\partial\bar{V}_a}\left[\begin{array}{l}\sin\{F_1(\boldsymbol{k},t)/2\}\cosh\{F_2(\boldsymbol{k},t)/2\}\sinh\{F_3(\boldsymbol{k},t)/2\}\\ +\cos\{F_1(\boldsymbol{k},t)/2\}\sinh\{F_2(\boldsymbol{k},t)/2\}\cosh\{F_3(\boldsymbol{k},t)/2\}\end{array}\right]\\
&\quad-\frac{1}{2}\frac{\partial F_1(\boldsymbol{k},t)}{\partial\bar{V}_a}\Lambda_{1,2}(\boldsymbol{k},t)-\frac{1}{2}\frac{\partial F_3(\boldsymbol{k},t)}{\partial\bar{V}_a}\Lambda_{1,3}(\boldsymbol{k},t),
\end{aligned}
\tag{G.41}
$$

$$
\begin{aligned}
\frac{\partial\Lambda_{1,2}(\boldsymbol{k},t)}{\partial\bar{V}_a} &= \frac{1}{2}\frac{\partial F_1(\boldsymbol{k},t)}{\partial\bar{V}_a}\left[\begin{array}{l}\cos\{F_1(\boldsymbol{k},t)/2\}\cosh\{F_2(\boldsymbol{k},t)/2\}\cosh\{F_3(\boldsymbol{k},t)/2\}\\ +\sin\{F_1(\boldsymbol{k},t)/2\}\sinh\{F_2(\boldsymbol{k},t)/2\}\sinh\{F_3(\boldsymbol{k},t)/2\}\end{array}\right]\\
&\quad-\frac{1}{2}\frac{\partial F_2(\boldsymbol{k},t)}{\partial\bar{V}_a}\left[\begin{array}{l}\cos\{F_1(\boldsymbol{k},t)/2\}\cosh\{F_2(\boldsymbol{k},t)/2\}\sinh\{F_3(\boldsymbol{k},t)/2\}\\ -\sin\{F_1(\boldsymbol{k},t)/2\}\sinh\{F_2(\boldsymbol{k},t)/2\}\cosh\{F_3(\boldsymbol{k},t)/2\}\end{array}\right]\\
&\quad+\frac{1}{2}\frac{\partial F_3(\boldsymbol{k},t)}{\partial\bar{V}_a}\left[\begin{array}{l}\sin\{F_1(\boldsymbol{k},t)/2\}\cosh\{F_2(\boldsymbol{k},t)/2\}\sinh\{F_3(\boldsymbol{k},t)/2\}\\ -\cos\{F_1(\boldsymbol{k},t)/2\}\sinh\{F_2(\boldsymbol{k},t)/2\}\cosh\{F_3(\boldsymbol{k},t)/2\}\end{array}\right]\\
&= -\frac{1}{2}\frac{\partial F_2(\boldsymbol{k},t)}{\partial\bar{V}_a}\left[\begin{array}{l}\cos\{F_1(\boldsymbol{k},t)/2\}\cosh\{F_2(\boldsymbol{k},t)/2\}\sinh\{F_3(\boldsymbol{k},t)/2\}\\ -\sin\{F_1(\boldsymbol{k},t)/2\}\sinh\{F_2(\boldsymbol{k},t)/2\}\cosh\{F_3(\boldsymbol{k},t)/2\}\end{array}\right]\\
&\quad+\frac{1}{2}\frac{\partial F_1(\boldsymbol{k},t)}{\partial\bar{V}_a}\Lambda_{1,1}(\boldsymbol{k},t)+\frac{1}{2}\frac{\partial F_3(\boldsymbol{k},t)}{\partial\bar{V}_a}\Lambda_{1,4}(\boldsymbol{k},t),
\end{aligned}
\tag{G.42}
$$

$$
\begin{aligned}
\frac{\partial \Lambda_{1,3}(\boldsymbol{k},t)}{\partial \bar{V}_a} &= \frac{1}{2}\frac{\partial F_1(\boldsymbol{k},t)}{\partial \bar{V}_a}\left[\begin{array}{c} \sin\{F_1(\boldsymbol{k},t)/2\}\cosh\{F_2(\boldsymbol{k},t)/2\}\sinh\{F_3(\boldsymbol{k},t)/2\} \\ -\cos\{F_1(\boldsymbol{k},t)/2\}\sinh\{F_2(\boldsymbol{k},t)/2\}\cosh\{F_3(\boldsymbol{k},t)/2\} \end{array}\right] \\
&\quad -\frac{1}{2}\frac{\partial F_2(\boldsymbol{k},t)}{\partial \bar{V}_a}\left[\begin{array}{c} \sin\{F_1(\boldsymbol{k},t)/2\}\cosh\{F_2(\boldsymbol{k},t)/2\}\cosh\{F_3(\boldsymbol{k},t)/2\} \\ +\cos\{F_1(\boldsymbol{k},t)/2\}\sinh\{F_2(\boldsymbol{k},t)/2\}\sinh\{F_3(\boldsymbol{k},t)/2\} \end{array}\right] \\
&\quad -\frac{1}{2}\frac{\partial F_3(\boldsymbol{k},t)}{\partial \bar{V}_a}\left[\begin{array}{c} \cos\{F_1(\boldsymbol{k},t)/2\}\cosh\{F_2(\boldsymbol{k},t)/2\}\cosh\{F_3(\boldsymbol{k},t)/2\} \\ +\sin\{F_1(\boldsymbol{k},t)/2\}\sinh\{F_2(\boldsymbol{k},t)/2\}\sinh\{F_3(\boldsymbol{k},t)/2\} \end{array}\right] \\
&= -\frac{1}{2}\frac{\partial F_2(\boldsymbol{k},t)}{\partial \bar{V}_a}\left[\begin{array}{c} \sin\{F_1(\boldsymbol{k},t)/2\}\cosh\{F_2(\boldsymbol{k},t)/2\}\cosh\{F_3(\boldsymbol{k},t)/2\} \\ +\cos\{F_1(\boldsymbol{k},t)/2\}\sinh\{F_2(\boldsymbol{k},t)/2\}\sinh\{F_3(\boldsymbol{k},t)/2\} \end{array}\right] \\
&\quad +\frac{1}{2}\frac{\partial F_1(\boldsymbol{k},t)}{\partial \bar{V}_a}\Lambda_{1,4}(\boldsymbol{k},t) - \frac{1}{2}\frac{\partial F_3(\boldsymbol{k},t)}{\partial \bar{V}_a}\Lambda_{1,1}(\boldsymbol{k},t),
\end{aligned}
\tag{G.43}
$$

and

$$
\begin{aligned}
\frac{\partial \Lambda_{1,4}(\boldsymbol{k},t)}{\partial \bar{V}_a} &= \frac{1}{2}\frac{\partial F_1(\boldsymbol{k},t)}{\partial \bar{V}_a}\left[\begin{array}{c} \cos\{F_1(\boldsymbol{k},t)/2\}\cosh\{F_2(\boldsymbol{k},t)/2\}\sinh\{F_3(\boldsymbol{k},t)/2\} \\ +\sin\{F_1(\boldsymbol{k},t)/2\}\sinh\{F_2(\boldsymbol{k},t)/2\}\cosh\{F_3(\boldsymbol{k},t)/2\} \end{array}\right] \\
&\quad -\frac{1}{2}\frac{\partial F_2(\boldsymbol{k},t)}{\partial \bar{V}_a}\left[\begin{array}{c} \cos\{F_1(\boldsymbol{k},t)/2\}\cosh\{F_2(\boldsymbol{k},t)/2\}\cosh\{F_3(\boldsymbol{k},t)/2\} \\ -\sin\{F_1(\boldsymbol{k},t)/2\}\sinh\{F_2(\boldsymbol{k},t)/2\}\sinh\{F_3(\boldsymbol{k},t)/2\} \end{array}\right] \\
&\quad +\frac{1}{2}\frac{\partial F_3(\boldsymbol{k},t)}{\partial \bar{V}_a}\left[\begin{array}{c} \sin\{F_1(\boldsymbol{k},t)/2\}\cosh\{F_2(\boldsymbol{k},t)/2\}\cosh\{F_3(\boldsymbol{k},t)/2\} \\ -\cos\{F_1(\boldsymbol{k},t)/2\}\sinh\{F_2(\boldsymbol{k},t)/2\}\sinh\{F_3(\boldsymbol{k},t)/2\} \end{array}\right] \\
&= -\frac{1}{2}\frac{\partial F_2(\boldsymbol{k},t)}{\partial \bar{V}_a}\left[\begin{array}{c} \cos\{F_1(\boldsymbol{k},t)/2\}\cosh\{F_2(\boldsymbol{k},t)/2\}\cosh\{F_3(\boldsymbol{k},t)/2\} \\ -\sin\{F_1(\boldsymbol{k},t)/2\}\sinh\{F_2(\boldsymbol{k},t)/2\}\sinh\{F_3(\boldsymbol{k},t)/2\} \end{array}\right] \\
&\quad -\frac{1}{2}\frac{\partial F_1(\boldsymbol{k},t)}{\partial \bar{V}_a}\Lambda_{1,3}(\boldsymbol{k},t) + \frac{1}{2}\frac{\partial F_3(\boldsymbol{k},t)}{\partial \bar{V}_a}\Lambda_{1,2}(\boldsymbol{k},t).
\end{aligned}
\tag{G.44}
$$

Thus, by combining above equations, we can calculate QFI as is discussed below Eq. (G.40).

G.3 Lower bound of Classical Fisher Information

In Schrödinger picture, suppose that we want to measure the physical operator $\hat{M}$ given by

$$\begin{aligned}\hat{M} &= \sum_{\boldsymbol{k}\neq 0}\left\{\mathbb{M}_1(k)+\mathbb{M}_2(k)\,\hat{b}_1^\dagger(\boldsymbol{k})\,\hat{b}_1(\boldsymbol{k})+\mathbb{M}_3(k)\,\hat{b}_1^\dagger(\boldsymbol{k})\,\hat{b}_1^\dagger(-\boldsymbol{k})+\mathbb{M}_3(k)\,\hat{b}_1(\boldsymbol{k})\,\hat{b}_1(-\boldsymbol{k})\right\}\\ &= \sum_{\boldsymbol{k}>0}\left\{2\mathbb{M}_1(k)-\mathbb{M}_2(k)+2\mathbb{M}_2(k)\,\hat{K}_z(\boldsymbol{k})+4\mathbb{M}_3(k)\,\hat{K}_x(\boldsymbol{k})\right\}. \qquad (G.45)\end{aligned}$$

Since

$$\left[\hat{K}_y(\boldsymbol{k}_1),\hat{K}_z(\boldsymbol{k}_2)\right]=i\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\hat{K}_x(\boldsymbol{k}_2),\quad \left[\hat{K}_y(\boldsymbol{k}_1),\left[\hat{K}_y(\boldsymbol{k}_1),\hat{K}_z(\boldsymbol{k}_2)\right]\right]=-\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\hat{K}_z(\boldsymbol{k}_2), \qquad (G.46)$$

$$\begin{aligned}&\underbrace{[\hat{K}_y(\boldsymbol{k}_1),[\hat{K}_y(\boldsymbol{k}_1),\cdots,[\hat{K}_y(\boldsymbol{k}_1)}_{2n\text{ times}},\hat{K}_z(\boldsymbol{k}_2)]\cdots]]=\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}(-1)^n\,\hat{K}_z(\boldsymbol{k}_2),\\ &\underbrace{[\hat{K}_y(\boldsymbol{k}_1),[\hat{K}_y(\boldsymbol{k}_1),\cdots,[\hat{K}_y(\boldsymbol{k}_1)}_{2n+1\text{ times}},\hat{K}_z(\boldsymbol{k}_2)]\cdots]]=\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}i(-1)^n\,\hat{K}_x(\boldsymbol{k}_2), \qquad (G.47)\end{aligned}$$

which gives

$$\begin{aligned}&e^{-iF_2(\boldsymbol{k}_1,\tilde{t})\hat{K}_y(\boldsymbol{k}_1)}\hat{K}_z(\boldsymbol{k})\,e^{iF_2(\boldsymbol{k}_1,\tilde{t})\hat{K}_y(\boldsymbol{k}_1)}\\ &=\hat{K}_z(\boldsymbol{k})+\delta_{\boldsymbol{k},\boldsymbol{k}_1}\sum_{j=0}^{\infty}\frac{(-i)^{2j+1}}{(2j+1)!}F_2^{2j+1}(\boldsymbol{k}_1,\tilde{t})\,i(-1)^j\,\hat{K}_x(\boldsymbol{k})+\delta_{\boldsymbol{k},\boldsymbol{k}_1}\sum_{j=1}^{\infty}\frac{(-i)^{2j}}{(2j)!}F_2^{2j}(\boldsymbol{k}_1,\tilde{t})(-1)^j\,\hat{K}_z(\boldsymbol{k})\\ &=\hat{K}_z(\boldsymbol{k})+\delta_{\boldsymbol{k},\boldsymbol{k}_1}\sum_{j=0}^{\infty}\frac{1}{(2j+1)!}F_2^{2j+1}(\boldsymbol{k},\tilde{t})\,\hat{K}_x(\boldsymbol{k})+\delta_{\boldsymbol{k},\boldsymbol{k}_1}\sum_{j=1}^{\infty}\frac{1}{(2j)!}F_2^{2j}(\boldsymbol{k},\tilde{t})\,\hat{K}_z(\boldsymbol{k})\\ &=(1-\delta_{\boldsymbol{k},\boldsymbol{k}_1})\,\hat{K}_z(\boldsymbol{k})+\delta_{\boldsymbol{k},\boldsymbol{k}_1}\left[\cosh\left\{F_2(\boldsymbol{k},\tilde{t})\right\}\hat{K}_z(\boldsymbol{k})+\sinh\left\{F_2(\boldsymbol{k},\tilde{t})\right\}\hat{K}_x(\boldsymbol{k})\right]. \qquad (G.48)\end{aligned}$$

From Eq. (G.48),

$$\begin{aligned}&e^{iF_3(\boldsymbol{k}_1,\tilde{t})\hat{K}_y(\boldsymbol{k}_1)}\hat{K}_z(\boldsymbol{k})\,e^{-iF_3(\boldsymbol{k}_1,\tilde{t})\hat{K}_y(\boldsymbol{k}_1)}\\ &=(1-\delta_{\boldsymbol{k},\boldsymbol{k}_1})\,\hat{K}_z(\boldsymbol{k})+\delta_{\boldsymbol{k},\boldsymbol{k}_1}\left[\cosh\left\{F_3(\boldsymbol{k},\tilde{t})\right\}\hat{K}_z(\boldsymbol{k})-\sinh\left\{F_3(\boldsymbol{k},\tilde{t})\right\}\hat{K}_x(\boldsymbol{k})\right]\\ &=(1-\delta_{\boldsymbol{k},\boldsymbol{k}_1})\,\hat{K}_z(\boldsymbol{k})+\delta_{\boldsymbol{k},\boldsymbol{k}_1}\left[-\frac{1}{2}\sinh\left\{F_3(\boldsymbol{k},\tilde{t})\right\}\left\{\hat{K}_+(\boldsymbol{k})+\hat{K}_-(\boldsymbol{k})\right\}+\cosh\left\{F_3(\boldsymbol{k},\tilde{t})\right\}\hat{K}_z(\boldsymbol{k})\right]. \qquad (G.49)\end{aligned}$$

From

$$[\underbrace{\hat{K}_y(\boldsymbol{k}_1),[\hat{K}_y(\boldsymbol{k}_1),\cdots,[\hat{K}_y(\boldsymbol{k}_1)}_{2n\text{ times}},\hat{K}_\pm(\boldsymbol{k}_2)]\cdots]]=(-1)^n\frac{1}{2}\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\left\{\hat{K}_+(\boldsymbol{k}_2)+\hat{K}_-(\boldsymbol{k}_2)\right\},$$
$$[\underbrace{\hat{K}_y(\boldsymbol{k}_1),[\hat{K}_y(\boldsymbol{k}_1),\cdots,[\hat{K}_y(\boldsymbol{k}_1)}_{2n+1\text{ times}},\hat{K}_\pm(\boldsymbol{k}_2)]\cdots]]=i(-1)^n\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\hat{K}_z(\boldsymbol{k}_2),\tag{G.50}$$

we can get

$$\begin{aligned}
&e^{iF_3(\boldsymbol{k}_1,t)\hat{K}_y(\boldsymbol{k}_1)}\hat{K}_\pm(\boldsymbol{k})\,e^{-iF_3(\boldsymbol{k}_1,t)\hat{K}_y(\boldsymbol{k}_1)}\\
&=\hat{K}_\pm(\boldsymbol{k})+\sum_{n=1}^{\infty}\frac{i^n}{n!}F_3^n(\boldsymbol{k}_1,t)\,[\underbrace{\hat{K}_y(\boldsymbol{k}_1),[\hat{K}_y(\boldsymbol{k}_1),\cdots,[\hat{K}_y(\boldsymbol{k}_1)}_{n\text{ times}},\hat{K}_\pm(\boldsymbol{k})]\cdots]]\\
&=\hat{K}_\pm(\boldsymbol{k})+\sum_{n=0}^{\infty}\frac{i^{2n+1}}{(2n+1)!}F_3^{2n+1}(\boldsymbol{k}_1,t)\,i(-1)^n\delta_{\boldsymbol{k},\boldsymbol{k}_1}\hat{K}_z(\boldsymbol{k})\\
&\quad+\sum_{n=1}^{\infty}\frac{i^{2n}}{(2n)!}F_3^{2n}(\boldsymbol{k}_1,t)(-1)^n\frac{1}{2}\delta_{\boldsymbol{k},\boldsymbol{k}_1}\left\{\hat{K}_+(\boldsymbol{k})+\hat{K}_-(\boldsymbol{k})\right\}\\
&=\hat{K}_\pm(\boldsymbol{k})-\delta_{\boldsymbol{k},\boldsymbol{k}_1}\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}F_3^{2n+1}(\boldsymbol{k},t)\,\hat{K}_z(\boldsymbol{k})+\frac{1}{2}\delta_{\boldsymbol{k},\boldsymbol{k}_1}\sum_{n=1}^{\infty}\frac{1}{(2n)!}F_3^{2n}(\boldsymbol{k},t)\left\{\hat{K}_+(\boldsymbol{k})+\hat{K}_-(\boldsymbol{k})\right\}\\
&=(1-\delta_{\boldsymbol{k},\boldsymbol{k}_1})\hat{K}_\pm(\boldsymbol{k})+\delta_{\boldsymbol{k},\boldsymbol{k}_1}\left[\frac{1}{2}\cosh\{F_3(\boldsymbol{k},t)\}\left\{\hat{K}_+(\boldsymbol{k})+\hat{K}_-(\boldsymbol{k})\right\}-\sinh\{F_3(\boldsymbol{k},t)\}\hat{K}_z(\boldsymbol{k})\right].
\end{aligned}\tag{G.51}$$

From

$$\left[\hat{K}_x(\boldsymbol{k}_1),\hat{K}_z(\boldsymbol{k}_2)\right]=-i\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\hat{K}_y(\boldsymbol{k}_2),\quad\left[\hat{K}_x(\boldsymbol{k}_1),\left[\hat{K}_x(\boldsymbol{k}_1),\hat{K}_z(\boldsymbol{k}_2)\right]\right]=-\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\hat{K}_z(\boldsymbol{k}_2),\tag{G.52}$$

$$[\underbrace{\hat{K}_x(\boldsymbol{k}_1),[\hat{K}_x(\boldsymbol{k}_1),\cdots,[\hat{K}_x(\boldsymbol{k}_1)}_{2n\text{ times}},\hat{K}_z(\boldsymbol{k}_2)]\cdots]]=\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}(-1)^n\hat{K}_z(\boldsymbol{k}_2),$$
$$[\underbrace{\hat{K}_x(\boldsymbol{k}_1),[\hat{K}_x(\boldsymbol{k}_1),\cdots,[\hat{K}_x(\boldsymbol{k}_1)}_{2n+1\text{ times}},\hat{K}_z(\boldsymbol{k}_2)]\cdots]]=\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}(-i)(-1)^n\hat{K}_y(\boldsymbol{k}_2),\tag{G.53}$$

which gives

$$
\begin{aligned}
&e^{-iF_1\left(\boldsymbol{k}_1,\tilde{t}\right)\hat{K}_x(\boldsymbol{k}_1)}\hat{K}_z\left(\boldsymbol{k}\right)e^{iF_1\left(\boldsymbol{k}_1,\tilde{t}\right)\hat{K}_x(\boldsymbol{k}_1)}\\
&\quad=\hat{K}_z\left(\boldsymbol{k}\right)+\delta_{\boldsymbol{k},\boldsymbol{k}_1}\sum_{j=0}^{\infty}\frac{\left(-i\right)^{2j+1}}{(2j+1)!}F_1^{2j+1}\left(\boldsymbol{k}_1,\tilde{t}\right)\left(-i\right)\left(-1\right)^j\hat{K}_y\left(\boldsymbol{k}\right)+\delta_{\boldsymbol{k},\boldsymbol{k}_1}\sum_{j=1}^{\infty}\frac{\left(-i\right)^{2j}}{(2j)!}F_1^{2j}\left(\boldsymbol{k}_1,\tilde{t}\right)\left(-1\right)^j\hat{K}_z\left(\boldsymbol{k}\right)\\
&\quad=\hat{K}_z\left(\boldsymbol{k}\right)-\delta_{\boldsymbol{k},\boldsymbol{k}_1}\sum_{j=0}^{\infty}\frac{1}{(2j+1)!}F_1^{2j+1}\left(\boldsymbol{k},\tilde{t}\right)\hat{K}_y\left(\boldsymbol{k}\right)+\delta_{\boldsymbol{k},\boldsymbol{k}_1}\sum_{j=1}^{\infty}\frac{1}{(2j)!}F_1^{2j}\left(\boldsymbol{k},\tilde{t}\right)\hat{K}_z\left(\boldsymbol{k}\right)\\
&\quad=\left(1-\delta_{\boldsymbol{k},\boldsymbol{k}_1}\right)\hat{K}_z\left(\boldsymbol{k}\right)+\delta_{\boldsymbol{k},\boldsymbol{k}_1}\left[\cosh\left\{F_1\left(\boldsymbol{k},\tilde{t}\right)\right\}\hat{K}_z\left(\boldsymbol{k}\right)-\sinh\left\{F_1\left(\boldsymbol{k},\tilde{t}\right)\right\}\hat{K}_y\left(\boldsymbol{k}\right)\right].
\end{aligned}
\tag{G.54}
$$

By rewriting $\mathbb{A}_1\left(k\right):=2\mathbb{M}_1\left(k\right)-\mathbb{M}_2\left(k\right)$, $\mathbb{A}_2\left(k\right):=2\mathbb{M}_2\left(k\right)$, and $\mathbb{A}_3\left(k\right):=4\mathbb{M}_3\left(k\right)$, from Eq. (G.10), Eq. (G.11), Eq. (G.48), Eq. (G.51), and Eq. (G.54),

$$
\begin{aligned}
&\left\{\prod_{\boldsymbol{k}_1>0}e^{iF_3(\boldsymbol{k}_1,t)\hat{K}_y(\boldsymbol{k}_1)}e^{iF_2(\boldsymbol{k}_1,t)\hat{K}_x(\boldsymbol{k}_1)}e^{iF_1(\boldsymbol{k}_1,t)\hat{K}_z(\boldsymbol{k}_1)}\right\}\hat{M}\left\{\prod_{\boldsymbol{k}_2>0}e^{-iF_1(\boldsymbol{k}_2,t)\hat{K}_z(\boldsymbol{k}_2)}e^{-iF_2(\boldsymbol{k}_2,t)\hat{K}_x(\boldsymbol{k}_2)}e^{-iF_3(\boldsymbol{k}_2,t)\hat{K}_y(\boldsymbol{k}_2)}\right\}\\
&\quad=\sum_{\boldsymbol{k}>0}\mathbb{A}_1\left(k\right)+\sum_{\boldsymbol{k}>0}\mathbb{A}_3\left(k\right)\cos\left\{F_1\left(\boldsymbol{k},t\right)\right\}\left[\cosh\left\{F_3\left(\boldsymbol{k},t\right)\right\}\hat{K}_x\left(\boldsymbol{k}\right)-\sinh\left\{F_3\left(\boldsymbol{k},t\right)\right\}\hat{K}_z\left(\boldsymbol{k}\right)\right]\\
&\qquad+\sum_{\boldsymbol{k}>0}\left[\mathbb{A}_2\left(k\right)\sinh\left\{F_2\left(\boldsymbol{k},\tilde{t}\right)\right\}-\mathbb{A}_3\left(k\right)\sin\left\{F_1\left(\boldsymbol{k},t\right)\right\}\cosh\left\{F_2\left(\boldsymbol{k},t\right)\right\}\right]\hat{K}_y\left(\boldsymbol{k}\right)\\
&\qquad+\sum_{\boldsymbol{k}>0}\left[\mathbb{A}_2\left(k\right)\cosh\left\{F_2\left(\boldsymbol{k},\tilde{t}\right)\right\}-\mathbb{A}_3\left(k\right)\sin\left\{F_1\left(\boldsymbol{k},t\right)\right\}\sinh\left\{F_2\left(\boldsymbol{k},t\right)\right\}\right]\\
&\qquad\qquad\times\left[-\sinh\left\{F_3\left(\boldsymbol{k},\tilde{t}\right)\right\}\hat{K}_x\left(\boldsymbol{k}\right)+\cosh\left\{F_3\left(\boldsymbol{k},\tilde{t}\right)\right\}\hat{K}_z\left(\boldsymbol{k}\right)\right]\\
&\quad=\sum_{\boldsymbol{k}>0}\mathbb{A}_1\left(k\right)-\frac{1}{2}\sum_{\boldsymbol{k}>0}\mathbb{A}_2\left(k\right)\left[\cosh\left\{F_2\left(\boldsymbol{k},\tilde{t}\right)\right\}\sinh\left\{F_3\left(\boldsymbol{k},\tilde{t}\right)\right\}+i\sinh\left\{F_2\left(\boldsymbol{k},\tilde{t}\right)\right\}\right]\hat{K}_+\left(\boldsymbol{k}\right)\\
&\qquad+\frac{1}{2}\sum_{\boldsymbol{k}>0}\mathbb{A}_3\left(k\right)\left[\begin{array}{c}\sin\left\{F_1\left(\boldsymbol{k},t\right)\right\}\sinh\left\{F_2\left(\boldsymbol{k},t\right)\right\}\sinh\left\{F_3\left(\boldsymbol{k},\tilde{t}\right)\right\}\\+\cos\left\{F_1\left(\boldsymbol{k},t\right)\right\}\cosh\left\{F_3\left(\boldsymbol{k},t\right)\right\}+i\sin\left\{F_1\left(\boldsymbol{k},t\right)\right\}\cosh\left\{F_2\left(\boldsymbol{k},t\right)\right\}\end{array}\right]\hat{K}_+\left(\boldsymbol{k}\right)\\
&\qquad-\frac{1}{2}\sum_{\boldsymbol{k}>0}\mathbb{A}_2\left(k\right)\left[\cosh\left\{F_2\left(\boldsymbol{k},\tilde{t}\right)\right\}\sinh\left\{F_3\left(\boldsymbol{k},\tilde{t}\right)\right\}-i\sinh\left\{F_2\left(\boldsymbol{k},\tilde{t}\right)\right\}\right]\hat{K}_-\left(\boldsymbol{k}\right)\\
&\qquad+\frac{1}{2}\sum_{\boldsymbol{k}>0}\mathbb{A}_3\left(k\right)\left[\begin{array}{c}\sin\left\{F_1\left(\boldsymbol{k},t\right)\right\}\sinh\left\{F_2\left(\boldsymbol{k},t\right)\right\}\sinh\left\{F_3\left(\boldsymbol{k},\tilde{t}\right)\right\}\\+\cos\left\{F_1\left(\boldsymbol{k},t\right)\right\}\cosh\left\{F_3\left(\boldsymbol{k},t\right)\right\}-i\sin\left\{F_1\left(\boldsymbol{k},t\right)\right\}\cosh\left\{F_2\left(\boldsymbol{k},t\right)\right\}\end{array}\right]\hat{K}_-\left(\boldsymbol{k}\right)\\
&\qquad-\sum_{\boldsymbol{k}>0}\mathbb{A}_3\left(k\right)\left[\sin\left\{F_1\left(\boldsymbol{k},t\right)\right\}\sinh\left\{F_2\left(\boldsymbol{k},t\right)\right\}\cosh\left\{F_3\left(\boldsymbol{k},\tilde{t}\right)\right\}+\cos\left\{F_1\left(\boldsymbol{k},t\right)\right\}\sinh\left\{F_3\left(\boldsymbol{k},t\right)\right\}\right]\hat{K}_z\left(\boldsymbol{k}\right)\\
&\qquad+\sum_{\boldsymbol{k}>0}\mathbb{A}_2\left(k\right)\cosh\left\{F_2\left(\boldsymbol{k},\tilde{t}\right)\right\}\cosh\left\{F_3\left(\boldsymbol{k},\tilde{t}\right)\right\}\hat{K}_z\left(\boldsymbol{k}\right).
\end{aligned}
\tag{G.55}
$$

For convenience, let

$$\begin{aligned}
\mathbb{B}_2(\boldsymbol{k},t) = &-\frac{1}{2}\mathbb{A}_2(k)\left[\cosh\left\{F_2\left(\boldsymbol{k},\tilde{t}\right)\right\}\sinh\left\{F_3\left(\boldsymbol{k},\tilde{t}\right)\right\}+i\sinh\left\{F_2\left(\boldsymbol{k},\tilde{t}\right)\right\}\right] \\
&+\frac{1}{2}\mathbb{A}_3(k)\left[\begin{array}{l}\sin\{F_1(\boldsymbol{k},t)\}\sinh\{F_2(\boldsymbol{k},t)\}\sinh\left\{F_3\left(\boldsymbol{k},\tilde{t}\right)\right\} \\ +\cos\{F_1(\boldsymbol{k},t)\}\cosh\{F_3(\boldsymbol{k},t)\}+i\sin\{F_1(\boldsymbol{k},t)\}\cosh\{F_2(\boldsymbol{k},t)\}\end{array}\right] \\
= &-\mathbb{M}_2(k)\left[\cosh\left\{F_2\left(\boldsymbol{k},\tilde{t}\right)\right\}\sinh\left\{F_3\left(\boldsymbol{k},\tilde{t}\right)\right\}+i\sinh\left\{F_2\left(\boldsymbol{k},\tilde{t}\right)\right\}\right] \\
&+2\mathbb{M}_3(k)\left[\begin{array}{l}\sin\{F_1(\boldsymbol{k},t)\}\sinh\{F_2(\boldsymbol{k},t)\}\sinh\left\{F_3\left(\boldsymbol{k},\tilde{t}\right)\right\} \\ +\cos\{F_1(\boldsymbol{k},t)\}\cosh\{F_3(\boldsymbol{k},t)\}+i\sin\{F_1(\boldsymbol{k},t)\}\cosh\{F_2(\boldsymbol{k},t)\}\end{array}\right],
\end{aligned} \tag{G.56}$$

and

$$\begin{aligned}
\mathbb{B}_3(\boldsymbol{k},t) &= \frac{1}{2}\mathbb{A}_2(k)\cosh\left\{F_2\left(\boldsymbol{k},\tilde{t}\right)\right\}\cosh\left\{F_3\left(\boldsymbol{k},\tilde{t}\right)\right\} \\
&\quad -\frac{1}{2}\mathbb{A}_3(k)\left[\begin{array}{l}\sin\{F_1(\boldsymbol{k},t)\}\sinh\{F_2(\boldsymbol{k},t)\}\cosh\left\{F_3\left(\boldsymbol{k},\tilde{t}\right)\right\} \\ +\cos\{F_1(\boldsymbol{k},t)\}\sinh\{F_3(\boldsymbol{k},t)\}\end{array}\right] \\
&= \mathbb{M}_2(k)\cosh\left\{F_2\left(\boldsymbol{k},\tilde{t}\right)\right\}\cosh\left\{F_3\left(\boldsymbol{k},\tilde{t}\right)\right\} \\
&\quad -2\mathbb{M}_3(k)\left[\begin{array}{l}\sin\{F_1(\boldsymbol{k},t)\}\sinh\{F_2(\boldsymbol{k},t)\}\cosh\left\{F_3\left(\boldsymbol{k},\tilde{t}\right)\right\} \\ +\cos\{F_1(\boldsymbol{k},t)\}\sinh\{F_3(\boldsymbol{k},t)\}\end{array}\right].
\end{aligned} \tag{G.57}$$

Then we get

$$\langle\Psi_S(t)|\,\hat{M}\,|\Psi_S(t)\rangle = \sum_{\boldsymbol{k}>0}\{\mathbb{A}_1(k)+\mathbb{B}_3(\boldsymbol{k},t)\}, \tag{G.58}$$

$$\begin{aligned}
\langle\Psi_S(t)|\,\hat{M}^2\,|\Psi_S(t)\rangle &= \sum_{\boldsymbol{k}_1>0}\sum_{\boldsymbol{k}_2>0}\langle\mathrm{vac}|\left\{\mathbb{A}_1(k_1)+\mathbb{B}_2^*(\boldsymbol{k}_1,t)\,\hat{K}_-(\boldsymbol{k}_1)+2\mathbb{B}_3(\boldsymbol{k}_1,t)\,\hat{K}_z(\boldsymbol{k}_1)\right\} \\
&\qquad \times\left\{\mathbb{A}_1(k_2)+\mathbb{B}_2(\boldsymbol{k}_2,t)\,\hat{K}_+(\boldsymbol{k}_2)+2\mathbb{B}_3(\boldsymbol{k}_2,t)\,\hat{K}_z(\boldsymbol{k}_2)\right\}|\mathrm{vac}\rangle \\
&= \left\{\sum_{\boldsymbol{k}>0}\mathbb{A}_1(k)\right\}^2+2\sum_{\boldsymbol{k}_1>0}\mathbb{A}_1(k_1)\sum_{\boldsymbol{k}_2>0}\mathbb{B}_3(\boldsymbol{k}_2,t)+\left\{\sum_{\boldsymbol{k}>0}\mathbb{B}_3(\boldsymbol{k},t)\right\}^2+\sum_{\boldsymbol{k}>0}|\mathbb{B}_2(\boldsymbol{k},t)|^2 \\
&= \left\{\langle\Psi_S(t)|\,\hat{M}\,|\Psi_S(t)\rangle\right\}^2+\sum_{\boldsymbol{k}>0}|\mathbb{B}_2(\boldsymbol{k},t)|^2,
\end{aligned} \tag{G.59}$$

and the lower bound of classical Fisher information $I_C\left(\bar{V}_a, t\right)$ is

$$I_C\left(\bar{V}_a, t\right) = \frac{1}{\sum_{\boldsymbol{k}>0} \left|\mathbb{B}_2\left(\boldsymbol{k}, t\right)\right|^2} \left\{\sum_{\boldsymbol{k}>0} \frac{\partial \mathbb{B}_3\left(\boldsymbol{k}, t\right)}{\partial \bar{V}_a}\right\}^2. \tag{G.60}$$

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초 록

본 논문은 보즈-아인슈타인 응집체의 기초 물리 및 계측학 연구로의 활용 가능성에 대해 다룬다.

첫번째 주제에서는 감쇄에 의한 영향을 반영한 보즈-아인슈타인 응집체의 평균장 이론에 등장하는 감쇄 지수를 측정할 수 있는 새로운 실험 방법에 대한 제안에 대해 다룬다. 보즈-아인슈타인 응집체의 평균장 이론 [1, 2]은 보즈-아인슈타인의 특성을 연구하는 데 널리 사용되고 있지만, 보즈-아인슈타인 응집체의 집단적 진동 (collective oscillation)이 제한된 시간 동안만 존재하는 점을 설명하지 못한다. 이를 보완하기 위해 피타에프스키는 감쇄 지수 γ_p를 도입하였고 [3] 이후 스칼라 ^{23}Na 보즈-아인슈타인 응집체의 실험 데이터 [5]를 바탕으로 버넷의 연구팀이 $\gamma_p \simeq 0.03$ 이라는 추정 결과를 발표하였다 [4]. 이 감쇄 지수 도입을 스칼라 보즈-아인슈타인 응집체에 대해 이론적으로 설명하기 위한 시도가 있었으나 [6] 아직까지는 γ_p의 값을 0.03으로 맞추기 위해 인위적으로 비례 상수를 도입한 불완전한 설명만 존재한다 [7, 8]. 또한 이 감쇄 지수는 보즈-아인슈타인 응집체의 종류에 따라 달라질 가능성이 있고, 스핀을 가진 보즈-아인슈타인 응집체에서는 이 감쇄 지수가 스핀에 따라 달라질 가능성이 있는데도 적절한 설명 없이 위에서 언급한 추정값(0.03)을 그대로 사용하는 경향이 있다 [9, 10].

피지컬 리뷰 A에 게재된 본 저자의 논문 [11]을 바탕으로, 저자는 스핀의 방향이 균일한 보즈-아인슈타인 응집체의 스핀의 방향이 외부 자기장에 의해 바뀌는 시간을 측정하면 이 감쇄 지수를 측정할 수 있다는 결과를 얻었다. 이 결과는 감쇄 지수가 보즈-아인슈타인 응집체의 스핀과 무관하다는 기존의 가설 [9, 10]을 바탕으로

하였는데, 그 과정에서 본 저자는 강자성체를 연구하는 데 있어서 널리 사용되지만 이론적으로 유도되지는 않았던 란다우-리프쉬츠-길버트 방정식을 이론적으로 유도할 수 있음을 보였다. 또한 강자성체의 자기이력현상을 설명하는 스토너-볼파르트 모델도 위의 가정에 감쇄 현상이 없다는 가정을 추가하면 이론적으로 유도할 수 있음을 보였다. 스핀의 방향이 모두 균일한 보즈-아인슈타인 응집체가 만들어질 수 있다는 것이 입증되었으므로 [12], 위에서 소개한 감쇄 지수를 실험적으로 측정할 수 있는 새로운 방법은 현실성이 전혀 없지는 않을 것으로 보인다.

두번째 주제에서는 초저온 화학 반응이 일어나는 보즈-아인슈타인 응집체에 외부 자극이 가해졌을 때 생성되는 보즈-아인슈타인 분자 응집체의 갯수를 측정하여 외부 자극의 크기를 추정할 수 있다는 내용에 대해 다룬다. 보즈-아인슈타인 응집체를 이용하여 가속도의 크기를 측정하거나 [13] 중력파를 검출하는 센서를 만들거나 [14, 15, 16, 17] 중력장의 공간에 따른 변화량을 밀리미터 단위로 측정하거나 [18] 암흑 물질을 검출하는 센서를 만들 수 있다는 [19] 이론적 제안들은 존재하였으나, 이들은 고전적 피셔 정보량을 계산하지 않았고 이로 인해 해당 측정의 정밀도가 예상보다 낮을 수 있다는 가능성이 존재한다. 또한 이들은 보즈-아인슈타인 응집체의 포논의 갯수를 측정하는 방식을 기반으로 하는데, 아직까지는 오직 초유체 헬륨 II에서만 포논의 갯수를 측정할 수 있었고 [20] 보즈-아인슈타인 응집체의 포논의 갯수를 측정했다는 보고는 존재하지 않는다.(이론적으로도 보즈-아인슈타인 응집체의 포논의 갯수를 측정하는 것은 어려운 것으로 알려져 있다 [21]).

곧 투고할 본 저자의 연구를 바탕으로, 공간상 균일하지만 시간에 따라 시스템의 밀도를 바꾸는 외부 자극이 초저온 화학 반응이 일어나는 보즈-아인슈타인 응집체에 가해진 상황에 대한 이론적 연구를 소개한다. 본 저자는 해당 상황에서 생성되는 보즈-아인슈타인 응집체 분자의 갯수를 측정하여 외부 자극의 크기를 측정할 때의 양자 피셔 정보량과 고전적 피셔 정보량의 하한값을 계산하였고, 이를 토대로 해당 측정 방식의 정밀도가 보즈-아인슈타인 응집체를 사용했을 때 얻을 수 있는 정밀도의 이론적 한계치에 근접한다는 결과를 얻었다. 초저온 화학 반응체을 통해 생성되는 보즈-아인슈타인 응집체 분자의 갯수는 측정할 수 있으므로 [22, 23, 24, 25], 이를 통해 외부 자극의 크기를 측정하는 방식은 보즈-아인슈타인 응집체의 포논의

갯수를 측정하는 방식을 기반으로 하는 기존의 제안들보다 실현될 가능성이 높을 것으로 기대된다.

주요어: 보즈-아인슈타인 응집체, 감쇄 계수, 스토너-볼파르트 스위칭, 초저온 화학 반응, 피셔 정보

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