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이학 석사 학위논문

Energy Landscape of the
Two-component
Curie–Weiss–Potts Model
with Three Spins

(스핀이 3개인 Curie–Weiss–Potts 모델의
에너지 경관)

2022년 8월

서울대학교 대학원

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2022년 4월

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Energy Landscape of the Two-component Curie–Weiss–Potts Model with Three Spins

A dissertation
submitted in partial fulfillment
of the requirements for the degree of
Master of Science
to the faculty of the Graduate School of
Seoul National University

by

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August 2022

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Abstract

In this paper, we investigate the energy landscape of the two-component spin systems, known as the Curie–Weiss–Potts model, which is a generalization of the Curie–Weiss model consisting of $q \geq 3$ spins. In the energy landscape of a multi-component model, the most important element is the relative strength between the inter-component interaction strength and the component-wise interaction strength. If the inter-component interaction is stronger than the component-wise interaction, we can expect all the components to be synchronized in the course of metastable transition. However, if the inter-component interaction is relatively weaker, then the components will be desynchronized in the course of metastable transition. For the two-component Curie–Weiss model, the phase transition from synchronization to desynchronization has been precisely characterized in studies owing to its mean-field nature. The purpose of this paper is to extend this result to the Curie–Weiss–Potts model with three spins. We observe that the nature of the phase transition for the three-spin case is entirely different from the two-spin case of the Curie–Weiss model, and the proof as well as the resulting phase diagram is fundamentally different and exceedingly complicated.

Key words: Phase transition, Spin system, Curie–Weiss–Potts model, Energy landscape, Multi-component model

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Chapter 1

Introduction

The Ising model is a ferromagnetic spin system consisting of two spins and plays a vital role in the mathematical study of stochastic interacting systems. In particular, its rich phase transition behaviors have been rigorously studied over the last century. One of the simplest Ising model is the Curie–Weiss model, which is an Ising model defined on a complete graph. Owing to its mean-field feature, it is possible to completely characterize the energy landscape of the Curie–Weiss model; essentially, everything is known for this model. We refer the reader to a monograph [18] for a comprehensive discussion of these classic results.

The Potts model is a ferromagnetic spin system consisting of $q \geq 3$ spins and hence, can be regarded as a simple extension of the Ising model. A surprising fact is that in most cases, the Potts model exhibits qualitatively different and more complex behaviors than the Ising model. For example, the characterization of the critical temperature for the Potts model on a lattice requires a non-trivial argument and is obtained in [1], and the phase transition at this temperature is known to be continuous as in the Ising model if $q \leq 4$ (cf. [5]). For $q > 4$, the phase transition is discontinuous (cf. [6]).

The Potts model on a complete graph is called the Curie–Weiss–Potts model, and it has been investigated in [11, 19, 20]. Even if it is a mean-

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field model, allowing us to perform dimension reduction on q variables representing the empirical magnetization vector, the analysis of the energy landscape is not a simple task. The phase transition of the scaling limit of the mean-field free energy of the Curie–Weiss–Potts model has been studied in [9]. The energy landscape was first studied in [3] from the viewpoint of equivalent and nonequivalent ensembles of phase transitions. For $q = 3$, the complete analysis of the energy landscape is carried out in [13, 14] and the metastability of the Glauber dynamics has been studied based on it in [14]. The cut-off phenomenon for the high-temperature regime has been proved in [4] for all $q \geq 3$. Extending the computation for a model with $q \geq 4$ is far more complicated and has been carried out recently in [15]. It has been observed in that study that the structure of the energy landscape for the model with $q \geq 5$ differs from that of the model with $q = 3, 4$.

Recently, investigations on multi-component Curie–Weiss or Curie–Weiss–Potts models have garnered much interest. These models are defined on a complete graph; however, the vertices are divided into several groups of macroscopically non-negligible sizes. The interaction strength between two sites depends on the groups to which these sites belong. Then, the empirical magnetization for each component interacts with the other components, and therefore rich behaviors according to the temperature and interaction parameters are expected. This has been investigated in recent literature. For example, [12] presented the law of large numbers and large deviation principle for the empirical magnetizations for the multi-component Curie–Weiss–Potts model. The corresponding central limit theorem and moderate deviation principle have been investigated in [10], and the central limit theorem for the joint distribution for empirical magnetization in the two-component Curie–Weiss–Potts model has been analyzed in [16] by inspecting the limiting behavior of the free energy. As observed in these studies, the Curie–Weiss–Potts model maintains the mean-field feature; however, the structure of the energy landscape might be qualitatively different from that of the one-component model provided that the inter-component interaction parameters are considerably different from

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the component-wise interaction. The present study seeks to quantitatively characterize the borderline for this behavior in terms of the relative interaction strengths between the component, and this is done by analyzing the energy landscape of the simplest possible multi-component Curie–Weiss–Potts model, namely the two-component model with three spins. We will verify that even in this simplest possible case, with a mean-field structure, the energy landscape is unbelievably complicated and therefore its complete characterization is a highly demanding task. We remark that this investigation has been carried out for the two-component Curie–Weiss model in [2], but our work shows that not only the proof but also the results for the Curie–Weiss–Potts model are fundamentally different.

Chapter 2

Main Results

2.1 Two-component Curie–Weiss–Potts Models

We first define the two-component Curie–Weiss–Potts model. For the simplicity of notation, we assume that two components $\Lambda_N^{(1)} = \{1, \dots, N\}$ and $\Lambda_N^{(2)} = \{1, \dots, N\}$ are of the same size. We consider a spin system on $\Lambda_N = \Lambda_N^{(1)} \cup \Lambda_N^{(2)}$. We denote the set of spins by $\mathcal{S} = \{1, \dots, q\}$ and define

$$\Omega = \mathcal{S}^{\Lambda_N^{(1)}} \times \mathcal{S}^{\Lambda_N^{(2)}}$$

to be the set of spin configurations. Each configuration $\boldsymbol{\sigma}$ in Ω is denoted by $\boldsymbol{\sigma} = (\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)})$ where

$$\boldsymbol{\sigma}^{(k)} = (\sigma_1^{(k)}, \dots, \sigma_N^{(k)}) \quad ; \quad k = 1, 2,$$

and where $\sigma_i^{(k)}$ denotes the spin assigned at site i of the k -th component $\Lambda_N^{(k)}$.

Now we define the Curie–Weiss–Potts (CWP) measure on Ω . We fix

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$J_{11}, J_{22}, J_{12} \geq 0$ and define the CWP Hamiltonian $\mathbb{H}_N : \Omega \rightarrow \mathbb{R}$ by

$$\mathbb{H}_N(\boldsymbol{\sigma}) = -\frac{1}{N} \sum_{k=1,2} \sum_{1 \leq i < j \leq N} J_{kk} \cdot \mathbf{1}\{\sigma_i^{(k)} = \sigma_j^{(k)}\} - \frac{1}{N} \sum_{i,j=1}^N J_{12} \cdot \mathbf{1}\{\sigma_i^{(1)} = \sigma_j^{(2)}\}. \quad (2.1.1)$$

We denote the Gibbs measure on Ω associated with the Hamiltonian \mathbb{H}_N at the inverse temperature $\beta > 0$ by $\mu_{N,\beta}$:

$$\mu_{N,\beta}(\boldsymbol{\sigma}) = \frac{1}{Z_{N,\beta}} e^{-\beta \mathbb{H}_N(\boldsymbol{\sigma})},$$

where $Z_{N,\beta}$ is the partition function defined by

$$Z_{N,\beta} = \sum_{\boldsymbol{\sigma} \in \Omega} e^{-\beta \mathbb{H}_N(\boldsymbol{\sigma})}.$$

The measure $\mu_{N,\beta}$ is called the CWP measure, and its energy landscape is what we seek to determine.

In order to simplify the question of the phase transition according to the relative interaction strength, we assume that

$$J_{11} = J_{22} = \frac{1}{1+J} \quad \text{and} \quad J_{12} = \frac{J}{1+J}$$

for some $J > 0$, where J denotes the relative strength of the inter-component interaction with respect to the component-wise interaction. With this notation, (2.1.1) can be rewritten as

$$\mathbb{H}_N(\boldsymbol{\sigma}) = -\frac{1}{N} \sum_{k=1,2} \sum_{1 \leq i < j \leq N} \frac{1}{1+J} \cdot \mathbf{1}\{\sigma_i^{(k)} = \sigma_j^{(k)}\} - \frac{1}{N} \sum_{i,j=1}^N \frac{J}{1+J} \cdot \mathbf{1}\{\sigma_i^{(1)} = \sigma_j^{(2)}\}.$$

2.2 Distribution of Empirical Magnetization

For $\sigma \in \Omega$, we denote by¹ $\mathbf{r}^{(k)}(\sigma) = (r_1^{(k)}(\sigma), \dots, r_q^{(k)}(\sigma))$, $k = 1, 2$, the empirical magnetization of the component $\Lambda_N^{(k)}$, i.e.,

$$r_u^{(k)}(\sigma) = \frac{1}{N} \sum_{i=1}^N \mathbf{1}\{\sigma_i^{(k)} = u\} \quad ; \quad u \in \mathcal{S}.$$

We write $\mathbf{r}(\sigma) = (\mathbf{r}^{(1)}(\sigma), \mathbf{r}^{(2)}(\sigma))$. Our main concern in this paper is the distribution of the empirical magnetization $\mathbf{r}(\sigma)$ under the CWP measure $\mu_{N,\beta}$ and we shall analyze its qualitative and quantitative behavior as J and β vary.

The main feature of the mean-field type model as in the CWP model is the dimension reduction. To explain this in more detail, let Ξ be the $(q-1)$ -dimensional simplex defined by

$$\Xi = \left\{ \mathbf{x} = (x_u)_{1 \leq u \leq q} \in [0, 1]^q : \sum_{u=1}^q x_u = 1 \right\}$$

and let $\Xi_N = \Xi \cap (N^{-1}\mathbb{Z})^q$. Then, we have $\mathbf{r}(\sigma) \in \Xi_N^2 \subset \Xi^2$ for all $\sigma \in \Omega$.

Notation 2.2.1. Since $r_q^{(k)}(\sigma) = 1 - \sum_{u=1}^{q-1} r_u^{(k)}(\sigma)$, there is no risk of confusion in regarding

$$\mathbf{r}^{(k)}(\sigma) = (r_1^{(k)}(\sigma), \dots, r_{q-1}^{(k)}(\sigma)).$$

Similarly, we can regard

$$\Xi = \left\{ \mathbf{x} = (x_u)_{1 \leq u \leq q-1} \in [0, 1]^{q-1} : \sum_{u=1}^{q-1} x_u \leq 1 \right\}.$$

Hence, we shall use these two alternative expressions of the coordinate

¹For notational simplicity, we do not stress the dependency of this object to N ; the same convention will be used throughout this paper.

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vectors interchangeably depending on the context.

Owing to the symmetry among the sites of the same component, we can reduce the complicated spin spaces Ω into Ξ_N^2 by looking at the empirical magnetization. To rigorously explain this, we denote by $\nu_{N,\beta}(\cdot)$ the measure on Ξ_N^2 representing the distribution of empirical magnetization $\mathbf{r}(\boldsymbol{\sigma})$ under $\mu_{N,\beta}$, i.e., for $\mathbf{x} = (\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) \in \Xi_N^2$,

$$\nu_{N,\beta}(\mathbf{x}) = \sum_{\boldsymbol{\sigma} \in \Omega: \mathbf{r}(\boldsymbol{\sigma}) = \mathbf{x}} \mu_{N,\beta}(\boldsymbol{\sigma}) = \sum_{\boldsymbol{\sigma} \in \Omega: \mathbf{r}(\boldsymbol{\sigma}) = \mathbf{x}} \frac{1}{Z_{N,\beta}} e^{-\beta \mathbb{H}_N(\boldsymbol{\sigma})}.$$

Then, we have the following expression for $\nu_{N,\beta}(\cdot)$.

Proposition 2.2.2. *We can write*

$$\nu_{N,\beta}(\mathbf{x}) = \frac{1}{\widehat{Z}_{N,\beta,J}} \exp \left\{ -N \frac{\beta}{1+J} \left(F_{\beta,J}(\mathbf{x}) + \frac{1}{N} G_{N,\beta}(\mathbf{x}) \right) \right\}$$

for some constant $\widehat{Z}_{N,\beta,J} > 0$ where

$$\begin{aligned} F_{\beta,J}(\mathbf{x}) &= H(\mathbf{x}) + \frac{1+J}{\beta} S(\mathbf{x}) \quad \text{and} \\ G_{N,\beta,J}(\mathbf{x}) &= \frac{1+J}{2\beta} \log \left(\prod_{k=1,2} \prod_{j=1}^q x_i^{(k)} \right) + O(N^{-(q-1)}), \end{aligned}$$

where

$$\begin{aligned} H(\mathbf{x}) &= - \sum_{k=1,2} \sum_{i=1}^q \frac{1}{2} (x_i^{(k)})^2 - J \sum_{i=1}^q x_i^{(1)} x_i^{(2)} \quad \text{and} \\ S(\mathbf{x}) &= \sum_{k=1,2} \sum_{i=1}^q x_i^{(k)} \log x_i^{(k)}. \end{aligned}$$

We give a proof in Appendix A. We note that $F_{\beta,J}$ indeed corresponds to the free-energy of the spin system and dominate the energy landscape

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associated with the empirical magnetization. The role of $G_{N,\beta,J}$ is limited to the sub-exponential factor, and hence will be neglected in the remainder of this paper. Our primary concern is the analysis of the function $F_{\beta,J}$.

2.3 Two Extreme Cases

In this section, we first discuss two extreme cases for $q = 3$ to explain the main objective of this paper.

1. $J = \infty$ (i.e., $J_{11} = J_{22} = 0$ and $J_{12} = 1$) ; no component-wise interaction,
2. $J = 0$ (i.e., $J_{12} = 0$ and $J_{11} = J_{22} = 1$) ; no inter-component interaction.

In both cases, the calculations are similar. However, there is a considerable difference between the two cases. To introduce these results, we define a function $\xi : (0, \frac{1}{2}) \rightarrow [0, \infty)$ by

$$\xi(x) := \frac{1}{1-3x} \log \frac{1-2x}{x}. \quad (2.3.1)$$

The function $\xi(x)$ has a unique global minimum; thus, we denote this value by $\beta_1 > 0$. We also define

$$\beta_2 := \frac{2(q-1)}{q-2} \log(q-1) = 4 \log 2 \quad \text{and} \quad \beta_3 := 3,$$

which will be used later. Note that the β_2 is introduced in [3, 7, 8, 15] and $\beta_1 \approx 2.7465 < \beta_2 \approx 2.7726 < \beta_3$. For a given $\beta > \beta_1$, the equation $\beta = \xi(x)$ has two solutions because $\xi(x)$ is convex. In this case, we denote the smaller solution by $x_s = x_s(\beta)$ and the larger one by $x_l = x_l(\beta)$. Then, we define sets

$$S = S(\beta) = \{\text{permutations of } (x_s, x_s, 1 - 2x_s)\},$$

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$$L = L(\beta) = \{\text{permutations of } (x_l, x_l, 1 - 2x_l)\}.$$

Furthermore, we define the set of this kind for the two-component case as

$$\begin{aligned} S^2 &= S^2(\beta) := \{(\mathbf{x}_{s,2}, \mathbf{x}_{s,2}) \in \Xi_N^2 : \mathbf{x}_{s,2} \in S\}, \\ L^2 &= L^2(\beta) := \{(\mathbf{x}_{l,2}, \mathbf{x}_{l,2}) \in \Xi_N^2 : \mathbf{x}_{l,2} \in L\}. \end{aligned}$$

We denote the point $(\mathbf{x}_{s,2}, \mathbf{x}_{s,2}) \in S^2$ by \mathbf{x}_s , and the point $(\mathbf{x}_{l,2}, \mathbf{x}_{l,2}) \in L^2$ by \mathbf{x}_l . Now, we can define the following definitions.

Definition 2.3.1. The two components are said to be *synchronized* if all the local minima and lowest saddles of $F_{\beta, J}$ belong to either S^2 or L^2 . Otherwise, the two components are said to be *desynchronized*.

Therefore, under the synchronization regime, the two components are synchronized in the course of metastable transition from a local minima to a global minima. More precisely, it is well known that the metastable transition from a local minima \mathbf{m} to another on \mathbf{m}' passing through a lowest saddle σ between them follows (in the macroscopic scale) the so-called instanton path defined by

$$\dot{\mathbf{x}}_t = \nabla F_{\beta, J}(\mathbf{x}_t) \ ; \ \lim_{x \rightarrow -\infty} \mathbf{x}_t = \sigma \text{ and } \lim_{x \rightarrow +\infty} \mathbf{x}_t = \mathbf{m} \text{ (or } \mathbf{m}' \text{),}$$

especially when we consider heat-bath Glauber dynamics. One can notice that, if the starting configuration σ is synchronized in the sense that its two coordinates are same, then the same property holds for \mathbf{x}_t for all $t > 0$ by the symmetry of the potential $F_{\beta, J}$. Rigorous verification of this fact follows from elementary computations and therefore we omit the detail. This implies that the synchronization property persists along the course of entire metastable transition. Of course, on the other hand, the two components are desynchronized in the course of metastable transition if we are in the desynchronization regime. Then, we have the following results for the extreme cases.

Theorem 2.3.2. *Suppose that $q = 3$.*

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1. *If there is no component-wise interaction in each component, then the two components are synchronized.*
2. *If there is no inter-component interaction between the two components, then the two components are desynchronized.*

2.4 Synchronization–Desynchronization Phase Transition

In the previous section, we defined synchronization between the components and stated the results of the two extreme cases. Naturally, it can be expected that synchronization occurs when J is sufficiently large. However, when J is sufficiently small, one expected that synchronization to be broken. In this section, we introduce results on the boundaries of the synchronization when $q = 2$ and $q = 3$, respectively. The Ising case (i.e., $q = 2$) can be handled as in [2] and it can be restated as the following theorem, and more comprehensive results are stated in Theorem 3.0.4.

Theorem 2.4.1. *Suppose that $q = 2$. Define a function $\zeta_1 : [2, \infty) \rightarrow [0, \infty)$ by*

$$\zeta_1(\beta) := \frac{\beta - 2}{\beta + 2}. \quad (2.4.1)$$

If $J > \zeta_1(\beta)$, then the two components are synchronized. However, if $J < \zeta_1(\beta)$, then the two components are desynchronized.

Nevertheless, for $q \geq 3$, the synchronization–desynchronization phase transition is very complicated to analyze. We explain this for the case $q = 3$. We define functions $\psi_1, \psi_2, \psi_3 : [0, \infty) \rightarrow [0, \infty)$ by

$$\psi_1(\beta) = \frac{\beta - \frac{1}{x_l}}{\beta + \frac{1}{x_l}} \cdot \mathbf{1}\{\beta \geq \beta_3\}, \quad \psi_2(\beta) = \frac{\beta - \frac{1}{3x_l(1-2x_l)}}{\beta + \frac{1}{3x_l(1-2x_l)}} \cdot \mathbf{1}\{\beta_1 \leq \beta \leq \beta_3\}, \quad \text{and} \quad (2.4.2)$$

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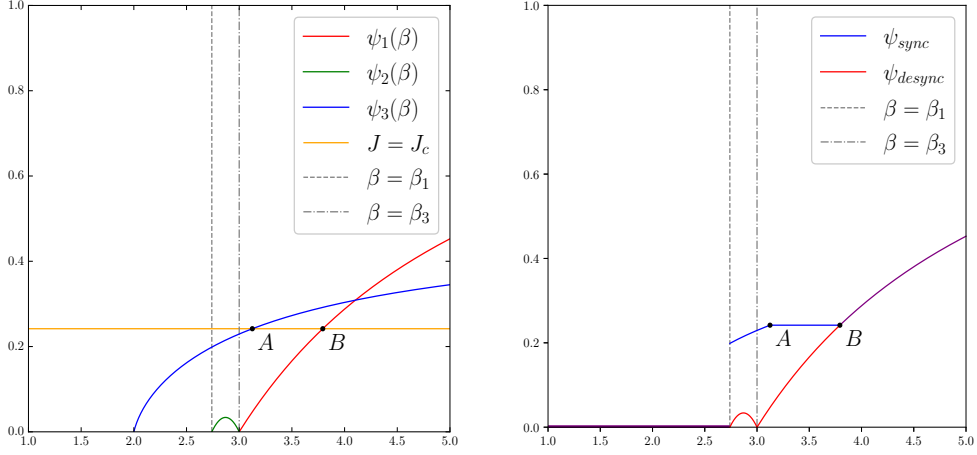


Figure 2.4.1: These are diagrams of functions of the inverse temperature β , and the β -value of $A \approx 3.1255$ and $B \approx 3.8290$. The purple curve represents the overlapping of functions ψ_s and ψ_d .

$$\psi_3(\beta) = \frac{\beta - 3 + \sqrt{25\beta^2 - 50\beta + 1}}{2(1 + 6\beta)}.$$

A constant $J_c \approx 0.2419$ will be specified later in Theorem 4.3.5. Then, we have the following main results.

Theorem 2.4.2 (Main Theorem). *We define functions $\psi_s, \psi_d : [0, \infty) \rightarrow [0, \infty)$ by*

$$\psi_s(\beta) = \begin{cases} 0 & \beta \leq \beta_1, \\ \max[\psi_1(\beta), \min\{J_c, \psi_3(\beta)\}] & \beta > \beta_1, \end{cases} \quad \text{and} \quad \psi_d(\beta) = \psi_1(\beta) + \psi_2(\beta).$$

If $J > \psi_s(\beta)$, then the two components are synchronized. However, if $J < \psi_d(\beta)$, then the two components are desynchronized.

We conjecture that there exists $\psi : [0, \infty) \rightarrow [0, \infty)$ such that the synchronization happens if and only if $J > \psi(\beta)$. We proved this by verifying that $\psi(\beta) = 0$ for $\beta \leq \beta_1$ and $\psi(\beta) = \psi_1(\beta)$ for $\beta > \beta_c$ where $\beta_c \approx 3.8290$. This type of behavior might hold for any multi-component Curie–Weiss–

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Potts model, but conditioned on the analysis and results of this paper, it will not be easy to characterize it. This question will be pursued future studies.

Chapter 3

Ising Model on Bipartite Graph

In this section, we deal with the Ising case. We do this because we present a more comprehensive proof and provide simplified versions of the calculations when $q = 3$. This emphasizes that there is a significant difference between $q = 2$ and $q = 3$.

From Proposition 2.2.2, the function $F_{\beta, J}$ for $q = 2$ is given by

$$\begin{aligned} F_{\beta, J}(x, y) = & -\frac{1}{2} \{x^2 + (1-x)^2 + y^2 + (1-y)^2\} - J\{xy + (1-x)(1-y)\} \\ & + \frac{1+J}{\beta} \{x \log x + (1-x) \log(1-x) + y \log y + (1-y) \log(1-y)\}. \end{aligned}$$

Here, we used the notation x for $x_1^{(1)}$ and y for $x_1^{(2)}$ to simplify the expression. To investigate the critical points of $F_{\beta, J}(x)$, computing the partial derivatives of the function yields

$$\frac{\partial F_{\beta, J}}{\partial x}(x, y) = (1-2x) + J(1-2y) + \frac{1+J}{\beta} \log\left(\frac{x}{1-x}\right) = 0 \quad (3.0.1)$$

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$$\frac{\partial F_{\beta, J}}{\partial y}(x, y) = (1 - 2y) + J(1 - 2x) + \frac{1 + J}{\beta} \log \left(\frac{y}{1 - y} \right) = 0.$$

An elementary computation shows that the Hessian matrix of the function $F_{\beta, J}$ is

$$\nabla^2 F_{\beta, J}(x, y) = \begin{pmatrix} \frac{1+J}{\beta x(1-x)} - 2 & -2J \\ -2J & \frac{1+J}{\beta y(1-y)} - 2 \end{pmatrix}. \quad (3.0.2)$$

Since the function $(1 - 2x) + J(1 - 2y) + \frac{1+J}{\beta} \log \left(\frac{x}{1-x} \right)$ is point symmetric at $(\frac{1}{2}, \frac{1}{2})$, substituting $x = \frac{s+1}{2}$ and $y = \frac{t+1}{2}$ for $-1 \leq s, t \leq 1$ into equations in (3.0.1) yields

$$\Theta(s) = t \quad \text{and} \quad \Theta(t) = s, \quad (3.0.3)$$

where

$$\Theta(s) = \frac{1}{J} \left(-s + \frac{1+J}{\beta} \log \frac{1+s}{1-s} \right).$$

Now, we define a function $\zeta_2, : [2, \infty) \rightarrow [0, \infty)$ by

$$\zeta_2(\beta) = \frac{\sqrt{\beta(\beta-2)} - 2 \log \{(\sqrt{\beta} + \sqrt{\beta-2})/\sqrt{2}\}}{\sqrt{\beta(\beta-2)} + 2 \log \{(\sqrt{\beta} + \sqrt{\beta-2})/\sqrt{2}\}}. \quad (3.0.4)$$

Recall that ζ_1 was defined in (2.4.1). By elementary computations, we obtain that

$$\zeta_1(2) = \zeta_2(2) = 0, \quad \zeta_1(\beta) > \zeta_2(\beta) \text{ for all } \beta > 2,$$

and

$$\lim_{\beta \rightarrow \infty} \zeta_1(\beta) = \lim_{\beta \rightarrow \infty} \zeta_2(\beta) = 1.$$

Then, we have the following proposition.

Proposition 3.0.1. *Consider the system of equations (3.0.3).*

1. *If $\Theta'(0) > 1$, then (3.0.3) has only one solution at the origin.*
2. *If $-1 < \Theta'(0) < 1$, then (3.0.3) has three intersections.*

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3. If $\Theta'(0) < -1$, then (3.0.3) has at least five intersections. Moreover, if

$$\Theta(\gamma) < -\gamma, \text{ where } \gamma := \sqrt{\frac{\beta - 2}{\beta}}, \quad (3.0.5)$$

then (3.0.3) has nine intersections.

Proof. The graphs in (3.0.3) is plotted similarly to Figure 3 in [2] depending on the value of $\Theta'(0)$. Note that $\Theta(s)$ is point symmetric at $(0,0)$ and $\Theta'(s)$ has a unique minimum at $s = 0$.

(1) and (2) are straightforward.

(3) If $\Theta'(0) < -1$, then (3.0.3) begin to intersect the line $t = -s$ at two points, except the origin. Note that $\Theta'(\gamma) = 1$. If we denote the positive solution of $\Theta(s) = -s$ by s_+ , then the condition (3.0.5) means that $\gamma < s_+$. Thus, (3.0.3) has two more intersections in the fourth quadrant, and by symmetry, there are two more intersections in the second quadrant. This completes the proof. \square

From the definitions of functions ζ_1 and ζ_2 , and the properties of function Θ , we have the following lemma.

Lemma 3.0.2. *Consider the functions ζ_1, ζ_2 , and Θ . Then, we have the following equivalence conditions.*

1. $\beta < 2$ if and only if $\Theta'(0) > 1$.

Suppose that $\beta > 2$.

(2) $J > \zeta_1(\beta)$ if and only if $-1 < \Theta'(0) < 1$. However, $J < \zeta_1(\beta)$ if and only if $\Theta'(0) < -1$

(3) $J < \zeta_2(\beta)$ if and only if the inequality (3.0.5) holds.

Proof. The results can be obtained from straightforward computations. \square

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Lemma 3.0.3. *Suppose that (a, b) is a critical point of $F_{\beta, J}(x, y)$ and let λ_1, λ_2 be the eigenvalues of the Hessian matrix $\nabla^2 F_{\beta, J}(a, b)$. Then,*

1. $\lambda_1 + \lambda_2 > 0$ if and only if $\Theta'(2a - 1) + \Theta'(2b - 1) > 0$.
2. $\lambda_1 \lambda_2 > 0$ if and only if $\Theta'(2a - 1)\Theta'(2b - 1) > 1$.

Proof. The derivative of the $\Theta(s)$ is

$$\Theta'(s) = \frac{1}{J} \left(\frac{1+J}{\beta} \frac{2}{1-s^2} - 1 \right).$$

By basic concepts in linear algebra and (3.0.2), we can derive the following.

(1) The first result is calculated as

$$\begin{aligned} \lambda_1 + \lambda_2 &= \text{tr}(\nabla^2 F_{\beta, J}(a, b)) = \frac{1+J}{\beta a(1-a)} + \frac{1+J}{\beta b(1-b)} - 4 \\ &= 2J(\Theta'(2a - 1) + \Theta'(2b - 1)), \end{aligned}$$

(2) The second result is calculated as

$$\begin{aligned} \lambda_1 \lambda_2 &= \det(\nabla^2 F_{\beta, J}(a, b)) = \left(\frac{1+J}{\beta a(1-a)} - 2 \right) \left(\frac{1+J}{\beta b(1-b)} - 2 \right) - 4J^2 \\ &= 4J^2(\Theta'(2a - 1)\Theta'(2b - 1) - 1). \end{aligned}$$

This completes the proof. □

Now, we can categorize all the critical points of $F_{\beta, J}$ as done in [2]. We have further found the function (3.0.4), which is the boundary of the phase transition, and compared the value of the local minima.

Theorem 3.0.4 (Classifications of the critical points of $F_{\beta, J}$).

The critical points of $F_{\beta, J}$ can be classified as follows.

1. If $\beta < 2$, then $F_{\beta, J}$ has unique local minimum at $(\frac{1}{2}, \frac{1}{2})$.

Now, suppose that $\beta > 2$.

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- (2) If $J > \zeta_1(\beta)$, then $F_{\beta, J}$ has two global minima and a saddle point at $(\frac{1}{2}, \frac{1}{2})$.
- (3) If $\zeta_2(\beta) < J < \zeta_1(\beta)$, then $F_{\beta, J}$ has two global minima, two saddle points, and a local maximum at $(\frac{1}{2}, \frac{1}{2})$.
- (4) If $J < \zeta_2(\beta)$, then $F_{\beta, J}$ has two global minima, two local minima, four saddle points, and a local maximum at $(\frac{1}{2}, \frac{1}{2})$.

Proof. (1), (2) and (3) are straightforward by Proposition 3.0.1, Lemma 3.0.2 and 3.0.3.

(4) By the aforementioned proposition and lemmas, there are four local minima, four saddle points and a local maximum at $(\frac{1}{2}, \frac{1}{2})$. Thus, it remains only to compare the values of the four local minima. We denote the intersections of $\Theta(s)$ and $t = s$ by (s_1, s_1) and $(-s_1, -s_1)$. Moreover, we denote the intersections of $\Theta(s)$ and $t = -s$ by $(s_2, -s_2)$ and $(-s_2, s_2)$. It is obvious that $s_1 > s_2$. If we set $x_1 = \frac{1+s_1}{2}$ and $x_2 = \frac{1+s_2}{2}$, then by Lemma 3.0.3, $F_{\beta, J}$ has local minima at (x_1, x_1) , $(1 - x_1, 1 - x_1)$, $(x_2, 1 - x_2)$, and $(1 - x_2, x_2)$. By symmetry, we have

$$F_{\beta, J}(x_1, x_1) = F_{\beta, J}(1 - x_1, 1 - x_1) \quad \text{and} \quad F_{\beta, J}(x_2, 1 - x_2) = F_{\beta, J}(1 - x_2, x_2).$$

In this case, we claim that

$$F_{\beta, J}(x_1, x_1) < F_{\beta, J}(x_2, 1 - x_2). \quad (3.0.6)$$

From the definitions of s_1 and s_2 , we obtain

$$Js_1 = \frac{1+J}{\beta} \log \frac{1+s_1}{1-s_1} - s_1 \quad \text{and} \quad -Js_2 = \frac{1+J}{\beta} \log \frac{1+s_2}{1-s_2} - s_2. \quad (3.0.7)$$

An elementary calculation using the equations in (3.0.7) shows that

$$F_{\beta, J}(x_2, 1 - x_2) - F_{\beta, J}(x_1, x_1) = \frac{1+J}{\beta} (f(s_1) - f(s_2)),$$

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where $f(x) = \frac{x}{2} \log \frac{1+x}{1-x} - 2 \left\{ \left(\frac{1+x}{2} \right) \log \left(\frac{1+x}{2} \right) + \left(\frac{1-x}{2} \right) \log \left(\frac{1-x}{2} \right) \right\}$. Since the function $f(x)$ is increasing on $x > 0$, and since $s_1 > s_2$, we conclude that (3.0.6) holds. Therefore, $F_{\beta, J}$ has global minima at (x_1, x_1) and $(1 - x_1, 1 - x_1)$, and has local minima at $(x_2, 1 - x_2)$ and $(1 - x_2, x_2)$. This completes the proof. \square

Chapter 4

Potts Model on Bipartite Graph with Three Spins

In this section, we prove Theorem 2.4.2, which is the main result. In section 4.1, we deduce the necessary conditions for the critical points of the function $F_{\beta, J}$ and calculate the eigenvalues of its corresponding Hessian matrix. In section 4.2, we prove Theorem 2.3.2 by considering the two extreme cases, respectively, in Theorem 4.2.2 and Theorem 4.2.1. In section 4.3, 4.4, and 4.5, we find the synchronization and desynchronization boundaries in low, medium, and high temperatures, respectively.

4.1 Critical Points of $F_{\beta, J}$

In this section, we will find the necessary conditions for the critical points of $F_{\beta, J}$ and define two functions, Φ and Ψ , which are derived from them. Then, we will present the relationships between the functions Φ, Ψ and the eigenvalues of the Hessian matrix of $F_{\beta, J}$ in Lemma 4.1.2. From Proposi-

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tion 2.2.2, the function $F_{\beta, J}$ for $q = 3$ is given by

$$F_{\beta, J}(\mathbf{x}, \mathbf{y}) = -\frac{1}{2} \sum_{i=1}^3 (x_i^2 + y_i^2) - J \sum_{i=1}^3 x_i y_i + \frac{1+J}{\beta} \sum_{i=1}^3 (x_i \log x_i + y_i \log y_i) \quad (4.1.1)$$

Here, we used the notations x_i for $x_i^{(1)}$ and y_i for $x_i^{(2)}$ to simplify the expression. To find critical points, the first order derivatives of $F_{\beta, J}$ must be zero:

$$\begin{aligned} \frac{\partial F_{\beta, J}}{\partial x_k}(\mathbf{x}, \mathbf{y}) &= -(x_k - x_3) - J(y_k - y_3) + \frac{1+J}{\beta} (\log x_k - \log x_3) = 0, \\ \frac{\partial F_{\beta, J}}{\partial y_k}(\mathbf{x}, \mathbf{y}) &= -(y_k - y_3) - J(x_k - x_3) + \frac{1+J}{\beta} (\log y_k - \log y_3) = 0. \end{aligned}$$

for $1 \leq k, l \leq 3$. Thus, we can obtain the equations

$$\begin{aligned} -x_k - Jy_k + \frac{1+J}{\beta} \log x_k &= -x_l - Jy_l + \frac{1+J}{\beta} \log x_l, \\ -y_k - Jx_k + \frac{1+J}{\beta} \log y_k &= -y_l - Jx_l + \frac{1+J}{\beta} \log y_l, \end{aligned}$$

for $1 \leq k, l \leq 3$. Since each side of the above equations are symmetric and negative, consider the following equations:

$$y = \Phi(x) + u \quad \text{and} \quad x = \Phi(y) + v, \quad (4.1.2)$$

where u, v are real positive numbers and the function Φ is defined by

$$\Phi(x) := \frac{1}{J} \left(-x + \frac{1+J}{\beta} \log x \right).$$

We need to analyze the solutions of (4.1.2) according to the values of u and v , because they are the candidates for the coordinates of critical points. Assume first that $u = v$. When $u = v = 0$, the graphs in (4.1.2) do not intersect since the graph $y = \Phi(x)$ is under the x -axis and the graph $x = \Phi(y)$ is to the left of the y -axis. If we increase u gradually, $y = \Phi(x) + u$

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will be tangent to $y = x$ at $x = \frac{1}{\beta}$ before intersecting with $x = \Phi(y) + u$ at two points on the line $y = x$. We denote these points by $\mathbf{P} = (P, P)$ and $\mathbf{Q} = (Q, Q)$. It is important to check whether these intersections are in the area $[0, 1] \times [0, 1]$, otherwise these points are meaningless, because they represent the ratio of each spin. In addition, the sum of three of them has to be equal to one, that is, $P + 2Q = 1$ or $2P + Q = 1$. There is no need to consider the case when $u \neq v$ and there are two intersections because the sum of the x -coordinates and the sum of the y -coordinates cannot be equal to one at the same time.

Note that the smaller the $J > 0$, the sharper is the graph of $y = \Phi(x) + u$. Since the function $y = \Phi(x) + u$ is concave, (4.1.2) can have at most four intersections. We denote these intersections by $\mathbf{P} = (P, P)$, $\mathbf{R} = (R, S)$, $\mathbf{S} = (S, R)$, and $\mathbf{Q} = (Q, Q)$ with $P \leq R \leq Q \leq S$. See Figure 4.1.1. The critical points must satisfy that both the sum of their x -coordinates and the sum of their y -coordinates should be equal to one, respectively. For example, if we choose \mathbf{P} , \mathbf{R} and \mathbf{S} , then $P + R + S = 1$. In general, if $u \neq v$, then we denote these intersections by $\mathbf{P} = (P_1, P_2)$, $\mathbf{R} = (R_1, R_2)$, $\mathbf{S} = (S_1, S_2)$, and $\mathbf{Q} = (Q_1, Q_2)$, with $P_1 \leq R_1 \leq Q_1 \leq S_1$. In this case, if we choose \mathbf{P} , \mathbf{R} and \mathbf{S} , then the coordinates should satisfy $P_1 + R_1 + S_1 = P_2 + R_2 + S_2 = 1$.

Remark 4.1.1. The points P_k, Q_k, R_k and S_k for $k = 1, 2$ are functions of the variables u, v, J and β .

For now, we assume that the critical points of $F_{\beta, J}$ are of the form $(s, s, 1 - 2s, t, t, 1 - 2t)$ up to permutations. Such critical points satisfy the following equations:

$$\begin{aligned} y &= \Phi(x) + u \quad \text{and} \quad 1 - 2y = \Phi(1 - 2x) + u, \\ x &= \Phi(y) + u \quad \text{and} \quad 1 - 2x = \Phi(1 - 2y) + u. \end{aligned}$$

Subtracting the second equation from the first equation yields

$$y = \Psi(x) \quad \text{and} \quad x = \Psi(y), \tag{4.1.3}$$

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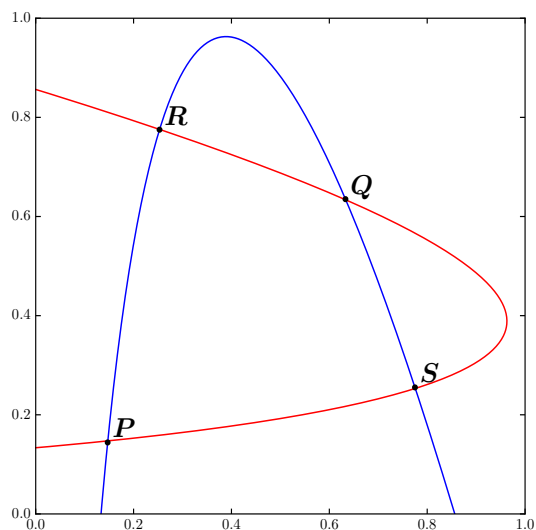


Figure 4.1.1: The graphs of $y = \Phi(x) + u$ (blue curve) and $x = \Phi(y) + u$ (red curve)

where

$$\Psi(x) := \frac{1}{3} (\Phi(x) - \Phi(1 - 2x) + 1) = \frac{1}{3J} \left(1 - 3x + \frac{1+J}{\beta} \log \frac{x}{1-2x} + J \right).$$

Note that the function $\Psi(x)$ always passes through $(\frac{1}{3}, \frac{1}{3})$. From the equation $\Psi(x) = x$, we obtain the inverse temperature

$$\beta = \xi(x). \tag{4.1.4}$$

Recall that $\xi(x)$ was defined in (2.3.1). The derivative of $\xi(x)$ is

$$\frac{d\xi(x)}{dx} = \frac{1 - 3x + 3x(1 - 2x) \log \frac{x}{1-2x}}{x(2x - 1)(3x - 1)^2}, \tag{4.1.5}$$

(4.1.5) has a unique solution and we denote this by $m_1 \approx 0.2076$. Then,

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$\xi(x)$ has a global minimum at m_1 and by the definition of β_1 , we obtain

$$\beta_1 = \frac{1}{1 - 3m_1} \log \frac{1 - 2m_1}{m_1}.$$

Recall that given $\beta > \beta_1$, we denoted the smaller solution of (4.1.4) by $x_s = x_s(\beta)$ and the larger one by $x_l = x_l(\beta)$. Then, we can derive the relationships between the functions Φ, Ψ and the eigenvalues of the Hessian matrix.

Lemma 4.1.2. *Suppose that the critical points of $F_{\beta, J}$ are of the form $(s, s, 1-2s, t, t, 1-2t)$ up to permutations, where $0 \leq s, t \leq 1$. Let $\lambda_1, \lambda_2, \lambda_3$, and λ_4 be the eigenvalues of the Hessian matrix of $F_{\beta, J}$. After reordering the eigenvalues, we have*

1. $\lambda_1 + \lambda_2 > 0$ if and only if $\Phi'(s) + \Phi'(t) > 0$ and $\lambda_1 \lambda_2 > 0$ if and only if $\Phi'(s)\Phi'(t) > 1$,
2. $\lambda_3 + \lambda_4 > 0$ if and only if $\Psi'(s) + \Psi'(t) > 0$ and $\lambda_3 \lambda_4 > 0$ if and only if $\Psi'(s)\Psi'(t) > 1$.

Proof. Note that

$$\Phi'(x) = \frac{1}{J} \left(\frac{1+J}{\beta x} - 1 \right) \quad \text{and} \quad \Psi'(x) = \frac{1}{J} \left(\frac{1+J}{\beta} \frac{1}{3x(1-2x)} - 1 \right).$$

Before giving the proof, we first find the characteristic polynomial of the Hessian matrix of $F_{\beta, J}$. Then, we compute the determinant of the following matrix:

$$\nabla^2(F_{\beta, J}) - \lambda I = \begin{pmatrix} \mathbf{A}_x & \mathbf{B} \\ \mathbf{B} & \mathbf{A}_y \end{pmatrix},$$

where

$$\mathbf{A}_x = \begin{pmatrix} \frac{1+J}{\beta} \left(\frac{1}{x_1} + \frac{1}{x_3} \right) - 2 - \lambda & \frac{1+J}{\beta} \frac{1}{x_3} - 1 \\ \frac{1+J}{\beta} \frac{1}{x_3} - 1 & \frac{1+J}{\beta} \left(\frac{1}{x_2} + \frac{1}{x_3} \right) - 2 - \lambda \end{pmatrix},$$

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$$\mathbf{A}_y = \begin{pmatrix} \frac{1+J}{\beta} \left(\frac{1}{y_1} + \frac{1}{y_3} \right) - 2 - \lambda & \frac{1+J}{\beta} \frac{1}{y_3} - 1 \\ \frac{1+J}{\beta} \frac{1}{y_3} - 1 & \frac{1+J}{\beta} \left(\frac{1}{y_2} + \frac{1}{y_3} \right) - 2 - \lambda \end{pmatrix},$$

and $\mathbf{B} = \begin{pmatrix} -2J & -J \\ -J & -2J \end{pmatrix}$. By the assumption that $x_1 = x_2 = s$, $y_1 = y_2 = t$, $x_3 = 1 - 2s$ and $y_3 = 1 - 2t$, \mathbf{A}_x and \mathbf{A}_y become symmetric matrices. In this case, $\mathbf{B}\mathbf{A}_y = \mathbf{A}_y\mathbf{B}$ and hence the determinant of the Hessian matrix becomes $\det(\mathbf{A}_x\mathbf{A}_y - \mathbf{B}^2)$. An elementary computation shows that

$$\begin{aligned} & \det \left(\nabla^2 (F_{\beta, J}(s, s, 1 - 2s, t, t, 1 - 2t)) - \lambda I \right) \\ &= \{ \lambda^2 + (2 - S_1 - T_1)\lambda + S_1T_1 - S_1 - T_1 + 1 - J^2 \} \\ & \quad \times \{ \lambda^2 + (6 - S_1 - T_1 - 2S_2 - 2T_2)\lambda + S_1T_1 + 2S_1T_2 \\ & \quad + 2S_2T_1 + 4S_2T_2 - 3S_1 - 3T_1 - 6S_2 - 6T_2 + 9 - 9J^2 \}, \end{aligned}$$

where $S_1 = \frac{1+J}{\beta s}$, $S_2 = \frac{1+J}{\beta(1-2s)}$ and $T_1 = \frac{1+J}{\beta t}$, $T_2 = \frac{1+J}{\beta(1-2t)}$. We denote the solutions of the first factor by λ_1, λ_2 and the solutions of the second factor by λ_3, λ_4 . Then, we may derive the following.

(1) The first result is calculated as

$$\begin{aligned} \lambda_1 + \lambda_2 &= S_1 + T_1 - 2 = J(\Phi'(s) + \Phi'(t)), \\ \lambda_1\lambda_2 &= S_1T_1 - S_1 - T_1 + 1 - J^2 = J^2(\Phi'(s)\Phi'(t) - 1), \end{aligned}$$

(2) The second result is calculated as

$$\begin{aligned} \lambda_3 + \lambda_4 &= S_1 + T_1 + 2S_2 + 2T_2 - 6 = 3J(\Psi'(s) + \Psi'(t)), \\ \lambda_3\lambda_4 &= S_1T_1 + 2S_1T_2 + 2S_2T_1 + 4S_2T_2 - 3S_1 - 3T_1 - 6S_2 - 6T_2 + 9 - 9J^2, \\ &= 9J^2(\Psi'(s)\Psi'(t) - 1). \end{aligned}$$

This completes the proof. □

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4.2 Two Extreme Cases: $J = \infty$ and $J = 0$

In this section, we prove Theorem 2.3.2 by categorizing all the critical points of $F_{\beta, J}$ according to the value of the inverse temperature β in both cases $J = \infty$ (i.e., $J_{11} = J_{22} = 0$ and $J_{12} = 1$), a case without component-wise interaction and $J = 0$ (i.e., $J_{12} = 0$ and $J_{11} = J_{22} = 1$), a case without inter-component interaction. For convenience, we denote the point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ by \mathbf{q}_3 . First, we consider the case when $J = \infty$.

Theorem 4.2.1. *Suppose that $J = \infty$. (i.e., $J_{11} = J_{22} = 0$ and $J_{12} = 1$), that is, there is no component-wise interaction in each block. Then, we have the following.*

1. *If $0 < \beta < \beta_1$, then there is only one local minimum at \mathbf{q}_3 .*
2. *If $\beta_1 < \beta < \beta_2$, then there is a global minimum at \mathbf{q}_3 , three local minima, and three saddle points.*
3. *If $\beta_2 < \beta < \beta_3$, then there is a local minimum at \mathbf{q}_3 , three global minima, and three saddle points.*
4. *If $\beta > \beta_3$, then there are three global minima and three saddle points.*

Proof. The function (4.1.1) becomes

$$F_{\beta}(\mathbf{x}, \mathbf{y}) = -\sum_{i=1}^3 x_i y_i + \frac{1}{\beta} \left(\sum_{i=1}^3 x_i \log x_i + y_i \log y_i \right). \quad (4.2.1)$$

The critical points must satisfy the following equations:

$$\begin{aligned} \frac{\partial F_{\beta}}{\partial x_k}(\mathbf{x}, \mathbf{y}) &= -(y_k - y_3) + \frac{1}{\beta} (\log x_k - \log x_3) = 0, \\ \frac{\partial F_{\beta}}{\partial y_k}(\mathbf{x}, \mathbf{y}) &= -(x_k - x_3) + \frac{1}{\beta} (\log y_k - \log y_3) = 0, \end{aligned} \quad (4.2.2)$$

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for $1 \leq k \leq 3$. Since the equations in (4.2.2) are symmetric, we consider the following system of equations:

$$y = \frac{1}{\beta} \log x + u \quad \text{and} \quad x = \frac{1}{\beta} \log y + v, \quad (4.2.3)$$

where u, v are real positive numbers. The function $\frac{1}{\beta} \log x$ is concave and increases; thus (4.2.3) has at most two solutions. Moreover, if $x_k = x_l$, then by (4.2.2), we have $y_k = y_l$, for $1 \leq k, l \leq 3$. Thus, all the critical points are of the form $(s, s, 1 - 2s, t, t, 1 - 2t)$ up to permutations. In fact, all of them are of the form $(s, s, 1 - 2s, s, s, 1 - 2s)$. This is because, from the equations

$$\begin{aligned} s &= \frac{1}{\beta} \log t + u \quad \text{and} \quad 1 - 2s = \frac{1}{\beta} \log(1 - 2t) + u, \\ t &= \frac{1}{\beta} \log s + v \quad \text{and} \quad 1 - 2t = \frac{1}{\beta} \log(1 - 2s) + v, \end{aligned}$$

we obtain

$$1 - 3s = \frac{1}{\beta} \log \frac{1 - 2t}{t} \quad \text{and} \quad 1 - 3t = \frac{1}{\beta} \log \frac{1 - 2s}{s}. \quad (4.2.4)$$

By subtracting the equations in (4.2.4), we have

$$1 - 3s + \frac{1}{\beta} \log \frac{1 - 2s}{s} = 1 - 3t + \frac{1}{\beta} \log \frac{1 - 2t}{t}.$$

Since the function $1 - 3x + \frac{1}{\beta} \log \frac{1 - 2x}{x}$ is decreasing, we have $s = t$.

Thus, all the critical points are of the form $(s, s, 1 - 2s, s, s, 1 - 2s)$ and from (4.2.4), we obtain the inverse temperature $\beta > 0$:

$$\beta = \frac{1}{1 - 3s} \log \frac{1 - 2s}{s}. \quad (4.2.5)$$

Hence, if $\beta > \beta_1$, then there are two types of critical points up to permu-

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tations, except \mathbf{q}_3 :

$$\mathbf{p}_1 = (x_s, x_s, 1-2x_s, x_s, x_s, 1-2x_s) \text{ and } \mathbf{p}_2 = (x_l, x_l, 1-2x_l, x_l, x_l, 1-2x_l),$$

A straightforward calculation gives the eigenvalues of the Hessian of (4.2.1) that

$$\lambda_1 = \frac{1}{\beta s} - 1, \quad \lambda_2 = \frac{1}{\beta s} + 1, \quad (4.2.6)$$

$$\lambda_3 = \frac{1}{\beta s(1-2s)} - 3, \text{ and } \lambda_4 = \frac{1}{\beta s(1-2s)} + 3 \quad (4.2.7)$$

Since λ_2 and λ_4 are always positive, we only need to know when $\lambda_1 > 0$ and $\lambda_3 > 0$. By substituting (4.2.5) into (4.2.6) and (4.2.7), we obtain that $\lambda_1 > 0$ if and only if $0 < s < \frac{1}{3}$, and $\lambda_3 > 0$ if and only if $0 < s < m_1$ or $\frac{1}{3} < s < \frac{1}{2}$. Therefore, regardless of the value of $\beta > \beta_1$, we can conclude that \mathbf{p}_1 is a local minimum, and \mathbf{p}_2 is a saddle point. The point \mathbf{q}_3 is a local minimum if $\beta < \beta_3$, but it is neither a local minimum nor a saddle point if $\beta > \beta_3$.

It remains to compare the local minima when $\beta_1 < \beta < \beta_2$ and $\beta_2 < \beta < \beta_3$. By symmetry, the function values at \mathbf{p}_1 up to permutations are the same. A straightforward computation yields

$$F_{\beta, J}(\mathbf{p}_1) - F_{\beta, J}(\mathbf{q}_3) = \tilde{F}(x_s),$$

where

$$\tilde{F}(x) = -6x^2 + 4x - \frac{2}{3} + \frac{2}{\beta} (2x \log x + (1-2x) \log(1-2x) + \log 3), \quad (4.2.8)$$

and $\beta = \xi(x)$. The function $\tilde{F}(x)$ is positive on $(\frac{1}{6}, \frac{1}{3})$ and is negative on $(0, \frac{1}{6})$ and $(\frac{1}{3}, \frac{1}{2})$. If $\beta_1 < \beta < \beta_2$, then $x_s \in (\frac{1}{6}, \frac{1}{3})$; hence $F_{\beta, J}(\mathbf{p}_1) > F_{\beta, J}(\mathbf{q}_3)$. However, if $\beta_2 < \beta < \beta_3$, then $x_s \in (0, \frac{1}{6})$; thus $F_{\beta, J}(\mathbf{p}_1) < F_{\beta, J}(\mathbf{q}_3)$. This completes the proof. \square

Next, we prove the other extreme case when $J = 0$.

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Theorem 4.2.2. *Suppose that $J = 0$. (i.e., $J_{11} = J_{22} = 1$ and $J_{12} = 0$), that is, there is no inter-component interaction between the two blocks. Then, we have the followings.*

1. *If $0 < \beta < \beta_1$, then there is only one local minimum at \mathbf{q}_3 .*
2. *If $\beta_1 < \beta < \beta_2$, then there is a global minimum at \mathbf{q}_3 , nine local minima and 18 saddle points.*
3. *If $\beta_2 < \beta < \beta_3$, then there is a local minimum at \mathbf{q}_3 , nine global minima and 18 saddle points*
4. *If $\beta > \beta_3$, then there is a local maximum at \mathbf{q}_3 , nine global minima, and 18 saddle points.*

Proof. The function (4.1.1) becomes a function of β :

$$F_\beta(\mathbf{x}, \mathbf{y}) = -\frac{1}{2} \sum_{i=1}^3 (x_i^2 + y_i^2) + \frac{1}{\beta} \sum_{i=1}^3 (x_i \log x_i + y_i \log y_i).$$

The critical points must satisfy the following equations:

$$\begin{aligned} \frac{\partial F_\beta}{\partial x_k}(\mathbf{x}, \mathbf{y}) &= -(x_k - x_3) + \frac{1}{\beta} (\log x_k - \log x_3) = 0, \\ \frac{\partial F_\beta}{\partial y_k}(\mathbf{x}, \mathbf{y}) &= -(y_k - y_3) + \frac{1}{\beta} (\log y_k - \log y_3) = 0, \end{aligned} \quad (4.2.9)$$

for $1 \leq k \leq 3$. Clearly, \mathbf{q}_3 is a critical point of F_β . First, we determine the form of all the critical points of F_β . Since the equations in (4.2.9) are symmetric, we consider the equation

$$x - \frac{1}{\beta} \log x = u, \quad \text{and} \quad y - \frac{1}{\beta} \log y = v \quad (4.2.10)$$

where $u, v > 0$. The function $x - \frac{1}{\beta} \log x$ is convex; thus (4.2.10) each of x and y has at most two solutions. Thus, all the critical points are of the

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form $(s, s, 1-2s, t, t, 1-2t)$ up to permutations and from (4.2.9), we obtain the inverse temperature

$$\beta = \xi(s) = \xi(t). \quad (4.2.11)$$

The function $\xi(x)$ has a unique minimum at $x = m_1$ and its value is β_1 . Hence, there is no critical point of the form $(s, s, 1-2s, t, t, 1-2t)$ when $\beta < \beta_1$. However, when $\beta > \beta_1$, from (4.2.11), s can be either $x_s(\beta)$ or $x_l(\beta)$; the same is true for t . Hence, if $\beta > \beta_1$, then there are four types of critical points up to permutations, except \mathbf{q}_3 :

$$\begin{aligned} \mathbf{p}_1 &= (x_s, x_s, 1-2x_s, x_s, x_s, 1-2x_s), & \mathbf{p}_2 &= (x_l, x_l, 1-2x_l, x_l, x_l, 1-2x_l), \\ \mathbf{p}_3 &= (x_s, x_s, 1-2x_s, x_l, x_l, 1-2x_l), & \mathbf{p}_4 &= (x_l, x_l, 1-2x_l, x_s, x_s, 1-2x_s). \end{aligned}$$

Since both the first and the last three coordinates of the aforementioned critical points behave independently, there are nine permutations for each. By substituting $J = 0$ in the proof of Lemma (4.1.2), we have

$$\lambda_1 = \frac{1}{\beta s} - 1, \quad \lambda_2 = \frac{1}{\beta t} - 1,$$

$$\lambda_3 = \frac{1}{\beta s(1-2s)} - 3, \quad \text{and} \quad \lambda_4 = \frac{1}{\beta t(1-2t)} - 3.$$

From the proof of Theorem 4.2.1, we know that $\lambda_1 > 0$ if and only if $0 < s < \frac{1}{3}$, and $\lambda_3 > 0$ if and only if $0 < s < m_1$ or $\frac{1}{3} < s < \frac{1}{2}$. The same is true for λ_2 and λ_4 . Therefore, regardless of the value of β , we can conclude that \mathbf{p}_1 is a local minimum, \mathbf{p}_2 is neither a local minimum nor a saddle point, and \mathbf{p}_3 and \mathbf{p}_4 are saddle points. The point \mathbf{q}_3 is a local minimum if $\beta < \beta_3$, but is a local maximum if $\beta > \beta_3$. The comparison of local minima is the same as the proof of Theorem 4.2.1. This completes the proof. \square

Remark 4.2.3. For any $q \geq 3$, the CWP model has a first-order phase transition at the critical temperature $\beta = \beta_2$, which was covered in [3, 7, 8, 15, 17]. We remark that the set of global minima is replaced at $\beta = \beta_2$ for

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the two-component case, and this critical temperature β_2 is independent of variable J . The third statement in Theorem 4.4.2 cover the general case of J .

4.3 General Case: Low-Temperature Regime ($\beta > \beta_3$)

In this section, we will find all the local minima and lowest saddles of $F_{\beta, J}$ which belong to either S^2 or L^2 , and the phase transition boundary for the low-temperature part of Theorem 2.4.2. Recall that we defined the function ψ_1 in (2.4.2). Then, the function ψ_1 has the following properties.

Proposition 4.3.1. *The function ψ_1 is a continuous function such that*

$$0 < \psi_1(\beta) < 1, \quad \psi_1(\beta_3) = 0, \quad \text{and} \quad \lim_{\beta \rightarrow \infty} \psi_1(\beta) = 1.$$

Proof. Note that x_l is a continuous function of β , and $x_l \in (\frac{1}{3}, \frac{1}{2})$ if $\beta > \beta_3$. Since $\psi_1(\beta)$ is a composition of continuous functions, it is continuous. It is obvious that $\psi_1(\beta) < 1$. Since $x_l > \frac{1}{3}$, we have $\beta = \frac{1}{1-3x_l} \log \frac{1-2x_l}{x_l} > \frac{1}{x_l}$, which means that $\psi_1(\beta) > 0$. Note that $\lim_{\beta \rightarrow \beta_3} x_l = \frac{1}{3}$ and $\lim_{\beta \rightarrow \infty} x_l = \frac{1}{2}$. This implies that $\psi_1(\beta_3) = \lim_{\beta \rightarrow \beta_3} \psi_1(\beta) = 0$ and $\lim_{\beta \rightarrow \infty} \psi_1(\beta) = 1$. This completes the proof. \square

Proposition 4.3.2. *Suppose that $u = v$ in (4.1.2) and there are two intersections: $\mathbf{P} = (P, P)$ and $\mathbf{Q} = (Q, Q)$ with $P < Q$. Suppose further that $\Phi'(Q) = -1$. Then, $J > \psi_1(\beta)$ if and only if $P + 2Q > 1$. In particular, $J = \psi_1(\beta)$ if and only if $P + 2Q = 1$.*

Proof. Suppose that $P + 2Q > 1$. Since $P < Q$, we have $Q > \frac{1}{3}$. The condition $\Phi'(Q) = -1$ is equivalent to $Q = \frac{1+J}{\beta(1-J)}$. By the definitions of P and Q , we have

$$P - \frac{1}{\beta} \log P = Q - \frac{1}{\beta} \log Q. \quad (4.3.1)$$

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The function $x - \frac{1}{\beta} \log x$ is convex and has a minimum at $x = \frac{1}{\beta}$; hence $P < \frac{1}{\beta} < Q$. Since $x - \frac{1}{\beta} \log x$ is decreasing on $(0, \frac{1}{\beta})$, from the inequality $1 - 2Q < P$ and (4.3.1), we obtain

$$\beta < \frac{1}{1 - 3Q} \log \frac{1 - 2Q}{Q}. \quad (4.3.2)$$

Thus, $Q > x_l = x_l(\beta)$, or equivalently, we obtain

$$J > \psi_1(\beta) = \frac{\beta - \frac{1}{x_l}}{\beta + \frac{1}{x_l}}.$$

On the other hand, we assume that $J > \psi_1(\beta)$, which is equivalent to $Q > x_l$. Since $x_l \in (\frac{1}{3}, \frac{1}{2})$ for $\beta > \beta_3$, we have

$$\psi_1(\beta) - \frac{\beta - 3}{\beta + 3} = \frac{2\beta(3 - 1/x_l)}{(\beta + 1/x_l)(\beta + 3)} \geq 0 \quad \text{for all } \beta > 0.$$

Hence, $J > \frac{\beta - 3}{\beta + 3}$ and this implies that $Q > \frac{1}{3}$. Since $Q > x_l$, (4.3.2) holds. By (4.3.1), the inequality (4.3.2) becomes

$$1 - 2Q - \frac{1}{\beta} \log(1 - 2Q) < Q - \frac{1}{\beta} \log Q = P - \frac{1}{\beta} \log P.$$

This implies that $1 - 2Q < P$. This completes the proof. \square

Note that by symmetry, the function values at \mathbf{x}_s or \mathbf{x}_l up to permutations are the same, respectively. Then, we have the following theorem for the low-temperature regime.

Theorem 4.3.3. *Suppose that $\beta > \beta_3$. Then, we have the following results.*

1. *If $J > \psi_1(\beta)$, then $F_{\beta, J}$ has three local minima at $\mathbf{x}_s \in S^2$ and three saddle points at $\mathbf{x}_l \in L^2$.*
2. *If $J < \psi_1(\beta)$, then there must be a lowest saddle of $F_{\beta, J}$ that does not belong to either S^2 or L^2 .*

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Proof. (1) From the definitions of x_s and x_l , we have

$$\beta = \frac{1}{1-3x_s} \log \frac{1-2x_s}{x_s} = \frac{1}{1-3x_l} \log \frac{1-2x_l}{x_l}.$$

Note that the function $\frac{1}{x} - \frac{1}{1-3x} \log \frac{1-2x}{x}$ is positive on $0 < x < \frac{1}{3}$. Since $x_s < \frac{1}{3}$ and since $x_l > \frac{1}{3}$, we obtain the inequality

$$x_s < \frac{1}{\beta} < x_l. \quad (4.3.3)$$

To analyze the critical points, we need to investigate the slopes of Φ and Ψ at $x = x_s$ and $x = x_l$. When we look at the graph $\Psi(x) = x$, it has solutions in the order of x_s , $\frac{1}{3}$ and x_l . It follows that $\Psi'(x_s) > 1$ and $\Psi'(x_l) > 1$. Since $\Phi'(\frac{1}{\beta}) = 1$, the inequality (4.3.3) implies that $\Phi'(x_s) > 1$ and $\Phi'(x_l) < 1$. $F_{\beta, J}$ The assumption $J > \psi_1(\beta)$ is equivalent to

$$x_l < \frac{1+J}{\beta(1-J)}. \quad (4.3.4)$$

The right hand side of (4.3.4) is the point where the function Φ has a slope of -1 . Thus, we have $-1 < \Phi'(x_l) < 1$ and by Lemma 4.1.2, $F_{\beta, J}$ has local minima at \mathbf{x}_s and saddle points at \mathbf{x}_l . This proves (1).

(2) The condition $J < \psi_1(\beta)$ is equivalent to $x_l > \frac{1+J}{\beta(1-J)}$. This means that $\Phi'(x_l) < -1$. Thus, by Lemma 4.1.2, \mathbf{x}_l is no longer a saddle point, and hence, there is no saddle point belonging to either S^2 or L^2 . However, according to Morse's theory, when there are two or more local minima, there must be a saddle point with the lowest level connecting them. Thus, there must be a saddle point that does not belong to either S^2 or L^2 . This completes the proof. \square

Lemma 4.3.4. *Suppose that $u = v$ in (4.1.2) and suppose that (4.1.2) intersects only two points at $\mathbf{P} = (P, P)$ and $\mathbf{Q} = (Q, Q)$ with $P < Q$. Then $2P + Q$ and $P + 2Q$ are increasing functions with respect to u .*

Proof. Note that Q is increasing but P is decreasing with respect to u .

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Thus, it suffices to show that $2P + Q$ is increasing with respect to u . From the definitions of P and Q , we have

$$\begin{aligned} P &= \Phi(P) + u = \frac{1}{J} \left(-P + \frac{1+J}{\beta} \log P \right) + u, \\ Q &= \Phi(Q) + u = \frac{1}{J} \left(-Q + \frac{1+J}{\beta} \log Q \right) + u. \end{aligned} \quad (4.3.5)$$

We can replace u with $\frac{1+J}{J}u$ and P, Q with $\beta P, \beta Q$, respectively. Hence, the equations in (4.3.5) become

$$\beta P - \log \beta P = \beta u - \log \beta \quad \text{and} \quad \beta Q - \log \beta Q = \beta u - \log \beta. \quad (4.3.6)$$

After scaling and translating the equations in (4.3.6), it suffices to consider the system of equations

$$P - \log P = u \quad \text{and} \quad Q - \log Q = u, \quad (4.3.7)$$

By differentiating (4.3.7) with respect to u , the sum of the derivatives of P and Q is

$$2 \frac{\partial P}{\partial u} + \frac{\partial Q}{\partial u} = \frac{2P}{P-1} + \frac{Q}{Q-1} = \frac{3PQ - 2P - Q}{(P-1)(Q-1)}.$$

Since P is decreasing and Q is increasing with respect to u , we have $P-1 < 0$ and $Q-1 > 0$. Since $0 \leq P, Q \leq 1$, we have $2P + Q - 3PQ \geq 0$. This completes the proof. \square

Theorem 4.3.5. *Suppose that $u = v$ in (4.1.2) and suppose that the graphs in (4.1.2) intersects at four points at $\mathbf{P} = (P, P)$, $\mathbf{R} = (R, S)$, $\mathbf{S} = (S, R)$, and $\mathbf{Q} = (Q, Q)$ with $P \leq R \leq Q \leq S$. Let J_c be the positive root of the function $(1+x) \left(\frac{1}{1-x} - \frac{1}{2+x} \right) - \log \frac{2+x}{1-x}$.*

1. *If $J \geq J_c$, then $P + R + S$ increases with respect to u .*
2. *The derivative $\frac{\partial(P+R+S)}{\partial u}$ increases with respect to u . That is, the func-*

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tion $(P + R + S)(u)$ is convex.

Proof. (1) Since

$$\Phi\left(\frac{(1+J)x}{\beta}\right) + u = \frac{1+J}{\beta} \left(\frac{1}{J}(-x + \log x) + \log \frac{1+J}{\beta} + \frac{\beta u}{1+J} \right),$$

by scaling and translating the function $\Phi(x)$ and the variable u , it suffices to consider the system of equations:

$$\tilde{\Phi}(P) + \frac{w}{J} = P, \quad \tilde{\Phi}(R) + \frac{w}{J} = S \quad \text{and} \quad \tilde{\Phi}(S) + \frac{w}{J} = R,$$

where

$$\tilde{\Phi}(x) = \frac{1}{J}(-x + \log x). \quad (4.3.8)$$

An elementary computation shows that $\frac{\partial(P+R+S)}{\partial w} > 0$ is equivalent to

$$(\tilde{P} - J)(\tilde{R} + \tilde{S} + 2J) - (J^2 - \tilde{R}\tilde{S}) > 0, \quad (4.3.9)$$

where $\tilde{P} := -1 + \frac{1}{P}$, $\tilde{R} := -1 + \frac{1}{R}$ and $\tilde{S} := -1 + \frac{1}{S}$. We denote the geometric mean of R and S by $\Gamma_{RS} := \sqrt{RS}$, and $\tilde{\Gamma}_{RS} := -1 + \frac{1}{\Gamma_{RS}}$. Then, we can rearrange (4.3.9) as

$$\begin{aligned} & (\tilde{P} - J)\{2(J + \tilde{\Gamma}_{RS}) + (\tilde{R} - \tilde{\Gamma}_{RS}) + (\tilde{S} - \tilde{\Gamma}_{RS})\} \\ & + \{(\tilde{R} - \tilde{\Gamma}_{RS}) + \tilde{\Gamma}_{RS}\}\{(\tilde{S} - \tilde{\Gamma}_{RS}) + \tilde{\Gamma}_{RS}\} - J^2 > 0. \end{aligned} \quad (4.3.10)$$

Note that

$$\begin{aligned} \tilde{R} - \tilde{\Gamma}_{RS} &= \left(-1 + \frac{1}{R}\right) - \left(-1 + \frac{1}{\Gamma_{RS}}\right) = \frac{1}{\sqrt{R}} \left(\frac{1}{\sqrt{R}} - \frac{1}{\sqrt{S}} \right), \\ \tilde{S} - \tilde{\Gamma}_{RS} &= \left(-1 + \frac{1}{S}\right) - \left(-1 + \frac{1}{\Gamma_{RS}}\right) = \frac{1}{\sqrt{S}} \left(\frac{1}{\sqrt{S}} - \frac{1}{\sqrt{R}} \right). \end{aligned}$$

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Substituting the aforementioned equations in (4.3.10), we obtain

$$(J + \tilde{\Gamma}_{RS})(2\tilde{P} + \tilde{\Gamma}_{RS} - 3J) + (\tilde{P} - J - 1) \left(\frac{1}{\sqrt{R}} - \frac{1}{\sqrt{S}} \right)^2 > 0. \quad (4.3.11)$$

Then, by Lemma 4.3.6, we acquire the desired results.

(2) We claim that (4.3.11) increases with respect to u . Since R and S are getting farther as u increases, it suffices to prove that both \tilde{P} and $\tilde{\Gamma}_{RS}$ are increasing. \tilde{P} is increasing since P is decreasing. If we show that RS is decreasing, then $\tilde{\Gamma}_{RS}$ increases with respect to u , achieving the required result. An elementary computation shows that

$$\frac{\partial(RS)}{\partial u} = \frac{\partial R}{\partial u}S + \frac{\partial S}{\partial u}R = \frac{1}{J^2 - \tilde{R}\tilde{S}} (2 - (1 - J)(R + S)).$$

By considering the graphs in (4.1.2), we observe $R + S$ increasing with respect to u . Since $R + S$ has a unique minimum at $R = S = Q = \frac{1}{1-J}$, we have $\frac{\partial(RS)}{\partial u} < 0$. This completes the proof. \square

Lemma 4.3.6. *Under the notations introduced in Theorem 4.3.5, we have*

1. $J + \tilde{\Gamma}_{RS} \geq 0$,
2. $\tilde{P} > 3J$,
3. $\tilde{P} - J - 1 \geq 0$ if $J \geq J_c$.

Proof. (1) From the definitions of R and S , we have

$$S = \tilde{\Phi}(R) + \frac{w}{J} \quad \text{and} \quad R = \tilde{\Phi}(S) + \frac{w}{J}.$$

Subtracting the first equation from the second equation yields

$$(1 - J)(S - R) = \log S - \log R = \int_R^S \frac{1}{t} dt.$$

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Since the sum of the area of the trapezoid with vertices $(R, 0)$, $(\Gamma_{RS}, 0)$, $(R, \frac{1}{R})$, $(\Gamma_{RS}, \frac{1}{\Gamma_{RS}})$, and the other one with vertices $(\Gamma_{RS}, 0)$, $(S, 0)$, $(\Gamma_{RS}, \frac{1}{\Gamma_{RS}})$, $(S, \frac{1}{S})$ is greater than the definite integral $\int_R^S \frac{1}{t} dt$, we obtain

$$\begin{aligned} (1 - J)(S - R) &= \int_R^S \frac{1}{t} dt \leq \frac{1}{2} \left(\frac{1}{R} + \frac{1}{\Gamma_{RS}} \right) (\Gamma_{RS} - R) \\ &\quad + \frac{1}{2} \left(\frac{1}{\Gamma_{RS}} + \frac{1}{S} \right) (S - \Gamma_{RS}) \\ &= \frac{1}{\Gamma_{RS}} (S - R). \end{aligned}$$

This proves (1).

(2) Similar to aforementioned proof, by comparing the area of the trapezoid and the definite integral, we have

$$(1 + J)(Q - P) = \log Q - \log P = \int_P^Q \frac{1}{t} dt \leq \frac{1}{2} \left(\frac{1}{P} + \frac{1}{Q} \right) (Q - P).$$

Thus, $\tilde{P} + \tilde{Q} \geq 2J$. However, $\tilde{Q}/J = \tilde{\Phi}'(Q) < -1$, which is equivalent to $\tilde{Q} < -J$. This proves (2).

(3) Define a function $h : [0, \infty) \rightarrow \mathbb{R}$ by $h(x) = (1 + J)x - \log x$. By the definitions of P and Q , we have $h(P) = h(Q)$. The condition $A - J - 1 \geq 0$ is equivalent to $P \leq \frac{1}{2+J}$. Note that the function $h(x)$ is decreasing on $[0, \frac{1}{1+J})$ and is increasing on $[\frac{1}{1+J}, \infty)$. Thus, we need to show that

$$h(P) \geq h\left(\frac{1}{2+J}\right).$$

We know that $Q \geq \frac{1}{1-J}$; hence, we obtain $h(P) = h(Q) \geq h\left(\frac{1}{1-J}\right)$. Consider the following function:

$$h\left(\frac{1}{1-J}\right) - h\left(\frac{1}{2+J}\right) = (1+J) \left(\frac{1}{1-J} - \frac{1}{2+J} \right) - \log \frac{2+J}{1-J}. \quad (4.3.12)$$

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The right-hand side of (4.3.12) is nonnegative if $J \geq J_c$, where J_c is the positive root of (4.3.12). This implies that $h(P) = h(Q) \geq h\left(\frac{1}{1-J}\right) \geq h\left(\frac{1}{2+J}\right)$ if $J \geq J_c$. This completes the proof. \square

Theorem 4.3.7. *Suppose that $u \neq v$ in (4.1.2) and there are four intersections at $\mathbf{P} = (P_1, P_2)$, $\mathbf{R} = (R_1, R_2)$, $\mathbf{S} = (S_1, S_2)$, and $\mathbf{Q} = (Q_1, Q_2)$, with $P_1 \leq R_1 \leq Q_1 \leq S_1$. Then, for a fixed v ,*

1. $P_2 + R_2 + S_2$ increases with respect to u .
2. $R_1 + S_1 + Q_1$ increases with respect to u .
3. $P_1 + S_1 + Q_1$ increases with respect to u .
4. $P_2 + R_2 + Q_2 > P_2 + R_2 + S_2$.

Proof. Without loss of generality, we may assume that $u > v$. By setting $u := v + w$, for $w > 0$, the equations in (4.1.2) become

$$y = \Phi(x) + v + w \quad \text{and} \quad x = \Phi(y) + v,$$

where $v, w > 0$. As in the proof of Theorem 4.3.5, by scaling and translating the function Φ , it suffices to consider

$$y = \tilde{\Phi}(x) + \frac{v+w}{J} \quad \text{and} \quad x = \tilde{\Phi}(y) + \frac{v}{J},$$

where $\tilde{\Phi}$ was defined in (4.3.8). We will use the notation $\tilde{P}_i = -1 + \frac{1}{P_i}$ for $i = 1, 2$, and the same applies to the other functions.

(1) Since R_2 is increasing with respect to w , it suffices to show that $P_2 + S_2$ is increasing with respect to w . By the definitions of P_1, P_2, S_1 , and S_2 , we have

$$P_2 = \tilde{\Phi}(P_1) + \frac{v+w}{J} \quad \text{and} \quad P_1 = \tilde{\Phi}(P_2) + \frac{v}{J}, \quad (4.3.13)$$

$$S_2 = \tilde{\Phi}(S_1) + \frac{v+w}{J} \quad \text{and} \quad S_1 = \tilde{\Phi}(S_2) + \frac{v}{J}. \quad (4.3.14)$$

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Differentiating (4.3.13) with respect to w , we obtain

$$\frac{\partial P_1}{\partial w} = \frac{\tilde{P}_2}{J^2 - \tilde{P}_1 \tilde{P}_2}, \quad \text{and} \quad \frac{\partial P_2}{\partial w} = \frac{1}{J^2 - \tilde{P}_1 \tilde{P}_2}.$$

Similar equations can be obtained for functions S_1 and S_2 by differentiating (4.3.14). Since P_2 is decreasing and S_2 is increasing with respect to w , we have $J^2 - \tilde{P}_1 \tilde{P}_2 < 0$ and $J^2 - \tilde{S}_1 \tilde{S}_2 > 0$. Thus, we have to show that

$$\frac{\partial P_2}{\partial w} + \frac{\partial S_2}{\partial w} \geq 0, \quad \text{or equivalently,} \quad \tilde{P}_1 \tilde{P}_2 + \tilde{S}_1 \tilde{S}_2 - 2J^2 \geq 0.$$

From (4.3.13) and (4.3.14), we obtain the following:

$$\begin{aligned} J(S_2 - P_2) &= -S_1 + \log S_1 + P_1 - \log P_1 = \int_{P_1}^{S_1} \left(-1 + \frac{1}{t} \right) dt, \\ J(S_1 - P_1) &= -S_2 + \log S_2 + P_2 - \log P_2 = \int_{P_2}^{S_2} \left(-1 + \frac{1}{t} \right) dt. \end{aligned}$$

Since the area of the trapezoid with vertices $(P_1, 0)$, $(S_1, 0)$, $(P_1, \frac{1}{P_1})$, and $(S_1, \frac{1}{S_1})$ is greater than the definite integral $\int_{P_1}^{S_1} \frac{1}{t} dt$, we have

$$\begin{aligned} &\int_{P_1}^{S_1} \left(-1 + \frac{1}{t} \right) dt \\ &\leq \frac{1}{2} \left(\frac{1}{P_1} + \frac{1}{S_1} \right) (S_1 - P_1) - (S_1 - P_1) \\ &= \frac{1}{2} (\tilde{P}_1 + \tilde{S}_1) (S_1 - P_1). \end{aligned}$$

Thus, we obtain

$$J(S_2 - P_2) \leq \frac{1}{2} (\tilde{P}_1 + \tilde{S}_1) (S_1 - P_1). \quad (4.3.15)$$

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Similarly, we have

$$J(S_1 - P_1) \leq \frac{1}{2}(\tilde{P}_2 + \tilde{S}_2)(S_2 - P_2). \quad (4.3.16)$$

Multiplying (4.3.15) and (4.3.16), we obtain the inequality

$$J^2 \leq \frac{1}{4}(\tilde{P}_1 + \tilde{S}_1)(\tilde{P}_2 + \tilde{S}_2). \quad (4.3.17)$$

Using (4.3.17), we have the following estimation:

$$\begin{aligned} 2(\tilde{P}_1\tilde{P}_2 + \tilde{S}_1\tilde{S}_2 - 2J^2) &= 2(\tilde{P}_1\tilde{P}_2 + \tilde{S}_1\tilde{S}_2 - 2J^2) \\ &\quad - (\tilde{P}_1 + \tilde{S}_1)(\tilde{P}_2 + \tilde{S}_2) + (\tilde{P}_1 + \tilde{S}_1)(\tilde{P}_2 + \tilde{S}_2) \\ &= (\tilde{P}_1 - \tilde{S}_1)(\tilde{P}_2 - \tilde{S}_2) \\ &\quad + (\tilde{P}_1 + \tilde{S}_1)(\tilde{P}_2 + \tilde{S}_2) - 4J^2 \\ &= \left(\frac{1}{P_1} - \frac{1}{S_1}\right) \left(\frac{1}{P_2} - \frac{1}{S_2}\right) \\ &\quad + (\tilde{P}_1 + \tilde{S}_1)(\tilde{P}_2 + \tilde{S}_2) - 4J^2 \geq 0, \end{aligned}$$

since $P_1 < P_2 < S_2 < S_1$. This proves (1).

(2) Since S_1 is increasing with respect to w , it suffices to show that $R_1 + Q_1$ is increasing with respect to w . Hence, we need to show that

$$\frac{\partial R_1}{\partial w} + \frac{\partial Q_1}{\partial w} \geq 0, \text{ or equivalently, } J^2(\tilde{R}_2 + \tilde{Q}_2) - \tilde{R}_2\tilde{Q}_2(\tilde{R}_1 + \tilde{Q}_1) \leq 0. \quad (4.3.18)$$

Since $R_2 > 1$ and $Q_2 > 1$, we have $\tilde{R}_2 < 0$, $\tilde{Q}_2 < 0$. Hence, if $\tilde{R}_1 + \tilde{Q}_1 \geq 0$, then (4.3.18) holds. Now, we assume that $\tilde{R}_1 + \tilde{Q}_1 < 0$. By the same argument in the proof of (1), we have

$$J(R_2 - Q_2) = \int_{Q_1}^{R_1} \left(-1 + \frac{1}{t}\right) dt \geq -\frac{1}{2}(\tilde{R}_1 + \tilde{Q}_1)(Q_1 - R_1),$$

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$$J(Q_1 - R_1) = \int_{R_2}^{Q_2} \left(-1 + \frac{1}{t} \right) dt \geq -\frac{1}{2}(\tilde{R}_2 + \tilde{Q}_2)(R_2 - Q_2).$$

Since $Q_1 - R_1 > 0$ and $R_2 - Q_2 > 0$, multiplying the aforementioned inequalities, we obtain

$$J^2 \geq \frac{1}{4}(\tilde{R}_1 + \tilde{Q}_1)(\tilde{R}_2 + \tilde{Q}_2). \quad (4.3.19)$$

By (4.3.19), an elementary computation yields

$$J^2 (\tilde{R}_2 + \tilde{Q}_2) - \tilde{R}_2 \tilde{Q}_2 (\tilde{R}_1 + \tilde{Q}_1) \leq \frac{1}{4}(\tilde{R}_2 - \tilde{Q}_2)^2 (\tilde{R}_1 + \tilde{Q}_1) \leq 0.$$

This proves (2).

(3) It suffices to verify that $P_1 + Q_1$ is increasing with respect to u . Thus, we have to show that

$$\frac{\partial P_1}{\partial w} + \frac{\partial Q_1}{\partial w} \geq 0 \text{ or equivalently, } J^2(\tilde{P}_2 + \tilde{Q}_2) - \tilde{P}_2 \tilde{Q}_2 (\tilde{P}_1 + \tilde{Q}_1) \geq 0. \quad (4.3.20)$$

Since $P_1 < P_2 < 1 < Q_2 < Q_1$, we have $\tilde{P}_1, \tilde{P}_2 > 0$ and $\tilde{Q}_1, \tilde{Q}_2 < 0$. By the same argument in the proof of (2), we obtain

$$\begin{aligned} J(Q_2 - P_2) &= \int_{P_1}^{Q_1} \left(-1 + \frac{1}{t} \right) dt \leq \frac{1}{2}(\tilde{P}_1 + \tilde{Q}_1)(Q_1 - P_1), \\ J(Q_1 - P_1) &= \int_{P_2}^{Q_2} \left(-1 + \frac{1}{t} \right) dt \leq \frac{1}{2}(\tilde{P}_2 + \tilde{Q}_2)(Q_2 - P_2). \end{aligned}$$

Each left-hand side of the aforementioned equations is positive; thus, we obtain $\tilde{P}_1 + \tilde{Q}_1 > 0$ and $\tilde{P}_2 + \tilde{Q}_2 > 0$. Therefore, (4.3.20) holds. This proves (3).

(4) This is obviously true. This completes the proof. \square

4.4 General Case: Middle-Temperature Regime

$$(\beta_1 < \beta < \beta_3)$$

In this section, we investigate all the local minima and lowest saddles of $F_{\beta, J}$ belonging to either S^2 or L^2 for the middle-temperature and compare the values of the local minima according to the temperature. Furthermore, we specify the synchronization boundary and the desynchronization boundary, respectively. Recall the definition of the function ψ_2 is in (2.4.2). Then, the function ψ_2 has the following properties.

Proposition 4.4.1. *The function ψ_2 is a continuous function such that*

$$0 < \psi_2(\beta) < \frac{1}{10} \quad \text{and} \quad \psi_2(\beta_1) = \psi_2(\beta_3) = 0.$$

Proof. Note that x_l is a continuous function of β and $x_l \in (m_1, \frac{1}{3})$ if $\beta_1 < \beta < \beta_3$. Since ψ_2 is a composition of continuous functions, it is continuous. Since $m_1 < x_l < \frac{1}{3}$, we have

$$\beta - \frac{1}{3x_l(1-2x_l)} = \frac{1}{1-3x_l} \log \frac{1-2x_l}{x_l} - \frac{1}{3x_l(1-2x_l)} > 0.$$

Thus, $\psi_2(\beta) > 0$ for $\beta_1 < \beta < \beta_3$. By the definition of β , the inequality $\psi_2(\beta) < \frac{1}{10}$ is equivalent to

$$\frac{9}{1-3x_l} \log \frac{1-2x_l}{x_l} < \frac{11}{3x_l(1-2x_l)}. \quad (4.4.1)$$

An elementary calculation shows that (4.4.1) holds when $0 < x_l < \frac{1}{3}$; hence, $\psi_2(\beta) < \frac{1}{10}$. From the numerator of (4.1.5), m_1 satisfies the equation

$$1 - 3m_1 + 3m_1(1 - 2m_1) \log \frac{m_1}{1 - 2m_1} = 0.$$

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By the definition of β_1 , 4.4 becomes

$$\beta_1 = \frac{1}{1 - 3m_1} \log \frac{1 - 2m_1}{m_1} = \frac{1}{3m_1(1 - 2m_1)}.$$

Since $\lim_{\beta \rightarrow \beta_1} x_l = m_1$, we have $\psi_2(\beta_1) = \lim_{\beta \rightarrow \beta_1} \psi_2(\beta) = 0$. In addition, $\lim_{\beta \rightarrow \beta_3} x_l = \frac{1}{3}$ implies that $\psi_2(\beta_3) = \lim_{\beta \rightarrow \beta_3} \psi_2(\beta) = 0$. This completes the proof. \square

Theorem 4.4.2. *Suppose that $\beta_1 < \beta < \beta_3$. Then, we have the following results.*

1. *If $J > \psi_2(\beta)$, then there is a local minimum at \mathbf{q}_3 , three local minima at $\mathbf{x}_s \in S^2$ and three saddle points at $\mathbf{x}_l \in L^2$.*
2. *If $J < \psi_2(\beta)$, then there is a local minimum at \mathbf{q}_3 , three local minima at $\mathbf{x}_s \in S^2$ and six saddle points of the form $(s, s, 1 - 2s, t, t, 1 - 2t)$ and $(t, t, 1 - 2t, s, s, 1 - 2s)$ with $s < t$ up to permutations*
3. *(Comparison of local minima) If $\beta_1 < \beta < \beta_2$, then $F_{\beta, J}(\mathbf{q}_3) < F_{\beta, J}(\mathbf{x}_s)$, and if $\beta_2 < \beta < \beta_3$, then $F_{\beta, J}(\mathbf{q}_3) > F_{\beta, J}(\mathbf{x}_s)$.*

Proof. Note that the assumption $\beta_1 < \beta < \beta_3$ implies that $\Psi'(x_l) < 1$ and $x_s < x_l < \frac{1}{3} < \frac{1}{\beta}$.

(1) If $J > \psi_2(\beta)$, which is equivalent to $\Psi'(x_l) > -1$, then the function $y = \Psi(x)$ intersects only at (x_s, x_s) , (x_l, x_l) , and $(\frac{1}{3}, \frac{1}{3})$ with $x = \Psi(y)$. By observing the graphs in (4.1.3), it follows that $\Psi'(\frac{1}{3}) > 1$, and $\Psi'(x_s) > 1$. Moreover, we know that $-1 < \Psi'(x_l) < 1$. Since $\Phi'(\frac{1}{\beta}) = 1$ and since $x_s < x_l < \frac{1}{3} < \frac{1}{\beta}$, it follows that $\Phi'(\frac{1}{3}) > 1$, $\Phi'(x_s) > 1$, and $\Phi'(x_l) > 1$. Therefore, $F_{\beta, J}$ has local minima at \mathbf{q}_3 and $\mathbf{x}_s \in S^2$, and it has saddle points at $\mathbf{x}_l \in L^2$ by Lemma 4.1.2. This proves (1).

(2) If $J < \psi_2(\beta)$, which is equivalent to $\Psi'(x_l) < -1$, then $y = \Psi(x)$ and $x = \Psi(y)$ intersect not only at the original three points, but also at (s, t) and (t, s) with $s < x_l < t < \frac{1}{3}$. Since $\Psi'(x_l) < -1$, \mathbf{x}_l is no longer a saddle point of $F_{\beta, J}$ by Lemma 4.1.2. We claim that $(s, s, 1 - 2s, t, t, 1 - 2t)$ and

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$(t, t, 1-2t, s, s, 1-2s)$ are six new saddle points of $F_{\beta, J}$ up to permutations. Since $s < \frac{1}{\beta}$ and $t < \frac{1}{\beta}$, we have $\Phi'(s) > 1$ and $\Phi'(t) > 1$. By the inverse function theorem, we obtain $\Psi'(s)\Psi'(t) < 1$. Therefore, $F_{\beta, J}$ has saddle points at $(s, s, 1-2s, t, t, 1-2t)$ and $(t, t, 1-2t, s, s, 1-2s)$ by Lemma 4.1.2. The arguments for \mathbf{q}_3 and \mathbf{x}_s are the same as (1). This proves (2).

(3) A straightforward computation yields

$$F_{\beta, J}(\mathbf{x}_s) - F_{\beta, J}(\mathbf{q}_3) = (1 + J)\tilde{F}(x_s),$$

where $\tilde{F}(x)$ was defined in (4.2.8). The rest of the argument is the same as the proof of Theorem 4.2.1. This completes the proof. \square

Recall that we defined the function ψ_3 in (2.4.2). Then, we have the following theorem.

Theorem 4.4.3. *Under the assumptions in Theorem 4.3.5, if $J > \psi_3(\beta)$, then the function $(P + R + S)(u) > 1$ for all $u > 0$.*

Proof. By Theorem 4.3.5, we know that $(P + R + S)(u)$ is convex; thus it has a unique minimum. We claim that the minimum value is greater than one when $J > \psi_3(\beta)$. For convenience, we set $\tilde{P}_{\beta, J} := -1 + \frac{1+J}{\beta P}$, and the same applies to the other functions. A straightforward calculation shows that

$$P'(u) = \frac{J}{J - \tilde{P}_{\beta, J}}, \quad R'(u) = \frac{J(J + \tilde{S}_{\beta, J})}{J^2 - \tilde{R}_{\beta, J}\tilde{S}_{\beta, J}}, \quad \text{and} \quad S'(u) = \frac{J(J + \tilde{R}_{\beta, J})}{J^2 - \tilde{R}_{\beta, J}\tilde{S}_{\beta, J}};$$

hence,

$$\begin{aligned} (P + R + S)'(u) = 0 & \text{ is equivalent to} \\ (\tilde{P}_{\beta, J} - J)(\tilde{R}_{\beta, J} + \tilde{S}_{\beta, J} + 2J) - (J^2 - \tilde{R}_{\beta, J}\tilde{S}_{\beta, J}) &= 0. \end{aligned}$$

Applying the method of Lagrange multipliers to

$$G(P, R, S) = P + R + S,$$

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$$H(P, R, S) = (\tilde{P}_{\beta,J} - J)(\tilde{R}_{\beta,J} + \tilde{S}_{\beta,J} + 2J) - (J^2 - \tilde{R}_{\beta,J}\tilde{S}_{\beta,J}) = 0,$$

yields the following equations:

$$\frac{1}{P^2}(2J + \tilde{R}_{\beta,J} + \tilde{S}_{\beta,J}) = \frac{1}{R^2}(\tilde{P}_{\beta,J} + \tilde{S}_{\beta,J} - J) = \frac{1}{S^2}(\tilde{P}_{\beta,J} + \tilde{R}_{\beta,J} - J). \quad (4.4.2)$$

Calculating the equation of the second and the third in (4.4.2), we obtain

$$R + S = \frac{\frac{1+J}{\beta}}{J + 2 - \frac{1+J}{\beta P}}.$$

Therefore, G becomes a function of P :

$$G(P, R, S) = P + R + S = \frac{(J + 2)\frac{1+J}{\beta}}{\left(J + 2 - \frac{1+J}{\beta P}\right)\frac{1+J}{\beta P}}. \quad (4.4.3)$$

The denominator of (4.4.3) is a quadratic function of $\frac{1+J}{\beta P}$; hence it has a maximum value at $\frac{1+J}{\beta P} = \frac{J+2}{2}$. However, if we apply the argument in the proof of (2) in Lemma 4.3.6 without scaling and translating $\Phi(x)$, then we obtain $\tilde{P}_{\beta,J} > 3J$. This implies that $\frac{1+J}{\beta P} \geq 3J + 1$. Since $3J + 1 \geq \frac{J+2}{2}$, (4.4.3) has a minimum at $\frac{1+J}{\beta P} = 3J + 1$, and its value of G is $\frac{(1+J)(2+J)}{\beta(1-2J)(1+3J)}$. Therefore, if

$$\frac{(1+J)(2+J)}{\beta(1-2J)(1+3J)} > 1, \quad (4.4.4)$$

then $(P + R + S)(u) > 1$ for all u . Solving the inequality (4.4.4) for J , we obtain $J > \psi_3(\beta)$. This completes the proof. \square

4.5 General Case: High-Temperature Regime

$$(\beta < \beta_1)$$

In this section, we show that the two components are synchronized in high-temperature by proving that there is only one local minimum at \mathbf{q}_3 .

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Theorem 4.5.1. *If $0 < \beta < \beta_1$, then there is only one global minimum at \mathbf{q}_3 .*

Proof. The fact that \mathbf{q}_3 is a local minimum is straightforward by Lemma 4.1.2. Since $\beta < \beta_1$, by the definitions of S^2 and L^2 , there is no critical point belonging to either S^2 or L^2 . We claim that there is no other kind of critical points other than \mathbf{q}_3 . First, we assume that $u = v$ in (4.1.2). When (4.1.2) has only one solution at $x = \frac{1}{\beta}$, the value $\frac{1}{\beta} > \frac{1}{3}$. After increasing u minutely, which allows (4.1.2) to have two intersections at $\mathbf{P} = (P, P)$ and $\mathbf{Q} = (Q, Q)$, by Lemma 4.3.4, $2P + Q$ increases with respect to u . We increase u a little more so that (4.1.2) have four intersections at $\mathbf{P} = (P, P)$, $\mathbf{R} = (R, S)$, $\mathbf{S} = (S, R)$, and $\mathbf{Q} = (Q, Q)$ with $P < R < Q < S$. In this case, $P + R + S > 2P + Q > \frac{3}{\beta} > 1$. Moreover, $R + S + Q > 1$. That is, there is no critical point of the form (P, R, S, P, S, R) or (Q, R, S, Q, S, R) up to permutations. Now, without loss of generality, we may assume that $u > v$. In this case, by Theorem 4.3.7, each sum of the coordinates of all possible combinations of the three intersections in (4.1.2) is greater than one. This completes the proof. \square

Finally, we prove the main theorem which is Theorem 2.4.2.

Proof of Theorem 2.4.2.

We prove the synchronization part first. By Theorem 4.5.1, the two components are synchronized in high-temperature ($\beta < \beta_1$). Proposition 4.3.2, and Theorems 4.3.5 and 4.3.7 imply that if $J > \max(\psi_1(\beta), J_c)$, then there are no local minima and lowest saddles of $F_{\beta, J}$ that does not belong to either S^2 or L^2 , that is, the two components are synchronized. In addition, Theorem 4.3.7 and 4.4.3 imply that if $J > \psi_3(\beta)$, then there are no local minima and lowest saddles of $F_{\beta, J}$ other than S^2 or L^2 . Therefore, the two components are synchronized when $J > \psi_s(\beta)$. However, if $J < \psi_d(\beta)$, then by the second statement of Theorems 4.3.3 and 4.4.2, the two components are desynchronized. This completes the proof. \square

Appendix A

Proof of Proposition 2.2.2

Here, we present a proof of Proposition 2.2.2.

Proof of Proposition 2.2.2. For $\mathbf{x} \in \Xi_N^2$, from the definition of \mathbb{H}_N ,

$$\begin{aligned} & \sum_{\boldsymbol{\sigma} \in \Omega: \mathbf{r}(\boldsymbol{\sigma}) = \mathbf{x}} \frac{1}{Z_{N, \beta}} e^{-\beta \mathbb{H}_N(\boldsymbol{\sigma})}, \\ &= \frac{N!}{(Nx_1^{(1)})! \cdots (Nx_q^{(1)})!} \cdot \frac{N!}{(Nx_1^{(2)})! \cdots (Nx_q^{(2)})!} \\ & \times \frac{1}{Z_{N, \beta}} \exp \left[\frac{\beta}{N(1+J)} \left\{ \sum_{k=1, 2} \sum_{i=1}^q \frac{1}{2} \left\{ (Nx_i^{(k)})(Nx_i^{(k)} - 1) \right\} + J \sum_{i=1}^q Nx_i^{(1)} Nx_i^{(2)} \right\} \right], \end{aligned}$$

and by Stirling's formula, we have

$$\begin{aligned} & \approx \frac{\exp \{ -\beta / (1+J) \}}{(\sqrt{2\pi N})^{2(q-1)} \sqrt{\prod_{k=1, 2} \prod_{i=1}^q x_i^{(k)}} Z_{N, \beta}} \\ & \times \exp \left[N \frac{\beta}{1+J} \left\{ \sum_{k=1, 2} \sum_{i=1}^q \frac{1}{2} (x_i^{(k)})^2 + J \sum_{i=1}^q x_i^{(1)} x_i^{(2)} - \frac{1+J}{\beta} \left(\sum_{k=1, 2} \sum_{i=1}^q x_i^{(k)} \log x_i^{(k)} \right) \right\} \right], \\ &= \frac{1}{\widehat{Z}_{N, \beta, J}} \exp \left\{ -N \frac{\beta}{1+J} \left(F_{\beta, J}(\mathbf{x}) + \frac{1}{N} G_{N, \beta}(\mathbf{x}) \right) \right\}, \end{aligned}$$

APPENDIX A. PROOF OF PROPOSITION ??

where

$$F_{\beta, J}(\mathbf{x}) = - \sum_{k=1, 2} \sum_{i=1}^q \frac{1}{2} (x_i^{(k)})^2 - J \sum_{i=1}^q x_i^{(1)} x_i^{(2)} + \frac{1+J}{\beta} \left(\sum_{k=1, 2} \sum_{i=1}^q x_i^{(k)} \log x_i^{(k)} \right),$$

$$G_{N, \beta, J}(\mathbf{x}) = \frac{1+J}{2\beta} \log \left(\prod_{k=1, 2} \prod_{j=1}^q x_i^{(k)} \right) + O(N^{-(q-1)}).$$

We decompose the function $F_{\beta, J}$ into an energy part H and entropy part S . That is,

$$F_{\beta, J}(\mathbf{x}) = H(\mathbf{x}) + \frac{1+J}{\beta} S(\mathbf{x}),$$

where

$$H(\mathbf{x}) = - \sum_{k=1, 2} \sum_{i=1}^q \frac{1}{2} (x_i^{(k)})^2 - J \sum_{i=1}^q x_i^{(1)} x_i^{(2)} \quad \text{and} \quad S(\mathbf{x}) = \sum_{k=1, 2} \sum_{i=1}^q x_i^{(k)} \log x_i^{(k)}.$$

□

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국문초록

이 논문에서 3개 이상의 스핀으로 구성된 Curie-Weiss 모델의 일반화인 Curie-Weiss-Potts 모델로 알려진 두 개의 구성 요소를 가지는 스핀 시스템의 에너지 경관을 조사합니다. 다중 구성 요소 모델의 에너지 환경에서 가장 중요한 요소는 구성 요소 간 상호 작용 강도와 구성 요소별 상호 작용 강도 간의 상대적 강도입니다. 구성 요소 간의 상호 작용이 구성 요소별 상호 작용보다 강하면 준안정 전환 과정에서 모든 구성 요소가 동기화될 것으로 기대할 수 있습니다. 그러나 구성 요소 간 상호 작용이 상대적으로 약하면 준 안정 전환 과정에서 구성 요소가 동기화되지 않습니다. 두 개의 구성 요소를 가지는 Curie-Weiss 모델의 경우 동기화에서 비동기화로의 위상 전이는 평균 필드 특성으로 인해 연구에서 정확하게 특성화되었습니다. 이 논문의 목적은 이 결과를 3개의 스핀이 있는 Curie-Weiss-Potts 모델로 확장하는 것입니다. 우리는 3-스핀의 경우에 대한 상전이의 특성이 Curie-Weiss 모델의 2-스핀 경우와 완전히 다르며 결과 위상 다이어그램과 함께 증명이 근본적으로 다르고 매우 복잡하다는 것을 관찰했습니다.

주요어휘: 상전이, 스핀 체계, Curie-Weiss-Potts 모델, 에너지 경관, 다중 구성 요소 모델

학번: 2019-27865