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Calderón-Zygmund estimates for elliptic equations with nonstandard growth

(비표준 성장조건을 가진 타원형 편미분방정식의 칼데론-지그문드 추정)

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Calderón-Zygmund estimates for elliptic equations with nonstandard growth

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Abstract

Calderón-Zygmund estimates for elliptic equations with nonstandard growth

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We investigate a certain kind of regularity results so-called Calderón-Zygmund estimates for the various kind of elliptic equations in divergence form and functionals. Several generalizations of p-Laplace equation are considered in this thesis. First, we study the following Orlicz growth problems: equations involving a more general form of nonlinearity, and equations with measurable nonlinearities. We also study general double phase problems and their extensions to $p(x)$-Laplace: equations for non-uniformly elliptic problems with BMO nonlinearity, ω-minimizers of functionals for double phase problems with variable powers $p(x)$ and $q(x)$, equations for Orlicz double phase problems with variable exponents.

The next topic under consideration is to establish the global Calderón-Zygmund theory for the elliptic equations with degenerate/singular coefficients. The coefficients are matrix weights whose absolute values belong to Muckenhoupt class. We first prove maximal regularity for Laplace and p-Laplace equations with degenerate weights, assuming that the boundary of the domain is Lipschitz. We find the sharp relation between the exponent of higher integrability and the smallness parameters, which will be shown by an example in this thesis. Finally, we consider the equations with matrix weights and measurable nonlinearities under the setting of the Reifenberg flat domain and prove global weighted gradient estimates.

Key words: Calderón-Zygmund theory, Orlicz growth, variable exponent, double phase problems, degenerate weights, Muckenhoupt class
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Contents

Chapter 1

Introduction

This thesis concerns Calderón-Zygmund type estimates, which are originated from the pioneering works of [59, 60], for weak solutions of elliptic equations of divergence form or minimizers of integral functionals involving elliptic operators. The Calderón-Zygmund theory deals with the relations between the integrability of the gradient of solutions or functionals and those of the associated datum. Since considered in [59] for the linear case and [139] for the nonlinear case, the theory has been developed and many regularity results have been provided.

Let us consider the following elliptic equation:

$$-\mathrm{div}(c(x)|Du|^{p-2}Du) = -\mathrm{div}(|F|^{p-2}F) \quad \text{in } \Omega, \tag{1.0.1}$$

where $p > 1$, $\Omega \subset \mathbb{R}^n$ is a bounded and open domain with $n \geq 2$, $F = (f_1, \dots, f_n) : \Omega \to \mathbb{R}^n$ is a given vector-valued function with $|F| \in L^p(\Omega)$, and the coefficient function $c(x) : \Omega \to \mathbb{R}$ satisfies the following uniform ellipticity condition

$$0 < \nu \leq c(x) \leq L < \infty \tag{1.0.2}$$

for positive constants ν and L. For a weak solution $u \in W^{1,p}(\Omega)$ of (1.0.1), we want to obtain that $F \in L^\gamma$ implies $Du \in L^\gamma$ for all $\gamma > p$ with the standard form of estimate. However, as in [177], the implication $F \in L^\gamma \Rightarrow Du \in L^\gamma$ fails in general, and so the VMO assumption for $c(x)$ (see [89, 152]) and the small BMO assumption for $c(x)$ (see [51]) are considered later to prove the relation. In this thesis, we generalize the nonlinearity $|Du|^{p-2}Du$ in (1.0.1) to

Orlicz growth and general double phase problem, and also generalize the uniform ellipticity (1.0.2) of $c(x)$ to degenerate ellipticity to extend the Calderón-Zygmund theory for a larger class of problems.

1.1 Elliptic equations with Orlicz growth

There has been a historical progress of studying the regularity theory of nonlinear p-Laplacian type equations of divergence form over last several decades such that there is almost no possibility to mention all the works that have been done up to now. We refer some pioneering results in this direction, see for instance [10, 16, 18, 28, 29, 55, 58, 139, 140, 152, 159, 160, 179, 180, 193, 203] and references therein.

The problems with Orlicz growth and generalized Orlicz growth are central topics as natural generalizations of p-Laplacian problems which have been an object of intensive studies over last decades. Besides the papers aforementioned, there is a wide literature on regularity properties of elliptic/parabolic equations of p-Laplacian or φ-Laplacian type, see for instance, Lipschitz regularity for elliptic/parabolic equations [23, 50, 98, 118], Potential estimates [20, 128], higher integrability [71, 131], Hölder continuity [97, 100, 133, 134, 136], Calderón-Zygmund estimates [40, 70, 135, 205], and so on.

Equations with measurable nonlinearities. We first investigate the validity of Calderón–Zygmund type estimates for solutions of elliptic equations when the behavior of the assigned nonlinearity is irregular in one of the variables. Our result is natural continuation of the recent observation that even the coefficient of the equation is fairly general discontinuous in one direction so that the coefficient has a jumping from the constant, yet the solutions can attain a certain degree of uniform regularity estimate. The problem under consideration has a deep relationship with natural substances having a big jump property in one direction. In this spirit, we refer to the problems related to composite materials [110, 142, 162, 163], linear laminates [67, 102, 109], transmission problems [19, 113, 114] and the references therein.

We consider the following general elliptic equation

$$\operatorname{div}\left(b_1(x_1)b_2(x')\frac{\varphi'(|Du|)}{|Du|}Du\right) = \operatorname{div}\left(\frac{\varphi'(|F|)}{|F|}F\right) \quad \text{in } \Omega, \tag{1.1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded and open domain with $n \geq 2$, $F = (f_1, \dots, f_n) : \Omega \to \mathbb{R}^n$ is a given vector-valued function with $|F| \in L^1(\Omega)$, $x = (x_1, x') \in \mathbb{R}^n$, $b_1 : \mathbb{R} \to \mathbb{R}$, $b_2 : \mathbb{R}^{n-1} \to \mathbb{R}$ are measurable functions such that $\nu \leq b_1(\cdot) \leq L$ and $\nu \leq b_2(\cdot) \leq L$ with constants $0 < \nu \leq L < \infty$. Here, we denote by $\varphi' : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ to mean a $C^1((0,\infty)) \cap C([0,\infty))$ function satisfying $\varphi'(0) = 0$ and

$$0 < \kappa_1 - 1 \leq \frac{t\varphi''(t)}{\varphi'(t)} \leq \kappa_2 - 1 < \infty \tag{1.1.2}$$

for some constants $\kappa_1, \kappa_2 > 1$. If $\varphi(t) = t^p$ for $p > 1$, then $\kappa_1 = \kappa_2 = p$ and so our problem is a natural generalization of the p–Laplace equation. Typical examples of the function φ include

$$\varphi(t) = t^p + a_0 t^q \quad (1 < p \leq q,\ a_0 \geq 0) \quad \text{and} \quad \varphi(t) = t^p \log(e + t) \quad (p > 1).$$

The rate of growth and decay of the function φ varies but is controllable in terms of the constant κ_1 and κ_2 as in (1.1.2). We refer to the noteworthy results [32, 68, 74, 82, 97, 98, 100, 135, 136] concerning the nonlinear problem with Orlicz growth, and [24, 87, 181] regarding the related problems.

With the function space $W^{1,\varphi}(\Omega)$ to be introduced in Chapter 2, the purpose of the present section is to prove the following implication

$$\varphi(|F|) \in L^\gamma_{\mathrm{loc}} \quad \Rightarrow \quad \varphi(|Du|) \in L^\gamma_{\mathrm{loc}} \quad \text{for each } \gamma > 1 \tag{1.1.3}$$

for a weak solution $u \in W^{1,\varphi}(\Omega)$ of (1.1.1). In [177], the author shows that if there is no regularity assumption, (1.1.3) fails in general even when $\varphi(t) = t^2$, the case of Laplace equation. Indeed, the VMO condition for both $b_1(x_1)$ and $b_2(x')$ are considered in [89], and the small BMO condition for both $b_1(x_1)$ and $b_2(x')$ are considered in [51, 58]. Now it is natural to ask that such a smallness assumption is indeed the minimal one. One may conjecture that

- considering the paper [67], (1.1.3) should hold when there is no regularity assumption in $x_1 \mapsto b_1(x_1)$.
- On the other hand, according to the paper [177], (1.1.3) fails in general when there is no regularity assumption in both $x_1 \mapsto b_1(x_1)$ and $x' \mapsto b_2(x')$.

Following this viewpoint, a possible minimal condition is that it is only measurable in $x_1 \mapsto b_1(x_1)$ and has a small BMO for $x' \mapsto b_2(x')$. In [54, 103, 148], the authors are able to show the implication (1.1.3) with such a partial BMO assumption for the linear problem. Later, the authors of [36] prove (1.1.3) for the nonlinear problem with linear growth, while the p–Laplace case is studied in [149]. For a further regularity results under the partial BMO assumption, we refer to [104, 150] for parabolic problems, [37] for Riesz potential estimates for parabolic equations, [201] for Morrey regularity for the elliptic equations.

The main difficulty in considering this problem arises from the fact that the homogeneity does not hold for the function φ and that we are only able to use the property (1.1.2). Moreover, compared to p–Laplacian problem, we should argue with the unified approach for both the case $1 < p < 2$ (subquadratic case) and $p > 2$ (superquadratic case) in order to carry out the resulting delicate and complicated computations. We overcome these difficulty by developing some analytic tools in the literature to deal with Orlicz growth in order to employ the Moser type iteration argument along with the Caccioppoli type estimate and Sobolev-Poincaré inequality.

Equations with u-dependence. Next, we shall deal with the global gradient estimates of a weak solution to the following Dirichlet problem:

$$\begin{cases} -\operatorname{div}\left(a(x,u)\dfrac{G'(|Du|)}{|Du|}Du\right) = -\operatorname{div}\left(\dfrac{G'(|F|)}{|F|}F\right) & \text{in } \Omega \\ \qquad\qquad\qquad\qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1.4}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded domain with possibly nonsmooth boundary $\partial\Omega$ and G is an N-function in the sense of the definition introduced in Chapter 2, $a : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ with $\nu \leq a(\cdot) \leq L$ with constants $0 < \nu \leq L < \infty$, whereas $F : \Omega \to \mathbb{R}^n$ is a given vector field such that $F \in L^G(\Omega; \mathbb{R}^n)$. The main purpose of this section is to prove that any bounded weak solution u to the equation (1.1.4) satisfies the following implication

$$G(|F|) \in L^\gamma(\Omega) \Longrightarrow G(|Du|) \in L^\gamma(\Omega) \quad \text{for any} \quad \gamma > 1 \tag{1.1.5}$$

under the most general structure and minimal regularity assumptions on $a(x,u)$ and $\partial\Omega$. To go further, we briefly overview the previous known results related to our purpose in the only sense of Calderón-Zygmund theory case by case:

1. In the case of $G(t) \equiv t^p$ for $p > 1$, our problem is reduced to a nonlinear elliptic problem with p-growth, which is considered in [47, 184].

2. If $a(x,u) \equiv$ constant, then the global Calderón-Zygmund estimate over the whole domain $\mathbb{R}^n$ have been achieved in [204] and the same result was proved over bounded non-smooth domains [33].

3. When $a(x,u)$ has no u-dependence, the Lipschitz regularity has been proved in [72] for equations and [73] for systems, respectively. In this case, Calderón-Zygmund estimates over non-smooth domain have been obtained in [70].

In particular, assuming the Lipschitz continuity for $z \mapsto a(x,z)$, the authors of [184] proved the local Calderón-Zygmund type estimates, whereas in [47] only the uniform continuity is assumed in z variable, and the global Calderón-Zygmund estimates have been obtained based on [51]. We point out that we provide the results globally in a unified way under the general Orlicz setting. The main difficulties for obtaining the desired result are the lack of homogeneity properties naturally appearing from the presence of solution-dependence in z-variable in the nonlinearity. To overcome them, we here interplay the minimal regularity assumptions offered in (3.2.3) and (3.2.4) with a new parameter K in (3.2.20), a dilated size of the associated domain under a correct scaling and normalization as in Remark 3.2.5, so that we are able to adopt the method so-called maximal function free technique introduced first in [3] in order to derive the desired global estimate.

1.2 General double phase problems

In this section, we investigate elliptic equations which have the prototype of

$$\begin{aligned} &\operatorname{div}(|Du|^{p-2}Du + a(x)|Du|^{q-2}Du) \\ &\qquad = \operatorname{div}(|F|^{p-2}F + a(x)|F|^{q-2}F) \quad \text{in } \Omega, \end{aligned} \tag{1.2.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain ($n \geq 2$), $F = (f_1, \dots, f_n) : \Omega \to \mathbb{R}^n$ is a given vector field such that $|F| \in L^1(\Omega)$, the constants p, q and a Hölder continuous function $a(\cdot) : \Omega \to [0,\infty)$ satisfy

$$1 < p < q < \infty, \tag{1.2.2}$$

$$a(\cdot) \in C^{0,\alpha}(\Omega) \quad \text{for some } \alpha \in (0,1] \tag{1.2.3}$$

and

$$\frac{q}{p} \leq 1 + \frac{\alpha}{n}. \tag{1.2.4}$$

Double phase problem is originally connected to the study of homogenization theory and the Lavrentiev phenomenon, as in [206, 208, 209]. Since then, there have been a lot of progress and regularity results in the realm of double phase problem. In [17, 117] it is ascertained that (1.2.2)–(1.2.4) are unavoidable conditions not only for the absence of Lavrentiev phenomenon but also for the higher integrability of the gradient of the weak solution. In recent years several notable results are known, see [22, 77, 78] for the $C^{1,\alpha}$-regularity of minimizers for the double phase functionals, and [79, 84] for the gradient estimates of solutions to the equations related to (1.2.1). Now we see many research activities for this type of the problem (1.2.1), see [41, 69, 76, 85, 130, 165, 188, 189, 199], and several types of generalizations, including [88, 83] for multi–phase problem, [38, 65, 182, 194, 198] for double phase problem with variable exponents, and [13, 43] for Orlicz double/multi–phase problem. We also refer [21, 24, 87, 86, 133, 136, 181] for further generalization and studies.

The equation (1.2.1) is regarded to involving a non-uniformly elliptic operator since the ratio between the highest and the lowest eigenvalue of the matrix $\partial_z \left[|z|^{p-2}z + a(x)|z|^{q-2}z\right]$ with $x \in \Omega$ and $z \in \mathbb{R}^n$ could be comparable to

$$1 + R^{\alpha}|z|^{q-p} \approx \frac{|z|^{p-2} + \sup_{B_R} a(x)|z|^{q-2}}{|z|^{p-2} + \inf_{B_R} a(x)|z|^{q-2}},$$

if a ball $B_R(x_0)$ intersects the zero set $\{a(x) = 0\}$ and $a(x) \approx |x - x_0|^{\alpha}$. The above ratio is not bounded with respect to z-variable so that the related operator is non-uniformly elliptic.

Double phase problems with BMO nonlinearity. With the assumptions (1.2.2)–(1.2.4) and the notation

$$H(x,t) = t^p + a(x)t^q \quad (x \in \Omega,\ t \geq 0), \tag{1.2.5}$$

we deal with the following equation of the form:

$$\operatorname{div}(A(x, Du)) = \operatorname{div}(|F|^{p-2}F + a(x)|F|^{q-2}F) \quad \text{in } \Omega. \tag{1.2.6}$$

Here, $F = (f_1, \dots, f_n) : \Omega \to \mathbb{R}^n$ is a given vector field such that $H(x, |F|) \in L^1(\Omega)$, and the given nonlinearity $A(x, z) : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory vector field which is specified later in 3.3.1. A weak solution u of (1.2.6) belongs to the Musielak-Orlicz space $W^{1,H}(\Omega)$ which is specifically defined in Chapter 2. Under a certain smallness assumption on $x \mapsto A(x, z)$, we want to find the validity of the following implication for $u \in W^{1,H}(\Omega)$:

$$H(x, |F|) \in L^\gamma(\Omega) \quad \Longrightarrow \quad H(x, |Du|) \in L^\gamma_{\mathrm{loc}}(\Omega) \quad (\forall \gamma > 1). \tag{1.2.7}$$

Following the spirit of the paper [58, 51] together with [89, 177], it is known that imposing the small BMO condition to the nonlinearity A is one of the natural smallness conditions for obtaining the relation (1.2.7). There are many relevant results for the p-Laplacian case, see [6, 56, 103, 175, 176]. Also, the small BMO condition is considered for the weighted Laplacian and p-Laplacian problem, see [16, 61, 191]. Furthermore, this condition is properly extended and imposed for several kinds of nonstandard growth problems such as $p(x)$-Laplace [45, 151], generalized p-Laplace [70] and the borderline case of double phase problem [42] for proving Calderón-Zygmund type estimate like (1.2.7). The purpose of this section is to establish an optimal Calderón-Zygmund theory for the double phase problem under this kind of small BMO assumption for the nonlinearity A as in Definition 3.3.1.

Compared to the another nonstandard growth problems, double phase problem exhibits the drastic phase transition as the value of the function $a(x)$ changes. Thus a known technique, now being considered to be classical, for obtaining the gradient estimates is the difference quotient technique, developed in [79] and used later in [13, 41, 83]. This technique enables us to achieve much higher integrability estimates for the gradient of solutions of a suitable reference problem to provide the desired comparison estimates. On the other hand, in order to apply this approach, we need to assume C^α-continuity of the nonlinearity A for the x-variable. Here we consider a new approach which needs only from the small higher integrability result, together with the extrapolation results based on [129, 136], and so thereby we only impose the small BMO assumption to our problem. Meanwhile, along with this small BMO assumption, we assume an extra structure condition as

$$q - p < \frac{\nu}{L}. \tag{1.2.8}$$

This assumption is necessary when we consider the uniform ellipticity condition for a typical p-Laplacian problem on the nonlinearity A. We discuss the legitimacy of the assumption (1.2.8) in Remark 3.3.3 below.

ω-minimizers of functionals to $p(x)$, $q(x)$ double phase. We concerns the integral functionals involving non-uniformly elliptic operators. The functional under consideration is

$$\mathcal{P}(w,\Omega) := \int_\Omega \left(f_1(x,Dw) + a(x)f_2(x,Dw)\right)\,dx \tag{1.2.9}$$

whose model case is when $f_1(x,z) = |z|^{p(x)}$ and $f_2(x,z) = |z|^{q(x)}$. Here Ω is a bounded open domain in $\mathbb{R}^n$ for $n \geq 2$ and the continuous functions $p(x), q(x), a(x) : \Omega \to \mathbb{R}$ are assumed to satisfy

$$\begin{aligned} 0 \leq a(x) \in C^{0,\alpha}(\Omega), \quad 1 < \gamma_1 \leq p(x) \leq q(x) \leq \gamma_2 < \infty, \\ \frac{q(x)}{p(x)} \leq 1 + \frac{\alpha}{n} \end{aligned} \tag{1.2.10}$$

for some constants $\alpha \in (0,1], \gamma_1, \gamma_2$ and for every $x \in \Omega$. Additionally, we assume that $p(x)$ and $q(x)$ are log-Hölder continuous in Ω, i.e., there exists a constant $c_{p(\cdot),q(\cdot)} > 0$ such that

$$|p(x) - p(y)| + |q(x) - q(y)| \leq \frac{c_{p(\cdot),q(\cdot)}}{-\log|x-y|} \tag{1.2.11}$$

for every $x, y \in \Omega$ with $|x-y| \leq \frac{1}{2}$.

For a given nonhomogeneous term $F = (f^1, \cdots, f^n) : \Omega \to \mathbb{R}^n$, $\tilde{H}(x,F) \in L^1(\Omega)$ where

$$\tilde{H}(x,z) = |z|^{p(x)} + a(x)|z|^{q(x)} \quad (x \in \Omega, z \in \mathbb{R}^n)\,, \tag{1.2.12}$$

the main goal of this section is to establish an optimal Calderón-Zygmund theory for ω-minimizers of the functional

$$\mathcal{F}(w,\Omega) := \mathcal{P}(w,\Omega) - \int_\Omega \left\langle |F|^{p(x)-2}F + a(x)|F|^{q(x)-2}F, Dw \right\rangle\,dx \tag{1.2.13}$$

among $w \in W^{1,1}(\Omega)$ with $\tilde{H}(x,Dw) \in L^1(\Omega)$, in the sense of variable exponent Lebesgue spaces. More precisely, we suppose that for a non-decreasing function $\tilde{\mu}(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ and a continuous function $\gamma(\cdot) : \Omega \to \mathbb{R}$,

$$\begin{gathered} 1 < \gamma_1 \leq \gamma(x) \leq \gamma_2 < \infty, \quad |\gamma(x) - \gamma(y)| \leq \tilde{\mu}(|x-y|), \\ \tilde{\mu}(r) \log \frac{1}{r} \leq c_\gamma, \end{gathered} \tag{1.2.14}$$

and we want to identify minimal regularity assumptions on the associated energy densities $f_1(x,z)$ and $f_2(x,z)$ under which an ω-minimizer $u \in W^{1,1}(\Omega)$ to $\mathcal{F}(w,\Omega)$ satisfies the desired implication

$$H(x,F) \in L^{\gamma(\cdot)}(\Omega) \quad \Longrightarrow \quad H(x,Du) \in L^{\gamma(\cdot)}_{\mathrm{loc}}(\Omega). \tag{1.2.15}$$

We will describe a detailed and precise notion of ω-minimizer later in Definition 3.4.1. For the case that u is a minimizer to $\mathcal{F}(w,\Omega)$ and $\gamma(\cdot) \equiv \gamma$, it is proved in a recent paper [38] that the relation (1.2.15) holds true. The aim of the present section is to show that it still holds even to a generalized minimizer such as ω-minimizer.

The study on generalized minimizers in the literature has been made in many research areas such as geometric measure theory [7], C^α-regularity [101, 133, 144, 188], higher integrability [131], singular sets [154, 155], and Calderón-Zygmund estimates [46, 49, 187]. A main difficulty in establishing the desired regularity estimates is that an ω-minimizer does not necessarily satisfy the Euler-Lagrange equation of the assigned functional (1.2.13) and so the regularity results obtained from the equations can not be directly applied to our variational problem. In this section we are using Taylor's formula and considering minimizers or solutions of appropriate reference problems in order to prove the implication (1.2.15) with the desired Calderón-Zygmund type estimate. To this end, we first show that (1.2.10) and (1.2.11) are unavoidable for the absence of Lavrentiev phenomenon regarding (1.2.9), and then prove a higher integrability for the gradient to the energy functional. Our result contributes to the theory of Calderón-Zygmund estimates to be more applicable in other areas such as the various concept of generalized minimizers. In particular, we clarify the dependence of the constants in the main result and we give a comprehensive investigation of the comparison estimates which makes the proof of [38] in a rigorous and clear way.

Orlicz double phase problems with variable exponents. This section aims to investigate the gradient estimates for weak solutions of elliptic equations of the divergence form with general non-standard growth conditions. The initial model equation under consideration is of the form

$$
\begin{aligned}
-\operatorname{div}&\left(\frac{G^{p(x)}(|Du|)}{|Du|^2}Du + a(x)\frac{H^{q(x)}(|Du|)}{|Du|^2}Du\right)\\
&= -\operatorname{div}\left(\frac{G^{p(x)}(|F|)}{|F|^2}F + a(x)\frac{H^{q(x)}(|F|)}{|F|^2}F\right) \quad \text{in} \quad \Omega,
\end{aligned}
\tag{1.2.16}
$$

which is primarily defined for $u \in W^{1,1}(\Omega)$, here $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded open domain and $F : \Omega \to \mathbb{R}^n$ is a given vector field. The functions G, H appearing in the equation (1.2.16) belong to $\mathcal{N}$ in the sense of Definition 2.1.1, and the functions in the exponents $p(\cdot), q(\cdot) : \Omega \to [1, \infty)$ are bounded and log-Hölder continuous functions in the following way that

$$1 \leq p(x), q(x) \leq m_{pq} \quad \text{for every} \quad x \in \Omega, \tag{1.2.17}$$

and

$$|p(x) - p(y)| + |q(x) - q(y)| \leq \frac{M_{pq}}{-\log|x-y|} \tag{1.2.18}$$

for some non-negative constants m_{pq} and M_{pq}, whenever $x, y \in \Omega$ with $|x - y| \leq 1/2$, whereas the coefficient function $a : \Omega \to [0, \infty)$ satisfies

$$0 \leq a(\cdot) \in C^{0,\alpha}(\Omega), \quad \alpha \in (0, 1]. \tag{1.2.19}$$

We shall assume that the functions presented above satisfy the central assumption in this section:

$$\kappa := \sup_{x \in \Omega} \sup_{t > 0} \frac{H^{q(x)}(t)}{G^{p(x)}(t) + G^{(1+\frac{\alpha}{n})p(x)}(t)} < \infty. \tag{1.2.20}$$

Denoting

$$\Psi(x, z) := G^{p(x)}(|z|) + a(x)H^{q(x)}(|z|) \text{ for every } x \in \Omega \text{ and } z \in \mathbb{R}^n \text{ or } z \in \mathbb{R},$$

our interest lies in finding the optimal condition under which the following local Calderón-Zygmund type relation

$$\Psi(x, F) \in L^{\gamma}(\Omega) \Longrightarrow \Psi(x, Du) \in L^{\gamma}_{\mathrm{loc}}(\Omega) \tag{1.2.21}$$

holds for every $\gamma > 1$.

The problem extensively covers the following ones:

1. $p(x)$-Laplacian: $\Psi(x, z) = |z|^{p(x)}$, e.g., [1, 2, 35, 44, 94, 99].
2. Double phase: $\Psi(x, z) = |z|^p + a(x)|z|^q$, e.g., [34, 41, 77, 78, 79, 84, 188, 199].
3. Orlicz growth: $\Psi(x, z) = G(|z|)$, e.g., [11, 40, 70, 97, 100, 129].
4. Double phase with variable exponents: $\Psi(x, z) = |z|^{p(x)} + a(x)|z|^{q(x)}$, e.g., [38, 39, 65, 194, 198].
5. Orlicz double phase: $\Psi(x, z) = G(|z|) + a(x)H(|z|)$, e.g., [43, 12, 13].
6. Orlicz growth with variable exponents: $\Psi(x, z) = G^{p(x)}(|z|)$, e.g., [124, 186].

The all significant examples aforementioned fall in a realm of the functionals with nonstandard growth treated first in a series of papers [171, 172, 173]. Over the decades the problems with nonstandard growth have been the object of intensive studies, see for instance [115, 116, 117, 121] reference therein. There are two keywords in this section: first one is $p(x)$-Laplacian and the other one is double phase or Orlicz double phase.

Zhikov was the first who introduced $p(x)$-growth functionals in [143] which go beyond the p-Laplace problem by investigating that the integrand in the energy functional can be varied depending on each point of the domain to deal with the generalizations of the p-Laplace problems. For example, $p(x)$-Laplacian problems are considered in many models coming from non-Newtonian fluids [4], homogenization theory [207] and electrorheological fluids [196]. The functions $p(\cdot)$ and $q(\cdot)$ are assumed to be continuous and to enjoy the assumption (1.2.18) below, which is not avoidable even when we consider $p(x)$-Laplacian problems. In the case of $G(t) = t^{p_m}$ and $H(t) = t^{q_m}$ for some constants $1 < p_m, q_m$, our problem can be reduced to double phase with variable exponents type problem examined in [38], where in order to obtain the Calderón-Zygmund type estimates it has been shown that the minimal required condition on the variable exponents is

$$\frac{p_m p(x)}{q_m q(x)} \leq 1 + \frac{\alpha}{n} \quad \text{for every} \quad x \in \Omega. \tag{1.2.22}$$

On the other hand, in the case of the functions $p(\cdot) \equiv q(\cdot) \equiv 1$, then our problem (1.2.16) can be curtailed to Orlicz double phase problem which was investigated in [12, 43]. In particular, in such a case it can become double phase problem when $G(t) = t^{p_m}$ and $H(t) = t^{q_m}$ for some constants $1 < p_m \leq q_m$. The double phase problem was also introduced first by Zhikov [206, 207] in order to provide models of strongly anisotropic materials in the framework of homogenization and nonlinear elasticity and later applied in the image restoration [130]. Recently, in [12] it has been proved that necessary and sufficient condition to have Calderón-Zygmund type implication like (1.2.21) is

$$\sup_{t>0} \frac{H(t)}{G(t) + G^{1+\frac{\alpha}{n}}(t)} < +\infty. \tag{1.2.23}$$

In the view of the conditions (1.2.22) and (1.2.23) our central assumption (1.2.20) is needed and not avoidable. Under this assumption we are able to obtain the desired Calderón-Zygmund estimates (1.2.21) and to have the absence of Lavrentiev phenomenon [208], see Theorem 3.5.5. Besides the papers mentioned above, there is a richness of literature concerning general Musielak-Orlicz growth problems, see for instance [132, 133, 136] and references therein.

We remark that one of the most difficult parts for proving the relation (1.2.21) under the assumption (1.2.20) is to obtain the higher integrability estimates and freeze the exponent functions of the nonlinearity in a proper way during the comparison process. We also point out that the methods, that have been used in earlier papers, cannot be directly employed in our case. Moreover, because of the lack of homogeneity properties for the equations of the Orlicz double phase with variable exponents type, we adopt the so-called maximal function-free technique initially introduced in the work [3], alongside a method of approximation developed and employed in [58, 51] and the references therein for each variant, in order to obtain the interior Calderón-Zygmund estimates (3.5.14).

1.3 Elliptic equations with degenerate weights

We consider the weighted elliptic equations which have the prototype of the form

$$\begin{aligned} -\mathrm{div}(\mathbb{M}^2(x)Du) &= -\mathrm{div}(\mathbb{M}^2(x)F) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.3.1}$$

Here, $\Omega \subset \mathbb{R}^n$ is a bounded domain with $n \geq 2$, $1 < p < \infty$, $F : \Omega \to \mathbb{R}^n$ is a given vector-valued function, $\mathbb{M} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is a given symmetric and positive definite matrix-valued weight satisfying

$$|\mathbb{M}(x)|\,|\mathbb{M}^{-1}(x)| \leq \Lambda \quad (x \in \mathbb{R}^n) \tag{1.3.2}$$

for some constant $\Lambda \geq 1$, where $|\cdot|$ is the spectral norm.

Let us define the scalar weight

$$\omega(x) = |\mathbb{M}(x)|. \tag{1.3.3}$$

Supposing that ω^2 is an $\mathcal{A}_2$-Muckenhoupt weight (see Chapter 2) and $F \in L^2_\omega(\Omega) := L^2(\Omega, \omega^2\,dx)$, we prove the following global estimate

$$|F|\omega \in L^q(\Omega) \quad \Longrightarrow \quad |Du|\omega \in L^q(\Omega) \quad (\forall q > 2) \tag{1.3.4}$$

under the suitable assumption of $\partial\Omega$. When $\mathbb{M}$ is the identity matrix, our result is related to [59, 60], from which the linear Calderón-Zygmund theory originates. If $\mathbb{M}(x)$ is assumed to be only measurable, but uniformly elliptic in the sense that

$$\lambda_{\min}|\xi|^2 \leq \left\langle \mathbb{M}^2(x)\xi, \xi \right\rangle \leq \lambda_{\max}|\xi|^2 \tag{1.3.5}$$

for any $x \in \Omega$ and $\xi \in \mathbb{R}^n$, a local version of the result (1.3.4) is proved in [177] for $q \in [2, 2+\epsilon)$ for some small $\epsilon > 0$. To obtain the estimate for all $q \in (1,\infty)$, one needs additional regularity assumption on $\mathbb{M}$. In [89] the assumption $\mathbb{M} \in \mathrm{VMO}$ is made to prove (1.3.4) for $q \in (1,\infty)$ for $\partial\Omega \in C^{1,1}$ and in [9] for $\partial\Omega \in C^1$. A global result on $\mathbb{R}^n$ is obtained in [141] and a local result for the case of systems is proved in [90] for $\mathbb{M} \in \mathrm{VMO}$. The condition $\mathbb{M} \in \mathrm{VMO}$ is relaxed to a small BMO condition. The global results for bounded domains are obtained in a series of papers [27, 31, 51].

Muckenhoupt weights have extensive applications in the field of analysis including partial differential equations and harmonic analysis, see [94, 120, 122, 127, 129, 137, 138] and references therein. In particular, the regularity of elliptic and parabolic equations with degenerate/singular coefficients has been exhaustively investigated along with the research of establishing uniform weighted norm inequalities for the purpose of identifying minimal requirements on the matrix weight and the nonlinearity for the optimal regularity theory to be valid in the literature as in, for instance, [16, 61, 106, 107, 108, 191] and references therein. This allows us to analyze the behaviors and properties of solutions of such wide ranging problems even when the coefficient has singularity or degeneracy in some region of the domain. Those problems are usually considered as a generalization of elliptic equations with a uniform ellipticity and the theory developed in this direction is a natural outgrowth of another thing under a certain regularity condition of singular or degenerate coefficients connecting to the associated uniformly elliptic operator.

Sharp global gradient estimates. We study the following degenerate elliptic equation of the form

$$\begin{aligned} -\mathrm{div}(\mathbb{A}(x)\nabla u) &= -\mathrm{div}(\mathbb{A}(x)F) \qquad &&\text{in } \Omega, \\ u &= 0 &&\text{on } \partial\Omega, \end{aligned} \tag{1.3.6}$$

in the linear case, and of the form

$$\begin{aligned} -\mathrm{div}(|\mathbb{M}(x)\nabla u|^{p-2}\mathbb{M}^2(x)\nabla u) &= -\mathrm{div}(|\mathbb{M}(x)F|^{p-2}\mathbb{M}^2(x)F) \ &&\text{in } \Omega, \\ u &= 0 &&\text{on } \partial\Omega, \end{aligned} \tag{1.3.7}$$

in the non-linear case. We often write $\mathbb{M}(x)$ to emphasize the dependence of the weight on x. Here, $\Omega \subset \mathbb{R}^n$ is a bounded domain with $n \geq 2$, $1 < p < \infty$, $F : \Omega \to \mathbb{R}^n$ is a given vector-valued function, $\mathbb{M} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is a given symmetric and positive definite matrix-valued weight satisfying (1.3.2), and $\mathbb{A}(x) := \mathbb{M}^2(x)$. This condition says that $\mathbb{M}$ has a uniformly bounded condition number. Note that a right-hand side of the form $-\mathrm{div}G$ with $G : \Omega \to \mathbb{R}^n$ can be immediately rewritten in the above form in terms of F. Note that (1.3.6) is a special case of (1.3.7) for $p = 2$. The condition (1.3.2) in this case reads as

$$|\mathbb{A}(x)|\,|\mathbb{A}^{-1}(x)| \leq \Lambda^2 \quad (x \in \mathbb{R}^n). \tag{1.3.8}$$

Let us define the scalar weight

$$\omega(x) = |\mathbb{M}(x)| = \sqrt{|\mathbb{A}(x)|}. \tag{1.3.9}$$

If ω^p is an $\mathcal{A}_p$-Muckenhoupt weight (see Chapter 2) and $F \in L^p_\omega(\Omega) := L^p(\Omega, \omega^p\, dx)$, then there exists a unique weak solution $u \in W^{1,p}_{0,\omega}(\Omega)$ of (1.3.7), which means

$$\int_\Omega |\mathbb{M}(x)\nabla u|^{p-2}\mathbb{M}^2(x)\nabla u \cdot \nabla\phi\, dx = \int_\Omega |\mathbb{M}(x)F|^{p-2}\mathbb{M}^2(x)F \cdot \nabla\phi\, dx \tag{1.3.10}$$

for all $\phi \in W^{1,p}_{0,\omega}(\Omega)$. Moreover, we have the following standard energy estimate

$$\int_\Omega |\nabla u|^p \omega^p\, dx \leq c \int_\Omega |F|^p \omega^p\, dx \tag{1.3.11}$$

with $c = c(n, p, \Lambda)$, see [16, 61].

We investigate the validity of the following global maximal regularity estimates

$$\int_\Omega |\nabla u|^q \omega^q\, dx \leq c \int_\Omega |F|^q \omega^q\, dx \tag{1.3.12}$$

for every $q \in (1, \infty)$ in the linear case (1.3.6), and for every $q \in [p, \infty)$ in the non-linear case (1.3.7). The positive constant c is independent of F and u, under minimal extra assumptions on both the boundary of Ω and the weight $\mathbb{M}$ in addition to (1.3.2). We pay special attention to the optimal dependence of the parameters of the boundary and of the coefficients on q. Estimates of this type are also known under the name of global *non-linear Calderón-Zygmund estimates*. Our main results are presented in Theorem 4.1.3 and 4.1.4.

In this section we are also interested in the degenerate case, where (1.3.5) fails. The most simple example of which is $\mathbb{A}(x) = |x|^{\pm\epsilon}\,\mathrm{id}$ with $\epsilon > 0$ small. Instead of (1.3.5), we assume

$$\Lambda^{-2}\mu(x)|\xi|^2 \leq \langle \mathbb{A}(x)\xi, \xi\rangle \leq \mu(x)|\xi|^2 \tag{1.3.13}$$

where $\mu(x) := |\mathbb{A}(x)| = |\mathbb{M}(x)|^2 = \omega^2(x)$. In [120], it is proved that if μ belongs to the Muckenhoupt class $\mathcal{A}_2$, then the solution u of (1.3.6) is Hölder continuous. Gradient estimates are obtained in [61] under (1.3.13), $\mu \in \mathcal{A}_2$ and a smallness assumption in terms of a weighted BMO norm of $\mathbb{A}$. They yield $|F|^q\mu \in L^1_{\mathrm{loc}} \Rightarrow |\nabla u|^q\mu \in L^1_{\mathrm{loc}}$ for all $q \in (1,\infty)$, including the case $\mu(x) = |x|^{\pm\epsilon}\,\mathrm{id}$ for small $\epsilon > 0$. The global result is obtained in [191] and the local result for the case of systems is proved in [62]. In the recent paper [16], the authors prove a new type of gradient estimates with the implication that $(|F|\omega)^q \in L^1_{\mathrm{loc}} \Rightarrow (|\nabla u|\omega)^q \in L^1_{\mathrm{loc}}$ for all $q \in (1,\infty)$, assuming (1.3.13) and the smallness condition for the BMO norm of $\log \mathbb{A}$ as follows:

$$\sup_{B \Subset \Omega} \fint_B |\log \mathbb{A}(x) - (\log \mathbb{A})_B\,|\,dx \leq \frac{\delta}{q} \tag{1.3.14}$$

for some $\delta = \delta(n,p,\Lambda)$, where

$$(f)_B = \fint_B f\,dx$$

for an integrable function $f : \mathbb{R}^n \to \mathbb{R}^{n\times n}_{\mathrm{sym}}$. Here, we can define $\log \mathbb{A} : \mathbb{R}^n \to \mathbb{R}^{n\times n}_{\mathrm{sym}}$, the logarithm of the matrix-valued weight $\mathbb{A}$, since $\mathbb{A}$ is positive definite almost everywhere. This novel log–BMO condition of [16] not only includes the degenerate weights of the form $\mathbb{A}(x) = |x|^{\pm\epsilon}\,\mathrm{id}$ for small $\epsilon > 0$, but also has the optimality in terms of the obtainable integrability exponent q. The condition of the logarithm of a matrix weight in BMO is natural, since in the scalar weight case, $\mu \in \mathcal{A}_p$ for some $p \geq 1$ implies $\log(\mu) \in \mathrm{BMO}$, and conversely, for any $p \geq 1$ there exists $\delta = \delta(p)$ such that if $[\mu]_{\mathrm{BMO}} \leq \delta$, then we have $e^\mu \in \mathcal{A}_p$.

Compared to [61], where $\mu\,dx$ is treated as a measure, the degenerate weight μ or better ω in [16] plays the role of a multiplier. Also here we treat ω as a multiplier, which seems also important for the optimal dependency of q on the constants.

Now, consider the p-Laplacian case. If we write $A(\xi) := |\xi|^{p-2}\xi$ and $\mathcal{A}(x,\xi) := |\mathbb{M}(x)\xi|^{p-2}\mathbb{M}^2(x)\xi$, then (1.3.7) is equivalent to

$$-\mathrm{div}\mathcal{A}(\cdot,\nabla u) = -\mathrm{div}\mathcal{A}(\cdot,F). \tag{1.3.15}$$

Writing $\mathbb{M}^2 = \mathbb{A}$ and $\mathcal{A}(\cdot,F) = G$ for $\mathbb{A} : \Omega \to \mathbb{R}^{n\times n}_{\mathrm{sym}}$ and $G : \Omega \to \mathbb{R}^n$ we can write (1.3.15) as

$$-\mathrm{div}\left(\langle \mathbb{A}\nabla u, \nabla u\rangle^{\frac{p-2}{2}}\, \mathbb{A}\nabla u\right) = -\mathrm{div} G. \tag{1.3.16}$$

Then u is the minimizer of the following functional:

$$\begin{aligned}\mathcal{P}(v) &:= \frac{1}{p}\int_\Omega \langle \mathbb{A}\nabla v, \nabla v\rangle^{\frac{p}{2}}\, dx - \int_\Omega G\cdot\nabla v\, dx \\ &= \frac{1}{p}\int_\Omega |\mathbb{M}\nabla v|^p\, dx - \int_\Omega |\mathbb{M}F|^{p-2}\mathbb{M}F\cdot(\mathbb{M}\nabla v)\, dx.\end{aligned}$$

If $p \in (1,\infty)$ and $\mathbb{M} = \mathrm{id}$, then $A(\nabla u) = \mathcal{A}(\cdot, \nabla u)$. In this case, the Hölder continuity of u and ∇u is investigated in [161, 203], and the gradient regularity estimates were obtained in [91, 140]. In the recent years, there have been many research activities for the gradient estimates in terms of $A(\nabla u)$. The BMO type estimate with the implication that $G \in \mathrm{BMO} \Rightarrow A(\nabla u) \in \mathrm{BMO}$ is shown in [91] for $p > 2$ and [95] for $1 < p < \infty$. In [95], the implication $G \in C^{0,\alpha} \Rightarrow A(\nabla u) \in C^{0,\alpha}$ for small $\alpha > 0$ is proved. A local pointwise estimate is proved in [28] and extended to the global one in [29]. Estimates in Besov space and Triebel-Lizorkin spaces up to differentiability one for $n = 2$ and $p > 2$ are shown in [18]. Besov space regularity for ∇u is also considered in [14, 75]. The result $A(\nabla u) \in W^{1,2}$ when $\mathrm{div} G \in L^2$ is obtained in [61] for scalar equations for $p > 1$ and for vectorial systems in [62] for $p > \frac{3}{2}$ and for $p > 2(2-\sqrt{2}) \approx 1.1715$ in [15]. Gradient potential estimates are studied for equations in [156, 157] and for systems in [57, 112, 158, 159].

Now, we pay attention to the weighted case. The local version of (1.3.12) is proved for $1 < p < \infty$, with a uniformly elliptic weight $\mathbb{M}$ as in (1.3.5) with $\mathbb{M} \in \mathrm{VMO}$ in [152]. Since $\mathbb{M}$ is uniformly elliptic, we have $\omega(x) \eqsim 1$, so the results reduce to the transfer of L^q-regularity from F to ∇u. The global estimate is obtained in [153] with a $C^{1,\alpha}$ domain Ω for $\alpha \in (0,1]$. The assumption $\mathbb{M} \in \mathrm{VMO}$ has been weakened to the one that $\mathbb{M}$ has a small BMO-norm, as shown in [53, 56, 70, 164]. Under similar assumptions it is possible to replace the L^q-regularity transfer by $L^q(\sigma\, dx)$-regularity transfer for suitable Muckenhoupt weights σ, see [48, 175, 176, 192]. Note that the weight σ is not related to the weight ω of the equation.

Now, we introduce Lipschitz domains along with our optimal regularity assumption for the boundary of the domain.

Definition 1.3.1. *Let $\delta \in [0, \frac{1}{2n}]$ and $R > 0$ be given. Then Ω is called (δ, R)–Lipschitz if for each $x_0 \in \partial\Omega$, there exists a coordinate system $\{x_1, \ldots, x_n\}$ and Lipschitz map $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $x_0 = 0$ in this coordinate system, and there holds*

$$\Omega \cap B_R(x_0) = \{x = (x_1, \ldots, x_n) = (x', x_n) \in B_R(x_0) : x_n > \psi(x')\} \quad (1.3.17)$$

and

$$\|\nabla\psi\|_\infty \leq \delta. \quad (1.3.18)$$

Imposing a Lipschitz condition for the boundary of the domain appears in many papers, the regularity and the asymptotic behavior of caloric function [8], homogenization [146], oblique derivative problem [166, 168, 170], Hölder continuity of solutions for Robin boundary condition [185], regularity results for elliptic Dirichlet problem [197], Calderón-Zygmund estimates [30, 53]. We would like to point out that in [152] $C^{1,\alpha}$ regularity with $\alpha \in (0, 1]$ is assumed for $\partial\Omega$. It was observed in [9] (in the linear case) that $\partial\Omega \in C^1$ is enough. Our Lipschitz assumption for the boundary is weaker than both $C^{1,\alpha}$ and C^1 assumption on $\partial\Omega$, so Theorem 4.1.3 and 4.1.4 can be both applied in particular to $C^{1,\alpha}$ and C^1-domains. Our assumption is indeed an optimal one to be discussed later. The sharp relation between the smallness parameter of the boundary and the integrability exponent q is, as far as we know, new in the literature, even in the unweighted, linear case.

In principle we use a standard perturbation argument combined with the regularity of p-harmonic functions. This argument is for example developed in [139] and [58], and used in [152]. However, we modify this technique such that it is possible to obtain optimal estimates in terms of the smallness of oscillation parameter $|\log \mathbb{M}|_{\mathrm{BMO}}$ and the boundary regularity parameter $\|\nabla\psi\|_\infty$. In particular, we obtain a linear dependence for the reciprocal of the integrability exponent instead of an exponential one. This is one of the main novelties of this result.

The approach from [139] and [58] can be reduced to redistributional estimates in terms of maximal operator of the gradient. However this technique always introduces an exponential dependence of q on the smallness parameter δ. We avoid this problem by using a qualitative version of the global Fefferman-Stein inequality $\|f\|_{L^q(\mathbb{R}^n)} \leq cq\|\mathcal{M}_1^\sharp f\|_{L^q(\mathbb{R}^n)}$. The important feature is the linear dependency on the exponent q. This allows us to extract the sharp dependency of $|\log \mathbb{M}|_{\mathrm{BMO}}$ and $\|\nabla\psi\|_\infty$.

The interior maximal regularity with optimal constants was already described in [16]. In this thesis we extend those results up to the boundary with an optimal dependence and the boundary parameters. To this end, we first use the localization argument adapted to our boundary comparison estimate and provide the pointwise sharp maximal function estimate for the localized function of u. As an auxiliary step we provide $C^{1,\alpha}$-regularity and the decay estimates up to boundary for the solutions of the reference problems. To this end, we employ the reflection principle of the reference problems, which is one of the intrinsic property in the divergence type equation, see [174].

Weighted elliptic equations with measurable nonlinearity. We consider a general elliptic equation with singular/degenerate nonlinearity in divergence form

$$\begin{cases} \operatorname{div}(\mathbb{M}(x)A(x,\mathbb{M}(x)Du)) &= \operatorname{div}(\mathbb{M}^2(x)F) & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3.19}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain with nonsmooth boundary $\partial\Omega$ and $A(x,\xi) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory vector field with the assumption (4.2.2). Assuming (1.3.2) and ω^2 being $\mathcal{A}_2$-Muckenhoupt weight, the purpose is to prove that the implication

$$|\mathbb{M}(x)F| \in L^\gamma(\Omega) \implies |\mathbb{M}(x)Du| \in L^\gamma(\Omega) \tag{1.3.20}$$

is valid for every $\gamma > 2$ with the global Calderón-Zygmund type estimate

$$\int_\Omega |\mathbb{M}Du|^\gamma \, dx \leq c \int_\Omega |\mathbb{M}F|^\gamma \, dx \tag{1.3.21}$$

for some constant $c = c(\mathtt{data}, \gamma) > 0$. We ask what further minimal extra assumptions on Ω, A and $\mathbb{M}$ other than the mentioned structure assumptions described in (4.2.2)–(4.2.3) will allow us to obtain this estimate (1.3.20). Needless to say, as γ is close to 2, we do not need any extra assumptions. However, as γ is away from 2 and getting larger, we need to impose a suitable smallness assumption on $(x_2, \cdots, x_n) \mapsto A(x_1, x_2, \cdots, x_n, \xi)$, uniformly in x_1 and ξ even when $\mathbb{M}$ is the id matrix I_n, as we have seen from earlier works including [36, 37, 103]. We also mention notable related results [54, 105, 148, 149, 150] and references therein for various types of elliptic and parabolic problems for the case that $\mathbb{M}$ is a constant matrix.

Here in this section we are mainly focusing on the general case that $\mathbb{M}$ is a variable matrix weight. In this case of elliptic equations with degenerate weights, Hölder continuity is studied in [120] while an optimal gradient estimate in weighted Lebesgue spaces is investigated in [61] for the linear problem and in [16] for the nonlinear problem, respectively, in the spirit of Muckenhoupt matrix weights. We would like to mention a series of interesting works [106, 107, 108] when $\mathbb{M}(x) = x_1^\alpha I_n$ with α being in a suitable range in $\mathbb{R}$.

Returning to our problem (4.2.1), we observe from the basic relationship between Muckenhoupt weight and BMO(bounded mean oscillation) that $\log \mathbb{M}$ is in the BMO class, and so it is naturally expected that the minimal condition is a suitable small BMO condition on $\log \mathbb{M}$. Indeed, in the very interesting paper [16] in which the case of

$$A\left(x, \mathbb{M}(x)\xi\right) = \left(\mathbb{M}(x)\xi \cdot \mathbb{M}(x)\xi\right)^{\frac{p-2}{2}} \mathbb{M}(x)\xi$$

for $p > 1$ is considered, a local Calderón-Zygmund type estimate is proved under a small BMO condition on $\log \mathbb{M}$. Our present work is motivated from the variational problem [16]. However there are two main differences. Initially, our problem is not necessarily of variational form, as we are enlarging our inventory to include very general nonlinearities $A(x, \xi)$ which are depending on also x-variables. The other is that we extend the interior gradient estimates to study the higher integrability of weak solutions up to the nonsmooth boundary. Additionally, it is now well understood from [36, 37] that if $\mathbb{M} = I_n$, a minimal condition on (A, Ω) is the following:

1. An optimal regularity requirement on $x \mapsto A(x, \xi)$ is that it is merely measurable in one variable while it has a small BMO condition on the other variables.

2. A minimal geometric assumption on $\partial\Omega$ is sufficiently flat in Reifenberg sense.

We are again making the same assumptions on the triple $(\Omega, \mathbb{M}, A)$ as in $\mathbb{M}$ from [16] and as in the couple (Ω, A) from [36, 37, 103], respectively, in order to prove that the implication (1.3.20) is still available for the full range of $\gamma \in [2, \infty)$. A main difficulty comes from the inherent connectivity and complexity of the matrix weight $\mathbb{M}$ and the nonlinearity $A(x, \xi)$ as well as the nonsmooth boundary $\partial\Omega$. One idea is that given a large $\gamma > 2$ one can find a small universal constant $\delta > 0$ so that if the BMO semi-norm of $\log \mathbb{M}$ is less than δ, then $\mathbb{M}$ is in the $\mathcal{A}_\gamma$ class. Along with this basic observation, we are making a systematic analysis of competitive interplay between associated matrix weights and nonlinearities in view of the utility of assigned regularity assumptions on $(\Omega, \mathbb{M}, A)$.

More studies also need to be done to understand the measurability of the matrix weight $\mathbb{M}(x)$ in one of the variables as well as a precise dependence of the smallness parameter δ, in particular in terms of γ, though it seems unclear as this smallness assumption in the other variables except one variable is closely associated to both A and $\mathbb{M}$ as well as the choice of a point near the very irregular boundary and a size of the localized domain under consideration. We leave these issues to be investigated in the future.

The domain Ω under consideration in this section is usually called by a (δ, R)-Reifenberg flat domain. Its definition is as follows.

Definition 1.3.2. *We say that Ω is (δ, R)-Reifenberg flat if for every $r \in (0, R]$ and $x_0 \in \partial\Omega$, there exists a new coordinate system $\{y_1, \cdots, y_n\}$ with the origin at x_0 such that*

$$B_r(0) \cap \{y : y_n > \delta r\} \subset B_r(0) \cap \Omega \subset B_r(0) \cap \{y : y_n > -\delta r\}$$

holds in this coordinate system.

The boundary of this domain goes beyond the Lipschitz category with a small Lipschitz constant, and allows a fractal boundary such as Koch snowflake. Later, this is considered in many literatures in the field of the regularity theory for partial differential equations, see [48, 51, 52, 56, 175, 176] and references therein. For further studies, we refer to [81, 145, 202].

The problem is deeply related to composite material. We refer [26, 37, 163] for the further studies to this topic. A property of matter such as conductivity or density can be discontinuously changed in nature, and in this spirit our assumption describes and allows the situation that there are big jumps of the property of the matter in x_1-direction.

Chapter 2

Preliminaries

Throughout the thesis, let $\Omega \subset \mathbb{R}^n$ be an open and bounded set with $n \geq 2$ and $B_\rho(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < \rho\}$ be an open ball in $\mathbb{R}^n$ centered at $x_0 \in \mathbb{R}^n$ with radius $\rho > 0$. If the center is clear from the context, we shall write it by $B_\rho \equiv B_\rho(x_0)$. We also offer the following notations.

- $x = (x_1, x_2, \dots, x_n) = (x_1, x') \in \mathbb{R}^n$.
- $B'_\rho(y') := \{x' \in \mathbb{R}^{n-1} : |x' - y'| < \rho\}$ and $Q_\rho(y) := (-\rho, \rho) \times B'_\rho(0) + y$.
- $Q^+_\rho(y) := Q_\rho(y) \cap \{x \in \mathbb{R}^n : x_1 > 0\}$ and $\Omega_\rho(y) := Q_\rho(y) \cap \Omega$.
- $T_\rho := Q_\rho(0) \cap \{x \in \mathbb{R}^n : x_1 = 0\}$ and $\partial_w \Omega_\rho(y) := Q_\rho(y) \cap \partial\Omega$.

We occasionally use the simple notations such as $B' := B'(0)$, $Q_\rho := Q_\rho(0)$, $Q^+_\rho := Q^+_\rho(0)$, and $\Omega_\rho := \Omega_\rho(0)$, when the center point is zero. For a ball B, let r_B be the radius and x_B be the center of B. For $x = (x_1, \dots, x_n)$ write $B^+_r(x) = B_r(x) \cap \{y = (y_1, \dots, y_n) \in \mathbb{R}^n : y_n \geq x_n\}$. For an open set U having finite and positive measure, and a function f we abbreviate

$$(f)_U := \fint_U f(x)\,dx = \frac{1}{|U|} \int_U f(x)\,dx.$$

We write χ_U for the characteristic function of the set U.

We denote by c a generic positive constant, which could vary from line to line; special constants will be denoted by symbols such as c_1, c_2, c_*, and so on. Moreover, relevant dependencies on parameters will be emphasized by using brackets, that is, for example $c \equiv c(n, s(G), s(H), m_{pq}, L)$ means that

c is a constant depending only on constants $n, s(G), s(H), m_{pq}, L$. We also write $f \lesssim g$ when $f \leq cg$, and write $f \approx g$ when $f \lesssim g$ and $g \lesssim f$ hold. For $1 < p < \infty$, $p' = \frac{p}{p-1}$ means the conjugate exponent of p.

For an integrable map $f : \mathcal{B} \subset \Omega \to \mathbb{R}^N$ $(N \geq 1)$ and a measurable subset $\mathcal{B} \subset \mathbb{R}^n$ having finite and positive measure, we denote by

$$[f]_{0,\beta;\mathcal{B}} := \sup_{x,y\in\mathcal{B}, x\neq y} \frac{|f(x) - f(y)|}{|x-y|^\beta}, \quad [f]_{0,\beta} := [f]_{0,\beta;\Omega},$$

$$\|f\|_{0,\beta;\mathcal{B}} := \|f\|_{L^\infty(\mathcal{B})} + [f]_{0,\beta;\mathcal{B}} \quad \text{and} \quad \|f\|_{0,\beta;\Omega} := \|f\|_{0,\beta}$$

for any $\beta \in (0,1]$.

We say $\mu : \mathbb{R}^n \to [0,\infty)$ is a weight if μ is positive a.e. For $1 < p < \infty$, a weight $\mu \in L^1_{\text{loc}}(\mathbb{R}^n)$ belongs to the class of Muckenhoupt weights $\mathcal{A}_p$ if

$$[\mu]_{\mathcal{A}_p} := \sup_{B_r \subset \mathbb{R}^n} \left(\fint_{B_r} \mu \, dx \right) \left(\fint_{B_r} \mu^{-\frac{1}{p-1}} \, dx \right)^{p-1} < \infty.$$

For $1 < p < \infty$ and a weight $\omega \in L^p_{\text{loc}}(\mathbb{R}^n)$ with $\omega^{-1} \in L^{p'}_{\text{loc}}(\mathbb{R}^n)$, we define the weighted Lebesgue spaces

$$L^p_\omega(\mathbb{R}^n) = L^p(\Omega, d\omega) := \{ f : \mathbb{R}^n \to \mathbb{R}^k \colon \omega f \in L^p(\mathbb{R}^n) \} \quad (k = 1, n),$$

equipped with the norm $\|f\|_{p,\omega} := \|f\omega\|_p$. In particular, we treat the weight ω as a multiplier. The dual space of $L^p_\omega(\mathbb{R}^n)$ is $L^{p'}_{1/\omega}(\mathbb{R}^n)$. Both $L^p_\omega(\mathbb{R}^n)$ and $L^{p'}_{1/\omega}(\mathbb{R}^n)$ are Banach spaces and continuously embedded into $L^1_{\text{loc}}(\mathbb{R}^n)$. Let $W^{1,p}_\omega(\Omega)$ be the weighted Sobolev space which consists of functions $u \in W^{1,1}(\Omega)$ such that $u, |\nabla u| \in L^p_\omega(\Omega)$, equipped with the norm $\|u\|_{W^{1,p}_\omega(\Omega)} = \|u\|_{L^p_\omega(\Omega)} + \|\nabla u\|_{L^p_\omega(\Omega)}$. Let $W^{1,p}_{0,\omega}(\Omega)$ denote the subspace of $W^{1,p}_\omega(\Omega)$ of functions with zero traces on $\partial\Omega$.

We write $\mathbb{R}^{n\times n}_{\text{sym}}$ for symmetric, real-valued matrices. We denote $\mathbb{R}^{n\times n}_{\geq 0}$ by the cone of symmetric, real-valued and positive semidefinite matrices. The collection of positive definite matrices is denoted by $\mathbb{R}^{n\times n}_{>0}$. For $\mathbb{X}, \mathbb{Y} \in \mathbb{R}^{n\times n}_{\text{sym}}$, we write $\mathbb{X} \geq \mathbb{Y}$ provided $\mathbb{X} - \mathbb{Y} \in \mathbb{R}^{n\times n}_{\geq 0}$. Let $\mathbb{M} \colon \mathbb{R}^n \to \mathbb{R}^{n\times n}_{\geq 0}$ be a (matrix-valued) weight if $\mathbb{M}$ is positive definite a.e., and $\omega \colon \mathbb{R}^n \to [0,\infty)$ be a (scalar) weight if ω is positive a.e.. For $\mathbb{L} \in \mathbb{R}^{n\times n}$, let $|\mathbb{L}|$ denote the spectral norm, which means $|\mathbb{L}| = \sup_{|\xi|\le 1} |\mathbb{L}\xi|$. If $\mathbb{L}$ is symmetric, then $|\mathbb{L}| = \sup_{|\xi|\le 1} \langle \mathbb{L}\xi, \xi\rangle$.

We consider the matrix exponential $\exp\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}^{n\times n}_{>0}$, with its unique inverse mapping $\log\colon \mathbb{R}^{n\times n}_{>0} \to \mathbb{R}^{n\times n}_{\mathrm{sym}}$. Thus, we can define $\log \mathbb{M}\colon \mathbb{R}^n \to \mathbb{R}^{n\times n}_{\mathrm{sym}}$, since $\mathbb{M}\colon \mathbb{R}^n \to \mathbb{R}^{n\times n}_{\mathrm{sym}}$ is positive definite a.e. We now define the logarithmic means

$$\begin{aligned}\langle \omega\rangle_U^{\log} &:= \exp\left(⨍_U \log\omega\right),\\ \langle \mathbb{M}\rangle_U^{\log} &:= \exp\left(⨍_U \log\mathbb{M}\right),\end{aligned}$$

for some subset $U \subset \mathbb{R}^n$. The logarithmic mean has the following compatibility property under taking reciprocal:

$$\left\langle \frac{1}{\omega}\right\rangle_U^{\log} = \exp\left(-⨍_U \log\omega\right) = \frac{1}{\langle\omega\rangle_U^{\log}}.$$

Moreover, using $\log(\mathbb{M}^{-1}) = -\log\mathbb{M}$ and $(\exp(\mathbb{L}))^{-1} = \exp(-\mathbb{L})$, we also obtain

$$\left\langle \mathbb{M}^{-1}\right\rangle_U^{\log} = \exp\left(-\left(\log(\mathbb{M})\right)_U\right) = \left(\exp\left(\log(\mathbb{M})\right)_U\right)^{-1} = (\langle\mathbb{M}\rangle_U^{\log})^{-1}.$$

If μ is an $\mathcal{A}_p$-Muckenhoupt weight, then the maximal operator is bounded on $L^p(\mathbb{R}^n,\mu)$ for $1<p<\infty$. We point out some properties for a Muckenhoupt weight related to its logarithmic means. If ω^p is an $\mathcal{A}_p$-Muckenhoupt weight, then from Jensen's inequality,

$$\begin{aligned}\left(⨍_B \omega^p\,dx\right)^{\frac{1}{p}} &\le [\omega^p]_{\mathcal{A}_p}^{\frac{1}{p}} \langle\omega\rangle_B^{\log},\\ \left(⨍_B \omega^{-p'}\,dx\right)^{\frac{1}{p'}} &\le [\omega^p]_{\mathcal{A}_p}^{\frac{1}{p}} \left\langle\omega^{-1}\right\rangle_B^{\log} = \frac{[\omega^p]_{\mathcal{A}_p}^{\frac{1}{p}}}{\langle\omega\rangle_B^{\log}}.\end{aligned} \tag{2.0.1}$$

Conversely, if (2.0.1) holds, then ω^p is an $\mathcal{A}_p$-Muckenhoupt weight, since we have $\langle\omega\rangle_B^{\log}\left\langle\omega^{-1}\right\rangle_B^{\log} = 1$.

The next lemma is classical one which will be employed later on.

Lemma 2.0.1 ([126]). *Let $h : [\rho_0, \rho_1] \to \mathbb{R}$ be a non-negative and bounded* *function,* $\theta \in (0,1)$*,* $A, B \geq 0$*, and* $\gamma_1, \gamma_2 \geq 0$*. Assume that*

$$h(t) \leq \theta h(s) + \frac{A}{(s-t)^{\gamma_1}} + \frac{B}{(s-t)^{\gamma_2}}$$

holds for $0 < \rho_0 \leq t < s \leq \rho_1$*. Then the following inequality holds with* $c \equiv c(\theta, \gamma_1, \gamma_2)$ *:*

$$h(\rho_0) \leq \frac{cA}{(\rho_1-\rho_0)^{\gamma_1}} + \frac{cB}{(\rho_1-\rho_0)^{\gamma_2}}.$$

Finally, we display a lemma for the difference quotient from [119, Chapter 5].

Lemma 2.0.2. *We have the followings.*

(1) Let $1 \leq p < \infty$ *and* $u \in W^{1,p}(U)$*. For each* $V \Subset U$*,*

$$\|D^h u\|_{L^p(V)} \leq c\|Du\|_{L^p(U)} \tag{2.0.2}$$

for some constant $c = c(n,p) > 0$ *and all* $h \in \mathbb{R}$ *with* $0 < |h| < \frac{1}{2}\operatorname{dist}(V, \partial U)$*, where*

$$\begin{aligned} D_i^h u(x) &= \frac{u(x+he_i) - u(x)}{h}, \\ D^h u(x) &= (D_1^h u(x), D_2^h u(x), \ldots, D_n^h u(x)) \quad (x \in V). \end{aligned}$$

(2) Let $1 < p < \infty$ *and* $u \in L^p(V)$*. Suppose that for some* $\tilde{c}$*, we have*

$$\|D^h u\|_{L^p(V)} \leq \tilde{c}$$

for all $0 < |h| < \frac{1}{2}\operatorname{dist}(V, \partial U)$*. Then there holds* $u \in W^{1,p}(V)$ *with*

$$\|Du\|_{L^p(V)} \leq c, \tag{2.0.3}$$

where $c = c(n, p, \tilde{c}) > 0$*.*

2.1 Musielak-Orlicz functions and spaces

Throughout the thesis, we occasionally use the notion of Musielak-Orlicz functions and spaces which is introduced in this section.

Definition 2.1.1. *We say that a measurable function $\Phi : \Omega\times[0,\infty) \to [0,\infty)$ is a Musielak-Orlicz function if, for any fixed $x \in \Omega$, $\Phi(x,\cdot)$ is an increasing convex function such that*

$$\Phi(x,0) = 0,\ \lim_{t\to\infty} \Phi(x,t) = \infty,\ \lim_{t\to 0^+} \frac{\Phi(x,t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{\Phi(x,t)}{t} = \infty.$$

We denote by $\mathcal{N}(\Omega)$ to mean the set of Musielak-Orlicz functions $\Phi : \Omega \times [0,\infty) \to [0,\infty)$ satisfying the following two conditions:

1. *For any fixed $x \in \Omega$, $\Phi(x,\cdot) \in C^1([0,\infty)) \cap C^2((0,\infty))$,*
2. *There exists a constant $s(\Phi) \geq 1$ with*

$$\frac{1}{s(\Phi)} \leq \frac{t\partial^2_{tt}\Phi(x,t)}{\partial_t\Phi(x,t)} \leq s(\Phi),$$

uniformly for all $x \in \Omega$ and $t > 0$. We shall call this number $s(\Phi)$ by an index of Φ. We shall denote by $\mathcal{N}$ a set of functions $\Phi \in \mathcal{N}(\Omega)$ that does not depend on $x \in \Omega$.

Remark 2.1.2. *Let $\Phi \in \mathcal{N}(\Omega)$ with $s(\Phi) \geq 1$. It can be easily seen that*

$$1 + \frac{1}{s(\Phi)} \leq \frac{t\partial_t\Phi(x,t)}{\Phi(x,t)} \leq 1 + s(\Phi),$$

and then

$$t^2\partial^2_{tt}\Phi(x,t) \approx t\partial_t\Phi(x,t) \approx \Phi(x,t),$$

uniformly for all $x \in \Omega$ and $t > 0$, where all implied constants only depend on the index $s(\Phi)$.

Definition 2.1.3. *Let Φ be a Musielak-Orlicz function.*

1. *We say that Φ satisfies the Δ_2-condition, denoted by $\Phi \in \Delta_2$, if there is a positive number $\Delta_2(\Phi)$ such that $\Phi(x,2t) \leq \Delta_2(\Phi)\,\Phi(x,t)$ for all $x \in \Omega$ and $t \geq 0$.*

2. *We say that Φ satisfies the ∇_2-condition, denoted by $\Phi \in \nabla_2$, if there is a positive number $\nabla_2(\Phi) > 1$ such that $\Phi(x, \nabla_2(\Phi)\, t) \geq 2\nabla_2(\Phi)\, \Phi(x,t)$ for all $x \in \Omega$ and $t \geq 0$.*

3. *We write $\Phi \in \Delta_2 \cap \nabla_2$ if $\Phi \in \Delta_2$ and $\Phi \in \nabla_2$.*

The Musielak-Orlicz class $K^\Phi(\Omega; \mathbb{R}^N)$, $N \geq 1$, is a set of all measurable functions $v : \Omega \to \mathbb{R}^N$ such that

$$\int_\Omega \Phi(x, |v(x)|)\, dx < +\infty.$$

The Musielak-Orlicz space $L^\Phi(\Omega; \mathbb{R}^N)$ is the vector space generated by the class $K^\Phi(\Omega; \mathbb{R}^N)$. If $\Phi \in \Delta_2$, then $K^\Phi(\Omega; \mathbb{R}^N) = L^\Phi(\Omega; \mathbb{R}^N)$ and it is a Banach space with the Luxemburg norm

$$\|v\|_{L^\Phi(\Omega;\mathbb{R}^N)} = \inf\left\{\sigma > 0 : \int_\Omega \Phi\left(x, \frac{|v(x)|}{\sigma}\right)\, dx \leq 1\right\}.$$

The Musielak-Orlicz-Sobolev space $W^{1,\Phi}(\Omega; \mathbb{R}^N)$ is the one consisting of all measurable functions $v \in L^\Phi(\Omega; \mathbb{R}^N)$ such that its weak gradient vector Dv belongs to $L^\Phi(\Omega; \mathbb{R}^{Nn})$. For $v \in W^{1,\Phi}(\Omega; \mathbb{R}^N)$, its norm is defined by

$$\|v\|_{W^{1,\Phi}(\Omega;\mathbb{R}^N)} = \|v\|_{L^\Phi(\Omega;\mathbb{R}^N)} + \|Dv\|_{L^\Phi(\Omega;\mathbb{R}^{Nn})}.$$

As usual, the space $W_0^{1,\Phi}(\Omega; \mathbb{R}^N)$ is understood as the closure of $C_0^\infty(\Omega; \mathbb{R}^N)$ in $W^{1,\Phi}(\Omega; \mathbb{R}^N)$. For $N = 1$, we simply write $L^\Phi(\Omega) := L^\Phi(\Omega; \mathbb{R})$ and $W^{1,\Phi}(\Omega) := W^{1,\Phi}(\Omega; \mathbb{R})$. For a further discussion of the Musielak-Orlicz space, Orlicz space and the associated Sobolev space, we refer the readers to [5, 94, 129, 183].

We end this chapter with the additional properties of Musielak-Orlicz functions.

Lemma 2.1.4 ([43])**.** *Let $\Phi \in \mathcal{N}(\Omega)$ with $s(\Phi) \geq 1$. Then,*

1. *$\Phi \in \Delta_2 \cap \nabla_2$, and the constants $\Delta_2(\Phi), \nabla_2(\Phi)$ depend only on $s(\Phi)$.*

2. *For every fixed $x \in \Omega$, $\Phi(x, \Lambda t) \leq \Lambda^{s(\Phi)+1}\Phi(x,t)$ for any $\Lambda \geq 1$ and $t \geq 0$.*

3. *For every fixed $x \in \Omega$, $\Phi(x, \lambda t) \leq \lambda^{\frac{1}{s(\Phi)}+1}\Phi(x,t)$ for any $0 < \lambda \leq 1$ and $t \geq 0$.*

Lemma 2.1.5 ([12]). *Let $\Phi, \tilde{\Phi} \in \mathcal{N}(\Omega)$ with $s(\Phi), s(\tilde{\Phi}) \geq 1$. Then,*

1. *For any non-negative numbers a, b with $a + b > 0$, $a\Phi + b\tilde{\Phi} \in \mathcal{N}(\Omega)$ with $s(\Phi + \tilde{\Phi}) := s(\Phi) + s(\tilde{\Phi})$ and $\Phi\tilde{\Phi} \in \mathcal{N}(\Omega)$ with $s(\Phi\tilde{\Phi}) := 4s(\Phi)s(\tilde{\Phi})(s(\Phi) + s(\tilde{\Phi}))$.*
2. *For any number $d \geq 1$, $\Phi^d \in \mathcal{N}(\Omega)$ with $s(\Phi^d) := s(\Phi) + (d-1)(s(\Phi) + 1)$.*
3. *For any number $d \geq 0$, $\Phi_d(x,t) := t^d\Phi(x,t) \in \mathcal{N}(\Omega)$ with $s(\Phi_d) := d + 3[s(\Phi)]^2$.*
4. *There exists $\theta_\Phi \in (0,1)$ depending only on $s(\Phi)$ such that $\Phi^{\theta_\Phi} \in \mathcal{N}(\Omega)$ with an index depending only on $s(\Phi)$.*

Lemma 2.1.6 ([12]). *Let $\Phi \in \mathcal{N}(\Omega)$ with $s(\Phi) \geq 1$. Then there is a positive constant $c \equiv c(s(\Phi))$ such that*

$$s\frac{\Phi(x,t)}{t} + t\frac{\Phi(x,s)}{s} \approx s\partial_t\Phi(x,t) + t\partial_t\Phi(x,s) \leq \varepsilon\Phi(x,s) + \frac{c}{\varepsilon^{s(\Phi)}}\Phi(x,t)$$

holds, whenever $x \in \Omega$, $s, t \geq 0$ and $0 < \varepsilon \leq 1$.

Chapter 3

Calderón-Zygmund estimates for nonstandard growth problems

3.1 Local estimates with measurable nonlinearities under Orlicz growth

In this section, we are concerned with weak solutions of elliptic equations involving measurable nonlinearities with Orlicz growth to address what would be the weakest regularity condition on the associated nonlinearity for the Calderón–Zygmund theory. We prove that the gradient of weak solution is as integrable as the nonhomogeneous term under the assumption that the nonlinearity is only measurable in one of the variables while it has a small BMO assumption in the other variables. To this end, we develop a nonlinear Moser type iteration argument for such a homogeneous reference problem with one variable–dependent nonlinearity under Orlicz growth to establish $W^{1,q}$–regularity for every $q > 1$.

Our results open a new path into the comprehensive understanding of the problem with nonstandard growth in the literature of optimal regularity theory in highly nonlinear elliptic and parabolic equations.

3.1.1 Hypothesis and main results

We consider the following general elliptic equation

$$\operatorname{div}(A(x, Du)) = \operatorname{div}\left(\frac{\varphi'(|F|)}{|F|}F\right) \quad \text{in } \Omega, \tag{3.1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded and open domain with $n \geq 2$, $F = (f_1, \ldots, f_n) : \Omega \to \mathbb{R}^n$ is a given vector-valued function with $|F| \in L^1(\Omega)$, and $A(x, \xi) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory vector field which is $C^1 \setminus \{0\}$-regular for ξ–variable and satisfies

$$\begin{cases} |A(x,\xi)| + |\partial_\xi A(x,\xi)||\xi| \leq L\varphi'(|\xi|) \\ \langle \partial_\xi A(x,\xi)\zeta, \zeta\rangle \geq \nu \dfrac{\varphi(|\xi|)}{|\xi|^2}|\zeta|^2 \end{cases} \tag{3.1.2}$$

for any a.e. $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n \setminus \{0\}$ and $\zeta \in \mathbb{R}^n$ with some constants $0 < \nu \leq L < \infty$. Here, we denote by $\varphi' : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ to mean a $C^1((0,\infty)) \cap C([0,\infty))$ function satisfying $\varphi'(0) = 0$ and

$$0 < \kappa_1 - 1 \leq \frac{t\varphi''(t)}{\varphi'(t)} \leq \kappa_2 - 1 < \infty \tag{3.1.3}$$

for some constants $\kappa_1, \kappa_2 > 1$.

We define a function $\theta(A, Q_r(y))$ on $Q_r(y)$ by

$$\theta(A, Q_r(y))(x) = \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{\left|A(x_1, x', \xi) - \bar{A}_{B'_r(y')}(x_1, \xi)\right|}{\varphi'(|\xi|)}, \tag{3.1.4}$$

where

$$\bar{A}_{B'_r(y')}(x_1, \xi) = \fint_{B'_r(y')} A(x_1, x', \xi)\, dx'$$

is the integral average of $A(x_1, \cdot, \xi)$ on $B'_r(y')$ for each fixed $x_1 \in (y_1 - r, y_1 + r)$ and $\xi \in \mathbb{R}^n \setminus \{0\}$. Then one can observe from (3.1.2) and (3.1.4) that

$$|\theta(A, Q_r(y))(x)| \leq 2L \quad \text{for a.e. } x \in Q_r(y).$$

For given $\delta \in (0,1)$ and $R > 0$, we say that A *is* (δ, R)*-vanishing of codimen-*

sion 1 if

$$\sup_{0<r\le R}\sup_{y\in\mathbb{R}^n}\fint_{Q_r(y)}\theta(A,Q_r(y))(x)\,dx\le\delta. \tag{3.1.5}$$

A typical example for $A(x,\xi)$ satisfying (3.1.5) is $A(x,\xi)=a_1(x_1)a_2(x')\frac{\varphi'(|\xi|)}{|\xi|}\xi$, where $a_1(\cdot):\mathbb{R}\to\mathbb{R}$ with $\sqrt{\nu}\le a_1(\cdot)\le\sqrt{L}$, and $a_2(\cdot):\mathbb{R}^{n-1}\to\mathbb{R}$ with $\sqrt{\nu}\le a_2(\cdot)\le\sqrt{L}$ and $[a_2]_{\mathrm{BMO}(\mathbb{R}^{n-1})}\le\delta/\sqrt{L}$.

Then the statement of the main theorem is the following.

Theorem 3.1.1. *For a given $\gamma>1$, assume $\varphi(|F|)\in L^{\gamma}_{\mathrm{loc}}(\Omega)$ with* (3.1.2) *and* (3.1.3)*. Let $u\in W^{1,\varphi}(\Omega)$ be a weak solution of* (3.1.1)*. Then there exists a small constant $\delta=\delta(n,\kappa_1,\kappa_2,\nu,L,\gamma)\in(0,1]$ such that if A is (δ,R)-vanishing of codimension 1, then $\varphi(|Du|)\in L^{\gamma}_{\mathrm{loc}}(\Omega)$ and we have the estimate*

$$\fint_{Q_R}\varphi(|Du|)^{\gamma}\,dx\le c\left(\fint_{Q_{2R}}\varphi(|Du|)\,dx\right)^{\gamma}+c\fint_{Q_{2R}}\varphi(|F|)^{\gamma}\,dx, \tag{3.1.6}$$

whenever $Q_{2R}\Subset\Omega$ with $c=c(n,\kappa_1,\kappa_2,\nu,L,\gamma)>0$.

Remark 3.1.2. *The essence of proving* (3.1.6) *is to show that if $q>1$ is any given number, $F\equiv 0$ holds and $A(x,\xi)\equiv A(x_1,\xi)$ satisfies* (3.1.2) *and* (3.1.3)*, then $\varphi(|Du|)\in L^{q}_{\mathrm{loc}}(\Omega)$ with the estimate*

$$\fint_{Q_R}\varphi(|Du|)^{q}\,dx\le c\left(\fint_{Q_{2R}}\varphi(|Du|)\,dx\right)^{q}, \tag{3.1.7}$$

whenever $Q_{2R}\Subset\Omega$ with $c=c(n,\kappa_1,\kappa_2,\nu,L,q)>0$. Since we do not assume that the map $x_1\mapsto A(x_1,\xi)$ has a small BMO condition, we cannot apply the perturbation argument as in [58]. *Here we argue directly a Moser type iteration for the regularized problem to derive the uniform $W^{1,q}$–estimate* (3.1.7)*. This regularization will be justified by an usual approximation argument.*

3.1.2 L^q-estimates for the reference problem

First, we record basic properties of the function φ which will be used in this subsection. The function φ satisfying (3.1.3) is usually called an N–function. For the precise definition and properties of N–functions, we refer to [5, 93]. The function φ has the following properties.

Remark 3.1.3 (Properties of the N–function φ)**.** *From* (3.1.3)*, we see that* $(\varphi')^{-1}(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *exists,*

$$\varphi'(t), \varphi(t) \text{ and } \varphi^{-1}(t) \text{ are increasing,} \tag{3.1.8}$$

$$t\varphi''(t) \eqsim \varphi'(t) \quad (t \geq 0), \tag{3.1.9}$$

$$t\varphi'(t) \eqsim \varphi(t) \quad (t \geq 0), \tag{3.1.10}$$

$$\min\{s^{\kappa_1 - 1}, s^{\kappa_2 - 1}\}\varphi'(t) \leq \varphi'(st) \leq \max\{s^{\kappa_1 - 1}, s^{\kappa_2 - 1}\}\varphi'(t) \ (s, t \geq 0), \tag{3.1.11}$$

$$\min\{s^{\kappa_1}, s^{\kappa_2}\}\varphi(t) \leq \varphi(st) \leq \max\{s^{\kappa_1}, s^{\kappa_2}\}\varphi(t) \quad (s, t \geq 0), \tag{3.1.12}$$

and

$$\min\{s^{\frac{1}{\kappa_1}}, s^{\frac{1}{\kappa_2}}\}\varphi^{-1}(t) \leq \varphi^{-1}(st) \leq \max\{s^{\frac{1}{\kappa_1}}, s^{\frac{1}{\kappa_2}}\}\varphi^{-1}(t) \quad (s, t \geq 0). \tag{3.1.13}$$

Note that in (3.1.9) *and* (3.1.10)*, the implicit constants depend only on* κ_1 *and* κ_2*.*

Now we define the conjugate of φ by

$$\varphi^*(t) := \sup_{s>0}\{st - \varphi(s)\} \quad (t \geq 0).$$

Then the following properties of φ^* are known.

Remark 3.1.4 (Properties of the conjugate function φ^*)**.** *We have*

$$\varphi^*(t) = \int_0^t (\varphi')^{-1}(s)\, ds, \tag{3.1.14}$$

$$\varphi^*(t) : [0, \infty) \to [0, \infty) \text{ is convex and increasing,} \tag{3.1.15}$$

$$(\varphi^*)^* \equiv \varphi,$$

$$(\varphi^*)'(t) = (\varphi')^{-1}(t) \quad (t \geq 0),$$

$$t(\varphi^*)'(t) \eqsim \varphi^*(t) \quad (t \geq 0) \tag{3.1.16}$$

with the implicit constant $c = c(\kappa_1, \kappa_2)$, *and*

$$\min\{s^{\frac{\kappa_1}{\kappa_1-1}}, s^{\frac{\kappa_2}{\kappa_2-1}}\}\varphi^*(t) \leq \varphi^*(st) \leq \max\{s^{\frac{\kappa_1}{\kappa_1-1}}, s^{\frac{\kappa_2}{\kappa_2-1}}\}\varphi^*(t) \tag{3.1.17}$$

for any $s, t \geq 0$. *Moreover, by* (3.1.10), (3.1.14) *and* (3.1.16), *we get the following inequality*

$$\varphi^*(\varphi'(t)) \eqsim \varphi(t) \quad (t \geq 0) \tag{3.1.18}$$

with the implicit constant $c = c(\kappa_1, \kappa_2) > 0$. *Since* $(\varphi^*)^* \equiv \varphi$, *changing the role of* φ *and* φ^* *and using* (3.1.13), *we have an analogous relation of* (3.1.18)*:*

$$\varphi((\varphi^*)')(t) \eqsim \varphi^*(t) \quad (t \geq 0) \tag{3.1.19}$$

with $c = c(\kappa_1, \kappa_2) > 0$.

We also need useful inequalities involving Young's inequality for φ.

Remark 3.1.5. *By the definition of* $\varphi^*(t)$, (3.1.12) *and* (3.1.17), *we can see that for* $\bar{\varepsilon} \in (0, 1]$ *and* $s, t \geq 0$,

$$st = (\bar{\varepsilon}s)\left(\frac{t}{\bar{\varepsilon}}\right) \leq \varphi(\bar{\varepsilon}s) + \varphi^*\left(\frac{t}{\bar{\varepsilon}}\right) \leq \bar{\varepsilon}^{\kappa_1}\varphi(s) + \left(\frac{1}{\bar{\varepsilon}}\right)^{\frac{\kappa_1}{\kappa_1-1}}\varphi^*(t)$$

and so for any $\varepsilon > 0$, *the following Young's inequality holds:*

$$st \leq \varepsilon\varphi(s) + c(\varepsilon)\varphi^*(t) \quad (s, t \geq 0) \tag{3.1.20}$$

with $c(\varepsilon) = c(\kappa_1, \kappa_2, \varepsilon) > 0$. *On the other hand, combining* (3.1.20) *and* (3.1.18) *yields*

$$s\varphi'(t) \leq c(\varepsilon)\varphi(s) + \varepsilon\varphi(t) \quad (\varepsilon \in (0, 1)\ s, t \geq 0). \tag{3.1.21}$$

For $s \le t$, by (3.1.9), (3.1.10) *and* (3.1.21), *it holds that*

$$s\varphi''(t) \approx s\frac{\varphi'(t)}{t} \lesssim \frac{c(\varepsilon)\varphi(s)+\varepsilon\varphi(t)}{t} \approx \frac{c(\varepsilon)\varphi(s)}{t} + \varepsilon\varphi'(t) \lesssim c(\varepsilon)\varphi'(s) + \varepsilon\varphi'(t)$$

for any $\varepsilon \in (0,1)$, and so we have the following type of inequality:

$$s\varphi''(t) \lesssim c(\varepsilon)\varphi'(s) + \varepsilon\varphi'(t) \quad (\varepsilon \in (0,1),\ s \le t). \tag{3.1.22}$$

The following triangle inequalities are occasionally used in this section.

Lemma 3.1.6. *For any $s, t \ge 0$ and $\varepsilon \in (0,1)$, we have*

$$\begin{aligned} \varphi'(s+t) &\le (1+\varepsilon)\varphi'(s) + c(\varepsilon)\varphi'(t), \\ \varphi(s+t) &\le (1+\varepsilon)\varphi(s) + c(\varepsilon)\varphi(t) \end{aligned} \tag{3.1.23}$$

and

$$\varphi^*(s+t) \le (1+\varepsilon)\varphi^*(s) + c(\varepsilon)\varphi^*(t) \quad (s,t \ge 0) \tag{3.1.24}$$

with $c(\varepsilon) = c(\kappa_1, \kappa_2, \varepsilon) > 0$. Moreover, it holds that

$$|\varphi(s+t) - \varphi(s)| \le \varepsilon\varphi(s) + c(\varepsilon)\varphi(t). \tag{3.1.25}$$

Proof. Let $\varepsilon \in (0,1)$ and $s, t \ge 0$ be given. To show (3.1.23), we consider the following alternatives with $\theta \in (0,1)$, which is a small parameter determined later:

$$\text{either} \quad \theta s \le t \quad \text{or} \quad \theta s > t.$$

If $\theta s \le t$, then $s \le \frac{1}{\theta} t$ and so (3.1.8) and (3.1.11) yield

$$\varphi'(s+t) \le \varphi'\left(\left(1+\tfrac{1}{\theta}\right)t\right) \le \left(1+\tfrac{1}{\theta}\right)^{\kappa_2-1}\varphi'(t).$$

If $\theta s > t$, then again by (3.1.8) and (3.1.11) we have

$$\varphi'(s+t) \le \varphi'((1+\theta)s) \le (1+\theta)^{\kappa_2-1}\varphi'(s).$$

Summing up the above two estimates, we obtain

$$\varphi'(s+t) \le (1+\theta)^{\kappa_2-1}\varphi'(s) + \left(1+\tfrac{1}{\theta}\right)^{\kappa_2-1}\varphi'(t).$$

Now choosing $\theta = \theta(\kappa_2, \varepsilon) \in (0,1)$ such that $(1+\theta)^{\kappa_2 - 1} \leq 1 + \varepsilon$, we have the left-hand side of (3.1.23). The right-hand side of (3.1.23) and (3.1.24) are proved similarly. Finally, (3.1.25) is a direct consequence of (3.1.23). □

We also derive a lemma which will be used in Section 3.1.3.

Lemma 3.1.7. *Let $X, Y \in \mathbb{R}^n$ and $\gamma \geq 1$. Then for any $\kappa > 0$ we have*

$$\varphi(|X|) \geq \kappa \quad \Rightarrow \quad \varphi(|X|) \leq c\varphi(|X-Y|) + c(\gamma)\kappa^{1-\gamma}\varphi(|Y|)^{\gamma}$$

with $c = c(n, \kappa_1, \kappa_2)$ and $c(\gamma) = c(n, \kappa_1, \kappa_2, \gamma)$.

Proof. Using (3.1.8) and (3.1.23), we observe

$$\varphi(|X|) \leq \varphi(|X-Y| + |Y|) \leq c\varphi(|X-Y|) + c\varphi(|Y|). \tag{3.1.26}$$

If $\varphi(|X-Y|) \leq \varphi(|Y|)$ holds, then by (3.1.26) it follows that $\kappa \leq \varphi(|X|) \leq c\varphi(|Y|)$. Since $\gamma \geq 1$, we have

$$\varphi(|Y|) \leq c(\gamma)\kappa^{1-\gamma}\varphi(|Y|)^{\gamma}$$

and so $\varphi(|X|) \leq c(\gamma)\kappa^{1-\gamma}\varphi(|Y|)^{\gamma}$, then the conclusion follows. If $\varphi(|Y|) \leq \varphi(|X-Y|)$ holds, then (3.1.26) implies the conclusion directly. □

We now start to prove higher integrability estimates for the reference problem (3.1.28) with respect to our problem (3.1.1). For the N–function φ with (3.1.3), let a Carathéodory vector field $\bar{A}(x_1, \xi) = (\bar{A}_1(x_1, \xi), \ldots, \bar{A}_n(x_1, \xi)) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be $C^1(\mathbb{R}^n \setminus \{0\})$-regular for ξ variable and satisfy

$$\begin{cases} |\bar{A}(x_1, \xi)| + |D_\xi \bar{A}(x_1, \xi)||\xi| \leq L\varphi'(|\xi|) \\ \left\langle D_\xi \bar{A}(x_1, \xi)\zeta, \zeta \right\rangle \geq \nu \dfrac{\varphi(|\xi|)}{|\xi|^2}|\zeta|^2 \end{cases} \tag{3.1.27}$$

for every $\xi, \zeta \in \mathbb{R}^n$, $x_1 \in \mathbb{R}$ and some constants $0 < \nu \leq L < \infty$.

For $0 < r < 1$, we consider the following homogeneous problem

$$\operatorname{div} \bar{A}(x_1, Dv) = 0 \quad \text{in} \quad Q_{4r}. \tag{3.1.28}$$

The main theorem that we are going to assert in this section is the following L^q-estimate for $\varphi(|Dv|)$, where $v \in W^{1,\varphi}(Q_{4r})$ is a weak solution of (3.1.28), which means that

$$\int_{Q_{4r}} \bar{A}(x_1, Dv) D\eta \, dx = 0$$

for every $\eta \in W_0^{1,\varphi}(Q_{4r})$.

Theorem 3.1.8. *Let v be a weak solution of* (3.1.28) *with the assumptions* (3.1.3) *and* (3.1.27)*. Then for every $q > 1$, we have*

$$\left(\fint_{Q_r} \varphi^q(|Dv|) \, dx \right)^{\frac{1}{q}} \leq c \fint_{Q_{4r}} \varphi(|Dv|) \, dx,$$

where $c = c(n, \kappa_1, \kappa_2, \nu, L, q)$.

To prove the above theorem, we first consider the regularized problems. Define $\phi \in C_c^\infty(\mathbb{R}^k)$ $(k = 1, n)$ as a standard mollifier:

$$\phi(x) = \begin{cases} c_0 \exp\left(\dfrac{1}{|x|^2 - 1} \right) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

where $c_0 = c_0(k)$ is the constant such that

$$\int_{\mathbb{R}^k} \phi(x) \, dx = 1. \tag{3.1.29}$$

Let $0 < \epsilon < r$ and $\bar{A}_\epsilon(x_1, \xi)$ be a mollification of $\bar{A}(x_1, \xi)$ in the following way:

$$\begin{aligned} \bar{A}_\epsilon(x_1, \xi) &= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \bar{A}(x_1 - \epsilon z_1, \xi - \epsilon \eta) \phi(\eta) \phi(z_1) \, d\eta \, dz_1 \\ &= \int_{-1}^{1} \int_{B_1} \bar{A}(x_1 - \epsilon z_1, \xi - \epsilon \eta) \phi(\eta) \phi(z_1) \, d\eta \, dz_1. \end{aligned} \tag{3.1.30}$$

Then $\bar{A}_\epsilon(x_1, \xi)$ is $C^1(\mathbb{R} \times \mathbb{R}^n)$-regular in $x_1 \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$. For the ellipticity and growth conditions of $\bar{A}_\epsilon(x_1, \xi)$, we have the following lemma.

Lemma 3.1.9. *We have*

$$\begin{cases} |\bar{A}_\epsilon(x_1,\xi)| + |D_\xi \bar{A}_\epsilon(x_1,\xi)|(|\xi|+\epsilon) \le cL\varphi'(|\xi|+\epsilon) \\ \left\langle D_\xi \bar{A}_\epsilon(x_1,\xi)\zeta,\zeta\right\rangle \ge c\nu\varphi''(|\xi|+\epsilon)|\zeta|^2 \end{cases} \tag{3.1.31}$$

for every $x_1 \in \mathbb{R}$ and $\xi, \zeta \in \mathbb{R}^n$ with some $c = c(n,\kappa_1,\kappa_2) > 0$.

Proof. The proof is motivated from [116]. To derive the first inequality of (3.1.31), by (3.1.8) and (3.1.27) we have

$$\begin{aligned} |\bar{A}_\epsilon(x_1,\xi)| &\le \int_{-1}^{1}\int_{B_1} |\bar{A}(x_1-\epsilon z_1, \xi-\epsilon\eta)|\phi(\eta)\phi(z_1)\,d\eta dz_1 \\ &\le L\int_{-1}^{1}\int_{B_1} \varphi'(|\xi-\epsilon\eta|)\phi(\eta)\phi(z_1)\,d\eta dz_1 \\ &\le cL\int_{-1}^{1}\int_{B_1} \varphi'(|\xi|+\epsilon)\phi(\eta)\phi(z_1)\,d\eta dz_1 \le cL\varphi'(|\xi|+\epsilon) \end{aligned}$$

with $c = c(n,\kappa_1,\kappa_2) > 0$. Also, to estimate $|D_\xi \bar{A}_\epsilon(x_1,\xi)|$, we consider two alternatives:

$$\text{either} \quad |\xi| > 2\epsilon \quad \text{or} \quad |\xi| \le 2\epsilon.$$

If $|\xi| > 2\epsilon$ holds, then $|\xi - \epsilon y| \approx |\xi| + \epsilon$ for $y \in B_1$ so that by (3.1.8) and (3.1.27), we obtain

$$\begin{aligned} |D_\xi \bar{A}_\epsilon(x_1,\xi)| &\le \int_{-1}^{1}\int_{B_1} \left|D_\xi \bar{A}(x_1-\epsilon z_1, \xi-\epsilon\eta)\right| \phi(\eta)\phi(z_1)\,d\eta dz_1 \\ &\le cL\int_{-1}^{1}\int_{B_1} \frac{\varphi'(|\xi-\epsilon\eta|)}{|\xi-\epsilon\eta|}\phi(\eta)\phi(z_1)\,d\eta dz_1 \\ &\le cL\int_{-1}^{1}\int_{B_1} \frac{\varphi'(|\xi|+\epsilon)}{|\xi|+\epsilon}\phi(\eta)\phi(z_1)\,d\eta dz_1 \le cL\frac{\varphi'(|\xi|+\epsilon)}{|\xi|+\epsilon}. \end{aligned}$$

In case of $|\xi| \le 2\epsilon$, $\epsilon \gtrsim \epsilon + |\xi|$ holds and so (3.1.8) and (3.1.27) yield

$$
\begin{aligned}
|D_\xi \bar{A}_\epsilon(x_1,\xi)| &= \left| \frac{1}{\epsilon}\int_{-1}^{1}\int_{B_1} \bar{A}(x_1-\epsilon z_1, \xi - \epsilon\eta)\phi'(\eta)\phi(z_1)\,d\eta dz_1 \right| \\
&\le \frac{L}{\epsilon}\int_{-1}^{1}\int_{B_1} \varphi'(|\xi-\epsilon\eta|)\phi'(\eta)\phi(z_1)\,d\eta dz_1 \\
&\le \frac{cL}{|\xi|+\epsilon}\int_{-1}^{1}\int_{B_1} \varphi'(|\xi|+\epsilon)\phi'(\eta)\,d\eta dz_1 \le cL\frac{\varphi'(|\xi|+\epsilon)}{|\xi|+\epsilon}
\end{aligned}
$$

with $c = c(n,\kappa_1,\kappa_2) > 0$. Thus the first inequality of (3.1.31) holds.

To show the second inequality of (3.1.31), by (3.1.9), (3.1.10) and (3.1.27) we observe that

$$
\begin{aligned}
\left\langle D_\xi \bar{A}_\epsilon(x_1,\xi)\zeta,\zeta \right\rangle &= \left\langle \left(\int_{-1}^{1}\int_{B_1} D_\xi \bar{A}(x_1-\epsilon z_1,\xi-\epsilon\eta)\phi(\eta)\phi(z_1)\,d\eta dz_1 \right)\zeta,\zeta \right\rangle \\
&\ge c\nu \left(\int_{-1}^{1}\int_{B_1} \varphi''(|\xi-\epsilon\eta|)\phi(\eta)\phi(z_1)\,d\eta dz_1 \right)|\zeta|^2.
\end{aligned}
$$

Here, simple computations together with (3.1.8), (3.1.9) and (3.1.11) give us that

$$
\begin{aligned}
&\int_{B_1} \varphi''(|\xi-\epsilon\eta|)\phi(\eta)\,d\eta = \int_{B_1} \varphi''(||\xi|^2+\epsilon^2|\eta|^2 - 2\epsilon\langle \xi,\eta\rangle|^{\frac12})\phi(\eta)\,d\eta \\
&\quad\gtrsim \nu\int_{(B_1\setminus B_{\frac12})\cap\{\langle\xi,\eta\rangle\le 0\}} \frac{\varphi'(||\xi|^2+\epsilon^2|\eta|^2-2\epsilon\langle\xi,\eta\rangle|^{\frac12})}{||\xi|^2+\epsilon^2|\eta|^2-2\epsilon\langle\xi,\eta\rangle|^{\frac12}}\phi(\eta)\,d\eta \\
&\quad\gtrsim \nu\int_{(B_1\setminus B_{\frac12})\cap\{\langle\xi,\eta\rangle\le 0\}} \frac{\varphi'(||\xi|^2+\epsilon^2|\eta|^2|^{\frac12})}{||\xi|^2+\epsilon^2|\eta|^2|^{\frac12}}\phi(\eta)\,d\eta \\
&\quad\gtrsim \nu\int_{(B_1\setminus B_{\frac12})\cap\{\langle\xi,\eta\rangle\le 0\}} \frac{\varphi'(||\xi|^2+\frac14\epsilon^2|^{\frac12})}{||\xi|^2+\epsilon^2|^{\frac12}}\phi(\eta)\,d\eta \\
&\quad\gtrsim \nu\int_{(B_1\setminus B_{\frac12})\cap\{\langle\xi,\eta\rangle\le 0\}} \varphi''(|\xi|+\epsilon)\phi(\eta)\,d\eta \\
&\quad\gtrsim \nu\left(\int_{B_1\setminus B_{\frac12}} \phi(\eta)\,d\eta\right)\varphi''(|\xi|+\epsilon) \ge c\nu\varphi''(|\xi|+\epsilon)
\end{aligned}
$$

with implicit constants $c = c(n, \kappa_1, \kappa_2) > 0$, and so the conclusion follows. $\square$

Remark 3.1.10. *Under the conclusion of Lemma 3.1.9, we obtain the following inequality by the same proof as [11, 16, 18]. For each $x_1 \in \mathbb{R}$ and $\xi_1, \xi_2 \in \mathbb{R}^n$ we have*

$$\begin{aligned} &\varphi(|\xi_1 - \xi_2| + \epsilon) \\ &\quad \le \varepsilon \varphi(|\xi_1| + \epsilon) + c(\varepsilon) \left\langle \bar{A}_\epsilon(x_1, \xi_1) - \bar{A}_\epsilon(x_1, \xi_2), \xi_1 - \xi_2 \right\rangle \end{aligned} \tag{3.1.32}$$

for any $\varepsilon \in (0,1)$ with $c(\varepsilon) = c(\kappa_1, \kappa_2, \varepsilon) > 0$.

We also need the following approximation lemma.

Lemma 3.1.11. *Let $v_\epsilon \in W^{1,\varphi}(Q_{2r})$ be the weak solution of*

$$\begin{cases} \mathrm{div} \bar{A}_\epsilon(x_1, Dv_\epsilon) = 0 & \text{in } Q_{2r}, \\ v_\epsilon = v & \text{on } \partial Q_{2r}, \end{cases} \tag{3.1.33}$$

where $v \in W^{1,\varphi}(Q_{4r})$ is a weak solution of (3.1.28). Then we have

$$\lim_{\epsilon \to 0} \int_{Q_{2r}} \varphi(|Dv_\epsilon - Dv|) \, dx = 0.$$

Proof. Testing $v_\epsilon - v \in W_0^{1,\varphi}(Q_{2r})$ to (3.1.33) and (3.1.28), we have

$$\begin{aligned} &\int_{Q_{2r}} \left\langle \bar{A}_\epsilon(x_1, Dv_\epsilon) - \bar{A}_\epsilon(x_1, Dv), Dv_\epsilon - Dv \right\rangle dx \\ &\qquad = \int_{Q_{2r}} \left\langle \bar{A}(x_1, Dv) - \bar{A}_\epsilon(x_1, Dv), Dv_\epsilon - Dv \right\rangle dx. \end{aligned}$$

Then together with (3.1.32), we observe that

$$\begin{aligned} &\int_{Q_{2r}} \varphi(|Dv_\epsilon - Dv| + \epsilon) \, dx \\ &\quad \le c(\varepsilon) \int_{Q_{2r}} \left\langle \bar{A}(x_1, Dv) - \bar{A}_\epsilon(x_1, Dv), Dv_\epsilon - Dv \right\rangle dx \\ &\qquad + c\,\varepsilon \int_{Q_{2r}} \varphi(|Dv| + \epsilon) \, dx \end{aligned} \tag{3.1.34}$$

for any $\varepsilon \in (0,1)$. On the other hand, due to (3.1.20) we have

$$\begin{aligned}
&\int_{Q_{2r}} \left\langle \bar{A}(x_1, Dv) - \bar{A}_\epsilon(x_1, Dv), Dv_\epsilon - Dv \right\rangle dx \\
&\quad \le \int_{Q_{2r}} |\bar{A}(x_1, Dv) - \bar{A}_\epsilon(x_1, Dv)||Dv_\epsilon - Dv|\, dx \\
&\quad \le c(\varepsilon) \int_{Q_{2r}} \varphi^*(|\bar{A}(x_1, Dv) - \bar{A}_\epsilon(x_1, Dv)|)\, dx \\
&\qquad + c\,\varepsilon \int_{Q_{2r}} \varphi(|Dv_\varepsilon - Dv|)\, dx \\
&\quad =: c(\varepsilon) I_1 + c\,\varepsilon I_2
\end{aligned} \tag{3.1.35}$$

with the same $\varepsilon \in (0,1)$ as in (3.1.34). Here, for each a.e. $x \in Q_{2r}$, by (3.1.29) we can use Jensen's inequality with the measure $\phi(\eta)\phi(z_1)\, d\eta\, dz_1$ to find

$$\begin{aligned}
&\varphi^*(|\bar{A}(x_1, Dv(x)) - \bar{A}_\epsilon(x_1, Dv(x))|) \\
&\lesssim c\varphi^*\left(\int_{-1}^1 \int_{B_1} \left|\bar{A}(x_1, Dv(x)) - \bar{A}(x_1 - \epsilon z_1, Dv(x) - \epsilon\eta)\right| \phi(\eta)\phi(z_1)\, d\eta\, dz_1\right) \\
&\lesssim \int_{-1}^1 \int_{B_1} \varphi^*\left(\left|\bar{A}(x_1, Dv(x)) - \bar{A}(x_1 - \epsilon z_1, Dv(x) - \epsilon\eta)\right|\right) \phi(\eta)\phi(z_1)\, d\eta dz_1 \\
&\lesssim \int_{-1}^1 \int_{B_1} \varphi^*\left(\frac{\left|\bar{A}(x_1, Dv(x)) - \bar{A}(x_1 - \epsilon z_1, Dv(x) - \epsilon\eta)\right|}{\varphi'(|Dv(x)| + \epsilon)} \varphi'(|Dv(x)| + \epsilon)\right) \\
&\quad \times \phi(\eta)\phi(z_1)\, d\eta dz_1 \\
&\underset{\substack{(3.1.27)\\(3.1.31)}}{\lesssim} \varphi(|Dv(x)| + \epsilon) \\
&\quad \times \int_{-1}^1 \int_{B_1} \left(\frac{\left|\bar{A}(x_1, Dv(x)) - \bar{A}(x_1 - \epsilon z_1, Dv(x) - \epsilon\eta)\right|}{2L\varphi'(|Dv(x)| + \epsilon)}\right) d\eta dz_1
\end{aligned}$$

with implicit constants $c = c(n, \kappa_1, \kappa_2, \nu, L)$. Here, using (3.1.27), for each a.e. $x \in Q_{2r}$ we see that

$$\int_{-1}^1 \int_{B_1} \left|\bar{A}(x_1, Dv(x)) - \bar{A}(x_1 - \epsilon z_1, Dv(x) - \epsilon\eta)\right| d\eta dz_1 \to 0 \quad \text{as} \quad \epsilon \to 0.$$

Hence it follows that

$$\varphi^*(|\bar{A}(x_1, Dv(x)) - \bar{A}_\epsilon(x_1, Dv(x))|) \to 0 \quad \text{as} \quad \epsilon \to 0$$

for a.e. $x \in Q_{2r}$. Moreover, using (3.1.18), (3.1.27), (3.1.31) and $0 < \epsilon < 1$, we have

$$\varphi^*(|\bar{A}(x_1, Dv(x)) - \bar{A}_\epsilon(x_1, Dv(x))|) \le c\varphi(|Dv(x)| + 1)$$

with some $c = c(n, \kappa_1, \kappa_2, \nu, L) > 0$. Then in (3.1.35), Lebesgue's dominated convergence theorem together with the above two displays yields that

$$I_1 \to 0 \quad \text{as} \quad \epsilon \to 0.$$

For I_2, testing $v_\epsilon - v \in W_0^{1,\varphi}(Q_{2r})$ to (3.1.33) and then using (3.1.23) we obtain

$$\begin{aligned}\int_{Q_{2r}} \varphi(|Dv_\epsilon - Dv|)\, dx &\le c\int_{Q_{2r}} \varphi(Dv_\epsilon)\, dx + c\int_{Q_{2r}} \varphi(Dv)\, dx \\ &\le c\int_{Q_{2r}} \varphi(Dv)\, dx\end{aligned}$$

with $c = c(n, \kappa_1, \kappa_2, \nu, L)$. Then merging (3.1.34) and (3.1.35), and combining the above two results, we have

$$\lim_{\epsilon \to 0} \int_{Q_{2r}} \varphi(|Dv_\epsilon - Dv|)\, dx \le c\varepsilon \int_{Q_{2r}} \varphi(|Dv|)\, dx.$$

Since $\varepsilon \in (0, 1)$ was arbitrary, we have the conclusion. □

Now we define some functions which are used for our Caccioppoli type estimate, see Lemma 3.1.16. With two parameters $M > 1$ and $\beta \in (0, 1]$ to be determined later, we write

$$\bar{A}_\epsilon(x_1, \xi) = (\bar{A}^1_\epsilon(x_1, \xi), \cdots, \bar{A}^n_\epsilon(x_1, \xi)),$$

$$g_\epsilon(x_1, \xi; M) = \varphi(|\xi'| + \epsilon) + M\varphi^*(|\bar{A}^1_\epsilon(x_1, \xi)| + \varphi'(\epsilon))$$

and

$$\hat{g}_\epsilon(x_1,\xi;\beta) = |\xi'| + \epsilon + \frac{\beta\varphi^{-1}(g_\epsilon(x_1,\xi;M))}{\varphi'(\varphi^{-1}(g_\epsilon(x_1,\xi;M)))}(|\bar{A}^1_\epsilon(x_1,\xi)| + \varphi'(\epsilon))$$

for all $x_1 \in \mathbb{R}$ and $\xi = (\xi_1,\xi') \in \mathbb{R}\times\mathbb{R}^{n-1}$. Then the following lemma holds.

Lemma 3.1.12. *There exists $M = M(n,\kappa_1,\kappa_2,\nu,L)$ such that*

$$g_\epsilon(x_1,\xi;M) \eqsim \varphi(|\xi|+\epsilon) \tag{3.1.36}$$

and

$$\beta(|\xi|+\epsilon) \lesssim \hat{g}_\epsilon(x_1,\xi;\beta) \lesssim |\xi|+\epsilon \tag{3.1.37}$$

with the implicit constant $c = c(n,\kappa_1,\kappa_2,\nu,L) > 0$, whenever $\xi \in \mathbb{R}^n$ and $\beta \in (0,1]$.

Proof. First, the inequality $g_\epsilon(x_1,\xi;M) \lesssim \varphi(|\xi|+\epsilon)$ follows from (3.1.17), (3.1.18) and (3.1.31):

$$\begin{aligned} g_\epsilon(x_1,\xi;M) &\underset{(3.1.31)}{\leq} \varphi(|\xi'|+\epsilon) + M\varphi^*(cL\varphi'(|\xi|+\epsilon)+\varphi'(\epsilon)) \\ &\underset{(3.1.17)}{\leq} \varphi(|\xi|+\epsilon) + cM\varphi^*(\varphi'(|\xi|+\epsilon)) \\ &\underset{(3.1.18)}{\leq} cM\varphi(|\xi|+\epsilon) \end{aligned} \tag{3.1.38}$$

with $c = c(n,\kappa_1,\kappa_2,\nu,L) > 0$.

To show $g_\epsilon(x_1,\xi;M) \gtrsim \varphi(|\xi|+\varepsilon)$, we first prove a lower bound of $|\bar{A}^1_\epsilon|$. Denote $0 = (0_1,0')$ with $0_1 \in \mathbb{R}$ and $0' \in \mathbb{R}^{n-1}$. By triangle inequality, we have

$$\begin{aligned} |\bar{A}^1_\epsilon(x_1,\xi)| &\geq -|\bar{A}^1_\epsilon(x_1,\xi) - \bar{A}^1_\epsilon(x_1,\xi_1,0')| + |\bar{A}^1_\epsilon(x_1,\xi_1,0')| \\ &=: -I_1 + I_2. \end{aligned} \tag{3.1.39}$$

For I_1, by (3.1.9) and (3.1.31) we estimate

$$\begin{aligned} I_1 &= \left| \int_0^1 \frac{d}{dt}[\bar{A}^1_\epsilon(x_1, t\xi + (1-t)(\xi_1, 0'))]\, dt \right| \\ &\leq \int_0^1 \left| D_\xi \bar{A}^1_\epsilon(x_1, t\xi + (1-t)(\xi_1, 0')) \right| |(0_1, \xi')|\, dt \\ &\leq L|\xi'| \int_0^1 \varphi''(|t\xi + (1-t)(\xi_1, 0')| + \epsilon)\, dt \\ &\leq c|\xi'| \int_0^1 \varphi''(|\xi_1| + |t\xi'| + \epsilon)\, dt \\ &= c\, [\varphi'(|\xi_1| + |t\xi'| + \epsilon)]_0^1 = c_0\varphi'(|\xi_1| + |\xi'| + \epsilon) - c_0\varphi'(|\xi_1| + \epsilon) \end{aligned} \tag{3.1.40}$$

with $c_0 = c_0(n, \kappa_1, \kappa_2, \nu, L) > 0$. Here, we see that for any $\theta \in (0, 1)$,

$$\begin{aligned} \text{if } |\xi'| \leq \theta(|\xi_1| + \epsilon) \Rightarrow \varphi'(|\xi_1| + |\xi'| + \epsilon) &\underset{(3.1.8)}{\leq} \varphi'((1+\theta)(|\xi_1| + \epsilon)) \\ &\underset{(3.1.11)}{\leq} (1+\theta)^{\kappa_2}\varphi'(|\xi_1| + \epsilon), \end{aligned}$$

$$\begin{aligned} \text{if } |\xi'| > \theta(|\xi_1| + \epsilon) \Rightarrow \varphi'(|\xi_1| + |\xi'| + \epsilon) &\underset{(3.1.8)}{\leq} \varphi'((\tfrac{1}{\theta} + 1)(|\xi'| + \epsilon)) \\ &\underset{(3.1.11)}{\leq} (\tfrac{1}{\theta} + 1)^{\kappa_2}\varphi'(|\xi'| + \epsilon). \end{aligned}$$

Using the above inequalities, and observing that $\theta \in (0, 1)$ implies $(1+\theta)^{\kappa_2} - 1 \leq c(\kappa_2)\theta$, we obtain

$$\begin{aligned} I_1 &\leq c_0\varphi'(|\xi_1| + |\xi'| + \epsilon) - c_0\varphi'(|\xi_1| + \epsilon) \\ &\leq c_0[(1+\theta)^{\kappa_2} - 1]\varphi'(|\xi_1| + \epsilon) + c_0(\tfrac{1}{\theta} + 1)^{\kappa_2}\varphi'(|\xi'| + \epsilon) \\ &\leq c_0 c(\kappa_2)\theta\varphi'(|\xi_1| + \epsilon) + c_0(\tfrac{1}{\theta} + 1)^{\kappa_2}\varphi'(|\xi'| + \epsilon). \end{aligned} \tag{3.1.41}$$

On the other hand, for I_2, we observe that

$$\begin{aligned}
&\bar{A}^1_\epsilon(x_1,\xi_1,0')\xi_1\\
&= \left\langle \bar{A}^1_\epsilon(x_1,\xi_1,0') - \bar{A}^1_\epsilon(x_1,0_1,0'),(\xi_1,0')\right\rangle + \left\langle \bar{A}^1_\epsilon(x_1,0_1,0'),(\xi_1,0')\right\rangle\\
&= \left\langle \int_0^1 \frac{d}{dt}(\bar{A}^1_\epsilon(x_1,t\xi_1,0'))\,dt,(\xi_1,0')\right\rangle + \left\langle \bar{A}^1_\epsilon(x_1,0_1,0'),(\xi_1,0')\right\rangle\\
&= \int_0^1 \left\langle D_\xi \bar{A}^1_\epsilon(x_1,t\xi_1,0')(\xi_1,0'),(\xi_1,0')\right\rangle\,dt + \left\langle \bar{A}^1_\epsilon(x_1,0_1,0'),(\xi_1,0')\right\rangle
\end{aligned}$$

and so by (3.1.31), there holds

$$\begin{aligned}
\bar{A}^1_\epsilon(x_1,\xi_1,0')\xi_1 &\geq c\int_0^1 \varphi''(|t\xi_1|+\epsilon)|\xi_1|^2\,dt - c\varphi'(\epsilon)|\xi_1|\\
&\geq c\int_{\frac{1}{2}}^1 \varphi''(|t\xi_1|+\epsilon)|\xi_1|^2\,dt - c\varphi'(\epsilon)|\xi_1|\\
&\geq c|\xi_1|^2\left(\min_{t\in[\frac{1}{2},1]}\varphi''(|t\xi_1|+\epsilon)\right) - c\varphi'(\epsilon)|\xi_1|\\
&\geq c|\xi_1|^2\left(\frac{1}{2}\right)^{\kappa_2}\varphi''(|\xi_1|+\epsilon) - c\varphi'(\epsilon)|\xi_1|.
\end{aligned}\tag{3.1.42}$$

Note that the last inequality is obtained by the following observations. For any $t\in\left[\frac{1}{2},1\right]$, we see that

$$\begin{aligned}
&\varphi''(|t\xi_1|+\epsilon) \underset{(3.1.9)}{\gtrsim} \frac{\varphi'(|t\xi_1|+\epsilon)}{|t\xi_1|+\epsilon}\\
&\quad\underset{(3.1.8)}{\gtrsim} \frac{\varphi'(\frac{1}{2}|\xi_1|+\epsilon)}{|\xi_1|+\epsilon} \underset{(3.1.11)}{\gtrsim} \left(\frac{1}{2}\right)^{\kappa_2}\frac{\varphi'(|\xi_1|+\epsilon)}{|\xi_1|+\epsilon} \underset{(3.1.9)}{\gtrsim} \left(\frac{1}{2}\right)^{\kappa_2}\varphi''(|\xi_1|+\epsilon).
\end{aligned}$$

Then from (3.1.42), (3.1.9) and (3.1.22) with small $\varepsilon=\varepsilon(n,\kappa_1,\kappa_2,\nu,L)\in(0,1)$, we have

$$\begin{aligned}
I_2 &\geq c|\xi_1|\varphi''(|\xi_1|+\epsilon) - L\varphi'(\epsilon)\\
&\geq c(|\xi_1|+\epsilon)\varphi''(|\xi_1|+\epsilon) - c\epsilon\varphi''(|\xi_1|+\epsilon) - L\varphi'(\epsilon)\\
&\geq c\varphi'(|\xi_1|+\epsilon) - \varepsilon\varphi'(|\xi_1|+\epsilon) - c(\varepsilon)\varphi'(\epsilon) - L\varphi'(\epsilon)\\
&\geq c_1\varphi'(|\xi_1|+\epsilon) - c_2\varphi'(\epsilon)
\end{aligned}\tag{3.1.43}$$

for $c_1 = c_1(n, \kappa_1, \kappa_2, \nu, L) > 0$ and $c_2 = c_2(n, \kappa_1, \kappa_2, \nu, L) \geq 1$. Summing up (3.1.39), (3.1.41) and (3.1.43), it follows that

$$\begin{aligned} &|\bar{A}^1_\epsilon(x_1, \xi)| \\ &\quad \geq [c_1 - c_0 c(\kappa_2)\theta]\varphi'(|\xi_1| + \epsilon) - c_0(\tfrac{1}{\theta} + 1)^{\kappa_2}\varphi'(|\xi'| + \epsilon) - c_2\varphi'(\epsilon). \end{aligned} \tag{3.1.44}$$

Now by choosing $\theta = \theta(n, \kappa_1, \kappa_2, \nu, L) \in (0, 1)$ sufficiently small such that $c_1 - c_0 c(\kappa_2)\theta \geq \frac{c_1}{2}$, we have

$$|\bar{A}^1_\epsilon(x_1, \xi)| + \varphi'(\epsilon) \geq \tfrac{c_1}{2c_2}\varphi'(|\xi_1| + \epsilon) - \tfrac{c_0}{c_2}(\tfrac{1}{\theta} + 1)^{\kappa_2}\varphi'(|\xi'| + \epsilon)$$

and so

$$|\bar{A}^1_\epsilon(x_1, \xi)| + \varphi'(\epsilon) + c_3\varphi'(|\xi'| + \epsilon) \geq c\varphi'(|\xi_1| + \epsilon) + c\varphi'(|\xi'| + \epsilon) \tag{3.1.45}$$

with $c_3 = \frac{2c_0}{c_2}(\frac{1}{\theta} + 1)^{\kappa_2}$ and $c = c(n, \kappa_1, \kappa_2, \nu, L) > 0$.

For the left-hand side of the above inequality, by taking φ^* and using (3.1.17), (3.1.18) and (3.1.24) we obtain

$$\begin{aligned} &\varphi^*\left([|\bar{A}^1_\epsilon(x_1, \xi)| + \varphi'(\epsilon)] + c_3\varphi'(|\xi'| + \epsilon)\right) \\ &\qquad \leq c\varphi^*\left([|\bar{A}^1_\epsilon(x_1, \xi)| + \varphi'(\epsilon)]\right) + c\max\left\{c_3^{\frac{\kappa_1}{\kappa_1 - 1}}, c_3^{\frac{\kappa_2}{\kappa_2 - 1}}\right\}\varphi^*\left(\varphi'(|\xi'| + \epsilon)\right) \\ &\qquad \leq c\varphi^*\left([|\bar{A}^1_\epsilon(x_1, \xi)| + \varphi'(\epsilon)]\right) + c\max\left\{c_3^{\frac{\kappa_1}{\kappa_1 - 1}}, c_3^{\frac{\kappa_2}{\kappa_2 - 1}}\right\} c_4\varphi(|\xi'| + \epsilon) \\ &\qquad \leq c g_\epsilon(x_1, \xi; M) \end{aligned}$$

with $c_4 = c_4(n, \kappa_1, \kappa_2, \nu, L)$, provided

$$M = M(n, \kappa_1, \kappa_2, \nu, L) := \max\left\{c_3^{\frac{\kappa_1}{\kappa_1 - 1}}, c_3^{\frac{\kappa_2}{\kappa_2 - 1}}\right\} c_4.$$

For the right-hand side of (3.1.45), by taking φ^* and using (3.1.8), (3.1.11) and (3.1.18), we observe

$$\varphi^*(c\varphi'(|\xi_1| + \epsilon) + c\varphi'(|\xi'| + \epsilon)) \geq c\varphi^*(\varphi'(|\xi| + \epsilon)) \geq c\varphi(|\xi| + \epsilon). \tag{3.1.46}$$

Thus by (3.1.45)–(3.1.46), we have $g_\epsilon(x_1, \xi; M) \gtrsim \varphi(|\xi| + \epsilon)$ and so (3.1.36) holds.

Now we start to prove (3.1.37). By (3.1.8), (3.1.9), (3.1.13), (3.1.31) and

(3.1.36), we have

$$\begin{aligned}\hat{g}_\epsilon(x_1,\xi;\beta) &\lesssim |\xi'| + \epsilon + \frac{\beta(|\xi|+\epsilon)}{\varphi'(|\xi|+\epsilon)}(\varphi'(|\xi|+\epsilon)+\varphi'(\epsilon))\\ &\lesssim |\xi'| + \epsilon + \frac{\varphi'(|\xi|+\epsilon)}{\varphi''(|\xi|+\epsilon)} \lesssim |\xi| + \epsilon.\end{aligned}$$

On the other hand, note that by (3.1.19), $\varphi^{-1}(\varphi^*)(t) \approx (\varphi^*)'(t)$ holds for $t \geq 0$. Then together with (3.1.8), (3.1.13), (3.1.16) and (3.1.19), one can find

$$\begin{aligned}&\varphi^{-1}(g_\epsilon(x_1,\xi;M))(|\bar{A}^1_\epsilon(x_1,\xi)|+\varphi'(\epsilon))\\ &\quad\gtrsim \varphi^{-1}(M\varphi^*(|\bar{A}^1_\epsilon(x_1,\xi)|+\varphi'(\epsilon)))(|\bar{A}^1_\epsilon(x_1,\xi)|+\varphi'(\epsilon))\\ &\quad\gtrsim \varphi^{-1}(\varphi^*(|\bar{A}^1_\epsilon(x_1,\xi)|+\varphi'(\epsilon)))(|\bar{A}^1_\epsilon(x_1,\xi)|+\varphi'(\epsilon))\\ &\quad\gtrsim (\varphi^*)'(|\bar{A}^1_\epsilon(x_1,\xi)|+\varphi'(\epsilon))(|\bar{A}^1_\epsilon(x_1,\xi)|+\varphi'(\epsilon))\\ &\quad\gtrsim \varphi^*(|\bar{A}^1_\epsilon(x_1,\xi)|+\varphi'(\epsilon)).\end{aligned}$$

Thus together with (3.1.36) and (3.1.10), we have

$$\begin{aligned}&\varphi'(\varphi^{-1}(g_\epsilon(x_1,\xi;M)))\hat{g}_\epsilon(x_1,\xi;\beta)\\ &\qquad= \varphi'(\varphi^{-1}(g_\epsilon(x_1,\xi;M)))(|\xi'|+\epsilon)\\ &\qquad\quad+\beta\varphi^{-1}(g_\epsilon(x_1,\xi;M))(|\bar{A}^1_\epsilon(x_1,\xi)|+\varphi'(\epsilon))\\ &\qquad\gtrsim \varphi(|\xi'|+\epsilon)+\beta\varphi^*(|\bar{A}^1_\epsilon(x_1,\xi)|+\varphi'(\epsilon))\\ &\qquad\gtrsim \beta g_\epsilon(x_1,\xi;M)\\ &\qquad\gtrsim \beta\varphi'(\varphi^{-1}(g_\epsilon(x_1,\xi;M)))(|\xi|+\epsilon),\end{aligned}$$

which implies (3.1.37). □

Remark 3.1.13. *Since $\bar{A}_\epsilon(x_1,\xi)$ is $C^1(\mathbb{R}\times\mathbb{R}^n)$-regular, we observe from Lemma 3.1.9 that $Dv_\epsilon \in L^\infty_{\mathrm{loc}}(Q_{2r})$. We refer to [167, 169] for the proof.*

The following lemma is a higher order differentiability result for $D_{x'}v_\epsilon$ of the regularized problem (3.1.33). See also [66] for the related results.

Lemma 3.1.14. *Let $x_0 \in Q_r$ and $0 < \rho < \frac{1}{4}r$. Then we have $DD_{x'}v_\epsilon \in L^2(Q_\rho(x_0))$.*

Proof. Write $Q_{j\rho} = Q_{j\rho}(x_0)$ for $j = 1, 2, 3$. We select a smooth cutoff function ϕ satisfying $0 \leq \phi \leq 1$, $\phi \equiv 1$ on Q_ρ, $\phi \equiv 0$ on $\mathbb{R}^n \setminus Q_{2\rho}$ and $|D\phi| \leq \frac{2}{\rho}$. Now let $|h| \in (0,\rho)$ be small, choose $k \in \{2,\dots,n\}$ and write

$$\varphi(x) = -D_k^{-h}(\phi^2(x)D_k^h v_\epsilon(x)) \quad (x \in Q_\rho).$$

Then we have

$$0 = \int_{Q_{3\rho}} \left\langle \bar{A}_\epsilon(x_1, Dv_\epsilon), D[-D_k^{-h}(\phi^2 D_k^h v_\epsilon)] \right\rangle dx.$$

Using integration by parts for difference quotient gives us that

$$\begin{aligned} 0 &= \int_{Q_{3\rho}} \left\langle D_k^h \bar{A}_\epsilon(x_1, Dv_\epsilon), D(\phi^2 D_k^h v_\epsilon) \right\rangle dx \\ &= \int_{Q_{3\rho}} \Big(\phi^2 \left\langle D_k^h \bar{A}_\epsilon(x_1, Dv_\epsilon), D_k^h Dv_\epsilon \right\rangle \\ &\qquad\qquad +2\phi D_k^h v_\epsilon \left\langle D_k^h \bar{A}_\epsilon(x_1, Dv_\epsilon), D\phi \right\rangle\Big) \, dx \\ &=: I_1 + I_2. \end{aligned} \tag{3.1.47}$$

Here, we compute

$$\begin{aligned} D_k^h \bar{A}_\epsilon(x_1, Dv_\epsilon) &= \frac{\bar{A}_\epsilon(x_1, Dv_\epsilon(x+he_k)) - \bar{A}_\epsilon(x_1, Dv_\epsilon(x))}{h} \\ &= \frac{1}{h}\int_0^1 \frac{d}{ds}\bar{A}_\epsilon(x_1, \{(1-s)Dv_\epsilon(x) + sDv_\epsilon(x+he_k)\})\, ds \\ &= \left(\int_0^1 D_\xi \bar{A}_\epsilon(x_1, \{(1-s)Dv_\epsilon(x) + sDv_\epsilon(x+he_k)\})\, ds\right)(D_k^h Dv_\epsilon(x)) \\ &=: \bar{A}_k^h(x)(D_k^h Dv_\epsilon(x)). \end{aligned}$$

Then by (3.1.31), it follows that

$$\begin{aligned} I_1 &= \int_{Q_{3\rho}} \phi^2 \left\langle \bar{A}_k^h(x)(D_k^h Dv_\epsilon), D_k^h Dv_\epsilon \right\rangle dx \\ &\geq c\int_{Q_{3\rho}} \phi^2 \underbrace{\left(\int_0^1 \varphi''(|(1-s)Dv_\epsilon(x) + sDv_\epsilon(x+he_k)| + \epsilon)\, ds\right)}_{=:I_3} |D_k^h Dv_\epsilon|^2 \, dx \end{aligned}$$

with $c = c(n,\kappa_1,\kappa_2,\nu,L) > 0$. Furthermore, by Young's inequality with

$\tau \in (0,1)$ and (3.1.31), we have

$$\begin{aligned} |I_2| &= \left| \int_{Q_{3\rho}} 2\phi D_k^h v_\epsilon \left\langle \bar{A}_k^h(x)(D_k^h D v_\epsilon), D\phi \right\rangle \, dx \right| \\ &\leq c\tau \int_{Q_{3\rho}} \phi^2 |\bar{A}_k^h(x)||D_k^h D v_\epsilon|^2 \, dx + \frac{c}{\tau\rho} \int_{Q_{3\rho}} |\bar{A}_k^h(x)||D_k^h v_\epsilon|^2 \, dx \\ &\leq c\tau \int_{Q_{3\rho}} \phi^2 I_3 |D_k^h D v_\epsilon|^2 \, dx + \frac{c}{\tau\rho} \int_{Q_{3\rho}} I_3 |D_k^h v_\epsilon|^2 \, dx. \end{aligned} \tag{3.1.48}$$

By merging (3.1.47)–(3.1.48), and selecting τ small enough to find

$$\int_{Q_\rho} I_3 |D_k^h D v_\epsilon|^2 \, dx \leq \int_{Q_{3\rho}} \phi^2 I_3 |D_k^h D v_\epsilon|^2 \, dx \leq \frac{c}{\rho} \int_{Q_{3\rho}} I_3 |D_k^h v_\epsilon|^2 \, dx \tag{3.1.49}$$

with $c = c(n, \kappa_1, \kappa_2, \nu, L) > 0$.

Here, we observe from (3.1.8) and (3.1.9) that

$$\begin{aligned} I_3 &\geq c \int_0^1 \frac{\varphi'(|(1-s)Dv(x) + sDv(x+he_k)| + \epsilon)}{|(1-s)Dv(x) + sDv(x+he_k)| + \epsilon} \, ds \\ &\geq c \int_0^1 \frac{\varphi'(\epsilon)}{\|Dv_\epsilon\|_{L^\infty(Q_{2\rho}(x_0))} + \epsilon} \, ds \\ &= \frac{c\varphi'(\epsilon)}{\|Dv_\epsilon\|_{L^\infty(Q_{2\rho}(x_0))} + \epsilon} \end{aligned}$$

and

$$\begin{aligned} I_3 &\leq c \int_0^1 \frac{\varphi'(|(1-s)Dv(x) + sDv(x+he_k)| + \epsilon)}{|(1-s)Dv(x) + sDv(x+he_k)| + \epsilon} \, ds \\ &\leq c \int_0^1 \frac{\varphi'(\|Dv_\epsilon\|_{L^\infty(Q_{2\rho}(x_0))} + \epsilon)}{\epsilon} \, ds \\ &= \frac{c\varphi'(\|Dv_\epsilon\|_{L^\infty(Q_{2\rho}(x_0))} + \epsilon)}{\epsilon} \end{aligned} \tag{3.1.50}$$

with $c = c(n, \kappa_1, \kappa_2, \nu, L) > 0$.

Summing up (3.1.49)–(3.1.50) and applying (3.1.10), we have

$$\begin{aligned}
&\int_{Q_\rho} |D_k^h D v_\epsilon|^2\, dx \\
&\quad \leq \frac{c\varphi'(\|Dv_\epsilon\|_{L^\infty(Q_{2\rho}(x_0))}+\epsilon)(\|Dv_\epsilon\|_{L^\infty(Q_{2\rho}(x_0))}+\epsilon)}{\varphi'(\epsilon)\epsilon\rho} \int_{Q_{3\rho}} |D_k^h v_\epsilon|^2\, dx \\
&\quad \leq \frac{c\varphi(\|Dv_\epsilon\|_{L^\infty(Q_{2\rho}(x_0))}+\epsilon)}{\varphi(\epsilon)\rho} \int_{Q_{3\rho}} |D_k^h v_\epsilon|^2\, dx.
\end{aligned}$$

Then by (2.0.2) and (2.0.3), it holds that

$$\int_{Q_\rho} |D_k^h D v_\epsilon|^2\, dx \leq \frac{c\varphi(\|Dv_\epsilon\|_{L^\infty(Q_{2\rho}(x_0))}+\epsilon)}{\varphi(\epsilon)\rho} \int_{Q_{3\rho}} |D v_\epsilon|^2\, dx$$

and so considering all cases $k \in \{2, \ldots, n\}$, we find

$$\int_{Q_\rho} |D D_{x'} v_\epsilon|^2\, dx \leq \frac{c\varphi(\|Dv_\epsilon\|_{L^\infty(Q_{2\rho}(x_0))}+\epsilon)}{\varphi(\epsilon)\rho} \int_{Q_{3\rho}} |D v_\epsilon|^2\, dx.$$

Then $DD_{x'}v_\epsilon \in L^2(Q_\rho(x_0))$ holds. □

From now on, we write

$$g = g(x) := g_\epsilon(x_1, Dv_\epsilon(x); M) \quad \text{and} \quad \hat{g} = \hat{g}(x) := \hat{g}_\epsilon(x_1, Dv_\epsilon(x); \beta),$$

where M is given in Lemma 3.1.12, while β is to be determined later in Lemma 3.1.16. Moreover, we define

$$E_{1,i} := D_i(|D_{x'}v_\epsilon| + \epsilon)$$

and

$$E_{2,i} := D_i\left(\frac{\varphi^{-1}(g)}{\varphi'(\varphi^{-1}(g))}(|\bar{A}_\epsilon^1(x_1, Dv_\epsilon)| + \varphi'(\epsilon))\right)$$

so that the following holds:

$$D_i\hat{g} = E_{1,i} + \beta E_{2,i} \quad (1 \leq i \leq n). \tag{3.1.51}$$

Then for $D\hat{g}$, we have the following lemma.

Lemma 3.1.15. *Let $x_0 \in Q_r$ and $0 < \rho < \frac{1}{4}r$. Then we have $D\hat{g} \in L^2(Q_\rho(x_0))$ and*

$$|D\hat{g}(x)| \le c\,|DD_{x'}v_\epsilon(x)| \quad \text{a.e. in } \ x \in Q_\rho(x_0) \tag{3.1.52}$$

for some positive constant c depending only on $n, \kappa_1, \kappa_2, \nu, L$ but independent of β.

Proof. We first claim that

$$|Dg| \le c\,\varphi'(|Dv_\epsilon| + \epsilon)|DD_{x'}v_\epsilon| \quad \text{in } Q_\rho(x_0) \tag{3.1.53}$$

with $c = c(n, \kappa_1, \kappa_2, \nu, L) > 0$. Indeed, by (3.1.9), Remark 3.1.13 and Lemma 3.1.14, we have

$$\begin{aligned}
&\varphi''(|Dv_\epsilon| + \epsilon)|DD_{x'}v_\epsilon| \\
&\qquad \le c\frac{\varphi'(|Dv_\epsilon| + \epsilon)}{|Dv_\epsilon| + \epsilon}|DD_{x'}v_\epsilon| \\
&\qquad \le c\frac{\varphi'(\|Dv_\epsilon\|_{L^\infty} + \epsilon)}{\epsilon}|DD_{x'}v_\epsilon| \in L^2 \quad \text{in } Q_\rho(x_0).
\end{aligned} \tag{3.1.54}$$

Then for $1 < k \le n$, by (3.1.31) we obtain

$$\begin{aligned}
\left|D_k[\bar{A}^1_\epsilon(x_1, Dv_\epsilon)]\right| &= \left|\sum_{1\le j\le n} D_{\xi_j}\bar{A}^1_\epsilon(x_1, Dv_\epsilon)D_{kj}v_\epsilon\right| \\
&\le c\varphi''(|Dv_\epsilon| + \epsilon)|DD_{x'}v_\epsilon| \in L^2 \quad \text{in } Q_\rho(x_0).
\end{aligned} \tag{3.1.55}$$

On the other hand, since v_ϵ is a weak solution of (3.1.33), together with (3.1.31) it holds that

$$\begin{aligned}
\left|D_1[\bar{A}^1_\epsilon(x_1, Dv_\epsilon)]\right| &= \left|-\sum_{1<i\le n} D_i[\bar{A}^i_\epsilon(x_1, Dv_\epsilon)]\right| \\
&= \left|\sum_{1<i\le n}\sum_{1\le j\le n} D_{\xi_j}\bar{A}^i_\epsilon(x_1, Dv_\epsilon)D_{ij}v_\epsilon\right| \\
&\le c\varphi''(|Dv_\epsilon| + \epsilon)|DD_{x'}v_\epsilon| \in L^2 \quad \text{in } Q_\rho(x_0).
\end{aligned} \tag{3.1.56}$$

Then by (3.1.55) and (3.1.56), we obtain

$$\left|D[\bar{A}^1_\epsilon(x_1, Dv_\epsilon)]\right| \le c\varphi''(|Dv_\epsilon| + \epsilon)|DD_{x'}v_\epsilon|. \tag{3.1.57}$$

Now together with (3.1.57), Lemma 3.1.9 yields

$$\begin{aligned}
|Dg| \le c &\left|\varphi'(|D_{x'}v_\epsilon| + \epsilon)\frac{D_{x'}v_\epsilon}{|D_{x'}v_\epsilon|}|DD_{x'}v_\epsilon|\right| \\
&\quad + c(\varphi^*)'(|\bar{A}^1_\epsilon(x_1, Dv_\epsilon)| + \varphi'(\epsilon))\frac{\left|\bar{A}^1_\epsilon(x_1, Dv_\epsilon)\right|}{|\bar{A}^1_\epsilon(x_1, Dv_\epsilon)| + \varphi'(\epsilon)}\left|D(\bar{A}^1_\epsilon(x_1, Dv_\epsilon))\right| \\
&\le c\,\varphi'(|D_{x'}v_\epsilon| + \epsilon)|DD_{x'}v_\epsilon| \\
&\quad + c\,(\varphi^*)'(|\bar{A}^1_\epsilon(x_1, Dv_\epsilon)| + \varphi'(\epsilon))\varphi''(|Dv_\epsilon| + \epsilon)|DD_{x'}v_\epsilon|.
\end{aligned}$$

Here, by (3.1.9), (3.1.10), (3.1.16), (3.1.17), (3.1.18) and (3.1.31), it holds that

$$\begin{aligned}
(\varphi^*)'(|\bar{A}^1_\epsilon(x_1, Dv_\epsilon)| &+ \varphi'(\epsilon))\varphi''(|Dv_\epsilon| + \epsilon) \\
&\underset{(3.1.31)}{\le} c(\varphi^*)'(\varphi'(|Dv_\epsilon| + \epsilon))\varphi''(|Dv_\epsilon| + \epsilon) \\
&\underset{(3.1.16)}{\le} c\frac{\varphi^*(\varphi'(|Dv_\epsilon| + \epsilon))}{\varphi'(|Dv_\epsilon| + \epsilon)}\varphi''(|Dv_\epsilon| + \epsilon) \\
&\underset{(3.1.18)}{\le} c\frac{\varphi(|Dv_\epsilon| + \epsilon)\varphi''(|Dv_\epsilon| + \epsilon)}{\varphi'(|Dv_\epsilon| + \epsilon)} \\
&\underset{\substack{(3.1.9)\\(3.1.10)}}{\le} c\varphi'(|Dv_\epsilon| + \epsilon)
\end{aligned}$$

with $c = c(n, \kappa_1, \kappa_2, \nu, L) > 0$, and so (3.1.53) follows.

To show (3.1.52), we observe that

$$|E_{1,i}| \le \left|\frac{D_{x'}v_\epsilon}{|D_{x'}v_\epsilon|}DD_{x'}v_\epsilon\right| = |DD_{x'}v_\epsilon|$$

and

$$|E_{2,i}| \le \left|D_i\left(\frac{\varphi^{-1}(g)}{\varphi'(\varphi^{-1}(g))}(|\bar{A}^1_\epsilon(x_1, Dv_\epsilon)| + \varphi'(\epsilon))\right)\right| \le I_1 + I_2 + I_3,$$

where

$$
\begin{aligned}
I_1 &= \frac{(\varphi^{-1})'(g)|Dg|(|\bar{A}^1_\epsilon(x_1, Dv_\epsilon)| + \varphi'(\epsilon))}{\varphi'(\varphi^{-1}(g))},\\
I_2 &= \frac{\varphi^{-1}(g)|D\bar{A}^1_\epsilon(x_1, Dv_\epsilon)|}{\varphi'(\varphi^{-1}(g))},\\
I_3 &= \frac{\varphi^{-1}(g)(|\bar{A}^1_\epsilon(x_1, Dv_\epsilon)| + \varphi'(\epsilon))\varphi''(\varphi^{-1}(g))(\varphi^{-1})'(g)|Dg|}{\varphi'(\varphi^{-1}(g))^2}.
\end{aligned}
$$

Here, by inverse function theorem for φ, (3.1.31), (3.1.36) and (3.1.53), we have

$$
\begin{aligned}
I_1 &\le c\frac{(\varphi^{-1})'(\varphi(|Dv_\epsilon| + \epsilon))\varphi'(|Dv_\epsilon| + \epsilon)|DD_{x'}v_\epsilon|\varphi'(|Dv_\epsilon| + \epsilon)}{\varphi'(\varphi^{-1}(g))}\\
&\le c(\varphi^{-1})'(\varphi(|Dv_\epsilon| + \epsilon))\varphi'(|Dv_\epsilon| + \epsilon)|DD_{x'}v_\epsilon|\\
&\le c\frac{\varphi'(|Dv_\epsilon| + \epsilon)}{\varphi'(|Dv_\epsilon| + \epsilon)}|DD_{x'}v_\epsilon| \le c|DD_{x'}v_\epsilon|
\end{aligned}
$$

with $c = c(n, \kappa_1, \kappa_2, \nu, L) > 0$. For I_2, owing to (3.1.9), (3.1.36) and (3.1.57), there holds

$$
\begin{aligned}
I_2 &\le c\frac{\varphi^{-1}(g)\varphi''(|Dv_\epsilon| + \epsilon)|DD_{x'}v_\epsilon|}{\varphi'(\varphi^{-1}(g))}\\
&\le c\frac{(|Dv_\epsilon| + \epsilon)\varphi''(|Dv_\epsilon| + \epsilon)|DD_{x'}v_\epsilon|}{\varphi'(|Dv_\epsilon| + \epsilon)} \le c|DD_{x'}v_\epsilon|.
\end{aligned}
$$

Finally, for I_3, by (3.1.9) we have $\varphi^{-1}(g)\varphi''(\varphi^{-1}(g)) \approx \varphi'(\varphi^{-1}(g))$. Then by (3.1.57) it follows that

$$
I_3 \le c\frac{\varphi^{-1}(g)\varphi''(\varphi^{-1}(g))}{\varphi'(\varphi^{-1}(g))}I_1 \le c|DD_{x'}v_\epsilon|.
$$

Therefore, we have

$$
|E_{2,i}| \le c|DD_{x'}v_\epsilon| \tag{3.1.58}
$$

with $c = c(n, \kappa_1, \kappa_2, \nu, L) > 0$. Then the lemma follows. □

Now we prove the following Caccioppoli type estimates.

Lemma 3.1.16. *Let $x_0 \in Q_r$ and $0 < \rho < \frac{1}{4}r$. For a given $\alpha \in [0, 4q]$, there exists $\beta \in (0,1)$ depending on $n, \kappa_1, \kappa_2, \nu, L$ and q such that for any $\eta \in C_0^\infty(Q_\rho(x_0))$ we have*

$$\begin{aligned} &\fint_{Q_{2r}} \eta^2 \varphi^\alpha(\hat{g}) \varphi''(|Dv_\epsilon| + \epsilon) |DD_{x'} v_\epsilon|^2 \, dx \\ &\qquad\qquad \leq c \fint_{Q_{2r}} \varphi^{\alpha+1}(|Dv_\epsilon| + \epsilon) |D\eta|^2 \, dx \end{aligned} \tag{3.1.59}$$

for $c = c(n, \kappa_1, \kappa_2, \nu, L) > 0$.

Proof. Fix $1 < k \leq n$. For each $1 \leq i, j \leq n$, by (3.1.31), (3.1.54) and (3.1.57) we obtain

$$\begin{aligned} |D_k \bar{A}^i_\epsilon(x_1, Dv_\epsilon)| &= \left| D_{\xi_j} \bar{A}^i_\epsilon(x_1, Dv_\epsilon) D_{kj} v_\epsilon \right| \\ &\leq c\varphi''(|Dv_\epsilon| + \epsilon) |DD_{x'} v_\epsilon| \in L^2(Q_\rho(x_0)) \end{aligned} \tag{3.1.60}$$

with $\epsilon > 0$. Then testing $D_k \phi \in C_c^\infty(Q_\rho(x_0))$ to (3.1.33) and using integration by parts, we have

$$\begin{aligned} 0 &= \int_{Q_{2r}} D_k \left[\bar{A}^i_\epsilon(x_1, Dv_\epsilon) \right] D_i \phi \, dx \\ &= \int_{Q_{2r}} D_{\xi_j} \bar{A}^i_\epsilon(x_1, Dv_\epsilon) D_{kj} v_\epsilon D_i \phi \, dx. \end{aligned} \tag{3.1.61}$$

Note that in (3.1.61), we omit the summation over $1 \leq i, j \leq n$. Due to (3.1.60), we have (3.1.61) for all $\phi \in W_0^{1,2}(Q_\rho(x_0))$.

Now let $\eta \in C_c^\infty(Q_\rho(x_0))$ be a smooth cutoff function with $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $Q_{\frac{\rho}{2}}(x_0)$, $\eta \equiv 0$ on $Q_{2r} \setminus Q_\rho(x_0)$ and $|D\eta| \leq \frac{4}{\rho}$. Note that

Remark 3.1.13 and Lemma 3.1.14 $\quad\Rightarrow\quad D_k v_\epsilon \in W^{1,2}(Q_\rho(x_0)) \cap L^\infty(Q_\rho(x_0))$,
Lemma 3.1.12, Remark 3.1.13 and Lemma 3.1.15
$\quad\Rightarrow\quad \hat{g} \in W^{1,2}(Q_\rho(x_0)) \cap L^\infty(Q_\rho(x_0))$.

Test $\phi = D_k v_\epsilon \varphi^\alpha(\hat{g}) \eta^2 \in W_0^{1,2}(Q_\rho(x_0))$ for (3.1.61) to obtain that

$$0 = \int_{Q_{2r}} D_{\xi_j} \bar{A}^i_\epsilon(x_1, Dv_\epsilon) D_{kj} v_\epsilon D_i (D_k v_\epsilon \varphi^\alpha(\hat{g}) \eta^2) \, dx. \tag{3.1.62}$$

Here, we compute

$$\begin{aligned} &D_i(D_k v_\epsilon \varphi^\alpha(\hat{g})\eta^2) \\ &\quad = D_{ki} v_\epsilon \varphi^\alpha(\hat{g})\eta^2 \\ &\qquad + \alpha D_k v_\epsilon \varphi^{\alpha-1}(\hat{g})\varphi'(\hat{g}) D_i(\hat{g})\eta^2 + 2 D_k v_\epsilon \varphi^\alpha(\hat{g})\eta D_i \eta. \end{aligned} \tag{3.1.63}$$

Then taking into account (3.1.51), (3.1.62)–(3.1.63) and summing up all integers k such that $1 < k \le n$, we have

$$I_1 + I_2 = -I_3 - I_4, \tag{3.1.64}$$

where

$$\begin{aligned} I_1 &= \sum_{1<k\le n} \int_{Q_{2r}} \eta^2 \varphi^\alpha(\hat{g}) D_{\xi_j} \bar{A}^i_\epsilon(x_1, Dv_\epsilon) D_{kj} v_\epsilon D_{ki} v_\epsilon, \\ I_2 &= \alpha \sum_{1<k\le n} \int_{Q_{2r}} \eta^2 \varphi^{\alpha-1}(\hat{g})\varphi'(\hat{g}) D_{\xi_j} \bar{A}^i_\epsilon(x_1, Dv_\epsilon) D_{kj} v_\epsilon D_k v_\epsilon E_{1,i}\, dx, \\ I_3 &= \alpha\beta \sum_{1<k\le n} \int_{Q_{2r}} \eta^2 \varphi^{\alpha-1}(\hat{g})\varphi'(\hat{g}) D_{\xi_j} \bar{A}^i_\epsilon(x_1, Dv_\epsilon) D_{kj} v_\epsilon D_k v_\epsilon E_{2,i}\, dx, \\ I_4 &= \sum_{1<k\le n} \int_{Q_{2r}} 2\eta \varphi^\alpha(\hat{g}) D_{\xi_j} \bar{A}^i_\epsilon(x_1, Dv_\epsilon) D_{kj} v_\epsilon D_k v_\epsilon D_i \eta\, dx. \end{aligned}$$

To estimate I_1, by Lemma 3.1.9 we have

$$c_5 \int_{Q_{2r}} \eta^2 \varphi^\alpha(\hat{g}) \varphi''(|Dv_\epsilon| + \epsilon) |DD_{x'} v_\epsilon|^2\, dx \le I_1 \tag{3.1.65}$$

for some $c_5 = c_5(n, \kappa_1, \kappa_2, \nu, L) > 0$. To deal with I_2, using Lemma 3.1.9 and recalling that the summation is taken over $1 \le i, j \le n$, we yield the estimate

$$\begin{aligned} &\sum_{1<k\le n} D_{\xi_j} \bar{A}^i_\epsilon(x_1, Dv_\epsilon) D_{kj} v_\epsilon D_k v_\epsilon E_{1,i} \\ &\qquad = \sum_{1<k\le n} D_{\xi_j} \bar{A}^i_\epsilon(x_1, Dv_\epsilon) D_{kj} v_\epsilon D_k v_\epsilon \frac{D_{x'} v_\epsilon}{|D_{x'} v_\epsilon|} D_i D_{x'} v_\epsilon \\ &\qquad \ge c \varphi''(|Dv_\epsilon| + \epsilon) |DD_{x'} v_\epsilon|^2 |D_{x'} v_\epsilon| \end{aligned}$$

and so for $c_6 = c_6(n, \kappa_1, \kappa_2, \nu, L)$,

$$I_2 \geq c_6 \int_{Q_{2r}} \eta^2 \varphi^{\alpha-1}(\hat{g})\varphi'(\hat{g})\varphi''(|Dv_\epsilon| + \epsilon)|DD_{x'}v_\epsilon|^2 |D_{x'}v_\epsilon|\, dx. \tag{3.1.66}$$

We now consider I_3. Observe that by (3.1.58), it follows that

$$\beta \left| \sum_{1<k\leq n} D_{kj}v_\epsilon D_k v_\epsilon \right| |E_{2,j}| \leq c\beta |DD_{x'}v_\epsilon|^2 |D_{x'}v_\epsilon|$$

and so by Lemma 3.1.9 and $\alpha \in [0, 4q]$, together with the constant $c_7 = c_7(n, \kappa_1, \kappa_2, \nu, L)$, we obtain

$$|I_3| \leq c_7 q\beta \int_{Q_{2r}} \eta^2 \varphi^{\alpha-1}(\hat{g})\varphi'(\hat{g})\varphi''(|Dv_\epsilon| + \epsilon)|DD_{x'}v_\epsilon|^2 |D_{x'}v_\epsilon|\, dx. \tag{3.1.67}$$

To estimate I_4, by Young's inequality with $\varepsilon \in (0, 1)$, (3.1.9), (3.1.10), (3.1.37) and Lemma 3.1.9, we have

$$\begin{aligned}
I_4 &\leq c \int_{Q_{2r}} \eta \varphi^\alpha(\hat{g}) |D_{\xi_j} \bar{A}^i_\epsilon(x_1, Dv_\epsilon)||D_{kj}v_\epsilon||D_k v_\epsilon||D_i \eta|\, dx \\
&\leq \varepsilon \int_{Q_{2r}} \eta^2 \varphi^\alpha(\hat{g})\varphi''(|Dv_\epsilon| + \epsilon)|DD_{x'}v_\epsilon|^2\, dx \\
&\quad + c(\varepsilon) \int_{Q_{2r}} \varphi^\alpha(\hat{g})\varphi''(|Dv_\epsilon| + \epsilon)|Dv_\epsilon|^2 |D\eta|^2\, dx \\
&\leq \varepsilon \int_{Q_{2r}} \eta^2 \varphi^\alpha(\hat{g})\varphi''(|Dv_\epsilon| + \epsilon)|DD_{x'}v_\epsilon|^2\, dx \\
&\quad + c(\varepsilon) \int_{Q_{2r}} \varphi^{\alpha+1}(|Dv_\epsilon| + \epsilon)|D\eta|^2\, dx
\end{aligned} \tag{3.1.68}$$

for any $\varepsilon \in (0, 1]$. Merging the estimates (3.1.65), (3.1.66), (3.1.67) and

(3.1.68) into (3.1.64), it holds that

$$
\begin{aligned}
c_5 \int_{Q_{2r}} & \eta^2 \varphi^{\alpha}(\hat{g}) \varphi''(|Dv_\epsilon| + \epsilon) |DD_{x'} v_\epsilon|^2 \, dx \\
& + c_6 \int_{Q_{2r}} \eta^2 \varphi^{\alpha-1}(\hat{g}) \varphi'(\hat{g}) \varphi''(|Dv_\epsilon| + \epsilon) |DD_{x'} v_\epsilon|^2 |D_{x'} v_\epsilon| \, dx \\
& \qquad \le c_7 q \beta \int_{Q_{2r}} \eta^2 \varphi^{\alpha-1}(\hat{g}) \varphi'(\hat{g}) \varphi''(|Dv_\epsilon| + \epsilon) |DD_{x'} v_\epsilon|^2 |D_{x'} v_\epsilon| \, dx \\
& \qquad\quad + \varepsilon \int_{Q_{2r}} \eta^2 \varphi^{\alpha}(\hat{g}) \varphi''(|Dv_\epsilon| + \epsilon) |DD_{x'} v_\epsilon|^2 \, dx \\
& \qquad\quad + c(\varepsilon) \int_{Q_{2r}} \varphi^{\alpha+1}(|Dv_\epsilon| + \epsilon) |D\eta|^2 \, dx.
\end{aligned}
$$

Choosing $\varepsilon \le \frac{c_5}{2}$ and $\beta = \beta(n, \kappa_1, \kappa_2, \nu, L, q)$ sufficiently small such that $c_7 q \beta \le \frac{c_6}{2}$, we have the desired estimate (3.1.59). $\square$

Now we prove the reverse Hölder's inequality.

Lemma 3.1.17. *We have*

$$
\left(\fint_{Q_r} \varphi^q(|Dv_\epsilon| + \epsilon) \, dx \right)^{\frac{1}{q}} \le c \fint_{Q_{2r}} \varphi(|Dv_\epsilon| + \epsilon) \, dx \tag{3.1.69}
$$

with $c = c(n, \kappa_1, \kappa_2, \nu, L, q) > 0$.

Proof. Define

$$
\chi = \begin{cases} \frac{n}{n-2} & n > 2, \\ 2 & n = 2. \end{cases}
$$

Let $x_0 \in Q_r$ and $0 < \rho < \frac{1}{4} r$. For any $\eta \in C_c^\infty(Q_\rho(x_0))$ with $0 \le \eta \le 1$, $\eta \equiv 1$ on $Q_{\frac{\rho}{2}}(x_0)$, $\eta \equiv 0$ on $Q_{2r} \setminus Q_\rho(x_0)$ and $|D\eta| \le \frac{4}{\rho}$, we first claim that for any $\alpha \in [0, 4q]$, we have

$$
\left(\fint_{Q_{\frac{\rho}{2}}(x_0)} \left[\varphi^{\alpha+1}(|Dv_\epsilon| + \epsilon) \right]^{\chi} \, dx \right)^{\frac{1}{\chi}} \le c \fint_{Q_\rho(x_0)} \varphi^{\alpha+1}(|Dv_\epsilon| + \epsilon) \, dx, \tag{3.1.70}
$$

if the right-hand side is finite. Indeed, observe that by triangle inequality, (3.1.9), (3.1.10), (3.1.37) and (3.1.52),

$$\begin{aligned}
&\left|D\left(\varphi^{\frac{\alpha+1}{2}}(\hat{g})\eta\right)\right|^2\\
&\le c\left|\frac{\alpha+1}{2}\varphi^{\frac{\alpha-1}{2}}(\hat{g})\varphi'(\hat{g})(D\hat{g})\eta\right|^2+c\left|\varphi^{\frac{\alpha+1}{2}}(\hat{g})D\eta\right|^2\\
&\le c(\alpha+1)^2\varphi^{\alpha-1}(\hat{g})(\varphi'(\hat{g}))^2\eta^2|DD_{x'}v_\epsilon|^2+c\varphi^{\alpha+1}(\hat{g})|D\eta|^2\\
&\le c(q+1)^2\varphi^{\alpha}(\hat{g})\varphi''(|Dv_\epsilon|+\epsilon)\eta^2|DD_{x'}v_\epsilon|^2\\
&\quad+c\varphi^{\alpha+1}(|Dv_\epsilon|+\epsilon)|D\eta|^2.
\end{aligned}\tag{3.1.71}$$

By Sobolev-Poincaré inequality and Lemma 3.1.16 we have

$$\begin{aligned}
\left(\fint_{Q_\rho(x_0)}\left[\frac{\varphi^{\frac{\alpha+1}{2}}(\hat{g})\eta}{\rho}\right]^{2\chi}dx\right)^{\frac{1}{\chi}} &\le c\fint_{Q_\rho(x_0)}\left|D\left(\varphi^{\frac{\alpha+1}{2}}(\hat{g})\eta\right)\right|^2 dx\\
&\le c\fint_{Q_\rho(x_0)}\varphi^{\alpha+1}(|Dv_\epsilon|+\epsilon)|D\eta|^2\,dx\\
&\le c\fint_{Q_\rho(x_0)}\left[\frac{\varphi^{\frac{\alpha+1}{2}}(|Dv_\epsilon|+\epsilon)}{\rho}\right]^2 dx.
\end{aligned}$$

Then by (3.1.37) and $\alpha\in[0,4q]$ there holds

$$\begin{aligned}
\left(\fint_{Q_{\frac{\rho}{2}}(x_0)}\left[\varphi^{\alpha+1}(|Dv_\epsilon|+\epsilon)\right]^{\chi}dx\right)^{\frac{1}{\chi}} &\le c\left(\fint_{Q_\rho(x_0)}\left[\varphi^{\frac{\alpha+1}{2}}(\hat{g})\eta\right]^{2\chi}dx\right)^{\frac{1}{\chi}}\\
&\le c\fint_{Q_\rho(x_0)}\varphi^{\alpha+1}(|Dv_\epsilon|+\epsilon)\,dx
\end{aligned}$$

with $c=c(n,\kappa_1,\kappa_2,\nu,L,q)>0$, and so we have (3.1.70).

Let $\beta=\beta(n,\kappa_1,\kappa_2,\nu,L,q)\in(0,1]$ be the constant in Lemma 3.1.16. Since $\chi>1$, there exists a integer $m_0=m_0(n,q)\ge 0$ such that

$$\chi^{m_0}<q\le\chi^{m_0+1}.$$

By taking $\alpha = \chi^m - 1 \leq 4q$ for $m = 0, \cdots, m_0$, we find from (3.1.70) that

$$\left(\fint_{Q_{2^{-m_0-1}\rho}(x_0)} \left[\varphi^{\alpha+1}(|Dv_\epsilon| + \epsilon) \right]^\chi \, dx \right)^{\frac{1}{(\alpha+1)\chi}} \leq c \fint_{Q_\rho(x_0)} \varphi(|Dv_\epsilon| + \epsilon) \, dx,$$

and so it follows that

$$\left(\fint_{Q_{2^{-m_0-1}\rho}(x_0)} \varphi^q(|Dv_\epsilon| + \epsilon) \, dx \right)^{\frac{1}{q}} \leq c \fint_{Q_\rho(x_0)} \varphi(|Dv_\epsilon| + \epsilon) \, dx.$$

Since the above estimate is invariant under scaling and translation, by the covering argument, (3.1.69) follows. □

Now we give the proof of main theorem in this section.

Proof of Theorem 3.1.8. From (3.1.69), it follows that

$$\limsup_{\epsilon \to 0} \left(\fint_{Q_r} \varphi^q(|Dv_\epsilon| + \epsilon) \, dx \right)^{\frac{1}{q}} \leq c \limsup_{\epsilon \to 0} \left(\fint_{Q_{2r}} \varphi(|Dv_\epsilon| + \epsilon) \, dx \right) \tag{3.1.72}$$

with $c = c(n, \kappa_1, \kappa_2, \nu, L, q) > 0$. For the right-hand side of (3.1.72), by Lemma 3.1.11 and (3.1.25), for any $\varepsilon \in (0, 1)$ we have

$$\begin{aligned} \limsup_{\epsilon \to 0} & \fint_{Q_{2r}} |\varphi(|Dv_\epsilon| + \epsilon) - \varphi(|Dv|)| \, dx \\ & \leq c(\varepsilon) \limsup_{\epsilon \to 0} \fint_{Q_{2r}} \varphi(|Dv_\epsilon - Dv| + \epsilon) \, dx + \varepsilon \fint_{Q_{2r}} \varphi(|Dv|) \, dx \\ & \leq \varepsilon \fint_{Q_{2r}} \varphi(|Dv|) \, dx. \end{aligned}$$

Since $\varepsilon \in (0, 1)$ was arbitrary, we have

$$\limsup_{\epsilon \to 0} \fint_{Q_{2r}} \varphi(|Dv_\epsilon| + \epsilon) \, dx = \fint_{Q_{2r}} \varphi(|Dv|) \, dx.$$

For the left-hand side of (3.1.72), since $\epsilon \in (0, 1)$ and Lemma 3.1.11 holds, we observe

$$\begin{aligned}\left(\fint_{Q_r} \varphi^q(|Dv_\epsilon|)\,dx\right)^{\frac{1}{q}} &\leq \left(\fint_{Q_r} \varphi^q(|Dv_\epsilon|+\epsilon)\,dx\right)^{\frac{1}{q}} \\ &\leq c\fint_{Q_{2r}} \varphi(|Dv_\epsilon|+\epsilon)\,dx \\ &\leq c\fint_{Q_{2r}} \varphi(|Dv|+1)\,dx,\end{aligned} \tag{3.1.73}$$

and so $\{Dv_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $L^{\varphi^q}(Q_r)$ which is a reflexive Banach space by [129, Theorem 6.1.4]. Then we have a subsequence $\{\epsilon_j\}_{j=1}^{\infty}$ with $\epsilon_j \to 0$ as $j\to\infty$ and $f \in L^{\varphi^q}(Q_r)$ such that

$$\begin{aligned}&Dv_{\epsilon_j} \rightharpoonup f \text{ in } L^{\varphi^q}(Q_r) \text{ and so} \\ &\int_{Q_r} \varphi^q(|f|)\,dx \leq \liminf_{j\to\infty} \int_{Q_r} \varphi^q(|Dv_{\epsilon_j}|)\,dx.\end{aligned} \tag{3.1.74}$$

Then $Dv_{\epsilon_j} \rightharpoonup f$ in $L^{\varphi}(Q_r)$ as $q>1$. But then, according to Lemma 3.1.11 and (3.1.74), f must be Dv a.e. in Q_r by the uniqueness of weak limit. Consequently we conclude that

$$\begin{aligned}&Dv_{\epsilon_j} \rightharpoonup Dv \text{ in } L^{\varphi^q}(Q_r) \text{ and so} \\ &\int_{Q_r} \varphi^q(|Dv|)\,dx \leq \liminf_{j\to\infty} \int_{Q_r} \varphi^q(|Dv_{\epsilon_j}|)\,dx.\end{aligned}$$

Thus we have the conclusion by letting $j\to\infty$ in (3.1.73). □

3.1.3 Proof of Theorem 3.1.1

We give in this section the proof of Theorem 3.1.1. To do this, let us provide first comparison estimates to be essentially used in the proof of Theorem 3.1.1. Assume $F \in L^{\varphi^\gamma}_{\text{loc}}(\Omega,\mathbb{R}^n)$ for $\gamma>1$ and consider that $u \in W^{1,\varphi}(\Omega)$ is a weak solution of (3.1.1), which means that

$$\int_\Omega A(x,Du)\cdot D\eta\,dx = \int_\Omega \frac{\varphi'(|F|)}{|F|}F\cdot D\eta\,dx$$

for every $\eta \in W_0^{1,\varphi}(\Omega)$. Let $Q_{16r} \subset \Omega$ with $0 < 16r \leq R$ and $h \in W^{1,\varphi}(Q_{8r})$ be the weak solution of

$$\begin{cases} \operatorname{div} A(x, Dh) &= 0 \quad \text{in } Q_{8r}, \\ h &= u \quad \text{on } \partial Q_{8r}, \end{cases} \tag{3.1.75}$$

and $v \in W^{1,\varphi}(Q_{4r})$ be the weak solution of

$$\begin{cases} \operatorname{div} \bar{A}(x_1, Dv) &= 0 \quad \text{in } Q_{4r}, \\ v &= h \quad \text{on } \partial Q_{4r}, \end{cases} \tag{3.1.76}$$

where

$$\bar{A}(x_1, \xi) = \fint_{Q'_{4r}} A(x_1, x', \xi)\, dx'$$

is the integral average of $A(x_1, \cdot, \xi)$ over Q'_{4r} for fixed $x_1 \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$. Then we have the following.

Lemma 3.1.18. *For any $\varepsilon \in (0,1)$, there exists a $\delta = \delta(n, \kappa_1, \kappa_2, \nu, L, \varepsilon) > 0$ such that if*

$$\fint_{Q_{4r}} \theta(A, Q_{4r}(y))(x)\, dx \leq \delta, \tag{3.1.77}$$

then we have

$$\fint_{Q_{4r}} \varphi(|Du - Dv|)\, dx \leq c_8 \varepsilon \left(\fint_{Q_{8r}} \varphi(|Du|)\, dx + \fint_{Q_{8r}} \frac{\varphi(|F|)}{\delta}\, dx \right) \tag{3.1.78}$$

for some $c_8 = c_8(n, \kappa_1, \kappa_2, \nu, L) > 0$. On the other hand, for any $\gamma > 1$ there holds

$$\left(\fint_{Q_r} \varphi^{2\gamma}(|Dv|)\, dx \right)^{\frac{1}{2\gamma}} \leq c_9 \fint_{Q_{8r}} \varphi(|Du|)\, dx \tag{3.1.79}$$

for some $c_9 = c_9(n, \kappa_1, \kappa_2, \nu, L, \gamma) > 0$.

Proof. The proof of (3.1.78) is the same as in [11, Section 3] and [70, Section 5]. We test $v - h \in W_0^{1,\varphi}(Q_{4r})$ to (3.1.76) and $h - u \in W_0^{1,\varphi}(Q_{8r})$ to (3.1.75), respectively, and use Young's inequality to find that

$$\fint_{Q_{4r}} \varphi(|Dv|)\,dx \lesssim \fint_{Q_{4r}} \varphi(|Dh|)\,dx \lesssim \fint_{Q_{8r}} \varphi(|Du|)\,dx.$$

Then (3.1.79) follows from Theorem 3.1.8. $\square$

Now we revisit the maximal function-free technique developed in [3] to prove the estimate on super-level sets of $\varphi(|Du|)$ in Lemma 3.1.19 below. We select real numbers r_1 and r_2 such that $R \le r_1 < r_2 \le 2R$. Let $Q_{2R} \Subset \Omega$ and λ_0 be such that

$$\lambda_0 := \left(\frac{80R}{r_2 - r_1}\right)^n \fint_{Q_{2R}} \left[\varphi(|Du|) + \frac{\varphi(|F|)}{\delta}\right] dx \tag{3.1.80}$$

with $\delta \in (0,1)$ being a free parameter to be chosen later. Then one can see that

$$\fint_{Q_\rho(y)} \left[\varphi(|Du|) + \frac{\varphi(|F|)}{\delta}\right] dx \le \lambda_0 \quad \left(y \in Q_{r_1}, \frac{r_2 - r_1}{40} \le \rho \le R\right). \tag{3.1.81}$$

Lemma 3.1.19. *For any $\varepsilon \in (0,1)$, there exists a $\delta = \delta(n, \kappa_1, \kappa_2, \nu, L, \varepsilon) > 0$ such that if*

$$\sup_{0<\rho\le R} \sup_{y \in Q_{2R}} \fint_{Q_\rho(y)} \theta(A, Q_\rho(y))(x)\,dx \le \delta, \tag{3.1.82}$$

then for any $\lambda \ge \lambda_0$ and large $N > 1$, we have

$$\begin{aligned}&\int_{Q_{r_1} \cap \{\varphi(|Du|) > N\gamma\}} \varphi(|Du|)\,dx \\ &\le c(\varepsilon + N^{1-2\gamma}) \left(\int_{Q_{r_2} \cap \{\varphi(|Du|) > \frac{\lambda}{3}\}} \varphi(|Du|)\,dx + \int_{Q_{r_2} \cap \{\varphi(|F|) > \frac{\delta\lambda}{3}\}} \frac{\varphi(|F|)}{\delta}\,dx\right).\end{aligned}$$

Proof. Fix $\lambda \ge \lambda_0$, and let us define the upper-level set

$$E(\lambda) = \left\{x \in Q_{r_1} : \varphi(|Du|) + \frac{\varphi(|F|)}{\delta} > \lambda\right\}.$$

If $E(\lambda) = \emptyset$, then one can see that $\{x \in Q_{r_1} : \varphi(|Du|) > N\lambda\} = \emptyset$ and so the conclusion holds. Thus we assume $E(\lambda) \neq \emptyset$.

With the help of Lebesgue differentiation theorem, for a.e. $y \in E(\lambda)$ we have

$$\lim_{\rho \to 0^+} \fint_{Q_\rho(y)} \left[\varphi(|Du|) + \frac{\varphi(|F|)}{\delta} \right] dx > \lambda.$$

Using (3.1.81) and the fact that $\lambda \geq \lambda_0$, we see that there exists a positive radius $\rho_y \in \left(0, \frac{r_2 - r_1}{40}\right)$ such that

$$\fint_{Q_{\rho_y}(y)} \left[\varphi(|Du|) + \frac{\varphi(|F|)}{\delta} \right] dx = \lambda \tag{3.1.83a}$$

and

$$\fint_{Q_\rho(y)} \left[\varphi(|Du|) + \frac{\varphi(|F|)}{\delta} \right] dx \leq \lambda \quad (\rho_y < \forall \rho \leq R). \tag{3.1.83b}$$

From (3.1.83a), we observe

$$\begin{aligned} |Q_{\rho_y}(y)| &= \frac{1}{\lambda} \int_{Q_{\rho_y}(y)} \varphi(|Du|) + \frac{\varphi(|F|)}{\delta} \, dx \\ &\leq \frac{2|Q_{\rho_y}(y)|}{3} + \frac{1}{\lambda} \int_{Q_{\rho_y}(y) \cap \{\varphi(|Du|) > \frac{\lambda}{3}\}} \varphi(|Du|) \, dx \\ &\quad + \frac{1}{\lambda} \int_{Q_{\rho_y}(y) \cap \{\varphi(|F|) > \frac{\delta\lambda}{3}\}} \frac{\varphi(|F|)}{\delta} \, dx, \end{aligned}$$

and so

$$\begin{aligned} |Q_{\rho_y}(y)| \leq \frac{c}{\lambda} &\int_{Q_{\rho_y}(y) \cap \{\varphi(|Du|) > \frac{\lambda}{3}\}} \varphi(|Du|) \, dx \\ &+ \frac{c}{\lambda} \int_{Q_{\rho_y}(y) \cap \{\varphi(|F|) > \frac{\delta\lambda}{3}\}} \frac{\varphi(|F|)}{\delta} \, dx. \end{aligned} \tag{3.1.84}$$

Next, we observe from (3.1.83b) that

$$\fint_{Q_{40\rho_y}(y)} \varphi(|Du|) \, dx \leq \lambda \quad \text{and} \quad \fint_{Q_{40\rho_y}(y)} \varphi(|F|) \, dx \leq \delta\lambda.$$

Fix any $\varepsilon \in (0,1)$. Then by Lemma 3.1.18, we have

$$\begin{aligned} &\fint_{Q_{20\rho_y}(y)} \varphi(|Du - Dv|)\,dx \le c\varepsilon\lambda \\ &\text{and} \quad \left(\fint_{Q_{5\rho_y}(y)} \varphi^{2\gamma}(|Dv|)\,dx\right)^{\frac{1}{2\gamma}} \le c\lambda \end{aligned} \tag{3.1.85}$$

provided (3.1.82) holds.

Fix any large $N > 1$ to be determined in (3.1.91). Then we have

$$\begin{aligned} &\varphi(|Du(x)|) \ge N\lambda \\ &\quad \Rightarrow \quad \varphi(|Du(x)|) \le c\varphi(|Du(x) - Dv(x)|) + c(\gamma)(N\lambda)^{1-2\gamma}\varphi(|Dv(x)|)^{2\gamma}. \end{aligned}$$

With the help of (3.1.85), it follows that

$$\begin{aligned} &\int_{Q_{5\rho_y}(y)\cap\{\varphi(|Du|)>N\lambda\}} \varphi(|Du|)\,dx \\ &\quad \le c\int_{Q_{5\rho_y}(y)\cap\{\varphi(|Du|)>N\lambda\}} \left[\varphi(|Du - Dv|) + (N\lambda)^{1-2\gamma}\varphi(|Dv|)^{2\gamma}\right]\,dx \\ &\quad \le c\left(\varepsilon\lambda + (N\lambda)^{1-2\gamma}\lambda^{2\gamma}\right)|Q_{\rho_y}(y)| \\ &\quad = c(\varepsilon + N^{1-2\gamma})\lambda|Q_{\rho_y}(y)|. \end{aligned} \tag{3.1.86}$$

Now we use Vitali covering lemma to obtain a covering $\{Q_{5\rho_m}(y_m)\}_{m=1}^{\infty}$ of $\{x \in Q_{r_1} : \varphi(|Du(x)|) > N\lambda\} \subset E(\lambda)$ with

$$\begin{aligned} &y_m \in E(\lambda), \quad \rho_m \in \left(0, \frac{r_2 - r_1}{40}\right) \\ &\text{and} \quad \{Q_{\rho_m}(y_m)\}_{m=1}^{\infty} \quad \text{are mutually disjoint.} \end{aligned} \tag{3.1.87}$$

Since $40\rho_m \le r_2 - r_1$, we notice that $\cup_{m=1}^{\infty} Q_{\rho_m}(y_m) \subset Q_{r_2}$. Hence it follows from (3.1.84), (3.1.86) and (3.1.87) that

$$
\begin{aligned}
&\int_{Q_{r_1}\cap\{\varphi(|Du|)>N\lambda\}} \varphi(|Du|)\,dx \\
&\le \sum_{m=1}^{\infty} \int_{Q_{5\rho_m}(y_m)\cap\{\varphi(|Du|)>N\lambda\}} \varphi(|Du|)\,dx \\
&\le c(\varepsilon+N^{1-2\gamma})\lambda \sum_{m=1}^{\infty} |Q_{\rho_m}(y_m)| \\
&\le c(\varepsilon+N^{1-2\gamma}) \left(\int_{Q_{r_2}\cap\{\varphi(|Du|)>\frac{\lambda}{3}\}} \varphi(|Du|)\,dx + \int_{Q_{r_2}\cap\{\varphi(|F|)>\frac{\delta\lambda}{3}\}} \frac{\varphi(|F|)}{\delta}\,dx \right),
\end{aligned}
$$

which is the conclusion of the lemma. $\square$

We are now ready to prove Theorem 2.1 using Fubini's theorem.

Proof of Theorem 2.1. We define the truncated functions by

$$
[\varphi(|Du|)]_t := \min\{\varphi(|Du|), t\} \quad (t \ge 0).
$$

According to Lemma 3.1.19, we have that for $t \ge 2\lambda_0$

$$
\begin{aligned}
&\int_{2\lambda_0}^{t} \lambda^{\gamma-2} \int_{Q_{r_1}\cap\{\varphi(|Du|)>N\lambda\}} \varphi(|Du|)\,dxd\lambda \\
&\le c(\varepsilon+N^{1-2\gamma}) \left[\int_{2\lambda_0}^{t} \lambda^{\gamma-2} \int_{Q_{r_2}\cap\{\varphi(|Du|)>\frac{\lambda}{3}\}} \varphi(|Du|)\,dxd\lambda \right. \\
&\qquad \left. + \int_{2\lambda_0}^{t} \lambda^{\gamma-2} \int_{Q_{r_2}\cap\{\varphi(|F|)>\frac{\delta\lambda}{3}\}} \frac{\varphi(|F|)}{\delta}\,dxd\lambda \right],
\end{aligned}
\tag{3.1.88}
$$

provided A is (δ, R)–vanishing of codimension 1. For the left-hand side of the above display, we use change of variables and Fubini's theorem to observe

that

$$\begin{aligned}
&\int_{2\lambda_0}^{t} \lambda^{\gamma-2} \int_{Q_{r_1}\cap\{\varphi(|Du|)>N\lambda\}} \varphi(|Du|)\, dx d\lambda \\
&= \frac{1}{N^{\gamma-1}} \int_{2N\lambda_0}^{Nt} \lambda^{\gamma-2} \int_{Q_{r_1}\cap\{\varphi(|Du|)>\lambda\}} \varphi(|Du|)\, dx d\lambda \\
&= \frac{1}{N^{\gamma-1}} \int_{Q_{r_1}} \varphi(|Du|) \int_{2N\lambda_0}^{\min\{\varphi(|Du|),Nt\}} \lambda^{\gamma-2}\, d\lambda dx \\
&= \frac{1}{N^{\gamma-1}(\gamma-1)} \int_{Q_{r_1}} \varphi(|Du|) \Big[\min\{\varphi(|Du|), Nt\}^{\gamma-1} \\
&\qquad\qquad -(2N\lambda_0)^{\gamma-1}\Big]\ dx \\
&\geq \frac{1}{N^{\gamma-1}(\gamma-1)} \int_{Q_{r_1}} \varphi(|Du|) \Big[\min\{\varphi(|Du|), t\}^{\gamma-1} - (2N\lambda_0)^{\gamma-1}\Big]\, dx.
\end{aligned} \tag{3.1.89}$$

For the right-hand side of (3.1.88), by the similar computation as above, we have

$$\begin{aligned}
&\int_{2\lambda_0}^{t} \lambda^{\gamma-2} \int_{Q_{r_2}\cap\{\varphi(|Du|)>\frac{\lambda}{3}\}} \varphi(|Du|)\, dx d\lambda \\
&\qquad = \frac{3^{\gamma-1}}{\gamma-1} \int_{Q_{r_2}} \varphi(|Du|) \Big[\min\{\varphi(|Du|), 3t\}^{\gamma-1} - (6\lambda_0)^{\gamma-1}\Big]\ dx \\
&\qquad \leq \frac{3^{\gamma}}{\gamma-1} \int_{Q_{r_2}} \varphi(|Du|) \min\{\varphi(|Du|), t\}^{\gamma-1}\, dx
\end{aligned}$$

and

$$\begin{aligned}
&\int_{2\lambda_0}^{t} \lambda^{\gamma-2} \int_{Q_{r_2}\cap\{\varphi(|F|)>\frac{\delta\lambda}{3}\}} \frac{\varphi(|F|)}{\delta}\, dx d\lambda \\
&\qquad \leq \frac{1}{\delta^{\lambda-1}} \int_{0}^{\infty} \lambda^{\gamma-2} \int_{Q_{r_2}\cap\{\varphi(|F|)>\lambda\}} \frac{\varphi(|F|)}{\delta}\, dx d\lambda \\
&\qquad \leq c \int_{Q_{r_2}} \left(\frac{\varphi(|F|)}{\delta}\right)^{\gamma} dx.
\end{aligned} \tag{3.1.90}$$

Combining all the estimates (3.1.89)–(3.1.90) with (3.1.88), we find that

$$\begin{aligned}
&\fint_{Q_{r_1}} \varphi(|Du|)\,[\min\{\varphi(|Du|),t\}]^{\gamma-1}\,dx \\
&\qquad \le \bar{c}3^{\gamma}(\varepsilon N^{\gamma-1}+N^{-\gamma}) \\
&\qquad\quad \times \left(\fint_{Q_{r_2}} \varphi(|Du|)\,[\min\{\varphi(|Du|),t\}]^{\gamma-1}\,dx + \fint_{Q_{r_2}} \left(\frac{\varphi(|F|)}{\delta}\right)^{\gamma} dx\right) \\
&\qquad\quad + c(2\lambda_0)^{\gamma-1}\fint_{Q_{r_2}} \varphi(|Du|)\,dx
\end{aligned}$$

for some positive constants $\bar{c}=\bar{c}(n,\kappa_1,\kappa_2,\nu,L,\gamma)$ and $c=c(n,\kappa_1,\kappa_2,\nu,L,\gamma)$. We make

$$\bar{c}3^{\gamma}(\varepsilon N^{\gamma-1}+N^{-\gamma}) \le \frac{1}{2}, \tag{3.1.91}$$

by first selecting the constant $N=N(n,\kappa_1,\kappa_2,\nu,L,\gamma)>1$ sufficiently large, and then choosing $\varepsilon=\varepsilon(n,\kappa_1,\kappa_2,\nu,L,\gamma)\in(0,1)$ sufficiently small. Accordingly, we can find a small $\delta=\delta(n,\kappa_1,\kappa_2,\nu,L,\gamma)>0$ from Lemma 3.1.19. Consequently, we have

$$\begin{aligned}
&\fint_{Q_{r_1}} \varphi(|Du|)\,[\min\{\varphi(|Du|),t\}]^{\gamma-1}\,dx \\
&\qquad \le \frac{1}{2}\fint_{Q_{r_2}} \varphi(|Du|)\,[\min\{\varphi(|Du|),t\}]^{\gamma-1}\,dx \\
&\qquad\quad + c\fint_{Q_{r_2}} \varphi(|F|)^{\gamma}\,dx + c\lambda_0^{\gamma-1}\fint_{Q_{r_2}} \varphi(|Du|)\,dx \\
&\qquad \le \frac{1}{2}\fint_{Q_{r_2}} \varphi(|Du|)\,[\min\{\varphi(|Du|),t\}]^{\gamma-1}\,dx + c\fint_{Q_{r_2}} \varphi(|F|)^{\gamma}\,dx \\
&\qquad\quad + c\left(\frac{80R}{r_2-r_1}\right)^{n(\gamma-1)}\left(\fint_{Q_{2R}}\left[\varphi(|Du|)+\frac{1}{\delta}\varphi(|F|)\right]dx\right)^{\gamma}.
\end{aligned}$$

Now from Lemma 2.0.1, we discover

$$\begin{aligned}
&\fint_{Q_R} \varphi(|Du|)\,[\min\{\varphi(|Du|), t\}]^{\gamma-1}\,dx \\
&\qquad \le c \fint_{Q_{2R}} \varphi(|F|)^{\gamma}\,dx + c\left(\fint_{Q_{2R}} \left[\varphi(|Du|) + \frac{1}{\delta}\varphi(|F|)\right]\,dx\right)^{\gamma} \\
&\qquad \le c\left(\fint_{Q_{2R}} \varphi(|Du|)\,dx\right)^{\gamma} + c\fint_{Q_{2R}} \varphi(|F|)^{\gamma}\,dx.
\end{aligned}$$

Finally, letting $t \to \infty$, the conclusion of Theorem 3.1.1 follows. □

3.2 Global estimates for a general class of quasilinear elliptic equations with Orlicz growth

This section is devoted to providing an optimal global Calderón-Zygmund theory for quasilinear elliptic equations of a very general form with Orlicz growth on bounded nonsmooth domains under minimal regularity assumptions of the nonlinearity $A = A(x, u, Du)$ in the first and second variables (x, z) as well as on the boundary of the domain. Our result improves known regularity results in the literature regarding nonlinear elliptic operators depending on a given bounded weak solution.

3.2.1 Hypothesis and main results

We shall deal with the global gradient estimates of a weak solution to the following Dirichlet problem:

$$\begin{cases} -\mathrm{div} A(x,u,Du) = -\mathrm{div}\left(\dfrac{G'(|F|)}{|F|}F\right) & \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.2.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \ge 2)$ is a bounded domain with possibly nonsmooth boundary $\partial\Omega$ and G is an N-function in the sense of the definition introduced in Section 3.2.2, whereas $F : \Omega \to \mathbb{R}^n$ is a given vector field such that $F \in L^G(\Omega; \mathbb{R}^n)$. Throughout the section, we shall assume that the vector field $A : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory map satisfying the following structure assumptions with fixed constants $0 < \nu \le L < \infty$:

$$\begin{cases} \nu \dfrac{G(|\xi|)}{|\xi|^2}|\eta|^2 \le \langle D_\xi A(x,z,\xi)\eta, \eta\rangle \\ |A(x,z,\xi)| + |\xi||D_\xi A(x,z,\xi)| \le L\dfrac{G(|\xi|)}{|\xi|} \end{cases} \tag{3.2.2}$$

for every $x \in \Omega$, $(z,\xi) \in \mathbb{R} \times (\mathbb{R}^n \setminus \{0\})$ and $\eta \in \mathbb{R}^n$.

In order to achieve the desired result, we need to ask minimal smoothness condition on $A(x,z,\xi)$ with respect to x and z-variables, and a proper geometric structure on $\partial\Omega$. Based on works [33, 55], we suppose that

1. *Continuity with respect to second variable of A:* For every $M > 0$, there exists a non-decreasing function $\omega_M : [0,\infty) \to [0,\infty)$ such that

$$\lim_{\rho \to 0^+} \omega_M(\rho) = 0 \tag{3.2.3}$$

   and

$$|A(x,z_1,\xi) - A(x,z_2,\xi)| \le \omega_M(|z_1 - z_2|)G'(|\xi|) \tag{3.2.4}$$

   holds for a.e. $x \in \mathbb{R}^n$, $z_1, z_2 \in [-M,M]$ and $\xi \in \mathbb{R}^n$.

2. *(δ,R)-vanishing of A:* For every $M > 0$, there exist $R > 0$ and $\delta > 0$ depending on M such that

$$\sup_{z\in[-M,M]} \sup_{0<\rho\le R} \sup_{y\in\mathbb{R}^n} \fint_{B_\rho(y)} \theta(B_\rho(y))(x,z)\,dx \le \delta, \tag{3.2.5}$$

   where $\theta(B_\rho(y))(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ are defined by

$$\theta(B_\rho(y))(x,z) = \sup_{\xi\in\mathbb{R}^n\setminus\{0\}} \frac{|A(x,z,\xi) - (A(\cdot,z,\xi))_{B_\rho(y)}|}{G'(|\xi|)}$$

   with the notation

$$(A(\cdot,z,\xi))_{B_\rho(y)} = \fint_{B_\rho(y)} A(x,z,\xi)\,dx.$$

3. *(δ, R)-Reifenberg flatness of Ω:* For every $r \in (0, R]$ and $x_0 \in \partial\Omega$, there exists a new coordinate system $\{y_1, \cdots, y_n\}$ with the origin at x_0 such that

$$B_r(0) \cap \{y : y_n > \delta r\} \subset B_r(0) \cap \Omega \subset B_r(0) \cap \{y : y_n > -\delta r\} \quad (3.2.6)$$

holds in this coordinate system.

The continuity assumption (3.2.3)-(3.2.4) on the second variable of the nonlinearity A is a minimal one in the setting of Orlicz growth. Roughly speaking, the (δ, R)-vanishing property (3.2.5) exhibits a kind of smallness in terms of BMO, which allows x-discontinuity of the nonlinearity. The geometric structure (3.2.6) means that the boundary of Ω can be locally dominated by hyperplanes with proper chosen scale. In fact, a set having rough fractal boundaries such as the Koch snowflake with smallness of the angle of the spike with respect to the horizontal is included in the class of the Reifenberg flat domains and in particular, domains with C^1-smooth boundary or boundary that can be locally expressed as a graph of a Lipschitz function with small Lipschitz constant are also members of the Reifenberg flat class.

Remark 3.2.1. *If Ω is (δ, R)-Reifenberg flat, then it holds that*

$$\sup_{y\in\Omega} \sup_{0<r\le R/2} \frac{|B_r(y)|}{|\Omega \cap B_r(y)|} \le \left(\frac{2}{1-4\delta}\right)^n < 4^n. \quad (3.2.7)$$

Now we are ready to state the main theorem.

Theorem 3.2.2. *Let $u \in W^{1,G}(\Omega) \cap L^\infty(\Omega)$ be a weak solution of (3.2.1) with $\|u\|_{L^\infty(\Omega)} \le M$ under the assumptions (3.2.2)-(3.2.4). Suppose that $G(|F|) \in L^\gamma(\Omega)$ for some $\gamma > 1$. Then there exists $\delta = \delta(n, s_G, \nu, L, M, \omega_M(\cdot), \gamma) > 0$ such that if the conditions (3.2.5) and (3.2.6) hold for some R, then there holds that $G(|Du|) \in L^\gamma(\Omega)$ with the estimate*

$$\int_\Omega G^\gamma(|Du|)\, dx \le c \int_\Omega G^\gamma(|F|)\, dx \quad (3.2.8)$$

for some constant $c \equiv c(n, s_G, \nu, L, M, \omega_M(\cdot), |\Omega|, \gamma)$.

3.2.2 Proof of Theorem 3.2.2

Before the proof of main theorem, we state the definition of an N-function G introduced in the previous section.

Definition 3.2.3. $\Phi : [0,\infty) \to [0,\infty)$ *is said to be an N-function with index s_Φ if $\Phi \in C^1([0,\infty)) \cap C^2((0,\infty))$ is an increasing convex function such that $\lim\limits_{t\to 0^+} \frac{\Phi(t)}{t} = 0$, $\lim\limits_{t\to\infty} \frac{\Phi(t)}{t} = \infty$ and there exists a positive constant $s_\Phi \geq 1$ satisfying*

$$\frac{1}{s_\Phi} \leq \frac{t\Phi''(t)}{\Phi'(t)} \leq s_\Phi \tag{3.2.9}$$

for uniformly all $t > 0$.

Clearly, Φ satisfies Δ_2 and ∇_2 conditions (see for details [5, 94, 129, 195]). As a consequence of (3.2.9), we easily observe that

$$\frac{1}{s_\Phi} + 1 \leq \frac{t\Phi'(t)}{\Phi(t)} \leq s_\Phi + 1 \quad \text{for every} \quad t > 0 \tag{3.2.10}$$

and

$$\min\left\{\lambda^{s_\Phi+1}, \lambda^{\frac{1}{s_\Phi}+1}\right\}\Phi(t) \leq \Phi(t\lambda) \leq \max\left\{\lambda^{s_\Phi+1}, \lambda^{\frac{1}{s_\Phi}+1}\right\}\Phi(t) \tag{3.2.11}$$

for all $s, t > 0$. Let Φ be an N-function with the index s_Φ. We also need the following Young's inequality [12], which will be used frequently later. There exists a positive constant $c \equiv c(s_\Phi)$ such that

$$s\frac{\Phi(t)}{t} + t\frac{\Phi(s)}{s} \approx s\Phi'(t) + t\Phi'(s) \leq \varepsilon\Phi(s) + \frac{c}{\varepsilon^{s_\Phi}}\Phi(t) \tag{3.2.12}$$

holds for all $s, t > 0$ and $0 < \varepsilon \leq 1$.

Given an N-function Φ with index s_Φ, we also define a vector field $V_\Phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n$ by

$$V_\Phi(\xi) := \left(\frac{\Phi'(|\xi|)}{|\xi|}\right)^{\frac{1}{2}}\xi. \tag{3.2.13}$$

Then the following known fact will be used frequently (see for instance [92]):

$$\begin{aligned}|V_\Phi(\xi_1)-V_\Phi(\xi_2)|^2 &\approx \Phi''(|\xi_1|+|\xi_2|)|\xi_1-\xi_2|^2\\ &\approx \frac{\Phi'(|\xi_1|+|\xi_2|)}{|\xi_1|+|\xi_2|}|\xi_1-\xi_2|^2\end{aligned} \tag{3.2.14}$$

for every $\xi_1, \xi_2 \in \mathbb{R}^n$ with $|\xi_1|+|\xi_2|>0$, where all the implied constant depend only on s_Φ. Moreover, we shall also use the following useful inequality several times afterwards.

Lemma 3.2.4. *Let Φ be an N-function with the index s_Φ. Then there exists a constant $c \equiv c(s_\Phi)$ such that*

$$\Phi(|\xi_1-\xi_2|) \le \varepsilon\Phi(|\xi_1|)+\frac{c}{\varepsilon}|V_\Phi(\xi_1)-V_\Phi(\xi_2)|^2 \tag{3.2.15}$$

holds, whenever $\varepsilon \in (0,1)$ and $\xi_1, \xi_2 \in \mathbb{R}^n$ with $|\xi_1|+|\xi_2|>0$.

Proof. Firstly, using (3.2.11), we observe that

$$\Phi(|\xi_1-\xi_2|) \le \Phi(2|\xi_1|)+\Phi(2|\xi_2|) \le 2^{s_\Phi+1}\left(\Phi(|\xi_1|)+\Phi(|\xi_2|)\right) \tag{3.2.16}$$

holds for every $\xi_1, \xi_2 \in \mathbb{R}^n$. Then, it can be easily seen that

$$\begin{aligned}\Phi(|\xi_1-\xi_2|) &\le c\frac{\Phi'(|\xi_1-\xi_2|)}{[\Phi''(|\xi_1|+|\xi_2|)]^{1/2}}[\Phi''(|\xi_1|+|\xi_2|)]^{1/2}|\xi_1-\xi_2|\\ &\le \tau\frac{[\Phi'(|\xi_1-\xi_2|)]^2}{\Phi''(|\xi_1|+|\xi_2|)}+\frac{c}{\tau}\Phi''(|\xi_1|+|\xi_2|)|\xi_1-\xi_2|^2\\ &\le c\tau\Phi(|\xi_1|+|\xi_2|)+\frac{c}{\tau}|V_\Phi(\xi_1)-V_\Phi(\xi_2)|^2\end{aligned}$$

hold for some constant $c \equiv c(s_\Phi)$ and every $\tau>0$, where we have applied the property that the function Φ' is increasing and Young's inequality together with the properties (3.2.9), (3.2.10) and (3.2.14). Now using (3.2.16) and (3.2.11) in the resulting term of the last display and recalling $\tau>0$ is a free parameter, we find

$$\Phi(|\xi_1-\xi_2|) \le \tau\Phi(|\xi_1-\xi_2|)+\tau\Phi(|\xi_1|)+\frac{c}{\tau}|V_\Phi(\xi_1)-V_\Phi(\xi_2)|^2$$

with some constant $c \equiv c(s_\Phi)$. In particular, we have

$$\Phi(|\xi_1 - \xi_2|) \leq \frac{\tau}{1-\tau}\Phi(|\xi_1|) + \frac{c}{\tau(1-\tau)}|V_\Phi(\xi_1) - V_\Phi(\xi_2)|^2.$$

Finally, replacing $\tau := \frac{\varepsilon}{1+\varepsilon}$ for any $\varepsilon \in (0,1)$ in the last display, we arrive at the desired inequality (3.2.15). $\square$

Using this vector field defined in (3.2.13), it's convenient to formulate the monotonicity properties of A appeared in (3.2.1) as follows:

$$\langle A(x,z,\xi_1) - A(x,z,\xi_2), \xi_1 - \xi_2\rangle \approx |V_G(\xi_1) - V_G(\xi_2)|^2 \tag{3.2.17}$$

and

$$\langle A(x,z,\xi), \xi\rangle \approx |V_G(\xi)|^2 \approx G(|\xi|) \tag{3.2.18}$$

for all $x \in \Omega$, $z \in \mathbb{R}$ and $\xi, \xi_1, \xi_2 \in \mathbb{R}^n \setminus \{0\}$.

Remark 3.2.5. *Here we introduce the scaling invariant properties of the equation* (3.2.1). *Let $u \in W_0^{1,G}(\Omega)$ be a weak solution of* (3.2.1) *under the assumptions* (3.2.2) *and* (3.2.3)-(3.2.6). *For fixed $x_0 \in \Omega$, $r > 0$, and $\lambda > 0$, we define*

$$\tilde{A}(x,z,\xi) := \frac{A(x_0 + rx, \lambda r z, \lambda\xi)}{G'(\lambda)}, \quad \tilde{G}(t) := \frac{G(\lambda t)}{G(\lambda)},$$
$$\tilde{u}(x) := \frac{u(x_0 + rx)}{r\lambda}, \quad \textit{and} \quad \tilde{F}(x) := \frac{F(x_0 + rx)}{\lambda}$$

for every $x \in \tilde{\Omega} := \left\{\frac{y - x_0}{r} : y \in \Omega\right\}$, $\xi \in \mathbb{R}^n$, $z \in \mathbb{R}$ and $t \geq 0$. Then the followings hold:

1. *$\tilde{G}$ is an increasing convex function satisfying the condition* (3.2.9) *that means $\tilde{G}$ is an N-function with the index s_G.*

2. *The newly defined nonlinearity $\tilde{A}$ satisfies the following structure as-*

sumptions:

$$\begin{cases} \left\langle D_\xi \tilde{A}(x,z,\xi)\eta,\eta \right\rangle \geq \tilde{\nu}\dfrac{(\tilde{G})'(|\xi|)}{|\xi|}|\eta|^2, \\ |\tilde{A}(x,z,\xi)| + |\xi||D_\xi \tilde{A}(x,z,\xi)| \leq \tilde{L}\dfrac{\tilde{G}(|\xi|)}{|\xi|} \end{cases}$$

with some constants $\tilde{\nu} = \tilde{\nu}(\nu, s_G)$ *and* $\tilde{L} = \tilde{L}(L, s_G)$*, whenever* $x, \eta \in \mathbb{R}^n$*,* $\xi \in \mathbb{R}^n \setminus \{0\}$ *and* $z \in \mathbb{R}$*.*

3. *Clearly,* $\tilde{u} \in W_0^{1,\tilde{G}}(\tilde{\Omega})$*,* $\tilde{F} \in L^{\tilde{G}}(\tilde{\Omega};\mathbb{R}^n)$*, and they satisfy*

$$\int_{\tilde{\Omega}} \left\langle \tilde{A}(x,\tilde{u},D\tilde{u}), D\varphi \right\rangle dx = \int_{\tilde{\Omega}} \left\langle \frac{\tilde{G}(|\tilde{F}|)}{|\tilde{F}|}\tilde{F}, D\varphi \right\rangle dx$$

for all $\varphi \in W_0^{1,\tilde{G}}(\tilde{\Omega})$*.*

4. *By the very definition* $\|\tilde{u}\|_{L^\infty(\tilde{\Omega})} \leq M/(\lambda r)$*, it holds that*

$$|\tilde{A}(x,z_1,\xi) - \tilde{A}(x,z_2,\xi)| \leq \omega_M(\lambda r|z_1 - z_2|)\tilde{G}'(|\xi|). \tag{3.2.19}$$

5. *If* A *is* (δ, R)*-vanishing, then* $\tilde{A}$ *is* $(\delta, \frac{R}{r})$*-vanishing, and if* Ω *is* (δ, R)*-Reifenberg flat, then* $\tilde{\Omega}$ *is* $(\delta, \frac{R}{r})$*-Reifenberg flat.*

Based on Remark 3.2.5, we shall proceed a series of comparison estimates in the scaled version with the parameters (ρ, K) from the original given two parameters (r, λ), where the free parameter K will be determined afterwards. In what follows, for $\rho = 1, 2, 3, 4$ or K, we denote

$$B_\rho = B_\rho(0), \quad \tilde{\Omega}_\rho := \tilde{\Omega} \cap B_\rho, \quad B_\rho^+(0) := \{x \in B_\rho : x_n > 0\},$$

where $\tilde{\Omega}$ has been defined by Remark 3.2.5. Before we start the comparison estimates, let us provide a Poincaré type inequality for functions of $W_0^{1,\tilde{G}}(\tilde{\Omega}_\rho)$.

Proposition 3.2.6. *There exists a constant* $c \equiv c(n, s_G)$ *such that*

$$\fint_{\tilde{\Omega}_\rho} \tilde{G}(|f|)\,dx \leq c\left(\rho^{\frac{1}{s_G}+1} + \rho^{s_G+1}\right) \fint_{\tilde{\Omega}_\rho} \tilde{G}(|Df|)\,dx$$

holds, whenever $f \in W_0^{1,\tilde{G}}(\tilde{\Omega}_\rho)$.

Proof. By redefining $f \equiv 0$ on $B_\rho \setminus \tilde{\Omega}$, one can assume that $f \in W^{1,\tilde{G}}(B_\rho)$. Then we are able to apply Poincaré inequality [92, Theorem 7]. In turn, it yields that

$$\fint_{B_\rho} \tilde{G}\left(\frac{|f|}{\rho}\right) dx \leq c \fint_{B_\rho} \tilde{G}\left(|Df|\right) dx$$

with some constant $c \equiv c(n, s_G)$. Therefore, using (3.2.11) and recalling that $f \equiv 0$ on $B_\rho \setminus \tilde{\Omega}$, we obtain the desired estimate. □

We only consider the boundary case since the interior case can be handled in a similar way. By the scaling invariance property and the definition of the Reifenberg flat domain introduced in Remark 3.2.5, it suffices to proceed the comparison estimates for $\tilde{u}$ instead of u. Let $K \geq 4$ be a free parameter which will be chosen in Lemma 3.2.7, and $\tilde{u}$ be a localized solution in $\tilde{\Omega}_K$ of the equation (3.2.1) as follows:

$$\begin{cases} -\mathrm{div}\tilde{A}(x, \tilde{u}, D\tilde{u}) = -\mathrm{div}\left(\dfrac{\tilde{G}'(|\tilde{F}|)}{|\tilde{F}|}\tilde{F}\right) \quad \text{in } \tilde{\Omega}_K, \\ \tilde{u} = 0 \quad \text{on } \partial\tilde{\Omega} \cap B_K. \end{cases} \tag{3.2.20}$$

Throughout this section we suppose that

$$\|\tilde{u}\|_{L^\infty(\tilde{\Omega})} \leq M/(Kr), \tag{3.2.21}$$

$$\fint_{\tilde{\Omega}_K} \tilde{G}(|D\tilde{u}|)\, dx \leq 1, \tag{3.2.22}$$

and

$$\fint_{\tilde{\Omega}_K} \tilde{G}(|\tilde{F}|)\, dx \leq \delta. \tag{3.2.23}$$

We further assume that

$$\begin{aligned} &\fint_{\tilde{\Omega}_K} \theta(B_K)(x, (\tilde{u})_{\tilde{\Omega}_5})\, dx \leq \delta \\ &\text{and} \quad B_K \cap \{x : x_n > 0\} \subset \tilde{\Omega}_K \subset B_K \cap \{x : x_n > -2K\delta\}. \end{aligned} \tag{3.2.24}$$

First comparison estimate: Under the assumptions (3.2.21)-(3.2.23), let $w \in W^{1,\tilde{G}}(\tilde{\Omega}_K)$ be the weak solution of the following homogeneous problem:

$$\begin{cases} -\mathrm{div}\tilde{A}(x,\tilde{u},Dw) = 0 \quad \text{in } \tilde{\Omega}_K \\ w = \tilde{u} \quad \text{on } \partial\tilde{\Omega}_K. \end{cases} \tag{3.2.25}$$

Then we collect the known facts regarding w.

1. *(Energy estimate)* There exists a constant $c \equiv c(n, s_G, \nu, L)$ such that

$$\fint_{\tilde{\Omega}_K} \tilde{G}(|Dw|)\,dx \le \fint_{\tilde{\Omega}_K} \tilde{G}(|D\tilde{u}|)\,dx \le c. \tag{3.2.26}$$

2. *(Comparison estimate)* By taking $w - \tilde{u}$ as a test function to the equations (3.2.20) and (3.2.25), respectively, following the proof of [70, Lemma 5.3] and applying Lemma 3.2.4, for any $\tau, \tau_1 \in (0,1)$, we discover that

$$\begin{aligned} &\fint_{\tilde{\Omega}_K} \tilde{G}(|D\tilde{u} - Dw|)\,dx \\ &\le \frac{c}{\tau_1}\fint_{\tilde{\Omega}_K} |V_{\tilde{G}}(D\tilde{u}) - V_{\tilde{G}}(Dw)|^2\,dx + \tau_1 \fint_{\tilde{\Omega}_K} \tilde{G}(|D\tilde{u}|)\,dx \\ &\le \frac{c}{\tau_1}\fint_{\tilde{\Omega}_K} \tilde{G}'(|\tilde{F}|)|D\tilde{u} - Dw|\,dx + \tau_1 \fint_{\tilde{\Omega}_K} \tilde{G}(|D\tilde{u}|)\,dx \\ &\le \frac{c}{\tau_1}\left(\frac{\delta}{\tau^{s_G}} + \tau\right) + c\tau_1 \end{aligned}$$

for some constant $c \equiv c(n, s_G, \nu, L)$, where we have applied (3.2.12) for $\tilde{G}$ and (3.2.26). As a result, by choosing small $\tau_1 := \delta^{\frac{1}{2(1+s_G)}}$ and $\tau := \delta^{\frac{1}{1+s_G}}$ in the last display, we find that

$$\fint_{\tilde{\Omega}_K} \tilde{G}(|D\tilde{u} - Dw|)\,dx \le c\delta^{\frac{1}{2(1+s_G)}}. \tag{3.2.27}$$

3. *(Higher integrability)* According to the proof of the higher integrability for (3.2.25) [70, Lemma 5.6] with (3.2.26), there exists a small constant

$\sigma_0 \equiv \sigma_0(n, s_G, \nu, L) > 0$ such that

$$\left(\fint_{\tilde{\Omega}_3} \tilde{G}^{1+\sigma_0}(|Dw|)\,dx\right)^{\frac{1}{1+\sigma_0}} \le c \fint_{\tilde{\Omega}_K} \tilde{G}(|Dw|)\,dx \le c \tag{3.2.28}$$

with some constant $c \equiv c(n, s_G, \nu, L)$.

4. *(Oscillation estimate)* Using [167, Corollary 1.5], there exists a constant $\beta \in (0,1)$, depending only on n, s_G, ν and L such that

$$\operatorname*{osc}_{\tilde{\Omega}_3} w \le c \left(\frac{3}{K}\right)^{\beta} \|w\|_{L^\infty(\tilde{\Omega}_K)} \tag{3.2.29}$$

holds for some $c \equiv c(n, s_G, \nu, L)$.

Second comparison estimate: We consider a function $v \in W^{1,\tilde{G}}(\tilde{\Omega}_3)$ being the weak solution of the following problem:

$$\begin{cases} -\mathrm{div}\tilde{A}(x, (\tilde{u})_{\tilde{\Omega}_3}, Dv) = 0 & \text{in } \tilde{\Omega}_3 \\ v = w & \text{on } \partial\tilde{\Omega}_3. \end{cases} \tag{3.2.30}$$

Then we provide a comparison estimate between functions w and v in the next lemma.

Lemma 3.2.7. *For any $\varepsilon \in (0,1)$, there exist two constants $\delta \in (0, 1/8)$ and $K \ge 4$ depending only on $n, s_G, \nu, L, M, \omega_M(\cdot)$ and ε such that if $w \in W^{1,\tilde{G}}(\tilde{\Omega}_K)$ is the weak solution of* (3.2.25) *and $v \in W^{1,\tilde{G}}(\tilde{\Omega}_3)$ is the weak solution of* (3.2.30) *under the assumptions* (3.2.21)-(3.2.24)*, then the following comparison estimate holds:*

$$\fint_{\tilde{\Omega}_3} \tilde{G}(|Dw - Dv|)\,dx \le \varepsilon. \tag{3.2.31}$$

Proof. First we show that, for any $\tau \in (0,1)$, we have

$$\begin{aligned} &\fint_{\tilde{\Omega}_3} \tilde{G}(|Dw - Dv|)\,dx \\ &\le \tau \fint_{\tilde{\Omega}_3} \tilde{G}(|Dw|)\,dx + \frac{c}{\tau^{s_G+1}} \fint_{\tilde{\Omega}_3} \omega_M(Kr|\tilde{u} - (\tilde{u})_{\tilde{\Omega}_3}|)\tilde{G}(|Dw|)\,dx \end{aligned} \tag{3.2.32}$$

with $c \equiv c(n, s_G, \nu, L, M, \omega_M(\cdot))$. Indeed, taking $w - v$ as a test function in both (3.2.25) and (3.2.30), we see

$$\fint_{\tilde{\Omega}_3} \left\langle \tilde{A}(x, \tilde{u}, Dw), Dw - Dv \right\rangle dx = \fint_{\tilde{\Omega}_3} \left\langle \tilde{A}(x, (\tilde{u})_{\tilde{\Omega}_3}, Dv), Dw - Dv \right\rangle dx.$$

On the other hand, recalling (3.2.19) and Young's inequality (3.2.12), we obtain that

$$\begin{aligned}
&\fint_{\tilde{\Omega}_3} \left\langle \tilde{A}(x, (\tilde{u})_{\tilde{\Omega}_3}, Dw) - \tilde{A}(x, \tilde{u}, Dw), Dw - Dv \right\rangle dx \\
&\quad \le \fint_{\tilde{\Omega}_3} \omega_M(Kr|\tilde{u} - (\tilde{u})_{\tilde{\Omega}_3}|) \frac{\tilde{G}(|Dw|)}{|Dw|} |Dw - Dv| \, dx \\
&\quad \le \frac{c}{\tau_1^{s_G}} \fint_{\tilde{\Omega}_3} \omega_M(Kr|\tilde{u} - (\tilde{u})_{\tilde{\Omega}_3}|) \tilde{G}(|Dw|) \, dx \\
&\qquad + \tau_1 \fint_{\tilde{\Omega}_3} \omega_M(Kr|\tilde{u} - (\tilde{u})_{\tilde{\Omega}_3}|) \tilde{G}(|Dw - Dv|) \, dx
\end{aligned}$$

for some constant $c \equiv c(n, s_G, \nu, L)$ and for any $\tau_1 \in (0, 1)$. Recalling that $|\tilde{u} - (\tilde{u})_{\tilde{\Omega}_3}| \le 2M/(Kr)$ by the assumption (3.2.21) and using Lemma 3.2.4, the property (3.2.17) and (3.2.19), we have that

$$\begin{aligned}
&\fint_{\tilde{\Omega}_3} \tilde{G}(|Dw - Dv|) \, dx \\
&\quad \le \frac{\tau}{2} \fint_{\tilde{\Omega}_3} \tilde{G}(|Dw|) \, dx \\
&\qquad + \frac{c}{\tau} \fint_{\tilde{\Omega}_3} \left\langle \tilde{A}(x, (\tilde{u})_{\tilde{\Omega}_3}, Dw) - \tilde{A}(x, (\tilde{u})_{\tilde{\Omega}_3}, Dv), Dw - Dv \right\rangle dx \\
&\quad = \frac{\tau}{2} \fint_{\tilde{\Omega}_3} \tilde{G}(|Dw|) \, dx \\
&\qquad + \frac{c}{\tau} \fint_{\tilde{\Omega}_3} \left\langle \tilde{A}(x, (\tilde{u})_{\tilde{\Omega}_3}, Dw) - \tilde{A}(x, \tilde{u}, Dw), Dw - Dv \right\rangle dx \\
&\quad \le \frac{\tau}{2} \fint_{\tilde{\Omega}_3} \tilde{G}(|Dw|) \, dx + \frac{c_*}{\tau_1^{s_G} \tau} \fint_{\tilde{\Omega}_3} \omega_M(Kr|\tilde{u} - (\tilde{u})_{\tilde{\Omega}_3}|) \tilde{G}(|Dw|) \, dx \\
&\qquad + \frac{c_* \, \omega_M(2M) \tau_1}{\tau} \fint_{\tilde{\Omega}_3} \tilde{G}(|Dw - Dv|) \, dx
\end{aligned}$$

holds for every $\tau, \tau_1 \in (0,1)$ with some constants $c_* \equiv c_*(n, s_G, \nu, L)$. Choosing $\tau_1 := \tau/(2c_*\omega_M(2M))$ after some manipulations, we arrive at (3.2.32).

We next show that there exists a constant $c_0 \equiv c_0(n, s_G, \nu, L, M, \omega_M(\cdot))$ such that

$$\fint_{\tilde{\Omega}_3} \tilde{G}(|Dw - Dv|)\, dx \le c_0 \tilde{G}\left(\frac{M}{Kr}\right). \tag{3.2.33}$$

For this, let $\eta \in C_0^\infty(B_4)$ be a cut-off function such that $0 \le \eta \le 1$, $\eta \equiv 1$ in B_3 and $|D\eta| \le 2$. By taking $\eta^{s_G+1} w$ as a test function in the equation (3.2.25), using (3.2.2) and Young's inequality (3.2.12), we have that

$$\begin{aligned}
\fint_{\tilde{\Omega}_4} \eta^{s_G+1} \tilde{G}(|Dw|)\, dx &\le c \fint_{\tilde{\Omega}_4} \left\langle \tilde{A}(x, \tilde{u}, Dw), \eta^{s_G+1} Dw \right\rangle dx \\
&= -c(s_G+1) \fint_{\tilde{\Omega}_4} \left\langle \tilde{A}(x, \tilde{u}, Dw), \eta^{s_G} w D\eta \right\rangle dx \\
&\le c \fint_{\tilde{\Omega}_4} \eta^{s_G} |w| \frac{(\tilde{G})'(|Dw|)}{|Dw|} |D\eta|\, dx \\
&\le \fint_{\tilde{\Omega}_4} \eta^{s_G} \left((\tau\eta)\tilde{G}(|Dw|) + \frac{c}{(\tau\eta)^{s_G}} \tilde{G}(|w||D\eta|) \right) dx \\
&\le \tau \fint_{\tilde{\Omega}_4} \eta^{s_G+1} \tilde{G}(|Dw|)\, dx + c(\tau) \fint_{\tilde{\Omega}_4} \tilde{G}(|w|)\, dx
\end{aligned}$$

with some constant $c(\tau) \equiv c(n, s_G, \nu, L, \tau)$. By choosing $\tau := 1/2$ and observing that $\eta \equiv 1$ on $\tilde{\Omega}_3$ in the last display, we conclude

$$\fint_{\tilde{\Omega}_3} \tilde{G}(|Dw|)\, dx \le c \fint_{\tilde{\Omega}_4} \tilde{G}(|w|)\, dx.$$

Using the last display and recalling $|\tilde{u} - (\tilde{u})_{\tilde{\Omega}_3}| \le 2M/(Kr)$ in (3.2.32) with fixed $\tau \equiv 1/2$, we conclude that

$$\begin{aligned}
\fint_{\tilde{\Omega}_3} \tilde{G}(|Dw - Dv|)\, dx &\le \omega_M(2M) \fint_{\tilde{\Omega}_3} \tilde{G}(|Dw|)\, dx + c \fint_{\tilde{\Omega}_3} \tilde{G}(|Dw|)\, dx \\
&\le c \fint_{\tilde{\Omega}_4} \tilde{G}(|w|)\, dx
\end{aligned}$$

with $c \equiv c(n, s_G, \nu, L, M, \omega_M(\cdot))$. Therefore, (3.2.33) follows from the maximum principle that

$$\|w\|_{L^\infty(\tilde{\Omega}_4)} \le \|w\|_{L^\infty(\tilde{\Omega}_K)} \le M/(Kr).$$

If $c_0\tilde{G}(\frac{M}{Kr}) \le \varepsilon$, then the conclusion directly holds by (3.2.33). So, we consider the remaining case that

$$c_0\tilde{G}\left(\frac{M}{Kr}\right) > \varepsilon \tag{3.2.34}$$

holds. Recalling that $\omega_M(\cdot)$ is a modulus of continuity, we can choose a small $\delta_0 \equiv \delta_0(s_G, M, \omega_M(\cdot), \tau) \in (0,1)$ such that

$$0 < \omega_M(\rho) < \tau^{s_G+2} \quad \text{for all } \rho \in (0, \delta_0).$$

On the other hand, for any $\alpha \in (0, \alpha_0)$ with $\alpha_0 := \frac{1}{1+s_G}\log_{\delta_0}\frac{1}{2}$, one can see that

$$[\tilde{G}(\delta_0)]^\alpha = \left[\frac{G(K\delta_0)}{G(K)}\right]^\alpha \ge \delta_0^{\alpha(1+s_G)} \ge \frac{1}{2},$$

where we have used (3.2.11). Then in the view of the last two display it follows that

$$\omega_M(\rho) \le \tau^{s_G+2} + 2\omega_M(2M)\tilde{G}(\rho)^\alpha \quad \text{for all } \rho \in \left(0, \frac{2M}{Kr}\right]. \tag{3.2.35}$$

Using the last display in (3.2.32) and absorbing the terms, we find that

$$\begin{aligned} &\fint_{\tilde{\Omega}_3} \tilde{G}(|Dw - Dv|)\,dx \\ &\le c\tau \fint_{\tilde{\Omega}_3} \tilde{G}(|Dw|)\,dx + \frac{c}{\tau^{s_G+1}} \fint_{\tilde{\Omega}_3} \tilde{G}(Kr|\tilde{u} - (\tilde{u})_{\tilde{\Omega}_3}|)^\alpha \tilde{G}(|Dw|)\,dx. \end{aligned} \tag{3.2.36}$$

Now we estimate the second term in the above display. For this, we first take $\alpha \le \min\left\{\frac{\sigma_0}{1+\sigma_0}, \alpha_0\right\}$ in such a way that $\alpha \equiv \alpha(n, s_G, \nu, L, M, \omega_M(\cdot), \tau)$, where σ_0 is the higher integrability exponent that has been defined in (3.2.28). Therefore, using Hölder's inequality and higher integrability (3.2.28), we find

that

$$
\begin{aligned}
&\fint_{\tilde\Omega_3} \tilde G(Kr|\tilde u-(\tilde u)_{\tilde\Omega_3}|)^{\alpha}\tilde G(|Dw|)\,dx \\
&\qquad \le \left(\fint_{\tilde\Omega_3} \tilde G(Kr|\tilde u-(\tilde u)_{\tilde\Omega_3}|)\,dx\right)^{\alpha}\left(\fint_{\tilde\Omega_3} \tilde G(|Dw|)^{\frac{1}{1-\alpha}}\,dx\right)^{1-\alpha} \\
&\qquad \le c\left(\fint_{\tilde\Omega_3} \tilde G(Kr|\tilde u-(\tilde u)_{\tilde\Omega_3}|)\,dx\right)^{\alpha}\left(\fint_{\tilde\Omega_3} \tilde G(|Dw|)^{1+\sigma_0}\,dx\right)^{\frac{1}{1+\sigma_0}} \\
&\qquad \le c\left(\fint_{\tilde\Omega_3} \tilde G(Kr|\tilde u-(\tilde u)_{\tilde\Omega_3}|)\,dx\right)^{\alpha}
\end{aligned}
\tag{3.2.37}
$$

for some $c \equiv c(n, s_G, \nu, L, M, \omega_M(\cdot))$, where our choice of α guarantees the validity of $\frac{1}{1-\alpha} \le 1+\sigma_0$. Now, it follows from the triangle and Jensen's inequalities that

$$
\begin{aligned}
&\fint_{\tilde\Omega_3} \tilde G(Kr|\tilde u-(\tilde u)_{\tilde\Omega_3}|)\,dx \\
&\qquad \le c\fint_{\tilde\Omega_3} \tilde G(Kr|\tilde u-w|)\,dx + c\fint_{\tilde\Omega_3} \tilde G(Kr|w-(w)_{\tilde\Omega_3}|)\,dx \\
&\qquad\quad + c\fint_{\tilde\Omega_3} \tilde G\left(Kr|(w)_{\tilde\Omega_3}-(\tilde u)_{\tilde\Omega_3}|\right)\,dx \\
&\qquad \le c_*\fint_{\tilde\Omega_3} \tilde G(Kr|\tilde u-w|)\,dx + c_*\fint_{\tilde\Omega_3} \tilde G(Kr|w-(w)_{\tilde\Omega_3}|)\,dx \\
&\qquad =: c_*(I_3+I_4)
\end{aligned}
\tag{3.2.38}
$$

for some $c_* \equiv c_*(n, s_G)$. By recalling (3.2.34) and (3.2.11), notice that

$$
\varepsilon < c_0 G\left(\frac{M}{Kr}\right) \le c\,G\left(\frac{1}{Kr}\right) \le c\left(\frac{1}{Kr}\right)^{\frac{1}{s_G}+1}
$$

holds for some constant $c \equiv c(n, s_G, \nu, L, M)$ if $Kr \ge 1$. Therefore, one can see that

$$
\frac{c}{\varepsilon^{s_G}} > \frac{c}{2\varepsilon^{s_G}} + 1 \ge c\left((Kr)^{1+s_G}+1\right) \tag{3.2.39}
$$

with some constant $c \equiv c(n, s_G, \nu, L, M)$. The last display together with

(3.2.27), (3.2.11) and applying Proposition 3.2.6 implies that

$$
\begin{aligned}
I_3 &\le c \fint_{\tilde{\Omega}_4} \max\left\{(Kr)^{s_G+1}, (Kr)^{1/s_G+1}\right\} \tilde{G}(|\tilde{u}-w|)\,dx \\
&\le c \fint_{\tilde{\Omega}_4} \left(1+(Kr)^{s_G+1}\right) \tilde{G}(|\tilde{u}-w|)\,dx \\
&\le \frac{c}{\varepsilon^{s_G}} \fint_{\tilde{\Omega}_K} \tilde{G}(|\tilde{u}-w|)\,dx \\
&\le \frac{c(1+K^{s_G+1})}{\varepsilon^{s_G}} \fint_{\tilde{\Omega}_K} \tilde{G}(|D\tilde{u}-Dw|)\,dx \\
&\le \frac{c\delta^{\frac{1}{2(1+s_G)}}}{\varepsilon^{s_G}} K^{n+s_G+1}
\end{aligned}
$$

for some $c \equiv c(n, s_G, \nu, L, M, \omega_M(\cdot))$. The remaining term I_4 in (3.2.38) can be controlled using the oscillation estimate (3.2.29) as follows:

$$
\begin{aligned}
I_4 &= \fint_{\tilde{\Omega}_3} \tilde{G}(Kr|w-(w)_{\tilde{\Omega}_3}|)\,dx \le c \fint_{\tilde{\Omega}_3} \tilde{G}\left(Kr\left(\frac{3}{K}\right)^{\beta} \|w\|_{L^\infty(\tilde{\Omega}_K)}\right) dx \\
&\le c \fint_{\tilde{\Omega}_3} \tilde{G}(M)\left(\frac{3}{K}\right)^{\beta(1/s_G+1)} dx \le \frac{c}{K^{\beta(1/s_G+1)}}
\end{aligned}
$$

for some $c \equiv c(n, s_G, \nu, L, M)$, where we have used the fact that $\|w\|_{L^\infty(\tilde{\Omega}_K)} \le \|w\|_{L^\infty(\tilde{\Omega})} \le \frac{M}{Kr}$ and (3.2.11). Inserting the last two displays into (3.2.38), we conclude that

$$
\fint_{\tilde{\Omega}_3} \tilde{G}(|\tilde{u}-(\tilde{u})_{\tilde{\Omega}_5}|)\,dx \le c\left(\frac{c\delta^{\frac{1}{2(1+s_G)}}}{\varepsilon^{s_G}} K^{n+s_G+1} + \frac{1}{K^{\beta(1/s_G+1)}}\right)
$$

with some constant $c \equiv c(n, s_G, \nu, L, M, \omega_M(\cdot))$. This estimate together with

(3.2.36) implies that the following inequality

$$
\begin{aligned}
&\fint_{\tilde{\Omega}_3} \tilde{G}(|Dw - Dv|)\,dx \\
&\quad \leq c_2 \left(\tau + c(\tau) \left(\frac{c\delta^{\frac{1}{2(1+s_G)}}}{\varepsilon^{s_G}} K^{n+s_G+1} + \frac{1}{K^{\beta(1/s_G+1)}} \right)^{\alpha} \right) \\
&\quad \leq c_2\tau + c_3(\tau) \left(\frac{\delta^{\frac{1}{2(1+s_G)}}}{\varepsilon^{s_G}} \right)^{\alpha} K^{(n+s_G+1)\alpha} + c_3(\tau) \left(\frac{1}{K} \right)^{\alpha\beta(1/s_G+1)}.
\end{aligned}
$$

holds for any numbers $\tau \in (0,1)$, $K \geq 4$ and $\delta \in (0, 1/8)$ to be chosen in a few lines, where the dependencies of constants are as follows: $c_2 \equiv c_2(n, s_G, \nu, L, M, \omega_M(\cdot))$, $c_3(\tau) \equiv c_3(\tau)(n, s_G, \nu, L, M, \omega_M(\cdot), \tau)$, $\alpha \equiv \alpha(n, s_G, \nu, L, M, \omega_M(\cdot), \tau)$ and $\beta \equiv \beta(n, s_G, \nu, L)$. Choosing τ small, K sufficiently large and then finally select δ sufficiently small in such a way that the following inequalities hold:

$$
\begin{aligned}
&c_2\tau \leq \frac{\varepsilon}{3}, \quad c_3(\tau) \left(\frac{1}{K} \right)^{\alpha\beta(1/s_G+1)} \leq \frac{\varepsilon}{3} \\
&\text{and} \quad c_3(\tau) \left(\frac{\delta^{\frac{1}{2(1+s_G)}}}{\varepsilon^{s_G}} \right)^{\alpha} K^{(n+s_G+1)\alpha} \leq \frac{\varepsilon}{3}.
\end{aligned}
$$

Therefore, the claim (3.2.31) follows. □

Third comparison estimate: Finally, we consider the limiting homogeneous equation:

$$
\begin{aligned}
&\begin{cases} -\mathrm{div}\bar{A}(Dh) = 0 & \text{in } B_2^+ \\ h = 0 & \text{on } B_2 \cap \{x_n = 0\} \end{cases} \\
&\quad \text{with} \quad \bar{A}(\xi) := \fint_{B_2^+} \tilde{A}(x, (\tilde{u})_{\tilde{\Omega}_3}, \xi)\,dx.
\end{aligned}
\tag{3.2.40}
$$

Under the assumptions of Lemma 3.2.7, there exists a weak solution $h \in W^{1,\tilde{G}}(B_2^+)$ to the equation (3.2.40) such that

$$
\|\tilde{G}(|D\bar{h}|)\|_{L^\infty(\tilde{\Omega}_1)} \leq c_b \quad \text{and} \quad \fint_{\tilde{\Omega}_2} \tilde{G}(|Dv - D\bar{h}|)\,dx \leq \varepsilon \tag{3.2.41}
$$

for some constant $c_b \equiv c_b(n, s_G, \nu, L) > 1$, where $\bar{h}$ is the null extension of h from B_2^+ to B_2. Indeed, the first inequality of (3.2.41) follows from [169, Theorem 1.2]. Then following the proof of [70, Lemma 5.4], we obtain the second inequality of (3.2.41). In the case of interior estimates, we consider the following equation:

$$\begin{cases} -\mathrm{div}\bar{A}(Dh) = 0 & \text{in } B_2 \\ h = v & \text{on } \partial B_2 \end{cases} \qquad \text{with } \bar{A}(\xi) := \fint_{B_2} \tilde{A}(x, (\tilde{u})_{B_3}, \xi)\, dx. \tag{3.2.42}$$

We are also able to obtain the exactly same estimates as (3.2.41), with $h = \bar{h}$. Taking into account (3.2.40)-(3.2.42) and combining all of the estimates obtained in (3.2.27), (3.2.41) and the one by Lemma 3.2.7 based on the triangle inequality, we can now achieve the following lemma.

Lemma 3.2.8. *For any $\varepsilon \in (0,1)$, there exist constants $K \geq 4$ and $\delta \in (0, 1/8)$, both depending on $n, s_G, \nu, L, M, \omega_M(\cdot)$ and ε such that if $\tilde{u} \in W^{1,\tilde{G}}(\tilde{\Omega}_K)$ is a weak solution of* (3.2.20) *under the assumptions* (3.2.21)-(3.2.24)*, there exists the weak solution $h \in W^{1,\tilde{G}}(B_2)$ of* (3.2.42) *for the interior case or a weak solution $h \in W^{1,\tilde{G}}(B_2^+)$ of* (3.2.40) *for the boundary case such that*

$$\|\tilde{G}(|D\bar{h}|)\|_{L^\infty(\tilde{\Omega}_1)} \leq c_b \quad \textit{and} \quad \fint_{\tilde{\Omega}_2} \tilde{G}(|D\tilde{u} - D\bar{h}|)\, dx \leq \varepsilon$$

for some constant $c_b \equiv c_b(n, s_G, \nu, L) > 1$, where $\bar{h} \in W^{1,\tilde{G}}(\tilde{\Omega}_2)$ is equal to h for the interior case, and $\bar{h}$ is the zero extension of h from B_2^+ to B_2 for the boundary case.

Proof of Theorem 1.1. The proof is based on the techniques employed in [70, Theorem 2.5] and initially introduced in [3]. For $\lambda > 0$, we define

$$E(u, \lambda) := \{x \in \Omega : G(|Du(x)|) > \lambda\}$$

and

$$H_y(\rho) := \fint_{\Omega_\rho(y)} \left[G(|Du|) + \frac{1}{\delta} G(|F|) \right] dx \quad (y \in \Omega, \rho > 0).$$

Here, $\delta \in (0, 1/8)$ will be determined later, depending only on

$n, s_G, \nu, L, M, \omega_M(\cdot)$ and γ. By the Lebesgue differentiation theorem, we have

$$\lim_{r\to 0} H_y(r) \geq G(|Du(x)|) > \lambda \tag{3.2.43}$$

for a.e. $y \in E(u,\lambda)$. Let ε be a positive number to be determined by the end of the proof and consider a number $K \geq 4$ as given in Lemma 3.2.8 depending on $n, s_G, \nu, L, M, \omega_M(\cdot)$ and ε, where in what follows we fix $M := \|u\|_{L^\infty(\Omega)}$. From now on, we only consider the values of λ satisfying the following bounds:

$$\begin{aligned} \lambda &\geq \left((8000K)^n \frac{|\Omega|}{|B_R|} + 1\right)\lambda_0 \\ \text{for} \quad \lambda_0 &:= \fint_\Omega \left[G(|Du|) + \frac{1}{\delta} G(|F|)\right] dx, \end{aligned} \tag{3.2.44}$$

where R is a number coming from the assumptions of Theorem 3.2.2. It can be easily seen that for a given $y \in E(u,\lambda)$,

$$\begin{aligned} &H_y(r) \leq \frac{|B_r(y)|}{|\Omega_r(y)|}\frac{|\Omega|}{|B_r(y)|}\lambda_0 \leq (8000K)^n \frac{|\Omega|}{|B_R|}\lambda_0 < \lambda \\ &\text{for any } r \in \left[\frac{R}{2000K}, \frac{R}{2}\right]. \end{aligned} \tag{3.2.45}$$

By (3.2.43) and (3.2.45), for a.e. $y \in E(u,\lambda)$, there exists $r_y \in \left(0, \frac{R}{2000K}\right)$ such that

$$H_y(r_y) = \lambda \quad \text{and} \quad H_y(r) < \lambda \text{ for all } r \in (r_y, R/2].$$

Now by the Vitali covering lemma, there exist mutually disjoint open balls $\{B_{r_i}(y_i)\}_{i=1}^\infty$ for $y_i \in E(u,\lambda)$ and $r_i \in (0, \frac{R}{2000K})$ such that

$$E(u,\lambda) \subset \bigcup_{i=1}^\infty \Omega_{5r_i}(y_i) \cup \{\text{a set of measure zero}\}, \tag{3.2.46}$$

$$\fint_{\Omega_{r_i}(y_i)} \left[G(|Du|) + \frac{1}{\delta}G(|F|)\right] dx = \lambda, \tag{3.2.47}$$

and

$$\fint_{\Omega_r(y_i)} \left[G(|Du|) + \frac{1}{\delta} G(|F|) \right] dx < \lambda \quad \text{for all} \quad r \in (r_i, R/2]. \tag{3.2.48}$$

Then since $\frac{R}{5r_i} > 5K > 5$, following the proof of [70, Section 6] and applying (3.2.48) and Lemma 3.2.8 with the scaling invariant property introduced in Section 3.2.2, we see that there exists $h_i \in W^{1,\infty}(\Omega_{5r_i}(y_i))$ such that

$$\begin{aligned} &\|G(Dh_i)\|_{L^\infty(\Omega_{5r_i}(y_i))} < c(n)c_b\lambda \\ \text{and} \quad &\fint_{\Omega_{5r_i}(y_i)} G(|Du - Dh_i|)\, dx < c(n)\varepsilon\lambda, \end{aligned} \tag{3.2.49}$$

where $c_b > 1$ is the same one determined by Lemma 3.2.8.

Let $c_m := 2^{s_G+2} \cdot c(n)c_b \geq 1$. For a.e $x \in E(u, c_m\lambda) \cap \Omega_{5r_i}(y_i)$, after elementary manipulations, we find

$$\begin{aligned} G(|Du(x)|) &\leq 2^{s_G+1} G(|Du(x) - Dh_i(x)|) + 2^{s_G+1} G(|Dh_i(x)|) \\ &\leq 2^{s_G+1} G(|Du(x) - Dh_i(x)|) + 2^{s_G+1} c(n)c_b\lambda \\ &\leq 2^{s_G+1} G(|Du(x) - Dh_i(x)|) + \frac{1}{2} G(|Du(x)|). \end{aligned}$$

In particular, we have

$$G(|Du(x)|) \leq 2^{s_G+2} G(|Du(x) - Dh_i(x)|) \quad \text{a.e} \quad x \in E(u, c_m\lambda) \cap \Omega_{5r_i}(y_i).$$

Therefore, using (3.2.49), it follows that

$$\begin{aligned} \int_{E(u,c_m\lambda)\cap\Omega_{5r_i}(y_i)} G(|Du|)\, dx &\leq c \int_{\Omega_{5r_i}(y_i)} G(|Du - Dh_i|)\, dx \\ &\leq c_4 |\Omega_{r_i}(y_i)| \varepsilon\lambda \end{aligned} \tag{3.2.50}$$

for some constant $c_4 \equiv c_4(n, s_G, \nu, L)$. On the other hand, (3.2.47) implies

that

$$\begin{aligned} |\Omega_{r_i}(y_i)| \le \frac{2}{\lambda} \int_{E\left(u,\frac{\lambda}{4}\right)\cap\Omega_{r_i}(y_i)} G(|Du|)\,dx \\ + \frac{2}{\delta\lambda} \int_{E\left(F,\frac{\delta\lambda}{4}\right)\cap\Omega_{r_i}(y_i)} G(|F|)\,dx, \end{aligned} \tag{3.2.51}$$

where we denote by

$$E(F,\lambda) := \{x \in \Omega : G(|F(x)|) > \lambda\}.$$

Now, inserting (3.2.51) into (3.2.50), it yields that

$$\begin{aligned} &\int_{E(u,c_m\lambda)\cap\Omega_{5r_i}(y_i)} G(|Du|)\,dx \\ &\le 2c_4\varepsilon \int_{E\left(u,\frac{\lambda}{4}\right)\cap\Omega_{r_i}(y_i)} G(|Du|)\,dx + \frac{2c_4\varepsilon}{\delta} \int_{E\left(F,\frac{\delta\lambda}{4}\right)\cap\Omega_{r_i}(y_i)} G(|F|)\,dx. \end{aligned}$$

Meanwhile, the choice of $c_m \ge 1$ and (3.2.46) imply that

$$E(u,c_m\lambda) \subset \bigcup_{i=1}^{\infty} (E(u,c_m\lambda)\cap\Omega_{5r_i}(y_i)) \cup \{\text{a set of measure zero}\}.$$

Combining the last two displays, we conclude

$$\begin{aligned} \int_{E(u,c_m\lambda)} G(|Du|)\,dx &\le \sum_{i=1}^{\infty} \int_{E(u,c_m\lambda)\cap\Omega_{5r_i}(y_i)} G(|Du|)\,dx \\ &\le 2c_4\varepsilon \sum_{i=1}^{\infty} \left(\int_{E\left(u,\frac{\lambda}{4}\right)\cap\Omega_{r_i}(y_i)} G(|Du|)\,dx + \frac{1}{\delta} \int_{E\left(F,\frac{\delta\lambda}{4}\right)\cap\Omega_{r_i}(y_i)} G(|F|)\,dx \right) \\ &\le 2c_4\varepsilon \left(\int_{E\left(u,\frac{\lambda}{4}\right)} G(|Du|)\,dx + \frac{1}{\delta} \int_{E\left(F,\frac{\delta\lambda}{4}\right)} G(|F|)\,dx \right), \end{aligned}$$

where we have used the fact that $\{B_{r_i}(y_i)\}_{i=1}^{\infty}$ is a collection of mutually disjoint balls. After arguing similarly as it has been done in [70, Section 6] and choosing $\varepsilon \in (0,1)$ so small such that $2c_4(c(n)c_b)^{\gamma-1}\varepsilon < 1$, we have

$$\fint_{\Omega} [G(|Du|)]^{\gamma}\, dx \le c\lambda_0^{\gamma} + c \fint_{\Omega} [G(|F|)]^{\gamma}\, dx \tag{3.2.52}$$

for some $c \equiv c(n, s_G, \nu, L, M, \omega_M(\cdot), |\Omega|, \gamma)$. Then we find small $\delta \equiv \delta(n, s_G, \nu, L, M, \omega_M(\cdot), \gamma) > 0$ from Lemma 3.2.8. Here, we note that the following energy estimate holds for the problem (3.2.1):

$$\int_{\Omega} G(|Du|)\, dx \le c \int_{\Omega} G(|F|)\, dx$$

for some $c \equiv c(n, s_G, \nu, L)$. This together with Jensen's inequality yields

$$\lambda_0^{\gamma} = \left(\fint_{\Omega} \left[G(|Du|) + \frac{1}{\delta} G(|F|) \right] dx \right)^{\gamma} \le c \fint_{\Omega} G(|F|)^{\gamma}\, dx.$$

Inserting the above inequality into (3.2.52), it yields (3.2.8), which completes the proof. □

3.3 Local estimates for non-uniformly elliptic problems with BMO nonlinearity

In this section, we provide a new approach to obtain Calderón-Zygmund type estimates for non-uniformly elliptic equations with discontinuous nonlinearities of double phase growth. This approach, which is based on a small higher integrability result for the gradient of weak solutions to the associated homogeneous problems together with extrapolation from Muckenhoupt weights, enables us to find a proper comparison estimate of approximation by imposing merely a small BMO assumption on the nonlinearity with respect to the x-variable. As a consequence, we are able to prove an optimal regularity theory for a larger class of double phase problems with discontinuous nonlinearities in the literature.

3.3.1 Hypothesis and main results

To present our gradient estimates, we display the precise structure assumptions of the problem. With the assumptions (1.2.2)–(1.2.4) and the notation

$$H(x,t) = t^p + a(x)t^q \quad (x \in \Omega,\ t \geq 0), \tag{3.3.1}$$

we actually deal with the following equation of the form:

$$\operatorname{div}(A(x,Du)) = \operatorname{div}(|F|^{p-2}F + a(x)|F|^{q-2}F) \quad \text{in } \Omega. \tag{3.3.2}$$

Here, $F = (f_1, \ldots, f_n) : \Omega \to \mathbb{R}^n$ is a given vector field such that $H(x,|F|) \in L^1(\Omega)$, and the given nonlinearity $A(x,z) : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory vector field which is $C^1(\mathbb{R}^n \setminus \{0\})$–regular for z variable and satisfies

$$\begin{cases} |A(x,z)||z| + |\partial_z A(x,z)||z|^2 \leq LH(x,|z|) \\ \nu \dfrac{H(x,|z|)}{|z|^2}|\xi|^2 \leq \langle \partial_z A(x,z)\xi, \xi\rangle \end{cases} \tag{3.3.3}$$

for any $\xi \in \mathbb{R}^n$ with some constants $0 < \nu \leq L < \infty$. A weak solution u of (3.3.2) belongs to the Musielak-Orlicz space $W^{1,H}(\Omega)$ which is specifically defined in Chapter 2.

We consider a smallness assumption on $x \mapsto \frac{A(x,z)}{|z|^{p-1}+a(x)|z|^{q-1}}$ in the BMO sense, uniformly in z, as we now state.

Definition 3.3.1. *We define*

$$\begin{aligned} &\theta(A; B_r(y))(x) \\ &:= \sup_{z \in \mathbb{R}^n \setminus \{0\}} \left| \frac{A(x,z)}{|z|^{p-1} + a(x)|z|^{q-1}} - \left(\frac{A(\cdot,z)}{|z|^{p-1} + a(\cdot)|z|^{q-1}} \right)_{B_r(y)} \right| \\ &\leq 2L. \end{aligned} \tag{3.3.4}$$

With two parameters $R \in (0, \frac{1}{2})$ and $\delta \in (0, \frac{1}{8})$, we call that A is (δ, R)-vanishing if the following holds:

$$\sup_{0<r\leq R} \sup_{B_r(y)\subset\Omega} \fint_{B_r(y)} \theta(A; B_r(y))(x)\, dx \leq \delta. \tag{3.3.5}$$

With the convenient notation

$$\mathtt{data} = \{n, p, q, \alpha, \|a\|_{0,\alpha}, \nu, L, \|H(\cdot, |Du|)\|_{L^1(\Omega)}\},$$

we now state our main theorem.

Theorem 3.3.2. *Under the assumptions* (1.2.2)–(1.2.4) *and* (3.3.3)*, suppose* $H(x,|F|) \in L^\gamma(\Omega)$ *for some* $\gamma > 1$. *Then together with the assumption* (1.2.8)*, there exists a constant* $\delta = \delta(\boldsymbol{data}, \gamma) > 0$ *such that if* A *is* (δ, R)*-vanishing for some small* $R > 0$*, any weak solution* $u \in W^{1,H}(\Omega)$ *of* (3.3.2) *satisfies* $H(x, |Du|) \in L^\gamma_{\mathrm{loc}}(\Omega)$. *Moreover, for any* $\Omega_0 \Subset \Omega$ *there exists a radius* $R = R(\boldsymbol{data}, \mathrm{dist}(\Omega_0, \partial\Omega), \gamma, \|H(x,|F|)\|_{L^\gamma(\Omega)}) > 0$ *such that for each* $B_r(y) \subset \Omega_0$ *with* $0 < 2r < R$ *and* $B_R(y) \subset \Omega$*, we have*

$$\begin{aligned}
&\fint_{B_r(y)} H(x, |Du|)^\gamma \, dx \\
&\quad \le c \left(\fint_{B_{2r}(y)} H(x, |Du|) \, dx \right)^\gamma + c \fint_{B_{2r}(y)} H(x, |F|)^\gamma \, dx
\end{aligned} \tag{3.3.6}$$

for some constant $c = c(\boldsymbol{data}, \mathrm{dist}(\Omega_0, \partial\Omega), \gamma, \|H(x,|F|)\|_{L^\gamma(\Omega)})$.

We then explain why (1.2.8) is needed when treating double phase growth problems with discontinuous coefficients.

Remark 3.3.3. *Let* $\Omega = B_R(0)$ *for some* $R > 0$*, and let the constants* n, p, q, α *satisfy* (1.2.2)–(1.2.4). *Suppose that the function* $a(x) \in C^{0,\alpha}(B_R(0))$ *defined in* (1.2.3) *is such that*

$$\begin{cases} a(x) = 0 & \text{if} \quad x \in B_R^+(0) := B_R(0) \cap \{x \in \mathbb{R}^n : x_n \ge 0\} \\ a(x) > 0 & \text{if} \quad x \in B_R^-(0) := B_R(0) \cap \{x \in \mathbb{R}^n : x_n < 0\}. \end{cases}$$

With F *being a given vector field as above, we now consider the following equation:*

$$-\mathrm{div}[(1 + a(x)|Du|^{q-p})\tilde{A}(x, Du)] = -\mathrm{div}[|F|^{p-2}F + a(x)|F|^{q-2}F] \ \text{ in } B_R(0),$$

where $\tilde{A}(x,z) : B_R(0) \times \mathbb{R}^n \to \mathbb{R}^n$ *is a Carathéodory vector field with* $\tilde{A}(x, \cdot)$

being $C^1(\mathbb{R}^n \setminus \{0\})$-regular such that

$$\begin{cases} |\tilde{A}(x,z)||z| + |\partial_z \tilde{A}(x,z)||z|^2 \le L|z|^p \\ \nu|z|^{p-2}|\xi|^2 \le \left\langle \partial_z \tilde{A}(x,z)\xi, \xi \right\rangle \end{cases} \tag{3.3.7}$$

for any $\xi \in \mathbb{R}^n$ and some constants $0 < \nu \le L < \infty$. It is indeed necessary to assume (3.3.7) *on $\tilde{A}(x,z)$ for $x \in B_R^+(0)$, since $a(x) = 0$ in this set.*

Now let us denote $\Phi : B_R(0) \times (\mathbb{R}^+ \cup \{0\}) \to \mathbb{R}$ and $B : B_R(0) \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$\Phi(x,t) = 1 + a(x)t^{q-p} \quad \text{and} \quad B(x,z) := \Phi(x,|z|)\tilde{A}(x,z).$$

It is also necessary to assume the following ellipticity, considering the case of $x \in B_R^-(0)$:

$$\begin{cases} |B(x,z)||z| + |\partial_z B(x,z)||z|^2 \le \tilde{L}H(x,|z|) \\ \tilde{\nu}\dfrac{H(x,|z|)}{|z|^2}|\xi|^2 \le \langle \partial_z B(x,z)\xi, \xi \rangle \end{cases} \tag{3.3.8}$$

for any $\xi \in \mathbb{R}^n$ and some constants $0 < \tilde{\nu} \le \tilde{L} < \infty$. But in this case,

$$\partial_z B(x,z) = \partial_t \Phi(x,|z|)\frac{z}{|z|} \otimes \tilde{A}(x,z) + \Phi(x,z)\partial_z \tilde{A}(x,z)$$

and so by (3.3.7),

$$\begin{aligned} &\langle \partial_z B(x,z)\xi, \xi \rangle \\ &\ge -|\partial_t \Phi(x,|z|)||\tilde{A}(x,z)||\xi|^2 + \Phi(x,z)|\partial_z \tilde{A}(x,z)||\xi|^2 \\ &\ge -L(q-p)a(x)|z|^{q-p-1}|z|^{p-1}|\xi|^2 + \nu(1 + a(x)|z|^{q-p})|z|^{p-2}|\xi|^2 \\ &\ge (\nu - L(q-p))(1 + a(x)|z|^{q-p})|z|^{p-2}|\xi|^2 \\ &\ge (\nu - L(q-p))\frac{H(x,|z|)}{|z|^2}|\xi|^2. \end{aligned}$$

Thus to hold true (3.3.8)*, we need to assume that $\tilde{\nu} := \nu - L(q-p) > 0$, which is exactly our additional structure assumption* (1.2.8)*. We also point out that* (1.2.8) *is to be used in Lemma 3.3.9 and its proof.*

3.3.2 Preliminaries and basic regularity results

To use the extrapolation in the proof, we also record the conditions (A0), (A1), $(\mathrm{aInc})_{s_1}$ and $(\mathrm{aDec})_{s_2}$ for constants $s_1, s_2 > 0$ concerning a given function $\varphi(x,t) : \Omega \times [0,\infty) \to [0,\infty)$.

- (A0) There exists a constant $M \geq 1$ such that $M^{-1} \leq \varphi(x,1) \leq M$ for every $x \in \Omega$.
- $(\mathrm{aInc})_{s_1}$ The map $t \mapsto \frac{\varphi(x,t)}{t^{s_1}}$ is almost increasing with constant $M \geq 1$, uniformly in $x \in \Omega$, i.e., for any $0 < t_1 < t_2 < \infty$, there holds
$$\frac{\varphi(x,t_1)}{t_1^{s_1}} \leq M \frac{\varphi(x,t_2)}{t_2^{s_1}}.$$
- $(\mathrm{aDec})_{s_2}$ The map $t \mapsto \frac{\varphi(x,t)}{t^{s_2}}$ is almost decreasing with constant $M \geq 1$, uniformly in $x \in \Omega$, i.e., for any $0 < t_1 < t_2 < \infty$, there holds
$$\frac{\varphi(x,t_2)}{t_2^{s_2}} \leq M \frac{\varphi(x,t_1)}{t_1^{s_2}}.$$
- (A1) Assuming $\varphi(x,t)$ satisfies $(\mathrm{aInc})_1$, $\varphi(x,t)$ is said to satisfy (A1) condition if there exists a constant $M \geq 1$ such that for any $B_r \Subset \Omega$ with $|B_r| < 1$,
$$\varphi^+_{B_r}(t) \leq M \varphi^-_{B_r}(t) \quad \text{for all } t > 0 \quad \text{with} \quad \varphi^-_{B_r}(t) \in [1, |B_r|^{-1}], \tag{3.3.9}$$
where $\varphi^+_{B_r}(t) := \sup_{x \in B_r} \varphi(x,t)$ and $\varphi^-_{B_r}(t) := \inf_{x \in B_r} \varphi(x,t)$.

We sometimes regard H as the following one, other than (3.3.1):
$$H(x,z) = |z|^p + a(x)|z|^q \quad (x \in \Omega,\ z \in \mathbb{R}^n). \tag{3.3.10}$$

For the comparison estimates, we also need the following higher integrability results. We refer to [84, Theorem 4] for the proof.

Lemma 3.3.4. *Let $u \in W^{1,H}(\Omega)$ be a weak solution of* (3.3.2) *under the assumptions* (1.2.2)–(1.2.4), (3.3.3) *and $H(x,F) \in L^\gamma(\Omega)$ for some $\gamma > 1$. Then there exists a constant $\sigma_0 = \sigma_0(\texttt{data}) \leq \gamma - 1$ such that $H(x,Du) \in$*

$L^{1+\sigma_0}_{\mathrm{loc}}(\Omega)$. *Moreover, for any* $B_{2\rho} \subset \Omega$ *and* $\sigma \in (0, \sigma_0]$ *we have the estimate*

$$\left(\fint_{B_\rho} H(x, Du)^{1+\sigma} \, dx \right)^{\frac{1}{1+\sigma}} \leq c \fint_{B_{2\rho}} H(x, Du) \, dx + c \left(\fint_{B_{2\rho}} H(x, F)^{1+\sigma} \, dx \right)^{\frac{1}{1+\sigma}}$$

for some $c = c(\boldsymbol{data})$. *Moreover, for any* $\Omega_0 \Subset \Omega$ *and* $\sigma \in (0, \sigma_0]$, *we have*

$$\|H(x, Du)^{1+\sigma}\|_{L^1(\Omega_0)} \leq c \tag{3.3.11}$$

for some constant $c = c(\boldsymbol{data}, \operatorname{dist}(\Omega_0, \partial\Omega), \|H(x, F)\|_{L^\gamma(\Omega)})$.

Next, we provide the following lemma which will be frequently used for our comparison estimates.

Lemma 3.3.5 ([16, 18]). *For each* $x \in \Omega$ *and* $z_1, z_2 \in \mathbb{R}^n$ *we have*

$$H(x, z_1 - z_2) \leq \varepsilon H(x, z_1) + c(\varepsilon) \langle A(x, z_1) - A(x, z_2), z_1 - z_2 \rangle \tag{3.3.12}$$

for any $\varepsilon \in (0, 1)$ *with* $c(\varepsilon) = c(p, q, \varepsilon)$.

In [16, 18], they considered the case when there is no x dependence for A and H. However, we can follow the same argument for each fixed x and prove Lemma 3.3.5.

We now give the Lipschitz regularity estimate for a reference problem. With n, p, q from (1.2.2)–(1.2.4) and with a constant $\bar{a} \geq 0$, we denote

$$\bar{H}(z) = |z|^p + \bar{a}|z|^q \quad (z \in \mathbb{R}^n).$$

A given $C^1(\mathbb{R}^n \setminus \{0\})$ vector field $\bar{A} : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to satisfy

$$\begin{cases} |\bar{A}(z)||z| + |\partial_z \bar{A}(z)||z|^2 \leq \bar{L}\bar{H}(z) \\ \bar{\nu} \dfrac{\bar{H}(z)}{|z|^2} |\xi|^2 \leq \langle \partial_z \bar{A}(z)\xi, \xi \rangle \end{cases}$$

for $z \in \mathbb{R}^n \setminus \{0\}$, $\xi \in \mathbb{R}^n$, where $0 < \bar{\nu} \leq \bar{L}$. With $U \subset \Omega$ being an open set, let $v \in W^{1,\bar{H}}(U)$ be a weak solution of

$$\operatorname{div}\bar{A}(Dv) = 0 \quad \text{in } U. \tag{3.3.13}$$

We then state the Lipschitz estimate for v without proof. For the proof, we refer [169].

Lemma 3.3.6 ([79, 169]). *Let $v \in W^{1,\bar{H}}(U)$ is a weak solution of* (3.3.13), *then there holds $Dv \in L^\infty_{\mathrm{loc}}(U)$. Moreover, for any $B_{2r} \subset U$ we have the estimate*

$$\sup_{x\in B_r} \bar{H}(Dv(x)) \le c_1 \fint_{B_{2r}} \bar{H}(Dv(x))\,dx$$

for an appropriate constant $c_1 = c_1(n,p,q,\bar{\nu},\bar{L})$ which is independent of $\bar{a}$.

3.3.3 Comparison estimates and the proof of Theorem 3.3.2

We start to provide the comparison estimates. From now on, we always assume (1.2.2)–(1.2.4) and (3.3.3). Let $R \in (0,\frac{1}{2})$ and $\delta \in (0,\frac{1}{8})$, and fix $\Omega_0 \Subset \Omega$. We assume $B_{8r} = B_{8r}(y) \subset \Omega_0$ with $8r \le R$ and $B_R(y) \subset \Omega$. With a solution u under consideration to the problem (3.3.2), let $h \in W^{1,H}(B_{4r})$ be the weak solution of

$$\begin{cases} -\operatorname{div}A(x,Dh) = 0 & \text{in } B_{4r} \\ h \in u + W_0^{1,H}(B_{4r}). \end{cases} \tag{3.3.14}$$

Then we list here some estimates for h as follows. For proofs, see [13, 38, 84].

Lemma 3.3.7 (Comparison estimate). *Let $\lambda \ge 1$ be given. For any $\varepsilon \in (0,1)$, there exists a constant $\delta_0 = \delta_0(\texttt{data},\varepsilon) \in (0,1)$ such that if for $\delta \in [0,\delta_0]$,*

$$\fint_{B_{4r}} H(x,Du)\,dx \le \lambda \quad \textit{and} \quad \fint_{B_{4r}} H(x,F)\,dx \le \delta\lambda$$

hold, then we have

$$\fint_{B_{4r}} H(x, Dh)\, dx \le c \fint_{B_{4r}} H(x, Du)\, dx \le c\lambda \tag{3.3.15}$$

and

$$\fint_{B_{4r}} H(x, Du - Dh)\, dx \le \varepsilon\lambda \tag{3.3.16}$$

for some $c = c(n, p, q, \nu, L)$.

For later use, with σ_0 mentioned in Lemma 3.3.4, we have the following lemma:

Lemma 3.3.8 (Higher integrability for h)**.** *There exists a small constant* $\sigma_1 = \sigma_1(\boldsymbol{data}) \le \sigma_0$ *such that* $H(x, Dh) \in L^{1+\sigma_1}_{\mathrm{loc}}(B_{4r})$*. Moreover, for any* $B_{2\rho} \subset B_{4r}$ *and* $\sigma \in (0, \sigma_1]$ *there holds*

$$\left(\fint_{B_\rho} H(x, Dh)^{1+\sigma}\, dx \right)^{\frac{1}{1+\sigma}} \le c \fint_{B_{2\rho}} H(x, Dh)\, dx$$

for some $c = c(\boldsymbol{data})$.

Note that (3.3.15) is used for the proof of Lemma 3.3.8, when we keep track of the exact dependence of σ_1 and c.

We next move for further comparison estimates. Let $K \ge 4$ be a free parameter to be determined later, and define

$$a_0 = a_{0,B_{2r}} = \begin{cases} \inf\limits_{x \in B_{2r}} a(x) & \text{if } \inf\limits_{x \in B_{2r}} a(x) > K[a]_{0,\alpha} r^\alpha & (3.3.17) \\ 0 & \text{if } \inf\limits_{x \in B_{2r}} a(x) \le K[a]_{0,\alpha} r^\alpha. & (3.3.18) \end{cases}$$

We denote

$$H_0(t) = t^p + a_0 t^q \quad (t \ge 0) \quad \text{or} \quad H_0(z) = |z|^p + a_0|z|^q \quad (z \in \mathbb{R}^n)$$

and define $A_0 : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$A_0(x, z) = A(x, z) \frac{H_0(|z|)}{H(x, |z|)}.$$

Now we write

$$\bar{A}_0(z) = \fint_{B_{4r}} A_0(x,z)\,dx. \tag{3.3.19}$$

Then we have the following lemma.

Lemma 3.3.9. *Together with the assumption* (1.2.8), *we have*

$$\begin{cases} |\bar{A}_0(z)||z| + |\partial_z \bar{A}_0(z)||z|^2 \le \tilde{L} H_0(|z|) \\ \tilde{\nu} \dfrac{H_0(|z|)}{|z|^2}|\xi|^2 \le \left\langle \partial_z \bar{A}_0(z)\xi, \xi \right\rangle \end{cases} \tag{3.3.20}$$

for some $\tilde{\nu} = \tilde{\nu}(p,q,\nu,L)$ *and* $\tilde{L} = \tilde{L}(p,q,\nu,L)$ *with* $0 < \tilde{\nu} \le \tilde{L} < \infty$. *Also, it holds that*

$$|A(x,z) - \bar{A}_0(z)| \le L(a(x) - a_0)|z|^{q-1} + \frac{H_0(|z|)}{|z|}\theta(B_{4r})(x), \tag{3.3.21}$$

where $\theta(B_{4r})(x) := \theta(A; B_{4r})(x)$ *as in* (3.3.4).

Proof. Throughout the proof, $H(x,t)$ and $H_0(t)$ are understood as $H(x,t) = t^p + a(x)t^q$ and $H_0(t) = t^p + a_0 t^q$. First, we compute

$$\partial_z A_0(x,z) = (\partial_z A(x,z))\frac{H_0(|z|)}{H(x,|z|)} + \frac{H_0' H - H_0 H'}{H^2}(x,|z|)\frac{z}{|z|} \otimes A(x,z).$$

Here, we have

$$\begin{aligned} \left|\frac{H_0' H - H_0 H'}{H^2}(x,|z|)\right| &\le \left|\frac{(p-q)a(x)|z|^{p+q-1}}{H(x,|z|)^2}\right| \\ \le \left|\frac{(p-q)a(x)|z|^{p+q-1}}{a(x)|z|^{p+q}}\right| &\frac{H_0(|z|)}{H(x,|z|)} \le (q-p)\frac{H_0(|z|)}{|z|H(x,|z|)}, \end{aligned} \tag{3.3.22}$$

and so by (3.3.3), there holds

$$\begin{aligned}|\partial_z A_0(x,z)| &\leq |\partial_z A(x,z)|\frac{H_0(|z|)}{H(x,|z|)} + \left|\frac{H_0'H - H_0H'}{H^2}(x,|z|)\right||A(x,z)| \\ &\leq L\frac{H_0(|z|)}{|z|^2} + (q-p)H_0(|z|)\frac{|A(x,z)|}{|z|H(x,|z|)} \\ &\leq L\frac{H_0(|z|)}{|z|^2} + L(q-p)\frac{H_0(|z|)}{|z|^2} \\ &\leq L(1+q-p)\frac{H_0(|z|)}{|z|^2}.\end{aligned}$$

Moreover, we observe from (3.3.3) that

$$|A_0(x,z)| \leq |A(x,z)|\frac{H_0(|z|)}{H(x,|z|)} \leq L\frac{H_0(|z|)}{|z|}.$$

Hence we obtain

$$|A_0(x,z)||z| + |\partial_z A_0(x,z)||z|^2 \leq \tilde{L}H_0(|z|) \tag{3.3.23}$$

with $\tilde{L} = L(2+q-p)$. Now the conclusion $(3.3.20)_1$ follows from (3.3.19) and (3.3.23).

To show $(3.3.20)_2$, we first compute

$$\begin{aligned}\langle \partial_z A_0(x,z)\xi,\xi\rangle &= \left\langle \partial_z A(x,z)\frac{H_0(|z|)}{H(x,|z|)}\xi,\xi\right\rangle \\ &\quad + \left\langle \frac{H_0'H - H_0H'}{H^2}(x,|z|)\frac{z}{|z|}\otimes A(x,z)\xi,\xi\right\rangle \\ &=: I_1 + I_2.\end{aligned} \tag{3.3.24}$$

But then, by (3.3.3) we estimate

$$I_1 \geq \nu\frac{H_0(|z|)}{|z|^2}|\xi|^2 \tag{3.3.25}$$

and by (3.3.22) and (3.3.3), it follows that

$$\begin{aligned} I_2 &\geq -\left|\frac{H_0'H - H_0H'}{H^2}(x,|z|)\right| |A(x,z)||\xi|^2 \\ &\geq -(q-p)\frac{H_0(|z|)}{|z|H(x,|z|)}|A(x,z)||\xi|^2 \\ &\geq -L(q-p)\frac{H_0(|z|)}{|z|^2}|\xi|^2. \end{aligned}$$

Now applying (1.2.8), for $\tilde{\nu} := \nu - L(q-p) > 0$, we discover

$$\langle \partial_z A_0(x,z)\xi,\xi\rangle \geq (\nu - L(q-p))\frac{H_0(|z|)}{|z|}|\xi|^2 \geq \tilde{\nu}\frac{H_0(|z|)}{|z|}|\xi|^2. \tag{3.3.26}$$

Then the conclusion $(3.3.20)_2$ follows from (3.3.19) and (3.3.26).

Finally, to obtain (3.3.21), first observe that

$$\begin{aligned} &|A(x,z) - \bar{A}_0(z)| \\ &\quad\leq |A(x,z) - A_0(x,z)| + |A_0(x,z) - \bar{A}_0(z)| \\ &\quad= |A(x,z)|\left(\frac{(a(x)-a_0)|z|^q}{H(x,|z|)}\right) + \frac{H_0(|z|)}{|z|}\left(\frac{|z|(A_0(x,z) - \bar{A}_0(z))}{H_0(|z|)}\right) \\ &\quad\leq L(a(x)-a_0)|z|^{q-1} + \frac{H_0(|z|)}{|z|}\left(\frac{|z|(A_0(x,z) - \bar{A}_0(z))}{H_0(|z|)}\right). \end{aligned}$$

Here, by the definition of θ as in (3.3.5), we have

$$\left|\frac{|z|(A_0(x,z) - \bar{A}_0(z))}{H_0(|z|)}\right| = \left|\frac{|z|A(x,z)}{H(x,|z|)} - \fint_{B_{4r}} \frac{|z|A(y,z)}{H(y,|z|)}\,dy\right| \leq \theta(B_{4r})(x),$$

which gives (3.3.21). □

We next let $v \in W^{1,H_0}(B_{2r})$ be the weak solution of

$$\begin{cases} -\mathrm{div}\bar{A}_0(Dv) = 0 & \text{in } B_{2r} \\ v \in h + W_0^{1,H_0}(B_{2r}). \end{cases} \tag{3.3.27}$$

When (3.3.17) holds, the comparison estimate is proved similarly as in [79], but the case of (3.3.18) is delicate to handle. As mentioned in Chapter 1, we are going to avoid using the difference quotient method. Instead, we provide two propositions which enable us to deal with the comparison estimate for the case of (3.3.18). First one is the boundary higher integrability for v, with the boundary data h. Note that σ_1 is as in Lemma 3.3.8.

Proposition 3.3.10. *Let $v \in W^{1,H_0}(B_{2r})$ be the weak solution of (3.3.27) and $h \in W^{1,H}(B_{4r})$ be the weak solution of (3.3.14). If (3.3.18) holds, then $H(x, Dv) \in L^{1+\sigma_1}(B_{2r})$. Moreover, for any $\sigma \in [0, \sigma_1]$ we have*

$$\fint_{B_{2r}} H(x, Dv)^{1+\sigma}\, dx \leq c \fint_{B_{2r}} H(x, Dh)^{1+\sigma}\, dx + c \tag{3.3.28}$$

for some $c = c(\boldsymbol{data}, \operatorname{dist}(\Omega_0, \partial\Omega), \|H(x, F)\|_{L^\gamma(\Omega)}) \geq 1$ which is independent of r. Especially, if $\sigma = 0$, then we have

$$\fint_{B_{2r}} H(x, Dv)\, dx \leq c \fint_{B_{3r}} H(x, Dh)\, dx + c \tag{3.3.29}$$

$c = c(\boldsymbol{data}) \geq 1$.

Proof. The proof is similar to the one of [136, Lemma 4.15]. We first prove (3.3.28). Note from (3.3.18) that $H_0(t) = t^p$. Then one can show that for any $\mu(\cdot) \in \mathcal{A}_{1+\sigma}$,

$$\int_{B_r} |Dv|^{p(1+\sigma)} \mu(x)\, dx \leq c([\mu]_{\mathcal{A}_{1+\sigma}}) \int_{B_r} |Dh|^{p(1+\sigma)} \mu(x)\, dx, \tag{3.3.30}$$

where $c([\mu]_{\mathcal{A}_{1+\sigma}}) = c(\mathtt{data}, [\mu]_{\mathcal{A}_{1+\sigma}}) > 0$. For the proof, see [136] together with [48, 79]. Now for each $j = 1, 2, \ldots,$ define

$$\Phi_j(x, t) := \min\left\{ (t + a(x) t^{\frac{q}{p}})^{1+\sigma}, j t^{1+\sigma} \right\}.$$

Then for each Φ_j, one can assert the conditions (A0), (A1), $(\text{aInc})_{1+\sigma}$ and $(\text{aDec})_{\frac{q}{p}(1+\sigma)}$ in Section 3.3.2 for a universal constant $M = M(n, \|a\|_{0,\alpha})$ which is independent of j and σ. We only show (A1) condition, especially (3.3.9). Indeed, by (1.2.4), one can see that $r^\alpha r^{-n\left(\frac{q-p}{p}\right)} \leq 1$ and so for any $t \in [0, |B_r|^{-1}]$, we have

$$r^\alpha t^{\frac{q-p}{p}} \leq c(n).$$

Then there holds

$$r^{\alpha}t^{\frac{q}{p}} \le c(n)t. \tag{3.3.31}$$

For $a_{B_r}^+ := \sup_{x\in B_r} a(x)$ and $a_{B_r}^- := \inf_{x\in B_r} a(x)$, (3.3.31) implies

$$(r^{\alpha}t^{\frac{q}{p}})^{1+\sigma} \le c(n)(t + a_{B_r}^- t^{\frac{q}{p}})^{1+\sigma}. \tag{3.3.32}$$

Thus denoting by

$$\tilde{H}(x,t) := (t + a(x)t^{\frac{q}{p}})^{1+\sigma},$$

$$\tilde{H}_{B_r}^+(t) := \sup_{x\in B_r} \tilde{H}(x,t) = \left(t + a_{B_r}^+ t^{\frac{q}{p}}\right)^{1+\sigma}$$

and

$$\tilde{H}_{B_r}^-(t) := \inf_{x\in B_r} \tilde{H}(x,t) = \left(t + a_{B_r}^- t^{\frac{q}{p}}\right)^{1+\sigma},$$

it follows from (1.2.3) and (3.3.32) that

$$\begin{aligned}
\tilde{H}_{B_r}^+(t) - \tilde{H}_{B_r}^-(t) &= \left(t + a_{B_r}^+ t^{\frac{q}{p}}\right)^{1+\sigma} - \left(t + a_{B_r}^- t^{\frac{q}{p}}\right)^{1+\sigma} \\
&\le c\left[\left(t + a_{B_r}^+ t^{\frac{q}{p}}\right) - \left(t + a_{B_r}^- t^{\frac{q}{p}}\right)\right]^{1+\sigma} + c\left(t + a_{B_r}^- t^{\frac{q}{p}}\right)^{1+\sigma} \\
&\le c(r^{\alpha}t^{\frac{q}{p}})^{1+\sigma} + c\left(t + a_{B_r}^- t^{\frac{q}{p}}\right)^{1+\sigma} \\
&\le c(n)\left(t + a_{B_r}^- t^{\frac{q}{p}}\right)^{1+\sigma} = c(n)\tilde{H}_{B_r}^-(t).
\end{aligned}$$

Therefore, for any $x \in B_r$ and $t \in [0, |B_r|^{-1}]$, we have

$$\tilde{H}_{B_r}^+(t) \eqsim \tilde{H}_{B_r}^-(t) \eqsim (t + a(x)t^{\frac{q}{p}})^{1+\sigma} \tag{3.3.33}$$

with an implicit constant depending only on n. Let us denote

$$\Phi_{j,B_r}^+(t) = \sup_{x\in B_r} \min\left\{(t + a(x)t^{\frac{q}{p}})^{1+\sigma}, jt^{1+\sigma}\right\}$$

and

$$\Phi^-_{j,B_r}(t) = \inf_{x\in B_r} \min\left\{(t+a(x)t^{\frac{q}{p}})^{1+\sigma}, jt^{1+\sigma}\right\}.$$

Since $0 \le t^{1+\sigma} \le \Phi^-_{j,B_r}(t)$, $\Phi^-_{j,B_r}(t) \in [1, |B_r|^{-1}]$ implies $t \in [0, |B_r|^{-1}]$. Then by (3.3.33), for all $x \in B_r$ and $\Phi^-_{j,B_r}(t) \in [1, |B_r|^{-1}]$, there holds

$$\Phi_j(x,t) \eqsim \Phi^+_{j,B_r}(t) \eqsim \Phi^-_{j,B_r}(t)$$

with an implicit constant depending only on n. Then (A1) is proved for $\Phi_j(x,t)$ for each $j = 1, 2, \dots$, with $M = M(n)$ which is independent of j and σ. Now we apply the extrapolation with $\Phi_j(x,t)$, take $j \to \infty$, and apply the argument of the proof of [136, Lemma 4.15] to find that

$$\begin{aligned} &\fint_{B_{2r}} H(x,Dv)^{1+\sigma}\,dx \\ &\quad \le c\left[\left(\int_{B_{2r}} H(x,Dh)^{1+\sigma}\,dx\right)^{\frac{q}{p}-1} + 1\right] \\ &\qquad \times \left(\fint_{B_{2r}} H(x,Dh)^{1+\sigma}\,dx + 1\right) \end{aligned} \tag{3.3.34}$$

for some $c = c(\mathtt{data}) \ge 1$. But then, we use Lemma 3.3.8, the energy estimate (3.3.15), Hölder's inequality, and (3.3.11), to discover

$$\begin{aligned} \int_{B_{2r}} H(x,Dh)^{1+\sigma}\,dx &\le cr^n \fint_{B_{2r}} H(x,Dh)^{1+\sigma}\,dx \\ &\le cr^n \left(\fint_{B_{4r}} H(x,Dh)\,dx + 1\right)^{1+\sigma} \\ &\le cr^n \left(\fint_{B_{4r}} H(x,Du)\,dx + 1\right)^{1+\sigma} \\ &\le cr^n \fint_{B_{4r}} H(x,Du)^{1+\sigma}\,dx + 1 \\ &\le c(\mathtt{data}, \operatorname{dist}(\Omega_0, \partial\Omega), \|H(x,F)\|_{L^\gamma(\Omega)}), \end{aligned}$$

and so we obtain the estimate (3.3.28).

Now it remains to show (3.3.29). By Hölder's inequality, (3.3.34) with

$\sigma = \zeta = \min\{-\frac{1}{\log r}, \sigma_1\}$, and Lemma 3.3.8, there holds

$$\begin{aligned}
\fint_{B_{2r}} H(x, Dv)\, dx &\le \left(\fint_{B_{2r}} H(x, Dv)^{1+\zeta}\, dx \right)^{\frac{1}{1+\zeta}} \\
&\le c \left[\left(\int_{B_{2r}} H(x, Dh)^{1+\zeta}\, dx \right)^{\frac{q}{p}-1} + 1 \right]^{\frac{1}{1+\zeta}} \\
&\quad \times \left(\fint_{B_{2r}} H(x, Dh)^{1+\zeta}\, dx \right)^{\frac{1}{1+\zeta}} \\
&\le c \left[\left(\int_{B_{2r}} H(x, Dh)^{1+\zeta}\, dx \right)^{\left(\frac{q}{p}-1\right)\frac{1}{1+\zeta}} + 1 \right] \\
&\quad \times \left(\fint_{B_{3r}} H(x, Dh)\, dx + 1 \right).
\end{aligned} \tag{3.3.35}$$

Here, by Lemma 3.3.8 and the energy estimate (3.3.15), it follows that

$$\begin{aligned}
\left(\int_{B_{2r}} H(x, Dh)^{1+\zeta}\, dx \right)^{\frac{1}{1+\zeta}} &\le c r^{\frac{n}{1+\zeta}} \left(\fint_{B_{2r}} H(x, Dh)^{1+\zeta}\, dx \right)^{\frac{1}{1+\zeta}} \\
&\le c r^{\frac{n}{1+\zeta}} \left(\fint_{B_{4r}} H(x, Dh)\, dx + 1 \right) \\
&= c r^{\frac{n}{1+\zeta}} r^{-n} \left(\int_{B_{4r}} H(x, Dh)\, dx + 1 \right) \\
&\le c(\mathtt{data}) r^{-\frac{\zeta n}{1+\zeta}}
\end{aligned} \tag{3.3.36}$$

and

$$r^{-\frac{\zeta n}{1+\zeta}} \le r^{\frac{n}{\log r}} = e^{\log r \frac{n}{\log r}} = e^n \le c. \tag{3.3.37}$$

Therefore, combining (3.3.35)–(3.3.37), we have the conclusion (3.3.29) with $c = c(\mathtt{data}) \ge 1$. $\square$

We now handle the case of (3.3.18) in the following proposition.

Proposition 3.3.11. *If* (3.3.18) *holds, then there exists* $\sigma_2 = \sigma_2(\mathtt{data}) > 0$

such that

$$\fint_{B_{2r}} a(x)|Dh|^q\,dx \le cKr^{\sigma_2}\left(\fint_{B_{4r}} H(x,Dh)\,dx+1\right) \tag{3.3.38}$$

and

$$\fint_{B_{2r}} a(x)|Dv|^q\,dx \le cKr^{\sigma_2}\left(\fint_{B_{4r}} H(x,Dh)\,dx+1\right) \tag{3.3.39}$$

for some positive constant $c = c(\texttt{data}, \operatorname{dist}(\Omega_0,\partial\Omega), \|H(x,F)\|_{L^\gamma(\Omega)})$.

Proof. First we claim the following type of higher integrability: for

$$\tilde q := \left(1+\frac{n\sigma_1}{n+\alpha}\right)q,$$

we have

$$\fint_{B_{2r}} a(x)|Dh|^{\tilde q}\,dx \le c\left(\fint_{B_{3r}} H(x,Dh)\,dx\right)^{1+\sigma_1}+c \tag{3.3.40}$$

for some $c = c(\texttt{data})$. Indeed, there holds $(\tilde q - q)\frac{1+\sigma_1}{\sigma_1} \le p(1+\sigma_1)$ by (1.2.4), and so by Hölder's inequality and Lemma 3.3.8, we have

$$\begin{aligned}
&\fint_{B_{2r}} a(x)|Dh|^{\tilde q}\,dx = \fint_{B_{2r}} a(x)|Dh|^q|Dh|^{\tilde q-q}\,dx\\
&\le \left(\fint_{B_{2r}} (a(x)|Dh|)^{q(1+\sigma_1)}\,dx\right)^{\frac{1}{1+\sigma_1}}\left(\fint_{B_{2r}} |Dh|^{(\tilde q-q)\frac{1+\sigma_1}{\sigma_1}}\,dx\right)^{\frac{\sigma_1}{1+\sigma_1}}\\
&\le \left(\fint_{B_{2r}} (a(x)|Dh|)^{q(1+\sigma_1)}\,dx\right)^{\frac{1}{1+\sigma_1}}\left(\fint_{B_{2r}} |Dh|^{p(1+\sigma_1)}\,dx+1\right)^{\frac{\sigma_1}{1+\sigma_1}} \qquad (3.3.41)\\
&\le \fint_{B_{2r}} H(x,Dh)^{1+\sigma_1}\,dx+1\\
&\le c\left(\fint_{B_{3r}} H(x,Dh)\,dx\right)^{1+\sigma_1}+c.
\end{aligned}$$

Thus the claim is proved.

Now, for $\tilde{\sigma} \in (0,1]$ being such that $\frac{q}{1-\tilde{\sigma}} = \tilde{q}$, i.e., with

$$\tilde{\sigma} = \frac{n\sigma_1}{\alpha + n(1+\sigma_1)}, \tag{3.3.42}$$

we will show

$$r^{\alpha\tilde{\sigma}} \fint_{B_{2r}} a(x)^{1-\tilde{\sigma}} |Dh|^q \, dx \leq c r^{\sigma_2} \fint_{B_{4r}} H(x, Dh) \, dx \tag{3.3.43}$$

for some $\sigma_2 = \sigma_2(\mathtt{data})$ and $c = c(\mathtt{data})$. Indeed, by Hölder's inequality and (3.3.40) we have

$$\begin{aligned}
&\fint_{B_{2r}} a(x)^{1-\tilde{\sigma}} |Dh|^q \, dx \\
&\quad = \fint_{B_{2r}} \left(a(x) |Dh|^{\frac{q}{1-\tilde{\sigma}}} \right)^{1-\tilde{\sigma}} \, dx \\
&\quad \leq \left(\fint_{B_{2r}} a(x) |Dh|^{\frac{q}{1-\tilde{\sigma}}} \, dx \right)^{1-\tilde{\sigma}} \\
&\quad \leq c \left(\fint_{B_{3r}} H(x, Dh) \, dx \right)^{(1-\tilde{\sigma})(1+\sigma_1)} + c \\
&\quad \leq c \left(\fint_{B_{4r}} H(x, Dh) \, dx \right)^{(1-\tilde{\sigma})(1+\sigma_1)-1} \left(\fint_{B_{3r}} H(x, Dh) \, dx \right) + c.
\end{aligned} \tag{3.3.44}$$

But then, since

$$\begin{aligned}
(1-\tilde{\sigma})(1+\sigma_1) - 1 &= \left(\frac{n+\alpha}{\alpha + n(1+\sigma_1)} \right) (1+\sigma_1) - 1 \\
&= \frac{\alpha + \alpha\sigma_1 + n(1+\sigma_1) - \alpha - n(1+\sigma_1)}{\alpha + n(1+\sigma_1)} \\
&= \frac{\alpha\sigma_1}{\alpha + n(1+\sigma_1)} > 0,
\end{aligned}$$

we see by the energy estimate (3.3.15), Hölder's inequality and (3.3.11) that

$$\begin{aligned}
&\left(\fint_{B_{4r}} H(x,Dh)\,dx\right)^{(1-\tilde{\sigma})(1+\sigma_1)-1} \\
&\qquad\le \left(\fint_{B_{4r}} H(x,Du)\,dx\right)^{(1-\tilde{\sigma})(1+\sigma_1)-1} \\
&\qquad\le c\left(\fint_{B_{4r}} H\left(x,Du\right)^{1+\sigma_1}\,dx\right)^{(1-\tilde{\sigma})-\frac{1}{1+\sigma_1}} \\
&\qquad\le cr^{-n\left[(1-\tilde{\sigma})-\frac{1}{1+\sigma_1}\right]}\left(\int_{B_{4r}} H\left(x,Du\right)^{1+\sigma_1}\,dx\right)^{(1-\tilde{\sigma})-\frac{1}{1+\sigma_1}} \\
&\qquad\le cr^{-n\left[(1-\tilde{\sigma})-\frac{1}{1+\sigma_1}\right]}
\end{aligned} \tag{3.3.45}$$

for some positive constant $c = c(\mathtt{data}, \operatorname{dist}(\Omega_0,\partial\Omega), \|H(x,F)\|_{L^\gamma(\Omega)})$.

Now we observe (3.3.42) from that

$$\begin{aligned}
\sigma_2 &:= \alpha\tilde{\sigma} - n\left(1-\tilde{\sigma}-\frac{1}{1+\sigma_1}\right) \\
&> \alpha\tilde{\sigma} - n\left[(1-\tilde{\sigma})(1+\sigma_1)-1\right] \\
&= \alpha\tilde{\sigma} - n\left(-\tilde{\sigma}+\sigma_1-\tilde{\sigma}\sigma_1\right) \\
&= \left[\alpha+n(1+\sigma_1)\right]\tilde{\sigma} - n\sigma_1 = 0.
\end{aligned}$$

Hence combining (3.3.44) and (3.3.45), we find

$$\begin{aligned}
&r^{\alpha\tilde{\sigma}}\fint_{B_{2r}} a(x)^{1-\tilde{\sigma}}|Dh|^q\,dx \\
&\quad\le cr^{\alpha\tilde{\sigma}}\left[\left(\fint_{B_{4r}} H(x,Dh)\,dx\right)^{(1-\tilde{\sigma})(1+\sigma_1)-1}\left(\fint_{B_{3r}} H(x,Dh)\,dx\right)+c\right] \\
&\quad\le cr^{\sigma_2}\left(\fint_{B_{3r}} H(x,Dh)\,dx+c\right)
\end{aligned}$$

for some $c = c(\mathtt{data}, \operatorname{dist}(\Omega_0,\partial\Omega), \|H(x,F)\|_{L^\gamma(\Omega)})$, which is (3.3.43). Now considering

$$\begin{aligned} a(x) &\le \sup_{x\in B_r} a(x) \\ &\le c[a]_{0,\alpha}r^\alpha + \inf_{x\in B_r} a(x) \le (c+K)[a]_{0,\alpha}r^\alpha \le cK[a]_{0,\alpha}r^\alpha, \end{aligned} \tag{3.3.46}$$

we have (3.3.38).

As for (3.3.39), like the estimates in (3.3.41) together with using (3.3.28), we have

$$\begin{aligned} &\fint_{B_{2r}} a(x)|Dv|^{\tilde q}\,dx \\ &\le \left(\fint_{B_{2r}} (a(x)|Dv|)^{q(1+\sigma_1)}\,dx\right)^{\frac{1}{1+\sigma_1}} \left(\fint_{B_{2r}} |Dv|^{(\tilde q-q)\frac{1+\sigma_1}{\sigma_1}}\,dx\right)^{\frac{\sigma_1}{1+\sigma_1}} \\ &\le \left(\fint_{B_{2r}} (a(x)|Dv|)^{q(1+\sigma_1)}\,dx\right)^{\frac{1}{1+\sigma_1}} \left(\fint_{B_{2r}} |Dv|^{p(1+\sigma_1)}\,dx+1\right)^{\frac{\sigma_1}{1+\sigma_1}} \\ &\le \fint_{B_{2r}} H(x,Dv)^{1+\sigma_1}\,dx+1 \\ &\underset{(3.3.28)}{\le} c\fint_{B_{2r}} H(x,Dh)^{1+\sigma_1}\,dx+c \\ &\le c\left(\fint_{B_{3r}} H(x,Dh)\,dx\right)^{1+\sigma_1}+c. \end{aligned} \tag{3.3.47}$$

Thus by Hölder's inequality, we see that

$$\begin{aligned} \fint_{B_{2r}} a(x)^{1-\tilde\sigma}|Dv|^q\,dx &= \fint_{B_{2r}} \left(a(x)|Dv|^{\frac{q}{1-\tilde\sigma}}\right)^{1-\tilde\sigma}\,dx \\ &\le \left(\fint_{B_{2r}} a(x)|Dv|^{\frac{q}{1-\tilde\sigma}}\,dx\right)^{1-\tilde\sigma}+c \\ &\underset{(3.3.47)}{\le} c\left(\fint_{B_{3r}} H(x,Dh)\,dx\right)^{(1-\tilde\sigma)(1+\sigma_1)}+c. \end{aligned}$$

We then apply the same argument as in (3.3.45)–(3.3.46) to derive (3.3.39). □

We also need the following higher integrability results for the problem

(3.3.27). Note that σ_1 is defined in Lemma 3.3.8. The proof is the same as in that of Lemma 3.3.8.

Lemma 3.3.12. *There exists a constant $\sigma_2 = \sigma_2(\boldsymbol{data}) \leq \sigma_1$ such that $H(x, Dv) \in L^{1+\sigma_2}_{\mathrm{loc}}(B_{2r})$. Moreover, for any $B_{2\rho} \subset B_{2r}$ and $\sigma \in (0, \sigma_2]$ there holds*

$$\left(\fint_{B_\rho} H(x, Dv)^{1+\sigma}\, dx\right)^{\frac{1}{1+\sigma}} \leq c \fint_{B_{2\rho}} H(x, Dv)\, dx + c \tag{3.3.48}$$

for some $c = c(\boldsymbol{data})$.

Now we provide the comparison estimates for w and h defined in (3.3.27) and (3.3.14), respectively. In case of (3.3.18), we additionally apply Proposition 3.3.10 and Proposition 3.3.11 for our proof. Recall that δ_0 is defined in Lemma 3.3.7 and σ_2 is in Proposition 3.3.11.

Lemma 3.3.13. *Under the assumptions and conclusions of Lemma 3.3.7, we further assume* (1.2.8)*. Then there exists a constant $\delta_1 = \delta_1(\boldsymbol{data}, \varepsilon) \leq \delta_0$ such that if A is (δ, R)-vanishing for some $R \in (0, 1)$ and $\delta \in [0, \delta_1]$, then we have*

$$\fint_{B_{2r}} H(x, Dv)\, dx \leq c \fint_{B_{3r}} H(x, Dh)\, dx \leq c\lambda \tag{3.3.49}$$

for some constant $c = c(\boldsymbol{data})$, and

$$\fint_{B_{2r}} H(x, Dh - Dv)\, dx \leq \left[\varepsilon + c(\varepsilon)\left(\frac{1}{K} + Kr^{\sigma_2}\right)\right]\lambda \tag{3.3.50}$$

with any $K \geq 4$, where $c(\varepsilon) = c(\boldsymbol{data}, \mathrm{dist}(\Omega_0, \partial\Omega), \|H(x, F)\|_{L^\gamma(\Omega)}, \varepsilon)$.

Proof. We first prove (3.3.49). To this end, by testing $v - h \in W^{1,H_0}_0(B_{2r})$ to (3.3.27) and applying the same argument for obtaining (3.3.15), we have

$$\fint_{B_{2r}} H_0(Dv)\, dx \leq c \fint_{B_{2r}} H_0(Dh)\, dx. \tag{3.3.51}$$

We first consider the case of (3.3.17). Since $K \geq 4$, one can observe that for

$x \in B_{2r}$,

$$\begin{aligned} a(x) &\le \sup_{x \in B_{2r}} a(x) \\ &\le a_0 + \operatorname*{osc}_{x \in B_{2r}} a(x) \\ &\le a_0 + 4[a]_{0,\alpha} r^{\alpha} \le \left(\frac{4}{K} + 1\right) a_0 \le 2a_0 \le 2a(x). \end{aligned} \tag{3.3.52}$$

Then together with (3.3.15), it follows that

$$\fint_{B_{2r}} H(x, Dv)\, dx \le 2 \fint_{B_{2r}} H(x, Dh)\, dx \le c\lambda.$$

If (3.3.18) holds, then by (3.3.29) in Proposition 3.3.10,

$$\fint_{B_{2r}} H(x, Dv)\, dx \le c \fint_{B_{3r}} H(x, Dh)\, dx$$

holds for $c = c(\mathtt{data})$. Then by (3.3.15), we have (3.3.49).

Next we will show (3.3.50). First, note that if (3.3.18) holds, $v \in W^{1,H}(B_{2r})$ by Proposition 3.3.10. If (3.3.17) holds, again $v \in W^{1,H}(B_{2r})$ by (3.3.52). Then testing $h - v \in W^{1,H}(B_{2r}) \subset W^{1,H_0}(B_{2r})$ to both (3.3.14) and (3.3.27), we have

$$\begin{aligned} I_1 &= \fint_{B_{2r}} \langle A(x, Dh) - A(x, Dv), Dh - Dv \rangle \, dx \\ &= \fint_{B_{2r}} \left\langle \bar{A}_0(Dv) - A(x, Dv), Dh - Dv \right\rangle \, dx = I_2. \end{aligned} \tag{3.3.53}$$

For I_1, applying (3.3.12) and (3.3.49), there holds

$$\fint_{B_{2r}} H(x, Dh - Dv)\, dx \le c(\varepsilon_0) I_1 + \varepsilon_0 \lambda$$

for any $\varepsilon_0 \in (0,1)$. For I_2, by (3.3.21) and Young's inequality, we estimate

$$\begin{aligned} I_2 &\le c \fint_{B_{2r}} \left[(a(x)-a_0)|Dv|^{q-1} + \frac{H_0(Dv)}{|Dv|}\theta(B_{4r})(x) \right] |Dh - Dv|\, dx \\ &\le c \fint_{B_{2r}} (a(x)-a_0)(|Dv|^q + |Dh|^q)\, dx \\ &\quad + c \fint_{B_{2r}} \frac{H_0(Dv)}{|Dv|}\theta(B_{4r})(x)|Dh-Dv|\, dx \\ &=: I_3 + I_4. \end{aligned}$$

For I_3, if (3.3.17) holds, by (3.3.52) we have

$$a(x) - a_0 \le 4[a]_{0,\alpha} r^{\alpha} \le \frac{4a(x)}{K} \underset{(3.3.52)}{\le} \frac{8a_0}{K} \tag{3.3.54}$$

for any $x \in B_{2r}$. Then applying (3.3.51), it follows that

$$I_3 \le \frac{8c}{K} \fint_{B_{2r}} a_0(|Dv|^q + |Dh|^q)\, dx \le \frac{8c}{K} \fint_{B_{2r}} (H_0(Dv) + H_0(Dh))\, dx \le \frac{c}{K}\lambda$$

for some $c = c(\texttt{data})$. On the other hand, if (3.3.18) holds, we apply (3.3.28) in Proposition 3.3.11 to I_3, to see that

$$\begin{aligned} I_3 &= c \fint_{B_{2r}} a(x)(|Dv|^q + |Dh|^q)\, dx \\ &\le cKr^{\sigma_2} \left(\fint_{B_{4r}} H(x, Dh)\, dx + 1 \right) \le cKr^{\sigma_2}\lambda. \end{aligned}$$

Here, the constant c depends on $c = c(\texttt{data}, \operatorname{dist}(\Omega_0, \partial\Omega), \|H(x,F)\|_{L^{\gamma}(\Omega)})$. Thus in any case, we have

$$I_3 \le \left(\frac{c}{K} + cKr^{\sigma_2} \right)\lambda.$$

To estimate I_4, we have

$$I_4 \le \varepsilon_1 \fint_{B_{2r}} H_0(Dh - Dv)\, dx + c(\varepsilon_1) \fint_{B_{2r}} \theta(B_{4r}) H_0(Dv)\, dx =: I_5 + I_6$$

for all $\varepsilon_1 \in (0,1)$ with some $c(\varepsilon_1) = c(p,q,\varepsilon_1)$, where we have used Young's inequality and the fact that

$$\theta(B_{4r}) \geq \max\left\{\theta(B_{4r})^{\frac{p}{p-1}}, \theta(B_{4r})^{\frac{q}{q-1}}\right\}.$$

For I_6, first note from (3.3.4) that $\theta(B_{4r}) \leq 2L$. Then by Hölder's inequality and (3.3.48), we have

$$\begin{aligned} I_6 &\leq c(\varepsilon_1)\left(\fint_{B_{2r}} \theta_0(B_{4r})^{\frac{1+\sigma_2}{\sigma_2}}\,dx\right)^{\frac{\sigma_2}{1+\sigma_2}}\left(\fint_{B_{2r}} H_0(Dv)^{1+\sigma_2}\,dx\right)^{\frac{1}{1+\sigma_2}} \\ &\leq c(\varepsilon_1)L^{\frac{1}{1+\sigma_2}}\left(\fint_{B_{2r}} \theta(B_{4r})\,dx\right)^{\frac{\sigma_2}{1+\sigma_2}}\left(\fint_{B_{3r}} (H_0(Dv)+1)\,dx\right) \\ &\leq c(\varepsilon_1)\delta^{\frac{\sigma_2}{1+\sigma_2}}\left(\fint_{B_{3r}} (H_0(Dv)+1)\,dx\right), \end{aligned} \tag{3.3.55}$$

provided A is (δ,R)-vanishing for some $R \in (0,1)$ and $\delta > 0$.

Combining all the estimates (3.3.53)–(3.3.55), we have

$$\begin{aligned} \fint_{B_{2r}} H(x, Dh - Dv)\,dx \leq{}& \left[\varepsilon_0 + \tilde{c}(\varepsilon_0)\left(\frac{1}{K} + Kr^{\sigma_2}\right)\right]\lambda \\ &+ c(\varepsilon_0)\left(\varepsilon_1 \fint_{B_{2r}} H_0(Dh-Dv)\,dx + c(\varepsilon_1)\delta^{\frac{\sigma_2}{1+\sigma_2}}\lambda\right), \end{aligned}$$

where $\tilde{c}(\varepsilon_0) = \tilde{c}(\mathtt{data}, \operatorname{dist}(\Omega_0,\partial\Omega), \|H(x,F)\|_{L^\gamma(\Omega)}, \varepsilon_0)$, $c(\varepsilon_0) = c(p,q,\varepsilon_0)$ and $c(\varepsilon_1) = c(p,q,\varepsilon_1)$. We now take $\varepsilon_0 = \frac{\varepsilon}{4}$, $\varepsilon_1 \leq \frac{1}{2c(\varepsilon_0)}$ and $\delta_1 = \delta_1(\mathtt{data},\varepsilon)$ sufficiently small so that $c(\varepsilon_0)c(\varepsilon_1)\delta_1^{\frac{\sigma_2}{1+\sigma_2}} \leq \frac{\varepsilon}{4}$. Then for $\delta \leq \delta_1$, we discover

$$\fint_{B_{2r}} H(x, Dh - Dv)\,dx \leq \left[\varepsilon + \tilde{c}(\varepsilon)\left(\frac{1}{K} + Kr^{\sigma_2}\right)\right]\lambda,$$

which is (3.3.50). □

We combine all the comparison estimates made in Lemma 3.3.7, Lemma 3.3.13 and Lemma 3.3.6, to derive the following key lemma. Recall that δ_1 is in Lemma 3.3.13 and σ_2 is in Proposition 3.3.11.

Lemma 3.3.14. *Assume* (1.2.8) *and let* $\lambda \geq 1$ *be given. Then for any* $\varepsilon \in$

$(0,1)$, there exists a small positive constant $\delta_1 = \delta_1(\boldsymbol{data}, \varepsilon)$ such that if

$$\fint_{B_{4r}} H(x, Du)\, dx \le \lambda, \quad \fint_{B_{4r}} H(x, F)\, dx \le \delta\lambda$$

and A is (δ, R)-vanishing for some $R \in (0,1)$ and $\delta \in [0, \delta_1]$, then for any $K \ge 4$ we have

$$\fint_{B_r} H(x, Du - Dv)\, dx \le 3\left[\varepsilon + c(\varepsilon)\left(\frac{1}{K} + K r^{\sigma_2}\right)\right]\lambda \tag{3.3.56}$$

with some $\sigma_2 = \sigma_2(\boldsymbol{data}) > 0$ and
$c(\varepsilon) = c(\boldsymbol{data}, \operatorname{dist}(\Omega_0, \partial\Omega), \|H(x,F)\|_{L^\gamma(\Omega)}, \varepsilon) > 0$. Also, there holds

$$\sup_{x \in B_r} H(x, Dv(x)) \le c\lambda \tag{3.3.57}$$

for some $c = c(\boldsymbol{data})$.

Proof. Note that (3.3.56) follows from (3.3.16), (3.3.50) and triangle inequality. To show (3.3.57), we fix $K = 10$ and divide the proof into two cases as (3.3.17) and (3.3.18). If (3.3.17) holds, then it follows from (3.3.52), Lemma 3.3.6 and (3.3.49) that

$$\sup_{x \in B_r} H(x, Dv(x)) \le 2 \sup_{x \in B_r} H_0(Dv(x)) \le c \fint_{B_{2r}} H_0(Dv(x))\, dx \le c\lambda.$$

If (3.3.18) holds, then $H_0(t) = t^p$, and then by Lemma 3.3.6 with $\bar{H}(t) = t^p$, we have

$$\sup_{x \in B_r} |Dv(x)|^p \le c \fint_{B_{2r}} |Dv(x)|^p\, dx. \tag{3.3.58}$$

Now using (3.3.18), (3.3.58), (1.2.4), (3.3.49) and (3.3.15) in order, we find

$$
\begin{aligned}
\sup_{x\in B_r}(a(x)|Dv(x)|^q) &\le cr^\alpha(\sup_{x\in B_r}|Dv(x)|^p)^{\frac{q}{p}}\\
&\le cr^\alpha\left(\fint_{B_{2r}}|Dv(x)|^p\,dx\right)^{\frac{q}{p}}\\
&\le c\left(\int_{B_{2r}}|Dv(x)|^p\,dx\right)^{\frac{q-p}{p}}\left(\fint_{B_{2r}}|Dv(x)|^p\,dx\right)\\
&\le c\left(\int_{B_{4r}}H(x,Du)\,dx\right)^{\frac{q-p}{p}}\left(\fint_{B_{2r}}|Dv(x)|^p\,dx\right)\\
&\le c(\|H(x,Du)\|_{L^1(\Omega)})\lambda,
\end{aligned}
$$

which implies (3.3.57). □

Now we are all set in position to give the proof of our main result.

Proof of Theorem 3.3.2. The proof is based on [38, 79]. Fix $\Omega_0 \Subset \Omega$. For a chosen $\gamma \in (1,\infty)$, let $H(x,F) \in L^\gamma(\Omega)$. Then we have $H(x,F) \in L^\gamma(B_{2r})$ for $B_{8r} \subset \Omega_0$ with $8r \le R$ and $B_R(y) \subset \Omega$. Select two radii $r \le r_1 < r_2 \le 2r$ and define

$$
\lambda_0 := \frac{20^n r_2^n}{(r_2-r_1)^n}\fint_{B_{r_2}}\left(H(x,Du)+\frac{H(x,F)}{\delta}\right)dx \tag{3.3.59}
$$

for $\delta > 0$ to be determined later. We write

$$
E(s,\lambda) := \{x\in B_s : H(x,Du(x)) > \lambda\} \quad \text{for} \quad \lambda > \lambda_0+1 \text{ and } r\le s\le 2r,
$$

and define

$$
\Psi_y(\rho) = \fint_{B_\rho(y)}\left(H(x,Du)+\frac{H(x,F)}{\delta}\right)dx \quad \text{for} \quad B_\rho(y)\subset B_r.
$$

Then Lebesgue differentiation theorem says that for a.e. $y \in E(s,\lambda)$, it holds that

$$
\lim_{\rho\to 0}\Psi_y(\rho) = H(y,Du(y)) + \frac{H(y,F(y))}{\delta} > \lambda. \tag{3.3.60}
$$

On the other hand, for any $y \in B_{r_1}$ and $\rho \in \left[\frac{r_2-r_1}{20}, r_2 - r_1\right]$, we observe

$$\Psi_y(\rho) \le \frac{20^n r_2^n}{(r_2-r_1)^n} \fint_{B_{r_2}} \left(H(x, Du) + \frac{H(x,F)}{\delta} \right) dx = \lambda_0 < \lambda. \tag{3.3.61}$$

Then (3.3.60) and (3.3.61) imply that for a.e. $y \in E(r_1, \lambda)$, there exists a radius $\rho_y \in \left(0, \frac{r_2-r_1}{20}\right)$ such that

$$\Psi_y(\rho_y) = \lambda \quad \text{and} \quad \Psi_y(\rho) < \lambda \quad \text{for any } \rho \in (\rho_y, r_2 - r_1].$$

Hence, Vitali covering lemma provides us with a countable family of mutually disjoint balls $\{B_{\rho_i}(y_i)\}_{i=1}^{\infty}$ with $y_i \in E(r_1, \lambda)$ and $\rho_i \in \left(0, \frac{r_2-r_1}{20}\right)$ such that

$$E(r_1, \lambda) \subset \bigcup_{i=1}^{\infty} B_{5\rho_i}(y_i) \cup N \quad (N : \text{a measure zero set}),$$

$$\Psi_{y_i}(\rho_i) = \lambda \tag{3.3.62}$$

and

$$\Psi_{y_i}(\rho) < \lambda \quad \text{for each} \quad \rho \in (\rho_i, r_2 - r_1].$$

Then we are under the setting of Lemma 3.3.14, which implies that for any $\varepsilon \in (0,1)$, there exists a constant $\delta_1 = \delta_1(\mathtt{data}, \varepsilon) > 0$ such that if

$$\fint_{B_{20\rho_i}(y_i)} H(x, Du)\, dx \le \lambda \quad \text{and} \quad \fint_{B_{20\rho_i}(y_i)} H(x, F)\, dx \le \delta\lambda,$$

and A is (δ, R)-vanishing for some $R \in (0,1)$ and $\delta \in [0, \delta_1]$, then we have a function $v_i \in W^{1,\infty}(B_{5\rho_i}(y_i))$ satisfying

$$\fint_{B_{5\rho_i}(y_i)} H(x, Du - Dv_i)\, dx \le 3\left[\varepsilon + c(\varepsilon)\left(\frac{1}{K} + KR^{\sigma_2}\right)\right]\lambda \tag{3.3.63}$$

for any $K \ge 4$ and some $c(\varepsilon) = c(\mathtt{data}, \operatorname{dist}(\Omega_0, \partial\Omega), \|H(x,F)\|_{L^\gamma(\Omega)}, \varepsilon)$ and

$\sigma_2 = \sigma_2(\texttt{data})$. Also, we have

$$\sup_{x \in B_{5\rho_i}(y_i)} H(x, Dv_i) \le c^* \lambda$$

for some $c^* = c^*(\texttt{data})$ which is independent of i and λ.

Now for $c_2 = 2^{q+1} c^*$, we perform the integration of $H(x, Du)$ over $E(r_1, c_2\lambda)$. We see that for a.e. $x \in E(r_1, c_2\lambda) \cap B_{5\rho_i}(y_i)$ with $B_{20\rho_i}(y_i) \subset B_{r_2}$,

$$\begin{aligned} H(x, Du) &\le 2^q H(x, Du - Dv_i) + 2^q H(x, Dv_i) \\ &\le 2^q H(x, Du - Dv_i) + 2^q c^* \lambda \\ &\le 2^q H(x, Du - Dv_i) + \frac{1}{2} H(x, Du), \end{aligned}$$

which implies

$$H(x, Du) \le 2^{q+1} H(x, Du - Dv_i).$$

Then in light of (3.3.63), we have

$$\begin{aligned} \int_{E(r_1, c_2\lambda) \cap B_{5\rho_i}(y_i)} H(x, Du)\, dx &\le 2^{q+1} \int_{B_{5\rho_i}(y_i)} H(x, Du - Dv_i)\, dx \\ &\le \bar{c} \cdot 2^{q+1} 5^n |B_{\rho_i}(y_i)| \left[\varepsilon + c(\varepsilon) \left(\frac{1}{K} + K R^{\sigma_2} \right) \right] \lambda \end{aligned} \tag{3.3.64}$$

for an appropriate constant $\bar{c} = \bar{c}(\texttt{data})$. Using (3.3.62), one can easily see that

$$\begin{aligned} |B_{\rho_i}(y_i)| \le \frac{2}{\lambda} \Bigg(&\int_{B_{\rho_i}(y_i) \cap E(r_2, \frac{\lambda}{4})} H(x, Du)\, dx \\ &+ \int_{B_{\rho_i}(y_i) \cap \{H(x,F) > \frac{\delta\lambda}{4}\}} \frac{H(x, F)}{\delta}\, dx \Bigg). \end{aligned} \tag{3.3.65}$$

Plugging (3.3.65) to (3.3.64), we find

$$
\begin{aligned}
&\int_{E(r_1,c_2\lambda)\cap B_{5\rho_i}(y_i)} H(x,Du)\,dx \\
&\le c_3 S(\varepsilon,R,K) \\
&\quad\times\left(\int_{B_{\rho_i}(y_i)\cap E(r_2,\frac{\lambda}{4})} H(x,Du)\,dx + \int_{B_{\rho_i}(y_i)\cap\{H(x,F)>\frac{\delta\lambda}{4}\}} \frac{H(x,F)}{\delta}\,dx\right) \\
&\le c_3 S(\varepsilon,R,K)\left(\int_{E(r_2,\frac{\lambda}{4})} H(x,Du)\,dx + \int_{\{H(x,F)>\frac{\delta\lambda}{4}\}} \frac{H(x,F)}{\delta}\,dx\right)
\end{aligned}
$$

for some constants $c_3 = \bar{c}\cdot 2^{q+2}5^n$ and $S(\varepsilon,R,K) = \varepsilon + c(\varepsilon)\left(\frac{1}{K} + KR^{\sigma_2}\right)$, where for the last inequality we have used the fact that $\{B_{\rho_i}(y_i)\}_{i=1}^{\infty}$ is mutually disjoint.

Now denoting

$$[H(x,Du)]_t := \min\{H(x,Du),t\},$$

and arguing similarly as in [79, Section 4, Step 11] or [38], we discover

$$
\begin{aligned}
&\fint_{B_{r_1}} [H(x,Du)]_t^{\gamma-1} H(x,Du)\,dx \\
&\quad\le c_3 S(\varepsilon,R,K)(4c_1)^{\gamma-1}\fint_{B_{r_2}} [H(x,Du)]_t^{\gamma-1}H(x,Du)\,dx \\
&\qquad + c_3 S(\varepsilon,R,K)\frac{(4c_1)^{\gamma-1}}{\delta^{\gamma}}\fint_{B_{r_2}} [H(x,F)]^{\gamma}dx + c_1^{\gamma-1}\lambda_0^{\gamma}.
\end{aligned}
$$

We now recall that $S(\varepsilon,R,K) = \varepsilon + c(\varepsilon)\left(\frac{1}{K} + KR^{\sigma_2}\right)$, $\sigma_2 = \sigma_2(\mathtt{data})$ is given in Proposition 3.3.11, $c(\varepsilon) = c(\mathtt{data}, \operatorname{dist}(\Omega_0,\partial\Omega), \|H(x,F)\|_{L^{\gamma}(\Omega)}, \varepsilon)$ and $K \ge 4$ is a free parameter. We first select

$$\varepsilon = \varepsilon(\mathtt{data},\gamma) \in (0,1),$$

and

$$K = K(\mathtt{data}, \operatorname{dist}(\Omega_0,\partial\Omega), \gamma, \|H(x,F)\|_{L^{\gamma}(\Omega)}) \ge 4. \tag{3.3.66}$$

Then we choose a small positive constant

$$R = R(\texttt{data}, \operatorname{dist}(\Omega_0, \partial\Omega), \gamma, \|H(x,F)\|_{L^\gamma(\Omega)}) \tag{3.3.67}$$

in order to satisfy

$$0 < c_3 S(\varepsilon, R, K)(4c_2)^{\gamma-1} < \frac{1}{2}.$$

Accordingly there exists $\delta = \delta(\texttt{data}, \gamma) > 0$ from Lemma 3.3.14. Now we recall the definition of λ_0 as in (3.3.59) to find

$$\begin{aligned}
&\fint_{B_{r_1}} [H(x,Du)]_t^{\gamma-1} H(x,Du)\,dx \\
&\qquad \le \frac{1}{2} \fint_{B_{r_2}} [H(x,Du)]_t^{\gamma-1} H(x,Du)\,dx + c(\gamma) \fint_{B_r} H(x,F)^\gamma\,dx \\
&\qquad\quad + c^\gamma \frac{20^{n\gamma} r^{n\gamma}}{(r_2-r_1)^{n\gamma}} \left\{ \fint_{B_r} \left(H(x,Du) + H(x,F)\right)\,dx \right\}^\gamma,
\end{aligned}$$

where $c = c(\texttt{data})$ and $c(\gamma) = c(\texttt{data}, \gamma)$. We use the technical lemma [126, Lemma 6.1] to conclude

$$\begin{aligned}
\fint_{B_r} [H(x,Du)]_t^{\gamma-1} H(x,Du)\,dx \le c^\gamma &\left\{ \fint_{B_{2r}} \left(H(x,Du) + H(x,F)\right)\,dx \right\}^\gamma \\
&+ c(\gamma) \fint_{B_{2r}} H(x,F)^\gamma dx.
\end{aligned}$$

Letting $t \to \infty$ and using Jensen's inequality, we obtain

$$\fint_{B_r} H(x,Du)^\gamma\,dx \le c^\gamma \left(\fint_{B_{2r}} H(x,Du)\,dx \right)^\gamma + c(\gamma) \fint_{B_{2r}} H(x,F)^\gamma\,dx,$$

which is (3.3.6). Now the assertion that $H(x,Du) \in L^\gamma_{\mathrm{loc}}(\Omega)$ follows from a standard covering argument. The proof is completed.

3.4 Local estimates of ω–minimizers to double phase variational problems with variable exponents

In this section, we are concerned with an optimal regularity for ω-minimizers to double phase variational problems with variable exponents where the associated energy density is allowed to be discontinuous. We identify basic structure assumptions on the density for the absence of Lavrentiev phenomenon and higher integrability. Moreover we establish a local Calderón-Zygmund theory for such generalized minimizers under minimal regularity requirements regarding such double phase functionals.

3.4.1 Hypothesis and main results

The functional under consideration is

$$\mathcal{P}(w,\Omega) := \int_\Omega \left(f_1(x,Dw) + a(x) f_2(x,Dw)\right) \, dx. \tag{3.4.1}$$

Here, Ω is a bounded open domain in $\mathbb{R}^n$ for $n \geq 2$ and the continuous functions $p(x), q(x), a(x) : \Omega \to \mathbb{R}$ are assumed to satisfy

$$\begin{aligned} 0 \leq a(x) \in C^{0,\alpha}(\Omega), \quad & 1 < \gamma_1 \leq p(x) \leq q(x) \leq \gamma_2 < \infty, \\ & \frac{q(x)}{p(x)} \leq 1 + \frac{\alpha}{n} \end{aligned} \tag{3.4.2}$$

for some constants $\alpha \in (0,1], \gamma_1, \gamma_2$ and for every $x \in \Omega$. Additionally, we assume that there exists a constant $c_{p(\cdot),q(\cdot)} > 0$ such that

$$|p(x) - p(y)| + |q(x) - q(y)| \leq \frac{c_{p(\cdot),q(\cdot)}}{-\log|x-y|} \tag{3.4.3}$$

for every $x, y \in \Omega$ with $|x-y| \leq \frac{1}{2}$. Let $F = (f^1, \cdots, f^n) : \Omega \to \mathbb{R}^n$ be a given nonhomogeneous term such that $H(x,F) \in L^1(\Omega)$, where

$$H(x,z) = |z|^{p(x)} + a(x)|z|^{q(x)} \quad (x \in \Omega, z \in \mathbb{R}^n). \tag{3.4.4}$$

We suppose that for a non-decreasing function $\tilde{\mu}(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ and a continuous function $\gamma(\cdot) : \Omega \to \mathbb{R}$,

$$\begin{gathered} 1 < \gamma_1 \le \gamma(x) \le \gamma_2 < \infty, \quad |\gamma(x) - \gamma(y)| \le \tilde{\mu}(|x-y|), \\ \tilde{\mu}(r) \log \frac{1}{r} \le c_\gamma, \end{gathered} \tag{3.4.5}$$

and we want to identify minimal regularity assumptions on the associated energy densities $f_1(x,z)$ and $f_2(x,z)$ under which an ω-minimizer $u \in W^{1,1}(\Omega)$ to $\mathcal{F}(w,\Omega)$ satisfies the desired implication

$$H(x,F) \in L^{\gamma(\cdot)}(\Omega) \quad \Longrightarrow \quad H(x,Du) \in L^{\gamma(\cdot)}_{\mathrm{loc}}(\Omega). \tag{3.4.6}$$

We next describe basic structure assumptions regarding our energy functional (3.4.1). Suppose that $f_1, f_2 : \Omega \times \mathbb{R}^n \to \mathbb{R}$ are Carathéodory functions, C^2-regular for second variable $z \in \mathbb{R}^n$ and satisfy

$$\begin{cases} \nu|z|^{p(x)} \le f_1(x,z) \le L|z|^{p(x)} \\ \nu|z|^{q(x)} \le f_2(x,z) \le L|z|^{q(x)} \end{cases} \tag{3.4.7}$$

and

$$\begin{cases} \nu|z|^{p(x)-2}|\eta|^2 \le \langle D_z^2 f_1(x,z)\eta,\eta\rangle \le L|z|^{p(x)-2}|\eta|^2 \\ \nu|z|^{q(x)-2}|\eta|^2 \le \langle D_z^2 f_2(x,z)\eta,\eta\rangle \le L|z|^{q(x)-2}|\eta|^2 \end{cases} \tag{3.4.8}$$

for all $x \in \Omega$, $z \in \mathbb{R}^n$, $\eta \in \mathbb{R}^n$ with constants $0 < \nu \le L$. We write

$$f(x,z) = f_1(x,z) + a(x) f_2(x,z).$$

We now define ω-minimizer. For a radius $r > 0$ and $y \in \Omega$, let us write $B_r = B_r(y) = \{x \in \mathbb{R}^n : |x-y| < r\}$.

Definition 3.4.1. *Let $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous and non-decreasing function. We say that a function $u \in W^{1,H}_{\mathrm{loc}}(\Omega)$ is a (local) ω-minimizer for the functional $\mathcal{F}$ if for every ball $B_r \Subset \Omega$ and every $w \in W^{1,H}(B_r)$ with $w - u \in W^{1,H}_0(B_r)$, we have*

$$\mathcal{F}(u,B_r) \le (1+\omega(r))\mathcal{F}(w,B_r). \tag{3.4.9}$$

To get the desired regularity estimates, we further impose additional assumptions on $\omega, p(\cdot), q(\cdot),\ f_1$ and f_2. Throughout this section, the constant of two parameters $R \in (0, \frac{1}{2})$ and $\delta \in (0, \frac{1}{8})$ are to be determined later. First we assume

$$\omega(R) \leq \delta. \tag{3.4.10}$$

Since ω is non-decreasing, $\omega(r) \leq \delta$ holds for every $r \in (0, R)$. Suppose also that there exists a non-decreasing function $\mu : [0, \infty) \to [0, \infty)$ such that $\mu(0) = 0$,

$$\begin{aligned} |p(x) - p(y)| + |q(x) - q(y)| \leq \mu\left(|x - y|\right) \\ \text{and} \qquad \sup_{0<r\leq R} \mu(r) \log \frac{1}{r} \leq \delta. \end{aligned} \tag{3.4.11}$$

Moreover, we assume

$$\sup_{0<r\leq R} \sup_{B_r(y)\subset\Omega} \fint_{B_r(y)} \left[\theta_1(B_r(y))(x) + \theta_2(B_r(y))(x)\right] dx \leq \delta, \tag{3.4.12}$$

where

$$\theta_1(B_r(y))(x) = \sup_{z\in\mathbb{R}^n\setminus\{0\}} \left| \frac{f_1(x,z)}{|z|^{p(x)}} - \left(\frac{f_1(\cdot,z)}{|z|^{p(\cdot)}}\right)_{B_r(y)} \right| \leq 2L \tag{3.4.13}$$

and

$$\theta_2(B_r(y))(x) = \sup_{z\in\mathbb{R}^n\setminus\{0\}} \left| \frac{f_2(x,z)}{|z|^{q(x)}} - \left(\frac{f_2(\cdot,z)}{|z|^{q(\cdot)}}\right)_{B_r(y)} \right| \leq 2L. \tag{3.4.14}$$

Definition 3.4.2. *We say that $(\omega, p(\cdot), q(\cdot), f_1, f_2)$ is (δ, R)-vanishing if the conditions* (3.4.10)*,* (3.4.11) *and* (3.4.12) *hold.*

One can see that in [2] and [177], it is considered that the condition $(\omega, p(\cdot), q(\cdot), f_1, f_2)$ being (δ, R)-vanishing is necessary. We point out that f_1 and f_2 are allowed to be nearly discontinuous in x variable, even if the condition (3.4.12) holds. The assumption (3.4.12) means that the maps $x \mapsto \frac{f_1(x,z)}{|z|^{p(x)}}$ and $x \mapsto \frac{f_2(x,z)}{|z|^{q(x)}}$ have small BMO semi-norms which are less than or equal to δ uniformly in z variable.

We denote

$$\texttt{data} \equiv \texttt{data}(n, \nu, L, \gamma_1, \gamma_2, \alpha, \|a\|_{0,\alpha}, \mu(\cdot), \tilde{\mu}(\cdot), c_\gamma, \|H(x, Du)\|_{L^1(\Omega)})$$

and state the main result of this section.

Theorem 3.4.3. *Assume* (3.4.2), (3.4.3), (3.4.5), (3.4.7), (3.4.8) *and* $H(x, F) \in L^{\gamma(\cdot)}(\Omega)$. *Then for any* ω*-minimizer* $u \in W^{1,H}(\Omega)$ *of* $\mathcal{F}$ *in* (3.4.1), *there exists* $\delta = \delta(\boldsymbol{data}, \|H(x, F)\|_{L^{\gamma(\cdot)}(\Omega)})$ *such that if* $(\omega, p(\cdot), q(\cdot), f_1, f_2)$ *is* (δ, R)*-vanishing for some small* $R > 0$, *then* $H(x, Du) \in L^{\gamma(\cdot)}_{\mathrm{loc}}(\Omega)$. *Moreover, for any* $\Omega_0 \Subset \Omega$ *there exists* $R = R(\boldsymbol{data}, \omega(\cdot), \mathrm{dist}(\Omega_0, \partial\Omega), \|H(x, F)\|_{L^{\gamma(\cdot)}(\Omega)})$ *such that for all* $B_r(y) \Subset \Omega$ *with* $y \in \Omega_0$ *and* $0 < 2r < R$,

$$\begin{aligned} &\left(\fint_{B_{\frac{r}{2}}(y)} H(x, Du)^{\gamma(\cdot)} \, dx \right) \\ &\quad \leq c \left(\fint_{B_{2r}(y)} H(x, Du) \, dx \right)^{\gamma_-} + c \left(\fint_{B_{2r}(y)} H(x, F)^{\gamma(\cdot)} \, dx \right) + c \end{aligned} \tag{3.4.15}$$

for some constant $c = c(\boldsymbol{data}, \mathrm{dist}(\Omega_0, \partial\Omega), \|H(x, F)\|_{L^{\gamma(\cdot)}(\Omega)})$, *where* $\gamma_- = \inf_{x \in B_r(y)} \gamma(x)$.

3.4.2 Proof of Theorem 3.4.3

In order to give perturbation argument required in this section, the following higher integrability lemmas are essential. Also, the lemmas itself provide the fact that without the regularity assumptions (3.4.10), (3.4.11) and (3.4.12), we can prove the implication (3.4.6) for very small $\gamma_1 > 0$ depending on the data.

Lemma 3.4.4. *Assume* (3.4.2), (3.4.3), *and* (3.4.7). *Let* $H(x, F) \in L^{\gamma_1}_{\mathrm{loc}}(\Omega)$ *for some* $\gamma_1 > 1$ *and* $u \in W^{1,H}(\Omega)$ *be an* ω*-minimizer of* $\mathcal{F}$ *satisfying*

$$\int_\Omega H(x, Du) \, dx + 1 \leq M \tag{3.4.16}$$

for some constant M. *We further assume that a positive constant* ρ *satisfies*

$$\rho \leq \frac{1}{4M} \quad \textit{and} \quad \mu(4\rho) \leq \min\left\{ \sqrt{\frac{n + \gamma_1}{n + 1}} - 1, 1 \right\}. \tag{3.4.17}$$

Then there exists $\sigma_0 = \sigma_0(n, \nu, L, \gamma_1, \gamma_2, \alpha, \|a\|_{0,\alpha}, \|H(x, Du)\|_{L^1(\Omega)}, c_{p(\cdot),q(\cdot)}) \in (0, 1)$ *with* $1 + \sigma_0 < \gamma_1$ *such that if* $\sigma \in (0, \sigma_0]$, *then we have*

$$\begin{aligned}
&\fint_{B_\rho} H(x, Du)^{1+\sigma}\, dx \\
&\quad \le c \left(\fint_{B_{2\rho}} H(x, Du)\, dx \right)^{1+\sigma} + c \fint_{B_{2\rho}} H(x, F)^{1+\sigma}\, dx + c
\end{aligned} \tag{3.4.18}$$

for some constant $c = c(n, \nu, L, \gamma_1, \gamma_2, \alpha, \|a\|_{0,\alpha}, c_{p(\cdot),q(\cdot)})$, *whenever* $B_{2\rho} \Subset \Omega$.

Proof. Let $p^+ = \sup_{x \in B_{2\rho}} p(x)$, $q^+ = \sup_{x \in B_{2\rho}} q(x)$ and $s = \sqrt{\frac{n+\gamma_1}{n+1}} > 1$. Consider concentric balls $B_{\rho_1} \subset B_{\rho_2} \subset B_{2\rho}$ with $\rho \le \rho_1 < \rho_2 \le 2\rho$. Let $\eta \in C_0^\infty(B_{2\rho})$ be a cut-off function such that $0 \le \eta \le 1, \eta \equiv 1$ on B_{ρ_1}, $\eta \equiv 0$ on $B_{2\rho} \setminus B_{\rho_2}$ and $|D\eta| \le \frac{2}{\rho_2 - \rho_1}$. Taking $w = u - \eta(u - (u)_{B_{2\rho}})$ in (3.4.9), triangle inequality and Young's inequality with $\left(p(x), \frac{p(x)}{p(x)-1}\right)$ and with $\left(q(x), \frac{q(x)}{q(x)-1}\right)$ yield

$$\begin{aligned}
&\int_{B_{\rho_2}} H(x, Du)\, dx \\
&\quad \le c \int_{B_{\rho_2}} H(x, (1-\eta)Du - (D\eta)(u - (u)_{B_{2\rho}}))\, dx \\
&\qquad + c \int_{B_{\rho_2}} \left(|F|^{p(x)-1} + a(x)|F|^{q(x)-1} \right) \left(|Du| + \left| \frac{u - (u)_{B_{2\rho}}}{\rho_2 - \rho_1} \right| \right) dx \\
&\quad \le c \int_{B_{\rho_2} \setminus B_{\rho_1}} H(x, Du)\, dx + c \int_{B_{\rho_2}} H\left(x, \frac{u - (u)_{B_{2\rho}}}{\rho_2 - \rho_1} \right) dx \\
&\qquad + c \int_{B_{\rho_2}} H(x, F)\, dx.
\end{aligned}$$

Filling-hole method gives

$$\begin{aligned}\int_{B_{\rho_1}} & H(x, Du)\,dx \\ &\leq \frac{c-1}{c}\int_{B_{\rho_2}} H(x, Du)\,dx + c\int_{B_{\rho_2}} H\left(x, \frac{u-(u)_{B_{2\rho}}}{\rho_2-\rho_1}\right)\,d \\ &+ c\int_{B_{\rho_2}} H(x, F)\,dx.\end{aligned}$$

By Lemma 2.0.1, it follows that

$$\int_{B_\rho} H(x, Du)\,dx \leq c\int_{B_{2\rho}} H\left(x, \frac{u-(u)_{B_{2\rho}}}{\rho}\right)\,dx + c\int_{B_{2\rho}} H(x, F)\,dx.$$

Thus, we get the following Caccioppoli type inequality:

$$\begin{aligned}\fint_{B_\rho} H(x, Du)\,dx \leq c\fint_{B_{2\rho}} & \left(\left|\frac{u-(u)_{B_{2\rho}}}{\rho}\right|^{p^+} + a(x)\left|\frac{u-(u)_{B_{2\rho}}}{\rho}\right|^{q^+}\right)\,dx \\ &+ c\fint_{B_{2\rho}} H(x, F)\,dx + c.\end{aligned}$$

Now, following the proof of [38, Lemma 4.1], we have

$$\begin{aligned}\fint_{B_\rho} H(x, Du)\,dx \leq c & \left(\fint_{B_{2\rho}} H(x, Du)^{\frac{1}{s}}\,dx\right)^s \\ &+ c\fint_{B_{2\rho}} H(x, F)\,dx + c.\end{aligned} \tag{3.4.19}$$

Finally by Gehring's lemma [2, Theorem 4], we obtain (3.4.18). □

We also consider another type of higher integrability. Assume that the functions $\xi_1, \xi_2 : B_\rho(y) \to \mathbb{R}$ satisfy

$$0 \leq \xi_1(x) \leq (1+\sigma)p(x) \quad \text{and} \quad q(x) \leq \xi_2(x) \leq q(x) + \sigma p(x), \tag{3.4.20}$$

where ρ and σ are the same as in Lemma 3.4.4. The proof is similar to [38, Lemma 4.3].

Lemma 3.4.5. *Under the assumptions and conclusions of Lemma 3.4.4, we obtain*

$$\begin{aligned}&\fint_{B_\rho} \left(|Du|^{\xi_1(x)} + a(x)|Du|^{\xi_2(x)}\right)\, dx\\ &\qquad \le c\left(\fint_{B_{2\rho}} H(x,Du)\,dx\right)^{1+\sigma} + c\fint_{B_{2\rho}} H(x,F)^{1+\sigma}\,dx + c\end{aligned}$$

for some constant $c = c(n,\nu,L,\gamma_1,\gamma_2,\|a\|_{0,\alpha},c_{p(\cdot),q(\cdot)}) > 0$.

Note that we still obtain (3.4.18) with the assumption (3.4.11) instead of (3.4.3). Then σ_0 and c are independent of $c_{p(\cdot),q(\cdot)}$ in this case.

We first make comparison estimates used in the proof of the main result. Fix $y \in \Omega_0$, $r \le \frac{R}{4}$ with R to be determined later in (3.4.28), (3.4.51) and (3.4.75), and assume $B_{4r} = B_{4r}(y) \Subset \Omega$ in this section. Let $u \in W^{1,H}(\Omega)$ be an ω-minimizer of $\mathcal{F}$ and $h \in u + W_0^{1,H}(B_{4r})$ be the minimizer of the functional

$$\begin{aligned}&\mathcal{F}_0(Dh) := \int_{B_{4r}} f(x,Dh)\,dx \le \int_{B_{4r}} f(x,Dh+D\varphi)\,dx\\ &\quad \text{for all } \varphi \in W_0^{1,H}(B_{4r}).\end{aligned} \tag{3.4.21}$$

We refer to [63, 64, 126, 178] for a discussion on the regularity for minimizers of variational integrals.

Note that h is the weak solution of the following Dirichlet problem:

$$\begin{cases} -\mathrm{div}\,(D_z f(x,Dh)) = 0 & \text{in } B_{4r} \\ \qquad\qquad\qquad\quad h = u & \text{on } \partial B_{4r}. \end{cases} \tag{3.4.22}$$

With u and h above and σ_0 given in Lemma 3.4.4, we prove the following comparison estimates:

Lemma 3.4.6. *Let* $\lambda \ge 1$. *Then for each* $\varepsilon > 0$, *there exists a small* $\delta = \delta(n,\nu,L,\gamma_1,\gamma_2,\varepsilon) > 0$ *such that if*

$$\fint_{B_{4r}} H(x,Du)\,dx \le \lambda, \quad \left(\fint_{B_{4r}} H(x,F)^{1+\frac{\sigma_0}{4}}\,dx\right)^{\frac{4}{4+\sigma_0}} \le \delta\lambda, \tag{3.4.23}$$

and $\omega(4r) \leq \delta$ hold, then we have

$$\fint_{B_{4r}} H(x, Dh)\,dx \leq c_1\lambda \quad \textit{and} \quad \fint_{B_{4r}} H(x, Du - Dh)\,dx \leq \varepsilon\lambda, \tag{3.4.24}$$

where $c_1 = c_1(n, \nu, L, \gamma_1, \gamma_2) \geq 1$.

Proof. By (3.4.7), $\omega(4r) \leq \delta$ and (3.4.21), we have

$$\begin{aligned}\fint_{B_{4r}} H(x, Dh)\,dx &\leq \frac{1}{\nu}\fint_{B_{4r}} f(x, Dh)\,dx \\ &\leq \frac{1}{\nu}\fint_{B_{4r}} f(x, Du)\,dx \leq \frac{L}{\nu}\fint_{B_{4r}} H(x, Du)\,dx,\end{aligned} \tag{3.4.25}$$

which is the first inequality of (3.4.24). Now by Taylor's formula of f, the conditions (3.4.7) and (3.4.8), we obtain

$$\begin{aligned}\frac{1}{c}\Big((|z_1| + |z_2|)^{p(x)-2}\,|z_1 - z_2|^2 + a(x)\,(|z_1| + |z_2|)^{q(x)-2}\,|z_1 - z_2|^2\Big) \\ \leq f(x, z_1) - f(x, z_2) - \langle D_z f(x, z_2), z_1 - z_2\rangle\end{aligned}$$

with $c = c(n, \nu, L, \gamma_1, \gamma_2) \geq 1$. Then plugging $z_1 = Du$, $z_2 = Dh$ and testing (3.4.22) with a test function $u - h \in W_0^{1,H}(B_{4r})$, it follows by Hölder's inequality, Young's inequality, (3.4.7), (3.4.9), $\omega(4r) \leq \delta$, (3.4.23) and the first inequality of (3.4.24) that

$$\begin{aligned}
I_0 :=& \fint_{B_{4r}} \Big((|Du| + |Dh|)^{p(x)-2} + a(x)\,(|Du| + |Dh|)^{q(x)-2} \Big) |Du - Dh|^2 \, dx \\
&\leq c \fint_{B_{4r}} f(x, Du) - f(x, Dh) \, dx \\
&\leq c\omega(4r) \fint_{B_{4r}} f(x, Dh) \, dx \\
&\quad + c \fint_{B_{4r}} \left\langle |F|^{p(x)-2} F + a(x)|F|^{q(x)-2} F, Du - Dh \right\rangle \, dx \\
&\quad - c\omega(4r) \fint_{B_{4r}} \left\langle |F|^{p(x)-2} F + a(x)|F|^{q(x)-2} F, Dh \right\rangle \, dx \\
&\leq c\delta \fint_{B_{4r}} H(x, Dh) \, dx + c\kappa \fint_{B_{4r}} [H(x, Du) + H(x, Dh)] \, dx \\
&\quad + c(\kappa) \fint_{B_{4r}} H(x, F) \, dx \\
&\leq (c\delta + c\kappa + c(\kappa)\delta)\lambda
\end{aligned}$$

for any $\kappa \in (0, 1)$, where $c(\kappa)$ depends on γ_1, γ_2 and κ.

Denote

$$\begin{aligned}
A_1 &= \{x \in B_{4r} : p(x) \geq 2 \ \text{ and } \ q(x) \geq 2\}, \\
A_2 &= \{x \in B_{4r} : p(x) < 2 \ \text{ and } \ q(x) \geq 2\}, \\
A_3 &= \{x \in B_{4r} : p(x) < 2 \ \text{ and } \ q(x) < 2\}.
\end{aligned}$$

Since $(|Du| + |Dh|)^{p(x)-2} \geq |Du - Dh|^{p(x)-2}$ and $(|Du| + |Dh|)^{q(x)-2} \geq |Du - Dh|^{q(x)-2}$ on A_1, we have

$$\fint_{A_1} H(x, Du - Dh) \, dx \leq I_0.$$

On A_3, by Young's inequality with $\left(\frac{2}{2-p(x)}, \frac{2}{p(x)}\right)$ and $\left(\frac{2}{2-q(x)}, \frac{2}{q(x)}\right)$, for any

$\kappa_1 \in (0,1)$,

$$\begin{aligned}
&\fint_{A_3} H(x, Du - Dh)\,dx \\
&= \fint_{A_3} \Big\{ (|Du| + |Dh|)^{\frac{p(x)(2-p(x))}{2}} (|Du| + |Dh|)^{\frac{p(x)(p(x)-2)}{2}} |D(u-h)|^{p(x)} \\
&\quad + a(x) (|Du| + |Dh|)^{\frac{q(x)(2-q(x))}{2}} (|Du| + |Dh|)^{\frac{q(x)(q(x)-2)}{2}} |D(u-h)|^{q(x)} \Big\}\,dx \\
&\le \kappa_1 \fint_{A_3} \Big((|Du| + |Dh|)^{p(x)} + a(x) (|Du| + |Dh|)^{q(x)} \Big)\,dx \\
&\quad + c(\kappa_1) \fint_{A_3} \Big((|Du| + |Dh|)^{p(x)-2} + a(x) (|Du| + |Dh|)^{q(x)-2} \Big) \\
&\qquad \times |D(u-h)|^2\,dx \\
&\le \kappa_1 c \fint_{A_3} \left(H(x, Du) + H(x, Dh) \right)\,dx + c(\kappa_1) I_0.
\end{aligned}$$

By (3.4.23) and (3.4.25), we have

$$\fint_{A_3} H(x, Du - Dh)\,dx \le \kappa_1 c \lambda + c(\kappa_1) I_0. \tag{3.4.26}$$

Similarly, the estimate on A_2 can be proceeded as (3.4.26). Consequently we find

$$\fint_{B_{4r}} H(x, Du - Dh)\,dx \le \kappa_1 c_* \lambda + c(\kappa_1) I_0$$

and so

$$\fint_{B_{4r}} H(x, Du - Dh)\,dx \le \left[\kappa_1 c_* + c(\kappa_1)\kappa + c(\kappa_1)\left(1 + c(\kappa)\right)\delta\right]\lambda$$

for any $\kappa_1 \in (0,1)$, where c_* depends on $n, \nu, L, \gamma_1, \gamma_2$, while $c(\kappa_1)$ depends on $n, \nu, L, \gamma_1, \gamma_2,\ \kappa_1$.

Now choose $\kappa_1 = \frac{\varepsilon}{3c_*}$ and $\kappa = \frac{\varepsilon}{3c(\kappa_1)}$, and then select δ such that $c(\kappa_1)(1 + c(\kappa))\delta \le \frac{\varepsilon}{3}$. This yields the second inequality of (3.4.24). $\square$

We next discuss each minimizer of the corresponding freezing functionals and higher integrabilities of them. To this end, let M be the number given in Lemma 3.4.4. Then by (3.4.25), we have

$$\int_{B_{4r}} H(x, Dh)\, dx \le c_1 \left(\|H(x, Du)\|_{L^1(\Omega)} + 1\right) \tag{3.4.27}$$

for some constant c_1 as in Lemma 3.4.6. By (3.4.7) and [80, Proposition 2.32], we obtain

$$|D_z f_1(x, z)| \le \tilde{L}|z|^{p(x)-1} \quad \text{and} \quad |D_z f_2(x, z)| \le \tilde{L}|z|^{q(x)-1}$$

for a positive constant $\tilde{L} = \tilde{L}(n, \nu, L, \gamma_1, \gamma_2)$. With this $\tilde{L}$, we consider a sufficiently small radius $R > 0$ such that

$$\begin{aligned} &R \le \frac{1}{4c_1\left(\|H(x, Du)\|_{L^1(\Omega)} + 1\right)} < \frac{1}{4}, \\ &\text{and} \quad \mu(2R) \le \min\left\{\sqrt{\frac{n+\gamma_1}{n+1}} - 1, \frac{\sigma_0}{4}, \frac{\nu}{8(L+\tilde{L})}\right\} \le \frac{1}{2}. \end{aligned} \tag{3.4.28}$$

Then $R = \rho$ satisfies (3.4.17) and so r satisfies also (3.4.17) for all $4r \le R$.

Denote

$$H_1(x, z) = |z|^{p_2} + a(x)|z|^{q_2} \quad (x \in B_{4r},\ z \in \mathbb{R}^n),$$

where $p_2 = \sup\limits_{x \in B_{4r}} p(x)$ and $q_2 = \sup\limits_{x \in B_{4r}} q(x)$. Define two functions $\tilde{f}_1, \tilde{f}_2 : B_{4r} \times \mathbb{R}^n \to \mathbb{R}$ and $\bar{f}_1, \bar{f}_2 : \mathbb{R}^n \to \mathbb{R}$ by

$$\begin{cases} \tilde{f}_1(x, z) = f_1(x, z)|z|^{p_2 - p(x)} \\ \tilde{f}_2(x, z) = f_2(x, z)|z|^{q_2 - q(x)} \end{cases} \quad \text{and} \quad \begin{cases} \bar{f}_1(z) = \fint_{B_{4r}} \tilde{f}_1(x, z)\, dx \\ \bar{f}_2(z) = \fint_{B_{4r}} \tilde{f}_2(x, z)\, dx. \end{cases}$$

Then by [46, Eq. (3.15)], we have

$$\begin{cases} \nu|z|^{p_2} \le \bar{f}_1(z) \le L|z|^{p_2} \\ \nu|z|^{q_2} \le \bar{f}_2(z) \le L|z|^{q_2} \end{cases} \tag{3.4.29}$$

and

$$\begin{cases} \frac{\nu}{8}|z|^{p_2-2}|\eta|^2 \le \left\langle D^2\bar{f}_1(z)\eta,\eta\right\rangle \le 2L|z|^{p_2-2}|\eta|^2 \\ \frac{\nu}{8}|z|^{q_2-2}|\eta|^2 \le \left\langle D^2\bar{f}_2(z)\eta,\eta\right\rangle \le 2L|z|^{q_2-2}|\eta|^2. \end{cases} \tag{3.4.30}$$

Now let us write

$$\bar{f}(x,z) = \bar{f}_1(z) + a(x)\bar{f}_2(z).$$

Note that for any $x \in \Omega$,

$$\begin{aligned} \sup_{z\in\mathbb{R}^n\setminus\{0\}} \frac{|\tilde{f}_1(x,z)-\bar{f}_1(z)|}{|z|^{p_2}} &= \theta_1(B_{4r})(x), \\ \sup_{z\in\mathbb{R}^n\setminus\{0\}} \frac{|\tilde{f}_2(x,z)-\bar{f}_2(z)|}{|z|^{q_2}} &= \theta_2(B_{4r})(x) \end{aligned} \tag{3.4.31}$$

as in [46, Eq. (3.17)].

Let $h_1 \in h + W_0^{1,H_1}(B_{3r})$ be the minimizer of the functional

$$\begin{aligned} \mathcal{F}_1(Dh_1) := \int_{B_{3r}} \bar{f}(x,Dh_1)\,dx \le \int_{B_{3r}} \bar{f}(x,Dh_1+D\varphi)\,dx \\ \text{for all } \varphi \in W_0^{1,H_1}(B_{3r}). \end{aligned} \tag{3.4.32}$$

Then h_1 is the weak solution of the following Dirichlet problem:

$$\begin{cases} -\mathrm{div}(D_z\bar{f}(x,Dh_1)) = 0 & \text{in } B_{3r} \\ h_1 = h & \text{on } \partial B_{3r}. \end{cases} \tag{3.4.33}$$

Now we exhibit some estimates which follow from the higher integrability results Lemma 3.4.4 and Lemma 3.4.5: with (3.4.28) and $(p(\cdot),q(\cdot))$ being $(\delta,4r)$-vanishing, we have

$$\fint_{B_{3r}} H_1(x,Dh_1)\,dx \le c\fint_{B_{4r}} H(x,Dh)\,dx + c, \tag{3.4.34}$$

$$\left(\fint_{B_{3r}} H_1(x,Dh)^{1+\frac{\sigma_0}{4}}\,dx\right)^{\frac{4}{4+\sigma_0}} \le c\fint_{B_{4r}} H(x,Dh)\,dx + c, \tag{3.4.35}$$

$$\left(\fint_{B_{3r}} H_1(x, Dh_1)^{1+\sigma_2}\,dx\right)^{\frac{1}{1+\sigma_2}} \leq c\fint_{B_{4r}} H(x, Dh)\,dx + c, \tag{3.4.36}$$

where σ_0 is as same as in Lemma 3.4.4, the constant $\sigma_2 = \sigma_2(n, \nu, L, \gamma_1, \gamma_2, \alpha, \|a\|_{0,\alpha}, \|H(x, Du)\|_{L^1(\Omega)}) \leq \frac{\sigma_0}{4}$ and the generic constants c depend on $n, \nu, L, \gamma_1, \gamma_2, \alpha, \|a\|_{0,\alpha}$ and $\|H(x, Du)\|_{L^1(\Omega)}$. To prove these, first observe that

$$p_2 \leq p(x)\,(1 + 2\mu(8r)) \leq p(x)(1 + \sigma_0) \tag{3.4.37}$$

and

$$\begin{aligned} q_2 &\leq q(x)\,(1 + \mu(8r)) \\ &\leq q(x) + p(x)\left(1 + \frac{\alpha}{n}\right)\mu(8r) \\ &\leq q(x) + 2p(x)\mu(8r) \leq q(x) + \sigma_0 p(x). \end{aligned} \tag{3.4.38}$$

Hence by (3.4.37) and (3.4.38), the choice $\xi_1(x) = p_2$, $\xi_2(x) = q_2$ and $\sigma = 2\mu(8r) \in (0, \sigma_0]$ is applicable for Lemma 3.4.5. Thus we have

$$\fint_{B_{3r}} H_1(x, Dh)\,dx \leq c\left(\fint_{B_{4r}} H(x, Dh)\,dx\right)^{1+2\mu(8r)} + c. \tag{3.4.39}$$

Here, note that by (3.4.27) and (3.4.28), if $(p(\cdot), q(\cdot))$ is $(\delta, 4r)$-vanishing,

$$\begin{aligned} \left(\fint_{B_{4r}} H(x, Dh)\,dx\right)^{2\mu(8r)} &\leq c\left(\frac{M}{r^n}\right)^{2\mu(8r)} \leq c\left(\frac{1}{r}\right)^{2(n+1)\mu(8r)} \\ &\leq ce^{-2(n+1)(\log r)\mu(8r)} \leq ce^{2(n+1)\delta} \leq c. \end{aligned} \tag{3.4.40}$$

Thus together with (3.4.39) and (3.4.40), we have

$$\fint_{B_{3r}} H_1(x, Dh)\,dx \leq c\fint_{B_{4r}} H(x, Dh)\,dx + c. \tag{3.4.41}$$

Now, using (3.4.32) with $\varphi = h_1 - h$, one can see that

$$\begin{aligned}\fint_{B_{3r}} H_1(x, Dh_1)\,dx &\le \frac{1}{\nu}\fint_{B_{3r}} \bar{f}(x, Dh_1)\,dx \\ &\le \frac{1}{\nu}\fint_{B_{3r}} \bar{f}(x, Dh)\,dx \le \frac{L}{\nu}\fint_{B_{3r}} H_1(x, Dh)\,dx.\end{aligned} \tag{3.4.42}$$

Then by (3.4.41) and (3.4.42), we have (3.4.34). Now by (3.4.19) with h instead of u and (3.4.41), we observe

$$\begin{aligned}\fint_{B_{3r}} H_1(x, Dh)\,dx &\le c\fint_{B_{\frac{10}{3}r}} H(x, Dh)\,dx + c \\ &\le c\left(\fint_{B_{\frac{11}{3}r}} H(x, Dh)^{\frac{1}{s}}\,dx\right)^s + c \\ &\le c\left(\fint_{B_{\frac{11}{3}r}} (|Dh|^{p_2} + a(x)|Dh|^{q_2})^{\frac{1}{s}}\,dx\right)^s + c \\ &= c\left(\fint_{B_{\frac{11}{3}r}} H_1(x, Dh)^{\frac{1}{s}}\,dx\right)^s + c,\end{aligned}$$

where s is as in the proof of Lemma 3.4.4. Then we have by Gehring's lemma [2, Theorem 4],

$$\left(\fint_{B_{3r}} H_1(x, Dh)^{1+\sigma}\,dx\right)^{\frac{1}{1+\sigma}} \le c\fint_{B_{\frac{11}{3}r}} H_1(x, Dh)\,dx + c$$

for all $\tilde{\sigma} \in (0, \sigma_0]$, where σ_0 is exactly same as in Lemma 3.4.4. Especially, we have

$$\left(\fint_{B_{3r}} H_1(x, Dh)^{1+\frac{\sigma_0}{4}}\,dx\right)^{\frac{4}{4+\sigma_0}} \le c\fint_{B_{\frac{11}{3}r}} H_1(x, Dh)\,dx + c. \tag{3.4.43}$$

Now together with (3.4.41), we have (3.4.35). Now (3.4.36) follows from [84, Theorem 3] and (3.4.41).

Next let us introduce an inequality which will be used in the following lemma. Let $0 < \beta_1 \leq \beta_2 < \infty$ and $s_1 > 1$. Then there exists a constant $c(s_1, \beta_1, \beta_2) > 0$ such that for any function $f \in L^1(\Omega)$ and $\beta \in [\beta_1, \beta_2]$,

$$\fint_{\Omega} |f| \left[\log \left(e + \frac{|f|}{\fint_{\Omega} |f|\, dx} \right) \right]^{\beta} dx \leq c(s_1, \beta_1, \beta_2) \left(\fint_{\Omega} |f|^{s_1}\, dx \right)^{\frac{1}{s_1}}. \tag{3.4.44}$$

(See [2].)

Now we prove the following comparison estimates:

Lemma 3.4.7. *Under the assumptions and conclusions of Lemma 3.4.6, there exists a small* $\delta = \delta(n, \nu, L, \gamma_1, \gamma_2, \alpha, \|a\|_{0,\alpha}, \|H(x, Du)\|_{L^1(\Omega)}, \varepsilon) > 0$ *such that if* $(p(\cdot), q(\cdot), f_1, f_2)$ *is* $(\delta, 4r)$*-vanishing with* $R \geq 4r$ *satisfying* (3.4.28)*, then there exists* $h_1 \in W^{1,H_1}(B_{3r})$ *such that*

$$\fint_{B_{3r}} H_1(x, Dh_1)\, dx \leq c\lambda \quad \text{and} \quad \fint_{B_{3r}} H_1(x, Dh - Dh_1)\, dx \leq \varepsilon\lambda \tag{3.4.45}$$

hold for some constant $c = c(n, \nu, L, \gamma_1, \gamma_2, \alpha, \|a\|_{0,\alpha})$.

Proof. The first inequality of (3.4.45) follows by (3.4.41) and Lemma 3.4.6.

By Taylor's formula of $\bar{f}$ and the conditions (3.4.29) and (3.4.30), we obtain

$$\begin{aligned} &\frac{1}{c} \left((|z_1| + |z_2|)^{p_2 - 2} |z_1 - z_2|^2 + a(x) (|z_1| + |z_2|)^{q_2 - 2} |z_1 - z_2|^2 \right) \\ &\qquad \leq \bar{f}(x, z_1) - \bar{f}(x, z_2) - \left\langle D_z \bar{f}(x, z_2), z_1 - z_2 \right\rangle \end{aligned} \tag{3.4.46}$$

with $c = c(n, \nu, L, \gamma_1, \gamma_2) \geq 1$. Applying $z_1 = Dh$, $z_2 = Dh_1$ and (3.4.33) with the test function $h - h_1 \in W_0^{1,H_1}(B_{3r})$, it follows that

$$\begin{aligned}
I_1 &:= \fint_{B_{3r}} \left((|Dh| + |Dh_1|)^{p_2-2} + a(x)\,(|Dh| + |Dh_1|)^{q_2-2} \right) |Dh - Dh_1|^2 \, dx \\
&\le c \fint_{B_{3r}} \bar{f}(x, Dh) - \bar{f}(x, Dh_1) - \left\langle D_z \bar{f}(x, Dh_1), Dh - Dh_1 \right\rangle dx \\
&= c \fint_{B_{3r}} \left(\bar{f}(x, Dh) - \tilde{f}(x, Dh) \right) dx + c \fint_{B_{3r}} \left(\tilde{f}(x, Dh) - f(x, Dh) \right) dx \\
&\quad + c \fint_{B_{3r}} \left(f(x, Dh) - f(x, Dh_1) \right) dx \\
&\quad + c \fint_{B_{3r}} \left(f(x, Dh_1) - \tilde{f}(x, Dh_1) \right) dx \\
&\quad + c \fint_{B_{3r}} \left(\tilde{f}(x, Dh_1) - \bar{f}(x, Dh_1) \right) dx \\
&=: I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned}$$

Estimates I_2 and I_6: Since (f_1, f_2) is $(\delta, 4r)$-vanishing, together with (3.4.31), Hölder's inequality, (3.4.34), (3.4.35) and (3.4.36), we obtain

$$\begin{aligned}
|I_2| + |I_6| &\le c \fint_{B_{3r}} \theta(B_{4r}) \left[H_1(x, Dh) + H_1(x, Dh_1) \right] dx \\
&\le c \left(\fint_{B_{3r}} \theta(B_{4r})^{\frac{1+\sigma_2}{\sigma_2}} \, dx \right)^{\frac{\sigma_2}{1+\sigma_2}} \lambda \\
&\le c \left(L^{\frac{1}{\sigma_2}} \fint_{B_{3r}} \theta(B_{4r}) \, dx \right)^{\frac{\sigma_2}{1+\sigma_2}} \lambda \le c \delta^{\frac{\sigma_2}{1+\sigma_2}} \lambda.
\end{aligned}$$

Estimates I_3 and I_5: Write

$$\begin{aligned}
I_3 &= \fint_{B_{3r}} \left(\tilde{f}_1(x, Dh) - f_1(x, Dh) \right) dx \\
&\quad + \fint_{B_{3r}} a(x) \left(\tilde{f}_2(x, Dh) - f_2(x, Dh) \right) dx =: I_{3,p} + I_{3,q}.
\end{aligned}$$

Now let us estimate $I_{3,q}$ first. To this end, by mean value theorem and Fubini's theorem, we find

$$
\begin{aligned}
I_{3,q} &\le L \fint_{B_{3r}} a(x) \left[\int_0^1 (q_2 - q(x)) |\log(|Dh|)| Dh|^{(q_2-q(x))t} \, dt \right] |Dh|^{q(x)} \, dx \\
&\le c\mu(8r) \int_0^1 \fint_{B_{3r}} a(x) |\log(|Dh|)||Dh|^{(q_2-q(x))t+\frac{\gamma_1}{2}} |Dh|^{q(x)-\frac{\gamma_1}{2}} \, dx dt.
\end{aligned}
$$

For every $\alpha > 0$ and $\beta > 1$ we have

$$
t^\alpha |\log t| \le \begin{cases} \dfrac{e^\alpha}{\alpha} & \text{if} \quad 0 < t \le e, \\ 2t^\alpha \log\left(e + t^{\frac{\beta}{2}}\right) & \text{if} \quad e < t, \end{cases}
$$

and for any $t_1, t_2 > 0$ we see $\log(e + t_1 t_2) \le \log(e + t_1) + \log(e + t_2)$. Then we estimate

$$
\begin{aligned}
&|\log(|Dh|)||Dh|^{(q_2-q(x))t+\frac{\gamma_1}{2}} |Dh|^{q(x)-\frac{\gamma_1}{2}} \\
&\quad \le 2|\log(e + |Dh|^{p_2})||Dh|^{q_2} + \frac{2e^{\gamma_2}}{\gamma_1} |Dh|^{q(x)-\frac{\gamma_1}{2}} \\
&\quad \le 2\log\left(e + \frac{|Dh|^{p_2}}{(H_1(x,Dh))_{B_{3r}}}\right)|Dh|^{q_2} + 2\log\left(e + (H_1(x,Dh))_{B_{3r}}\right)|Dh|^{q_2} \\
&\qquad + c(\gamma_1,\gamma_2)|Dh|^{q(x)} \\
&\quad \le 2\log\left(e + \frac{H_1(x,Dh)}{(H_1(x,Dh))_{B_{3r}}}\right)|Dh|^{q_2} + 2\log\left(e + (H_1(x,Dh))_{B_{3r}}\right)|Dh|^{q_2} \\
&\qquad + c(\gamma_1,\gamma_2)|Dh|^{q(x)}.
\end{aligned}
$$

Thus, it follows by (3.4.28), (3.4.35) and (3.4.44) that

$$\begin{aligned}|I_{3,q}| &\le c\mu(8r)\fint_{B_{3r}} a(x)|Dh|^{q_2}\log\left(e+\frac{H_1(x,Dh)}{(H_1(x,Dh))_{B_{3r}}}\right)dx\\&\quad + c\mu(8r)\fint_{B_{3r}} a(x)|Dh|^{q_2}\log\left(e+(H_1(x,Dh))_{B_{3r}}\right)dx\\&\quad + c\mu(8r)\fint_{B_{3r}} a(x)|Dh|^{q(x)}\,dx\\&\le c\mu(8r)\left(\fint_{B_{3r}} H_1(x,Dh)^{\left(1+\frac{\sigma_0}{4}\right)}\,dx\right)^{\frac{4}{4+\sigma_0}}\\&\quad + c\mu(8r)\log\left(\frac{1}{r}\right)\fint_{B_{3r}} H_1(x,Dh)\,dx + c\mu(8r)\fint_{B_{3r}} H(x,Dh)\,dx\\&\le c\mu(8r)\log\left(\frac{1}{r}\right)\left(\fint_{B_{3r}} H_1(x,Dh)^{\left(1+\frac{\sigma_0}{4}\right)}\,dx\right)^{\frac{4}{4+\sigma_0}} \le c\delta\lambda,\end{aligned}$$

since $(p(\cdot),q(\cdot))$ is $(\delta,4r)$-vanishing. By substituting $p(x)$ for $q(x)$ and $p(x)$, and 1 for $a(x)$, with the same argument as above, we see

$$|I_{3,p}| \le c\delta\lambda.$$

By the similar argument for $I_{3,p}$ and $I_{3,q}$, we have

$$|I_5| \le c\delta\lambda.$$

Here, note that we have to use (3.4.36) instead of (3.4.35).

Estimate I_4: (3.4.21) yields $I_4 \le 0$.

Estimate I_1: Similar to the proof of Lemma 3.4.6, we have

$$\fint_{B_{3r}} H_1(x,Dh-Dh_1)\,dx \le \kappa_2 c\lambda + c(\kappa_2)I_1$$

for any $\kappa_2 \in (0,1)$, where $c(\kappa_2)$ depends on $n,\nu,L,\gamma_1,\gamma_2$ and κ_2. Therefore, we have

$$\fint_{B_{3r}} H_1(x,Dh-Dh_1)\,dx \le \kappa_2 c\lambda + c(\kappa_2)\left(c\delta^{\frac{\sigma_0}{4+\sigma_0}} + 4c\delta\right)\lambda.$$

Now choose $\kappa_2 = \frac{\varepsilon}{2c}$ and then select δ small enough so that

$$c(\kappa_2)\left(c\delta^{\frac{\sigma_0}{4+\sigma_0}} + 4c\delta\right) \le \frac{\varepsilon}{2}.$$

This finishes the proof of the lemma. $\square$

Now, let $v \in W^{1,H_1}(B_{2r})$ be the weak solution of

$$\begin{cases} \operatorname{div}(D_z\bar{f}(x_M, Dv)) = 0 & \text{in } B_{2r} \\ \qquad\qquad\qquad v = h_1 & \text{on } \partial B_{2r}, \end{cases} \tag{3.4.47}$$

where $x_M \in \overline{B_{2r}}$ is such that $a(x_M) = \sup_{x\in B_{2r}} a(x)$. Now we refer the following comparison estimates.

Lemma 3.4.8 ([79, 84])**.** *Assume $H(x,F) \in L^{\gamma(\cdot)}(\Omega)$ with* (3.4.5)*. Under the assumptions and conclusions of Lemma 3.4.7, there exists a small $\delta = \delta(n,\nu,L,\gamma_1,\gamma_2,\alpha,\|a\|_{0,\alpha},\|H(x,Du)\|_{L^1(\Omega)},\varepsilon) > 0$ such that*

$$\sup_{x\in B_r} H_1(x_M, Dv) \le c\lambda \tag{3.4.48}$$

for some constant $c = c(n,\nu,L,\gamma_1,\gamma_2,\alpha,\|a\|_{0,\alpha})$ and

$$\fint_{B_{\frac{3r}{2}}} H_1(x, Dh_1 - Dv)\,dx \le \left(2\varepsilon + \frac{\bar{c}}{K} + c^*(K)r^{\sigma_3}\right)\lambda$$

with $K \ge 4$, $\bar{c} = \bar{c}(n,\nu,L,\gamma_1,\gamma_2)$,
$c^*(K) = c^*(\boldsymbol{data}, \operatorname{dist}(\Omega_0,\partial\Omega), \|H(x,F)\|_{L^{\gamma(\cdot)}(\Omega)}, K)$ and
$\sigma_3 = \sigma_2(n,\nu,L,\gamma_1,\gamma_2,\alpha,\|a\|_{0,\alpha},\|H(x,Du)\|_{L^1(\Omega)})$.

We combine all the estimates in Lemma 3.4.6, 3.4.7 and 3.4.8 to reach the desired comparison estimates.

Lemma 3.4.9. *Let $\lambda \ge 1$, $K \ge 4$ and assume $H(x,F) \in L^{\gamma(\cdot)}(\Omega)$ with* (3.4.5)*. Then for any $\varepsilon > 0$, there exists a small*
$\delta = \delta(n,\nu,L,\gamma_1,\gamma_2,\alpha,\|a\|_{0,\alpha},\|H(x,Du)\|_{L^1(\Omega)},\varepsilon) > 0$ such that if

$$\fint_{B_{4r}} H(x,Du)\,dx \le \lambda, \quad \left(\fint_{B_{4r}} H(x,F)^{1+\frac{\sigma_0}{4}}\,dx\right)^{\frac{4}{4+\sigma_0}} \le \delta\lambda, \tag{3.4.49}$$

and $(\omega, p(\cdot), q(\cdot), f_1, f_2)$ *is* $(\delta, 4r)$*-vanishing with* $R \geq 4r$ *satisfying* (3.4.28), *then there exist* $h \in W^{1,H}(B_r)$, $h_1 \in W^{1,H_1}(B_r)$ *and* $v \in W^{1,\infty}(B_r)$ *such that*

$$\fint_{B_r} H(x, Du - Dh)\, dx \leq 4^n \varepsilon \lambda,$$
$$\fint_{B_r} H_1(x, Dh - Dv)\, dx \leq 4^n S(\varepsilon, R, K)\lambda,$$

and

$$\sup_{x \in B_r} H_1\left(x_M, Dv(x)\right) \leq c_4 \lambda.$$

Here, $S(\varepsilon, R, K) = 4\varepsilon + \frac{c_2}{K} + c_3 R^{\sigma_3}$*, where* $c_2 = c_2(n, \nu, L, \gamma_1, \gamma_2)$*,*
$c_3 = c_3(\boldsymbol{data}, \operatorname{dist}(\Omega_0, \partial\Omega), \|H(x,F)\|_{L^{\gamma(\cdot)}(\Omega)}, K)$*,*
$\sigma_3 = \sigma_3(n, \nu, L, \gamma_1, \gamma_2, \alpha, \|a\|_{0,\alpha}, \|H(x, Du)\|_{L^1(\Omega)})$ *and*
$c_4 = c_4(n, \nu, L, \gamma_1, \gamma_2, \alpha, \|a\|_{0,\alpha})$*.*

Now we are ready to prove Theorem 3.4.3. Let $H(x, F) \in L^{\gamma(\cdot)}(\Omega)$ with (3.4.5) and $u \in W^{1,H}(\Omega)$ be an ω-minimizer of $\mathcal{F}$ in (3.4.1). For $R \geq 4r$ with (3.4.28), let $B_{4r}(y) = B_{4r} \Subset \Omega$. Define $\lambda_0 > 0$ by

$$\lambda_0 := \fint_{B_{2r}} H(x, Du)\, dx + \frac{1}{\delta}\left\{\left(\fint_{B_{2r}} H(x,F)^{1+\sigma_0}\, dx\right)^{\frac{1}{1+\sigma_0}} + 1\right\}, \tag{3.4.50}$$

where σ_0 is given in Lemma 3.4.4 and $\delta \in (0, \frac{1}{8})$ will be determined later depending on $\mathtt{data}$ and $\|H(x,F)\|_{L^{\gamma(\cdot)}(\Omega)}$. With $4r \leq R$ satisfying (3.4.28), choose r_1 and r_2 such that $\frac{r}{2} \leq r_1 < r_2 \leq r$ and write

$$E(s, \lambda) = \{x \in B_s : H(x, Du(x))^{\frac{\gamma(x)}{\gamma_-}} > \lambda\},$$
$$\mathcal{E}(s, \lambda) = \{x \in B_s : H(x, F(x))^{\frac{\gamma(x)}{\gamma_-}} > \lambda\}$$

for $\lambda > 0$ and $\frac{r}{2} \leq s \leq r$, where $\gamma_- := \inf_{x \in B_{2r}} \gamma(x)$ and $\gamma_+ := \sup_{x \in B_{2r}} \gamma(x)$. Let

$$A := \left(\frac{20}{r_2 - r_1}\right)^n \geq 1.$$

Now we give the following lemma obtained from an exit time argument and

Vitali covering lemma.

Lemma 3.4.10. *Assume for c_1 as in Lemma 3.4.6,*

$$4r \le R \le \frac{1}{4c_1\left(\|H(x,Du)\|_{L^1(\Omega)}+1\right)} < \frac{1}{4} \quad \textit{and} \quad \tilde{\mu}(R) \le \frac{\sigma_0}{8} < \frac{1}{8}. \tag{3.4.51}$$

Then there exist
$\tilde{c} = \tilde{c}(n, \nu, L, \gamma_1, \gamma_2, \|a\|_{0,\alpha}, c_\gamma, \|H(x,Du)\|_{L^1(\Omega)}, \|H(x,F)\|_{L^{\gamma(\cdot)}(\Omega)})$ *such that if* $\lambda > \tilde{c}A\lambda_0 \ge 1$, *then there is a countable collection of mutually disjoint open balls* $\{B_{\rho_i}(y^i)\}_{i=1}^\infty$ *with* $y^i \in E(r_1, \lambda)$ *and* $\rho_i \in \left(0, \frac{r_2-r_1}{20}\right)$ *such that*

$$E(r_1,\lambda) \subset \bigcup_{i=1}^\infty B_{5\rho_i}(y^i) \cup \textit{(a negligible set)}, \tag{3.4.52}$$

$$\fint_{B_{\rho_i(y^i)}} H(x,Du)^{\frac{\gamma(x)}{\gamma_-}}\,dx + \frac{1}{\delta}\left(\fint_{B_{\rho_i(y^i)}} H(x,F)^{\left(1+\frac{\sigma_0}{4}\right)\frac{\gamma(x)}{\gamma_-}}\,dx\right)^{\frac{4}{4+\sigma_0}} = \lambda \tag{3.4.53}$$

and

$$\fint_{B_\rho(y^i)} H(x,Du)^{\frac{\gamma(x)}{\gamma_-}}\,dx + \frac{1}{\delta}\left(\fint_{B_\rho(y^i)} H(x,F)^{\left(1+\frac{\sigma_0}{4}\right)\frac{\gamma(x)}{\gamma_-}}\,dx\right)^{\frac{4}{4+\sigma_0}} < \lambda \tag{3.4.54}$$

for each $\rho \in (\rho_i, r_2 - r_1]$.

Proof. First, note that for every $B_\rho \subset B_r$,

$$\begin{aligned}\left(\fint_{B_\rho} H(x,Du)\,dx\right)^{\tilde{\mu}(2\rho)} &= \rho^{-n\tilde{\mu}(2\rho)}\left(\int_{B_\rho} H(x,Du)\,dx\right)^{\tilde{\mu}(2\rho)} \underset{(3.4.51)}{\le} \rho^{-(n+1)\tilde{\mu}(2\rho)} \\ &\le e^{-(n+1)(\log\rho)\tilde{\mu}(2\rho)} \underset{(3.4.5)}{\le} e^{(n+1)c_\gamma} \le c \end{aligned} \tag{3.4.55}$$

with $c = c(n, c_\gamma)$ and

$$\begin{aligned}\left(\fint_{B_\rho} H(x,F)^{1+\sigma_0}\,dx\right)^{\frac{\tilde{\mu}(2\rho)}{1+\sigma_0}} &= \rho^{\frac{-n\tilde{\mu}(2\rho)}{1+\sigma_0}}\left(\int_{B_\rho} H(x,F)^{1+\sigma_0}\,dx\right)^{\frac{\tilde{\mu}(2\rho)}{1+\sigma_0}} \\ &\le e^{\frac{-n(\log\rho)\tilde{\mu}(2\rho)}{1+\sigma_0}}\left(\|H(x,F)\|_{L^{\gamma(\cdot)}(\Omega)}+1\right) \\ &\le ce^{\frac{nc_\gamma}{1+\sigma_0}} \le c \qquad (3.4.56)\end{aligned}$$

with $c = c(n, c_\gamma, \|H(x,F)\|_{L^{\gamma(\cdot)}(\Omega)})$.

For each $B_\rho(\tilde{y}) \subset B_r$, define

$$\Phi_{\tilde{y}}(\rho) = \fint_{B_\rho(\tilde{y})} H(x,Du)^{\frac{\gamma(x)}{\gamma_-}}\,dx + \frac{1}{\delta}\left(\fint_{B_\rho(\tilde{y})} H(x,F)^{\left(1+\frac{\sigma_0}{4}\right)\frac{\gamma(x)}{\gamma_-}}\,dx\right)^{\frac{4}{4+\sigma_0}}.$$

Then by Lebesgue differentiation theorem, for almost every $\tilde{y} \in E(s,\lambda)$,

$$\lim_{\rho\to 0}\Phi_{\tilde{y}}(\rho) = H(\tilde{y}, Du(\tilde{y}))^{\frac{\gamma(\tilde{y})}{\gamma_-}} + \frac{1}{\delta}H(\tilde{y}, F(\tilde{y}))^{\frac{\gamma(\tilde{y})}{\gamma_-}} > \lambda. \qquad (3.4.57)$$

If $\tilde{y} \in B_{r_1}$, for $\rho \in \left[\frac{r_2-r_1}{20}, r_2 - r_1\right]$, by Hölder's inequality, we have

$$
\begin{aligned}
\Phi_{\tilde{y}}(\rho) &\leq A \fint_{B_{\frac{r}{2}}} H(x,Du)^{\frac{\gamma(x)}{\gamma_-}}\,dx + \frac{1}{\delta}\left\{A \fint_{B_{\frac{r}{2}}} H(x,F)^{\left(1+\frac{\sigma_0}{4}\right)\frac{\gamma(x)}{\gamma_-}}\,dx\right\}^{\frac{4}{4+\sigma_0}} \\
&= A \fint_{B_{\frac{r}{2}}} H(x,Du)^{\frac{1}{2}} H(x,Du)^{\frac{1}{2}+\frac{\gamma(x)-\gamma_-}{\gamma_-}}\,dx \\
&\quad + \frac{A}{\delta}\left(\fint_{B_{\frac{r}{2}}} H(x,F)^{\frac{1}{2}+\frac{\sigma_0}{8}} H(x,F)^{\frac{1}{2}+\frac{\sigma_0}{8}+\frac{\gamma(x)-\gamma_-}{\gamma_-}\left(1+\frac{\sigma_0}{4}\right)}\,dx\right)^{\frac{4}{4+\sigma_0}} \\
&\leq A\left(\fint_{B_{\frac{r}{2}}} H(x,Du)\,dx\right)^{\frac{1}{2}}\left(\fint_{B_{\frac{r}{2}}} H(x,Du)^{1+2\frac{\gamma(x)-\gamma_-}{\gamma_-}}\,dx\right)^{\frac{1}{2}} \\
&\quad + \frac{A}{\delta}\left(\fint_{B_{\frac{r}{2}}} H(x,F)^{1+\frac{\sigma_0}{4}}\,dx\right)^{\frac{4}{2(4+\sigma_0)}} \\
&\quad \times \left(\fint_{B_{\frac{r}{2}}} H(x,F)^{1+\frac{\sigma_0}{4}+\frac{2(\gamma(x)-\gamma_-)}{\gamma_-}\left(1+\frac{\sigma_0}{4}\right)}\right)^{\frac{4}{2(4+\sigma_0)}} \\
&\leq A2^{\frac{n}{2}}\left(\fint_{B_r} H(x,Du)\,dx\right)^{\frac{1}{2}}\left(\fint_{B_{\frac{r}{2}}} H(x,Du)^{1+2\tilde{\mu}(2r)}\,dx + 1\right)^{\frac{1}{2}} \\
&\quad + \frac{A2^{\frac{n}{2}}}{\delta}\left(\fint_{B_r} H(x,F)^{1+\sigma_0}\,dx\right)^{\frac{1}{2(1+\sigma_0)}} \\
&\quad \times \left(\fint_{B_{\frac{r}{2}}} H(x,F)^{(1+2\tilde{\mu}(2r))\left(1+\frac{\sigma_0}{4}\right)}\,dx + 1\right)^{\frac{4}{2(4+\sigma_0)}}.
\end{aligned}
\tag{3.4.58}
$$

Here, by (3.4.5), Lemma 3.4.4, (3.4.55), (3.4.56) and Hölder's inequality,

$$
\begin{aligned}
&\left(\fint_{B_{\frac{r}{2}}} H(x,Du)^{1+2\tilde{\mu}(2r)}\,dx+1\right)^{\frac{1}{2}}\\
&\qquad\le c\left(\fint_{B_r} H(x,Du)\,dx\right)^{\frac{1+2\tilde{\mu}(2r)}{2}} + c\left(\fint_{B_r} H(x,F)^{1+2\tilde{\mu}(2r)}\,dx\right)^{\frac{1}{2}} + c\\
&\qquad\le c\left(\fint_{B_r} H(x,Du)\,dx\right)^{\frac{1}{2}} + c\left(\fint_{B_r} H(x,F)^{1+\sigma_0}\,dx\right)^{\frac{1+2\tilde{\mu}(2r)}{2(1+\sigma_0)}} + c\\
&\qquad\le c\left(\fint_{B_r} H(x,Du)\,dx\right)^{\frac{1}{2}} + c\left(\fint_{B_r} H(x,F)^{1+\sigma_0}\,dx\right)^{\frac{1}{2(1+\sigma_0)}} + c
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(\fint_{B_{\frac{r}{2}}} H(x,F)^{(1+2\tilde{\mu}(2r))\left(1+\frac{\sigma_0}{4}\right)}\,dx+1\right)^{\frac{4}{2(4+\sigma_0)}}\\
&\qquad\le\left(\fint_{B_{\frac{r}{2}}} H(x,F)^{\left(1+\frac{\sigma_0}{4}\right)^2}\,dx\right)^{\frac{8+16\tilde{\mu}(2r)}{(4+\sigma_0)^2}} + c\\
&\qquad\le\left(\fint_{B_{\frac{r}{2}}} H(x,F)^{1+\sigma_0}\,dx\right)^{\frac{\frac{1}{2}+\tilde{\mu}(2r)}{1+\sigma_0}} + c\\
&\qquad\le c\left(\fint_{B_{\frac{r}{2}}} H(x,F)^{1+\sigma_0}\,dx\right)^{\frac{1}{2(1+\sigma_0)}} + c
\end{aligned}
$$

with $c = c(n,\nu,L,\gamma_1,\gamma_2,\|a\|_{0,\alpha},c_\gamma,\|H(x,Du)\|_{L^1(\Omega)},\|H(x,F)\|_{L^{\gamma(\cdot)}(\Omega)})$.

Hence,

$$
\begin{aligned}
&\Phi_{\tilde{y}}(\rho) \\
&\le cA\left(\fint_{B_r} H(x,Du)\,dx\right)^{\frac{1}{2}} \\
&\qquad\times\left\{\left(\fint_{B_r} H(x,Du)\,dx\right)^{\frac{1}{2}} + c\left(\fint_{B_r} H(x,F)^{1+\sigma_0}\,dx\right)^{\frac{1}{2(1+\sigma_0)}} + c\right\} \\
&\quad + \frac{cA}{\delta}\left(\fint_{B_r} H(x,F)^{1+\sigma_0}\,dx\right)^{\frac{1}{2(1+\sigma_0)}}\left\{\left(\fint_{B_{\frac{r}{2}}} H(x,F)^{1+\sigma_0}\,dx\right)^{\frac{1}{2(1+\sigma_0)}} + c\right\} \\
&\le cA\fint_{B_{\frac{r}{2}}} H(x,Du)\,dx + \frac{cA}{\delta}\left\{\left(\fint_{B_{\frac{r}{2}}} H(x,F)^{1+\sigma_0}\,dx\right)^{\frac{1}{1+\sigma_0}} + 1\right\} \le \tilde{c}A\lambda_0,
\end{aligned}
\tag{3.4.59}
$$

where $\tilde{c} = \tilde{c}(n,\nu,L,\gamma_1,\gamma_2,\|a\|_{0,\alpha},c_\gamma,\|H(x,Du)\|_{L^1(\Omega)},\|H(x,F)\|_{L^{\gamma(\cdot)}(\Omega)})$. Thus if $\tilde{c}A\lambda_0 < \lambda$, since $\Phi_{\tilde{y}}(\rho)$ is continuous, (3.4.57) and (3.4.59) imply that for almost every $\tilde{y} \in E(r_1,\lambda)$, there is a small number $\rho_{\tilde{y}} \in \left(0, \frac{r_2-r_1}{20}\right)$ such that

$$
\Phi_{\tilde{y}}(\rho_{\tilde{y}}) = \lambda \quad \text{and} \quad \Phi_{\tilde{y}}(\rho) < \lambda \quad \text{for all } \rho \in (\rho_{\tilde{y}}, r_2 - r_1].
$$

Therefore by the Vitali covering lemma to $\{B_{\rho_i}(y^i)\}_{i=1}^{\infty}$ with $y^i \in E(r_1,\lambda)$, where y^i are the Lebesgue points of $H(x,Du)$ and $H(x,F)^{1+\sigma_0}$, we obtain (3.4.52), (3.4.53) and (3.4.54). $\square$

Now we apply Lemma 3.4.9. By (3.4.54),

$$
\fint_{B_{20\rho_i}(y^i)} H(x,Du)^{\frac{\gamma(x)}{\gamma_-}}\,dx \le \lambda
$$
$$
\text{and} \quad \left(\fint_{B_{20\rho_i}(y^i)} H(x,F)^{(1+\sigma_0)\frac{\gamma(x)}{\gamma_-}}\,dx\right)^{\frac{1}{1+\sigma_0}} \le \delta\lambda.
$$

Then we have

$$
\fint_{B_{20\rho_i}(y^i)} H(x,Du)\,dx \le \fint_{B_{20\rho_i}(y^i)} \left(H(x,Du)^{\frac{\gamma(x)}{\gamma_-}} + 1\right)dx \le 2\lambda \tag{3.4.60}
$$

and

$$\begin{aligned}&\left(\fint_{B_{20\rho_i}(y^i)}\left(\frac{H(x,F)}{\delta}\right)^{1+\sigma_0}dx\right)^{\frac{1}{1+\sigma_0}}\\&\le\left(\fint_{B_{20\rho_i}(y^i)}\left(\frac{H(x,F)}{\delta}\right)^{(1+\sigma_0)\frac{\gamma(x)}{\gamma_-}}dx+1\right)^{\frac{1}{1+\sigma_0}}\le c\lambda.\end{aligned}\tag{3.4.61}$$

Denote $H_1^i(x,z)=|z|^{p_2^i}+a(x)|z|^{q_2^i}$ with $p_2^i=\sup_{x\in B_{20\rho_i}(y^i)}p(x)$ and $q_2^i=\sup_{x\in B_{20\rho_i}(y^i)}q(x)$ and let $x_M^i\in\overline{B_{20\rho_i}(y^i)}$ satisfy $a(x_M^i)=\sup_{B_{20\rho_i}(y^i)}a(x)$. By Lemma 3.4.9, for any $\varepsilon\in(0,1)$, we find sufficiently small positive number $\delta=\delta(n,\nu,L,\gamma_1,\gamma_2,\|a\|_{0,\alpha},\|H(x,Du)\|_{L^1(\Omega)},\gamma,\varepsilon)$ such that if $(\omega,p(\cdot),q(\cdot),f_1,f_2)$ is (δ,R)-vanishing, there exist $h_i\in W^{1,H_1^i}\left(B_{5\rho_i}(y^i)\right)$ and $v_i\in W^{1,\infty}\left(B_{5\rho_i}(y^i)\right)$ with the estimate

$$\fint_{B_{5\rho_i}}H(x,Du-Dh_i)\,dx\le 4^n\varepsilon\lambda,\quad \fint_{B_{5\rho_i}}H_1^i(x,Dh_i-Dv_i)\,dx\le 4^nS(\varepsilon,R,K)\lambda$$

and

$$\sup_{x\in B_{5\rho_i}}H_1^i(x_M^i,Dv_i(x))\le c_4\lambda,$$

where the constant c_4 is given in Lemma 3.4.9 and so independent of i and λ. Then we have

$$\fint_{B_{5\rho_i}}H(x,Du-Dh_i)^{\frac{\gamma(x)}{\gamma_-}}\,dx\le c\varepsilon^{\frac12}\lambda,\tag{3.4.62}$$

$$\fint_{B_{5\rho_i}}H_1^i(x,Dh_i-Dv_i)^{\frac{\gamma(x)}{\gamma_-}}\,dx\le cS(\varepsilon,R,K)^{\frac12}\lambda\tag{3.4.63}$$

and

$$\sup_{x\in B_{5\rho_i}}H_1^i(x_M^i,Dv_i(x))^{\frac{\gamma(x)}{\gamma_-}}\le c\lambda,\tag{3.4.64}$$

where the generic constants c depend on $n, \nu, L, \gamma_1, \gamma_2, \|a\|_{0,\alpha}, c_\gamma, \|H(x,Du)\|_{L^1(\Omega)}, \|H(x,F)\|_{L^{\gamma(\cdot)}(\Omega)}$ in (3.4.62), (3.4.63) and (3.4.64). Indeed, by Hölder's inequality, Lemma 3.4.4 to u and h_i, (3.4.51), (3.4.55), (3.4.56), (3.4.60) and (3.4.61),

$$
\begin{aligned}
&\fint_{B_{5\rho_i}} H(x, Du - Dh_i)^{\frac{\gamma(x)}{\gamma_-}}\, dx = \fint_{B_{5\rho_i}} H(x, Du - Dh_i)^{\frac{1}{2}+\left(\frac{\gamma(x)}{\gamma_-}-\frac{1}{2}\right)}\, dx \\
&\le \left(\fint_{B_{5\rho_i}} H(x, Du - Dh_i)\, dx\right)^{\frac{1}{2}} \left(\fint_{B_{5\rho_i}} H(x, Du - Dh_i)^{2\frac{\gamma(x)}{\gamma_-}-1}\, dx\right)^{\frac{1}{2}} \\
&\le \varepsilon^{\frac{1}{2}}\lambda^{\frac{1}{2}} \left(\fint_{B_{5\rho_i}} H(x, Du - Dh_i)^{2\frac{\gamma(x)}{\gamma_-}-1}\, dx\right)^{\frac{1}{2}} \\
&\le \varepsilon^{\frac{1}{2}}\lambda^{\frac{1}{2}} \left(\fint_{B_{5\rho_i}} (H(x, Du) + H(x, Dh_i))^{1+2\tilde{\mu}(2r)}\, dx + 1\right)^{\frac{1}{2}} \\
&\le c\varepsilon^{\frac{1}{2}}\lambda^{\frac{1}{2}} \left\{\fint_{B_{10\rho_i}} (H(x, Du) + H(x, Dh_i))\, dx \right. \\
&\qquad \left. + \left(\fint_{B_{10\rho_i}} H(x, F)^{1+\sigma_0}\, dx\right)^{\frac{1}{1+\sigma_0}} + 1\right\}^{\frac{1}{2}} \\
&\le c\varepsilon^{\frac{1}{2}}\lambda
\end{aligned}
$$

and so (3.4.62) holds. Similarly,

$$
\begin{aligned}
\fint_{B_{5\rho_i}} H_1^i(x, Dh_i - Dv_i)^{\frac{\gamma(x)}{\gamma_-}}\,dx &= \fint_{B_{5\rho_i}} H_1^i(x, Dh_i - Dv_i)^{\frac{1}{2}+\left(\frac{\gamma(x)}{\gamma_-}-\frac{1}{2}\right)}\,dx \\
&\le \left(\fint_{B_{5\rho_i}} H_1^i(x, Dh_i - Dv_i)\,dx\right)^{\frac{1}{2}} \left(\fint_{B_{5\rho_i}} H_1^i(x, Dh_i - Dv_i)^{2\frac{\gamma(x)}{\gamma_-}-1}\,dx\right)^{\frac{1}{2}} \\
&\le S(\varepsilon, R, K)^{\frac{1}{2}}\lambda^{\frac{1}{2}} \left(\fint_{B_{5\rho_i}} H_1^i(x, Dh_i - Dv_i)^{2\frac{\gamma(x)}{\gamma_-}-1}\,dx\right)^{\frac{1}{2}} \\
&\le S(\varepsilon, R, K)^{\frac{1}{2}}\lambda^{\frac{1}{2}} \left(\fint_{B_{5\rho_i}} \left(H_1^i(x, Dh_i) + H_1^i(x, Dv_i)\right)^{1+2\tilde{\mu}(2r)}\,dx + 1\right)^{\frac{1}{2}} \\
&\le cS(\varepsilon, R, K)^{\frac{1}{2}}\lambda,
\end{aligned}
$$

thus we have (3.4.63). Finally,

$$
\begin{aligned}
\sup_{x\in B_{5\rho_i}} H_1^i(x_M^i, Dv_i(x))^{\frac{\gamma(x)}{\gamma_-}} &\le \sup_{x\in B_{5\rho_i}} H_1^i(x_M^i, Dv_i(x))^{\frac{\gamma_+}{\gamma_-}} + 1 \\
&\le \left(\sup_{x\in B_{5\rho_i}} H_1^i(x_M^i, Dv_i(x))\right)^{\frac{\gamma_+}{\gamma_-}} + 1 \\
&\le c\lambda^{\frac{\gamma_+}{\gamma_-}}
\end{aligned}
$$

so we have (3.4.64).

Let $c_5 = 2 \cdot 4^{\gamma_2 - 1}(c_4 + 2\|a\|_{0,\alpha} + 2) > 1$. Since $E(r_1, c_5\lambda) \subset E(r_1, \lambda)$, observe that

$$
\begin{aligned}
&\int_{E(r_1, c_5\lambda)} H(x, Du)^{\frac{\gamma(x)}{\gamma_-}}\,dx \\
&\quad \le \sum_{i=1}^{\infty} \left(\int_{E(r_1, c_5\lambda)\cap B_{5\rho_i}(y^i)} H(x, Du)^{\frac{\gamma(x)}{\gamma_-}}\,dx\right).
\end{aligned}
\tag{3.4.65}
$$

For almost every $x \in E(r_1, c_5\lambda) \cap B_{5\rho_i}(y^i)$ with $B_{80\rho_i}(y^i) \subset B_{r_2}$, it follows

that

$$
\begin{aligned}
&H(x, Du) \\
&\leq 4^{\gamma_2 - 1} \left(H(x, Du - Dh_i) + H(x, Dh_i - Dv_i) + H(x, Dv_i)\right) \\
&\leq 4^{\gamma_2 - 1} \big(H(x, Du - Dh_i) + H_1^i(x, Dh_i - Dv_i) + H_1^i(x_M^i, Dv_i) + 2\|a\|_{0,\alpha} + 2\big) \\
&\leq 4^{\gamma_2 - 1} \big(H(x, Du - Dh_i) + H_1^i(x, Dh_i - Dv_i)\big) + 4^{\gamma_2 - 1}(c_2 + 2\|a\|_{0,\alpha} + 2)\lambda \\
&\leq 4^{\gamma_2 - 1} \big(H(x, Du - Dh_i) + H_1^i(x, Dh_i - Dv_i)\big) + \frac{1}{2} H(x, Du)
\end{aligned}
$$

with (3.4.63). Then we have

$$
H(x, Du) \leq 2 \cdot 4^{\gamma_2 - 1} \big(H(x, Du - Dh_i) + H_1^i(x, Dh_i - Dv_i)\big). \tag{3.4.66}
$$

and so

$$
\begin{aligned}
&H(x, Du)^{\frac{\gamma(x)}{\gamma_-}} \\
&\leq 4^{\frac{{\gamma_2}^2}{\gamma_1}} \left(H(x, Du - Dh_i)^{\frac{\gamma(x)}{\gamma_-}} + H_1^i(x, Dh_i - Dv_i)^{\frac{\gamma(x)}{\gamma_-}}\right)
\end{aligned} \tag{3.4.67}
$$

Thus, it follows by (3.4.62), (3.4.63) and (3.4.67) that

$$
\begin{aligned}
&\int_{E(r_1, c_5\lambda) \cap B_{5\rho_i}(y^i)} H(x, Du)^{\frac{\gamma(x)}{\gamma_-}} \, dx \\
&\quad \leq 4^{\frac{{\gamma_2}^2}{\gamma_1}} \left(\int_{B_{5\rho_i}(y^i)} H(x, Du - Dh_i)^{\frac{\gamma(x)}{\gamma_-}} \, dx \right. \qquad (3.4.68) \\
&\qquad\qquad \left. + \int_{B_{5\rho_i}(y^i)} H_1^i(x, Dh_i - Dv_i)^{\frac{\gamma(x)}{\gamma_-}} \, dx \right) \\
&\quad \leq c \cdot 4^{\frac{{\gamma_2}^2}{\gamma_1}} 5^n |B_{\rho_i}(y^i)| \left(\varepsilon^{\frac{1}{2}} + S(\varepsilon, R, K)^{\frac{1}{2}}\right) \lambda, \qquad (3.4.69)
\end{aligned}
$$

where $c = c(n, \nu, L, \gamma_1, \gamma_2, \|a\|_{0,\alpha}, c_\gamma, \|H(x, Du)\|_{L^1(\Omega)}, \|H(x, F)\|_{L^{\gamma(\cdot)}(\Omega)})$. To estimate $|B_{\rho_i}(y^i)|$, we have from (3.4.53) that

$$\begin{aligned} &\fint_{B_{\rho_i}(y^i)} H(x,Du)^{\frac{\gamma(x)}{\gamma_-}}\,dx \geq \frac{\lambda}{2} \\ \text{or} \quad &\fint_{B_{\rho_i}(y^i)} H(x,F)^{\left(1+\frac{\sigma_0}{4}\right)\frac{\gamma(x)}{\gamma_-}}\,dx \geq \left(\frac{\delta\lambda}{2}\right)^{\frac{4+\sigma_0}{4}} \end{aligned} \tag{3.4.70}$$

hold. The first inequality of (3.4.70) implies that

$$|B_{\rho_i}(y^i)| \leq \frac{2}{\lambda}\int_{E(r_2,\frac{\lambda}{4})\cap B_{\rho_i}(y^i)} H(x,Du)^{\frac{\gamma(x)}{\gamma_-}}\,dx + \frac{|B_{\rho_i}(y^i)|}{2}$$

and so

$$|B_{\rho_i}(y^i)| \leq \frac{4}{\lambda}\int_{E(r_2,\frac{\lambda}{4})\cap B_{\rho_i}(y^i)} H(x,Du)^{\frac{\gamma(x)}{\gamma_-}}\,dx.$$

Likewise, the second inequality of (3.4.70) implies that

$$|B_{\rho_i}(y^i)| \leq \frac{2^{2+\sigma_0}}{(\delta\lambda)^{1+\sigma_0}}\int_{\mathcal{E}(r_2,2^{-1-\frac{1}{1+\sigma_0}}\delta\lambda)\cap B_{\rho_i}(y^i)} H(x,F)^{(1+\sigma_0)\frac{\gamma(x)}{\gamma_-}}\,dx.$$

Therefore, we have

$$\begin{aligned} |B_{\rho_i}(y^i)| \leq \frac{4}{\lambda}&\int_{E(r_2,\frac{\lambda}{4})\cap B_{\rho_i}(y^i)} H(x,Du)^{\frac{\gamma(x)}{\gamma_-}}\,dx \\ &+ \frac{2^{2+\sigma_0}}{(\delta\lambda)^{1+\sigma_0}}\int_{\mathcal{E}(r_2,2^{-1-\frac{1}{1+\sigma_0}}\delta\lambda)\cap B_{\rho_i}(y^i)} H(x,F)^{(1+\sigma_0)\frac{\gamma(x)}{\gamma_-}}\,dx. \end{aligned} \tag{3.4.71}$$

By (3.4.68) and (3.4.71), we see that

$$\begin{aligned} &\int_{E(r_1,c_5\lambda)\cap B_{5\rho_i}(y^i)} H(x,Du)^{\frac{\gamma(x)}{\gamma_-}}\,dx \\ &\leq cS(\varepsilon,R,K)\int_{E(r_2,\frac{\lambda}{4})\cap B_{\rho_i}(y^i)} H(x,Du)^{\frac{\gamma(x)}{\gamma_-}}\,dx \\ &\quad + \frac{cS(\varepsilon,R,K)}{\lambda^{\sigma_0}}\int_{\mathcal{E}(r_2,2^{-1-\frac{1}{1+\sigma_0}}\delta\lambda)\cap B_{\rho_i}(y^i)} \left(\frac{H(x,F)}{\delta}\right)^{(1+\sigma_0)\frac{\gamma(x)}{\gamma_-}}\,dx \end{aligned} \tag{3.4.72}$$

for each i. Since $B_{\rho_i}(y^i)$ is mutually disjoint, by (3.4.65) and (3.4.72) we have

$$\begin{aligned}
&\int_{E(r_1,\lambda)} H(x,Du)^{\frac{\gamma(x)}{\gamma_-}}\,dx \\
&\le cS(\varepsilon,R,K)\int_{E(r_2,\frac{\lambda}{4c_5})} H(x,Du)^{\frac{\gamma(x)}{\gamma_-}}\,dx \\
&\quad + \frac{cS(\varepsilon,R,K)}{\lambda^{\sigma_0}}\int_{\mathcal{E}(r_2,\frac{2^{-1-\frac{1}{1+\sigma_0}}\delta\lambda}{c_5})} \left(\frac{H(x,F)}{\delta}\right)^{(1+\sigma_0)\frac{\gamma(x)}{\gamma_-}}\,dx
\end{aligned} \tag{3.4.73}$$

for any $\lambda \ge \tilde{c}c_5A\lambda_0$.

Define the truncated functions

$$\left[H(x,Du)^{\frac{\gamma(x)}{\gamma_-}}\right]_t = \min\left\{H(x,Du)^{\frac{\gamma(x)}{\gamma_-}}, t\right\} \quad (t\ge 0). \tag{3.4.74}$$

Then for $t \ge 2\tilde{c}c_5A\lambda_0$,

$$\begin{aligned}
&\int_{\tilde{c}c_5A\lambda_0}^{t} \lambda^{\gamma_- -2}\int_{E(r_1,\lambda)} H(x,Du)^{\frac{\gamma(x)}{\gamma_-}}\,dxd\lambda \\
&\le cS(\varepsilon,R,K)\int_{\tilde{c}c_5A\lambda_0}^{t} \lambda^{\gamma_- -2}\int_{E(r_2,\frac{\lambda}{4})} H(x,Du)^{\frac{\gamma(x)}{\gamma_-}}\,dxd\lambda \\
&\quad + cS(\varepsilon,R,K)\int_{\tilde{c}c_5A\lambda_0}^{t} \lambda^{\gamma_- -\sigma_0-2}\int_{\mathcal{E}(r_2,\frac{2^{-1-\frac{1}{1+\sigma_0}}\delta\lambda}{c_5})} \left(\frac{H(x,F)}{\delta}\right)^{(1+\sigma_0)\frac{\gamma(x)}{\gamma_-}}\,dxd\lambda.
\end{aligned}$$

By change of variables and Fubini's theorem,

$$\begin{aligned}
\int_{\tilde{c}c_5A\lambda_0}^{t} \lambda^{\gamma_- -2}&\int_{E(r_1,\lambda)} H(x,Du)^{\frac{\gamma(x)}{\gamma_-}}\,dxd\lambda \\
&= \frac{1}{\gamma_1 - 1}\int_{B_{r_1}} [H(x,Du)^{\frac{\gamma(x)}{\gamma_-}}]_t^{\gamma_- -1} H(x,Du)^{\frac{\gamma(x)}{\gamma_-}}\,dx \\
&\quad - \int_0^{\tilde{c}c_5A\lambda_0} \lambda^{\gamma_- -2}\int_{E(r_1,\lambda)} H(x,Du)^{\frac{\gamma(x)}{\gamma_-}}\,dxd\lambda,
\end{aligned}$$

$$\begin{aligned}\int_{\tilde{c}c_5A\lambda_0}^{t} & \lambda^{\gamma_- -2}\int_{E(r_2,\frac{\lambda}{4c_5})} H(x,Du)^{\frac{\gamma(x)}{\gamma_-}}\,dxd\lambda \\ & \le \frac{1}{\gamma_1-1}\int_{B_{r_2}} [H(x,Du)^{\frac{\gamma(x)}{\gamma_-}}]_{\frac{t}{4c_5}}^{\gamma_- -1} H(x,Du)^{\frac{\gamma(x)}{\gamma_-}}\,dx \\ & \le \frac{1}{\gamma_1-1}\int_{B_{r_2}} [H(x,Du)^{\frac{\gamma(x)}{\gamma_-}}]_{t}^{\gamma_- -1} H(x,Du)^{\frac{\gamma(x)}{\gamma_-}}\,dx\end{aligned}$$

and

$$\begin{aligned}\int_{\tilde{c}c_5A\lambda_0}^{t} & \lambda^{\gamma_- -\sigma_0-2}\int_{\mathcal{E}(r_2,\frac{2^{-1-\frac{1}{1+\sigma_0}}\delta\lambda}{c_5})} \left(\frac{H(x,F)}{\delta}\right)^{(1+\sigma_0)\frac{\gamma(x)}{\gamma_-}}\,dxd\lambda \\ & \le \int_0^{\infty}\lambda^{\gamma_- -\sigma_0-2}\int_{\mathcal{E}(r_2,\frac{\delta\lambda}{4c_5})} \left(\frac{H(x,F)}{\delta}\right)^{(1+\sigma_0)\frac{\gamma(x)}{\gamma_-}}\,dxd\lambda \\ & \le c\int_{B_{r_2}}\left(\frac{H(x,F)}{\delta}\right)^{\gamma(x)}\,dx.\end{aligned}$$

Moreover, by the definition of λ_0, Lemma 3.4.4, and (3.4.55) and (3.4.56), we find

$$\begin{aligned}\int_{0}^{\tilde{c}c_5A\lambda_0} & \lambda^{\gamma_- -2}\int_{E(r_1,\lambda)} H(x,Du)^{\frac{\gamma(x)}{\gamma_-}}\,dxd\lambda \\ & \le \int_0^{\tilde{c}c_5A\lambda_0}\lambda^{\gamma_- -2}d\lambda\int_{B_{r_2}} H(x,Du)^{\frac{\gamma(x)}{\gamma_-}}\,dx \\ & \le \frac{(\tilde{c}c_5A\lambda_0)^{\gamma_- -1}}{\gamma_1-1}\int_{B_{r_2}} H(x,Du)^{\frac{\gamma(x)}{\gamma_-}}\,dx \\ & \le \frac{(\tilde{c}c_5A\lambda_0)^{\gamma_- -1}|B_{r_2}|}{\gamma_1-1}\left(\fint_{B_{r_2}} H(x,Du)^{1+\tilde{\mu}(r)}\,dx+1\right) \\ & \le \frac{c(\tilde{c}c_5A\lambda_0)^{\gamma_-}|B_{r_2}|}{\gamma_1-1}.\end{aligned}$$

Consequently, we discover

$$\begin{aligned}
&\fint_{B_{r_1}} [H(x,Du)^{\frac{\gamma(x)}{\gamma_-}}]_t^{\gamma_- -1} H(x,Du)^{\frac{\gamma(x)}{\gamma_-}}\,dx \\
&\qquad \le cS(\varepsilon,R,K)\fint_{B_{r_2}} [H(x,Du)^{\frac{\gamma(x)}{\gamma_-}}]_t^{\gamma_- -1} H(x,Du)^{\frac{\gamma(x)}{\gamma_-}}\,dx \\
&\qquad\quad + cS(\varepsilon,R,K)\fint_{B_{r_2}} \left(\frac{H(x,F)}{\delta}\right)^{\gamma(x)} dx + c(\tilde{c}c_5)^{\gamma_-}\left(\frac{\lambda_0}{(r_2-r_1)^n}\right)^{\gamma_-},
\end{aligned}$$

where $c = c(\mathtt{data}, \mathrm{dist}(\Omega_0,\partial\Omega), \gamma_1, \|H(x,F)\|_{L^{\gamma(\cdot)}(\Omega)}) \ge 1$. Choose $0<\varepsilon<1$, $K \ge 4$ and then $0<R<1$ such that
$R = R(\mathtt{data}, \omega(\cdot), \mathrm{dist}(\Omega_0,\partial\Omega), \gamma_1, \|H(x,F)\|_{L^{\gamma(\cdot)}(\Omega)})$ in order to have

$$0 < cS(\varepsilon,R,K) < \frac{1}{2}. \tag{3.4.75}$$

Then we find a small $\delta = \delta(\mathtt{data}) > 0$ from Lemma 3.4.9. In light of Lemma 2.0.1, we obtain

$$\fint_{B_{\frac{r}{2}}} [H(x,Du)^{\frac{\gamma(x)}{\gamma_-}}]_t^{\gamma_- -1} H(x,Du)^{\frac{\gamma(x)}{\gamma_-}}\,dx \le c\lambda_0^{\gamma_-} + c\fint_{B_{2r}} H(x,F)^{\gamma(x)}\,dx.$$

Letting $t\to\infty$ and then recalling the definition of λ_0 in (3.4.50), Hölder's inequality, we have

$$\begin{aligned}
&\fint_{B_{\frac{r}{2}}} H(x,Du)^{\gamma(x)}\,dx \\
&\le c\left(\fint_{B_{2r}} H(x,Du)\,dx + \left(\fint_{B_{2r}} H(x,F)^{1+\sigma_0}\,dx\right)^{\frac{1}{1+\sigma_0}} + 1\right)^{\gamma_-} \\
&\quad + c\fint_{B_{2r}} H(x,F)^{\gamma(x)}\,dx \\
&\le c\left(\fint_{B_{2r}} H(x,Du)\,dx\right)^{\gamma_-} + c\fint_{B_{2r}} H(x,F)^{\gamma(x)}\,dx + c,
\end{aligned}$$

and so we arrive at the required Calderón-Zygmund estimates (3.4.15). Now $H(x,Du) \in L^{\gamma(\cdot)}(\Omega_0)$ is obtained by a standard covering argument. The proof is complete.

3.5 Local estimates for Orlicz double phase problems with variable exponents

In the present section, optimal regularity estimates are established for the gradient of solutions to non-uniformly elliptic equations of Orlicz double phase with variable exponents type in divergence form under sharp conditions on such highly nonlinear operators for the Calderón-Zygmund theory.

3.5.1 Hypothesis and main results

The functions in the exponents $p(\cdot), q(\cdot) : \Omega \to [1, \infty)$ are bounded and log-Hölder continuous functions in the following way that

$$1 \leq p(x), q(x) \leq m_{pq} \quad \text{for every} \quad x \in \Omega, \tag{3.5.1}$$

and

$$|p(x) - p(y)| + |q(x) - q(y)| \leq \frac{M_{pq}}{-\log |x-y|} \tag{3.5.2}$$

for some non-negative constants m_{pq} and M_{pq}, whenever $x, y \in \Omega$ with $|x - y| \leq 1/2$, whereas the coefficient function $a : \Omega \to [0, \infty)$ satisfies

$$0 \leq a(\cdot) \in C^{0,\alpha}(\Omega), \quad \alpha \in (0, 1]. \tag{3.5.3}$$

We shall assume that the functions presented above satisfy the central assumption in this section:

$$\kappa := \sup_{x \in \Omega} \sup_{t>0} \frac{H^{q(x)}(t)}{G^{p(x)}(t) + G^{(1+\frac{\alpha}{n})p(x)}(t)} < \infty. \tag{3.5.4}$$

We consider weak solutions of the equation

$$-\mathrm{div} A(x, Du) = -\mathrm{div} B(x, F) \quad \text{in} \quad \Omega, \tag{3.5.5}$$

where the vector field $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is represented by

$$A(x, z) = A_1(x, z) + a(x) A_2(x, z) \tag{3.5.6}$$

for every $x \in \Omega$ and $z \in \mathbb{R}^n$, in which $A_1, A_2 : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ are Carathéodory vector fields and differentiable with respect to second variable $z \in \mathbb{R}^n \setminus \{0\}$ satisfying the following structure assumptions

$$\begin{cases} |z||\partial_z A_1(x,z)| + |A_1(x,z)| \leq L\dfrac{G^{p(x)}(|z|)}{|z|} \\ |z||\partial_z A_2(x,z)| + |A_2(x,z)| \leq L\dfrac{H^{q(x)}(|z|)}{|z|}, \end{cases} \tag{3.5.7}$$

and

$$\begin{cases} \langle \partial_z A_1(x,z)\xi, \xi\rangle \geq \nu\dfrac{G^{p(x)}(|z|)}{|z|^2}|\xi|^2 \\ \langle \partial_z A_2(x,z)\xi, \xi\rangle \geq \nu\dfrac{H^{q(x)}(|z|)}{|z|^2}|\xi|^2 \end{cases} \tag{3.5.8}$$

with fixed constants $0 < \nu \leq L < \infty$, whenever $z \in \mathbb{R}^n \setminus \{0\}, \xi \in \mathbb{R}^n$ and $x \in \Omega$. The map $B : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ appearing on the right-hand side of the equation (3.5.5) is a Carathéodory vector field such that

$$|B(x,z)| \leq L\left(\frac{G^{p(x)}(|z|) + a(x)H^{q(x)}(|z|)}{|z|}\right) \tag{3.5.9}$$

for all $x \in \Omega$ and $z \in \mathbb{R}^n \setminus \{0\}$.

To go further on, we need to define a notion of (δ, R)-vanishing condition.

Definition 3.5.1. *With small numbers $\delta \in (0, 1/8)$ and $R \in (0,1)$, we say that the quadruple $(p(\cdot), q(\cdot), A_1, A_2)$ is (δ, R)-vanishing if the following two conditions are satisfied:*

1. *There is a non-decreasing concave function $\omega : [0,\infty) \to [0,\infty)$ such that*

$$|p(x) - p(y)| + |q(x) - q(y)| \leq \omega(|x-y|), \qquad \omega(0) = 0 \tag{3.5.10}$$

for every $x, y \in \Omega$, with

$$\sup_{0<\rho\leq R} \omega(\rho)\log\frac{1}{\rho} \leq \delta. \tag{3.5.11}$$

2. *The following inequality holds true:*

$$\sup_{0<\rho\leq R}\sup_{B_\rho(y)\subset\Omega}\fint_{B_\rho(y)}[\theta(A_1,B_\rho(y))(x)+\theta(A_2,B_\rho(y))(x)]\,dx\leq\delta, \tag{3.5.12}$$

where, with fixed $y\in\Omega$, the maps $\theta\left(A_1,B_\rho(y)\right)(\cdot),\theta\left(A_2,B_\rho(y)\right)(\cdot):\Omega\to[0,\infty)$ are given by

$$\begin{aligned}&\theta\left(A_1,B_\rho(y)\right)(x)\\&\quad:=\sup_{z\in\mathbb{R}^n\setminus\{0\}}\left|\frac{A_1(x,z)}{G^{p(x)-1}(|z|)G'(|z|)}-\left(\frac{A_1(\cdot,z)}{G^{p(\cdot)-1}(|z|)G'(|z|)}\right)_{B_\rho(y)}\right|\end{aligned}$$

and

$$\begin{aligned}&\theta\left(A_2,B_\rho(y)\right)(x)\\&\quad:=\sup_{z\in\mathbb{R}^n\setminus\{0\}}\left|\frac{A_2(x,z)}{H^{q(x)-1}(|z|)H'(|z|)}-\left(\frac{A_2(\cdot,z)}{H^{q(\cdot)-1}(|z|)H'(|z|)}\right)_{B_\rho(y)}\right|\end{aligned}$$

for every $x\in\Omega$.

Remark 3.5.2. *In fact, the smallness of the quantity described in* (3.5.12) *says that the mappings $x\mapsto\frac{A_1(x,z)}{G^{p(x)-1}(|z|)G'(|z|)}$ and $x\mapsto\frac{A_2(x,z)}{H^{q(x)-1}(|z|)H'(|z|)}$ have a small BMO (Bounded mean oscillation) condition, uniformly in z variable, that are naturally considered in earlier works [16, 51, 58, 89] and references therein, as an minimal condition for the Calderón-Zygmund type estimates.*

Remark 3.5.3. *The structure assumptions* (3.5.7) *together with Remark 2.1.2 imply that*

$$\theta\left(A_1,B_\rho(y)\right)(x)\leq 2L\quad\textit{and}\quad\theta\left(A_2,B_\rho(y)\right)(x)\leq 2L$$

for every $x\in\Omega$, whenever $B_\rho(y)\subset\Omega$ is a ball. Moreover, with a number $d\geq1$, we also notice the following obvious but useful inequality:

$$\begin{aligned}&\fint_{B_\rho(y)}[\theta(A_1,B_\rho(y))(x)+\theta(A_2,B_\rho(y))(x)]^d\,dx\\&\leq(4L)^{d-1}\fint_{B_\rho(y)}[\theta(A_1,B_\rho(y))(x)+\theta(A_2,B_\rho(y))(x)]\,dx.\end{aligned}$$

For the abbreviation of notations, we shall use a set of parameters which is data of our problem for a solution u of the equation (3.5.5) as follows:

$$\texttt{data} \equiv \left(n, \kappa, s(G), s(H), \nu, L, m_{pq}, \alpha, \|a\|_{0,\alpha}, \omega(\cdot), \|\Psi(x, Du)\|_{L^1(\Omega)}\right).$$

Denoting

$$\Psi(x,z) := G^{p(x)}(|z|) + a(x)H^{q(x)}(|z|) \quad \text{for every } x \in \Omega \text{ and } z \in \mathbb{R}^n \text{ or } z \in \mathbb{R},$$

we are ready to state the main result of this section.

Theorem 3.5.4. *Let $u \in W^{1,\Psi}(\Omega)$ be a weak solution to* (3.5.5) *with $\Psi(x,F) \in L^\gamma(\Omega)$ for some $\gamma > 1$ under the assumptions* (3.5.1)-(3.5.4) *and* (3.5.7)-(3.5.9). *Then there exists $\delta \equiv \delta(\texttt{data}, \gamma) \in (0, 1/8)$ such that if $(p(\cdot), q(\cdot), A_1, A_2)$ is (δ, R)-vanishing for some small $R > 0$, then the following implication holds:*

$$\Psi(x,F) \in L^\gamma(\Omega) \Longrightarrow \Psi(x, Du) \in L^\gamma_{\mathrm{loc}}(\Omega). \tag{3.5.13}$$

Moreover, for every open subset $\Omega_0 \Subset \Omega$, there exists a radius R depending only on $\texttt{data}$, $\mathrm{dist}(\Omega_0, \partial\Omega)$,$\gamma$ *and* $\|\Psi(x,F)\|_{L^\gamma(\Omega)}$ *such that the following inequality*

$$\begin{aligned} &\left(\fint_{B_{r/2}(x_0)} [\Psi(x, Du)]^\gamma \, dx\right)^{\frac{1}{\gamma}} \\ &\quad \le c \fint_{B_r(x_0)} \Psi(x, Du)\, dx + c\left(\fint_{B_r(x_0)} [\Psi(x,F)]^\gamma \, dx + 1\right)^{\frac{1}{\gamma}} \end{aligned} \tag{3.5.14}$$

holds for some constant $c \equiv c(\texttt{data}, \mathrm{dist}(\Omega_0, \partial\Omega), \gamma, \|\Psi(x,F)\|_{L^\gamma(\Omega)})$, whenever $B_r(x_0) \Subset \Omega_0$ is a ball with $0 < r < R$.

The above theorem is a considerable generalization of the work [38], where local gradient estimates for the case $G(t) = t^{p_m}$ and $H(t) = t^{q_m}$ with some numbers $p_m, q_m > 1$ were investigated. Moreover, it covers the results from [12] when the exponent functions $p(\cdot) \equiv q(\cdot) \equiv 1$.

Furthermore, we can always assume that

$$G(1) = H(1) = 1, \tag{3.5.15}$$

otherwise we restate the problem under the new settings by considering $G(t) := \frac{G(t)}{G(1)}$ and $H(t) := \frac{H(t)}{H(1)}$ for every $t \geq 0$.

3.5.2 Absence of Lavrentiev phenomenon and Sobolev-Poincaré type inequality

In the sequel we prove an approximation property for functions of $W^{1,\Psi}(\Omega)$, the so-called absence of Lavrentiev phenomenon.

Theorem 3.5.5. *Under the assumptions* (3.5.1)-(3.5.4), *for any function* $v \in W^{1,\Psi}(\Omega)$ *and a ball* $B \Subset \Omega$, *there exists a sequence of functions* $\{v_k\}_{k=1}^{\infty} \subset W^{1,\infty}(B)$ *such that*

$$\begin{aligned} & v_k \to v \ \ \textit{in} \ \ W^{1,G^{p(\cdot)}}(B) \\ & \textit{and} \quad \lim_{k\to\infty} \int_B \Psi(x, Dv_k)\, dx = \int_B \Psi(x, Dv)\, dx. \end{aligned} \tag{3.5.16}$$

Proof. Fix a ball $B \Subset \Omega$ and take small enough $\varepsilon_0 \in (0,1)$ such that $B \equiv B_r \Subset B_{r+\varepsilon_0} \Subset \Omega$. Let $\rho \in C_0^{\infty}(B_1(0))$ be a standard mollifier with $\int_{\mathbb{R}^n} \rho\, dx = 1$. Set $\rho_\varepsilon(x) := \frac{1}{\varepsilon^n}\rho\left(\frac{x}{\varepsilon}\right)$ for $x \in B_\varepsilon(0)$ with $0 < \varepsilon < \varepsilon_0$. Then

$$\rho_\varepsilon \in C_0^{\infty}(B_\varepsilon), \qquad \int_{\mathbb{R}^n} \rho_\varepsilon\, dx = 1, \quad 0 \leq \rho_\varepsilon \leq c(n)\varepsilon^{-n}.$$

Let us denote by

$$\begin{aligned} a_\varepsilon(x) := \inf_{y \in B_\varepsilon(x)} a(y), \quad p_\varepsilon(x) := \inf_{y \in B_\varepsilon(x)} p(y) \\ \text{and} \quad q_\varepsilon(x) := \inf_{y \in B_\varepsilon(x)} q(y), \end{aligned} \tag{3.5.17}$$

and consider the mollified functions by

$$v_\varepsilon(x) = (v * \rho_\varepsilon)(x) \quad \text{and} \quad \Psi_\varepsilon(x,z) = G^{p(x)}(|z|) + a_\varepsilon(x) H^{q(x)}(|z|)$$

for every $x \in B$ and $z \in \mathbb{R}^n$. First, using the assumption (3.5.2), we observe that

$$\begin{aligned} \varepsilon^{-(p(x)-p_\varepsilon(x))} + \varepsilon^{-(q(x)-q_\varepsilon(x))} &\leq e^{\left(\log\frac{1}{\varepsilon}\right)(p(x)-p_\varepsilon(x))} + e^{\left(\log\frac{1}{\varepsilon}\right)(q(x)-q_\varepsilon(x))} \\ &\leq 2e^{M_{pq}} \end{aligned} \tag{3.5.18}$$

holds true for every $x \in B$. Therefore, the last display directly implies

$$\varepsilon^{-\frac{n}{p_\varepsilon(x)}} = \varepsilon^{-\frac{n}{p(x)}\left(\frac{p(x)-p_\varepsilon(x)}{p_\varepsilon(x)}\right)}\varepsilon^{-\frac{n}{p(x)}} \leq e^{nM_{pq}}\varepsilon^{-\frac{n}{p(x)}} \leq c(n, M_{pq})\varepsilon^{-\frac{n}{p(x)}}.$$

By the very definition of convolution together with Hölder's and Jensen's inequalities, we have

$$\begin{aligned}
G(|Dv_\varepsilon(x)|) = G(|(Dv * \rho_\varepsilon)(x)|) &\leq \int_{\mathbb{R}^n} G(|Dv(x-y)|)\rho_\varepsilon(y)\,dy \\
&\leq c \fint_{B_\varepsilon(x)} G(|Dv(y)|)\,dy \leq c\left(\fint_{B_\varepsilon(x)} G^{p_\varepsilon(x)}(|Dv(y)|)\,dy\right)^{\frac{1}{p_\varepsilon(x)}} \\
&\leq c\left(\fint_{B_\varepsilon(x)} G^{p(y)}(|Dv(y)|)\,dy + 1\right)^{\frac{1}{p_\varepsilon(x)}} \\
&\leq c\varepsilon^{-\frac{n}{p_\varepsilon(x)}}\left(\int_\Omega G^{p(y)}(|Dv(y)|)\,dy + 1\right) \leq c\varepsilon^{-\frac{n}{p_\varepsilon(x)}}
\end{aligned}$$

for some $c \equiv c(n, m_{pq}, \|\Psi(x, Dv)\|_{L^1(\Omega)})$. Combining the last two displays, we see that

$$G(|Dv_\varepsilon(x)|) \leq c\varepsilon^{-\frac{n}{p(x)}} \tag{3.5.19}$$

holds with some constant $c \equiv c(n, m_{pq}, M_{pq}, \|\Psi(x, Dv)\|_{L^1(\Omega)})$ for every $x \in B$. Moreover, again recalling the definition of p_ε and q_ε in (3.5.17) together with (3.5.18) and (3.5.19), we have

$$\begin{aligned}
G^{p(x)}(|Dv_\varepsilon(x)|) &= G^{p_\varepsilon(x)}(|Dv_\varepsilon(x)|)G^{p(x)-p_\varepsilon(x)}(|Dv_\varepsilon(x)|) \\
&\leq c\varepsilon^{-\frac{n}{p(x)}(p(x)-p_\varepsilon(x))}G^{p_\varepsilon(x)}(|Dv_\varepsilon(x)|) \\
&\leq cG^{p_\varepsilon(x)}(|Dv_\varepsilon(x)|)
\end{aligned} \tag{3.5.20}$$

and

$$
\begin{aligned}
&H^{q(x)}(|Dv_\varepsilon(x)|) \\
&= H^{q_\varepsilon(x)}(|Dv_\varepsilon(x)|)H^{q(x)-q_\varepsilon(x)}(|Dv_\varepsilon(x)|) \\
&\leq c\left(G^{p(x)}(|Dv_\varepsilon(x)|) + G^{\left(1+\frac{\alpha}{n}\right)p(x)}(|Dv_\varepsilon(x)|)\right)^{\frac{q(x)-q_\varepsilon(x)}{q(x)}} \\
&\quad \times H^{q_\varepsilon(x)}(|Dv_\varepsilon(x)|) \\
&\leq c\left(\varepsilon^{-n} + \varepsilon^{-n\left(1+\frac{\alpha}{n}\right)}\right)^{\frac{q(x)-q_\varepsilon(x)}{q(x)}} H^{q_\varepsilon(x)}(|Dv_\varepsilon(x)|) \\
&\leq cH^{q_\varepsilon(x)}(|Dv_\varepsilon(x)|)
\end{aligned} \tag{3.5.21}
$$

with some constant $c \equiv c(n, \kappa, m_{pq}, M_{pq}, \alpha, \|\Psi(x, Dv)\|_{L^1(\Omega)})$ for every $x \in B$, where we have used the assumption (3.5.4) and some elementary manipulations. Therefore, applying again Jensen's inequality, we obtain

$$
\begin{aligned}
G^{p_\varepsilon(x)}(|Dv_\varepsilon(x)|) &= G^{p_\varepsilon(x)}\left(\left|\int_{B_\varepsilon(x)} Dv(y)\rho_\varepsilon(x-y)\,dy\right|\right) \\
&\leq \int_{B_\varepsilon(x)} G^{p_\varepsilon(x)}(|Dv(y)|)\rho_\varepsilon(x-y)\,dy \\
&\leq \int_{B_\varepsilon(x)} \Psi(y, Dv(y))\rho_\varepsilon(x-y)\,dy + c \\
&\leq \left[\Psi(\cdot, Dv(\cdot)) * \rho_\varepsilon\right](x) + c
\end{aligned} \tag{3.5.22}
$$

and

$$
\begin{aligned}
a_\varepsilon(x)H^{q_\varepsilon(x)}(|Dv_\varepsilon(x)|) &= a_\varepsilon(x)H^{q_\varepsilon(x)}\left(\left|\int_{B_\varepsilon(x)} Dv(y)\rho_\varepsilon(x-y)\,dy\right|\right) \\
&\leq a_\varepsilon(x)\int_{B_\varepsilon(x)} H^{q_\varepsilon(x)}(|Dv(y)|)\rho_\varepsilon(x-y)\,dy \\
&\leq \int_{B_\varepsilon(x)} \Psi(y, Dv(y))\rho_\varepsilon(x-y)\,dy + c \\
&\leq \left[\Psi(\cdot, Dv(\cdot)) * \rho_\varepsilon\right](x) + c
\end{aligned} \tag{3.5.23}
$$

for some constant c independent of ε. To proceed further, let us observe

$$\begin{aligned}\Psi(x, Dv_\varepsilon(x)) &\leq \Psi_\varepsilon(x, Dv_\varepsilon(x)) + |a(x) - a_\varepsilon(x)| H^{q(x)}(|Dv_\varepsilon(x)|) \\ &\leq \Psi_\varepsilon(x, Dv_\varepsilon(x)) + [a]_{0,\alpha} \varepsilon^\alpha H^{q(x)}(|Dv_\varepsilon(x)|).\end{aligned} \tag{3.5.24}$$

Using again the assumption (3.5.4) together with (3.5.19), we have

$$\begin{aligned}H^{q(x)}(|Dv_\varepsilon(x)|) &\leq \kappa \left(1 + G^{\frac{\alpha}{n} p(x)}(|Dv_\varepsilon(x)|)\right) G^{p(x)}(|Dv_\varepsilon(x)|) \\ &\leq c(1 + \varepsilon^{-\alpha}) G^{p(x)}(|Dv_\varepsilon(x)|) \leq c(1 + \varepsilon^{-\alpha}) \Psi_\varepsilon(x, Dv_\varepsilon(x)).\end{aligned}$$

Then inserting the last display into (3.5.24), we find

$$\begin{aligned}\Psi(x, Dv_\varepsilon(x)) &\leq \Psi_\varepsilon(x, Dv_\varepsilon(x)) + c\varepsilon^\alpha (1 + \varepsilon^{-\alpha}) \Psi_\varepsilon(x, Dv_\varepsilon(x)) \\ &\leq c \Psi_\varepsilon(x, Dv_\varepsilon(x))\end{aligned}$$

with some constant $c \equiv c(n, \kappa, m_{pq}, M_{pq}, \alpha, [a]_{0,\alpha}, \|\Psi(x, Dv)\|_{L^1(\Omega)})$. Now taking (3.5.20)-(3.5.23) into account in the last display, we conclude with

$$\Psi(x, Dv_\varepsilon(x)) \leq c \left[\Psi(\cdot, Dv(\cdot)) * \rho_\varepsilon\right](x) + c$$

for some constant $c \equiv c(n, \kappa, m_{pq}, M_{pq}, \alpha, [a]_{0,\alpha}, \|\Psi(x, Dv)\|_{L^1(\Omega)})$ independent of ε, whenever $x \in B$. Since

$$\left[\Psi(\cdot, Dv(\cdot)) * \rho_\varepsilon\right](x) \to \Psi(x, Dv(x)) \quad \text{strongly in} \quad L^1(B),$$

we are able to apply a variant of Lebesgue's dominated convergence theorem for a sequence of functions $\{v_{\varepsilon_k}\} \subset C_0^\infty(B_{r+\varepsilon_0})$ with some choice of $\varepsilon_k \to 0$. As a result, $v_k \to v$ in $W^{1,G^{p(\cdot)}}(B)$ and this ensures the existence of a sequence satisfying our desired convergence (3.5.16). The proof is complete. $\square$

Let us now consider a Sobolev-Poincaré type inequality related to an Orlicz function with variable exponent, which plays an important role afterwards. In the following, let $b(\cdot) : B_r \to [0, \infty)$ be a continuous function such that

$$\begin{aligned}1 \leq b_i := \inf_{y \in B_r} b(y) \leq b(x) \leq \sup_{y \in B_r} b(y) =: b_s < \infty \\ \text{and} \quad |b(x) - b(y)| \leq \omega(|x - y|)\end{aligned} \tag{3.5.25}$$

holds, whenever $x, y \in B_r$ with $|x - y| < 1/2$, where $\omega(\cdot)$ is a modulus continuous function such that

$$\omega(0)=0, \qquad \sup_{0<\rho\le r}\omega(\rho)\log\frac{1}{\rho}\le 1 \quad \text{and} \quad \omega(2r)\le\sqrt{\frac{n}{n-1}}-1. \tag{3.5.26}$$

Lemma 3.5.6. *Let $\Phi\in\mathcal{N}$ with $s(\Phi)\ge 1$ and let $b(\cdot):B_r\to[0,\infty)$ be a function as defined in (3.5.25)-(3.5.26). Then, for any $d\in\left[1,\frac{n}{n-1}\right)$, there exists $\theta\equiv\theta(n,s(\Phi),b_i,b_s,d)\in(0,1)$ such that*

$$\begin{aligned}&\left[\fint_{B_r}\Phi^{db(x)}\left(\left|\frac{v-(v)_{B_r}}{r}\right|\right)dx\right]^{\frac{1}{d}}\\ &\quad\le c\left[\left(\int_{B_r}\Phi^{b(x)}(|Dv|)\,dx\right)^{\omega(2r)}+1\right]\left[\fint_{B_r}\Phi^{\theta b(x)}(|Dv|)\,dx\right]^{\frac{1}{\theta}}+c\end{aligned} \tag{3.5.27}$$

holds for some constant $c\equiv c(n,s(\Phi),b_i,b_s,d)$, whenever $v\in W^{1,\Phi^{b(\cdot)}}(B_r)$. Moreover, the above inequality still holds for every $v\in W_0^{1,\Phi^{b(\cdot)}}(B_r)$ if $v-(v)_{B_r}$ is replaced by v.

Proof. First we notice the following classical formula that

$$|v(x)-(v)_{B_r}|\le c(n)\int_{B_r}\frac{|Dv(y)|}{|x-y|^{n-1}}\,dy$$

holds for a.e. $x\in B_r$, whenever $v\in W^{1,1}(B_r)$, see for instance [125, Lemma 7.16]. Using the last formula and the property that the function $\Phi^{db(x)}(\cdot)$ is increasing for any fixed $x\in B_r$, and then applying Lemma 2.1.4, we have

$$\begin{aligned}I:=\fint_{B_r}\Phi^{db(x)}\left(\left|\frac{v-(v)_{B_r}}{r}\right|\right)dx&\le c\fint_{B_r}\Phi^{db(x)}\left(\int_{B_r}\frac{|Dv(y)|}{r|x-y|^{n-1}}\,dy\right)dx\\ &\le c\fint_{B_r}\Phi^{db_s}\left(\int_{B_r}\frac{|Dv(y)|}{r|x-y|^{n-1}}\,dy\right)dx+c\end{aligned}$$

with $c\equiv c(n,s(\Phi),d)$. Now by Lemma 2.1.5, there exists $\theta\equiv\theta(n,b_i,s(\Phi),d)\in\left(\sqrt{\frac{n-1}{n}}d,1\right)$ such that $\Phi^{b_i\theta}\in\mathcal{N}$ with $s(\Phi^{b_i\theta})$ depending only on $n,b_i,s(\Phi),d$. Let $E:=\fint_{B_r}\Phi^{b_i\theta}(|Dv|)\,dx$. One can always assume $E>0$, otherwise (3.5.27) becomes trivial. Recalling the fact that $\int_{B_r}\frac{1}{r|x-y|^{n-1}}\,dy\le c(n)$, where the constant $c(n)$ is independent of $x\in B_r$ and the ball B_r, we apply Jensen's inequality for the convex function $\Phi^{b_i\theta}(\cdot)$ with respect to the measure $r^{-1}|x-y|^{-(n-1)}\,dy$. In turn, it yields that

$$\begin{aligned} I &\le c \fint_{B_r} \left(\int_{B_r} \Phi^{b_i\theta}(|Dv(y)|) r^{-1}|x-y|^{-(n-1)}\,dy \right)^{\frac{b_s d}{b_i\theta}} dx + c \\ &= c r^{\frac{b_s(n-1)d}{b_i\theta}} E^{\frac{b_s d}{b_i\theta}} \fint_{B_r} \left(\fint_{B_r} \frac{\Phi^{b_i\theta}(|Dv(y)|)}{|x-y|^{n-1}} E^{-1}\,dy \right)^{\frac{b_s d}{b_i\theta}} dx + c \\ &\le c r^{\frac{b_s(n-1)d}{b_i\theta}} E^{\frac{b_s d}{b_i\theta}} \fint_{B_r} \fint_{B_r} \frac{\Phi^{b_i\theta}(|Dv(y)|)}{|x-y|^{\frac{(n-1)b_s d}{b_i\theta}}} E^{-1}\,dy dx + c, \end{aligned} \tag{3.5.28}$$

where in the last inequality of above display we have applied again Jensen's inequality to the convex function $t \mapsto t^{\frac{b_s d}{b_i\theta}}$ with respect to the probability measure $E^{-1}\Phi^{b_i\theta}(|Dv(y)|)\,dy$. Note also that

$$\begin{aligned} \fint_{B_r} \frac{1}{|x-y|^{\frac{(n-1)b_s d}{b_i\theta}}}\,dx &\le \frac{1}{|B_r|} \int_{B_{2r}(y)} \frac{1}{|x-y|^{\frac{(n-1)b_s d}{b_i\theta}}}\,dx \\ &\le c(n, s(\Phi), d) r^{-\frac{(n-1)b_s d}{b_i\theta}} \end{aligned} \tag{3.5.29}$$

by observing that

$$\begin{aligned} \frac{(n-1)b_s d}{b_i\theta} &= \frac{(n-1)(b_s-b_i)d}{b_i\theta} + \frac{(n-1)d}{\theta} \\ &< (n-1)\omega(2r)\left(\frac{n}{n-1}\right)^{\frac{1}{2}} + (n-1)\left(\frac{n}{n-1}\right)^{\frac{1}{2}} \le n, \end{aligned}$$

where we have used our choice of θ, (3.5.25) and (3.5.26). Merging the estimate (3.5.29) into (3.5.28) and using Hölder's inequality together with

(3.5.25) and (3.5.26), we conclude with

$$
\begin{aligned}
I &\le E^{\frac{b_s d}{b_i \theta}} + c \le c \left(\fint_{B_r} \Phi^{b(x)\theta}(|Dv|)\,dx \right)^{\frac{b_s d}{b_i \theta}} + c \\
&\le c r^{-\frac{n(b_s - b_i)d}{b_i}} \left(\int_{B_r} \Phi^{b(x)}(|Dv|)\,dx \right)^{\frac{(b_s - b_i)d}{b_i}} \left(\fint_{B_r} \Phi^{b(x)\theta}(|Dv|)\,dx \right)^{\frac{d}{\theta}} + c \\
&\le c \left[\left(\int_{B_r} \Phi^{b(x)}(|Dv|)\,dx \right)^{\omega(2r)} + 1 \right]^d \left(\fint_{B_r} \Phi^{b(x)\theta}(|Dv|)\,dx \right)^{\frac{d}{\theta}} + c
\end{aligned}
$$

for some $c \equiv c(n, s(\Phi), b_i, b_s, d)$. Obviously our desired inequality (3.5.27) follows from the last estimate. □

3.5.3 Higher integrability

Before proving the higher integrability, for a given $\Phi \in \mathcal{N}(\Omega)$, we define a vector field $V_\Phi : \Omega \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}^n$ as follows:

$$
V_\Phi(x, z) := \left[\frac{\partial_t \Phi(x, |z|)}{|z|} \right]^{\frac{1}{2}} z.
$$

Using these maps, it is convenient to formulate the monotonicity properties of the vector field $A(\cdot,\cdot)$ in (3.5.6), i.e., the following inequality holds:

$$
|V_\Psi(x, z_1) - V_\Psi(x, z_2)|^2 \le c \, \langle A(x, z_1) - A(x, z_2), z_1 - z_2 \rangle \tag{3.5.30}
$$

with some constant $c \equiv c(n, s(G), s(H), m_{pq}, \nu, L)$, whenever $z_1, z_2 \in \mathbb{R}^n \setminus \{0\}$ and $x \in \Omega$. We also shall use the following facts frequently, for any $\Phi \in \mathcal{N}(\Omega)$ with $s(\Phi) \ge 1$, that

$$
\begin{aligned}
|V_\Phi(x, z_1) - V_\Phi(x, z_2)|^2 &\approx \partial_{tt}^2 \Phi(x, |z_1| + |z_2|)|z_1 - z_2|^2 \\
&\approx \frac{\partial_t \Phi(x, |z_1| + |z_2|)}{|z_1| + |z_2|} |z_1 - z_2|^2,
\end{aligned}
$$

$$
\left\langle \partial_t \Phi(x, |z_1|) \frac{z_1}{|z_1|} - \partial_t \Phi(x, |z_2|) \frac{z_2}{|z_2|}, z_1 - z_2 \right\rangle \approx |V_\Phi(x, z_1) - V_\Phi(x, z_2)|^2
$$

and

$$\Phi(x, z_1 - z_2) \leq c(\tau)|V_\Phi(x, z_1) - V_\Phi(x, z_2)|^2 + \tau\Phi(x, z_1) \tag{3.5.31}$$

for every $\tau > 0$ with some constant $c(\tau) \equiv c(s(\Phi), \tau)$, whenever $x \in \Omega$ and $z_1, z_2 \in \mathbb{R}^n$ with $|z_1| + |z_2| > 0$, where all the implied constants depend only on n and $s(\Phi)$ (see [12, 92]).

Remark 3.5.7. *We notice that $\Psi \in \mathcal{N}(\Omega)$ with the index $s(\Psi)$ depending only on $s(G), s(H)$ and m_{pq} by Lemma 2.1.4 and Lemma 2.1.5, which means that*

$$\frac{1}{s(\Psi)} \leq \frac{t\partial_{tt}^2\Psi(x,t)}{\partial_t\Psi(x,t)} \leq s(\Psi)$$

holds for all $x \in \Omega$ and $t > 0$. Also we note the following elementary but useful inequality by Lemma 2.1.4 as

$$\Psi(x, t_1 + t_2) \leq 2^{s(\Psi)+1}\left(\Psi(x, t_1) + \Psi(x, t_2)\right) \tag{3.5.32}$$

for all $x \in \Omega$ and $t_1, t_2 \in \mathbb{R}$.

In the present section we provide a higher integrability of solutions to the equation (3.5.5) and its homogeneous equation.

Lemma 3.5.8. *Let $u \in W^{1,\Psi}(\Omega)$ be a weak solution to the equation* (3.5.5) *under the assumptions* (3.5.1)-(3.5.4) *and* (3.5.7)-(3.5.9)*. We also assume that $\Psi(x, F) \in L^\gamma(\Omega)$ for some $\gamma > 1$. Then there exists a positive higher integrability exponent $\sigma_0 \equiv \sigma_0(\boldsymbol{data}, \gamma) < \gamma - 1$ such that $\Psi(x, Du) \in L^{1+\sigma_0}_{\mathrm{loc}}(\Omega)$. Moreover, there exists a constant $c \equiv c(\boldsymbol{data})$ such that*

$$\begin{aligned}
&\left(\fint_{B_\rho} [\Psi(x, Du)]^{1+\sigma}\,dx\right)^{\frac{1}{1+\sigma}} \\
&\quad \leq c\fint_{B_{2\rho}} \Psi(x, Du)\,dx + c\left(\fint_{B_{2\rho}} [\Psi(x, F)]^{1+\sigma}\,dx\right)^{\frac{1}{1+\sigma}} + c
\end{aligned} \tag{3.5.33}$$

holds for every $\sigma \in (0, \sigma_0]$*, whenever* $B_{2\rho} \subset \Omega$ *is a ball such that*

$$\sup_{0<\tau\leq 4\rho} \omega(\tau) \log \frac{1}{\tau} \leq 1 \quad \textit{and} \quad \omega(4\rho) \leq \sqrt{\frac{n}{n-1}} - 1 \tag{3.5.34}$$

for $\omega(\cdot)$ *being a modulus continuous function introduced in* (3.5.10)*. In particular, for every* $\Omega_0 \Subset \Omega$ *and* $\sigma \in (0, \sigma_0]$*, there exists a constant* $c \equiv c(\boldsymbol{data}, \operatorname{dist}(\Omega_0, \Omega), \|\Psi(x, F)\|_{L^\gamma(\Omega)})$ *such that*

$$\|\Psi(x, Du)\|_{L^{1+\sigma}(\Omega_0)} \leq c. \tag{3.5.35}$$

Proof. Let $\eta \in C_0^\infty(B_{2\rho})$ be a cut-off function such that $\chi_{B_\rho} \leq \eta \leq \chi_{B_{2\rho}}$ and $|D\eta| \leq 4/\rho$. We take $\varphi := \eta^{s(\Psi)+1}\left(u - (u)_{B_{2\rho}}\right)$ as a test function in the equation (3.5.5), where $s(\Psi)$ is the index of Ψ depending only on $s(G), s(H)$ and m_{pq} by Remark 3.5.7, to observe that

$$\begin{aligned}
I_0 &:= \fint_{B_{2\rho}} \eta^{s(\Psi)+1} \langle A(x, Du), Du\rangle \, dx \\
&= -(s(\Psi)+1) \fint_{B_{2\rho}} \eta^{s(\Psi)} (u - (u)_{B_{2\rho}}) \langle A(x, Du), D\eta\rangle \, dx \\
&\quad + \fint_{B_{2\rho}} \eta^{s(\Psi)+1} \left\langle B(x, F), Du\right\rangle \, dx \\
&\quad + (s(\Psi)+1) \fint_{B_{2\rho}} \eta^{s(\Psi)} (u - (u)_{B_{2\rho}}) \left\langle B(x, F), D\eta\right\rangle \, dx \\
&=: I_{01} + I_{02} + I_{03}.
\end{aligned}$$

Clearly, by the monotonicity property (3.5.30), we have

$$\fint_{B_{2\rho}} \eta^{s(\Psi)+1} \Psi(x, Du) \, dx \leq c I_0$$

for some $c \equiv c(s(G), s(H), \nu, L, m_{pq})$. Applying Lemma 2.1.6, for every $\varepsilon \in (0, 1)$, we find

$$
\begin{aligned}
I_{01} &\leq c \fint_{B_{2\rho}} \eta^{s(\Psi)} |A(x, Du)| \left| \frac{u - (u)_{B_{2\rho}}}{\rho} \right| dx \\
&\leq c \fint_{B_{2\rho}} \eta^{s(\Psi)} \frac{\Psi(x, Du)}{|Du|} \left| \frac{u - (u)_{B_{2\rho}}}{\rho} \right| dx \\
&\leq c \fint_{B_{2\rho}} \eta^{s(\Psi)} (\varepsilon\eta) \Psi(x, Du)\, dx + c \fint_{B_{2\rho}} \eta^{s(\Psi)} \frac{1}{(\varepsilon\eta)^{s(\Psi)}} \Psi\left(x, \frac{u - (u)_{B_{2\rho}}}{\rho}\right) dx \\
&\leq c\varepsilon \fint_{B_{2\rho}} \eta^{s(\Psi)+1} \Psi(x, Du)\, dx + \frac{c}{\varepsilon^{s(\Psi)}} \fint_{B_{2\rho}} \Psi\left(x, \frac{u - (u)_{B_{2\rho}}}{\rho}\right) dx
\end{aligned}
$$

with $c \equiv c(s(G), s(H), L, m_{pq})$. Similarly, using again Lemma 2.1.6, for every $\varepsilon \in (0, 1)$, we have

$$
I_{02} \leq \varepsilon \fint_{B_{2\rho}} \eta^{s(\Psi)+1} \Psi(x, Du)\, dx + \frac{c}{\varepsilon^{s(\Psi)}} \fint_{B_{2\rho}} \eta^{s(\Psi)+1} \Psi\left(x, F\right)\, dx
$$

and

$$
I_{03} \leq c \fint_{B_{2\rho}} \eta^{s(\Psi)} \Psi(x, F)\, dx + c \fint_{B_{2\rho}} \eta^{s(\Psi)} \Psi\left(x, \frac{u - (u)_{B_{2\rho}}}{\rho}\right) dx
$$

for some constant $c \equiv c(s(G), s(H), L, m_{pq})$. Combining the last three displays and choosing small enough ε after some standard manipulations, we have

$$
\fint_{B_\rho} \Psi(x, Du)\, dx \leq c \fint_{B_{2\rho}} \Psi\left(x, \frac{u - (u)_{B_{2\rho}}}{\rho}\right) dx + c \fint_{B_{2\rho}} \Psi(x, F)\, dx \tag{3.5.36}
$$

for some $c \equiv c(n, s(G), s(H), \nu, L, m_{pq})$.

Now we estimate the term on the right-hand side of the above display.

For this, using the assumption (3.5.4), we estimate as follows:

$$
\begin{aligned}
I_1 &:= \fint_{B_{2\rho}} \Psi\left(x, \frac{u-(u)_{B_{2\rho}}}{\rho}\right) dx \\
&= \fint_{B_{2\rho}} \Bigg(G^{p(x)}\left(\left|\frac{u-(u)_{B_{2\rho}}}{\rho}\right|\right) \\
&\qquad\qquad + (a(x)-a_i(B_{2\rho}))H^{q(x)}\left(\left|\frac{u-(u)_{B_{2\rho}}}{\rho}\right|\right)\Bigg) dx \\
&\quad + \fint_{B_{2\rho}} a_i(B_{2\rho}) H^{q(x)}\left(\left|\frac{u-(u)_{B_{2\rho}}}{\rho}\right|\right) dx \\
&\le c_* \fint_{B_{2\rho}} G^{p(x)}\left(\left|\frac{u-(u)_{B_{2\rho}}}{\rho}\right|\right) dx \\
&\qquad\qquad + c_*\rho^\alpha \fint_{B_{2\rho}} G^{p(x)\left(1+\frac{\alpha}{n}\right)}\left(\left|\frac{u-(u)_{B_{2\rho}}}{\rho}\right|\right) dx \\
&\quad + c_* a_i(B_{2\rho}) \fint_{B_{2\rho}} H^{q(x)}\left(\left|\frac{u-(u)_{B_{2\rho}}}{\rho}\right|\right) dx \\
&=: c_*(I_{11}+I_{12}+I_{13})
\end{aligned} \tag{3.5.37}
$$

for some $c_* \equiv c_*(\kappa, [a]_{0,\alpha})$, where $a_i(B_{2\rho}) := \inf_{x\in B_{2\rho}} a(x)$. Now we estimate the terms appearing in the last display. For I_{11}, applying Lemma 3.5.6 with $\Phi \equiv G$, $b(\cdot) \equiv p(\cdot)$ and $d \equiv 1$, there exists $\theta_1 \equiv \theta_1(n, s(G), m_{pq}) \in (0,1)$ such that

$$
\begin{aligned}
I_{11} &\le c\left[\left(\int_{B_{2\rho}} G^{p(x)}(|Du|)\, dx\right)^{\omega(4\rho)} + 1\right] \\
&\qquad\qquad \times \left[\fint_{B_{2\rho}} G^{\theta_1 p(x)}(|Du|)\, dx\right]^{\frac{1}{\theta_1}} + c \\
&\le c\left(\fint_{B_{2\rho}} \Psi^{\theta_1}(x, Du)\, dx\right)^{\frac{1}{\theta_1}} + c
\end{aligned} \tag{3.5.38}
$$

for some constant $c \equiv c(n, \kappa, s(G), m_{pq}, \|\Psi(x,Du)\|_{L^1(\Omega)})$. For the estimate on I_{12}, let us first observe that $1+\frac{\alpha}{n} < \frac{n}{n-1}$. Therefore, we are able to apply

Lemma 3.5.6 for $\Phi \equiv G$, $b(\cdot) \equiv p(\cdot)$ and $d \equiv 1 + \frac{\alpha}{n}$. In turn, there exists $\theta_2 \equiv \theta_2(n, s(G), \alpha, m_{pq}) \in (0,1)$ such that

$$\begin{aligned}
I_{12} &\le c\rho^\alpha \left[\left(\int_{B_{2\rho}} G^{p(x)}(|Du|)\,dx \right)^{\left(1+\frac{\alpha}{n}\right)\omega(4\rho)} + 1 \right] \\
&\qquad \times \left(\fint_{B_{2\rho}} G^{\theta_2 p(x)}(|Du|)\,dx \right)^{\left(1+\frac{\alpha}{n}\right)\frac{1}{\theta_2}} + c\rho^\alpha \\
&\le c \left[\left(\int_{B_{2\rho}} G^{p(x)}(|Du|)\,dx \right)^{\frac{\alpha}{n}+\left(1+\frac{\alpha}{n}\right)\omega(4\rho)} \right. \\
&\qquad \left. + \left(\int_{B_{2\rho}} G^{p(x)}(|Du|)\,dx \right)^{\frac{\alpha}{n}} \right] \\
&\qquad \times \left(\fint_{B_{2\rho}} G^{\theta_2 p(x)}(|Du|)\,dx \right)^{\frac{1}{\theta_2}} + c \\
&\le c \left(\fint_{B_{2\rho}} \Psi^{\theta_2}(x, Du)\,dx \right)^{\frac{1}{\theta_2}} + c
\end{aligned} \tag{3.5.39}$$

with some constant $c \equiv c(n, \kappa, s(G), \alpha, m_{pq}, \|\Psi(x, Du)\|_{L^1(\Omega)})$. Finally, to estimate I_{13}, we consider two cases depending on smallness of the quantity $a_i(B_{2\rho})$, which means that if $a_i(B_{2\rho}) \le 4\rho^\alpha$, then using the assumption (3.5.4), we see that

$$I_{13} \le c(\kappa)\,(I_{11} + I_{12})\,.$$

In particular, in this case the estimates obtained in (3.5.38)-(3.5.39) imply that

$$I_{13} \le c \left(\fint_{B_{2\rho}} \Psi^{\theta_1}(x, Du)\,dx \right)^{\frac{1}{\theta_1}} + c \left(\fint_{B_{2\rho}} \Psi^{\theta_2}(x, Du)\,dx \right)^{\frac{1}{\theta_2}} + c \tag{3.5.40}$$

for some $c \equiv c(n, \kappa, s(G), \alpha, m_{pq}, \|\Psi(x, Du)\|_{L^1(\Omega)})$. Now we consider the remaining case $a_i(B_{2\rho}) > 4\rho^{\alpha}$. In this case we observe

$$[a_i(B_{2\rho})]^{-\omega(4\rho)} \leq [4\rho]^{-\alpha\omega(4\rho)} \leq e^{\alpha}$$

by the assumption (3.5.34). Then using the last display and applying Lemma 3.5.6 for $\Phi \equiv H$, $b(\cdot) \equiv q(\cdot)$ and $d \equiv 1$, there exists $\theta_3 \equiv \theta_3(n, s(H), m_{pq}) \in (0,1)$ such that

$$\begin{aligned}
I_{13} &\leq c a_i(B_{2\rho}) \left[\left(\int_{B_{2\rho}} H^{q(x)}(|Du|)\,dx \right)^{\omega(4\rho)} + 1 \right] \\
&\quad \times \left(\fint_{B_{2\rho}} H^{\theta_3 q(x)}(|Du|)\,dx \right)^{\frac{1}{\theta_3}} + c \\
&= c \left[[a_i(B_{2\rho})]^{-\omega(4\rho)} \left(\int_{B_{2\rho}} a_i(B_{2\rho}) H^{q(x)}(|Du|)\,dx \right)^{\omega(4\rho)} + 1 \right] \qquad (3.5.41) \\
&\quad \times \left(\fint_{B_{2\rho}} [a_i(B_{2\rho})]^{\theta_3} H^{\theta_3 q(x)}(|Du|)\,dx \right)^{\frac{1}{\theta_3}} + c \\
&\leq c \left(\fint_{B_{2\rho}} \Psi^{\theta_3}(x, Du)\,dx \right)^{\frac{1}{\theta_3}} + c
\end{aligned}$$

for some $c \equiv c(n, s(H), \alpha, m_{pq}, \|a\|_{L^\infty(\Omega)}, \|\Psi(x, Du)\|_{L^1(\Omega)})$. Taking (3.5.38)-(3.5.39) and (3.5.41) into account, we conclude that

$$\fint_{B_{2\rho}} \Psi\left(x, \frac{u-(u)_{B_{2\rho}}}{\rho}\right) dx \leq c \left(\fint_{B_{2\rho}} \Psi^{\theta}(x, Du)\,dx \right)^{\frac{1}{\theta}} + c \qquad (3.5.42)$$

for some $c \equiv c(\texttt{data})$, where $\theta := \max\{\theta_1, \theta_2, \theta_3\} \in (0,1)$ depending only on $n, s(G), s(H), \alpha$ and m_{pq}. Inserting the estimate in the last display into (3.5.36), we find an exponent $\theta \equiv \theta(n, s(G), s(H), m_{pq}, \alpha) \in (0,1)$ such that

$$\fint_{B_{\rho}} \Psi(x, Du)\,dx \leq c \left(\fint_{B_{2\rho}} [\Psi(x, Du)]^{\theta}\,dx \right)^{\frac{1}{\theta}} + c \fint_{B_{2\rho}} \Psi(x, F)\,dx + c$$

for some $c \equiv c(\texttt{data})$. Now we apply Gehring's lemma to obtain

$$\left(\fint_{B_\rho} [\Psi(x, Du)]^{1+\sigma} \, dx \right)^{\frac{1}{1+\sigma}}$$
$$\leq c \fint_{B_{2\rho}} \Psi(x, Du) \, dx + c \left(\fint_{B_{2\rho}} [\Psi(x, F)]^{1+\sigma} \, dx \right)^{\frac{1}{1+\sigma}} + c$$

for every $\sigma \in (0, \sigma_0]$, where $\sigma_0 \equiv \sigma_0(\texttt{data}, \gamma) < \gamma - 1$ and $c \equiv c(\texttt{data})$, which proves (3.5.33). Clearly, by a standard covering argument, (3.5.35) follows. $\square$

Throughout this section, let u be a weak solution of the equation (3.5.5). Our purpose here is to prove a higher integrability of the solution of the following Dirichlet boundary value problem:

$$\begin{cases} -\mathrm{div} A(x, Dw) = 0 & \text{in} \quad B_{32r} \equiv B_{32r}(y_0) \\ w \in u + W_0^{1,\Psi}(B_{32r}) \end{cases} \tag{3.5.43}$$

for a ball $B_{32r} \Subset \Omega_0 \Subset \Omega$ such that

$$\sup_{0<\rho\leq 64r} \omega(\rho) \log \frac{1}{\rho} \leq 1 \quad \text{and} \quad \omega(64r) \leq \sqrt{\frac{n}{n-1}} - 1 \tag{3.5.44}$$

with $\omega(\cdot)$ being a modulus continuous function that has been introduced in (3.5.10).

Lemma 3.5.9. *Let $w \in W^{1,\Psi}(B_{32r})$ be the weak solution to the equation* (3.5.43) *under the assumptions* (3.5.1)-(3.5.4), (3.5.7)-(3.5.8) *and* (3.5.44). *Then there exist a positive exponent $\sigma_1 \equiv \sigma_1(\texttt{data}, \gamma) \in (0, 1)$ with $\sigma_1 \leq \sigma_0$ and $c \equiv c(\texttt{data})$ such that*

1. *For every ball $B_{2\rho} \equiv B_{2\rho}(y) \Subset B_{32r}$ and $\sigma \in (0, \sigma_1]$, it holds that*

$$\left[\fint_{B_\rho} [\Psi(x, Dw)]^{1+\sigma} \, dx \right]^{\frac{1}{1+\sigma}} \leq c \left(\fint_{B_{2\rho}} \Psi(x, Dw) \, dx + 1 \right). \tag{3.5.45}$$

2. *The following energy estimate*

$$\fint_{B_{32r}} [\Psi(x, Dw)]^{1+\sigma}\, dx \le c \fint_{B_{32r}} [\Psi(x, Du)]^{1+\sigma}\, dx + c \tag{3.5.46}$$

holds for every $\sigma \in (0, \sigma_1]$.

Proof. Let us start with testing the equation (3.5.43) by $u - w \in W_0^{1,\Psi}(B_{32r})$. In turn, by (3.5.8) and then (3.5.7) and Lemma 2.1.6 it yields that

$$\begin{aligned}\fint_{B_{32r}} \Psi(x, Dw)\, dx &\le c \fint_{B_{32r}} \langle A(x, Dw), Dw\rangle\, dx \\ &\le c \fint_{B_{32r}} |A(x, Dw)||Du|\, dx \\ &\le c \fint_{B_{32r}} \frac{\Psi(x, Dw)}{|Dw|}|Du|\, dx \\ &\le c\tau \fint_{B_{32r}} \Psi(x, Dw)\, dx + \frac{c}{\tau^{s(\Psi)}} \fint_{B_{32r}} \Psi(x, Du)\, dx\end{aligned} \tag{3.5.47}$$

for any $\tau \in (0, 1)$ and some constant $c \equiv c(n, s(G), s(H), \nu, L, m_{pq})$. By taking τ small enough in the last display after arranging the terms, we have the energy estimate

$$\fint_{B_{32r}} \Psi(x, Dw)\, dx \le c \fint_{B_{32r}} \Psi(x, Du)\, dx \tag{3.5.48}$$

for some $c \equiv c(n, s(G), s(H), \nu, L, m_{pq})$.

Arguing similarly as in the proof of Lemma 3.5.8, we obtain the following Caccioppoli type inequality:

$$\fint_{B_\rho} \Psi(x, Dw)\, dx \le c \fint_{B_{2\rho}} \Psi\left(x, \frac{w - (w)_{B_{2\rho}}}{\rho}\right)\, dx$$

for some constant $c \equiv c(n, s(G), s(H), \nu, L, m_{pq})$. Therefore, by this inequality and with similar computations as in (3.5.37)-(3.5.42) under the assumption (3.5.44), there exists $\theta \equiv \theta(n, s(G), s(H), \alpha, m_{pq}) \in (0,1)$ such that

$$\fint_{B_\rho} \Psi(x, Dw)\, dx \le c \left(\fint_{B_{2\rho}} \Psi^\theta(x, Dw)\, dx \right)^{\frac{1}{\theta}} + c \tag{3.5.49}$$

for some $c \equiv c(\texttt{data})$. Now we apply Gehring's lemma to obtain

$$\left(\fint_{B_\rho} \Psi^{1+\sigma}(x, Dw)\, dx \right)^{\frac{1}{1+\sigma}} \le c \fint_{B_{2\rho}} \Psi(x, Dw)\, dx + c$$

for every $\sigma \in (0, \sigma_1]$, where $\sigma_1 \equiv \sigma_1(\texttt{data})$ and $c \equiv c(\texttt{data})$. This yields (3.5.45). To show (3.5.46), we need to prove a version of the last inequality near the boundary of B_{32r}. For this, let $B_{2\rho}(y) \subset \mathbb{R}^n$ be a ball such that $y \in B_{32r}$ and $\frac{1}{10} < \frac{|B_{2\rho}(y) \setminus B_{32r}|}{|B_{2\rho}(y)|}$. We take a test function by $\varphi \equiv \eta^{s(\Psi)+1}(u - w)$, where $\eta \in C_0^\infty(B_{2\rho})$ is a standard cut-off function as before so that $\chi_{B_\rho} \le \eta \le \chi_{B_{2\rho}}$ and $|D\eta| \le 4/\rho$. This choice of φ is admissible since $\operatorname{supp} \varphi \Subset B_{32r} \cap B_{2\rho}(y)$. Arguing similarly as we have done above, we see that

$$\begin{aligned}
&\int_{B_{32r} \cap B_{2\rho}(y)} \eta^{s(\Psi)+1} \Psi(x, Dw)\, dx \\
&\quad \le c \int_{B_{32r} \cap B_{2\rho}(y)} \eta^{s(\Psi)} \frac{\Psi(x, Dw)}{|Dw|} \left| \frac{w-u}{\rho} \right| dx \\
&\qquad + c \int_{B_{32r} \cap B_{2\rho}(y)} \eta^{s(\Psi)} \frac{\Psi(x, Dw)}{|Dw|} |Du|\, dx \\
&\quad \le c \int_{B_{32r} \cap B_{2\rho}(y)} \eta^{s(\Psi)} \left((\varepsilon\eta) \Psi(x, Dw) + \frac{1}{(\varepsilon\eta)^{s(\Psi)}} \Psi\left(x, \frac{w-u}{\rho}\right) \right) dx \\
&\qquad + c \int_{B_{32r} \cap B_{2\rho}(y)} \eta^{s(\Psi)} \left((\varepsilon\eta) \Psi(x, Dw) + \frac{1}{(\varepsilon\eta)^{s(\Psi)}} \Psi\left(x, Du\right) \right) dx.
\end{aligned}$$

Again choosing ε small enough and reabsorbing the terms, we find that

$$\begin{aligned}
&\fint_{B_{32r} \cap B_{2\rho}(y)} \eta^{s(\Psi)+1} \Psi(x, Dw)\, dx \\
&\qquad\qquad \le c \fint_{B_{32r} \cap B_{2\rho}(y)} \Psi\left(x, \frac{w-u}{\rho}\right) dx + c \fint_{B_{32r} \cap B_{2\rho}(y)} \Psi\left(x, Du\right) dx
\end{aligned}$$

for some constant $c \equiv c(n, s(G), s(H), \nu, L, m_{pq})$. Redefining $w - u \equiv 0$ on $B_{2\rho}(y) \setminus B_{32r}$, we are able to repeat the same proof for the estimate in (3.5.49). In turn, there exists $\theta_b \equiv \theta_b(n, s(G), s(H), \alpha, m_{pq}) \in (0, 1)$ such that

$$\begin{aligned}
\fint_{B_{32r} \cap B_{2\rho}(y)} \Psi\left(x, \frac{w-u}{\rho}\right) dx &\leq c \left(\fint_{B_{32r} \cap B_{2\rho}(y)} [\Psi(x, Dw - Du)]^{\theta_b} \, dx \right)^{\frac{1}{\theta_b}} + c \\
&\leq c \left(\fint_{B_{32r} \cap B_{2\rho}(y)} [\Psi(x, Dw)]^{\theta_b} \, dx \right)^{\frac{1}{\theta_b}} \\
&\quad + c \fint_{B_{32r} \cap B_{2\rho}(y)} [\Psi(x, Du)] \, dx + c
\end{aligned}$$

for some constant $c \equiv c(\texttt{data})$, where for the last inequality we have used (3.5.32) and Hölder's inequality. Combining the last two displays, we have

$$\fint_{B_\rho(y)} [V(x)]^{\frac{1}{\theta_m}} \, dx \leq c \left(\fint_{B_{2\rho}(y)} V(x) \, dx \right)^{\frac{1}{\theta_m}} + c \fint_{B_{2\rho}(y)} U(x) \, dx + c$$

for some $c \equiv c(\texttt{data})$, where $\theta_m = \max\{\theta, \theta_b\}$,

$$V(x) := [\Psi(x, Dw)]^{\theta_m} \chi_{B_{32r}}(x) \quad \text{and} \quad U(x) := \Psi(x, Du) \chi_{B_{32r}}(x)$$

for every ball $B_{2\rho}(y) \subset \mathbb{R}^n$ satisfying either $B_{2\rho}(y) \subset B_{32r}$ or $\frac{1}{10} < \frac{|B_{2\rho}(y) \setminus B_{32r}|}{|B_{2\rho}(y)|}$ with $y \in B_{32r}$. Applying a variant of Gehring's lemma and a standard covering argument together with Lemma 3.5.8, we arrive at the desired estimate (3.5.46). □

As a consequence of the above lemma, we need another type of higher integrability results. For a given ball $B_{2\rho}(y) \Subset B_{32r}$, let $\tilde{p}, \tilde{q} : B_{2\rho}(y) \to [0, \infty)$ be functions satisfying the following bounds:

$$0 \leq \tilde{p}(x) \leq (1 + \sigma) p(x) \quad \text{and} \quad q(x) \leq \tilde{q}(x) \leq q(x) \left(1 + \frac{n\sigma}{n + \alpha}\right) \tag{3.5.50}$$

for some $\sigma \in (0, \sigma_1]$, where $\sigma_1 \equiv \sigma_1(\texttt{data})$ is a higher integrability exponent determined by Lemma 3.5.9.

Lemma 3.5.10. *Under the assumptions and conclusions of Lemma 3.5.9 and notations introduced in* (3.5.50)*, it holds that*

$$\fint_{B_\rho(y)} \left(G^{\tilde{p}(x)}(|Dw|) + a(x)H^{\tilde{q}(x)}(|Dw|)\right)\,dx \\ \leq c\left\{\left(\fint_{B_{2\rho}(y)} \Psi(x,Dw)\,dx\right)^{1+\sigma} + 1\right\}$$

for some constant $c \equiv c(\boldsymbol{data})$.

Proof. By Lemma 3.5.9, it follows that

$$\begin{aligned}\fint_{B_\rho(y)} G^{\tilde{p}(x)}(|Dw|)\,dx &\leq \fint_{B_\rho(y)} G^{p(x)(1+\sigma)}(|Dw|)\,dx + 1 \\ &\leq c\left\{\left(\fint_{B_{2\rho}(y)} \Psi(x,Dw)\,dx\right)^{1+\sigma} + 1\right\}.\end{aligned} \tag{3.5.51}$$

By (3.5.50), for every $x \in B_{2\rho}(y)$ and $t \geq 0$, we also notice

$$\begin{aligned}H^{\tilde{q}(x)-q(x)}(t) &\leq \left[\kappa\left(G^{p(x)}(t) + G^{\left(1+\frac{\alpha}{n}\right)p(x)}(t)\right)\right]^{\frac{\tilde{q}(x)-q(x)}{q(x)}} \\ &\leq c\left(1 + G^{\left(1+\frac{\alpha}{n}\right)p(x)}(t)\right)^{\frac{\tilde{q}(x)-q(x)}{q(x)}} \\ &\leq c\left(1 + G^{\left(1+\frac{\alpha}{n}\right)p(x)\frac{\tilde{q}(x)-q(x)}{q(x)}}(t)\right) \\ &\leq c\left(1 + G^{\sigma p(x)}(t)\right)\end{aligned} \tag{3.5.52}$$

for some $c \equiv c(\kappa)$. Then, using Hölder's inequality together with the assumption (3.5.50), Lemma 3.5.9, the estimates (3.5.51) and (3.5.52), we conclude

that

$$
\begin{aligned}
\fint_{B_\rho(y)} a(x) H^{\tilde{q}(x)}(|Dw|)\,dx &= \fint_{B_\rho(y)} a(x) H^{q(x)}(|Dw|) H^{\tilde{q}(x)-q(x)}(|Dw|)\,dx \\
&\le \left(\fint_{B_\rho(y)} \left[a(x) H^{q(x)}(|Dw|)\right]^{1+\sigma} \,dx \right)^{\frac{1}{1+\sigma}} \\
&\quad \times \left(\fint_{B_\rho(y)} H^{(\tilde{q}(x)-q(x))\frac{1+\sigma}{\sigma}}(|Dw|)\,dx \right)^{\frac{\sigma}{1+\sigma}} \\
&\le c \left(\fint_{B_{2\rho}(y)} [\Psi(x,Dw)]^{1+\sigma}\,dx + 1 \right)^{\frac{1}{1+\sigma}} \\
&\quad \times \left(\fint_{B_\rho(y)} \left[G^{p(x)}(|Dw|)\right]^{1+\sigma} \,dx + 1 \right)^{\frac{\sigma}{1+\sigma}} \\
&\le c \left(\left(\fint_{B_{2\rho}(y)} \Psi(x,Dw)\,dx \right)^{1+\sigma} + 1 \right)
\end{aligned}
$$

for some constant $c \equiv c(\mathtt{data})$. This completes the proof. □

3.5.4 Comparison estimates

Throughout this section, let us fix a ball $B_{32r} \equiv B_{32r}(y_0) \Subset \Omega_0 \Subset \Omega$ with r being a small number depending on $\mathtt{data}$ to be determined later, and let also $u \in W^{1,\Psi}(\Omega)$ be a weak solution to the equation (3.5.5). In the following, we shall discuss a series of comparison estimates until we arrive at the limiting equation.

Lemma 3.5.11. *Let $w \in W^{1,\Psi}(B_{32r})$ be the solution to the equation* (3.5.43) *under the assumptions* (3.5.7)-(3.5.9). *Then for every $\varepsilon > 0$, there exists a small number $\delta \equiv \delta(n, s(G), s(H), \nu, L, m_{pq}, \varepsilon)$ such that if*

$$
\fint_{B_{32r}} \Psi(x, Du)\,dx \le \lambda \quad \textit{and} \quad \fint_{B_{32r}} \Psi(x, F)\,dx \le \delta\lambda
$$

hold for some $\lambda \geq 1$, then we have

$$\fint_{B_{32r}} \Psi(x, Dw)\, dx \leq c\lambda \quad \textit{and} \quad \fint_{B_{32r}} \Psi(x, Du - Dw)\, dx \leq \varepsilon\lambda \tag{3.5.53}$$

for some constant $c \equiv c(n, s(G), s(H), \nu, L, m_{pq})$.

Proof. Let us start with testing the equation (3.5.43) by $u - w \in W_0^{1,\Psi}(B_{32r})$. Arguing similarly to (3.5.47)-(3.5.48), we see that

$$\fint_{B_{32r}} \Psi(x, Dw)\, dx \leq c \fint_{B_{32r}} \Psi(x, Du)\, dx \leq c\lambda \tag{3.5.54}$$

for some $c \equiv c(n, s(G), s(H), \nu, L, m_{pq})$. This gives the validity of the first inequality of (3.5.53). To show the second estimate in (3.5.53), recalling that $u - w \in W_0^{1,\Psi}(B_{32r})$ is admissible as a test function to (3.5.5), we have

$$\fint_{B_{32r}} \langle A(x, Du) - A(x, Dw), Du - Dw \rangle\, dx = \fint_{B_{32r}} \langle B(x, F), Du - Dw \rangle\, dx.$$

By using (3.5.30) and (3.5.31) together with the last equality, we see that

$$\begin{aligned}
\fint_{B_{32r}} \Psi(x, Du - Dw)\, dx &\leq c_{\tau_1} \fint_{B_{32r}} |V_\Psi(x, Du) - V_\Psi(x, Dw)|^2\, dx \\
&\quad + \tau_1 \fint_{B_{32r}} \Psi(x, Du)\, dx \\
&\leq c_{\tau_1} \fint_{B_{32r}} \langle A(x, Du) - A(x, Dw), Du - Dw \rangle\, dx \\
&\quad + \tau_1 \fint_{B_{32r}} \Psi(x, Du)\, dx \\
&\leq c_{\tau_1} \fint_{B_{32r}} |B(x, F)||Du - Dw|\, dx + \tau_1\lambda \\
&\leq c_{\tau_1} \fint_{B_{32r}} \frac{\Psi(x, F)}{|F|}|Du - Dw|\, dx + \tau_1\lambda \\
&\leq \tau_2 c_{\tau_1} \fint_{B_{32r}} \Psi(x, Du - Dw)\, dx + c_{\tau_1\tau_2}\delta\lambda + \tau_1\lambda
\end{aligned}$$

holds with $c_{\tau_1} \equiv c_{\tau_1}(n, s(G), s(H), \nu, L, m_{pq}, \tau_1)$ and $c_{\tau_1\tau_2} \equiv c_{\tau_1\tau_2}(n, s(G), s(H), \nu, L, m_{pq}, \tau_1, \tau_2)$, whenever $\tau_1, \tau_2 \in (0, 1)$ are arbitrary numbers, where in the last inequality of the above display we have applied Lemma 2.1.6. By choosing $\tau_2 := \frac{1}{2c_{\tau_1}}$ in the last display, we have

$$\fint_{B_{32r}} \Psi(x, Du - Dw)\, dx \le c_{\tau_1}\delta\lambda + \tau_1\lambda.$$

Finally, taking small $\tau_1 \le \varepsilon/2$ and $\delta \le \varepsilon/(2c_{\tau_1})$ in the above display, the second inequality of (3.5.53) follows. $\square$

From now on, let us fix the auxiliary notations as

$$p_i := \inf_{x\in B_{32r}} p(x),\ p_s := \sup_{x\in B_{32r}} p(x),\ q_i := \inf_{x\in B_{32r}} q(x) \text{ and } q_s := \sup_{x\in B_{32r}} q(x).$$

Define a function $\Psi_s : \Omega \times \mathbb{R}^n \to \mathbb{R}$ by

$$\Psi_s(x, z) := G^{p_s}(|z|) + a(x)H^{q_s}(|z|) \tag{3.5.55}$$

for every $x \in \Omega$ and $z \in \mathbb{R}^n$.

Remark 3.5.12. *Let us remark several important observations regarding Ψ_s.*

1. *Applying Lemma 2.1.5, for every $t > 0$, we observe that*

$$\begin{aligned} \frac{1}{s(G^{p_s})} &\le \frac{(G^{p_s})''(t)t}{(G^{p_s})'(t)} \le s(G^{p_s}) \\ \text{and} \quad \frac{1}{s(H^{q_s})} &\le \frac{(H^{q_s})''(t)t}{(H^{q_s})'(t)} \le s(H^{q_s}) \end{aligned} \tag{3.5.56}$$

   with $s(G^{p_s}) = s(G) + (m_{pq} - 1)(s(G) + 1)$ and $s(H^{q_s}) = s(H) + (m_{pq} - 1)(s(H) + 1)$.

2. *By (3.5.56), for every $x \in \Omega$ and $t > 0$, we have*

$$\frac{1}{s(\Psi_s)} \le \frac{\partial^2_{tt}\Psi_s(x, t)t}{\partial_t\Psi_s(x, t)} \le s(\Psi_s) \tag{3.5.57}$$

   with $s(\Psi_s) := s(G) + s(H) + (m_{pq} - 1)(s(G) + s(H) + 2)$.

3. We can also easily see that

$$\sup_{t>0} \frac{[H(t)]^{q_s}}{[G(t)]^{p_s} + [G(t)]^{\left(1+\frac{\alpha}{n}\right)p_s}} \leq (\kappa+1)^{m_{pq}}. \tag{3.5.58}$$

We shall consider the vector fields $\tilde{A}_{1,B_{32r}}, \tilde{A}_{2,B_{32r}} : B_{32r} \times \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\begin{aligned} \tilde{A}_{1,B_{32r}}(x,z) &= G^{p_s - p(x)}(|z|)A_1(x,z) \\ \text{and} \quad \tilde{A}_{2,B_{32r}}(x,z) &= H^{q_s - q(x)}(|z|)A_2(x,z). \end{aligned} \tag{3.5.59}$$

Using the structure assumptions (3.5.7), (3.5.8) and recalling Remark 2.1.2, the standard manipulations yield that

$$\begin{cases} |z||\partial_z \tilde{A}_{1,B_{32r}}(x,z)| + |\tilde{A}_{1,B_{32r}}(x,z)| \leq L_s \dfrac{G^{p_s}(|z|)}{|z|} \\ \langle \partial_z \tilde{A}_{1,B_{32r}}(x,z)\xi, \xi\rangle \geq \nu_s \dfrac{G^{p_s}(|z|)}{|z|^2}|\xi|^2 \end{cases} \tag{3.5.60}$$

and

$$\begin{cases} |z||\partial_z \tilde{A}_{2,B_{32r}}(x,z)| + |\tilde{A}_{2,B_{32r}}(x,z)| \leq L_s \dfrac{H^{q_s}(|z|)}{|z|} \\ \langle \partial_z \tilde{A}_{2,B_{32r}}(x,z)\xi, \xi\rangle \geq \nu_s \dfrac{H^{q_s}(|z|)}{|z|^2}|\xi|^2 \end{cases} \tag{3.5.61}$$

for every $x \in B_{32r}$, $z \in \mathbb{R}^n \setminus \{0\}$ and $\xi \in \mathbb{R}^n$, provided

$$\omega(64r) \leq \min\left\{1, \frac{\nu}{(s(G)+1)(s(H)+1)}\right\},$$

where $L_s = L(2m_{pq}(1 + s(G) + s(H)) + 1)$ and $\nu_s = \nu/2$.

To proceed further, we need to consider another type of the higher integrability of w. In what follows let σ_0 and σ_1 be universal higher integrability exponents depending on `data`, which have been determined by Lemma 3.5.8 and 3.5.9, respectively.

Proposition 3.5.13. *Under the assumptions and conclusions of Lemma*

3.5.9, suppose also that

$$\begin{aligned}&\sup_{0<\rho\le 64r}\omega(\rho)\log\frac{1}{\rho}\le 1 \quad \textit{and}\\ &\omega(64r)\le\min\left\{1,\frac{\nu}{(s(G)+1)(s(H)+1)},\frac{n\sigma_1}{n+\alpha},\sqrt{\frac{n}{n-1}}-1\right\}.\end{aligned} \tag{3.5.62}$$

Then there exist constants $\tilde{\sigma}_1\equiv\tilde{\sigma}_1(\texttt{data})$ *and* $c\equiv c(\texttt{data})$ *such that*

$$\left(\fint_{B_{16r}}\Psi_s^{1+\sigma}(x,Dw)\,dx\right)^{\frac{1}{1+\sigma}}\le c\left(\fint_{B_{32r}}\Psi(x,Dw)\,dx+1\right) \tag{3.5.63}$$

holds for every $\sigma\in(0,\tilde{\sigma}_1]$.

Proof. First, by the assumption (3.5.62), we observe that

$$p_s\le p(x)\left(1+\frac{n+\alpha}{n}\omega(64r)\right)\le p(x)(1+\sigma_1)$$

and

$$q_s\le q(x)\left(1+\frac{n}{n+\alpha}\frac{n+\alpha}{n}\omega(64r)\right)\le q(x)\left(1+\frac{n\sigma_1}{n+\alpha}\right).$$

Thus we apply Lemma 3.5.10 for $\tilde{p}(x)\equiv p_s$, $\tilde{q}(x)\equiv q_s$ and $\sigma\equiv\frac{n+\alpha}{n}\omega(2\rho)\in(0,\sigma_1]$ to obtain

$$\begin{aligned}\fint_{B_\rho}\Psi_s(x,Dw)\,dx&\le c\left(\fint_{B_{2\rho}}\Psi(x,Dw)\,dx\right)^{1+\sigma}+c\\ &\le c\left(\fint_{B_{2\rho}}\Psi(x,Dw)\,dx\right)^{1+\frac{n+\alpha}{n}\omega(2\rho)}+c\end{aligned} \tag{3.5.64}$$

for some constant $c\equiv c(\texttt{data})$. Therefore, by (3.5.62) and (3.5.54), we notice

that

$$\left(\fint_{B_{2\rho}} \Psi(x,Dw)\,dx\right)^{\frac{n+\alpha}{n}\omega(2\rho)} \le c\left(\frac{\|\Psi(x,Dw)\|_{L^1(B_{32r})}}{\rho^n}\right)^{\frac{n+\alpha}{n}\omega(2\rho)}$$
$$\le ce^{(n+\alpha)(\log\frac{1}{2\rho})\omega(2\rho)} \le c(\mathtt{data}).$$

Inserting the last display into (3.5.64), we obtain that

$$\fint_{B_\rho} \Psi_s(x,Dw)\,dx \le c\fint_{B_{2\rho}} \Psi(x,Dw)\,dx + c$$

for some $c \equiv c(\mathtt{data})$, whenever $B_{2\rho} \subset B_{32r}$ is a ball. Therefore, using a standard covering argument, we find that

$$\fint_{B_{16r}} \Psi_s(x,Dw)\,dx \le c\fint_{B_{64r/3}} \Psi(x,Dw)\,dx + c. \tag{3.5.65}$$

On the other hand, by taking $\eta^{s(\Psi)+1}(w-(w)_{B_{64r/3}}) \in W_0^{1,\Psi}(B_{2r})$ as a test function to the equation (3.5.43), where $\eta \in C_0^\infty(B_{64r/3}(y))$ is a cut-off function such that $\chi_{B_{64r/3}(y)} \le \eta \le \chi_{B_{80r/3}(y)}$ and $|D\eta| \le 12/r$, and following the similar proof for obtaining (3.5.49) in Lemma 3.5.9, there exists $\theta \equiv \theta(n, s(G), s(H), \alpha, m_{pq}) \in (0,1)$ such that

$$\fint_{B_{64r/3}} \Psi(x,Dw)\,dx \le c\left(\fint_{B_{80r/3}} \Psi^\theta(x,Dw)\,dx\right)^{\frac{1}{\theta}} + c.$$

for some $c \equiv c(\mathtt{data})$. Recalling the definition of Ψ_s in (3.5.55) and the above inequality together with (3.5.65) yields

$$\le c\left(\fint_{B_{80r/3}} \Psi^\theta(x,Dw)\,dx\right)^{\frac{1}{\theta}} + c \le c\left(\fint_{B_{32r}} \Psi_s^\theta(x,Dw)\,dx\right)^{\frac{1}{\theta}} + c.$$

Then by applying Gehring's lemma, there exists a positive number $\tilde{\sigma}_1 \equiv$

$\tilde{\sigma}_1(\mathtt{data})$ such that

$$\left(\fint_{B_{16r}} \Psi_s^{1+\sigma}(x, Dw)\,dx\right)^{\frac{1}{1+\sigma}} \leq c \fint_{B_{32r}} \Psi_s(x, Dw)\,dx + c$$

with some constant $c \equiv c(\mathtt{data})$ for every $\sigma \in (0, \tilde{\sigma}_1]$. $\square$

Now let $\tilde{v} \in W^{1,\Psi_s}(B_{16r})$ be the weak solution of the following Dirichlet boundary value problem:

$$\begin{cases} -\mathrm{div}\tilde{A}_{B_{32r}}(x, D\tilde{v}) = 0 & \text{in } B_{16r} \\ \tilde{v} \in w + W_0^{1,\Psi_s}(B_{16r}), \end{cases} \tag{3.5.66}$$

where

$$\tilde{A}_{B_{32r}}(x, z) := \tilde{A}_{1,B_{32r}}(x, z) + a(x)\tilde{A}_{2,B_{32r}}(x, z) \tag{3.5.67}$$

for every $x \in B_{32r}$ and $z \in \mathbb{R}^n$, in which the vector fields $\tilde{A}_{1,B_{32r}}$ and $\tilde{A}_{2,B_{32r}}$ have been introduced in (3.5.59).

Lemma 3.5.14. *Let $\tilde{v} \in W^{1,\Psi_s}(B_{16r})$ be the weak solution to the equation (3.5.66) under the assumption (3.5.1)-(3.5.4), (3.5.7)-(3.5.8) and (3.5.62). There exists a positive number $\sigma_2 \leq \min\{\sigma_1, \tilde{\sigma}_1\}$ depending only on $\mathtt{data}$ and γ such that*

$$\fint_{B_{16r}} [\Psi_s(x, D\tilde{v})]^{1+\sigma}\,dx \leq c \fint_{B_{16r}} [\Psi_s(x, Dw)]^{1+\sigma}\,dx + c \tag{3.5.68}$$

with some constant $c \equiv c(\mathtt{data})$ for every $\sigma \in (0, \sigma_2]$.

Proof. Firstly, the standard energy estimate and (3.5.63) imply that

$$\int_{B_{16r}} \Psi_s(x, D\tilde{v})\,dx \leq c \int_{B_{16r}} \Psi_s(x, Dw)\,dx \leq c \int_{B_{16r}} \Psi(x, Du)\,dx + c \leq c$$

holds with some constant $c \equiv c(\mathtt{data})$. For a ball $B_{2\rho} \equiv B_{2\rho}(y) \Subset B_{16r}$, let $\eta \in C_0^\infty(B_{2\rho}(y))$ be a standard cut-off function satisfying $\chi_{B_\rho(y)} \leq \eta \leq \chi_{B_{2\rho}(y)}$ and $|D\eta| \leq 4/\rho$. Let us take the function $\varphi = \eta^{s(\Psi_s)+1}\left(\tilde{v} - (\tilde{v})_{B_{2\rho}}\right)$ as a test function in the equation (3.5.66), where $s(\Psi_s)$ has been defined in (3.5.57). This choice of φ is admissible due to the assumption (3.5.62). Then using

(3.5.60) and (3.5.61) and Lemma 2.1.6 to Ψ_s, we have

$$
\begin{aligned}
&\int_{B_{2\rho}} \eta^{s(\Psi_s)+1} \Psi_s(x, D\tilde{v})\, dx \\
&\qquad \le c \int_{B_{2\rho}} \eta^{s(\Psi_s)} \frac{\Psi_s(x, D\tilde{v})}{|D\tilde{v}|} \left| \frac{\tilde{v} - (\tilde{v})_{B_{2\rho}}}{\rho} \right| dx \\
&\qquad \le c \int_{B_{2\rho}} \eta^{s(\Psi_s)} \left((\varepsilon\eta)\Psi(x, D\tilde{v}) + \frac{1}{(\varepsilon\eta)^{s(\Psi_s)}} \Psi_s\left(x, \frac{\tilde{v} - (\tilde{v})_{B_{2\rho}}}{\rho}\right)\right) dx.
\end{aligned}
$$

Choosing ε sufficiently small in the last display, we conclude that

$$
\fint_{B_\rho} \Psi_s(x, D\tilde{v})\, dx \le c \fint_{B_{2\rho}} \Psi_s\left(x, \frac{\tilde{v} - (\tilde{v})_{B_{2\rho}}}{\rho}\right) dx
$$

for a constant $c \equiv c(n, s(G), s(H), \nu, L, m_{pq}, \alpha)$. Repeating the argument in the proof of Lemma 3.5.8 with Ψ_s, there exists $\theta_s \equiv \theta_s(n, s(G), s(H), m_{pq}) \in (0,1)$ such that

$$
\begin{aligned}
\fint_{B_\rho} \Psi_s(x, D\tilde{v})\, dx &\le c \fint_{B_{2\rho}} \Psi_s\left(x, \frac{\tilde{v} - (\tilde{v})_{B_{2\rho}}}{\rho}\right) dx \\
&\le c \left(\fint_{B_{2\rho}} [\Psi_s(x, D\tilde{v})]^{\theta_s}\, dx \right)^{\frac{1}{\theta_s}}
\end{aligned}
\tag{3.5.69}
$$

holds for some constant $c \equiv c(\mathtt{data})$ whenever $B_{2\rho} \Subset B_{16r}$ is a ball. Now we prove a version of the last inequality near the boundary of B_{16r}. For this, let $B_{2\rho}(y) \subset \mathbb{R}^n$ be a ball such that $y \in B_{16r}$ and $\frac{1}{10} < \frac{|B_{2\rho}(y) \setminus B_{16r}|}{|B_{2\rho}(y)|}$. We take a test function by $\varphi \equiv \eta^{s(\Psi_s)+1}(w - \tilde{v})$, where $\eta \in C_0^\infty(B_{2\rho})$ is a standard cut-off function as before so that $\chi_{B_\rho} \le \eta \le \chi_{B_{2\rho}}$ and $|D\eta| \le 4/\rho$. This choice of φ is admissible since supp $\varphi \Subset B_{16r} \cap B_{2\rho}(y)$. Arguing similarly as we have done before, we see that

$$
\begin{aligned}
&\int_{B_{16r}\cap B_{2\rho}(y)} \eta^{s(\Psi_s)+1}\Psi_s(x, D\tilde{v})\,dx \\
&\leq c\int_{B_{16r}\cap B_{2\rho}(y)} \eta^{s(\Psi_s)}\frac{\Psi_s(x, D\tilde{v})}{|D\tilde{v}|}\left|\frac{\tilde{v}-w}{\rho}\right|\,dx \\
&\qquad + c\int_{B_{16r}\cap B_{2\rho}(y)} \eta^{s(\Psi_s)}\frac{\Psi_s(x, D\tilde{v})}{|D\tilde{v}|}|Dw|\,dx \\
&\leq c\int_{B_{16r}\cap B_{2\rho}(y)} \eta^{s(\Psi_s)}\left((\varepsilon\eta)\Psi_s(x, D\tilde{v}) + \frac{1}{(\varepsilon\eta)^{s(\Psi_s)}}\Psi_s\left(x, \frac{\tilde{v}-w}{\rho}\right)\right)\,dx \\
&\qquad + c\int_{B_{16r}\cap B_{2\rho}(y)} \eta^{s(\Psi_s)}\left((\varepsilon\eta)\Psi_s(x, D\tilde{v}) + \frac{1}{(\varepsilon\eta)^{s(\Psi_s)}}\Psi_s\left(x, Dw\right)\right)\,dx.
\end{aligned}
$$

Again choosing ε small enough and reabsorbing the terms, we find that

$$
\begin{aligned}
&\fint_{B_{16r}\cap B_{2\rho}(y)} \eta^{s(\Psi_s)+1}\Psi_s(x, D\tilde{v})\,dx \\
&\qquad \leq c\fint_{B_{16r}\cap B_{2\rho}(y)} \Psi_s\left(x, \frac{\tilde{v}-w}{\rho}\right)\,dx + c\fint_{B_{16r}\cap B_{2\rho}(y)} \Psi_s\left(x, Dw\right)\,dx
\end{aligned}
$$

for some constant $c \equiv c(n, s(G), s(H), \nu, L, m_{pq})$. Redefining $\tilde{v} - w \equiv 0$ on $B_{2\rho}(y) \setminus B_{16r}$ and following again the proof of Lemma 3.5.8, there exists $\theta_s \equiv \theta_s(n, s(G), s(H), m_{pq}, \alpha) \in (0, 1)$ as appearing in (3.5.69) such that

$$
\begin{aligned}
&\fint_{B_{16r}\cap B_{2\rho}(y)} \Psi_s\left(x, \frac{\tilde{v}-w}{\rho}\right)\,dx \\
&\quad \leq \left(\fint_{B_{16r}\cap B_{2\rho}(y)} [\Psi_s(x, D\tilde{v} - Dw)]^{\theta_s}\,dx\right)^{\frac{1}{\theta_s}} \\
&\quad \leq \left(\fint_{B_{16r}\cap B_{2\rho}(y)} [\Psi_s(x, D\tilde{v})]^{\theta_s}\,dx\right)^{\frac{1}{\theta_s}} + c\fint_{B_{16r}\cap B_{2\rho}(y)} [\Psi_s(x, Dw)]\,dx
\end{aligned}
$$

for some constant $c \equiv c(\mathtt{data})$, where for the last inequality we have used again (3.5.32) and Hölder's inequality. Combining the last two displays, we have

$$\fint_{B_\rho(y)} [V(x)]^{\frac{1}{\theta_s}}\,dx \le c\left(\fint_{B_{2\rho}(y)} V(x)\,dx\right)^{\frac{1}{\theta_s}} + c\fint_{B_{2\rho}(y)} U(x)\,dx$$

for some $c \equiv c(\mathtt{data})$, where

$$V(x) := [\Psi_s(x, D\tilde{v})]^{\theta_s}\chi_{B_{16r}}(x) \quad \text{and} \quad U(x) := \Psi_s(x, Dw)\chi_{B_{16r}}(x)$$

for every ball $B_{2\rho}(y) \subset \mathbb{R}^n$ satisfying either $B_{2\rho}(y) \subset B_{16r}$ or $\frac{1}{10} < \frac{|B_{2\rho}(y)\setminus B_{16r}|}{|B_{2\rho}(y)|}$ with $y \in B_{16r}$. Applying a variant of Gehring's lemma and a standard covering argument, we arrive at the desired estimate (3.5.68). □

Moreover, we also need some elementary properties regarding log function, see for instance [2].

1. For any $s, t \ge 0$, it holds that

$$\log(e + st) \le \log(e + s) + \log(e + t), \tag{3.5.70}$$

   where e is Euler's constant.

2. For any $0 < \beta_1 \le \beta \le \beta_2$ and $0 < t < e$, there exists $c(\beta_1, \beta_2) > 0$ such that

$$t^\beta |\log t| \le c(\beta_1, \beta_2). \tag{3.5.71}$$

3. For any $0 < \beta_1 \le \beta_2 < \infty$ and $s_1 > 1$, there exists $c(s_1, \beta_1, \beta_2) > 0$ such that

$$\begin{aligned}\fint_\Omega |f| \left[\log\left(e + \frac{|f|}{\fint_\Omega |f|\,dx}\,dx\right)\right]^\beta dx \\ \le c(s_1, \beta_1, \beta_2)\left(\fint_\Omega |f|^{s_1}\,dx\right)^{\frac{1}{s_1}},\end{aligned} \tag{3.5.72}$$

   whenever $\beta \in [\beta_1, \beta_2]$ and $f \in L^1(\Omega)$.

Then we shall deal with the second comparison estimates which are essential parts of our comparison process.

Lemma 3.5.15. *Let $\tilde{v} \in W^{1,\Psi_s}(B_{16r})$ be the weak solution to (3.5.66) under assumptions and conclusions of Lemma 3.5.14. Then for every $\varepsilon > 0$, there exists a small number $\delta \equiv \delta(\boldsymbol{\mathtt{data}}, \gamma, \varepsilon)$ such that if*

$$\fint_{B_{32r}} \Psi(x, Du)\, dx \le \lambda \tag{3.5.73}$$

and

$$\sup_{0<\rho\le 64r} \omega(\rho) \log \frac{1}{\rho} \le \delta \tag{3.5.74}$$

for some $\lambda \ge 1$, then we have

$$\fint_{B_{16r}} \Psi_s(x, D\tilde{v})\, dx \le c\lambda \tag{3.5.75}$$

for some constant $c \equiv c(n, s(G), s(H), \nu, L, m_{pq}, \|a\|_{0,\alpha})$ and

$$\fint_{B_{16r}} \Psi_s(x, Dw - D\tilde{v})\, dx \le \varepsilon\lambda. \tag{3.5.76}$$

Proof. The standard energy estimates together with Proposition 3.5.13, we have

$$\begin{aligned} \fint_{B_{16r}} \Psi_s(x, D\tilde{v})\, dx &\le c \fint_{B_{16r}} \Psi_s(x, Dw)\, dx \\ &\le c \fint_{B_{32r}} \Psi(x, Du)\, dx + c \le c\lambda \end{aligned} \tag{3.5.77}$$

for some constant $c \equiv c(\mathtt{data})$. To show (3.5.76), first let us observe that the following equality

$$\begin{aligned} I_1 :&= \fint_{B_{16r}} \langle \tilde{A}_{B_{32r}}(x, Dw) - \tilde{A}_{B_{32r}}(x, D\tilde{v}), D\tilde{v} - Dw \rangle\, dx \\ &= \fint_{B_{16r}} \langle \tilde{A}_{B_{32r}}(x, Dw) - A(x, Dw), D\tilde{v} - Dw \rangle\, dx =: I_2 \end{aligned} \tag{3.5.78}$$

holds by the admissibility of $\tilde{v} - w$ in the equation (3.5.43). The structure properties (3.5.60)-(3.5.61), (3.5.31) and the last display give us that

$$\fint_{B_{16r}} \Psi_s(x, D\tilde{v} - Dw)\, dx \leq \tau c\lambda + c_\tau I_1 \tag{3.5.79}$$

for every $\tau \in (0,1)$, where $c \equiv c(\mathtt{data})$ and $c_\tau \equiv c_\tau(\mathtt{data}, \tau)$. Recalling the definition of the vector field $\tilde{A}_{B_{32r}}$ introduced in (3.5.66) and using the structure assumptions (3.5.7) and (3.5.8), we estimate I_2 as follows:

$$\begin{aligned} I_2 &\leq c_* \fint_{B_{16r}} \left| G^{p_s}(|Dw|) - G^{p(x)}(|Dw|) \right| \frac{|Dw - D\tilde{v}|}{|Dw|}\, dx \\ &\quad + c_* \fint_{B_{16r}} a(x) \left| H^{q_s}(|Dw|) - H^{q(x)}(|Dw|) \right| \frac{|Dw - D\tilde{v}|}{|Dw|}\, dx \\ &=: c_* \left(I_3 + I_4 \right). \end{aligned} \tag{3.5.80}$$

For the simplicity, let us denote by

$$\begin{aligned} \mathcal{H}_1 &:= \{x \in B_{16r} : |Dw(x)| \geq |D\tilde{v}(x)| \quad \text{and} \quad 0 < H(|Dw(x)|) \leq 1\}, \\ \mathcal{H}_2 &:= \{x \in B_{16r} : |Dw(x)| \geq |D\tilde{v}(x)| \quad \text{and} \quad 1 < H(|Dw(x)|)\}, \\ \mathcal{H}_3 &:= \{x \in B_{16r} : |Dw(x)| < |D\tilde{v}(x)| \quad \text{and} \quad 0 < H(|Dw(x)|) \leq 1\}, \\ \mathcal{H}_4 &:= \{x \in B_{16r} : |Dw(x)| < |D\tilde{v}(x)| \quad \text{and} \quad 1 < H(|Dw(x)|)\}. \end{aligned}$$

Now applying the mean value theorem, the second term in (3.5.80) can be estimated as

$$
\begin{aligned}
I_4 \le{}& \frac{\omega(64r)}{|B_{16r}|} \int_{B_{16r}\cap\{|Dw|>0\}} a(x) H^{t_x(q_s-q(x))+q(x)}(|Dw|)|\log H(|Dw|)| \\
&\times \frac{|Dw - D\tilde{v}|}{|Dw|}\, dx \\
\le{}& 2\frac{\omega(64r)}{|B_{16r}|} \int_{\mathcal{H}_1} a(x) H^{t_x(q_s-q(x))+q(x)}(|Dw|)|\log H(|Dw|)|\, dx \\
&+ 2\frac{\omega(64r)}{|B_{16r}|} \int_{\mathcal{H}_2} a(x) H^{t_x(q_s-q(x))+q(x)}(|Dw|)|\log H(|Dw|)|\, dx \\
&+ 2\frac{\omega(64r)}{|B_{16r}|} \int_{\mathcal{H}_3} a(x) H^{t_x(q_s-q(x))+q(x)}(|Dw|)|\log H(|Dw|)|\frac{|D\tilde{v}|}{|Dw|}\, dx \\
&+ 2\frac{\omega(64r)}{|B_{16r}|} \int_{\mathcal{H}_4} a(x) H^{t_x(q_s-q(x))+q(x)}(|Dw|)|\log H(|Dw|)|\frac{|D\tilde{v}|}{|Dw|}\, dx \\
=:{}& 2\left(I_{41} + I_{42} + I_{43} + I_{44}\right)
\end{aligned}
\tag{3.5.81}
$$

for some $t_x \in [0,1]$ depending on $x \in B_{16r}$. Now we estimate the integrals appearing in the last display. First using (3.5.71) with the observation that $1 \le t_x(q_s - q(x)) + q(x) \le m_{pq}$ for every $x \in B_{16r}$, we have

$$
I_{41} \le c(m_{pq})\omega(64r) \le c(m_{pq})\delta\lambda. \tag{3.5.82}
$$

For every $x \in \mathcal{H}_2$, recalling (3.5.15) and $H \in \mathcal{N}$, we see

$$
\begin{aligned}
&H^{t_x(q_s-q(x))+q(x)}(|Dw(x)|)\log H(|Dw(x)|) \\
&\qquad\le (1+s(H))H^{q_s}(|Dw(x)|)\log(|Dw(x)|) \\
&\qquad\le c\, H^{q_s}(|Dw(x)|)\left[\log(e + G^{p_s}(|Dw(x)|))\right]
\end{aligned}
$$

for some $c \equiv c(n,\kappa,s(G),s(H),\alpha,m_{pq})$. Therefore, using the last display and (3.5.70), (3.5.72), (3.5.54), (3.5.63), (3.5.74) and Hölder's inequality, it implies that

$$
\begin{aligned}
I_{42} &\leq \frac{c\omega(64r)}{|B_{16r}|}\int_{\mathcal{H}_2}\left(a(x)H^{q_s}(|Dw|)\log\left(e+G^{p_s}(|Dw|)\right)+c\right)\,dx \\
&\leq c\omega(64r)\fint_{B_{16r}} a(x)H^{q_s}(|Dw|)\log\left(e+\frac{\Psi_s(x,Dw)}{\fint_{B_{16r}}\Psi_s(x,Dw)dx}\right)dx \\
&\quad + c\omega(64r)\fint_{B_{8r}} a(x)H^{q_s}(|Dw|)\log\left(e+\fint_{B_{16r}}\Psi_s(x,Dw)dx\right)dx \\
&\quad + c\omega(64r) \qquad\qquad (3.5.83)\\
&\leq c\omega(64r)\left(\fint_{B_{16r}}\Psi_s^{1+\sigma_2}(x,Dw)\,dx\right)^{\frac{1}{1+\sigma_2}} \\
&\quad + c\omega(64r)\log\left(\frac{1}{r}\right)\fint_{B_{16r}}\Psi_s(x,Dw)\,dx + c\omega(64r) \\
&\leq c\omega(64r)\log\left(\frac{1}{r}\right)\left[\left(\fint_{B_{16r}}\Psi_s^{1+\sigma_2}(x,Dw)\,dx\right)^{\frac{1}{1+\sigma_2}}+1\right] \leq c\delta\lambda
\end{aligned}
$$

with some constant $c = c(\texttt{data}, \gamma)$, where we have applied the following inequality

$$
\begin{aligned}
\fint_{B_{16r}}\Psi_s(x,D\tilde{v})\,dx &\leq c\fint_{B_{16r}}\Psi_s(x,Dw)\,dx \\
&= \frac{c}{|B_{16r}|}\int_{B_{16r}}\Psi_s(x,Dw)\,dx \qquad (3.5.84)\\
&\leq \frac{c}{r^n}\|\Psi(x,Du)\|_{L^1(\Omega)} + c \leq \frac{c}{r^n}
\end{aligned}
$$

with $c \equiv c(\texttt{data})$, which is valid by (3.5.77) and (3.5.63). Applying Lemma 2.1.5, there exists $\theta_H \in (0,1)$ depending only on $s(H)$ such that $H^{\theta_H} \in \mathcal{N}$.

Then we write I_{43} in the following form:

$$
\begin{aligned}
I_{43} &\leq \frac{\omega(64r)}{|B_{16r}|} \int_{\mathcal{H}_3} a(x) H^{\theta_H(t_x(q_s-q(x))+q(x))}(|Dw|) \frac{|D\tilde{v}|}{|Dw|} \\
&\quad \times H^{(1-\theta_H)(t_x(q_s-q(x))+q(x))}(|Dw|) |\log H(|Dw|)| \, dx \\
&\leq \frac{c\omega(64r)}{|B_{16r}|} \int_{\mathcal{H}_3} a(x) H^{\theta_H(t_x(q_s-q(x))+q(x))}(|Dw|) \frac{|D\tilde{v}|}{|Dw|} \, dx \\
&\leq \frac{c\omega(64r)}{|B_{16r}|} \int_{\mathcal{H}_3} a(x) H^{\theta_H}(|Dw|) \frac{|D\tilde{v}|}{|Dw|} \, dx
\end{aligned}
\tag{3.5.85}
$$

for some constant $c \equiv c(s(H), m_{pq})$, where we have used (3.5.71) with the observation that

$$
1-\theta_H \leq (1-\theta_H)\left(t_x(q_s-q(x))+q(x)\right) \leq (1-\theta_H) m_{pq} \quad \text{for every } x \in B_{16r}.
$$

In order to estimate I_{43} further, we apply Lemma 2.1.6 for H^{θ_H} in the resulting term of (3.5.85). In turn, it yields that

$$
\begin{aligned}
I_{43} &\leq \frac{c\omega(64r)}{|B_{16r}|} \int_{\mathcal{H}_3} \left(a(x) H^{\theta_H}(|Dw|) + a(x) H^{\theta_H}(|D\tilde{v}|) \right) \, dx \\
&\leq c\omega(64r) \fint_{B_{16r}} \|a\|_{L^\infty(\Omega)}^{1-\theta_H} \left[a(x) H(|Dw|)\right]^{\theta_H} \, dx \\
&\quad + c\omega(64r) \fint_{B_{16r}} \|a\|_{L^\infty(\Omega)}^{\frac{q_s-\theta_H}{q_s}} \left[a(x)\right]^{\frac{\theta_H}{q_s}} \left[H(|D\tilde{v}|)\right]^{\theta_H} \, dx \\
&\leq c\omega(64r) \fint_{B_{16r}} \left[a(x) H(|Dw|)\right]^{\theta_H} \, dx \\
&\quad + c\omega(64r) \fint_{B_{16r}} \left[a(x)\right]^{\frac{\theta_H}{q_s}} \left[H(|D\tilde{v}|)\right]^{\theta_H} \, dx \\
&\leq c\omega(64r) \left(\fint_{B_{16r}} a(x) H(|Dw|) \, dx \right)^{\theta_H} \\
&\quad + c\omega(64r) \left(\fint_{B_{16r}} a(x) H^{q_s}(|D\tilde{v}|) \, dx \right)^{\frac{\theta_H}{q_s}}
\end{aligned}
$$

for some constant $c \equiv c(s(H), m_{pq}, \|a\|_{L^\infty(\Omega)})$, where in the inequalities of the above display we have used some elementary manipulations, and then Hölder's inequality. Finally, recalling the energy estimate (3.5.77), we find the desired estimate as

$$I_{43} \le c\omega(64r)\lambda \le c\delta\lambda \tag{3.5.86}$$

with $c \equiv c(n, s(H), \nu, L, m_{pq}, \alpha, \|a\|_{0,\alpha})$. It remains to estimate I_{44} in (3.5.81) and this can be handled in a similar way as we have done for I_{42}. First, using Remark 2.1.2, for every $x \in \mathcal{H}_4$, we have

$$\begin{aligned} H^{t_x(q_s-q(x))+q(x)}(|Dw(x)|)\frac{|D\tilde{v}(x)|}{|Dw(x)|} &\le cH^{q_s-1}(|Dw(x)|)H^{'}(|Dw(x)|)|D\tilde{v}(x)| \\ &\le cH^{q_s-1}(|D\tilde{v}(x)|)H^{'}(|D\tilde{v}(x)|)|D\tilde{v}(x)| \\ &\le cH^{q_s}(|D\tilde{v}(x)|) \end{aligned}$$

and

$$H^{q_s}(|D\tilde{v}(x)|)\log H(|D\tilde{v}(x)|) \le cH^{q_s}(|D\tilde{v}(x)|)\log\left(e + G^{p_s}(|D\tilde{v}(x)|)\right)$$

with some constant $c \equiv c(n, \kappa, s(G), s(H), \alpha, m_{pq})$. Therefore, taking the last two displays into account and arguing similarly as in (3.5.83), we discover that

$$\begin{aligned} I_{44} &\le \frac{\omega(64r)}{|B_{16r}|}\int_{\mathcal{H}_4} a(x)H^{q_s}(|D\tilde{v}|)\log H(|D\tilde{v}|)\,dx \\ &\le c\omega(64r)\fint_{B_{16r}} a(x)H^{q_s}(|D\tilde{v}|)\log\left(e + G^{p_s}(|D\tilde{v}|)\right)\,dx \\ &\overset{(3.5.70)}{\le} c\omega(64r)\fint_{B_{16r}} a(x)H^{q_s}(|D\tilde{v}|)\log\left(e + \frac{\Psi_s(x, D\tilde{v})}{\fint_{B_{16r}}\Psi_s(x, D\tilde{v})\,dx}\right)\,dx \\ &\quad + c\omega(64r)\fint_{B_{16r}} a(x)H^{q_s}(|D\tilde{v}|)\log\left(e + \fint_{B_{16r}}\Psi_s(x, D\tilde{v})\,dx\right)\,dx \\ &\overset{(3.5.68)}{\le} c\omega(64r)\log\left(\frac{1}{r}\right)\left[\left(\fint_{B_{16r}}\Psi_s^{1+\sigma_2}(x, Dw)\,dx\right)^{\frac{1}{1+\sigma_2}} + 1\right] \\ &\overset{(3.5.54),(3.5.63),(3.5.74)}{\le} c\delta\lambda \end{aligned} \tag{3.5.87}$$

for some $c \equiv c(\mathtt{data}, \gamma)$. Inserting the estimates obtained in (3.5.82)-(3.5.83) and (3.5.86)-(3.5.87) into (3.5.79), we find

$$I_4 \leq c\delta\lambda \quad \text{for some} \quad c \equiv c(\texttt{data}, \gamma). \tag{3.5.88}$$

In a similar way as we have treated for I_4, we also can see that

$$I_3 \leq c\delta\lambda \quad \text{for some} \quad c \equiv c(\texttt{data}, \gamma). \tag{3.5.89}$$

Merging (3.5.88) and (3.5.89) into (3.5.78), and then into (3.5.79), we find that

$$\fint_{B_{16r}} \Psi_s(x, D\tilde{v} - Dw)\, dx \leq c\,(\tau\lambda + c_\tau \delta\lambda)$$

with some constant $c_\tau \equiv c(\texttt{data}, \gamma, \tau)$ for every $\tau \in (0,1)$. Therefore, choosing small enough τ and δ depending on $\texttt{data}, \gamma$ and ε, the desired comparison estimate (3.5.76) follows. $\square$

First let us define the vector fields $\bar{A}_{1,B_{32r}}, \bar{A}_{2,B_{32r}} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\begin{cases} \bar{A}_{1,B_{32r}}(z) := \fint_{B_{32r}} \tilde{A}_{1,B_{32r}}(x,z)\, dx \\ \bar{A}_{2,B_{32r}}(z) := \fint_{B_{32r}} \tilde{A}_{2,B_{32r}}(x,z)\, dx. \end{cases} \tag{3.5.90}$$

Clearly, the vector fields $\bar{A}_{1,B_{32r}}$ and $\bar{A}_{2,B_{32r}}$ belong to $C^1(\mathbb{R}^n \setminus \{0\})$ and satisfy the following structure conditions:

$$\begin{cases} |z||\partial_z \bar{A}_{1,B_{32r}}(z)| + |\bar{A}_{1,B_{32r}}(z)| \leq L_s \dfrac{G^{p_s}(|z|)}{|z|} \\ \langle \partial_z \bar{A}_{1,B_{32r}}(z)\xi, \xi\rangle \geq \nu_s \dfrac{G^{p_s}(|z|)}{|z|^2}|\xi|^2 \end{cases} \tag{3.5.91}$$

and

$$\begin{cases} |z||\partial_z \bar{A}_{2,B_{32r}}(z)| + |\bar{A}_{2,B_{32r}}(z)| \leq L_s \dfrac{H^{q_s}(|z|)}{|z|} \\ \langle \partial_z \bar{A}_{2,B_{32r}}(z)\xi, \xi\rangle \geq \nu_s \dfrac{H^{q_s}(|z|)}{|z|^2}|\xi|^2 \end{cases} \tag{3.5.92}$$

for every $z \in \mathbb{R}^n \setminus \{0\}$ and $\xi \in \mathbb{R}^n$, provided

$$\omega(64r) \le \min\left\{1, \frac{\nu}{(s(G)+1)(s(H)+1)}\right\},$$

where $L_s = L(2m_{pq}(1+s(G)+s(H))+1)$ and $\nu_s = \nu/2$. Recalling the definition of $\tilde{A}_{1,B_{32r}}$ in (3.5.59) and $\bar{A}_{1,B_{32r}}$ in (3.5.90), we observe that

$$\begin{aligned} &\frac{|\tilde{A}_{1,B_{32r}}(x,z) - \bar{A}_{1,B_{32r}}(z)|}{G^{p_s-1}(|z|)G'(|z|)} \\ &= \left| \frac{\tilde{A}_{1,B_{32r}}(x,z)}{G^{p_s-1}(|z|)G'(|z|)} - \fint_{B_{32r}} \frac{\tilde{A}_{1,B_{32r}}(\tilde{x},z)}{G^{p_s-1}(|z|)G'(|z|)}\, d\tilde{x} \right| \\ &= \left| \frac{A_1(x,z)}{G^{p(x)-1}(|z|)G'(|z|)} - \fint_{B_{32r}} \frac{A_1(\tilde{x},z)}{G^{p(\tilde{x})-1}(|z|)G'(|z|)}\, d\tilde{x} \right| \end{aligned}$$

for $x \in B_{32r}$ and $z \in \mathbb{R}^n \setminus \{0\}$. Therefore, by (3.5.12), it holds that

$$\begin{aligned} &\fint_{B_{32r}} \sup_{z\in\mathbb{R}^n\setminus\{0\}} \frac{|\tilde{A}_{1,B_{32r}}(x,z) - \bar{A}_{1,B_{32r}}(z)|}{G^{p_s-1}(|z|)G'(|z|)}\, dx \\ &\qquad = \fint_{B_{32r}} \theta(A_1, B_{32r}(y_0))(x)\, dx \le \delta. \end{aligned} \tag{3.5.93}$$

Arguing similarly, we also see

$$\begin{aligned} &\fint_{B_{32r}} \sup_{z\in\mathbb{R}^n\setminus\{0\}} \frac{|\tilde{A}_{2,B_{32r}}(x,z) - \bar{A}_{2,B_{32r}}(z)|}{H^{q_s-1}(|z|)H'(|z|)}\, dx \\ &\qquad = \fint_{B_{32r}} \theta(A_2, B_{32r}(y_0))(x)\, dx \le \delta. \end{aligned} \tag{3.5.94}$$

In what follows we denote by

$$\bar{A}_{B_{32r}}(x,z) := \bar{A}_{1,B_{32r}}(z) + a(x)\bar{A}_{2,B_{32r}}(z) \quad \text{for every} \quad x \in \Omega,\ z \in \mathbb{R}^n.$$

We now consider a function $\bar{v} \in W^{1,\Psi_s}(B_{8r})$ as the weak solution of the following Dirichlet boundary value problem:

$$\begin{cases} -\mathrm{div}\bar{A}_{B_{32r}}(x, D\bar{v}) = 0 \quad \text{in} \quad B_{8r} \\ \bar{v} \in \tilde{v} + W_0^{1,\Psi_s}(B_{8r}). \end{cases} \tag{3.5.95}$$

Arguing similarly as for Lemma 3.5.14, we are also able to prove the following higher integrability.

Lemma 3.5.16. *Let $\bar{v} \in W^{1,\Psi_s}(B_{8r})$ be the weak solution to* (3.5.95) *under the assumptions of Lemma 3.5.15. Then there exists a higher integrability exponent $\sigma_3 \le \sigma_2$ depending on $\boldsymbol{data}$ and γ such that*

$$\left(\fint_{B_{8r}} [\Psi_s(x, D\bar{v})]^{1+\sigma} \, dx \right)^{\frac{1}{1+\sigma}} \le c \left(\fint_{B_{8r}} [\Psi_s(x, D\tilde{v})]^{1+\sigma} \, dx \right)^{\frac{1}{1+\sigma}} + c \tag{3.5.96}$$

with some constant $c \equiv c(\boldsymbol{data})$ for every $\sigma \in (0, \sigma_3]$.

Taking the conditions (3.5.91), (3.5.92) and Remark 3.5.12 into account, we are able to apply [12, Theorem 5.1] to have the following higher differentiability.

Lemma 3.5.17. *Under the assumptions of Lemma 3.5.15, let $\bar{v}$ be the weak solution to* (3.5.95)*. Then it holds that*

$$G^{p_s}(|D\bar{v}|) \in L_{\mathrm{loc}}^{\frac{n}{n-2\beta}}(B_{8r}) \cap W_{\mathrm{loc}}^{\beta,2}(B_{8r}) \tag{3.5.97}$$

for every $\beta < \alpha/2$.

In the following we shall deal with the third comparison estimates.

Lemma 3.5.18. *Under the assumptions and conclusions of Lemma 3.5.15, let $\bar{v} \in W^{1,\Psi_s}(B_{8r})$ be the weak solution to* (3.5.95)*. Then for every $\varepsilon > 0$, there exists $\delta = \delta(\boldsymbol{data}, \gamma, \varepsilon) > 0$ such that if*

$$\fint_{B_{32r}(y_0)} [\theta(A_1, B_{32r}(y_0))(x) + \theta(A_2, B_{32r}(y_0))(x)] \, dx \le \delta, \tag{3.5.98}$$

then there exists a constant $c \equiv c(n, s(G), s(H), \nu, L, m_{pq}, \alpha, \|a\|_{0,\alpha})$ such that

$$\fint_{B_{8r}} \Psi_s(x, D\bar{v}) \, dx \le c\lambda \tag{3.5.99}$$

and

$$\fint_{B_{8r}} \Psi_s(x, D\tilde{v} - D\bar{v})\, dx \leq \varepsilon\lambda. \tag{3.5.100}$$

Proof. First, taking $\bar{v} - \tilde{v} \in W_0^{1,\Psi_s}(B_{8r})$ as a test function in (3.5.95) and using some standard manipulations that we have already employed in the previous lemmas, we find

$$\fint_{B_{8r}} \Psi_s(x, D\bar{v})\, dx \leq c \fint_{B_{8r}} \Psi_s(x, D\tilde{v})\, dx \leq c\lambda \tag{3.5.101}$$

with $c = c(n, s(G), s(H), \nu, L, m_{pq}, \alpha, \|a\|_{0,\alpha})$, which proves (3.5.99). On the other hand, testing $\bar{v} - \tilde{v}$ in the equation (3.5.95), it can be written as

$$\begin{aligned} J_1 &:= \fint_{B_{8r}} \langle \bar{A}_{B_{32r}}(x, D\bar{v}) - \bar{A}_{B_{32r}}(x, D\tilde{v}), D\bar{v} - D\tilde{v} \rangle\, dx \\ &= \fint_{B_{8r}} \langle \tilde{A}_{B_{32r}}(x, D\tilde{v}) - \bar{A}_{B_{32r}}(x, D\tilde{v}), D\bar{v} - D\tilde{v} \rangle\, dx =: J_2. \end{aligned} \tag{3.5.102}$$

Again by (3.5.31), for every $\tau_1 \in (0, 1)$, we see

$$\fint_{B_{8r}} \Psi_s(x, D\tilde{v} - D\bar{v})\, dx \leq c\tau_1\lambda + c_{\tau_1} J_1 \tag{3.5.103}$$

with $c_{\tau_1} \equiv c(\texttt{data}, \tau_1)$. Before we go on further, using Hölder's inequality and the assumption (3.5.98) together with Remark 3.5.3, for a higher integrability exponent $\sigma_2 > 0$ determined by Lemma 3.5.14, we observe that

$$\begin{aligned}
&\fint_{B_{8r}} [\theta(A_1, B_{32r}(y_0))(x) + \theta(A_2, B_{32r}(y_0))(x)]\, \Psi_s(x, D\tilde{v})\, dx \\
&\qquad \leq \left(\fint_{B_{8r}} [\theta(A_1, B_{32r}(y_0))(x) + \theta(A_2, B_{32r}(y_0))(x)]^{\frac{1+\sigma_2}{\sigma_2}}\, dx \right)^{\frac{\sigma_2}{1+\sigma_2}} \\
&\qquad\quad \times \left(\fint_{B_{8r}} [\Psi_s(x, D\tilde{v})]^{1+\sigma_2}\, dx \right)^{\frac{1}{1+\sigma_2}} \\
&\qquad \leq c(4L)^{\frac{1}{\sigma_2}} \delta^{\frac{\sigma_2}{1+\sigma_2}} \left(\fint_{B_{16r}} [\Psi_s(x, Dw)]^{1+\sigma_2}\, dx \right)^{\frac{1}{1+\sigma_2}} \\
&\qquad \leq c\delta^{\frac{\sigma_2}{1+\sigma_2}} \left(\fint_{B_{32r}} \Psi(x, Dw)\, dx + 1 \right) \\
&\qquad \leq c\delta^{\frac{\sigma_2}{1+\sigma_2}} \lambda
\end{aligned} \tag{3.5.104}$$

for some $c \equiv c(\mathtt{data})$, where we have applied Lemma 3.5.14 and Proposition 3.5.13. We now estimate J_2 based on the assumption (3.5.98). In turn, we have that

$$
\begin{aligned}
J_2 &\le \fint_{B_{8r}} |\tilde{A}_{1,B_{32r}}(x, D\tilde{v}) - \bar{A}_{1,B_{32r}}(D\tilde{v})||D\tilde{v} - D\bar{v}|\,dx \\
&\quad + \fint_{B_{8r}} a(x)|\tilde{A}_{2,B_{32r}}(x, D\tilde{v}) - \bar{A}_{2,B_{32r}}(D\tilde{v})||D\tilde{v} - D\bar{v}|\,dx \\
&\le \fint_{B_{8r}} \theta(A_1, B_{32r}(y_0))(x) G^{p_s-1}(|D\tilde{v}|) G'(|D\tilde{v}|)|D\tilde{v} - D\bar{v}|\,dx \\
&\quad + \fint_{B_{8r}} \theta(A_2, B_{32r}(y_0))(x) a(x) H^{q_s-1}(|D\tilde{v}|) H'(|D\tilde{v}|)|D\tilde{v} - D\bar{v}|\,dx \\
&\le \tau_2 \fint_{B_{8r}} [\theta(A_1, B_{32r}(y_0))(x) + \theta(A_2, B_{32r}(y_0))(x)]\, \Psi_s(x, D\tilde{v} - D\bar{v})\,dx \\
&\quad + c_{\tau_2} \fint_{B_{8r}} [\theta(A_1, B_{32r}(y_0))(x) + \theta(A_2, B_{32r}(y_0))(x)]\, \Psi_s(x, D\tilde{v})\,dx \\
&\le 4L\tau_2 \fint_{B_{8r}} \Psi_s(x, D\tilde{v} - D\bar{v})\,dx \\
&\quad + c_{\tau_2} \fint_{B_{8r}} [\theta(A_1, B_{32r}(y_0))(x) + \theta(A_2, B_{32r}(y_0))(x)]\, \Psi_s(x, D\tilde{v})\,dx
\end{aligned}
$$

for every $\tau_2 \in (0,1)$ and some constant $c_{\tau_2} \equiv c_{\tau_2}(s(G), s(H), L, \tau)$, where we have applied Lemma 2.1.6 for G^{p_s} and H^{q_s} and Remark 3.5.3. Applying (3.5.104) in the last display, we conclude that

$$
J_2 \le 4L\tau_2 \fint_{B_{8r}} \Psi_s(x, D\tilde{v} - D\bar{v})\,dx + c_{\tau_2}\delta^{\frac{\sigma_2}{1+\sigma_2}}\lambda
$$

with $c_{\tau_2} \equiv c_{\tau_2}(\mathtt{data}, \tau_2)$. Plugging the above display into (3.5.103), we find that

$$
\fint_{B_{8r}} \Psi_s(x, D\tilde{v} - D\bar{v})\,dx \le c\tau_1\lambda + c_{\tau_1}\tau_2 \fint_{B_{8r}} \Psi_s(x, D\tilde{v} - D\bar{v})\,dx + c_{\tau_1\tau_2}\delta^{\frac{\sigma_2}{1+\sigma_2}}\lambda
$$

holds for every $\tau_1, \tau_2 \in (0,1)$, where $c_{\tau_1} \equiv c_{\tau_1}(\mathtt{data}, \tau_1)$ and $c_{\tau_1\tau_2} \equiv c_{\tau_1\tau_2}(\mathtt{data}, \tau_1, \tau_2)$. First we choose small enough $\tau_2 \le \frac{1}{2c_{\tau_1}}$ to obtain

$$
\fint_{B_{8r}} \Psi_s(x, D\tilde{v} - D\bar{v})\,dx \le c\tau_1\lambda + c_{\tau_1}\delta^{\frac{\sigma_2}{1+\sigma_2}}\lambda
$$

for any $\tau_1 \in (0,1)$ and some constant $c_{\tau_1} \equiv c_{\tau_1}(\mathtt{data}, \tau_1)$. Finally, selecting small enough $\tau_1 \equiv \tau_1(\mathtt{data}, \varepsilon)$, and then $\delta \equiv \delta(\mathtt{data}, \gamma, \varepsilon)$ again sufficiently small, we arrive at the desired comparison estimate (3.5.100). The proof is complete. $\square$

Let $x_m \in \overline{B_{4r}}$ be a point such that $a(x_m) = \sup_{x \in B_{4r}} a(x)$. Now we consider a function $h \in W^{1,\Psi_{s,m}}(B_{4r})$ as the weak solution of

$$\begin{cases} -\mathrm{div}\bar{A}_{B_{32r}}(x_m, Dh) = 0 & \text{in } B_{4r} \\ h \in \bar{v} + W_0^{1,\Psi_{s,m}}(B_{4r}), \end{cases}$$

where

$$\Psi_{s,m}(t) = G^{p_s}(t) + a(x_m)H^{q_s}(t) \quad \text{for every} \quad t \geq 0.$$

The existence of h is guaranteed by Lemma 3.5.17 since $\bar{v} \in W^{1,\Psi_{s,m}}_{\mathrm{loc}}(B_{8r})$. At this stage, proofs of [12, Theorem 2.1] and [169] imply the following important result.

Lemma 3.5.19. *Under the assumptions of Lemma 3.5.18, for every $\varepsilon \in (0,1)$ and $\Theta \geq 4$, it holds that*

$$\sup_{x \in B_r} \Psi_s(x_m, Dh(x)) \leq c\lambda$$

for some $c \equiv c(\boldsymbol{data}, \|\Psi(x,F)\|_{L^\gamma(\Omega)}) > 0$, and that

$$\fint_{B_r} \Psi_s(x, D\bar{v} - Dh)\, dx \leq \left(\varepsilon + 2c_0(\Theta)r^{s_0} + \frac{c_1}{\Theta}\right)\lambda =: S(\varepsilon, r, \Theta)\lambda \quad (3.5.105)$$

holds, where the dependence of constants are as follows: $s_0 \equiv s_0(\boldsymbol{data}) \in (0,1)$, $c_0(\Theta) \equiv c_0(\boldsymbol{data}, \mathrm{dist}(\Omega_0, \partial\Omega), \|\Psi(x,F)\|_{L^\gamma(\Omega)}, \Theta)$ and $c_1 \equiv c_1(n, s(G), s(H), \nu, L, m_{pq})$.

Summarizing all the comparison estimates discussed in Lemmas 3.5.11-3.5.19, we can conclude the following most important part of the present section.

Lemma 3.5.20. *Let $\lambda \geq 1$ be a given number and $B_{32r}(y_0) \Subset \Omega_0 \Subset \Omega$ be a given ball. Then for every $\varepsilon > 0$ and $\Theta \geq 4$, there exists $\delta \equiv \delta(\boldsymbol{data}, \gamma, \varepsilon) > 0$*

such that if

$$\fint_{B_{32r}(y_0)} [\theta(A_1, B_{32r}(y_0))(x) + \theta(A_2, B_{32r}(y_0))(x)] \, dx \leq \delta,$$

$$\fint_{B_{32r}(y_0)} \Psi(x, Du)\, dx \leq \lambda, \quad \fint_{B_{32r}(y_0)} \Psi(x, F)\, dx \leq \delta\lambda,$$

$$\sup_{0<\rho\leq 64r} \omega(\rho) \log \frac{1}{\rho} \leq \delta \quad \textit{and}$$
$$\omega(64r) \leq \min\left\{1, \frac{\nu}{(s(G)+1)(s(H)+1)}, \frac{n\sigma_1}{n+\alpha}, \sqrt{\frac{n}{n-1}} - 1\right\},$$

then there exist $w \in W^{1,\Psi_s}(B_{32r})$ *and* $h \in W^{1,\Psi_s}(B_{4r})$ *such that*

$$\fint_{B_r} \Psi(x, Du - Dw)\, dx \leq \varepsilon\lambda, \quad \fint_{B_r} \Psi_s(x, Dw - Dh)\, dx \leq S(\varepsilon, r, \Theta)\lambda$$

and

$$\sup_{x\in B_r} \Psi_s(x_m, Dh(x)) \leq c\lambda$$

for some constant $c \equiv c(\texttt{data}, \|\Psi(x,F)\|_{L^\gamma(\Omega)})$, *where* $S(\varepsilon, r, \Theta)$ *is the same one that has been defined in* (3.5.105) *and* Ψ_s *is the same one as in* (3.5.55).

3.5.5 Proof of Theorem 3.5.4

In the present section we shall provide the proof of Theorem 3.5.4. Our proof based on the so-called maximal function-free technique introduced in [3]. Suppose that $\Psi(x, F) \in L^\gamma(\Omega)$ for some $\gamma > 1$. Let $B_r \equiv B_r(x_0) \Subset \Omega$ be a ball with $r \leq R/64$ for some $R > 0$ to be determined later in (3.5.109) and (3.5.116). Choose radii r_1, r_2 such that $r/2 \leq r_1 < r_2 \leq r$ and consider the super-level sets

$$E(s;\lambda) := \{x \in B_s(x_0) : \Psi(x, Du) > \lambda\} \quad (r/2 \leq s \leq r,\ \lambda > 0)\,.$$

For each ball $B_\rho(y_0) \subset B_r$, we define

$$T(B_\rho(y_0)) := \fint_{B_\rho(y_0)} \left(\Psi(x, Du) + \frac{1}{\delta}\Psi(x, F) \right) dx$$

for some $\delta \in (0, 1/8)$ to be determined later.

Then one can see that for almost every $y_0 \in E(s; \lambda)$ and $r/2 \le s \le r$,

$$\lim_{\rho \to 0} T(B_\rho(y_0)) > \lambda.$$

On the other hand, for $y_0 \in B_{r_1}$ and $\rho \in \left[\frac{r_2 - r_1}{160}, r_2 - r_1\right]$ we have

$$T(B_\rho(y_0)) \le \frac{160^n r_2^n}{(r_2 - r_1)^n} \fint_{B_{r_2}} \left(\Psi(x, Du) + \frac{1}{\delta}\Psi(x, F) \right) dx =: \lambda_0. \tag{3.5.106}$$

From now on, we only consider

$$\lambda > \lambda_0.$$

Then in the view of last three displays, for almost every $y_0 \in E(r_1; \lambda)$, there is a small radius $\rho_{y_0} \in \left(0, \frac{r_2 - r_1}{160}\right)$ such that

$$T(B_{\rho_{y_0}}(y_0)) = \lambda \quad \text{and} \quad T(B_\rho(y_0)) < \lambda \quad \text{for all } \rho \in (\rho_{y_0}, r_2 - r_1]. \tag{3.5.107}$$

Since (3.5.107) holds for almost every $y_0 \in E(r_1; \lambda)$, the set of balls $\{B_{\rho_{y_0}}(y_0)\}$ covers $E(r_1; \lambda)$ up to a negligible set. Hence by the Vitali covering lemma, there is a family of mutually disjoint countable balls $\{B_{\rho_{y_k}}(y_k)\}_{k=1}^\infty$ such that

$$E(r_1; \lambda) \subset \bigcup_{k=1}^\infty B_{5\rho_{y_k}}(y_k)$$

and

$$\begin{gathered} T(B_{\rho_{y_k}}(y_k)) = \lambda \quad \text{and} \quad T(B_\rho(y_k)) < \lambda \\ \text{for every } \rho \in (\rho_{y_k}, r_2 - r_1] \end{gathered} \tag{3.5.108}$$

for each $k \in \mathbb{N}$. From now on we denote

$$B_k := B_{\rho_k}(y_k) \quad \text{and} \quad \rho_k := 5\rho_{y_k}.$$

We notice that

$$32B_k \subset B_{r_2} \quad \text{and} \quad \rho_k = 5\rho_{y_k} \le \frac{r_2 - r_1}{32}.$$

Now we will employ Lemma 3.5.20. Before that, let us denote by

$$\Psi_{s,k}(x,z) := G^{p_{s,k}}(|z|) + a(x)H^{q_{s,k}}(|z|)$$

for every $x \in \Omega$ and $z \in \mathbb{R}^n$, where

$$p_{s,k} = \sup_{x \in 32B_k} p(x) \quad \text{and} \quad q_{s,k} = \sup_{x \in 32B_k} q(x)$$

and let $x_{m,k} \in \overline{32B_k}$ be a point such that

$$a(x_{m,k}) = \sup_{x \in 32B_k} a(x).$$

By (3.5.108), we have

$$\fint_{32B_k} \Psi(x, Du)\,dx \le \lambda \quad \text{and} \quad \fint_{32B_k} \Psi(x, F)\,dx \le \delta\lambda.$$

Thus by Lemma 3.5.20, there exists a small $\delta \equiv \delta(\mathtt{data}, \gamma, \varepsilon)$ such that if $(p(\cdot), q(\cdot), A_1, A_2)$ is (δ, R)-vanishing and

$$\omega(R) \le \min\left\{1, \frac{\nu}{(s(G)+1)(s(H)+1)}, \frac{n\sigma_1}{n+\alpha}, \sqrt{\frac{n}{n-1}} - 1\right\} \tag{3.5.109}$$

holds, then there exist functions $w_k \in W^{1,\Psi_{s,k}}(B_k)$ and $h_k \in W^{1,\infty}(B_k)$ such that

$$\begin{aligned} &\fint_{B_k} \Psi(x, Du - Dw_k)\,dx \le \varepsilon\lambda, \\ &\fint_{B_k} \Psi_{s,k}(x, Dw_k - Dh_k)\,dx \le S(\varepsilon, R, \Theta)\lambda \end{aligned} \tag{3.5.110}$$

and

$$\sup_{x \in B_k} \Psi_{s,k}(x_{m,k}, Dh_k(x)) \le c_l\lambda \tag{3.5.111}$$

for some constants $c_l \equiv c_l(\texttt{data}, \|\Psi(x,F)\|_{L^\gamma(\Omega)}) > 0$, where

$$S(\varepsilon, R, \Theta) := 2\varepsilon + 2c_0(\Theta)R^{s_0} + \frac{2c_1}{\Theta} \tag{3.5.112}$$

for any numbers $\varepsilon \in (0,1)$ and $\Theta \geq 4$ to be chosen later, while the dependence of the other constants are as follows: $s_0 \equiv s_0(\texttt{data}) \in (0, 1/2)$, $c_0(\Theta) \equiv c_0(\texttt{data}, \mathrm{dist}(\Omega_0, \partial\Omega), \|\Psi(x,F)\|_{L^\gamma(\Omega)}, \Theta)$ and
$c_1 \equiv c_1(n, s(G), s(H), \nu, L, m_{pq})$. We here notice that all constants appearing in the last display are independent of k and λ.

Let $t_l := 2\cdot 4^{m_{pq}-1}(c_l + 2\|a\|_{L^\infty(\Omega)} + 2)$ for the constant c_l being determined in (3.5.111). Since $E(r_1; t_l\lambda) \subset E(r_1;\lambda)$, we have

$$\int_{E(r_1;t_l\lambda)} \Psi(x, Du)\, dx \leq \sum_{k=1}^{\infty} \left(\int_{E(r_1;t_l\lambda)\cap B_k} \Psi(x, Du)\, dx \right). \tag{3.5.113}$$

Therefore, for almost every $x \in E(r_1; d_l\lambda) \cap B_k$, by (3.5.111) and elementary manipulations, it holds that

$$\begin{aligned}
\Psi(x, Du) &\leq 4^{m_{pq}-1}\left[\Psi(x, Du - Dw_k) + \Psi(x, Dw_k - Dh_k) + \Psi(x, Dh_k)\right] \\
&\leq 4^{m_{pq}-1}\Big[\Psi(x, Du - Dw_k) + \Psi_{s,k}(x, Dw_k - Dh_k) \\
&\quad + \Psi_{s,k}(x_{m,k}, Dh_k) + 2\|a\|_{L^\infty(\Omega)} + 2\Big] \\
&\leq 4^{m_{pq}-1}\left(\Psi(x, Du - Dw_k) + \Psi_{s,k}(x, Dw_k - Dh_k)\right) \\
&\quad + 4^{m_{pq}-1}(c_l + 2\|a\|_{L^\infty(\Omega)} + 2)\lambda \\
&\leq 4^{m_{pq}-1}\left(\Psi(x, Du - Dw_k) + \Psi_{s,k}(x, Dw_k - Dh_k)\right) + \frac{1}{2}\Psi(x, Du),
\end{aligned}$$

and so

$$\Psi(x, Du) \leq 2\cdot 4^{m_{pq}-1}\left(\Psi(x, Du - Dw_k) + \Psi_{s,k}(x, Dw_k - Dh_k)\right) \tag{3.5.114}$$

holds for almost every $x \in E(r_1; d_l\lambda) \cap B_k$. Thus by (3.5.110) and (3.5.114), for any $k \in \mathbb{N}$, we obtain

$$\begin{aligned}
&\int_{E(r_1;t_l\lambda)\cap B_k} \Psi(x,Du)\,dx \\
&\quad \le 2\cdot 4^{m_{pq}-1}\left(\int_{B_k}\Psi(x,Du-Dw_k)\,dx \right.\\
&\qquad\qquad \left. + \int_{B_k}\Psi_{s,k}(x,Dw_k-Dh_k)\,dx\right) \\
&\quad \le 2\cdot 4^{m_{pq}-1}5^n|B_{\rho_{y_k}}(y_k)|\left(\varepsilon + S(\varepsilon,R,\Theta)\right)\lambda.
\end{aligned} \tag{3.5.115}$$

Clearly, we see

$$\begin{aligned}
&|B_{\rho_{y_k}}(y_k)| \\
&\quad \le \frac{1}{\lambda}\left[\int_{E(r_2;\frac{\lambda}{4})\cap B_{\rho_{y_k}}(y_k)}\Psi(x,Du)\,dx + \frac{1}{\delta}\int_{\{x\in B_{\rho_{y_k}}(y_k):\Psi(x,F)>\frac{\delta\lambda}{4}\}}\Psi(x,F)\,dx\right] \\
&\qquad + \frac{|B_{\rho_{y_k}}(y_k)|}{2},
\end{aligned}$$

and so

$$\begin{aligned}
&|B_{\rho_{y_k}}(y_k)| \\
&\quad \le \frac{2}{\lambda}\left(\int_{E(r_2;\frac{\lambda}{4})\cap B_{\rho_{y_k}}(y_k)}\Psi(x,Du)\,dx + \frac{1}{\delta}\int_{\{x\in B_{\rho_{y_k}}(y_k):\Psi(x,F)>\frac{\delta\lambda}{4}\}}\Psi(x,F)\,dx\right).
\end{aligned}$$

Merging the above inequality into (3.5.115) and absorbing ε to $S(\varepsilon,R,\Theta)$, we get

$$\begin{aligned}
&\int_{E(r_1;t_l\lambda)\cap B_k}\Psi(x,Du)\,dx \\
&\qquad \le 8^{m_{pq}}5^n S(\varepsilon,R,\Theta)\left(\int_{E(r_2;\frac{\lambda}{4})\cap B_{\rho_{y_k}}(y_k)}\Psi(x,Du)\,dx\right. \\
&\qquad\qquad\qquad \left. + \frac{1}{\delta}\int_{\{x\in B_{\rho_{y_k}}(y_k):\Psi(x,F)>\frac{\delta\lambda}{4}\}}\Psi(x,F)\,dx\right).
\end{aligned}$$

Recalling that $\{B_{\rho_{y_k}}(y_k)\}_{k=1}^{\infty}$ is mutually disjoint, merging the last inequality into (3.5.113), we find

$$\begin{aligned}\int_{E(r_1;t_l\lambda)} &\Psi(x, Du)\,dx \\ &\le 8^{m_{pq}} 5^n S(\varepsilon, R, \Theta) \left(\int_{E(r_2;\frac{\lambda}{4})} \Psi(x, Du)\,dx + \frac{1}{\delta} \int_{\mathcal{E}(r_2;\frac{\delta\lambda}{4})} \Psi(x, F)\,dx \right),\end{aligned}$$

where we denote by

$$\mathcal{E}(s;\lambda) := \{x \in B_s(x_0) : \Psi(x, F) > \lambda\} \quad \text{for } r/2 \le s \le r \text{ and } \lambda > 0.$$

In other words, we have

$$\begin{aligned}\int_{E(r_1;\lambda)} &\Psi(x, Du)\,dx \\ &\le 8^{m_{pq}} 5^n S(\varepsilon, R, \Theta) \left(\int_{E(r_2;\frac{\lambda}{4t_l})} \Psi(x, Du)\,dx + \frac{1}{\delta} \int_{\mathcal{E}(r_2;\frac{\delta\lambda}{4t_l})} \Psi(x, F)\,dx \right)\end{aligned}$$

for any $\lambda \ge t_l\lambda_0$. To proceed further we define the truncated functions by

$$[\Psi(x, Du)]_t := \min\{\Psi(x, Du), t\} \quad (t \ge 0).$$

For $t \ge 2t_l\lambda_0$, we have

$$\begin{aligned}\int_{t_l\lambda_0}^{t} &\lambda^{\gamma-2} \int_{E(r_1;\lambda)} \Psi(x, Du)\,dxd\lambda \\ &\le c\, S(\varepsilon, R, \Theta) \int_{t_l\lambda_0}^{t} \lambda^{\gamma-2} \int_{E(r_2;\frac{\lambda}{4t_l})} \Psi(x, Du)\,dxd\lambda \\ &\quad + c\, \frac{S(\varepsilon, R, \Theta)}{\delta} \int_{t_l\lambda_0}^{t} \lambda^{\gamma-2} \int_{\mathcal{E}(r_2;\frac{\delta\lambda}{4t_l})} \Psi(x, F)\,dxd\lambda.\end{aligned}$$

By change of variables and Fubini's theorem, we see

$$\begin{aligned}
\int_{t_l\lambda_0}^{t} \lambda^{\gamma-2} \int_{E(r_1;\lambda)} \Psi(x, Du)\, dxd\lambda & \\
&= \frac{1}{\gamma-1} \int_{B_{r_1}} [\Psi(x, Du)]_t^{\gamma-1} \Psi(x, Du)\, dx \\
&\quad - \int_0^{t_l\lambda_0} \lambda^{\gamma-2} \int_{E(r_1;\lambda)} \Psi(x, Du)\, dxd\lambda,
\end{aligned}$$

$$\begin{aligned}
\int_{t_l\lambda_0}^{t} \lambda^{\gamma-2} \int_{E(r_2;\frac{\lambda}{4t_l})} \Psi(x, Du)\, dxd\lambda & \\
&\le \frac{1}{\gamma-1} \int_{B_{r_2}} [\Psi(x, Du)]_{\frac{t}{4t_l}}^{\gamma-1} \Psi(x, Du)\, dx \\
&\le \frac{1}{\gamma-1} \int_{B_{r_2}} [\Psi(x, Du)]_t^{\gamma-1} \Psi(x, Du)\, dx
\end{aligned}$$

and

$$\begin{aligned}
\int_{t_l\lambda_0}^{t} \lambda^{\gamma-2} \int_{\mathcal{E}(r_2;\frac{\delta\lambda}{4t_l})} \Psi(x, F)\, dxd\lambda &\le \int_0^{\infty} \lambda^{\gamma-2} \int_{\mathcal{E}(r_2;\frac{\delta\lambda}{4t_l})} \Psi(x, F)\, dxd\lambda \\
&\le c \int_{B_{r_2}} \Psi^{\gamma}(x, F)\, dx.
\end{aligned}$$

Moreover, we also notice that

$$\begin{aligned}
\int_0^{t_l\lambda_0} \lambda^{\gamma-2} \int_{E(r_1;\lambda)} \Psi(x, Du)\, dxd\lambda &\le \int_0^{t_l\lambda_0} \lambda^{\gamma-2} d\lambda \int_{B_{r_2}} \Psi(x, Du)\, dx \\
&\le \frac{(t_l\lambda_0)^{\gamma-1}}{\gamma-1} \int_{B_{r_2}} \Psi(x, Du)\, dx.
\end{aligned}$$

Therefore, taking the estimates in last four displays into account, it follows

that

$$
\begin{aligned}
&\fint_{B_{r_1}} [\Psi(x,Du)]_t^{\gamma-1}\Psi(x,Du)\,dx \\
&\qquad \le c_2 S(\varepsilon,R,\Theta) \fint_{B_{r_2}} [\Psi(x,Du)]_t^{\gamma-1}\Psi(x,Du)\,dx \\
&\qquad\quad + c\frac{S(\varepsilon,R,\Theta)}{\delta}\fint_{B_{r_2}} [\Psi(x,F)]^{\gamma}\,dx + t_l^{\gamma-1}{\lambda_0}^{\gamma-1}\fint_{B_{r_2}} \Psi(x,Du)\,dx
\end{aligned}
$$

with the constant $c_2 = c_2(\mathtt{data},\gamma) \ge 1$.

Choose $\varepsilon(\mathtt{data},\gamma) = 1/(8c_2) \in (0,1)$, and so we obtain $\delta \equiv \delta(\mathtt{data},\gamma) > 0$ from Lemma 3.5.20. Now select $\Theta \ge 4$ and then $0 < R < 1$ to satisfy

$$
0 < c_2 S(\varepsilon,R,\Theta) \le \frac{1}{2}, \tag{3.5.116}
$$

where $S(\varepsilon,R,\Theta)$ is defined in (3.5.112). Then by (3.5.109) together with the above display, we obtain $R \equiv R(\mathtt{data},\mathrm{dist}(\Omega_0,\partial\Omega),\gamma,\|\Psi(x,F)\|_{L^\gamma(\Omega)}) > 0$. In turn, recalling the definition of λ_0 in (3.5.106) and applying Young's inequality with conjugate exponents $\left(\frac{\gamma}{\gamma-1},\gamma\right)$, we have

$$
\begin{aligned}
&\fint_{B_{r_1}} [\Psi(x,Du)]_t^{\gamma-1}\Psi(x,Du)\,dx \\
&\qquad \le \frac{1}{2}\fint_{B_{r_2}} [\Psi(x,Du)]_t^{\gamma-1}\Psi(x,Du)\,dx + c\fint_{B_r} [\Psi(x,F)]^{\gamma}\,dx \\
&\qquad\quad + c\frac{r^{n(\gamma-1)}}{(r_2-r_1)^{n(\gamma-1)}}\left(\fint_{B_r} \Psi(x,Du)\,dx + \fint_{B_r} \Psi(x,F)\,dx\right)^{\gamma}
\end{aligned}
$$

for some constant $c = c(\mathtt{data},\mathrm{dist}(\Omega_0,\partial\Omega),\gamma,\|\Psi(x,F)\|_{L^\gamma(\Omega)})$. At this point, we apply Lemma 2.0.1 with $\gamma_1 \equiv n(\gamma-1)$, $\gamma_2 \equiv 0$ for a function

$$
h(s) := \fint_{B_s} [\Psi(x,Du)]_t^{\gamma-1}\Psi(x,Du)\,dx
$$

being non-negative and bounded on $[r/2,r]$ in order to obtain the following estimate:

$$\begin{aligned}&\fint_{B_{r/2}} [\Psi(x, Du)]_t^{\gamma-1} \Psi(x, Du)\, dx \\&\qquad \le c \left(\fint_{B_r} \Psi(x, Du)\, dx \right)^{\gamma} + c \fint_{B_r} [\Psi(x, F)]^{\gamma}\, dx\end{aligned}$$

with again some constant $c = c(\mathtt{data}, \operatorname{dist}(\Omega_0, \partial\Omega), \gamma, \|\Psi(x, F)\|_{L^{\gamma}(\Omega)})$. Finally, taking $t \to \infty$ in the last display, we conclude with the desired Calderón-Zygmund estimate (3.5.14). Clearly, (3.5.13) follows from a standard covering argument. We finish the proof.

Chapter 4

Global gradient estimates for elliptic equations with degenerate matrix weights

4.1 Global maximal regularity for equations with degenerate weights

In this section, we are concerned with global maximal regularity estimates for elliptic equations with degenerate weights. We consider both the linear case and the non-linear case. We show that higher integrability of the gradients can be obtained by imposing a local small oscillation condition on the weight and a local small Lipschitz condition on the boundary of the domain. Our results are new in the linear and non-linear case. We show by example that the relation between the exponent of higher integrability and the smallness parameters is sharp even in the linear or the unweighted case.

4.1.1 Hypothesis and main results

We study the following degenerate elliptic equation of the form

$$\begin{aligned} -\operatorname{div}(\mathbb{A}(x)\nabla u) &= -\operatorname{div}(\mathbb{A}(x)F) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{4.1.1}$$

in the linear case, and of the form

$$\begin{aligned} -\mathrm{div}(|\mathbb{M}(x)\nabla u|^{p-2}\mathbb{M}^2(x)\nabla u) &= -\mathrm{div}(|\mathbb{M}(x)F|^{p-2}\mathbb{M}^2(x)F) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{4.1.2}$$

in the non-linear case. We often write $\mathbb{M}(x)$ to emphasize the dependence of the weight on x.

Here, $\Omega \subset \mathbb{R}^n$ is a bounded domain with $n \geq 2$, $1 < p < \infty$, $F : \Omega \to \mathbb{R}^n$ is a given vector-valued function, $\mathbb{M} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is a given symmetric and positive definite matrix-valued weight satisfying

$$|\mathbb{M}(x)|\,|\mathbb{M}^{-1}(x)| \leq \Lambda \quad (x \in \mathbb{R}^n) \tag{4.1.3}$$

for some constant $\Lambda \geq 1$, where $|\cdot|$ is the spectral norm, and $\mathbb{A}(x) := \mathbb{M}^2(x)$. This condition says that $\mathbb{M}$ has a uniformly bounded condition number. Note that a right-hand side of the form $-\mathrm{div}G$ with $G : \Omega \to \mathbb{R}^n$ can be immediately rewritten in the above form in terms of F. Note that (4.1.1) is a special case of (4.1.2) for $p = 2$. The condition (4.1.3) in this case reads as

$$|\mathbb{A}(x)|\,|\mathbb{A}^{-1}(x)| \leq \Lambda^2 \quad (x \in \mathbb{R}^n). \tag{4.1.4}$$

Let us define the scalar weight

$$\omega(x) = |\mathbb{M}(x)| = \sqrt{|\mathbb{A}(x)|}. \tag{4.1.5}$$

Now, we introduce Lipschitz domains along with our optimal regularity assumption for the boundary of the domain.

Definition 4.1.1. *Let $\delta \in [0, \frac{1}{2n}]$ and $R > 0$ be given. Then Ω is called (δ, R)–Lipschitz if for each $x_0 \in \partial\Omega$, there exists a coordinate system $\{x_1, \ldots, x_n\}$ and Lipschitz map $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $x_0 = 0$ in this coordinate system, and there holds*

$$\Omega \cap B_R(x_0) = \{x = (x_1, \ldots, x_n) = (x', x_n) \in B_R(x_0) : x_n > \psi(x')\} \tag{4.1.6}$$

and

$$\|\nabla\psi\|_\infty \leq \delta. \tag{4.1.7}$$

Our optimal regularity assumption for $\mathbb{M}$ is a small BMO assumption on its logarithm. This condition is also used in [16] for the interior estimates.

Definition 4.1.2. *We say that* $\log \mathbb{M}$ *is* (δ, R)*-vanishing if*

$$|\log \mathbb{M}|_{\mathrm{BMO}(\mathbb{R}^n)} := \sup_{y\in\mathbb{R}^n} \sup_{0<r\le R} \fint_{B_r(y)} |\log \mathbb{M}(x) - (\log \mathbb{M})_{B_r(y)}|\, dx \le \delta. \quad (4.1.8)$$

Now, we state the main theorems.

Theorem 4.1.3 (Linear case)**.** *Define* ω *as* (4.1.5)*, and assume* (4.1.3) *and* $F \in L^q_\omega(\Omega)$ *for* $q \in (1,\infty)$ *in* (4.1.1)*. Then there exists a constant* $\delta = \delta(n,\Lambda) \in (0, \frac{1}{2})$ *such that if for some* R*,*

$$\log \mathbb{A} \ \textit{is} \ \left(\delta \min\left\{\frac{1}{q}, 1-\frac{1}{q}\right\}, R\right) \textit{–vanishing and} \quad (4.1.9a)$$

$$\Omega \ \textit{is} \ \left(\delta \min\left\{\frac{1}{q}, 1-\frac{1}{q}\right\}, R\right) \textit{–Lipschitz,} \quad (4.1.9b)$$

then the weak solution $u \in W^{1,2}_{0,\omega}(\Omega)$ *of* (4.1.1) *satisfies* $\nabla u \in L^q_\omega(\Omega)$ *and we have the estimate*

$$\int_\Omega (|\nabla u|\omega)^q \, dx \le c \int_\Omega (|F|\omega)^q \, dx \quad (4.1.10)$$

for some $c = c(n,\Lambda,\Omega,q)$*.*

For the non-linear case, we have the following result.

Theorem 4.1.4 (Non-linear case)**.** *Define* ω *as* (4.1.5)*, and assume* (4.1.3) *and* $F \in L^q_\omega(\Omega)$ *for* $q \in [p,\infty)$ *in* (4.1.2)*. Then there exists a constant* $\delta = \delta(n,p,\Lambda) \in (0, \frac{1}{2})$ *such that if for some* R*,*

$$\log \mathbb{M} \ \textit{is} \ \left(\frac{\delta}{q}, R\right) \textit{–vanishing and } \Omega \textit{ is } \left(\frac{\delta}{q}, R\right) \textit{–Lipschitz,} \quad (4.1.11)$$

then the weak solution $u \in W^{1,p}_{0,\omega}(\Omega)$ *of* (4.1.2) *satisfies* $\nabla u \in L^q_\omega(\Omega)$ *and we have the estimate*

$$\int_\Omega (|\nabla u|\omega)^q \, dx \le c \int_\Omega (|F|\omega)^q \, dx \quad (4.1.12)$$

for some $c = c(n, p, \Lambda, \Omega, q)$.

4.1.2 Notation and preliminary results

Let Ω be $(\delta, 4R)$–Lipschitz and $x_0 \in \partial\Omega$. Then there exists a Lipschitz map $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ as in Definition 4.1.1. By translation, without loss of generality we assume $x_0 = 0$ and $\psi(0) = 0$. We define $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ as

$$\Psi(x', x_n) = (x', x_n - \psi(x')) \quad \text{for } (x', x_n) \in \mathbb{R}^n \tag{4.1.13}$$

and so there hold $\Psi(\partial\Omega \cap B_{4R}(0)) \subset \{(y', y_n) : y_n = 0\}$, $\Psi(\overline{\Omega} \cap B_{4R}(0)) \subset \{(y', y_n) : y_n \geq 0\}$ and $\Psi(0) = 0$. The mapping Ψ is invertible, with a Lipschitz continuous inverse Ψ^{-1}. We easily obtain

$$\nabla\Psi(x', x_n) = \begin{pmatrix} I & 0 \\ -\nabla\psi(x') & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & 1 & 0 \\ -\partial_{x_1}\psi(x') & \dots & -\partial_{x_{n-1}}\psi(x') & 1 \end{pmatrix} \tag{4.1.14}$$

for $(x', x_n) \in \mathbb{R}^n$, where the right-hand side of (4.1.14) is an $n \times n$ matrix. In particular, $\det(\nabla\Psi(x)) = 1$ and so $|\Psi(B)| = |B|$ for each ball $B \subset \mathbb{R}^n$. Note that $|\operatorname{id} - (\nabla\Psi)(x)| \leq n\|\nabla\psi\|_\infty$.

Now, we provide some geometric properties related to the maps ψ and Ψ which will be used throughout the section.

Remark 4.1.5. *From now on, we implicitly use the following properties. If we assume*

$$\Omega \ \textit{ is } \ (\delta, 4R)\text{–}\textit{Lipschitz with } \ \delta \in \left[0, \frac{1}{2n}\right] \ \textit{ and } \ R > 0, \tag{4.1.15}$$

Then for any induced map Ψ *from the Lipschitz map* ψ *assigned to given* $x_0 \in \partial\Omega$, *we have* $|\operatorname{id} - \nabla\psi| \leq \frac{1}{2}$,

$$\tfrac{1}{2}B \subset \Psi(B) \subset 2B \tag{4.1.16}$$

for all ball $B \subset \mathbb{R}^n$, and the following measure density properties also hold:

$$\sup_{0<r\leq 4R} \sup_{y\in\Omega} \frac{|B_r(y)|}{|\Omega \cap B_r(y)|} \leq 4^n \tag{4.1.17}$$

and

$$\inf_{0<r\leq 4R} \inf_{y\in\partial\Omega} \frac{|B_r(y) \cap \Omega^c|}{|B_r(y)|} \geq 4^{-n}. \tag{4.1.18}$$

We first consider a weighted Poincaré inequality with partial zero boundary values. The corresponding mean value version is given in [16].

Proposition 4.1.6 (Weighted Poincaré inequality at boundary)**.** *Let $1 < p < \infty$ and $\theta \in (0,1)$ be such that $\theta p \geq \max\{1, \frac{np}{n+p}\}$. Moreover, let $B_r = B_r(x_0)$ with $x_0 \in \Omega$ and $B_{\frac{3}{2}r}(x_0) \not\subset \Omega$. Assume that Ω is $(\delta, 3r)$-Lipschitz with $\delta \in [0, \frac{1}{2n}]$ and that ω is a weight on B_{3r} with*

$$\sup_{B'\subset B_{3r}} \left(\fint_{B'} \omega^p \, dx \right)^{\frac{1}{p}} \left(\fint_{B'} \omega^{-(\theta p)'} \, dx \right)^{\frac{1}{(\theta p)'}} \leq c_1. \tag{4.1.19}$$

Then for any $v \in W^{1,p}_{\omega}(B_{2r} \cap \Omega)$ with $v = 0$ on $\partial\Omega \cap B_{2r}$,

$$\left(\fint_{B_{2r}\cap\Omega} \left| \frac{v}{r} \right|^p \omega^p \, dx \right)^{\frac{1}{p}} \leq c \left(\fint_{B_{2r}\cap\Omega} (|\nabla v|\omega)^{\theta p} \, dx \right)^{\frac{1}{\theta p}} \tag{4.1.20}$$

holds with $c = c(n, p, c_1)$.

Proof. Since $v \in W^{1,p}_{\omega}(\Omega_{2r})$ with $v = 0$ on $\partial\Omega \cap B_{2r}$, we can take the zero extension of v on the set $B_{2r} \setminus \Omega$. Since $B_{\frac{3r}{2}}(x_0) \not\subset \Omega$, by (4.1.17) and (4.1.18) in Remark 4.1.5, for $A := B_{2r} \setminus \Omega$, $|A| \eqsim |B_{2r} \cap \Omega|$ holds. Then $(v)_A = 0$ holds, and so by Remark 4.1.5, Proposition 3 in [16] and Jensen's inequality, we have

$$
\begin{aligned}
&\left(\fint_{B_{2r}\cap\Omega}\left|\frac{v}{r}\right|^p \omega^p\,dx\right)^{\frac{1}{p}}\\
&\lesssim \left(\fint_{B_{2r}}\left|\frac{v-(v)_{B_{2r}}}{r}\right|^p \omega^p\,dx\right)^{\frac{1}{p}} + \left(\fint_{B_{2r}}\left|\frac{(v)_{B_{2r}}-(v)_A}{r}\right|^p \omega^p\,dx\right)^{\frac{1}{p}}\\
&\lesssim \left(\fint_{B_{2r}}(|\nabla v|\omega)^{\theta p}\,dx\right)^{\frac{1}{\theta p}} + \left[\fint_{B_{2r}}\left(\fint_A\left|\frac{v(y)-(v)_{B_{2r}}}{r}\right|\,dy\right)^p \omega^p\,dx\right]^{\frac{1}{p}}.
\end{aligned}
$$

Here, using $|A| \eqsim |B_{2r}\cap\Omega| \eqsim |B_{2r}|$, Hölder's inequality, (4.1.19) and Proposition 3 in [16], it follows that

$$
\begin{aligned}
&\fint_{B_{2r}}\left(\fint_A\left|\frac{v(y)-(v)_{B_{2r}}}{r}\right|\,dy\right)^p \omega(x)^p\,dx\\
&\lesssim \fint_{B_{2r}}\left[\left(\fint_{B_{2r}}\left|\frac{v(y)-(v)_{B_{2r}}}{r}\right|^p \omega(y)^p\,dy\right)^{\frac{1}{p}}\left(\fint_{B_{2r}}\omega(y)^{-p'}\,dy\right)^{\frac{1}{p'}}\right]^p \omega(x)^p\,dx\\
&\eqsim \left(\fint_{B_{2r}}\left|\frac{v-(v)_{B_{2r}}}{r}\right|^p \omega^p\,dy\right)\left(\fint_{B_{2r}}\omega^{-p'}\,dy\right)^{p-1}\left(\fint_{B_{2r}}\omega^p\,dx\right)\\
&\lesssim \left(\fint_{B_{2r}}(|\nabla v|\omega)^{\theta p}\,dx\right)^{\frac{1}{\theta}}.
\end{aligned}
$$

Now, since $v = |\nabla v| = 0$ on $B_{2r}\backslash\Omega$ and $|B_{2r}\cap\Omega| \eqsim |B_{2r}|$, we have (4.1.20). $\square$

Let us collect a few auxiliary results from [16] that will be used later. It follows from [16, (3.24)] with $\omega = |\mathbb{M}|$ that

$$
\begin{aligned}
\Lambda^{-1}\langle|\mathbb{M}|\rangle_B^{\log} \le |\langle\mathbb{M}\rangle_B^{\log}| \le \langle|\mathbb{M}|\rangle_B^{\log},\\
\Lambda^{-2}\langle|\mathbb{A}|\rangle_B^{\log} \le |\langle\mathbb{A}\rangle_B^{\log}| \le \langle|\mathbb{A}|\rangle_B^{\log}.
\end{aligned}
\tag{4.1.21}
$$

Moreover, by monotonicity of the scalar versions of exp and log we have

$$
\langle|\mathbb{A}|\rangle_B^{\log} \le (|\mathbb{A}|)_B. \tag{4.1.22}
$$

Lemma 4.1.7. *[16, Lemma 4] For a matrix-valued weight* $\mathbb{M}$ *and* $\omega = |\mathbb{M}|$

we have

$$\fint_B |\log \omega(x) - (\log \omega)_B\,|\,dx \leq 2\fint_B |\log \mathbb{M}(x) - (\log \mathbb{M})_B\,|\,dx$$

and so $|\log \omega|_{\mathrm{BMO}(B)} \leq 2|\log \mathbb{M}|_{\mathrm{BMO}(B)}$.

The next results provides a qualitative John-Nirenberg type inequality.

Lemma 4.1.8. *[16, Proposition 5] There exist constants* $\kappa_1 = \kappa_1(n, \Lambda) > 0$ *and* $c_2 = c_2(n, \Lambda) > 0$ *such that the following holds: If* $t \geq 1$ *and* $\mathbb{M}$ *is a matrix-valued weight with* $|\log \mathbb{M}|_{\mathrm{BMO}(B)} \leq \frac{\kappa_1}{t}$, *then we have*

$$\left(\fint_B \left(\frac{|\mathbb{M}(x) - \langle \mathbb{M} \rangle_B^{\log}|}{|\langle \mathbb{M} \rangle_B^{\log}|}\right)^t dx\right)^{\frac{1}{t}} \leq c_2 t |\log \mathbb{M}|_{\mathrm{BMO}(B)}.$$

The same holds with ω *instead of* $\mathbb{M}$.

The following results is a minor modification of [16, Proposition 6].

Lemma 4.1.9. *Let* κ_1 *and* c_2 *be as in Lemma 4.1.8. Then with a constant* $\beta = \beta(n, \Lambda) = \min\{\kappa_1, 1/c_2\} > 0$, *the following holds for all weights* ω.

1. *If* $|\log \omega|_{\mathrm{BMO}(B)} \leq \frac{\beta}{\gamma}$ *with* $\gamma \geq 1$, *then there holds*

$$\left(\fint_B \omega^\gamma\,dx\right)^{\frac{1}{\gamma}} \leq 2\,\langle \omega \rangle_B^{\log} \quad \text{and} \quad \left(\fint_B \omega^{-\gamma}\,dx\right)^{\frac{1}{\gamma}} \leq 2\frac{1}{\langle \omega \rangle_B^{\log}}.$$

2. *If* $|\log \omega|_{\mathrm{BMO}(B)} \leq \beta \min\{\frac{1}{p}, \frac{1}{p'}\}$ *with* $1 < p < \infty$, *then* ω^p *is an* $\mathcal{A}_p$*–Muckenhoupt weight and*

$$[\omega^p]_{\mathcal{A}_p}^{\frac{1}{p}} = \sup_{B' \subset B} \left(\fint_{B'} \omega^p\,dx\right)^{\frac{1}{p}} \left(\fint_{B'} \omega^{-p'}\,dx\right)^{\frac{1}{p'}} \leq 4.$$

3. *Let* $1 < p < \infty$ *and* $\theta \in (0, 1)$ *be such that* $\theta p > 1$. *If* $|\log \omega|_{\mathrm{BMO}(B)} \leq \beta \min\{\frac{1}{p}, 1 - \frac{1}{\theta p}\}$, *then*

$$\sup_{B' \subset B} \left(\fint_{B'} \omega^p\,dx\right)^{\frac{1}{p}} \left(\fint_{B'} \omega^{-(\theta p)'}\,dx\right)^{\frac{1}{(\theta p)'}} \leq 4.$$

Proof. The proof is the same as in [16, Proposition 6] with minimal changes due to the localized versions. □

Remark 4.1.10. *Using the relation* $\log(\mathbb{M}^{-1}) = -\log(\mathbb{M})$ *and* $\log(\omega^{-1}) = -\log(\omega)$*, we can apply Lemma 4.1.8 and Lemma 4.1.9 also to* $\mathbb{M}^{-1}$ *and* ω^{-1}.

We now define a specific N-function

$$\phi(t) := \frac{1}{p}t^p.$$

Then we denote

$$A(\xi) := \frac{\phi'(|\xi|)}{|\xi|}\xi = |\xi|^{p-2}\xi,$$
$$V(\xi) := \sqrt{\frac{\phi'(|\xi|)}{|\xi|}}\xi = |\xi|^{\frac{p-2}{2}}\xi.$$

Let $\tilde{\phi}^*$ be the conjugate of an N-function $\tilde{\phi}$ as follows:

$$\tilde{\phi}^*(t) := \sup_{s\geq 0}(ts - \tilde{\phi}(s)), \quad t \geq 0,$$

and so $\phi^*(t) = \frac{1}{p'}t^{p'}$.

We also need the shifted N-functions as introduced in [92, 96, 93, 16]. For $a \geq 0$ we define ϕ_a as

$$\phi_a(t) := \int_0^t \frac{\phi'(a \vee s)}{a \vee s} s\, ds. \tag{4.1.23}$$

Here $s_1 \vee s_2 := \max\{s_1, s_2\}$ for $s_1, s_2 \in \mathbb{R}$. We call a the *shift*. So for $t \leq a$, then function $\phi_a(t)$ is quadratic in t. One can see that $\phi_0 = \phi$ holds, and $a \eqsim b$ implies $\phi_a(t) \eqsim \phi_b(t)$. Also, we have

$$\phi_a(t) \eqsim (a \vee t)^{p-2}t^2, \tag{4.1.24}$$
$$\phi_a'(t) \eqsim (a \vee t)^{p-2}t, \tag{4.1.25}$$
$$(\phi_a)^* = (\phi^*)_{\phi'(a)}, \tag{4.1.26}$$
$$(\phi_{|\xi|})^* = (\phi^*)_{\phi'(|\xi|)} \tag{4.1.27}$$

with constants depending only on p. Moreover, for $a \geq 0$, the collection of

ϕ_a and $(\phi_a)^*$ satisfy the Δ_2-condition with a Δ_2-constant independent of a.

We also have Young's inequality. For every $\epsilon > 0$ there exists $c(\epsilon) = c(\epsilon, p) \geq 1$ such that for all $s, t, a \geq 0$

$$st \leq c(\epsilon)(\phi_a)^*(s) + \epsilon\phi_a(t). \tag{4.1.28}$$

Here, $c(\epsilon) \eqsim \max\{\epsilon^{-1}, \epsilon^{-\frac{1}{p-1}}\}$. Similarly, considering the relations $\phi_a(t) \eqsim \phi_a'(t)t$ and $(\phi_a)^* \eqsim (t\phi_a'(t))$, we have

$$\begin{aligned} \phi_a'(s)t &\leq c(\epsilon)\phi_a(s) + \epsilon\phi_a(t), \\ \phi_a'(s)t &\leq \epsilon\phi_a(s) + \tilde{c}(\epsilon)\phi_a(t) \end{aligned} \tag{4.1.29}$$

for all $s, t, a \geq 0$, where $\tilde{c}(\epsilon) \eqsim \max\{\epsilon^{-1}, \epsilon^{1-p}\}$. Moreover, the following relation holds for $a \geq 0$:

$$\phi_a(\lambda a) \eqsim \begin{cases} \lambda^2\phi(a), & \text{for } \lambda < 1, \\ \phi(\lambda a) & \text{for } \lambda \geq 1. \end{cases} \tag{4.1.30}$$

We emphasize the relation between A, V and ϕ_a as in the following:

Lemma 4.1.11 ([93, Lemma 41])**.** *For all $P, Q \in \mathbb{R}^n$ we have*

$$\begin{aligned} (A(P) - A(Q)) \cdot (P - Q) &\eqsim |V(P) - V(Q)|^2 \\ &\eqsim \phi_{|Q|}(|P - Q|) \eqsim (\phi^*)_{|A(Q)|}(|A(P) - A(Q)|), \end{aligned}$$

$$A(Q) \cdot Q = |V(Q)|^2 \eqsim \phi_{|Q|}(|Q|) \eqsim \phi(|Q|)$$

and

$$|A(P) - A(Q)| \eqsim (\phi_{|Q|})'(|P - Q|) \eqsim \phi'_{|P| \vee |Q|}(|P - Q|),$$

where the implicit constants depend only on p.

We usually use the following *change of shift*:

Lemma 4.1.12 (Change of shift, [93, Corollary 44])**.** *For $\epsilon > 0$, there exists*

$c_\epsilon = c_\epsilon(\epsilon, p)$ such that for all $P, Q \in \mathbb{R}^n$ we have

$$\phi_{|P|}(t) \le c_\epsilon \phi_{|Q|}(t) + \epsilon |V(P) - V(Q)|^2,$$
$$\left(\phi_{|P|}\right)^*(t) \le c_\epsilon \left(\phi_{|Q|}\right)^*(t) + \epsilon |V(P) - V(Q)|^2,$$

where $c_\epsilon = c(\epsilon, p)$.

Also, we need the following *removal of shift*.

Lemma 4.1.13 (Removal of shift, [16, Lemma 13])**.** *For all $a \in \mathbb{R}^n$, $t \ge 0$ and $\epsilon \in (0, 1]$, we have*

$$\phi'_{|a|}(t) \le \phi'\left(\frac{t}{\epsilon}\right) \vee (\epsilon \phi'(|a|)), \tag{4.1.31}$$

$$\phi_{|a|}(t) \le \epsilon \phi(|a|) + c\epsilon \phi\left(\frac{t}{\epsilon}\right), \tag{4.1.32}$$

$$(\phi_{|a|})^*(t) \le \epsilon \phi(|a|) + c\epsilon \phi^*\left(\frac{t}{\epsilon}\right) \tag{4.1.33}$$

with $c = c(p)$.

4.1.3 Global maximal regularity estimates

We now provide global maximal regularity estimates for the weak solutions of our weighted p-Laplace equation for the linear case $p = 2$ as well as the non-linear case $p \in (1, \infty)$. Let $\Omega \subset \mathbb{R}^n$ be (δ, R)–Lipschitz and $\mathbb{M} : \mathbb{R}^n \to \mathbb{R}^{n\times n}_{>0}$ be a degenerate elliptic matrix-valued weight with uniformly bounded condition number (4.1.3). Recall, that $\omega(x) := |\mathbb{M}(x)|$. Note, that (4.1.3) is equivalent to

$$\Lambda^{-1}\omega(x)|\xi| \le |\mathbb{M}(x)\xi| \le \omega(x)|\xi| \qquad \text{for all } \xi \in \mathbb{R}^n \tag{4.1.34}$$

and also

$$\Lambda^{-1}\omega(x)\mathrm{Id} \le \mathbb{M}(x) \le \omega(x)\mathrm{Id} \qquad \text{for all } x \in \Omega. \tag{4.1.35}$$

If we assume that $\log \mathbb{M}$ has a small BMO-norm, i.e., assume

$$|\log \mathbb{M}|_{\mathrm{BMO}(\Omega)} \le \kappa, \tag{4.1.36}$$

we also have $|\log \omega|_{\mathrm{BMO}(\Omega)} \leq 2\kappa$ by Lemma 4.1.7. Suppose that κ is so small such that by Lemma 4.1.9, ω^p is an $\mathcal{A}_p$-Muckenhoupt weight. Then $C_0^\infty(\Omega)$ is dense in $W^{1,p}_{0,\omega}(\Omega)$. Now, let $u \in W^{1,p}_{0,\omega}(\Omega)$ be the weak solution of (4.1.2) with $F \in L^p_\omega(\Omega)$, i.e., if we denote

$$\mathcal{A}(x,\xi) := |\mathbb{M}(x)\xi|^{p-2}\mathbb{M}^2(x)\xi = \mathbb{M}(x)A(\mathbb{M}(x)\xi),$$

then

$$\int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla\xi \, dx = \int_\Omega \mathcal{A}(x,F)\cdot\nabla\xi \, dx \tag{4.1.37}$$

for all $\xi \in W^{1,p}_{0,\omega}(\Omega)$. Since ω^p is an $\mathcal{A}_p$-Muckenhoupt weight, the existence and uniqueness of u is guaranteed by standard arguments from the calculus of variations.

We start with the standard Caccioppoli estimates associated with our degenerate p-Laplacian problem. We fix a ball $B_0 := B_R(x_0)$ with $x_0 \in \partial\Omega$. Then since Ω is $(\delta, 4R)$–Lipschitz, there exists a coordinate system $\{x_1,\dots,x_n\}$ such that $x_0 = 0$ in this coordinate system, and with the assigned Lipschitz map $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ we have (4.1.6) with $4R$ instead of R. Let $u \in W^{1,p}_\omega(4B_0 \cap \Omega)$ be a weak solution of

$$\begin{aligned} -\mathrm{div}\mathcal{A}(x,\nabla u) &= -\mathrm{div}\mathcal{A}(x,F) && \text{in } 4B_0\cap\Omega,\\ u &= 0 && \text{on } \partial\Omega\cap(4B_0). \end{aligned} \tag{4.1.38}$$

From now on, let $B_r = B_r(\tilde{x})$ denote an arbitrary ball with $\tilde{x} \in \overline{\Omega}$ and $4B_r \subset 2B_0$. Denoting $a\Omega_r = aB_r \cap 2B_0 \cap \Omega$ for $a \in \mathbb{R}_+$, we have the following:

Proposition 4.1.14 (Caccioppoli inequality)**.** *Let $u \in W^{1,p}_\omega(4B_0 \cap \Omega)$ be a weak solution of* (4.1.38) *and $B_r = B_r(\tilde{x})$ denote an arbitrary ball with $\tilde{x} \in \overline{\Omega}$ and $4B_r \subset 2B_0$.*

(1) (Interior case) If $2B_r \subset \Omega$, then we have

$$\fint_{\Omega_r} |\nabla u|^p \omega^p \, dx \leq c \fint_{2\Omega_r} \left| \frac{u-(u)_{2\Omega_r}}{r} \right|^p \omega^p \, dx + c \fint_{2\Omega_r} |F|^p \omega^p \, dx. \tag{4.1.39}$$

(2) (Boundary case) Assume (4.1.15)*. If* $2B_r \not\subset \Omega$*, then we have*

$$\fint_{\Omega_r} |\nabla u|^p \omega^p \, dx \le c \fint_{2\Omega_r} \left|\frac{u}{r}\right|^p \omega^p \, dx + c \fint_{2\Omega_r} |F|^p \omega^p \, dx. \tag{4.1.40}$$

In both cases $c = c(n, p, \Lambda)$*.*

Proof. First, (4.1.39) follows from [16, Proposition 8]. To show (4.1.40), let η be a smooth cut-off function with $\chi_{B_r} \le \eta \le \chi_{2B_r}$ and $|\nabla \eta| \le \frac{c}{r}$. Testing $\eta^p u \in W^{1,p}_{0,\omega}(2\Omega_r)$ in (4.1.38), we get

$$\int_{2\Omega_r} |\mathbb{M}\nabla u|^{p-2} \mathbb{M}\nabla u \cdot \mathbb{M}\nabla(\eta^p u) \, dx = \int_{2\Omega_r} |\mathbb{M}F|^{p-2} \mathbb{M}F \cdot \mathbb{M}\nabla(\eta^p u) \, dx.$$

Using (4.1.34) we have

$$\begin{aligned} \int_{2\Omega_r} \eta^p |\nabla u|^p \omega^p \, dx &\lesssim \int_{2\Omega_r} \eta^{p-1} |\nabla u|^{p-1} \left|\frac{u}{r}\right| \omega^p \, dx \\ &\quad + \int_{2\Omega_r} \eta^p |F|^{p-1} |\nabla u| \omega^p \, dx + \int_{2\Omega_r} \eta^{p-1} |F|^{p-1} \left|\frac{u}{r}\right| \omega^p \, dx. \end{aligned}$$

By Young's inequality, absorb the term $\eta^p |\nabla u|^p \omega^p$ into left-hand side, it follows that

$$\int_{\Omega_r} |\nabla u|^p \omega^p \, dx \lesssim \int_{2\Omega_r} \left|\frac{u}{r}\right|^p \omega^p \, dx + \int_{2\Omega_r} |F|^p \omega^p \, dx.$$

Now, by (4.1.17), we have $|\Omega_r| \eqsim |B_r| \eqsim |2B_r| \eqsim |2\Omega_r|$. Then (4.1.40) follows. $\square$

Now, we can provide the reverse Hölder's inequality.

Lemma 4.1.15. *Assume* (4.1.15)*. There exists* $\kappa_2 = \kappa_2(n, p, \Lambda) > 0$ *and* $\theta \in (0, 1)$ *such that if* $|\log \mathbb{M}|_{\mathrm{BMO}(3B_r)} \le \kappa_2$*, then*

$$\fint_{\Omega_r} |\nabla u|^p \omega^p \, dx \le c \left(\fint_{2\Omega_r} (|\nabla u| \omega)^{\theta p} \, dx \right)^{\frac{1}{\theta}} + c \fint_{2\Omega_r} |F|^p \omega^p \, dx$$

holds with $c = c(n, p, \Lambda)$*.*

Proof. If $\frac{3}{2}B_r \subset \Omega$, then we can select small κ_2 such that Lemma 4.1.9 (c) holds true to use Proposition 3 in [16]. This and Proposition 4.1.14 proves the claim.

If $\frac{3}{2}B_r \not\subset \Omega$, using Proposition 4.1.6 instead of Proposition 3 in [16], we again prove the claim. □

Now, we have the following higher integrability.

Corollary 4.1.16 (Higher integrability). *Assume* (4.1.15). *There exist* $\kappa_2 = \kappa_2(n,p,\Lambda) > 0$ *and* $s = s(n,p,\Lambda) \in (1,2)$ *such that if* $|\log \mathbb{M}|_{\mathrm{BMO}(3B_r)} \leq \kappa_2$, *then*

$$\left(\fint_{\Omega_r} (|\nabla u|^p \omega^p)^s \, dx\right)^{\frac{1}{s}} \leq c \fint_{2\Omega_r} |\nabla u|^p \omega^p \, dx + c \left(\fint_{2\Omega_r} (|F|^p \omega^p)^s \, dx\right)^{\frac{1}{s}}$$

holds with $c = c(n,p,\Lambda)$.

Proof. After extending $\nabla u = F = 0$ in $2B_r \setminus \Omega$, and considering $|\Omega_r| \eqsim |B_r|$ and $|2\Omega_r| \eqsim |2B_r|$, Gehring's lemma (e.g. [126, Theorem 6.6]) implies the conclusion. □

In this section, we only prove boundary comparison estimates, since the interior estimates are proved in [16]. Let us assume $\mathbb{M}F \in L^q(\Omega)$. Choose a cut-off function $\eta \in C_0^\infty(B_0)$ with

$$\chi_{\frac{1}{2}B_0} \leq \eta \leq \chi_{B_0} \quad \text{and} \quad \|\nabla \eta\|_\infty \leq c/R. \tag{4.1.41}$$

Define z on $\mathbb{R}^n$ as follows: first let z on $B_0 \cap \Omega$ be such that

$$z := u\eta^{p'} \tag{4.1.42}$$

and take the zero extension for z on $\mathbb{R}^n \setminus (B_0 \cap \Omega)$, if necessary. Also, we denote

$$g := \eta^{p'} \nabla u - \nabla z = -u \nabla(\eta^{p'}) = -up'\eta^{p'-1}\nabla \eta. \tag{4.1.43}$$

Then we have the following estimate:

$$|g| \lesssim \frac{|u|}{R}. \tag{4.1.44}$$

For the convenience of notation, we write

$$\begin{aligned}\mathbb{M}_{B_r} &:= \langle \mathbb{M} \rangle_{B_r}^{\log},\\ \omega_{B_r} &:= \langle \omega \rangle_{B_r}^{\log},\end{aligned}$$

and so

$$\begin{aligned}\mathcal{A}_{B_r}(\xi) &:= |\mathbb{M}_{B_r}\xi|^{p-2}\mathbb{M}_{B_r}^2\xi = \mathbb{M}_{B_r}A(\mathbb{M}_{B_r}\xi),\\ \mathcal{V}(x,\xi) &:= V(\mathbb{M}(x)\xi),\\ \mathcal{V}_{B_r}(\xi) &:= V(\mathbb{M}_{B_r}\xi).\end{aligned}$$

Then we have the following relations for all $\xi \in \mathbb{R}^n$:

$$\mathcal{A}(x,\xi)\cdot\xi = |\mathcal{V}(x,\xi)|^2, \tag{4.1.45}$$

$$\mathcal{A}_{B_r}(\xi)\cdot\xi = |\mathcal{V}_{B_r}(\xi)|^2, \tag{4.1.46}$$

$$|\mathcal{A}_{B_r}(\xi)| \lesssim \omega_{B_r}^p|\xi|^{p-1} \tag{4.1.47}$$

and by [16, Section 3],

$$\Lambda^{-1}\omega_{B_r}|\xi| \le |\mathbb{M}_{B_r}\xi| \le \omega_{B_r}|\xi|, \tag{4.1.48}$$

$$\Lambda^{-1}\omega_{B_r} \le |\mathbb{M}_{B_r}| \le \omega_{B_r}. \tag{4.1.49}$$

Summing up the above result, we have [16, Lemma 16] as follows: for all $\xi \in \mathbb{R}^n$ and all $x \in B_r$ there holds

$$|\mathcal{A}_{B_r}(\xi) - \mathcal{A}(x,\xi)| \lesssim \frac{|\mathbb{M} - \mathbb{M}_{B_r}(x)|}{|\mathbb{M}_{B_r}|}\left(|\mathcal{A}_{B_r}(\xi)| + |\mathcal{A}(x,\xi)|\right). \tag{4.1.50}$$

Before introducing the reference problem, we provide the following lemma for the well-posedness:

Lemma 4.1.17. *Assuming* (4.1.15)*, there exists $\kappa_3 = \kappa_3(n,p,\Lambda) > 0$ and $s = s(n,p,\Lambda) \in (1,2)$ such that if $|\log \mathbb{M}|_{\mathrm{BMO}(4B_r)} \le \kappa_3$, then with $2\Omega_r = 2B_r \cap B_0 \cap \Omega$ there holds*

$$\fint_{\Omega_r} (|\nabla u|\omega_{B_r})^p\,dx \lesssim \fint_{2\Omega_r} (|\nabla u|\omega)^p\,dx + \left(\fint_{2\Omega_r} (|F|\omega)^{ps}\,dx\right)^{\frac{1}{s}}. \tag{4.1.51}$$

Proof. Using Hölder's inequality, $|\Omega_r| \eqsim |B_r|$ and Lemma 4.1.9, we have

$$\begin{aligned}
\fint_{\Omega_r} (|\nabla u|\omega_{B_r})^p \, dx &\le \left(\fint_{\Omega_r} (|\nabla u|\omega)^{ps} \, dx\right)^{\frac{1}{s}} \left(\fint_{\Omega_r} (\omega_{B_r}\omega^{-1})^{ps'} \, dx\right)^{\frac{1}{s'}} \\
&\lesssim \left(\fint_{\Omega_r} (|\nabla u|\omega)^{ps} \, dx\right)^{\frac{1}{s}} \left(\fint_{B_r} (\omega_{B_r}\omega^{-1})^{ps'} \, dx\right)^{\frac{1}{s'}} \\
&\lesssim \left(\fint_{\Omega_r} (|\nabla u|\omega)^{ps} \, dx\right)^{\frac{1}{s}} .
\end{aligned}$$

Then Corollary 4.1.16 yields the conclusion. □

Now, let $h \in W^{1,p}_{\omega_{B_r}}(\Omega_r)$ be the weak solution of

$$\begin{aligned}
-\mathrm{div}\,(\mathcal{A}_{B_r}(\nabla h)) &= 0 \quad \text{in } \Omega_r, \\
h &= z \quad \text{on } \partial\Omega_r.
\end{aligned} \tag{4.1.52}$$

Then h is the unique minimizer of

$$w \mapsto \int_{\Omega_r} \phi(|\mathbb{M}_{B_r}\nabla w|) \, dx \tag{4.1.53}$$

with boundary data $w = z$ on $\partial\Omega_r$. Now, we provide the first comparison estimate. Recall that $B_r = B_r(\tilde{x})$, $B_0 = B_R(x_0)$, $4B_r \subset 2B_0$, $\tilde{x} \in \overline{\Omega}$ and z, h are given by (4.1.42) and (4.1.52), respectively. Moreover, as [16, Eq. (3.25)], we have

$$\begin{aligned}
&-\mathrm{div}\,(\mathcal{A}_{B_r}(\nabla z) - \mathcal{A}_{B_r}(\nabla h)) \\
&\quad = -\mathrm{div}\,(\mathcal{A}_{B_r}(\nabla z) - \mathcal{A}(\cdot, \nabla z)) - \mathrm{div}\,(\mathcal{A}(\cdot, \nabla z) - \mathcal{A}(\cdot, \nabla z + g)) \\
&\qquad - \eta^p \mathrm{div}\,(\mathcal{A}(\cdot, F)) - \nabla(\eta^p) \cdot \mathcal{A}(\cdot, \nabla u)
\end{aligned} \tag{4.1.54}$$

on Ω_r, in the distributional sense.

Proposition 4.1.18 (First comparison at boundary)**.** *Assuming* (4.1.15)*, let h be as in* (4.1.52) *and z be as in* (4.1.42)*. There exist $s > 1$ and $\kappa_4 = \kappa_4(n, p, \Lambda) \in (0, 1)$, such that if $|\log \mathbb{M}|_{\mathrm{BMO}(4B_r)} \le \kappa_4$ holds, then for every*

$\epsilon \in (0,1)$ *we have*

$$\begin{aligned}
&\fint_{\Omega_r} |\mathcal{V}_{B_r}(\nabla h) - \mathcal{V}_{B_r}(\nabla z)|^2 \, dx \\
&\le c \left(|\log \mathbb{M}|^2_{\mathrm{BMO}(B_r)} + \epsilon \right) \left(\fint_{\Omega_r} (|\nabla z|^p \omega^p)^s \, dx \right)^{\frac{1}{s}} \\
&\quad + c\, C^*(\epsilon) \left(\fint_{2\Omega_r} \left(\frac{|u|^p}{R^p} \omega^p \right)^s \, dx \right)^{\frac{1}{s}} + c\, C^*(\epsilon) \left(\fint_{2\Omega_r} (|F|^p \omega^p)^s \, dx \right)^{\frac{1}{s}}
\end{aligned} \tag{4.1.55}$$

for some $c = c(n, p, \Lambda)$, *and* $C^*(\epsilon) = \max \left\{ \epsilon^{1-p}, \epsilon^{-\frac{1}{p-1}} \right\}$.

Proof. The proof is similar to the one of [16, Proposition 17]. Observe that $|\log \mathbb{M}|_{\mathrm{BMO}(4B_r)} \le \kappa_4$ implies that (4.1.52) is well-defined. Testing $z - h$ to (4.1.52) and (4.1.38), by (4.1.54) it follows that

$$\begin{aligned}
I_0 &:= \fint_{\Omega_r} (\mathcal{A}_{B_r}(\nabla z) - \mathcal{A}_{B_r}(\nabla h)) \cdot (\nabla z - \nabla h) \, dx \\
&= \fint_{\Omega_r} (\mathcal{A}_{B_r}(\nabla z) - \mathcal{A}(x, \nabla z)) \cdot (\nabla z - \nabla h) \, dx \\
&\quad + \fint_{\Omega_r} (\mathcal{A}(x, \nabla z) - \mathcal{A}(x, \nabla z + g)) \cdot (\nabla z - \nabla h) \, dx \\
&\quad + \fint_{\Omega_r} \mathcal{A}(x, F) \cdot (\nabla(\eta^p z) - \nabla(\eta^p h)) \, dx \\
&\quad + \fint_{\Omega_r} \nabla(\eta^p) \cdot \mathcal{A}(x, \nabla u)(z - h) \, dx =: I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{4.1.56}$$

By Lemma 4.1.11, we have

$$I_0 \eqsim \fint_{\Omega_r} |\mathcal{V}_{B_r}(\nabla h) - \mathcal{V}_{B_r}(\nabla z)|^2 \, dx \eqsim \fint_{\Omega_r} \phi_{|\nabla z|}(|\nabla z - \nabla h|) \omega^p_{B_r} \, dx. \tag{4.1.57}$$

To estimate I_1, arguing as in the proof of [16, Proposition 17], we have

$$\begin{aligned} I_1 &= \fint_{\Omega_r} (\mathcal{A}_{B_r}(\nabla z) - \mathcal{A}(x, \nabla z)) \cdot (\nabla z - \nabla h)\, dx \\ &\leq \sigma \fint_{\Omega_r} \phi_{|\nabla z|}(|\nabla z - \nabla h|) \omega_{B_r}^p \, dx \\ &\quad + c(\sigma) \fint_{\Omega_r} \left(\frac{|\mathbb{M} - \mathbb{M}_{B_r}|}{|\mathbb{M}_{B_r}|} \right)^2 \phi(|\nabla z|) \left(\frac{\omega_{B_r}^p}{\omega^p} + \frac{\omega^p}{\omega_{B_r}^p} + \frac{\omega^{p'}}{\omega_{B_r}^{p'}} \right) \omega^p \, dx \\ &= I_{1,1} + I_{1,2} \end{aligned} \tag{4.1.58}$$

for any $\sigma \in (0,1)$. Now, $I_{1,1}$ is absorbed to the left-hand side I_0 by choosing $\sigma = \sigma(n,p,\Lambda)$ sufficiently small, and so $c(\sigma) = c$. For $I_{1,2}$, we first assume $|\log \mathbb{M}|_{\mathrm{BMO}(B_r)} \leq \kappa_1 = \kappa_1(n,p,\Lambda)$ and then use Lemma 4.1.8, together with $|\Omega_r| \eqsim |B_r|$ (the measure density of Ω_r to B_r) from (4.1.15) and (4.1.17), to have

$$\begin{aligned} I_{1,3} &:= \left(\fint_{\Omega_r} \left(\frac{|\mathbb{M} - \mathbb{M}_{B_r}|}{|\mathbb{M}_{B_r}|} \right)^{4s'} dx \right)^{\frac{1}{2s'}} \\ &\lesssim \left(\fint_{B_r} \left(\frac{|\mathbb{M} - \mathbb{M}_{B_r}|}{|\mathbb{M}_{B_r}|} \right)^{4s'} dx \right)^{\frac{1}{2s'}} \lesssim |\log \mathbb{M}|_{\mathrm{BMO}(B_r)}^2. \end{aligned} \tag{4.1.59}$$

Also, assume $|\log \mathbb{M}|_{\mathrm{BMO}(B_r)} \leq \kappa_4$ for some small $\kappa_4 = \kappa_4(n,p,\Lambda)$ and then use Lemma 4.1.9, together with $|\Omega_r| \eqsim |B_r|$ from (4.1.17) with the help of (4.1.15) to have

$$\begin{aligned} I_{1,4} &:= \fint_{\Omega_r} \left(\frac{\omega_{B_r}^p}{\omega^p} + \frac{\omega^p}{\omega_{B_r}^p} + \frac{\omega^{p'}}{\omega_{B_r}^{p'}} \right)^{2s'} dx \\ &\lesssim \fint_{B_r} \left(\frac{\omega_{B_r}^p}{\omega^p} + \frac{\omega^p}{\omega_{B_r}^p} + \frac{\omega^{p'}}{\omega_{B_r}^{p'}} \right)^{2s'} dx \leq c. \end{aligned} \tag{4.1.60}$$

Then by Hölder's inequality and the above two displays, we obtain

$$\begin{aligned} I_{1,2} &\leq c\, I_{1,3} I_{1,4}^{\frac{1}{2s'}} \left(\fint_{\Omega_r} (|\nabla z|^p \omega^p)^s \, dx \right)^{\frac{1}{s}} \\ &\lesssim c\, |\log \mathbb{M}|^2_{\mathrm{BMO}(B_r)} \left(\fint_{\Omega_r} (|\nabla z|^p \omega^p)^s \, dx \right)^{\frac{1}{s}}. \end{aligned} \tag{4.1.61}$$

From now on, we only specify the necessary tools to provide each resulting estimates, since we mainly follow the proof of [16, Proposition 17], and when we use the measure density property, we apply the similar manipulation as above. First, by (4.1.44), Lemma 4.1.12 and Lemma 4.1.9 together with the measure density property of Ω_r to B_r, we estimate I_2 as follows:

$$\begin{aligned} I_2 \leq\ & \sigma \fint_{\Omega_r} \phi_{|\nabla z|}(|\nabla z - \nabla h|) \omega_{B_r}^p \, dx \\ & + c_\sigma \epsilon \left(\fint_{\Omega_r} (|\nabla z|^p \omega^p)^s \, dx \right)^{\frac{1}{s}} + c_\sigma \epsilon^{-\frac{1}{p-1}} \left(\fint_{\Omega_r} \left(\frac{|u|^p}{R^p} \omega^p \right)^s dx \right)^{\frac{1}{s}} \end{aligned} \tag{4.1.62}$$

for any $\sigma, \epsilon \in (0,1)$. To estimate I_3, using Young's inequality and $0 \leq \eta \leq 1$, we have

$$\begin{aligned} I_3 &\lesssim \epsilon^{-\frac{1}{p-1}} \fint_{\Omega_r} \frac{\omega^{p'}}{\omega_{B_r}^{p'}} \eta^p |F|^p \omega^p \, dx \\ &\quad + \epsilon \fint_{\Omega_r} |\nabla z - \nabla h|^p \omega_{B_r}^p \, dx + \epsilon \fint_{\Omega_r} \left| \frac{z-h}{r} \right|^p \omega_{B_r}^p \, dx \\ &=: I_{3,1} + I_{3,2} + I_{3,3}. \end{aligned} \tag{4.1.63}$$

Extending $z-h$ as 0 in $B_r \setminus \Omega$, and using Proposition 3 in [16], we have $I_{3,3} \lesssim I_{3,2}$. We employ triangle inequality, minimizing property of h in (4.1.53), (4.1.48), Hölder's inequality and Lemma 4.1.9 together with $|\Omega_r| \eqsim |B_r|$, to obtain

$$I_{3,2} \leq \epsilon c \left(\fint_{\Omega_r} (|\nabla z|^p \omega^p)^s \, dx \right)^{\frac{1}{s}}. \tag{4.1.64}$$

Using also Hölder's inequality and Lemma 4.1.9 with $|\Omega_r| \approx |B_r|$, we have

$$I_{3,1} \lesssim c\epsilon^{-\frac{1}{p-1}} \left(\fint_{\Omega_r} (|F|^p \omega^p)^s \, dx \right)^{\frac{1}{s}}. \tag{4.1.65}$$

Finally, to estimate I_4, instead of dividing the case into $p > 2$ and $1 < p \leq 2$ as in [16], we consider the cases in a unified way. By $p'(p-2+\frac{1}{p}) = p-1$ and $\eta^{p'} \nabla u = \nabla z + g$, we first see that

$$\begin{aligned}
I_4 &\leq \fint_{\Omega_r} |\nabla(\eta^p)||\mathcal{A}(x, \nabla u)||z-h| \, dx \\
&\lesssim \fint_{\Omega_r} |\nabla \eta||\eta^{p'} \nabla u|^{p-2+\frac{1}{p}} |\nabla u|^{1-\frac{1}{p}} |z-h| \omega^p \, dx \\
&\lesssim \fint_{\Omega_r} \frac{r}{R} |\nabla z + g|^{p-2+\frac{1}{p}} |\nabla u|^{1-\frac{1}{p}} \left| \frac{z-h}{r} \right| \omega^p \, dx \\
&\lesssim \epsilon \fint_{\Omega_r} |\nabla z + g|^p \omega^p \, dx \\
&\quad + C^*(\epsilon) \fint_{\Omega_r} \left(\frac{r}{R} \right)^{\frac{p^2}{p-1}} |\nabla u|^p \omega^p \, dx + \epsilon \fint_{\Omega_r} \left| \frac{z-h}{r} \right|^p \omega^p \, dx
\end{aligned} \tag{4.1.66}$$

for any $\epsilon \in (0, 1]$, where for the last step we have used Young's inequality for the exponents $\left(\frac{p}{p-2+\frac{1}{p}}, \frac{p^2}{p-1}, p \right)$ and $0 \leq \eta_1 \leq 1$. Here, $C^*(t) : (0, 1] \to \mathbb{R}^+$ is such that

$$C^*(t) = \max \left\{ t^{1-p}, t^{-\frac{1}{p-1}} \right\}, \tag{4.1.67}$$

which is a continuous function on $(0, 1]$ for each fixed $p \in (1, \infty)$. Recalling (4.1.44), we have

$$\fint_{\Omega_r} |\nabla z + g|^p \omega^p \, dx \lesssim \fint_{\Omega_r} |\nabla z|^p \omega^p \, dx + \fint_{\Omega_r} \left| \frac{u}{R} \right|^p \omega^p \, dx. \tag{4.1.68}$$

Also, since $\frac{p^2}{p-1} > p$ on $p \in (1, \infty)$ and $r \le R$ hold, there holds

$$\begin{aligned} \fint_{\Omega_r} \left(\frac{r}{R}\right)^{\frac{p^2}{p-1}} |\nabla u|^p \omega^p \, dx &\lesssim \left(\frac{r}{R}\right)^p \fint_{\Omega_r} |\nabla u|^p \omega^p \, dx \\ &\lesssim \left(\frac{r}{R}\right)^p \fint_{2\Omega_r} \left|\frac{u}{r}\right|^p \omega^p \, dx + \fint_{2\Omega_r} |F|^p \omega^p \, dx \\ &\lesssim \fint_{2\Omega_r} \left|\frac{u}{R}\right|^p \omega^p \, dx + \fint_{2\Omega_r} |F|^p \omega^p \, dx. \end{aligned} \tag{4.1.69}$$

Extending $z - h$ as 0 in $B_r \setminus \Omega$, and using Proposition 3 in [16], Hölder's inequality, Lemma 4.1.9 together with $|\Omega_r| \eqsim |B_r|$, triangle inequality, minimizing property of h in (4.1.53) and (4.1.48) yields

$$\fint_{\Omega_r} \left|\frac{z-h}{r}\right|^p \omega^p \, dx \lesssim \left(\fint_{\Omega_r} (|\nabla z|^p \omega^p)^s \, dx\right)^{\frac{1}{s}}, \tag{4.1.70}$$

as in [16]. Note that the argument used in [16] for (4.1.70) can be applied in all cases $p \in (1, \infty)$. Thus it follows that

$$\begin{aligned} I_3 \lesssim \epsilon &\left(\fint_{\Omega_r} (|\nabla z|^p \omega^p)^s \, dx\right)^{\frac{1}{s}} \\ &+ C^*(\epsilon) \fint_{2\Omega_r} \left|\frac{u}{R}\right|^p \omega^p \, dx + C^*(\epsilon) \fint_{2\Omega_r} |F|^p \omega^p \, dx. \end{aligned} \tag{4.1.71}$$

Summing up the above all estimates, we have

$$
\begin{aligned}
&\fint_{\Omega_r} |\mathcal{V}_{B_r}(\nabla h) - \mathcal{V}_{B_r}(\nabla z)|^2 \, dx \\
&\quad \lesssim \sigma \fint_{\Omega_r} \phi_{|\nabla z|}(|\nabla z - \nabla h|)\omega_{B_r}^p \, dx \\
&\qquad + c(\sigma)(\,|\log \mathbb{M}|^2_{\mathrm{BMO}(B_r)} + \epsilon) \left(\fint_{\Omega_r} (|\nabla z|^p \omega^p)^s \, dx \right)^{\frac{1}{s}} \\
&\qquad + c(\sigma)\big(\epsilon^{-\frac{1}{p-1}} + C^*(\epsilon)\big) \left(\fint_{2\Omega_r} \left(\frac{|u|^p}{R^p}\omega^p \right)^s dx \right)^{\frac{1}{s}} \\
&\qquad + c(\sigma)\big(\epsilon^{-\frac{1}{p-1}} + C^*(\epsilon)\big) \left(\fint_{2\Omega_r} (|F|^p \omega^p)^s \, dx \right)^{\frac{1}{s}} \\
&\quad = I_5 + I_6 + I_7 + I_8.
\end{aligned} \tag{4.1.72}
$$

By Lemma 4.1.11 and (4.1.34), $\phi_{|\nabla z|}(|\nabla z - \nabla h|)\omega_{B_r}^p \eqsim |\mathcal{V}_{B_r}(\nabla h) - \mathcal{V}_{B_r}(\nabla z)|^2$. Then by choosing $\sigma \in (0,1)$ sufficiently small depending on n, p and Λ, I_5 is absorbed to the left-hand side. Finally, $\epsilon^{-\frac{1}{p-1}} \leq C^*(\epsilon)$ holds when $\epsilon \in (0,1)$, and so the estimate (4.1.55) holds true. $\square$

Now, we give the second comparison estimate. With $y = \Psi(x)$, we define

$$
\tilde{h}(y) := h(\Psi^{-1}(y)). \tag{4.1.73}
$$

Let $\tilde{v} = \tilde{v}(y) \in W^{1,p}(\Psi(\frac{1}{2}\Omega_r))$ be the weak solution of

$$
\begin{aligned}
-\mathrm{div}_y(|\mathbb{M}_{B_r}\nabla_y \tilde{v}|^{p-2}\mathbb{M}_{B_r}^2 \nabla_y \tilde{v}) = 0 \quad &\text{in } \Psi(\tfrac{1}{2}\Omega_r), \\
\tilde{v} = \tilde{h} \quad &\text{on } \partial(\Psi(\tfrac{1}{2}\Omega_r)).
\end{aligned} \tag{4.1.74}
$$

Since Ψ is a homeomorphism, together with (4.1.6) we have $\partial(\Psi(\frac{1}{2}\Omega_r)) = \Psi(\partial(\frac{1}{2}\Omega_r))$. Denoting $v(x) = \tilde{v}(y) = \tilde{v}(\Psi(x))$, we have

$$
\nabla v(x) = \nabla \tilde{v}(\Psi(x)) = (\nabla \Psi)(x) \nabla_y \tilde{v}(\Psi(x)) = (\nabla \Psi)(x) \nabla_y \tilde{v}(y). \tag{4.1.75}
$$

Denoting $T(x) = (\nabla\Psi)^{-1}(x)$, (4.1.74) transforms into

$$\begin{aligned} -\mathrm{div}(T^t(x)|\mathbb{M}_{B_r}T(x)\nabla v|^{p-2}\mathbb{M}_{B_r}^2 T(x)\nabla v) = 0 \quad & \text{in } \tfrac{1}{2}\Omega_r, \\ v(x) = h(x) \qquad\qquad\qquad\qquad & \text{on } \partial(\tfrac{1}{2}\Omega_r), \end{aligned} \tag{4.1.76}$$

where $T^t(x)$ abbreviation stands for transpose. Or equivalently, denoting

$$\mathcal{A}_\Psi(x,\xi) := T^t(x)|\mathbb{M}_{B_r}T(x)\xi|^{p-2}\mathbb{M}_{B_r}^2 T(x)\xi, \tag{4.1.77}$$

we have

$$\begin{aligned} -\mathrm{div}(\mathcal{A}_\Psi(x,\nabla v)) = 0 \qquad & \text{in } \tfrac{1}{2}\Omega_r, \\ v(x) = h(x) \quad & \text{on } \partial(\tfrac{1}{2}\Omega_r). \end{aligned} \tag{4.1.78}$$

The problems (4.1.76) and (4.1.78) can be derived also from the weak formulation of the equation. At this time we have also used that $\det(\nabla\Psi) = 1$ for the change of coordinate in the integrals. The natural function space for v is $W^{1,p}_{\omega_{B_r}}(\frac{1}{2}\Omega_r)$ and v is the unique minimizer of

$$w \mapsto \int_{\frac{1}{2}\Omega_r} \phi(|\mathbb{M}_{B_r}T(x)\nabla w|)\,dx \tag{4.1.79}$$

subject to the boundary condition $v = h$ on $\partial(\frac{1}{2}\Omega_r)$.

Now, we need the following lemma.

Lemma 4.1.19. *Assume* (4.1.15)*. For all $\xi \in \mathbb{R}^n$ and $x \in \frac{1}{2}\Omega_r$ we have*

$$|T(x)| \eqsim |T^{-1}(x)| \eqsim c \tag{4.1.80}$$

and

$$|\mathcal{A}_B(\xi) - \mathcal{A}_\Psi(x,\xi)| \leq c\|\nabla\psi\|_\infty \omega_B^p \min\left\{|\xi|^{p-1}, |T(x)\xi|^{p-1}\right\} \tag{4.1.81}$$

for some $c = c(n,p,\Lambda)$.

Proof. First, since $(T(x) - \mathrm{id})^2 = ((\nabla\Psi(x))^{-1} - \mathrm{id})^2 = 0$, we have $T^{-1}(x) - \mathrm{id} = \mathrm{id} - T(x)$. Then

$$|T^{-1}(x) - \mathrm{id}\,| = |\,\mathrm{id} - T(x)| \leq n\|\nabla\psi\|_\infty \tag{4.1.82}$$

holds, and so together with $\|\nabla\psi\|_\infty \le \frac{1}{2n}$, it follows that

$$\frac{1}{2} \le |\operatorname{id}| - n\|\nabla\psi\|_\infty \le |T(x)| \le |\operatorname{id}| + n\|\nabla\psi\|_\infty \le \frac{3}{2}. \tag{4.1.83}$$

Hence we have $|T(x)| \eqsim c$. Similarly, we have $|T^{-1}(x)| \eqsim c$.

On the other hand, observe that by (4.1.48),

$$\begin{aligned}
&|\mathcal{A}_B(\xi) - \mathcal{A}_\Psi(x,\xi)| \\
&= \big||\mathbb{M}_B\xi|^{p-2}\mathbb{M}_B^2\xi - |\mathbb{M}_B T(x)\xi|^{p-2}T^t(x)\mathbb{M}_B^2 T(x)\xi\big| \\
&\le \big|\mathbb{M}_B|\mathbb{M}_B\xi|^{p-2}\mathbb{M}_B\xi - \mathbb{M}_B|\mathbb{M}_B T(x)\xi|^{p-2}\mathbb{M}_B T(x)\xi\big| \\
&\quad + |\mathbb{M}_B - T^t(x)\mathbb{M}_B| \cdot \big||\mathbb{M}_B T(x)\xi|^{p-2}\mathbb{M}_B T(x)\xi\big| \\
&\lesssim \omega_B \cdot |A(\mathbb{M}_B\xi) - A(\mathbb{M}_B T(x)\xi)| + \omega_B \cdot |\operatorname{id} - T^t(x)| \cdot |\mathbb{M}_B T(x)\xi|^{p-1}.
\end{aligned}$$

Here, by (4.1.80) and (4.1.48),

$$\begin{aligned}
|\mathbb{M}_{B_r}\xi| &= |\mathbb{M}_{B_r}T^{-1}(x)T(x)\xi| \\
&\lesssim |\mathbb{M}_{B_r}| \cdot |T^{-1}(x)| \cdot |T(x)\xi| \lesssim |\mathbb{M}_{B_r}| \cdot |T(x)\xi| \lesssim |\mathbb{M}_{B_r}T(x)\xi|
\end{aligned}$$

and similarly $|\mathbb{M}_{B_r}T(x)\xi| \lesssim |\mathbb{M}_{B_r}T(x)T^{-1}(x)\xi| = |\mathbb{M}_{B_r}\xi|$ holds. Thus we have $|\mathbb{M}_{B_r}T(x)\xi| \eqsim |\mathbb{M}_{B_r}\xi|$. Then together with (4.1.82) and Lemma 4.1.11, there holds

$$\begin{aligned}
&|A(\mathbb{M}_{B_r}\xi) - A(\mathbb{M}_{B_r}T(x)\xi)| \eqsim \phi'_{|\mathbb{M}_{B_r}\xi| \vee |\mathbb{M}_{B_r}T(x)\xi|}\left(|\mathbb{M}_{B_r}\xi - \mathbb{M}_{B_r}T(x)\xi|\right) \\
&\eqsim \left(|\mathbb{M}_{B_r}\xi| \vee |\mathbb{M}_{B_r}T(x)\xi| \vee |\mathbb{M}_{B_r}\xi - \mathbb{M}_{B_r}T(x)\xi|\right)^{p-2} |\mathbb{M}_{B_r}\xi - \mathbb{M}_{B_r}T(x)\xi| \\
&\lesssim \omega_{B_r}|\operatorname{id} - T(x)||\xi| \left(|\mathbb{M}_{B_r}\xi| \vee |\mathbb{M}_{B_r}T(x)\xi| \vee |\mathbb{M}_{B_r}\xi - \mathbb{M}_{B_r}T(x)\xi|\right)^{p-2} \\
&\lesssim \omega_{B_r}|\operatorname{id} - T(x)||\xi| \frac{\left(|\mathbb{M}_{B_r}\xi| + |\mathbb{M}_{B_r}T(x)\xi|\right)^{p-1}}{|\mathbb{M}_{B_r}\xi|} \\
&\lesssim \Lambda|\operatorname{id} - T(x)| \left(|\mathbb{M}_{B_r}\xi| + |\mathbb{M}_{B_r}T(x)\xi|\right)^{p-1} \\
&\lesssim \|\nabla\psi\|_\infty |\mathbb{M}_{B_r}T(x)\xi|^{p-1}.
\end{aligned}$$

Thus, together with $|\operatorname{id} - T^t(x)| = |(\operatorname{id} - T(x))^t| = |\operatorname{id} - T(x)| \le n\|\nabla\psi\|_\infty$, we have

$$|\mathcal{A}_{B_r}(\xi) - \mathcal{A}_\Psi(x,\xi)| \lesssim \omega_{B_r}\|\nabla\psi\|_\infty |\mathbb{M}_{B_r}T(x)\xi|^{p-1} \lesssim \omega_{B_r}^p \|\nabla\psi\|_\infty |T(x)\xi|^{p-1}.$$

Finally, since $\omega_{B_r}|T(x)\xi| \eqsim |\mathbb{M}_{B_r}T(x)\xi| \eqsim |\mathbb{M}_{B_r}\xi| \eqsim \omega_{B_r}|\xi|$, we get the conclusion. $\square$

Now, we can compute the comparison estimate.

Proposition 4.1.20 (Second comparison at boundary)**.** *Assuming* (4.1.15)*, let v be as in* (4.1.76) *and h be as in* (4.1.52)*. There exists $\kappa_4 = \kappa_4(n, p, \Lambda)$ such that if $|\log \mathbb{M}|_{\mathrm{BMO}(4B_r)} \le \kappa_4$, then for some $c = c(n, p, \Lambda)$, we have*

$$\fint_{\frac{1}{2}\Omega_r} |\mathcal{V}_{B_r}(\nabla v) - \mathcal{V}_{B_r}(\nabla h)|^2 \, dx \le c\|\nabla\psi\|_\infty^2 \fint_{\Omega_r} (|\nabla h|^p \omega_{B_r}^p) \, dx. \tag{4.1.84}$$

Proof. Test $v - h$ to (4.1.52) and (4.1.76) to have

$$\begin{aligned}\fint_{\frac{1}{2}\Omega_r} &(\mathcal{A}_{B_r}(\nabla v) - \mathcal{A}_{B_r}(\nabla h)) \cdot (\nabla v - \nabla h) \, dx \\ &= \fint_{\frac{1}{2}\Omega_r} (\mathcal{A}_{B_r}(\nabla v) - \mathcal{A}_{\Psi}(x, \nabla v)) \cdot (\nabla v - \nabla h) \, dx.\end{aligned} \tag{4.1.85}$$

We apply Lemma 4.1.11, (4.1.81) and then use Young's inequality to obtain

$$\begin{aligned}\fint_{\frac{1}{2}\Omega_r} &|\mathcal{V}_{B_r}(\nabla v) - \mathcal{V}_{B_r}(\nabla h)|^2 \, dx \\ &\eqsim \fint_{\frac{1}{2}\Omega_r} \phi_{|\nabla v|}(|\nabla v - \nabla h|)\omega_{B_r}^p \, dx \\ &\lesssim \fint_{\frac{1}{2}\Omega_r} |\mathcal{A}_{\Psi}(x, \nabla v) - \mathcal{A}_{B_r}(\nabla v)||\nabla v - \nabla h| \, dx \\ &\lesssim \fint_{\frac{1}{2}\Omega_r} \|\nabla\psi\|_\infty \omega_{B_r}^p \phi'(|\nabla v|)|\nabla v - \nabla h| \, dx \\ &\le \sigma \fint_{\frac{1}{2}\Omega_r} \phi_{|\nabla v|}(|\nabla v - \nabla h|)\omega_{B_r}^p \, dx \\ &\quad + c(\sigma) \fint_{\frac{1}{2}\Omega_r} (\phi_{|\nabla v|})^*(\|\nabla\psi\|_\infty \phi'(|\nabla v|))\omega_{B_r}^p \, dx =: I_1 + I_2\end{aligned} \tag{4.1.86}$$

for any $\sigma \in (0, 1)$. Then I_1 is absorbed to the left-hand side by choosing σ sufficiently small depending on n, p and Λ. To estimate I_2, we use (4.1.30),

$(\phi^*)_{\phi'(|a|)}(\phi'(|a|)) \eqsim \phi(|a|)$ and then (4.1.80) to have

$$\begin{aligned} I_2 &\lesssim \fint_{\frac{1}{2}\Omega_r} (\phi^*)_{\phi'(|\nabla v|)}(\|\nabla\psi\|_\infty \phi'(|\nabla v|))\omega_{B_r}^p \, dx \\ &\lesssim \|\nabla\psi\|_\infty^2 \fint_{\frac{1}{2}\Omega_r} (\phi^*)_{\phi'(|\nabla v|)}(\phi'(|\nabla v|))\omega_{B_r}^p \, dx \\ &\lesssim \|\nabla\psi\|_\infty^2 \fint_{\frac{1}{2}\Omega_r} \phi(|\nabla v|)\omega_{B_r}^p \, dx \\ &\lesssim \|\nabla\psi\|_\infty^2 \fint_{\frac{1}{2}\Omega_r} \phi(|T(x)\nabla v|)\omega_{B_r}^p \, dx. \end{aligned} \tag{4.1.87}$$

Now, we use minimizing property of ∇v together with (4.1.48), (4.1.80) and $|\Omega_r| \eqsim |\frac{1}{2}\Omega_r|$ to have

$$\begin{aligned} \fint_{\frac{1}{2}\Omega_r} \phi(|T(x)\nabla v|)\omega_{B_r}^p \, dx &\lesssim \fint_{\frac{1}{2}\Omega_r} \phi(|T(x)\nabla h|)\omega_{B_r}^p \, dx \\ &\lesssim \fint_{\Omega_r} \phi(|\nabla h|)\omega_{B_r}^p \, dx. \end{aligned} \tag{4.1.88}$$

Summing up the above estimates, we obtain (4.1.84). □

Before providing decay estimates of $\mathcal{V}(\cdot, \nabla z)$, we discuss some regularity results and corresponding estimates related to $\tilde{v}$ and v which are defined in (4.1.74) and (4.1.78), respectively. First, we have the following estimates which imply Lipschitz regularity and $C^{1,\alpha}$ regularity of $\tilde{v}$.

Proposition 4.1.21. *Assuming* (4.1.15), *let $\tilde{v}$ be the solution of* (4.1.52). *Then there holds*

$$\sup_{\frac{1}{32}B_r \cap \mathbb{R}^n_+} |\nabla\tilde{v}|^p \omega_{B_r}^p \le c \fint_{\frac{1}{4}B_r \cap \mathbb{R}^n_+} |\nabla\tilde{v}|^p \omega_{B_r}^p \, dy. \tag{4.1.89}$$

Moreover, there exist $\alpha = \alpha(n, p, \Lambda) \in (0, 1)$ and $c = c(n, p, \Lambda) > 0$ such that

$$\begin{aligned} &\fint_{\lambda B_r \cap \mathbb{R}^n_+} |\mathcal{V}_{B_r}(\nabla\tilde{v}) - (\mathcal{V}_{B_r}(\nabla\tilde{v}))_{\lambda B_r \cap \mathbb{R}^n_+}|^2 \, dy \\ &\qquad \le c\lambda^{2\alpha} \fint_{\frac{1}{8}B_r \cap \mathbb{R}^n_+} |\mathcal{V}_{B_r}(\nabla\tilde{v}) - (\mathcal{V}_{B_r}(\nabla\tilde{v}))_{\frac{1}{8}B_r \cap \mathbb{R}^n_+}|^2 \, dy \end{aligned} \tag{4.1.90}$$

holds for all $\lambda \in (0, \frac{1}{80})$.

Proof. We first show (4.1.90). Throughout the proof of (4.1.90), let us write the center of the ball B_r as y_{B_r}.
Step 1. If $\frac{1}{8}B_r \subset \{y \in \mathbb{R}^n : y_n \geq 0\}$, then it directly follows from [16, Proposition 15].
Step 2. Now, we consider $z_{B_r} \in \{y \in \mathbb{R}^n : y_n = 0\}$. We have by (4.1.49) that

$$\Lambda^{-1}\omega_{B_r} \,\mathrm{id} \leq \mathbb{M}_{B_r} \leq \omega_{B_r} \,\mathrm{id}\,.$$

Since the equation (4.1.74) and estimate (4.1.90) are invariant under normalization, without loss of generality we let $\omega_{B_r} = 1$. Also, assume that B_r is centered at 0, i.e., $z_{B_r} = 0$.

Since $\mathbb{M}_{B_r}$ is symmetric, there is an orthogonal matrix Q and a diagonal matrix $\mathbb{D}_{B_r}$ such that $\mathbb{M}_{B_r} = Q\mathbb{D}_{B_r}Q^*$. Then $\tilde{w}_0(y) := \tilde{v}(Qy)$ is a solution of (4.1.74) with $\mathbb{D}_{B_r}$ instead of $\mathbb{M}_{B_r}$. Notice that the boundary of the domain is also rotated, and for $\mathbb{D}_{B_r}$ we have

$$\Lambda^{-1}\,\mathrm{id} \leq \mathbb{D}_{B_r} \leq \mathrm{id}\,. \tag{4.1.91}$$

Now, we apply an anisotropic scaling $y \mapsto \mathbb{D}_{B_r}^{-1}y$. This turns estimates on half balls (with the rotated flat part) into estimates on half-ellipses (with the rotated flat part) of uniformly bounded eccentricity depending on n and Λ. Thus, after properly rotating the coordinate axis to make the rotated flat part to the subset of $\{y \in \mathbb{R}^n : y_n = 0\}$, we can take the odd extension to (4.1.74) to obtain (4.1.90).

In detail, let $\tilde{Q}$ be an $n \times n$ orthogonal matrix which maps $\mathbb{D}_{B_r}^{-1}Q^*(\{y \in \mathbb{R}^n : y_n = 0\})$ to $\{y \in \mathbb{R}^n : y_n = 0\}$. In other words, $\tilde{Q}$ satisfies

$$\tilde{Q}(\mathbb{D}_{B_r}^{-1}Q^*(\{y \in \mathbb{R}^n : y_n = 0\})) = \{y \in \mathbb{R}^n : y_n = 0\}. \tag{4.1.92}$$

Now, using this $n \times n$ orthogonal matrix $\tilde{Q}$, define

$$\tilde{w}(y) := \tilde{v}(\tilde{Q}\mathbb{D}_{B_r}Q^*y). \tag{4.1.93}$$

Then we have

$$\begin{aligned}(Q\mathbb{D}_{B_r}^{-1}(\tilde{Q})^*)^*\mathbb{M}_{B_r}^2(Q\mathbb{D}_{B_r}^{-1}(\tilde{Q})^*) &= \tilde{Q}\mathbb{D}_{B_r}^{-1}Q^*\mathbb{M}_{B_r}^2 Q\mathbb{D}_{B_r}^{-1}(\tilde{Q})^* \\ &= \tilde{Q}\mathbb{D}_{B_r}^{-1}\mathbb{D}_{B_r}^2\mathbb{D}_{B_r}^{-1}(\tilde{Q})^* = \mathrm{id}\end{aligned} \tag{4.1.94}$$

and for $t = Q\mathbb{D}_{B_r}^{-1}(\tilde{Q})^* y$, we have

$$\tilde{w}(Q\mathbb{D}_{B_r}^{-1}(\tilde{Q})^* y) = \tilde{v}(y) = \tilde{w}(t) \tag{4.1.95}$$

and so

$$\nabla_y \tilde{v}(y) = \nabla_y \tilde{w}(t) = (Q\mathbb{D}_{B_r}^{-1}(\tilde{Q})^*)\nabla_t \tilde{w}(t). \tag{4.1.96}$$

Therefore, (4.1.74) defined in $\Psi(\frac{1}{2}\Omega_r) \subset B_r$ transforms into

$$\begin{aligned}
&\operatorname{div}_y(|\mathbb{M}_{B_r}\nabla\tilde{v}|^{p-2}\mathbb{M}_{B_r}^2\nabla\tilde{v}) = \operatorname{div}_y(\langle \mathbb{M}_{B_r}\nabla\tilde{v}, \mathbb{M}_{B_r}\nabla\tilde{v}\rangle^{\frac{p-2}{2}}\mathbb{M}_{B_r}^2\nabla\tilde{v}) \\
&= \operatorname{div}_t\left((Q\mathbb{D}_{B_r}^{-1}(\tilde{Q})^*)^*\left\langle \mathbb{M}_{B_r}Q\mathbb{D}_{B_r}^{-1}(\tilde{Q})^*\nabla_t\tilde{w}, \mathbb{M}_{B_r}Q\mathbb{D}_{B_r}^{-1}(\tilde{Q})^*\nabla_t\tilde{w}\right\rangle^{\frac{p-2}{2}}\right. \\
&\qquad\qquad \left. \cdot\mathbb{M}_{B_r}^2 Q\mathbb{D}_{B_r}^{-1}(\tilde{Q})^*\nabla_t\tilde{w}\right) \\
&= \operatorname{div}_t\left(\left\langle (\mathbb{M}_{B_r}Q\mathbb{D}_{B_r}^{-1}(\tilde{Q})^*)^*\mathbb{M}_{B_r}Q\mathbb{D}_{B_r}^{-1}(\tilde{Q})^*\nabla_t\tilde{w}, \nabla_t\tilde{w}\right\rangle^{\frac{p-2}{2}}\nabla_t\tilde{w}\right) \\
&= \operatorname{div}_t(|\nabla_t\tilde{w}|^{p-2}\nabla_t\tilde{w}) = 0
\end{aligned}$$

defined in $t \in Q\mathbb{D}_{B_r}^{-1}(\tilde{Q})^*(\{y \in \mathbb{R}^n : y_n = 0\})$. Note that since $\Psi(\frac{1}{2}\Omega_r) \subset \{y \in \mathbb{R} : y_n = 0\}$ and (4.1.92) hold, we can employ [174] and apply the odd extension for $\operatorname{div}_t(|\nabla_t\tilde{w}|^{p-2}\nabla_t\tilde{w}) = 0$ and get the analogous estimate to (4.1.90) for w with half-ellipses instead of half-balls. Using the relation (4.1.94) and $\mathbb{M}_{B_r} = Q\mathbb{D}_{B_r}Q^*$ for changing w to v, and then using the fact that all balls can be covered by slightly enlarged ellipses and vice versa, the estimate (4.1.90) is also true for half balls. Then we have (4.1.90).

Step 3. Now, we consider the general case, i.e., $\frac{1}{4}B_r \not\subset \{y \in \mathbb{R}^n : y_n \geq 0\}$ and $z = z_{B_r} \notin \{y \in \mathbb{R}^n : y_n = 0\}$ holds. We employ the argument of [152, Lemma 3.7]. Denote $z = (z_1, \ldots, z_{n-1}, z_n)$, $\bar{z} = (z_1, \ldots, z_{n-1}, 0)$, and recall that $0 < \lambda < \frac{1}{80}$ and $z_n > 0$ since $\tilde{x} \in \Omega$, where $\tilde{x}$ is the center of the ball B_r with $4B_r \subset 2B_0$. Let us specify the exact center of the balls in this step. In particular, we write $B_r = B_r(z)$.

Case 1. $z_n > \frac{r}{10}$. In this case we have

$$\lambda B_r(z) \subset \tfrac{1}{10}B_r(z) \subset \tfrac{z_n}{r}B_r(z) \subset \mathbb{R}^n_+. \tag{4.1.97}$$

By the interior estimates in [100, Theorem 6.4], we obtain

$$
\begin{aligned}
&\fint_{\lambda B_r(z)} |\mathcal{V}_{B_r}(\nabla \tilde{v}) - (\mathcal{V}_{B_r}(\nabla \tilde{v}))_{\lambda B_r(z)}|^2 \, dy \\
&\lesssim \left(\frac{\lambda}{1/10}\right)^{2\alpha} \fint_{\frac{1}{10} B_r(z)} |\mathcal{V}_{B_r}(\nabla \tilde{v}) - (\mathcal{V}_{B_r}(\nabla \tilde{v}))_{\frac{1}{10} B_r(z)}|^2 \, dy \\
&\lesssim \left(\frac{\lambda}{1/10}\right)^{2\alpha} \left(\frac{1}{|\frac{1}{10} B_r(z)|} \int_{\frac{1}{4} B_r(z) \cap \mathbb{R}^n_+} |\mathcal{V}_{B_r}(\nabla \tilde{v}) - (\mathcal{V}_{B_r}(\nabla \tilde{v}))_{\frac{1}{4} B_r(z) \cap \mathbb{R}^n_+}|^2 \, dy \right) \\
&\lesssim \lambda^{2\alpha} \fint_{\frac{1}{4} B_r(z) \cap \mathbb{R}^n_+} |\mathcal{V}_{B_r}(\nabla \tilde{v}) - (\mathcal{V}_{B_r}(\nabla \tilde{v}))_{\frac{1}{4} B_r(z) \cap \mathbb{R}^n_+}|^2 \, dy.
\end{aligned}
\tag{4.1.98}
$$

Thus we obtain (4.1.90) in this case.

Case 2. $0 < z_n \leq \frac{r}{10}$. We divide the proof into two subcases.

Subcase 1. $0 < \lambda < \frac{z_n}{4r}$. In this subcase

$$
\lambda B_r(z) \subset \tfrac{z_n}{4r} B_r(z) \subset \tfrac{5 z_n}{4r} B_r^+(\bar{z}). \tag{4.1.99}
$$

By the interior estimates in [100, Theorem 6.4], we have

$$
\begin{aligned}
&\fint_{\lambda B_r(z)} |\mathcal{V}_{B_r}(\nabla \tilde{v}) - (\mathcal{V}_{B_r}(\nabla \tilde{v}))_{\lambda B_r(z)}|^2 \, dy \\
&\qquad \lesssim \left(\frac{4\lambda}{z_n/4r}\right)^{2\alpha} \fint_{\frac{z_n}{4r} B_r(z)} |\mathcal{V}_{B_r}(\nabla \tilde{v}) - (\mathcal{V}_{B_r}(\nabla \tilde{v}))_{\frac{z_n}{4r} B_r(z)}|^2 \, dy \\
&\qquad \lesssim \left(\frac{4\lambda}{z_n/4r}\right)^{2\alpha} \fint_{\frac{5 z_n}{4r} B_r^+(\bar{z})} |\mathcal{V}_{B_r}(\nabla \tilde{v}) - (\mathcal{V}_{B_r}(\nabla \tilde{v}))_{\frac{5 z_n}{4r} B_r^+(\bar{z})}|^2 \, dy.
\end{aligned}
\tag{4.1.100}
$$

Since $\bar{z} \in \{y \in \mathbb{R}^n : z_n = 0\}$, by **Step 2** above and then using $0 < z_n \leq \frac{r}{10}$, we have

$$\begin{aligned}
&\fint_{\frac{5z_n}{4r}B_r^+(\bar{z})} |\mathcal{V}_{B_r}(\nabla\tilde{v}) - (\mathcal{V}_{B_r}(\nabla\tilde{v}))_{\frac{5z_n}{4r}B_r^+(\bar{z})}|^2\,dy \\
&\lesssim \left(\frac{5z_n}{r}\right)^{2\alpha} \fint_{\frac{1}{8}B_r^+(\bar{z})} |\mathcal{V}_{B_r}(\nabla\tilde{v}) - (\mathcal{V}_{B_r}(\nabla\tilde{v}))_{\frac{1}{8}B_r^+(\bar{z})}|^2\,dy \\
&\lesssim \left(\frac{5z_n}{r}\right)^{2\alpha} \fint_{\frac{1}{4}B_r(z)\cap\mathbb{R}^n_+} |\mathcal{V}_{B_r}(\nabla\tilde{v}) - (\mathcal{V}_{B_r}(\nabla\tilde{v}))_{\frac{1}{4}B_r(z)\cap\mathbb{R}^n_+}|^2\,dy.
\end{aligned} \tag{4.1.101}$$

Combining the above two estimates, (4.1.100) and (4.1.101), we obtain (4.1.90) in this subcase.

Subcase 2. $\lambda \geq \frac{z_n}{4r}$. Since $\lambda \leq \frac{1}{40}$, we see that

$$\lambda B_r(z) \cap \mathbb{R}^n_+ \subset 5\lambda B_r^+(\bar{z}) \subset \tfrac{1}{8}B_r^+(\bar{z}) \subset \tfrac{1}{4}B_r(z)\cap\mathbb{R}^n_+. \tag{4.1.102}$$

Therefore, using the boundary estimate above in **Step 2**, we have

$$\begin{aligned}
&\fint_{\lambda B_r(z)\cap\mathbb{R}^n_+} |\mathcal{V}_{B_r}(\nabla\tilde{v}) - (\mathcal{V}_{B_r}(\nabla\tilde{v}))_{\lambda B_r(z)\cap\mathbb{R}^n_+}|^2\,dy \\
&\qquad \lesssim \fint_{20\lambda B_r^+(\bar{z})} |\mathcal{V}_{B_r}(\nabla\tilde{v}) - (\mathcal{V}_{B_r}(\nabla\tilde{v}))_{20\lambda B_r^+(\bar{z})}|^2\,dy \\
&\qquad \lesssim \left(\frac{5\lambda}{1/8}\right)^{2\alpha} \fint_{\frac{1}{8}B_r^+(\bar{z})} |\mathcal{V}_{B_r}(\nabla\tilde{v}) - (\mathcal{V}_{B_r}(\nabla\tilde{v}))_{\frac{1}{8}B_r^+(\bar{z})}|^2\,dy \\
&\qquad \lesssim \lambda^{2\alpha} \fint_{\frac{1}{4}B_r(z)\cap\mathbb{R}^n_+} |\mathcal{V}_{B_r}(\nabla\tilde{v}) - (\mathcal{V}_{B_r}(\nabla\tilde{v}))_{\frac{1}{4}B_r(z)\cap\mathbb{R}^n_+}|^2\,dy.
\end{aligned} \tag{4.1.103}$$

Merging all cases *Case 1*–*Case 2*, we have (4.1.90) in **Step 3**. Therefore, by **Step 1**–**Step 3**, we have (4.1.90).

To show (4.1.89), we employ the similar argument as above. If $\frac{1}{4}B_r \subset \{y \in \mathbb{R}^n : y_n \geq 0\}$, then it follows from [16, Proposition 15]. Now, when $z_{B_r} \in \{y \in \mathbb{R}^n : y_n = 0\}$, by employing the same matrix $\tilde{Q}$, $\mathbb{D}_{B_r}$, Q^* as above, we can apply [167, Lemma 5] and we have (4.1.89) in this case. In the general case, i.e., when $\frac{1}{4}B_r \not\subset \{y \in \mathbb{R}^n : y_n \geq 0\}$ and $z_{B_r} \notin \{y \in \mathbb{R}^n : y_n = 0\}$ holds, we divide the cases as same as above and apply the argument of [169] instead of [100]. Now, (4.1.89) is obtained. □

Now, we transform the above estimates for $\tilde{v}$ to the estimates for v.

Proposition 4.1.22. *Assuming* (4.1.15), *let* v *be the solution of* (4.1.78). *Then there holds*

$$\sup_{\frac{1}{32}\Omega_r} |\nabla v|^p \omega_{B_r}^p \le c \fint_{\frac{1}{4}\Omega_r} |\nabla v|^p \omega_{B_r}^p \, dx. \tag{4.1.104}$$

Moreover, there exist $\alpha = \alpha(n, p, \Lambda) \in (0, 1)$ *and* $c = c(n, p, \Lambda) > 0$ *such that*

$$\begin{aligned} &\fint_{\lambda\Omega_r} |\mathcal{V}_{B_r}(\nabla v) - (\mathcal{V}_{B_r}(\nabla v))_{\lambda\Omega_r}|^2 \, dx \\ &\quad \le c\lambda^{2\alpha} \fint_{\frac{1}{4}\Omega_r} |\mathcal{V}_{B_r}(\nabla v) - (\mathcal{V}_{B_r}(\nabla v))_{\frac{1}{4}\Omega_r}|^2 \, dx \\ &\qquad + c\|\nabla \psi\|_\infty^2 \lambda^{-n} \fint_{\frac{1}{4}\Omega_r} |\nabla v|^p \omega_{B_r}^p \, dx \end{aligned} \tag{4.1.105}$$

holds for all $\lambda \in (0, \frac{1}{80})$.

Proof. To obtain (4.1.105), using (4.1.16) with the help of (4.1.15), there holds

$$\begin{aligned} &\fint_{\lambda\Omega_r} |\mathcal{V}_{B_r}(\nabla v) - (\mathcal{V}_{B_r}(\nabla v))_{\lambda\Omega_r}|^2 \, dx \\ &\quad = \fint_{\Psi(\lambda\Omega_r)} |\mathcal{V}_{B_r}((\nabla\Psi)\nabla\tilde{v}) - (\mathcal{V}_{B_r}((\nabla\Psi)\nabla\tilde{v}))_{\Psi(\lambda\Omega_r)}|^2 \, dy \\ &\quad \lesssim \fint_{\Psi(4\lambda\Omega_r)} |\mathcal{V}_{B_r}((\nabla\Psi)\nabla\tilde{v}) - (\mathcal{V}_{B_r}(\nabla\tilde{v}))_{\Psi(4\lambda\Omega_r)}|^2 \, dy \\ &\quad \lesssim \fint_{\Psi(4\lambda\Omega_r)} |\mathcal{V}_{B_r}((\nabla\Psi)\nabla\tilde{v}) - \mathcal{V}_{B_r}(\nabla\tilde{v})|^2 \, dy \\ &\qquad + \fint_{\Psi(4\lambda\Omega_r)} |\mathcal{V}_{B_r}(\nabla\tilde{v}) - (\mathcal{V}_{B_r}(\nabla\tilde{v}))_{\Psi(4\lambda\Omega_r)}|^2 \, dy =: I_1 + I_2. \end{aligned} \tag{4.1.106}$$

For I_1, by Lemma 4.1.11 and Lemma 4.1.19, we obtain

$$
\begin{aligned}
I_1 &\lesssim \fint_{\Psi(4\lambda\Omega_r)} |\mathcal{A}((\nabla\Psi)\nabla\tilde{v}) - \mathcal{A}_{B_r}(\nabla\tilde{v})| \cdot |(\nabla\Psi)\nabla\tilde{v} - \nabla\tilde{v}|\, dy \\
&\lesssim \fint_{\Psi(4\lambda\Omega_r)} \omega_{B_r}^p \|\nabla\psi\|_\infty |(\nabla\Psi)\nabla\tilde{v}|^{p-1} \cdot \|\nabla\psi\|_\infty |\nabla\tilde{v}|\, dy \\
&\lesssim \|\nabla\psi\|_\infty^2 \fint_{\Psi(4\lambda\Omega_r)} \omega_{B_r}^p |(\nabla\Psi)\nabla\tilde{v}|^p |(\nabla\Psi)^{-1}|\, dy \\
&\lesssim \|\nabla\psi\|_\infty^2 \fint_{4\lambda\Omega_r} |\nabla v|^p \omega_{B_r}^p\, dx \\
&\lesssim \|\nabla\psi\|_\infty^2 \lambda^{-n} \fint_{\frac{1}{4}\Omega_r} |\nabla v|^p \omega_{B_r}^p\, dx.
\end{aligned}
\tag{4.1.107}
$$

On the other hand, for I_2, we apply (4.1.16) from (4.1.15), and (4.1.90) to have

$$
\begin{aligned}
I_2 &\lesssim \fint_{8\lambda B_r\cap\mathbb{R}^n_+} |\mathcal{V}_{B_r}(\nabla\tilde{v}) - (\mathcal{V}_{B_r}(\nabla\tilde{v}))_{8\lambda B_r\cap\mathbb{R}^n_+}|^2\, dy \\
&\lesssim \lambda^{2\alpha} \fint_{\frac{1}{8}B_r\cap\mathbb{R}^n_+} |\mathcal{V}_{B_r}(\nabla\tilde{v}) - (\mathcal{V}_{B_r}(\nabla\tilde{v}))_{\frac{1}{8}B_r\cap\mathbb{R}^n_+}|^2\, dy \\
&\lesssim \lambda^{2\alpha} \fint_{\Psi^{-1}(\frac{1}{8}B_r\cap\mathbb{R}^n_+)} |\mathcal{V}_{B_r}((\nabla\Psi)^{-1}\nabla v) - (\mathcal{V}_{B_r}(\nabla v))_{\frac{1}{4}\Omega_r}|^2\, dx \\
&\lesssim \lambda^{2\alpha} \fint_{\frac{1}{4}\Omega_r} |\mathcal{V}_{B_r}((\nabla\Psi)^{-1}\nabla v) - \mathcal{V}_{B_r}(\nabla v)|^2\, dx \\
&\quad + \lambda^{2\alpha} \fint_{\frac{1}{4}\Omega_r} |\mathcal{V}_{B_r}(\nabla v) - (\mathcal{V}_{B_r}(\nabla v))_{\frac{1}{4}\Omega_r}|^2\, dx \\
&=: \lambda^{2\alpha}(I_{2,1} + I_{2,2}).
\end{aligned}
\tag{4.1.108}
$$

To obtain (4.1.105), we only have to estimate $I_{2,1}$. By the similar argument as (4.1.107), we have

$$\begin{aligned}
I_{2,1} &\lesssim \fint_{\frac{1}{4}\Omega_r} |\mathcal{A}((\nabla\Psi)^{-1}\nabla v) - \mathcal{A}_{B_r}(\nabla v)| \cdot \|\nabla\psi\|_\infty |\nabla v|\, dx \\
&\lesssim \|\nabla\psi\|_\infty^2 \fint_{\frac{1}{4}\Omega_r} \omega_{B_r}^p |(\nabla\Psi)^{-1}\nabla v|^p |\nabla\Psi|\, dx \\
&\lesssim \|\nabla\psi\|_\infty^2 \fint_{\frac{1}{4}\Omega_r} |\nabla v|^p \omega_{B_r}^p\, dx.
\end{aligned} \tag{4.1.109}$$

Summing up all estimates (4.1.106)–(4.1.109), we conclude (4.1.105).

To show (4.1.104), by using (4.1.80) and (4.1.16), there holds

$$\begin{aligned}
\sup_{x\in\frac{1}{32}\Omega_r} |\nabla v(x)|^p \omega_{B_r}^p &\le \sup_{y\in\Psi(\frac{1}{32}\Omega_r)} |(\nabla\Psi)\nabla\tilde{v}(y)|^p \omega_{B_r}^p \\
&\lesssim \sup_{y\in\frac{1}{16}B_r\cap\mathbb{R}^n_+} |\nabla\tilde{v}(y)|^p \omega_{B_r}^p
\end{aligned} \tag{4.1.110}$$

and by (4.1.89), we have

$$\begin{aligned}
\sup_{y\in\frac{1}{16}B_r\cap\mathbb{R}^n_+} |\nabla\tilde{v}(y)|^p \omega_{B_r}^p &\lesssim \fint_{\frac{1}{8}B_r\cap\mathbb{R}^n_+} |\nabla\tilde{v}|^p \omega_{B_r}^p\, dy \\
&\lesssim \fint_{\Psi^{-1}(\frac{1}{8}B_r\cap\mathbb{R}^n_+)} |(\nabla\Psi)^{-1}\nabla v|^p \omega_{B_r}^p\, dx \\
&\lesssim \fint_{\frac{1}{4}\Omega_r} |\nabla v|^p \omega_{B_r}^p\, dx.
\end{aligned} \tag{4.1.111}$$

Summing up the above two inequalities, we have (4.1.104). □

With the help of the above estimates for v, we give the following decay estimate of $\mathcal{V}(\cdot,\nabla z)$. Recall that for $B_0 = B_R(x_0)$ with $x_0 \in \partial\Omega$, let $B_r = B_r(\tilde{x})$ with $\tilde{x} \in \overline{\Omega}$ and $4B_r \subset 2B_0$. Also, z on $B_0 \cap \Omega$ is such that $z := u\eta^{p'}$ and we take the zero extension for z on $\mathbb{R}^n \setminus (B_0 \cap \Omega)$, if necessary.

Proposition 4.1.23 (Decay estimate at boundary)**.** *Let z, u and F be as in* (4.1.38) *and* (4.1.42). *There exist $\lambda = \lambda(n,p,\Lambda) \in (0, \frac{1}{80})$, $s = s(n,p,\Lambda) > 1$ and $\kappa_5 = \kappa_5(n,p,\Lambda) \in (0,1)$ such that the following holds: If $|\log \mathbb{M}|_{\mathrm{BMO}(4B_r)} \le$*

κ_5, and (4.1.15) *hold, then for every $\epsilon \in (0,1)$ there holds*

$$
\begin{aligned}
&\fint_{\lambda B_r} |\mathcal{V}(x,\nabla z) - (\mathcal{V}(\cdot,\nabla z))_{\lambda B_r}|^2 \, dx \\
&\qquad \leq \frac{1}{4}\fint_{B_r} |\mathcal{V}(x,\nabla z) - (\mathcal{V}(\cdot,\nabla z))_{B_r}|^2 \, dx \\
&\qquad\quad + c \left(|\log \mathbb{M}|^2_{\mathrm{BMO}(B_r)} + \|\nabla\psi\|^2_\infty + \epsilon \right) \left(\fint_{B_r} |\mathcal{V}(x,\nabla z)|^{2s} \, dx \right)^{\frac{1}{s}} \\
&\qquad\quad + c\, C^*(\epsilon) \left(\fint_{2B_r} \left(\frac{\chi_{2B_0\cap\Omega}|u|^p}{R^p} \omega^p \right)^s dx \right)^{\frac{1}{s}} \\
&\qquad\quad + c\, C^*(\epsilon) \left(\fint_{2B_r} |\chi_{2B_0\cap\Omega} \mathcal{V}(x,F)|^{2s} \, dx \right)^{\frac{1}{s}}
\end{aligned}
\tag{4.1.112}
$$

with $c = c(n,p,\Lambda)$ and $C^*(\epsilon)$ defined in (4.1.67).

Proof. We first assume $|\log \mathbb{M}|_{\mathrm{BMO}(4B_r)} \leq \kappa_4$ with κ_4 from Proposition 4.1.20. Since $z = |\nabla z| = 0$ in $\lambda B_r \setminus \Omega$ and $|\lambda\Omega_r| \eqsim |\lambda B_r|$ hold from (4.1.17) because of (4.1.15) and $\tilde{x} \in \overline{\Omega}$, we have

$$
\begin{aligned}
I_1 &:= \fint_{\lambda B_r} |\mathcal{V}(x,\nabla z) - (\mathcal{V}(\cdot,\nabla z))_{\lambda B_r}|^2 \, dx \\
&\lesssim \fint_{\lambda B_r} |\mathcal{V}(x,\nabla z) - (\mathcal{V}(\cdot,\nabla v))_{\lambda \Omega_r}|^2 \, dx \\
&\lesssim \fint_{\lambda \Omega_r} |\mathcal{V}(x,\nabla z) - (\mathcal{V}(\cdot,\nabla v))_{\lambda \Omega_r}|^2 \, dx \\
&\lesssim \fint_{\lambda \Omega_r} |\mathcal{V}(x,\nabla v) - (\mathcal{V}(\cdot,\nabla v))_{\lambda \Omega_r}|^2 \, dx \\
&\qquad + \fint_{\lambda \Omega_r} |\mathcal{V}(x,\nabla z) - \mathcal{V}(x,\nabla v)|^2 \, dx \\
&=: I_2 + I_3.
\end{aligned}
\tag{4.1.113}
$$

We start to estimate I_3. There holds

$$\begin{aligned} I_3 &\lesssim \lambda^{-n} \fint_{\frac{1}{2}\Omega_r} |\mathcal{V}(x, \nabla z) - \mathcal{V}(x, \nabla v)|^2 \, dx \\ &\lesssim \lambda^{-n} \fint_{\frac{1}{2}\Omega_r} |\mathcal{V}_{B_r}(\nabla z) - \mathcal{V}_{B_r}(\nabla v)|^2 \, dx \\ &\quad + \lambda^{-n} \fint_{\frac{1}{2}\Omega_r} |\mathcal{V}(x, \nabla z) - \mathcal{V}_{B_r}(\nabla z)|^2 \, dx \\ &\quad + \lambda^{-n} \fint_{\frac{1}{32}\Omega_r} |\mathcal{V}(x, \nabla v) - \mathcal{V}_{B_r}(\nabla v)|^2 \, dx \\ &=: I_{3,1} + I_{3,2} + I_{3,3}. \end{aligned} \tag{4.1.114}$$

For $I_{3,2}$ and $I_{3,3}$, since $\tilde{x} \in \overline{\Omega}$ and (4.1.15) holds, $|\Omega_r| \eqsim |B_r|$ also holds. Then we can apply the similar argument of the proof as in Proposition 18 in [16]. By (4.1.34), (4.1.48), (4.1.49), Lemma 4.1.11, (4.1.30), and Hölder's inequality with exponents $(2s', s, 2s')$, we have

$$\begin{aligned} I_{3,2} \lesssim \lambda^{-n} &\left(\fint_{\Omega_r} \left(\frac{|\mathbb{M}_{B_r} - \mathbb{M}|}{|\mathbb{M}_{B_r}|} \right)^{4s'} dx \right)^{\frac{1}{2s'}} \left(\fint_{\Omega_r} (|\nabla z|^p \omega^p)^s \, dx \right)^{\frac{1}{s}} \\ &\times \left(1 + \left(\fint_{\Omega_r} \left(\frac{\omega_{B_r}^p}{\omega^p} \right)^{2s'} dx \right)^{\frac{1}{2s'}} \right). \end{aligned} \tag{4.1.115}$$

Together with $z = |\nabla z| = 0$ in $B_r \setminus \Omega$ and $|\Omega_r| \eqsim |B_r|$ due to (4.1.15), we obtain

$$I_{3,2} \lesssim \lambda^{-n} \left(\fint_{\Omega_r} (|\nabla z|^p \omega^p)^s \, dx \right)^{\frac{1}{s}}. \tag{4.1.116}$$

Note that by $4B_r \subset 2B_0$, $\chi_{2B_0 \cap \Omega}$ can be inserted in the integrand of the above estimate. On the other hand, similar to $I_{3,2}$, we continue to estimate

$I_{3,3}$ as follows:

$$\begin{aligned}
I_{3,3} &= \lambda^{-n} \fint_{\frac{1}{32}\Omega_r} |V(\mathbb{M}\nabla v) - V(\mathbb{M}_{B_r}\nabla v)|^2 \, dx \\
&\lesssim \lambda^{-n} \fint_{\frac{1}{32}\Omega_r} \left(\frac{|\mathbb{M} - \mathbb{M}_{B_r}|}{|\mathbb{M}_{B_r}|} \right)^2 \left(|\nabla v|^p \omega^p + |\nabla v|^p \omega_{B_r}^p \right) \, dx \\
&\lesssim \lambda^{-n} \left(\sup_{\frac{1}{32}\Omega_r} |\nabla v|^p \omega_{B_r}^p \right) \left(\fint_{\frac{1}{2}\Omega_r} \left(\frac{|\mathbb{M} - \mathbb{M}_{B_r}|}{|\mathbb{M}_{B_r}|} \right)^2 \left(\frac{\omega^p}{\omega_{B_r}^p} + 1 \right) dx \right) \\
&= \lambda^{-n} I_{3,3,1} I_{3,3,2}.
\end{aligned} \tag{4.1.117}$$

Here, (4.1.104) in Proposition 4.1.22, (4.1.80) and (4.1.48) imply

$$\begin{aligned}
I_{3,3,1} &:= \sup_{\frac{1}{32}\Omega_r} |\nabla v|^p \omega_{B_r}^p \\
&\lesssim \fint_{\frac{1}{4}\Omega_r} |\nabla v|^p \omega_{B_r}^p \, dx \\
&\lesssim \fint_{\frac{1}{4}\Omega_r} |\mathbb{M}_{B_r} T(x) \nabla v|^p |T^{-1}(x)|^p \, dx \lesssim \fint_{\frac{1}{2}\Omega_r} |\mathbb{M}_{B_r} T(x) \nabla v|^p \, dx
\end{aligned} \tag{4.1.118}$$

and the minimizing property of v and h, together with (4.1.80) and (4.1.48) give us that

$$\begin{aligned}
I_{3,3,1} &\lesssim \fint_{\frac{1}{2}\Omega_r} |\mathbb{M}_{B_r} T(x) \nabla h|^p \, dx \\
&\lesssim \fint_{\Omega_r} |\mathbb{M}_{B_r} \nabla h|^p \, dx \lesssim \fint_{\Omega_r} |\nabla z|^p \omega_{B_r}^p \, dx.
\end{aligned} \tag{4.1.119}$$

Then we use Hölder's inequality, $|\Omega_r| \eqsim |B_r|$ from (4.1.15), Lemma 4.1.9 and $z = |\nabla z| = 0$ in $B_r \setminus \Omega$ to obtain

$$\begin{aligned}
I_{3,3,1} &\lesssim \left(\fint_{\Omega_r} (|\nabla z|^p \omega^p)^s \, dx \right)^{\frac{1}{s}} \left(\fint_{\Omega_r} \left(\frac{\omega_{B_r}^p}{\omega^p} \right)^{\frac{s}{s-1}} \, dx \right)^{1-\frac{1}{s}} \\
&\lesssim \left(\fint_{B_r} (|\nabla z|^p \omega^p)^s \, dx \right)^{\frac{1}{s}},
\end{aligned} \tag{4.1.120}$$

provided $|\log \mathbb{M}|_{\mathrm{BMO}(4B_r)} \leq \kappa_4$ holds. On the other hand, by Hölder's inequality, Lemma 4.1.8 and Lemma 4.1.9 we obtain

$$\begin{aligned} I_{3,3,2} &\lesssim \left(\fint_{\frac{1}{2}\Omega_r} \left(\frac{|\mathbb{M} - \mathbb{M}_{B_r}|}{|\mathbb{M}_{B_r}|} \right)^4 dx \right)^{\frac{1}{2}} \left(\left(\fint_{\frac{1}{2}\Omega_r} \left(\frac{\omega^p}{\omega^p_{B_r}} \right)^2 dx \right)^{\frac{1}{2}} + 1 \right) \\ &\lesssim |\log \mathbb{M}|^2_{\mathrm{BMO}(B_r)}. \end{aligned} \tag{4.1.121}$$

Summing up, there holds

$$I_{3,3} \lesssim \lambda^{-n} |\log \mathbb{M}|^2_{\mathrm{BMO}(B_r)} \left(\fint_{B_r} |\mathcal{V}(x, \nabla z)|^{2s} \, dx \right)^{\frac{1}{s}}. \tag{4.1.122}$$

For $I_{3,1}$, we first apply Proposition 4.1.18 and Proposition 4.1.20, use $z = |\nabla z| = 0$ in $2B_r \setminus \Omega$ and $|2\Omega_r| \eqsim |2B_r|$ with the help of (4.1.15), and argue similarly to (4.1.119)–(4.1.120) for the integral of $|\nabla h|^p \omega^p_{B_r}$ term. The resulting estimate is as follows:

$$\begin{aligned} I_{3,1} &\lesssim \lambda^{-n} \fint_{\frac{1}{2}\Omega_r} |\mathcal{V}_{B_r}(\nabla z) - \mathcal{V}_{B_r}(\nabla h)|^2 \, dx + \lambda^{-n} \fint_{\frac{1}{2}\Omega_r} |\mathcal{V}_{B_r}(\nabla h) - \mathcal{V}_{B_r}(\nabla v)|^2 \, dx \\ &\lesssim \lambda^{-n} (|\log \mathbb{M}|^2_{\mathrm{BMO}(B_r)} + \epsilon) \fint_{\Omega_r} (|\nabla z|^p \omega^p) \, dx + \lambda^{-n} \|\nabla \psi\|^2_\infty \fint_{\Omega_r} (|\nabla h|^p \omega^p_{B_r}) \, dx \\ &\quad + \lambda^{-n} C^*(\epsilon) \left(\fint_{2\Omega_r} \left(\frac{|u|^p}{R^p} \omega^p \right)^s dx \right)^{\frac{1}{s}} + \lambda^{-n} C^*(\epsilon) \left(\fint_{2\Omega_r} |\mathcal{V}(x, F)|^{2s} \, dx \right)^{\frac{1}{s}} \\ &\lesssim \lambda^{-n} (|\log \mathbb{M}|^2_{\mathrm{BMO}(B_r)} + \|\nabla \psi\|^2_\infty + \epsilon) \left(\fint_{B_r} |\mathcal{V}(x, \nabla z)|^{2s} \, dx \right)^{\frac{1}{s}} \\ &\quad + \lambda^{-n} C^*(\epsilon) \left(\fint_{2B_r} \left(\frac{\chi_{2B_0 \cap \Omega} |u|^p}{R^p} \omega^p \right)^s dx \right)^{\frac{1}{s}} \\ &\quad + \lambda^{-n} C^*(\epsilon) \left(\fint_{2B_r} |\chi_{2B_0 \cap \Omega} \mathcal{V}(x, F)|^{2s} \, dx \right)^{\frac{1}{s}}. \end{aligned}$$

Consequently, we have

$$
\begin{aligned}
I_3 &\lesssim \lambda^{-n}\left(|\log \mathbb{M}|^2_{\mathrm{BMO}(4B_r)} + \|\nabla\psi\|^2_\infty + \epsilon\right)\left(\fint_{B_r} |\mathcal{V}(x,\nabla z)|^{2s}\,dx\right)^{\frac{1}{s}} \\
&\quad + \lambda^{-n}C^*(\epsilon)\left(\fint_{2B_r}\left(\frac{\chi_{2B_0\cap\Omega}|u|^p}{R^p}\omega^p\right)^s dx\right)^{\frac{1}{s}} \\
&\quad + \lambda^{-n}C^*(\epsilon)\left(\fint_{2B_r} |\chi_{2B_0\cap\Omega}\mathcal{V}(x,F)|^{2s}\,dx\right)^{\frac{1}{s}}.
\end{aligned}
\tag{4.1.123}
$$

For I_2, we have

$$
\begin{aligned}
I_2 &\lesssim \fint_{\lambda\Omega_r} |\mathcal{V}_{B_r}(\nabla v) - (\mathcal{V}_{B_r}(\nabla v))_{\lambda\Omega_r}|^2\,dx \\
&\quad + \fint_{\lambda\Omega_r} |\mathcal{V}(x,\nabla v) - \mathcal{V}_{B_r}(\nabla v)|^2\,dx \\
&\lesssim \fint_{\lambda\Omega_r} |\mathcal{V}_{B_r}(\nabla v) - (\mathcal{V}_{B_r}(\nabla v))_{\lambda\Omega_r}|^2\,dx \\
&\quad + \lambda^{-n}\fint_{\frac{1}{2}\Omega_r} |\mathcal{V}(x,\nabla v) - \mathcal{V}_{B_r}(\nabla v)|^2\,dx \\
&=: I_{2,1} + I_{2,2}.
\end{aligned}
\tag{4.1.124}
$$

With the help of (4.1.105) in Proposition 4.1.22, it follows that

$$
\begin{aligned}
I_{2,1} &\lesssim \lambda^{2\alpha}\fint_{\frac{1}{4}\Omega_r} |\mathcal{V}_{B_r}(\nabla v) - (\mathcal{V}_{B_r}(\nabla v))_{\frac{1}{4}\Omega_r}|^2\,dx \\
&\quad + \|\nabla\psi\|^2_\infty \lambda^{-n}\fint_{\frac{1}{4}\Omega_r} |\nabla v|^p \omega^p_{B_r}\,dx.
\end{aligned}
\tag{4.1.125}
$$

Triangle inequalities yield

$$
\begin{aligned}
I_{2,1} &\lesssim \lambda^{2\alpha} \fint_{\Omega_r} |\mathcal{V}(x, \nabla z) - (\mathcal{V}(\cdot, \nabla z))_{B_r}|^2 \, dx \\
&\quad + \lambda^{2\alpha} \fint_{\frac{1}{2}\Omega_r} |\mathcal{V}_{B_r}(\nabla z) - \mathcal{V}_{B_r}(\nabla v)|^2 \, dx \\
&\quad + \lambda^{2\alpha} \fint_{\frac{1}{2}\Omega_r} |\mathcal{V}(x, \nabla z) - \mathcal{V}_{B_r}(\nabla z)|^2 \, dx \\
&\quad + \lambda^{2\alpha} \fint_{\frac{1}{2}\Omega_r} |\mathcal{V}(x, \nabla v) - \mathcal{V}_{B_r}(\nabla v)|^2 \, dx \\
&\quad + \|\nabla \psi\|_\infty^2 \lambda^{-n} \fint_{\frac{1}{4}\Omega_r} |\nabla v|^p \omega_{B_r}^p \, dx \\
&=: I_{2,1,0} + I_{2,1,1} + I_{2,1,2} + I_{2,1,3} + I_{2,1,4}.
\end{aligned}
\tag{4.1.126}
$$

To estimate $I_{2,1,0}$, by $z = |\nabla z| = 0$ in $B_r \setminus \Omega$ and $|\Omega_r| \eqsim |B_r|$ due to (4.1.15), we have

$$
I_{2,1,0} \lesssim \lambda^{2\alpha} \fint_{B_r} |\mathcal{V}(x, \nabla z) - (\mathcal{V}(x, \nabla z))_{B_r}|^2 \, dx. \tag{4.1.127}
$$

Besides, using the similar argument for $I_{3,1}, I_{3,2}$ and $I_{3,3}$, we estimate $I_{2,1,1}, I_{2,1,2}$ and $I_{2,1,3}$, respectively, with replacing the factor λ^{-n} with $\lambda^{2\alpha}$. For $I_{2,1,4}$, by the same argument as in (4.1.118)–(4.1.120), we have

$$
I_{2,1,4} \lesssim \|\nabla \psi\|_\infty^2 \lambda^{-n} \left(\fint_{B_r} |\mathcal{V}(x, \nabla z)|^{2s} \, dx \right)^{\frac{1}{s}}. \tag{4.1.128}
$$

On the other hand, for $I_{2,2}$ we apply the same estimate for $I_{3,3}$. Finally, we have

$$\begin{aligned}
I_1 &\le I_2 + I_3 \\
&\le c\lambda^{2\alpha} \fint_{B_r} |\mathcal{V}(x,\nabla z) - (\mathcal{V}(\cdot,\nabla z))_{B_r}|^2 \, dx \\
&\quad + c\lambda^{-n} \left(|\log \mathbb{M}|^2_{\mathrm{BMO}(4B_r)} + \|\nabla \psi\|^2_\infty + \epsilon \right) \\
&\qquad \times \left(\fint_{B_r} |\mathcal{V}(x,\nabla z)|^{2s} \, dx \right)^{\frac{1}{s}} \\
&\quad + c\lambda^{-n} C^*(\epsilon) \left(\fint_{2B_r} \left(\frac{\chi_{2B_0\cap\Omega}|u|^p}{R^p} \omega^p \right)^s \, dx \right)^{\frac{1}{s}} \\
&\quad + c\lambda^{-n} C^*(\epsilon) \left(\fint_{2B_r} |\chi_{2B_0\cap\Omega} \mathcal{V}(x,F)|^{2s} \, dx \right)^{\frac{1}{s}}
\end{aligned} \tag{4.1.129}$$

for some $c = c(n,p,\Lambda)$. We select a small $\lambda = \lambda(n,p,\Lambda) \in (0, \frac{1}{80})$ such that $c\lambda^{2\alpha} \le \frac{1}{4}$ holds, so that we get (4.1.112). □

We define the Hardy-Littlewood maximal function and the sharp maximal function for $f \in L^1_{\mathrm{loc}}$ and $\rho \in [1,\infty)$ by

$$\begin{aligned}
\mathcal{M}_\rho f(x) &:= \sup_{r>0} \left(\fint_{B_r(x)} |f|^\rho \, dy \right)^{\frac{1}{\rho}}, \\
\mathcal{M}^\sharp_\rho f(x) &:= \sup_{r>0} \left(\fint_{B_r(x)} |f - (f)_{B_r(x)}|^\rho \, dy \right)^{\frac{1}{\rho}}.
\end{aligned} \tag{4.1.130}$$

Now, we employ Proposition 4.1.23 to show the pointwise sharp maximal function estimate, which is more adaptable form to our gradient estimates. Recall that for $B_0 = B_R(x_0)$ with $x_0 \in \partial\Omega$.

Proposition 4.1.24. *Let z, u and F be as in* (4.1.38) *and* (4.1.42)*. There exists $s = s(n,p,\Lambda) > 1$ and $\kappa_5 = \kappa_5(n,p,\Lambda)$ such that the following holds: If $|\log \mathbb{M}|_{\mathrm{BMO}(4B_0)} \le \kappa_5$ and* (4.1.15) *hold, then for a.e. $x \in \mathbb{R}^n$ and any*

$\epsilon \in (0,1]$, *there holds*

$$\begin{aligned}
&\mathcal{M}_2^\sharp\left(\mathcal{V}(\cdot,\nabla z)\right)(x)\\
&\leq c\left(|\log \mathbb{M}|_{\mathrm{BMO}(4B_0)} + \|\nabla\psi\|_\infty + \epsilon\right)\mathcal{M}_{2s}\left(\mathcal{V}(\cdot,\nabla z)\right)(x)\\
&\quad + c\,C^*(\epsilon^2)R^{-\frac{p}{2}}\left(\mathcal{M}_s\left(\chi_{4B_0\cap\Omega}|u|^p\omega^p\right)(x)\right)^{\frac{1}{2}}\\
&\quad + c\,C^*(\epsilon^2)\mathcal{M}_{2s}\left(\chi_{4B_0\cap\Omega}\mathcal{V}(\cdot,F)\right)(x)\\
&\quad + c\,\frac{R^n}{(R+|x-x_0|)^n}\left(\fint_{B_0}|\mathcal{V}(y,\nabla z)-(\mathcal{V}(\cdot,\nabla z))_{B_0}|^2\,dy\right)^{\frac{1}{2}}
\end{aligned} \tag{4.1.131}$$

for $c = c(n,p,\Lambda) > 0$.

Proof. Let κ_5 and s be as in Proposition 4.1.23. Since $\mathcal{V}(\cdot,\nabla z) \in L^2(\mathbb{R}^n)$, $\mathcal{V}(\cdot,F) \in L^2(4B_0\cap\Omega)$ and $|u|^p\omega^p \in L^s(4B_0\cap\Omega)$ by Proposition 4.1.6, all terms in (4.1.131) are finite for a.e. x. Choose $x \in \mathbb{R}^n$ and denote

$$\begin{aligned}
I &:= \mathcal{M}_2^\sharp(\mathcal{V}(\cdot,\nabla z))(x)\\
&= \sup_{r>0}\left(\fint_{B_r(x)}|\mathcal{V}(y,\nabla z)-(\mathcal{V}(\cdot,\nabla z))_{B_r(x)}|^2\,dy\right)^{\frac{1}{2}}.
\end{aligned} \tag{4.1.132}$$

We divide the case for $r \in (0,\infty)$ as follows:

(1) $J_1 := \{r > 0 : B_r(x)\cap B_0\cap\Omega = \emptyset\}$

(2) $J_2 := \{r > 0 : \frac{4}{\lambda}B_r(x) \subset 4B_0 \text{ and } x \in \overline{\Omega}\}$

(3) $J_3 := \{r > 0 : \frac{4}{\lambda}B_r(x) \subset 4B_0 \text{ and } x \notin \overline{\Omega}\}$

(4) $J_4 := \{r > 0 : B_r(x)\cap B_0\cap\Omega \neq \emptyset \text{ and } \frac{4}{\lambda}B_r(x) \not\subset 4B_0\}$.

For $k = 1,2,3,4$ let us denote

$$I_k := \sup_{r\in J_k}\fint_{B_r(x)}|\mathcal{V}(y,\nabla z)-(\mathcal{V}(\cdot,\nabla z))_{B_r(x)}|\,dy. \tag{4.1.133}$$

We immediately find $I_1 = 0$ since $z = 0$ in $\mathbb{R}^n\setminus(B_0\cap\Omega)$. For I_2, we apply

Proposition 4.1.23 with $B_r = \lambda^{-1}B_r(x)$ and ϵ^2 instead of ϵ to have

$$\begin{aligned} I_2 \leq \frac{1}{4}I &+ c\left(|\log \mathbb{M}|_{\mathrm{BMO}(4B_0)} + \|\nabla\psi\|_\infty + \epsilon\right)\mathcal{M}_{2s}(\mathcal{V}(\cdot,\nabla z))(x) \\ &+ c\,C^*(\epsilon^2)R^{-\frac{p}{2}}\left(\mathcal{M}_s\left(\chi_{4B_0}|u|^p\omega^p\right)(x)\right)^{\frac{1}{2}} \\ &+ c\,C^*(\epsilon^2)\mathcal{M}_{2s}(\chi_{4B_0}\mathcal{V}(\cdot,F))(x). \end{aligned} \tag{4.1.134}$$

For I_3, when $r \in J_1$, then $I_3 \equiv 0$. If $r \in J_3 \setminus J_1$, then for $x = (x_1,\dots,x_n)$, denote $\tilde{x} = (x_1,\dots,x_n+r)$ and consider $2B_r(\tilde{x})(\supset B_r(x))$. Then since $\tilde{x} \in \overline{\Omega}$ and $\frac{2}{\lambda}B_r(\tilde{x}) \subset 4B_0$, we can apply Proposition 4.1.23 similarly as above with $B = \lambda^{-1}B_r(\tilde{x})$ and ϵ^2 instead of ϵ and obtain

$$\begin{aligned} I_3 \leq \frac{1}{4}I &+ c\left(|\log \mathbb{M}|_{\mathrm{BMO}(4B_0)} + \|\nabla\psi\|_\infty + \epsilon\right)\mathcal{M}_{2s}(\mathcal{V}(\cdot,\nabla z))(x) \\ &+ c\,C^*(\epsilon^2)R^{-\frac{p}{2}}\left(\mathcal{M}_s\left(\chi_{4B_0}|u|^p\omega^p\right)(x)\right)^{\frac{1}{2}} \\ &+ c\,C^*(\epsilon^2)\mathcal{M}_{2s}(\chi_{4B_0}\mathcal{V}(\cdot,F))(x). \end{aligned} \tag{4.1.135}$$

For I_4, since $r \in J_4$ implies $r \geq cR$, and so together with $\operatorname{supp} z \subset \overline{B_0}$, we have

$$I_4 \leq c\frac{R^n}{(R+|x-x_0|)^n}\left(\fint_{B_0}|\mathcal{V}(y,\nabla z) - (\mathcal{V}(\cdot,\nabla z))_{B_0}|^2\,dy\right)^{\frac{1}{2}}. \tag{4.1.136}$$

Merging the above estimates, taking the supremum for all $r > 0$, and absorbing $\frac{1}{4}I$ in the estimates of I_2 and I_3 to the left-hand side, the conclusion holds. □

Now, we prove Theorem 4.1.4, the non-linear case. To extract the sharp dependency of q, we apply the following global Fefferman-Stein inequality.

Lemma 4.1.25. *[16, Theorem 20] Let $q > 1$. Then for all $f \in L^q(\mathbb{R}^n)$ and $g \in L^{q'}(\mathbb{R}^n)$, we have*

$$\|f\|_{L^q(\mathbb{R}^n)} \leq cq\|\mathcal{M}_1^\sharp f\|_{L^q(\mathbb{R}^n)} \tag{4.1.137}$$

and

$$\|\mathcal{M}_1 g\|_{L^{q'}(\mathbb{R}^n)} \leq cq\|g\|_{L^{q'}(\mathbb{R}^n)} \tag{4.1.138}$$

for some $c = c(n)$.

We also need the following lemma, which is from [95].

Lemma 4.1.26. *Let $B \subset \mathbb{R}^n$ be a ball and $g, h : B \to \mathbb{R}$ be such that $g, h \in L^1(B)$. Suppose that for some $\theta \in (0,1)$, we have*

$$\fint_{\tilde{B}} |g| \, dx \le c_0 \left(\fint_{2\tilde{B}} |g|^\theta \, dx \right)^{\frac{1}{\theta}} + \fint_{2\tilde{B}} |h| \, dx \tag{4.1.139}$$

for any $2\tilde{B} \subset B$. Then for any $\gamma \in (0,1)$, there holds

$$\fint_{B} |g| \, dx \le c_1 \left(\fint_{2B} |g|^\gamma \, dx \right)^{\frac{1}{\gamma}} + c_1 \fint_{2B} |h| \, dx \tag{4.1.140}$$

for some constant $c_1 = c_1(c_0, \gamma, \theta)$. Here, c_1 is an increasing function on c_0.

Now, we prove gradient estimate results for the local boundary case, when there is a priori assumption $u \in W^{1,q}_\omega(4B_0 \cap \Omega)$.

Proposition 4.1.27 (Local boundary estimate)**.** *Assume* (4.1.15) *and let $u \in W^{1,q}_\omega(\Omega)$ be a weak solution of* (4.1.38) *with $F \in L^q_\omega(\Omega)$ for $q \in (p, \infty)$. Then there exists $\delta = \delta(n, p, \Lambda)$ such that for any balls B with $x_B \in \partial\Omega$, $r_B \le 4R$ and all $q \in [p, \infty)$ with*

$$|\log \mathbb{M}|_{\mathrm{BMO}(8B)} + \|\nabla \psi\|_\infty \le \frac{\delta}{q}, \tag{4.1.141}$$

there holds

$$\begin{aligned} &\left(\fint_{\frac{1}{2}B \cap \Omega} (|\nabla u|^p \omega^p)^\rho \, dx \right)^{\frac{1}{\rho}} \\ &\quad \le \bar{c} \fint_{4B \cap \Omega} (|\nabla u| \omega)^p \, dx + \bar{c} \left(\fint_{4B \cap \Omega} (|F|^p \omega^p)^\rho \, dx \right)^{\frac{1}{\rho}} \end{aligned} \tag{4.1.142}$$

for some $\bar{c} = \bar{c}(n, \Lambda, q)$ which is continuous on q.

Proof. Define $z = u\eta^{p'}$ as in the previous subsection with $|\nabla u|\omega \in L^q(\Omega)$. Let $B_0 = B_R(x_0)$ with $x_0 \in \partial\Omega$ and

$$\rho := \frac{q}{p} \ge 1.$$

We first claim the following type of reverse Hölder's inequality:

$$\begin{aligned}\left(\fint_{\frac{1}{2}B_0\cap\Omega}(|\nabla u|^p\omega^p)^\rho\,dx\right)^{\frac{1}{\rho}} &\le c(q)\left(\fint_{4B_0\cap\Omega}(|\nabla u|^p\omega^p)^{\theta\rho}\,dx\right)^{\frac{1}{\theta\rho}} \\ &\quad + c(q)\left(\fint_{4B_0\cap\Omega}(|F|^p\omega^p)^\rho\,dx\right)^{\frac{1}{\rho}}\end{aligned} \tag{4.1.143}$$

for some $\theta\in(0,1)$. If $1\le\rho\le s$ where s is defined in Corollary 4.1.16, then the conclusion directly follows from Corollary 4.1.16. Hence we only consider the case $\rho>s$. To prevent constants blowing up as ρ close to 1, we change s to s^2 so that $1<s<s^2<\rho$ holds.

Let $\epsilon\le\min\{\frac{\kappa_5}{p},\frac{1}{n}\}$, where κ_5 is as in Proposition 4.1.24. Under the assumptions $|\log\mathbb{M}|_{\mathrm{BMO}(4B_0)}\le\epsilon$ and $\|\nabla\psi\|_\infty\le\epsilon$, taking $L^{2\rho}(B_0\cap\Omega)$ norm $\|\cdot\|_{2\rho}$ to Proposition 4.1.24, we have

$$\begin{aligned}I &:= \|\mathcal{M}_2^\sharp(\mathcal{V}(\cdot,\nabla z))\|_{2\rho} \\ &\le c\left(|\log\mathbb{M}|_{\mathrm{BMO}(4B_0)}+\|\nabla\psi\|_\infty+\epsilon\right)\|\mathcal{M}_{2s}(\mathcal{V}(\cdot,\nabla z))\|_{2\rho} \\ &\quad + c\,C^*(\epsilon^2)R^{-\frac{p}{2}}\|\mathcal{M}_s(\chi_{4B_0\cap\Omega}|u|^p\omega^p)^{\frac{1}{2}}\|_{2\rho} \\ &\quad + c\,C^*(\epsilon^2)\|\mathcal{M}_{2s}(\chi_{4B_0\cap\Omega}\mathcal{V}(\cdot,F))\|_{2\rho} \\ &\quad + c\left\|\frac{R^n}{(R+|\cdot-x_0|)^n}\right\|_{2\rho}\left(\fint_{B_0}|\mathcal{V}(x,\nabla z)-(\mathcal{V}(\cdot,\nabla z))_{B_0}|^2\,dx\right)^{\frac{1}{2}} \\ &=: I_1+I_2+I_3+I_4.\end{aligned} \tag{4.1.144}$$

Since $|\nabla u|^p\omega^p\in L^\rho(B_0)$, it follows that $\mathcal{V}(\cdot,\nabla z)\in L^{2q}(\mathbb{R}^n)$ and so $I<\infty$. First, using Lemma 4.1.25 together with $\mathcal{M}_{2s}(g)=(\mathcal{M}(|g|^{2s}))^{\frac{1}{2s}}$ and $\frac{2\rho}{2s}\ge s\ge1$, we obtain

$$\begin{aligned}\|\mathcal{M}_{2s}(\mathcal{V}(\cdot,\nabla z))\|_{2\rho} &\le c_s\frac{2\rho}{2\rho-1}\|\mathcal{V}(\cdot,\nabla z)\|_{2\rho} \\ &\le c_s\frac{(2\rho)^2}{2\rho-1}\|\mathcal{M}_1^\sharp(\mathcal{V}(\cdot,\nabla z))\|_{2\rho} \\ &\le c_s\frac{(2\rho)^2}{2\rho-1}\|\mathcal{M}_2^\sharp(\mathcal{V}(\cdot,\nabla z))\|_{2\rho}.\end{aligned} \tag{4.1.145}$$

Thus for I_1, one can see that

$$I_1 \le c_3 q \left(|\log \mathbb{M}|_{\mathrm{BMO}(4B_0)} + \|\nabla \psi\|_\infty + \epsilon \right) I \tag{4.1.146}$$

for some $c_3 = c_3(n, p, \Lambda)$. Here, we choose

$$\delta = \min \left\{ \frac{1}{6c_3}, \frac{\kappa_5}{p}, \frac{1}{n} \right\} \quad \text{and} \quad \epsilon = \frac{\delta}{q}$$

so that

$$I_1 \le \frac{1}{2} I \tag{4.1.147}$$

holds. Then we are able to absorb I_1 to I. For the remaining term I_2, I_3 and I_4, by (4.1.138) one can see that

$$\begin{aligned} I_2 &\le c\, C^*(\tfrac{1}{q^2}) R^{-\frac{p}{2}} \left\| \mathcal{M}_s [(\chi_{4B_0 \cap \Omega} |u|^p \omega^p)^{\frac{1}{2}}] \right\|_{2\rho} \\ &\le c\, C^*(\tfrac{1}{q^2}) R^{-\frac{p}{2}} \tfrac{\rho}{\rho - s} \left\| \mathcal{M}_1 [(\chi_{4B_0 \cap \Omega} |u|^p \omega^p)^s] \right\|_{\frac{\rho}{s}}^{\frac{1}{2s}} \\ &\le c\, C^*(\tfrac{1}{q^2}) R^{-\frac{p}{2}} \left\| (\chi_{4B_0 \cap \Omega} |u|^p \omega^p)^s \right\|_{\frac{\rho}{s}}^{\frac{1}{2s}} \\ &\le c\, C^*(\tfrac{1}{q^2}) \left(\int_{4B_0 \cap \Omega} \left(\frac{|u|^p}{R^p} \omega^p \right)^\rho dx \right)^{\frac{1}{2\rho}} \end{aligned} \tag{4.1.148}$$

with $c = c(n, p, \Lambda)$, and similarly,

$$I_3 \le c\, C^*(\tfrac{1}{q^2}) \left(\int_{4B_0 \cap \Omega} |\mathcal{V}(x, F)|^{2\rho} \, dx \right)^{\frac{1}{2\rho}}. \tag{4.1.149}$$

For I_4, if we assume (4.1.15), then $|B_0 \cap \Omega| \eqsim |B_0|$ holds. Then together with the fact that $z = |\nabla z| = 0$ in $B_0 \setminus \Omega$, we have

$$I_4 \le c |B_0|^{\frac{1}{2\rho}} \left(\fint_{B_0 \cap \Omega} |\mathcal{V}(x, \nabla z)|^2 \, dx \right)^{\frac{1}{2}}. \tag{4.1.150}$$

On the other hand, by Lemma 4.1.25 there holds

$$I = \|\mathcal{M}_2^\sharp \mathcal{V}(\cdot, \nabla z)\|_{2\rho} \geq c\|\mathcal{M}_1^\sharp \mathcal{V}(\cdot, \nabla z)\|_{2\rho} \geq \frac{c}{q}\|\mathcal{V}(\cdot, \nabla z)\|_{2\rho}. \tag{4.1.151}$$

Summing up, we have

$$\begin{aligned}\|\mathcal{V}(\cdot, \nabla z)\|_{2\rho} \leq cqC^*(\tfrac{1}{q^2}) &\left(\int_{4B_0\cap\Omega} \left(\frac{|u|^p}{R^p}\omega^p\right)^\rho dx\right)^{\frac{1}{2\rho}} \\ &+ cqC^*(\tfrac{1}{q^2}) \left(\int_{4B_0\cap\Omega} |\mathcal{V}(x, F)|^{2\rho}\, dx\right)^{\frac{1}{2\rho}} \\ &+ cq|B_0|^{\frac{1}{2\rho}} \left(\fint_{B_0\cap\Omega} |\mathcal{V}(x, \nabla z)|^2\, dx\right)^{\frac{1}{2}}.\end{aligned} \tag{4.1.152}$$

If we assume (4.1.15), then $|B_0 \cap \Omega| \eqsim |B_0|$ and the above estimate implies

$$\begin{aligned}\left(\fint_{B_0\cap\Omega} (|\nabla z|^p\omega^p)^\rho\, dx\right)^{\frac{1}{\rho}} \leq c(qC^*(\tfrac{1}{q^2}))^2 &\left(\fint_{4B_0\cap\Omega} \left(\frac{|u|^p}{R^p}\omega^p\right)^\rho dx\right)^{\frac{1}{\rho}} \\ &+ c(qC^*(\tfrac{1}{q^2}))^2 \left(\fint_{4B_0\cap\Omega} (|F|^p\omega^p)^\rho\, dx\right)^{\frac{1}{\rho}} \\ &+ cq\fint_{B_0\cap\Omega} |\mathcal{V}(x, \nabla z)|^2\, dx.\end{aligned} \tag{4.1.153}$$

Since $z = u\eta^{p'}$ as in (4.1.42), it follows that

$$\begin{aligned}\left(\fint_{\frac{1}{2}B_0\cap\Omega} (|\nabla u|^p\omega^p)^\rho\, dx\right)^{\frac{1}{\rho}} \leq c(q) &\left(\fint_{4B_0\cap\Omega} \left(\frac{|u|^p}{R^p}\omega^p\right)^\rho dx\right)^{\frac{1}{\rho}} \\ &+ c(q) \left(\fint_{4B_0\cap\Omega} (|F|^p\omega^p)^\rho\, dx\right)^{\frac{1}{\rho}} \\ &+ c(q)\fint_{B_0\cap\Omega} |\nabla u|^p\omega^p\, dx,\end{aligned} \tag{4.1.154}$$

where $c(q) \eqsim (qC^*(\frac{1}{q^2}))^2$ which is a continuous and increasing function on q. Then since $|\log \mathbb{M}|_{\mathrm{BMO}(4B_0)} \leq \kappa_5$ and (4.1.15) holds, together with Lemma 4.1.9, we can apply Proposition 4.1.6. Consequently, we have (4.1.143).

Now, using (4.1.143), we next claim that

$$\begin{aligned}\left(\fint_{\frac{1}{2}B}(\chi_\Omega|\nabla u|^p\omega^p)^\rho\,dx\right)^{\frac{1}{\rho}} \le c(q)\left(\fint_{4B}(\chi_\Omega|\nabla u|^p\omega^p)^{\theta\rho}\,dx\right)^{\frac{1}{\theta\rho}}\\ + c(q)\left(\fint_{4B}(\chi_\Omega|F|^p\omega^p)^\rho\,dx\right)^{\frac{1}{\rho}}\end{aligned} \tag{4.1.155}$$

for all $B \subset \mathbb{R}^n$. Indeed, if $4B \subset \Omega$, then we employ [16, Proposition 22] to obtain (4.1.155). If $4B \subset (\mathbb{R}^n \backslash \Omega)$, then (4.1.155) becomes trivial since $\chi_\Omega = 0$ on $\mathbb{R}^n \backslash \Omega$. Finally, if $4B \not\subset \Omega$ and $4B \not\subset (\mathbb{R}^n \backslash \Omega)$, we use the similar argument of **Step 3** in the proof of Proposition 4.1.21 and so we have (4.1.155).

Now, Lemma 4.1.26 gives us that

$$\begin{aligned}&\left(\fint_{\frac{1}{2}B_0\cap\Omega}(|\nabla u|^p\omega^p)^\rho\,dx\right)^{\frac{1}{\rho}}\\ &\le \bar{c}(q)\left(\fint_{4B_0\cap\Omega}|\nabla u|\omega\,dx\right)^p + \bar{c}(q)\left(\fint_{4B_0\cap\Omega}(|F|^p\omega^p)^\rho\,dx\right)^{\frac{1}{\rho}},\end{aligned} \tag{4.1.156}$$

where $\bar{c}(q)$ is still a continuous and increasing function on q. This proves (4.1.142). □

Proof of Theorem 4.1.4. Applying the argument of the proof of [16, Theorem 2], we can eliminate the assumption $u \in W^{1,q}_\omega(4B_0 \cap \Omega)$ in the statement of Proposition 4.1.27. Note that here we used the fact that $\bar{c}(q)$ is continuous and increasing function on q. Now, by considering Proposition 4.1.27 for the boundary case and [16, Theorem 2] for the interior case, using the covering argument, together with the assumption

$$\log \mathbb{M} \text{ is } \left(\frac{\delta}{q}, R\right)\text{–vanishing and } \Omega \text{ is } \left(\frac{\delta}{q}, R\right)\text{–Lipschitz,} \tag{4.1.157}$$

we get

$$\int_\Omega (|\nabla u|^p\omega^p)^\rho\,dx \le c^*\left(\int_\Omega |\nabla u|^p\omega^p\,dx\right)^{\frac{\rho}{p}} + c^*\int_\Omega (|F|^p\omega^p)^\rho\,dx, \tag{4.1.158}$$

where $c^* = c^*(n, p, \Lambda, \Omega, R, q)$. Then the standard energy estimate as in

(1.3.11) and Hölder's inequality imply (4.1.12). $\square$

Here we prove Theorem 4.1.3 using the duality argument.

Proof of Theorem 4.1.3. We only show in the case $1 < q < 2$, since the case $q \geq 2$ follows from Theorem 4.1.4 with $p = 2$. As the previous argument, we first prove the local boundary case, and then employ the result of [16, Theorem 1] as the interior case to use the standard covering argument. Recall that

$$-\mathrm{div}(\mathbb{A}(x)\nabla u) = -\mathrm{div}(\mathbb{A}(x)F) \tag{4.1.159}$$

and that $B_0 = B_R(x_0)$ with $x_0 \in \partial\Omega$. Define $H \in L^{q'}_{\omega}(B_0)$ with the following property:

$$\left(\fint_{2B_0} (|H|\omega)^{q'}\,dx\right)^{\frac{1}{q'}} \leq 1. \tag{4.1.160}$$

Let $aB_0 \cap \Omega := B_{aR}(x_0) \cap \Omega$ for $a > 0$ and $w \in W^{1,2}_{0,\omega}(2B_0 \cap \Omega)$ be the weak solution of

$$\begin{aligned} -\mathrm{div}(\mathbb{A}(x)\nabla w) &= -\mathrm{div}(\mathbb{A}(x)\chi_{2B_0}H) && \text{in } 4B_0 \cap \Omega, \\ w &= 0 && \text{on } \partial(4B_0 \cap \Omega). \end{aligned} \tag{4.1.161}$$

Under the assumption that

$$|\log \mathbb{M}|_{\mathrm{BMO}(4B_0)} \leq \delta\left(1 - \frac{1}{q}\right) \quad \text{and} \quad \|\nabla\psi\|_{\infty} \leq \delta\left(1 - \frac{1}{q}\right), \tag{4.1.162}$$

by (4.1.156) with the exponent $q' \geq 2$ and Hölder's inequality, it follows that

$$\begin{aligned} &\left(\fint_{2B_0\cap\Omega} (|\nabla w|\omega)^{q'}\,dx\right)^{\frac{1}{q'}} \\ &\quad \leq c\left(\fint_{4B_0\cap\Omega} (|\nabla w|\omega)^{2}\,dx\right)^{\frac{1}{2}} + c\left(\fint_{2B_0\cap\Omega} (|H|\omega)^{q'}\,dx\right)^{\frac{1}{q'}}. \end{aligned} \tag{4.1.163}$$

Here, testing w itself in (4.1.161), we have

$$\fint_{4B_0\cap\Omega}(|\nabla w|\omega)^2\,dx \le c\fint_{2B_0\cap\Omega}(|H|\omega)^2\,dx \le c\left(\fint_{2B_0\cap\Omega}(|H|\omega)^{q'}\,dx\right)^{\frac{2}{q'}} \tag{4.1.164}$$

and so there holds

$$\left(\fint_{2B_0\cap\Omega}(|\nabla w|\omega)^{q'}\,dx\right)^{\frac{1}{q'}} \le c\left(\fint_{2B_0\cap\Omega}(|H|\omega)^{q'}\,dx\right)^{\frac{1}{q'}} \le c. \tag{4.1.165}$$

Let $\eta \in C_0^\infty(2B_0)$ be a smooth cut-off function with $\chi_{B_0} \le \eta \le \chi_{2B_0}$ and $\|\nabla\eta\|_\infty \le c/R$. From (4.1.161), we have

$$\begin{aligned} I :=& \fint_{2B_0\cap\Omega}\mathbb{A}(x)\nabla(\eta^2 u)\cdot H\,dx \\ =& \fint_{2B_0\cap\Omega}\mathbb{A}(x)\nabla(\eta^2 u)\cdot \nabla w\,dx \\ =& \fint_{2B_0\cap\Omega}\mathbb{A}(x)\nabla u\cdot \nabla(\eta^2 w)\,dx \\ &+ \fint_{2B_0\cap\Omega}\mathbb{A}(x)u\nabla(\eta^2)\cdot\nabla w\,dx - \fint_{2B_0\cap\Omega}\mathbb{A}(x)w\nabla u\cdot\nabla(\eta^2)\,dx \\ =:& I_1+I_2+I_3. \end{aligned} \tag{4.1.166}$$

To estimate I_1, using the equation for u in (4.1.38), there holds

$$\begin{aligned} I_1 &= \fint_{2B_0\cap\Omega}\mathbb{A}(x)F\cdot\nabla(\eta^2 w)\,dx \\ &\le c\fint_{2B_0\cap\Omega}\omega^2|F||\nabla(\eta^2 w)|\,dx \\ &\le c\left(\fint_{2B_0\cap\Omega}(\omega|F|)^q\,dx\right)^{\frac{1}{q}}\left(\fint_{2B_0\cap\Omega}(\omega|\nabla(\eta^2 w)|)^{q'}\,dx\right)^{\frac{1}{q'}}. \end{aligned} \tag{4.1.167}$$

With the help of triangle inequality and Proposition 4.1.6, we have

$$|I_1| \le \left(\fint_{2B_0\cap\Omega}(\omega|F|)^q\,dx\right)^{\frac{1}{q}}\left(\fint_{2B_0\cap\Omega}(\omega|\nabla w|)^{q'}\,dx\right)^{\frac{1}{q'}}. \tag{4.1.168}$$

For I_2, by Hölder's inequality and Proposition 4.1.6, we have

$$\begin{aligned}|I_2| &\le c\left(\fint_{2B_0\cap\Omega}\left(\omega\frac{|u|}{R}\right)^q dx\right)^{\frac{1}{q}}\left(\fint_{2B_0\cap\Omega}(\omega|\nabla w|)^{q'}\,dx\right)^{\frac{1}{q'}}\\ &\le c\left(\fint_{2B_0\cap\Omega}(\omega|\nabla u|)^{\theta p}\,dx\right)^{\frac{1}{\theta p}}\left(\fint_{2B_0\cap\Omega}(\omega|\nabla w|)^{q'}\,dx\right)^{\frac{1}{q'}}\end{aligned}\tag{4.1.169}$$

for some $\theta\in(\frac{1}{q},1)$. Similarly, for I_3, there holds

$$\begin{aligned}|I_3| &\le c\left(\fint_{2B_0\cap\Omega}(\omega|\nabla u|)^{\theta_2 p}\,dx\right)^{\frac{1}{\theta_2 p}}\left(\fint_{2B_0\cap\Omega}\left(\omega\frac{|w|}{R}\right)^{(\theta_2 q)'} dx\right)^{\frac{1}{(\theta_2 q)'}}\\ &\le c\left(\fint_{2B_0\cap\Omega}(\omega|\nabla u|)^{\theta_2 q}\,dx\right)^{\frac{1}{\theta_2 q}}\left(\fint_{2B_0\cap\Omega}(\omega|\nabla w|)^{q'}\,dx\right)^{\frac{1}{q'}}.\end{aligned}\tag{4.1.170}$$

Now, without loss of generality we assume $\theta=\theta_2$. Consequently, with (4.1.165) we have

$$\begin{aligned}|I| &\le c\left[\left(\fint_{2B_0\cap\Omega}(\omega|\nabla u|)^{\theta q}\,dx\right)^{\frac{1}{\theta q}}+\left(\fint_{2B_0\cap\Omega}(\omega|F|)^{q}\,dx\right)^{\frac{1}{q}}\right]\\ &\quad\times\left(\fint_{2B_0\cap\Omega}(\omega|\nabla w|)^{q'}\,dx\right)^{\frac{1}{q'}}\\ &\le c\left(\fint_{2B_0\cap\Omega}(\omega|\nabla u|)^{\theta q}\,dx\right)^{\frac{1}{\theta q}}+c\left(\fint_{2B_0\cap\Omega}(\omega|F|)^{q}\,dx\right)^{\frac{1}{q}}.\end{aligned}\tag{4.1.171}$$

Since H was an arbitrary function with (4.1.160) and $(L^{q'}_{\omega})^*=L^q_{\omega^{-1}}$ holds, we obtain

$$\begin{aligned}&\left(\fint_{2B_0\cap\Omega}\left(|\mathbb{A}\nabla(\eta^2u)|\omega^{-1}\right)^q\,dx\right)^{\frac{1}{q}}\\ &\qquad\le c\left(\fint_{2B_0\cap\Omega}(\omega|\nabla u|)^{\theta q}\,dx\right)^{\frac{1}{\theta q}}+c\left(\fint_{2B_0\cap\Omega}(\omega|F|)^{q}\,dx\right)^{\frac{1}{q}}.\end{aligned}\tag{4.1.172}$$

Since $\mathbb{A}\nabla(\eta^2 u) = \mathbb{A}\nabla u$ on $2B_0 \cap \Omega$ and $|\mathbb{A}\nabla u| \eqsim \omega^2|\nabla u|$ hold, we conclude

$$\begin{aligned}&\left(\fint_{B_0\cap\Omega} (\omega|\nabla u|)^q\,dx\right)^{\frac{1}{q}}\\&\quad\leq c\left(\fint_{2B_0\cap\Omega} (\omega|\nabla u|)^{\theta q}\,dx\right)^{\frac{1}{\theta q}} + c\left(\fint_{2B_0\cap\Omega} (\omega|F|)^q\,dx\right)^{\frac{1}{q}},\end{aligned} \tag{4.1.173}$$

which is analogous to (4.1.143). Now, by applying the argument of the proof of Proposition 4.1.27, we can change the exponent θq to 1. Then similar to the proof of Theorem 4.1.4, using the covering argument, together with the assumptions

$$\log\mathbb{A} \text{ is } \left(\delta\min\left\{\frac{1}{q}, 1-\frac{1}{q}\right\}, R\right)\text{–vanishing and} \tag{4.1.174}$$

$$\Omega \text{ is } \left(\delta\min\left\{\frac{1}{q}, 1-\frac{1}{q}\right\}, R\right)\text{–Lipschitz,} \tag{4.1.175}$$

we get (4.1.10). This proves Theorem 4.1.3. □

4.1.4 Sharpness and smallness conditions

In this section we discuss the sharpness of our smallness condition. We have shown in Theorem 4.1.3 and Theorem 4.1.4 that reciprocal of the exponent q of higher integrability is linearly connected to the smallness condition on $\log\mathbb{A}$ and $\partial\Omega$. In this section we show that this linear dependence is the best possible. It has been shown already in [16, Section 4] that the smallness on $\log\mathbb{A}$ is necessary by means of analyzing the counterexample introduced by Meyers [177]. Therefore, we concentrate in this article on the sharpness of the condition on Ω. Since the effect already occurs in the unweighted case, we assume that $\mathbb{M} = \mathrm{id}$. We provide a two dimensional example, but the principle generalizes to higher dimensions as well.

Example 4.1.28. *For $n = 2$ and $\epsilon \in (0,1)$, we consider the following type of the domain:*

$$\Omega = \{x = (x_1, x_2) \in B_1(0) : x_2 > -\epsilon|x_1|\}.$$

Then Ω is $(\epsilon, 1)$-Lipschitz. Moreover, the assigned Lipschitz map for the origin is $\psi : \mathbb{R} \to \mathbb{R}$ such that $\psi(x_1) = -\epsilon|x_1|$ and so $\|\nabla\psi\|_\infty \leq \epsilon$.

Now, for $\alpha := \frac{\pi/2}{\pi/2+\tan^{-1}\epsilon}$ we define in polar coordinates $x = r(\cos\phi, \sin\phi)$

$$u(x) = u(r, \phi) = \cos\big(\alpha(\phi - \tfrac{\pi}{2})\big)r^\alpha.$$

Then u is a solution of the equation

$$\begin{aligned} \Delta u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = -\epsilon|x_1|\} \cap B_1(0). \end{aligned}$$

Indeed, we have

$$\begin{aligned} \Delta u &= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} \\ &= \Big(\alpha(\alpha-1)r^{\alpha-2} + \frac{\alpha}{r}r^{\alpha-1} - \frac{\alpha^2}{r^2}\Big)\cos\big(\alpha(\phi - \tfrac{\pi}{2})\big)r^\alpha = 0. \end{aligned}$$

Moreover, with $e_r := (\cos\phi, \sin\phi)$ and $e_\phi := (-\sin\phi, \cos\phi)$ that

$$\begin{aligned} |\nabla u| &= \left|\frac{\partial u}{\partial r}e_r + \frac{\partial u}{r\partial\phi}e_\phi\right| \\ &= \big|\alpha\cos\big(\alpha(\phi - \tfrac{\pi}{2})\big)r^{\alpha-1}e_r - \alpha\sin\big(\alpha(\phi - \tfrac{\pi}{2})\big)r^{\alpha-1}e_\phi\big| = \alpha r^{\alpha-1}. \end{aligned}$$

Assume that $q > 2$. Then $\nabla u \in L^{q,\infty}(B_1(0))$ (Lorentz space or weak Lebesgue space) is equivalent to $(\alpha-1)q \geq -2$. This simplifies to

$$\nabla u \in L^{q,\infty}(B_1(0)) \quad \Leftrightarrow \quad \tan^{-1}\epsilon \leq \frac{\pi}{q-2}.$$

Thus,

$$\nabla u \in L^{q}(B_1(0)) \quad \Leftrightarrow \quad \tan^{-1}\epsilon < \frac{\pi}{q-2}.$$

Note that for small ϵ, we have $\tan^{-1}\epsilon \approx \epsilon$. This implies that the smallness assumptions (4.1.9b) *and* (4.1.11) *are optimal.*

From now on we compare our smallness condition to other type of conditions as found in [61, 62]. Let us assume that $\mathbb{A} : \mathbb{R}^n \to \mathbb{R}^{n\times n}_{>0}$ with bounded condition number with $|\mathbb{A}||\mathbb{A}^{-1}| \leq \Lambda^2$. In this section we compare our smallness condition on the weight in terms of $\log \mathbb{M}$ with the smallness condition on $\mathbb{M}$ from Cao, Mengesha and Phan in [61, 62]. In [61], they introduced the quantity[1]

$$|\mathbb{A}|_{\mathrm{BMO}^2_\mu} := \sup_B \left(\frac{1}{\mu(B)} \int_B |\mathbb{A}(x) - \langle \mathbb{A} \rangle_B|^2 \mu^{-1}(x)\, dx \right)^{\frac{1}{2}} \tag{4.1.176}$$

in order to measure the oscillations of $\mathbb{A}$, where $\mu(x) := |\mathbb{A}(x)|$ and the supremum is taken over all balls. In addition to the smallness conditions Cao, Mengesha and Phan assume that $\mu := |\mathbb{A}| \in \mathcal{A}_2$.

In [62] they use the simpler quantity

$$|\mathbb{A}|_{\mathrm{BMO}_\mu} := \sup_B \frac{1}{\mu(B)} \int_B |\mathbb{A}(x) - \langle \mathbb{A} \rangle_B|\, dx. \tag{4.1.177}$$

Note that by Hölder's inequality

$$|\mathbb{A}|_{\mathrm{BMO}_\mu} \leq |\mathbb{A}|_{\mathrm{BMO}^2_\mu}.$$

In contrast our measure of oscillations is

$$|\log \mathbb{A}|_{\mathrm{BMO}} = \sup_B \fint_B |\log \mathbb{A} - \langle \log \mathbb{A} \rangle_B|\, dx. \tag{4.1.178}$$

Due to $1 \leq |\mathbb{A}||\mathbb{A}^{-1}| \leq \Lambda^2$ we have

$$\langle \mu \rangle_B \left\langle \mu^{-1} \right\rangle_B \leq \langle |\mathbb{A}| \rangle_B \left\langle |\mathbb{A}^{-1}| \right\rangle_B \leq \Lambda^2 \langle \mu \rangle_B \left\langle \mu^{-1} \right\rangle_B. \tag{4.1.179}$$

The following lemma shows that our smallness condition on $|\log \mathbb{A}|_{\mathrm{BMO}}$ is weaker than the smallness condition on $|\mathbb{A}|_{\mathrm{BMO}^2_\mu}$ combined with the $\mathcal{A}_2$-condition.

Lemma 4.1.29. *Let $\mathbb{A} : \mathbb{R}^n \to \mathbb{R}^{n\times n}_{>0}$ be a weight with $|\mathbb{A}||\mathbb{A}^{-1}| \leq \Lambda^2$ and*

[1]Cao, Mengesha and Phan do not use $\mu(x) := |\mathbb{A}(x)|$, but treat μ as an independent function that satisfies the equivalence $\mu(x) \eqsim |\mathbb{A}(x)|$. However, choosing $\mu(x) := |\mathbb{A}(x)|$ is always an equivalent valid choice.

$\mu := |\mathbb{A}| \in \mathcal{A}_2$. *Then for all balls $B \subset \mathbb{R}^n$ there holds*

$$\begin{aligned}
&\fint_B |\log \mathbb{A} - \langle \log \mathbb{A} \rangle_B| \, dx \\
&\qquad \leq 4 \, \langle |\mathbb{A}| \rangle_B \left\langle |\mathbb{A}^{-1}| \right\rangle_B \left(\frac{1}{\mu(B)} \int_B |\mathbb{A}(x) - \langle \mathbb{A} \rangle_B|^2 \mu^{-1}(x) \, dx \right)^{\frac{1}{2}}.
\end{aligned}$$

Moreover, $|\log \mathbb{A}|_{\mathrm{BMO}} \leq 4 \, \Lambda^2 [\mu]_2 |\mathbb{A}|_{\mathrm{BMO}^2_\mu}$.

Proof. We begin with

$$\begin{aligned}
\fint_B |\log \mathbb{A} - \langle \log \mathbb{A} \rangle_B| \, dx &\leq 2 \inf_{\mathbb{A}_0 \in \mathbb{R}^{n \times n}_{>0}} \fint_B |\log \mathbb{A} - \log \mathbb{A}_0| \, dx \\
&\leq 2 \fint_B |\log \mathbb{A} - \log(\langle \mathbb{A} \rangle_B)| \, dx.
\end{aligned}$$

It has been shown e.g. in [123, Example 1] that for all $\mathbb{G}, \mathbb{H} \in \mathbb{R}^{n \times n}_{>0}$ there holds

$$|\log \mathbb{G} - \log \mathbb{H}| \leq \max\{|\mathbb{G}^{-1}|, |\mathbb{H}^{-1}|\} |\mathbb{G} - \mathbb{H}|.$$

This implies

$$\begin{aligned}
\fint_B |\log \mathbb{A} - \langle \log \mathbb{A} \rangle_B| \, dx &\leq 2 \fint_B \max\{|\mathbb{A}^{-1}|, |\langle \mathbb{A} \rangle_B^{-1}|\} |\mathbb{A} - \langle \mathbb{A} \rangle_B| \, dx \\
&\leq 2 \fint_B |\mathbb{A}^{-1}| |\mathbb{A} - \langle \mathbb{A} \rangle_B| \, dx + 2 |\langle \mathbb{A} \rangle_B^{-1}| \fint_B |\mathbb{A} - \langle \mathbb{A} \rangle_B| \, dx \\
&=: \mathrm{I} + \mathrm{II}.
\end{aligned}$$

By Hölder's inequality we obtain

$$\begin{aligned}
\mathrm{I} &= \fint_B |\mathbb{A}^{-1}| |\mathbb{A} - \langle \mathbb{A} \rangle_B| \, dx \\
&\leq \left(\fint_B |\mathbb{A}^{-1}| \, dx \right)^{\frac{1}{2}} \left(\fint_B |\mathbb{A} - \langle \mathbb{A} \rangle_B|^2 |\mathbb{A}|^{-1} \, dx \right)^{\frac{1}{2}} \\
&\leq \langle |\mathbb{A}| \rangle_B \left\langle |\mathbb{A}^{-1}| \right\rangle_B \left(\frac{1}{\langle |\mathbb{A}| \rangle_B} \int_B |\mathbb{A} - \langle \mathbb{A} \rangle_B|^2 |\mathbb{A}|^{-1} \, dx \right)^{\frac{1}{2}}.
\end{aligned}$$

On the other hand by Hölder's inequality

$$\mathrm{I} \le |\langle \mathbb{A}\rangle_B^{-1}| \fint_B |\mathbb{A} - \langle \mathbb{A}\rangle_B| \, dx \le |\langle \mathbb{A}\rangle_B^{-1}| |\langle \mathbb{A}\rangle_B| \bigg(\fint_B |\mathbb{A} - \langle \mathbb{A}\rangle_B|^2 |\mathbb{A}|^{-1} \, dx \bigg)^{\frac{1}{2}}.$$

Due to [25, Exercise 1.5.10] the mapping $\mathbb{A} \mapsto \mathbb{A}^{-1}$ is convex on $\mathbb{R}^{n\times n}_{>0}$. Thus, by Jensen's inequality $0 < \langle \mathbb{A}\rangle_B^{-1} \le \langle \mathbb{A}^{-1}\rangle_B$ and as a consequence $\big|\langle \mathbb{A}\rangle_B^{-1}\big| \le \big|\langle \mathbb{A}^{-1}\rangle_B\big| \le \langle |\mathbb{A}^{-1}|\rangle_B$. Using this fact, we obtain for II the same estimate as for I. Combining all estimates proves the first claim. Taking the supremum over all balls B and using (4.1.179) proves the second claim. □

On the other hand we will show now that if $|\log \mathbb{A}|_{\mathrm{BMO}}$ is small enough, then it controls $|\mathbb{A}|_{\mathrm{BMO}^2_\mu}$ in a linear way.

Lemma 4.1.30. *Let $\mathbb{A} : \mathbb{R}^n \to \mathbb{R}^{n\times n}_{>0}$ be a weight such $|\mathbb{A}||\mathbb{A}^{-1}| \le \Lambda^2$. Then there exists $\delta = \delta(n, \Lambda) > 0$ such that the following holds: If $|\log \mathbb{A}|_{\mathrm{BMO}} \le \delta$, then for all balls $B \subset \mathbb{R}^n$*

$$\bigg(\frac{|B|}{\mu(B)} \fint_B |\mathbb{A}(x) - \langle \mathbb{A}\rangle_B|^2 \mu^{-1}(x) \, dx \bigg)^{\frac{1}{2}} \le c(n, \Lambda) |\log \mathbb{A}|_{\mathrm{BMO}(B)}.$$

In particular, $|\mathbb{A}|_{\mathrm{BMO}^2_\mu(B)} \le c(n, \Lambda)|\log \mathbb{A}|_{\mathrm{BMO}(B)}$.

Proof. Let $|\log \mathbb{A}|_{\mathrm{BMO}(B)} \le \delta$. We choose $\delta > 0$ so small such that we can apply Lemma 4.1.8 (for $t = 4$) and Lemma 4.1.9 (for $\gamma = 2$). By Lemma 4.1.9 2 we obtain $\mu = |\mathbb{A}| \in \mathcal{A}_2$ with $[\mu]_{\mathcal{A}_2} = [|\mathbb{A}|]_{\mathcal{A}_2} \le 16$. Thus, with (4.1.179) we have $\langle |\mathbb{A}|\rangle_B \langle |\mathbb{A}^{-1}|\rangle_B \le 16\,\Lambda^2$. Recall that

$$\mu(B) = \int_B \mu \, dx = \int_B |\mathbb{A}| \, dx = |B| \, \langle |\mathbb{A}|\rangle_B \,.$$

For all $\mathbb{A}_0 \in \mathbb{R}^{n\times n}_{>0}$ we estimate with Jensen's inequality in the third step

$$
\begin{aligned}
&\left(\frac{1}{\mu(B)}\int_B |\mathbb{A}(x) - \fint_{\mathbb{A}} {}_B|^2 \mu^{-1}(x)\,dx\right)^{\frac{1}{2}} \\
&\quad = \left(\frac{1}{\langle|\mathbb{A}|\rangle_B}\fint_B |\mathbb{A}(x) - \langle\mathbb{A}\rangle_B|^2 |\mathbb{A}(x)|^{-1}\,dx\right)^{\frac{1}{2}} \\
&\quad \leq \left(\frac{1}{\langle|\mathbb{A}|\rangle_B}\fint_B |\mathbb{A}(x) - \mathbb{A}_0|^2 |\mathbb{A}(x)|^{-1}\,dx\right)^{\frac{1}{2}} \\
&\qquad + \left(\frac{1}{\langle|\mathbb{A}|\rangle_B}\fint_B |\langle\mathbb{A} - \mathbb{A}_0\rangle_B|^2 |\mathbb{A}(x)|^{-1}\,dx\right)^{\frac{1}{2}} =: \mathrm{I} + \mathrm{II}.
\end{aligned}
$$

With Jensen's inequality, Hölder's inequality and $\mu^{-1} = |\mathbb{A}|^{-1} \leq |\mathbb{A}^{-1}|$ we obtain

$$
\begin{aligned}
\mathrm{II} &\leq \fint_B |\mathbb{A}(x) - \mathbb{A}_0|\,dx \left(\frac{\langle|\mathbb{A}|^{-1}\rangle_B}{\langle|\mathbb{A}|\rangle_B}\right)^{\frac{1}{2}} \\
&\leq \left(\fint_B |\mathbb{A}(x) - \mathbb{A}_0|^2 |\mathbb{A}(x)|^{-1}\,dx\right)^{\frac{1}{2}} \langle|\mathbb{A}|\rangle_B^{\frac{1}{2}} \left(\frac{\langle|\mathbb{A}|^{-1}\rangle_B}{\langle|\mathbb{A}|\rangle_B}\right)^{\frac{1}{2}} \\
&= \left(\langle|\mathbb{A}|\rangle_B \langle|\mathbb{A}^{-1}|\rangle_B\right)^{\frac{1}{2}} \left(\frac{1}{\langle|\mathbb{A}|\rangle_B}\fint_B |\mathbb{A}(x) - \mathbb{A}_0|^2 |\mathbb{A}(x)|^{-1}\,dx\right)^{\frac{1}{2}} \\
&= \left(\langle|\mathbb{A}|\rangle_B \langle|\mathbb{A}^{-1}|\rangle_B\right)^{\frac{1}{2}} \mathrm{I} \leq 4\,\Lambda\,\mathrm{I}.
\end{aligned}
$$

Overall, we obtain

$$
\begin{aligned}
&\left(\frac{1}{\mu(B)}\int_B |\mathbb{A}(x) - \langle\mathbb{A}\rangle_B|^2 \mu^{-1}(x)\,dx\right)^{\frac{1}{2}} \\
&\qquad \leq (1 + 4\,\Lambda) \underbrace{\inf_{\mathbb{A}_0 \in \mathbb{R}^{n\times n}_{>0}} \left(\frac{1}{\langle|\mathbb{A}|\rangle_B}\fint_B |\mathbb{A}(x) - \mathbb{A}_0|^2 |\mathbb{A}(x)|^{-1}\,dx\right)^{\frac{1}{2}}}_{=:\mathrm{III}}.
\end{aligned}
$$

Now, choosing $\mathbb{A}_0 = \langle \mathbb{A} \rangle_B^{\log}$ and using Hölder's inequality we obtain

$$\begin{aligned} \mathrm{III} &\leq \left(\frac{1}{\langle |\mathbb{A}| \rangle_B} \fint_B \mathbb{A}(x) - \langle \mathbb{A} \rangle_B^{\log} |^2 |\mathbb{A}(x)|^{-1} \, dx \right)^{\frac{1}{2}} \\ &\leq \frac{|\langle \mathbb{A} \rangle_B^{\log}| \left\langle |\mathbb{A}|^{-2} \right\rangle_B^{\frac{1}{4}}}{\langle |\mathbb{A}| \rangle_B^{\frac{1}{2}}} \left(\fint_B \frac{|\mathbb{A}(x) - \langle \mathbb{A} \rangle_B^{\log} |^4}{|\langle \mathbb{A} \rangle_B^{\log}|^4} \, dx \right)^{\frac{1}{4}}. \end{aligned}$$

With Lemma 4.1.9 1 (with $-\gamma = -2$), (4.1.21) and (4.1.22) we obtain

$$\frac{|\langle \mathbb{A} \rangle_B^{\log}| \left\langle |\mathbb{A}|^{-2} \right\rangle_B^{\frac{1}{4}}}{\langle |\mathbb{A}| \rangle_B^{\frac{1}{2}}} \leq \frac{2 \, |\langle \mathbb{A} \rangle_B^{\log}|}{\langle |\mathbb{A}| \rangle_B^{\frac{1}{2}} \, (\langle |\mathbb{A}| \rangle_B^{\log})^{\frac{1}{2}}} \leq 2.$$

This and Lemma 4.1.8 (with $t = 2$) gives

$$\mathrm{III} \leq 2 \, c(n, \Lambda) \, |\log \mathbb{A}|_{\mathrm{BMO}(B)}.$$

Collecting the estimates proves the claim. □

Remark 4.1.31. *We shown that if $|\log \mathbb{A}|_{\mathrm{BMO}(B)}$ is small enough, then it controls $|\mathbb{A}|_{\mathrm{BMO}^2_\mu(B)}$ and therefore also $|\mathbb{A}|_{\mathrm{BMO}_\mu(B)}$. On the other hand we know from Lemma 4.1.29 that $|\log \mathbb{A}|_{\mathrm{BMO}(B)}$ is directly controlled by $|\mathbb{A}|_{\mathrm{BMO}^2_\mu(B)}$. Based on standard John-Nirenberg estimates, it is possible to show that sufficient smallness of $|\mathbb{A}|_{\mathrm{BMO}_\mu(B)}$ implies that $|\log \mathbb{A}|_{\mathrm{BMO}^2_\mu(B)}$ can be linearly controlled by $|\mathbb{A}|_{\mathrm{BMO}_\mu(B)}$.*

So overall, once one of the three quantities $|\log \mathbb{A}|_{\mathrm{BMO}(B)}$, $|\mathbb{A}|_{\mathrm{BMO}_\mu(B)}$ and $|\mathbb{A}|_{\mathrm{BMO}^2_\mu(B)}$ is small, then they are all comparable. This allows to transfer results state in one language to the others. For example, smallness of $|\log \mathbb{A}|_{\mathrm{BMO}(B)}$ implies the validity of the estimates in [61] and we obtain $\|\nabla u\|_{L^q(\mu\, dx)} \lesssim \|F\|_{L^q(\mu\, dx)}$ for $q > p = 2$. However, the smallness of $|\log \mathbb{A}|_{\mathrm{BMO}(B)}$ depends (negative) exponentially on q, see the discussion in [16, Remark 23].

4.2 Global estimates for equations with matrix weights and measurable nonlinearities

We study general elliptic equations with singular/degenerate matrix weights and measurable nonlinearities on nonsmooth bounded domains to obtain a global Calderón-Zygmund type estimate under possibly minimal assumptions that the logarithm of the matrix weight has a small BMO norm, the nonlinearity is allowed to be merely measurable in one variable but has a small BMO norm in the other variables and that the boundary of the domain is sufficiently flat in Reifenberg sense.

4.2.1 Hypothesis and main results

We consider a general elliptic equation with singular/degenerate nonlinearity in divergence form

$$\begin{cases} \operatorname{div}(\mathbb{M}(x)A(x,\mathbb{M}(x)Du)) = \operatorname{div}(\mathbb{M}^2(x)F) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.2.1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain with nonsmooth boundary $\partial\Omega$. The Carathéodory vector field $A(x,\xi) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is $C^1(\mathbb{R}^n)$-regular for ξ-variable and satisfies

$$\begin{cases} |A(x,\xi)| + |\partial_\xi A(x,\xi)||\xi| \leq L|\xi| \\ \nu|\zeta|^2 \leq \langle \partial_\xi A(x,\xi)\zeta, \zeta\rangle \end{cases} \tag{4.2.2}$$

for any $\zeta \in \mathbb{R}^n$, a.e. $x \in \mathbb{R}^n$ and some constants $0 < \nu \leq L < \infty$. A main point in this section is that we are treating with a symmetric and positive definite matrix-valued weight $\mathbb{M}(x) : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ satisfying

$$|\mathbb{M}(x)||\mathbb{M}^{-1}(x)| \leq \Lambda \tag{4.2.3}$$

for some constant $\Lambda > 0$. With this basic structure on Ω, A and $\mathbb{M}$, throughout this section we write

$$\omega(x) = |\mathbb{M}(x)| \quad \text{and} \quad \texttt{data} = \{n, \Lambda, \nu, L, |\Omega|\}. \tag{4.2.4}$$

The nonhomogeneous term $F = (f_1, \dots, f_n) : \Omega \to \mathbb{R}^n$ is a given vector-valued function with $|\mathbb{M}F| \in L^\gamma(\Omega)$ for some $\gamma \in [2, \infty)$. Then with (4.2.2) and (4.2.3), we will see later in Section 4.2.2 that there is a unique weak solution u of (4.2.1) in $W_0^{1,2}(\Omega, d\omega^2)$ and we have the standard energy estimate

$$\int_\Omega |\mathbb{M}(x)Du|^2 \, dx \leq c \int_\Omega |\mathbb{M}(x)F|^2 \, dx \tag{4.2.5}$$

with $c = c(n, \Lambda, \nu, L) > 0$, provided $\omega(x)^2$ belongs to $\mathcal{A}_2$-Muckenhoupt class. We will return to some issues including preliminaries of Muckenhoupt class and weighted Sobolev spaces, and the existence and uniqueness of the problem (4.2.1) with the estimate (4.2.5). Assuming (1.3.2) and ω^2 being $\mathcal{A}_2$-Muckenhoupt weight, the purpose is to prove that the implication

$$|\mathbb{M}(x)F| \in L^\gamma(\Omega) \implies |\mathbb{M}(x)Du| \in L^\gamma(\Omega) \tag{4.2.6}$$

is valid for every $\gamma > 2$ with the global Calderón-Zygmund type estimate

$$\int_\Omega |\mathbb{M}Du|^\gamma \, dx \leq c \int_\Omega |\mathbb{M}F|^\gamma \, dx \tag{4.2.7}$$

for some constant $c = c(\mathtt{data}, \gamma) > 0$.

With the precise notation and assumptions to be presented in detail in the next section, we now state our main theorem.

Theorem 4.2.1. *Assume* (4.2.2), (4.2.3), $\mathbb{M}^2 \in \mathcal{A}_2$ *and let* $|\mathbb{M}F| \in L^\gamma(\Omega)$ *for some* $\gamma \geq 2$. *Then there exists* $\delta = \delta(\mathtt{data}, \gamma) > 0$ *such that if* $(\Omega, \mathbb{M}, A)$ *is* (δ, R)*–vanishing of codimension 1, then the weak solution* $u \in W_0^{1,2}(\Omega, d\omega^2)$ *of* (4.2.1) *satisfies* $|\mathbb{M}Du| \in L^\gamma(\Omega)$ *with the estimate* (4.2.7).

More studies also need to be done to understand the measurability of the matrix weight $\mathbb{M}(x)$ in one of the variables as well as a precise dependence of the smallness parameter δ, in particular in terms of γ, though it seems unclear as this smallness assumption in the other variables except one variable is closely associated to both A and $\mathbb{M}$ as well as the choice of a point near the very irregular boundary and a size of the localized domain under consideration. We leave these issues to be investigated in the future.

4.2.2 Preliminaries and basic definitions

For $x = (x_1, x') \in \mathbb{R}^n$, $y_0 = (y_{0,1}, y_0') \in \mathbb{R}^n$ and $\rho \in (0, R]$, we define

$$\theta(\mathbb{M}, Q_\rho(y_0))(x) := |\log \mathbb{M}(x) - (\log \mathbb{M})_{Q_\rho(y_0)}|. \tag{4.2.8}$$

Also, we write

$$\theta(A, Q_\rho(y_0))(x) := \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{|A(x_1, x', \xi) - \bar{A}_{B'_\rho(y'_0)}(x_1, \xi)|}{|\xi|} \leq 2L, \tag{4.2.9}$$

where

$$\bar{A}_{B'_\rho(y'_0)}(x_1, \xi) := \fint_{B'_\rho(y'_0)} A(x_1, x', \xi)\, dx'.$$

Then we introduce the following condition.

Definition 4.2.2. *Let $\delta \in (0, \frac{1}{8})$ and $R \in (0, 1)$ be given. We say that $(\Omega, \mathbb{M}, A)$ is (δ, R)-vanishing of codimension 1, if for any $y \in \Omega$ and every $r \in (0, R]$ together with*

$$\operatorname{dist}(y, \partial\Omega) = \min_{z_0 \in \partial\Omega} \operatorname{dist}(y, z_0) > \sqrt{2} r,$$

there is a coordinate system depending on y and r, whose variables are still denoted by $x = (x_1, x')$, such that in this coordinate system, y is the origin and there holds

$$\begin{gathered}\fint_{Q_\rho(x_0)} (|\theta(A, Q_\rho(x_0))(x)|^2 + |\theta(\mathbb{M}, Q_\rho(x_0))(x)|^2)\, dx \leq \delta^2 \\ \textit{for every} \quad Q_\rho(x_0) \subset Q_r.\end{gathered} \tag{4.2.10}$$

Also, for any $y \in \Omega$ and every $r \in (0, R]$ together with

$$\operatorname{dist}(y, \partial\Omega) = |y - z_0| \leq \sqrt{2} r$$

for some $z_0 \in \partial\Omega$, there is a new coordinate system depending on y and r, whose variables are still denoted by $x = (x_1, x')$, such that in this coordinate *system, z_0 is the origin and there hold*

$$Q_{3r} \cap \{(x_1, x') : x_1 > 3\delta r\} \subset \Omega_{3r} \subset Q_{3r} \cap \{(x_1, x') : x_1 > -3\delta r\} \tag{4.2.11}$$

and

$$\fint_{Q_\rho(x_0)} (|\theta(A, Q_\rho(x_0))(x)|^2 + |\theta(\mathbb{M}, Q_\rho(x_0))(x)|^2)\, dx \le \delta^2 \quad \text{for every} \quad Q_\rho(x_0) \subset Q_{3r}. \tag{4.2.12}$$

If Ω satisfies (4.2.11) with $\delta \le \frac{1}{8}$, then it is well-known that the following measure density conditions hold:

$$\sup_{0<r\le R} \sup_{y\in\Omega} \frac{|Q_r(y)|}{|\Omega \cap Q_r(y)|} \approx \inf_{0<r\le R} \inf_{y\in\partial\Omega} \frac{|Q_r(y) \cap \Omega^c|}{|Q_r(y)|} \approx 1 \tag{4.2.13}$$

with the implicit constant $c = c(n)$. For further studies, we refer to [81, 145, 202].

We first introduce the weighted Sobolev-Poincaré inequality as in [16].

Lemma 4.2.3. *Let $n \ge 2$. For any $\theta \in (\frac{n}{n+2}, 1]$, we have the following lemmas:*

(1) (Interior case) Let $Q_{2r}(x_0)$ be a cylinder in $\mathbb{R}^n$. If μ is a scalar weight with

$$\sup_{Q_\rho(y)\subset Q_{2r}(x_0)} \left(\fint_{Q_\rho(y)} \mu^2\, dx \right)^{\frac{1}{2}} \left(\fint_{Q_\rho(y)} \mu^{-(2\theta)'}\, dx \right)^{\frac{1}{(2\theta)'}} \le c_{sp} \tag{4.2.14}$$

for some $c_{sp} > 0$, then for every $v \in W^{1,2}(Q_r(x_0), d\mu^2)$ we have

$$\fint_{Q_r(x_0)} \left| \frac{v - (v)_{Q_r(x_0)}}{r} \right|^2 \mu^2\, dx \le c \left(\fint_{Q_r(x_0)} (|Dv|\mu)^{2\theta}\, dx \right)^{\frac{1}{\theta}} \tag{4.2.15}$$

for some $c = c(n, c_{sp}) > 0$.

(2) (Boundary case) Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying

$$|Q_{4r}(x_0) \setminus \Omega| \ge \alpha |Q_{4r}(x_0)| \tag{4.2.16}$$

for some $\alpha > 0$. If μ is a scalar weight with

$$\sup_{Q_\rho(y)\subset Q_{5r}(x_0)} \left(\fint_{Q_\rho(y)} \mu^2 \, dx \right)^{\frac{1}{2}} \left(\fint_{Q_\rho(y)} \mu^{-(2\theta)'} \, dx \right)^{\frac{1}{(2\theta)'}} \le c_{sp} \tag{4.2.17}$$

for some $c_{sp} > 0$, then for every $v \in W^{1,2}(\Omega_{4r}(x_0), d\mu^2)$ with $v = 0$ on $\partial\Omega \cap Q_{4r}(x_0)$ we have

$$\frac{1}{|Q_{4r}|} \int_{\Omega_{4r}(x_0)} \left|\frac{v}{r}\right|^2 \mu^2 \, dx \le c \left(\frac{1}{|Q_{4r}|} \int_{\Omega_{4r}(x_0)} (|Dv|\mu)^{2\theta} \, dx \right)^{\frac{1}{\theta}} \tag{4.2.18}$$

for some $c = c(n, c_{sp}, \alpha) > 0$.

Proof. First, (4.2.15) follows from [16, Proposition 3] together with [111]. To show (4.2.18), let us abbreviate $Q_{4r} = Q_{4r}(x_0)$ and $\Omega_{4r} := \Omega_{4r}(x_0)$. We extend v as zero on the set $K := Q_{4r} \setminus \Omega$. Then we have $(v)_K = 0$, thus employing (4.2.17) and Proposition 3 of [16], there holds

$$\begin{aligned}
&\frac{1}{|Q_{4r}|} \int_{\Omega_{4r}} \left|\frac{v}{r}\right|^2 \mu^2 \, dx \\
&\le \fint_{Q_{4r}} \left| \frac{v - (v)_{Q_{4r}}}{r} \right|^2 \mu^2 \, dx + \fint_{Q_{4r}} \left| \frac{(v)_{Q_{4r}} - (v)_K}{r} \right|^2 \mu^2 \, dx \\
&\le c \left[\fint_{Q_{4r}} (|Dv|\mu)^{2\theta} \, dx \right]^{\frac{1}{\theta}} + \fint_{Q_{4r}} \left(\fint_K \left| \frac{v(y) - (v)_{Q_{4r}}}{r} \right| dy \right)^2 \mu^2 \, dx
\end{aligned}$$

for some $c = c(n, c_{sp}) > 0$. For the last integral on the right-hand side, we apply (4.2.16). Using Hölder's inequality, (4.2.17) and Proposition 3 in [16] yields that

$$
\begin{aligned}
&\fint_{Q_{4r}} \left(\fint_K \left| \frac{v(y) - (v)_{Q_{4r}}}{r} \right| dy \right)^2 \mu^2 \, dx \\
&\qquad \le \fint_{Q_{4r}} \left\{ \left(\fint_{Q_{4r}} \left| \frac{v(y) - (v)_{Q_{4r}}}{r} \right|^2 \mu^2 \, dy \right)^{\frac{1}{2}} \left(\fint_{Q_{4r}} \mu^{-2} \, dy \right)^{\frac{1}{2}} \right\}^2 \mu^2 \, dx \\
&\qquad \le c \left(\fint_{Q_{4r}} \left| \frac{v - (v)_{Q_{4r}}}{r} \right|^2 \mu^2 \, dy \right) \left(\fint_{Q_{4r}} \mu^{-2} \, dy \right) \left(\fint_{Q_{4r}} \mu^2 \, dx \right) \\
&\qquad \le c \left(\fint_{Q_{4r}} (|Dv|\mu)^{2\theta} \, dx \right)^{\frac{1}{\theta}}
\end{aligned}
$$

for some $c = c(n, c_{sp}, \alpha) > 0$. Then (4.2.18) follows, since $|Dv| = 0$ on $Q_{4r} \setminus \Omega$. $\square$

Remark 4.2.4. *We also deduce the following Poincaré type inequality on a (δ, R)-Reifenberg flat domain $\Omega \subset \mathbb{R}^n$ for $\delta \in (0, \frac{1}{8})$ and $R \in (0, 1)$. If μ is a scalar weight with*

$$
\sup_{Q_\rho(y) \subset \mathbb{R}^n} \left(\fint_{Q_\rho(y)} \mu^2 \, dx \right) \left(\fint_{Q_\rho(y)} \mu^{-2} \, dx \right) \le c_{sp2} \tag{4.2.19}
$$

for some $c_{sp2} > 0$, i.e., $\mu^2 \in \mathcal{A}_2$, then for every $v \in W_0^{1,2}(\Omega, d\mu^2)$ we have

$$
\int_\Omega |v|^2 \mu^2 \, dx \le c \int_\Omega (|Dv|\mu)^2 \, dx \tag{4.2.20}
$$

for some $c = c(n, c_{sp2}, |\Omega|) > 0$.

We next introduce the following lemmas related to the logarithm of a matrix-valued weight $\mathbb{L}$.

Lemma 4.2.5. *[16, Proposition 5] Let $Q_r(y_0) \subset \mathbb{R}^n$ be a cylinder and $p \ge 1$ be given. There exists a constant $c_1 = c_1(n) > 0$ such that if $\mathbb{L}$ is a matrix-*

valued weight with $[\log \mathbb{L}]_{\mathrm{BMO}(Q_r(y_0))} \le \frac{c_1}{p}$*, then we have*

$$\left[\fint_{Q_r(y_0)} \left(\frac{|\mathbb{L} - \bar{\mathbb{L}}_{Q_r(y_0)}|}{|\bar{\mathbb{L}}_{Q_r(y_0)}|}\right)^p dx\right]^{\frac{1}{p}} \le c_2 p [\log \mathbb{L}]_{\mathrm{BMO}(Q_r(y_0))}$$

with some $c_2 = c_2(n) > 0$*, where* $\bar{\mathbb{L}}_{Q_r(y_0)} := \exp((\log \mathbb{L})_{Q_r(y_0)})$.

Lemma 4.2.6. *[16, Proposition 6] For any matrix-valued weight* $\mathbb{L}$*, there exists a constant* $c_3 = c_3(n) > 0$ *such that the followings hold.*

(1) If $[\log \mathbb{L}]_{\mathrm{BMO}(Q_r(y_0))} \le \frac{c_3}{p}$ *with* $p \ge 1$*, then we have*

$$\left[\fint_{Q_r(y_0)} \left(\frac{|\mathbb{L}|}{|\bar{\mathbb{L}}_{Q_r(y_0)}|}\right)^p dx\right]^{\frac{1}{p}} + \left[\fint_{Q_r(y_0)} \left(\frac{|\bar{\mathbb{L}}_{Q_r(y_0)}|}{|\mathbb{L}|}\right)^p dx\right]^{\frac{1}{p}} \le 4,$$

where $\bar{\mathbb{L}}_{Q_r(y_0)} := \exp((\log \mathbb{L})_{Q_r(y_0)})$.

(2) If $[\log \mathbb{L}]_{\mathrm{BMO}(Q_r(y_0))} \le c_3 \min\{\frac{1}{p}, \frac{1}{p'}\}$ *with* $1 < p < \infty$*, then* $[|\mathbb{L}|^p]_{\mathcal{A}_p}^{\frac{1}{p}} \le 4$ *and so* $|\mathbb{L}|^p \in \mathcal{A}_p$.

(3) Let $\theta \in (\frac{1}{2}, 1)$ *be given. If* $|\log \mathbb{L}|_{\mathrm{BMO}(Q_r(y_0))} \le c_3(1 - \frac{1}{2\theta})$*, then we have*

$$\sup_{Q_\rho(y) \subset Q_r(y_0)} \left(\fint_{Q_\rho(y)} |\mathbb{L}|^2 \, dx\right)^{\frac{1}{2}} \left(\fint_{Q_\rho(y)} |\mathbb{L}|^{-(2\theta)'} \, dx\right)^{\frac{1}{(2\theta)'}} \le 4.$$

We now provide the existence of a solution to the problem (4.2.1). Before that, we give useful inequalities. Under the assumptions (4.2.2) and (4.2.3), we have the following inequalities. For the proof, see [11, 16, 18].

- For each $\xi_1, \xi_2 \in \mathbb{R}^n$,

$$\nu |\xi_1 - \xi_2|^2 \le \langle A(x, \xi_1) - A(x, \xi_2), \xi_1 - \xi_2 \rangle . \tag{4.2.21}$$

- For any $\xi \in \mathbb{R}^n$,

$$\Lambda^{-1} \omega |\xi| \le |\mathbb{M}\xi| \le \omega |\xi|, \tag{4.2.22}$$

where $\omega = |\mathbb{M}|$.

We remark that the inequality (4.2.22) implies that $\omega|f| \in L^p$ if and only if $|\mathbb{M}f| \in L^p$ for any vector valued function $f : \mathbb{R}^n \to \mathbb{R}^n$.

Lemma 4.2.7. *Let $\Omega \subset \mathbb{R}^n$ be a (δ, R)-Reifenberg flat domain for $\delta \in (0, \frac{1}{8})$ and $R \in (0, 1)$. Suppose that $\mathbb{M}$ is a matrix-valued weight with $\omega^2 = |\mathbb{M}|^2 \in \mathcal{A}_2$ and (4.2.3). Also, assume (4.2.2) and $F \in L^2(\Omega, d\omega^2)$. Then there exists a unique weak solution $u \in W_0^{1,2}(\Omega, d\omega^2)$ of the problem (4.2.1) with the energy estimate*

$$\int_\Omega |\mathbb{M}Du|^2 \, dx \leq \tilde{c} \int_\Omega |\mathbb{M}F|^2 \, dx \tag{4.2.23}$$

with some $\tilde{c} = \tilde{c}(\boldsymbol{data}) > 0$.

Proof. Note that $\operatorname{div}(\mathbb{M}^2(x)F) \in (W_0^{1,2}(\Omega, d\omega^2))'$, the dual space of $W_0^{1,2}(\Omega, d\omega^2)$. Moreover, since $\omega^2 \in \mathcal{A}_2$, (4.2.22) and (4.2.2) hold, by the standard theory of monotone operators (see [200, II.2.]), there exists a unique solution $u \in W_0^{1,2}(\Omega, d\omega^2)$ satisfying (4.2.1).

We now show (4.2.23). Testing $u \in W_0^{1,2}(\Omega, d\omega^2)$ to (4.2.1), there holds

$$\int_\Omega A(x, \mathbb{M}Du) \cdot \mathbb{M}Du \, dx = \int_\Omega \mathbb{M}F \cdot \mathbb{M}Du \, dx.$$

Then by (4.2.21), and Young's inequality,

$$\begin{aligned}
\int_\Omega |\mathbb{M}Du|^2 \, dx &\leq c \int_\Omega A(x, \mathbb{M}Du) \cdot \mathbb{M}Du \, dx \\
&\leq c \int_\Omega |\mathbb{M}F||\mathbb{M}Du| \, dx \\
&\leq \frac{1}{2} \int_\Omega |\mathbb{M}Du|^2 \, dx + c \int_\Omega |\mathbb{M}F|^2 \, dx.
\end{aligned}$$

Then (4.2.23) follows. □

4.2.3 Proof of Theorem 4.2.1

In this section we derive comparison estimates with reference problems. Recall that $\mathbb{M} : \mathbb{R}^n \to \mathbb{R}^{n \times n}_{\geq 0}$ is a matrix-valued weight with the assumption (4.2.3) and that $\omega := |\mathbb{M}|$. Also (4.2.2) is enforced. Here we only compute the boundary comparison estimates, as we can deduce the interior estimates in a similar way. According to our main assumption that $(\Omega, \mathbb{M}, A)$ is (δ, R)-vanishing of codimension 1, with the choice of a size $0 < 5r < R$ and a point in Ω, we are now under the following setting:

$$Q_{5r}^{+} \subset \Omega_{5r} \subset Q_{5r} \cap \{(x_1, x') : x_1 > -10\delta r\}, \tag{4.2.24}$$

$$\begin{aligned} &\fint_{Q_\rho(x_0)} |\theta(A, Q_\rho(x_0))(x)|^2\, dx \le \delta^2 \\ \text{and} \quad &\fint_{Q_\rho(x_0)} |\theta(\mathbb{M}, Q_\rho(x_0))(x)|^2\, dx \le \delta^2, \quad \forall Q_\rho(x_0) \subset Q_{5r}. \end{aligned} \tag{4.2.25}$$

Here, $\delta \in (0, \frac{1}{8})$ will be determined later. We also denote

$$\begin{aligned} &\bar{A}_{B'}(x_1, \xi) := \fint_{B'_{4r}} A(x_1, x', \xi)\, dx', \\ &\overline{\mathbb{M}}_Q := \exp\left(\log \mathbb{M}(\cdot)\right)_{Q_{4r}}, \quad \text{and} \quad \bar{\omega}_Q := \exp\left(\log \omega(\cdot)\right)_{Q_{4r}}. \end{aligned}$$

Then we have the following properties from (4.2.2) and (4.2.3).

- There holds

$$\begin{cases} |\bar{A}_{B'}(x_1, \xi)| + |\partial_\xi \bar{A}_{B'}(x_1, \xi)||\xi| \le L|\xi| \\ \nu|\zeta|^2 \le \left\langle \partial_\xi \bar{A}_{B'}(x_1, \xi)\zeta, \zeta \right\rangle \end{cases} \tag{4.2.26}$$

  for a.e. $x_1 \in \mathbb{R}$ and for all $\zeta \in \mathbb{R}^n$.

- (See [16]) We have

$$\Lambda^{-1}\bar{\omega}_Q|\xi| \le |\overline{\mathbb{M}}_Q \xi| \le \bar{\omega}_Q|\xi| \quad \text{for all } \xi \in \mathbb{R}^n. \tag{4.2.27}$$

Let $u \in W^{1,2}(\Omega_{4r}, d\omega^2)$ satisfy the problem

$$\begin{cases} \operatorname{div}(\mathbb{M}(x)A(x, \mathbb{M}(x)Du)) = \operatorname{div}(\mathbb{M}^2(x)F) & \text{in } \Omega_{4r}, \\ u = 0 & \text{on } \partial_w \Omega_{4r}. \end{cases} \tag{4.2.28}$$

We then suppose that for some $\lambda \geq 1$,

$$\fint_{\Omega_{4r}} |\mathbb{M}(x)Du|^2 \, dx \leq \lambda \quad \text{and} \quad \fint_{\Omega_{4r}} |\mathbb{M}(x)F|^2 \, dx \leq \delta\lambda. \tag{4.2.29}$$

Next, we sequentially consider the following problems:

$$\begin{cases} \operatorname{div}(\mathbb{M}(x)A(x,\mathbb{M}(x)Dh)) &= 0 \quad \text{in } \Omega_{4r}, \\ h &= u \quad \text{on } \partial\Omega_{4r}, \end{cases} \tag{4.2.30}$$

and

$$\begin{cases} \operatorname{div}(\overline{\mathbb{M}}_Q \bar{A}_{B'}(x_1,\overline{\mathbb{M}}_Q Dw)) &= 0 \quad \text{in } \Omega_{2r}, \\ w &= h \quad \text{on } \partial\Omega_{2r}. \end{cases} \tag{4.2.31}$$

We now show the following higher integrability results of $\mathbb{M}Dh$ in the problem (4.2.30).

Lemma 4.2.8. *Let $h \in W^{1,2}(\Omega_{4r}, d\omega^2)$ be the weak solution of (4.2.30). Then there exists $\delta = \delta(n) > 0$ such that if (4.2.24) and $(4.2.25)_2$ hold, then there is a constant $\sigma = \sigma(n,\Lambda,\nu,L) \in (0,1)$ such that*

$$\left(\fint_{\Omega_{2r}} |\mathbb{M}Dh|^{2(1+\sigma)} \, dx\right)^{\frac{1}{1+\sigma}} \leq c \fint_{\Omega_{4r}} |\mathbb{M}Dh|^2 \, dx \tag{4.2.32}$$

holds with $c = c(n,\Lambda,\nu,L) > 0$.

Proof. Let η be a smooth cut-off function with $0 \leq \eta \leq 1$, $\eta = 1$ in Q_{2r}, $\eta = 0$ in $\mathbb{R}^n \setminus Q_{4r}$, and $|D\eta| \leq \frac{c}{r}$. Testing $\eta^2 h \in W_0^{1,2}(\Omega_{4r}, d\omega^2)$ in (4.2.30) and using (4.2.2) and Young's inequality, we have

$$\begin{aligned} \int_{\Omega_{4r}} \eta^2 |\mathbb{M}Dh|^2 \, dx &\leq c \int_{\Omega_{4r}} \eta^2 A(x,\mathbb{M}Dh) \cdot \mathbb{M}Dh \, dx \\ &\leq c \int_{\Omega_{4r}} \eta |h| |A(x,\mathbb{M}Dh)| |\mathbb{M}| |D\eta| \, dx \\ &\leq c \int_{\Omega_{4r}} \eta |\mathbb{M}Dh| \left|\frac{h}{r}\right| \omega \, dx \\ &\leq \frac{1}{2} \int_{\Omega_{4r}} \eta^2 |\mathbb{M}Dh|^2 \, dx + c \int_{\Omega_{4r}} \left|\frac{h}{r}\right|^2 \omega^2 \, dx, \end{aligned}$$

which follows the weighted Caccioppoli estimate

$$\frac{1}{|Q_{2r}|}\int_{\Omega_{2r}} |\mathbb{M}Dh|^2\,dx \le \frac{c}{|Q_{4r}|}\int_{\Omega_{4r}} \left|\frac{h}{r}\right|^2 \omega^2\,dx \tag{4.2.33}$$

with some constant $c = c(n,\Lambda,\nu,L) > 0$.

From (4.2.24) and $\delta < \frac{1}{8}$, we figure out $|Q_{4r}\setminus\Omega| \ge (\frac{11}{16})^n|Q_{4r}|$. In addition, using Lemma 4.2.6 (3), we choose $\delta = \delta(n) > 0$ sufficiently small so that the condition (4.2.17) holds true with $(4.2.25)_2$. Now, we are under the assumptions in Lemma 4.2.3 (2). Then applying Lemma 4.2.3 (2), (4.2.33) deduces the reverse Hölder's inequality

$$\fint_{\Omega_{2r}} |\mathbb{M}Dh|^2\,dx \le c\left(\fint_{\Omega_{4r}} |\mathbb{M}Dh|^{\frac{2n}{n+2}}\,dx\right)^{\frac{n+2}{n}}.$$

Here, we also used (4.2.22) and (4.2.24). Then Gehring's lemma, e.g. [126], yields (4.2.32). □

We prove that the weak solution h to (4.2.30) is of $W^{1,2}(\Omega_{2r}, d\overline{\omega}_Q^2)$, which guarantees that the problem (4.2.31) with the boundary value h is well-posed.

Lemma 4.2.9. *Let $h \in W^{1,2}(\Omega_{4r}, d\omega^2)$ be the weak solution to* (4.2.30)*. Then there exists $\delta = \delta(n,\Lambda,\nu,L) > 0$ such that if* (4.2.24) *and* $(4.2.25)_2$ *hold, then h belongs to $W^{1,2}(\Omega_{2r}, d\overline{\omega}_Q^2)$ together with the estimate*

$$\fint_{\Omega_{2r}} |\overline{\mathbb{M}}_Q Dh|^2\,dx \le c\fint_{\Omega_{4r}} |\mathbb{M}Dh|^2\,dx$$

for some $c = c(n,\lambda,\nu,L) > 0$.

Proof. We first choose $\delta > 0$ sufficiently small so that conclusions of Lemma 4.2.8 hold. For $\sigma > 0$ as in Lemma 4.2.8, Hölder's inequality implies

$$\begin{aligned}&\fint_{\Omega_{2r}} |\overline{\mathbb{M}}_Q Dh|^2\,dx\\ &\quad\le \left(\fint_{\Omega_{2r}} |\mathbb{M}Dh|^{2(1+\sigma)}\,dx\right)^{\frac{1}{(1+\sigma)}}\left(\fint_{\Omega_{2r}} |\overline{\mathbb{M}}_Q\mathbb{M}^{-1}|^{2(1+\sigma)'}\,dx\right)^{\frac{1}{(1+\sigma)'}}.\end{aligned}$$

Meanwhile, using $|\Omega_{2r}| \eqsim |Q_{2r}|$, (4.2.3) and selecting δ smaller, Lemma 4.2.6

(1) implies

$$\left(\fint_{\Omega_{2r}} |\overline{\mathbb{M}}_Q \mathbb{M}^{-1}|^{2(1+\sigma)'}\, dx\right)^{\frac{1}{(1+\sigma)'}} \leq c \left(\fint_{Q_{2r}} |\overline{\mathbb{M}}_Q \mathbb{M}^{-1}|^{2(1+\sigma)'}\, dx\right)^{\frac{1}{(1+\sigma)'}} \leq c.$$

This and (4.2.32) follow that

$$\fint_{\Omega_{2r}} |\overline{\mathbb{M}}_Q Dh|^2\, dx \leq c \left(\fint_{\Omega_{2r}} |\mathbb{M} Dh|^{2(1+\sigma)}\, dx\right)^{\frac{1}{1+\sigma}} \leq c \fint_{\Omega_{4r}} |\mathbb{M} Dh|^2\, dx.$$

$\square$

We next consider the problems (4.2.31). Let us define a map $a(x_1, \eta)$: $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ such that

$$a(x_1, \eta) := \frac{\overline{\mathbb{M}}_Q \overline{A}_{B'}(x_1, \overline{\mathbb{M}}_Q \eta)}{|\overline{\mathbb{M}}_Q|^2}.$$

We also mention that $\overline{\mathbb{M}}_Q$ is a symmetric and positive definite constant matrix. Then we have

$$\partial_\eta a(x_1, \eta) = \frac{\overline{\mathbb{M}}_Q}{|\overline{\mathbb{M}}_Q|^2} \cdot \partial_\xi \overline{A}_{B'}(x_1, \overline{\mathbb{M}}_Q \eta) \cdot \overline{\mathbb{M}}_Q$$

for a.e. $x_1 \in \mathbb{R}$ and for all $\eta \in \mathbb{R}^n$. By (4.2.26) and (4.2.27), it follows that

$$\begin{aligned} \langle \partial_\eta a(x_1, \eta)\zeta, \zeta \rangle &= \frac{1}{|\overline{\mathbb{M}}_Q|^2} \left\langle \partial_\eta \overline{A}_{B'}(x_1, \overline{\mathbb{M}}_Q \eta) \overline{\mathbb{M}}_Q \zeta, \overline{\mathbb{M}}_Q \zeta \right\rangle \\ &\geq \frac{\nu}{|\overline{\mathbb{M}}_Q|^2} |\overline{\mathbb{M}}_Q \zeta|^2 \geq \frac{\nu}{\Lambda^2} |\zeta|^2. \end{aligned}$$

Then together with (4.2.26), one can see that

$$\begin{cases} |a(x_1, \eta)| + |\partial_\eta a(x_1, \eta)||\eta| \leq L|\eta| \\ \dfrac{\nu}{\Lambda^2} |\zeta|^2 \leq \langle \partial_\eta a(x_1, \eta)\zeta, \zeta \rangle . \end{cases} \tag{4.2.34}$$

Moreover, $W^{1,2}(\Omega_{2r}, d\bar{\omega}_Q^2) = W^{1,2}(\Omega_{2r})$. Then the problem (4.2.31) is converted into

$$\begin{cases} \operatorname{div} a(x_1, Dw) &= 0 \quad \text{in } \Omega_{2r}, \\ w &= h \quad \text{on } \partial\Omega_{2r}. \end{cases} \tag{4.2.35}$$

Now, investigating properties of the problem (4.2.35) with (4.2.34), we obtain the following lemma for the weak solution $w \in W^{1,2}(\Omega_{2r}, d\bar{\omega}_Q^2)$ to (4.2.31).

Lemma 4.2.10. *Under the assumptions and conclusion of Lemma 4.2.9, let $w \in W^{1,2}(\Omega_{2r}, d\bar{\omega}_Q^2)$ be the weak solution to* (4.2.31)*. Then there exists $\delta = \delta(n, \Lambda, \nu, L) > 0$ such that if* (4.2.24) *and* $(4.2.25)_2$ *hold, then w belongs to $W^{1,2}(\Omega_{2r}, d\omega^2)$.*

Proof. We first recall that $|\mathbb{M}Dh| \in L^{2(1+\sigma)}(\Omega_{2r})$ for $\sigma \in (0,1)$ as in Lemma 4.2.8. By Hölder's inequality and Lemma 4.2.6 (1) with the selection of $\delta > 0$ smaller, we obtain that

$$\begin{aligned} \left(\fint_{\Omega_{2r}} |\overline{\mathbb{M}}_Q Dh|^{2(1+\sigma_1)} \, dx \right)^{\frac{1}{1+\sigma_1}} &\leq \left(\fint_{\Omega_{2r}} \left(\frac{|\overline{\mathbb{M}}_Q|}{|\mathbb{M}|} \right)^{\frac{2(2+\sigma)(1+\sigma)}{\sigma}} dx \right)^{\frac{\sigma}{(1+\sigma)(2+\sigma)}} \\ &\quad \times \left(\fint_{\Omega_{2r}} |\mathbb{M}Dh|^{2(1+\sigma)} \, dx \right)^{\frac{1}{1+\sigma}} \\ &\leq c \left(\fint_{\Omega_{2r}} |\mathbb{M}Dh|^{2(1+\sigma)} \, dx \right)^{\frac{1}{1+\sigma}}, \end{aligned} \tag{4.2.36}$$

where $\sigma_1 := \frac{\sigma}{2}$. Since $\overline{\mathbb{M}}_Q$ is a positive definite constant matrix with (4.2.27), this means that $|Dh| \in L^{2(1+\sigma_1)}(\Omega_{2r})$. We now remark that the domain Ω_{2r} satisfies a uniform measure density condition from (4.2.24), and w is also the weak solution of (4.2.35) satisfying (4.2.34) with the boundary value h. Hence, the weak solution w has a global higher integrability for the gradient such that $|Dw| \in L^{2(1+\tau)}(\Omega_{2r})$ for some $0 < \tau \leq \sigma_1$, see [147, 190] for the proof. From (4.2.27), we get $|\overline{\mathbb{M}}_Q Dw| \in L^{2(1+\tau)}(\Omega_{2r})$. Again with Hölder's inequality and Lemma 4.2.6 (1), we deduce $|\mathbb{M}Dw| \in L^2(\Omega_{2r})$ provided that $\delta > 0$ is sufficiently small. Therefore, the conclusion holds. $\square$

We also provide a useful lemma. Note that $\theta(A, Q_{4r})$ is defined in (4.2.9).

Lemma 4.2.11. *For $x \in Q_{4r}$ and $\xi \in \mathbb{R}^n$, we have*

$$\begin{aligned}&|\mathbb{M}A(x, \mathbb{M}\xi) - \overline{\mathbb{M}}_Q \bar{A}_{B'}(x_1, \overline{\mathbb{M}}_Q\xi)| \\ &\quad \leq c|\mathbb{M} - \overline{\mathbb{M}}_Q|(\omega + \bar{\omega}_Q)|\xi| + \bar{\omega}_Q^2 \theta(A, Q_{4r})(x)|\xi|.\end{aligned}$$

Proof. We compute by triangle inequality, (4.2.2), (4.2.9), and (4.2.27),

$$\begin{aligned}&|\mathbb{M}A(x, \mathbb{M}\xi) - \overline{\mathbb{M}}_Q \bar{A}_{B'}(x_1, \overline{\mathbb{M}}_Q\xi)| \\ &\leq |\mathbb{M} - \overline{\mathbb{M}}_Q||A(x, \mathbb{M}\xi)| \\ &\quad + \bar{\omega}_Q|A(x, \mathbb{M}\xi) - A(x, \overline{\mathbb{M}}_Q\xi)| + \bar{\omega}_Q|A(x, \overline{\mathbb{M}}_Q\xi) - \bar{A}_{B'}(x_1, \overline{\mathbb{M}}_Q\xi)| \\ &\leq c|\mathbb{M} - \overline{\mathbb{M}}_Q|\omega|\xi| \\ &\quad + \bar{\omega}_Q|A(x, \mathbb{M}\xi) - A(x, \overline{\mathbb{M}}_Q\xi)| + \bar{\omega}_Q^2 \theta(A, Q_{4r})(x)|\xi|. \end{aligned} \tag{4.2.37}$$

Meanwhile, we have from (4.2.2) that

$$|A(x, \mathbb{M}\xi) - A(x, \overline{\mathbb{M}}_Q\xi)| \leq L|\mathbb{M} - \overline{\mathbb{M}}_Q||\xi|.$$

Inserting this into (4.2.37) we obtain the conclusion. □

Now we derive the following comparison estimate.

Lemma 4.2.12. *Assume that $u \in W^{1,2}(\Omega_{4r}, d\omega^2)$ satisfies the problem (4.2.28). Then for any $\varepsilon \in (0,1)$ there is a constant $\delta = \delta(n, \Lambda, \nu, L, \varepsilon) \in (0,1)$ such that if (4.2.29) holds for some $\lambda \geq 1$ under (4.2.24) and (4.2.25), then there is a function $v \in W^{1,2}(\Omega_r)$ having*

$$\fint_{\Omega_r} |\mathbb{M}Du - \overline{\mathbb{M}}_Q Dv|^2 \, dx \leq \varepsilon\lambda \quad \text{and} \quad \left\| |\overline{\mathbb{M}}_Q Dv| \right\|^2_{L^\infty(\Omega_r)} \leq m_b\lambda \tag{4.2.38}$$

for some $m_b = m_b(n, \Lambda, \nu, L) \geq 1$.

Proof. We first compare u with the weak solution $h \in W^{1,2}(\Omega_{4r}, d\omega^2)$ to the problem (4.2.30). Testing $u - h \in W_0^{1,2}(\Omega_{4r}, d\omega^2)$ to (4.2.28) and (4.2.30), we obtain that

$$\fint_{\Omega_{4r}} \langle A(x, \mathbb{M}Du), \mathbb{M}(Du - Dh)\rangle \, dx = \fint_{\Omega_{4r}} \langle \mathbb{M}F, \mathbb{M}(Du - Dh)\rangle \, dx \tag{4.2.39}$$

and

$$\fint_{\Omega_{4r}} \langle A(x, \mathbb{M}Dh), \mathbb{M}(Du - Dh)\rangle \, dx = 0, \tag{4.2.40}$$

since $\mathbb{M}(x)$ is symmetric. According to (4.2.2), (4.2.21) and (4.2.29), (4.2.40) deduces that

$$\fint_{\Omega_{4r}} |\mathbb{M}Dh|^2 \, dx \leq c \fint_{\Omega_{4r}} |\mathbb{M}Du|^2 \, dx \leq c\lambda \tag{4.2.41}$$

for some $c = c(n, \Lambda, \nu, L) \geq 0$. Moreover, we subtract (4.2.40) from (4.2.39) and then we apply (4.2.21), Young's inequality and (4.2.29), to find

$$\begin{aligned}\fint_{\Omega_{4r}} |\mathbb{M}(Du - Dh)|^2 \, dx &\leq \frac{1}{2} \fint_{\Omega_{4r}} |\mathbb{M}(Du - Dh)|^2 \, dx + c \fint_{\Omega_{4r}} |\mathbb{M}F|^2 \, dx \\ &\leq \frac{1}{2} \fint_{\Omega_{4r}} |\mathbb{M}(Du - Dh)|^2 \, dx + c\delta\lambda\end{aligned}$$

with some $c = c(n, \Lambda, \nu, L) > 0$. This follows

$$\fint_{\Omega_{4r}} |\mathbb{M}(Du - Dh)|^2 \, dx \leq c\delta\lambda. \tag{4.2.42}$$

We second compare h with the weak solution $w \in W^{1,2}(\Omega_{2r}, d\overline{\omega}_Q^2)$ to the problem (4.2.31). We observe $\phi_1 := h - w \in W_0^{1,2}(\Omega_{2r}, d\omega^2) \cap W_0^{1,2}(\Omega_{2r}, d\overline{\omega}_Q^2)$ from Lemma 4.2.9 and Lemma 4.2.10. Testing ϕ_1 to (4.2.30) and (4.2.31), we have

$$\fint_{\Omega_{2r}} \langle A(x, \mathbb{M}Dh), \mathbb{M}(Dh - Dw)\rangle \, dx = 0 \tag{4.2.43}$$

and

$$\fint_{\Omega_{2r}} \left\langle \overline{A}_{B'}(x, \overline{\mathbb{M}}_Q Dw), \overline{\mathbb{M}}_Q(Dh - Dw)\right\rangle \, dx = 0, \tag{4.2.44}$$

since $\overline{\mathbb{M}}_Q$ is also symmetric. Then (4.2.44) and (4.2.26) induce

$$\fint_{\Omega_{2r}} |\overline{\mathbb{M}}_Q Dw|^2 \, dx \le c \fint_{\Omega_{2r}} |\overline{\mathbb{M}}_Q Dh|^2 \, dx,$$

which implies by Lemma 4.2.9 and (4.2.41) that

$$\fint_{\Omega_{2r}} |\overline{\mathbb{M}}_Q Dw|^2 \, dx \le c\lambda \tag{4.2.45}$$

for some $c = c(n, \Lambda, \nu, L) > 0$. Moreover, with (4.2.43) and (4.2.44) we have

$$\begin{aligned}
&\fint_{\Omega_{2r}} \left\langle \bar{A}_{B'}(x_1, \overline{\mathbb{M}}_Q Dw) - \bar{A}_{B'}(x_1, \overline{\mathbb{M}}_Q Dh), \overline{\mathbb{M}}_Q(Dw - Dh) \right\rangle \, dx \\
&\qquad = \fint_{\Omega_{2r}} \left\langle \mathbb{M}A(x, \mathbb{M}Dh) - \overline{\mathbb{M}}_Q \bar{A}_{B'}(x_1, \overline{\mathbb{M}}_Q Dh), Dw - Dh \right\rangle \, dx.
\end{aligned}$$

Since (4.2.21) holds replacing A to $\bar{A}_{B'}$, this leads to

$$\begin{aligned}
&\fint_{\Omega_{2r}} |\overline{\mathbb{M}}_Q(Dw - Dh)|^2 \, dx \\
&\qquad \le c \fint_{\Omega_{2r}} |\mathbb{M}A(x, \mathbb{M}Dh) - \overline{\mathbb{M}}_Q \bar{A}_{B'}(x_1, \overline{\mathbb{M}}_Q Dh)||Dw - Dh| \, dx
\end{aligned}$$

for some $c = c(n, \Lambda, \nu, L) > 0$. We now apply Lemma 4.2.11, (4.2.27) and

Young's inequality, to see that

$$
\begin{aligned}
&\fint_{\Omega_{2r}} |\overline{\mathbb{M}}_Q(Dw - Dh)|^2 \, dx \\
&\le c \fint_{\Omega_{2r}} \left(|\mathbb{M} - \overline{\mathbb{M}}_Q|(\omega + \bar{\omega}_Q) + \bar{\omega}_Q^2 \theta(A, Q_{4r})(x) \right) |Dh||Dw - Dh| \, dx \\
&\le c \fint_{\Omega_{2r}} \left(\frac{|\mathbb{M} - \overline{\mathbb{M}}_Q|}{|\overline{\mathbb{M}}_Q|} \left(1 + \frac{\bar{\omega}_Q}{\omega}\right) + \frac{\bar{\omega}_Q}{\omega} \theta(A, Q_{4r})(x) \right) |\mathbb{M}Dh| \qquad (4.2.46) \\
&\quad \times |\overline{\mathbb{M}}_Q(Dw - Dh)| \, dx \\
&\le \frac{1}{2} \fint_{\Omega_{2r}} |\overline{\mathbb{M}}_Q(Dw - Dh)|^2 \, dx + c \fint_{\Omega_{2r}} \left[\frac{|\mathbb{M} - \overline{\mathbb{M}}_Q|}{|\overline{\mathbb{M}}_Q|} \left(1 + \frac{\bar{\omega}_Q}{\omega}\right) \right]^2 |\mathbb{M}Dh|^2 \, dx \\
&\quad + c \fint_{\Omega_{2r}} \left[\frac{\bar{\omega}_Q}{\omega} \theta(A, Q_{4r})(x) \right]^2 |\mathbb{M}Dh|^2 \, dx. \qquad (4.2.47)
\end{aligned}
$$

By the way, using Hölder's inequality with exponents $(t, 2t', 2t')$ with $t = 1 + \sigma$, we employ Lemma 4.2.5, Lemma 4.2.6, Lemma 4.2.8 and (4.2.25), to have

$$
\begin{aligned}
&\fint_{\Omega_{2r}} \left(\frac{|\mathbb{M} - \overline{\mathbb{M}}_Q|}{|\overline{\mathbb{M}}_Q|} \right)^2 \left(1 + \frac{\bar{\omega}_Q}{\omega}\right)^2 |\mathbb{M}Dh|^2 \, dx \\
&\quad \le c \left[\fint_{Q_{2r}} \left(\frac{|\mathbb{M} - \overline{\mathbb{M}}_Q|}{|\overline{\mathbb{M}}_Q|} \right)^{8t'} dx \right]^{\frac{1}{2t'}} \\
&\qquad \times \left[\fint_{\Omega_{2r}} \left(1 + \frac{\bar{\omega}_Q}{\omega}\right)^{4t'} dx \right]^{\frac{1}{2t'}} \left[\fint_{\Omega_{2r}} |\mathbb{M}Dh|^{2t} \, dx \right]^{\frac{1}{t}} \\
&\quad \le c\delta \fint_{\Omega_{4r}} |\mathbb{M}Dh|^2 \, dx, \qquad (4.2.48)
\end{aligned}
$$

for some $c = c(n, \Lambda, \nu, L) > 0$, provided $\delta = \delta(n, \Lambda, \nu, L) > 0$ sufficiently small. Furthermore, again using Hölder's inequality with exponents $(t, 2t', 2t')$, we employ Lemma 4.2.6, Lemma 4.2.8, (4.2.9) and (4.2.25), to have

$$\begin{aligned}
&\fint_{\Omega_{2r}} \left(\frac{\bar{\omega}_Q}{\omega}\right)^2 [\theta(A, Q_{4r})(x)]^2 \, |\overline{\mathbb{M}}_Q Dh|^2 \, dx \\
&\leq \left[\fint_{\Omega_{2r}} \left(\frac{\bar{\omega}_Q}{\omega}\right)^{4t'} dx\right]^{\frac{1}{2t'}} \left[\fint_{\Omega_{2r}} [\theta(A, Q_{4r})(x)]^{4t'} \, dx\right]^{\frac{1}{2t'}} \left[\fint_{\Omega_{2r}} |\mathbb{M}Dh|^{2t} \, dx\right]^{\frac{1}{t}} \\
&\leq cL^{2-\frac{1}{t'}} \left[\fint_{\Omega_{2r}} \theta(A, Q_{4r})(x)^2 \, dx\right]^{\frac{1}{2t'}} \left[\fint_{\Omega_{2r}} |\mathbb{M}Dh|^{2t} \, dx\right]^{\frac{1}{t}} \\
&\leq c\delta^{\frac{1}{t'}} \fint_{\Omega_{4r}} |\mathbb{M}Dh|^2 \, dx
\end{aligned} \tag{4.2.49}$$

for some $c = c(n, \Lambda, \nu, L) > 0$, provided $\delta = \delta(n, \Lambda, \nu, L) > 0$ sufficiently small. Now, combining all estimates (4.2.46)–(4.2.49), we arrive from (4.2.41)

$$\fint_{\Omega_{2r}} |\overline{\mathbb{M}}_Q(Dw - Dh)|^2 \, dx \leq c\delta^{\frac{\sigma}{1+\sigma}} \lambda \tag{4.2.50}$$

for some $c = c(n, \Lambda, \nu, L) > 0$.

We next consider the problems (4.2.35) as a substitute of the problem (4.2.31), since $w \in W^{1,2}(\Omega_{2r})$ is the weak solution to (4.2.35). We also observe from (4.2.45)

$$\fint_{\Omega_{2r}} |Dw|^2 \, dx \leq |\overline{\mathbb{M}}_Q|^{-2} \fint_{\Omega_{2r}} |\overline{\mathbb{M}}_Q Dw|^2 \, dx \leq c|\overline{\mathbb{M}}_Q|^{-2} \lambda =: c^*. \tag{4.2.51}$$

Then employing the results of [36, Section 5], there exists a function $v \in W^{1,2}(\Omega_r)$ such that

$$\big\| |Dv| \big\|^2_{L^\infty(\Omega_r)} \leq cc^* \quad \text{and} \quad \fint_{\Omega_r} |Dw - Dv|^2 \, dx \leq \varepsilon_1 c^* \tag{4.2.52}$$

with $c = c(n, \Lambda, \nu, L) > 0$ for any $\varepsilon_1 > 0$, selecting $\delta = \delta(n, \Lambda, \nu, L, \varepsilon_1) > 0$ sufficiently small. Then (4.2.51) and (4.2.52) give us that

$$\big\| |\overline{\mathbb{M}}_Q Dv| \big\|^2_{L^\infty(\Omega_r)} \leq |\overline{\mathbb{M}}_Q|^2 \big\| |Dv| \big\|^2_{L^\infty(\Omega_r)} \leq m_b \lambda \tag{4.2.53}$$

and

$$\fint_{\Omega_r} |\overline{\mathbb{M}}_Q(Dw - Dv)|^2 \, dx \le |\overline{\mathbb{M}}_Q|^2 \fint_{\Omega_r} |Dw - Dv|^2 \, dx \le c\varepsilon_1 \lambda \tag{4.2.54}$$

for some constants $m_b = m_b(n, \Lambda, \nu, L) \ge 1$ and $c = c(n, \Lambda, \nu, L) > 0$.

We finally combine (4.2.42), (4.2.50) and (4.2.54), in order to have

$$\begin{aligned}
&\fint_{\Omega_r} |\mathbb{M}Du - \overline{\mathbb{M}}_Q Dv|^2 \, dx \\
&\qquad \le c \fint_{\Omega_r} |\mathbb{M}(Du - Dh)|^2 \, dx + c \fint_{\Omega_r} \left|\left(\mathbb{M} - \overline{\mathbb{M}}_Q\right) Dh\right|^2 \, dx \\
&\qquad\quad + c \fint_{\Omega_r} |\overline{\mathbb{M}}_Q(Dh - Dw)|^2 \, dx + c \fint_{\Omega_r} |\overline{\mathbb{M}}_Q(Dw - Dv)|^2 \, dx \\
&\qquad \le c(\delta^{\frac{\sigma}{1+\sigma}} + \varepsilon_1)\lambda + c \fint_{\Omega_r} \left|\left(\mathbb{M} - \overline{\mathbb{M}}_Q\right) Dh\right|^2 \, dx.
\end{aligned} \tag{4.2.55}$$

Meanwhile, we derive by Hölder's inequality, Lemma 4.2.5, Lemma 4.2.6 (1), Lemma 4.2.8 and (4.2.41),

$$\begin{aligned}
&\fint_{\Omega_r} \left|\left(\mathbb{M} - \overline{\mathbb{M}}_Q\right) Dh\right|^2 \, dx \\
&\le \left(\fint_{\Omega_r} \left| \frac{\mathbb{M} - \overline{\mathbb{M}}_Q}{\overline{\mathbb{M}}_Q} \right|^{\frac{4(1+\sigma)}{\sigma}} \, dx \right)^{\frac{\sigma}{2(1+\sigma)}} \left(\fint_{\Omega_r} \left(\frac{|\overline{\mathbb{M}}_Q|}{|\mathbb{M}|} \right)^{\frac{4(1+\sigma)}{\sigma}} \, dx \right)^{\frac{\sigma}{2(1+\sigma)}} \\
&\quad \times \left(\fint_{\Omega_r} |\mathbb{M}Dh|^{2(1+\sigma)} \, dx \right)^{\frac{1}{1+\sigma}} \\
&\le c\delta^2 \fint_{\Omega_{4r}} |\mathbb{M}Dh|^2 \, dx \le c\delta\lambda,
\end{aligned}$$

provided $\delta > 0$ sufficiently small. With (4.2.55), this yields that

$$\fint_{\Omega_r} |\mathbb{M}Du - \overline{\mathbb{M}}_Q Dv|^2 \, dx \le c(\delta^{\frac{\sigma}{1+\sigma}} + \varepsilon_1)\lambda \tag{4.2.56}$$

for some $c = c(n, \Lambda, \nu, L) > 0$. Choosing $\delta^{\frac{\sigma}{1+\sigma}} < \varepsilon_1$, we eventually conclude the result, since $\varepsilon_1 > 0$ is an arbitrary number. $\square$

We next provide the interior version of Lemma 4.2.12.

Lemma 4.2.13. *Let $\rho < \frac{R}{4}$ and $Q_{4\rho}(x_0) \subset \Omega$. Assume that $u \in W^{1,2}(Q_{4\rho}(x_0), d\omega^2)$ satisfies the problem (4.2.28). Then for any $\varepsilon \in (0,1)$ there is a constant $\delta = \delta(n, \Lambda, \nu, L, \varepsilon) \in (0,1)$ such that if $(\Omega, \mathbb{M}, A)$ is (δ, R)-vanishing of codimension 1, and it holds*

$$\fint_{Q_{4\rho}(x_0)} |\mathbb{M}(x)Du|^2\,dx \leq \lambda \quad \textit{and} \quad \fint_{Q_{4\rho}(x_0)} |\mathbb{M}(x)F|^2\,dx \leq \delta\lambda \tag{4.2.57}$$

for some $\lambda \geq 1$, then there is a function $v \in W^{1,2}(Q_\rho(x_0))$ having

$$\begin{aligned} &\fint_{Q_\rho(x_0)} |\mathbb{M}Du - \overline{\mathbb{M}}_Q Dv|^2\,dx \leq \varepsilon\lambda \\ \textit{and} \quad &\left\| |\overline{\mathbb{M}}_Q Dv| \right\|^2_{L^\infty(Q_\rho(x_0))} \leq m_a\lambda \end{aligned} \tag{4.2.58}$$

for some $m_a = m_a(n, \Lambda, \nu, L) \geq 1$.

In this section, we prove our main theorem. We first construct a suitable collection of mutually disjoint countable cylinders by Vitali covering lemma, to cover the upper level set of $|\mathbb{M}Du|$ with the enlarged cylinders of the collection and derive our gradient estimate for the main theorem. With the comparison estimates obtained previously, we control the measure of the upper level sets of $|\mathbb{M}Du|$ in the cylinders and then finally deduce the gradient estimate that we want. This technique was introduced in [3].

Proof of Theorem 4.2.1.. Suppose that $(\Omega, \mathbb{M}, A)$ is (δ, R)–vanishing of codimension 1 for some $\delta \in (0, \frac{1}{8})$ which is to be determined later depending only on `data` and γ. We define

$$\lambda_\Omega := \fint_\Omega \left(|\mathbb{M}Du|^2 + \frac{|\mathbb{M}F|^2}{\delta} \right) dx, \tag{4.2.59}$$

$$E_u(\lambda) := \{x \in \Omega : |\mathbb{M}Du|^2 > \lambda\} \quad \text{for} \quad \lambda > \left(12000^n \frac{|\Omega|}{|Q_R|} + 1 \right) \lambda_\Omega$$

and

$$\Gamma_y(\rho) := \fint_{\Omega_\rho(y)} \left(|\mathbb{M}Du|^2 + \frac{|\mathbb{M}F|^2}{\delta} \right) dx \quad \text{for} \quad y \in \Omega \text{ and } \rho > 0.$$

Then Lebesgue differentiation theorem gives us that

$$\lim_{\rho \to 0} \Gamma_y(\rho) = |\mathbb{M}(y)Du(y)|^2 + \frac{|\mathbb{M}(y)F(y)|^2}{\delta} > \lambda \quad \text{for a.e. } y \in E_u(\lambda). \tag{4.2.60}$$

Besides, for any $\rho \in \left[\frac{R}{3000}, \frac{R}{2}\right]$, we have from (4.2.11)

$$\Gamma_y(\rho) \leq \frac{|Q_\rho(y)|}{|\Omega_\rho(y)|} \frac{|\Omega|}{|Q_\rho(y)|} \lambda_\Omega \leq 12000^n \frac{|\Omega|}{|Q_R|} \lambda_\Omega. \tag{4.2.61}$$

Then we obtain from (4.2.60) and (4.2.61) that for a.e. $y \in E_u(\lambda)$, there exists $\rho_y \in \left(0, \frac{R}{3000}\right)$ such that

$$\Gamma_y(\rho_y) = \lambda \quad \text{and} \quad \Gamma_y(\rho) < \lambda \quad \text{for any } \rho \in (\rho_y, \tfrac{R}{2}].$$

By Vitali covering lemma, we know the existence of a collection of mutually disjoint cylinders $\{Q_{\rho_j}(y_j)\}_{j=1}^\infty$ with $y_j \in E_u(\lambda)$ and $\rho_j \in \left(0, \frac{R}{3000}\right)$ such that

$$E_u(\lambda) \subset \bigcup_{i=1}^\infty \Omega_{5\rho_j}(y_j) \cup (\text{negligible set}),$$

$$\Gamma_{y_j}(\rho_j) = \fint_{\Omega_{\rho_j}(y_j)} \left(|\mathbb{M}Du|^2 + \frac{|\mathbb{M}F|^2}{\delta} \right) dx = \lambda \tag{4.2.62}$$

and

$$\Gamma_{y_j}(\rho) = \fint_{\Omega_\rho(y_j)} \left(|\mathbb{M}Du|^2 + \frac{|\mathbb{M}F|^2}{\delta} \right) dx < \lambda \quad \forall \rho \in (\rho_j, \tfrac{R}{2}]. \tag{4.2.63}$$

From the above display, one can see that

$$\fint_{\Omega_{20\rho_j}(y_j)} |\mathbb{M}Du|^2\,dx < \lambda \quad \text{and} \quad \fint_{\Omega_{20\rho_j}(y_j)} |\mathbb{M}F|^2\,dx < \delta\lambda. \tag{4.2.64}$$

Let $\varepsilon \in (0,1)$ be given. If $Q_{20\rho_j}(y_j) \subset \Omega$, then by (4.2.64) we have

$$\fint_{Q_{20\rho_j}(y_j)} |\mathbb{M}Du|^2\,dx < \lambda \quad \text{and} \quad \fint_{Q_{20\rho_j}(y_j)} |\mathbb{M}F|^2\,dx < \delta\lambda.$$

Now we use Lemma 4.2.13 so that there is a constant $\delta = \delta(n, \Lambda, \nu, L, \varepsilon) \in (0,1)$ and a function $v_{a_j} \in W^{1,2}(Q_{5\rho_j}(y_j))$ such that we have

$$\begin{aligned} &\fint_{Q_{5\rho_j}(y_j)} |\mathbb{M}Du - \overline{\mathbb{M}}_Q Dv_{a_j}|^2\,dx \le \varepsilon\lambda \\ \text{and} \quad &\sup_{x \in Q_{5\rho_j}(y_j)} |\overline{\mathbb{M}}_Q Dv_{a_j}(x)|^2 \le m_a\lambda \end{aligned} \tag{4.2.65}$$

with $m_a = m_a(n, \Lambda, \nu, L) \ge 1$. Next, we consider the case of $Q_{20\rho_j}(y_j) \not\subset \Omega$. By Definition 4.2.2, there is a coordinate system such that in this coordinate system

$$Q^+_{800\rho_j} \subset \Omega_{800\rho_j} \subset Q_{800\rho_j} \cap \{(x_1, x') : x_1 > -1600\delta\rho_j\}, \tag{4.2.66}$$

$$\begin{aligned} \fint_{Q_\rho(x_0)} |\theta(A, Q_\rho(x_0))(x)|^2 + |\theta(\mathbb{M}, Q_\rho(x_0))(x)|^2\,dx \le \delta^2, \\ \forall Q_\rho(x_0) \subset Q_{800\rho_j}, \end{aligned} \tag{4.2.67}$$

$$\Omega_{5\rho_j}(z_j) \subset \Omega_{160\rho_j} \quad \text{and} \quad \Omega_{640\rho_j} \subset \Omega_{1500\rho_j}(z_j), \tag{4.2.68}$$

where we denote y_j by z_j in the new coordinate system. Here, we remark that this new coordinate system is obtained by rotation and translation. In view of (4.2.66)-(4.2.68) and (4.2.63), we are under the assumptions of Lemma 4.2.12, replacing r by $160\rho_j$. Now, we apply Lemma 4.2.12, to find a function v_{b_j} such that

$$\begin{aligned} &\fint_{\Omega_{160\rho_j}} |\mathbb{M}Du - \overline{\mathbb{M}}_Q Dv_{b_j}|^2\,dx \le \varepsilon\lambda \\ \text{and} \quad &\sup_{x\in\Omega_{160\rho_j}} |\overline{\mathbb{M}}_Q Dv_{b_j}(x)|^2 \le m_b\lambda \end{aligned} \tag{4.2.69}$$

with $m_b = m_b(n, \Lambda, \nu, L) \ge 1$, by selecting $\delta > 0$ smaller. Observing (4.2.68) and (4.2.69), we recover the original coordinate system, to get

$$\begin{aligned} &\fint_{\Omega_{5\rho_j}(y_j)} |\mathbb{M}Du - \overline{\mathbb{M}}_Q Dv_{b_j}|^2\,dx \le \varepsilon\lambda \\ \text{and} \quad &\sup_{x\in\Omega_{5\rho_j}(y_j)} |\overline{\mathbb{M}}_Q Dv_{b_j}(x)|^2 \le m_b\lambda. \end{aligned} \tag{4.2.70}$$

Let v_j be either v_{a_j} or v_{b_j} and let $\bar{c} := \max\{m_a, m_b\} \ge 1$. We first see from (4.2.65) and (4.2.70) that for a.e. $x \in E_u(4\bar{c}\lambda) \cap \Omega_{5\rho_j}(y_j)$,

$$\begin{aligned} |\mathbb{M}Du|^2 &\le 2|\mathbb{M}Du - \overline{\mathbb{M}}_Q Dv_j|^2 + 2|\overline{\mathbb{M}}_Q Dv_j|^2 \\ &\le 2|\mathbb{M}Du - \overline{\mathbb{M}}_Q Dv_j|^2 + \frac{1}{2}|\mathbb{M}Du|^2, \end{aligned}$$

which implies

$$|\mathbb{M}Du|^2 \le 4|\mathbb{M}Du - \overline{\mathbb{M}}_Q Dv_j|^2.$$

Then in light of (4.2.65) and (4.2.70), this follows

$$\begin{aligned} \int_{E_u(4\bar{c}\lambda)\cap\Omega_{5\rho_j}(y_j)} |\mathbb{M}Du|^2\,dx &\le 4\int_{\Omega_{5\rho_j}(y_j)} |\mathbb{M}Du - \overline{\mathbb{M}}_Q Dv_j|^2\,dx \\ &\le c|\Omega_{\rho_j}(y_j)|\varepsilon\lambda \end{aligned} \tag{4.2.71}$$

for some $c \ge 0$. Using (4.2.62), one can easily see that

$$\begin{aligned} &|\Omega_{\rho_j}(y_j)| \\ &\le \frac{2}{\lambda}\left(\int_{E_u(\frac{\lambda}{4})\cap\Omega_{\rho_j}(y_j)} |\mathbb{M}Du|^2\,dx + \int_{E_F(\frac{\delta\lambda}{4})\cap\Omega_{\rho_j}(y_j)} \frac{|\mathbb{M}F|^2}{\delta}\,dx\right), \end{aligned} \tag{4.2.72}$$

where the following notation is used

$$E_F(\lambda) := \{x \in \Omega : |\mathbb{M}F|^2 > \lambda\}.$$

Plugging (4.2.72) to (4.2.71), since $\{B_{\rho_i}(y_i)\}_{i=1}^{\infty}$ is mutually disjoint, we find that

$$\begin{aligned}
&\int_{E_u(4\bar{c}\lambda)} |\mathbb{M}Du|^2\,dx \\
&\leq \sum_{j=1}^{\infty} \int_{E_u(4\bar{c}\lambda)\cap\Omega_{5\rho_j}(y_j)} |\mathbb{M}Du|^2\,dx \\
&\leq c\varepsilon \sum_{j=1}^{\infty} \left(\int_{E_u(\frac{\lambda}{4})\cap\Omega_{\rho_j}(y_j)} |\mathbb{M}Du|^2\,dx + \int_{E_F(\frac{\delta\lambda}{4})\cap\Omega_{\rho_j}(y_j)} \frac{|\mathbb{M}F|^2}{\delta}\,dx \right) \\
&\leq c\varepsilon \left(\int_{E_u(\frac{\lambda}{4})} |\mathbb{M}Du|^2\,dx + \int_{E_F(\frac{\delta\lambda}{4})} \frac{|\mathbb{M}F|^2}{\delta}\,dx \right)
\end{aligned}$$

for some constants $c = c(\mathtt{data})$. Using the similar argument as in [70] and selecting $\varepsilon = \varepsilon(\mathtt{data}, \gamma) \in (0,1)$,

$$\fint_{\Omega} |\mathbb{M}Du|^{\gamma}\,dx \leq c\lambda_{\Omega}^{\frac{\gamma}{2}} + c \fint_{\Omega} |\mathbb{M}F|^{\gamma}\,dx,$$

where $c = c(\mathtt{data})$ and $c(\gamma) = c(\mathtt{data}, \gamma)$. Then there exists $\delta = \delta(\mathtt{data}, \gamma) > 0$ from Lemma 4.2.12 and Lemma 4.2.13 . Now by Jensen's inequality and (4.2.23), we have

$$\lambda_{\Omega}^{\frac{\gamma}{2}} = \left(\fint_{\Omega} \left(|\mathbb{M}Du|^2 + |\mathbb{M}F|^2 \right)\,dx \right)^{\frac{\gamma}{2}} \leq c \left(\fint_{\Omega} |\mathbb{M}F|^2\,dx \right)^{\frac{\gamma}{2}} \leq c \fint_{\Omega} |\mathbb{M}F|^{\gamma}\,dx.$$

Then the proof is completed. $\square$

Bibliography

[1] E. Acerbi and G. Mingione, *Regularity results for a class of functionals with non-standard growth*, Arch. Ration. Mech. Anal. **156** (2001), no. 2, 121–140.

[2] ———, *Gradient estimates for the $p(x)$-Laplacean system*, J. Reine Angew. Math. **584** (2005), 117–148.

[3] ———, *Gradient estimates for a class of parabolic systems*, Duke Math. J. **136** (2007), no. 2, 285–320.

[4] E. Acerbi, G. Mingione, and G. A. Seregin, *Regularity results for parabolic systems related to a class of non-Newtonian fluids*, Ann. Inst. H. Poincaré Anal. Non Linéaire **21** (2004), no. 1, 25–60.

[5] R. A. Adams and J. J. F. Fournier, *Sobolev spaces*, second ed., Pure and Applied Mathematics (Amsterdam), vol. 140, Elsevier/Academic Press, Amsterdam, 2003.

[6] K. Adimurthi and N. C. Phuc, *An end-point global gradient weighted estimate for quasilinear equations in non-smooth domains*, Manuscripta Math. **150** (2016), no. 1-2, 111–135.

[7] F. J. Almgren, Jr., *Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints*, Mem. Amer. Math. Soc. **4** (1976), no. 165, viii+199.

[8] I. Athanasopoulos, L. Caffarelli, and S. Salsa, *Caloric functions in Lipschitz domains and the regularity of solutions to phase transition problems*, Ann. of Math. (2) **143** (1996), no. 3, 413–434.

[9] P. Auscher and M. Qafsaoui, *Observations on $W^{1,p}$ estimates for divergence elliptic equations with VMO coefficients*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) **5** (2002), no. 2, 487–509.

[10] B. Avelin, T. Kuusi, and G. Mingione, *Nonlinear Calderón-Zygmund theory in the limiting case*, Arch. Ration. Mech. Anal. **227** (2018), no. 2, 663–714.

[11] S. Baasandorj, S. Byun, and H. Lee, *Global gradient estimates for a general class of quasilinear elliptic equations with Orlicz growth*, Proc. Amer. Math. Soc. **149** (2021), no. 10, 4189–4206.

[12] S. Baasandorj, S. Byun, and J. Oh, *Calderón-Zygmund estimates for generalized double phase problems*, J. Funct. Anal. **279** (2020), no. 7, 108670, 57.

[13] ———, *Gradient estimates for multi-phase problems*, Calc. Var. Partial Differential Equations **60** (2021), no. 3, 1–48.

[14] A. L. Baisón, A. Clop, R. Giova, J. Orobitg, and A. Passarelli di Napoli, *Fractional differentiability for solutions of nonlinear elliptic equations*, Potential Anal. **46** (2017), no. 3, 403–430.

[15] A. K. Balci, A. Cianchi, L. Diening, and V. G. Maz'ya, *A pointwise differential inequality and second-order regularity for nonlinear elliptic systems*, Mathematische Annalen (online first) (2021).

[16] A. K. Balci, L. Diening, R. Giova, and A. Passarelli di Napoli, *Elliptic equations with degenerate weights*, SIAM J. Math. Anal. **54** (2022), no. 2, 2373–2412.

[17] A. K. Balci, L. Diening, and M. Surnachev, *New examples on Lavrentiev gap using fractals*, Calc. Var. Partial Differential Equations **59** (2020), no. 5, Paper No. 180, 34.

[18] A. K. Balci, L. Diening, and M. Weimar, *Higher order Calderón-Zygmund estimates for the p-Laplace equation*, J. Differential Equation **268** (2020), no. 2, 590–635.

[19] J. Bao and J. Xiong, *Sharp regularity for elliptic systems associated with transmission problems*, Potential Anal. **39** (2013), no. 2, 169–194.

[20] P. Baroni, *Riesz potential estimates for a general class of quasilinear equations*, Calc. Var. Partial Differential Equations **53** (2015), no. 3-4, 803–846.

[21] P. Baroni, M. Colombo, and G. Mingione, *Harnack inequalities for double phase functionals*, Nonlinear Anal. **121** (2015), 206–222.

[22] ———, *Regularity for general functionals with double phase*, Calc. Var. Partial Differential Equations **57** (2018), no. 2, Art. 62, 48.

[23] P. Baroni and C. Lindfors, *The Cauchy-Dirichlet problem for a general class of parabolic equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire **34** (2017), no. 3, 593–624.

[24] L. Beck and G. Mingione, *Lipschitz bounds and non-uniform ellipticity*, Comm. Pure Appl. Math. **73** (2020), 944–1034.

[25] Rajendra Bhatia, *Positive definite matrices*, Princeton Series in Applied Mathematics, Princeton University Press, Princeton, NJ, 2007. MR 2284176

[26] E. Bonnetier and M. Vogelius, *An elliptic regularity result for a composite medium with "touching" fibers of circular cross-section*, SIAM J. Math. Anal. **31** (2000), no. 3, 651–677.

[27] D. Breit, *New regularity theorems for non-autonomous variational integrals with (p, q)-growth*, Calc. Var. Partial Differential Equations **44** (2012), no. 1-2, 101–129.

[28] D. Breit, A. Cianchi, L. Diening, T. Kuusi, and S. Schwarzacher, *Pointwise Calderón-Zygmund gradient estimates for the p-Laplace system*, J. Math. Pures Appl. (9) **114** (2018), 146–190.

[29] D. Breit, A. Cianchi, L. Diening, and S. Schwarzacher, *Global Schauder estimates for the p-Laplace system*, Arch. Ration. Mech. Anal. **243** (2022), no. 1, 201–255.

[30] S. Byun, *Elliptic equations with BMO coefficients in Lipschitz domains*, Trans. Amer. Math. Soc. **357** (2005), no. 3, 1025–1046.

[31] S. Byun, H. Chen, M. Kim, and L. Wang, *L^p regularity theory for linear elliptic systems*, Discrete Contin. Dyn. Syst. **18** (2007), no. 1, 121–134.

[32] S. Byun, N. Cho, and H. Lee, *Maximal differentiability for a general class of quasilinear elliptic equations with right-hand side measures*, Int. Math. Res. Not. IMRN, to appear.

[33] S. Byun and Y. Cho, *Nonlinear gradient estimates for generalized elliptic equations with nonstandard growth in nonsmooth domains*, Nonlinear Anal. **140** (2016), 145–165.

[34] S. Byun, Y. Cho, and J. Oh, *Gradient estimates for double phase problems with irregular obstacles*, Nonlinear Anal. **177** (2018), no. part A, 169–185.

[35] S. Byun, Y. Cho, and J. Ok, *Global gradient estimates for nonlinear obstacle problems with nonstandard growth*, Forum Math. **28** (2016), no. 4, 729–747.

[36] S. Byun and Y. Kim, *Elliptic equations with measurable nonlinearities in nonsmooth domains*, Adv. Math. **288** (2016), 152–200.

[37] ———, *Riesz potential estimates for parabolic equations with measurable nonlinearities*, Int. Math. Res. Not. IMRN (2018), no. 21, 6737–6779.

[38] S. Byun and H. Lee, *Calderón-Zygmund estimates for elliptic double phase problems with variable exponents*, J. Math. Anal. Appl. **501** (2021), no. 1, Paper No. 124015, 31.

[39] ———, *Gradient estimates of ω-minimizers to double phase variational problems with variable exponents*, Q. J. Math. **72** (2021), no. 4, 1191–1221.

[40] S. Byun, S. Liang, and J. Ok, *Irregular double obstacle problems with Orlicz growth*, J. Geom. Anal. **30** (2020), no. 2, 1965–1984.

[41] S. Byun and J. Oh, *Global gradient estimates for non-uniformly elliptic equations*, Calc. Var. Partial Differential Equations **56** (2017), no. 2, Art. 46, 36.

[42] ———, *Global gradient estimates for the borderline case of double phase problems with BMO coefficients in nonsmooth domains*, J. Differential Equations **263** (2017), no. 2, 1643–1693.

[43] ———, *Regularity results for generalized double phase functionals*, Anal. PDE **13** (2020), no. 5, 1269–1300.

[44] S. Byun and J. Ok, *On $W^{1,q(\cdot)}$-estimates for elliptic equations of $p(x)$-Laplacian type*, J. Math. Pures Appl. (9) **106** (2016), no. 3, 512–545.

[45] S. Byun, J. Ok, and S. Ryu, *Global gradient estimates for elliptic equations of $p(x)$-Laplacian type with BMO nonlinearity*, J. Reine Angew. Math. **715** (2016), 1–38.

[46] S. Byun, J. Ok, and Y. Youn, *Global gradient estimates for spherical quasi-minimizers of integral functionals with $p(x)$-growth*, Nonlinear Anal. **177** (2018), no. part A, 186–208.

[47] S. Byun, D. K. Palagachev, and P. Shin, *Global Sobolev regularity for general elliptic equations of p-Laplacian type*, Calc. Var. Partial Differential Equations **57** (2018), no. 5, Paper No. 135, 19.

[48] S. Byun and S. Ryu, *Global weighted estimates for the gradient of solutions to nonlinear elliptic equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire **30** (2013), no. 2, 291–313.

[49] S. Byun, S. Ryu, and P. Shin, *Calderón-Zygmund estimates for ω-minimizers of double phase variational problems*, Appl. Math. Lett. **86** (2018), 256–263.

[50] S. Byun and H. So, *Lipschitz regularity for a general class of quasilinear elliptic equations in convex domains*, J. Math. Anal. Appl. **453** (2017), no. 1, 32–47.

[51] S. Byun and L. Wang, *Elliptic equations with BMO coefficients in Reifenberg domains*, Comm. Pure Appl. Math. **57** (2004), no. 10, 1283–1310.

[52] ———, *The conormal derivative problem for elliptic equations with BMO coefficients on Reifenberg flat domains*, Proc. London Math. Soc. (3) **90** (2005), no. 1, 245–272.

[53] ———, *L^p-estimates for general nonlinear elliptic equations*, Indiana Univ. Math. J. **56** (2007), no. 6, 3193–3221.

[54] ———, *Elliptic equations with measurable coefficients in Reifenberg domains*, Adv. Math. **225** (2010), no. 5, 2648–2673.

[55] ———, *Nonlinear gradient estimates for elliptic equations of general type*, Calc. Var. Partial Differential Equations **45** (2012), no. 3-4, 403–419.

[56] S. Byun, L. Wang, and S. Zhou, *Nonlinear elliptic equations with BMO coefficients in Reifenberg domains*, J. Funct. Anal. **250** (2007), no. 1, 167–196.

[57] S. Byun and Y. Youn, *Potential estimates for elliptic systems with subquadratic growth*, J. Math. Pures Appl. (9) **131** (2019), 193–224.

[58] L. A. Caffarelli and I. Peral, *On $W^{1,p}$ estimates for elliptic equations in divergence form*, Comm. Pure Appl. Math. **51** (1998), no. 1, 1–21.

[59] A. P. Calderon and A. Zygmund, *On the existence of certain singular integrals*, Acta Math. **88** (1952), 85–139.

[60] A. P. Calderón and A. Zygmund, *On singular integrals*, Amer. J. Math. **78** (1956), 289–309.

[61] D. Cao, T. Mengesha, and T. Phan, *Weighted-$W^{1,p}$ estimates for weak solutions of degenerate and singular elliptic equations*, Indiana Univ. Math. J. **67** (2018), no. 6, 2225–2277.

[62] ———, *Gradient estimates for weak solutions of linear elliptic systems with singular-degenerate coefficients*, Nonlinear dispersive waves and fluids, Contemp. Math., vol. 725, Amer. Math. Soc., Providence, RI, 2019, pp. 13–33.

[63] M. Carozza, J. Kristensen, and A. Passarelli di Napoli, *Higher differentiability of minimizers of convex variational integrals*, Ann. Inst. H. Poincaré Anal. Non Linéaire **28** (2011), no. 3, 395–411.

[64] ———, *Regularity of minimizers of autonomous convex variational integrals*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **13** (2014), no. 4, 1065–1089.

[65] M. Cencelj, V. D. Rădulescu, and D. D. Repovš, *Double phase problems with variable growth*, Nonlinear Anal. **177** (2018), no. part A, 270–287.

[66] S. Challal and A. Lyaghfouri, *Second order regularity for the A-Laplace operator*, Mediterr. J. Math. **7** (2010), no. 3, 283–296.

[67] M. Chipot, D. Kinderlehrer, and G. Vergara-Caffarelli, *Smoothness of linear laminates*, Arch. Ration. Mech. Anal. **96** (1986), no. 1, 81–96.

[68] I. Chlebicka, *Gradient estimates for problems with Orlicz growth*, Nonlinear Anal. **194** (2020), 111364, 32.

[69] I. Chlebicka and C. De Filippis, *Removable sets in non-uniformly elliptic problems*, Ann. Mat. Pura Appl. (4) **199** (2020), no. 2, 619–649.

[70] Y. Cho, *Global gradient estimates for divergence-type elliptic problems involving general nonlinear operators*, J. Differential Equations **264** (2018), no. 10, 6152–6190.

[71] A. Cianchi and N. Fusco, *Gradient regularity for minimizers under general growth conditions*, J. Reine Angew. Math. **507** (1999), 15–36.

[72] A. Cianchi and V. Maz'ya, *Global Lipschitz regularity for a class of quasilinear elliptic equations*, Comm. Partial Differential Equations **36** (2011), no. 1, 100–133.

[73] ______, *Global boundedness of the gradient for a class of nonlinear elliptic systems*, Arch. Ration. Mech. Anal. **212** (2014), no. 1, 129–177.

[74] A. Cianchi and V. G. Maz'ya, *Second-order two-sided estimates in nonlinear elliptic problems*, Arch. Ration. Mech. Anal. **229** (2018), no. 2, 569–599.

[75] A. Clop, R. Giova, and A. Passarelli di Napoli, *Besov regularity for solutions of p-harmonic equations*, Adv. Nonlinear Anal. **8** (2019), no. 1, 762–778.

[76] F. Colasuonno and M. Squassina, *Eigenvalues for double phase variational integrals*, Ann. Mat. Pura Appl. (4) **195** (2016), no. 6, 1917–1959.

[77] M. Colombo and G. Mingione, *Bounded minimisers of double phase variational integrals*, Arch. Ration. Mech. Anal. **218** (2015), no. 1, 219–273.

[78] ———, *Regularity for double phase variational problems*, Arch. Ration. Mech. Anal. **215** (2015), no. 2, 443–496.

[79] ———, *Calderón-Zygmund estimates and non-uniformly elliptic operators*, J. Funct. Anal. **270** (2016), no. 4, 1416–1478.

[80] B. Dacorogna, *Direct methods in the calculus of variations*, second ed., Applied Mathematical Sciences, vol. 78, Springer, New York, 2008.

[81] G. David and T. Toro, *Reifenberg parameterizations for sets with holes*, Mem. Amer. Math. Soc. **215** (2012), no. 1012, vi+102.

[82] C. De Filippis, *On the regularity of the ω-minima of ϕ-functionals*, Nonlinear Anal. **194** (2020), 111464, 25.

[83] ———, *Optimal gradient estimates for multi-phase integrals*, Math. Eng. **4** (2022), no. 5, Paper No. 043, 36.

[84] C. De Filippis and G. Mingione, *A borderline case of Calderón-Zygmund estimates for nonuniformly elliptic problems*, St. Petersburg Math. J. **31** (2020), 455–477.

[85] ———, *Manifold constrained non-uniformly elliptic problems*, J. Geom. Anal. **30** (2020), no. 2, 1661–1723.

[86] ———, *On the regularity of minima of non-autonomous functionals*, J. Geom. Anal. **30** (2020), no. 2, 1584–1626.

[87] ———, *Lipschitz bounds and nonautonomous integrals*, Arch. Ration. Mech. Anal. **242** (2021), no. 2, 973–1057.

[88] C. De Filippis and J. Oh, *Regularity for multi-phase variational problems*, J. Differential Equations **267** (2019), no. 3, 1631–1670.

[89] G. Di Fazio, *L^p estimates for divergence form elliptic equations with discontinuous coefficients*, Boll. Un. Mat. Ital. A (7) **10** (1996), no. 2, 409–420.

[90] G. Di Fazio, M. S. Fanciullo, and P. Zamboni, *L^p estimates for degenerate elliptic systems with VMO coefficients*, Algebra i Analiz **25** (2013), no. 6, 24–36.

[91] E. DiBenedetto and J. Manfredi, *On the higher integrability of the gradient of weak solutions of certain degenerate elliptic systems*, Amer. J. Math. **115** (1993), no. 5, 1107–1134.

[92] L. Diening and F. Ettwein, *Fractional estimates for non-differentiable elliptic systems with general growth*, Forum Math. **20** (2008), no. 3, 523–556.

[93] L. Diening, M. Fornasier, R. Tomasi, and M. Wank, *A relaxed Kačanov iteration for the p-Poisson problem*, Numer. Math. **145** (2020), no. 1, 1–34.

[94] L. Diening, P. Harjulehto, P. Hästö, and M. Růžička, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, vol. 2017, Springer, Heidelberg, 2011.

[95] L. Diening, P. Kaplický, and S. Schwarzacher, *BMO estimates for the p-Laplacian*, Nonlinear Anal. **75** (2012), no. 2, 637–650.

[96] L. Diening and C. Kreuzer, *Linear convergence of an adaptive finite element method for the p-Laplacian equation*, SIAM J. Numer. Anal. **46** (2008), no. 2, 614–638.

[97] L. Diening, D. Lengeler, B. Stroffolini, and A. Verde, *Partial regularity for minimizers of quasi-convex functionals with general growth*, SIAM J. Math. Anal. **44** (2012), no. 5, 3594–3616.

[98] L. Diening, T. Scharle, and S. Schwarzacher, *Regularity for parabolic systems of Uhlenbeck type with Orlicz growth*, J. Math. Anal. Appl. **472** (2019), no. 1, 46–60.

[99] L. Diening and S. Schwarzacher, *Global gradient estimates for the $p(\cdot)$-Laplacian*, Nonlinear Anal. **106** (2014), 70–85.

[100] L. Diening, B. Stroffolini, and A. Verde, *Everywhere regularity of functionals with φ-growth*, Manuscripta Math. **129** (2009), no. 4, 449–481.

[101] A. Dolcini, L. Esposito, and N. Fusco, *$C^{0,\alpha}$ regularity of ω-minima*, Boll. Un. Mat. Ital. A (7) **10** (1996), no. 1, 113–125.

[102] H. Dong, *Gradient estimates for parabolic and elliptic systems from linear laminates*, Arch. Ration. Mech. Anal. **205** (2012), no. 1, 119–149.

[103] H. Dong and D. Kim, *Elliptic equations in divergence form with partially BMO coefficients*, Arch. Ration. Mech. Anal. **196** (2010), no. 1, 25–70.

[104] ______, *Parabolic and elliptic systems in divergence form with variably partially BMO coefficients*, SIAM J. Math. Anal. **43** (2011), no. 3, 1075–1098.

[105] ______, *L_q-estimates for stationary Stokes system with coefficients measurable in one direction*, Bull. Math. Sci. **9** (2019), no. 1, 1950004, 30.

[106] H. Dong and T. Phan, *Parabolic and elliptic equations with singular or degenerate coefficients: the Dirichlet problem*, Trans. Amer. Math. Soc. **374** (2021), no. 9, 6611–6647.

[107] ______, *Regularity for parabolic equations with singular or degenerate coefficients*, Calc. Var. Partial Differential Equations **60** (2021), no. 1, Paper No. 44, 39.

[108] ______, *Weighted mixed-norm L_p-estimates for elliptic and parabolic equations in non-divergence form with singular coefficients*, Rev. Mat. Iberoam. **37** (2021), no. 4, 1413–1440.

[109] H. Dong and J. Xiong, *Boundary gradient estimates for parabolic and elliptic systems from linear laminates*, Int. Math. Res. Not. IMRN (2015), no. 17, 7734–7756.

[110] H. Dong and L. Xu, *Gradient estimates for divergence form elliptic systems arising from composite material*, SIAM J. Math. Anal. **51** (2019), no. 3, 2444–2478.

[111] I. Drelichman and R. G. Durán, *Improved Poincaré inequalities with weights*, J. Math. Anal. Appl. **347** (2008), no. 1, 286–293.

[112] F. Duzaar and G. Mingione, *Gradient estimates via linear and nonlinear potentials*, J. Funct. Anal. **259** (2010), no. 11, 2961–2998.

[113] J. Elschner, J. Rehberg, and G. Schmidt, *Optimal regularity for elliptic transmission problems including C^1 interfaces*, Interfaces Free Bound. **9** (2007), no. 2, 233–252.

[114] L. Escauriaza, E. B. Fabes, and G. Verchota, *On a regularity theorem for weak solutions to transmission problems with internal Lipschitz boundaries*, Proc. Amer. Math. Soc. **115** (1992), no. 4, 1069–1076.

[115] L. Esposito, F. Leonetti, and G. Mingione, *Regularity for minimizers of functionals with p-q growth*, NoDEA Nonlinear Differential Equations Appl. **6** (1999), no. 2, 133–148.

[116] ______, *Regularity results for minimizers of irregular integrals with (p, q) growth*, Forum Math. **14** (2002), no. 2, 245–272.

[117] ______, *Sharp regularity for functionals with (p, q) growth*, J. Differential Equations **204** (2004), no. 1, 5–55.

[118] L. Esposito, G. Mingione, and C. Trombetti, *On the Lipschitz regularity for certain elliptic problems*, Forum Math. **18** (2006), no. 2, 263–292.

[119] L. C. Evans, *Partial differential equations*, second ed., Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 2010.

[120] E. B. Fabes, C. E. Kenig, and R. P. Serapioni, *The local regularity of solutions of degenerate elliptic equations*, Comm. Partial Differential Equations **7** (1982), no. 1, 77–116.

[121] I. Fonseca, J. Malý, and G. Mingione, *Scalar minimizers with fractal singular sets*, Arch. Ration. Mech. Anal. **172** (2004), no. 2, 295–307.

[122] José García-Cuerva and José L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland Mathematics Studies, vol. 116, North-Holland Publishing Co., Amsterdam, 1985, Notas de Matemática [Mathematical Notes], 104.

[123] A. G. Ghazanfari, *Refined Heinz operator inequalities and norm inequalities*, Oper. Matrices **15** (2021), no. 1, 239–252.

[124] F. Giannetti, A. Passarelli di Napoli, M. Ragusa, and A. Tachikawa, *Partial regularity for minimizers of a class of non autonomous functionals with nonstandard growth*, Calc. Var. Partial Differential Equations **56** (2017), no. 6, Paper No. 153, 29.

[125] D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001, Reprint of the 1998 edition.

[126] E. Giusti, *Direct methods in the calculus of variations*, World Scientific Publishing Co., Inc., River Edge, NJ, 2003.

[127] Loukas Grafakos, *Modern Fourier analysis*, second ed., Graduate Texts in Mathematics, vol. 250, Springer, New York, 2009.

[128] P. Harjulehto and P. Hästö, *The Riesz potential in generalized Orlicz spaces*, Forum Math. **29** (2017), no. 1, 229–244.

[129] ______, *Orlicz spaces and generalized Orlicz spaces*, Lecture Notes in Mathematics, vol. 2236, Springer, Cham, 2019.

[130] P. Harjulehto and P. Hästö, *Double phase image restoration*, J. Math. Anal. Appl. **501** (2021), no. 1, 123832.

[131] P. Harjulehto, P. Hästö, and A. Karppinen, *Local higher integrability of the gradient of a quasiminimizer under generalized Orlicz growth conditions*, Nonlinear Anal. **177** (2018), no. part B, 543–552.

[132] P. Harjulehto, P. Hästö, and R. Klén, *Generalized Orlicz spaces and related PDE*, Nonlinear Anal. **143** (2016), 155–173.

[133] P. Harjulehto, P. Hästö, and M. Lee, *Hölder continuity of ω-minimizers of functionals with generalized Orlicz growth*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **22** (2021), no. 2, 549–582.

[134] P. Harjulehto, P. Hästö, and O. Toivanen, *Hölder regularity of quasiminimizers under generalized growth conditions*, Calc. Var. Partial Differential Equations **56** (2017), no. 2, Paper No. 22, 26.

[135] P. Hästö and J. Ok, *Higher integrability for parabolic systems with Orlicz growth*, J. Differential Equations **300** (2021), 925–948.

[136] ———, *Maximal regularity for local minimizers of non-autonomous functionals*, J. Eur. Math. Soc. (JEMS) **24** (2022), no. 4, 1285–1334.

[137] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1993, Oxford Science Publications.

[138] J. Isralowitz and K. Moen, *Matrix weighted Poincaré inequalities and applications to degenerate elliptic systems*, Indiana Univ. Math. J. **68** (2019), no. 5, 1327–1377.

[139] T. Iwaniec, *On L^p-integrability in PDEs and quasiregular mappings for large exponents*, Ann. Acad. Sci. Fenn. Ser. A I Math. **7** (1982), no. 2, 301–322.

[140] ———, *Projections onto gradient fields and L^p-estimates for degenerated elliptic operators*, Studia Math. **75** (1983), no. 3, 293–312.

[141] T. Iwaniec and C. Sbordone, *Riesz transforms and elliptic PDEs with VMO coefficients*, J. Anal. Math. **74** (1998), 183–212.

[142] Y. Jang and Y. Kim, *Gradient estimates for solutions of elliptic systems with measurable coefficients from composite material*, Math. Methods Appl. Sci. **41** (2018), no. 16, 7007–7031.

[143] V. V. Jikov, S. M. Kozlov, and O. A. Oleĭnik, *Homogenization of differential operators and integral functionals*, Springer-Verlag, Berlin, 1994, Translated from the Russian by G. A. Yosifian [G. A. Iosif'yan].

[144] A. Karppinen and M. Lee, *Hölder continuity of the minimizer of an obstacle problem with generalized orlicz growth*, arXiv preprint arXiv:2006.08244 (2020).

[145] C. Kenig and T. Toro, *Free boundary regularity for harmonic measures and Poisson kernels*, Ann. of Math. (2) **150** (1999), no. 2, 369–454.

[146] C. E. Kenig and Z. Shen, *Homogenization of elliptic boundary value problems in Lipschitz domains*, Math. Ann. **350** (2011), no. 4, 867–917.

[147] T. Kilpeläinen and P. Koskela, *Global integrability of the gradients of solutions to partial differential equations*, Nonlinear Anal. **23** (1994), no. 7, 899–909.

[148] D. Kim and N. V. Krylov, *Elliptic differential equations with coefficients measurable with respect to one variable and VMO with respect to the others*, SIAM J. Math. Anal. **39** (2007), no. 2, 489–506.

[149] Y. Kim, *Gradient estimates for elliptic equations with measurable nonlinearities*, J. Math. Pures Appl. (9) **114** (2018), 118–145.

[150] Y. Kim and S. Ryu, *Global gradient estimates for parabolic equations with measurable nonlinearities*, Nonlinear Anal. **164** (2017), 77–99.

[151] Y. Kim, L. Wang, C. Zhang, and S. Zhou, *Global gradient estimates for $p(x)$-Laplace equation in non-smooth domains*, Commun. Pure Appl. Anal. **13** (2014), no. 6, 2559–2587.

[152] J. Kinnunen and S. Zhou, *A local estimate for nonlinear equations with discontinuous coefficients*, Comm. Partial Differential Equations **24** (1999), no. 11-12, 2043–2068.

[153] S. Kinnunen, J.and Zhou, *A boundary estimate for nonlinear equations with discontinuous coefficients*, Differential Integral Equations **14** (2001), no. 4, 475–492.

[154] J. Kristensen and G. Mingione, *The singular set of ω-minima*, Arch. Ration. Mech. Anal. **177** (2005), no. 1, 93–114.

[155] ———, *The singular set of Lipschitzian minima of multiple integrals*, Arch. Ration. Mech. Anal. **184** (2007), no. 2, 341–369.

[156] T. Kuusi and G. Mingione, *New perturbation methods for nonlinear parabolic problems*, J. Math. Pures Appl. (9) **98** (2012), no. 4, 390–427.

[157] ———, *Linear potentials in nonlinear potential theory*, Arch. Ration. Mech. Anal. **207** (2013), no. 1, 215–246.

[158] ———, *A nonlinear Stein theorem*, Calc. Var. Partial Differential Equations **51** (2014), no. 1-2, 45–86.

[159] ———, *Vectorial nonlinear potential theory*, J. Eur. Math. Soc. (JEMS) **20** (2018), no. 4, 929–1004.

[160] O. A. Ladyzhenskaya and N. N. Ural'tseva, *Linear and quasilinear equations of elliptic type*, Izdat. Nauka, Moscow, 1973, Second edition, revised.

[161] J. L. Lewis, *Regularity of the derivatives of solutions to certain degenerate elliptic equations*, Indiana Univ. Math. J. **32** (1983), no. 6, 849–858.

[162] Y. Li and L. Nirenberg, *Estimates for elliptic systems from composite material*, vol. 56, 2003, Dedicated to the memory of Jürgen K. Moser, pp. 892–925.

[163] Y. Y. Li and M. Vogelius, *Gradient estimates for solutions to divergence form elliptic equations with discontinuous coefficients*, Arch. Ration. Mech. Anal. **153** (2000), no. 2, 91–151.

[164] S. Liang, M. Cai, and S. Zheng, *Global regularity in Lorentz spaces for nonlinear elliptic equations with $L^{p(\cdot)}\log L$-growth*, J. Math. Anal. Appl. **467** (2018), no. 1, 67–94.

[165] S. Liang and S. Zheng, *Calderón-Zygmund estimate for asymptotically regular non-uniformly elliptic equations*, J. Math. Anal. Appl. **484** (2020), no. 2, 123749, 17.

[166] G. M. Lieberman, *Oblique derivative problems in Lipschitz domains. I. Continuous boundary data*, Boll. Un. Mat. Ital. B (7) **1** (1987), no. 4, 1185–1210.

[167] ———, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal. **12** (1988), no. 11, 1203–1219.

[168] ———, *Oblique derivative problems in Lipschitz domains. II. Discontinuous boundary data*, J. Reine Angew. Math. **389** (1988), 1–21.

[169] ———, *The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations*, Comm. Partial Differential Equations **16** (1991), no. 2-3, 311–361.

[170] ______, *Higher regularity for nonlinear oblique derivative problems in Lipschitz domains*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **1** (2002), no. 1, 111–151.

[171] P. Marcellini, *On the definition and the lower semicontinuity of certain quasiconvex integrals*, Ann. Inst. H. Poincaré Anal. Non Linéaire **3** (1986), no. 5, 391–409.

[172] ______, *Regularity of minimizers of integrals of the calculus of variations with nonstandard growth conditions*, Arch. Rational Mech. Anal. **105** (1989), no. 3, 267–284.

[173] ______, *Regularity and existence of solutions of elliptic equations with* p, q*-growth conditions*, J. Differential Equations **90** (1991), no. 1, 1–30.

[174] O. Martio, *Reflection principle for solutions of elliptic partial differential equations and quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A I Math. **6** (1981), no. 1, 179–187.

[175] T. Mengesha and N. C. Phuc, *Weighted and regularity estimates for nonlinear equations on Reifenberg flat domains*, J. Differential Equation **250** (2011), no. 5, 2485–2507.

[176] ______, *Global estimates for quasilinear elliptic equations on Reifenberg flat domains*, Arch. Ration. Mech. Anal. **203** (2012).

[177] N. G. Meyers, *An* L^p*-estimate for the gradient of solutions of second order elliptic divergence equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **17** (1963), 189–206.

[178] G. Mingione, *Regularity of minima: an invitation to the dark side of the calculus of variations*, Appl. Math. **51** (2006), no. 4, 355–426.

[179] ______, *The Calderón-Zygmund theory for elliptic problems with measure data*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **6** (2007), no. 2, 195–261.

[180] ______, *Gradient estimates below the duality exponent*, Math. Ann. **346** (2010), no. 3, 571–627.

[181] G. Mingione and V. Rădulescu, *Recent developments in problems with nonstandard growth and nonuniform ellipticity*, J. Math. Anal. Appl. **501** (2021), no. 1, Paper No. 125197, 41.

[182] Y. Mizuta, E. Nakai, T. Ohno, and T. Shimomura, *Campanato-Morrey spaces for the double phase functionals with variable exponents*, Nonlinear Anal. **197** (2020), 111827, 19.

[183] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Mathematics, vol. 1034, Springer-Verlag, Berlin, 1983.

[184] T. Nguyen and T. Phan, *Interior gradient estimates for quasilinear elliptic equations*, Calc. Var. Partial Differential Equations **55** (2016), no. 3, Art. 59, 33.

[185] R. Nittka, *Regularity of solutions of linear second order elliptic and parabolic boundary value problems on Lipschitz domains*, J. Differential Equations **251** (2011), no. 4-5, 860–880.

[186] J. Ok, *Gradient estimates for elliptic equations with $L^{p(\cdot)}\log L$ growth*, Calc. Var. Partial Differential Equations **55** (2016), no. 2, Art. 26, 30.

[187] ———, *Calderón-Zygmund estimates for ω-minimizers*, J. Differential Equations **263** (2017), no. 5, 3090–3109.

[188] ———, *Regularity of ω-minimizers for a class of functionals with non-standard growth*, Calc. Var. Partial Differential Equations **56** (2017), no. 2, Paper No. 48, 31.

[189] ———, *Regularity for double phase problems under additional integrability assumptions*, Nonlinear Anal. **194** (2020), 111408, 13.

[190] M. Parviainen, *Global higher integrability for parabolic quasiminimizers in nonsmooth domains*, Calc. Var. Partial Differential Equations **31** (2008), no. 1, 75–98.

[191] T. Phan, *Weighted Calderón-Zygmund estimates for weak solutions of quasi-linear degenerate elliptic equations*, Potential Anal. **52** (2020), no. 3, 393–425.

[192] N. C. Phuc, *Weighted estimates for nonhomogeneous quasilinear equations with discontinuous coefficients*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **10** (2011), no. 1, 1–17.

[193] ______, *Nonlinear Muckenhoupt-Wheeden type bounds on Reifenberg flat domains, with applications to quasilinear Riccati type equations*, Adv. Math. **250** (2014), 387–419.

[194] M. Ragusa and A. Tachikawa, *Regularity for minimizers for functionals of double phase with variable exponents*, Adv. Nonlinear Anal. **9** (2020), no. 1, 710–728.

[195] M. M. Rao and Z. D. Ren, *Theory of Orlicz spaces*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 146, Marcel Dekker, Inc., New York, 1991.

[196] M. Růžička, *Flow of shear dependent electrorheological fluids*, C. R. Acad. Sci. Paris Sér. I Math. **329** (1999), no. 5, 393–398.

[197] G. Savaré, *Regularity results for elliptic equations in Lipschitz domains*, J. Funct. Anal. **152** (1998), no. 1, 176–201.

[198] X. Shi, V. Rădulescu, D. Repovš, and Q. Zhang, *Multiple solutions of double phase variational problems with variable exponent*, Adv. Calc. Var. **13** (2020), no. 4, 385–401.

[199] P. Shin, *Calderón-Zygmund estimates for general elliptic operators with double phase*, Nonlinear Anal. **194** (2020), 111409, 16.

[200] R. E. Showalter, *Monotone operators in Banach space and nonlinear partial differential equations*, Mathematical Surveys and Monographs, vol. 49, American Mathematical Society, Providence, RI, 1997.

[201] H. Tian and S. Zheng, *Morrey regularity for nonlinear elliptic equations with partial BMO nonlinearities under controlled growth*, Nonlinear Anal. **180** (2019), 1–19.

[202] T. Toro, *Doubling and flatness: geometry of measures*, Notices Amer. Math. Soc. **44** (1997), no. 9, 1087–1094.

[203] K. Uhlenbeck, *Regularity for a class of non-linear elliptic systems*, Acta Math. **138** (1977), no. 3-4, 219–240.

[204] A. Verde, *Calderón-Zygmund estimates for systems of ϕ-growth*, J. Convex Anal. **18** (2011), no. 1, 67–84.

[205] F. Yao and S. Zhou, *Calderón-Zygmund estimates for a class of quasi-linear elliptic equations*, J. Funct. Anal. **272** (2017), no. 4, 1524–1552.

[206] V. V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Izv. Akad. Nauk SSSR Ser. Mat. **50** (1986), no. 4, 675–710.

[207] ———, *Lavrentiev phenomenon and homogenization for some variational problems*, C. R. Acad. Sci. Paris Sér. I Math. **316** (1993), no. 5, 435–439.

[208] ———, *On Lavrentiev's phenomenon*, Russian J. Math. Phys. **3** (1995), no. 2, 249–269.

[209] ———, *On some variational problems*, Russian J. Math. Phys. **5** (1997), no. 1, 105–116.

국문초록

이 학위논문에서는 다양한 종류의 발산형 타원 방정식과 범함수에 대해 칼데론-지그문드 추정이라 불리는 정칙성 결과를 조사한다. p-라플라스 방정식의 여러 일반화가 고려되는데, 우선 오리츠 증가 조건을 갖는 문제와 관련하여 좀 더 일반적인 형태의 비선형성을 포함하는 방정식과, 측정 가능한 비선형성이 있는 방정식을 연구한다. 또한 일반적인 이중 위상 문제와 이의 변수지수로의 확장을 고려한다. 구체적으로 BMO 비선형성이 있는 비균일 타원 문제에 대한 방정식, 변수지수를 갖는 이중 위상 문제에 대한 범함수의 오메가-최소자, 변수 지수가 있는 오리츠 이중 위상 문제에 대한 방정식을 다룬다.

다음으로 축퇴/특이 계수가 있는 타원 방정식에 대한 대역적 칼데론-지그문드 이론을 수립한다. 여기서 계수는 행렬 가중치로서 그 크기가 무켄호프트 류에 속한다. 우선 립쉬츠 영역에서 축퇴/특이 가중치를 사용하여 라플라스 및 p-라플라스 방정식에 대한 극대 정칙성을 증명한다. 더 높은 적분가능성에 대한 지수와 작은 매개변수 가정 사이의 예리한 관계도 추가적으로 밝혔다. 마지막으로, 라이펜버그 영역에서 행렬 가중치와 측정 가능한 비선형성을 포함하는 방정식을 고려하고, 대역적 가중 그래디언트 추정치를 증명한다.

주요어휘: 칼데론-지그문드 이론, 오리츠 증가, 변수 지수, 이중 위상 문제, 퇴화 가중치, 무켄호프트 류
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