이학박사 학위논문

Combinatorial interpretation of Howe duality of symplectic type

(사교 타입 하우 쌍대성의 조합론적 해석)

2022년 8월

서울대학교 대학원

수리과학부

허 태 혁

Combinatorial interpretation of Howe duality of symplectic type

(사교 타입 하우 쌍대성의 조합론적 해석)

지도교수 권 재 훈

이 논문을 이학박사 학위논문으로 제출함

2022년 **4**월

서울대학교 대학원

수리과학부

허 태 혁

허 태 혁의 이학박사 학위논문을 인준함

2022년 **6**월

위 원 장	이 승 진	(인)
부위원장	권 재 훈	(인)
위 원	서 의 린	(인)
위 원	오 영 탁	(인)
위 원	김 장 수	(인)

Combinatorial interpretation of Howe duality of symplectic type

A dissertation
submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
to the faculty of the Graduate School of
Seoul National University

by

Taehyeok Heo

Dissertation Director : Professor Jae-Hoon Kwon

Department of Mathematical Sciences
Seoul National University

August 2022

# Abstract

There is a well-known bijection called the Robinson-Schensted-Knuth correspondence, which explains the Howe duality for a pair of general linear groups. It is given by a combinatorial algorithm for semistandard tableaux, which is closely related to irreducible representations of general linear groups. In this thesis, we give a combinatorial interpretation of a Howe duality associated with a pair of a symplectic group and a Lie (super)algebra. We establish a symplectic analogue of the RSK correspondence via symplectic tableaux models: spinor model and King tableaux, which are related to representations of symplectic groups and Lie (super)algebras. We introduce a symplectic analogue of jeu de taquin sliding for spinor model to define an insertion tableau in a uniform way that does not depend on the set of letters for tableaux and assign a King tableau as its recording tableau. We give new bijective proofs of well-known identities for irreducible symplectic characters as a corollary.

Key words: crystal graph, Howe duality, Robinson-Schensted-Knuth correspondene, jeu de taquin, spinor model
Student Number: 2015-20283

Contents

Chapter 1

Introduction

Duality is one of the important themes in representation theory. While a duality in representation theory often means a functor between some categories of modules, it can be also given by the multiplicity-free decomposition of some spaces into irreducible bimodules. For example, we have the following $(GL_n(\mathbb{C}), S_k)$-bimodule decomposition, which is called the Schur-Weyl duality [45]:

$$(\mathbb{C}^n)^{\otimes k} \cong \bigoplus_{\lambda} V_{GL_n(\mathbb{C})}(\lambda) \otimes S_k^{\lambda},$$

where $\mathbb{C}^n$ is the vector representation of $GL_n(\mathbb{C})$, λ is a partition of k with $\ell(\lambda) \leq n$, and $V_{GL_n(\mathbb{C})}(\lambda)$ is the highest weight $GL_n(\mathbb{C})$-module of highest weight corresponding to λ, and S_k^{λ} is the Specht module over S_k corresponding to λ. Note that the decomposition indeed yields a functor between the category of $GL_n(\mathbb{C})$-modules and that of S_k-modules sending $V_{GL_n(\mathbb{C})}(\lambda)$ to S_k^{λ}, which is often referred to the Schur-Weyl duality functor.

1.1 Howe duality

Besides the Schur-Weyl duality, there are many dualities. One important method to obtain a duality is the theory of reductive dual pairs due to Howe [10, 11], which we often call Howe duality. In this section, we briefly describe some of its results, which are motivations for this thesis.

For positive integers n and r, we have the $(GL_n(\mathbb{C}), GL_r(\mathbb{C}))$-bimodule decomposition of a symmetric algebra

$$\mathcal{S}(\mathbb{C}^n \otimes \mathbb{C}^r) \cong \bigoplus_{\lambda} V_{GL_n(\mathbb{C})}(\lambda) \otimes V_{GL_r(\mathbb{C})}(\lambda), \tag{1.1}$$

where λ is a partition with $\ell(\lambda) \leq \min\{n, r\}$. It is generalized to the $(\mathfrak{g}, GL_r(\mathbb{C}))$-bimodule decomposition, where $\mathfrak{g}$ is a Lie (super)algebra of type A, with irreducible unitarizable highest weight $\mathfrak{g}$-modules $V_{\mathfrak{g}}(\lambda)$ instead of $V_{GL_n(\mathbb{C})}(\lambda)$ (see [4, 6, 10, 14] and references therein for more details).

There is also a duality associated to a pair $(\mathfrak{g}, \mathrm{Sp}_{2\ell}(\mathbb{C}))$, where $\mathfrak{g}$ is a Lie (super)algebra of classical type. For later use, let us explain more precisely. Let $\mathbb{Z}_+$ be the set of nonnegative integers. Let $\mathscr{P}$ be the set of partitions and $\mathscr{P}_n$ ($n \in \mathbb{Z}_+$) be the subset of $\mathscr{P}$ such that the length $\ell(\lambda)$ of $\lambda \in \mathscr{P}$ is less than or equal to n. Denote by $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ the transpose of $\lambda \in \mathscr{P}$, where λ'_i is the number of boxes in the ith column from the left, and by $|\lambda| = \lambda_1 + \lambda_2 + \cdots$ the size of λ.

Let $\mathcal{A}$ be a $\mathbb{Z}_2$-graded linearly ordered set and let $\mathscr{E}_{\mathcal{A}}$ be the super exterior algebra generated by the superspace with a linear basis indexed by $\mathcal{A}$. Then $\mathscr{F}_{\mathcal{A}} = \mathscr{E}^*_{\mathcal{A}} \otimes \mathscr{E}_{\mathcal{A}}$ is a semisimple module over a classical Lie (super)algebra $\mathfrak{g}_{\mathcal{A}}$, the type of which depends on $\mathcal{A}$, and the ℓ-fold tensor power $\mathscr{F}_{\mathcal{A}}^{\otimes \ell}$ ($\ell \geq 1$) is a $(\mathfrak{g}_{\mathcal{A}}, \mathrm{Sp}_{2\ell}(\mathbb{C}))$-bimodule with the following multiplicity-free bimodule decomposition:

$$\mathscr{F}_{\mathcal{A}}^{\otimes \ell} \cong \bigoplus_{(\lambda,\ell) \in \mathscr{P}(\mathrm{Sp})_{\mathcal{A}}} V_{\mathfrak{g}_{\mathcal{A}}}(\lambda, \ell) \otimes V_{\mathrm{Sp}_{2\ell}(\mathbb{C})}(\lambda), \tag{1.2}$$

where the direct sum is over a set $\mathscr{P}(\mathrm{Sp})_{\mathcal{A}}$ of pairs $(\lambda, \ell) \in \mathscr{P} \times \mathbb{Z}_+$ with $\ell(\lambda) \leq \ell$. Here $V_{\mathrm{Sp}_{2\ell}(\mathbb{C})}(\lambda)$ is the irreducible $\mathrm{Sp}_{2\ell}(\mathbb{C})$-module corresponding to λ, and $V_{\mathfrak{g}_{\mathcal{A}}}(\lambda, \ell)$ is the irreducible highest weight $\mathfrak{g}_{\mathcal{A}}$-module corresponding to $V_{\mathrm{Sp}_{2\ell}(\mathbb{C})}(\lambda)$ appearing in $\mathscr{F}_{\mathcal{A}}^{\otimes \ell}$ (see Remark 6.1.5). Note that every selection of $\mathcal{A}$ does not necessarily define the Lie (super)algebra $\mathfrak{g}_{\mathcal{A}}$ and give the decomposition (1.2). Further known results are found in [10–12, 29, 44].

1.2 Robinson-Schensted-Knuth correspondence

The decomposition (1.1) has a nice combinatorial interpretation known as the Robinson-Schensted-Knuth (simply RSK) correspondence [21], which is given by a bijection between combinatorial objects parameterizing the bases for both sides of (1.1).

The RSK correspondence is well-studied and generalized in many directions in combinatorics, but it still has fundamental significance in representation theory. Precisely speaking, it is not only a bijection but also an isomorphism of bicrystals. A (bi-)crystal is introduced by Kashiwara [16] to understand a combinatorial structure of irreducible representations of quantum groups. It is shown in [30] that the RSK correspondence is an isomorphism of $(\mathfrak{gl}_n, \mathfrak{gl}_r)$-bicrystals. It is also possible to generalize (1.1) to a pair $(\mathfrak{g}, \mathfrak{gl}_r)$, where $\mathfrak{g}$ is a Lie (super)algebra of type A, and give a bicrystal isomorphism [3, 10, 23, 24] (It can be proved without difficulty).

The correspondence is also closely related to symmetric functions. Since a crystal has a weight function, we can consider the weight generating function for crystals, i.e., the formal power sum of monomials determined by the weight function. Then the Cauchy type identity, one of the most important identities in the theory of symmetric functions, is deduced as a corollary of the correspondence.

1.3 Main results

In this thesis, we construct the following bijection, which explains the Howe duality (1.2) of symplectic type.

$$\begin{aligned} \mathbf{F}^{\ell}_{\mathcal{A}} &\longrightarrow \bigsqcup_{\lambda \in \mathscr{P}(\mathrm{Sp})_{\mathcal{A}}} \mathbf{T}_{\mathcal{A}}(\lambda, \ell) \times \mathbf{K}(\lambda, \ell) \\ \mathbf{T} &\longmapsto \qquad (\mathtt{P}(\mathbf{T}), \mathtt{Q}(\mathbf{T})) \end{aligned} \tag{1.3}$$

The main combinatorial object is a spinor model, which is introduced by Kwon [25]. The set $\mathbf{F}^{\ell}_{\mathcal{A}}$ is the set of (2ℓ)-tuples of $\mathcal{A}$-semistandard column tableaux. The spinor model $\mathbf{T}_{\mathcal{A}}(\lambda, \ell)$ is the subset of $\mathbf{F}^{\ell}_{\mathcal{A}}$ satisfying certain configuration conditions associated with $(\lambda, \ell) \in \mathscr{P}(\mathrm{Sp})_{\mathcal{A}}$. It gives the character of the irreducible $\mathfrak{g}_{\mathcal{A}}$-module $V_{\mathfrak{g}_{\mathcal{A}}}(\lambda, \ell)$ appearing in (1.2). In addition, when $\mathcal{A}$ is a finite set of degree 0, $\mathbf{T}_{\mathcal{A}}(\lambda, \ell)$ has a one-to-one correspondence with the set of Kashiwara-Nakashima (simply KN) tableaux [18], which are one of the well-known combinatorial objects describing irreducible characters for classical Lie algebras. In this sense, the spinor model is considered a super analogue of KN tableaux. We remark that the spinor model has several applications including branching multiplicities for classical groups [13, 27], crystal bases of quantum superalgebras of orthosymplectic type [25, 26], and generalized exponents [13, 32].

To obtain the insertion tableau $\mathtt{P}(\mathbf{T})$, we define a symplectic jeu de taquin for a spinor model, which is a symplectic analogue of the usual jeu de taquin for semistandard tableaux due to Schützenberger [40]. Since there is a bijection between KN tableaux and the spinor model for a finite set $\mathcal{A}$ of even degree. We have a symplectic jeu de taquin for the spinor model induced from the one for KN tableaux by Sheats [41]. A key observation to define a symplectic jeu de taquin to the spinor model for arbitrary $\mathcal{A}$ is that the induced symplectic jeu de taquin can be described as applying a sequence of Kashiwara operators with respect to a $\mathfrak{gl}_{2\ell}$-crystal structure on $\mathbf{F}^{\ell}_{\mathcal{A}}$. In fact, applying the sequence to $\mathbf{T}$ is described in terms of jeu de taquin sliding of type A [30]. This implies that the algorithm (or sequence of Kashiwara operators) does *not* depend on the choice of $\mathcal{A}$, and enables us to define a symplectic analogue of jeu de taquin uniformly using the jeu de taquin of type A (cf. [2, 37]).

On the other hand, we use the symplectic RSK correspondence due to Lecouvey [31] to give an oscillating tableau $Q(\mathbf{T})$ as a recording tableau for $\mathtt{P}(\mathbf{T})$. Recently, Lee [33] gave a bijection between oscillating tableaux and King tableaux [20], which are another mostly used combinatorial objects for irreducible modules over symplectic Lie algebras. Finally, using this bijection, we obtain a tableau $\mathtt{Q}(\mathbf{T})$ in $\mathbf{K}(\lambda, \ell)$, the set of King tableaux of shape λ, which corresponds to the oscillating tableau $Q(\mathbf{T})$.

1.4 Organization

The organization of the thesis is as follows.

- In Chapters 2, 3, and 4, we give preliminaries including crystals and combinatorics of tableaux. Especially, combinatorics of (usual) semistandard tableaux is covered in Chapter 3, and combinatorics of $\mathcal{A}$-semistandard tableaux, a generalization of semistandard tableaux, is covered in Chapter 4.

- In Chapters 5 and 6, we review two main combinatorial models for irreducible characters of symplectic type and their properties. We consider KN tableaux in Chapter 5 and the spinor model in Chapter 6 together with the relation between them.

- In Chapter 7, we define a symplectic jeu de taquin for the spinor model. It is compatible with a symplectic jeu de taquin for KN tableaux when $\mathcal{A}$ is a finite set with $\mathcal{A} = \mathcal{A}_0$ and it is defined as a sequence of Kashiwara operators regardless of choices of $\mathcal{A}$, or as applying jeu de taquin sliding of type A.

- In Chapter 8, we introduce a set of oscillating tableaux which is a one-to-one correspondence with the set of King tableaux of a given shape.
- In Chapter 9, we prove the main result and its character identity. We recover well-known classical identities for irreducible characters of symplectic groups.

This thesis is based on the works in [8].

Chapter 2

Crystals

In this chapter, we review the notion of crystals and their properties. The theory of crystal base introduced by Kashiwara is one of the fundamental tools to understand the representations of the quantized universal enveloping algebras $U_q(\mathfrak{g})$. The crystal base is important in the sense that it reflects a combinatorial behavior of modules, and has many applications including the irreducible character problem and the decomposition of tensor products. This chapter is organized as follows.

- In Section 2.1, we define the quantum groups and their representations.
- In Section 2.2, we recall the notion of crystals and give examples.
- In Section 2.3, we briefly summarize the crystal base of an irreducible highest weight module.

This chapter is based on [9, 17] and references therein.

2.1 Quantum groups and their representation

Let $\mathfrak{g}$ be the symmetrizable Kac-Moody algebra associated to a Cartan datum $(A, P, P^\vee, \Pi, \Pi^\vee)$ (cf. [15]), where $A = (a_{ij})$ is a generalized Cartan matrix indexed by a set I. Especially, let $\Pi = \{\alpha_i \in P : i \in I\}$ be the set of simple roots, and $\Pi^\vee = \{h_i \in P^\vee : i \in I\}$ be the set of simple coroots. We have a canonical pairing $\langle \cdot, \cdot \rangle : P^\vee \times P \to \mathbb{Q}$ such that $\langle h_i, \alpha_j \rangle = a_{ij}$ for $i, j \in I$. Let $Q = \bigoplus_i \mathbb{Z}\alpha_i$ and $Q^+ = \bigoplus_i \mathbb{Z}_+\alpha_i$ and $P^+ \subseteq P$ be the set of integral dominant weights, i.e., $\lambda \in P^+$ if and only if $\langle h, \lambda \rangle \in \mathbb{Z}_+$ for all $h \in P^\vee$. Fix a diagonal matrix $D = \mathrm{diag}(s_i \,|\, i \in I)$ such that DA is symmetric.

On the other hand, there exists a non-degenerate symmetric bilinear form $(\cdot,\cdot)$ on the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Let ϖ_i be the ith fundamental weight ($i \in I$), i.e., $(\varpi_i, \alpha_j) = \delta_{i,j}$ for $i, j \in I$. Then we have $\langle h_i, \alpha_j \rangle = \frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$ for $i, j \in I$.

For an indeterminate q, define the q-integer by

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

for $n \in \mathbb{N}$. Let $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$ for $n \in \mathbb{N}$ and $[0]_q! = 1$. Define the q-binomial coefficients by

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[n]_q![m-n]_q!}$$

for nonnegative integer $m \geq n \geq 0$.

Definition 2.1.1. The quantum group or quantized enveloping algebra $U_q(\mathfrak{g})$ associated with $\mathfrak{g}$ is the associative $\mathbb{Q}(q)$-algebra with 1 generated by q^h ($h \in P^\vee$), e_i and f_i ($i \in I$) with the following relations:

(1) $q^0 = 1, q^h q^{h'} = q^{h+h'}$ for $h, h' \in P^\vee$,

(2) $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i,\ q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i$ for $i \in I, h \in P^\vee$,

(3) $e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$ for $i, j \in I$,

(4) $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} e_i^{1-a_{ij}-k}\, e_j\, e_i^k = 0$ for $i \neq j$,

(5) $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} f_i^{1-a_{ij}-k}\, f_j\, f_i^k = 0$ for $i \neq j$,

where $q_i = q^{s_i}$ and $K_i = q_i^{h_i}$.

The quantum group $U_q(\mathfrak{g})$ has the root space decomposition

$$U_q(\mathfrak{g}) = \bigoplus_{\alpha \in Q} U_q(\mathfrak{g})_\alpha,$$

where $U_q(\mathfrak{g})_\alpha = \{\, u \in U_q(\mathfrak{g}) \,|\, q^h u q^{-h} = q^{\langle h,\alpha \rangle} u$ for all $h \in P^\vee \,\}$. In addition, let $U_q^\pm$ be the subalgebra of $U_q(\mathfrak{g})$ generated by all e_i, f_i, respectively, and U_q^0 be the subalgebra of

$U_q(\mathfrak{g})$ generated by q^h. Then

$$U_q(\mathfrak{g}) = U_q^- \otimes U_q^0 \otimes U_q^+,$$

which is called the triangular decomposition of $U_q(\mathfrak{g})$. Note that $U_q^{\pm} = \bigoplus_{\alpha \in Q^+} U_q(\mathfrak{g})_{\pm\alpha}$ and $U_q^0 = U_q(\mathfrak{g})_0$.

Now we consider representations of $U_q(\mathfrak{g})$. A $U_q(\mathfrak{g})$-module V is called a weight module if V admits a weight space decomposition

$$V = \bigoplus_{\mu \in P} V_\mu,$$

where $V_\mu = \{\, v \in V \mid q^h v = q^{\langle h, \mu \rangle} v \text{ for all } h \in P^\vee \,\}$. In this case, we call a nonzero vector $v \in V_\mu$ a weight vector of weight μ. If $V_\mu \neq 0$, we say that μ is a weight of V and V_μ is the μ-weight space of V.

A weight module V over $U_q(\mathfrak{g})$ is integrable if all e_i and f_i $(i \in I)$ are locally nilpotent on V, i.e., for any nonzero $v \in V$, there exists $N \in \mathbb{Z}_+$ such that $e_i^N v = f_i^N v = 0$.

Definition 2.1.2. The category $\mathcal{O}_{\mathrm{int}}^q$ consists of $U_q(\mathfrak{g})$-modules M satisfying the following conditions:

(1) M is a weight module and integrable, and

(2) the set $\mathrm{wt}(M)$ of weights of M is finitely dominated, i.e., there exists a finite number of elements $\lambda_1, \ldots, \lambda_s \in P$ such that

$$\mathrm{wt}(M) \subseteq \bigcup_{i=1}^{s} (\lambda_i - Q^+).$$

The morphisms are taken to be usual $U_q(\mathfrak{g})$-module homomorphisms.

Note that $\mathcal{O}_{\mathrm{int}}^q$ is closed under taking direct sums or tensor products of finitely many $U_q(\mathfrak{g})$-modules. Also, it is known that $\mathcal{O}_{\mathrm{int}}^q$ is semisimple.

We say that a weight module M is a highest weight module of highest weight $\lambda \in P$ if there exists a vector $v \in M_\lambda$ such that M is generated by v and $e_i v = \mathbf{0}$ for all $i \in I$. In this case, we call v a highest weight vector of M, which is unique up to constant multiple. By construction, $M = U_q^- v$ and $\mathrm{wt}(M) \subseteq \lambda - Q^+$. For example, for $\lambda \in P^+$, let $V(\lambda)$ be the $U_q(\mathfrak{g})$-module generated by the element v_λ with the relations

$$q^h v_\lambda = q^{\langle h,\lambda\rangle} v_\lambda, \quad e_i v_\lambda = \mathbf{0}, \quad f_i^{1+\langle h_i,\lambda\rangle} v_\lambda = \mathbf{0}$$

for $h \in P^\vee$ and $i \in I$. By its defining relations, it is clear that $V(\lambda)$ is in $\mathcal{O}^q_{\mathrm{int}}$.

2.2 Crystals

We assume the notation in Section 2.1.

Definition 2.2.1. A $\mathfrak{g}$-crystal is a set B endowed with the functions $\mathrm{wt} : B \to P$, $\widetilde{e}_i, \widetilde{f}_i : B \to B \cup \{\mathbf{0}\}$, $\varepsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{-\infty\}$ $(i \in I)$ satisfying the followings: for $i \in I$,

(1) $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b)\rangle$ for $b \in B$,

(2) $\mathrm{wt}(\widetilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$, $\varepsilon_i(\widetilde{e}_i b) = \varepsilon_i(b) - 1$, and $\varphi_i(\widetilde{e}_i b) = \varphi_i(b) + 1$ if $\widetilde{e}_i b \in B$,

(3) $\mathrm{wt}(\widetilde{f}_i b) = \mathrm{wt}(b) - \alpha_i$, $\varepsilon_i(\widetilde{f}_i b) = \varepsilon_i(b) + 1$, and $\varphi_i(\widetilde{f}_i b) = \varphi_i(b) - 1$ if $\widetilde{f}_i b \in B$,

(4) $b' = \widetilde{f}_i b$ if and only if $b = \widetilde{e}_i b'$ for $b, b' \in B$, and

(5) if $\varphi_i(b) = -\infty$, then $\widetilde{e}_i b = \widetilde{f}_i b = \mathbf{0}$.

Here, $\mathbf{0}$ is a formal symbol and $-\infty$ is the smallest element in $\mathbb{Z} \cup \{-\infty\}$ with $(-\infty) + (-\infty) = -\infty$ and $(-\infty) + n = -\infty$ for all $n \in \mathbb{Z}$. We call $\widetilde{e}_i, \widetilde{f}_i$ $(i \in I)$ crystal operators.

When there is no confusion, a $\mathfrak{g}$-crystal can be simply called a crystal.

For a crystal B, the crystal graph of B is an I-directed graph whose vertex set is B and $b \xrightarrow{i} b'$ if and only if $b' = \widetilde{f}_i b$ for $i \in I$ and $b, b' \in B$. To describe crystals, we often present their crystal graphs.

Example 2.2.2.

(1) For $\lambda \in P$, the singleton set $T_\lambda = \{t_\lambda\}$ is a crystal with

$$\mathrm{wt}(t_\lambda) = \lambda \text{ and } \varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty.$$

(2) For $i \in I$, $\mathbf{B}_i = \{b_i(n) : n \in \mathbb{Z}\}$ is a crystal whose crystal graph is

$$\cdots \xrightarrow{i} b_i(1) \xrightarrow{i} b_i(0) \xrightarrow{i} b_i(-1) \xrightarrow{i} b_i(-2) \xrightarrow{i} \cdots$$

with $\mathrm{wt}(b_i(n)) = n\alpha_i$, $\varepsilon_i(b_i(n)) = -n$, $\varphi_i(b_i(n)) = n$, and $\varepsilon_j(b_i(n)) = \varphi_j(b_i(n)) = -\infty$ for all $j \neq i$.

(3) For $\lambda \in P^+$, let $B(\lambda)$ be a crystal basis of $V(\lambda)$ (cf. Section 2.3). For example, if we take $\mathfrak{g} = \mathfrak{sl}_2$ and $m \in \mathbb{Z}_+$ under the identification P^+ with $\mathbb{Z}_+$, then $B(m)$ has the following crystal graph

$$v \longrightarrow fv \longrightarrow f^{(2)}v \longrightarrow \cdots \longrightarrow f^{(m)}v$$

with $\mathrm{wt}(f^{(k)}v) = m - 2k$, $\varepsilon_i(f^{(k)}v) = k$, and $\varphi_i(f^{(k)}v) = m - k$ for $0 \leq k \leq m$. Here, v is the highest weight vector of $B(\lambda)$, i.e., $\widetilde{e}_i v = \mathbf{0}$ for all $i \in I$.

A crystal B is called semi-regular (or semi-normal) if for $i \in I$ and $b \in B$, we have

$$\varepsilon_i(b) = \max\{k \in \mathbb{Z}_+ | \, \widetilde{e}_i^k(b) \neq 0\}, \quad \varphi_i(b) = \max\{k \in \mathbb{Z}_+ | \, \widetilde{f}_i^k(b) \neq 0\}.$$

Note that the crystal $B(\lambda)$ ($\lambda \in P^+$) is semi-regular, but $\mathbf{B}_i$ is not. A crystal B is called regular (or normal) if B is isomorphic to a direct sum of $B(\lambda)$'s for $\lambda \in P^+$, explained in the next paragraph.

Let B_1 and B_2 be crystals. The direct sum of B_1 and B_2, denoted by $B_1 \oplus B_2$, is the disjoint union of B_1 and B_2, where $\widetilde{e}_i, \widetilde{f}_i, \varepsilon_i, \varphi_i$, and wt are naturally induced from those on B_1 and B_2. Note that the crystal graph of $B_1 \oplus B_2$ is the union of two crystal graphs. The tensor product of B_1 and B_2, denoted by $B_1 \otimes B_2$, is the product $B_1 \times B_2$ as a set with the following maps:

(1) $\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2)$,

(2) $\varepsilon_i(b_1 \otimes b_2) = \max\{\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, \mathrm{wt}(b_1)\rangle\}$,

(3) $\varphi_i(b_1 \otimes b_2) = \max\{\varphi_i(b_1) + \langle h_i, \mathrm{wt}(b_2), \varphi_i(b_2)\rangle\}$,

(4) $\widetilde{e}_i(b_1 \otimes b_2) = \begin{cases} \widetilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \widetilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$

(5) $\widetilde{f}_i(b_1 \otimes b_2) = \begin{cases} \widetilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \widetilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$

Here, we write $(b_1, b_2) \in B_1 \otimes B_2$ as $b_1 \otimes b_2$ and assume $b_1 \otimes \mathbf{0} = \mathbf{0} \otimes b_2 = \mathbf{0}$.

Example 2.2.3. Let u, v be the highest weight vectors of $\mathfrak{sl}_2$-crystals $B(3)$ and $B(2)$, respectively. Then the crystal graph of $B(3) \otimes B(2)$ is given as follows.

$$
\begin{array}{ccccccc}
u \otimes v & \longrightarrow & fu \otimes v & \longrightarrow & f^{(2)}u \otimes v & \longrightarrow & f^{(3)} \otimes v \\
 & & & & & & \downarrow \\
u \otimes fv & \longrightarrow & fu \otimes fv & \longrightarrow & f^{(2)}u \otimes fv & & f^{(3)} \otimes fv \\
 & & & & \downarrow & & \downarrow \\
u \otimes f^{(2)}v & \longrightarrow & fu \otimes f^{(2)}v & & f^{(2)}u \otimes f^{(2)}v & & f^{(3)} \otimes f^{(2)}v
\end{array}
$$

In general, there is a combinatorial rule called the i-signature, which describes the action of crystal operators on a tensor product of more than two crystals. Let $B_1, \ldots, B_r$ be crystals and consider $\widetilde{x}_i(b_1 \otimes \cdots \otimes b_r)$ for $b_1 \otimes \cdots \otimes b_r \in B_1 \otimes \cdots \otimes B_r$ ($x = e$ or f).

(1) Assign each b_j to a sequence of $+$ and $-$ of the form $\underbrace{- - \cdots -}_{\varepsilon_i(b_j)}\underbrace{+ + \cdots +}_{\varphi_i(b_j)}$.

(2) Find a pair of $+$ and $-$ such that $+$ is left to $-$ and there is no sign between them.

(3) Replace the pairs in (2) by a symbol $\cdot$, which is ignored when we count the sign.

(4) Repeat the steps (2) and (3) above until there is no $+$ left to $-$.

Then the crystal operator $\widetilde{e}_i$ (resp. $\widetilde{f}_i$) acts on the component b_j, which contains the right-most $-$ (resp. left-most $+$).

Example 2.2.4. For $i = 1, 2, 3, 4$, let v_i be the highest weight vector of $\mathfrak{sl}_2$-crystal B_i which is isomorphic to $B(4), B(0), B(2), B(3)$, respectively, and consider the element $b = fv_1 \otimes v_2 \otimes fv_3 \otimes f^{(2)}v_4 \in B_1 \otimes B_2 \otimes B_3 \otimes B_4$.

$$
\begin{array}{rcccc}
 & fv_1 & v_2 & fv_3 & f^{(2)}v_4 \\
 & - + + \color{red}{+} & \cdot & \color{red}{-}\color{blue}{+} & \color{blue}{-} - + \\
\longrightarrow & - + \color{red}{+} \cdot & \cdot & \cdot\cdot & \cdot \color{red}{-} + \\
\longrightarrow & - + \cdot\cdot & \cdot & \cdot\cdot & \cdot\cdot +
\end{array}
$$

As we can observe, we obtain $\widetilde{e}(b) = v_1 \otimes v_2 \otimes fv_3 \otimes f^{(2)}v_4$, $\widetilde{f}(b) = f^{(2)}v_1 \otimes v_2 \otimes fv_3 \otimes f^{(2)}v_4$, and $\widetilde{f}^2(b) = f^{(2)}v_1 \otimes v_2 \otimes fv_3 \otimes f^{(3)}v_4$.

Definition 2.2.5. Let B_1 and B_2 be crystals.

(1) A crystal morphism $\phi : B_1 \to B_2$ is a map $\phi : B_1 \cup \{\mathbf{0}\} \to B_2 \cup \{\mathbf{0}\}$ such that

   (a) $\phi(0) = 0$,

   (b) if $b \in B_1$ and $f(b) \in B_2$, then $\mathrm{wt}(\phi(b)) = \mathrm{wt}(b)$, $\varepsilon_i(\phi(b)) = \varepsilon_i(b)$, and $\varphi_i(\phi(b)) = \varphi_i(b)$ for all $i \in I$,

   (c) if $b, b' \in B_1$ and $\phi(b), \phi(b') \in B_2$ with $b' = \widetilde{f}_i b$, then $\phi(\widetilde{f}_i b) = \widetilde{f}_i \phi(b)$ and $\phi(\widetilde{e}_i b') = \widetilde{e}_i \phi(b')$.

(2) A crystal morphism $\phi : B_1 \to B_2$ is strict if it commutes with $\widetilde{e}_i$ and $\widetilde{f}_i$ for all i.

(3) A crystal morphism $\phi : B_1 \to B_2$ is an embedding if $\phi : B_1 \cup \{\mathbf{0}\} \to B_2 \cup \{\mathbf{0}\}$ is injective.

(4) A crystal morphism $\phi : B_1 \to B_2$ is an isomorphism if $\phi : B_1 \cup \{\mathbf{0}\} \to B_2 \cup \{\mathbf{0}\}$ is bijective.

For $b \in B$, denote by $C(b)$ the connected component of B containing b. For $\mathfrak{g}$-crystals B_1 and B_2, we say that $b_1 \in B_1$ is $\mathfrak{g}$-crystal equivalent to $b_2 \in B_2$ if there exists a crystal isomorphism $\phi : C(b_1) \to C(b_2)$ sending b_1 to b_2. In this case, we write $b_1 \overset{\mathfrak{g}}{\equiv} b_2$.

2.3 Crystal bases

Let M be an integrable $U_q(\mathfrak{g})$-module. For each $i \in I$, any weight vector $u \in M_\lambda$ can be written in the form

$$u = \sum_{n=0}^{N} f_i^{(n)} u_n$$

where $N \in \mathbb{Z}_+$ and $f_i^{(n)} = \frac{f^n}{[n]_{q_i}!}$ and $u_n \in \ker e_i \cap M_{\lambda + n\alpha_i}$ for $0 \le n \le N$. Furthermore, each u_n is uniquely determined by u and $u_n \neq 0$ only if $\lambda(h_i) + n \ge 0$. Then we define

$$\widetilde{e}_i u = \sum_{n=1}^{N} f_i^{(n-1)} u_n, \quad \widetilde{f}_i = \sum_{n=0}^{N} f_i^{(n+1)} u_n.$$

Note that $\widetilde{e}_i u \in M_{\lambda+\alpha_i}$ and $\widetilde{f}_i u \in M_{\lambda-\alpha_i}$ for $u \in M_\lambda$. We call $\widetilde{e}_i, \widetilde{f}_i$ the Kashiwara operators.

Let

$$\mathbf{A} = \left\{ \frac{f}{g} \,\middle|\, f, g \in \mathbb{Q}[q], h(0) \neq 0 \right\}.$$

Definition 2.3.1. An **A**-lattice $\mathcal{L}$ of M is called a crystal lattice if

(1) $\mathcal{L} = \bigoplus_{\lambda \in P} \mathcal{L}_\lambda$, where $\mathcal{L}_\lambda = \mathcal{L} \cap M_\lambda$ for all $\lambda \in P$,

(2) $\widetilde{e}_i \mathcal{L} \subseteq \mathcal{L}$, $\widetilde{f}_i \mathcal{L} \subseteq \mathcal{L}$ for all $i \in I$.

By (2), Kashiwara operators on M induces the $\mathbb{Q}$-linear operators on $\mathcal{L}/q\mathcal{L}$, for which we use same notations.

Definition 2.3.2. A crystal basis of M is a pair $(\mathcal{L}, \mathcal{B})$ such that

(1) $\mathcal{L}$ is a crystal lattice of M,

(2) $\mathcal{B}$ is a $\mathbb{Q}$-basis of $\mathcal{L}/q\mathcal{L}$,

(3) $\mathcal{B} = \sqcup_\lambda \mathcal{B}_\lambda$, where $\mathcal{B}_\lambda = \mathcal{B} \cap \mathcal{L}_\lambda / q\mathcal{L}_\lambda$,

(4) $\widetilde{e}_i \mathcal{B} \subseteq \mathcal{B} \cup \{\mathbf{0}\}$, $\widetilde{f}_i \mathcal{B} \subseteq \mathcal{B} \cup \{\mathbf{0}\}$ for all $i \in I$, and

(5) for any $b, b' \in \mathcal{B}$ and $i \in I$, $\widetilde{e}_i b = b'$ if and only if $\widetilde{f}_i b' = b$.

Here, $\mathbf{0}$ is the zero element of $\mathcal{L}/q\mathcal{L}$.

Set $\mathrm{wt}(b) = \lambda$ for $b \in \mathcal{B}_\lambda$ and

$$\varepsilon_i(b) = \max\{k \in \mathbb{Z}_+ |\, \widetilde{e}_i^k(b) \neq 0\}, \quad \varphi_i(b) = \max\{k \in \mathbb{Z}_+ |\, \widetilde{f}_i^k(b) \neq 0\}$$

for $i \in I$ and $b \in \mathcal{B}$. It is obvious by definition that $\varepsilon_i(\widetilde{e}_i b) = \varepsilon_i(b) - 1$, and $\varphi_i(\widetilde{e}_i b) = \varphi_i(b) + 1$ if $\widetilde{e}_i b \in B$, $\varepsilon_i(\widetilde{f}_i b) = \varepsilon_i(b) + 1$, and $\varphi_i(\widetilde{f}_i b) = \varphi_i(b) - 1$ if $\widetilde{f}_i b \in B$. In addition, we can check that $\varphi_i(b) - \varepsilon_i(b) = \langle h_i, \lambda \rangle$ for $b \in \mathcal{B}_\lambda$ by the theory of $U_q(\mathfrak{sl}_2)$-modules. Thus, $\mathcal{B}$ satisfies the conditions in Definition 2.2.1 and hence $\mathcal{B}$ is a crystal.

The existence of a crystal base of $V(\lambda)$ for $\lambda \in P^+$ was proved by Kashiwara [16] as follows. Take the highest weight vector v_λ of $V(\lambda)$ and let $\mathcal{L}(\lambda)$ be the **A**-submodule of $V(\lambda)$ spanned by the vectors of the form $\widetilde{f}_{i_1} \cdots \widetilde{f}_{i_r} v_\lambda$ for $r \geq 0$ and $i_k \in I$ and set

$$\mathcal{B}(\lambda) = \{\widetilde{f}_{i_1} \cdots \widetilde{f}_{i_r} v_\lambda + q\mathcal{L}(\lambda) \in \mathcal{L}(\lambda)/q\mathcal{L}(\lambda) \,|\, r \geq 0, i_k \in I\} \setminus \{0\}.$$

Theorem 2.3.3 ([16])**.** *The pair $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ is a crystal base of $V(\lambda)$.*

Suppose an integrable $U_q(\mathfrak{g})$-module M_i has a crystal base $(\mathcal{L}_i, \mathcal{B}_i)$ for $i = 1, 2$. Then it is clear that the direct product $M_1 \oplus M_2$ has a crystal base $(\mathcal{L}_1 \oplus \mathcal{L}_2, \mathcal{B}_1 \sqcup \mathcal{B}_2)$. It is also known that the tensor product $M_1 \otimes M_2$ has a crystal base $(\mathcal{L}_1 \otimes \mathcal{L}_2, \mathcal{B}_1 \times \mathcal{B}_2)$. In this case, we use the notation $\mathcal{B}_1 \otimes \mathcal{B}_2$ as crystals instead of $B_1 \times B_2$ and its element is denoted by $b_1 \otimes b_2 = (b_1, b_2) \in \mathcal{B}_1 \otimes \mathcal{B}_2$. We remark that the crystal $\mathcal{B}_1 \otimes \mathcal{B}_2$ satisfies the conditions of tensor products of crystals, which means that our notation makes sense.

Now we consider the uniqueness of crystal bases. To do this, we define the notion of isomorphism of crystal bases.

Definition 2.3.4. Let M be an integrable $U_q(\mathfrak{g})$-module and $(\mathcal{L}_i, \mathcal{B}_i)$ be crystal bases for $i = 1, 2$. We say that two crystal bases $(\mathcal{L}_1, \mathcal{B}_1)$ and $(\mathcal{L}_2, \mathcal{B}_2)$ are isomorphic if there exists an $\mathbf{A}$-linear isomorphism $\psi : \mathcal{L}_1 \to \mathcal{L}_2$ such that

(1) ψ commutes with all $\widetilde{e}_i$ and $\widetilde{f}_i$ for all $i \in I$, and

(2) the induced $\mathbb{Q}$-linear isomorphism $\overline{\psi} : \mathcal{L}_1/q\mathcal{L}_1 \to \mathcal{L}_2/q\mathcal{L}_2$ defines a bijection $\overline{\psi} : \mathcal{B}_1 \cup \{0\} \to \mathcal{B}_2 \cup \{0\}$ that commutes with all $\widetilde{e}_i$ and $\widetilde{f}_i$ $(i \in I)$.

Then we have the following theorem.

Theorem 2.3.5 ([16])**.** *Let M be an integrable $U_q(\mathfrak{g})$-module. Then there exists a unique crystal base $(\mathcal{L}, \mathcal{B})$ of M. In particular, if M is isomorphic to $\bigoplus_i V(\lambda_i)$ for some $\lambda_i \in P^+$, then there exists an isomorphism of crystal bases*

$$(\mathcal{L}, \mathcal{B}) \longmapsto \left(\bigoplus_i \mathcal{L}(\lambda_i),\ \bigsqcup_i \mathcal{B}(\lambda_i) \right).$$

Chapter 3

Combinatorics of tableaux

In this chapter, we review combinatorics of semistandard tableaux and their crystal-theoretical meaning, which is closely related to representation theory of general linear groups. This chapter is organized as follows.

- In Section 3.1, we recall the notion of semistandard tableaux and its crystal structure.
- In Section 3.2, we review Schensted's bumping (or insertion) algorithm [39] and consider its relation with Knuth relations.
- In Section 3.3, we review the RSK correspondence and its crystal-theoretical meaning.

This chapter is based on [7, 38] and references therein.

3.1 Semistandard tableaux

A Young diagram for $\lambda \in \mathscr{P}$ is a collection of boxes, arranged in left-justified rows with λ_i boxes in the ith row for $i \geq 1$. We identify a partition with its Young diagram. A tableau of shape $\lambda \in \mathscr{P}$ is a filling of the Young diagram for λ with numbers.

For a skew diagram λ/μ, a semistandard tableau is a tableau of shape λ/μ with entries in $\mathbb{N}$ satisfying the following conditions:

(1) all entries in each row are weakly increasing from left to right,

(2) all entries in each column are strictly increasing from top to bottom.

In particular, when the shape of a semistandard tableau is single-columned, we call it a column tableau and write $\mathrm{ht}(T) = c$, the height of a column tableau T of shape (1^c). Denote by $SST_n(\lambda/\mu)$ be the set of semistandard tableaux of shape λ/μ with entries in $\{1, \ldots, n\}$.

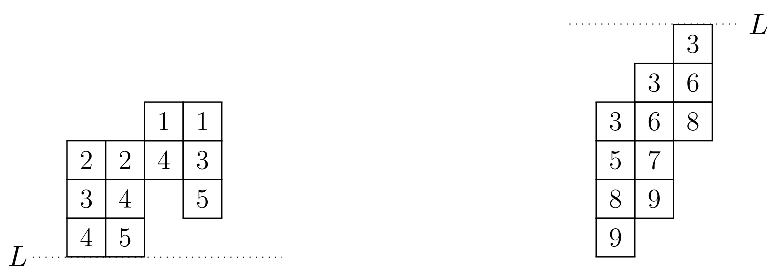
We assume that all tableaux are placed on the plane $\mathbb{P}_L$ with a horizontal line L and that some edges of tableaux are aligned with the line L. To emphasize the alignment, we introduce some notations. Let $U_1, \ldots, U_r$ be column tableaux and denote by

$$\lfloor U_1, \ldots, U_r \rfloor \quad \text{(resp. } \lceil U_1, \ldots, U_r \rceil\text{)}$$

the tableau whose ith column from the left is U_i and all bottom (resp. top) edges of U_i lie in L. For $(u_1, \ldots, u_r) \in \mathbb{Z}_+^r$, denote by

$$\lfloor U_1, \ldots, U_r \rfloor_{(u_1, \ldots, u_r)} \quad \text{(resp. } \lceil U_1, \ldots, U_r \rceil^{(u_1, \ldots, u_r)}\text{)}$$

the tableau obtained from $\lfloor U_1, \ldots, U_r \rfloor$ (resp. $\lceil U_1, \ldots, U_r \rceil$) by shifting each U_i by u_i positions up (resp. down). Note that these tableaux are not necessarily of skew shapes nor semistandard. For example, the left tableau is not semistandard nor of skew shape, whereas the right one is a semistandard tableau of skew shape.



$\lfloor U_1, U_2, U_3, U_4 \rfloor_{(0,0,2,1)}$ $\qquad$ $\lceil V_1, V_2, V_3 \rceil^{(2,1,0)}$

For $n \geq 2$, consider the crystal $\mathcal{B}(\varpi_1)$ of the vector representation $V(\varpi_1)$ of $U_q(\mathfrak{gl}_n)$ with the following crystal graph with $\mathrm{wt}(i) = \epsilon_i$. Here, we take $P = \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$.

$$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-1} n$$

For $n \in \mathbb{Z}_+$, let W_n be the set of words with letters in $\{1, \ldots, n\}$, where $W_0 = \{\emptyset\}$ is a singleton set. By identifying $\mathcal{B}(\varpi_1)$ as $\{1, \ldots, n\}$, we also identify $w_1 \otimes \cdots \otimes w_r \in \mathcal{B}(\varpi_1)^{\otimes r}$ with a word $w_1 \cdots w_r \in W_n$. Under the identification, the set W_n is a $\mathfrak{gl}_n$-crystal, where the element $\emptyset \in W_0$ satisfies $\mathrm{wt}(\emptyset) = 0, \widetilde{e}_i(\emptyset) = \widetilde{f}_i(\emptyset) = \mathbf{0}$ and $\varepsilon_i(\emptyset) = \varphi_i(\emptyset) = 0$ for all $i \in I$.

For a semistandard tableau T, let $w(T)$ be the word of T obtained by reading entries of T column by column from right to left, and from top to bottom in each column (cf. [7, 9]). Then we give a $\mathfrak{gl}_n$-crystal structure on $SST_n(\lambda)$ for $\lambda \in \mathscr{P}$ via their reading words, i.e., for a semistandard tableau T, $\mathrm{wt}(T) = \mathrm{wt}(w(T))$, $\widetilde{x}_i(T) = \widetilde{x}_i(w(T))$ for $x = e$ or f and $i \in I$, $\varepsilon_i(T) = \varepsilon_i(w(T)), \varphi_i(T) = \varphi_i(w(T))$ for $i \in I$. In this case, it is proved in [18] that the $\mathfrak{gl}_n$-crystal $SST_n(\lambda)$ for $\lambda \in \mathscr{P}$ is isomorphic to the crystal base $\mathcal{B}(w_\lambda)$ of highest weight $\mathfrak{gl}_n$-module of highest weight $w_\lambda = \sum_{i=1}^n \lambda_i \epsilon_i \in P^+$.

Note that one can define the character of a $\mathfrak{g}$-crystal as follows. Let $\mathbb{Z}[P]$ be the group algebra of P with a basis $\{e^\mu \,|\, \mu \in P\}$, where the multiplication is given by $e^\mu e^\nu = e^{\mu+\nu}$ for $\mu, \nu \in P$. For a crystal B and $\mu \in P$, write $B_\mu = \{b \in B | \mathrm{wt}(b) = \mu\}$ and define the character of B by

$$\mathrm{ch} B = \sum_{\mu \in P} |B_\mu| e^\mu = \sum_{b \in B} e^{\mathrm{wt}(b)}$$

if all B_μ are finite. Note that $\mathrm{ch}\, \mathcal{B}(\lambda)$ is well-defined for all $\lambda \in P^+$ since $|\mathcal{B}(\lambda)_\mu| = \dim V(\lambda)_\mu$ is finite-dimensional for $\mu \in \mathrm{wt}\, V(\lambda)$.

Now suppose that $\mathfrak{g} = \mathfrak{gl}_n$ and write $e^{\epsilon_i} = x_i$ for $1 \le i \le n$, where $x_1, \ldots, x_n$ are formal commuting variables. For $T \in SST_n(\lambda)$, let $\mathbf{x}^T = \prod_{i=1}^n x_i^{m_i} = e^{\mathrm{wt}(T)}$, where m_i is the number of i of T. Then the character of $SST_n(\lambda)$ or $\mathcal{B}(w_\lambda)$ is

$$\sum_{T \in SST_n(\lambda)} \mathbf{x}^T = s_\lambda(x_1, \ldots, x_n),$$

the Schur polynomial in n variables corresponding to λ.

3.2 Insertion algorithm

For a semistandard tableau T and a letter $a \in \mathbb{N}$, define the column insertion of a into T, denoted by $a \to T$, to be the tableau obtained as follows :

(1) If a is larger than all entries x in the first (or left-most) column of T, then put a at

the bottom of the first column and stop the algorithm.

(2) Otherwise, let a' be the smallest entry in the first column of T such that $a' \geq a$. Then replace a' by a in the first column of T.

(3) Apply (1) and (2) to the next column of T with a'. Repeat the process until the algorithm stops.

Note that $a \to T$ is again semistandard. In addition, if $T \in SST_n(\lambda)$ and $a \in \{1, \ldots, n\}$, then $(a \to T) \in SST_n(\mu)$ for some $\mu \supsetneq \lambda$ with $|\mu| = |\lambda| + 1$.

Example 3.2.1.

$$3 \to \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 2 & 4 & 5 & \\ \cline{1-3} 3 & & & \\ \cline{1-1} 4 & & & \\ \cline{1-1} \end{array} \quad = \quad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 2 & \color{red}{3} & \color{red}{4} & \color{red}{5} \\ \hline \color{red}{3} & & & \\ \cline{1-1} 4 & & & \\ \cline{1-1} \end{array}$$

Here, the red-colored letters mean the letters a and a' in the insertion algorithm.

In general, define $(w \to T) = (w_r \to (\cdots \to (w_1 \to T) \cdots))$ for a word $w = w_1 \ldots w_r$ and $(S \to T) = (w(S) \to T)$ for a semistandard tableau S. For a word $w = w_1 \ldots w_r$, let

$$P(w) = (w_r \to (\cdots \to (w_2 \to w_1) \cdots))$$

and we call it the insertion tableau of w.

We can characterize the insertion tableau $P(w)$ in another way using an equivalence relation between words. Define the Knuth transformation on words by

(a) $\cdots xzy \cdots \longmapsto \cdots zxy \cdots$ if $x < y \leq z$,

(b) $\cdots yzx \cdots \longmapsto \cdots yxz \cdots$ if $x \leq y < z$

and define an equivalence relation on the set of words, called the Knuth equivalence, by $w \equiv w'$ if and only if one is obtained from the other by applying a sequence of Knuth transformations and their inverses. For a semistandard tableau T and $a \in \mathbb{N}$, we can observe that $w(a \to T) \equiv w(T)a$ and we obtain $w(P(w)) \equiv w$ by applying the above fact repeatedly. The tableau $P(w)$ is the unique semistandard tableau in terms of the Knuth equivalence.

Theorem 3.2.2. *The insertion tableau $P(w)$ is the unique semistandard tableau whose word is Knuth equivalent to w.*

Remark 3.2.3. It is known in [34] that $w \equiv w'$ if and only if $w \overset{\mathfrak{gl}_n}{\equiv} w'$ and these stories are fundamentally connected to the crystal theory. In particular, there exists a $\mathfrak{gl}_n$-crystal isomorphism between W_n and $\bigoplus_\lambda SST_n(\lambda)$ sending w to $P(w)$.

3.3 Robinson-Schensted-Knuth correspondence

A biword is a two-rowed array with positive integers of the form $w = \begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix}$ with $i_1 \leq i_2 \leq \cdots \leq i_k$ and $j_t \geq j_{t+1}$ if $i_t = i_{t+1}$. Let $\mathbf{M}$ be the set of biwords and $\mathbf{M}_{n\times r}$ be the subset of biwords whose entries in the upper row are in $\{1, \ldots, r\}$ and whose entries in the lower row are in $\{1, \ldots, n\}$. Note that when $i_t = t$ for all t, a biword is just the word obtained by reading its lower row.

For a biword w, define two tableaux $P(w)$ and $Q(w)$ as follows:

(1) $P(w) = P(j_1 j_2 \cdots j_k)$ with $P_t = P(j_1 \cdots j_t)$ for $1 \leq t \leq k = \ell(w)$,

(2) $Q(w)$ is the tableau of same shape as $P(w)$ such that a box $\mathrm{sh}(P_t)/\mathrm{sh}(P_{t-1})$ is filled with i_t for $1 \leq t \leq k$, where $P_0 = \emptyset$.

Then we can check that $Q(w)$ is semistandard. We say that $P(w)$ (resp. $Q(w)$) is the insertion (resp. recording) tableau of w. It is known that the map sending $w \in \mathbf{M}_{n\times r}$ to the pair $(P(w), Q(w))$ is a bijection, which is known as the RSK correspondence.

Theorem 3.3.1 ([21]). *For $n, r \geq 1$, we have a bijection*

$$\mathbf{M}_{n\times r} \longrightarrow \bigsqcup_{\lambda \in \mathscr{P}} SST_n(\lambda) \times SST_r(\lambda)$$

sending w to $(P(w), Q(w))$.

Remark 3.3.2. For $M \in \mathbf{M}_{n\times r}$, let $m(i,j)$ be the number of occurrences of $\begin{pmatrix} i \\ j \end{pmatrix}$ for $1 \leq i \leq r$, $1 \leq j \leq n$ and assign a vector

$$v_M = \prod_{i,j} (v_j \otimes w_i)^{\otimes m(i,j)} \in \mathcal{S}(\mathbb{C}^n \otimes \mathbb{C}^r),$$

where $\{v_1, \ldots, v_n\}$ (resp. $\{w_1, \ldots, w_r\}$) is a basis of $\mathbb{C}^n$ (resp. $\mathbb{C}^r$). Then Theorem 3.3.1 is a combinatorial interpretation of the decomposition (1.1) since $SST_n(\lambda) \cong \mathcal{B}(\omega_\lambda)$ parametrizes a linear basis of $V_{GL_n(\mathbb{C})}(\lambda)$.

Example 3.3.3. Let $w = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 \\ 4 & 3 & 5 & 1 & 4 & 2 & 2 & 4 & 3 & 3 \end{pmatrix} \in \mathbf{M}_{5\times 4}$. Then we obtain

$$P(w) = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 4 \\ \hline 2 & 3 & 4 & 5 \\ \cline{1-4} 3 \\ \cline{1-1} \end{array}\,,$$

$$Q(w) = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 3 & 4 \\ \hline 2 & 3 & 3 & 4 \\ \cline{1-4} 4 \\ \cline{1-1} \end{array}\,.$$

Note that the following is the sequence of $P(w_1 \cdots w_i)$.

$$\begin{array}{|c|} \hline 4 \\ \hline \end{array}\,, \begin{array}{|c|c|} \hline 3 & 4 \\ \hline \end{array}\,, \begin{array}{|c|c|} \hline 3 & 4 \\ \hline 5 \\ \cline{1-1} \end{array}\,, \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 5 \\ \cline{1-1} \end{array}\,, \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 4 & 5 \\ \cline{1-2} \end{array}\,, \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 4 & 5 \\ \hline \end{array}\,, \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 2 & 4 & 5 \\ \cline{1-3} \end{array}\,,$$

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 2 & 4 & 5 \\ \cline{1-3} 4 \\ \cline{1-1} \end{array}\,, \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 2 & 4 & 4 & 5 \\ \hline 3 \\ \cline{1-1} \end{array}\,, \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 4 \\ \hline 2 & 3 & 4 & 5 \\ \cline{1-4} 3 \\ \cline{1-1} \end{array}$$

The RSK correspondence has a nice crystal-theoretic interpretation. A $\mathfrak{gl}_n$-crystal structure on $\mathbf{M}_{n\times r}$ is obtained by applying $\widetilde{e}_i$ and $\widetilde{f}_i$ to the lower row in biwords, in other words, $\widetilde{x}_i \begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix}$ for $x = e$ or f and $i = 1, \ldots, n-1$ is the biword obtained by taking $\widetilde{e}_i(j_1 j_2 \cdots j_k)$ as its lower low and rearranging columns if necessary. On the other hand, a $\mathfrak{gl}_n$-crystal structure on $SST_n(\lambda) \times SST_r(\lambda)$ is obtained by the restriction to the first component, in other words, $\widetilde{x}_i(P, Q) = (\widetilde{x}_i P, Q)$. By Remark 3.2.3, the map sending w to $P(w)$ is a $\mathfrak{gl}_n$-crystal isomorphism and hence the correspondence is a $\mathfrak{gl}_n$-crystal isomorphism.

There is also a dual RSK correspondence. We define dual biwords similarly except the condition that $j_t < j_{t+1}$ if $i_t = i_{t+1}$. Let $\mathbf{M}^*$ be the set of dual biwords and let $\mathbf{M}^*_{n\times r}$ be the subset defined similarly. For a dual biword w, $P(w)$ is defined in the same way while the recording tableau is of the transposed shape. More precisely, $Q(w)$ is a semistandard tableau of shape $\mathrm{sh}P(w)'$ such that a box $\mathrm{sh}(P_t)'/\mathrm{sh}(P_{t-1})'$ is filled with i_t for $1 \le t \le k$, where $P_0 = \emptyset$. Then we have the following bijection (and a $\mathfrak{gl}_n$-crystal isomorphism), which is called the dual RSK correspondence.

Theorem 3.3.4 ([21])**.** *For $n, r \geq 1$, we have a bijection*

$$\mathbf{M}^*_{n\times r} \quad \longrightarrow \quad \bigsqcup_{\lambda\in\mathscr{P}} SST_n(\lambda)\times SST_r(\lambda')$$

sending w to $(P(w), Q(w))$. Moreover, this bijection is a $\mathfrak{gl}_n$-crystal isomorphism, whose crystal structures on both sides are defined in a similar way as in the RSK correspondence.

Finally, we consider the character formulae obtained from two correspondences. We assign a biword (or a dual biword) w to a monomial

$$\prod_{i,j} x_i^{m_i} y_j^{n_j}$$

where m_i (resp. n_j) is the number of occurrences of i (resp. j) in the second (resp. first) row of w. Then two correspondences give the following identities, which are known as the Cauchy identities [35].

Corollary 3.3.5 (Cauchy identity)**.** *We have*

$$\prod_{i=1}^{n}\prod_{j=1}^{r}\frac{1}{1-x_iy_j} = \sum_{\ell(\lambda)\leq\min\{n,r\}} s_\lambda(\mathbf{x})s_\lambda(\mathbf{y}), \quad \prod_{i=1}^{n}\prod_{j=1}^{r}(1+x_iy_j) = \sum_{\substack{\ell(\lambda)\leq n\\ \ell(\lambda')\leq r}} s_\lambda(\mathbf{x})s_{\lambda'}(\mathbf{y}).$$

Chapter 4

Jeu de taquin and crystals

The word 'jeu de taquin' comes from French, which means the fifteen puzzle. As similar to the puzzle, this algorithm switches (or slides) an empty box of semistandard tableaux of skew shape with another box following certain rules. These slides explain the Knuth transformations, which play a fundamental role in the theory of symmetric functions. In this chapter, we generalize the notion of semistandard tableaux and discuss the relation between the associated jeu de taquin sliding and $\mathfrak{gl}_r$-crystals. This chapter is organized as follows.

- In Section 4.1, we introduce the notion of $\mathcal{A}$-semistandard tableaux, which is a generalization of semistandard tableaux.
- In Section 4.2, we recall the jeu de taquin (or sliding) algorithm due to Schützenberger [40], and its generalization.
- In Section 4.3, we consider the relation between jeu de taquin algorithm and a $\mathfrak{gl}_r$-crystal structure.

4.1 $\mathcal{A}$-semistandard tableaux

Let $\mathcal{A}$ be a linearly ordered set with a $\mathbb{Z}_2$-grading $\mathcal{A} = \mathcal{A}_0 \sqcup \mathcal{A}_1$. For $n \geq 1$, let

$$[n] = \{1 < 2 < \cdots < n\}, \quad [\overline{n}] = \{\overline{n} < \overline{n-1} < \cdots < \overline{1}\},$$

where we assume that all entries of these sets assumed to be of degree 0, and let

$$[n]' = \{1' < 2' < \cdots < n'\},$$

where we assume that all entries of this set are assumed to be of degree 1. For positive integers m and n, let

$$\mathbb{I}_{m|n} = \{1 < \cdots < m < 1' < \cdots < n'\}$$

with $(\mathbb{I}_{m|n})_0 = [m]$ and $(\mathbb{I}_{m|n})_1 = [n]'$. Define

$$\mathcal{I}_n = \{1 < \cdots < n < \overline{n} < \cdots < \overline{1}\} \tag{4.1}$$

with $(\mathcal{I}_n)_0 = \mathcal{I}_n$.

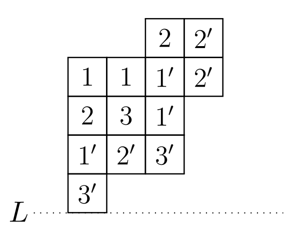
For a skew diagram λ/μ, let $SST_{\mathcal{A}}(\lambda/\mu)$ be the set of $\mathcal{A}$-semistandard tableaux of shape λ/μ, that is, tableaux of shape λ/μ with entries in $\mathcal{A}$ satisfying the following conditions:

(1) all entries in each row (resp. column) are weakly increasing from left to right (resp. from top to bottom),

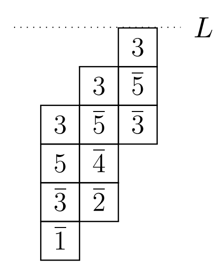
(2) entries in $\mathcal{A}_0$ (resp. $\mathcal{A}_1$) are strictly increasing in each column (resp. row).

In fact, we get $SST_{[n]}(\lambda/\mu) = SST_n(\lambda/\mu)$. If there is no confusion on $\mathcal{A}$, we sometimes call $T \in SST_{\mathcal{A}}(\lambda/\mu)$ a semistandard tableau of shape λ/μ. Note that we use the same notation $\lfloor U_1, \ldots, U_r \rfloor_{(u_1,\ldots,u_r)}$ and $\lceil U_1, \ldots, U_r \rceil^{(u_1,\ldots,u_r)}$ for $\mathcal{A}$-semistandard column tableaux $U_1, \ldots, U_r$ and $(u_1, \ldots, u_r) \in \mathbb{Z}_+^r$.

Example 4.1.1.



$\lfloor U_1, U_2, U_3, U_4 \rfloor_{(0,1,1,3)}$
$\in SST_{\mathbb{I}_{4|3}}((4,4,3,3,1)/(2))$



$\lceil U_1, U_2, U_3 \rceil^{(2,1,0)}$
$\in SST_{\mathcal{I}_5}((3,3,3,2,2,1)/(2,1))$

The RSK or dual RSK correspondence can be generalized to the case of $\mathcal{A}$-semistandard tableaux. For this, we summarize an insertion algorithm to define insertion tableaux (cf. [2, 37]). For an $\mathcal{A}$-semistandard tableau T and $a \in \mathcal{A}_0$ (resp. $a \in \mathcal{A}_1$), define the column insertion of a into T, denoted by $a \to T$, to be the tableau obtained as follows.

(1) If $a > x$ (resp. $a \geq x$) for all entries x in the first (or left-most) column of T, then put a at the bottom of the first column and stop the algorithm.

(2) Otherwise, let a' be the smallest entry in the first column of T such that $a' \geq a$ (resp. $a' > a$). Then replace a' by a in the first column of T.

(3) Apply two steps to the next column of T with a'. Repeat the process until the algorithm stops.

For a $\mathcal{A}$-semistandard tableau T, we define $w(T)$ to be the word with letters in $\mathcal{A}$ obtained by the same way as semistandard tableaux and then we also similarly define $w \to T$ for a word $w = w_1 \cdots w_r$ with letters in $\mathcal{A}$ and $S \to T$ for an $\mathcal{A}$-semistandard tableau S.

Example 4.1.2. Suppose that $\mathcal{A} = \mathbb{I}_{4|3}$.

$$3 \;\to\; \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 1' & 2' \\ \hline 2 & 2 & 3 & 2' & \\ \cline{1-4} 4 & 1' & 2' & 3' & \\ \cline{1-4} 1' & 3' & & & \\ \cline{1-2} 2' & & & & \\ \cline{1-1} \end{array} \quad = \quad \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 1' & 2' \\ \hline 2 & 2 & 3 & 2' & 3' \\ \hline 3 & 4 & 1' & 2' & \\ \cline{1-4} 1' & 3' & & & \\ \cline{1-2} 2' & & & & \\ \cline{1-1} \end{array}$$

Here, the red-colored letters mean the letters a and a' in the insertion algorithm.

We can generalize the RSK correspondences using the insertion above. Let us introduce the dual RSK correspondence. For $r \geq 1$, let

$$\mathbf{E}^r_{\mathcal{A}} = \bigsqcup_{(u_r,\ldots,u_1)\in\mathbb{Z}^r_+} SST_{\mathcal{A}}((1^{u_r})) \times \cdots \times SST_{\mathcal{A}}((1^{u_1}))$$

and define the insertion tableau $P(\mathbf{U})$ and the recording tableau $Q(\mathbf{U})$ for $\mathbf{U} = (U_r, \ldots, U_1) \in \mathbf{E}^r_{\mathcal{A}}$ as follows: for $1 \leq i \leq r$,

(1) $P(\mathbf{U}) = (U_r \to (\cdots \to (U_2 \to U_1)\cdots))$ with $P_i = P(U_r, \ldots, U_1)$,

(2) $Q(\mathbf{U})$ is an $[r]$-semistandard tableau such that $\mathrm{sh}(P_i)'/\mathrm{sh}(P_{i-1})'$ is filled with i.

In this case, if $P(\mathbf{U}) \in SST_{\mathcal{A}}(\lambda)$ for some $\lambda \in \mathscr{P}$, then $\lambda' \in \mathscr{P}_r$ and $Q(\mathbf{U}) \in SST_r(\lambda')$. Then the map

$$\kappa_{\mathcal{A}} : \mathbf{E}^r_{\mathcal{A}} \longrightarrow \bigsqcup_{\lambda' \in \mathscr{P}_r} SST_{\mathcal{A}}(\lambda) \times SST_r(\lambda') \tag{4.2}$$

sending $\mathbf{U}$ to $(P(\mathbf{U}), Q(\mathbf{U}))$ is a bijection.

Example 4.1.3. Suppose $\mathcal{A} = \mathbb{I}_{4|3}$ and $\mathbf{U} \in \mathbf{E}^6_{\mathcal{A}}$ is given as follows.

$$\mathbf{U} = \left(\begin{array}{|c|}\hline 2\\\hline 3\\\hline 1'\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 3\\\hline 1'\\\hline 2'\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 2'\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 1'\\\hline 1'\\\hline 3'\\\hline 3'\\\hline\end{array}, \begin{array}{|c|}\hline 2'\\\hline 2'\\\hline\end{array} \right)$$

Then

$$P(\mathbf{U}) = \begin{array}{ccccc} 1 & 1 & 2 & 1' & 2' \\ 2 & 2 & 3 & 2' & 3' \\ 3 & 4 & 1' & 2' & \\ 1' & 2' & 3' & & \\ 1' & & & & \end{array} \qquad Q(\mathbf{U}) = \begin{array}{ccccc} 1 & 1 & 2 & 2 & 2 \\ 2 & 2 & 3 & 4 & \\ 3 & 4 & 4 & 6 & \\ 4 & 5 & 5 & & \\ 6 & 6 & & & \end{array}.$$

Note that the following is the sequence of P_i's.

$$\begin{array}{c} 2' \\ 2' \end{array}, \quad \begin{array}{cc} 2 & 2' \\ 1' & 2' \\ 1' & \\ 3' & \\ 3' & \end{array}, \quad \begin{array}{ccc} 1 & 2 & 2' \\ 1' & 2' & \\ 1' & 3' & \\ 2' & & \\ 3' & & \end{array}, \quad \begin{array}{cccc} 1 & 1 & 2 & 2' \\ 3 & 1' & 2' & \\ 1' & 2' & 3' & \\ 1' & 3' & & \\ 2' & & & \end{array}, \quad \begin{array}{cccc} 1 & 1 & 2 & 2' \\ 2 & 3 & 1' & 2' \\ 4 & 1' & 2' & 3' \\ 1' & 3' & & \\ 2' & & & \end{array}, \quad \begin{array}{ccccc} 1 & 1 & 2 & 1' & 2' \\ 2 & 2 & 3 & 2' & 3' \\ 3 & 4 & 1' & 2' & \\ 1' & 2' & 3' & & \\ 1' & & & & \end{array}$$

We also obtain the Cauchy-type identity corresponding to (4.2). Let $\{x_a : a \in \mathcal{A}\}$ be a set of commuting formal variables indexed by $\mathcal{A}$. For $a \in \mathcal{A}$ and $T \in SST_{\mathcal{A}}(\lambda)$, let

$$\mathbf{x}^T_{\mathcal{A}} = \prod x_a^{m_a},$$

where m_a is the number of a in T. Let $s_\lambda(\mathbf{x}_{\mathcal{A}}) = \sum_{T \in SST_{\mathcal{A}}(\lambda)} \mathbf{x}^T_{\mathcal{A}}$ be a super Schur function corresponding to $\lambda \in \mathscr{P}$.

Corollary 4.1.4. *We have*

$$\prod_{j=1}^{r}\prod_{a\in\mathcal{A}_0}(1+x_ay_j)\prod_{a\in\mathcal{A}_1}(1-x_ay_j)^{-1} \;=\; \sum_{\lambda'\in\mathscr{P}} s_\lambda(\mathbf{x}_{\mathcal{A}})s_{\lambda'}(\mathbf{y}).$$

4.2 Jeu de taquin algorithm

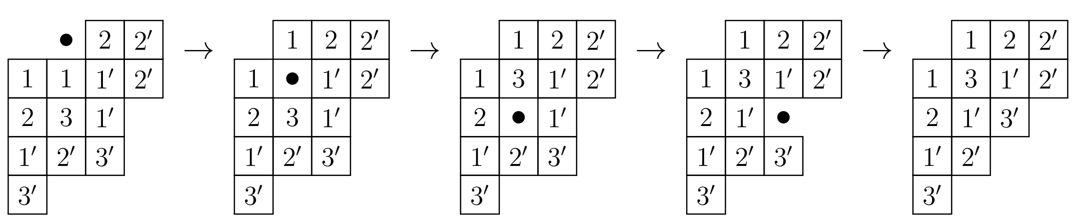
In [40], Schützenberger introduced an algorithm called a sliding algorithm or jeu de taquin on semistandard tableaux. One can easily generalize this to the case of $\mathcal{A}$-semistandard tableaux as follows (cf. [2, 37]).

A tableau is said to be punctured if one of its boxes contains an empty box and we say that a punctured tableau is $\mathcal{A}$-semistandard if it is $\mathcal{A}$-semistandard when we ignore empty boxes. Let T be a punctured $\mathcal{A}$-semistandard tableau with an empty box c and denote by a (resp. b) the letter right to (resp. below) c.

(1) When b is empty or $a \le b$ (the equality holds only when both are of degree 1), the elementary sliding step applied to c is the process of changing c and a. In this case, we say that a horizontal move occurs.

(2) When a is empty or $a \ge b$ (the equality holds only when both are of degree 0), the elementary sliding step applied to c is the process of changing c and b. In this case, we say that a vertical move occurs.

Note that the tableau obtained by applying the elementary sliding step to T is again $\mathcal{A}$-semistandard. Take an inner corner c of T as the empty box and denote by $\mathtt{jdt}(T, c)$ the $\mathcal{A}$-semistandard tableau obtained by applying elementary sliding steps repeatedly to c until c becomes an outer corner of T. For example, suppose $\mathcal{A} = \mathbb{I}_{4|3}$ and a semistandard tableau T is given with an inner corner c. To keep track of positions of c, we mark $\bullet$ at positions of c. Then we obtained $\mathtt{jdt}(T, c)$ as follows.



Since the elementary sliding steps are reversible, a tableau $\mathtt{jdt}(T, c)$ for an outer corner of c of T can be defined as the semistandard tableau obtained by applying the inverse algorithm.

By repeatedly applying $\mathtt{jdt}(\cdot, c)$ for inner corners c, we obtain a semistandard tableau of partition shape and denote it by $\mathtt{jdt}(T)$. Then $\mathtt{jdt}(T)$ is independent of the order of choices of inner corners and uniquely determined by T.

Remark 4.2.1. The Knuth equivalence $\equiv$ can be described using the elementary sliding steps. Here all letters are of degree 0.

(a) $xzy \equiv zxy$ if $x < y \leq z$: $\begin{array}{|c|c|} \cline{2-2} \multicolumn{1}{c|}{} & x \\ \hline y & z \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|} \hline x & z \\ \hline y & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array}$

(b) $yzx \equiv yxz$ if $x \leq y < z$: $\begin{array}{|c|c|} \cline{2-2} \multicolumn{1}{c|}{} & y \\ \hline x & z \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|} \hline x & y \\ \hline z & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array}$

We can easily check that the tableau $\mathtt{jdt}(T)$ is the same as the insertion tableau of $w(T)$. Thus, the jeu de taquin algorithm is another combinatorial tool to obtain insertion tableaux.

4.3 Jeu de taquin and $\mathfrak{gl}_r$-crystals

The jeu de taquin algorithm introduced in Section 4.2 can be described in terms of $\mathfrak{gl}_r$-crystal operators.

We first investigate the case of tableaux with two columns. For $a, b, c \in \mathbb{Z}_+$, let $\lambda(a,b,c) = (2^{b+c}, 1^a)/(1^b)$ be a skew partition with two columns and denote by $T^{\mathtt{L}}$ (resp. $T^{\mathtt{R}}$) the left (resp. right) column of $T \in SST_{\mathcal{A}}(\lambda(a,b,c))$. For $T = \lfloor T^{\mathtt{L}}, T^{\mathtt{R}} \rfloor_{(0,a)} \in SST_{\mathcal{A}}(\lambda(a,b,c))$, let

$$\mathfrak{r}_T = \max\{k \in \mathbb{Z}_+ \,:\, \lfloor T^{\mathtt{L}}, T^{\mathtt{R}} \rfloor_{(0,a-k)} \in SST_{\mathcal{A}}(\lambda(a-k, b-k, c+k))\}.$$

We remark that when we apply jeu de taquin to a corner of T, we have $\mathfrak{r}_T > 0$ if and only if a vertical move occurs by [25, Lemma 6.2]. Thus, we have $\mathfrak{r}_{\mathtt{jdt}(T,c)} = 0$ whenever $\mathfrak{r}_T = 0$.

For $T \in SST_{\mathcal{A}}(\lambda(a,b,c))$ with $\mathfrak{r}_T = 0$, set

(1) $\mathcal{E}T$ to be the tableau $\mathtt{jdt}(T, C)$ in $SST_{\mathcal{A}}(\lambda(a-1, b+1, c))$, where C is the empty box below $T^{\mathtt{R}}$ if $a > 0$,

(2) $\mathcal{F}T$ to be the tableau $\mathtt{jdt}(T, C')$ in $SST_{\mathcal{A}}(\lambda(a+1, b-1, c))$, where C' is the empty box above $T^{\mathtt{L}}$ (or the inner corner of T) if $b > 0$.

Here $\mathcal{E}T = \mathbf{0}$ and $\mathcal{F}T = \mathbf{0}$ if $a = 0$ and $b = 0$, respectively. Note that $\mathfrak{r}_{\mathcal{X}T} = 0$ unless $\mathcal{X}T = \mathbf{0}$ for $\mathcal{X} = \mathcal{E}$ or $\mathcal{F}$. Additionally, define

- $\varepsilon(T) = \max\{k \in \mathbb{Z}_+ \,|\, \mathcal{E}^k(b) \neq 0\}$,
- $\varphi(T) = \max\{k \in \mathbb{Z}_+ \,|\, \mathcal{F}^k(b) \neq 0\}$,
- $\mathrm{wt}(T) = \varphi(T) - \varepsilon(T)$.

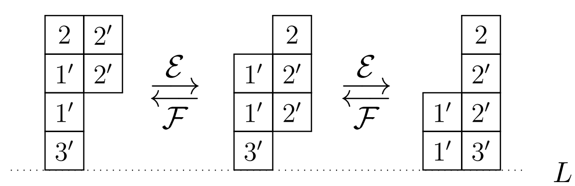
Then, for a given $T \in SST_{\mathcal{A}}(\lambda(a, b, c))$, the set $B = \{\, \mathcal{E}^k T \,|\, 0 \leq k \leq \varepsilon(T) \,\} \cup \{\, \mathcal{F}^s T \,|\, 0 \leq s \leq \varphi(T) \,\}$ forms a regular $\mathfrak{sl}_2$-crystal with respect to $\mathcal{E}$ and $\mathcal{F}$ with $\varepsilon(T) = a$ and $\varphi(T) = b$. Here we assume that $P = \mathbb{Z}$ with identifying $2 \in \mathbb{Z}$ with the simple root of $\mathfrak{sl}_2$. In this case, the crystal B is isomorphic to the regular $\mathfrak{sl}_2$-crystal of highest weight $a + b$.

Remark 4.3.1.

(1) For $T \in SST_{\mathcal{A}}(\lambda(a, b, c))$, we define $\mathcal{X}T = \mathcal{X}\left\lfloor T^{\mathtt{L}}, T^{\mathtt{R}} \right\rfloor_{(0, a-\mathfrak{r}_T)}$ for $\mathcal{X} = \mathcal{E}$ or $\mathcal{F}$. Then B is again a regular $\mathfrak{sl}_2$-crystal with $\varepsilon(T) = a - \mathfrak{r}_T$ and $\varphi(T) = b - \mathfrak{r}_T$. Thus, we always consider the case of $\mathfrak{r}_T = 0$ when we apply $\mathcal{E}, \mathcal{F}$.

(2) If we take $P = \mathbb{Z}\epsilon_1 \oplus \mathbb{Z}\epsilon_2$ for the weight lattice of $\mathfrak{gl}_2$ with $\epsilon_1 - \epsilon_2$ the simple root and define $\mathrm{wt}(T) = m_1\epsilon_1 + m_2\epsilon_2$ for $T \in SST_{\mathcal{A}}(\lambda(a, b, c))$, where m_1 (resp. m_2) is the number of boxes of $T^{\mathtt{R}}$ (resp. $T^{\mathtt{L}}$), then we may regard B as a $\mathfrak{gl}_2$-crystal.

Example 4.3.2. Suppose that $\mathcal{A} = \mathbb{I}_{4|3}$.



We can define a (regular) $\mathfrak{gl}_2$-crystal structure on $(U, V) \in SST_{\mathcal{A}}((1^u)) \times SST_{\mathcal{A}}((1^v))$ by

$$\mathcal{X}(U, V) = \begin{cases} ((\mathcal{X}T)^{\mathtt{L}}, (\mathcal{X}T)^{\mathtt{R}}) & \text{if } \mathcal{X}T \neq \mathbf{0}, \\ \mathbf{0} & \text{if } \mathcal{X}T = \mathbf{0}, \end{cases} \qquad (\mathcal{X} = \mathcal{E}, \mathcal{F}), \tag{4.3}$$

where T is the unique tableau in $SST_{\mathcal{A}}(\lambda(u-k,v-k,k))$ for some $0 \le k \le \min\{u,v\}$ such that $(T^{\mathtt{L}},T^{\mathtt{R}})=(U,V)$ and $\mathfrak{r}_T=0$. In general, we define a $\mathfrak{gl}_r$-crystal structure on $\mathbf{E}^r_{\mathcal{A}}$ $(r\ge 2)$ as follows: for $(U_r,\ldots,U_1)\in\mathbf{E}^r_{\mathcal{A}}$,

$$\mathcal{X}_i(U_r,\ldots,U_1)=\begin{cases}(U_r,\ldots,\mathcal{X}(U_{i+1},U_i),\ldots,U_1) & \text{if } \mathcal{X}(U_{i+1},U_i)\neq \mathbf{0},\\ \mathbf{0} & \text{if } \mathcal{X}(U_{i+1},U_i)=\mathbf{0},\end{cases} \tag{4.4}$$

for $\mathcal{X}=\mathcal{E},\mathcal{F}$ and $1\le i\le r-1$, and

- $\varepsilon_i(U_r,\ldots,U_1)=\max\{k\in\mathbb{Z}_+ \,|\, \mathcal{E}_i^k(b)\neq 0\}$ $(=\varepsilon(U_{i+1},U_i))$,
- $\varphi_i(U_r,\ldots,U_1)=\max\{k\in\mathbb{Z}_+ \,|\, \mathcal{F}_i^k(b)\neq 0\}$ $(=\varphi(U_{i+1},U_i))$,
- $\mathrm{wt}(U_r,\ldots,U_1)=\sum_{i=1}^r m_i\epsilon_i$ with $m_i=\mathrm{ht}(U_i)$ (cf. Remark 4.3.1 (2)).

Lemma 4.3.3. *Under the above hypothesis, $\mathbf{E}^r_{\mathcal{A}}$ is a regular $\mathfrak{gl}_r$-crystal.*

Proof. Let $\mathbf{M}_{\mathcal{A}\times[r]}$ be the set of matrices $\mathbf{m}=(m_{ab})$ with non-negative integral entries $(a\in\mathcal{A},\ b\in[r])$ satisfying (1) $m_{ab}\in\{0,1\}$ if $a\in\mathcal{A}_0$, (2) $\sum_{a,b}m_{ab}<\infty$. There is a natural bijection from $\mathbf{E}^r_{\mathcal{A}}$ to $\mathbf{M}_{\mathcal{A}\times[r]}$, where $(U_r,\ldots,U_1)\in\mathbf{E}^r_{\mathcal{A}}$ is sent to $\mathbf{m}=(m_{ab})$ such that m_{ab} is the number of occurrences of a in U_b.

Suppose $\mathbf{m}=(m_{ab})\in\mathbf{M}_{\mathcal{A}\times[r]}$ is given. For $a\in\mathcal{A}$, we may identify the a-th row of $\mathbf{m}$ with a unique tableau $T^{(a)}$ in $SST_{[r]}((u))$ (resp. $SST_{[r]}((1^u))$) if $a\in\mathcal{A}_0$ (resp. $a\in\mathcal{A}_1$), where $u=\sum_b m_{ab}$ and m_{ab} is the number of occurrences of b in $T^{(a)}$. We may define a regular $\mathfrak{gl}_r$-crystal structure on $\mathbf{M}_{\mathcal{A}\times[r]}$ by regarding $\mathbf{m}$ as $\overrightarrow{\bigotimes}_{a\in\mathcal{A}}T^{(a)}$. By similar arguments as in [24, 30], we can check that the associated operators $\widetilde{e}_i$ and $\widetilde{f}_i$ for $1\le i\le r-1$ coincide with $\mathcal{E}_i$ and $\mathcal{F}_i$, and the $\mathfrak{gl}_r$-weight is equal to wt. Hence $\mathbf{M}_{\mathcal{A}\times[r]}$ is a regular $\mathfrak{gl}_r$-crystal since it is a disjoint union of tensor products of regular $\mathfrak{gl}_r$-crystals $SST_{[r]}((u))$ and $SST_{[r]}((1^u))$. □

Example 4.3.4. Suppose $\mathcal{A}=\mathbb{I}_{4|3}$ and $\mathbf{U}\in\mathbf{E}^6_{\mathcal{A}}$ is the tableau given in Example 4.1.3. Then we have

$$Q(\mathcal{E}_5\mathbf{U})=\widetilde{e}_5Q(\mathbf{U})=\begin{array}{|c|c|c|c|c|}\hline 1&1&2&2&2\\\hline 2&2&3&4\\\cline{1-4} 3&4&4&5\\\cline{1-4} 4&5&5\\\cline{1-3} 6&6\\\cline{1-2}\end{array}.$$

We consider the dual RSK correspondence $\kappa_{\mathcal{A}}$ in (4.2) in connection with the $\mathfrak{gl}_r$-crystal structure on $\mathbf{E}^r_{\mathcal{A}}$. By the argument in the proof of Lemma 4.3.3, $\kappa_{\mathcal{A}}$ is a morphism of (regular) $\mathfrak{gl}_r$-crystals, which implies the following.

Lemma 4.3.5. *The bijection $\kappa_{\mathcal{A}}$ is an isomorphism of $\mathfrak{gl}_r$-crystals, where the right-hand side is a regular $\mathfrak{gl}_r$-crystal with respect to the second component.*

Remark 4.3.6. (1) The set $\mathbf{M}^*_{n\times r}$ of dual biwords in Theorem 3.3.4 can be identified with $\mathbf{E}^r_{[n]}$ under the following rule.

$$\begin{pmatrix} 1 & \cdots & 1 & 2 & \cdots & 2 & \cdots & r & \cdots & r \\ i_{1,1} & \cdots & i_{1,u_1} & i_{2,1} & \cdots & i_{2,u_2} & \cdots & i_{r,1} & \cdots & i_{r,u_r} \end{pmatrix} \in \mathbf{M}^*_{n\times r}$$

$$\longleftrightarrow \quad \begin{array}{|c|}\hline i_{1,1}\\ \hline \vdots \\ \hline i_{1,u_1}\\ \hline\end{array} \otimes \begin{array}{|c|}\hline i_{2,1}\\ \hline \vdots\\ \hline i_{2,u_2}\\ \hline\end{array} \otimes\cdots\otimes \begin{array}{|c|}\hline i_{r1}\\ \hline \vdots\\ \hline i_{r,u_r}\\ \hline\end{array} \quad \in \mathbf{E}^r_{[n]}$$

Hence $\kappa_{[n]}$ recovers the dual RSK correspondence in Theorem 3.3.4. Similarly, the set $\mathbf{M}_{n\times r}$ of biwords is identified with the set of r-tuples of row tableaux (semistandard tableaux of single-rowed shape) with letters in $[n]$ and we can recover the RSK correspondence using this identification.

(2) The bijection $\kappa_{[n]}$ is an isomorphism of $(\mathfrak{gl}_n, \mathfrak{gl}_r)$-bicrystal, i.e., if $\widetilde{e}_i, \widetilde{f}_i$ are the $\mathfrak{gl}_n$-crystal operators and $\widetilde{e}^*_j, \widetilde{f}^*_j$ are the $\mathfrak{gl}_r$-crystal operators, then $\widetilde{x}_i\widetilde{y}^*_j = \widetilde{y}^*_j\widetilde{x}_i$ for any $x, y \in \{e, f\}$ and any $i = 1, \ldots, n-1$, $j = 1, \ldots, r-1$. In general, it is shown in [22] that the bijection $\kappa_{\mathcal{A}}$ for $\mathcal{A} = \mathbb{I}_{m|n}$ is an isomorphism of $(\mathfrak{gl}_{m|n}, \mathfrak{gl}_r)$-bicrystals, where $\mathfrak{gl}_{m|n}$ is a general linear Lie superalgebra.

Finally, we describe the Weyl group action. The regularity of $SST_r(\lambda)$ results in a well-defined Weyl group action on $SST_r(\lambda)$. In this case, the Weyl group is isomorphic to $\mathfrak{S}_r$ and suppose

$$W = \langle s_1, \ldots, s_{r-1} \,|\, s_i^2 = 1,\ (s_i s_{i+1})^2 = 1 \rangle. \tag{4.5}$$

Note that the Weyl group action on a regular crystal B is given by

$$s_i(b) = \begin{cases} \widetilde{f}_i^{m}(T) & \text{if } m := \langle h_i, \mathrm{wt}(b)\rangle \geq 0, \\ \widetilde{e}_i^{-m}(T) & \text{if } m := \langle h_i, \mathrm{wt}(b)\rangle \leq 0. \end{cases}$$

for $b \in B$ and $i = 1, \ldots, r-1$. In our case, $s_i \in W$ acts on $T \in SST_r(\lambda)$ by

$$s_i(T) = \begin{cases} \widetilde{f}_i^{m_i - m_{i+1}}(T) & \text{if } m_i \geq m_{i+1}, \\ \widetilde{e}_i^{m_{i+1} - m_i}(T) & \text{if } m_i \leq m_{i+1}, \end{cases}$$

where m_i is the number of i in T for $1 \leq i \leq r$, and acts on $(U_r, \ldots, U_1) \in \mathbf{E}_{\mathcal{A}}^r$ by

$$s_i(U_r, \ldots, U_1) = \begin{cases} \mathcal{F}_i^{u_i - u_{i+1}}(U_r, \ldots, U_1) & \text{if } u_i \geq u_{i+1}, \\ \mathcal{E}_i^{u_{i+1} - u_i}(U_r, \ldots, U_1) & \text{if } u_i \leq u_{i+1}, \end{cases}$$

where $u_j = \mathrm{ht}(U_j)$ for $1 \leq j \leq r$.

Chapter 5

Kashiwara–Nakashima tableaux

In [18], it is shown that the crystals of integrable highest weight representations of the quantum group of classical types can be realized by a set of semistandard tableaux with certain configurations, which are now known as Kashiwara-Nakashima tableaux. Since KN tableaux of type A_{n-1} are just $[n]$-semistandard tableaux, they can be considered analogues for other classical types. In this chapter, we discuss KN tableaux of type C and their combinatorics. We deal with KN tableaux of type C or symplectic KN tableaux. This chapter is organized as follows.

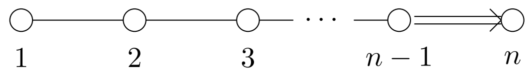
- In Section 5.1, we recall the definition of KN tableaux of type C and its related notations.
- In Section 5.2, we recall the symplectic analogues of Knuth equivalence and the RSK correspondence due to Lecouvey [31].
- In Section 5.3, we review the jeu de taquin algorithm for KN tableaux of type C introduced by Sheats [41].

This chapter is based on [31, 41] and references therein.

5.1 Kashiwara-Nakashima tableaux

Suppose $\mathfrak{g} = \mathfrak{sp}_{2n}$ for $n \geq 2$ and take $P = \bigoplus_{i=1}^{n} \mathbb{Z}\epsilon_i$ to be the weight lattice of $\mathfrak{g}$, $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leq i \leq n-1$, and $\alpha_n = 2\epsilon_n$ as simple roots. Then we have the following

Dynkin diagram of $\mathfrak{g}$.



Let $\mathcal{I}_n = \{1 < \cdots < n < \overline{n} < \cdots < \overline{1}\}$ as given in (4.1). We call the letters $1, \ldots, n$ unbarred (or positive) and the letters $\overline{1}, \ldots, \overline{n}$ barred (or negative). By convention, we set $\overline{\overline{z}} = z$.

We simply say that an $\mathcal{I}_n$-semistandard tableau of single-columned shape is an $\mathcal{I}_n$-column tableau. Let C be an $\mathcal{I}_n$-column tableau C with $h = \mathrm{ht}(C)$ and suppose that $w(C) = w_1 \cdots w_h$.

Definition 5.1.1. We call C admissible if there exists no $z = 1, \ldots, n$ such that $w_p = z$ and $w_q = \overline{z}$ with $1 \le p < q \le h$ and $(q - p) + z \le h$.

For $z = 1, \ldots, n$, let

$$N_C(z) = |\{x \in \mathcal{I}_n : x \text{ is an entry of } C,\;\; x \le z \text{ or } x \ge \overline{z}\}|.$$

Then the condition for C to be admissible is equivalent to $N(z) \le z$ for all $z = 1, \ldots, n$ since $|\{x : x \le z\}| = p$ and $|\{x : x \ge \overline{z}\}| = h - q + 1$.

We also have another characterization of admissible columns. For an $\mathcal{I}_n$-column tableau C, let

$$\{z_1 > \cdots > z_r\} = \{z \mid C \text{ contains both } z \text{ and } \overline{z}\,\}. \tag{5.1}$$

Then we have the (unique) set

$$\{t_1 > \cdots > t_s\} \subseteq \{1, \ldots, n\} \tag{5.2}$$

for some $s \le r$ such that

(1) $t_1 = \max\{t : t < z_1,\ C \text{ do not contain both } t \text{ and } \overline{t}\}$,

(2) $t_i = \max\{t : t < \min\{z_i, t_{i-1}\},\ C \text{ do not contain both } t \text{ and } \overline{t}\}$ for $i \ge 2$.

Definition 5.1.2. We say that C splits if $s = r$.

Lemma 5.1.3 ([41])**.** *An $\mathcal{I}_n$-column tableau is admissible if and only if it splits.*

For an admissible $\mathcal{I}_n$-column tableau C, let

- lC : the $\mathcal{I}_n$-column tableau obtained from C by replacing $\overline{z_i}$ by $\overline{t_i}$ $(1 \le i \le r)$,
- rC : the $\mathcal{I}_n$-column tableau obtained from C by replacing z_i by t_i $(1 \le i \le r)$,
- $\mathrm{spl}(C) = \lceil lC, rC \rceil$, which we call the split form of C.

For a given $\mathcal{I}_n$-semistandard tableau $T = \lceil C_1, \ldots, C_r \rceil$ with admissible $\mathcal{I}_n$-column tableaux C_i, let

$$\mathrm{spl}(T) = \lceil lC_1, rC_1, \ldots, lC_r, rC_r \rceil.$$

Next, we introduce the notion of coadmissible column tableaux, which is necessary to define a symplectic analogue of the jeu de taquin algorithm. For an $\mathcal{I}_n$-tableau C, set

$$N_C^*(z) = |\{x \in \mathcal{I}_n : x \text{ is an entry of } C, \ \ z \le x \le \overline{z}\}|$$

for $z = 1, \ldots, n$. We call C coadmissible if $N_C^*(z) \le n - z + 1$ for all z. There exists a bijection between the set of admissible $\mathcal{I}_n$-column tableaux and coadmissible $\mathcal{I}_n$-column tableaux. In fact, if an $\mathcal{I}_n$-column tableau C is admissible, then the corresponding coadmissible $\mathcal{I}_n$-column tableau C^* is obtained from C by replacing $z_i, \overline{z_i}$ by $t_i, \overline{t_i}$ $(1 \le i \le r)$, respectively.

Example 5.1.4. For the $\mathcal{I}_5$-column tableau C with $w(C) = 14\bar{5}\bar{4}\bar{2}$, we have $z_1 = 4$ and $t_1 = 3$. Thus, the following tableaux are obtained.

$$C = \begin{array}{|c|}\hline 1 \\ \hline 4 \\ \hline \bar{5} \\ \hline \bar{4} \\ \hline \bar{2} \\ \hline \end{array} \quad \longleftarrow \quad \mathrm{spl}(C) = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & 4 \\ \hline \bar{5} & \bar{5} \\ \hline \bar{4} & \bar{3} \\ \hline \bar{2} & \bar{2} \\ \hline \end{array} \quad \longrightarrow \quad C^* = \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \bar{5} \\ \hline \bar{3} \\ \hline \bar{2} \\ \hline \end{array}$$

Note that it is possible to recover C and C^* from $\mathrm{spl}(C)$ and vice versa.

Let C_1 and C_2 be admissible $\mathcal{I}_n$-column tableaux with $\mathrm{ht}(C_i) = m_i$ $(i = 1, 2)$ with $m_2 \ge m_1$. Define

$$C_2 \prec C_1 \quad \text{if } \lceil rC_2, lC_1 \rceil \text{ is } \mathcal{I}_n\text{-semistandard.} \tag{5.3}$$

Note that $C_2 \prec C_1$ if and only if $\mathrm{spl}\lceil C_2, C_1 \rceil$ is $\mathcal{I}_n$-semistandard since $\lceil lC_i, rC_i \rceil$ is semistandard for $i = 1, 2$.

Definition 5.1.5. For $\lambda \in \mathscr{P}_n$, let $\mathbf{KN}_n(\lambda)$ be the set of $T = \lceil C_r, \ldots, C_1 \rceil \in SST_{\mathcal{I}_n}(\lambda)$ such that all columns of T are admissible and $\mathrm{spl}(T)$ is $\mathcal{I}_n$-semistandard, equivalently, $C_{i+1} \prec C_i$ for all $1 \leq i \leq r-1$.

We call $\mathbf{KN}_n(\lambda)$ the set of KN tableaux of type C_n.

Remark 5.1.6. In [18], KN tableaux are defined as tableaux satisfying certain configuration conditions referred to an (a,b)-configuration. It is shown in [41, Theorem A.4] that the definition in [18] is equivalent to Definition 5.1.2.

Let $\mathcal{B}(\varpi_1)$ be the crystal base of the vector representation $V(\varpi_1)$ of $\mathfrak{sp}_{2n}$ whose crystal graph is the following:

$$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-1} n \xrightarrow{n} \overline{n} \xrightarrow{n-1} \overline{n-1} \xrightarrow{n-2} \cdots \xrightarrow{1} \overline{1}\,,$$

where $\mathrm{wt}(i) = \epsilon_i$, $\mathrm{wt}(\bar{i}) = -\epsilon_i$, and $\mathrm{wt}(0) = 0$. As in the case of type A, we give an $\mathfrak{sp}_{2n}$-crystal structure on $\mathbf{KN}_n(\lambda)$ for $\lambda \in \mathscr{P}_n$ as a subcrystal of $B_n(\varpi_1)^{\otimes r}$ for some r. Then $\mathbf{KN}_n(\lambda)$ is isomorphic to $\mathcal{B}(w_\lambda)$ [18] as $\mathfrak{sp}_{2n}$-crystals, where $w_\lambda = \sum_{i=1}^{n} \lambda_i \epsilon_i \in P^+$.

5.2 Symplectic RSK correspondence

In this section, we recall a symplectic analogue of the RSK correspondence [31]. To do this, we introduce a symplectic analogue of the Knuth equivalence relation. Let $Pl(C_n)$ be the quotient of the free monoid generated by $\mathcal{I}_n$ subject to following relations:

(R_1) $xzy \overset{C}{\equiv} zxy$ if $x < y \leq z$ with $z \neq \overline{x}$, $yzx \overset{C}{\equiv} yxz$ if $x \leq y < z$ with $z \neq \overline{x}$,

(R_2) $y\overline{(x-1)}(x-1) \overset{C}{\equiv} yx\overline{x}$ for $1 < x \leq n$ and $x \leq y \leq \overline{x}$,
$x\overline{x}y \overset{C}{\equiv} \overline{(x-1)}(x-1)y$ for $1 < x \leq n$ and $x \leq y \leq \overline{x}$, and

(R_3) if $w = w(C)$ for a non-admissible column C such that every proper subword is an admissible column word (the word of an admissible column), and $z \in \{1, \ldots, n\}$ is the smallest letter such that the pair $(z, \bar{z})$ occurs in w with $N(z) > z$, then $w \overset{C}{\equiv} \widetilde{w}$ where $\widetilde{w}$ is the column word obtained by removing the pair $(z, \bar{z})$ in w.

Denote by $\mathcal{W}_n$ the set of words with letters in $\mathcal{I}_n$. For $w, w' \in \mathcal{W}_n$, write $w \overset{C}{\equiv} w'$ if $w = w'$ in $Pl(C_n)$.

Theorem 5.2.1 ([31])**.** *For $w \in \mathcal{W}_n$, there exists a unique KN tableau T such that $w \overset{C}{\equiv} w(T)$, which we denote by $P(w)$.*

By definition, if $w = w(T)$ for some KN tableau T, then $P(w) = T$.

Remark 5.2.2. We have $w \overset{C}{\equiv} w'$ if and only if $w \overset{\mathfrak{sp}_{2n}}{\equiv} w'$ for $w, w' \in \mathcal{W}_n$ [31].

On the other hand, we can also construct $P(w)$ by a symplectic analogue of the insertion algorithm. For $x \in \mathcal{I}_n$ and $T = \lceil C_r, \ldots, C_1 \rceil \in \mathbf{KN}_n(\lambda)$, define $x \to T$, the insertion of x into T to be the tableau given as follows.

Case 1. Suppose that $w(C_r)x$ is the word of an admissible column C_r'. Then

$$(x \to T) := \lceil C_r', C_{r-1}, \ldots, C_1 \rceil.$$

Case 2. Suppose that $w(C_r)x$ is not the word of a column tableau and then there exists a letter x' such that $w(C_r)x \overset{C}{\equiv} x' w(C_r')$ for some admissible column C_r'. We consider the insertion of x' into $T' = \lceil C_{r-1}, \ldots, C_1 \rceil$. If it belongs to *Case 1*, then we have $(x' \to T')$ and let

$$(x \to T) = \lceil C_r', (x' \to T') \rceil. \tag{5.4}$$

Otherwise, repeat the above step until we get to *Case 1*.

Case 3. Suppose that $w(C_r)x$ is the word of a non-admissible column whose proper subwords are admissible. Suppose $\widetilde{w(C_r)x} = y_1 \cdots y_s$. Then let

$$(x \to T) := (y_s \to (y_{s-1} \to (\cdots \to (y_1 \to T') \cdots))), \tag{5.5}$$

where $T' = \lceil C_{r-1}, \ldots, C_1 \rceil$.

Then this algorithm terminates in finite steps and $x \to T$ is again a KN tableau. In addition, it does not occur that $w(C_r)x$ is the word of a non-admissible column such that some proper subwords are non-admissible, and *Case 3* does not occur during the insertion on the right-hand side of (5.4) and (5.5).

Remark 5.2.3. It is also described in [1] that which letter is bumped out in each step (or column).

Next, we define a recording tableau associated to $P(w)$ for $w \in \mathcal{W}_n$. An oscillating tableau is a sequence $Q = (Q_1, \ldots, Q_r)$ of partitions that Q_i and Q_{i+1} are different by exactly one box for $1 \leq i < r$, i.e., either $|Q_{j+1}/Q_j| = 1$ or $|Q_j/Q_{j+1}| = 1$. Denote by $|Q| = r$ the length of Q and by $\mathrm{sh}(Q) = Q_r$ the shape of Q. Note that an oscillating tableau Q such that $Q_1 \subsetneq Q_2 \subsetneq \cdots \subsetneq Q_r$ and $|Q_i/Q_{i-1}| = 1$ for $1 \leq i \leq r$ with $Q_0 = \emptyset$ is identified with a semistandard tableau of shape Q_r by filling Q_i/Q_{i-1} with i.

Let $w = w_1 \cdots w_r \in \mathcal{W}_n$ be given and let $P_i = P(w_1 \cdots w_i)$ for $1 \leq i \leq r$. Define $Q(w)$ to be the sequence $(Q_1, \ldots, Q_r)$ of partitions where $Q_i = \mathrm{sh}(P_i)$. For $\lambda \in \mathscr{P}_n$, let $OT_n(\lambda)$ be the set of oscillating tableaux $Q = (Q_1, \ldots, Q_r)$ such that $Q_i \in \mathscr{P}_n$ for all $1 \leq i \leq r$ with $Q_r = \lambda$. Then $Q(w) \in OT_n(\lambda)$ where $\lambda = \mathrm{sh}(P(w))$. We get an analogue of the Roninson-Schensted correspondence.

Theorem 5.2.4 ([31])**.** *For $n \geq 1$, we have a bijection*

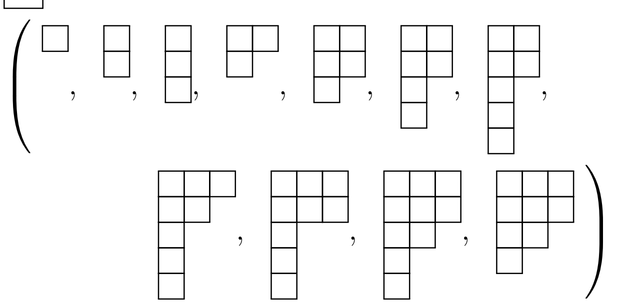
$$\kappa_n \ : \ \mathcal{W}_n \quad \longrightarrow \quad \bigsqcup_{\lambda \in \mathscr{P}} \mathbf{KN}_n(\lambda) \times OT_n(\lambda)$$

sending w to $(P(w), Q(w))$. Furthermore, it is an isomorphism of $\mathfrak{sp}_{2n}$-crystals, where the right-hand side is an $\mathfrak{sp}_{2n}$-crystal with respect to the first component.

Example 5.2.5. Let $w = 3\bar{5}\bar{3}3\bar{5}\bar{4}\bar{2}35\bar{3}\bar{1} \in \mathcal{W}_5$. Then we have $P(w)$ and $Q(w)$ is given by

$$P(w) \ = \ \begin{array}{|c|c|c|} \hline 3 & 3 & \bar{5} \\ \hline 5 & \bar{5} & \bar{2} \\ \hline \bar{4} & \bar{1} & \multicolumn{1}{c}{} \\ \cline{1-2} \bar{3} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

$Q(w) \ =$



where the associated sequence of P_i's is as follows.

$$
\begin{array}{|c|}\hline 3\\\hline\end{array}\,,\;
\begin{array}{|c|}\hline 3\\\hline \overline{5}\\\hline\end{array}\,,\;
\begin{array}{|c|}\hline 3\\\hline \overline{5}\\\hline \overline{3}\\\hline\end{array}\,,\;
\begin{array}{|c|c|}\hline 3 & \overline{5}\\\hline 4 \\\cline{1-1} \overline{4}\\\cline{1-1}\end{array}\,,\;
\begin{array}{|c|c|}\hline 2 & \overline{5}\\\hline 3 & \overline{2}\\\hline \overline{5}\\\cline{1-1}\end{array}\,,\;
\begin{array}{|c|c|}\hline 2 & \overline{5}\\\hline 3 & \overline{2}\\\hline \overline{5}\\\cline{1-1} \overline{4}\\\cline{1-1}\end{array}\,,\;
\begin{array}{|c|c|}\hline 2 & \overline{5}\\\hline 3 & \overline{2}\\\hline \overline{5}\\\cline{1-1} \overline{4}\\\cline{1-1} \overline{2}\\\cline{1-1}\end{array}\,,
$$

$$
\begin{array}{|c|c|c|}\hline 2 & 3 & \overline{5}\\\hline 3 & \overline{2}\\\cline{1-2} \overline{5}\\\cline{1-1} \overline{4}\\\cline{1-1} \overline{2}\\\cline{1-1}\end{array}\,,\;
\begin{array}{|c|c|c|}\hline 2 & 3 & \overline{5}\\\hline 3 & \overline{5} & \overline{2}\\\hline 5\\\cline{1-1} \overline{4}\\\cline{1-1} \overline{2}\\\cline{1-1}\end{array}\,,\;
\begin{array}{|c|c|c|}\hline 1 & 3 & \overline{5}\\\hline 3 & \overline{5} & \overline{2}\\\hline 5 & \overline{1}\\\cline{1-2} \overline{4}\\\cline{1-1} \overline{3}\\\cline{1-1}\end{array}\,,\;
\begin{array}{|c|c|c|}\hline 3 & 3 & \overline{5}\\\hline 5 & \overline{5} & \overline{2}\\\hline \overline{4} & \overline{1}\\\cline{1-2} \overline{3}\\\cline{1-1}\end{array}
$$

5.3 Symplectic jeu de taquin for KN tableaux

The jeu de taquin algorithm is another combinatorial tool to describe the Knuth equivalence and insertion tableaux in the case of type A. In this section, we review a symplectic analogue of jeu de taquin algorithm [41], which we simply say symplectic jeu de taquin (algorithm).

An $\mathcal{I}_n$-semistandard tableau T of shape λ/μ is called admissible if all columns of T are admissible and $\mathrm{spl}(T)$ is $\mathcal{I}_n$-semistandard, where

$$\mathrm{spl}(T) = \left\lceil lC_1, rC_1, \ldots, lC_r, rC_r \right\rceil^{(u_1,u_1,\ldots,u_r,u_r)}$$

for $T = \left\lceil C_1, \ldots, C_r \right\rceil^{(u_1,\ldots,u_r)}$ and $(u_1, \ldots, u_r) = \mu'$. Denote by $\mathbf{KN}_n(\lambda/\mu)$ the set of admissible tableaux of shape λ/μ.

To keep track of empty slots of skew shapes when we apply jeu de taquin, we put $\bullet$ in each empty slot. When we consider the split form of admissible tableaux, marks $\bullet$ are duplicated.

Step 1. For an admissible tableau $T = \left\lceil C_2, C_1 \right\rceil^{(c_2,c_1)}$ of skew shape with two columns, suppose that $\mathrm{spl}(T)$ is given as follow.

$$
\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
\hline
\multicolumn{1}{|c|}{\bullet} & \multicolumn{1}{c|}{\bullet} & \multicolumn{1}{c|}{b} & \multicolumn{1}{c|}{b'} \\
\hline
\multicolumn{1}{|c|}{a} & \multicolumn{1}{c|}{a'} & \vdots & \vdots \\
\cline{1-2}
\vdots & \vdots & &
\end{array}
\tag{5.6}
$$

Note that the boxes filled with a, a' and b, b' may be empty. Let $T' = \lceil C_2', C_1' \rceil^{(c_2-1, c_1)}$ be the tableau defined as follows.

(1) Suppose that $a' \leq b$ or the boxes $\boxed{b}\boxed{b'}$ is empty. Then $\mathrm{spl}(T')$ is obtained from $\mathrm{spl}(T)$ by switching $\boxed{a}\boxed{a'}$ and two marked empty slots.

(2) Suppose that $a' > b$ or the boxes $\boxed{a}\boxed{a'}$ is empty.

  (2-a) If $b \in [n]$, then C_2' is the column tableau obtained from C_2 by replacing the empty slot with $\boxed{b}$, and C_1' is the column tableau with an empty slot such that $(C_1')^*$ is obtained from C_1^* by replacing $\boxed{b}$ with the empty slot.

  (2-a) If $b \in [\overline{n}]$, then C_2' is the column tableau such that $(C_2')^*$ is obtained from C_2^* by replacing the empty slot with $\boxed{b}$, and C_1' is the column tableau with an empty slot obtained from C_1 by replacing $\boxed{b}$ with the empty slot.

It is proved in [41] that both C_1' and C_2' are admissible columns only except the column C_2' in the case of (2-a). The case (2-a) with non-admissible C_2' will be covered in *Step 2*.

Example 5.3.1.

$$T_1 = \begin{array}{|c|c|}\hline \bullet & 3 \\ \hline 3 & \overline{5} \\ \hline \overline{5} & \overline{3} \\ \hline \overline{4} & \\ \cline{1-1} \overline{2} & \\ \cline{1-1} \end{array} \qquad \mathrm{spl}(T_1) = \begin{array}{|c|c|c|c|}\hline \bullet & {\color{red}\bullet} & {\color{blue}2} & 3 \\ \hline 1 & {\color{blue}3} & \overline{5} & {\color{red}\overline{5}} \\ \hline {\color{blue}\overline{5}} & \overline{5} & \overline{3} & {\color{red}\overline{2}} \\ \hline {\color{blue}\overline{4}} & \overline{4} & & \\ \cline{1-2} {\color{blue}\overline{2}} & \overline{2} & & \\ \cline{1-2} \end{array} \qquad T_1' = \begin{array}{|c|c|}\hline {\color{blue}2} & {\color{red}\bullet} \\ \hline {\color{blue}3} & {\color{red}\overline{5}} \\ \hline {\color{blue}\overline{5}} & {\color{red}\overline{2}} \\ \hline {\color{blue}\overline{4}} & \\ \cline{1-1} {\color{blue}\overline{2}} & \\ \cline{1-1} \end{array}$$

$$T_2 = \begin{array}{|c|c|}\hline \bullet & 2 \\ \hline \bullet & 3 \\ \hline 3 & \overline{5} \\ \hline 5 & \overline{4} \\ \hline \overline{3} & \overline{2} \\ \hline \overline{1} & \\ \cline{1-1} \end{array} \qquad \mathrm{spl}(T_2) = \begin{array}{|c|c|c|c|}\hline \bullet & \bullet & 1 & {\color{blue}2} \\ \hline {\color{red}2} & 3 & 3 & {\color{blue}3} \\ \hline {\color{red}5} & 5 & {\color{blue}\overline{5}} & \overline{5} \\ \hline \bullet & {\color{blue}\bullet} & {\color{red}\overline{4}} & \overline{4} \\ \hline \overline{3} & {\color{red}\overline{2}} & {\color{blue}\overline{2}} & \overline{1} \\ \hline \overline{1} & {\color{red}\overline{1}} & & \\ \cline{1-2} \end{array} \qquad T_2' = \begin{array}{|c|c|}\hline \bullet & {\color{blue}2} \\ \hline {\color{red}3} & {\color{blue}3} \\ \hline {\color{red}5} & {\color{blue}\overline{5}} \\ \hline {\color{red}\overline{4}} & {\color{blue}\bullet} \\ \hline {\color{red}\overline{3}} & {\color{blue}\overline{2}} \\ \hline {\color{red}\overline{1}} & \\ \cline{1-1} \end{array}$$

Note that unlike the case of type A, letters in admissible tableaux may change during this step while the weight of the tableaux are invariant.

Step 2. For an admissible tableau $T = \lceil C_r, \ldots, C_1 \rceil^{(c_r, \ldots, c_1)}$ of skew shape, choose an inner corner c of T and suppose c lies in the i-th column from the right. Mark $\bullet$ at c to keep track of c. Apply *Step 1* to $\lceil C_i, C_{i-1} \rceil^{(c_i - 1, c_{i-1})}$ and get $\lceil C_i', C_{i-1}' \rceil^{(c_i', c_{i-1}')}$ such that C_{i-1}'

has the empty slot $\bullet$. Now, we repeat *Step 1* until the mark is placed at the bottom of the j-th column (or becomes an outer corner of T) to have

$$T' = \lceil \ldots, C_{i+1}, C'_i, \ldots, C'_j, C_{j-1}, \ldots \rceil^{(\ldots, c'_i, \ldots, c'_j, \ldots)}$$

for some $j \le i$, $C'_i, \ldots, C'_j$, and $c_i, \ldots, c'_j \in \mathbb{Z}_+$ ignoring the mark $\bullet$. It is shown in [41] that if T' has an non-admissible column, then it must be C'_i. Let C''_i be the unique admissible column such that $w(C''_i) \overset{C}{\equiv} w(C'_i)$, i.e., $w(C''_i) = \widetilde{w(C'_i)}$. Put

$$T'' = \lceil \ldots, C_{i+1}, C''_i, C'_{i-1}, \ldots, C'_j, C_{j-1}, \ldots \rceil^{(\ldots, c_i, \ldots, c'_j, \ldots)}.$$

Now we define

$$\mathtt{jdt}_{KN}(T, c) = \begin{cases} T' & \text{if } T' \text{ is admissible,} \\ T'' & \text{if } T' \text{ has a non-admissible column.} \end{cases} \tag{5.7}$$

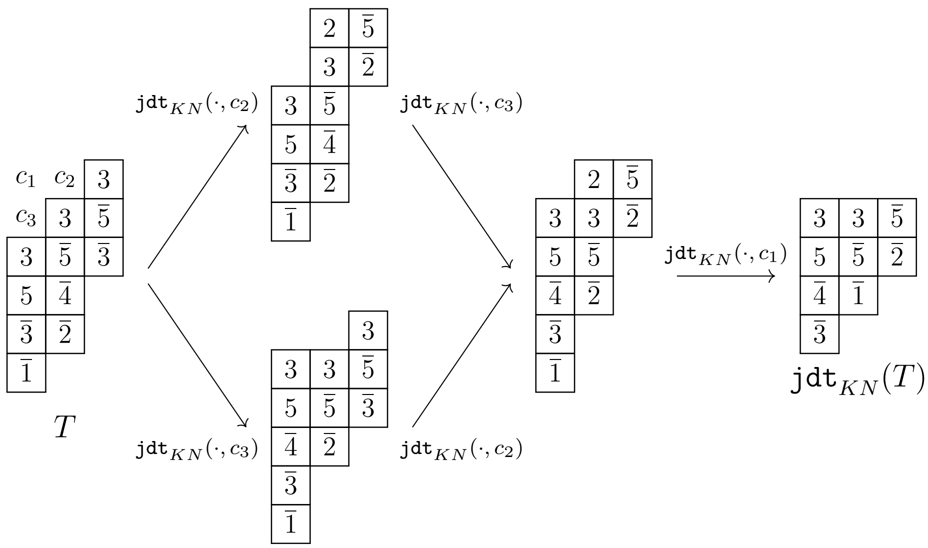
It is shown in [31, Theorem 6.3.8] that $w(\mathtt{jdt}_{KN}(T, c)) \overset{C}{\equiv} w(T)$. Note that if $\mathrm{sh}(T) = \lambda/\mu$, then

$$\mathrm{sh}(\mathtt{jdt}_{KN}(T, c)) = \begin{cases} \alpha/\mu & \text{for some } \alpha \subsetneq \lambda \text{ if } T' \text{ is non-admissible,} \\ \alpha/\beta & \text{for some } \alpha \subsetneq \lambda, \beta \subsetneq \mu \text{ if } T' \text{ is admissible.} \end{cases}$$

By applying $\mathtt{jdt}_{KN}(\,\cdot\,, c)$ consecutively to inner corners, we obtain a KN tableau, say $\mathtt{jdt}_{KN}(T)$, and then it is clear that $w(\mathtt{jdt}_{KN}(T)) \overset{C}{\equiv} w(T)$. By Theorem 5.2.1, the unique KN tableau T' such that $w(T') \overset{C}{\equiv} w(T)$ is $P(w(T))$ and we have $P(w(T)) = \mathtt{jdt}_{KN}(T)$. We remark that $\mathtt{jdt}_{KN}(T)$ is independent of orders of choices of inner corners by the uniqueness of $P(w(T))$.

Example 5.3.2. Let T be an admissible tableau with empty slots c_1, c_2, c_3 given below. Then we have $\mathtt{jdt}_{KN}(T)$ regardless of orders of choices of inner corners (cf. Example

5.2.5).


$\texttt{jdt}_{KN}(\cdot, c_2)$
$\texttt{jdt}_{KN}(\cdot, c_3)$
$\texttt{jdt}_{KN}(\cdot, c_3)$
$\texttt{jdt}_{KN}(\cdot, c_2)$
$\texttt{jdt}_{KN}(\cdot, c_1)$
c_1 c_2
c_3
T
$\texttt{jdt}_{KN}(T)$

Chapter 6

Spinor model

In this chapter, we discuss a spinor model, which is the main object of this thesis. A spinor model, which is introduced by Kwon [25, 26], is a tableau model for irreducible characters of classical type in (1.2). We give an explicit relation between the spinor model and KN tableaux of type C, which is essential to develop a jeu de taquin sliding for the spinor model with an arbitrary set of letters. This chapter is organized as follows.

- In Section 6.1, we recall the definition of a spinor model $\mathbf{T}_{\mathcal{A}}(\lambda, \ell)$.
- In Section 6.2, we give a Schur expansion of the character of $\mathbf{T}_{\mathcal{A}}(\lambda, \ell)$.
- In Section 6.3, we give a weight-preserving bijection between $\mathbf{T}_{[\bar{n}]}(\lambda, \ell)$ and $\mathbf{KN}_n(\mu)$ corresponding to (λ, ℓ), which is the key ingredient to define a jeu de taquin for the spinor model.
- In Section 6.4, we discuss a $\mathfrak{gl}_2$-crystal structure and the admissibility condition on $\mathcal{I}_n$-column tableaux.

6.1 Spinor model

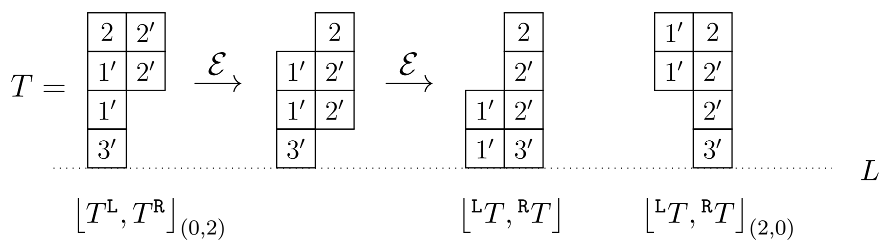
For a linearly ordered $\mathbb{Z}_2$-graded set $\mathcal{A}$, let

$$\mathbf{T}_{\mathcal{A}}(a) \;\; = \bigsqcup_{c \in \mathbb{Z}_+} SST_{\mathcal{A}}(\lambda(a, 0, c))$$

for $a \in \mathbb{Z}_+$. For $T \in \mathbf{T}_{\mathcal{A}}(a)$, we have $\mathcal{E}^a T \in SST_{\mathcal{A}}(\lambda(0,a,c))$ and set

$$^{\mathtt{L}}T = (\mathcal{E}^a T)^{\mathtt{L}}, \quad {}^{\mathtt{R}}T = (\mathcal{E}^a T)^{\mathtt{R}}.$$

Example 6.1.1. Suppose $\mathcal{A} = \mathbb{I}_{4|3}$ and $T \in SST_{\mathcal{A}}(\lambda(2,0,2))$ as below.



Definition 6.1.2.

(1) For $a_1, a_2 \in \mathbb{Z}_+$ with $a_2 \leq a_1$ and $(T_2, T_1) \in \mathbf{T}_{\mathcal{A}}(a_2) \times \mathbf{T}_{\mathcal{A}}(a_1)$, we define

$$T_2 \prec T_1 \quad \text{if } \left\lfloor {}^{\mathtt{R}}T_2, T_1^{\mathtt{L}} \right\rfloor \text{ and } \left\lfloor T_2^{\mathtt{R}}, {}^{\mathtt{L}}T_1 \right\rfloor_{(a_2,a_1)} \text{ are } \mathcal{A}\text{-semistandard.}$$

(2) Let $\mathscr{P}(\mathrm{Sp}) = \{\, (\lambda, \ell) \mid \ell \geq 1,\ \lambda \in \mathscr{P}_\ell \,\}$. For $(\lambda,\ell) \in \mathscr{P}(\mathrm{Sp})$, we define

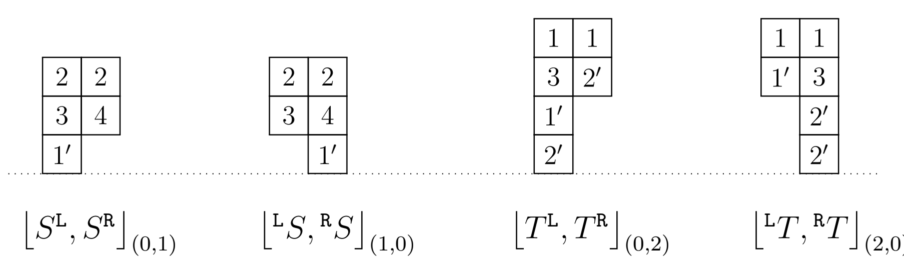
$$\mathbf{T}_{\mathcal{A}}(\lambda,\ell) = \{\, \mathbf{T} = (T_\ell, \ldots, T_1) \mid T_\ell \prec \cdots \prec T_1 \,\} \subseteq \mathbf{T}_{\mathcal{A}}(\lambda_\ell) \times \cdots \times \mathbf{T}_{\mathcal{A}}(\lambda_1).$$

We call $\mathbf{T}_{\mathcal{A}}(\lambda,\ell)$ a spinor model of shape (λ,ℓ) with respect to $\mathcal{A}$.

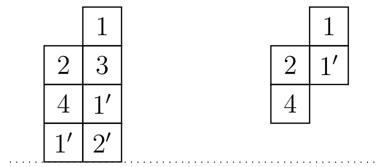
We put $\mathscr{P}(\mathrm{Sp})_{\mathcal{A}} = \{\, (\lambda,\ell) \in \mathscr{P}(\mathrm{Sp}) \,|\, \mathbf{T}_{\mathcal{A}}(\lambda,\ell) \neq \emptyset \,\}$.

Remark 6.1.3. We use the definition of $\mathbf{T}_{\mathcal{A}}(\lambda,\ell)$ in [25]. There is another definition of $\mathbf{T}_{\mathcal{A}}(\lambda,\ell)$ in [27], but almost the same result hold for this case.

Example 6.1.4. Let $\mathcal{A} = \mathbb{I}_{4|3}$, and take $S \in \mathbf{T}_{\mathcal{A}}(1)$ and $T \in \mathbf{T}_{\mathcal{A}}(2)$ as follows.



Then $S \prec T$ since $\lfloor {}^{\mathtt{R}}S, T^{\mathtt{L}} \rfloor$ and $\lfloor S^{\mathtt{R}}, {}^{\mathtt{L}}T \rfloor_{(1,2)}$ are $\mathcal{A}$-semistandard.



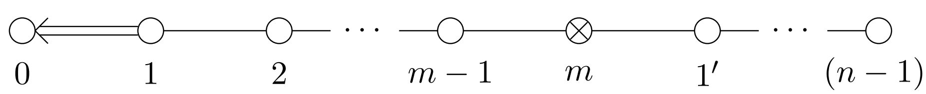
$\lfloor {}^{\mathtt{R}}S, T^{\mathtt{L}} \rfloor$ $\qquad$ $\lfloor S^{\mathtt{R}}, {}^{\mathtt{L}}T \rfloor_{(1,2)}$

Let t be a formal variable commuting with all x_a ($a \in \mathcal{A}$). Define the character of $\mathbf{T}_{\mathcal{A}}(\lambda, \ell)$ for $(\lambda, \ell) \in \mathscr{P}(\mathrm{Sp})_{\mathcal{A}}$ by

$$S_{(\lambda,\ell)}(\mathbf{x}_{\mathcal{A}}) = t^{\ell} \sum_{(T_\ell,\ldots,T_1)\in \mathbf{T}_{\mathcal{A}}(\lambda,\ell)} \mathbf{x}_{\mathcal{A}}^{T_\ell} \cdots \mathbf{x}_{\mathcal{A}}^{T_1} . \tag{6.1}$$

Remark 6.1.5. The character of $\mathbf{T}_{\mathcal{A}}(\lambda, \ell)$ has an important application in representation theory. Indeed, the spinor model is motivated by the $(\mathfrak{g}_{\mathcal{A}}, \mathrm{Sp}_{2\ell}(\mathbb{C}))$-duality (1.2) for a Lie (super)algebra $\mathfrak{g}_{\mathcal{A}}$.

Let us first recall the decomposition (1.2) for various choices of $\mathcal{A}$. If $\mathcal{A} = [\overline{m}]$, then we have the $(\mathfrak{sp}_{2m}, \mathrm{Sp}_{2\ell}(\mathbb{C}))$-duality, where $V_{\mathfrak{sp}_{2m}}(\lambda, \ell)$ is a finite-dimensional irreducible $\mathfrak{sp}_{2m}$-module. If $\mathcal{A} = [n]'$, then we have the $(\mathfrak{so}_{2n}, \mathrm{Sp}_{2\ell}(\mathbb{C}))$-duality, where $V_{\mathfrak{so}_{2n}}(\lambda, \ell)$ is an infinite-dimensional irreducible $\mathfrak{so}_{2n}$-module. See [10–12] for these dualities. In general, when $\mathcal{A} = \mathbb{I}_{m|n}$, we have the $(\mathfrak{spo}_{2m|2n}, \mathrm{Sp}_{2\ell}(\mathbb{C}))$-duality [5], which includes both of the above cases with $n = 0$ and $m = 0$, respectively. Here $\mathfrak{spo}_{2m|2n}$ is the orthosymplectic Lie superalgebra whose Dynkin diagram is given by



Dualities for an infinite $\mathbb{Z}_2$-graded set $\mathcal{A}$ can be found in [29, 44].

It is shown in [25] that $S_{(\lambda,\ell)}(\mathbf{x}_{\mathcal{A}})$ is equal to the character of $V_{\mathfrak{g}_{\mathcal{A}}}(\lambda, \ell)$ when $\mathcal{A} = \mathbb{I}_{m|n}$. Indeed, this will also follow from comparing the character identities of (1.2) and (1.3) for any $(\mathfrak{g}_{\mathcal{A}}, \mathrm{Sp}_{2\ell}(\mathbb{C}))$-duality (see Theorem 9.3.1).

6.2 Schur positivity

We can embed $T \in \mathbf{T}_{\mathcal{A}}(a)$ into $(T^{\mathtt{L}}, T^{\mathtt{R}}) \in \mathbf{E}^2_{\mathcal{A}}$. In general, we have a natural embedding of r-tuples of $\mathbf{T}_{\mathcal{A}}(a)$'s into $\mathbf{E}^{2\ell}_{\mathcal{A}}$. By composing $\kappa_{\mathcal{A}}$ (4.2) and this embedding restricted to $\mathbf{T}_{\mathcal{A}}(\lambda,\ell)$ for $(\lambda,\ell) \in \mathscr{P}(\mathrm{Sp})_{\mathcal{A}}$, we have the following

$$\begin{array}{rccc} \Phi_{\mathcal{A}} & : & \mathbf{T}_{\mathcal{A}}(\lambda,\ell) & \longrightarrow & \displaystyle\bigsqcup_{\mu} SST_{\mathcal{A}}(\mu) \times SST_{[2\ell]}(\mu') \\ & & \mathbf{T} = (T_\ell,\ldots,T_1) & \longmapsto & (P(\mathbf{T}), Q(\mathbf{T})), \end{array}$$

where μ is over all partitions with $\ell(\mu') \leq 2\ell$.

Let us explicitly describe the image of $\Phi_{\mathcal{A}}$. Recall the Weyl group action on $SST_{[2\ell]}(\mu')$. Let s_i be the simple reflection given in (4.5) for $1 \leq i \leq 2\ell-1$. For $Q \in SST_{[2\ell]}(\mu')$, define the weight of Q to be the sequence $(m_1,\ldots,m_{2\ell})$, and the i-signature of Q to be the pair $(\varepsilon_i(Q), \varphi_i(Q))$ for $1 \leq i \leq 2\ell-1$. For $\mu \in \mathscr{P}$ with $\ell(\mu') \leq 2\ell$ and $(\lambda,\ell) \in \mathscr{P}(\mathrm{Sp})_{\mathcal{A}}$, let $K_{\mu(\lambda,\ell)}$ be the set of $Q \in SST_{[2\ell]}(\mu')$ satisfying the following conditions:

(1) $m_{2k} - m_{2k-1} = \lambda_k$ for $1 \leq k \leq \ell$,

(2) $m_{2k} \geq m_{2k+2}$ for $1 \leq k \leq \ell-1$,

(3) the $(2k-1)$-signature of Q is $(\lambda_k, 0)$ for $1 \leq k \leq \ell$,

(4) the $(2k)$-signature of $s_{2k+1}Q$ is $(0, m_{2k} - m_{2k+2})$ for $1 \leq k \leq \ell-1$,

(5) the $(2k)$-signature of $s_{2k-1}Q$ is $(\lambda_k - \lambda_{k+1} - p, m_{2k} - m_{2k+2} - p)$ for some $p \geq 0$ and all $1 \leq k \leq \ell-1$.

Remark 6.2.1. Considering the action of s_i on $\mathbf{E}^{2\ell}_{\mathcal{A}}$, we have

$$s_{2k-1}(T^{\mathtt{L}}_\ell, T^{\mathtt{R}}_\ell, \ldots, T^{\mathtt{L}}_1, T^{\mathtt{R}}_1) = (T^{\mathtt{L}}_\ell, T^{\mathtt{R}}_\ell, \ldots, {}^{\mathtt{L}}T_k, {}^{\mathtt{R}}T_k, \ldots, T^{\mathtt{L}}_1, T^{\mathtt{R}}_1)$$

for $1 \leq k \leq \ell$. Indeed, it is related to the combinatorial R-matrix. For detailed explanation, see [25, Section 6].

Theorem 6.2.2. [25, Theorem 6.12] *For $(\lambda,\ell) \in \mathscr{P}(\mathrm{Sp})_{\mathcal{A}}$, $\Phi_{\mathcal{A}}$ induces a bijection*

$$\Phi_{\mathcal{A}} : \mathbf{T}_{\mathcal{A}}(\lambda,\ell) \longrightarrow \bigsqcup_{\mu} SST_{\mathcal{A}}(\mu) \times K_{\mu(\lambda,\ell)}$$

where the union is over the partitions μ such that $\ell(\mu') \leq 2\ell$.

Example 6.2.3. Suppose $\mathcal{A} = \mathbb{I}_{4|3}$ and $(\lambda, \ell) = ((3,2,1),3)$. Then for $\mathbf{T} \in \mathbf{T}_{\mathcal{A}}(\lambda, \ell)$ below, $\Phi_{\mathcal{A}}(\mathbf{T})$ is given as follows (cf. Example 4.1.3).

$$\mathbf{T} = \quad \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 & 4 \\ \hline 1' & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 & 2' \\ \hline 1' & \\ \hline 2' & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 2 & 2' \\ \hline 1' & 2' \\ \hline 1' & \\ \hline 3' & \\ \hline 3' & \\ \hline \end{array}$$

$$\Phi_{\mathcal{A}}(\mathbf{T}) = \left(\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 1' & 2' \\ \hline 2 & 2 & 3 & 2' & 3' \\ \hline 3 & 4 & 1' & 2' & \\ \hline 1' & 2' & 3' & & \\ \hline 1' & & & & \\ \hline \end{array} \;,\quad \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 2 \\ \hline 2 & 2 & 3 & 4 & \\ \hline 3 & 4 & 4 & 6 & \\ \hline 4 & 5 & 5 & & \\ \hline 6 & 6 & & & \\ \hline \end{array} \right)$$

6.3 Bijection between the spinor model and KN tableaux

Recall that $S_{(\lambda,\ell)}(\mathbf{x}_{\mathcal{A}})$ gives an irreducible character for $\mathfrak{sp}_{2n}$ when $\mathcal{A} = [\overline{n}]$. In this section, we give a bijection between $\mathbf{T}_{\mathcal{A}}(\lambda, \ell)$ and the corresponding set of KN tableaux [28].

From now on, we assume $\mathcal{A} = [\overline{n}]$ in this section and put

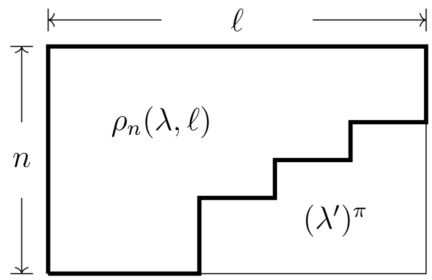
- $\mathscr{P}(\mathrm{Sp})_n := \mathscr{P}(\mathrm{Sp})_{[\overline{n}]} = \{\, (\lambda, \ell) \in \mathscr{P}(\mathrm{Sp}) \,|\, \lambda_1 \leq n \,\}$,
- $\mathbf{T}_n(a) = \mathbf{T}_{[\overline{n}]}(a) \quad$ for $0 \leq a \leq n$,
- $\mathbf{T}_n(\lambda, \ell) = \mathbf{T}_{[\overline{n}]}(\lambda, \ell) \quad$ for $(\lambda, \ell) \in \mathscr{P}(\mathrm{Sp})_n$.

On the other hand, for $(\lambda, \ell) \in \mathscr{P}(\mathrm{Sp})_n$, we put

$$\rho_n(\lambda, \ell) = (n - \lambda_\ell, n - \lambda_{\ell-1}, \ldots, n - \lambda_1)',$$

which is the conjugate of the rectangular complement of λ in (n^ℓ). The diagram for $\rho_n(\lambda, \ell)$ is given as the follow, where μ^π is the skew Young diagram obtained by 180°-

rotating the Young diagram for $\mu \in \mathscr{P}$.



For $U \in SST_{[\overline{n}]}((1^m))$, let U^c be the tableau in $SST_{[n]}((1^{n-m}))$ such that k is an entry of U^c if and only if $\overline{k}$ is not an entry of U for each $k = 1, \ldots, n$. For $T \in \mathbf{T}_n(a)$ $(0 \le a \le n)$, we define $T^{\mathtt{ad}}$ to be the column tableau obtained by putting ${}^{\mathtt{L}}T$ below $({}^{\mathtt{R}}T)^c$. It is proved in [28] that $T^{\mathtt{ad}}$ is an admissible column of height $n-a$, and the map $T \longmapsto T^{\mathtt{ad}}$ is a bijection from $\mathbf{T}_n(a)$ to $\mathbf{KN}_n((1^{n-a}))$ for $0 \le a \le n$. We remark that the coadmissible column $(T^{\mathtt{ad}})^*$ is obtained by putting $T^{\mathtt{R}}$ below $(T^{\mathtt{L}})^c$. For simplicity, write $(T^{\mathtt{ad}})^* = T^{\mathtt{ad}*}$.

Example 6.3.1. Suppose that $n = 5$ and $T \in \mathbf{T}_5(1)$ is given as follows. Then

$$T = \underset{T^{\mathtt{L}}\ \ T^{\mathtt{R}}}{\begin{array}{|c|c|}\hline \overline{4} & \overline{2} \\ \hline \overline{3} & \overline{1} \\ \hline \overline{1} & \\ \cline{1-1} \end{array}} \qquad \underset{{}^{\mathtt{L}}T\ \ {}^{\mathtt{R}}T}{\begin{array}{|c|c|}\hline \overline{3} & \overline{4} \\ \hline \overline{1} & \overline{2} \\ \hline & \overline{1} \\ \cline{2-2} \end{array}} \qquad \begin{array}{l} ({}^{\mathtt{R}}T)^c = \begin{array}{|c|}\hline 3 \\ \hline 5 \\ \hline \end{array} \quad T^{\mathtt{ad}} = \begin{array}{|c|}\hline 3 \\ \hline 5 \\ \hline \overline{3} \\ \hline \overline{1} \\ \hline \end{array} \\ \\ (T^{\mathtt{L}})^c = \begin{array}{|c|}\hline 2 \\ \hline 5 \\ \hline \end{array} \quad T^{\mathtt{ad}*} = \begin{array}{|c|}\hline 2 \\ \hline 5 \\ \hline \overline{2} \\ \hline \overline{1} \\ \hline \end{array} \end{array}.$$

Remark 6.3.2. The bijection $(\cdot)^{\mathtt{ad}} : \mathbf{T}_n(a) \to \mathbf{KN}_n((1^{n-a}))$ is actually an isomorphism of $\mathfrak{sp}_{2n}$-crystals.

For $U \in SST_{\mathcal{I}_n}((1^m))$, let U_+ and U_- be the subtableau of U consisting of all positive (or unbarred) and negative (or barred) letters, respectively. We can directly check the following lemma, which plays a crucial role in this thesis.

Lemma 6.3.3. *Let $T \in \mathbf{T}_n(a)$ be given with $0 \le a \le n$ and let $C = T^{\mathtt{ad}}$. Then*

$$(lC)_+ = C_+^* = (T^{\mathtt{L}})^c, \qquad (rC)_+ = C_+ = ({}^{\mathtt{R}}T)^c,$$
$$(lC)_- = C_- = {}^{\mathtt{L}}T, \qquad (rC)_- = C_-^* = T^{\mathtt{R}}.$$

Lemma 6.3.4. *Let $T_i \in \mathbf{T}_n(m_i)$ be given $(i = 1, 2)$ with $0 \le m_2 \le m_1$. Then we have $T_2^{\mathtt{ad}} \prec T_1^{\mathtt{ad}}$ if and only if $T_2 \prec T_1$.*

Proof. It is straightforward that Definition 6.1.2 (1) is equivalent to (5.3) by using Lemma 6.3.3. □

Proposition 6.3.5. *For $(\lambda, \ell) \in \mathscr{P}(\mathrm{Sp})_n$, we have a bijection*

$$\begin{array}{ccc} \mathbf{T}_n(\lambda, \ell) & \longrightarrow & \mathbf{KN}_n(\rho_n(\lambda, \ell)) \\ \mathbf{T} = (T_\ell, \ldots, T_1) & \longmapsto & \mathbf{T}^{\mathtt{ad}} := \left\lceil T_\ell^{\mathtt{ad}}, \ldots, T_1^{\mathtt{ad}} \right\rceil \end{array} . \tag{6.2}$$

Proof. Suppose that $\mathbf{T} = (T_\ell, \ldots, T_1) \in \mathbf{T}_n(\lambda, \ell)$ is given. Then $\mathbf{T}^{\mathtt{ad}} = \left\lceil T_\ell^{\mathtt{ad}}, \ldots, T_1^{\mathtt{ad}} \right\rceil$ is a tableau of shape $\rho_n(\lambda, \ell)$ whose columns are admissible. By Definition 5.1.5, $\mathbf{T}^{\mathtt{ad}} \in \mathbf{KN}_n(\rho_n(\lambda, \ell))$ if and only if $T_{i+1}^{\mathtt{ad}} \prec T_i^{\mathtt{ad}}$ for all $1 \le i \le \ell - 1$. By applying Lemma 6.3.4 to all adjacent columns, we see that $\mathbf{T} \in \mathbf{T}_n(\lambda, \ell)$ if and only if $\mathbf{T}^{\mathtt{ad}} \in \mathbf{KN}_n(\rho_n(\lambda, \ell))$. □

Note that $(\cdot)^{\mathtt{ad}} : \mathbf{T}_n(\lambda, \ell) \to \mathbf{KN}_n(\rho_n(\lambda, \ell))$ is also an isomorphism of $\mathfrak{sp}_{2n}$-crystals.

6.4 Admissibility and $\mathfrak{gl}_2$-strings

In this section, we investigate the admissibility of $\mathcal{I}_n$-column tableaux and the $\mathfrak{gl}_2$-crystal structure in connection with $\mathcal{E}$ and $\mathcal{F}$ (4.3).

Let

$$\mathbf{F}_n = \bigsqcup_{0 \le m \le 2n} SST_{\mathcal{I}_n}((1^m)) \tag{6.3}$$

be the set of $\mathcal{I}_n$-column tableaux, where $SST_{\mathcal{I}_n}((1^0))$ is the singleton set of the empty tableau. For $C \in SST_{\mathcal{I}_n}((1^m))$, we define $\mathcal{E}C$ to be the $\mathcal{I}_n$-column tableau C' of shape (1^{m-2}) such that

$$(C'_-, (C'_+)^c) = \mathcal{E}(C_-, (C_+)^c)$$

if $\mathcal{E}(C_-, (C_+)^c) \neq \mathbf{0}$, and $\mathcal{E}C = \mathbf{0}$ otherwise (see (4.3)). Also, we define $\mathcal{F}C$ to be the $\mathcal{I}_n$-column tableau of shape (1^{m+2}) in a similar way as in $\mathcal{E}C$. Then we have the following.

Lemma 6.4.1. *Under the above hypothesis,*

(1) $\mathbf{F}_n$ *is a regular* $\mathfrak{gl}_2$*-crystal with respect to* $\mathcal{E}$ *and* $\mathcal{F}$*,*

(2) $C \in \mathbf{F}_n$ *is admissible if and only if* $\mathcal{E}C = 0$*, and*

(3) we have a bijection

$$\begin{array}{ccc} \mathbf{F}_n & \longrightarrow & \bigsqcup_{0\leq a\leq n} \mathbf{KN}_n((1^{n-a})) \times \mathbb{Z}/(a+1)\mathbb{Z}\,, \\ C & \longmapsto & (T, \varepsilon) \end{array} \tag{6.4}$$

where $\varepsilon = \varepsilon(C)$*, and* $T = \mathcal{E}^{\max}C = \mathcal{E}^{\varepsilon}C$*. Here,* $\mathbb{Z}/(a+1)\mathbb{Z}$ *is understood as the set* $\{0, 1, \ldots, a\}$*.*

Proof. (1) It follows from the regular $\mathfrak{gl}_2$-crystal structure on $\mathbf{E}^2_{\mathcal{A}}$ by Lemma 4.3.3.

(2) Suppose $C \in SST_{\mathcal{I}_n}((1^m))$ with $\varepsilon = \varepsilon(C)$ and $\varphi = \varphi(C)$. As defined in (4.3), we may identify $(C_-, (C_+)^c)$ with the semistandard tableau U of skew shape $\lambda(\varepsilon, \varphi, c)$ for some $c \in \mathbb{Z}_+$ with $\mathfrak{r}_T = 0$. Take two sets $\{z_1 > \cdots > z_r\}$ and $\{t_1 > \cdots > t_s\}$ from C as in (5.1) and (5.2), respectively. By construction of these two sets, C contains both z_i and $\overline{z_i}$ for all i, but does not contain neither t_j nor $\overline{t_j}$ for all j. By the definition of U, $\overline{z_i}$ lies in C_- but not in $(C_+)^c$. Suppose that $\overline{z}$ is moved from the left to the right when we apply $\mathcal{E}$ to U. Considering the jeu de taquin sliding, we can check that $z = z_k$ for some k with $N(z_k) > z_k$. Moreover, z_k is such that $N(z_k) - z_k$ is largest. If there are at least two such elements, then z_k is the smallest one among them. In turn, C is admissible if and only if $\varepsilon = 0$.

(3) For given $C \in \mathbf{F}_n$, let (T, ε) be given in (6.4). By the argument in the proof of (2) and Proposition 6.3.5, $(T_-, (T_+)^c)$ is a semistandard tableau of skew shape $\lambda(0, a, c)$ for some $a, c \in \mathbb{Z}_+$ and hence $T \in \mathbf{KN}_n((1^{n-a}))$. Note that $0 \leq a \leq n$ from the condition on $\mathrm{ht}((T_+)^c)$. On the other hand, since $\mathbf{F}_n$ is a regular $\mathfrak{gl}_2$-crystal, we have $\varepsilon(C), \varphi(C) \geq 0$ and $\varepsilon(C) + \varphi(C) = \varepsilon(T) + \varphi(T) = a$, which implies $0 \leq \varepsilon \leq a$. Thus, the map (6.4) is well-defined. Moreover, since the jeu de taquin is reversible, the map (6.4) is bijective. $\square$

Example 6.4.2. For $C \in \mathbf{F}_5$ below, we have

$$C = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \overline{5} \\ \hline \overline{4} \\ \hline \end{array} \quad , \quad (C_-, (C_+)^c) = \begin{array}{|c|c|}\hline \overline{5} & \overline{5} \\ \hline \overline{4} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad , \quad \mathcal{E}(C_-, (C_+)^c) = \begin{array}{c|c|}\cline{2-2} \multicolumn{1}{c|}{} & \overline{5} \\ \hline \multicolumn{1}{|c|}{\overline{5}} & \overline{4} \\ \hline \end{array} \quad .$$

Then the pair corresponding to C under (6.4) is

$$(T, \varepsilon(C)) = \left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \overline{5} \\ \hline \end{array} \, , \ 1 \right) .$$

Remark 6.4.3. Using the similar argument as the proof of Lemma 6.4.1(2), the crystal operators $\mathcal{E}$ and $\mathcal{F}$ on $C \in \mathbf{F}_n$ are calculated directly as follows:

(1) $\mathcal{E}C$ is the $\mathcal{I}_n$-column tableau obtained by deleting z and $\overline{z}$ from C, where z is such that both z and $\overline{z}$ appear in C and the function $N(x) - x$ ($x = 1, \ldots, n$) has the maximum at $x = z$ with choosing the smallest z if there exists at least two such elements.

(2) $\mathcal{F}C$ is the $\mathcal{I}_n$-column tableau obtained by adding t and $\overline{t}$ to C, where neither t nor $\overline{t}$ do not appear in C and the function $N(x) - x$ ($x = 1, \ldots, n$) has the maximum at $x = t$ with choosing the largest t if there exists at least two such elements.

For example, suppose $n = 8$ and let T be an $\mathcal{I}_8$-column tableau given below. Then we can check to check $\mathcal{E}T$ is obtained from T by deleting 3 and $\overline{3}$ and $\mathcal{F}T$ is obtained by adding 7 and $\overline{7}$ to T.

$$T = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 5 \\ \hline 6 \\ \hline \overline{6} \\ \hline \overline{3} \\ \hline \overline{2} \\ \hline \end{array} \qquad \mathcal{E}T = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 5 \\ \hline 6 \\ \hline \overline{6} \\ \hline \overline{2} \\ \hline \end{array} \qquad \mathcal{F}T = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 5 \\ \hline 6 \\ \hline 7 \\ \hline \overline{7} \\ \hline \overline{6} \\ \hline \overline{3} \\ \hline \overline{2} \\ \hline \end{array}$$

Note that $N(2) - 2 = 1, N(3) - 3 = 2, N(6) - 6 = 2$, where $\{2, 3, 6\}$ is the set of z's (cf.

(5.1)), and $N(4) - 4 = 1, N(7) - 7 = 1, N(8) - 8 = 0$, where $\{4, 7, 8\}$ is the set of t's (cf. (5.2)).

Chapter 7

Symplectic jeu de taquin for spinor model

In this chapter, we define a symplectic jeu de taquin sliding for a spinor model. A key observation is that the symplectic jeu de taquin $\mathtt{jdt}_{KN}(\cdot)$ on KN tableaux can be described in terms of the $\mathfrak{gl}_{2\ell}$-crystal structure (4.4) on $\mathbf{E}^{2\ell}_{[\bar{n}]}$ under the bijection $(\cdot)^{\mathtt{ad}}$ in (7.1). This chapter is organized as follows.

- In Section 7.1, we define a spinor model of a skew shape and its n-conjugate.
- In Section 7.2, we introduce a symplectic jeu de taquin sliding for the spinor model of a skew shape.

7.1 Spinor model of a skew shape

Definition 7.1.1. Suppose $\mathbf{T} = (T_\ell, \ldots, T_1) \in \mathbf{T}_{\mathcal{A}}(a_\ell) \times \cdots \times \mathbf{T}_{\mathcal{A}}(a_1)$ is given for some $a_1, \ldots, a_\ell \in \mathbb{Z}_+$. For a skew diagram λ/μ with $\lambda, \mu \in \mathscr{P}_\ell$, we say that

(1) $\mathbf{T}$ is of shape λ/μ if for all $1 \le i \le \ell$,

$$a_i = \lambda_i - \mu_i, \quad \mathrm{ht}(T^{\mathtt{L}}_{i+1}) + \mu_{i+1} \le \mathrm{ht}(T^{\mathtt{L}}_i) + \mu_i.$$

(2) $\mathbf{T}$ of shape λ/μ is $\mathcal{A}$-admissible if for $1 \le i \le \ell - 1$,

$$\left\lfloor {}^{\mathtt{R}}T_{i+1}, T^{\mathtt{L}}_i \right\rfloor_{(\mu_{i+1}, \mu_i)} \text{ and } \left\lfloor T^{\mathtt{R}}_{i+1}, {}^{\mathtt{L}}T_i \right\rfloor_{(\lambda_{i+1}, \lambda_i)} \text{ are } \mathcal{A}\text{-semistandard.}$$

Denote by $\mathbf{T}_{\mathcal{A}}(\lambda/\mu,\ell)$ the set of $\mathcal{A}$-admissible tableaux $\mathbf{T}$ of shape λ/μ.

When μ is the empty partition, the condition for $\mathbf{T}_{\mathcal{A}}(\lambda/\mu,\ell)$ is exactly the same as that of $\mathbf{T}_{\mathcal{A}}(\lambda,\ell)$. In turn, we consider $\mathbf{T}_{\mathcal{A}}(\lambda/\mu,\ell)$ as a spinor model of skew shapes in this manner. For $\mathbf{T}\in\mathbf{T}_{\mathcal{A}}(\lambda/\mu,\ell)$, we put $\mathbf{T}$ in the plane $\mathbb{P}_L$ and denote

$$\lfloor \mathbf{T} \rfloor_{(\mu_\ell,\ldots,\mu_1)} = \lfloor T_\ell,\ldots,T_1 \rfloor_{(\mu_\ell,\ldots,\mu_1)} := \lfloor T_\ell^{\mathtt{L}}, T_\ell^{\mathtt{R}},\ldots,T_1^{\mathtt{L}},T_1^{\mathtt{R}} \rfloor_{(\mu_\ell,\lambda_\ell,\ldots,\mu_1,\lambda_1)}.$$

In general, $\mathbf{T}\in\mathbf{T}_{\mathcal{A}}(\lambda/\mu,\ell)$ does not correspond to a KN tableau of skew shape unless $\mathcal{A}=[\overline{n}]$. Furthermore, we may not use the algorithm $\mathtt{jdt}_{KN}(\cdot)$ since $\mathbf{T}^{\mathtt{ad}}$ is not defined in general. To overcome this problem, we introduce the notion of the n-conjugate of $\mathbf{T}$. Recall that a $\mathfrak{gl}_r$-crystal morphism $\kappa_{\mathcal{A}}$ (4.2) sends $\mathbf{T}\in\mathbf{E}^r_{\mathcal{A}}$ to $(P(\mathbf{T}),Q(\mathbf{T}))\in SST_{\mathcal{A}}(\nu)\times SST_r(\nu')$ for some $\nu\in\mathscr{P}_r$.

Definition 7.1.2. For $\mathbf{T}=(T_\ell,\ldots,T_1)\in\mathbf{T}_{\mathcal{A}}(a_\ell)\times\cdots\times\mathbf{T}_{\mathcal{A}}(a_1)$, suppose that $\kappa_{\mathcal{A}}(\mathbf{T})=\kappa_{\mathcal{A}}(T_\ell^{\mathtt{L}},T_\ell^{\mathtt{R}},\ldots,T_1^{\mathtt{L}},T_1^{\mathtt{R}})=(P(\mathbf{T}),Q(\mathbf{T}))$ with $\nu=\mathrm{sh}(P(\mathbf{T}))$.

For $n\geq\ell(\nu)$, we define the n-conjugate $\overline{\mathbf{T}}$ of $\mathbf{T}$ to be the unique tableau $\overline{\mathbf{T}}=(\overline{T_\ell},\ldots,\overline{T_1})\in\mathbf{T}_{[\overline{n}]}(a_\ell)\times\cdots\times\mathbf{T}_{[\overline{n}]}(a_1)$ such that

$$\kappa_{[\overline{n}]}(\overline{\mathbf{T}})=(H_\nu,\,Q(\mathbf{T})),$$

where $H_\nu\in SST_{[\overline{n}]}(\nu)$ is the highest weight vector in $SST_{[\overline{n}]}(\nu)$, that is, the i-th row is filled with $\overline{n-i+1}$ for $1\leq i\leq n$. Note that it is well-defined by the bijectivity of $\kappa_{[\overline{n}]}$.

Remark 7.1.3. If we choose another m for some $m\geq n$, then the m-conjugate of $\mathbf{T}$ is obtained from its n-conjugate by replacing $\overline{a}$ with $\overline{a+m-n}$ for all $1\leq a\leq n$.

Example 7.1.4. Suppose $\mathcal{A}=\mathbb{I}_{4|3}$ and take $\mathbf{T}=(T_3,T_2,T_1)\in\mathbf{T}_{\mathcal{A}}(1)\times\mathbf{T}_{\mathcal{A}}(1)\times\mathbf{T}_{\mathcal{A}}(2)$ below. Then we have the following data

$$\mathbf{T}=\left(\begin{array}{|c|c|}\hline 2&4\\\hline 3&2'\\\hline 1'&\\\hline\end{array}\,,\;\begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline 1'&2'\\\hline 3'&\\\hline\end{array}\,,\;\begin{array}{|c|c|}\hline 2&2'\\\hline 1'&2'\\\hline 1'&\\\hline 3'&\\\hline\end{array}\right),\quad \kappa_{\mathcal{A}}(T)=\left(\begin{array}{|c|c|c|c|c|}\hline 1&1&2&1'&2'\\\hline 2&2&3&2'&3'\\\hline 3&4&1'&2'&\\\hline 1'&2'&3'&&\\\hline 1'&&&&\\\hline\end{array}\,,\;\begin{array}{|c|c|c|c|c|}\hline 1&1&2&2&4\\\hline 2&2&3&5&\\\hline 3&3&4&6&\\\hline 4&4&5&&\\\hline 6&6&&&\\\hline\end{array}\right)$$

and the 5-conjugate $\overline{\mathbf{T}}$ is the below.

$$\overline{\mathbf{T}} = \left(\begin{array}{|c|c|} \hline \overline{5} & \overline{3} \\ \hline \overline{4} & \overline{2} \\ \hline \overline{2} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \, , \quad \begin{array}{|c|c|} \hline \overline{5} & \overline{5} \\ \hline \overline{4} & \overline{4} \\ \hline \overline{3} & \overline{3} \\ \hline \overline{1} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \, , \quad \begin{array}{|c|c|} \hline \overline{5} & \overline{5} \\ \hline \overline{4} & \overline{4} \\ \hline \overline{3} & \multicolumn{1}{c}{} \\ \cline{1-1} \overline{2} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right)$$

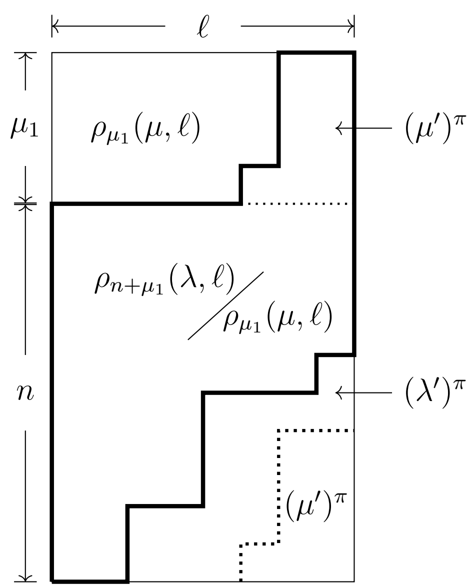
The following two lemmas explains why we need the notion of n-conjugates.

Lemma 7.1.5. *For* $\mathbf{T} = (T_\ell, \ldots, T_1) \in \mathbf{T}_{\mathcal{A}}(a_\ell) \times \cdots \times \mathbf{T}_{\mathcal{A}}(a_1)$*, we have* $\mathbf{T} \in \mathbf{T}_{\mathcal{A}}(\lambda/\mu, \ell)$ *if and only if* $\overline{\mathbf{T}} \in \mathbf{T}_{[\overline{n}]}(\lambda/\mu, \ell)$*.*

Proof. It follows from [25, Lemma 6.2] and $Q(\mathbf{T}) = Q(\overline{\mathbf{T}})$. $\square$

A key point is that whether a tableau $(T_\ell, \ldots, T_1)$ of shape λ/μ is $\mathcal{A}$-admissible depends only on its $\mathfrak{gl}_{2\ell}$-crystal structure, not on the choice of $\mathcal{A}$. On the other hand, we have a bijection $(\cdot)^{\mathsf{ad}}$ between $\mathbf{T}_{[\overline{n}]}(\lambda, \ell)$ and $\mathbf{KN}_n(\rho_n(\lambda, \ell))$. By applying $(\cdot)^{\mathsf{ad}}$ to each component, we obtain the following.

Lemma 7.1.6. *Let* $\mathbf{T}$ *be as above. We have* $\overline{\mathbf{T}} \in \mathbf{T}_{[\overline{n}]}(\lambda/\mu, \ell)$ *if and only if* $\left\lceil \overline{T}_\ell^{\mathsf{ad}}, \ldots, \overline{T}_1^{\mathsf{ad}} \right\rceil^{\rho_{\mu_1}(\mu,\ell)'} \in \mathbf{KN}_n(\rho_{n+\mu_1}(\lambda, \ell)/\rho_{\mu_1}(\mu, \ell))$*.*



Proof. Suppose that $\overline{\mathbf{T}}$ is $[\overline{n}]$-admissible of shape λ/μ. Let us introduce additional letters $u_1, \dots, u_m$ for a sufficiently large m and set

$$
\begin{aligned}
X &= \{\, u_m < \cdots < u_1 < 1 < \cdots < n \,\},\\
\overline{X} &= \{\, \overline{n} < \cdots < \overline{1} < \overline{u_1} < \cdots < \overline{u_m} \,\},\\
\mathcal{I}_{m,n} &= \{\, u_m < \cdots < u_1 < 1 < \cdots < n < \overline{n} < \cdots < \overline{1} < \overline{u}_1 < \cdots < \overline{u}_m \,\}
\end{aligned}
$$

with all entries of degree 0.

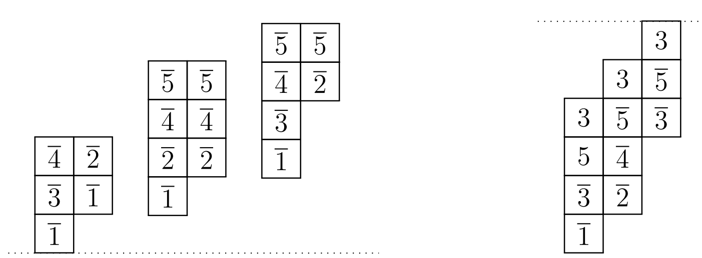
By attaching suitable boxes with entries in $\{u_1, \dots, u_m\}$ below $\overline{T}_i^{\mathtt{L}}$ for all i, we may obtain $\mathbf{S} = (S_\ell, \dots, S_1) \in \mathbf{T}_{\overline{X}}(\lambda, \ell)$. Then $\mathbf{S}^{\mathtt{ad}} = (S_\ell^{\mathtt{ad}}, \dots, S_1^{\mathtt{ad}})$ is a KN tableau of shape $\rho_{m+n}(\lambda, \ell)$ with respect to $\mathcal{I}_{m,n}$ by Proposition 6.3.5. In this case, $\mathbf{S}^{\mathtt{ad}}$ does not contain $\overline{u_j}$ for all $1 \le j \le m$. Moreover, $\left\lceil \overline{T}_\ell^{\mathtt{ad}}, \dots, \overline{T}_1^{\mathtt{ad}} \right\rceil^{\rho_{\mu_1}(\mu,\ell)}$ is obtained from $\mathbf{S}^{\mathtt{ad}}$ by ignoring the letters $u_1, \dots, u_m$. Since $\mathbf{S}^{\mathtt{ad}}$ is a KN tableau, $\left\lceil \overline{T}_\ell^{\mathtt{ad}}, \dots, \overline{T}_1^{\mathtt{ad}} \right\rceil^{\rho_{\mu_1}(\mu,\ell)}$ is $\mathcal{I}_n$-admissible of shape $\rho_{n+\mu_1}(\lambda, \ell)/\rho_{\mu_1}(\mu, \ell)$. Note that the admissibility is independent of the entries below $\overline{T}_i^{\mathtt{L}}$ because we ignore them when it comes to $\overline{T}_i^{\mathtt{ad}}$. The converse can be proved similarly. □

Using Lemma 6.3.3, we have an analogue of Proposition 6.3.5 for skew shapes.

Corollary 7.1.7. *We have a bijection*

$$
\begin{aligned}
\mathbf{T}_n(\lambda/\mu, \ell) &\longrightarrow \mathbf{KN}_n(\rho_{n+\mu_1}(\lambda, \ell)/\rho_{\mu_1}(\mu, \ell)) \;. \qquad (7.1)\\
\mathbf{T} = \left\lfloor T_\ell, \dots, T_1 \right\rfloor_{(\mu_\ell, \dots, \mu_1)} &\longmapsto \mathbf{T}^{\mathtt{ad}} := \left\lceil T_\ell^{\mathtt{ad}}, \dots, T_1^{\mathtt{ad}} \right\rceil^{\rho_{\mu_1}(\mu,\ell)'}
\end{aligned}
$$

Example 7.1.8. Let $\mathbf{T} = \left\lfloor T_3, T_2, T_1 \right\rfloor_{(0,1,2)} \in \mathbf{T}_5(\lambda/\mu, 3)$ be given, where $\lambda = (4,2,1)$ and $\mu = (2,1,0)$. Then $\mathbf{T}^{\mathtt{ad}} \in \mathbf{KN}_{(6,5,3)'/(2,1,0)'}$ is the follow.



$\mathbf{T} = \left\lfloor T_3, T_2, T_1 \right\rfloor_{(0,1,2)}$ $\qquad$ $\mathbf{T}^{\mathtt{ad}} = \left\lceil T_3^{\mathtt{ad}}, T_2^{\mathtt{ad}}, T_1^{\mathtt{ad}} \right\rceil^{(2,1,0)}$

7.2 Symplectic jeu de taquin for spinor model

In this section, we introduce an analogue of the jeu de taquin for $\mathbf{T} \in \mathbf{T}_{\mathcal{A}}(\lambda/\mu, \ell)$. We first consider the case when $\ell = 2$ which is a crucial step and then discuss the general case.

7.2.1 The case when $\ell = 2$

Suppose that $\mathbf{T} = (T_2, T_1) \in \mathbf{T}_{\mathcal{A}}(a_2) \times \mathbf{T}_{\mathcal{A}}(a_1)$ is given for some $a_1, a_2 \in \mathbb{Z}_+$. Let

$$d(T_1, T_2) = \min \left\{ d \,\middle|\, d \in \mathbb{Z}_+,\ \lfloor T_2, T_1 \rfloor_{(0,d)} \text{ is } \mathcal{A}\text{-admissible (of a skew shape)} \right\}.$$

Note that we have $T_2 \prec T_1$ if and only if $d(T_1, T_2) = 0$.

We also regard $\mathbf{T}$ as an element $(T_2^{\mathtt{L}}, T_2^{\mathtt{R}}, T_1^{\mathtt{L}}, T_1^{\mathtt{R}}) \in \mathbf{E}_{\mathcal{A}}^4$. Suppose that $d = d(T_1, T_2) > 0$. We define $\mathbf{T}'$ to be the tableau obtained by applying a sequence of crystal operators to $\mathbf{T}$ as follows: (recall that $\mathcal{E}_3^{a_2}\mathbf{T} = ({}^{\mathtt{L}}T_2, {}^{\mathtt{R}}T_2, T_1^{\mathtt{L}}, T_1^{\mathtt{R}})$ and $\mathcal{E}_1^{a_1}\mathbf{T} = (T_2^{\mathtt{L}}, T_2^{\mathtt{R}}, {}^{\mathtt{L}}T_1, {}^{\mathtt{R}}T_1)$ by Remark 6.2.1.)

Case 1. Suppose that $\lfloor {}^{\mathtt{R}}T_2, T_1^{\mathtt{L}} \rfloor_{(0,d-1)}$ is not $\mathcal{A}$-semistandard. Then we put

$$\mathbf{T}' = (U_4, U_3, U_2, U_1) = \begin{cases} \mathcal{F}_3^{a_2-1}\mathcal{E}_2\mathcal{E}_3^{a_2}\mathbf{T} & \text{if } \varepsilon_3(\mathcal{E}_2\mathcal{E}_3^{a_2}\mathbf{T}) = 0, \\ \mathcal{F}_3^{a_2}\mathcal{E}_2\mathcal{E}_3^{a_2}\mathbf{T} & \text{if } \varepsilon_3(\mathcal{E}_2\mathcal{E}_3^{a_2}\mathbf{T}) = 1. \end{cases} \tag{7.2}$$

Case 2. Suppose that $\lfloor {}^{\mathtt{R}}T_2, T_1^{\mathtt{L}} \rfloor_{(0,d-1)}$ is $\mathcal{A}$-semistandard, but $\lfloor T_2^{\mathtt{R}}, {}^{\mathtt{L}}T_1 \rfloor_{(0,d-1)}$ is not. Then we put

$$\mathbf{T}' = (U_4, U_3, U_2, U_1) = \mathcal{F}_1^{a_1+1}\mathcal{F}_2\mathcal{E}_1^{a_1}\mathbf{T}. \tag{7.3}$$

Lemma 7.2.1. *Under the above hypothesis, $\mathbf{T}'$ is well-defined.*

Proof. In *Case 1*, the condition that $\lfloor {}^{\mathtt{R}}T_2, T_1^{\mathtt{L}} \rfloor_{(0,d-1)}$ is not $\mathcal{A}$-semistandard implies $\varepsilon_2(\mathcal{E}_3^{a_2}\mathbf{T}) > 0$. Then we apply $\mathcal{E}_2$ to $\mathcal{E}_3^{a_2}\mathbf{T}$, which is non-zero, and let $\mathcal{E}_2\mathcal{E}_3^{a_2}\mathbf{T} = (V_4, V_3, V_2, V_1)$. By the jeu de taquin sliding, we have

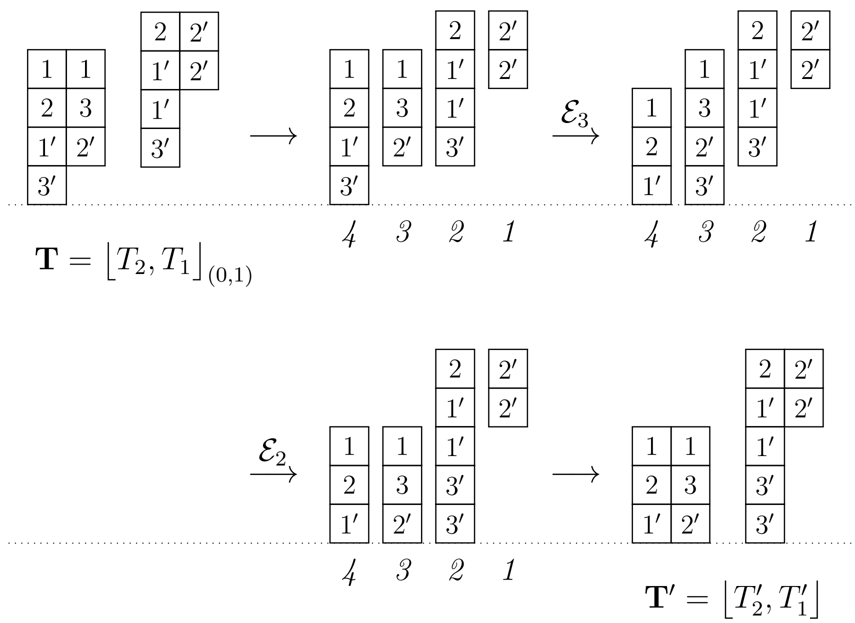
$$\mathcal{E}_2 \lfloor {}^{\mathtt{L}}T_2, {}^{\mathtt{R}}T_2, T_1^{\mathtt{L}}, T_1^{\mathtt{R}} \rfloor_{(0,0,d,d+a_1)} = \lfloor V_4, V_3, V_2, V_1 \rfloor_{(0,1,d-1,d+a_1)}$$

where $\lfloor V_4, V_3 \rfloor_{(0,1)}$ is $\mathcal{A}$-semistandard and hence $\varepsilon_3(\mathcal{E}_2\mathcal{E}_3^{a_2}\mathbf{T}) = 0$ or 1. Indeed, we have

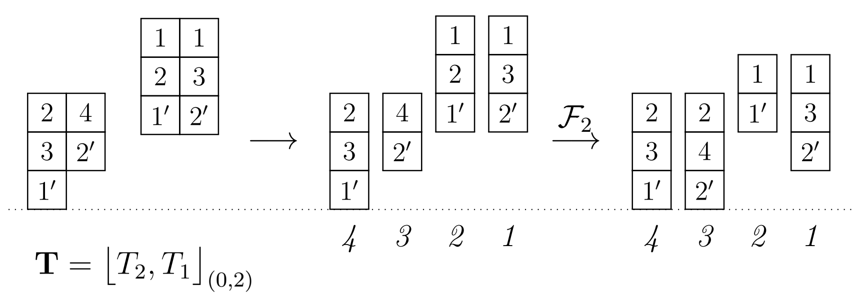
$\varepsilon_3(\mathcal{E}_2\mathcal{E}_3^{a_2}\mathbf{T}) = 0$ if $\lfloor V_4, V_3 \rfloor$ is $\mathcal{A}$-semistandard, and $\varepsilon_3(\mathcal{E}_2\mathcal{E}_3^{a^2}\mathbf{T}) = 1$ otherwise. Hence $\mathbf{T}'$ is well-defined. We can similarly check the well-definedness of $\mathbf{T}'$ in *Case 2*. □

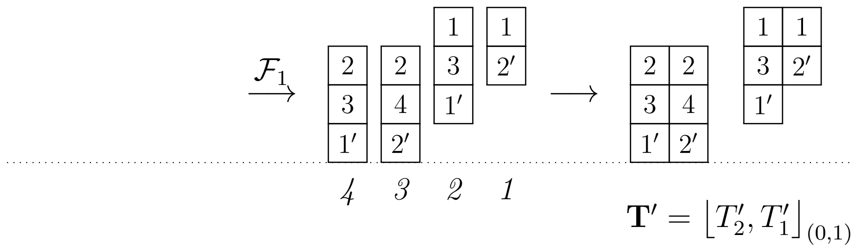
Example 7.2.2. Suppose that $\mathcal{A} = \mathbb{I}_{4|3}$.

(1) The following is an example of *Case 1* with $\varepsilon_3(\mathcal{E}_2\mathcal{E}_3^{a_2}\mathbf{T}) = 0$.

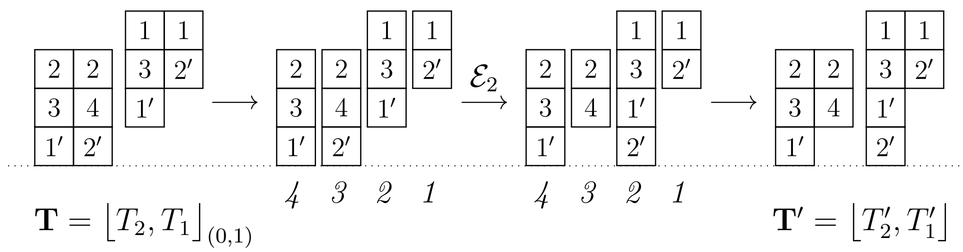


(2) The following is an example of *Case 2*.





(3) The following is an example of *Case 1* with $\varepsilon_3(\mathcal{E}_2\mathcal{E}_3^{a_2}\mathbf{T}) = 1$.



Proposition 7.2.3. *Under the above hypothesis, there exists a unique pair* $\mathbf{T}' = (T_2', T_1')$ *such that*

$$\mathbf{T}' = (T_2', T_1') \in \mathbf{T}_{\mathcal{A}}(a_2 + 2\varepsilon - 1) \times \mathbf{T}_{\mathcal{A}}(a_1 + 1),$$
$$d(T_1', T_2') \leq d(T_1, T_2) - 1,$$

where $\varepsilon = \varepsilon_3(\mathcal{E}_2\mathcal{E}_3^{a_2}\mathbf{T})$ *in Case 1, and* $\varepsilon = 0$ *otherwise.*

Proof. First, suppose that $\mathcal{A} = [\overline{n}]$ and let $d = d(T_1, T_2) > 0$. Since $\mathbf{T} = \lfloor T_2, T_1 \rfloor_{(0,d)}$ is $[\overline{n}]$-admissible, $\mathbf{T}^{\mathtt{ad}} = \lceil T_2^{\mathtt{ad}}, T_1^{\mathtt{ad}} \rceil^{(d,0)}$ is admissible by Lemma 7.1.6. For simplicity, write $T = \lceil C_2, C_1 \rceil^{(c_2,c_1)} = \lceil T_2^{\mathtt{ad}}, T_1^{\mathtt{ad}} \rceil^{(d,0)}$ and let c be the inner corner of T.

We claim that the algorithm (5.7) to have $T' = \mathtt{jdt}_{KN}(T, c)$ corresponds to either (7.2) or (7.3). First, we mark $\bullet$ at c to keep track of sliding and apply *Step 1.*(1) as far as possible to have (5.6). Since $d = d(T_1, T_2) > 0$, we should apply *Step 1.*(2) to (5.6) with $a' > b$.

Suppose that $b \in [\overline{n}]$. Then it is straightforward to see from Lemma 6.3.3 that applying *Step 1.*(2-b) and sliding $\bullet$ to the bottom of the column corresponds to (7.3). This implies that $(U_4, U_3) = ((T_2')^{\mathtt{L}}, (T_2')^{\mathtt{R}})$ for some $T_2' \in \mathbf{T}_n(a_2 - 1)$ and $(U_2, U_1) = ((T_1')^{\mathtt{L}}, (T_1')^{\mathtt{R}})$ for some $T_1' \in \mathbf{T}_n(a_1 + 1)$. Furthermore, since $T' = \lceil C_2', C_1' \rceil^{(c_2-1,c_1)}$, it follows from Lemma

7.1.6 that $\lfloor T_2', T_1' \rfloor_{(0,d-1)}$ is $\mathcal{A}$-admissible and

$$(\mathbf{T}')^{\mathtt{ad}} = T' = \lceil C_2', C_1' \rceil^{(c_2-1,c_1)}, \tag{7.4}$$

where $(\mathbf{T}')^{\mathtt{ad}}$ is given in Corollary 7.1.7. This implies that $d(T_1', T_2') \leq d-1$.

Next, suppose that $b \in [n]$. As similarly as above, exchanging $\bullet$ with b in *Step 1.*(2-a) to have C_1' corresponds to $\mathcal{E}_2\mathcal{E}_3^{a_2}\mathbf{T}$. If C_2' is admissible, which is equivalent to $\varepsilon_3(\mathcal{E}_2\mathcal{E}_3^{a_2}\mathbf{T}) = 0$, then the process to have C_2' corresponds to applying $\mathcal{F}_3^{a_2-1}$ to $\mathcal{E}_2\mathcal{E}_3^{a_2}\mathbf{T}$. As in the previous case, we have $(U_4, U_3) = ((T_2')^{\mathtt{L}}, (T_2')^{\mathtt{R}})$ and $(U_2, U_1) = ((T_1')^{\mathtt{L}}, (T_1')^{\mathtt{R}})$ for some $T_2' \in \mathbf{T}_n(a_2-1)$ and $T_1' \in \mathbf{T}_n(a_1+1)$, and $\lfloor T_2', T_1' \rfloor_{(0,d-1)}$ is $\mathcal{A}$-admissible with (7.4), which implies $d(T_1', T_2') \leq d(T_1, T_2) - 1$. On the other hand, if C_2' is not admissible, which is equivalent to $\varepsilon_3(\mathcal{E}_2\mathcal{E}_3^{a_2}\mathbf{T}) = 1$, then it is not difficult to see that the process to have C_2'' corresponds to applying $\mathcal{F}_3^{a_2}$ to $\mathcal{E}_2\mathcal{E}_3^{a_2}\mathbf{T}$. Hence, we have $(U_4, U_3) = ((T_2')^{\mathtt{L}}, (T_2')^{\mathtt{R}})$ for some $T_2' \in \mathbf{T}_n(a_2+1)$ and $(U_2, U_1) = ((T_1')^{\mathtt{L}}, (T_1')^{\mathtt{R}})$ for some $T_1' \in \mathbf{T}_n(a_1+1)$. In this case, we have $T' = \lceil C_2'', C_1' \rceil^{(c_2,c_1)}$, and hence $\lfloor T_2', T_1' \rfloor_{(0,d)}$ is $\mathcal{A}$-admissible. Moreover, it can be shown that $\lceil C_2'', C_1' \rceil^{(c_2-1,c_1)}$ is also admissible, hence $\lfloor T_2', T_1' \rfloor_{(0,d-1)}$ is $\mathcal{A}$-admissible with (7.4), which implies $d(T_1', T_2') \leq d(T_1, T_2) - 1$.

Finally, suppose that $\mathcal{A}$ is arbitrary. Take a sufficiently large n and let $\overline{\mathbf{T}}$ be the n-conjugate of $\mathbf{T}$. Let $\mathcal{X}$ be the composite of operators $\mathcal{E}_i$ and $\mathcal{F}_i$ in (7.2) or (7.3). By definition of $\overline{\mathbf{T}}$ and Lemma 4.3.5, we have

$$Q(\mathcal{X}\mathbf{T}) = \mathcal{X}Q(\mathbf{T}) = \mathcal{X}Q(\overline{\mathbf{T}}) = Q(\mathcal{X}\overline{\mathbf{T}}). \tag{7.5}$$

By applying the previous arguments to $\overline{\mathbf{T}}$, there exists $\overline{\mathbf{T}}' = (\overline{T}_2', \overline{T}_1')$ such that

$$\mathcal{X}\overline{\mathbf{T}} = \overline{\mathbf{T}}' \in \mathbf{T}_n(a_2 + 2\varepsilon - 1) \times \mathbf{T}_n(a_1+1).$$

By [25, Lemma 6.2] and (7.5), we have $\mathbf{T}' = (T_2', T_1')$ such that

$$\mathcal{X}\mathbf{T} = \mathbf{T}' \in \mathbf{T}_{\mathcal{A}}(a_2 + 2\varepsilon - 1) \times \mathbf{T}_{\mathcal{A}}(a_1+1)$$

and the n-conjugate of T_i' is $\overline{T}_i'$ for $i = 1, 2$. Furthermore, it follows from the argument for $\mathcal{A} = [\overline{n}]$ and Lemma 7.1.5 that $d(T_1', T_2') \leq d(T_1, T_2) - 1$. Note that our argument does not depend on the choice of n. □

Now we introduce the following notation.

Definition 7.2.4. For $\mathbf{T} = (T_2, T_1) \in \mathbf{T}_{\mathcal{A}}(a_2) \times \mathbf{T}_{\mathcal{A}}(a_1)$ with $d(T_1, T_2) > 0$, we define

$$\texttt{jdt}_{spin}(\mathbf{T}) = \mathbf{T}', \tag{7.6}$$

where $\mathbf{T}'$ is given in (7.2) and (7.3).

By the arguments in the proof of Proposition 7.2.3, we have following, which shows that $\texttt{jdt}_{spin}(\cdot)$ is a natural analogue of $\texttt{jdt}_{KN}(\cdot, c)$.

Corollary 7.2.5. *Under the above hypothesis, we have*

$$\overline{\texttt{jdt}_{spin}(\mathbf{T})} = \texttt{jdt}_{spin}(\overline{\mathbf{T}}),$$
$$\left(\texttt{jdt}_{spin}(\overline{\mathbf{T}})\right)^{\mathsf{ad}} = \texttt{jdt}_{KN}\left(\overline{\mathbf{T}}^{\mathsf{ad}}, c\right),$$

where $\overline{\,\cdot\,}$ *denotes the* n*-conjugate for a sufficiently large* n*,* $(\,\cdot\,)^{\mathsf{ad}}$ *is given in* (6.2) *or* (7.1)*, and* c *is the inner corner of* $\overline{\mathbf{T}}^{\mathsf{ad}}$*.*

7.2.2 The when $\ell \geq 2$

Take a skew diagram λ/μ with $\lambda, \mu \in \mathscr{P}_\ell$ and $\mathbf{T} = (T_\ell, \ldots, T_1) \in \mathbf{T}_{\mathcal{A}}(\lambda/\mu, \ell)$. For an inner corner c of λ/μ in the i-th row from the top, let us define symplectic the jeu de taquin sliding on $\mathbf{T}$ with respect to c.

For a sufficiently large n, take the n-conjugate $\overline{\mathbf{T}}$ of $\mathbf{T}$. Then we consider

$$\texttt{jdt}_{KN}\left(\overline{\mathbf{T}}^{\mathsf{ad}}, b\right),$$

which is defined in (5.7), where b is the inner corner of $\overline{\mathbf{T}}^{\mathsf{ad}}$ in the $(i+1)$-th column from the right and $(\,\cdot\,)^{\mathsf{ad}}$ is in (7.1). By (7.6) and Corollary 7.2.5, there exists a composite of operators $\mathcal{E}_i$ and $\mathcal{F}_i$, say $\mathcal{X}$, such that $\left(\mathcal{X}\,\overline{\mathbf{T}}\right)^{\mathsf{ad}} = \texttt{jdt}_{KN}\left(\overline{\mathbf{T}}^{\mathsf{ad}}, b\right)$.

Definition 7.2.6. Under the above hypothesis, we define

$$\texttt{jdt}_{spin}(\mathbf{T}, c) = \mathcal{X}\mathbf{T}. \tag{7.7}$$

Note that $\texttt{jdt}_{spin}(\mathbf{T},c)$ is independent of the choice of $\mathcal{X}$ since

$$\left(\overline{\texttt{jdt}_{spin}(\mathbf{T},c)}\right)^{\texttt{ad}} = \left(\overline{\mathcal{X}\,\mathbf{T}}\right)^{\texttt{ad}} = \left(\mathcal{X}\,\overline{\mathbf{T}}\right)^{\texttt{ad}} = \texttt{jdt}_{KN}\left(\overline{\mathbf{T}}^{\texttt{ad}}, b\right),$$

and $\overline{\cdot}$ and $(\,\cdot\,)^{\texttt{ad}}$ are injective on the connected component of $\mathbf{T}$. Indeed, suppose that we choose another sequence $\mathcal{X}'$ such that $(\mathcal{X}'\overline{\mathbf{T}})^{\texttt{ad}} = \texttt{jdt}_{KN}(\overline{\mathbf{T}}^{\texttt{ad}}, b)$. By the injectivity of $(\,\cdot\,)^{\texttt{ad}}$, $\mathcal{X}\overline{\mathbf{T}} = \mathcal{X}'\overline{\mathbf{T}}$ and then

$$Q(\mathcal{X}\mathbf{T}) = Q(\overline{\mathcal{X}\mathbf{T}}) = Q(\mathcal{X}\overline{\mathbf{T}}) = Q(\mathcal{X}'\overline{\mathbf{T}}) = Q(\overline{\mathcal{X}'\mathbf{T}}) = Q(\mathcal{X}'\mathbf{T}).$$

Since $P(\mathcal{X}\mathbf{T}) = P(\mathbf{T})$ holds, the bijection (4.2) gives $\mathcal{X}\mathbf{T} = \mathcal{X}'\mathbf{T}$.

Theorem 7.2.7. *Let λ/μ be a skew diagram with $\lambda,\mu \in \mathscr{P}_\ell$ and let $\mathbf{T} = (T_\ell,\ldots,T_1) \in \mathbf{T}_{\mathcal{A}}(\lambda/\mu,\ell)$ be given. There exists a unique $\texttt{P}(\mathbf{T}) \in \mathbf{T}_{\mathcal{A}}(\nu,\ell)$ for some $(\nu,\ell) \in \mathscr{P}(\mathrm{Sp})_{\mathcal{A}}$, which can be obtained from $\mathbf{T}$ by applying $\texttt{jdt}_{spin}(\,\cdot\,,c)$ finitely many times with respect to inner corners. In particular, if $\mathcal{A} = [\overline{n}]$, then we have*

$$\texttt{P}(\mathbf{T})^{\texttt{ad}} = P\left(w\left(\mathbf{T}^{\texttt{ad}}\right)\right),$$

where a tableau $P(w)$ for $w \in \mathcal{W}_n$ is defined in Theorem 5.2.1.

Proof. We first show the existence of $\texttt{P}(\mathbf{T})$. Take a sufficiently large n and let $\overline{\mathbf{T}}$ be the n-conjugate of $\mathbf{T}$. Let $\mathbf{U} = \overline{\mathbf{T}}$ and $\mathbf{V} = \overline{\mathbf{T}}^{\texttt{ad}}$. By applying (5.7), there exists a sequence $\mathbf{V} = \mathbf{V}_0,\ldots,\mathbf{V}_r$ of skew admissible tableaux such that

$$\mathbf{V}_{i+1} = \texttt{jdt}_{KN}(\mathbf{V}_i, b_i) \quad (1 \le i \le r-1), \tag{7.8}$$

for some inner corner b_i in $\mathrm{sh}(\mathbf{V}_i)$, and $\mathbf{V}_r \in \mathbf{KN}_\delta$ for some $\delta \in \mathscr{P}_n$ with $\delta_1 \le \ell$. By applying (7.1), we get a sequence $\mathbf{U} = \mathbf{U}_0,\ldots,\mathbf{U}_r$ such that $\mathbf{U}_i^{\texttt{ad}} = \mathbf{V}_i$. In this case, it follows from Corollary 7.2.5 that

$$\mathbf{U}_{i+1} = \texttt{jdt}_{spin}(\mathbf{U}_i, c_i) \quad (1 \le i \le r-1) \tag{7.9}$$

for some inner corners c_i in $\mathrm{sh}(\mathbf{U}_i)$. By (7.7) and Corollary 7.2.5, there exists a sequence $\mathbf{T} = \mathbf{T}_0,\ldots,\mathbf{T}_r$ such that

$$\mathbf{T}_{i+1} = \texttt{jdt}_{spin}(\mathbf{T}_i, c_i) \quad (1 \le i \le r-1)$$

and $\overline{\mathbf{T}}_i = \mathbf{U}_i$. Since $\mathbf{V}_r \in \mathbf{KN}_\delta$, we deduce from Lemmas 7.1.5 and 7.1.6 that $\mathbf{T}_r \in \mathbf{T}_\mathcal{A}(\nu, \ell)$ with $\rho_n(\nu, \ell) = \delta$. We put $\mathtt{P}(\mathbf{T}) = \mathbf{T}_r$.

Now consider the uniqueness of such $\mathtt{P}(\mathbf{T})$. For this, suppose that there exists another sequence $\mathbf{T} = \mathbf{T}'_0, \ldots, \mathbf{T}'_s$ such that $\mathbf{T}'_{i+1} = \mathtt{jdt}_{spin}(\mathbf{T}'_i, c'_i)$ for some c'_i $(1 \le i \le s-1)$ and $\mathbf{T}'_s \in \mathbf{T}_\mathcal{A}(\xi, \ell)$ for some $(\xi, \ell) \in \mathscr{P}(\mathrm{Sp})_\mathcal{A}$. Our claim is $\mathbf{T}_r = \mathbf{T}'_s$.

Put $\mathbf{U}'_i = \overline{\mathbf{T}'_i}$ and $\mathbf{V}'_i = (\mathbf{U}'_i)^{\mathtt{ad}}$ for $1 \le i \le s$ and then they also satisfy same relation (7.8) and (7.9) by Corollary 7.2.5. By the same reason as $\mathbf{V}_r$, $\mathbf{V}'_s \in \mathbf{KN}_{\rho_n(\xi,\ell)}$ and $\mathbf{V}'_s = \mathtt{P}(w(\mathbf{V}'_0)) = \mathtt{P}(w(\mathbf{V}_0)) = \mathbf{V}_r$ since $\mathbf{V}_0 = \mathbf{V}'_0 = \mathbf{V}$. It implies that $\xi = \nu$. On the other hand, let $\mathcal{X}$ and $\mathcal{X}'$ be composites of $\mathcal{E}_i$ and $\mathcal{F}_i$ $(1 \le i \le 2\ell - 1)$ such that

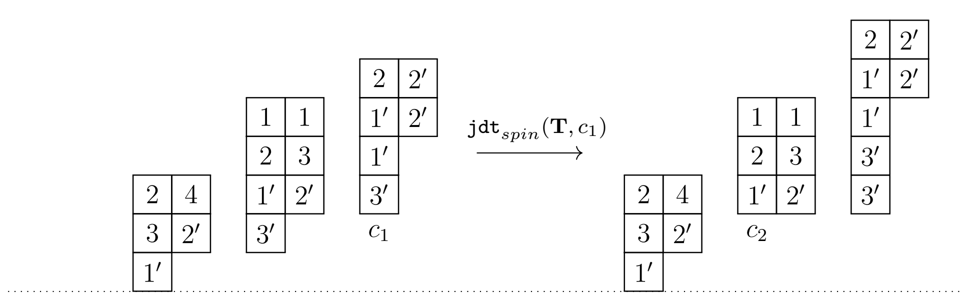
$$\mathcal{X}\mathbf{T} = \mathbf{T}_r, \quad \mathcal{X}'\mathbf{T} = \mathbf{T}'_s.$$

By (7.7), we have $\mathcal{X}\mathbf{U} = \mathbf{U}_r$ and $\mathcal{X}'\mathbf{U} = \mathbf{U}'_s$, respectively. Since $\mathbf{V}_r = \mathbf{V}'_s$, we have $\mathcal{X}\mathbf{U} = \mathbf{U}_r = \mathbf{U}'_s = \mathcal{X}'\mathbf{U}$, and

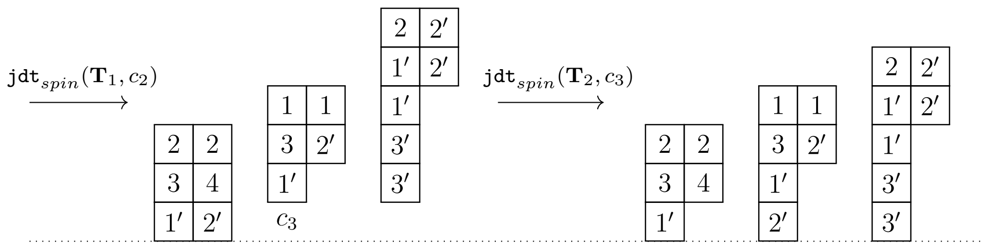
$$\begin{aligned} Q(\mathcal{X}\mathbf{T}) &= \mathcal{X}Q(\mathbf{T}) = \mathcal{X}Q(\mathbf{U}) = Q(\mathcal{X}\mathbf{U}) \\ &= Q(\mathcal{X}'\mathbf{U}) = \mathcal{X}'Q(\mathbf{U}) = \mathcal{X}Q(\mathbf{T}) = Q(\mathcal{X}'\mathbf{T}). \end{aligned}$$

Recall that $Q(\mathbf{T}) = Q(\mathbf{U})$ by definition of the n-conjugate. We already know that $P(\mathcal{X}\mathbf{T}) = P(\mathbf{T}) = P(\mathcal{X}'\mathbf{T})$ and we have $\mathbf{T}_r = \mathbf{T}'_s$ by Theorem 6.2.2. If we restrict to the case for $\mathcal{A} = [\overline{n}]$, then the last statement follows from Corollary 7.2.5. □

Example 7.2.8. Let $\mathbf{T} = \lfloor T_3, T_2, T_1 \rfloor_{(0,1,2)}$ be the tableau given in Example 7.1.4. Then $\mathtt{P}(\mathbf{T})$ can be obtained as follows (cf. Example 7.2.2).



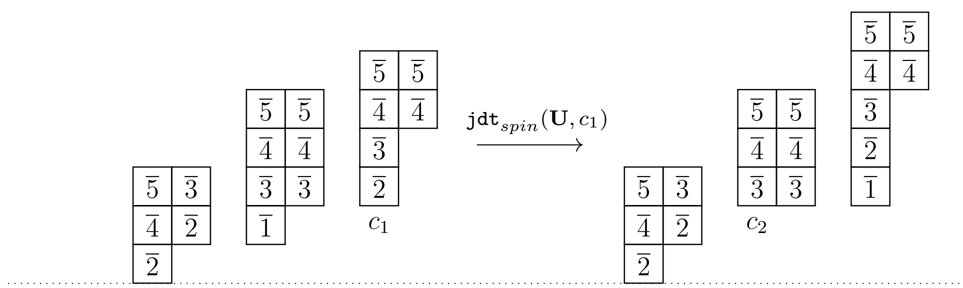
$\mathbf{T} = \lfloor T_3, T_2, T_1 \rfloor_{(0,1,2)}$ $\qquad$ $\mathbf{T}_1 = \lfloor T'_3, T'_2, T'_1 \rfloor_{(0,2,2)}$



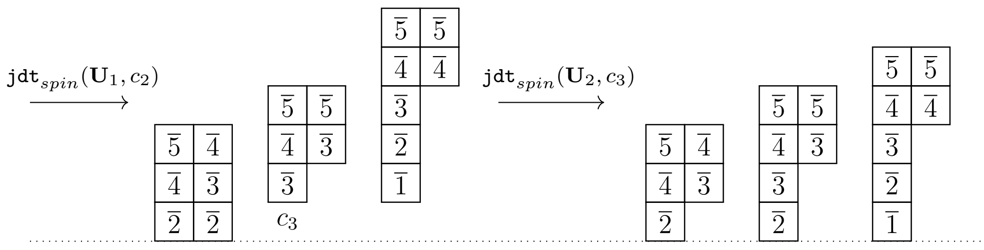
$\mathbf{T}_2 = \lfloor T_3'', T_2'', T_1'' \rfloor_{(0,1,1)}$ $\qquad$ $\mathtt{P}(\mathbf{T}) = \lfloor T_3''', T_2''', T_1''' \rfloor$

$$\mathtt{jdt}_{spin}(\mathbf{T}, c_1) = \mathcal{E}_2\mathcal{E}_3\mathbf{T}$$
$$\mathtt{jdt}_{spin}(\mathbf{T}_1, c_2) = \mathcal{F}_3\mathcal{F}_4\mathbf{T}_1$$
$$\mathtt{jdt}_{spin}(\mathbf{T}_2, c_3) = \mathcal{E}_4\mathbf{T}_2$$

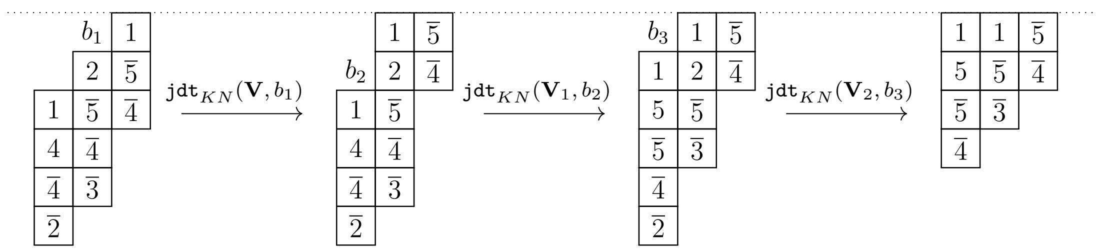
The corresponding jeu de taquin for the 5-conjugate $\mathbf{U}$ and $\mathbf{V} = \mathbf{U}^{\mathsf{ad}}$ is given as follows.



$\mathbf{U}$ $\qquad$ $\mathbf{U}_1$



$\mathbf{U}_2$ $\mathtt{P}(\mathbf{U})$



$\mathbf{V}$ $\mathbf{V}_1$ $\mathbf{V}_2$ $\mathtt{P}(\mathbf{V})$

Chapter 8

Oscillating tableaux and King tableaux

In this chapter, we introduce a set of oscillating tableaux $\mathbf{O}(\lambda, \ell)$ for $(\lambda, \ell) \in \mathscr{P}(\mathrm{Sp})_{\mathcal{A}}$. It will be used in the next chapter to define a recording tableau for $\mathtt{P}(\mathbf{T})$ obtained in the previous chapter. We also present a bijection in [33] from $\mathbf{O}(\lambda, \ell)$ to $\mathbf{K}(\lambda, \ell)$, the set of King tableaux of shape λ, which is a well-known combinatorial model for the symplectic character of $V_{\mathrm{Sp}_{2\ell}}(\lambda)$ [20]. This chapter is organized as follows.

- In Section 8.1, we define the set $\mathbf{O}(\lambda, \ell)$ of oscillating tableaux and discuss their properties.

- In Section 8.2, we recall the bijection between the set of oscillating tableaux and the set of King tableaux in [33].

8.1 Oscillating tableaux

Recall that oscillating tableau is a sequence $Q = (Q_1, \dots, Q_r)$ of partitions such that each adjacent partitions differs by exactly one box (cf. Section 5.2). We say that an oscillating tableau $Q = (Q_1, \dots, Q_r)$ is vertical if $Q_1 \subsetneq \cdots \subsetneq Q_s \supsetneq \cdots \supsetneq Q_r$ for some $1 \leq s \leq r$ and Q_r/Q_1 and Q_r/Q_s is a skew diagram of vertical strip.

Definition 8.1.1. Suppose $(\lambda, \ell) \in \mathscr{P}(\mathrm{Sp})$ is given. For $n \geq \lambda_1$, define $\mathbf{O}(\lambda, \ell; n)$ to be the set of a sequence of oscillating tableaux $Q = (Q^{(1)} : \cdots : Q^{(\ell)})$ such that

(1) Q is itself an oscillating tableau,

(2) $Q^{(i)} = (Q_{i,1}, \ldots, Q_{i,r_i})$ is a vertical oscillating tableau for $1 \leq i \leq \ell$,

(3) $\ell(Q_{i,j}) \leq n$ for $1 \leq i \leq \ell$ and $1 \leq j \leq r_i$, and

(4) $Q_{1,1} = \square$ and $Q_{\ell,r_\ell} = \rho_n(\lambda, \ell)$.

Let us consider a relation between $\mathbf{O}(\lambda, \ell; n)$ for different choices of n. More precisely, for $Q \in \mathbf{O}(\lambda, \ell; n)$, define $\sigma(Q) = \widehat{Q} = (\widehat{Q}^{(1)} : \cdots : \widehat{Q}^{(\ell)})$ to be the sequence such that $\widehat{Q}^{(i)}$ is a vertical oscillating tableau with $|\widehat{Q}^{(i)}| = |Q^{(i)}| + 1 = r_i + 1$ and

$$\widehat{Q}^{(i)} = ((i) \cup Q_{i-1,r_{i-1}}, (i) \cup Q_{i,1}, \ldots, (i) \cup Q_{i,r_i}) \quad (1 \leq i \leq \ell).$$

Since $Q^{(j)}$ is vertical for $1 \leq j \leq i-1$, we have $\ell(Q'_{i,k}) \leq i$ for $1 \leq k \leq r_i$, and denote by $(i) \cup Q_{i,k}$ the partition obtained by adding i to $Q_{i,k}$ as its first part, which is well-defined. It is straightforward to check that $\sigma(Q) \in \mathbf{O}(\lambda, \ell; n+1)$ and $\sigma : \mathbf{O}(\lambda, \ell; n) \longrightarrow \mathbf{O}(\lambda, \ell; n+1)$ is injective for $n \geq \lambda_1$. For example, if

$$(Q^{(1)} : Q^{(2)}) = \left(\ydiagram{1}\,,\ \ydiagram{1,1}\,,\ \ydiagram{1,1,1} : \ydiagram{2,1,1}\,,\ \ydiagram{2,2,1}\,,\ \ydiagram{2,2,1,1}\,,\ \ydiagram{2,2,1,1,1} \right),$$

then

$$(\widehat{Q}^{(1)}, \widehat{Q}^{(2)}) = \left(\ydiagram{1}\,,\ \ydiagram{1,1}\,,\ \ydiagram{1,1,1}\,,\ \ydiagram{1,1,1,1} : \ydiagram{2,1,1,1}\,,\ \ydiagram{2,2,1,1}\,,\ \ydiagram{2,2,2,1}\,,\ \ydiagram{2,2,2,1,1}\,,\ \ydiagram{2,2,2,1,1,1} \right).$$

By using σ, we can consider a stable limit of oscillating tableaux. Especially, we can define an equivalence relation using σ, which is defined on $\bigsqcup_{n \geq \lambda_1} \mathbf{O}(\lambda, \ell; n) \times \{n\}$ with the relation

$$(Q', m) \sim (Q, n) \text{ if and only if } \sigma^{m-n}(Q) = Q' \tag{8.1}$$

for $Q' \in \mathbf{O}(\lambda, \ell; m)$ and $Q \in \mathbf{O}(\lambda, \ell; n)$ with $m \geq n$. Then we define

$$\mathbf{O}(\lambda, \ell) = \{\, [Q, n] \,|\, Q \in \mathbf{O}(\lambda, \ell; n) \ (n \geq \lambda_1) \,\},$$

where $[Q, n]$ is the equivalence class of $Q \in \mathbf{O}(\lambda, \ell; n)$ with respect to (8.1). In this case, we call $[Q, n] \in \mathbf{O}(\lambda, \ell)$ an oscillating tableau of shape (λ, ℓ), which can be views as a stable limit of $Q \in \mathbf{O}(\lambda, \ell; n)$ as $n \to \infty$.

Remark 8.1.2. We identify $Q^{(i)} = (Q^{(1)} : \cdots : Q^{(\ell)}) \in \mathbf{O}(\lambda, \ell; n)$ with a tableau $U^{(i)} \in SST_{\mathcal{I}_n}((1^{r_i}))$ ($r_i = |Q^{(i)}|$) for $1 \leq i \leq \ell$ such that a (resp. $\overline{a}$) appears in $U^{(i)}$ if and only if a box is added (resp. removed) in the a-th row in $Q^{(i)}$. Under this identification, we view $Q^{(i)}$ as an element of the regular crystal $\mathbf{F}_n$ (6.3). Note that $\mathcal{E}Q^{(i)}$ (resp. $\mathcal{F}Q^{(i)}$) is obtained by removing (resp. adding) two components in $Q^{(i)}$ corresponding to the letters described in Remark 6.4.3 if it is not $\mathbf{0}$.

Next, we define the weight of an oscillating tableau in $\mathbf{O}(\lambda, \ell)$. Let $\mathbf{a} = (a_1, \ldots, a_\ell) \in \mathbb{Z}_+^\ell$. Take a sufficiently large n so that $n - a_i > 0$ for $1 \leq i \leq \ell$. Define $\mathbf{O}(\lambda, \ell; n)_{\mathbf{a}}$ to be the subset of $Q = (Q^{(1)} : \cdots : Q^{(\ell)}) \in \mathbf{O}(\lambda, \ell; n)$ such that

$$|Q^{(i)}| = (n - a_i) + 2\varepsilon(Q^{(i)}) \quad (1 \leq i \leq \ell), \tag{8.2}$$

where $Q^{(i)}$ is considered as an element of $\mathbf{F}_n$ by Remark 8.1.2.

Lemma 8.1.3. *Under the above hypothesis,*

(1) $\varphi(Q^{(i)}) + \varepsilon(Q^{(i)}) = a_i$ *for* $1 \leq i \leq \ell$,

(2) $\varepsilon(Q^{(i)}) = \varepsilon(\widehat{Q}^{(i)})$ *for* $1 \leq i \leq \ell$, *where* $\sigma(Q) = (\widehat{Q}^{(1)} : \cdots : \widehat{Q}^{(\ell)})$, *and*

(3) $\sigma\left(\mathbf{O}(\lambda, \ell; n)_{\mathbf{a}}\right) \subseteq \mathbf{O}(\lambda, \ell; n+1)_{\mathbf{a}}$.

Proof. (1) If $\mathcal{E}Q^{(i)} = \mathbf{0}$, then the tableau $U^{(i)} \in SST_{\mathcal{I}_n}((1^{n-a_i}))$ corresponding to $Q^{(i)}$ is admissible by Lemma 6.4.1 (2) and it is the highest weight vector of the regular $\mathfrak{sl}_2$-crystal of highest weight a_i by (8.2). In general, each $Q^{(i)} \in \mathbf{F}_n$ belongs to a regular highest weight $\mathfrak{sl}_2$-crystal. By Remark 6.4.3 and (8.2), the highest weight is a_i. By the description for the regular crystal, we have $\varphi(Q^{(i)}) + \varepsilon(Q^{(i)}) = a_i$ for $1 \leq i \leq \ell$.

(2) Let $\widehat{U}^{(i)}$ correspond to $\widehat{Q}^{(i)}$ by Remark 8.1.2. Then it is obtained from $U^{(i)}$ by replacing k (resp. $\overline{k}$) with $k+1$ (resp. $\overline{k+1}$) for $k \geq 1$, and adding the box $\boxed{1}$ at the top. Since $(\widehat{U}_+^{(i)})^c$ and $\widehat{U}_-^{(i)}$ are obtained by replacing $\overline{k}$ with $\overline{k+1}$ for all $k \geq 1$ without any change of shapes, it's done.

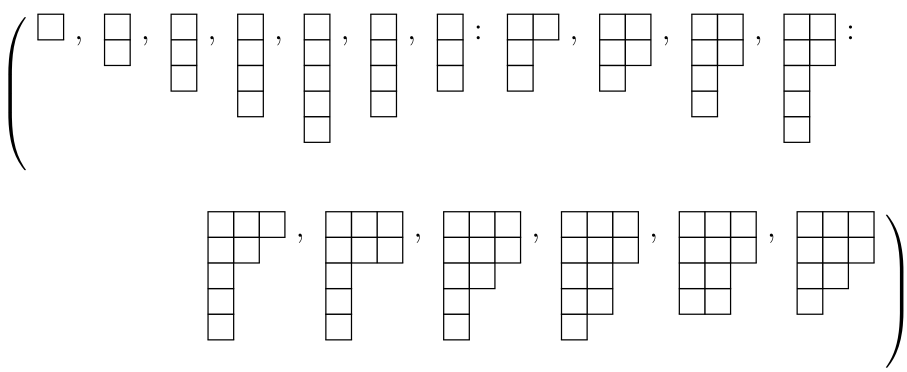
(3) By the construction in (2), we have $|\widehat{Q}^{(i)}| = |Q^{(i)}| + 1 = (n + 1 - a_i) + 2\varepsilon(Q^{(i)}) = (n + 1 - a_i) + 2\varepsilon(\widehat{Q}^{(i)})$ and so it's done. □

From Lemma 8.1.3, we have the following weight decomposition

$$\mathbf{O}(\lambda,\ell) = \bigsqcup_{\mathbf{a}\in\mathbb{Z}_+^{\ell}} \mathbf{O}(\lambda,\ell)_{\mathbf{a}}, \tag{8.3}$$

where $\mathbf{O}(\lambda,\ell)_{\mathbf{a}}$ is the set of equivalence classes $[Q,n]$ of $Q \in \mathbf{O}(\lambda,\ell;n)_{\mathbf{a}}$. We call $[Q,n] \in \mathbf{O}(\lambda,\ell)_{\mathbf{a}}$ an oscillating tableau of shape (λ,ℓ) with weight $\mathbf{a}$.

Example 8.1.4. Consider an oscillating tableau $Q = (Q^{(1)} : Q^{(2)} : Q^{(3)}) \in \mathbf{O}(\lambda,\ell;n)$ with $\lambda = (3,2,1), \ell = 3$, and $n = 5$.

$Q =$


If we consider each $Q^{(i)}$ as an element in $\mathbf{F}_n$, then we see that $\varepsilon(Q^{(1)}) = 2, \varepsilon(Q^{(2)}) = 0$, and $\varepsilon(Q^{(3)}) = 1$ and hence the weight of $[Q,n]$ is $\mathbf{a} = (2,1,1)$.

For $(\lambda,\ell) \in \mathscr{P}(\mathrm{Sp})$ and $n \geq \lambda_1$, let

$$\mathbf{O}_\circ(\lambda,\ell;n) = \left\{ Q \,\middle|\, Q = (Q^{(1)} : \cdots : Q^{(\ell)}) \in \mathbf{O}(\lambda,\ell;n),\ \varepsilon(Q^{(i)}) = 0 \ (1 \leq i \leq \ell) \right\}.$$

Set $\mathbf{O}_\circ(\lambda,\ell;n)_{\mathbf{a}} = \mathbf{O}_\circ(\lambda,\ell;n) \cap \mathbf{O}(\lambda,\ell;n)_{\mathbf{a}}$. By Lemma 8.1.3, we have $\sigma\left(\mathbf{O}_\circ(\lambda,\ell;n)_{\mathbf{a}}\right) \subset \mathbf{O}_\circ(\lambda,\ell;n{+}1)_{\mathbf{a}}$. Hence the decomposition (8.3) induces the following weight decomposition

$$\mathbf{O}_\circ(\lambda,\ell) = \bigsqcup_{\mathbf{a}\in\mathbb{Z}_+^{\ell}} \mathbf{O}_\circ(\lambda,\ell)_{\mathbf{a}},$$

where $\mathbf{O}_\circ(\lambda,\ell)_{\mathbf{a}}$ is the set of equivalence classes $[Q,n]$ of $Q \in \mathbf{O}_\circ(\lambda,\ell;n)_{\mathbf{a}}$. We call $[Q,n] \in \mathbf{O}_\circ(\lambda,\ell)$ an *admissible oscillating tableau of shape* (λ,ℓ).

Proposition 8.1.5 (cf. Lemma 6.4.1)**.** *For* $(\lambda,\ell) \in \mathscr{P}(\mathrm{Sp})$ *and* $\mathbf{a} = (a_1,\ldots,a_\ell) \in \mathbb{Z}_+^{\ell}$, *we have a bijection*

$$\begin{aligned} \mathbf{O}(\lambda,\ell)_{\mathbf{a}} &\longrightarrow \mathbf{O}_\circ(\lambda,\ell)_{\mathbf{a}} \times \mathbb{Z}/(\mathbf{a}+\mathbf{1})\mathbb{Z}\,, \\ [Q,n] &\longmapsto ([Q_\circ,n],\varepsilon(Q)) \end{aligned} \tag{8.4}$$

where $\mathbb{Z}/(\mathbf{a}+\mathbf{1})\mathbb{Z} = \mathbb{Z}/(a_1+1)\mathbb{Z}\times\cdots\times\mathbb{Z}/(a_\ell+1)\mathbb{Z}$*, and*

$$Q_\circ = (\mathcal{E}^{\max}Q^{(1)} : \cdots : \mathcal{E}^{\max}Q^{(\ell)}), \quad \varepsilon(Q) = \big(\varepsilon(Q^{(1)}),\ldots,\varepsilon(Q^{(\ell)})\big)$$

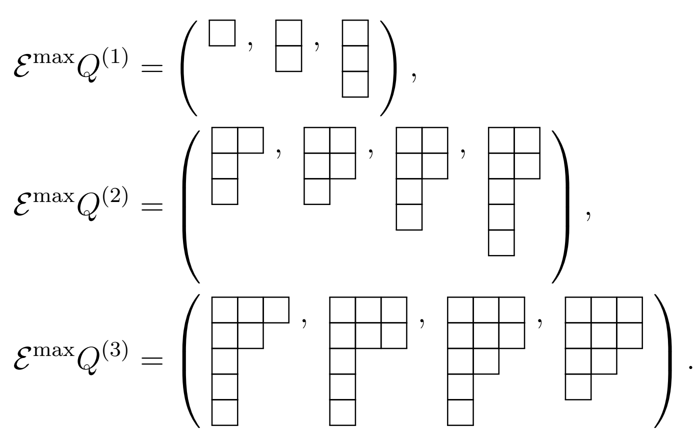
for $Q = (Q^{(1)} : \cdots : Q^{(\ell)}) \in \mathbf{O}(\lambda,\ell;n)_{\mathbf{a}}$.

Proof. Choose a sufficiently large n and then the map sending $Q \in \mathbf{O}(\lambda,\ell;n)_{\mathbf{a}}$ to $(Q_\circ,\varepsilon(Q))$ is a bijection by Lemma 6.4.1 (3). Moreover, Lemma 8.1.3 tells us that the above bijection sends $\sigma(Q)$ to $(\sigma(Q_\circ),\varepsilon(Q))$ with $\sigma(Q)_\circ = \sigma(Q_\circ)$. Thus, the bijection induces a well-defined bijection (8.4). □

Example 8.1.6. Let $[Q,5] \in \mathbf{O}(\lambda,\ell)$ be given in Example 8.1.4. Then the image of $[Q,5]$ under (8.4) is

$$[(\mathcal{E}^{\max}Q^{(1)} : \mathcal{E}^{\max}Q^{(2)} : \mathcal{E}^{\max}Q^{(3)}),(2,0,1)],$$

where



In addition, the weight of $Q_\circ$ is $(5-3,5-4,5-4) = (2,1,1)$, which coincides with that of Q.

8.2 King tableaux

For $\ell \geq 2$, let

$$\mathcal{J}_\ell = \{\, 1 < \overline{1} < 2 < \overline{2} < \cdots < \ell < \overline{\ell} \,\},$$

where we assume that all the entries are of degree 0. For $(\lambda, \ell) \in \mathscr{P}(\mathrm{Sp})$, let $\mathbf{K}(\lambda, \ell)$ be the set of $T \in SST_{\mathcal{J}_\ell}(\lambda)$ such that all entries in the ith row are larger than or equal to i for all $i \geq 1$. It is known as the set of *King tableaux of shape* λ [20]. For our notational coherence, let $\mathbf{K}(\lambda, \ell; n)$ for $n \geq \lambda_1$ denote the set $\mathbf{K}(\lambda, \ell)$, where the columns of the tableaux are enumerated by $n, n-1, \ldots, 1$ from the left.

Let $K \in \mathbf{K}(\lambda, \ell; n)$ be given. We define a sequence of vertical oscillating tableaux $Q(K; n) = (Q^{(1)} : \cdots : Q^{(\ell)})$ as follows: for $1 \leq i \leq \ell$ and $1 \leq j \leq n$,

(1) the letter i is contained in the jth column of K if and only if there is no step in $Q^{(i)}$ such that a box is added in the jth row,

(2) the letter $\overline{i}$ is contained in the jth column of K if and only if there is a step in $Q^{(i)}$ such that a box is deleted in the jth row.

Theorem 8.2.1. [33, Theorem 2.7] *For $\lambda \subseteq (n^\ell)$, we have a bijection*

$$\begin{array}{ccc} \mathbf{K}(\lambda, \ell; n) & \longrightarrow & \mathbf{O}(\lambda, \ell; n)\ . \\ K & \longmapsto & Q(K; n) \end{array}$$

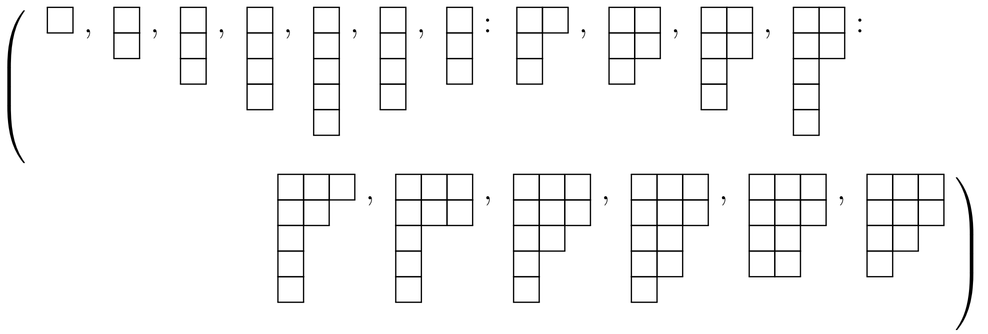
Remark 8.2.2. In [33], the bijection is described using a horizontal analogue of oscillating tableaux. By taking conjugate partitions to each element in oscillating tableaux, we obtain the bijection in Theorem 8.2.1.

Example 8.2.3. Let $\lambda = (3, 2, 1) \subseteq (5^3)$ with $n = 5$ and $\ell = 3$, and

$$K = \begin{array}{|c|c|c|} \hline \overline{1} & \overline{1} & 2 \\ \hline 3 & \overline{3} & \\ \cline{1-2} \overline{3} & & \\ \cline{1-1} \end{array} \quad \in \ \mathbf{K}(\lambda, \ell; n).$$

Then

$Q(K;n) =$



Corollary 8.2.4. *For $\lambda \subseteq (n^\ell)$, we have a bijection*

$$\begin{aligned} \mathbf{K}(\lambda,\ell) &\longrightarrow \mathbf{O}(\lambda,\ell) \\ K &\longmapsto [Q(K;n),n] \end{aligned} \quad . \tag{8.5}$$

Proof. By the construction of $Q(K;n)$, it is straightforward to check that $\sigma(Q(K;n)) = Q(K;n+1)$ for $K \in \mathbf{K}(\lambda,\ell;n)$. By Theorem 8.2.1, it's done. $\square$

Chapter 9

Combinatorial Howe duality of symplectic type

Finally, we are ready to state our main result, which gives a combinatorial realization of the duality (1.3). This chapter is organized as follows.

- In Section 9.1, we describe the Pieri rule for the spinor model.
- In Section 9.2, we give the RSK correspondence for the spinor model.
- In Section 9.3, we consider the character identity associated to the RSK correspondence and its application.

9.1 Pieri rule for the spinor model

For $\mathbf{a} = (a_1, \ldots, a_\ell) \in \mathbb{Z}_+^\ell$, let $\mathbf{T}_\mathcal{A}(\mathbf{a}) = \mathbf{T}_\mathcal{A}(a_\ell) \times \cdots \times \mathbf{T}_\mathcal{A}(a_1)$. Take $\mathbf{T} = (T_\ell, \ldots, T_1) \in \mathbf{T}_\mathcal{A}(\mathbf{a})$ and then regard it as an element in $\mathbf{T}_\mathcal{A}(\zeta/\eta, \ell)$ for some skew diagram ζ/η with $\zeta, \eta \in \mathscr{P}_\ell$ by properly shifting T_i up. By Theorem 7.2.7, there exists a unique $\mathtt{P}(\mathbf{T}) \in \mathbf{T}_\mathcal{A}(\lambda, \ell)$ for some $(\lambda, \ell) \in \mathscr{P}(\mathrm{Sp})_\mathcal{A}$, which is obtained by applying $\mathtt{jdt}_{spin}(\,\cdot\,, c)$ finitely many times with respect to inner corners c.

Now define a recording tableau $\mathtt{Q}(\mathbf{T})$ for $\mathtt{P}(\mathbf{T})$ in terms of oscillating tableaux as follows. Choose a sufficiently large n and take the n-conjugate $\overline{\mathbf{T}} = (\overline{T}_\ell, \ldots, \overline{T}_1)$ of $\mathbf{T}$. By Corollary 7.1.7, $\overline{\mathbf{T}}^{\mathtt{ad}} \in \mathbf{KN}_{\alpha/\beta}$ for some skew diagram α/β. Let

$$Q(\mathbf{T}; n) = Q(\overline{\mathbf{T}}^{\mathtt{ad}}),$$

where the right-hand side means the oscillating tableau $Q(w(\overline{T}^{\mathsf{ad}}))$ in Theorem 5.2.4.

Lemma 9.1.1. *Under the above hypothesis, we have*

(1) $Q(\mathbf{T};n) \in \mathbf{O}_\circ(\lambda,\ell;n)_{\mathbf{a}}$,

(2) $\sigma(Q(\mathbf{T};n)) = Q(\mathbf{T};n+1)$.

Proof. (1) Note that the $\mathcal{I}_n$-column tableau $\overline{T}_i^{\mathsf{ad}}$ has height $n-a_i$. Let $w^{(i)} = w_{i,1}\cdots w_{i,n-a_i} = w(\overline{T}_i^{\mathsf{ad}})$. For $1 \le i \le \ell$ and $1 \le k \le n-a_i$, put

$$Q_{i,k} = \mathrm{sh}P\left(w^{(1)}\cdots w^{(i-1)}w_{i,1}\cdots w_{i,k}\right), \tag{9.1}$$

where we assume that $w^{(0)}$ is the empty word. By Lemma 6.4.1 and Theorem 5.2.4, we have

- $Q^{(i)} = (Q_{i,1},\ldots,Q_{i,n-a_i})$ is a vertical oscillating tableau with $\varepsilon(Q^{(i)}) = 0$,
- $Q(\mathbf{T};n) = Q(w^{(1)}\cdots w^{(\ell)}) = (Q^{(1)}:\cdots:Q^{(\ell)})$,

which implies that $Q(\mathbf{T};n) \in \mathbf{O}_\circ(\lambda,\ell;n)_{\mathbf{a}}$.

(2) By definition of σ and (9.1), $\sigma(Q(\mathbf{T};n)) = Q(\widehat{w}^{(1)}\cdots\widehat{w}^{(\ell)})$, where $\widehat{w}^{(i)}$ is obtained from $w^{(i)}$ by replacing k (resp. $\overline{k}$) with $k+1$ (resp. $\overline{k+1}$ for $k \ge 1$ and adding 1 ahead of the first letter of $w^{(i)}$. By Remark 7.1.3 and Proposition 6.3.5, we can easily check that $\sigma(Q(\mathbf{T};n)) = Q(\mathbf{T};n+1)$. □

Define

$$Q_\circ(\mathbf{T}) = [Q(\mathbf{T};n),n] \in \mathbf{O}_\circ(\lambda,\ell)_{\mathbf{a}}, \tag{9.2}$$

which is well-defined by Lemma 9.1.1.

Theorem 9.1.2. *For* $\mathbf{a} = (a_1,\ldots,a_\ell) \in \mathbb{Z}_+^\ell$, *we have a bijection*

$$\begin{array}{ccc} \mathbf{T}_{\mathcal{A}}(\mathbf{a}) & \longrightarrow & \displaystyle\bigsqcup_{(\lambda,\ell)\in\mathscr{P}(\mathrm{Sp})_{\mathcal{A}}} \mathbf{T}_{\mathcal{A}}(\lambda,\ell)\times\mathbf{O}_\circ(\lambda,\ell)_{\mathbf{a}}\,. \\ \mathbf{T} & \longmapsto & (\mathtt{P}(\mathbf{T}),Q_\circ(\mathbf{T})) \end{array} \tag{9.3}$$

Proof. Let $\mathbf{T}$ be given. Choose a sufficiently large n and let $\overline{\mathbf{T}}$ be the n-conjugate of $\mathbf{T}$. Put $\mathbf{U} = \overline{\mathbf{T}}$ and $\mathbf{V} = \mathbf{U}^{\mathsf{ad}}$. By Corollary 7.2.5, we have the following commuting diagram:

$$\begin{array}{ccc}
\mathbf{T} & \xmapsto{\ \mathcal{Y}'\ } & \mathtt{P}(\mathbf{T}) \\
\big\downarrow{\scriptstyle \overline{\cdot}} & & \big\downarrow{\scriptstyle \overline{\cdot}} \\
\mathbf{U} & \xmapsto{\ \mathcal{Y}'\ } & \mathtt{P}(\mathbf{U}) \\
\big\downarrow{\scriptstyle \mathsf{ad}} & & \big\downarrow{\scriptstyle \mathsf{ad}} \\
\mathbf{V} & \xmapsto{\ \mathcal{Y}\ } & P(\mathbf{V})
\end{array} \tag{9.4}$$

where $\mathcal{Y}$ is a sequence of $\mathtt{jdt}_{KN}$'s (5.7), and $\mathcal{Y}'$ is the corresponding sequence of $\mathtt{jdt}_{spin}$'s (7.7). Recall $Q_\circ(\mathbf{T}) = [Q(\mathbf{V}), n]$.

To show that the map is injective, suppose $(\mathtt{P}(\mathbf{T}), Q_\circ(\mathbf{T})) = (\mathtt{P}(\mathbf{T}'), Q_\circ(\mathbf{T}'))$ for $\mathbf{T}, \mathbf{T} \in \mathbf{T}_{\mathcal{A}}(\mathbf{a})$. As similarly, $\mathbf{U}' = \overline{\mathbf{T}'}$ and $\mathbf{V}' = \mathbf{U}'^{\mathsf{ad}}$. Since $\mathtt{P}(\mathbf{T}) = \mathtt{P}(\mathbf{T}')$, we have $P(\mathbf{V}) = P(\mathbf{V}')$. In addition, since $Q_\circ(\mathbf{T}) = [Q(\mathbf{V}), n]$, we have $Q(\mathbf{V}) = Q(\mathbf{V}')$. We claim that

$$\begin{array}{ccc}
\mathbf{KN}_{(n-a_\ell)} \times \cdots \times \mathbf{KN}_{(n-a_1)} & \longrightarrow & \displaystyle\bigsqcup_{(\lambda,\ell)\in\mathscr{P}(\mathrm{Sp})_n} \mathbf{KN}_{\rho_n(\lambda,\ell)} \times \mathbf{O}_\circ(\lambda,\ell;n)_{\mathbf{a}} \\
\mathbf{V} & \longmapsto & (P(\mathbf{V}), Q(\mathbf{V}))
\end{array} \tag{9.5}$$

is a bijection. Theorem 5.2.4 states that $Q(\mathbf{V})$ is constant on its connected component and shows that the map is a morphism of $\mathfrak{sp}_{2n}$-crystals. Using the combinatorial rule of tensor product decomposition [36], we see that $\mathtt{Q}(\mathbf{V})$ uniquely determines a highest weight element in the connected component and hence the map is injective. On the other hand, for a given pair $(H_{\rho_n(\lambda,\ell)}, Q)$ on the right-hand side of (9.5), one can construct directly $\mathbf{T}$ such that $\widetilde{e}_i\mathbf{T} = \mathbf{0}$ for $1 \leq i \leq n$ and $\mathtt{Q}(\mathbf{T}) = Q$ again by [36]. This implies the surjectivity of (9.5). Hence, we have $\mathbf{V} = \mathbf{V}'$ and $\mathbf{T} = \mathbf{T}'$ by (9.4).

The surjectivity of the map follows from (9.4) and the bijection (9.5). □

Example 9.1.3. Let $\mathbf{T} = (T_3, T_2, T_1)$ be given in Example 7.1.4. By Example 7.2.8, we get

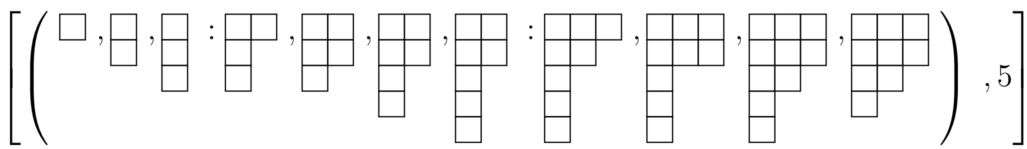
$$\mathtt{P}(\mathbf{T}) = \left(\begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 & 4 \\ \hline 1' & \\ \cline{1-1} \end{array}\ ,\ \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 & 2' \\ \hline 1' & \\ \cline{1-1} 2' & \\ \cline{1-1} \end{array}\ ,\ \begin{array}{|c|c|} \hline 2 & 2' \\ \hline 1' & 2' \\ \hline 1' & \\ \cline{1-1} 3' & \\ \cline{1-1} 3' & \\ \cline{1-1} \end{array} \right).$$

The bijection (7.1) gives

$$\overline{\mathbf{T}}^{\mathsf{ad}} = \begin{array}{ccc} & & 1 \\ & 2 & \overline{5} \\ 1 & \overline{5} & \overline{4} \\ 4 & \overline{4} & \\ \overline{4} & \overline{3} & \\ \overline{2} & & \end{array}$$

and we have (cf. Example 5.2.5)

$$Q_\circ(\mathbf{T}) =$$



.

9.2 RSK correspondence

We give an analogue of the RSK correspondence for the spinor model. Let $\mathbf{F}^\ell_\mathcal{A} = \mathbf{E}^{2\ell}_\mathcal{A}$ for $\ell \geq 1$.

Lemma 9.2.1. *We have a bijection*

$$\begin{aligned} \mathbf{F}^1_\mathcal{A} &\longrightarrow \bigsqcup_a \mathbf{T}_\mathcal{A}(a) \times \mathbb{Z}/(a+1)\mathbb{Z}\,, \\ \mathbf{T} &\longmapsto (\mathcal{F}^{\max}\mathbf{T}, \varphi(\mathbf{T})) \end{aligned}$$

where $\mathcal{F}^{\max}(\mathbf{T}) = \mathcal{F}^{\varphi(\mathbf{T})}\mathbf{T}$ *and the union is over* $a \in \mathbb{Z}_+$ *such that* $\mathbf{T}_\mathcal{A}(a) \neq \emptyset$.

Proof. Let $\mathbf{T} \in \mathbf{F}^1_\mathcal{A}$ be given. Then $\mathcal{F}^{\max}\mathbf{T} \in SST_\mathcal{A}(\lambda(a,0,c))$ for some $a, c \in \mathbb{Z}_+$ by Lemma 4.3.3. It means that the connected component of $\mathbf{T}$ is a regular $\mathfrak{sl}_2$-crystal with highest weight a. Then it is straightforward to check that the map is bijective by the fact that $\mathcal{E}$ and $\mathcal{F}$ are inverse to each other whenever they yield a nonzero element. □

Remark 9.2.2. Lemma 9.2.1 is essentially the same as Lemma 6.4.1 when $\mathcal{A} = [\overline{n}]$. The first components of the right-hand side of two bijection correspond to each other by Proposition 6.3.5, and the second ones correspond to each other by the complement to a, i.e., $\varepsilon(\mathbf{T}) + \varphi(\mathbf{T}) = a$ holds. Roughly speaking, the second component of the right-hand side in both bijections means how far from being admissible.

We generalize Lemma 9.2.1 in the case of ℓ-tuples of $\mathbf{F}^\ell_\mathcal{A}$.

Corollary 9.2.3. *We have a bijection*

$$\begin{aligned} \mathbf{F}^{\ell}_{\mathcal{A}} &\longrightarrow \bigsqcup_{\mathbf{a}} \mathbf{T}_{\mathcal{A}}(\mathbf{a}) \times \mathbb{Z}/(\mathbf{a}+\mathbf{1})\mathbb{Z}\,, \\ \mathbf{T} &\longmapsto (\mathcal{F}^{\max}\mathbf{T}, \varphi(\mathbf{T})) \end{aligned} \tag{9.6}$$

where

$$\begin{aligned} \mathcal{F}^{\max}\mathbf{T} &= (\mathcal{F}^{\max}(U_{2\ell}, U_{2\ell-1}), \ldots, \mathcal{F}^{\max}(U_2, U_1)), \\ \varphi(\mathbf{T}) &= (\varphi(U_{2\ell}, U_{2\ell-1}), \ldots, \varphi(U_2, U_1)), \end{aligned}$$

for $\mathbf{T} = (U_{2\ell}, \ldots, U_1) \in \mathbf{T}_{\mathcal{A}}(\mathbf{a})$ *and the union is over* $\mathbf{a} \in \mathbb{Z}^{\ell}_{+}$ *such that* $\mathbf{T}_{\mathcal{A}}(\mathbf{a}) \neq \emptyset$.

Example 9.2.4. Suppose that $\mathcal{A} = \mathbb{I}_{4|3}$. If

$$\mathbf{T} = \left(\begin{array}{|c|}\hline 3 \\ \hline 1' \\ \hline \end{array}\,, \begin{array}{|c|}\hline 2 \\ \hline 4 \\ \hline 2' \\ \hline \end{array}\,, \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 1' \\ \hline 3' \\ \hline \end{array}\,, \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 2' \\ \hline \end{array}\,, \begin{array}{|c|}\hline 1' \\ \hline 1' \\ \hline \end{array}\,, \begin{array}{|c|}\hline 2 \\ \hline 2' \\ \hline 2' \\ \hline 3' \\ \hline \end{array} \right) \in \mathbf{F}^{3}_{\mathcal{A}},$$

then we have

$$\mathcal{F}^{\max}\mathbf{T} = \left(\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 & 2' \\ \hline 1' & \\ \cline{1-1} \end{array}\,, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline 1' & 2' \\ \hline 3' & \\ \cline{1-1} \end{array}\,, \begin{array}{|c|c|}\hline 2 & 2' \\ \hline 1' & 2' \\ \hline 1' & \\ \cline{1-1} 3' & \\ \cline{1-1} \end{array} \right), \quad \varphi(\mathbf{T}) = (1, 0, 2).$$

Now, consider the composition of the following sequence of bijections.

$$\begin{aligned} \mathbf{F}^{\ell}_{\mathcal{A}} &\xrightarrow{(9.6)} \bigsqcup_{\mathbf{a} \in \mathbb{Z}^{\ell}_{+}} \mathbf{T}_{\mathcal{A}}(\mathbf{a}) \times \mathbb{Z}/(\mathbf{a}+\mathbf{1})\mathbb{Z} \\ &\xrightarrow{(9.3)} \bigsqcup_{\mathbf{a} \in \mathbb{Z}^{\ell}_{+}} \bigsqcup_{(\lambda,\ell) \in \mathscr{P}(\mathrm{Sp})_{\mathcal{A}}} \mathbf{T}_{\mathcal{A}}(\lambda, \ell) \times \mathbf{O}_{\circ}(\lambda, \ell)_{\mathbf{a}} \times \mathbb{Z}/(\mathbf{a}+\mathbf{1})\mathbb{Z} \\ &\xrightarrow{(8.4)} \bigsqcup_{(\lambda,\ell) \in \mathscr{P}(\mathrm{Sp})_{\mathcal{A}}} \mathbf{T}_{\mathcal{A}}(\lambda, \ell) \times \mathbf{O}(\lambda, \ell) \\ &\xrightarrow{(8.5)} \bigsqcup_{(\lambda,\ell) \in \mathscr{P}(\mathrm{Sp})_{\mathcal{A}}} \mathbf{T}_{\mathcal{A}}(\lambda, \ell) \times \mathbf{K}(\lambda, \ell) \end{aligned}$$

Let $(\mathtt{P}(\mathbf{T}), Q(\mathbf{T}))$ denote the image of $\mathbf{T} \in \mathbf{F}^{\ell}_{\mathcal{A}}$ under the composition of (9.6), (9.3), and (8.4). Let $(\mathtt{P}(\mathbf{T}), \mathtt{Q}(\mathbf{T}))$ denote the image of $(\mathtt{P}(\mathbf{T}), Q(\mathbf{T}))$ under (8.5). Hence we obtain the following correspondence, which is the main result in this thesis.

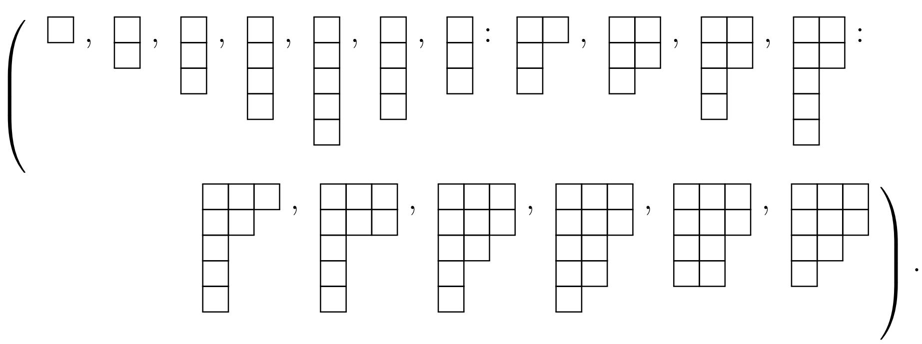
Theorem 9.2.5. *For $\ell \geq 1$, we have a bijection*

$$\begin{array}{ccc} \mathbf{F}^{\ell}_{\mathcal{A}} & \longrightarrow & \displaystyle\bigsqcup_{(\lambda,\ell)\in\mathscr{P}(\mathrm{Sp})_{\mathcal{A}}} \mathbf{T}_{\mathcal{A}}(\lambda,\ell) \times \mathbf{K}(\lambda,\ell)\, . \\ \mathbf{T} & \longrightarrow & (\mathtt{P}(\mathbf{T}), \mathtt{Q}(\mathbf{T})) \end{array}$$

Example 9.2.6. Let $\mathbf{T} \in \mathbf{F}^{3}_{\mathcal{A}}$ be the one given in Example 9.2.4. Combining Examples 9.2.4, 9.1.3, 7.2.2, 8.1.6, and 8.1.4, we have $(\mathtt{P}(\mathbf{T}), Q(\mathbf{T})) \in \mathbf{T}_{\mathcal{A}}(\lambda, 3) \times \mathbf{O}(\lambda, 3)$ for $\lambda = (3, 2, 1)$, where

$$\mathtt{P}(\mathbf{T}) = \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & 4 \\ \hline 1' \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & 2' \\ \hline 1' \\ \cline{1-1} 2' \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 2 & 2' \\ \hline 1' & 2' \\ \hline 1' \\ \cline{1-1} 3' \\ \cline{1-1} 3' \\ \cline{1-1} \end{array}$$

$Q(\mathbf{T}) =$



The oscillating tableau $Q(\mathbf{T})$ corresponds to a King tableau K in Example 8.2.3 under (8.5). Hence $\mathtt{Q}(\mathbf{T}) \in \mathbf{K}(\lambda, 3)$, where

$$\mathtt{Q}(\mathbf{T}) = \begin{array}{|c|c|c|}\hline \overline{1} & \overline{1} & 2 \\ \hline 3 & \overline{3} \\ \cline{1-2} \overline{3} \\ \cline{1-1} \end{array}\, .$$

Remark 9.2.7. When $\mathcal{A} = [\overline{n}]$, the right-hand side of the bijection in Theorem 9.2.5 has an $(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2\ell})$-bicrystal structure. On the other hand, $\mathbf{F}^{\ell}_{\mathcal{A}}$ is an $\mathfrak{sp}_{2n}$-crystal by (9.6), and the bijection is an isomorphism of $\mathfrak{sp}_{2n}$-crystals. However, we do not know yet how to define an $\mathfrak{sp}_{2\ell}$-crystal structure on $\mathbf{F}^{\ell}_{\mathcal{A}}$ directly so that the bijection is an isomorphism of $(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2\ell})$-bicrystals.

9.3 Symplectic Cauchy identity

Let $\mathbf{z} = \mathbf{z}_\ell = \{ z_1, \ldots, z_\ell \}$ be formal commuting variables, which commute with $\mathbf{x} = \mathbf{x}_{\mathcal{A}} = \{ x_a \,|\, a \in \mathcal{A} \}$ (cf. Section 6.1). For $\mathcal{A} = [n]$, write $\mathbf{x}_n = \mathbf{x}_{[n]} = \{ x_1, \ldots, x_n \}$. For $K \in \mathbf{K}(\lambda, \ell)$ with $(\lambda, \ell) \in \mathscr{P}(\mathrm{Sp})$, let $\mathbf{z}^K = \prod_{i \in [\ell]} z_i^{m_i - m_{\bar{i}}}$, where m_i (resp. $m_{\bar{i}}$) is the number of occurrences of i (resp. $\bar{i}$) in K. Then put

$$sp_\lambda(\mathbf{z}) = \sum_{K \in \mathbf{K}(\lambda,\ell)} \mathbf{z}^K.$$

It is well-known that $sp_\lambda(\mathbf{z})$ is the character of the irreducible highest weight module of $\mathrm{Sp}_{2\ell}(\mathbb{C})$ with the highest weight corresponding to λ.

Let $\mathbf{U} = (U_{2\ell}, \ldots, U_1) \in \mathbf{F}^{\ell}_{\mathcal{A}}$ be given with $u_i = \mathrm{ht}(U_i)$. Let $\mathbf{x}^{\mathbf{U}} = \prod_{i=1}^{2\ell} \mathbf{x}^{U_i}$ and $\mathbf{z}^{\mathbf{U}} = \prod_{i \in [\ell]} z_i^{u_{2i} - u_{2i-1}}$. Then we have

$$\mathrm{ch}\mathbf{F}^{\ell}_{\mathcal{A}} := \sum_{\mathbf{U}} \mathbf{x}^{\mathbf{U}} \mathbf{z}^{\mathbf{U}} = \prod_{j=1}^{\ell} \frac{\prod_{a \in \mathcal{A}_0} (1 + x_a z_j)(1 + x_a z_j^{-1})}{\prod_{a \in \mathcal{A}_1} (1 - x_a z_j)(1 - x_a z_j^{-1})}.$$

Theorem 9.3.1. *We have the following identity.*

$$t^\ell \prod_{j=1}^{\ell} \frac{\prod_{a \in \mathcal{A}_0} (1 + x_a z_j)(1 + x_a z_j^{-1})}{\prod_{a \in \mathcal{A}_1} (1 - x_a z_j)(1 - x_a z_j^{-1})} = \sum_{(\lambda,\ell) \in \mathscr{P}(\mathrm{Sp})_{\mathcal{A}}} S_{(\lambda,\ell)}(\mathbf{x}_{\mathcal{A}}) sp_\lambda(\mathbf{z})$$

Proof. Suppose $\mathbf{U} = (U_{2\ell}, \ldots, U_1) \in \mathbf{F}^{\ell}_{\mathcal{A}}$ is mapped to $(\mathbf{T}, K)$ by Theorem 9.2.5 and then it suffices to show that $\mathbf{x}^{\mathbf{U}} \mathbf{z}^{\mathbf{U}} = \mathbf{x}^{\mathbf{T}} \mathbf{z}^K$. Since $\mathbf{x}^{\mathbf{U}} = \mathbf{x}^{\mathbf{T}}$ is clear, it remains to show that $\mathbf{z}^{\mathbf{U}} = \mathbf{z}^K$.

Suppose that $\mathbf{U}$ is mapped to $(\mathbf{T}, \varphi)$ by (9.6) where $\mathbf{T} = (T_\ell, \ldots, T_1) \in \mathbf{T}_{\mathcal{A}}(\mathbf{a})$ for some $\mathbf{a} \in \mathbb{Z}_+^\ell$ and $\varphi = (\varphi_\ell, \ldots, \varphi_1) \in \mathbb{Z}/(\mathbf{a} + \mathbf{1})\mathbb{Z}$.

For a sufficiently large n, take the n-conjugate $\overline{\mathbf{T}} = (\overline{T}_\ell, \ldots, \overline{T}_1)$ of $\mathbf{T}$ and $\overline{\mathbf{T}}^{\mathsf{ad}} = (\overline{T}_\ell^{\mathsf{ad}}, \ldots, \overline{T}_1^{\mathsf{ad}})$. Suppose $\mathbf{T}$ is mapped to $(\mathtt{P}(\mathbf{T}), Q_0(\mathbf{T}))$ by (9.3), where $\mathtt{P}(\mathbf{T}) \in \mathbf{T}_{\mathcal{A}}(\lambda, \ell)$

for some $(\lambda,\ell)\in\mathscr{P}(\mathrm{Sp})$ and $Q_0(\mathbf{T})=[Q(\mathbf{T};n),n]\in\mathbf{O}_\circ(\lambda,\ell)_{\mathbf{a}}$ with

$$Q(\mathbf{T};n)=(Q^{(1)}:\cdots:Q^{(\ell)})$$

as given in (9.1). Note that $|Q^{(i)}|=n-a_i$ for $1\le i\le\ell$. If $(Q(\mathbf{T};n),\boldsymbol{\varphi})$ is mapped to $[Q',n]$ by (8.4), then $Q'=(Q'^{(1)}:\cdots:Q'^{(\ell)})\in\mathbf{O}(\lambda,\ell;n)_{\mathbf{a}}$. In particular, we have $|Q'^{(i)}|=|Q^{(i)}|+2\varphi_i=n-a_i+2\varphi_i$ for $1\le i\le\ell$.

Let $u_j=\mathrm{ht}(U_j)$ for $1\le j\le 2\ell$, and $t_i^{\pm}=\mathrm{ht}(\overline{T}_i^{\mathtt{ad}})_{\pm}$ for $1\le i\le\ell$. Suppose that (9.6) sends (U_{2i},U_{2i-1}) to (T_i,φ_i) for some $T_i\in\mathbf{T}_{\mathcal{A}}(a_i)$. By (6.2), we have $n-t_i^+=\mathrm{ht}(T_i^{\mathtt{L}})$ and $t_i^-=\mathrm{ht}(T_i^{\mathtt{R}})$. Since $T_i=\mathcal{F}^{\varphi_i}(U_{2i},U_{2i-1})$, we have $\mathrm{ht}(T_i^{\mathtt{L}})=u_{2i}+\varphi_i$ and $\mathrm{ht}(T_i^{\mathtt{R}})=u_{2i-1}-\varphi_i$. Then we have

$$a_i=\mathrm{ht}(T_i^{\mathtt{L}})-\mathrm{ht}(T_i^{\mathtt{R}})=u_{2i}-u_{2i-1}+2\varphi_i.$$

On the other hand, it is straightforward to see from the bijection in Theorem 8.2.1 that $|Q'^{(i)}|=(n-m_i)+m_{\overline{i}}$. Since $|Q'^{(i)}|=n-a_i+2\varphi_i$, we have

$$a_i-2\varphi_i=m_i-m_{\overline{i}}$$

for $1\le i\le\ell$. Hence $m_i-m_{\overline{i}}=u_{2i}-u_{2i-1}$ and this proves $\mathbf{z}^{\mathbf{U}}=\mathbf{z}^{K}$. □

Let us end this section with well-known identities that can be recovered from Theorem 9.3.1 under special choices of $\mathcal{A}$. First, assume that $\mathcal{A}=[\overline{n}]$. Let $P=\bigoplus_{i=1}^n\mathbb{Z}\epsilon_i$ be the weight lattice for $\mathfrak{sp}_{2n}$ in Section 5.1, and let $\mathbb{Z}[P]$ be its group ring with the formal linear basis $\{\,e^{\mu}\,|\,\mu\in P\,\}$. Note that $\varpi_n=\epsilon_1+\cdots+\epsilon_n$, the n-th fundamental weight. For $0\le a\le n$ and $T\in\mathbf{T}_n(a)$, define

$$\mathrm{wt}(T)=\varpi_n-\sum_{i=1}^n m_i\epsilon_i,$$

where m_i is the number of occurrences of $\overline{i}$ in T. For $\mathbf{a}=(a_1,\ldots,a_\ell)\in\mathbb{Z}_+^\ell$ and $\mathbf{T}=(T_\ell,\ldots,T_1)\in\mathbf{T}_n(\mathbf{a})$, define $\mathrm{wt}(\mathbf{T})=\sum_{i=1}^\ell\mathrm{wt}(T_i)$.

For a tableau K with letters in $\mathcal{I}_n$, let

$$\mathrm{wt}(K)=\sum_{i=1}^n(m_i-m_{\overline{i}})\epsilon_i,$$

where m_a is the number of occurrences of $a\in\mathcal{I}_n$ in K. It is easy to check that $\mathrm{wt}(\mathbf{T})=$

$\mathrm{wt}(\mathbf{T}^{\mathrm{ad}})$ for any $\mathbf{T} \in \mathbf{T}_n(\mathbf{a})$, and hence (6.2) and (7.1) are weight-preserving bijections.

By identifying $x_{\bar{i}} = x_i^{-1} = e^{-\epsilon_i} \in \mathbb{Z}[P]$ for $i \in [n]$ and $t = e^{\varpi_n} = x_1 \cdots x_n$ in (6.1),

$$S_{(\lambda,\ell)}(\mathbf{x}_{\mathcal{A}}) = \sum_{\mathbf{T}\in\mathbf{T}_n(\lambda,\ell)} t^\ell \mathbf{x}_{\mathcal{A}}^{\mathbf{T}} = \sum_{\mathbf{T}\in\mathbf{T}_n(\lambda,\ell)} e^{\mathrm{wt}(\mathbf{T})} = \sum_{\mathbf{T}^{\mathrm{ad}}\in\mathbf{KN}_{\rho_n(\lambda,\ell)}} e^{\mathrm{wt}(\mathbf{T}^{\mathrm{ad}})} = sp_{\rho_n(\lambda,\ell)}(\mathbf{x}_n).$$

The following identity follows immediately from Theorem 9.3.1 and the identity $x_i + x_i^{-1} + z_j + z_j^{-1} = x_i(1 + x_i^{-1}z_j)(1 + x_i^{-1}z_j^{-1})$.

Corollary 9.3.2 ([19])**.** *For $n, \ell \geq 1$, we have*

$$\prod_{i=1}^{n}\prod_{j=1}^{\ell}(x_i + x_i^{-1} + z_j + z_j^{-1}) = \sum_{\lambda\subseteq(n^\ell)} sp_{\rho_n(\lambda,\ell)}(\mathbf{x}_n) sp_\lambda(\mathbf{z}).$$

Next, assume that $\mathcal{A} = [n]'$. For $\ell \geq n$, there exists a bijection in [27, Theorem 6.5]

$$\mathbf{T}_{\mathcal{A}}(\lambda,\ell) \longrightarrow \bigsqcup_{\beta:\mathrm{even}} SST_{\mathcal{A}}(\lambda') \times SST_{\mathcal{A}}(\beta^\pi),$$

which gives the identity

$$S_{(\lambda,\ell)}(\mathbf{x}_{\mathcal{A}}) = t^\ell s_{\lambda'}(\mathbf{x}_{\mathcal{A}}) \sum_{\beta:\mathrm{even}} s_\beta(\mathbf{x}_{\mathcal{A}}) = t^\ell s_\lambda(\mathbf{x}_n) \sum_{\beta:\mathrm{even}} s_{\beta'}(\mathbf{x}_n).$$

Here we call a partition β even if all of its parts are even. Also, note that we have

$$s_\mu(\mathbf{x}_{\mathcal{A}}) = \sum_{\mathbf{T}\in SST_{\mathcal{A}}(\mu)} \mathbf{x}_{\mathcal{A}}^T = \sum_{\mathbf{T}\in SST_{[n]}(\mu')} \mathbf{x}_{[n]}^T = s_{\mu'}(\mathbf{x}_n)$$

for $\mu \in \mathscr{P}$ by identifying $x_{i'} = x_i$ for $i \in [n]$. By Theorem 9.3.1, we also recover the well-known classical identity due to Littlewood [35] and Weyl [45].

Corollary 9.3.3 ([35, 45])**.** *For $\ell \geq n \geq 1$, we have*

$$\begin{aligned}\prod_{i=1}^{n}\prod_{j=1}^{\ell}(1 - x_i z_j)^{-1}(1 - x_i z_j^{-1})^{-1} &= \sum_{\ell(\lambda)\leq n} sp_\lambda(\mathbf{z}) s_\lambda(\mathbf{x}_n) \prod_{1\leq i<j\leq n}(1 - x_i x_j)^{-1} \\ &= \sum_{\ell(\lambda)\leq n} sp_\lambda(\mathbf{z}) s_\lambda(\mathbf{x}_n) \sum_{\beta':\mathrm{even}} s_\beta(\mathbf{x}_n).\end{aligned}$$

Remark 9.3.4. The bijection in Theorem 9.2.5 even when reduced to the above cases is completely different from the ones in [43] and [42] for the identities in Corollaries 9.3.2 and 9.3.3, respectively, where the insertion algorithm in terms of the King tableaux is used.

Bibliography

[1] T. H. Baker, *An insertion scheme for C_n crystals*, Physical combinatorics (Kyoto, 1999), 2000, pp. 1–48. MR1768079

[2] A. Berele and A. Regev, *Hook Young diagrams with applications to combinatorics and to representations of Lie superalgebras*, Adv. in Math. **64** (1987), no. 2, 118–175. MR884183

[3] Shun-Jen Cheng and Weiqiang Wang, *Howe duality for Lie superalgebras*, Compositio Math. **128** (2001), no. 1, 55–94. MR1847665

[4] ———, *Lie subalgebras of differential operators on the super circle*, Publ. Res. Inst. Math. Sci. **39** (2003), no. 3, 545–600. MR2001187

[5] Shun-Jen Cheng and R. B. Zhang, *Howe duality and combinatorial character formula for orthosymplectic Lie superalgebras*, Adv. Math. **182** (2004), no. 1, 124–172. MR2028498

[6] I. B. Frenkel, *Representations of Kac-Moody algebras and dual resonance models*, Applications of group theory in physics and mathematical physics (Chicago, 1982), 1985, pp. 325–353. MR789298

[7] William Fulton, *Young tableaux*, London Mathematical Society Student Texts, vol. 35, Cambridge University Press, Cambridge, 1997. With applications to representation theory and geometry. MR1464693

[8] Taehyeok Heo and Jae-Hoon Kwon, *Combinatorial Howe duality of symplectic type*, J. Algebra **600** (2022), 1–44. MR4393875

[9] Jin Hong and Seok-Jin Kang, *Introduction to quantum groups and crystal bases*, Graduate Studies in Mathematics, vol. 42, American Mathematical Society, Providence, RI, 2002. MR1881971

[10] Roger Howe, *Remarks on classical invariant theory*, Trans. Amer. Math. Soc. **313** (1989), no. 2, 539–570. MR986027

[11] ———, *Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond*, The Schur lectures (1992) (Tel Aviv), 1995, pp. 1–182. MR1321638

[12] Roger Howe, Eng-Chye Tan, and Jeb F. Willenbring, *Stable branching rules for classical symmetric pairs*, Trans. Amer. Math. Soc. **357** (2005), no. 4, 1601–1626. MR2115378

[13] Il-Seung Jang and Jae-Hoon Kwon, *Flagged Littlewood-Richardson tableaux and branching rule for classical groups*, J. Combin. Theory Ser. A **181** (2021), Paper No. 105419, 51. MR4216338

[14] Victor Kac and Andrey Radul, *Representation theory of the vertex algebra* $W_{1+\infty}$, Transform. Groups **1** (1996), no. 1-2, 41–70. MR1390749

[15] Victor G. Kac, *Infinite-dimensional Lie algebras*, Third, Cambridge University Press, Cambridge, 1990. MR1104219

[16] M. Kashiwara, *On crystal bases of the Q-analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), no. 2, 465–516. MR1115118

[17] Masaki Kashiwara, *On crystal bases*, Representations of groups (Banff, AB, 1994), 1995, pp. 155–197. MR1357199

[18] Masaki Kashiwara and Toshiki Nakashima, *Crystal graphs for representations of the q-analogue of classical Lie algebras*, J. Algebra **165** (1994), no. 2, 295–345. MR1273277

[19] R. C. King, *Branching rules for classical Lie groups using tensor and spinor methods*, J. Phys. A **8** (1975), 429–449. MR411400

[20] ______, *Weight multiplicities for the classical groups*, Group theoretical methods in physics (Fourth Internat. Colloq., Nijmegen, 1975), 1976, pp. 490–499. Lecture Notes in Phys., Vol. 50. MR0480895

[21] Donald E. Knuth, *Permutations, matrices, and generalized Young tableaux*, Pacific J. Math. **34** (1970), 709–727. MR272654

[22] Jae-Hoon Kwon, *Crystal graphs for Lie superalgebras and Cauchy decomposition*, J. Algebraic Combin. **25** (2007), no. 1, 57–100. MR2284148

[23] ______, *Rational semistandard tableaux and character formula for the Lie superalgebra* $\widehat{\mathfrak{gl}}_{\infty|\infty}$, Adv. Math. **217** (2008), no. 2, 713–739. MR2370279

[24] ______, *Crystal graphs for general linear Lie superalgebras and quasi-symmetric functions*, J. Combin. Theory Ser. A **116** (2009), no. 7, 1199–1218. MR2527604

[25] ______, *Super duality and crystal bases for quantum ortho-symplectic superalgebras*, Int. Math. Res. Not. IMRN **23** (2015), 12620–12677. MR3431632

[26] ______, *Super duality and crystal bases for quantum ortho-symplectic superalgebras II*, J. Algebraic Combin. **43** (2016), no. 3, 553–588. MR3482440

[27] ______, *Combinatorial extension of stable branching rules for classical groups*, Trans. Amer. Math. Soc. **370** (2018), no. 9, 6125–6152. MR3814326

[28] ______, *Lusztig data of Kashiwara-Nakashima tableaux in types B and C*, J. Algebra **503** (2018), 222–264. MR3779994

[29] Ngau Lam and R. B. Zhang, *Quasi-finite modules for Lie superalgebras of infinite rank*, Trans. Amer. Math. Soc. **358** (2006), no. 1, 403–439. MR2171240

[30] Alain Lascoux, *Double crystal graphs*, Studies in memory of Issai Schur (Chevaleret/Rehovot, 2000), 2003, pp. 95–114. MR1985724

[31] Cédric Lecouvey, *Schensted-type correspondence, plactic monoid, and jeu de taquin for type C_n*, J. Algebra **247** (2002), no. 2, 295–331. MR1877856

[32] Cédric Lecouvey and Cristian Lenart, *Combinatorics of generalized exponents*, Int. Math. Res. Not. IMRN **16** (2020), 4942–4992. MR4139030

[33] Seung Jin Lee, *Crystal structure on king tableaux and semistandard oscillating tableaux*, arXiv preprint arXiv:1910.04459 (2019).

[34] Peter Littelmann, *A plactic algebra for semisimple Lie algebras*, Adv. Math. **124** (1996), no. 2, 312–331. MR1424313

[35] Dudley E. Littlewood, *The theory of group characters and matrix representations of groups*, AMS Chelsea Publishing, Providence, RI, 2006. Reprint of the second (1950) edition. MR2213154

[36] Toshiki Nakashima, *Crystal base and a generalization of the Littlewood-Richardson rule for the classical Lie algebras*, Comm. Math. Phys. **154** (1993), no. 2, 215–243. MR1224078

[37] Jeffrey B. Remmel, *The combinatorics of (k, l)-hook Schur functions*, Combinatorics and algebra (Boulder, Colo., 1983), 1984, pp. 253–287. MR777704

[38] Bruce E. Sagan, *The symmetric group*, Second, Graduate Texts in Mathematics, vol. 203, Springer-Verlag, New York, 2001. Representations, combinatorial algorithms, and symmetric functions. MR1824028

[39] C. Schensted, *Longest increasing and decreasing subsequences*, Canadian J. Math. **13** (1961), 179–191. MR121305

[40] M.-P. Schützenberger, *La correspondance de Robinson*, Combinatoire et représentation du groupe symétrique (Actes Table Ronde CNRS, Univ. Louis-Pasteur Strasbourg, Strasbourg, 1976), 1977, pp. 59–113. Lecture Notes in Math., Vol. 579. MR0498826

[41] Jeffrey T. Sheats, *A symplectic jeu de taquin bijection between the tableaux of King and of De Concini*, Trans. Amer. Math. Soc. **351** (1999), no. 9, 3569–3607. MR1466956

[42] Sheila Sundaram, *The Cauchy identity for* $\mathrm{Sp}(2n)$, J. Combin. Theory Ser. A **53** (1990), no. 2, 209–238. MR1041446

[43] Itaru Terada, *A Robinson-Schensted-type correspondence for a dual pair on spinors*, J. Combin. Theory Ser. A **63** (1993), no. 1, 90–109. MR1213133

[44] Weiqiang Wang, *Duality in infinite-dimensional Fock representations*, Commun. Contemp. Math. **1** (1999), no. 2, 155–199. MR1696098

[45] Hermann Weyl, *The Classical Groups. Their Invariants and Representations*, Princeton University Press, Princeton, N.J., 1939. MR0000255

국문초록

로빈슨-솀스테드-크누스 대응으로 불리는 잘 알려진 전단사 함수는 일반 선형군의 쌍에 대한 하우 쌍대성을 설명한다. 이것은 반표준 타블로와 관련된 조합론적 알고리즘으로 주어지며, 이는 일반 선형 군의 기약 표현과 밀접한 관련을 가지고 있다. 본 학위 논문에서는 사교군과 리 (초)대수의 쌍에 대한 하우 쌍대성에 대한 조합론적 해석을 부여한다. 우리는 사교 타블로 모델인 스피너 모델과 킹 타블로를 이용해 로빈슨-솀스테드-크누스 대응의 사교적 유사체를 설립한다. 이 때, 두 모델은 사교군과 리 (초)대수의 표현과 관련이 있다. 우리는 삽입 타블로를 정의하기 위해 스피너 모델에 대한 주 드 타킨 이동의 사교적 유사체를 타블로의 문자의 선택에 의존하지 않는 균일한 방식으로 정의하고, 킹 타블로를 삽입 타블로에 대응하는 기록 타블로로서 할당한다. 따름 정리로서, 우리는 기약 사교 특성 함수와 관련된 잘 알려진 등식들에 대한 새로운 조합론적 증명을 제시한다.

주요어휘: 결정 그래프, 하우 쌍대성, 로빈슨-솀스테드-크누스 대응, 주 드 타킨, 스피너 모델
학번: 2015-20283

감사의 글

본 학위 과정을 마칠 때까지 많은 가르침을 주신 권재훈 교수님께 가장 먼저 감사의 인사를 드립니다. 저의 대학원 생활을 돌이켜봤을 때 정말 다양한 방식으로 교수님께 걱정을 끼칠 일만 만들었던 것 같습니다. 졸업을 하게 되는 이 시점에서도 아직 해결하지 못한 문제들도 있지만, 교수님께서 많은 도움을 주신 덕분에 이렇게 학위를 무사히 마칠 수 있게 됐다고 생각하고 있습니다. 지금까지 제게 해주셨던 많은 말씀들을 기억하고 명심하며 앞으로의 생활을 잘 헤쳐나가도록 하겠습니다. 다시 한 번 감사 드립니다.

바쁘신 와중에도 저의 박사 학위 심사를 위해 시간을 내주신 이승진 교수님, 서의린 교수님, 오영탁 교수님, 김장수 교수님께도 감사의 인사를 드립니다. 특히나 자격 시험을 포함해서 본 학위를 받을 수 있게 많은 도움을 주신 이승진 교수님께 감사의 인사를 다시 한 번 드립니다. 그리고 학문적으로도 많은 가르침을 주시고 제게 좋은 말씀도 많이 해주셨던 이규환 교수님, 박의용 교수님, 김명호 교수님, 오세진 교수님께도 감사의 인사를 드립니다.

대학원 생활동안 즐거웠던 기억을 많이 만들어준 표현론 팀원들한테도 고맙다고 말하고 싶습니다. 가장 먼저 표현론 팀을 함께 시작했던 최승일 박사님부터 황병학 박사님, 일승이 형, 정우 형, 현세 형, 호빈이 형, 상준이 형, 신명이, 아끼또, 정이, 수홍이 모두 고맙습니다. 그리고 440호를 같이 쓰게 되면서 더 다양하게 만나게 됐던 김영훈 박사님, 이승재 박사님, 이석형 박사님, 이강산, 송아림, 박민희, 최동준, 윤상원 학생에게도 감사의 인사를 전합니다.

학부 때도 재미있던 많은 일을 함께 겪었던 윤환이 형, 필영이 형, 정원 누나, 정무 형, 강주 형, 부근이 형, 태희, 우남, 준영, 지호, 동규, 찬규, 은지 모두 고맙습니다. 특히나 대학원 생활이 힘들 때마다 제 찡찡거림을 모두 받아주셨던 윤환이 형과 우리 팀 가장 큰 형으로 졸업 때문에 이것저것 귀찮게 해서 미안한 일승이 형에게 정말로 고맙다고 하고 싶습니다. 대학원 준비는 다 같이 했지만 지금은 모두 다 각자의 길에서 열심히 일하고 있는, 대학원 준비 스터디를 빙자한 과방에서 놀고 먹었던 효진이 형, 창훈이, 수봉이, 국영이한테도 고맙습니다. 대학원 1년차 때 332호에서 모이다가 그 이후에는 제 방 427호에서 다 같이 모여서 수다를 열심히 떨어주던 탁원이 형, 자현이 형, 원태 형, 현수, 남경, 경현, 재린, 석창, 재훈, 병창, 창훈 모두 모두 고맙습니다.

마지막으로 본 학위를 마칠 때까지 계속해서 응원하고 가장 많이 고생해준 우리 가족, 아빠, 엄마, 형한테 사랑한다고 말하고 싶습니다. 저의 부주의함으로 혹시 언급되지 않았더라도 박사 졸업을 축하해주신 다른 많은 분들에게도 감사드립니다.