이학 박사 학위논문

Estimates of heat kernels for jump processes with degeneracy and critical killing

(퇴화와 임계 킬링이 있는 도약과정의 열핵에 대한 추정)

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이 논문을 이학 박사 학위논문으로 제출함

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Estimates of heat kernels for jump processes with degeneracy and critical killing

A dissertation
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Abstract

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Transition densities of Markov processes are of significant interest in both probability and analysis. The transition density $p(t,x,y)$ of a Markov process with generator $\mathcal{L}$ is the fundamental solution of the equation $\partial_t u = \mathcal{L}u$. Hence the transition density $p(t,x,y)$ is also called as the heat kernel of $\mathcal{L}$. However, an explicit expression of the heat kernel is rarely known. Due to the importance of heat kernels, there is a huge body of literature on the heat kernel estimates. The thesis consists of six parts concerning heat kernel estimates for Markov jump processes. The first part devotes to estimates for subordinators, namely, nondecreasing Lévy processes on $\mathbb{R}$. The second part considers heat kernels for non-local operators with critical killings. The third part studies subordinate killed Markov processes with help from the previous two parts. Motivated by the third part, in the fourth part, we study heat kernel estimates for jump processes with degeneracy and critical killing using Dirichlet form theory. The fifth part is concerned with the fundamental solution of general time fractional equations with Dirichlet boundary condition. In the last part, we study Dirichlet heat kernel estimates for isotropic unimodal Lévy processes with low intensity of small jumps.

Key words: Markov process, heat kernel estimate, nonlocal operator, Dirichlet form

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Contents

Chapter 1

Introduction

Transition densities of Markov processes are of significant interest in both probability and analysis. The transition density $p(t,x,y)$ of a Markov process with generator $\mathcal{L}$ is the fundamental solution of the equation $\partial_t u = \mathcal{L}u$. To be precise, let $X = \{X_t, t \geq 0; \mathbb{P}^x, x \in M\}$ be a strong Markov process on a locally compact separable Hausdorff space M whose transition semigroup $(P_t)_{t\geq 0}$ is a uniformly bounded strong continuous semigroup in some Banach space $(\mathbb{B}, \|\cdot\|)$. Typically, $\mathbb{B} = L^p(M;m)$ for some Radon measure m on M and $p \geq 1$. Denote by $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ the infinitesimal generator of the semigroup $(P_t)_{t\geq 0}$ in $\mathbb{B}$. Then it is well known that for any $f \in \mathcal{D}(\mathcal{L})$, the function $u(t,x) := \mathbb{E}^x[f(X_t)]$ is the unique solution in $\mathbb{B}$ to the equation

$$\begin{cases} \partial_t u(t,x) = \mathcal{L}u(t,x), & x \in M, \quad t > 0, \\ u(0,x) = f(x), & x \in M \end{cases} \tag{1.0.1}$$

in the following sense:

(i) $x \mapsto u(t,x)$ is in $\mathcal{D}(\mathcal{L})$ for each $t \geq 0$, $\sup_{t\geq 0}(\|u(t,\cdot)\| + \|\mathcal{L}u(t,\cdot)\|) < \infty$, and $t \mapsto \mathcal{L}u(t,\cdot)$ is continuous in $\mathbb{B}$;

(ii) $t \mapsto u(t,\cdot)$ is absolutely continuous and $\partial_t u(t,\cdot) = \mathcal{L}u(t,\cdot)$ in $\mathbb{B}$.

Suppose that the process X has a transition density function $p(t,x,y)$ with respect to a reference measure m. Then, for any $f \in \mathcal{D}(\mathcal{L})$, the unique solution to the equation (1.0.1) is given by

$$u(t,x) = \mathbb{E}^x[f(X_t)] = \int_M p(t,x,y)f(y)m(dy).$$

Hence, the transition density $p(t,x,y)$ is also known as the heat kernel of $\mathcal{L}$. An explicit expression of the heat kernel is rarely known. Instead, there is a long history on the heat kernel estimates. In the thesis, we are concerned with heat kernel estimates for Markov jump processes. We note that, when X is a jump process, its generator $\mathcal{L}$ is a non-local operator.

In a celebrated paper [45], Z.-Q. Chen and T. Kumagai proved that when M is a Ahlfors d-set and X is a pure jump process with the jump kernel $J(x,y)$ such that

$$C_1|x-y|^{-d-\alpha} \le J(x,y) \le C_2|x-y|^{-d-\alpha}, \quad x,y \in M, \tag{1.0.2}$$

for some constants $\alpha \in (0,2)$ and $C_1, C_2 > 0$, the heat kernel satisfies the following two-sided estimates for all $0 < t < 1$ and $x, y \in M$:

$$p(t,x,y) \simeq \min\left\{t^{-d/\alpha}, \frac{t}{|x-y|^{d+\alpha}}\right\}. \tag{1.0.3}$$

Here, $f \simeq g$ means that there exist constants $c_1, c_2 > 0$ such that $c_1 g(x) \le f(x) \le c_2 g(x)$. (See also the earlier work [11] by R. F. Bass and D. A. Levin for random walks on the lattice $\mathbb{Z}^d$.) Later, this result has been extended to mixed stable-like processes in [46] by the same authors, where M is a metric measure space whose volume function $V(x,r) := m(B(x,r))$ satisfies a uniform volume doubling assumption, and the jump kernel $J(x,y)$ is comparable to $(V(x,d(x,y)\phi(d(x,y))^{-1}$, under some growth condition and weak scaling property on the weight function ϕ (see [46, (1.11)-(1.14)]).

For jump processes with killing potential, R. Song proved in [120] that when X is a symmetric α-stable-like-process ($0 < \alpha < 2$) and the killing potential is in a suitable Kato class, the small time heat kernel estimates (1.0.3) still hold true. This result has been extended to some nonsymmetric processes in [125] by C. Wang. When X has a critical killing potential, the situation becomes more complicated. In [40], Z.-Q. Chen, P. Kim and R. Song proved in that the heat kernel $p(t,x,y)$ of a killed α-stable process $(0<\alpha<2)$ in a $C^{1,1}$ open subset D of $\mathbb{R}^d$ satisfies the following estimates for all $0<t<1$ and $x,y\in D$:

$$p(t,x,y) \simeq \min\left\{1,\, \frac{\delta_D(x)}{t^{1/\alpha}}\right\}^{\alpha/2} \min\left\{1,\, \frac{\delta_D(y)}{t^{1/\alpha}}\right\}^{\alpha/2} \min\left\{t^{-d/\alpha},\, \frac{t}{|x-y|^{d+\alpha}}\right\}, \tag{1.0.4}$$

where $\delta_D(x)$ denotes the distance between $x\in D$ and the complement D^c. (See also the work [19] by K. Bogdan, T. Grzywny and M. Ryznar for Dirichlet heat kernel estimates when D is a κ-fat open subset of $\mathbb{R}^d$.) The authors also proved in [36] that the heat kernel $p(t,x,y)$ of a censored α-stable process $(1<\alpha<2)$ in a $C^{1,1}$ open subset D of $\mathbb{R}^d$ satisfies that

$$p(t,x,y) \simeq \min\left\{1,\, \frac{\delta_D(x)}{t^{1/\alpha}}\right\}^{\alpha-1} \min\left\{1,\, \frac{\delta_D(y)}{t^{1/\alpha}}\right\}^{\alpha-1} \min\left\{t^{-d/\alpha},\, \frac{t}{|x-y|^{d+\alpha}}\right\} \tag{1.0.5}$$

for all $0<t<1$ and $x,y\in D$. Hence, the Dirichlet condition and censoring are examples of critical killings in the sense that heat kernel estimates take different form from ones for the free process (without killing) given in (1.0.3).

We note that, the condition (1.0.2) can be regarded as a non-local counterpart of the usual uniform ellipticity condition. For jump processes with degenerate jump kernel, only a few results exist in the literature. See the works [119] by R. Song for subordinate killed Brownian motions and [28] by X. Chen, T. Kumagai and J. Wang for random conductance models with long range jumps.

In the thesis, we study heat kernel estimates for jump processes with critical killings whose jump kernel may be degenerate. The thesis is divided into this introduction and six chapters. In Chapter 2, we study distributional properties of a large class of subordinators. In particular, we get heat kernel estimates for subordinators with Lévy density decaying in mixed polynomial orders. In Chapter 3, we prove factorization of heat kernels $p(t,x,y)$ in a subset D of M for a class of non-local operators. As a consequence, we obtain sharp two-sided heat kernel estimates for α-stable processes ($0 < \alpha < 2$) with critical killings in a $C^{1,1}$ open subset D of $\mathbb{R}^d$. In particular, we give an alternative and unified proof for (1.0.4) and (1.0.5). In Chapter 4, we obtain sharp estimates for the jump kernel, heat kernel and the Green function of subordinate killed Markov processes by using results in Chapters 2 and 3. We then give important examples of heat kernel estimates for jump processes with degeneracy and critical killing. In Chapter 5, we study heat kernel estimates for Hunt processes with degenerate jump kernel and critical killing potential. The objects therein are motivated by Chapter 4 in the spirit of stability theorems for heat kernel estimates initiated by D. G. Aronson [4] for local operators (see, e.g. [48] and [72] for non-local operators). In Chapter 6, we obtain estimates on the fundamental solution of general time fractional equation with Dirichlet boundary condition by using probabilistic representation introduced by Z.-Q. Chen [30]. In Chapter 7, we give Dirichlet heat kernel estimates for isotropic unimodal Lévy processes with low intensity of small jumps. In particular, we show that factorization of the Dirichlet heat kernel in a $C^{1,1}$ open subset D of $\mathbb{R}^d$ holds true for such processes.

1.1 Preliminary and notation

For the most part of this thesis, we will play with functions satisfying (weak) scaling property. A nonnegative function f defined on an interval $I \subset [0,\infty)$ is said to be satisfying *(weak) scaling property* if there exist constants $p_2 \ge p_1$ and $c_1, c_2 > 0$ such that

$$c_1\Big(\frac{r_2}{r_1}\Big)^{p_1} \le \frac{f(r_2)}{f(r_1)} \le c_2\Big(\frac{r_2}{r_1}\Big)^{p_2} \quad \text{for all } \ r_1, r_2 \in I, \ 0 < r_1 \le r_2. \qquad (1.1.1)$$

The constant p_1 (resp. p_2) is called the lower scaling index (resp. upper scaling index) of f. We give integral estimates for functions with scaling property. The following lemma will be used frequently throughout the thesis.

Lemma 1.1.1. [59, Lemma 5.1] *Let $f : I \to [0, \infty)$ be a function defined on an interval $I \subset [0, \infty)$. Suppose that f satisfies* (1.1.1) *with $p_1, p_2 \in \mathbb{R}$. Then for any $a > 1$, there exists $c_1 > 0$ such that for all $r, R \in I$, $ar \le R$,*

$$\int_r^R s^{-1} f(s) ds \ge c_1 \big(f(r) + f(R) \big). \tag{1.1.2}$$

(i) If we assume $p_1 > 0$, then, for any $a > 1$, there exists $c_2 > 0$ such that for all $r, R \in I$, $ar \le R$,

$$c_1 f(R) \le \int_r^R s^{-1} f(s) ds \le c_2 f(R).$$

(ii) If we assume $p_2 < 0$, then, for any $a > 1$, there exists $c_3 > 0$ such that for all $r, R \in I$, $ar \le R$,

$$c_1 f(r) \le \int_r^R s^{-1} f(s) ds \le c_3 f(r).$$

Notation: We will use the symbol ":=" to denote a definition, which is read as "is defined to be." We deonte by $\mathbb{R}^d$ the d-dimensional Euclidean space and $\mathbb{R}^d_+ := \big\{ x = (\widetilde{x}, x_d) \in \mathbb{R}^d \mid x_d > 0 \big\}$ the upper half plane. We write $\mathbf{e}_d := (\widetilde{0}, 1) \in \mathbb{R}^d$. For $a, b \in \mathbb{R}$, we set $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$, $a_+ := a \vee 0$ and $\lfloor a \rfloor := \max\{n \in \mathbb{Z} : n \le a\}$.

The constants C_i, α_i and β_i for $i \ge 0$, d_1, d_2 will retain throughout the section, whereas c, A, C, ϵ, δ, η and κ represent constants having insignificant values that may be changed from one appearance to another. The labeling of the constants $a_0, a_1, a_2, \dots$ and $c_0, c_1, c_2, \dots$ begins anew in the proof of each result. $C = C(a, b, \dots)$ denotes a generic constant depending on $a, b, \dots$. All these constants are positive finite. Recall that we write $f(x) \simeq g(x)$ if there exist constants $c_1, c_2 > 0$ such that $c_1 g(x) \le f(x) \le c_2 g(x)$ for the specified range of the argument x. Similarly, we write $f(x) \asymp g(cx)$ if there exist constants $c_i > 0$, $i = 1, \cdots, 4$ such that $c_1 g(c_2 x) \le f(x) \le c_3 g(c_4 x)$ for the specified range of x.

For a given metric space (M, ρ), we denote the open ball in M with center $x \in M$ and radius $r > 0$ by $B(x, r)$. For any subset D of (M, ρ), we write $\mathrm{diam}(D) = \sup_{x,y\in D} \rho(x, y)$ and $\delta_D(x) = \inf\{\rho(x, y) : y \in M \setminus D\}$. We define $\delta_\wedge(x, y) = \delta_D(x) \wedge \delta_D(y)$ and $\delta_\vee(x, y) = \delta_D(x) \vee \delta_D(y)$. For a function space $\mathbb{H}(U)$ on an open set U in M, we let $\mathbb{H}_c(U) := \{f \in \mathbb{H}(U) : f \text{ has compact support}\}$ and $\mathbb{H}_0(U) := \{f \in \mathbb{H}(U) : f \text{ vanishes at infinity}\}$.

Chapter 2

Estimates for subordinators

This chapter is concerned with distributional properties of subordinators. The results in this chapter are mainly based on [54, 55]. We first give basic estimates for subordinators whose tail of the Lévy measure is locally decaying in polynomial orders. We next establish tail probability estimates for three classes of subordinators: (1) ones with polynomially decaying tail, (2) ones with subexponentially decaying tail, and (3) truncated subordinators. Lastly, we study two-sided sharp estimates and the exact asymptotic behaviors of the transition density function for a large class of subordinators.

Let us begin with the definition of subordinator. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A stochastic process $S = (S_t : t \geq 0)$ with values in $[0, \infty)$ is called a *subordinator* if

(1) $S_0 = 0$ and $t \mapsto S_t$ is nondecreasing and right-continuous a.s.,

(2) for every $t, s \geq 0$, the increment $S_{t+s} - S_t$ has the same law as S_s and is independent of $(S_u : 0 \leq u \leq t)$,

(3) $\lim_{t \to 0} \mathbb{P}(|S_t| > \epsilon) = 0$ for all $\epsilon > 0$.

Consider an arbitrary subordinator S. For any rational number $p/q \geq 0$, since S has stationary independent increments, using the decomposition

$$S_{p/q} = S_{1/q} + (S_{2/q} - S_{1/q}) + \cdots + (S_{p/q} - S_{(p-1)/q}),$$

we see that the laws of $S_{p/q}$ and $(p/q)S_1$ are equal. Therefore, there exists a function $\phi : [0,\infty) \to [0,\infty)$ such that for any rational number $t \geq 0$,

$$\mathbb{E}e^{-\lambda S_t} = e^{-t\phi(\lambda)}, \quad \lambda \geq 0. \tag{2.0.1}$$

Moreover, by the right-continuity of S, (2.0.1) holds for all $t \geq 0$. The function ϕ is called the *Laplace exponent* of the subordinator S, and it characterizes the law of S in the sense that two subordinators with the same Laplace exponent have the same law. It is well known that ϕ is a Bernstein function, that is, a nonnegative C^∞ function such that $(-1)^{k-1}\phi^{(k)} \geq 0$ for all $k \geq 1$, with $\phi(0) = 0$ and there exist a unique constant $a \geq 0$ and a Borel measure ν on $(0,\infty)$ satisfying $\int_0^\infty (1 \wedge s)\nu(ds) < \infty$ such that

$$\phi(\lambda) = a\lambda + \int_0^\infty (1 - e^{-\lambda s})\nu(ds). \tag{2.0.2}$$

The constant a is called the drift and ν the Lévy measure of the subordinator S. Conversely, for any $a \geq 0$ and a Borel measure ν on $(0,\infty)$ satisfying $\int_0^\infty (1 \wedge s)\nu(ds) < \infty$, there exists a subordinator $S = (S_t)_{t\geq 0}$ such that (2.0.1) and (2.0.2) hold so that ν is the Lévy measure of S. We write $w(\lambda) = \nu((\lambda,\infty))$ for the tail of the Lévy measure ν.

Throughout this chapter, we suppose that S is a subordinator having the Laplace exponent ϕ with

$$w(0+) = \infty.$$

Moreover, by considering the subordinator $\widetilde{S} = (S_t - at : t \geq 0)$, we always assume that

$$\text{the drift } \ a = 0.$$

We introduce some auxiliary functions which will be used in the study of subordinators. Let $H : (0,\infty) \to (0,\infty)$ be defined by

$$H(\lambda) := \phi(\lambda) - \lambda\phi'(\lambda), \quad \lambda > 0.$$

The function H plays an important role in estimates for the distributions of subordinators. See, e.g. [81, 109]. Because ϕ is a Bernstein function, H is increasing and $H(0+) = 0$. Moreover, using the representation (2.0.2) and integration by parts formula, we observe that for all $\lambda > 0$,

$$\phi(\lambda) = \lambda \int_0^\infty w(s)e^{-\lambda s}ds \quad \text{and} \quad H(\lambda) = \lambda^2 \int_0^\infty sw(s)e^{-\lambda s}ds. \tag{2.0.3}$$

Hence, it holds that for all $\lambda > 0$,

$$\phi(\lambda) \geq H(\lambda) \geq \lambda^2 w(1/\lambda) \int_0^{1/\lambda} se^{-\lambda s}ds \geq \frac{w(1/\lambda)}{2e}. \tag{2.0.4}$$

In particular, since we have assumed $w(0+) = 0$, it holds that $\lim_{\lambda\to\infty} \phi(\lambda) = \lim_{\lambda\to\infty} H(\lambda) = \infty$. We also get from (2.0.3) that

$$\phi(\lambda r) \leq r\phi(\lambda) \quad \text{and} \quad H(\lambda r) \leq r^2 H(\lambda) \qquad \text{for all } \ \lambda > 0, \ r \geq 1 \tag{2.0.5}$$

and since w is nonincreasing,

$$\phi(\lambda) \simeq \lambda \int_0^{1/\lambda} w(s)ds \quad \text{and} \quad H(\lambda) \simeq \lambda^2 \int_0^{1/\lambda} sw(s)ds, \qquad \lambda > 0. \tag{2.0.6}$$

Next, we introduce the function $b : (0, \infty) \to (0, \infty)$ defined by

$$b(t) := (\phi' \circ H^{-1})(1/t) = \int_0^\infty se^{-H^{-1}(1/t)s}\nu(ds), \quad t > 0.$$

The function b also appears naturally in the study of subordinators, especially when describe the displacement with the highest probability of the given subordinator S at time t (see, e.g. Proposition 2.2.1 below). The function b is increasing, $b(0+) = 0$, and $\lim_{t\to\infty} b(t) = \phi'(0+)$. This implies that $t \mapsto tb(t)$ is also increasing and $\lim_{t\to\infty} tb(t) = +\infty$. Moreover, according to [54, Lemma 2.4(ii)], cf. also [55, (2.14)], it holds that

$$\frac{1}{\phi^{-1}(c_*/t)} \leq tb(t) \leq \frac{1}{\phi^{-1}(1/t)}, \quad c_* := \frac{e^2 - e}{e - 2}, \quad \text{for all } t > 0. \tag{2.0.7}$$

Moreover, we have general estimates for differences of the function b. The following result is particularly important when ϕ is not comparable to H.

Lemma 2.0.1. [55, Lemma 2.7] *For any $a_2 \geq a_1 > 0$, it holds that*

$$\begin{aligned} tb(t/a_1) - tb(t/a_2) &\leq \frac{2ea_2}{H^{-1}(a_2/t)} + \frac{e^{-1}tw\big(H^{-1}(a_2/t)^{-1}\big)}{H^{-1}(a_1/t)} \\ &\leq \frac{2e^2-4e+1}{e-2}\frac{a_2}{H^{-1}(a_1/t)} \end{aligned}$$

and

$$tb(t/a_1) - tb(t/(4a_1)) \geq \frac{1}{2}\frac{tH^{-1}(4a_1/t)^2\left|(\phi'' \circ H^{-1})(4a_1/t)\right|}{H^{-1}(4a_1/t)}$$

for all $t \geq 0$.

Finally, we introduce the function $\sigma = \sigma(t,r) : (0,\infty)\times(0,\infty) \to [0,\infty)$ defined by

$$\sigma = \sigma(t,r) := (\phi')^{-1}(r/t)\mathbf{1}_{(0,\phi'(0+))}(r/t), \quad r,t > 0.$$

Because ϕ' is decreasing, for each fixed $t > 0$, the map $r \mapsto \sigma(t,r)$ is decreasing with $\lim_{r\to 0}\sigma(t,r) = \infty$ and $\lim_{r\to\infty}\sigma(t,r) = 0$, while for each fixed $r > 0$, the map $t \mapsto \sigma(t,r)$ is increasing with $\lim_{t\to 0}\sigma(t,r) = 0$ and $\lim_{t\to\infty}\sigma(t,r) = \infty$. Further, by using the former and the fact that H is increasing, we conclude that

$$t(H\circ\sigma)(t,tb(t)) = 1 \quad \text{and} \quad t(H\circ\sigma)(t,r) > 1 \quad \text{for } r < tb(t). \tag{2.0.8}$$

The function σ plays a crucial role in estimating the left tail of S.

For the most part of this chapter, we assume the following (local) weak scaling property for the tail measure w.

Definition 2.0.2. Let $R_1 \in (0,\infty]$, $R_2 > 0$ and $\beta_2 \geq \beta_1 > 0$ be constants.

(i) We say that $\mathbf{Poly}_{R_1}(\beta_1, \beta_2)$ holds if there are $c_1, c_2 > 0$ such that

$$c_1\Big(\frac{r}{s}\Big)^{\beta_1} \le \frac{w(s)}{w(r)} \le c_2\Big(\frac{r}{s}\Big)^{\beta_2} \quad \text{for all } \; 0 < s \le r < R_1. \tag{2.0.9}$$

We say that $\mathbf{Poly}_{R_1,\le}(\beta_2)$ (resp. $\mathbf{Poly}_{R_1,\ge}(\beta_1)$) holds if the upper bound (resp. lower bound) in (2.0.9) holds.

(ii) We say that $\mathbf{Poly}^{\infty}_{R_2}(\beta_1, \beta_2)$ holds if (2.0.9) holds for all $r \ge s \ge R_2$. We say that $\mathbf{Poly}^{\infty}_{R_2,\le}(\beta_2)$ (resp. $\mathbf{Poly}^{\infty}_{R_2,\ge}(\beta_1)$) holds if the upper bound (resp. lower bound) in (2.0.9) holds for all $r \ge s \ge R_2$.

Remark 2.0.3. Since w is nonincreasing, if $\mathbf{Poly}^{\infty}_{R_2}(\beta_1, \beta_2)$ holds with some $R_2 > 0$, then it holds with every $R_2 > 0$. The same is true for $\mathbf{Poly}^{\infty}_{R_2,\le}(\beta_2)$ and $\mathbf{Poly}^{\infty}_{R_2,\ge}(\beta_1)$.

When S is a β-stable subordinator with Laplace exponent λ^β $(0 < \beta < 1)$, we see that $w(r) = c_1 r^{-\beta}$ for $r > 0$ so that $\mathbf{Poly}_{\infty}(\beta_1, \beta_2)$ holds.

We sometimes consider the following conditions to cover cases when the subordinator has exceedingly small tail.

Definition 2.0.4. (i) We say that $\mathbf{Sub}^{\infty}(\gamma, \theta)$ holds if there exist constants $\gamma \in (0, 1]$ and $\theta, c_1 > 0$ such that

$$w(r) \le c_1 \exp(-\theta r^{\gamma}) \quad \text{for all } \; r \ge 1.$$

(ii) We say that $\mathbf{Trun}^{\infty}_{R_2}$ holds if there exist constants $R_2, \beta_2 > 0$ and $K \ge 1$ such that $w(R_2) = 0$, $\mathbf{Poly}_{R_2/2,\le}(\beta_2)$ holds and

$$K^{-1}|r - s| \le w(s) - w(r) \le K|r - s| \quad \text{for all } \; R_2/4 \le s \le r \le R_2.$$

2.1 Preliminary results

In this section, we give some consequences of conditions $\mathbf{Poly}_{R_1}(\beta_1, \beta_2)$ and $\mathbf{Poly}^{\infty}_{R_2}(\beta_1, \beta_2)$.

Using (2.0.6), we get relations between these conditions and weak scaling properties of the functions ϕ and H. The proof of the next lemma can be found in [54, Lemma 2.1] and [55, Lemmas 2.3 and 2.4].

Lemma 2.1.1. *(i) If* $\mathbf{Poly}_{R_1,\le}(\beta_2)$ *holds, then for every* $r_0 > 0$*, there is* $c_1 > 0$ *such that for all* $r_0 \le s \le r$*,*

$$\frac{\phi(r)}{\phi(s)} \le c_1 \left(\frac{r}{s}\right)^{\beta_2 \wedge 1} \quad \text{and} \quad \frac{H(r)}{H(s)} \le c_1 \left(\frac{r}{s}\right)^{\beta_2 \wedge 2}. \tag{2.1.1}$$

Conversely, if there is $r_0 > 0$ *such that the first inequality in* (2.1.1) *holds with* $\beta_2 < 1$ *(resp. the second with* $\beta_2 < 2$*), then there is* $R_1 \in (0, \infty]$ *such that* $\mathbf{Poly}_{R_1,\le}(\beta_2)$ *holds true and*

$$\phi(1/r) \simeq w(r) \quad (\text{resp. } H(1/r) \simeq w(r)) \quad \text{for } 0 < r < R_1. \tag{2.1.2}$$

(ii) If $\mathbf{Poly}_{R_1,\ge}(\beta_1)$ *holds, then for every* $r_0 > 0$*, there is* $c_2 > 0$ *such that for all* $r_0 \le s \le r$*,*

$$\frac{\phi(r)}{\phi(s)} \ge c_2 \left(\frac{r}{s}\right)^{\beta_1 \wedge (1/2)} \quad \text{and} \quad \frac{H(r)}{H(s)} \ge c_2 \left(\frac{r}{s}\right)^{\beta_1 \wedge (3/2)}. \tag{2.1.3}$$

Conversely, if there is $r_0 > 0$ *such that the first inequality or the second in* (2.1.3) *holds true, then* $\mathbf{Poly}_{R_1,\ge}(\beta_1)$ *holds with some* $R_1 \in (0, \infty]$.

(iii) If $\mathbf{Poly}^{\infty}_{R_2,\le}(\beta_2)$ *holds, then for every* $r_0 > 0$*, there is* $c_1 > 0$ *such that* (2.1.1) *holds for all* $0 < s \le r \le r_0$*. Conversely, if there is* $r_0 > 0$ *such that the first inequality in* (2.1.1) *holds with* $\beta_2 < 1$ *(resp. the second with* $\beta_2 < 2$*), then for any* $R_2 > 0$*, the comparability* (2.1.2) *holds with the range* $r \ge R_2$ *and* $\mathbf{Poly}^{\infty}_{R_2,\le}(\beta_2)$ *holds.*

(iv) If $\mathbf{Poly}^{\infty}_{R_2,\ge}(\beta_1)$ *holds, then for every* $r_0 > 0$*, there is* $c_2 > 0$ *such that* (2.1.3) *holds. Conversely, if there is* $r_0 > 0$ *such that the first inequality or the second in* (2.1.3) *holds, then* $\mathbf{Poly}^{\infty}_{R_2,\le}(\beta_2)$ *holds with any* $R_2 > 0$.

Note that the constant β_2 in (2.1.1) can be arbitrarily large, but upper scaling indices of ϕ and H can not be larger than 2 by (2.0.5).

Recall that $\phi(\lambda) \ge H(\lambda) \ge (2e)^{-1}w(1/\lambda)$ for all $\lambda > 0$. In the following two lemmas, we give upper bounds for H and ϕ under $\mathbf{Poly}_{R_1,\le}(\beta_2)$ and $\mathbf{Poly}^{\infty}_{R_2,\le}(\beta_2)$.

Lemma 2.1.2. *If* $\mathbf{Poly}_{R_1,\le}(\beta_2)$ *holds, then there exists* $c_1 > 0$ *such that for all* $0 < r < R_1$,

$$H(r^{-1})^{\beta_2+1} \le c_1\phi(r^{-1})^{\beta_2}w(r). \tag{2.1.4}$$

Similarly, if $\mathbf{Poly}^{\infty}_{R_2,\le}(\beta_2)$ *holds, then for any* $r_0 > 0$, *there exists* $c_1 > 0$ *such that* (2.1.4) *holds for all* $r > r_0$.

Proof. Suppose that $\mathbf{Poly}_{R_1,\le}(\beta_2)$ holds. If $\beta_2 < 2$, then by Lemma 2.1.1(i), using the fact $\phi \ge H$, we get that for all $0 < r < R_1$,

$$H(r^{-1})^{\beta_2+1} \le c_1 H(r^{-1})^{\beta_2}w(t) \le c_1\phi(r^{-1})^{\beta_2}w(r).$$

Assume that $\beta_2 \ge 2$. Using (2.0.6) in the first and the third inequalities below, Hölder inequality in the second and Lemma 1.1.1(i) in the third, we get that for all $0 < r < R_1$,

$$\begin{aligned}
&H(r^{-1}) \le c_2 r^{-2}\int_0^r sw(s)ds \\
&\le c_2 r^{-2}\left(\int_0^r w(s)ds\right)^{\beta_2/(\beta_2+1)}\left(\int_0^r s^{\beta_2+1}w(s)ds\right)^{1/(\beta_2+1)} \\
&\le c_3 r^{-2}\big(r\phi(r^{-1})\big)^{\beta_2/(\beta_2+1)}\big(r^{\beta_2+2}w(r)\big)^{1/(\beta_2+1)} = c_3\phi(r^{-1})^{\beta_2/(\beta_2+1)}w(r)^{1/(\beta_2+1)}.
\end{aligned}$$

Next, suppose that $\mathbf{Poly}^{\infty}_{R_2,\le}(\beta_2)$ holds. When $\beta_2 < 2$, the result follows from Lemma 2.1.1(iii) similar to the above situation. Assume that $\beta_2 \ge 2$ and let $r_0 > 0$. Set $c_4 := \int_0^{r_0/2} sw(s)ds / \int_{r_0/2}^{r_0} sw(s)ds$. Then, using (2.0.6), Hölder inequality and Lemma 1.1.1(i) with help from Remark 2.0.3, we get that for all $r > r_0$,

$$H(r^{-1}) \le c_5 r^{-2}\int_0^\infty sw(s)ds \le (1+c_4)c_5 r^{-2}\int_{r_0/2}^r sw(s)ds$$

$$\le (1+c_4)c_5 r^{-2}\left(\int_0^r w(s)ds\right)^{\beta_2/(\beta_2+1)}\left(\int_{r_0/2}^r s^{\beta_2+1}w(s)ds\right)^{1/(\beta_2+1)}$$
$$\le c_6\phi(r^{-1})^{\beta_2/(\beta_2+1)}w(r)^{1/(\beta_2+1)}.$$

The proof is complete. □

Lemma 2.1.3. *If* $\mathbf{Poly}^{\infty}_{R_2,\le}(\beta_2)$ *holds, then for any* $r_0>0$*, there exists* $c_1>0$ *such that*

$$\phi(r^{-1})^{\beta_2+1}\le c_1 w(r)\quad \text{for all } \ r\ge r_0.$$

Proof. Let $r_0>0$. We first suppose that $\int_0^\infty w(s)ds<\infty$. Then by (2.0.6) and $\mathbf{Poly}^{\infty}_{R_2,\le}(\beta_2)$, it holds that for all $r\ge r_0$,

$$\phi(r^{-1})^{\beta_2+1}\le c_1\left(r^{-1}\int_0^\infty w(s)ds\right)^{\beta_2+1}\le \frac{c_2}{r^{\beta_2+1}}\le \frac{c_3}{r_0^{\beta_2+1}w(r_0)}w(r).$$

Now, suppose that $\int_0^\infty w(s)ds=\infty$. Then we see from (2.0.6) that there are comparison constants depend on r_0 such that $\phi(r^{-1})\simeq r^{-1}\int_{r_0/2}^r w(s)ds$ for $r>r_0$. Besides, since $\mathbf{Poly}^{\infty}_{R_2,\le}(\beta_2)$ holds, we see from Lemma 1.1.1(i) that $w(r)\asymp r^{-\beta_2-1}\int_{r_0/2}^r s^{\beta_2}w(s)ds$ for $r>r_0$. Using these two comparabilities and l'Hospital's rule, since w is nonincreasing, we deduce that

$$\limsup_{r\to\infty}\frac{w(r)}{\phi(r^{-1})^{\beta_2+1}}\le c_4\limsup_{r\to\infty}\frac{\int_{r_0/2}^r s^{\beta_2}w(s)ds}{\left(\int_{r_0/2}^r w(s)ds\right)^{\beta_2+1}}$$
$$\le c_4\limsup_{r\to\infty}\frac{r^{\beta_2}w(r)}{(\beta_2+1)w(r)\left(\int_{r_0/2}^r w(s)ds\right)^{\beta_2}}$$
$$\le c_4\limsup_{r\to\infty}\frac{r^{\beta_2}}{(\beta_2+1)(rw(r_0/2))^{\beta_2}}=\frac{c_4}{(\beta_2+1)w(r_0/2)^{\beta_2}}.$$

We have finished the proof. □

Using Lemma 2.1.2 and the inequality $x^ae^{-x}\le a^ae^{-a}$ for $x,a>0$, we get the following result.

Lemma 2.1.4. [59, Lemma 2.5] *Suppose that* $\mathbf{Poly}_{R_1,\le}(\beta_2)$ *holds. Then, for any* $\kappa > 0$*, there exists a constant* $c_1 = c_1(\kappa) > 0$ *such that*

$$\exp\big(-\kappa r H^{-1}(1/t)\big) \le c_1 t w(r) \quad \text{for all } \phi^{-1}(1/t)^{-1} \le r < R_1.$$

Below, we give some consequences of $\mathbf{Poly}_{R_1,\ge}(\beta_1)$. The proofs can be found in [59, Section 2].

Lemma 2.1.5. *Suppose that* $\mathbf{Poly}_{R_1,\ge}(\beta_1)$ *holds. Then for any* $a > 0$*, there exists* $c_1 = c_1(a) \in (0,1)$ *such that*

$$c_1\phi(\lambda) \le \lambda\phi'(\lambda) \le \phi(\lambda) \quad \text{for all } \lambda > a. \tag{2.1.5}$$

Moreover, if $\mathbf{Poly}_{\infty,\ge}(\beta_1)$ *holds, then* (2.1.5) *holds true for all* $\lambda > 0$.

Lemma 2.1.6. *Suppose* $\mathbf{Poly}_{R_1,\ge}(\beta_1)$ *holds. Then, for any* $a > 0$*, there exists* $\delta = \delta(a) > 0$ *such that*

$$\frac{\sigma(t,s)}{\sigma(t,r)} \ge 2^{-\delta}\Big(\frac{r}{s}\Big)^{\delta} \quad \text{for all } 0 < s \le r \le t\phi'(a). \tag{2.1.6}$$

Moreover, if $\mathbf{Poly}_{\infty,\ge}(\beta_1)$ *holds, then there exists* $\delta > 0$ *such that* (2.1.6) *holds true for all* $0 < s \le r < t\phi'(0+)$.

Lemma 2.1.7. *Suppose that* $\mathbf{Poly}_{R_1,\ge}(\beta_1)$ *holds. Then, for all* $\kappa, N > 0$ *and* $T > 0$*, there exists a constant* $C = C(T,\kappa,N) > 0$ *such that for all* $0 < t \le T$ *and* $0 < r \le \phi^{-1}(1/t)^{-1}$,

$$\exp\big(-\kappa t (H\circ\sigma)(t,r)\big) \le C(r\phi^{-1}(1/t))^{N}. \tag{2.1.7}$$

Moreover, if $\mathbf{Poly}_{\infty,\ge}(\beta_1)$ *holds, then for all* $\kappa, N > 0$*, there exists a constant* $C = C(\kappa,N) > 0$ *such that* (2.1.7) *holds true for all* $0 < r \le \phi^{-1}(1/t)^{-1}$.

2.2 Tail probability estimates

In this section, we give estimates for left and right tails of the subordinator S. Concerning the left tail of S, the following general result is obtained in [59] (see [81] for the original version).

Proposition 2.2.1. [59, Proposition 2.3] *There exist constants $C_1, C_2 > 0$ independent of S such that for all $t, r > 0$,*

$$C_1 \exp\big(-C_2 t(H \circ \sigma)(t,r)\big) \le \mathbb{P}(S_t \le r) \le e \exp\big(-t(H \circ \sigma)(t,r)\big). \quad (2.2.1)$$

Moreover, there exist comparability constants independent of S such that

$$\mathbb{P}(S_t \le tb(t)) \simeq \mathbb{P}(S_t \ge tb(t)) \simeq 1, \qquad t > 0.$$

See Theorems 2.3.4 and 2.3.6 below for left tail estimates on the transition density function, and Corollary 2.3.18 for its exact asymptotic behavior under some mild conditions.

As a consequence of Proposition 2.2.1, we get from (2.0.7) the following corollary.

Corollary 2.2.2. *Suppose that $\mathbf{Poly}_{R_1,\ge}(\beta_1)$ holds. Then for any $T > 0$, there exist constants $\delta \in (0,1)$ independent of T and $\epsilon = \epsilon(T) \in (0,1)$ such that*

$$\mathbb{P}\big(\epsilon\phi^{-1}(1/t)^{-1} \le S_t \le \phi^{-1}(1/t)^{-1}\big) \ge \delta, \quad t \in (0,T). \quad (2.2.2)$$

Moreover, if $\mathbf{Poly}_{\infty,\ge}(\beta_1)$ holds, then there exist $\epsilon, \delta \in (0,1)$ such that (2.2.2) *holds with $T = \infty$.*

Using Lemmas 2.1.7 and 1.1.1(i), we also deduce from Proposition 2.2.1 the following result.

Corollary 2.2.3. [59, Lemma 2.6] *Let $f : (0,\infty) \to (0,\infty)$ be a given function. Assume that $\mathbf{Poly}_{R_1,\ge}(\beta_1)$ holds and there exist constants $c_1, p > 0$ such that $s^p f(s) \le c_1 t^p f(t)$ for all $0 < s \le t$. Then for every $T > 0$, there exists a* *constant* $C = C(T, c_1, p) > 0$ *such that for any* $t \in (0, T]$,

$$\mathbb{E}[f(S_t) : S_t \le r] \le Cf(r) \exp\Big(-\frac{t}{2}(H \circ \sigma)(t, r)\Big), \quad 0 < r \le \phi^{-1}(1/t)^{-1}. \tag{2.2.3}$$

Moreover, if $\mathbf{Poly}_{\infty,\ge}(\beta_1)$ *holds, then there exists* $C = C(c_1, p) > 0$ *such that* (2.2.3) *holds for all* $t > 0$.

The following general right tail estimates are obtained in [109]. The lower estimate for the right tail comes from an observation that if a jump of size larger than r occurs before time t, then $S_t \ge r$ by the monotone property of subordinators and the upper estimate comes from Dynkin's formula.

Proposition 2.2.4. *For any* $a > 0$ *and all* $t, r > 0$ *satisfying* $t\phi(r^{-1}) \le a$, *it holds that*

$$\mathbb{P}(S_t \ge r) \ge e^{-2ea} tw(r). \tag{2.2.4}$$

On the other hand, there exists a constant $C_3 > 0$ *independent of* S *such that for all* $t, r > 0$ *satisfying* $t\phi(r^{-1}) \le 1/(2e)$,

$$\mathbb{P}(S_t \ge r) \le C_3 tH(1/r). \tag{2.2.5}$$

Proof. (2.2.5) follows from [109, Proposition 2.3]. By (2.0.4), we see that for all $t, r > 0$ satisfying $t\phi(r^{-1}) \le a$, $tw(r) \le 2et\phi(r^{-1}) \le 2ea$. Thus, using [109, Proposition 2.5] and the inequality $1 - e^{-x} \ge xe^{-x}$ for $x > 0$, we get that for all $t, r > 0$ satisfying $t\phi(r^{-1}) \le a$,

$$\mathbb{P}(S_t \ge r) \ge 1 - e^{-tw(r)} \ge tw(r)e^{-tw(r)} \ge e^{-2ea} tw(r).$$

□

Unlike the left tail estimates (2.2.1), lower and upper estimates for the right tail given in (2.2.4) and (2.2.5) take different forms. In [109], the author proved that if H satisfies weak upper scaling property with index $\delta < 2$ (see the condition (U) therein), then for small $r > 0$, $H(r)$ and $w(1/r)$ are comparable so that the estimates (2.2.4) and (2.2.5) are sharp. Below, we get sharp estimates for the right tail in more general situations by imposing scaling condition on the tail measure w. To do this, we need the following lemma which comes from the analytic continuation of the Laplace exponent.

Lemma 2.2.5. [54, Lemma 2.5] *Assume that w is finitely supported, that is, there exists a constant $R > 0$ such that $w(R) = 0$. Then, for every $\lambda \in \mathbb{R}$, $t > 0$ and $n \in \{0\} \cup \mathbb{N}$, we have that*

$$\mathbb{E}[(S_t)^n e^{\lambda S_t}] = \frac{d^n}{d\lambda^n} \exp\left(t \int_{(0,R]} (e^{\lambda s} - 1)\nu(ds) \right).$$

Theorem 2.2.6. *Assume that $\mathbf{Poly}_{R_1,\le}(\beta_2)$ holds. Then, there exists $C > 1$ such that for all $t > 0$, $0 < r < R_1$ satisfying $t\phi(r^{-1}) \le 1/2$,*

$$C^{-1}tw(r) \le \mathbb{P}(S_t \ge r) \le Ctw(r).$$

Moreover, if $\mathbf{Poly}_{R_1,\ge}(\beta_1)$ also holds, then there exist constants $C', K > 1$ such that for all $t > 0$, $0 < r < R_1/K$ satisfying $t\phi(r^{-1}) \le 1/2$,

$$C'^{-1}tw(r) \le \mathbb{P}(S_t \in [r, Kr]) \le C'tw(r). \tag{2.2.6}$$

Proof. By Proposition 2.2.4, we only need to prove the upper bound.

Fix $t > 0$ and $0 < r < R_1$ satisfying $t\phi(r^{-1}) \le 1/2$. Let $\epsilon := \log(5/4)/2$. We set

$$\mu^1(ds) := \mathbf{1}_{(0,\epsilon/H^{-1}(1/t)]}\, \nu(ds), \quad \mu^2(ds) := \mathbf{1}_{(\epsilon/H^{-1}(1/t),r]}\, \nu(ds),$$

and $\mu^3(ds) := \mathbf{1}_{(r,\infty)}\, \nu(ds)$. Denote by S^1, S^2 and S^3 the independent driftless subordinators with Lévy measures μ^1, μ^2 and μ^3, respectively. Then, we have $S_t \le S_t^1 + S_t^2 + S_t^3$ (note that it may happen that $r < \epsilon/H^{-1}(1/t)$). Hence, it holds that

$$\mathbb{P}(S_t \ge r) \le \mathbb{P}(S_t^1 \ge 3r/4) + \mathbb{P}(S_t^2 \ge r/4) + \mathbb{P}(S_t^3 > 0).$$

Note that $\mathbb{P}(S_t^3 > 0) = 1 - e^{-tw(r)} \le tw(r)$. Set $f_0(s) := w(s)\mathbf{1}_{(0,r]}(s) + w(r)r^2s^{-2}\mathbf{1}_{(r,\infty)}(s)$. Then, f_0 is nonincreasing and for every Borel set $A \subset \mathbb{R}$,

$$\mu^2(A) \le w(\mathrm{dist}(0,A))\mathbf{1}_{(0,r]}(\mathrm{dist}(0,A)) \le f_0(\mathrm{dist}(0,A)),$$

where $\mathrm{dist}(0,A) := \inf\{|y| : y \in A\}$. Moreover, using $\mathbf{Poly}_{R_1,\le}(\beta_2)$ and (2.0.4), we get that for all $u, s > 0$,

$$\int_u^\infty f_0\big(s \vee y - \frac{y}{2}\big)\mu^2(dy) \le f_0(s/2)w(u) \le c_1 f_0(s)H(1/u).$$

Therefore, by [83, Proposition 1 and Lemma 9] and $\mathbf{Poly}_{R_1,\le}(\beta_2)$, it holds that for every $x > 0$ and $\rho \in (0, x/3]$,

$$\mathbb{P}(S_t^2 \in [x-\rho, x+\rho]) \le c_2 t f_0(x/3) \le c_3 t f_0(4x).$$

It follows that

$$\begin{aligned}\mathbb{P}(S_t^2 \ge r/4) &\le \sum_{i=0}^\infty \mathbb{P}\big(S_t^2 \in [2^{i-2}r, 2\cdot 2^{i-2}r]\big) \le c_3 t \sum_{i=0}^\infty f_0(6\cdot 2^{i-2}r)\\ &= c_3 tw(r)\sum_{i=0}^\infty 6^{-2}2^{4-2i} = c_4 tw(r).\end{aligned}$$

Lastly, by using Markov's inequality, Lemma 2.2.5 and Lemma 2.1.4, since $r > 2tb(t)$ due to (2.0.7), we have that

$$\begin{aligned}&\mathbb{P}(S_t^1 \ge 3r/4) \le \mathbb{E}\left[\exp\left(-(3r/4)H^{-1}(1/t) + H^{-1}(1/t)S_t^1\right)\right]\\ &= \exp\bigg(-(3r/4)H^{-1}(1/t) + t\int_0^{\epsilon/H^{-1}(1/t)} (e^{H^{-1}(1/t)s} - 1)\nu(ds)\bigg)\\ &\le \exp\bigg(-(3r/4)H^{-1}(1/t) + e^{2\epsilon}tH^{-1}(1/t)\int_0^{\epsilon/H^{-1}(1/t)} se^{-H^{-1}(1/t)s}\nu(ds)\bigg)\\ &\le \exp\big(-(3r/4)H^{-1}(1/t) + (5/4)H^{-1}(1/t)tb(t)\big) \le \exp\big(-8^{-1}rH^{-1}(1/t)\big).\end{aligned}$$

We used the fact that $e^y - 1 \le ye^{-y}e^{2y}$ for all $y \ge 0$ in the third line. Hence, we deduce from Lemma 2.1.4 that $\mathbb{P}(S_t^1 \geq 3r/4) \leq c_5 t w(r)$ and hence the first assertion holds.

The second assertion follows from the first one and $\mathbf{Poly}_{R_1,\geq}(\beta_1)$. □

By a similar argument, we get the following result for large time t.

Theorem 2.2.7. *Assume that* $\mathbf{Poly}^{\infty}_{R_2,\leq}(\beta_2)$ *holds. Then, for every* $r_0 > 0$*, there exists* $C > 1$ *such that for all* $t > 0$*,* $r > r_0$ *satisfying* $t\phi(r^{-1}) \leq 1/2$*,*

$$C^{-1} t w(r) \leq \mathbb{P}(S_t \geq r) \leq C t w(r).$$

Moreover, if $\mathbf{Poly}^{\infty}_{R_2,\geq}(\beta_1)$ *also holds, then for every* $r_0 > 0$*, there exists a constant* $C' > 1$ *such that for all* $t > 0$*,* $r > r_0$ *satisfying* $t\phi(r^{-1}) \leq 1/2$*,*

$$C'^{-1} t w(r) \leq \mathbb{P}(S_t \in [r, kr]) \leq C' t w(r).$$

Next, we study right tail estimates for subordinators with extremely small tails. Observe that under $\mathbf{Sub}^{\infty}(\gamma, \theta)$ or $\mathbf{Trun}^{\infty}_{R_2}$, for every $r_0 > 0$, there are comparison constants such that $\phi(r^{-1}) \simeq r^{-1}$ for $r > r_0$ because of (2.0.6).

Theorem 2.2.8. *Assume that* $\mathbf{Sub}^{\infty}(\gamma, \theta)$ *holds. Then, for every* $r_0 > 0$*, there exist constants* $C, L > 0$ *such that for all* $r > r_0$ *and* $0 < t \leq Lr$*,*

$$\mathbb{P}(S_t \geq r) \leq C t \exp\big(-\frac{\theta}{2} r^{\gamma}\big).$$

Proof. Fix $r > r_0$ and $0 < t \leq Lr$ where the constant $L > 0$ will be chosen later. Set

$$\widehat{\mu}^1(ds) := \mathbf{1}_{(0,r]}\, \nu(ds) \quad \text{and} \quad \widehat{\mu}^2(ds) := \mathbf{1}_{(r,\infty)}\, \nu(ds).$$

We denote by $\widehat{S}^1$ and $\widehat{S}^2$ the independent driftless subordinators with Lévy measures $\widehat{\mu}^1$ and $\widehat{\mu}^2$, respectively. Then, we have $S_t = \widehat{S}^1_t + \widehat{S}^2_t$ and hence

$$\mathbb{P}(S_t \geq r) \leq \mathbb{P}(\widehat{S}^1_t \geq r) + \mathbb{P}(\widehat{S}^2_t > 0).$$

Because $\widehat{S}_t^2$ is a compound Poisson process, we get from $\mathbf{Sub}^\infty(\gamma,\theta)$ that

$$\mathbb{P}(\widehat{S}_t^2 > 0) = 1 - e^{-tw(r)} \le tw(r) \le c_1 t e^{-\theta r^\gamma}.$$

Next, using Markov's inequality and Lemma 2.2.5, we get that for

$$\begin{aligned}\mathbb{P}(\widehat{S}_t^1 \ge r) &\le r^{-1} e^{-\lambda r}\mathbb{E}\big[\widehat{S}_t^1 \exp(\lambda \widehat{S}_t^1)\big]\\ &= r^{-1}e^{-\lambda r} t\Big[\int_{(0,r]} s e^{\lambda s}\nu(ds)\Big]\exp\Big(t\int_{(0,r]}(e^{\lambda s}-1)\nu(ds)\Big). \quad (2.2.7)\end{aligned}$$

By integration by parts and $\mathbf{Sub}^\infty(\gamma,\theta)$, we see that

$$\begin{aligned}&\int_{(0,r]} s e^{\lambda s}\nu(ds) = -re^{\lambda r}w(r) + \int_0^r (1+\lambda s)w(s)e^{\lambda s}ds\\ &\le 2\lambda e^\lambda \int_0^1 w(s)ds + c_1\int_1^r (1+\lambda s)\exp\big(-\theta s^\gamma + \lambda s\big)ds \qquad (2.2.8)\end{aligned}$$

and

$$\begin{aligned}&\int_{(0,r]} (e^{\lambda s}-1)\nu(ds) = -(e^{\lambda r}-1)w(r) + \lambda\int_0^r w(s)e^{\lambda s}ds\\ &\le \lambda e^\lambda \int_0^1 w(s)ds + c_1\lambda\int_1^r s\exp\big(-\theta s^\gamma + \lambda s\big)ds. \qquad (2.2.9)\end{aligned}$$

Take $\lambda = 2\theta r^{\gamma-1}/3$. Then, because $\gamma \le 1$, we obtain that

$$\int_1^r s\exp\big(-\theta s^\gamma + \lambda s\big)ds \le \int_1^r s\exp\big(-\frac{\theta}{3}s^\gamma\big)ds \le c_2.$$

Thus, since $\lambda \le 2\theta r_0^{\gamma-1}/3$ and $\int_0^1 w(s)ds < \infty$, we deduce from (2.2.7) that

$$\mathbb{P}(\widehat{S}_t^1 \ge r) \le c_4 t r^{-1}\exp\big(-\lambda r + c_5 t\lambda e^\lambda\big) \le c_6 t r^{-1}\exp\big(-\frac{2\theta}{3}r^\gamma + c_7 t r^{\gamma-1}\big).$$

Set $L := \theta/(6c_7)$. Then, because $t \le Lr$, we conclude that

$$\mathbb{P}(\widehat{S}_t^1 \ge r) \le c_8 t\exp\big(-\frac{2\theta}{3}r^\gamma + c_7 L r^\gamma\big) \le c_8 t\exp\big(-\frac{\theta}{2}r^\gamma\big).$$

The proof is complete. □

When $\mathbf{Sub}^{\infty}(\gamma,\theta)$ holds with $\gamma<1$, we get upper bounds for $\mathbb{P}(S_t \geq r)$ which decrease with exactly the same rate as the bounds for w as $r\to\infty$.

Theorem 2.2.9. *Assume that $\mathbf{Sub}^{\infty}(\gamma,\theta)$ holds with $\gamma<1$. Then, for every $r_0>0$, there exist constants $A>0$ independent of r_0 and $C=C(r_0)>0$ such that for all $r>r_0$ and $0<t\leq r$,*

$$\mathbb{P}(S_t\geq r)\leq Ct\exp\big(-\theta r^{\gamma}+Atr^{\gamma-1}\big).$$

Proof. Fix $r>r_0$ and $0<t\leq r$. We define $\widetilde{\mu}^1(ds)=\mathbf{1}_{(0,r/2]}\,\nu(ds)$ and $\widetilde{\mu}^2(ds)=\mathbf{1}_{(r/2,\infty)}\,\nu(ds)$, and denote by $\widetilde{S}^1$ and $\widetilde{S}^2$ the independent driftless subordinators with Lévy measures $\widetilde{\mu}^1$ and $\widetilde{\mu}^2$, respectively. Then since $S_t=\widetilde{S}^1_t+\widetilde{S}^2_t$, we obtain that

$$\mathbb{P}(S_t\geq r)\leq\int_0^{r-r_0/2}\mathbb{P}(\widetilde{S}^2_t\geq r-u)\mathbb{P}(\widetilde{S}^1_t\in du)+\mathbb{P}(\widetilde{S}^1_t\geq r-r_0/2). \quad (2.2.10)$$

Let $\lambda:=\theta r^{\gamma-1}\in(0,\theta r_0^{\gamma-1})$. Then we see that

$$\int_1^{r/2}\exp\big(-\theta s^{\gamma}+\lambda s\big)ds\leq\int_1^{\infty}\exp\big(-\theta s^{\gamma}(1-2^{\gamma-1})\big)ds<\infty.$$

Hence, by (2.2.8) and (2.2.9) (with $r/2$ instead of r), we get that

$$\int_{(0,r/2]}se^{\lambda s}\nu(ds)\leq c_1\quad\text{and}\quad\int_{(0,r/2]}(e^{\lambda s}-1)\nu(ds)\leq c_1\lambda.$$

Using the above inequalities, Markov's inequality and Lemma 2.2.5, we deduce that for all $u>0$,

$$\begin{aligned}\mathbb{P}(\widetilde{S}^1_t\geq u)&\leq e^{-\lambda u}\mathbb{E}\big[\exp(\lambda\widetilde{S}^1_t)\big]=e^{-\lambda u}\exp\left(t\int_{(0,r/2]}(e^{\lambda s}-1)\nu(ds)\right)\\&\leq\exp\big(-\theta r^{\gamma-1}u+c_1\theta tr^{\gamma-1}\big) \quad (2.2.11)\end{aligned}$$

and

$$\begin{aligned}
\mathbb{P}(\widetilde{S}_t^1 \geq u) &\leq u^{-1}e^{-\lambda u}\mathbb{E}\big[\widetilde{S}_t^1 \exp(\lambda \widetilde{S}_t^1)\big] \\
&= tu^{-1}e^{-\lambda u}\left[\int_{(0,r/2]} se^{\lambda s}\nu(ds)\right]\exp\left(t\int_{(0,r/2]}(e^{\lambda s}-1)\nu(ds)\right) \\
&\leq c_1 tu^{-1}\exp\big(-\theta r^{\gamma-1}u + c_1\theta t r^{\gamma-1}\big).
\end{aligned}$$

In particular, since $r - r_0/2 > r_0/2$ and $r^{\gamma-1}r_0 < r_0^\gamma$, it holds that

$$\mathbb{P}(\widetilde{S}_t^1 \geq r - r_0/2) \leq c_3 t \exp\big(-\theta r^\gamma + c_1\theta t r^{\gamma-1}\big). \tag{2.2.12}$$

We note that, the above constant c_1 can be chosen independent of r_0.

On the other hand, we observe that $\widetilde{S}_t^2 = \sum_{i=1}^{N(t)} D_i$ where $N(t)$ is a Poisson random variable with mean $tw(r/2)$ and D_i are i.i.d. random variables with distribution $\mathbb{P}(D_i > u) = w(u \vee (r/2))/w(r/2)$. Thus, for all $u > 0$, since $t < r$ and $\sup_{x>0} xe^{-(2^{1-\gamma}-1)\theta x^\gamma} < \infty$, we get from $\mathbf{Sub}^\infty(\gamma,\theta)$ that

$$\begin{aligned}
\mathbb{P}(\widetilde{S}_t^2 \geq u) &\leq \mathbb{P}(N(t)=1)\mathbb{P}(D_1 \geq u) + \mathbb{P}(N(t)\geq 2) \\
&\leq tw(u \vee (r/2)) + (tw(r/2))^2 \\
&\leq c_4 t \exp\big(-\theta(u \vee (r/2))^\gamma\big) + c_4 tr \exp\big(-2^{1-\gamma}\theta r^\gamma\big) \\
&\leq c_4 t \exp\big(-\theta(u \vee (r/2))^\gamma\big) + c_5 t\exp(-\theta r^\gamma).
\end{aligned}$$

Hence, by integration by parts and (2.2.11), since $\gamma < 1$, we obtain that

$$\begin{aligned}
&\int_0^{r-r_0/2} \mathbb{P}(\widetilde{S}_t^2 \geq r-u)\mathbb{P}(\widetilde{S}_t^1 \in du) \\
&\leq c_4 t\int_0^{r/2} \exp\big(-\theta(r-u)^\gamma\big)\mathbb{P}(\widetilde{S}_t^1 \in du) \\
&\quad + c_4 t\int_{r/2}^{r-r_0/2} \exp\big(-\theta(r/2)^\gamma\big)\mathbb{P}(\widetilde{S}_t^1 \in du) + c_5 t\exp(-\theta r^\gamma)\int_0^\infty \mathbb{P}(\widetilde{S}_t^1 \in du) \\
&\leq c_4\gamma\theta t\int_0^{r/2}(r-u)^{\gamma-1}\exp\big(-\theta(r-u)^\gamma\big)\mathbb{P}(\widetilde{S}_t^1 \geq u)du + c_4 t\exp(-\theta r^\gamma) \\
&\quad + c_4 t\exp\big(-\theta(r/2)^\gamma\big)\mathbb{P}(\widetilde{S}_t^1 \geq r/2) + c_5 t\exp(-\theta r^\gamma)
\end{aligned}$$

$$\leq c_6\gamma\theta t \exp\big(c_1\theta t r^{\gamma-1}\big) r^{\gamma-1}\int_0^{r/2} \exp\big(-\theta(r-u)^\gamma - \theta r^{\gamma-1}u\big)du$$
$$+ c_7 t \exp\big(-\theta r^\gamma + c_1\theta t r^{\gamma-1}\big). \tag{2.2.13}$$

Define $f:(0,r/2)\to(0,\infty)$ by $f(u) := (r-u)^\gamma + r^{\gamma-1}u - (2^{1-\gamma}-1)r^{\gamma-1}u$. Then there exists $r_* \in (0,r/2)$ such that f is increasing on $(0,r_*)$ and decreasing on $(r_*, r/2)$. Hence, $\inf_{u\in(0,r/2)} f(u) = f(0)\wedge f(r/2) = r^\gamma$. Therefore, by using the change of variables $r^{\gamma-1}u = s$, we get that

$$\begin{aligned}
&r^{\gamma-1}\int_0^{r/2} \exp\big(-\theta(r-u)^\gamma - \theta r^{\gamma-1}u\big)du\\
&\leq r^{\gamma-1}\exp(-\theta r^\gamma)\int_0^{r/2}\exp\big(-(2^{1-\gamma}-1)r^{\gamma-1}u\big)du\\
&\leq \exp(-\theta r^\gamma)\int_0^\infty e^{-(2^{1-\gamma}-1)s}ds = c_8\exp(-\theta r^\gamma).
\end{aligned}$$

In the end, we get the desired result from (2.2.10), (2.2.12) and (2.2.13). □

Theorem 2.2.10. *Assume that* $\mathbf{Trun}^\infty_{R_2}$ *holds.*
(i) It holds that for all $0<t<1$ *and* $0<r<R_2/2$ *satisfying* $t\phi(r^{-1})\leq 1$,

$$\mathbb{P}(S_t \geq r) \simeq tw(r).$$

(ii) It holds that for all $0<t<1$ *and* $r\geq R_2/2$,

$$\mathbb{P}(S_t\geq r) \asymp (t+(nR_2-r)^n)t^n\exp\big(-cr\log r\big), \qquad n:=\lfloor r/R_2\rfloor + 1.$$

(iii) There exists a constant $L>0$ *such that for all* $r\geq R_2/2$ *and* $1\leq t\leq Lr$,

$$\mathbb{P}(S_t\geq r)\asymp \exp\big(-cr\log\frac{r}{t}\big).$$

Proof. (i) Since $\mathbf{Trun}^\infty_{R_2}$ implies $\mathbf{Poly}_{R_2/2,\leq}(\beta_2)$, the result follows from Theorem 2.2.6.
(ii) Fix $0<t<1$ and $r\geq R_2/2$. Then we set $n:=\lfloor r/R_2\rfloor+1$.

(Lower bound) Let U^1 and U^2 be the driftless subordinators with Lévy measures $\nu_1(ds) := \mathbf{1}_{(r/(n+1),\infty)}\,\nu(ds)$ and $\nu_2(ds) := \mathbf{1}_{(r/n,\infty)}\,\nu(ds)$, respectively. Note that $\nu_1((0,\infty)) \le \nu_2((0,\infty)) \le w(R_2/2)$. Since $\mathbb{P}(S_t \ge r) \ge \mathbb{P}(U_t^2 \ge r) \ge \mathbb{P}(U_t^1 \ge r)$, it holds that

$$\begin{aligned}2\mathbb{P}(S_t \ge r) &\ge \mathbb{P}(U_t^1 \ge r) + \mathbb{P}(U_t^2 \ge r)\\ &\ge \mathbb{P}\big(|\{0 \le s \le t : \Delta U_s^1 > r/(n+1)\}| \ge n+1\big)\\ &\quad + \mathbb{P}\big(|\{0 \le s \le t : \Delta U_s^2 > r/n\}| \ge n\big)\\ &\ge e^{-tw(R_2/2)}\left(\frac{t^{n+1}w(r/(n+1))^{n+1}}{(n+1)!} + \frac{t^n w(r/n)^n}{n!}\right).\end{aligned}$$

Since $r/n \ge r/(n+1) \ge R_2/4$ and $nR_2 \ge r$, we see from $\mathbf{Trun}_{R_2}^{\infty}$ that

$$w(r/(n+1)) = w(r/(n+1)) - w(R_2) \ge \frac{(n+1)R_2 - r}{K(n+1)} \ge \frac{R_2}{K(n+1)}$$

and $w(r/n) = w(r/n) - w(R_2) \ge (nR_2 - r)/(Kn)$. Then using Stirling's formula, since $n \simeq r$, $t < 1$ and $r > r_0$, we deduce that

$$\begin{aligned}\mathbb{P}(S_t \ge r) &\ge e^{-tw(R_2/2)}\left(\frac{t^{n+1}R_2^{n+1}}{2K^{n+1}(n+1)^{n+1}(n+1)!} + \frac{t^n(nR_2 - r)^n}{2K^n n^n n!}\right)\\ &\ge c_1(t + (nR_2 - r)^n)t^n \exp\big(-c_2 r\log r\big).\end{aligned}$$

(Upper bound) Let U^3 and U^4 be the driftless subordinators with Lévy measures $\nu_3(ds) := \mathbf{1}_{(0,R_2/9]}\,\nu(ds)$ and $\nu_4(ds) := \mathbf{1}_{(R_2/9,\infty)}\,\nu(ds)$, respectively. Then, $S_t = U_t^3 + U_t^4$ and $U_t^4 = \sum_{i=1}^{P(t)} J_i$ where $P(t)$ is a Poisson random variable with mean $tw(R_2/9)$ and J_i are i.i.d. random variables with distribution

$$F(u) := \mathbb{P}(J_1 \ge u) = w(u \vee (R_2/9))/w(R_2/9). \tag{2.2.14}$$

Hence, it holds that

$$\mathbb{P}(S_t \ge r) = \sum_{j=0}^{\infty}\mathbb{P}\big(U_t^3 + U_t^4 \ge r, P(t) = j\big)$$

$$\leq \mathbb{P}(U_t^3 \geq r) + \sum_{j=1}^{n} \mathbb{P}\big(U_t^3 + U_t^4 \geq r \,|\, P(t) = j\big)\mathbb{P}\big(P(t) = j\big) + \mathbb{P}(P(t) > n).$$

By Stirling's formula, since $P(t)$ is a Poisson random variable and $1 \leq n \simeq r$, we see that

$$\mathbb{P}(P(t) > n) \leq \frac{(tw(R_2/9))^{n+1}}{(n+1)!} \leq c_3 t^{n+1} \exp\big(-c_4 r \log r\big).$$

Next, using Markov's inequality and Lemma 2.2.5, we get that for all $u, \lambda > 0$,

$$\begin{aligned}\mathbb{P}(U_t^3 \geq u) &\leq e^{-\lambda u}\mathbb{E}\big[\exp\big(\lambda U_t^3\big)\big] = e^{-\lambda u} \exp\left(t \int_{(0,R_2/9]} (e^{\lambda s} - 1)\nu(ds)\right) \\ &\leq e^{-\lambda u} \exp\left(\lambda e^{\lambda R_2/9} t \int_{(0,R_2/9]} s\nu(ds)\right) = e^{-\lambda u} \exp\big(c_5 \lambda e^{\lambda R_2/9} t\big).\end{aligned}$$

Hence, by taking $\lambda = 9R_2^{-1}\log(u/(9c_5 t))$, we deduce that

$$\mathbb{P}(U_t^3 \geq u) \leq \big(9c_5 t/u\big)^{8u/R_2} \quad \text{for all } \; u > 0. \tag{2.2.15}$$

In particular, since $t < 1$, $n \simeq r$ and $n + 1 < 2 + r/R_2 < 8r/R_2$, we get that

$$\mathbb{P}(U_t^3 \geq r) \leq \big(9c_5 t/r\big)^{8r/R_2} \leq t^{n+1}\big(9c_5/r\big)^{8r/R_2} \leq c_6 t^{n+1} \exp\big(-c_7 r \log r\big).$$

Moreover, when $n \geq 3$, since $r \geq (n-1)R_2$ and $n \simeq r$, using the fact that the jump sizes of U^4 are at most R_2 by $\mathbf{Trun}^{\infty}_{R_2}$, and Stirling's formula, we deduce that

$$\begin{aligned}&\sum_{j=1}^{n-2} \mathbb{P}\big(U_t^3 + U_t^4 \geq r \,|\, P(t) = j\big)\mathbb{P}\big(P(t) = j\big) \\ &\leq \sum_{j=1}^{n-2} \frac{t^j w(R_2/9)^j}{j!} \mathbb{P}\big(U_t^3 \geq r - jR_2\big) \leq \sum_{j=1}^{n-2} \frac{t^j w(R_2/9)^j}{j!} \mathbb{P}\big(U_t^3 \geq (n-1-j)R_2\big) \\ &\leq \sum_{j=1}^{n-2} \frac{t^j w(R_2/9)^j}{j!} \left(\frac{9c_5 t}{(n-j-1)R_2}\right)^{8(n-1-j)} \leq e^{c_8 n} \sum_{j=1}^{n-2} \frac{t^{8(n-1)-7j}}{j!(n-j-1)^{8(n-j-1)}}\end{aligned}$$

$$\leq c_9 t^{n+1} e^{c_{10} r} \sum_{j=1}^{n-2} \frac{1}{j!(n-j-1)^{8(n-j-1)}}$$
$$\leq c_{11} t^{n+1} e^{c_{12} r} \sum_{j=1}^{n-2} \exp\big(- c_{13} j \log j - 8(n-j-1)\log(n-j-1)\big)$$
$$\leq c_{14} t^{n+1} e^{c_{11} r} n \exp\big(- c_{15}(n-1)\log(n-1)\big) \leq c_{16} t^{n+1} \exp\big(- c_{17} r \log r\big).$$

The seventh inequality holds by the fact that $4(a\log a + b \log b) \geq 2(a \vee b)\log(2(a \vee b)) \geq (a+b)\log(a+b)$ for all $a, b \geq 1$ satisfying $a \vee b \geq 2$.

It remains to bound probabilities $\mathbb{P}(U_t^3 + U_t^4 \geq r, P(t) = j)$ for $j = n-1$ (when $n \geq 2$) and $j = n$. Observe that by Stirling's formula, we have

$$\mathbb{P}(U_t^3 + U_t^4 \geq r \,|\, P(t) = n-1)\mathbb{P}(P(t) = n-1)$$
$$\leq \frac{t^{n-1} w(R_2/9)^{n-1}}{(n-1)!} \int_0^{(n-1)R_2} \mathbb{P}(U_t^3 \geq r-u) d_u \mathbb{P}\big(\sum_{i=1}^{n-1} J_i \leq u\big)$$
$$\leq c_{18} t^{n-1} e^{-c_{19} r \log r}\big(A_1 + \mathbb{P}(U_t^3 \geq R_2/4)\big),$$

where

$$A_1 := - \int_{(r-R_2/4)\wedge((n-1)R_2)}^{(n-1)R_2} \mathbb{P}(U_t^3 \geq r-u) d_u \mathbb{P}(\sum_{i=1}^{n-1} J_i \geq u).$$

Similarly, we also have that

$$\mathbb{P}\big(U_t^3 + U_t^4 \geq r \,|\, P(t) = n\big)\mathbb{P}\big(P(t) = n\big) \leq c_{20} t^n e^{-c_{21} r \log r}\big(A_2 + \mathbb{P}(U_t^3 \geq R_2/4)\big),$$

where

$$A_2 := - \int_{r-R_2/4}^{nR_2} \mathbb{P}(U_t^3 \geq r-u) d_u \mathbb{P}(\sum_{i=1}^{n} J_i \geq u).$$

By (2.2.15), since $t < 1$, we have $\mathbb{P}(U_t^3 \geq R_2/4) \leq c_{22} t^2$.

Now, we bound A_1 and A_2. Since $\lim_{\lambda\to\infty} \phi(\lambda)/\lambda = 0$, there exists $t_0 \in (0, R_2/4)$ such that $s\phi(s^{-1}) \leq 1/2$ for all $0 < s < t_0$. If $t \geq t_0$, then $A_1 \leq 1 \leq t_0^{-2} t^2$ and $A_2 \leq 1 \leq t_0^{-1} t$ and hence we are done. Suppose that $t < t_0$.

To bound A_1 and A_2, we set $K' := K + 2w(R_2/9)/R_2$ with the constant $K \geq 1$ in $\mathbf{Trun}^{\infty}_{R_2}$ and claim that

$$\mathbb{P}\big(\sum_{i=1}^{m} J_i \geq mR_2 - u\big) \leq \left(\frac{K'u}{w(R_2/9)}\right)^m \quad \text{for all } m \in \mathbb{N},\ u > 0. \qquad (2.2.16)$$

Indeed, (2.2.16) clearly holds for all $u \in \mathbb{N}$ and $u \geq R_2/2$ since $\mathbb{P}\big(\sum_{i=1}^{m} J_i \geq mR_2 - u\big) \leq 1$. When $m = 1$, by (2.2.14) and $\mathbf{Trun}^{\infty}_{R_2}$, we see that for all $u \in (0, R_2/2)$,

$$\mathbb{P}(J_1 \geq R_2 - u) = \frac{w(R_2 - u)}{w(R_2/9)} = \frac{w(R_2 - u) - w(R_2)}{w(R_2/9)} \leq \frac{K'u}{w(R_2/9)}. \qquad (2.2.17)$$

Suppose that (2.2.16) holds for m. Then, by using (2.2.17) and the induction hypothesis, we get that for all $u \in (0, R_2/2)$,

$$\begin{aligned}
&\mathbb{P}\big(\sum_{i=1}^{m+1} J_i \geq (m+1)R_2 - u\big) \\
&= \int_{\{\sum_{i=1}^m u_i \leq u\}} \mathbb{P}\big(J_{m+1} \geq R_2 - u + \sum_{i=1}^{m} u_i\big) d_{u_m}F(R_2 - u_m) \cdots d_{u_1}F(R_2 - u_1) \\
&\leq \frac{K'}{w(R_2/9)} \int_{\{\sum_{i=1}^m u_i \leq u\}} \big(u - \sum_{i=1}^{m} u_i\big) d_{u_m}F(R_2 - u_m) \cdots d_{u_1}F(R_2 - u_1) \\
&\leq \frac{K'u}{w(R_2/9)} \int_{\{\sum_{i=1}^m u_i \leq u\}} d_{u_m}F(R_2 - u_m) \cdots d_{u_1}F(R_2 - u_1) \\
&= \frac{K'u}{w(R_2/9)} \mathbb{P}\big(\sum_{i=1}^{m} J_i \geq mR_2 - u\big) \leq \left(\frac{K'u}{w(R_2/9)}\right)^{m+1}.
\end{aligned}$$

Therefore, (2.2.16) holds by induction.

We consider the following two cases separately.

Case 1. $(n - 3/4)R_2 \leq r < nR_2$.

In this case, we have $A_1 \leq \mathbb{P}(U_t^3 \geq R_2/4) \leq c_{22}t^2$ by (2.2.15). Besides, by

Theorem 2.2.6, since $t\phi(t^{-1}) \leq 1/2$, it holds that

$$A_2 \leq -\int_{r-R_2/4}^{r-t} \mathbb{P}(U_t^3 \geq r-u) d_u \mathbb{P}(\sum_{i=1}^n J_i \geq u) - \int_{r-t}^{nR_2} d_u \mathbb{P}(\sum_{i=1}^n J_i \geq u)$$
$$\leq -c_{23} t \int_{r-R_2/4}^{r-t} w(r-u) d_u \mathbb{P}(\sum_{i=1}^n J_i \geq u) + \mathbb{P}(\sum_{i=1}^n J_i \geq r-t).$$

Using integration by parts in the second line below, (2.2.16) in the third, the inequality $(a+b)^n \leq 2^n(a^n+b^n)$ for $a, b > 0$ in the fourth, and (2.0.4) and the facts that $n \simeq r$ and $t\phi(1/t) \leq 1/2$ in the last, we get that

$$-t\int_{r-R_2/4}^{r-t} w(r-u) d_u \mathbb{P}(\sum_{i=1}^n J_i \geq u) = t\int_t^{R_2/4} w(u) d_u \mathbb{P}(\sum_{i=1}^n J_i \geq r-u)$$
$$\leq tw(R_2/4)\mathbb{P}(\sum_{i=1}^n J_i \geq r - R_2/4) + t\int_t^{R_2/4} \mathbb{P}(\sum_{i=1}^n J_i \geq r-u)\nu(du)$$
$$\leq tw(R_2/4) + c_{26} t e^{c_{27} n} \int_t^{R_2/4} (u + nR_2 - r)^n \nu(du)$$
$$\leq tw(R_2/4) + c_{26} t e^{c_{27} n} 2^n \int_t^{R_2/4} u^n \nu(du) + c_{26} t e^{c_{27} n} 2^n (nR_2 - r)^n \int_t^{R_2/4} \nu(du)$$
$$\leq tw(R_2/4) + c_{26} t e^{c_{27} n} 2^n (R_2/4)^{n-1} \int_0^\infty u\nu(du) + c_{26} t w(t) e^{c_{27} n} 2^n (nR_2 - r)^n$$
$$\leq c_{28} e^{c_{29} r}(t + t\phi(1/t)(nR_2 - r)^n) \leq c_{28} e^{c_{29} r}(t + (nR_2 - r)^n).$$

Using (2.2.16), the inequality $(a+b)^n \leq 2^n(a^n+b^n)$ for $a, b > 0$ and the fact $n \simeq r$ again, we also get that

$$\mathbb{P}\big(\sum_{i=1}^n J_i \geq r-t\big) \leq c_{30} e^{c_{31} n}(t + nR_2 - r)^n \leq c_{30} e^{c_{31} r}(t + (nR_2 - r)^n).$$

The proof is complete in this case.

Case 2. $(n-1)R_2 \leq r < (n-3/4)R_2$.

Note that $(nR_2 - r) \simeq 1$ in this case. By Theorem 2.2.6 and (2.2.16), since $t\phi(1/t) \le 1/2$, $n \simeq r$ and $t < 1$, we see that when $n \ge 2$,

$$A_1 \le -\int_{r-R_2/4-t}^{(n-1)R_2-t} \mathbb{P}(U_t^3 \ge r-u)d_u\mathbb{P}(\sum_{i=1}^{n-1} J_i \ge u) + \mathbb{P}(\sum_{i=1}^{n-1} J_i \ge (n-1)R_2 - t)$$
$$\le -c_{32}t\int_{r-R_2/4-t}^{(n-1)R_2-t} w(r-u)d_u\mathbb{P}(\sum_{i=1}^{n-1} J_i \ge u) + c_{32}e^{c_{33}r}t.$$

Using similar arguments to the ones for Case 1, we obtain from integration by parts, (2.2.16) and the fact $n \simeq r$ that

$$\int_{r-R_2/4-t}^{(n-1)R_2-t} w(r-u)d_u\mathbb{P}(\sum_{i=1}^{n-1} J_i \ge u) = \int_{r-(n-1)R_2+t}^{R_2/4+t} w(u)d_u\mathbb{P}(\sum_{i=1}^{n-1} J_i \ge r-u)$$
$$\le w(R_2/4+t) + \int_{r-(n-1)R_2+t}^{R_2/4+t} \mathbb{P}(\sum_{i=1}^{n-1} J_i \ge r-u)\nu(du)$$
$$\le w(R_2/4+t) + c_{32}e^{c_{33}r}\int_{r-(n-1)R_2+t}^{R_2/4+t} u^{n-1}\nu(du) \le w(R_2/4+t) + c_{34}e^{c_{35}r}.$$

Since $A_2 \le 1$, we can also conclude the desired result in this case.

(iii) Pick any $L \in (0, 1 \wedge (R_2/2))$ such that $w(L) \ge 1$. Since $w(0+) = \infty$, we can always find such constant L. Fix $r \ge R_2/2$ and $0 < t \le Lr$. Let $k := \lfloor r/L \rfloor + 1$ and U^5 be the driftless subordinator with Lévy measure $\mathbf{1}_{(L,\infty)}\,\nu(ds)$. Since $S_t \ge U_t^5$ and the jump sizes of U^5 are at least L, using Stirling's formula and the inequality $x^{1/2} \le e^x$ for $x > 0$, we get

$$\mathbb{P}(S_t \ge r) \ge \mathbb{P}(U_t^5 \ge r) \ge \mathbb{P}(U^5 \text{ jumps } k \text{ times before time } t)$$
$$= e^{-tw(L)}\frac{(tw(L))^k}{k!} \ge e^{-tw(L)}\frac{e^{k-1}}{k^{1/2}}\left(\frac{t}{k}\right)^k \ge e^{-1-rLw(L)}\left(\frac{t}{k}\right)^k.$$

Since $r/L + 1 \ge k > r/L \ge t$ and $r \ge R_2/2 > L$, we have

$$e^{-rLw(L)}\left(\frac{t}{k}\right)^k \ge \exp\Big(-rLw(L) - (1+r/L)\log\frac{r+L}{Lt}\Big)$$
$$\ge \exp\Big(-rLw(L) - \frac{2r}{L}\log\frac{2r}{Lt}\Big) \ge \exp\Big(-c_{36}r\log\frac{r}{t}\Big).$$

and deduce the lower bound.

Next, we set $c_{37} := \int_0^{R_2} s\nu(ds)$ and $\lambda := R_2^{-1}\log(t/(2c_{37}r))$. By Markov's inequality and Lemma 2.2.5, since $t \le Lr \le r$, we obtain that

$$\begin{aligned}\mathbb{P}(S_t \ge r) &\le e^{-\lambda r}\mathbb{E}\big[e^{\lambda S_t}\big] = e^{-\lambda r}\exp\left(t\int_0^{R_2}(e^{\lambda s}-1)\nu(ds)\right)\\ &\le e^{-\lambda r}\exp\left(\lambda e^{\lambda R_2} t\int_0^{R_2} s\nu(ds)\right) = \exp\big(-\lambda r + c_{37}\lambda e^{\lambda R_2} t\big)\\ &= \exp\Big(\Big(-\frac{r}{R_2}+\frac{t^2}{2rR_2}\Big)\log\frac{t}{2c_{37}r}\Big) \le \exp\Big(-\frac{r}{2R_2}\log\frac{t}{2c_{37}r}\Big)\end{aligned}$$

and hence the upper bound holds. We have finished the proof. □

2.3 Transition density estimates

Recall that S is a driftless subordinator with Laplace exponent ϕ whose tail measure w satisfies that $w(0+) = \infty$. Throughout this section, we also assume that the Lévy measure ν has a density function $\nu(x)$ and the following condition holds: There exists a constant $T_0 \in [0,\infty)$ such that

$$\liminf_{x\to 0} x\nu(x) = 1/T_0 \quad \text{with the convention } 1/0=\infty. \tag{2.3.1}$$

(2.3.1) implies that

$$\liminf_{|\xi|\to\infty} \frac{\operatorname{Re}\phi(i\xi)}{\log(1+|\xi|)} \ge \frac{1}{T_0}. \tag{2.3.2}$$

Hence, under (2.3.1), by [79, (64) and (74)] (see also [99, ($\mathrm{HW}_{1/t}$)]), we get the existence of a continuous bounded transition desity of S_t for $t > T_0$.

Proposition 2.3.1. *For all $t > T_0$, the transition density $p(t,x)$ of S_t exists and is a continuous bounded function on $(0,\infty)$ as a function of x.*

In this section, we establish two-sided estimates and the exact asymptotic behaviors of the transition density $p(t,x)$. We consider the following conditions for the Lévy measure ν.

Definition 2.3.2. Let $R_1 \in (0,\infty]$, $R_2 > 0$ and $\beta_2 \geq \beta_1 > 0$ be constants.
(i) We say that $\mathbf{Poly}^*_{R_1}(\beta_1,\beta_2)$ holds if there are $c_1, c_2 > 0$ such that

$$c_1\Big(\frac{r}{s}\Big)^{1+\beta_1} \leq \frac{\nu(s)}{\nu(r)} \leq c_2\Big(\frac{r}{s}\Big)^{1+\beta_2} \quad \text{for all } \ 0 < s \leq r < R_1. \tag{2.3.3}$$

We say that $\mathbf{Poly}^*_{R_1,\leq}(\beta_2)$ (resp. $\mathbf{Poly}^*_{R_1,\geq}(\beta_1)$) holds if the upper bound (resp. lower bound) in (2.3.3) holds.
(ii) We say that $\mathbf{Reg}_{R_1}$ holds, if there are constants $c_1, \delta > 0$ such that

$$\frac{\nu(s)}{\nu(r)} \geq c_3\left(\frac{r}{s}\right)^{-\delta} \qquad \text{for all } \ 0 < s \leq r \leq R_1.$$

(iii) We say that $\mathbf{Dou}^\infty_{R_1}$ holds, if there are $c_1, c_2 > 0$ such that

$$c_1 \sup_{u \geq r} \nu(u) \leq \nu(r) \quad \text{and} \quad c_2\nu(r/2) \leq \nu(r) \quad \text{for all } \ r \geq R_1.$$

(iv) We say that $\mathbf{Poly}^{*,\infty}_{R_2}(\beta_1,\beta_2)$ holds if (2.3.3) holds for all $r \geq s \geq R_2$. We say that $\mathbf{Poly}^{*,\infty}_{R_2,\leq}(\beta_2)$ (resp. $\mathbf{Poly}^{*,\infty}_{R_2,\geq}(\beta_1)$) holds if the upper bound (resp. lower bound) in (2.3.3) holds for all $r \geq s \geq R_2$.

Remark 2.3.3. (i) If $\mathbf{Poly}^*_{R_1,\geq}(\beta_1)$ holds, then $\mathbf{Reg}_{R_1}$ holds and the condition (2.3.1) holds with $T_0 = 0$.
(ii) The constant β_1 in $\mathbf{Poly}^*_{R_1,\geq}(\beta_1)$ should be less than 1. Indeed, since

$$\infty > \int_0^{R_1/2} s\nu(s)ds \geq c_1(R_1/2)^{-1-\beta_1}\nu(R_1/2)\int_0^r s^{-\beta_1}ds,$$

it must hold that $\beta_1 < 1$.
(iii) The condition $\mathbf{Reg}_{R_1}$ is very mild. For instance, if the Lévy density is almost decreasing, then it holds trivially. Therefore, every subordinator whose Laplace exponent is a complete Bernstein function satisfies this assumption since its Lévy measure has a completely monotone density. (See [116, Chapter 16] for examples of complete Bernstein functions.)

Recall that $2eH(\lambda) \geq w(1/\lambda)$ for all $\lambda > 0$. Thus, $H^{-1}(1/t)^{-1} \geq w^{-1}(2e/t)$

for all $t > 0$. We define for $t > 0$ and $y \geq 0$,

$$D(t) := t \max_{s \in [w^{-1}(2e/t), H^{-1}(1/t)^{-1}]} sH(s^{-1}) \tag{2.3.4}$$

and

$$\theta(t,y) := \begin{cases} H^{-1}(1/t)^{-1} & \text{if } y < H^{-1}(1/t)^{-1}, \\ \min\{s \geq w^{-1}(2e/t) : tsH(s^{-1}) \leq y\} & \text{if } y \in [H^{-1}(1/t)^{-1}, D(t)], \\ w^{-1}(2e/t) & \text{if } y > D(t). \end{cases} \tag{2.3.5}$$

Note that the minimum in the definition of θ attained at some $s \in [w^{-1}(2e/t), H^{-1}(1/t)^{-1}]$. We observe that for each fixed $t > 0$, the map $y \mapsto \theta(t,y)$ is nonincreasing and for each fixed $y \geq 0$, $\lim_{t\to 0} \theta(t,y) = 0$ and $\lim_{t\to\infty} \theta(t,y) = \infty$.

Recall that $b(t) = (\phi' \circ H^{-1})(1/t)$ and $\sigma = \sigma(t,r) = (\phi'^{-1})(r/t)\mathbf{1}_{(0,\phi'(0+))}(r/t)$. The following theorems are the main results of this section. See Figure 2.1.

Theorem 2.3.4. *Assume that* $\mathbf{Poly}^*_{R_1}(\beta_1,\beta_2)$ *holds. Then, for every $T > 0$, there exist constants $c_1, c_2, c_3, c_5 > 1$ and $c_4 > 0$ such that the following estimates hold for all $t \in (0,T]$.*

(i) (Left tail estimates) *It holds that for all $x \in (0, tb(t)]$,*

$$\frac{c_1^{-1}}{\sqrt{t(-\phi''(\sigma))}} \exp\big(-tH(\sigma)\big) \leq p(t,x) \leq \frac{c_1}{\sqrt{t(-\phi''(\sigma))}} \exp\big(-tH(\sigma)\big). \tag{2.3.6}$$

In particular, it holds that for all $x \in (0, tb(t)]$,

$$c_2^{-1} H^{-1}(1/t) \exp\big(-2tH(\sigma)\big) \leq p(t,x) \leq c_2 H^{-1}(1/t) \exp\big(-\frac{t}{2}H(\sigma)\big). \tag{2.3.7}$$

(ii) (Right tail estimates) *Assume also that* $\sup_{r\geq R_1} \nu(r) < \infty$. *Then it holds that for all $y \in [0, R_1/2)$,*

$$c_3^{-1} H^{-1}(1/t) \min\left\{1, \frac{t\nu(y)}{H^{-1}(1/t)} + \exp\Big(-\frac{c_4 y}{\theta(t, y/(8e^2))}\Big)\right\}$$

$$\le p(t, tb(t)+y) \le c_3 H^{-1}(1/t) \min\left\{1,\ \frac{t\nu(y)}{H^{-1}(1/t)} + \exp\Big(-\frac{y}{8\theta(t, y/(8e^2))}\Big)\right\}, \tag{2.3.8}$$

where $\theta(t,y)$ is defined as (2.3.5). In particular, for all $y \in (D(t), R_1/2)$,

$$c_5^{-1} t\nu(y) \le p(t, tb(t)+y) \le c_5 t\nu(y). \tag{2.3.9}$$

Moreover, if $\mathbf{Dou}^{\infty}_{R_1}$ also holds, then (2.3.8) holds for all $y \in [0,\infty)$, and (2.3.9) holds for all $y \in (D(t),\infty)$.

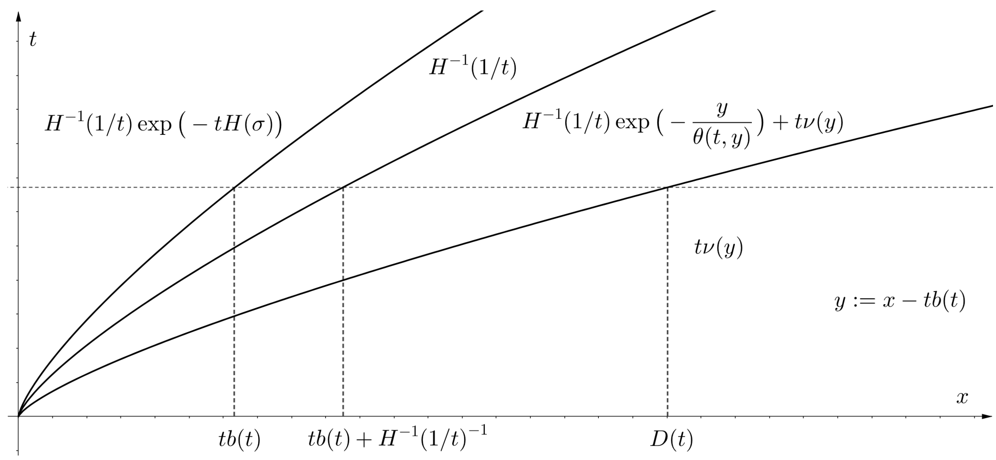

Figure 2.1: Dominant terms in estimates

Corollary 2.3.5. *Assume that $\mathbf{Poly}^{*}_{R_1}(\beta_1,\beta_2)$ holds with $\beta_2 < 2$. Then, for every $T>0$, there exists $c_1>1$ such that for all $t \in (0,T]$ and $y \in [0, R_1/2)$,*

$$c_1^{-1}\left(H^{-1}(1/t)\wedge t\nu(y)\right) \le p(t, tb(t)+y) \le c_1\left(H^{-1}(1/t)\wedge t\nu(y)\right). \tag{2.3.10}$$

Therefore, there exists $c_2>1$ such that for all $t\in(0,T]$ and $x \in (0, R_1/2)$,

$$c_2^{-1}\min\left\{\frac{\exp\big(-tH(\sigma)\big)}{\sqrt{t(-\phi''(\sigma))}},\ t\nu\big((x-tb(t))_+\big)\right\}$$
$$\le p(t,x) \le c_2 \min\left\{\frac{\exp\big(-tH(\sigma)\big)}{\sqrt{t(-\phi''(\sigma))}},\ t\nu\big((x-tb(t))_+\big)\right\}. \tag{2.3.11}$$

and

$$\begin{aligned} &c_2^{-1}H^{-1}(1/t)\min\left\{\exp\big(-2tH(\sigma)\big),\ \frac{t\nu\big((x-tb(t))_+\big)}{H^{-1}(1/t)}\right\} \\ &\le p(t,x)\le c_2H^{-1}(1/t)\min\left\{\exp\big(-\frac{t}{2}H(\sigma)\big),\ \frac{t\nu\big((x-tb(t))_+\big)}{H^{-1}(1/t)}\right\}. \quad (2.3.12)\end{aligned}$$

Moreover, if $\mathbf{Dou}_{R_1}^{\infty}$ *also holds, then* (2.3.10) *holds for all* $t \in (0,T]$ *and* $y \in [0,\infty)$*, and* (2.3.12)–(2.3.11) *hold for all* $t \in (0,T]$ *and* $x \in (0,\infty)$*.*

Below, we give large time transition density estimates. Recall that we have assumed (2.3.1) with the constant $T_0 \in [0,\infty)$.

Theorem 2.3.6. *Assume that* $\mathbf{Reg}_{R_1}$ *and* $\mathbf{Poly}_{R_2}^{*,\infty}(\beta_1,\beta_2)$ *hold.*
(i) There exist constants $T_1 > T_0$*,* $c_1, c_2, c_3, c_5 > 1$ *and* $c_4 > 0$ *such that for all* $t \in [T_1,\infty)$*,* (2.3.6) *holds for all* $x \in (0, tb(t)]$*,* (2.3.7) *holds for all* $x \in [tb(T_1), tb(t)]$*,* (2.3.8) *holds for all* $y \in [0,\infty)$ *and* (2.3.9) *holds for all* $y \in (D(t),\infty)$*.*
(ii) If $T_0 = 0$*, then for every* $T > 0$*, there are comparison constants such that for all* $t \in [T,\infty)$*,* (2.3.6) *holds for all* $x \in (0, tb(t)]$*,* (2.3.7) *holds for all* $x \in [tb(T), tb(t)]$*,* (2.3.8) *holds for all* $y \in [0,\infty)$ *and* (2.3.9) *holds for all* $y \in (D(t),\infty)$*.*

Corollary 2.3.7. *Assume that* $\mathbf{Reg}_{R_1}$ *and* $\mathbf{Poly}_{R_2}^{*,\infty}(\beta_1,\beta_2)$ *holds with* $\beta_2 < 2$*. Then, there exist constants* $T_1 > T_0$ *and* $c_1 > 1$ *such that* (2.3.10) *holds for all* $t \in [T_1,\infty)$ *and* $y \in [0,\infty)$*, and that* (2.3.11) ((2.3.12)*, respectively*) *holds for all* $t \in [T_1,\infty)$ *and* $x \in [tb(T_1),\infty)$ (*and* $x \in (0,\infty)$*, respectively*)*.*

Moreover, if $T_0 = 0$*, then for every* $T > 0$*, there are comparison constants such that* (2.3.10) *holds for all* $t \in [T,\infty)$ *and* $y \in [0,\infty)$*, and that* (2.3.11) ((2.3.12)*, respectively*) *holds for all* $t \in [T,\infty)$ *and* $x \in (0,\infty)$ (*and* $x \in [tb(T),\infty)$*, respectively*)*.*

When $\mathbf{Poly}_{R_2,\ge}^{*,\infty}(\beta_1)$ holds with $\beta_1 > 1$, we can find a monotone function which is easy to compute and can play the same role as the function θ. Define

$$\mathscr{H}(r) := \inf_{s\ge r}\frac{1}{sH(s^{-1})} \qquad \text{and} \qquad \mathscr{H}^{-1}(u) := \sup\{r\in\mathbb{R} : \mathscr{H}(r)\le u\}.$$

Using the above function $\mathscr{H}$, the large time right tail estimates in Theorem 2.3.6 can be simplified as follows. See Figure 2.2 also.

Note that $\phi'(0)$ is finite if $\mathbf{Poly}^{*,\infty}_{R_2,\geq}(\beta_1)$ holds with $\beta_1 > 1$. See (2.3.59) and the line below.

Corollary 2.3.8. *Assume that* $\mathbf{Reg}_{R_1}$ *and* $\mathbf{Poly}^{*,\infty}_{R_2}(\beta_1,\beta_2)$ *holds with* $\beta_1 > 1$*. Then, there exist constants* $T_1 > T_0$*,* $c_1 > 1$ *and* $c_2, c_3 > 0$ *such that for all* $t \in [T_1,\infty)$ *and* $y \in [0,\infty)$*,*

$$c_1^{-1}H^{-1}(1/t)\min\left\{1,\ \frac{t\nu(y)}{H^{-1}(1/t)} + \exp\Big(-\frac{c_2 y}{\mathscr{H}^{-1}(t/y)}\Big)\right\}$$
$$\leq p(t, t\phi'(0)+y) \leq c_1 H^{-1}(1/t)\min\left\{1,\ \frac{t\nu(y)}{H^{-1}(1/t)} + \exp\Big(-\frac{c_3 y}{\mathscr{H}^{-1}(t/y)}\Big)\right\}.$$

Moreover, if $T_0 = 0$*, then for every* $T > 0$*, there are comparison constants such that the above estimates hold for all* $t \in [T,\infty)$ *and* $y \in [0,\infty)$*.*

The above corollary may be considered as a counterpart of [5, Theorem 1.5(2)] where a similar result was obtained for symmetric jump processes. Since $\mathbf{Poly}^{*}_{R_1}(\beta_1,\beta_2)$ can not holds with $\beta_1 > 1$ by Remark 2.3.3(ii), there is no analogous result to Corollary 2.3.8 concerning small times estimates.

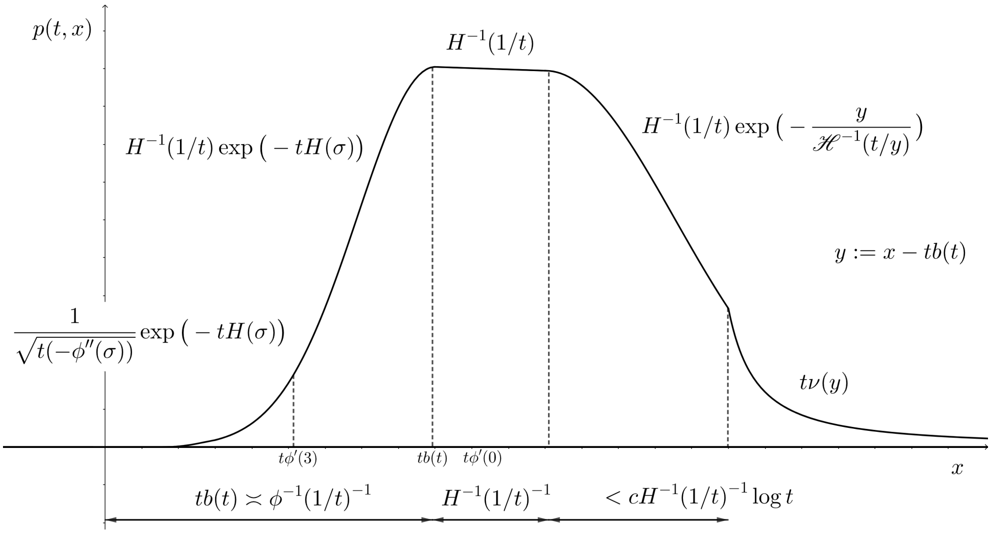

Figure 2.2: Large time estimates when $\beta_1 > 1$

In Figure 2.2, $p(t,x)$ satisfies (2.3.7) for all $x \in (0, tb(t)]$ if $\mathbf{Poly}^*_{R_1,\geq}(\beta_1)$ also holds. We also note that in Figure 2.2, the exponential term in right tail can be the dominant term in estimates only in an interval whose length is smaller than a constant multiple of $H^{-1}(1/t)^{-1}\log t$. This fact can be proved by using Lemma 2.3.14.

Our main theorems also cover the cases when $\beta_1 \leq 1$ and $\beta_2 \geq 2$. In such cases, the exponential term in the right tail estimates may have an effect on the estimates at specific times but no effect at other time values. (See, Section 4.2.)

2.3.1 Some consequences of $\mathbf{Poly}^*_{R_1}(\beta_1, \beta_2)$

Recall that $H(\lambda) := \phi(\lambda) - \lambda\phi'(\lambda)$ for $\lambda > 0$. Using the inequality $1 - e^{-x} - xe^{-x} \geq x^2/(2e)$ for $0 \leq x \leq 1$, we see that for every $\lambda > 0$,

$$H(\lambda) \geq \int_0^{1/\lambda} (1 - e^{-\lambda s} - \lambda s e^{-\lambda s})\nu(s)ds \geq \frac{\lambda^2}{2e}\int_0^{1/\lambda} s^2\nu(s)ds. \qquad (2.3.13)$$

Denote by $\phi^{(n)}$ the n-th derivative of the Laplace exponent ϕ. Using the expression (2.0.2), we get the following lemma.

Lemma 2.3.9. *Suppose that* $\mathbf{Reg}_{R_1}$ *holds.*
(i) For every $\lambda_0 > 0$*, there are constants* $c_n > 0$*,* $n = 1, 2, ...$ *such that*

$$e^{-1}\int_0^{\frac{1}{\lambda}} s^n\nu(s)ds \leq |\phi^{(n)}(\lambda)| \leq c_n \int_0^{\frac{1}{\lambda}} s^n\nu(s)ds \quad \textit{for all } \lambda \geq \lambda_0 \textit{ and } n \geq 1.$$

(ii) For every $\lambda_0 > 0$*, there are constants* $c'_n > 1$*,* $n = 1, 2, ...$ *such that*

$$c_n'^{-1}|\phi^{(n)}(2\lambda)| \leq |\phi^{(n)}(\lambda)| \leq c'_n|\phi^{(n)}(2\lambda)| \quad \textit{for all } \lambda \geq \lambda_0 \textit{ and } n \geq 1.$$

(iii) For every $\lambda_0 > 0$*, there are constants* $c''_n > 0$*,* $n = 1, 2, ...$ *such that*

$$|\lambda\phi^{(n+1)}(\lambda)| \leq c''_n|\phi^{(n)}(\lambda)| \quad \textit{for all } \lambda \geq \lambda_0 \textit{ and } n \geq 1.$$

Since $\mathbf{Poly}^{*}_{R_1,\geq}(\beta_1)$ implies $\mathbf{Reg}_{R_1}$, the results of Lemma 2.3.9 hold true under $\mathbf{Poly}^{*}_{R_1,\geq}(\beta_1)$. We get analogous results under $\mathbf{Poly}^{*,\infty}_{R_2,\geq}(\beta_1)$.

Lemma 2.3.10. *Suppose that* $\mathbf{Poly}^{*,\infty}_{R_2,\geq}(\beta_1)$ *holds.*
(i) For every $\lambda_0 > 0$*, there are constants* $c_n > 0$*,* $n = 1, 2, ...$ *such that*

$$e^{-1}\int_0^{\frac{1}{\lambda}} s^n\nu(s)ds \leq |\phi^{(n)}(\lambda)| \leq c_n\int_0^{\frac{1}{\lambda}} s^n\nu(s)ds \quad \textit{for all } 0<\lambda\leq\lambda_0 \textit{ and } n\geq 1.$$

(ii) For every $\lambda_0 > 0$*, there are constants* $c'_n > 1$*,* $n = 1, 2, ...$ *such that*

$$c_n'^{-1}|\phi^{(n)}(2\lambda)| \leq |\phi^{(n)}(\lambda)| \leq c'_n|\phi^{(n)}(2\lambda)| \quad \textit{for all } 0<\lambda\leq\lambda_0 \textit{ and } n\geq 1.$$

(iii) For every $\lambda_0 > 0$*, there are constants* $c''_n > 0$*,* $n = 1, 2, ...$ *such that*

$$|\lambda\phi^{(n+1)}(\lambda)| \leq c''_n|\phi^{(n)}(\lambda)| \quad \textit{for all } 0<\lambda\leq\lambda_0 \textit{ and } n\geq 1.$$

From the definition of the tail measure w, we deduce the following result from Lemma 2.1.1.

Lemma 2.3.11. *Suppose that* $\mathbf{Poly}^{*}_{R_1,\geq}(\beta_1)$ *holds.*
(i) There are constants $c_1, c_2 > 0$ *such that*

$$c_1 r\nu(2r) \leq w(r) \leq c_2 r\nu(r) \quad \textit{for all } 0<r<R_1/2.$$

Therefore, if $\mathbf{Poly}^{*}_{R_1}(\beta_1,\beta_2)$ *holds, then* $w(r) \simeq r\nu(r)$ *for* $r \in (0, R_1/2)$*.*
(ii) There is a constant $c_3 > 0$ *such that*

$$\frac{w(s)}{w(r)} \geq c_3\left(\frac{r}{s}\right)^{\beta_1} \quad \textit{for all } 0<r\leq R<R_1/2.$$

(iii) For every $r_0 > 0$*, there is a constant* $c_4 > 0$ *such that*

$$\frac{H(r)}{H(s)} \geq c_4\left(\frac{r}{s}\right)^{\beta_1} \quad \textit{for all } r_0\leq s\leq r.$$

In particular,

$$\frac{H^{-1}(t)}{H^{-1}(u)} \le c_4^{-1/\beta_1} \left(\frac{t}{u}\right)^{1/\beta_1} \quad \text{for all } H(r_0) \le u \le t.$$

(iv) For every $\lambda_0 > 0$*, there are comparison constants such that*

$$H(\lambda) \simeq \lambda^2 \int_0^{1/\lambda} s^2 \nu(s) ds \simeq \lambda^2(-\phi''(\lambda)) \quad \text{for } \lambda \ge \lambda_0.$$

Lemma 2.3.12. *Suppose that* $\mathbf{Poly}^{*,\infty}_{R_2,\ge}(\beta_1)$ *holds.*
(i) There are constants $c_1, c_2 > 0$ *such that*

$$c_1 r\nu(2r) \le w(r) \le c_2 r\nu(r) \quad \text{for all } r \ge R_2.$$

Therefore, if $\mathbf{Poly}^{*,\infty}_{R_2}(\beta_1, \beta_2)$ *holds, then* $w(r) \simeq r\nu(r)$ *for* $r \in [R_2, \infty)$*.*
(ii) There is a constant $c_3 > 0$ *such that*

$$\frac{w(s)}{w(r)} \ge c_3 \left(\frac{r}{s}\right)^{\beta_1} \qquad \text{for all } r \ge s \ge R_2.$$

(iii) For every $r_0 > 0$*, there is a constant* $c_3 > 0$ *such that*

$$\frac{H(r)}{H(s)} \ge c_3 \left(\frac{r}{s}\right)^{\beta_1 \wedge (3/2)} \quad \text{for all } 0 < s \le r \le R \le r_0.$$

In particular,

$$\frac{H^{-1}(t)}{H^{-1}(u)} \le c_3^{-((1/\beta_1)\vee(2/3))} \left(\frac{t}{u}\right)^{(1/\beta_1)\vee(2/3)} \quad \text{for all } 0 < u \le t \le H(r_0).$$

(iv) For every $\lambda_0 > 0$*, there are comparison constants such that*

$$H(\lambda) \simeq \lambda^2 \int_0^{1/\lambda} s^2 \nu(s) ds \simeq \lambda^2(-\phi''(\lambda)) \quad \text{for } 0 < \lambda \le \lambda_0.$$

Since the Lévy density $\nu(x)$ (locally) decays in polynomial orders, using the fact $\sup_{x>0} x^k e^{-x} < 0$ for all $k > 0$, we get the following two lemmas. See

[55, Lemmas 2.5 and 2.6] for the proofs.

Lemma 2.3.13. *(i) Suppose that* $\mathbf{Poly}^*_{R_1}(\beta_1, \beta_2)$ *hold. Then, for every* $a, T > 0$*, there exists* $c_1 > 0$ *such that for all* $t \in (0, T]$ *and* $y \in [H^{-1}(1/t)^{-1}, R_1/2)$*,*

$$\exp\Big(-\frac{ay}{w^{-1}(2e/t)}\Big) \le c_1 \frac{t\nu(y)}{H^{-1}(1/t)}. \tag{2.3.14}$$

Moreover, if $\mathbf{Dou}^\infty_{R_1}$ *also holds, then* (2.3.14) *holds true for all* $t \in (0, T]$ *and* $y \in [H^{-1}(1/t)^{-1}, \infty)$*.*
(ii) Suppose that $\mathbf{Poly}^{*,\infty}_{R_2}(\beta_1, \beta_2)$ *hold. Then, for every* $a, T > 0$*, there exists* $c_1 > 0$ *such that* (2.3.14) *holds for all* $t \in [T, \infty)$ *and* $y \in [H^{-1}(1/t)^{-1} \vee R_2, \infty)$*.*

Lemma 2.3.14. *Suppose that* $\mathbf{Poly}^{*,\infty}_{R_2}(\beta_1, \beta_2)$ *hold. Then, for every* $a, T > 0$*, there exist* $c_1 > 0$*,* $c_2 \ge 1$ *such that*

$$\exp\Big(-\frac{ay}{H^{-1}(1/t)^{-1}}\Big) \le c_1 \frac{t\nu(y)}{H^{-1}(1/t)}$$

for all $t \in [T, \infty)$ *and* $y \ge \big(c_2 H^{-1}(1/t)^{-1} \log(e+t)\big) \vee R_2$*.*

2.3.2 Left tail estimates

In this subsection, we study left tail estimates on $p(t, x)$. We first present a result established in [75] which holds under $\mathbf{Poly}^*_{R_1,\ge}(\beta_1)$.

Recall that $\sigma = \sigma(t, r) := (\phi'^{-1})(r/t)\mathbf{1}_{(0,\phi'(0+))}(r/t)$.

Proposition 2.3.15. *Suppose that* $\mathbf{Poly}^*_{R_1,\ge}(\beta_1)$ *holds. Then, for every* $T > 0$*, there exist constants* $M_0 > 0$ *and* $C > 1$ *such that for all* $t \in (0, T]$*,*

$$C^{-1}\frac{\exp\big(-tH(\sigma)\big)}{\sqrt{t(-\phi''(\sigma))}} \le p(t, x) \le C\frac{\exp\big(-tH(\sigma)\big)}{\sqrt{t(-\phi''(\sigma))}}, \quad x \in (0, tb(t/M_0)]. \tag{2.3.15}$$

Proof. By Lemma 2.3.11(iii, iv), we see that for every $x_0 > 0$, the condition $-\phi'' \in \mathrm{WLSC}(\alpha_1 - 2, c, x_0)$ in [75, Theorem 3.3] is satisfied with some constant $c > 0$. Since $x \mapsto \sigma(t, x)$ decreases for each fixed t, we have that, for all

$t \in (0, T]$ and $x \in (0, tb(t/M_0)]$,

$$\sigma(t, x) \geq ((\phi')^{-1} \circ b)(t/M_0) = H^{-1}(M_0/t) \geq H^{-1}(M_0/T).$$

Also, by the above inequality and Lemma 2.3.11(iv), it holds that

$$t\sigma^2(-\phi''(\sigma)) \geq c_1 tH(\sigma) \geq c_1 tH(H^{-1}(M_0/t)) = c_1 M_0.$$

Therefore, we obtain the result from [75, Theorem 3.3]. □

Now, we establish left tail probabilities under $\mathbf{Reg}_{R_1}$ and $\mathbf{Poly}^{*,\infty}_{R_2,\geq}(\beta_1)$. Define a function $\mathcal{M} : (0, \infty) \times (0, \infty) \times (-\infty, \infty) \to \mathbb{C}$ by

$$\mathcal{M}(s, z, u) := \phi(z + \frac{iu}{\sqrt{s(-\phi''(z))}}) - \phi(z) - \phi'(z)\frac{iu}{\sqrt{s(-\phi''(z))}}. \qquad (2.3.16)$$

In the setting of [75], the Laplace exponent ϕ should satisfy a lower weak scaling condition at infinity (i.e., the lower Matuszewska index (at infinity) of the function $\phi(\lambda)\mathbf{1}_{\{\lambda \geq 1\}}$ should be strictly bigger than 0) so that the map $u \mapsto e^{-t\mathcal{M}(t,\sigma,u)}$ for each fixed $t > 0$ decreases at least subexponentially. This property plays an important role in the proof of [75, Theorem 3.3]. Unlike [75], in our setting, the Laplace exponent ϕ can be slowly varying at infinity so that the map $u \mapsto e^{-t\mathcal{M}(t,\sigma,u)}$ can decay only in polynomial orders. Thus, we need significant modifications in the proof of the next proposition.

Proposition 2.3.16. *Suppose that* $\mathbf{Reg}_{R_1}$ *and* $\mathbf{Poly}^{*,\infty}_{R_2,\geq}(\beta_1)$ *hold. Then, there exist constants* $T_1 > T_0$, $M_0 > 0$ *and* $C > 1$, *where* T_0 *is the constant in* (2.3.1), *such that* (2.3.15) *holds for all* $t \in [T_1, \infty)$ *and* $x \in (0, tb(t/M_0)]$.

Moreover, if $T_0 = 0$, *then for every* $T > 0$, *there exist constants* $M_0 > 0$ *and* $C > 1$ *such that* (2.3.15) *holds for all* $t \in [T_1, \infty)$ *and* $x \in (0, tb(t/M_0)]$.

We need the following lemma in the proof of Proposition 2.3.16.

Lemma 2.3.17. *For every constants* $a > 0$ *and* $\delta \in (0, 1)$, *there exists a* *constant* $\xi_0 > 0$ *such that*

$$\int_0^a (1-\cos(\xi s))\frac{ds}{s} \geq (1-\delta)\log(1+\xi) \quad \textit{for all } \xi \geq \xi_0.$$

Proof. Let U be a Gamma subordinator whose Laplace exponent is $\log(1+\lambda)$. It is known that the Lévy measure of U has the density function $s^{-1}e^{-s}$. By the analytic continuation of Bernstein functions (see, e.g. [116, Proposition 3.6]), it holds that for all $\xi \in \mathbb{R}$,

$$\begin{aligned}\log\sqrt{1+\xi^2} = \mathrm{Re}\ \log(1+i\xi) &= \mathrm{Re}\int_0^\infty (1-e^{-i\xi s})s^{-1}e^{-s}ds \\ &= \int_0^\infty (1-\cos(\xi s))s^{-1}e^{-s}ds.\end{aligned}$$

For any constant $a > 0$, we see that

$$\int_a^\infty (1-\cos(\xi s))s^{-1}e^{-s}ds \leq 2\int_a^\infty s^{-1}e^{-s}ds < \infty$$

and hence

$$\begin{aligned}1 = \lim_{\xi\to\infty}\frac{\log\sqrt{1+\xi^2}}{\log(1+\xi)} &= \lim_{\xi\to\infty}\frac{\int_0^a(1-\cos(\xi s))s^{-1}e^{-s}ds}{\log(1+\xi)} \\ &\leq \lim_{\xi\to\infty}\frac{\int_0^a(1-\cos(\xi s))s^{-1}ds}{\log(1+\xi)}.\end{aligned}$$

This yields the desired result. □

Proof of Proposition 2.3.16. Recall that $\mathcal{M}$ is defined in (2.3.16). By the Fourier-Mellin inversion formula and a change of variables, we get that

$$\begin{aligned}p(t,x) &= \frac{e^{-t\phi(\sigma)+\sigma x}}{2\pi}\int_{-\infty}^\infty \exp\Big(-t\big(\phi(\sigma+iu)-\phi(\sigma)\big)+iux\Big)du \\ &= \frac{e^{-t(\phi(\sigma)-\sigma\phi'(\sigma))}}{2\pi}\int_{-\infty}^\infty \exp\Big(-t\big(\phi(\sigma+iu)-\phi(\sigma)-iu\phi'(\sigma)\big)\Big)du \\ &= \frac{e^{-tH(\sigma)}}{2\pi\sqrt{t(-\phi''(\sigma))}}\int_{-\infty}^\infty e^{-t\mathcal{M}(t,\sigma,u)}du, \qquad (2.3.17)\end{aligned}$$

whenever the last integral above converges. Note that the complex conjugate of $\mathcal{M}(t,\sigma,u)$ is $\mathcal{M}(t,\sigma,-u)$ so that $e^{-t\mathcal{M}(t,\sigma,u)}+e^{-t\mathcal{M}(t,\sigma,-u)} \in \mathbb{R}$ for all $t,\sigma>0$ and $u\in\mathbb{R}$. Hence, whenever $|e^{-t\mathcal{M}(t,\sigma,u)}|$ is integrable on $\mathbb{R}$ with respect to u, using the equality $\int_{-\infty}^{\infty} e^{-t\mathcal{M}(t,\sigma,u)}du = \int_0^\infty (e^{-t\mathcal{M}(t,\sigma,u)}+e^{-t\mathcal{M}(t,\sigma,-u)})du$, we deduce that $p(t,x)$ is a positive real number.

Let $T_1 \in (T_0,\infty)$ be a constant which will be chosen later. Pick a constant $\epsilon\in(0,1)$ such that $(1-\epsilon)T_1 > T_0$. By (2.3.1) and Lemma 2.3.17, there exist constants $\sigma_0,\xi_0>0$ such that

$$\nu(s) \ge \frac{(1-\epsilon/2)}{T_0}s^{-1} \quad \text{for all } \; s\in(0,\sigma_0^{-1}|\log(1-\epsilon/2)|) \tag{2.3.18}$$

and

$$\int_0^{|\log(1-\epsilon/2)|}(1-\cos(\xi s))\frac{ds}{s} \ge \frac{1-\epsilon}{(1-\epsilon/2)^2}\log(1+\xi) \quad \text{for all } \; \xi\ge\xi_0.$$

Using (2.3.18), (2.3.18) and a change of the variables, we get that for all $t\ge T_1$, $\sigma>0$ and $|u|>\xi_0(\sigma_0\vee\sigma)\sqrt{t(-\phi''(\sigma))}$,

$$\begin{aligned}
&\mathrm{Re}\, t\mathcal{M}(t,\sigma,u) = t\int_0^\infty e^{-\sigma s}\left(1-\cos\frac{us}{\sqrt{t(-\phi''(\sigma))}}\right)\nu(s)ds \\
&\ge \frac{(1-\epsilon/2)^2 t}{T_0}\int_0^{\frac{|\log(1-\epsilon/2)|}{\sigma_0\vee\sigma}}\left(1-\cos\frac{us}{\sqrt{t(-\phi''(\sigma))}}\right)\frac{ds}{s} \\
&= \frac{(1-\epsilon/2)^2 t}{T_0}\int_0^{|\log(1-\epsilon/2)|}\left(1-\cos\frac{us}{(\sigma_0\vee\sigma)\sqrt{t(-\phi''(\sigma))}}\right)\frac{ds}{s} \\
&\ge \frac{(1-\epsilon)t}{T_0}\log\left(1+\frac{u}{(\sigma_0\vee\sigma)\sqrt{t(-\phi''(\sigma))}}\right).
\end{aligned} \tag{2.3.19}$$

Since $(1-\epsilon)t/T_0 \ge (1-\epsilon)T_1/T_0 > 1$, we see from (2.3.19) that $|e^{-t\mathcal{M}(t,\sigma,u)}|$ is integrable on $\mathbb{R}$ with respect to u. Therefore, (2.3.17) holds true for all $t\ge T_1$ and $x\in(0,t\phi'(0))$.

Define for $t,\sigma>0$,

$$\mathcal{T}_0 = \mathcal{T}_0(t,\sigma) := (\sigma_0\vee\sigma)\sqrt{t(-\phi''(\sigma))} \quad \text{and} \quad \mathcal{T} = \mathcal{T}(t,\sigma) := \sigma\sqrt{t(-\phi''(\sigma))}.$$

Clearly, $\mathcal{T}_0 \geq \mathcal{T}$. For all $t > 0$, $M > 0$ and $x \in (0, tb(t/M))$, if $\sigma > \sigma_0$, then

$$\mathcal{T}^2 \geq \sigma^2 t \int_0^{1/\sigma} s^2 e^{-\sigma s} \nu(s) ds \geq c_1 \sigma^2 t \int_0^{1/\sigma} s^2 s^{-1} ds = \frac{c_1 t}{2} \tag{2.3.20}$$

by (2.3.1), and if $\sigma \leq \sigma_0$, then

$$\mathcal{T}^2 \geq c_2 t H(\sigma) \geq c_2 t (H \circ \phi'^{-1} \circ b)(t/M) = c_2 M \tag{2.3.21}$$

by Lemma 2.3.12(iv) and the monotone property of σ. We will prove that

$$\lim_{\mathcal{T} \to \infty} \int_{-\infty}^{\infty} e^{-t\mathcal{M}(t,\sigma,u)} du = \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2} du = \sqrt{2\pi}. \tag{2.3.22}$$

Assuming (2.3.22) for the moment. By (2.3.17), there exists $c_3 > 0$ such that (2.3.15) holds if $\mathcal{T} \geq c_3$. Then by taking $T_1 = 2c_3/c_1$ and $M_0 = c_3/c_2$, we deduce from (2.3.20) and (2.3.21) that (2.3.15) holds for all $t \geq T_1$ and $x \in (0, tb(t/M_0)$, and conclude that the first assertion holds true.

Now, we prove (2.3.22). First, we see from (2.3.19) that

$$\left| \int_{|u| > \xi_0 \mathcal{T}_0} e^{-t\mathcal{M}(t,\sigma,u)} du \right| \leq 2 \int_{\xi_0 \mathcal{T}_0}^{\infty} \left(1 + \frac{u}{\mathcal{T}_0}\right)^{-(1-\epsilon)T_1/T_0} du < \infty. \tag{2.3.23}$$

Next, by Taylor's theorem, it holds that

$$\begin{aligned} \left| t\mathcal{M}(t,\sigma,u) - \frac{1}{2}u^2 \right| &= \left| t\Big(\phi(\sigma + \frac{i\sigma}{\mathcal{T}}u) - \phi(\sigma) - \phi'(\sigma)\frac{i\sigma}{\mathcal{T}}u\Big) - \frac{1}{2}u^2 \right| \\ &\leq \frac{u^2}{2} \sup_{z \in [-|u|,|u|]} \left| \frac{t\sigma^2}{\mathcal{T}^2}\big(- \phi''(\sigma + \frac{i\sigma}{\mathcal{T}}z)\big) - 1 \right| \\ &= \frac{u^2}{2(-\phi''(\sigma))} \sup_{z \in [-|u|,|u|]} \left| - \phi''(\sigma + \frac{i\sigma}{\mathcal{T}}z) + \phi''(\sigma) \right|. \end{aligned}$$

Note that, since $|\sin x| \leq |x|$ for all $x \in \mathbb{R}$, we get that for all $|z| \leq |u|$,

$$\left| - \phi''(\sigma + \frac{i\sigma}{\mathcal{T}}z) + \phi''(\sigma) \right| \leq \int_0^{\infty} s^2 e^{-\sigma s} \left| \cos(\frac{\sigma z s}{\mathcal{T}}) - 1 - i\sin(\frac{\sigma z s}{\mathcal{T}}) \right| \nu(s) ds$$

$$= 2\int_0^\infty s^2 e^{-\sigma s}\Big|\sin(\frac{\sigma z s}{2\mathcal{T}})\Big|\nu(s)ds \le \frac{\sigma|u|}{\mathcal{T}}\int_0^\infty s^3 e^{-\sigma s}\nu(s)ds = \frac{\sigma|u|}{\mathcal{T}}\phi'''(\sigma).$$

Thus, we deduce that

$$\Big| t\mathcal{M}(t,\sigma,u) - \frac{1}{2}u^2 \Big| \le \frac{\sigma\phi'''(\sigma)}{2\mathcal{T}(-\phi''(\sigma))}|u|^3.$$

Combining with the fact that $|e^z - 1| \le |z|e^{|z|}$ for $z \in \mathbb{C}$, we obtain that for all $u \in \mathbb{R}$,

$$\begin{aligned}\Big| e^{-t\mathcal{M}(t,\sigma,u)} - e^{-\frac{1}{2}u^2} \Big| &= e^{-\frac{1}{2}u^2}\Big| \exp\Big(\frac{1}{2}u^2 - t\mathcal{M}(t,\sigma,u)\Big) - 1 \Big| \\ &\le \frac{\sigma\phi'''(\sigma)}{2\mathcal{T}(-\phi''(\sigma))}|u|^3 \exp\Big(-\frac{1}{2}u^2 + \frac{\sigma\phi'''(\sigma)}{2\mathcal{T}(-\phi''(\sigma))}|u|^3\Big). \end{aligned} \tag{2.3.24}$$

Below, we consider the cases $\sigma > \sigma_0$ and $\sigma \le \sigma_0$, separately.

(Case 1) Assume that $\sigma > \sigma_0$. By Lemma 2.3.9(iii), there exists $c_4 > 0$ such that $\sigma\phi'''(\sigma) \le c_4(-\phi''(\sigma))$. Let $\xi_1 = (2c_4)^{-1} \wedge \xi_0$. Using (2.3.24), we get that

$$\begin{aligned}\Big| \int_{|u|\le\xi_1\mathcal{T}} (e^{-t\mathcal{M}(t,\sigma,u)} - e^{-\frac{1}{2}u^2})du \Big| &\le \frac{c_5}{\mathcal{T}}\int_0^{\xi_1\mathcal{T}} u^3 \exp\Big(-\Big(\frac{1}{2} - \frac{c_4}{2\mathcal{T}}u\Big)u^2\Big)du \\ &\le \frac{c_5}{\mathcal{T}}\int_0^{\xi_1\mathcal{T}} u^3 \exp\Big(-\Big(\frac{1}{2} - \frac{c_4\xi_1}{2}\Big)u^2\Big)du \\ &\le \frac{c_5}{\mathcal{T}}\int_0^{\infty} u^3 \exp\Big(-\frac{1}{4}u^2\Big)du \le \frac{c_6}{\mathcal{T}}. \end{aligned} \tag{2.3.25}$$

On the other hand, note that $\sigma|u|/\mathcal{T} > \sigma_0\xi_1$ for $|u| > \xi_1\mathcal{T}$. Hence, by Lemma 2.3.9(i), since $1 - \cos r \ge \frac{\cos 1}{2}r^2$ for all $|r| \le 1$, we have that for all $|u| > \xi_1\mathcal{T}$,

$$\begin{aligned}\mathrm{Re}\, t\mathcal{M}(t,\sigma,u) &\ge t\int_0^{\mathcal{T}/(\sigma|u|)} \big(1 - \cos\frac{\sigma u s}{\mathcal{T}}\big)e^{-\sigma s}\nu(s)ds \\ &\ge t\frac{\cos 1}{2}\frac{\sigma^2 u^2}{\mathcal{T}^2}e^{-\mathcal{T}/|u|}\int_0^{\mathcal{T}/(\sigma|u|)} s^2\nu(s)ds \\ &\ge c_7 e^{-1/\xi_1} t\frac{\sigma^2 u^2}{\mathcal{T}^2}\big|\phi''(\sigma|u|/\mathcal{T})\big|. \end{aligned} \tag{2.3.26}$$

It follows that

$$\begin{aligned}
&\Big| \int_{\xi_1 \mathcal{T} < |u| \le \xi_0 \mathcal{T}_0} e^{-t\mathcal{M}(t,\sigma,u)} du \Big| \\
&\le 2\xi_0 \mathcal{T} \max_{\xi_1 \mathcal{T} < u \le \xi_0 \mathcal{T}} \exp\Big(- c_7 e^{-1/\xi_1} t \frac{\sigma^2 u^2}{\mathcal{T}^2} |\phi''(\sigma u/\mathcal{T})| \Big) \\
&\le 2\xi_0 \mathcal{T} \exp\big(- c_7 e^{-1/\xi_1} t \xi_1^2 \sigma^2 |\phi''(\sigma \xi_0)| \big) \\
&\le 2\xi_0 \mathcal{T} \exp\big(- c_8 t \sigma^2 |\phi''(\sigma)| \big) = 2\xi_0 \mathcal{T} \exp\big(- c_8 \mathcal{T}^2 \big). \qquad (2.3.27)
\end{aligned}$$

We used the fact that $\mathcal{T}_0 = \mathcal{T}$ under the assumption $\sigma > \sigma_0$ in the first inequality and Lemma 2.3.9(ii) in the third.

Eventually, by the triangle inequality and inequalities (2.3.23), (2.3.25) and (2.3.27), we obtain

$$\begin{aligned}
&\Big| \int_{\mathbb{R}} (e^{-t\mathcal{M}(t,\sigma,u)} - e^{-\frac{1}{2}u^2}) du \Big| \\
&\le \frac{c_6}{\mathcal{T}} + 2\xi_0 \mathcal{T} \exp\big(- c_8 \mathcal{T}^2 \big) + 2 \int_{\xi_0 \mathcal{T}}^{\infty} \Big(1 + \frac{u}{\mathcal{T}} \Big)^{-(1-\epsilon)T_1/T_0} du + 2 \int_{\xi_1 \mathcal{T}}^{\infty} e^{-\frac{1}{2}u^2} du \\
&\to 0 \qquad \text{as} \quad \mathcal{T} \to \infty. \qquad (2.3.28)
\end{aligned}$$

(Case 2) Assume that $\sigma \le \sigma_0$. We follow the proof for (Case 1). First, using Lemma 2.3.10(iii) instead of Lemma 2.3.9(iii), (2.3.25) still hold with possibly different constants ξ_1 and c_6. Next, note that $\sigma |u|/\mathcal{T} \le \xi_0 \sigma_0$ for $|u| \le \xi_0 \mathcal{T}_0$ in this case. Hence, by Lemma 2.3.10(i), (2.3.26) holds for all $|u| \le \xi_0 \mathcal{T}_0$ with some $c_7 > 0$. Also, by Lemma 2.3.12(iv), we see that $\mathcal{T}^2 \simeq tH(\sigma)$ and $\sigma^2 u^2 \mathcal{T}^{-2} |\phi''(\sigma|u|/\mathcal{T})| \simeq H(\sigma |u|/\mathcal{T})$ for $|u| \le \xi_0/\mathcal{T}_0$. Using these comparisons and (2.3.26) in the first line below, Lemma 2.3.12(iii) and a change of the variables in the second, the fact that $\sup_{x>\xi_1} x^{2/\beta_1'} e^{-x/2} < \infty$ in the third, and Lemma 2.3.12(iv) in the last, we get that for $\beta_1' := \beta_1 \wedge (3/2)$,

$$\Big| \int_{\xi_1 \mathcal{T} < |u| \le \xi_0 \mathcal{T}_0} e^{-t\mathcal{M}(t,\sigma,u)} du \Big| \le 2 \int_{\xi_1 \mathcal{T}}^{\xi_0 \mathcal{T}_0} \exp\big(- c_9 t H(\sigma u/\mathcal{T}) \big) du$$

$$\le 2\int_{\xi_1\mathcal{T}}^{\xi_0\mathcal{T}_0} \exp\big(-c_{10}tH(\sigma)u^{\beta_1'}\mathcal{T}^{-\beta_1'}\big)du = 2\mathcal{T}\int_{\xi_1}^{\xi_0\mathcal{T}_0/\mathcal{T}} \exp\big(-c_{10}tH(\sigma)u^{\beta_1'}\big)du$$
$$\le c_{11}\mathcal{T}\exp\big(-\frac{c_{10}}{2}tH(\sigma)\xi_1^{\beta_1'}\big)\int_{\xi_1}^{\infty}(tH(\sigma)u^{\beta_1'})^{-2/\beta_1'}du$$
$$\le c_{12}\mathcal{T}^{1-4/\beta_1'}\exp\big(-c_{13}\mathcal{T}^2\big). \tag{2.3.29}$$

Using (2.3.29) instead of (2.3.27), we can see that (2.3.28) is still valid. This finishes the proof for (2.3.22).

Now, we further assume that $T_0 = 0$ and prove the second assertion. Choose any $T > 0$. By (2.3.17), since the first assertion holds true, it suffices to show that there exist constants $c_{14} > 1$ and $M_0 \ge c_3^2/c_2$ such that for all $t \in [T, 2c_3/c_1]$ and $x \in (0, tb(t/M_0)]$,

$$c_{14}^{-1} \le \int_{-\infty}^{\infty} e^{-t\mathcal{M}(t,\sigma,u)}du \le c_{14}. \tag{2.3.30}$$

Note that (2.3.19) is still valid with possibly different constants ϵ, σ_0 and ξ_0. Hence, we have $\int_{-\infty}^{\infty} e^{-t\mathcal{M}(t,\sigma,u)}du \in \mathbb{R}$ for all $t \in [T, 2c_1^{-1}c_3]$ and $\sigma > 0$. Also, since inequalities (2.3.23), (2.3.25), (2.3.27) and (2.3.29) still work, by a similar argument to (2.3.28), we see that there exists $c_{15} > 0$ such that if $\mathcal{T} = \sigma\sqrt{t(-\phi''(\sigma))} \ge c_{15}$, then (2.3.30) holds. Therefore, it remains to prove that for the set $A := \{(t,\sigma) : t \in [T, 2c_3/c_1], \sigma > 0, 0 < \mathcal{T} < c_{15}\}$,

$$\inf_{(t,\sigma)\in A}\int_{-\infty}^{\infty} e^{-t\mathcal{M}(t,\sigma,u)}du \simeq \sup_{(t,\sigma)\in A}\int_{-\infty}^{\infty} e^{-t\mathcal{M}(t,\sigma,u)}du \simeq 1. \tag{2.3.31}$$

Recall that $\mathcal{T}^2 \ge c_2M_0$ if $\sigma \le \sigma_0$. By taking M_0 larger than $c_2^{-1}c_{15}^2$, we obtain $A \subset [T, 2c_3/c_1] \times [\sigma_0, \infty)$. Besides, since $T_0 = 0$, we see that

$$\begin{aligned}\lim_{\sigma\to\infty}\sigma^2(-\phi''(\sigma)) &\ge \lim_{\sigma\to\infty} e^{-1}\sigma^2\int_0^{1/\sigma} s(s\nu(s))ds \\ &\ge (2e)^{-1}\liminf_{\sigma\to\infty}\inf_{0<s<1/\sigma}(s\nu(s)) = \infty. \end{aligned}\tag{2.3.32}$$

Thus, there exists a constant $\sigma_1 > 0$ such that $\mathcal{T}^2 \ge T\sigma^2(-\phi''(\sigma)) \ge c_{15}^2$

for all $\sigma > \sigma_1$ and hence $A \subset [T, 2c_3/c_1] \times [\sigma_0, \sigma_1] =: A_0$. Clearly, $(t, \sigma) \mapsto \int_{-\infty}^{\infty} e^{-t\mathcal{M}(t,\sigma,u)} du$ is a continuous function on A_0. Therefore, we obtain (2.3.31) from the extreme value theorem. □

Using (2.3.17) and (2.3.22), we obtain the following corollary.

Corollary 2.3.18. *Suppose that* $\mathbf{Reg}_{R_1}$ *holds. Then, for every* $N > 0$*,*

$$\lim_{t\to\infty} p(t,x)\sqrt{t(-\phi''(\sigma))}\exp\big(tH(\sigma)\big) = (2\pi)^{-1/2} \quad \textit{uniformly in } x \in (0, N].$$

If we also assume that (2.3.1) *holds with* $T_0 = 0$*, then for every* $N > 0$*,*

$$\lim_{x\to 0} p(t,x)\sqrt{t(-\phi''(\sigma))}\exp\big(tH(\sigma)\big) = (2\pi)^{-1/2} \quad \textit{uniformly in } t \in [N, \infty).$$

Proof. Let $N > 0$. Fix a constant σ_0 so that (2.3.18) is satisfied with $\epsilon = 1/2$. Write $\mathcal{T} = \mathcal{T}(t, \sigma) := \sigma\sqrt{t(-\phi''(\sigma))}$ as in the proof of Proposition 2.3.16. Similar to (2.3.20), we see from (2.3.1) that $\mathcal{T}^2 \geq c_1 t/2$ if $\sigma > \sigma_0$. Since Lemma 2.3.9 holds under $\mathbf{Reg}_{R_1}$ only, we can use it and follow (Case 1) in the proof of Proposition 2.3.16 to see that (2.3.28) holds if $\sigma > \sigma_0$.

By the monotone property of σ, we see that $\sigma(t, x) \to \infty$ as $t \to \infty$ uniformly in $x \in (0, N]$. Hence, there exists a constant $t_N > 2T_0$ such that for all $t > t_N$ and $x \in (0, N]$, it holds $\sigma > \sigma_0$ so that $\mathcal{T}^2 \geq c_1 t/2$ and (2.3.28) holds. Therefore, by (2.3.17) and (2.3.22), the first assertion holds true.

Now, we further assume that $T_0 = 0$. Using the monotone property of σ again, we see that there exists a constant $x_N > 0$ such that $\sigma > \sigma_0$ for all $t \geq N$ and $x \in (0, x_N)$. Hence, (2.3.28) holds for all $t \geq N$ if $x < x_N$. Moreover, by (2.3.32), we get that $\lim_{x\to 0} \mathcal{T} = \infty$ uniformly in $t \in [N, \infty)$ since $t \mapsto \sigma$ is increasing. Therefore, the second assertion also holds true by (2.3.17) and (2.3.22). □

A similar result to Corollary 2.3.18 is obtained in [75, Section 3]. Note that since condition $\mathbf{Reg}_{R_1}$ is very mild, our result covers geometric stable subordinators and Gamma subordinators (see, [55, Example 3.4]), which are not covered in [75, Corollary 3.6].

2.3.3 Estimates on the transition density near the maximum value

Lemma 2.3.19. [55, Lemma 3.5] *Let $f : I \to [0,\infty)$ be a nondecreasing function defined on an interval $I \subset [0,\infty)$. Assume that there exist constants $a \in [0,\infty)$, $\beta, c_1 > 0$ such that*

$$\frac{f(r_2)}{f(r_1)} \geq c_1 \left(\frac{r_2}{r_1}\right)^{\beta} \qquad \text{for all } \ a < r_1 \leq r_2 \ \ (\text{resp. } \ 0 < r_1 \leq r_2 \leq a).$$

Then, for every $c_2 > 0$, there exists a constant $c_3 > 0$ such that

$$\int_a^\infty \exp\big(-c_2 t f(\xi)\big) d\xi \leq c_3 f^{-1}(1/t) \quad \text{for all } \ t \in (0, 1/f(a)),$$
$$\left(\text{resp. } \int_0^a \exp\big(-c_2 t f(\xi)\big) d\xi \leq c_3 f^{-1}(1/t) \quad \text{for all } \ t \in [1/f(a), \infty),\right)$$

where $f^{-1}(s) := \inf\{r \geq 0 : f(r) > s\}$ with the convention that $\inf \emptyset = \infty$.

Proposition 2.3.20. *(i) Suppose that $\mathbf{Poly}^{*}_{R_1,\geq}(\beta_1)$ holds. Then, for every $T > 0$, there exists a constant $C > 0$ such that for all $t \in (0,T]$,*

$$\sup_{x\in\mathbb{R}} p(t,x) \leq C H^{-1}(1/t). \tag{2.3.33}$$

(ii) Suppose that $\mathbf{Poly}^{*,\infty}_{R_2,\geq}(\beta_1)$ holds. Then, for every $T > T_0$, where T_0 is the constant in (2.3.1), there exists a constant $C > 0$ such that (2.3.33) holds for all $t \in [T,\infty)$.

Proof. Since the proofs are similar, we only give the proof for (ii) which is more delicate.
(ii) Fix $T_0 < T' < T$. By (2.3.1), there exists a constant $s_0 > 0$ such that $\nu(s) \geq 1/(2T_0 s)$ for all $s \in (0, s_0]$. By the Fourier inversion theorem, we get that for every $t \geq T$ and $x \in \mathbb{R}$,

$$p(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\xi x} e^{-t\phi(-i\xi)} d\xi$$

$$
\begin{aligned}
&\le \frac{1}{2\pi}\int_{|\xi|\le 1/s_0} |e^{-t\phi(-i\xi)}|d\xi + \frac{1}{2\pi}\int_{|\xi|>1/s_0} |e^{-(t-T')\phi(-i\xi)}||e^{-T'\phi(-i\xi)}|d\xi \\
&=: I_1 + I_2.
\end{aligned}
$$

By Lemma 2.3.12(iv), using the facts that $\cos y \le 1$ for all $y \in \mathbb{R}$ and $1-\cos s \ge s^2/4$ for all $|s| \le 1$, we get that for all $|\xi| \le 1/s_0$,

$$
\begin{aligned}
|e^{-t\phi(-i\xi)}| &= \exp\Big(-t\int_0^\infty (1-\cos(\xi s))\nu(s)ds\Big) \\
&\le \exp\Big(-t\int_0^{1/\xi} (1-\cos(\xi s))\nu(s)ds\Big) \\
&\le \exp\Big(-c_1 t\xi^2\int_0^{1/\xi} s^2\nu(s)ds\Big) \\
&\le e^{-c_2 tH(\xi)}.
\end{aligned}
$$

Hence, by Lemma 2.3.19,

$$
I_1 \le \frac{1}{\pi}\int_0^{1/s_0 \vee H^{-1}(1/T)} e^{-c_2 tH(\xi)}d\xi \le c_3 H^{-1}(1/t).
$$

On the other hand, since $T' > T_0$, we see from (2.3.2) that $|e^{-T'\phi(-i\xi)}| = e^{-T'\mathrm{Re}\,\phi(-i\xi)}$ is integrable on $\mathbb{R}$ with respect to ξ. Therefore, using the facts that $\cos y \le 1$ for all $y \in \mathbb{R}$ and $1-\cos s \ge s^2/4$ for all $|s| \le 1$ again, and (2.3.1), we get

$$
\begin{aligned}
I_2 &\le \frac{1}{2\pi}\Big(\sup_{|\xi|>1/s_0} |e^{-(t-T')\phi(-i\xi)}|\Big)\int_{|\xi|>1/s_0} |e^{-T'\phi(-i\xi)}|d\xi \\
&\le c_4 \sup_{|\xi|>1/s_0} \exp\Big(-(t-T')\int_0^\infty (1-\cos(\xi s))\nu(s)ds\Big) \\
&\le c_4 \sup_{|\xi|>1/s_0} \exp\Big(-c_1(t-T')\xi^2\int_0^{1/\xi} s^2\nu(s)ds\Big) \\
&\le c_4 \sup_{|\xi|>1/s_0} \exp\Big(-c_5(1-T'/T)t\xi^2\int_0^{1/\xi} s^2 s^{-1}ds\Big) \\
&= c_4 e^{-c_6 t}.
\end{aligned}
$$

Lemma 2.3.12(iii) implies that $H^{-1}(1/t) \geq c_8 t^{-((1/\beta_1)\vee(2/3))}$ for all $t \geq T$. Since $\sup_{t\geq T} t^{(1/\beta_1)\vee(2/3)} e^{-c_6 t} < \infty$, the desired result holds true. □

Now, we find a range of x which achieves the maximum value of $p(t,x)$. A similar result to the following proposition was established in [77, Theorem 5.3] which considers a class of Lévy processes whose Lévy measure dominates some symmetric measure. Note that since subordinators are nondecreasing, we can only push the y-variable to the positive direction in the following result, unlike [77, Theorem 5.3].

In the following proposition, we let $M_0 > 0$ be the constant in Proposition 2.3.15 in the first assertion, and the constant in Proposition 2.3.16 in the second.

Proposition 2.3.21. *(i) Suppose that* $\mathbf{Poly}^*_{R_1,\geq}(\beta_1)$ *holds. Then, for every* $N, T > 0$*, there exists a constant* $C > 1$ *such that*

$$C^{-1}H^{-1}(1/t) \leq p(t, tb(t/(2M_0)) + y) \leq CH^{-1}(1/t), \tag{2.3.34}$$

for all $t \in (0,T]$ *and* $0 \leq y \leq NH^{-1}(1/t)^{-1}$.
(ii) Suppose that $\mathbf{Reg}_{R_1}$ *and* $\mathbf{Poly}^{*,\infty}_{R_2,\geq}(\beta_1)$ *hold. Then, for every* $N > 0$*, there exists a constant* $C > 1$ *such that* (2.3.34) *holds for all* $t \in [2T_1, \infty)$ *and* $0 \leq y \leq NH^{-1}(1/t)^{-1}$ *with the constant* T_1 *in Proposition 2.3.16.*

Moreover, if $T_0 = 0$ *in* (2.3.1)*, then for every* $N, T > 0$*, there exists a constant* $C > 1$ *such that* (2.3.34) *holds for all* $t \in [T, \infty)$ *and* $0 \leq y \leq NH^{-1}(1/t)^{-1}$.

Proof. By Proposition 2.3.20, it remains to prove the lower bound. Since the proofs are similar, we only give the proof for (i).

Let $T > 0$. For $p \in [1,4]$, we observe that for all $t > 0$,

$$b(t/(pM_0)) \leq b(t/M_0) \quad \text{and} \quad \big((\phi')^{-1} \circ b\big)(t/(pM_0)) = H^{-1}(pM_0/t).$$

Hence, using Proposition 2.3.15 and Lemma 2.3.11, we get for all $t \in (0,T]$

and $p \in [1,4]$,

$$\begin{aligned} p\big(t, tb(t/(pM_0))\big) &\geq \frac{c_1 e^{-pM_0}}{\sqrt{t\,|(\phi'' \circ H^{-1})(pM_0/t)|}} \\ &\geq \frac{c_2 e^{-4M_0}}{\sqrt{pM_0}} H^{-1}(pM_0/t) \geq c_3 H^{-1}(1/t). \end{aligned} \tag{2.3.35}$$

By Lemma 2.0.1 and 2.3.11(iii),

$$tb(t/M_0) - tb(t/(4M_0)) \geq c_4 H^{-1}(1/t)^{-1} \quad \text{for all } \ t \in (0,T]$$

Thus, for all $t \in (0,T]$ and $u \in [0, c_4 H^{-1}(1/t)^{-1}]$, there is $p \in [1,4]$ such that $tb(t/M_0) - u = tb(t/(pM_0))$ by the intermediate value theorem. By (2.3.35), it follows that

$$p(t, tb(t/M_0) - u) \geq c_3 H^{-1}(1/t) \quad \text{for all } \ t \in (0,T], \ u \in [0, c_4 H^{-1}(1/t)^{-1}].$$

By the semigroup property, we deduce that for all $t \in (0,T]$ and $y \geq 0$,

$$\begin{aligned} p(2t, 2tb(t/M_0) + y) &= \int_{\mathbb{R}} p(t, tb(t/M_0) - u) p(t, tb(t/M_0) + y + u) du \\ &\geq \int_0^{c_4 H^{-1}(1/t)^{-1}} p(t, tb(t/M_0) - u) p(t, tb(t/M_0) + y + u) du \\ &\geq c_3 H^{-1}(1/t) \mathbb{P}\Big(y \leq S_t - tb(t/M_0) \leq y + c_4 H^{-1}(1/t)^{-1}\Big). \end{aligned}$$

Therefore, since $H^{-1}(1/t) \simeq H^{-1}(1/(2t))$ for $t \in (0,T]$ by Lemma 2.3.11(iii), to get the desired lower bound, it suffices to show that for each fixed $N > 0$, the following inequality holds true:

$$\inf_{t \in (0,T]} \ \inf_{y \in [0, NH^{-1}(1/t)^{-1}]} \mathbb{P}\Big(y \leq S_t - tb(t/M_0) \leq y + c_4 H^{-1}(1/t)^{-1}\Big) > 0. \tag{2.3.36}$$

Let $(t_n : n \geq 1)$ be a sequence of time variables realizing the infimum in (2.3.36). Since T is finite, after taking a subsequence, we can assume that t_n converges to $t_* \in [0,T]$. If $t_* \in (0,T]$, then since the support of the distribution of S_{t_*} is $(0,\infty)$, we obtain (2.3.36). Hence, we assume that $t_* = 0$ and all t_n are sufficiently small.

Define $\nu_n(s) := \mathbf{1}_{(0,H^{-1}(1/t_n)^{-1}]}\,\nu(s)$ and let $\tilde{S}_u$ be a driftless subordinator, whose Lévy measure is given by $\nu_n(s)ds$. Then, for all $u>0$, $S_u = \tilde{S}_u + P_u$, $\mathbb{P}$-a.s. where P is a compounded Poisson process, whose Lévy measure is given by $\mathbf{1}_{(H^{-1}(1/t_n)^{-1},\infty)}\,\nu(s)ds$. By (2.0.4), we have

$$\begin{aligned}\mathbb{P}(\tilde{S}_{t_n} = S_{t_n}) = \mathbb{P}(P_{t_n}=0) &= \exp\big(-t_n w\big(H^{-1}(1/t_n)^{-1}\big)\big)\\ &\geq \exp\big(-2et_n(H\circ H^{-1})(1/t_n)\big) = e^{-2e}.\end{aligned}$$

Hence, to prove (2.3.36), it is enough to show that

$$\liminf_{n\to\infty}\inf_{y\in[0,NH^{-1}(1/t_n)^{-1}]}\mathbb{P}\Big(y\leq \tilde{S}_{t_n} - t_nb(t_n/M_0)\leq y + c_4H^{-1}(1/t_n)^{-1}\Big) > 0. \tag{2.3.37}$$

Define $Z_n = H^{-1}(1/t_n)\big(\tilde{S}_{t_n} - t_nb(t_n/M_0)\big)$. Then, for $\xi\in\mathbb{R}$,

$$\begin{aligned}&\mathbb{E}\big[\exp(i\xi Z_n)\big] = \exp\Big(-i\xi t_nH^{-1}(1/t_n)b(t_n/M_0)\Big)\mathbb{E}\Big[\exp\big(i\xi H^{-1}(1/t_n)\tilde{S}_{t_n}\big)\Big]\\ &= \exp\left(-i\xi t_nH^{-1}(1/t_n)b(t_n/M_0) + t_n\int_0^\infty\Big(e^{i\xi H^{-1}(1/t_n)s}-1\Big)\nu_n(s)ds\right).\end{aligned}$$

Therefore, by a change of variables, we get $\mathbb{E}[\exp(i\xi Z_n)] = \exp(\Psi_n(\xi))$ for all $\xi\in\mathbb{R}$ and $n\geq 1$ where

$$\begin{aligned}\Psi_n(\xi) &= \int_0^\infty\left(e^{i\xi s} - 1 - \frac{i\xi s}{1+s^2}\right)\lambda_n(s)ds - i\xi\gamma_n,\\ \lambda_n(s) &= t_nH^{-1}(1/t_n)^{-1}\nu_n\big(H^{-1}(1/t_n)^{-1}s\big),\\ \gamma_n &= t_nH^{-1}(1/t_n)b(t_n/M_0) - \int_0^\infty\frac{s}{1+s^2}\lambda_n(s)ds.\end{aligned}$$

We claim that the family of random variables $\{Z_n : n\geq 1\}$ is tight. Indeed, according to [81, (3.2)], it holds that for all $n\geq 1$ and $R>1$,

$$\mathbb{P}(Z_n\geq R)\leq c_5\int_0^\infty\left(\frac{s^2}{R^2}\wedge 1\right)\lambda_n(s)ds$$

$$
\begin{aligned}
&+ c_5 R^{-1} \left| \gamma_n + \int_R^\infty \frac{s}{1+s^2} \lambda_n(s) ds - \int_0^R \frac{s^3}{1+s^2} \lambda_n(s) ds \right| \\
&=: c_5(I_1 + I_2).
\end{aligned}
$$

By a change of variables and (2.3.13), we see that

$$
\begin{aligned}
I_1 &= t_n \int_0^\infty \left(\frac{H^{-1}(1/t_n)^2 u^2}{R^2} \wedge 1 \right) \nu_n(u) du \\
&= R^{-2} t_n H^{-1}(1/t_n)^2 \int_0^{H^{-1}(1/t_n)^{-1}} u^2 \nu_n(u) du \\
&\le 2eR^{-2} t_n H(H^{-1}(1/t_n)) = 2eR^{-2}.
\end{aligned}
$$

Besides, by a change of variables, using the fact that the support of ν_n is contained in $(0, H^{-1}(1/t_n)^{-1}]$ in the first inequality below, the facts that for every $a > 0$, $1 - e^{-a} \le a$ and $\sup_{x>0} xe^{-ax} = e^{-1}a^{-1}$ in the second, and Lemmas 2.3.11 and (2.0.4) in the last, we get that

$$
\begin{aligned}
RI_2 &= \left| t_n H^{-1}(1/t_n) b(t_n/M_0) - \int_0^R s\lambda_n(s) ds \right| \\
&= t_n H^{-1}(1/t_n) \left| \int_0^\infty s e^{-sH^{-1}(M_0/t_n)} \nu(s) ds - \int_0^{RH^{-1}(1/t_n)^{-1}} s\nu_n(s) ds \right| \\
&\le t_n H^{-1}(1/t_n) \Big(\int_0^{H^{-1}(1/t_n)^{-1}} s\Big(1 - e^{-sH^{-1}(M_0/t_n)}\Big) \nu(s) ds \\
&\qquad + \int_{H^{-1}(1/t_n)^{-1}}^\infty s e^{-sH^{-1}(M_0/t_n)} \nu(s) ds \Big) \\
&\le t_n H^{-1}(1/t_n) \Big(H^{-1}(M_0/t_n) \int_0^{H^{-1}(1/t_n)^{-1}} s^2 \nu(s) ds \\
&\qquad + + H^{-1}(M_0/t_n)^{-1} w\big(H^{-1}(1/t_n)^{-1}\big) \Big) \\
&\le c_6 t_n H^{-1}(1/t_n) \left(H^{-1}(1/t_n)^{-1} (H \circ H^{-1})(1/t_n) \right) = c_6.
\end{aligned}
$$

Hence, $\mathbb{P}(Z_n \ge R) \le c_7 R^{-1}$ for all $n \ge 1$ and $R > 1$. By Prokhorov's theorem, this yields that there is a subsequence Z_{a_n} of Z_n which is weakly convergent to some random variable Z_*. Now, we can prove (2.3.37) by showing the following inequality:

$$\inf_{z\in[0,N]} \mathbb{P}(z \le Z_* \le z + c_4) > 0. \tag{2.3.38}$$

According to [115, Theorem 8.7], Z_* is a infinitely divisible random variable with the characteristic function

$$\Psi_*(\xi) = -\frac{1}{2}A_*\xi^2 - i\xi\gamma_* + \int_0^\infty \left(e^{i\xi s} - 1 - \frac{i\xi s}{1+s^2}\lambda_*(s)ds \right),$$

where the triplet $(A_*, \gamma_*, \lambda_*)$ is characterized by

(1) $\lim_{\epsilon\to 0} \limsup_{n\to\infty} \left| \int_0^\epsilon s^2\lambda_n(s)ds - A_* \right| = 0$;

(2) $\gamma_* = \lim_{n\to\infty} \gamma_n$;

(3) $\int_0^\infty f(s)\lambda_*(s)ds = \lim_{n\to\infty} \int_0^\infty f(s)\lambda_n(s)ds$ for any bounded continuous function f vanishing in a neighborhood of 0.

If $A_* > 0$, then the support of Z_* is $\mathbb{R}$ and hence (2.3.38) holds. Suppose that $A_* = 0$. Then, by using (1) and (3) in the above characterization, Lemma 2.3.11(iv), since H is nondecreasing, we get that for every $\eta \in (0, 1)$,

$$\begin{aligned}
\int_0^\eta s^2\lambda_*(s)ds &= \lim_{\epsilon\to 0+} \int_\epsilon^\eta s^2\lambda_*(s)ds = \lim_{\epsilon\to 0+} \lim_{n\to\infty} \left(\int_\epsilon^\eta s^2\lambda_n(s)ds \right) \\
&= \lim_{n\to\infty} \int_0^\eta s^2\lambda_n(s)ds = \lim_{n\to\infty} t_n H^{-1}(1/t_n)^2 \int_0^{\eta H^{-1}(1/t_n)^{-1}} u^2\nu(u)du \\
&\ge \lim_{n\to\infty} c_8\eta^2 t_n H(\eta^{-1}H^{-1}(1/t_n)) \ge c_8\eta^2 > 0.
\end{aligned}$$

Thus, by [113, Lemma 2.5], if $\int_0^1 s\lambda_*(s)ds = \infty$, then the support of Z_* is $\mathbb{R}$ so that (2.3.38) holds. Assume that $\int_0^1 s\lambda_*(s)ds < \infty$. Then we see from (3) in the characterization that $\limsup_{n\to\infty} \int_0^1 s\lambda_n(s)ds < \infty$. Hence, by using [113, Lemma 2.5] again, we see that the support of Z_* is $[-\chi, \infty)$ where $\chi = \lim_{n\to\infty} t_n H^{-1}(1/t_n)b(t_n/M_0) \ge 0$. Since the support of Z_* includes $(0, \infty)$ in any cases, we see that (2.3.38) holds. This finishes the proof. □

Now, we can omit the constant M_0 in Propositions 2.3.15 and 2.3.16.

Corollary 2.3.22. *(i) Suppose that* $\mathbf{Poly}^{*}_{R_1,\geq}(\beta_1)$ *holds. Then, for every* $N, T > 0$*, there are comparison constants such that for all* $t \in (0, T]$*,*

$$p(t,x) \simeq \frac{\exp\big(-tH(\sigma)\big)}{\sqrt{t(-\phi''(\sigma))}} \quad \textit{for } x \in (0, tb(t)], \tag{2.3.39}$$

$$p(t,x) \simeq H^{-1}(1/t) \quad \textit{for } x \in [tb(t),\, tb(t) + NH^{-1}(1/t)^{-1}]. \tag{2.3.40}$$

(ii) Suppose that $\mathbf{Reg}_{R_1}$ *and* $\mathbf{Poly}^{*;\infty}_{R_2,\geq}(\beta_1)$ *hold. Then, for every* $N > 0$*, there are comparison constants such that for all* $t \in [2T_1, \infty)$*,* (2.3.39) *and* (2.3.40) *hold with the constant* T_1 *in Proposition 2.3.16.*

Moreover, if $T_0 = 0$ *in* (2.3.1)*, then for every* $N, T > 0$*, there are comparison constants such that* (2.3.39) *and* (2.3.40) *hold for all* $t \in [T, \infty)$*.*

2.3.4 Right tail estimates

In this subsection, we get estimates on the transition density $p(t,x)$ when $x \geq tb(t)$. Recall the definitions of $D(t)$ and $\theta(t,y)$ from (2.3.4) and (2.3.5), respectively. From the definitions, we obtain

Lemma 2.3.23. *For all* $t > 0$ *and* $y \geq 0$*, it holds that*

$$t\theta(t,y)H(\theta(t,y)^{-1}) \leq y \vee H^{-1}(1/t)^{-1}.$$

In particular, we have

$$t\theta(t,y)H(\theta(t,y)^{-1}) = y \quad \textit{for all } t > 0,\ y \in [H^{-1}(1/t)^{-1}, D(t)].$$

The following theorem is the main result of this subsection.

Theorem 2.3.24. *(i) Suppose that* $\mathbf{Poly}^{*}_{R_1}(\beta_1, \beta_2)$ *holds and* $\sup_{r \geq R_1} \nu(r) < \infty$*. Then, for every* $T > 0$*, there exist constants* $c_1 > 0$*,* $C > 1$ *such that for* *all* $t \in (0, T]$ *and* $y \in [0, R_1/2)$,

$$C^{-1}H^{-1}(1/t)\min\left\{1,\ \frac{t\nu(y)}{H^{-1}(1/t)} + \exp\Big(-\frac{c_1 y}{\theta(t, y/(8e^2))}\Big)\right\}$$
$$\le p(t, tb(t)+y) \le CH^{-1}(1/t)\min\left\{1,\ \frac{t\nu(y)}{H^{-1}(1/t)} + \exp\Big(-\frac{y}{8\theta(t, y/(8e^2))}\Big)\right\}. \tag{2.3.41}$$

Moreover, if $\mathbf{Dou}^{\infty}_{R_1}$ *also holds, then* (2.3.41) *holds for all* $t \in (0, T]$ *and* $y \in [0, \infty)$.

(ii) Suppose that $\mathbf{Reg}_{R_1}$ *and* $\mathbf{Poly}^{*,\infty}_{R_2}(\beta_1, \beta_2)$ *hold. Then, there exist constants* $c_1 > 0$, $C > 1$ *such that* (2.3.41) *holds for all* $t \in [2T_1, \infty)$ *and* $y \in [0, \infty)$ *with the constant* T_1 *in Proposition 2.3.16.*

Moreover, if $T_0 = 0$ *in* (2.3.1), *then for every* $T > 0$, *there are constants* $c_1 > 0$, $C > 1$ *such that* (2.3.41) *holds for all* $t \in [T, \infty)$ *and* $y \in [0, \infty)$.

Proof. The result follows from Propositions 2.3.25 and 2.3.27 below. □

Proposition 2.3.25. *Under the setting of Theorem 2.3.24, the upper bound in* (2.3.41) *holds true.*

Proof. For convenience of notation, we let $\delta := 1/(8e^2)$.

(i) We first assume that $\mathbf{Poly}^{*}_{R_1}(\beta_1, \beta_2)$ holds and $\sup_{r \ge R_1} \nu(r) < \infty$ only. Let $T > 0$, $t \in (0, T]$ and $y \ge 0$. If $\delta y \le H^{-1}(1/t)^{-1}$, then $\exp(-y/\theta(t, \delta y)) \ge e^{-1/\delta}$ and hence the upper bound in (2.3.41) follows from Proposition 2.3.20. Hence, for the remainder part of the proof of (i), we assume $\delta y > H^{-1}(1/t)^{-1}$.

Set

$$\nu_1(s) := \mathbf{1}_{(0, \theta(t, \delta y)]}\,\nu(s) \quad \text{and} \quad \nu_2(s) := \nu(s) - \nu_1(s).$$

Denote by S^i and H_i the corresponding driftless subordinator and H-function with respect to the Lévy measure ν_i for $i = 1, 2$, respectively. We suppose that S^1 and S^2 are independent. By Proposition 2.3.1, for every $u > T_0$, S^1_u has a transition density function $p^1(u, \cdot)$. Recall that $T_0 = 0$ under $\mathbf{Poly}^{*}_{R_1}(\beta_1, \beta_2)$.

Since $S_t = S_t^1 + S_t^2$, it holds that

$$
\begin{aligned}
p(t, tb(t)+y) &= \int_{\mathbb{R}} p^1(t, tb(t)+y-z)\mathbb{P}(S_t^2 \in dz) \\
&= \left(\int_{z \le y/4} + \int_{z > y/4} \right) p^1(t, tb(t)+y-z)\mathbb{P}(S_t^2 \in dz) \\
&\le \sup_{z \ge 3y/4} p^1(t, tb(t)+z) + \sup_{z > y/4} \frac{\mathbb{P}(S_t^2 \in dz)}{dz} =: A_1 + A_2. \qquad (2.3.42)
\end{aligned}
$$

Step 1. First, we estimate A_1. By the semigroup property, for all $z \ge 3y/4$,

$$
\begin{aligned}
&p^1(t, tb(t)+z) \\
&= \left(\int_{u < z/2} + \int_{u \ge z/2} \right) p^1\big(t/2, tb(t)/2 + u\big) p^1\big(t/2, tb(t)/2 + z - u\big) du \\
&\le 2\mathbb{P}\left(S_{t/2}^1 \ge \frac{t}{2} b(t) + \frac{3y}{8} \right) \sup_{u \in \mathbb{R}} p^1(t/2, u).
\end{aligned}
$$

Using [48, Lemma 7.2], Proposition 2.3.20 and Lemma 2.3.11(iii), we get that

$$
\begin{aligned}
\sup_{u \in \mathbb{R}} p^1(t/2, u) &\le e^{2^{-1} t w(\theta(t, \delta y))} \sup_{u \in \mathbb{R}} p(t/2, u) \\
&\le c_1 e^e H^{-1}(2/t) \le c_2 H^{-1}(1/t). \qquad (2.3.43)
\end{aligned}
$$

On the other hand, by Markov's inequality and Lemma 2.2.5, it holds that for every $\lambda > 0$,

$$
\begin{aligned}
&\mathbb{P}\left(S_{t/2}^1 \ge \frac{t}{2} b(t) + \frac{3y}{8} \right) \le \mathbb{E}\left[\exp\left(\lambda S_{t/2}^1 - \frac{\lambda t}{2} b(t) - \frac{3\lambda y}{8} \right) \right] \\
&= e^{-3\lambda y/8} \exp\left(\frac{t}{2} \int_0^{\theta(t,\delta y)} (e^{\lambda s} - 1)\nu(s) ds - \frac{t}{2} \int_0^\infty \lambda s e^{-H^{-1}(1/t)s} \nu(s) ds \right) \\
&\le e^{-3\lambda y/8} \exp\left(\frac{\lambda t}{2} \int_0^{\theta(t,\delta y)} (e^{\lambda s} - e^{-H^{-1}(1/t)s}) s\nu(s) ds \right) \\
&\le e^{-3\lambda y/8} \exp\left(\frac{\lambda t}{2} (\lambda + H^{-1}(1/t)) \int_0^{\theta(t,\delta y)} e^{\lambda s} s^2 \nu(s) ds \right).
\end{aligned}
$$

We used the mean value theorem in the second and third inequalities. Thus, by letting $\lambda = \theta(t,\delta y)^{-1} \geq H^{-1}(1/t)$, using Lemma 2.3.23, we get that

$$\begin{aligned}\mathbb{P}\Big(S^1_{t/2} \geq \frac{t}{2}b(t) + \frac{3y}{8}\Big) &\leq \exp\Big(-\frac{3\lambda y}{8} + e\lambda^2 t\int_0^{1/\lambda} s^2\nu(s)ds\Big)\\ &\leq \exp\Big(-\frac{3\lambda y}{8} + 2e^2 tH(\lambda)\Big)\\ &= \exp\Big(-\frac{\lambda}{8}(3y - 16e^2 t\lambda^{-1}H(\lambda))\Big)\\ &\leq \exp\Big(-\frac{\lambda}{8}(3y - 16e^2\delta y)\Big)\\ &= \exp\Big(-\frac{y}{8\theta(t,\delta y)}\Big). \end{aligned} \tag{2.3.44}$$

We used (2.3.13) in the second inequality. Consequently, we deduce that

$$A_1 \leq 2c_2 H^{-1}(1/t)\exp\Big(-\frac{y}{8\theta(t,\delta y)}\Big). \tag{2.3.45}$$

Note that (2.3.42) and (2.3.45) hold for all $y > \delta^{-1}H^{-1}(1/t)^{-1}$ and we have not assumed $y < R_1/2$ yet.

Step 2. Next, we assume $y \in [0, R_1/2)$ and estimate A_2. Since S^2 is a compounded Poisson process, for every $z > 0$ and $\rho > 0$, we have that

$$\mathbb{P}(S^2_{t/2} \in (z, z+\rho)) = \sum_{n=1}^{\infty} e^{-tw(\theta(t,\delta y))/2}\,\frac{t^n\nu_2^{n*}(z,z+\rho)}{n!} \leq \sum_{n=1}^{\infty}\frac{t^n\nu_2^{n*}(z,z+\rho)}{n!}, \tag{2.3.46}$$

where ν_2^{n*} is the n-fold convolution of the measure ν_2. Define a function $f : (0,\infty) \to (0,\infty)$ by

$$f(r) := \begin{cases} \sup_{u\geq r}\nu(u) & \text{if } \ r < R_1/2,\\ \sup_{u\geq R_1/2}\nu(u) & \text{if } \ r \geq R_1/2. \end{cases} \tag{2.3.47}$$

Then f is nonincreasing and $\nu(r) \leq f(r)$ for all $r > 0$. Moreover, by $\mathbf{Poly}^*_{R_1}(\beta_1,\beta_2)$ and the assumption that $\sup_{r\geq R_1}\nu(r) < \infty$, we see that $f(r) \leq c_3\nu(r)$ for $r \in (0, R_1/2)$ and that $\sup_{r>0} f(r)/f(2r) = c_4 < \infty$.

Now, we prove that for every $n \geq 1$,

$$\nu_2^{n*}(z, z+\rho) \leq (4ec_4)^n t^{1-n} f(z)\rho \quad \text{for all } \; z, \rho > 0. \tag{2.3.48}$$

Cf. [83, Lemma 9 and Corollary 10]. Since $\nu(r) \leq f(r)$ for all $r > 0$, it holds that $\nu_2(z, z+\rho) \leq f(z)\rho$. Assume that (2.3.48) is true for $n \geq 1$. Using the induction hypothesis and Fubini's theorem in the first inequality below, the facts that f is nonincreasing and $\nu(r) \leq f(r)$ for all $r > 0$ in the second, and the fact that $\nu_2(\mathbb{R}) = w(\theta(t, \delta y)) \leq w(w^{-1}(2e/t)) = 2e/t$ in the last, we get

$$\begin{aligned}
&\nu_2^{(n+1)*}(z, z+\rho) = \left(\int_{u<z/2} + \int_{u \geq z/2} \right) \nu_2^{n*}(z-u, z-u+\rho)\nu_2(du) \\
&\leq \int_{u<z/2} (4ec_4)^n t^{1-n} f(z-u)\rho\nu_2(du) + \int_0^{z/2+\rho} \int_{(z-v)\vee(z/2)}^{z-v+\rho} \nu_2(du)\nu_2^{n*}(dv) \\
&\leq (4ec_4)^n t^{1-n} f(z/2)\rho\nu_2(\mathbb{R}) + f(z/2)\rho\nu_2(\mathbb{R})^n \\
&\leq \big(2e(4ec_4)^n + (2e)^n\big) c_4 t^{-n} f(z)\rho \leq (4ec_4)^{n+1} t^{-n} f(z)\rho.
\end{aligned}$$

Hence, we conclude that (2.3.48) holds by induction.

By (2.3.46), (2.3.48) and the monotone property and doubling property of f, since $y \in [0, R_1/2)$, we deduce that

$$\begin{aligned}
A_2 &= \sup_{z>y/4} \lim_{\rho \to 0} \rho^{-1} \mathbb{P}(S^2_{t/2} \in (z, z+\rho)) \\
&\leq \sup_{z>y/4} tf(z) \sum_{n=1}^{\infty} \frac{(4ec_4)^n}{n!} \leq e^{4ec_4} tf(y/4) \leq c_5 tf(y) \leq c_3c_5 t\nu(y).
\end{aligned} \tag{2.3.49}$$

Finally, we get the desired upper bound from (2.3.42), (2.3.45) and (2.3.49).

Now, we further assume that $\mathbf{Dou}^{\infty}_{R_1}$ holds and assume that $y > (R_1/2) \vee (\delta^{-1} H^{-1}(1/t)^{-1})$. Recall that (2.3.42) and (2.3.45) still hold for those values of y. Define $f_*(r) := \sup_{u \geq r} \nu(u)$. By $\mathbf{Poly}^*_{R_1}(\beta_1, \beta_2)$ and $\mathbf{Dou}^{\infty}_{R_1}$, it holds that $\nu(r) \simeq f_*(r)$ for $r > 0$, and $\sup_{r>0} f_*(r)/f_*(2r) = c_6 < \infty$. Then, by following the above proof given in *Step 2.*, we get $A_2 \leq e^{4ec_6} tf_*(y/4) \leq c_7\nu(y)$. Thus, (2.3.49) still holds for those values of y and this completes the proof.

(ii) We follow the proof of (i). Since $T_1 > T_0$, we see that S^1_u has a transition density function $p^1(u, \cdot)$ for all $u \geq T_1$ by Proposition 2.3.1. Hence, (2.3.42) still holds. Also, by using Proposition 2.3.20 and Lemma 2.3.12(iii), we get that (2.3.43) holds for all $t \in [2T_1, \infty)$. We can prove (2.3.44) by exactly the same way. Moreover, by using Lemma 2.3.12 instead of Lemma 2.3.11, and the function $f_*(r) := \sup_{u \geq r} \nu(u)$ instead of the function f given in (2.3.47), we can follow the proof in *Step 2.* This proves the proposition under the conditions $\mathbf{Reg}_{R_1}$ and $\mathbf{Poly}^{*,\infty}_{R_2}(\beta_1, \beta_2)$.

Furthermore, if $T_0 = 0$, then for every fixed $T > 0$, by Proposition 2.3.16, (2.3.42) holds for all $t \geq T$. Then, there is no difference in the proof for the last assertion. □

Now, we begin to prove the lower bound in Theorem 2.3.24. We first establish a preliminary jump type estimates for $p(t, x)$.

Proposition 2.3.26. *(i) Suppose that* $\mathbf{Poly}^{*}_{R_1,\geq}(\beta_1)$ *holds and* $\sup_{r \geq R_1} \nu(r) < \infty$. *Then, for every* $T > 0$, *there exists a constant* $c_1 > 0$ *such that for all* $t \in (0, T]$ *and* $y \in [0, R_1/2)$,

$$p(t, tb(t) + y) \geq c_1 H^{-1}(1/t) \min\left\{1, \frac{t\nu(y)}{H^{-1}(1/t)}\right\}. \qquad (2.3.50)$$

Moreover, if $\mathbf{Dou}^{\infty}_{R_1}$ *also holds, then* (2.3.50) *holds true for all* $t \in (0, T]$ *and* $y \in [0, \infty)$.

(ii) Suppose that $\mathbf{Reg}_{R_1}$ *and* $\mathbf{Poly}^{*,\infty}_{R_2,\geq}(\beta_1)$ *hold. Then, there exists a constant* $c_1 > 0$ *such that* (2.3.50) *holds for all* $t \in [2T_1, \infty)$ *and* $y \in [0, \infty)$ *with the constant* T_1 *in Proposition 2.3.16.*

Moreover, if $T_0 = 0$ *in* (2.3.1), *then for every* $T > 0$, *there exists* $c_1 > 0$ *such that* (2.3.50) *holds for all* $t \in [T, \infty)$ *and* $y \in [0, \infty)$.

Proof. (i) Let $T > 0$, $t \in (0, T]$ and $y \geq 0$. If $y \leq 2H^{-1}(1/t)^{-1}$, then (2.3.50) follows from Corollary 2.3.22. Hence, we assume $y > 2H^{-1}(1/t)^{-1}$.

With a constant $\epsilon \in (0, 1/2)$ which will be chosen later, we define

$$\mu_1(s) := \mathbf{1}_{(0,H^{-1}(1/t))^{-1}}\, \nu(s) + (1-\epsilon)\mathbf{1}_{[H^{-1}(1/t)^{-1},\infty)}\, \nu(s)$$

and $\mu_2(s) := \nu(s) - \mu_1(s)$. We denote by T^i the corresponding driftless subordinator with respect to the Lévy measure μ_i for $i = 1, 2$, respectively. We suppose that T^1 and T^2 are independent.

By Proposition 2.3.1, for all $u > T_0$, T^1_u has a transition density function $q^1(u, \cdot)$. Using $\mathbf{Poly}^*_{R_1,\geq}(\beta_1)$ and the condition that $\sup_{r\geq R_1} \nu(r) < \infty$ in the second inequality below, Lemma 2.3.11(i) in the third, (2.0.4) in the fourth and (2.0.5) in the fifth, since $t^{-1}H^{-1}(1/t) \geq T^{-1}H^{-1}(1/T)$, we get that

$$\begin{aligned}\sup_{s>0} \mu_2(s) &\leq \epsilon\big(\sup_{H^{-1}(1/t)^{-1}\leq s<R_2} \nu(s) + \sup_{s\geq R_2} \nu(s)\big)\\ &\leq \epsilon c_1 \nu(H^{-1}(1/t)^{-1}) + \epsilon c_1\\ &\leq \epsilon c_2 H^{-1}(1/t) w(H^{-1}(1/t)^{-1}/2) + \epsilon c_1\\ &\leq 2e\epsilon c_2 H^{-1}(1/t) H(2H^{-1}(1/t)) + \epsilon c_1\\ &\leq 8e\epsilon c_2 t^{-1} H^{-1}(1/t) + \epsilon c_1 \leq \epsilon c_3 t^{-1} H^{-1}(1/t).\end{aligned}$$

Therefore, by [10, Lemma 3.1(c)] and Corollary 2.3.22, we see that for all $z \in [0, H^{-1}(1/t)^{-1}]$,

$$q^1(t, tb(t) + z) \geq p(t, tb(t) + z) - t \sup_{s>0} |\mu_2(s)| \geq (c_4 - \epsilon c_3) H^{-1}(1/t).$$

By taking $\epsilon = c_4/(2c_3)$, we arrive at

$$q^1(t, tb(t) + z) \geq 2^{-1} c_4 H^{-1}(1/t) \quad \text{for all } z \in [0, H^{-1}(1/t)^{-1}].$$

Since $S_t = T^1_t + T^2_t$, T^2 is a compounded Poisson process and $y > 2H^{-1}(1/t)^{-1}$, using (2.0.4), it follows that

$$\begin{aligned}p(t, tb(t) + y) &= \int_{\mathbb{R}} q^1(t, tb(t) + y - z) \mathbb{P}(T^2_t \in dz)\\ &\geq 2^{-1} c_4 H^{-1}(1/t) \mathbb{P}\big(T^2_t \in [y - H^{-1}(1/t)^{-1}, y]\big)\\ &\geq 2^{-1} c_4 H^{-1}(1/t)\, \epsilon t \nu\big([y - H^{-1}(1/t)^{-1}, y]\big) e^{-\epsilon t w(H^{-1}(1/t)^{-1})}\\ &\geq 2^{-1} c_4 \epsilon e^{-2e\epsilon} t H^{-1}(1/t) H^{-1}(1/t)^{-1} \inf_{u\in[y-H^{-1}(1/t)^{-1}, y]} \nu(u)\end{aligned}$$

$$\geq c_5 t \inf_{u\in[y/2,y]} \nu(u).$$

By $\mathbf{Poly}^*_{R_1,\geq}(\beta_1)$, we can see that for $z \in (2H^{-1}(1/t)^{-1}, R_1/2)$,

$$\inf_{u\in[z/2,z]} \nu(u) \simeq \nu(z). \tag{2.3.51}$$

Moreover, if $\mathbf{Dou}^{\infty}_{R_1}$ also holds, then (2.3.51) holds for all $z > 2H^{-1}(1/t)^{-1}$. Hence, we get the results.

(ii) Let $N > 2$ be such that $NH^{-1}(1/T)^{-1} \geq R_2$. By Corollary 2.3.22, it suffices to prove (2.3.50) for $y > NH^{-1}(1/T)^{-1}$. This can be done by repeating the proof for (i). The proof for the second assertion is exactly the same. □

Proposition 2.3.27. *Under the setting of Theorem 2.3.24, the lower bound in* (2.3.41) *holds true.*

Proof. We prove (ii) first. Since the proof for the case when $T_0 = 0$ is easier, we only give the proof for the case when $T_0 > 0$.

Let $\rho := (16e^2 T_1 H(w^{-1}(e/T_1)^{-1}))^{-1} \wedge (4e^2)^{-1}$ with the constant T_1 in Proposition 2.3.16. Then, since the map $t \mapsto H(w^{-1}(2e/t)^{-1})$ is decreasing, it holds that

$$\frac{1}{8e^2\rho H(w^{-1}(2e/t)^{-1})} \geq 2T_1 \quad \text{for all } t \geq 2T_1. \tag{2.3.52}$$

By Lemma 2.3.13, Corollary 2.3.22 and Proposition 2.3.26, it remains to prove that there are constants $c_1, c_2 > 0$ such that for all $t \in [2T_1, \infty)$ and $y \in [2\rho^{-1}H^{-1}(1/t)^{-1}, 8e^2 D(t))$,

$$p(t, tb(t) + y) \geq c_1 H^{-1}(1/t) \exp\Big(- \frac{c_2 y}{\theta(t, y/(8e^2))}\Big).$$

Fix $t \in [2T_1, \infty)$, $y \in [2\rho^{-1}H^{-1}(1/t)^{-1}, 8e^2 D(t))$ and we simply denote $\theta := \theta(t, y/(8e^2))$. Then, since $2\rho^{-1} \geq 8e^2$, by Lemma 2.3.23, we have

$$8e^2 t\theta H(\theta^{-1}) = y. \tag{2.3.53}$$

Let $n = \lfloor \rho y/\theta \rfloor$. Since $\theta \le H^{-1}(1/t)^{-1}$, we have $n \ge \rho y H^{-1}(1/t) - 1 \ge 1$. We claim that there exist constants $\kappa_1 \in (0,1)$ and $\kappa_2 \in (1,\infty)$ independent of t and y such that

$$\kappa_1 H^{-1}(n/t)^{-1} \le y/n \le \kappa_2 H^{-1}(n/t)^{-1}. \tag{2.3.54}$$

Indeed, first note that (2.3.54) is equivalent to $H(\kappa_1 n/y) \le n/t \le H(\kappa_2 n/y)$. Since $\rho/\theta \le \rho w^{-1}(e/T_1)^{-1}$, by Lemma 2.3.12(iii), (2.0.5) and (2.3.53), there exists $c_3 \in (0,1)$ independent of t and y such that for every $\kappa \in [2, y/n]$,

$$\begin{aligned} H(\kappa n/y) &\ge c_3 \kappa^{\beta_1'} H(2n/y) \ge c_3 \kappa^{\beta_1'} H(\rho/\theta) \ge c_3 \kappa^{\beta_1'} \rho^2 H(\theta^{-1}) \\ &= \frac{c_3 \kappa^{\beta_1'} \rho^2 y}{8e^2 t\theta} \ge \frac{c_3 \kappa^{\beta_1'} \rho n}{8e^2 t}, \end{aligned}$$

where $\beta_1' = \beta_1 \wedge (3/2)$. Hence, if $y/n \ge (8e^2 c_3^{-1} \rho^{-1})^{1/\beta_1'}$, then the upper bound in (2.3.54) holds with any $\kappa_2 > (8e^2 c_3^{-1}\rho^{-1})^{1/\beta_1'}$. Otherwise, if $y/n < (8e^2 c_3^{-1}\rho^{-1})^{1/\beta_1'}$, then we obtain

$$(8e^2 c_3^{-1}\rho^{-1})^{1/\beta_1'} \ge \frac{y}{n} \ge \frac{\theta}{\rho} \ge \frac{w^{-1}(2e/t)}{\rho} \ge \frac{w^{-1}(e/T_1)}{\rho}.$$

This implies that $t \simeq 1$ so that $y \simeq \theta \simeq n \simeq 1$. Hence, by choosing κ_2 large enough, we deduce that the upper bound in (2.3.54) holds true. On the other hand, we see from Lemma 2.3.12(iii) and (2.3.53) that for every $\kappa \in (0,1)$,

$$\begin{aligned} H(\kappa n/y) &\le c_4 \kappa^{\beta_1'} H(n/y) \le c_4 \kappa^{\beta_1'} H(\rho/\theta) \le c_5 \kappa^{\beta_1'} \rho^{\beta_1'} H(\theta^{-1}) \\ &= \frac{c_5 \kappa^{\beta_1'} \rho^{\beta_1'} y}{8e^2 t\theta} \le \frac{c_5 \kappa^{\beta_1'} \rho^{\beta_1'-1} n}{4e^2 t}. \end{aligned}$$

Therefore, we also deduce that the lower bound in (2.3.54) holds true.

Set $z := y + tb(t) - tb(t/n)$. For $j \in \{1, ..., n-1\}$, define $z_j = jz/n$ and $A_j = (z_j - z/(2n), z_j + z/(2n))$. By Lemma 2.0.1, (2.3.54) and Lemma

2.3.12(ii), since $y \geq 2\rho^{-1}H^{-1}(1/t)^{-1}$, it holds that

$$\begin{aligned} y \leq z \leq y + \frac{2en}{H^{-1}(n/t)} + \frac{e^{-1}tw(H^{-1}(n/t)^{-1})}{H^{-1}(1/t)} \\ \leq (1+2e/\kappa_1)y + \frac{e^{-1}tw(y/(\kappa_2 n))}{H^{-1}(1/t)} \\ \leq (1+2e/\kappa_1)y + \frac{c_6 tw(\theta)}{H^{-1}(1/t)} \leq (1+2e/\kappa_1 + c_6\rho e)y. \end{aligned}$$

We used the definition that $\theta \geq w^{-1}(2e/t)$ in the last inequality. Then, by (2.3.54), we get that for any $j \in \{1,..,n-2\}$, $u \in A_j$ and $v \in A_{j+1}$,

$$|u-v| \leq \frac{z}{n} + |z_{j+1} - z_j| = \frac{2z}{n} \leq \frac{c_7 y}{n} \leq \frac{c_7\kappa_2}{H^{-1}(n/t)}.$$

Note that by (2.3.52) and (2.3.53),

$$\frac{t}{n} \geq \frac{t\theta}{\rho y} = \frac{1}{8e^2\rho H(\theta^{-1})} \geq \frac{1}{8e^2\rho H(w^{-1}(2e/t)^{-1})} \geq 2T_1.$$

Therefore, by Corollary 2.3.22, there exists $c_8 > 0$ independent of t and y such that for every $j \in \{1,...,n-2\}$,

$$p\left(t/n, (t/n)b(t/n) + v - u\right) \geq c_8 H^{-1}(n/t) \quad \text{for all } u \in A_j,\ v \in A_{j+1}.$$

Then, by the semigroup property and (2.3.54), we deduce that

$$\begin{aligned} & p(t, tb(t) + y) \\ & \geq \int_{A_1\times\cdots\times A_k} p\big(t/n, (t/n)b(t/n) + u_1\big) \prod_{k=1}^{n-2} p\big(t/n, (t/n)b(t/n) + u_{k+1} - u_k\big) \\ & \qquad \times p\big(t/n, (t/n)b(t/n) + z - u_{n-1}\big) du_1...du_{n-1} \\ & \geq (c_8 H^{-1}(n/t))^n (z/n)^{n-1} \geq (c_8 H^{-1}(n/t))^n (y/n)^{n-1} \\ & \geq (c_8\kappa_1)^{n-1} H^{-1}(n/t) \geq (c_8\kappa_1)^{-1} H^{-1}(1/t) \exp\big(-n\log(c_8\kappa_1)^{-1}\big). \end{aligned}$$

Since $n \leq \rho y/\theta$, we have finished the proof for (ii).

(i) We follow the proof for (ii). In this case, welet $\rho = (4e^2)^{-1}$. Since $\theta^{-1} \geq H^{-1}(1/t)^{-1} \geq H^{-1}(1/T)^{-1}$, by using Lemma 2.3.11 instead of Lemma 2.3.12, we obtain (2.3.54). Then, we get the result by exactly the same proof. We can also conclude that the second assertion in (i) is true in view of the second statement in Proposition 2.3.26. □

2.3.5 Proofs of Theorems 2.3.4, 2.3.6 and Corollaries 2.3.5, 2.3.7 and 2.3.8

In this subsection, we give the proofs for our main theorems of this section.

Lemma 2.3.28. *(i) Suppose that* $\mathbf{Poly}^*_{R_1,\geq}(\beta_1)$ *holds. Then, for every fixed* $T > 0$*, there exist* $c_1, c_2 > 0$ *such that for all* $t \in (0, T]$ *and* $x \in (0, tb(t)]$*,*

$$c_1 H^{-1}(1/t)e^{-2tH(\sigma)} \leq \frac{e^{-tH(\sigma)}}{\sqrt{t(-\phi''(\sigma))}} \leq c_2 H^{-1}(1/t)e^{-2^{-1}tH(\sigma)}. \qquad (2.3.55)$$

Moreover, if $\mathbf{Poly}^*_{\infty,\geq}(\beta_1)$ *holds, then* (2.3.55) *holds for all* $t \in (0, \infty)$ *and* $x \in (0, tb(t)]$*.*
(ii) Suppose that $\mathbf{Poly}^{*,\infty}_{R_2,\geq}(\beta_1)$ *holds. Then, for every fixed* $T > 0$*, there exist* $c_1, c_2 > 0$ *such that* (2.3.55) *holds for all* $t \in [T, \infty)$ *and* $x \in [tb(T), tb(t)]$*.*

Proof. (i) Note that for all $t \in (0, T]$ and $x \in (0, tb(t)]$, we have $\sigma \geq H^{-1}(1/t) \geq H^{-1}(1/T)$ and $tH(\sigma) \geq 1$. Hence, by Lemma 2.3.11(iii & iv),

$$c_1 H^{-1}(1/t)e^{-2tH(\sigma)} \leq \frac{e^{-tH(\sigma)}}{\sqrt{t(-\phi''(\sigma))}} \simeq \frac{\sigma e^{-tH(\sigma)}}{\sqrt{tH(\sigma)}} \leq \sigma e^{-tH(\sigma)}. \qquad (2.3.56)$$

Using Lemma 2.3.11(iii) and the fact that $x^k e^{-x} \leq k^k e^{-k}$ for all $x, k > 0$, we get that

$$\begin{aligned}\sigma e^{-tH(\sigma)} &= H^{-1}(1/t)\frac{\sigma}{H^{-1}(1/t)}e^{-tH(\sigma)} \leq c_2 H^{-1}(1/t)\left(\frac{H(\sigma)}{1/t}\right)^{1/\beta_1} e^{-tH(\sigma)} \\ &\leq c_3 H^{-1}(1/t)e^{-2^{-1}tH(\sigma)}. \qquad (2.3.57)\end{aligned}$$

This proves the first assertion. If we further assume that $R_1 = \infty$, then by combining Lemmas 2.3.11 and 2.3.12, we can see that (2.3.56) and (2.3.57) hold for all $t \in (0, \infty)$ and $x \in (0, tb(t)]$ since $tH(\sigma) \geq 1$ for those values of t and x. We have finished the proof for (i).
(ii) For all $t \in [T, \infty)$ and $x \in [tb(T), tb(t)]$, we have $\sigma \leq H^{-1}(1/T)$ and $tH(\sigma) \geq 1$. Hence, by using Lemma 2.3.12 instead of Lemma 2.3.11, we can follow the proof for (i) and conclude that (ii) also holds. □

Proof of Theorems 2.3.4 and 2.3.6. The results follow from Corollary 2.3.22, Theorem 2.3.24 and Lemmas 2.3.13 and 2.3.28. □

Proof of Corollaries 2.3.5 and 2.3.7. Since the proofs are similar, we only give the proof for Corollary 2.3.5. By [54, Lemma 2.1(iii)], since $\beta_2 < 2$, it holds that $w^{-1}(2e/t) \simeq H^{-1}(1/t)^{-1}$ for $t \in (0, T]$. It follows that $D(t) \simeq H^{-1}(1/t)^{-1}$ for $t \in (0, T]$. Thus, by Theorem 2.3.4(ii), (2.3.9) and Corollary 2.3.22, we obtain (2.3.10).

On the other hand, note that by (2.3.1), we have $\nu\big((x - tb(t))_+\big) = \nu(0) = \infty$ for all $x \leq tb(t)$. Thus, by joining (2.3.10) and (2.3.7) together, we also deduce (2.3.12). □

Proof of Corollary 2.3.8. Since the proofs for the case $T_0 = 0$ and the case $T_0 > 0$ are similar, we give the proof for the case $T_0 > 0$ only.

Let $T_1 > 0$ be the constant in Theorem 2.3.6(i) and $\beta_1' := \beta_1 \wedge (3/2)$. Note that $\beta_1' > 1$. By Lemma 2.3.12(iii), there exists $c_1 \in (0, 1)$ such that

$$H(\kappa\lambda) \geq c_1 \kappa^{\beta_1'} H(\lambda) \quad \text{for all } \kappa \geq 1,\ 0 < \lambda \leq \kappa^{-1}. \tag{2.3.58}$$

Moreover, by Lemma 2.3.12(ii & iv), we see that for every $t \geq T_1$,

$$\begin{aligned} 0 \leq t\phi'(0) - tb(t) &= t \int_0^{H^{-1}(1/t)} (-\phi''(\lambda)) d\lambda \\ &\leq c_2 \int_0^{H^{-1}(1/t)} \lambda^{-2} \frac{H(\lambda)}{H(H^{-1}(1/t))} d\lambda \end{aligned}$$

$$\leq c_3 H^{-1}(1/t)^{-\beta_1'} \int_0^{H^{-1}(1/t)} \lambda^{-2+\beta_1'} d\lambda$$
$$\leq c_4 H^{-1}(1/t)^{-1}. \tag{2.3.59}$$

Write $y_t = y + t\phi'(0) - tb(t)$. Define

$$F(t,y) = \min\left\{1,\ \frac{t\nu(y)}{H^{-1}(1/t)} + \exp\Big(-\frac{y}{\mathscr{H}^{-1}(t/y)}\Big)\right\},$$
$$G(t,y,c) = \min\left\{1,\ \frac{t\nu(y_t)}{H^{-1}(1/t)} + \exp\Big(-\frac{cy_t}{\theta(t,y_t/(8e^2))}\Big)\right\}.$$

By Theorem 2.3.6(i), since $t\phi'(0) + y = tb(t) + y_t$, it remains to prove that

$$F(t,y) \asymp G(t,y,c) \quad \text{for } t \geq T_1,\ y \geq 0. \tag{2.3.60}$$

We prove (2.3.60) by considering several cases. We use the following notations below.

$$\epsilon_1 := (c_1/(8e^2))^{1/(\beta_1'-1)} \in (0,1), \quad \kappa_1 := c_1^{-1/(\beta_1'-1)} > 1, \quad \theta := \theta(t, y_t/(8e^2)).$$

(1) Suppose that $0 \leq y_t < 8e^2 H^{-1}(1/t)^{-1}$. Then, we have $\theta = H^{-1}(1/t)^{-1} \geq y_t/(8e^2)$ and hence $G(t,y,1) \simeq 1$. We claim that it also holds that $F(t,y) \asymp 1$ which yields the desired result in this case. To prove this claim, we consider the following two cases separately.

(a) Suppose that $t \geq 1/H(\epsilon_1)$. Then we get from (2.3.58) that

$$\mathscr{H}(\epsilon_1 H^{-1}(1/t)^{-1}) \leq \frac{H^{-1}(1/t)}{\epsilon_1 H(\epsilon_1^{-1}H^{-1}(1/t))} \leq \frac{tH^{-1}(1/t)}{c_1\epsilon_1^{1-\beta_1'}} = \frac{tH^{-1}(1/t)}{8e^2} \leq \frac{t}{y_t} \leq \frac{t}{y}.$$

Thus, $\mathscr{H}^{-1}(t/y) \geq \epsilon_1 H^{-1}(1/t)^{-1} \geq \epsilon_1 y/(8e^2)$ so that $F(t,y) \simeq 1$.

(b) Suppose that $T_1 \leq t \leq 1/H(\epsilon_1)$. Then $y \leq y_t < 8e^2 H^{-1}(1/t)^{-1} \leq 8e^2/\epsilon_1$. Hence from the monotonicity, we get $\mathscr{H}^{-1}(t/y) \geq \mathscr{H}^{-1}(\epsilon_1 T_1/(8e^2)) \geq \epsilon_1 \mathscr{H}^{-1}(\epsilon_1 T_1/(8e^2)) y/(8e^2)$ which yields that $F(t,y) \simeq 1$.

(2) Suppose that $8e^2 H^{-1}(1/t)^{-1} \leq y_t < 8e^2 D(t)$. By Lemma 2.3.23, we have

$y_t = 8e^2 t\theta H(\theta^{-1})$. Denote by $\epsilon_2 = \epsilon_2(t,y) = \theta H^{-1}(1/t) \in (0,1]$ so that $\theta = \epsilon_2 H^{-1}(1/t)^{-1}$.

(a) Suppose that $y < c_4 H^{-1}(1/t)^{-1}$. Then by (2.3.58) and (2.3.59), we see that if $\theta \geq 1$, then

$$\frac{2c_4}{H^{-1}(1/t)} > y_t = 8e^2\theta \frac{H(\theta^{-1})}{H(H^{-1}(1/t))} \geq 8e^2 c_1 \theta \epsilon_2^{-\beta_1'} = \frac{8e^2 c_1 \epsilon_2^{-\beta_1'+1}}{H^{-1}(1/t)}.$$

Hence, if $\theta \geq 1$, then $\epsilon_2 \geq (4e^2 c_1/c_4)^{1/(\beta_1'-1)}$ and hence $y_t \simeq \theta \simeq H^{-1}(1/t)^{-1}$. From this, we can deduce that $F(t,y) \simeq G(t,y,1) \simeq 1$ in this case. Otherwise, if $\theta < 1$, then $w^{-1}(2e/t) \leq \theta < 1$ and hence $t < 2e/w(1)$. Then by a similar argument to the one given in (1 - b), we can also deduce that $F(t,y) \simeq G(t,y,1) \simeq 1$.

(b) Assume that $y \geq c_4 H^{-1}(1/t)^{-1}$. By the proof given in (1 - b), we may assume that $H^{-1}(1/t)^{-1} \geq R_2$ and $w^{-1}(2e/t) \geq \epsilon_1^{-1}$. Note that by (2.3.59), we have $y \leq y_t \leq 2y$ in this case. Then we get from $\mathbf{Poly}^{*,\infty}_{R_2}(\beta_1,\beta_2)$ that $\nu(y) \simeq \nu(y_t)$. Hence, it remains to prove that $\mathscr{H}^{-1}(t/y) \simeq \theta$. Using (2.3.58), since $\kappa_1 = c_1^{-1/(\beta_1'-1)}$, we see that

$$\mathscr{H}(\epsilon_1\theta) \leq \frac{1}{\epsilon_1\theta H(\epsilon_1^{-1}\theta^{-1})} \leq \frac{1}{c_1\epsilon_1^{1-\beta_3'}\theta H(\theta^{-1})} = \frac{1}{8e^2\theta H(\theta^{-1})} = \frac{t}{y_t} \leq \frac{t}{y}$$

and

$$\mathscr{H}(\kappa_1\theta) = \inf_{\kappa \geq \kappa_1} \frac{1}{\kappa\theta H(\kappa^{-1}\theta^{-1})} \geq \frac{c_1\kappa_1^{\beta_1'-1}}{\theta H(\theta^{-1})} = \frac{8e^2 t}{y_t} > \frac{t}{y}.$$

Therefore, we obtain $\epsilon_1\theta \leq \mathscr{H}^{-1}(t/y) \leq \kappa_1\theta$.

(3) Suppose that $y_t > 8e^2 D(t)$. If $y < c_4 H^{-1}(1/t)^{-1}$, then by the proof given in (2 - a), we get the result. Hence, we assume $y \geq c_4 H^{-1}(1/t)^{-1}$ so that $y_t \leq 2y$. By the proof given in (2 - b), it suffices to prove for $c_4 H^{-1}(1/t)^{-1} \geq R_2$ and $\nu(y) \simeq \nu(y_t)$. By Lemma 2.3.12(i) and (2.0.4), we see that $tH^{-1}(1/t)^{-1}\nu(y_t) \leq c_5 t y_t \nu(y_t) \leq c_6 t w(y_t) \leq c_7 t H(y_t^{-1}) \leq c_8$. Moreover, by Lemma 2.3.13 and

$\mathbf{Poly}^{*,\infty}_{R_2,\le}(\beta_2)$, for each fixed $a>0$, it holds that

$$\exp\Big(-\frac{ay_t}{\theta(t,y_t/(8e^2))}\Big)\le \exp\Big(-\frac{ac_4c_4^{-1}y}{w^{-1}(2e/t)}\Big)\le \frac{c_9t\nu(y/c_4)}{H^{-1}(1/t)}\le \frac{c_{10}t\nu(y)}{H^{-1}(1/t)}.$$

Thus, $G(t,y,1)\simeq t\nu(y)/H^{-1}(1/t)$ in this case. Hence, it remains to prove that there exists $c_{11}>0$ such that

$$\exp\Big(-\frac{y}{\mathscr{H}^{-1}(t/y)}\Big)\le c_{11}\frac{t\nu(y)}{H^{-1}(1/t)}. \tag{2.3.61}$$

If $w^{-1}(2e/t)\ge 1$, then since $\theta=w^{-1}(2e/t)$ in this case, by (2.3.58) and Lemma 2.3.23,

$$\mathscr{H}(\kappa_1 w^{-1}(2e/t))=\inf_{\kappa\ge\kappa_1}\frac{1}{\kappa\theta H(\kappa^{-1}\theta^{-1})}\ge\frac{c_1\kappa_1^{\beta_1'-1}}{\theta H(\theta^{-1})}\ge\frac{8e^2t}{y_t}>\frac{t}{y},$$

which implies that $\mathscr{H}^{-1}(t/y)\le\kappa_1 w^{-1}(2e/t)$. Hence, we get (2.3.61) from Lemma 2.3.13. On the other hand, if $w^{-1}(2e/t)<1$, then $t\simeq 1$. Since $\mathscr{H}$ is increasing, it follows that $\mathscr{H}^{-1}(t/y)\le c_{12}$. Let $c_{13}:=\sup_{u>0}u^{1+\beta_2}e^{-u/c_{12}}$. By $\mathbf{Poly}^{*,\infty}_{R_2,\le}(\beta_2)$, since $y\ge c_4H^{-1}(1/t)^{-1}\ge R_2$, it holds that $\nu(y)\ge c_{14}R_2^{1+\beta_2}\nu(R_2)y^{-1-\beta_2}\ge c_{13}c_{14}R_2^{1+\beta_2}\nu(R_2)e^{-y/c_{12}}$. This proves (2.3.61) and ends the proof. □

2.3.6 An example to varying transition density estimates

In this subsection, we give an example of subordinator whose transition density has the estimates given in Theorem 2.3.6 and the exponential term in the estimates only appears at specific time ranges.

Define an increasing sequence $(a_n)_{n\ge0}$ as follows:

$$a_0:=0,\qquad a_1:=3,\qquad a_{n+1}:=\exp(a_n^{3/2})\quad\text{for } n\ge1. \tag{2.3.62}$$

Using this $(a_n)_{n\geq 0}$, we define an increasing function $\psi : (0,\infty) \to (0,\infty)$ by

$$\psi(r) = \begin{cases} (4/3)r^{1/2} & \text{for } r \in (0, a_1], \\ r^4 + \psi(a_{2n-1}) - a_{2n-1}^4 & \text{for } r \in (a_{2n-1}, a_{2n}], \\ (4/3)r^{1/2} + \psi(a_{2n}) - (4/3)a_{2n}^{1/2} & \text{for } r \in (a_{2n}, a_{2n+1}]. \end{cases} \quad (2.3.63)$$

One can easily check that there exist $c_2 \geq c_1 > 0$ such that

$$c_1 \left(\frac{R}{r}\right)^{1/2} \leq \frac{\psi(R)}{\psi(r)} \leq c_2 \left(\frac{R}{r}\right)^4 \qquad \text{for all } 0 < r \leq R. \quad (2.3.64)$$

Let

$$\Phi(r) := \frac{r^2}{2\int_0^r s\psi(s)^{-1} ds}.$$

By [5, Lemma 2.4] and (2.3.64), there exists a constant $c_3 > 0$ such that

$$c_3 \left(\frac{R}{r}\right)^{1/2} \leq \frac{\Phi(R)}{\Phi(r)} \leq \left(\frac{R}{r}\right)^2 \qquad \text{for all } 0 < r \leq R. \quad (2.3.65)$$

Lemma 2.3.29. *For every $\epsilon \in (0,1)$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, the following estimates hold:*
(i) For every $r \in [a_{2n+1}^{1-\epsilon}, a_{2n+1}]$,

$$\frac{4}{3}r^{1/2} \leq \psi(r) \leq 2r^{1/2} \quad \text{and} \quad r^{1/2} \leq \Phi(r) \leq 2r^{1/2}.$$

(ii) For every $r \in [a_{2n}^{1-\epsilon}, a_{2n}]$,

$$\frac{1}{2}r^4 \leq \psi(r) \leq r^4 \quad \text{and} \quad \frac{2(1-\epsilon)r^2}{3\log r} \leq \Phi(r) \leq \frac{2r^2}{\log r}.$$

Proof. From the definition (2.3.62) of the sequence (a_n), by choosing N large enough, we can assume that $[a_{2n+1}^{1-\epsilon}, a_{2n+1}] \subset (a_{2n}, a_{2n+1}]$ and $[a_{2n+1}^{1-\epsilon}, a_{2n+1}] \subset (a_{2n}, a_{2n+1}]$ for all $n \geq N$.

First, we prove the assertions for ψ. From the construction, we have

$$\frac{4}{3}r^{1/2} \le \psi(r) \le r^4 \qquad \text{for all } \ r \ge 1. \tag{2.3.66}$$

Moreover, for all n large enough and $r \in [a_{2n+1}^{1-\epsilon}, a_{2n+1}]$, by (2.3.62),

$$\psi(r) \le \left(1 + \frac{a_{2n}^4}{(4/3)a_{2n+1}^{(1-\epsilon)/2}}\right)\frac{4}{3}r^{1/2} \le \Big(1 + a_{2n}^4 \exp\big(-2^{-1}(1-\epsilon)a_{2n}^{3/2}\big)\Big)\frac{4}{3}r^{1/2}.$$

Similarly, for all n large enough and $r \in [a_{2n}^{1-\epsilon}, a_{2n}]$,

$$\psi(r) \ge \left(1 - \frac{a_{2n-1}^4}{a_{2n}^{4(1-\epsilon)}}\right) r^4 \ge \Big(1 - a_{2n-1}^4 \exp\big(-4(1-\epsilon)a_{2n-1}^{3/2}\big)\Big) r^4.$$

Since $\lim_{x\to\infty} x^4 e^{-4(1-\epsilon)x^{3/2}} = \lim_{x\to\infty} x^4 e^{-2^{-1}(1-\epsilon)x^{3/2}} = 0$, we deduce the results for ψ.

Now, we prove the assertions for Φ. Fix $\epsilon' \in (0, 1-\epsilon)$. By using the results for ψ and (2.3.66), we can see that for all n large enough, it holds that for $r \in [a_{2n+1}^{1-\epsilon}, a_{2n+1}]$,

$$\begin{aligned}\frac{2}{3}r^{3/2}(1 - a_{2n+1}^{-3\epsilon'/2}) &\le \frac{2}{3}r^{3/2}\big(1 - (a_{2n+1}^{(1-\epsilon-\epsilon')}/r)^{3/2}\big) = \frac{2}{3}\big(r^{3/2} - a_{2n+1}^{3(1-\epsilon-\epsilon')/2}\big) \\ &= \int_{a_{2n+1}^{1-\epsilon-\epsilon'}}^r s^{1/2}ds \le 2\int_0^r s\psi(s)^{-1}ds \le \frac{3}{2}\int_0^r s^{1/2}ds = r^{3/2}.\end{aligned}$$

Since $\lim_{n\to\infty} a_{2n+1}^{-3\epsilon'/2} = 0$, it follows that for all n large enough and $r \in [a_{2n+1}^{1-\epsilon}, a_{2n+1}]$,

$$\frac{1}{2}r^{3/2} \le 2\int_0^r s\psi(s)^{-1}ds \le r^{3/2} \quad \text{and hence} \quad r^{1/2} \le \Phi(r) \le 2r^{1/2}. \tag{2.3.67}$$

Next, by (2.3.67), for all n large enough and $r \in [a_{2n}^{1-\epsilon}, a_{2n}]$, we get

$$\begin{aligned}\frac{1}{2}a_{2n-1}^{3/2} &\le 2\int_0^{a_{2n-1}} s\psi(s)^{-1}ds \le 2\int_0^r s\psi(s)^{-1}ds \\ &= 2\int_0^{a_{2n-1}} s\psi(s)^{-1}ds + 2\int_{a_{2n-1}}^r \frac{s}{s^4 + \psi(a_{2n-1}) - a_{2n-1}^4}ds. \end{aligned}\tag{2.3.68}$$

Note that for all n large enough, by (2.3.66),

$$\int_{a_{2n-1}}^{r} \frac{s}{s^4+\psi(a_{2n-1})-a_{2n-1}^4}ds \le \int_{a_{2n-1}}^{r} \frac{s}{(s-a_{2n-1})s^3+\psi(a_{2n-1})}ds$$
$$\le \frac{1}{\psi(a_{2n-1})}\int_{a_{2n-1}}^{a_{2n-1}+1} sds + \int_{a_{2n-1}+1}^{r} s^{-2}ds \le \frac{3}{4}a_{2n-1}^{-1/2}(a_{2n-1}+1)+1 \le a_{2n-1}^{1/2}.$$

Thus, by combining the above inequality with (2.3.67) and (2.3.68), we deduce that for all n large enough and $r \in [a_{2n}^{1-\epsilon}, a_{2n}]$,

$$\frac{\log r}{2} \le \frac{1}{2}a_{2n-1}^{3/2} \le 2\int_0^r \frac{s}{\psi(s)}ds \le (1+a_{2n-1}^{-1})a_{2n-1}^{3/2} \le \frac{3}{2}a_{2n-1}^{3/2} \le \frac{3\log r}{2(1-\epsilon)}$$

and hence
$$\frac{2(1-\epsilon)r^2}{3\log r} \le \Phi(r) \le \frac{2r^2}{\log r}.$$
The proof is completed. □

Let $t_{2n} = a_{2n}^2/(\log a_{2n})$ and $t_{2n+1} = a_{2n+1}^{1/2}$ for $n \ge 1$. Since $\exp(x^{3/2}) \ge 4x^4$ for $x \ge 10$, we have that $t_{n+1} \ge 4t_n$ for all $n \ge 2$. As a corollary to Lemma 2.3.29, we obtain the following estimates for the inverse functions of Φ and ψ, respectively.

Lemma 2.3.30. *(i) There are comparison constants such that*

$$\Phi^{-1}(t) \simeq \psi^{-1}(t) \simeq t^2 \quad \textit{for all } t \in [t_{2n+1}/2, t_{2n+1}],\ n \ge 1.$$

(ii) There are comparison constants such that

$$\Phi^{-1}(t) \simeq t^{1/2}(\log t)^{1/2} \ \textit{ and } \ \psi^{-1}(t) \simeq t^{1/4} \quad \textit{for all } t \in [t_{2n}/2, t_{2n}],\ n \ge 1.$$

Proof. (i) For each fixed $T > 0$, since $\Phi^{-1}(t) \simeq \psi^{-1}(t) \simeq t^2 \simeq 1$ for $t \in [t_3/2, T]$, it suffices to prove the desired comparisons only for n large enough.

For all large enough n and $t \in [t_{2n+1}/2, t_{2n+1}]$, by (2.3.64), (2.3.65) and Lemma 2.3.29(i), we have $\Phi(t^2) \simeq \Phi(a_{2n+1}) \simeq \psi(t^2) \simeq \psi(a_{2n+1}) \simeq t_{2n+1} \simeq t$. Then, we get the result from (2.3.64) and (2.3.65).

(ii) Similar to (i), it suffices to prove for n large enough. For all n large enough and $t \in [t_{2n}/2, t_{2n}]$, we see that $t^{1/2}(\log t)^{1/2} \simeq a_{2n}$ and $t^{1/4} \simeq a_{2n}^{1/2}(\log a_{2n})^{-1/4}$. Since $\Phi(a_{2n}) \simeq \psi(a_{2n}^{1/2}(\log a_{2n})^{-1/4}) \simeq t_{2n} \simeq t$ by Lemma 2.3.29(ii) with $\epsilon = 2/3$, we obtain the results. □

Construction of subordinator and its transition density estimates

With the function ψ defined by (2.3.63), we let S be a driftless subordinator whose Lévy measure $\nu(dr)$ is given by

$$\nu(dr) = \frac{1}{r\psi(r)}dr,$$

i.e., the Laplace exponent is given by $\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda s})\nu(ds)$. Since ν satisfies the condition (2.3.1) with $T_0 = 0$, S_t has a transition density function $p(t, x)$ for all $t > 0$. The following theorem is the main result in this example.

Recall that $b(t) = (\phi' \circ H^{-1})(1/t)$ for $t > 0$.

Theorem 2.3.31. *(i) There are comparison constants such that*

$$p(t, tb(t) + y) \simeq t^{-2} \wedge \frac{t}{y\psi(y)}$$

for all $t \in [t_{2n+1}/2, t_{2n+1}]$, $n \geq 1$ *and* $y \geq 0$.
(ii) There are comparison constants such that

$$p(t, tb(t) + y) \asymp t^{-1/2}(\log t)^{-1/2} \wedge \left(\frac{t}{y\psi(y)} + t^{-1/2}(\log t)^{-1/2} \exp\Big(- \frac{c\, y^2}{t \log t}\Big)\right)$$

for all $t \in [t_{2n}/2, t_{2n}]$, $n \geq 1$ *and* $y \geq 0$.

Remark 2.3.32. For all $t \in [t_{2n}/2, t_{2n}]$, $n \geq 1$ and $y \in [a_{2n}, a_{2n}(\log a_{2n})^{1/3}]$, since $\lim_{n\to\infty} a_{2n} = \infty$, we have

$$\begin{aligned} &t^{-1/2}(\log t)^{-1/2} \exp\Big(- \frac{c_1 y^2}{t \log t}\Big) \geq c_2 a_{2n}^{-1} \exp\Big(- \frac{c_3 y^2}{a_{2n}^2}\Big) \\ &\geq c_2 a_{2n}^{-1} \exp\Big(- c_3 (\log a_{2n})^{-1/3} \log a_{2n}\Big) = c_2 a_{2n}^{-1 - c_3(\log a_{2n})^{-1/3}} \geq c_4 a_{2n}^{-2}, \end{aligned}$$

while
$$\frac{t}{y\psi(y)} \le \frac{c_5 a_{2n}^2}{a_{2n}^{1+4}\log a_{2n}} = c_5 a_{2n}^{-3}(\log a_{2n})^{-1}.$$
Hence, we see that the exponential term is the dominating factor in heat kernel estimates at those intervals. Therefore, we deduce that the exponential term in (2.3.8) is indispensable in heat kernel estimates, although it does not appear in some other time ranges.

Proof of Theorem 2.3.31. By Lemmas 2.3.11 and 2.3.12, we have

$$H(r^{-1}) \simeq \Phi(r)^{-1} \quad \text{and} \quad w(r) \simeq \psi(r)^{-1} \quad \text{for all } r > 0 \tag{2.3.69}$$

and

$$H^{-1}(1/t) \simeq \Phi^{-1}(t)^{-1} \quad \text{and} \quad w^{-1}(2e/t) \simeq \psi^{-1}(t) \quad \text{for all } t > 0. \tag{2.3.70}$$

We simply denote θ for $\theta(t, y/(8e^2))$.
(i) By Lemma 2.3.30(i) and (2.3.70), it holds that

$$H^{-1}(1/t)^{-1} \simeq w^{-1}(2e/t) \simeq t^2 \quad \text{for all } t \in [t_{2n+1}/2, t_{2n+1}],\ n \ge 1.$$

Hence, for all $t \in [t_{2n+1}/2, t_{2n+1}]$ and $y \in [0, H^{-1}(1/t)^{-1}]$, we get $\theta \simeq t^2$ so that $e^{-cy/\theta} \asymp 1$. Moreover, by Lemma 2.3.13, for each fixed $a > 0$ and all $t \in [t_{2n+1}/2, t_{2n+1}]$, it holds that

$$\exp\big(-ay/\theta\big) \le \exp\Big(-\frac{ay}{w^{-1}(2e/t)}\Big) \le \frac{c_1 t\nu(y)}{H^{-1}(1/t)} \quad \text{for } y > H^{-1}(1/t)^{-1}.$$

Therefore, we get the result from Theorem 2.3.6.
(ii) By Lemma 2.3.30(ii) and (2.3.70), we see that for all $t \in [t_{2n}/2, t_{2n}]$, $n \ge 1$,

$$H^{-1}(1/t)^{-1} \simeq t^{1/2}(\log t)^{1/2} \quad \text{and} \quad w^{-1}(2e/t) \simeq t^{1/4}. \tag{2.3.71}$$

Then, by (2.3.69) and Lemma 2.3.29(ii), we obtain that for all $t \in [t_{2n}/2, t_{2n}]$,

$$D(t) \simeq t \max_{s \in [w^{-1}(2e/t), H^{-1}(1/t)^{-1}]} \frac{s}{\Phi(s)} \simeq t \max_{s \in [t^{1/4}, t^{1/2}(\log t)^{1/2}]} \frac{\log s}{s} \simeq t^{3/4} \log t.$$

From (2.3.66), for fixed $a > 0$, we see that for all $t \in [t_{2n}/2, t_{2n}]$ and $y \geq D(t)$,

$$\begin{aligned} \frac{1}{t^{1/2}(\log t)^{1/2}} \exp\Big(- \frac{ay^2}{t \log t}\Big) &\leq c_2 (t \log t)^{-1/2} \left(\frac{t \log t}{y^2} \right)^{11/2} \\ &= \frac{c_2 t}{y^5} \frac{t^4 \log^5 t}{y^6} \leq \frac{c_3 t}{y \psi(y)}. \end{aligned}$$

Hence, in view of Theorem 2.3.6 and Lemma 2.3.13, it remains to show that

$$y/\theta \simeq y^2/(t \log t) \quad \text{for all } t \in [t_{2n}/2, t_{2n}],\ y \in [H^{-1}(1/t)^{-1}, D(t)]. \quad (2.3.72)$$

Let $t \in [t_{2n}/2, t_{2n}]$ and $y \in [H^{-1}(1/t)^{-1}, D(t)]$. By (2.3.71), there are $c_4, c_5 > 0$ such that $c_4 t^{1/4} \leq \theta \leq c_5 t^{1/2} (\log t)^{1/2}$. Since $t\theta H(\theta^{-1}) = y$ by Lemma 2.3.23, using (2.3.69) and Lemma 2.3.29(ii), (as before, it suffices to consider large t only,) we get that

$$y\theta = t\theta^2 H(\theta^{-1}) \simeq t\theta^2/\Phi(\theta) \simeq t \log \theta \simeq t \log t.$$

This completes the proof for (2.3.72). □

Chapter 3

Estimates on heat kernels for non-local operators with critical killings

In this chapter, we study sharp two-sided heat kernel estimates for critical killing type perturbations of non-local operators in a smooth domain D. The results in this chapter is based on [58].

Stability of Dirichlet heat kernel estimates under certain Feynman-Kac transforms was studied in the recent paper [42]. To be precise, let X be a Hunt process on a Borel set $D \subset \mathbb{R}^d$ with the L^2-infinitesimal generator $\mathcal{L}$. Consider the following Feynman-Kac transform:

$$T_t f(x) = \mathbb{E}^x \left[\exp\left(-A_t\right) f(X_t)\right],$$

where A is a continuous additive functional of X with Revuz measure μ. Informally, the semigroup (T_t) has the L^2-infinitesimal generator $\mathcal{A}f(x) := (\mathcal{L} - \mu)f(x)$. Let $\alpha \in (0,2)$ and $\gamma \in [0, \alpha \wedge d)$, and define

$$q_\gamma(t,x,y) := \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^\gamma \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^\gamma \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right).$$

Suppose that X admits a jointly continuous transition density $p_D(t,x,y)$

with respect to the Lebesgue measure and that $p_D(t,x,y)$ is comparable to $q_\gamma(t,x,y)$ for $(t,x,y) \in (0,1] \times D \times D$. Examples of processes satisfying this assumption include killed symmetric stable processes in $C^{1,1}$ open sets D (with $\gamma = \alpha/2$, cf. [35]), and, when $\alpha \in (1,2)$, censored α-stable processes in any $C^{1,1}$ open sets D (with $\gamma = \alpha - 1$, cf. [36]).

Under the assumption that μ belongs to some Kato class, it is established in [42] that the semigroup (T_t) admits a continuous density $q^D(t,x,y)$ which is comparable to $q_\gamma(t,x,y)$ for all $(t,x,y) \in (0,1] \times D \times D$. Hence a Kato class perturbation preserves the (Dirichlet) heat kernel estimates and is in this sense subcritical. We also refer to earlier results (without boundary condition) [23, 120, 125].

Kato class perturbations of the Laplacian have been studied earlier and more thoroughly, e.g. [3, 14, 89, 118], with the same conclusion that Kato class perturbations preserve the (Dirichlet) heat kernel estimates. Since [7], it is known that for the Laplacian in $\mathbb{R}^d$, $d \geq 3$, the inverse square potential $\kappa(x) = c|x|^{-2}$, $c \geq -((d-2)/2)^2$ is critical, and, for the Dirichlet Laplacian in a domain D, the potential $\kappa(x) = c\delta_D(x)^{-2}$, $c \geq -1/4$, is critical. Criticality of the potentials above can be explained by Hardy's inequality. In both cases above, when $c < 0$, the potential κ above can be interpreted as creation, and, when $c > 0$, the potential κ can be interpreted as killing. Note that in both cases, the potential κ does not belong to the Kato class. The heat kernel estimates of critical perturbations of the (Dirichlet) Laplacian have been studied extensively, e.g., [8, 66, 80, 107, 108, 111].

In this chapter, we use probabilistic methods to study sharp two-sided heat kernel estimates for critical perturbations of the fractional Laplacian in a smooth domain D, as well as the fractional Laplacian in $\mathbb{R}^d$. When the potential involves both killing and creation, there is no Markov process associated with the corresponding Schrödinger type operator. Since our argument depends crucially on properties of Markov processes, we will only deal with killing type potentials.

This chapter is divided into two major parts. The first part is Section 3.1

and the setup is quite general there. We consider a Hunt process X on a locally compact separable metric space $(\mathfrak{X}, \rho)$. The process X is not necessarily symmetric and may not be conservative. Let X^D be the killed subprocess of X in an open subset D of $\mathfrak{X}$. Using the positive additive functional (A_t^μ) of X^D with Revuz measure μ, which is possibly critical, we define the Feynman-Kac semigroup of X^D associated with μ:

$$T_t^{\mu,D} f(x) = \mathbb{E}^x \left[\exp(-A_t^\mu) f(X_t^D)\right], \quad t \geq 0, x \in D\,.$$

The main result of the first part is a factorization formula involving tails of lifetimes for the transition density of the semigroup $T_t^{\mu,D}$ (see Theorem 3.1.21). The form of this factorization formula can be traced back to [18, 19, 123]. If one can get explicit two-sided estimates on the survival probabilities, then one can combine them with the approximate factorization to get explicit two-sided estimates on the heat kernel. This is the strategy employed in [19, 41]. We will also use this strategy in Section 3.2, and as a by-product, give an alternative and unified proof of the main results of [35, 36, 40].

The second part is Section 3.2. In this section we assume that $\mathfrak{X}$ is either the closure of a $C^{1,1}$ open subset D of $\mathbb{R}^d$ or $\mathbb{R}^d$ itself, $d \geq 2$, and we assume that the underlying process X is either a reflected α-stable(-like) process on $\overline{D}$ (or a non-local perturbation of it), or an α-stable process in $\mathbb{R}^d$ (or a drift perturbation of it). The critical potentials have been already described above and are essentially of the form either $c\delta_D(x)^{-\alpha}$ or $c|x|^{-\alpha}$. The goal of this section is to estimate the tail of the lifetime $\mathbb{P}^x(\zeta > t)$ in terms of $\delta_D(x)$ and $|x|$ respectively. Then, as was done in [19, 41], together with the factorization obtained in Theorem 3.1.20, this gives sharp two-sided estimates of the transition density of the Feynman-Kac semigroup. Section 3.2 also provides an alternative and unified proof of the main results of [35, 36, 40].

3.1 Factorization of Dirichlet heat kernels in metric measure spaces

3.1.1 Setup

Let $(\mathfrak{X}, \rho)$ be a locally compact separable metric space such that all bounded closed sets are compact and m is a Radon measure on $\mathfrak{X}$ with full support. Let $R_0 \in (0, \infty]$ be the largest number such that $\mathfrak{X} \setminus B(x, 2r) \neq \emptyset$ for all $x \in \mathfrak{X}$ and all $r < R_0$. We call R_0 the localization radius of $(\mathfrak{X}, \rho)$.

Let $V(x, r) := m(B(x, r))$. We assume that there exist constants $d_2 \geq d_1 > 0$ such that for every $M \geq 1$, there exists $\widetilde{C}_M \geq 1$ with the property that

$$\widetilde{C}_M^{-1} \left(\frac{R}{r}\right)^{d_1} \leq \frac{V(x,R)}{V(x,r)} \leq \widetilde{C}_M \left(\frac{R}{r}\right)^{d_2} \quad \text{for all } x \in \mathfrak{X} \text{ and } 0 < r \leq R < MR_0. \tag{3.1.1}$$

Note that the lower inequality in (3.1.1) implies

$$V(x, n_0 r) \geq 2V(x, r) \quad \text{for all } x \in \mathfrak{X} \text{ and } r \in (0, R_0/n_0), \tag{3.1.2}$$

where $n_0 := (2\widetilde{C}_1)^{1/d_1}$.

Now we spell out the assumptions on the processes we are going to work with. We assume that $X = (X_t, \mathbb{P}^x)$ is a Hunt process admitting a (strong) dual Hunt process $\widehat{X} = (\widehat{X}_t, \widehat{\mathbb{P}}^x)$ with respect to the measure m. For the definition of (strong) duality, see [15, Section VI.1]. We further assume that the transition semigroups (P_t) and $(\widehat{P}_t)$ of X and $\widehat{X}$ are both Feller and strongly Feller, and that all semipolar sets are polar. The condition that semipolar sets are polar is known as Hunt's hypothesis (H). This guarantees the duality between the killed processes when the original processes are duals (since X never hits irregular points). See [25, p.481] and the end of [60, Section 13.6].

In the sequel, all objects related to the dual process $\widehat{X}$ will be denoted by a hat. We also assume that X admits a strictly positive and jointly continuous transition density $p(t,x,y)$ with respect to m so that

$$P_t f(x) = \int_{\mathfrak{X}} p(t,x,y) f(y) m(dy) \quad \text{and} \quad \widehat{P}_t f(x) = \int_{\mathfrak{X}} p(t,y,x) f(y) m(dy).$$

We will make some assumptions on the transition density $p(t,x,y)$. To do this, we first introduce some notation.

Let $\Phi : (0,\infty) \to (0,\infty)$ be a strictly increasing function with $\Phi(0+) = 0$ and $\lim_{r\to\infty} \Phi(r) = \infty$ satisfying the following scaling condition: there exist constants $\delta_l, \delta_u \in (0,\infty)$, $a_l \in (0,1]$, $a_u \in [1,\infty)$ such that

$$a_l \left(\frac{R}{r}\right)^{\delta_l} \le \frac{\Phi(R)}{\Phi(r)} \le a_u \left(\frac{R}{r}\right)^{\delta_u}, \quad r \le R < R_0. \tag{3.1.3}$$

Remark 3.1.1. *Since the function Φ is strictly increasing, for every $\widetilde{R} \in (0,\infty)$, there exist $\widetilde{a}_l \in (0,1]$ and $\widetilde{a}_u \in [1,\infty)$ such that*

$$\widetilde{a}_l \left(\frac{R}{r}\right)^{\delta_l} \le \frac{\Phi(R)}{\Phi(r)} \le \widetilde{a}_u \left(\frac{R}{r}\right)^{\delta_u}, \quad 0 < r \le R \le \widetilde{R}. \tag{3.1.4}$$

We will use (3.1.4) instead of (3.1.3) whenever necessary. From (3.1.3) we can also get the scaling condition for the inverse of Φ:

$$a_u^{-1/\delta_u} \left(\frac{R}{r}\right)^{1/\delta_u} \le \frac{\Phi^{-1}(R)}{\Phi^{-1}(r)} \le a_l^{-1/\delta_l} \left(\frac{R}{r}\right)^{1/\delta_l}, \quad 0 < r \le R < \Phi(R_0). \tag{3.1.5}$$

Define for $t > 0$ and $x, y \in \mathfrak{X}$,

$$\widetilde{q}(t,x,y) := \frac{1}{V(x,\Phi^{-1}(t))} \wedge \frac{t}{V(x,\rho(x,y))\Phi(\rho(x,y))}. \tag{3.1.6}$$

Remark 3.1.2. *Since (3.1.1) holds true, it is easy to see that*

$$\widetilde{q}(t,x,y) \simeq \widetilde{q}(t,y,x) \simeq \frac{1}{V(x,\Phi^{-1}(t))} \wedge \frac{t}{V(y,\rho(x,y))\Phi(\rho(x,y))}.$$

See [48, Remark 1.12]. Moreover, by integrating $\widetilde{q}(t,x,y)$ over the set $\{y :$

$\rho(x,y) \le \Phi^{-1}(t))\}$, *one easily gets that for all* $t > 0$ *and* $x \in \mathfrak{X}$,

$$\int_{\mathfrak{X}} \widetilde{q}(t,x,y)\, m(dy) \ge 1\,. \tag{3.1.7}$$

We will assume that there exists a constant $C_0 \ge 1$ such that

$$C_0^{-1}\widetilde{q}(t,x,y) \le p(t,x,y) \le C_0\widetilde{q}(t,x,y), \quad (t,x,y) \in (0,\widetilde{T}) \times \mathfrak{X} \times \mathfrak{X} \tag{3.1.8}$$

for some $\widetilde{T} \in (0,\infty]$. Then (3.1.7) and the lower bound in (3.1.8) yield that

$$1 \le \int_{\mathfrak{X}} \widetilde{q}(t,x,y)\, m(dy) \le C_0 \quad \text{for all } (t,x) \in (0,\widetilde{T}) \times \mathfrak{X}.$$

The processes X and $\widehat{X}$ may not be conservative so the lifetimes may be finite. We add an extra point ∂ (which is called the cemetery point) to $\mathfrak{X}$ and assume our processes stay at the cemetery point after their lifetimes. When $\widetilde{T} = \infty$, we assume $R_0 = m(\mathfrak{X}) = \infty$. Note that, if $\widetilde{T} = \infty$ and both X and $\widehat{X}$ admit no killing inside $\mathfrak{X}$, then it follows that $R_0 = m(\mathfrak{X}) = \infty$, and X and $\widehat{X}$ are conservative (see the proof of [86, Proposition 2.5], which still works under the non-symmetric setting). All functions h on $\mathfrak{X}$ will be automatically extended to $\mathfrak{X} \cup \{\partial\}$ by setting $h(\partial) = 0$.

Remark 3.1.3. *When* $\widetilde{T} \in (0,\infty)$, *the value of* $\widetilde{T}$ *is not important. That is, when* $\widetilde{T} \in (0,\infty)$, *for every* $T > 0$ *there exists a constant* $\overline{C}_0 = \overline{C}_0(T) \ge 1$ *such that*

$$\overline{C}_0^{-1}\widetilde{q}(t,x,y) \le p(t,x,y) \le \overline{C}_0\widetilde{q}(t,x,y), \quad (t,x,y) \in (0,T) \times \mathfrak{X} \times \mathfrak{X}. \tag{3.1.9}$$

This is a consequence of the semigroup property of $p(t,x,y)$, (3.1.1), (3.1.5) *and* (3.1.8).

Let $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ and $(\widehat{\mathcal{L}}, \mathcal{D}(\widehat{\mathcal{L}}))$ be the generators of (P_t) and $(\widehat{P}_t)$ in $C_0(\mathfrak{X})$ respectively. We assume the following Urysohn-type condition.

Assumption A: There is a linear subspace $\mathcal{D}$ of $\mathcal{D}(\mathcal{L}) \cap \mathcal{D}(\widehat{\mathcal{L}})$ satisfying the following condition: For any compact K and open U with $K \subset U \subset \mathfrak{X}$, there is a nonempty collection $\mathcal{D}(K,U)$ of functions $f \in \mathcal{D}$ satisfying the conditions (i) $f(x) = 1$ for $x \in K$; (ii) $f(x) = 0$ for $x \in \mathfrak{X} \setminus U$; (iii) $0 \le f(x) \le 1$ for $x \in \mathfrak{X}$, and (iv) the boundary of the set $\{x : f(x) > 0\}$ has zero m measure.

Assumption **A** implies that there exists a kernel $J(x,dy) = J(x,y)m(dy)$, satisfying $J(x,\{x\}) = 0$ for all $x \in \mathfrak{X}$, such that X satisfies the following Lévy system formula (see [25, p.482]): for every stopping time T and function $f : \mathfrak{X} \times \mathfrak{X} \to [0,\infty]$ with the property that $f(x,x) = 0$ for all $x \in \mathfrak{X}$,

$$\mathbb{E}^x \sum_{s\in(0,T]} f(X_{s-}, X_s) = \mathbb{E}^x \int_0^T \int_{\mathfrak{X}} f(X_s, z) J(X_s, dz) ds.$$

The kernel $J(x,dy) = J(x,y)m(dy)$ is called the jump kernel of X. For all bounded continuous function f on $\mathfrak{X}$ and $x \in \mathfrak{X} \setminus \mathrm{supp}(f)$, it is known that J satisfies

$$\int_{\mathfrak{X}} f(y) J(x,dy) = \lim_{t\downarrow 0} \frac{\mathbb{E}^x f(X_t)}{t}.$$

Therefore, we have from (3.1.8) that

$$\frac{C_0^{-1}}{V(x,\rho(x,y))\Phi(\rho(x,y))} \le J(x,y) \le \frac{C_0}{V(x,\rho(x,y))\Phi(\rho(x,y))}. \tag{3.1.10}$$

Similarly, $\widehat{X}$ has a jump kernel $\widehat{J}(x,dy) = \widehat{J}(x,y)m(dy)$ with $\widehat{J}(x,y) = J(y,x)$.

There are plenty of examples of processes satisfying the assumptions of this subsection. Reflected stable-like processes in a closed d-set $D \subset \mathbb{R}^d$ satisfy the assumptions of this subsection, see [25, 42]. Unimodal Lévy processes in $\mathbb{R}^d$ with Lévy exponents satisfying weak upper and lower scaling conditions at infinity, in particular, isotropic stable processes, satisfy the assumptions of this subsection, see, for example, [20, 41]. Another typical example is given at the end of this section.

3.1.2 Interior estimates and scale-invariant parabolic Harnack inequality for X

Let $\tau_U^X := \inf\{t > 0 : X_t \notin U\}$ be the first exit time from U for X. For an open subset D of $\mathfrak{X}$, the killed process X^D is defined by $X_t^D = X_t$ if $t < \tau_D^X$ and $X_t^D = \partial$ if $t \geq \tau_D^X$, where ∂ is the cemetery point added to $\mathfrak{X}$. Similarly, we define the killed process $\widehat{X}^D$. It is well known that X^D and $\widehat{X}^D$ are strong duals of each other with respect to m_D, the restriction of m to D (see [25, p.481] and the end of [60, Section 13.6]).

For $t > 0$ and $x, y \in D$, define

$$p_D(t,x,y) = p(t,x,y) - \mathbb{E}^x\big[p(t-\tau_D^X, X_{\tau_D^X}, y) : \tau_D^X < t < \zeta^X\big], \qquad (3.1.11)$$

where ζ^X is the lifetime of X. By the strong Markov property, $p_D(t,x,y)$ is the transition density of X^D and, by the continuity of $p(t,x,y)$, (3.1.9), the Feller and the strong Feller properties of X and $\widehat{X}$, it is easy to see that $p_D(t,x,y)$ is jointly continuous (see [61, pp.34–35] and [96, Lemma 2.2 and Proposition 2.3]).

The following lemma is basically [9, Lemma 3.8], except that we require neither symmetry nor conservativeness.

Lemma 3.1.4. *Suppose that there exist positive constants r, t and p such that*

$$\mathbb{P}^x\big(X_s \notin B(x,r), s < \zeta^X\big) \leq p, \qquad x \in \mathfrak{X}, s \in [0,t].$$

Then

$$\mathbb{P}^x\big(\sup_{0\leq s\leq t} \rho(X_s, X_0) > 2r, t < \zeta^X\big) \leq 2p, \qquad x \in \mathfrak{X}.$$

Combining this lemma with (3.1.9) and (3.1.11), we can repeat the proof of [32, Proposition 2.3] word for word to get the following result. Note that conservativeness is not needed.

Proposition 3.1.5. *For every $a > 0$, there exist constants $c > 0$ and $\epsilon \in$*

$(0, 1/2)$ *such that for all* $x_0 \in \mathfrak{X}$ *and* $r \in (0, aR_0)$,

$$p_{B(x_0,r)}(t, x, y) \geq \frac{c}{V(x, \Phi^{-1}(t))} \quad \text{for } x, y \in B(x_0, \epsilon\Phi^{-1}(t)) \text{ and } t \in (0, \Phi(\epsilon r)].$$

Let $\Xi_s := (V_s, X_s)$ be the time-space process of X, where $V_s = V_0 - s$. The law of the time-space process $s \mapsto \Xi_s$ starting from (t, x) will be denoted as $\mathbb{P}^{(t,x)}$.

Definition 3.1.6. *A non-negative Borel function* $h(t, x)$ *on* $\mathbb{R} \times \mathfrak{X}$ *is said to be parabolic on* $(a, b] \times B(x_0, r)$ *with respect to* X *if for every relatively compact open subset* U *of* $(a, b] \times B(x_0, r)$,

$$h(t, x) = \mathbb{E}^{(t,x)}[h(\Xi_{\tau_U^\Xi}) : \tau_U^\Xi < \zeta^X]$$

for every $(t, x) \in U \cap ([0, \infty) \times \mathfrak{X})$, *where* $\tau_U^\Xi := \inf\{s > 0 : \Xi_s \notin U\}$.

Theorem 3.1.7. *For every* $a > 0$, *there exist constants* $c > 0$ *and* $c_1, c_2 \in (0, 1)$ *depending on* d, $\widetilde{T}$ *and* a *such that for all* $x_0 \in \mathbb{R}^d$, $t_0 \geq 0$, $R \in (0, aR_0)$ *and every non-negative function* u *on* $[0, \infty) \times \mathbb{R}^d$ *that is parabolic on* $(t_0, t_0 + 4c_1\Phi(R)] \times B(x_0, R)$ *with respect to* X *or* $\widehat{X}$,

$$\sup_{(t_1,y_1)\in Q_-} u(t_1, y_1) \leq c \inf_{(t_2,y_2)\in Q_+} u(t_2, y_2),$$

where $Q_- = (t_0 + c_1\Phi(R), t_0 + 2c_1\Phi(R)] \times B(x_0, c_2R)$ *and* $Q_+ = [t_0 + 3c_1\Phi(R), t_0 + 4c_1\Phi(R)] \times B(x_0, c_2R)$.

Proof. By (3.1.10), (3.1.1) and (3.1.3), we see that there exist $c_1, c_2 > 0$ such that for all $x \neq y \in \mathfrak{X}$ with $r \leq \rho(x, y)/2 < R_0$,

$$J(y, x) \leq c_1 J(x, y) \quad \text{and} \quad J(x, y) \leq \frac{c_2}{V(x, r)} \int_{B(x,r)} J(z, y)m(dz).$$

Using this, Proposition 3.1.5 and (3.1.8), we see that the proof of Theorem 3.1.7 is almost identical to the proof for the symmetric case in [47, Theorem

4.3]. We emphasize that the conservativeness is not used in the proofs of [47, Lemmas 3.7, 4.1, 4.2 and Theorem 4.3]. We omit the details. □ .

Theorem 3.1.7 clearly implies the elliptic Harnack inequality. Using Theorem 3.1.7, we have the following result. In the remainder of this section, D will always stand for an open subset of $\mathfrak{X}$.

Proposition 3.1.8. *For all $a, b > 0$, there exists $c = c(a, b) > 0$ such that for every open set $D \subset \mathfrak{X}$, $p_D(t, x, y) \geq c\widetilde{q}(t, x, y)$ for all $t \in (0, aR_0), x, y \in D$ with $\delta_D(x) \wedge \delta_D(y) \geq b\Phi^{-1}(t)$.*

3.1.3 3P inequality and Feynman-Kac perturbations

The following 3P inequality holds true.

Lemma 3.1.9. *For every $a \in (0, \infty)$, there exists $c > 0$ such that for all $0 < s < t < aR_0$,*

$$\frac{\widetilde{q}(s, x, z)\widetilde{q}(t-s, z, y)}{\widetilde{q}(t, x, y)} \leq c(\widetilde{q}(s, x, z) + \widetilde{q}(t-s, z, y)), \quad x, y, z \in \mathfrak{X}.$$

For an open set $D \subset \mathfrak{X}$, a measure μ on D is said to be a smooth measure of X^D with respect to the reference measure m_D if there is a positive continuous additive functional (PCAF) A of X^D such that for any bounded non-negative Borel function f on D,

$$\int_D f(x)\mu(dx) = \lim_{t \downarrow 0} \mathbb{E}^m \left[\frac{1}{t} \int_0^t f(X_s^D) dA_s \right],$$

cf. [114]. The additive functional A is called the PCAF of X^D with Revuz measure μ with respect to the reference measure m_D.

It is known (see [68]) that for any $x \in D$, $\alpha \geq 0$ and bounded nonnegative Borel function f on D,

$$\mathbb{E}^x \int_0^\infty e^{-\alpha t} f(X_t^D) dA_t = \int_0^\infty e^{-\alpha t} \int_D p_D(t, x, y) f(y) \mu(dy) dt,$$

and we have for any $x \in D$, $t > 0$ and non-negative Borel function f on D,

$$\mathbb{E}^x \int_0^t f(X_s^D) dA_s = \int_0^t \int_D p_D(s,x,y) f(y) \mu(dy) ds.$$

We first introduce our class of possibly critical perturbations. For an open set $D \subset \mathfrak{X}$, a smooth Radon measure μ of X^D, $t > 0$ and $a \geq 0$, we define

$$N_a^{D,\mu}(t) := \sup_{x \in \mathfrak{X}} \int_0^t \int_{z \in D: \delta_D(z) > a\Phi^{-1}(t)} \widetilde{q}(s,x,z) \mu(dz) ds.$$

Definition 3.1.10. *Let μ be a smooth measure for both X^D and $\widehat{X}^D$ with respect to the reference measure m_D and let $T \in (0,\infty]$. The measure μ is said to be in the class $\mathbf{K}_T(D)$ if*
(1) $\sup_{t<T} N_a^{D,\mu}(t) < \infty$ for all $a \in (0,1]$;
(2) $\lim_{t \to 0} N_0^{U,\mu}(t) = 0$ for every relatively compact open set U of D.

For $\mu \in \mathbf{K}_T(D)$, using condition (2) in the definition above, one can show that, for any relatively compact open subset U of D, $A_{t \wedge \tau_U^X}$ is a PCAF of X^U with Revuz measure μ_U, where μ_U is the measure μ restricted to U. See Proposition 3.3.3 in Appendix for the proof.

Remark 3.1.11. *By the semigroup property, it is easy to check that*

$$N_a^{D,\mu}(t) \leq N_a^{D,\mu}(s) + \overline{C}_0(T)^2 N_a^{D,\mu}(t-s), \quad 0 < s < t \leq T,$$

where $\overline{C}_0(T)$ is the constant in (3.1.9). *Thus, if μ is in the class $\mathbf{K}_1(D)$, then* $\sup_{t<T} N_a^{D,\mu}(t) < \infty$ *for all $a > 0$ and $T \in (0,\infty)$.*

For $\mu \in \mathbf{K}_1(D)$, we denote by A_t^μ the positive continuous additive functional of X^D with Revuz measure μ and denote by $\widehat{A}_t^\mu$ the positive continuous additive functional of $\widehat{X}^D$ with Revuz measure μ. For any non-negative Borel function f on D, we define for $t \geq 0$, $x \in D$,

$$T_t^{\mu,D} f(x) = \mathbb{E}^x \left[\exp(-A_t^\mu) f(X_t^D) \right], \quad \widehat{T}_t^{\mu,D} f(x) = \widehat{\mathbb{E}}^x \left[\exp(-\widehat{A}_t^\mu) f(\widehat{X}_t^D) \right].$$

The semigroup $(T_t^{\mu,D} : t \geq 0)$ (respectively $(\widehat{T}_t^{\mu,D} : t \geq 0)$) is called the Feynman-Kac semigroup of X^D (respectively $\widehat{X}^D$) associated with μ. By [127, Theorem 6.10(2)], $T_t^{\mu,D}$ and $\widehat{T}_t^{\mu,D}$ are duals of each other with respect to the measure m_D so that

$$\int_D T_t^{\mu,D} f(x)g(x)m(dx) = \int_D f(x)\widehat{T}_t^{\mu,D} g(x)m(dx). \qquad (3.1.12)$$

Let Y ($\widehat{Y}$, respectively) be a Hunt process on D corresponding to the transition semigroup $(T_t^{\mu,D})$ ($(\widehat{T}_t^{\mu,D})$, respectively). For an open subset $U \subset D$, we denote by Y^U ($\widehat{Y}^U$, respectively) the process Y ($\widehat{Y}$, respectively) killed upon exiting U.

Suppose that $U \subset D$ is a relatively compact open subset of D. Since for any relatively compact open set U, $A_t^{\mu,U} := A^\mu_{t\wedge\tau_U^X}$ is a positive continuous additive functional of X^U with Revuz measure μ_U, the transition semigroup of Y^U is $(T_t^{\mu_U,U})$. For simplicity, in the sequel we denote this semigroup as $(T_t^{\mu,U})$. Moreover, for any $t \geq 0, x \in U$,

$$T_t^{\mu,U} f(x) = \mathbb{E}^x\left[f(Y_t^U)\right] = \mathbb{E}^x\left[\exp\left(-A^\mu_{t\wedge\tau_U^X}\right)f(X_t^U)\right]$$

and

$$\mathbb{E}^x \int_0^t f(X_s^U)dA_s^{\mu,U} = \int_0^t \int_U p_U(s,x,y)f(y)\mu(dy)ds.$$

It follows from Definition 3.1.10(2) that, for all relatively compact open subset U of D, μ_U is in the standard Kato class of X^U, that is,

$$\lim_{t\to 0}\sup_{x\in U}\int_0^t\int_U p_U(s,x,y)\mu(dy)ds = 0.$$

Thus, according to the discussion in [42, Section 1.2], we have for any non-negative bounded Borel function f on U,

$$T_t^{\mu,U} f(x) = \mathbb{E}^x \left[f(X_t^U) \right] + \mathbb{E}^x \left[f(X_t^U) \sum_{n=1}^{\infty} (-1)^n \int_{0<s_1<\cdots<s_n<t} dA_{s_1}^{\mu,U} \cdots dA_{s_n}^{\mu,U} \right].$$

Define $p_U^0(t,x,y) := p_U(t,x,y)$ and, for $k \geq 1$,

$$p_U^k(t,x,y) = -\int_0^t \int_U p_U(s,x,z) p_U^{k-1}(t-s,z,y)\mu(dz)ds.$$

Then we set $q^U(t,x,y) := \sum_{k=0}^{\infty} p_U^k(t,x,y)$. By Lemma 3.1.9, we have that for any μ in $\mathbf{K}_1(D)$, any relatively compact open set U of D and any $(t,x,y) \in (0,1] \times U \times U$,

$$\begin{aligned}
&\int_0^t \int_U \widetilde{q}(t-s,x,z)\widetilde{q}(s,z,y)\mu(dz)ds \\
&\leq c\widetilde{q}(t,x,y) \sup_{u\in\mathfrak{X}} \int_0^t \int_U \widetilde{q}(s,u,z)\mu(dz)ds = c\widetilde{q}(t,x,y)N_0^{U,\mu}(t). \qquad (3.1.13)
\end{aligned}$$

Using (3.1.13), (3.1.8) and the semigroup property, one can show that $p_U^k(t,x,y)$ is continuous in (t,y) for each fixed x, continuous in (t,x) for each fixed y, and $\sum_{k=0}^{\infty} p_U^k(t,x,y)$ converges absolutely and uniformly so that $q^U(t,x,y)$ is continuous in (t,y) for each fixed x, and also continuous in (t,x) for each fixed y (for example, see [42]). Moreover, by repeating the discussion in [42, Section 1.2], one can conclude that

$$T_t^{\mu,U} f(x) = \int_U q^U(t,x,y)f(y)m(dy), \quad (t,x) \in (0,\infty) \times U.$$

Define $q^D(t,x,y) := \lim_{n\to\infty} q^{D_n}(t,x,y)$, where $D_n \subset D$ are bounded increasing open sets such that $\overline{D_n} \subset D_{n+1}$ and $\cup_{n=1}^{\infty} D_n = D$. Then, using the monotone convergence theorem and

$$q^{D_n}(t,x,y) \leq p_{D_n}(t,x,y) \leq p(t,x,y) \leq \overline{C}_0(T)\widetilde{q}(t,x,y), \quad t < T,$$

we see that $q^D(t,x,y)$ is the transition density of the process Y and $q^D(t,x,y) \leq$

$\overline{C}_0(T)\widetilde{q}(t,x,y)$ for $t < T$. Therefore, we obtain the following

Proposition 3.1.12. *Suppose that D is an open set in $\mathfrak{X}$ and $\mu \in \mathbf{K}_1(D)$. Then the Hunt process Y on D corresponding to the transition semigroup $(T_t^{\mu,D})$ has a transition density $q^D(t,x,y)$ with respect to m such that for each $T \in (0,\infty)$, $q^D(t,x,y) \le \overline{C}_0(T)\widetilde{q}(t,x,y)$ for $t < T$. Furthermore, if D is relatively compact, then $q^D(t,x,y)$ is continuous in (t,y) for each fixed x, and continuous in (t,x) for each fixed y. If $\mu \in \mathbf{K}_\infty(D)$ and $\widetilde{T} = \infty$, then the estimate $q^D(t,x,y) \le c\widetilde{q}(t,x,y)$ holds for every $t > 0$.*

3.1.4 Interior estimates for Y

In this subsection, we prove some interior estimates for the transition density $q^U(t,x,y)$, where U is an open subset of D. Recall that we assume $R_0 = m(\mathfrak{X}) = \infty$ when $\widetilde{T} = \infty$.

Theorem 3.1.13. *Suppose that $\mu \in \mathbf{K}_1(D)$. Then for every $T \in (0,\infty)$ and $a \in (0,1]$, there exists a constant $c := c(a,\Phi,C_0,M,\sup_{t\le T} N^{D,\mu}_{2^{-1}a}(t)) > 0$ such that for every open $U \subset D$,*

$$q^U(t,x,y) \ge c\widetilde{q}(t,x,y) \tag{3.1.14}$$

for all $t \in (0,T), x,y \in U$ with $\delta_U(x) \wedge \delta_U(y) \ge a\Phi^{-1}(t)$. Moreover, if $\mu \in \mathbf{K}_\infty(D)$ and $\widetilde{T} = \infty$, then (3.1.14) holds for all $t > 0$.

Proof. Fix $t \in (0,T)$, $x,y \in U$ with $\delta_U(x) \wedge \delta_U(y) \ge a\Phi^{-1}(t)$. Let V be a bounded open subset of U defined by

$$V := \{z \in U : \delta_U(z) > 2^{-1}a\Phi^{-1}(t)\} \cap B\big(x, \rho(x,y) + a\Phi^{-1}(t)\big).$$

Then, one can check that $x,y \in V$ and $\delta_V(x) \wedge \delta_V(y) \ge 2^{-2}a\Phi^{-1}(t)$. Note that $q^U(t,x,w) \ge q^V(t,x,w)$ for all $w \in V$ and $w \mapsto q^V(t,x,w)$ is continuous.

For $w \in V$, let

$$\widetilde{p}^1_V(t,x,w) := \int_0^t \left(\int_V p_V(t-s,x,z)p_V(s,z,w)\mu(dz)\right) ds.$$

Then for any bounded Borel function f on V, by the Markov property of X^V, we have

$$\mathbb{E}^x\left[A_t^{\mu_V} f(X_t^V)\right] = \mathbb{E}^x\left[\int_0^t \mathbb{E}^{X_s^V}[f(X_{t-s}^V)]dA_s^{\mu_V}\right] = \int_V \widetilde{p}_V^1(t,x,w)f(w)m(dw). \tag{3.1.15}$$

Since $\delta_D(z) \geq \delta_U(z) > 2^{-1}a\Phi^{-1}(t)$ for $z \in V$, by (3.1.11), (3.1.8), Lemma 3.1.9 and Proposition 3.1.8, we have that for $w \in B(y, 2^{-3}a\Phi^{-1}(t))$,

$$\begin{aligned}
&\widetilde{p}_V^1(t,x,w) \leq \overline{C}_0^2 \int_0^t \int_V \widetilde{q}(t-s,x,z)\widetilde{q}(s,z,w)\mu(dz)ds \\
&\leq \overline{C}_0^2 \int_0^t \int_{z\in D:\delta_D(z)>2^{-1}a\Phi^{-1}(t)} \widetilde{q}(t-s,x,z)\widetilde{q}(s,z,w)\mu(dz)ds \\
&\leq \overline{C}_0^2 c\left(\sup_{s\leq T} N_{2^{-1}a}^{D,\mu}(s)\right)\widetilde{q}(t,x,w) \leq \overline{C}_0^2 c\left(\sup_{s\leq T} N_{2^{-1}a}^{D,\mu}(s)\right) C_*^{-1} p_V(t,x,w) \\
&=: (k/2)p_V(t,x,w).
\end{aligned}$$

Hence, for $w \in B(y, 2^{-3}a\Phi^{-1}(t))$, we have $p_V(t,x,w) - k^{-1}\widetilde{p}_V^1(t,x,w) \geq 2^{-1}p_V(t,x,w)$, which implies that for any $r < 2^{-3}a\Phi^{-1}(t)$,

$$\frac{1}{2}\mathbb{E}^x[\mathbf{1}_{B(y,r)}(X_t^V)] \leq \mathbb{E}^x\left[\left(1 - A_t^{\mu_V}/k\right)\mathbf{1}_{B(y,r)}(X_t^V)\right]. \tag{3.1.16}$$

Using the fact that $1 - A_t^{\mu_V}/k \leq \exp\left(-A_t^{\mu_V}/k\right)$, we get that for any $r < 2^{-3}a\Phi^{-1}(t)$,

$$\frac{1}{V(y,r)}\mathbb{E}^x\left[\left(1 - A_t^{\mu_V}/k\right)\mathbf{1}_{B(y,r)}(X_t^V)\right] \leq \frac{1}{V(y,r)}\mathbb{E}^x\left[\exp(-A_t^{\mu_V}/k)\mathbf{1}_{B(y,r)}(X_t^V)\right].$$

Thus, by (3.1.16), (3.1.15) and Hölder's inequality, we have

$$\begin{aligned}
&\frac{1}{2}\frac{1}{V(y,r)}\mathbb{E}^x\left[\mathbf{1}_{B(y,r)}(X_t^V)\right] \leq \frac{1}{V(y,r)}\mathbb{E}^x\left[\exp(-A_t^{\mu_V}/k)\mathbf{1}_{B(y,r)}(X_t^V)\right] \\
&\leq \left(\frac{1}{V(y,r)}\mathbb{E}^x\left[\exp(-A_t^{\mu_V})\mathbf{1}_{B(y,r)}(X_t^V)\right]\right)^{1/k}\left(\frac{1}{V(y,r)}\mathbb{E}^x\left[\mathbf{1}_{B(y,r)}(X_t^V)\right]\right)^{1-1/k}.
\end{aligned}$$

Therefore,

$$\frac{1}{2^k}\frac{1}{V(y,r)}\mathbb{E}^x\left[\mathbf{1}_{B(y,r)}(X_t^V)\right] \le \frac{1}{V(y,r)}\mathbb{E}^x\left[\exp(-A_t^{\mu_V})\mathbf{1}_{B(y,r)}(X_t^V)\right].$$

Since $w \to q^V(t,x,w)$ is continuous by Proposition 3.1.12, we conclude by sending $r \downarrow 0$ and applying Proposition 3.1.8 again that for every $t \in (0,T], x,y \in U$ with $\delta_U(x)\wedge\delta_U(y) \ge a\Phi^{-1}(t)$,

$$q^U(t,x,y) \ge q^V(t,x,y) \ge 2^{-k}p_V(t,x,y) \ge c2^{-k}\widetilde{q}(t,x,y).$$

□

Let $\tau_U := \inf\{s>0 : Y_s \notin U\}$ and $\widehat{\tau}_U := \inf\{s>0 : \widehat{Y}_s \notin U\}$. Using Theorem 3.1.13, we have the following result.

Corollary 3.1.14. *(i) Suppose that $\mu \in \mathbf{K}_1(D)$. For any positive constants R, T and a, there exists $c_1 = c_1(a,T) > 0$ such that for all $t \in (0,T)$ and $B(x,\Phi^{-1}(t)) \subset D$,*

$$\inf_{z\in B(x,a\Phi^{-1}(t)/2)}\mathbb{P}^z(\tau_{B(x,a\Phi^{-1}(t))} > t) \wedge \inf_{z\in B(x,a\Phi^{-1}(t)/2)}\widehat{\mathbb{P}}^z(\widehat{\tau}_{B(x,a\Phi^{-1}(t))} > t) \ge c_1 \tag{3.1.17}$$

and

$$\mathbb{E}^x[\tau_{B(x,r)}] \wedge \widehat{\mathbb{E}}^x[\widehat{\tau}_{B(x,r)}] \ge c_1\Phi(r). \tag{3.1.18}$$

Moreover, there exist $r_1, c_2 > 0$ such that for all $r \in (0,r_1]$ and $B(x,r) \subset D$,

$$\mathbb{E}^x[\tau_{B(x,r)}] \vee \widehat{\mathbb{E}}^x[\widehat{\tau}_{B(x,r)}] \le c_2\Phi(r). \tag{3.1.19}$$

(ii) If $\mu \in \mathbf{K}_\infty(D)$ and $\widetilde{T} = \infty$ (and $R_0 = \infty$), then (3.1.17)–(3.1.19) *hold for all $r, t > 0$.*

3.1.5 Examples of critical potentials

In this subsection, we give two examples of critical potentials.

Example 3.1.15. *Suppose* $\mu(dz) = q(z)m(dz)$ *with* $0 \le q(z) \simeq 1/\Phi(\delta_D(z) \wedge 1)$. *Since* q *is bounded on every relatively compact open set* $U \subset D$, $N_0^{U,\mu}(t) \le C_0 t \|q\|_{L^\infty(U)} \to 0$ *as* $t \to 0$. *Moreover, for* $x \in D$, $a \in (0,1]$ *and* $t < 1$,

$$\begin{aligned}
&\int_0^t \int_{z \in D: \delta_D(z) > a\Phi^{-1}(t)} \widetilde{q}(s,x,z) q(z) m(dz) ds \\
\le& ct + c \int_0^t \int_{z \in D: 1 > \delta_D(z) > a\Phi^{-1}(t)} \Phi(\delta_D(z))^{-1} \widetilde{q}(s,x,z) m(dz) ds \\
\le& ct + c \frac{1}{\Phi(a\Phi^{-1}(t))} \int_0^t \int_D \widetilde{q}(s,x,z) m(dz) ds \le ct + c \frac{t}{\Phi(a\Phi^{-1}(t))} < c < \infty.
\end{aligned}$$

Thus $\sup_{t<1} N_a^{D,\mu}(t) < \infty$ *for all* $a \in (0,1]$ *and so* μ *is in the class* $\mathbf{K}_1(D)$.

Example 3.1.16. *Suppose* $\widetilde{T} = \infty$ *and* $\mu(dz) = q(z)m(dz)$ *with* $0 \le q(z) \simeq 1/\Phi(\delta_D(z))$. *Then* $R_0 = \infty$ *and for all* $a \in (0,1]$ *and* $t < \infty$,

$$N_a^{D,\mu}(t) \le c \frac{1}{\Phi(a\Phi^{-1}(t))} \sup_{x \in \mathfrak{X}} \int_0^t \int_D \widetilde{q}(s,x,z) m(dz) ds \le \frac{ct}{\Phi(a\Phi^{-1}(t))} < c < \infty.$$

Thus μ *is in the class* $\mathbf{K}_\infty(D)$.

3.1.6 Factorization of heat kernel in κ-fat open set

Recall that $\mathcal{D}(K,U)$ is the subset of $\mathcal{D}$ in Assumption **A**. Let

$$\begin{aligned}
A(z_0,p,q) &:= \{x \in \mathfrak{X} : p < \rho(x,z_0) < q\}, \\
\overline{A}(z_0,p,q) &:= \{x \in \mathfrak{X} : p \le \rho(x,z_0) \le q\}.
\end{aligned}$$

Note that, due to our assumption that all bounded closed sets are compact, $\overline{A}(z_0,p,q)$ is compact. Thus by Assumption **A**, for any $1/2 < b < a < 1$, the set $\mathcal{D}(\overline{A}(z_0,br,ar), A(z_0,r/2,r))$ is nonempty. We now add the final assumption saying that there exist proper bump functions in each nonempty set $\mathcal{D}(\overline{A}(z_0,br,ar), A(z_0,r/2,r))$ providing scale-invariant control on the action of the generator.

Assumption U: There exists $r_0 \in (0,\infty]$ such that for any $1/2 < b < a < 1$, there exists $c = c(a, b)$ such that for every $z_0 \in \mathfrak{X}$ and $r < r_0$,

$$\inf_{f \in \mathcal{D}(\overline{A}(z_0, br, ar), A(z_0, r/2, r))} \sup_{x \in \mathfrak{X}} \max(\mathcal{L}f(x), \widehat{\mathcal{L}}f(x)) \leq \frac{c}{\Phi(r)}.$$

This assumption is used in connection with Dynkin's formula in Lemma 3.1.18 to get a scale-invariant estimate of the exit probability.

Definition 3.1.17. *Let $0 < \kappa \leq 1/2$. We say that an open set D is κ-fat if there is $R_1 \in (0, \infty]$ such that for all $x \in \overline{D}$ and all $r \in (0, R_1)$, there is a ball $B(A_r(x), \kappa r) \subset D \cap B(x, r)$. The pair (R_1, κ) is called the characteristics of the κ-fat open set D.*

In the remainder of this subsection, $T > 0$ is a fixed constant and D is a fixed κ-fat open set with characteristics (R_1, κ). Without loss of generality, we can assume that $R_1 \leq R_0 \wedge r_0 \wedge r_1$, where r_1 is the constant in Corollary 3.1.14(i). For $(t, x) \in (0, T) \times D$, set $r_t = \Phi^{-1}(t)R_1/(3\Phi^{-1}(T)) \leq R_1/3$. An open neighborhood $\mathcal{U}(x, t)$ of $x \in D$ and an open ball $\mathcal{W}(x, t) \subset D \setminus \mathcal{U}(x, t)$ are defined as follows:

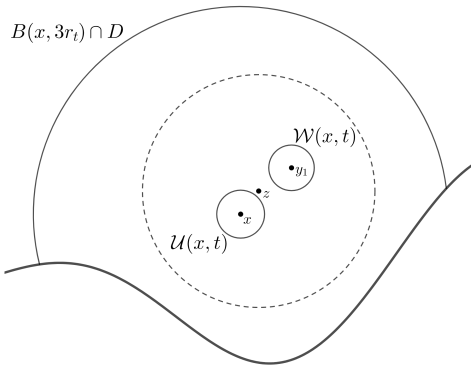

Figure 1. $\rho(x, z) \leq 3\kappa r_t/2$.

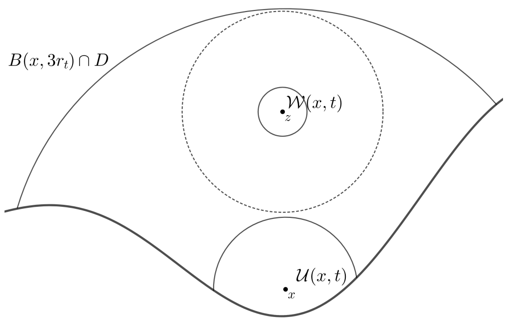

Figure 2. $\rho(x, z) > 3\kappa r_t/2$.

By the definition of κ-fat open set, we can find $z = z_{x,t} \in D$ such that $B(z, 3\kappa r_t) \subset B(x, 3r_t) \cap D$.

(i) If $\rho(x, z) \leq 3\kappa r_t/2$, we choose $y_1 \in \mathfrak{X}$ such that $\kappa r_t/n_0 \leq \rho(x, y_1) \leq \kappa r_t$, where $n_0 > 1$ is the constant in (3.1.2). Then we define $\mathcal{U}(x, t) =$

$B(x, \kappa r_t/(4n_0))$ and $\mathcal{W}(x,t) = B(y_1, \kappa r_t/(4n_0))$. We can easily check that $\mathcal{U}(x,t) \cup \mathcal{W}(x,t) \subset B(x, 3\kappa r_t/2) \subset B(z, 3\kappa r_t) \subset D$ and $\mathcal{U}(x,t) \cap \mathcal{W}(x,t) = \emptyset$.

(ii) If $\rho(x,z) > 3\kappa r_t/2$, we define $\mathcal{U}(x,t) = B(x, \kappa r_t) \cap D$ and $\mathcal{W}(x,t) = B(z, \kappa r_t/(4n_0))$.

Note that in either case, we have,

$$\kappa r_t/(2n_0) \le \rho(u,v) \le 4r_t \quad \text{for all} \;\; u \in \mathcal{U}(x,t) \text{ and } v \in \mathcal{W}(x,t). \tag{3.1.20}$$

See Figures 1 and 2 for some illustration of the sets $\mathcal{U}(x,t)$ and $\mathcal{W}(x,t)$.

It follows from [127, Theorem I.3.4] that the Lévy system of Y is the same as that of X, hence the following Lévy system formula is valid: for any $f : D \times D \to [0,\infty]$ vanishing on the diagonal and every stopping time S,

$$\mathbb{E}^x \sum_{t \in (0,S]} f(Y_{t-}, Y_t) = \mathbb{E}^x \int_0^S \int_D f(Y_t, z) J(Y_t, z) m(dz) dt. \tag{3.1.21}$$

Recall that $\tau_U = \inf\{s > 0 : Y_s \notin U\}$ and $\widehat{\tau}_U = \inf\{s > 0 : \widehat{Y}_s \notin U\}$. Note that $\mathbb{P}^x(Y_{\tau_{\mathcal{U}(x,t)}} \in D) = \mathbb{P}^x(\tau_{\mathcal{U}(x,t)} < \zeta)$, where ζ is the lifetime of Y.

Since the proofs for the dual processes are same, throughout this section, we give the proofs for Y only.

Lemma 3.1.18. *Suppose that $\mu \in \mathbf{K}_1(D)$. For all $(t,x) \in (0,T) \times D$ and $z = z_{x,t} \in D$ with $B(z, 3\kappa r_t) \subset B(x, 3r_t) \cap D$ and $\rho(x,z) > 3\kappa r_t/2$, we have*

$$\mathbb{P}^x(Y_{\tau_{\mathcal{U}(x,t)}} \in \mathcal{W}(x,t)) \simeq \mathbb{P}^x(Y_{\tau_{\mathcal{U}(x,t)}} \in D) \simeq t^{-1}\mathbb{E}^x[\tau_{\mathcal{U}(x,t)}]$$

and

$$\widehat{\mathbb{P}}^x(\widehat{Y}_{\widehat{\tau}_{\mathcal{U}(x,t)}} \in \mathcal{W}(x,t)) \simeq \widehat{\mathbb{P}}^x(\widehat{Y}_{\widehat{\tau}_{\mathcal{U}(x,t)}} \in D) \simeq t^{-1}\mathbb{E}^x[\widehat{\tau}_{\mathcal{U}(x,t)}],$$

where $\mathcal{U}(x,t)$ and $\mathcal{W}(x,t)$ are the open sets defined in the beginning of this subsection and the comparison constants depend only on $d_0, d, \delta_l, \delta_u, T, M, R_1$ and κ.

Proof. Fix $(t,x) \in (0,T) \times D$ and assume that $B(z, 3\kappa r_t) \subset B(x, 3r_t) \cap D$ and $\rho(x,z) > 3\kappa r_t/2$. Recall that $\mathcal{U}(x,t) = B(x, \kappa r_t) \cap D$ and $\mathcal{W}(x,t) =$

$B(z, \kappa r_t/(4n_0))$. Define $D_1 := B(x, 9\kappa r_t/8) \cap D$ and $D_2 := D \setminus D_1$. Take any

$$f \in \mathcal{D}\big(\overline{A}(z, \kappa r_t, 9\kappa r_t/8), A(z, 5\kappa r_t/8, 5\kappa r_t/4)\big).$$

Then, by Dynkin's formula for X (see [25, (2.11)] and the proof of [25, (4.6)]), we have

$$\begin{aligned}
\mathbb{P}^x(Y_{\tau_{\mathcal{U}(x,t)}} \in D_1) &= \mathbb{E}^x\Big[\, \exp\big(-A^\mu_{\tau^X_{\mathcal{U}(x,t)}}\big) : X_{\tau^X_{\mathcal{U}(x,t)}} \in D_1\Big] \\
&\le \mathbb{E}^x\left[f\big(X_{\tau^X_{\mathcal{U}(x,t)}}\big)\ \exp\big(-A^\mu_{\tau^X_{\mathcal{U}(x,t)}}\big)\right] - f(y) \\
&= \mathbb{E}^x\left[\int_0^{\tau^X_{\mathcal{U}(x,t)}} \mathcal{L}f(X_s)\exp(-A^\mu_s)ds\right] + \mathbb{E}^x\left[\int_0^{\tau^X_{\mathcal{U}(x,t)}} f(X_s)d\exp(-A^\mu_s)\right] \\
&\le \left(\sup_{z\in\mathfrak{X}} \mathcal{L}f(z)\right)\mathbb{E}^x\left[\int_0^{\tau^X_{\mathcal{U}(x,t)}} \exp(-A^\mu_s)ds\right] = \left(\sup_{z\in\mathfrak{X}} \mathcal{L}f(z)\right)\mathbb{E}^x[\tau_{\mathcal{U}(x,t)}].
\end{aligned}$$

By Assumption **U** and (3.1.3), taking infimum over f on both sides gives

$$\mathbb{P}^x(Y_{\tau_{\mathcal{U}(x,t)}} \in D_1) \le \frac{c_1}{\Phi(r_t)}\mathbb{E}^y[\tau_{\mathcal{U}(x,t)}] \le c_2 t^{-1}\mathbb{E}^y[\tau_{\mathcal{U}(x,t)}].$$

On the other hand, by (3.1.21), (3.1.10), (3.1.20), (3.1.1) and (3.1.3), we have that

$$\begin{aligned}
\mathbb{P}^x(Y_{\tau_{\mathcal{U}(x,t)}} \in \mathcal{W}(x,t)) &= \mathbb{E}^x\left[\int_0^{\tau_{\mathcal{U}(x,t)}}\int_{\mathcal{W}(x,t)} J(Y_s,w)m(dw)ds\right] \\
&\simeq \mathbb{E}^x[\tau_{\mathcal{U}(x,t)}]\int_{\mathcal{W}(x,t)} \frac{1}{V(x,r_t)\Phi(r_t)}m(dw) \simeq t^{-1}\mathbb{E}^x[\tau_{\mathcal{U}(x,t)}]
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{P}^x(Y_{\tau_{\mathcal{U}(x,t)}} \in D_2) &= \mathbb{E}^x\left[\int_0^{\tau_{\mathcal{U}(x,t)}}\int_{D_2} J(Y_s,w)m(dw)ds\right] \\
&\le c_3\mathbb{E}^x[\tau_{\mathcal{U}(x,t)}]\int_{D\setminus B(x,9\kappa r_t/8)} \frac{1}{V(x,\rho(x,w))\Phi(\rho(x,w))}m(dw) \\
&\le c_4 t^{-1}\mathbb{E}^x[\tau_{\mathcal{U}(x,t)}].
\end{aligned}$$

We used [48, Lemma 2.1] in the last inequality. Therefore, using the fact $\mathbb{P}^x(Y_{\tau_{\mathcal{U}(x,t)}} \in D) = \mathbb{P}^x(Y_{\tau_{\mathcal{U}(x,t)}} \in D_1) + \mathbb{P}^x(Y_{\tau_{\mathcal{U}(x,t)}} \in D)$, we get the desired result. □

Recall that ζ is the lifetime of Y. We denote by $\widehat{\zeta}$ the lifetime of $\widehat{Y}$.

Lemma 3.1.19. *Suppose that $\mu \in \mathbf{K}_1(D)$. For all $M, T \geq 1$, we have that, for all $t \in (0, T)$ and $x \in D$,*

$$\begin{aligned}\mathbb{P}^x(\zeta > t) &\simeq \mathbb{P}^x(\zeta > t/M) \simeq \mathbb{P}^x(\tau_{\mathcal{U}(x,t)} > t) \simeq t^{-1}\mathbb{E}^x[\tau_{\mathcal{U}(x,t)}] \\ &\simeq \mathbb{P}^x(Y_{\tau_{\mathcal{U}(x,t)}} \in D) \simeq \mathbb{P}^x(Y_{\tau_{\mathcal{U}(x,t)}} \in \mathcal{W}(x,t))\end{aligned}$$

and

$$\begin{aligned}\widehat{\mathbb{P}}^x(\widehat{\zeta} > t) &\simeq \widehat{\mathbb{P}}^x(\widehat{\zeta} > t/M) \simeq \widehat{\mathbb{P}}^x(\widehat{\tau}_{\mathcal{U}(x,t)} > t) \simeq t^{-1}\widehat{\mathbb{E}}^x[\widehat{\tau}_{\mathcal{U}(x,t)}] \\ &\simeq \mathbb{P}^x(\widehat{Y}_{\tau_{\mathcal{U}(x,t)}} \in D) \simeq \mathbb{P}^x(\widehat{Y}_{\tau_{\mathcal{U}(x,t)}} \in \mathcal{W}(x,t)),\end{aligned}$$

where $\mathcal{U}(x,t)$ and $\mathcal{W}(x,t)$ are the open sets defined in the beginning of this subsection and the comparison constants depend only on $d_0, d, \delta_l, \delta_u, T, M, R_1$ and κ.

Proof. Fix $t \in (0, T)$, $x \in D$ and set $r := r_t = \Phi^{-1}(t)R_1/3\Phi^{-1}(T)$.
Case (1): $\rho(x, z) \leq 3\kappa r/2$. By (3.1.17), we have

$$1 \geq \mathbb{P}^x(\zeta > t/M) \geq \mathbb{P}^x(\zeta > t) \geq \mathbb{P}^x(\tau_{\mathcal{U}(x,t)} > t) = \mathbb{P}^x(\tau_{B(x,\kappa r/(4n_0))} > t) \geq c.$$

On the other hand, by (3.1.20), (3.1.21), (3.1.1), (3.1.3) and (3.1.18),

$$\begin{aligned}1 &\geq \mathbb{P}^x(Y_{\tau_{\mathcal{U}(x,t)}} \in D) \geq \mathbb{P}^x(Y_{\tau_{\mathcal{U}(x,t)}} \in \mathcal{W}(x,t)) \\ &= \mathbb{E}^x\left[\int_0^{\tau_{\mathcal{U}(x,t)}} \int_{\mathcal{W}(x,t)} J(Y_s, v) m(dv) ds\right] \\ &\geq c_1 \frac{m(\mathcal{W}(x,t))\mathbb{E}^x[\tau_{\mathcal{U}(x,t)}]}{V(x, 3r)\Phi(3r)} \geq c_2 \frac{\mathbb{E}^x[\tau_{\mathcal{U}(x,t)}]}{\Phi(3r)} \geq c_3 t^{-1}\mathbb{E}^x[\tau_{B(x,\kappa r/(4n_0))}] \geq c_4.\end{aligned}$$

Therefore, we arrive at the assertion of the lemma in this case.

Case (2): $\rho(x,z) > 3\kappa r/2$. Using the Markov inequality, we get

$$\begin{aligned}\mathbb{P}^x(\zeta > t/M) &= \mathbb{P}^x(\zeta \geq \tau_{\mathcal{U}(x,t)} > t/M) + \mathbb{P}^x(\zeta > t/M \geq \tau_{\mathcal{U}(x,t)}) \\ &\leq \mathbb{P}^x(\tau_{\mathcal{U}(x,t)} > t/M) + \mathbb{P}^x(\zeta > \tau_{\mathcal{U}(x,t)}) \\ &\leq Mt^{-1}\mathbb{E}^x[\tau_{\mathcal{U}(x,t)}] + \mathbb{P}^x(Y_{\tau_{\mathcal{U}(x,t)}} \in D).\end{aligned}$$

Then by Lemma 3.1.18 we have

$$\mathbb{P}^x(\zeta > t/M) \leq c_5 t^{-1}\mathbb{E}^x[\tau_{\mathcal{U}(x,t)}] \simeq \mathbb{P}^x(Y_{\tau_{\mathcal{U}(x,t)}} \in \mathcal{W}(x,t)) \simeq \mathbb{P}^x(Y_{\tau_{\mathcal{U}(x,t)}} \in D).$$

Note that $B(x,(3-2\kappa)r)\cap D \supset \mathcal{U}(x,t)\cup\mathcal{W}(x,t)$ for every $(t,x) \in (0,T]\times D$. Thus by (3.1.17),

$$\begin{aligned}\mathbb{P}^x(\zeta > t/M) &\geq \mathbb{P}^x(\zeta > t) \geq \mathbb{P}^x(\tau_{B(x,3r)\cap D} > t) \\ &\geq \mathbb{E}^x\left[\inf_{w\in\mathcal{W}(x,t)} \mathbb{P}^w(\tau_{B(x,3r)\cap D} > t) : Y_{\tau_{\mathcal{U}(x,t)}} \in \mathcal{W}(x,t)\right] \\ &\geq \mathbb{E}^x\left[\inf_{w\in\mathcal{W}(x,t)} \mathbb{P}^w(\tau_{B(w,\kappa r)} > t) : Y_{\tau_{\mathcal{U}(x,t)}} \in \mathcal{W}(x,t)\right] \\ &\geq c_6\mathbb{P}^x(Y_{\tau_{\mathcal{U}(x,t)}} \in \mathcal{W}(x,t)).\end{aligned}$$

The proof is now complete. □

Theorem 3.1.20. *Let D be a κ-fat set with characteristics (R_1,κ). Suppose that $\mu \in \mathbf{K}_1(D)$. Then for all $T > 0$, there exists $c \geq 1$ such that for all $(t,x,y) \in (0,T)\times D\times D$,*

$$c^{-1} \leq \frac{q^D(t,x,y)}{\mathbb{P}^x(\zeta > t)\widehat{\mathbb{P}}^y(\widehat{\zeta} > t)\widetilde{q}(t,x,y)} \leq c.$$

Proof. Fix $t \in (0,T)$, $x,y \in D$ and set $r := \Phi^{-1}(t)R_1/(3\Phi^{-1}(T))$.

(1) We first prove the upper bound. By (3.1.8) and Lemma 3.1.19, if

$\rho(x,y) \le 4r$, then we have

$$\begin{aligned} q^D(t/2,x,y) &= \int_D q^D(t/4,x,w)q^D(t/4,w,y)m(dw) \\ &\le C_0 \int_D q^D(t/4,x,w)\widetilde{q}(t/4,w,y)m(dw) \\ &\le c_1 \mathbb{P}^x(\zeta > t/4) V(y,\Phi^{-1}(t))^{-1} \le c_2 \mathbb{P}^x(\zeta > t) p(t/2,x,y). \end{aligned}$$

Suppose that $\rho(x,y) > 4r$. Let $U_1 := \mathcal{U}(x,t)$ be the set defined before, $U_3 := \{u \in D : \rho(x,u) > \rho(x,y)/2\}$, and $U_2 := D \setminus (U_1 \cup U_3)$. Since $x \in U_1$, $y \in U_3$ and $U_1 \cap U_3 = \emptyset$, by the strong Markov property, we have

$$\begin{aligned} q^D(t/2,x,y) = \ &\mathbb{E}^x[q^D(t/2-\tau_{U_1}, Y_{\tau_{U_1}}, y) : \tau_{U_1} < t/2, Y_{\tau_{U_1}} \in U_2] \\ &+ \mathbb{E}^x[q^D(t/2-\tau_{U_1}, Y_{\tau_{U_1}}, y) : \tau_{U_1} < t/2, Y_{\tau_{U_1}} \in U_3] =: I + II. \end{aligned}$$

Note that for every $u \in U_2$, $\rho(u,y) \ge \rho(x,y) - \rho(x,u) \ge \rho(x,y)/2$ and hence

$$V(y,\rho(x,y)) \le V(u,\rho(x,y)+\rho(u,y)) \le V(u,3\rho(u,y)).$$

Then, using (3.1.8), (3.1.1) and (3.1.3), we get that for all $(s,u) \in (0,t/2]\times U_2$,

$$\begin{aligned} q^D(s,u,y) &\le \frac{c_3 s}{V(u,3\rho(u,y))\Phi(2\rho(u,y))} \le \frac{c_4 t}{V(y,\rho(x,y))\Phi(\rho(x,y))} \\ &\le c_5 p(t/2,x,y). \end{aligned}$$

Now it follows from Lemma 3.1.19 that

$$I \le c_5 p(t/2,x,y)\mathbb{P}^x(Y_{\tau_{U_1}} \in D) \simeq \mathbb{P}^x(\zeta > t)p(t/2,x,y).$$

On the other hand, for all $u \in U_1$ and $w \in U_3$, we have $\rho(u,x) \le r < \rho(x,y)/4$ and $\rho(u,w) \ge \rho(x,w) - \rho(x,u) \ge \rho(x,y)/2 - r \ge \rho(x,y)/4$, which implies that

$$V(x,\rho(x,y)) \le V(u,\rho(u,x)+\rho(x,y)) \le V(u,2\rho(x,y)) \le V(u,8\rho(u,w)).$$

Thus, by (3.1.21), (3.1.1), (3.1.4) and Lemma 3.1.19, using the assumption $\rho(x,y) > 4r$, we get

$$\begin{aligned}
II &= \int_0^{t/2} \int_{U_1} \int_{U_3} q^{U_1}(s,x,u) J(u,w) q^D(t/2-s,w,y) m(dw) m(du) ds \\
&\le c_6 \frac{1}{V(x,\rho(x,y))\Phi(\rho(x,y))} \int_0^{t/2} \mathbb{P}^x(\tau_{U_1} > s) \widehat{\mathbb{P}}^y(\widehat{\zeta} > t/2-s) ds \\
&\le c_6 \frac{1}{V(x,\rho(x,y))\Phi(\rho(x,y))} \int_0^{\infty} \mathbb{P}^x(\tau_{U_1} > s) ds \\
&= 2c_6 \frac{t/2}{V(x,\rho(x,y))\Phi(\rho(x,y))} t^{-1} \mathbb{E}^x[\tau_{U_1}] \simeq \mathbb{P}^x(\zeta > t) p(t/2,x,y).
\end{aligned}$$

Eventually, we deduce that whether $\rho(x,y) \le 4r$ or not, there exists $c_7 > 0$ independent of t, x, y such that $q^D(t/2,x,y) \le c_7 \mathbb{P}^x(\zeta > t) p(t/2,x,y)$, and, similarly, $q^D(t/2,x,y) \le c_7 \widehat{\mathbb{P}}^y(\widehat{\zeta} > t) p(t/2,x,y)$. Then, by the semigroup property and (3.1.8), we obtain that

$$\begin{aligned}
q^D(t,x,y) &= \int_D q^D(t/2,x,w) q^D(t/2,w,y) m(dw) \\
&\le c_7^2 \mathbb{P}^x(\zeta > t) \widehat{\mathbb{P}}^y(\widehat{\zeta} > t) \int_{\mathfrak{X}} p(t/2,x,w) p(t/2,w,y) m(dw) \\
&\le c_7^2 C_0 \mathbb{P}^x(\zeta > t) \widehat{\mathbb{P}}^y(\widehat{\zeta} > t) \widetilde{q}(t,x,y).
\end{aligned}$$

(2) For the lower bound, we use the notation $\mathcal{W}$ as before. By the semigroup property, we see that

$$\begin{aligned}
q^D(t,x,y) &= \int_D \int_D q^D(t/3,x,u) q^D(t/3,u,w) q^D(t/3,w,y) m(dw) m(du) \\
&\ge \int_{\mathcal{W}(x,t/3)} \int_{\mathcal{W}(y,t/3)} q^D(t/3,x,u) q^D(t/3,u,w) q^D(t/3,w,y) m(dw) m(du).
\end{aligned}$$

Observe that for all $(u,w) \in \mathcal{W}(x,t/3) \times \mathcal{W}(y,t/3)$,

$$\delta_D(u) \wedge \delta_D(w) \ge 4^{-1} \kappa (a_l/3)^{1/\delta_l} r, \quad |\rho(u,w) - \rho(x,y)| \le 6(3a_u)^{1/\delta_u} r. \tag{3.1.22}$$

Here is an explanation of the last inequality above, the others being similar.

By the triangle inequality and symmetry, it suffices to show that $\rho(u,x) \le 3(3a_u)^{1/\delta_u} r$. Since $\mathcal{W}(x,t/3) \subset B(x, 3r_{t/3})$, this will be so provided that $r_{t/3} \le (3a_u)^{1/\delta_u} r$. But this immediately follows from (3.1.5) by estimating $\Phi^{-1}(t/3)/\Phi^{-1}(t)$. By considering cases $\rho(x,y) > 12(3a_u)^{1/\delta_u} r$ and $\rho(x,y) \le 12(3a_u)^{1/\delta_u} r$ separately, we get from Theorem 3.1.13, (3.1.6) and (3.1.22) that for all $(u,w) \in \mathcal{W}(x,t/3) \times \mathcal{W}(y,t/3)$,

$$\begin{aligned} q^D(t/3,u,w) &\simeq \widetilde{q}(t/3,u,w) \simeq \left(\frac{1}{V(u,\Phi^{-1}(t))} \wedge \frac{t}{V(u,\rho(w,u))\Phi(\rho(w,u))} \right) \\ &\simeq \widetilde{q}(t,x,y). \end{aligned}$$

Let $c_8 := 8^{-1}\kappa(a_l/3)^{1/\delta_l}$. By Theorem 3.1.13, (3.1.6) and (3.1.22), for all $(s,u) \in (t/6,t/3) \times \mathcal{W}(x,t/3)$ and $w \in B(u, c_8 r)$, we have $q^D(s,w,u) \simeq \widetilde{q}(s,w,u) \simeq 1/V(u,r)$. Besides, by (3.1.20), (3.1.10), (3.1.1) and (3.1.3), we see that for all $u \in \mathcal{W}(x,t/3)$ and $(v,w) \in \mathcal{U}(x,t/3) \times B(u,c_8 r)$,

$$J(v,w) \simeq \frac{1}{V(v,r)\Phi(r)} \simeq \frac{1}{V(x,r)\Phi(r)}.$$

Therefore, by (3.1.21) and Lemma 3.1.19, we get that for all $u \in \mathcal{W}(x,t/3)$,

$$\begin{aligned} & q^D(t/3,x,u) \\ & \ge \mathbb{E}^x[q^D(t/3 - \tau_{\mathcal{U}(x,t/3)}, Y_{\tau_{\mathcal{U}(x,t/3)}}, u) : \tau_{\mathcal{U}(x,t/3)} < t/6, Y_{\tau_{\mathcal{U}(x,t/3)}} \in B(u,c_8 r)] \\ & \ge \int_0^{t/6} \int_{\mathcal{U}(x,t/3)} \int_{B(u,c_8 r)} q^{\mathcal{U}(x,t/3)}(s,x,v) J(v,w) q^D(t/3 - s, w, u) m(dw) m(dv) ds \\ & \ge \frac{c_9}{V(u,r)V(x,r)\Phi(r)} \int_0^{t/6} \int_{B(u,c_8 r)} \mathbb{P}^x(\tau_{\mathcal{U}(x,t/3)} > s) m(dw) ds \\ & \ge \frac{c_9 V(u,c_8 r)}{V(u,r)V(x,r)\Phi(r)} \mathbb{P}^x(\tau_{\mathcal{U}(x,t/3)} > t/6) \int_0^{t/6} ds \\ & \ge \frac{c_{10} t}{V(x,r) t} \mathbb{P}^x(\tau_{\mathcal{U}(x,t/3)} > t/6) \simeq \frac{1}{V(x,r)} \mathbb{P}^x(\zeta > t). \end{aligned}$$

Similarly for $w \in \mathcal{W}(y,t/3)$, $q^D(t/3,w,y) \ge c_{11} \frac{1}{V(y,r)} \widehat{\mathbb{P}}^y(\widehat{\zeta} > t)$.

Finally, using (3.1.1) and (3.1.3), we conclude that

$$\begin{aligned} q^D(t,x,y) &\geq c_{12} \inf_{u\in\mathcal{W}(x,t/3),\, w\in\mathcal{W}(y,t/3)} \big(q^D(t/3,x,u)q^D(t/3,u,w)q^D(t/3,w,y)\big) \\ &\quad \times \int_{\mathcal{W}(x,t/3)} m(du) \int_{\mathcal{W}(y,t/3)} m(dw) \\ &\geq c_{13}\mathbb{P}^x(\zeta > t)\widehat{\mathbb{P}}^y(\widehat{\zeta} > t)\widetilde{q}(t,x,y). \end{aligned}$$

We have finished the proof. □

Using Theorem 3.1.13 and Corollary 3.1.14(ii), the following global estimates can be proved by the same argument. We omit the proof.

Theorem 3.1.21. *Let D be a κ-fat set with characteristics (∞,κ). Suppose that $\mu \in \mathbf{K}_1(D)$ and $R_0 = m(\mathfrak{X}) = \widetilde{T} = r_0 = \infty$, where r_0 is the constant in Assumption* **U**. *Then there exists $c_1(\kappa) \geq 1$ such that for all $(t,x,y) \in (0,\infty)\times D \times D$,*

$$c_1^{-1} \leq \frac{q^D(t,x,y)}{\mathbb{P}^x(\zeta > t)\widehat{\mathbb{P}}^y(\widehat{\zeta} > t)\widetilde{q}(t,x,y)} \leq c_1.$$

Example 3.1.22. *Suppose that $(\mathfrak{X},\rho,m)$ is an unbounded Ahlfors regular n-space for some $n\in(0,\infty)$, that is, for all $x\in\mathfrak{X}$ and $r\in(0,1]$, $m(B(x,r)) \simeq r^n$. Assume that ρ is uniformly equivalent to the shortest-path metric in $\mathfrak{X}$. Suppose that there is a diffusion process ξ with a symmetric, continuous transition density $p^\xi(t,x,y)$ satisfying the following sub-Gaussian bounds*

$$\begin{aligned} \frac{c_1}{t^{n/d_w}} \exp\left(-c_2\left(\frac{\rho(x,y)^{d_w}}{t}\right)^{1/(d_w-1)}\right) &\leq p^\xi(t,x,y) \\ \leq \frac{c_3}{t^{n/d_w}} \exp\left(-c_4\left(\frac{\rho(x,y)^{d_w}}{t}\right)^{1/(d_w-1)}\right), \end{aligned}$$

for all $x,y\in\mathfrak{X}$ and $t\in(0,\infty)$. Here $d_w \geq 2$ is the walk dimension of the space $\mathfrak{X}$. Examples of ξ include Brownian motions on unbounded Riemannian manifolds, Brownian motions on Sierpinski gaskets, Sierpinski carpets *or more general fractals. Let $\alpha \in (0, d_w)$ and T be an (α/d_w)-stable subordinator independent of ξ. We define a process X by $X_t = \xi_{T_t}$. Then X is a symmetric Feller process. It is easy to check that X has a transition density $p(t,x,y)$ satisfying*

$$p(t,x,y) \simeq \left(t^{-n/\alpha} \wedge \frac{t}{\rho(x,y)^{n+\alpha}} \right), \tag{3.1.23}$$

for all $x, y \in \mathfrak{X}$ and $t \in (0,\infty)$. It follows from [25, Appendix A] that Assumptions **A** *and* **U** *above are also satisfied with $\Phi(r) = r^\alpha$. Therefore, by Theorem 3.1.20 and* (3.1.23)*, if D is a κ-fat open set in $\mathfrak{X}$ and $\mu \in \mathbf{K}_1(D)$, then for all $(t,x,y) \in (0,r_0) \times D \times D$,*

$$q^D(t,x,y) \simeq \mathbb{P}^x(\zeta > t)\mathbb{P}^y(\zeta > t) \left(t^{-n/\alpha} \wedge \frac{t}{\rho(x,y)^{n+\alpha}} \right).$$

3.2 Heat kernel estimates of regional fractional Laplacian with critical killing

An open subset $D \subset \mathbb{R}^d$ $(d \geq 2)$ is said to be a $C^{1,1}$ open set if there exist a localization radius $R_2 > 0$ and a constant $\Lambda > 0$ such that for every $z \in \partial D$, there is a $C^{1,1}$ function $\Gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\Gamma(0) = 0, \nabla\Gamma(0) = (\widetilde{0}, 0), \|\Gamma\|_\infty \leq \Lambda, |\nabla\Gamma(y) - \nabla\Gamma(z)| \leq \Lambda|y-z|$ and an orthonormal coordinate system $CS_z : x = (\tilde{x}, x_d) := (x_1, ..., x_{d-1}, x_d)$ with origin at z such that

$$D \cap B(z, R_0) = \{x \in B(0, R_0) \text{ in } CS_z : x_d > \Gamma(\tilde{x})\}.$$

A $C^{1,1}$ open set in $\mathbb{R}$ is the union of disjoint intervals so that the minimum of their lengths and the distances between them is positive.

In this section we assume that $d \geq 2$, $\mathfrak{X}$ is either the closure of a $C^{1,1}$ open subset D of $\mathbb{R}^d$ or $\mathbb{R}^d$ itself, and the underlying process is either a reflected α-stable process in $\overline{D}$ (or a non-local perturbation of it), or an α-stable process in $\mathbb{R}^d$ (or a drift perturbation of it). We investigate heat kernel estimates under critical killing. We first recall the definition of reflected α-stable processes.

Let $0 < \alpha < 2$ and $\mathcal{A}_{d,\alpha} = \alpha 2^{\alpha-1}\pi^{-d/2}\Gamma((d+\alpha)/2)\Gamma(1-\alpha/2)^{-1}$. Here Γ is the gamma function defined by $\Gamma(\lambda) := \int_0^\infty t^{\lambda-1}e^{-t}dt$, $\lambda > 0$. For a $C^{1,1}$ open subset D of $\mathbb{R}^d$, let $(\overline{\mathcal{E}}, \overline{\mathcal{F}})$ be the Dirichlet space on $L^2(D, dx)$ defined by

$$\begin{aligned}\overline{\mathcal{F}} &:= \left\{u \in L^2(D);\, \int_D\int_D \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\,dxdy < \infty\right\},\\ \overline{\mathcal{E}}(u,v) &:= \frac{1}{2}\mathcal{A}(d,\,-\alpha)\int_D\int_D \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+\alpha}}\,dxdy, \quad u,v\in\overline{\mathcal{F}}.\end{aligned}$$

It is well known that $W^{\alpha/2,2}(D) = \overline{\mathcal{F}}$ and the Sobolev norm $\|\cdot\|_{\alpha/2,2;D}$ is equivalent to $\sqrt{\overline{\mathcal{E}}_1}$ where $\overline{\mathcal{E}}_1 := \overline{\mathcal{E}} + (\,\cdot\,,\,\cdot\,)_{L^2(D)}$. As noted in [16], $(\,\overline{\mathcal{E}},\,\overline{\mathcal{F}}\,)$ is a regular Dirichlet form on $\overline{D}$ and its associated Hunt process X lives on $\overline{D}$. We call the process X a reflected α-stable process in $\overline{D}$. When D is the whole $\mathbb{R}^d$, X is simply an α-stable process.

It follows from [45] that X admits a strictly positive and jointly continuous transition density $p(t,x,y)$ with respect to the Lebesgue measure dx and that

$$p(t,x,y) \simeq \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right), \qquad (t,x,y) \in (0,1)\times\overline{D}\times\overline{D}.$$

When $\alpha \in (1,2)$, the killed process X^D is the censored stable process in D. When $\alpha \in (0,1]$, it follows from [16, Section 2] that, starting from inside D, the process X neither hits nor approaches ∂D at any finite time. Thus, the killed process X^D is simply X restricted to D (without killing).

We will see that, for all $\alpha \in (0,2)$, the killed isotropic α-stable process Z^D can be obtained from X^D through a Feynman-Kac perturbation of the form (3.2.6) with κ satisfying (3.2.5).

It follows from [36] that, when $\alpha \in (1,2)$, the transition density $p^X_D(t,x,y)$

of X^D has the following estimates:

$$p_D^X(t,x,y) \simeq \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha-1} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha-1} \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) \tag{3.2.1}$$

for $(t,x,y) \in (0,1) \times D \times D$.

It follows from [35] that the transition density $p_D^Z(t,x,y)$ of Z^D has the following estimates:

$$p_D^Z(t,x,y) \simeq \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha/2} \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) \tag{3.2.2}$$

for $(t,x,y) \in (0,1) \times D \times D$.

In Subsection 3.2.1, we will establish explicit (Dirichlet) heat kernel estimates under critical killing, which also provides an alternative and unified proof of (3.2.1) and (3.2.2). In Subsection 3.2.2, we consider non-local perturbations of $(\overline{\mathcal{E}}, \overline{\mathcal{F}})$ when D is a bounded $C^{1,1}$ open set. Subsection 3.2.3 covers the case $D = \mathbb{R}^d \setminus \{0\}$ and drift perturbations.

3.2.1 $C^{1,1}$ open set

In this subsection, we assume that D is a $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics (R_2, Λ), and that X is a reflected α-stable process in $\overline{D}$. Without loss of generality, we will always assume that $\Lambda \geq 1$. It is easy to check that the process X satisfies the assumptions in Subsection 3.1.1 and Assumption **U**.

Recall that $\mathbb{R}^d_+ := \{y = (\widetilde{y}, y_d) \in \mathbb{R}^d : y_d > 0\}$. For $d \geq 2$ and $p \in \mathbb{R}$, we define $w_p(y) = (y_d)_+^p$ for $y \in \mathbb{R}^d$. According to [16, (5.4)], we have for $z \in \mathbb{R}^d_+$,

$$\mathcal{A}_{d,\alpha} \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}^d_+, |y-z|>\epsilon} \frac{w_p(y) - w_p(z)}{|y-z|^{d+\alpha}} dy = C(d,\alpha,p) z_d^{p-\alpha}, \tag{3.2.3}$$

where $C(d,\alpha,p) := \mathcal{A}_{d,\alpha} \frac{\omega_{d-1}}{2} \beta(\frac{\alpha+1}{2}, \frac{d-1}{2}) \gamma(\alpha,p)$, $\beta(\cdot,\cdot)$ is the beta function, ω_{d-1} is the $(d-2)$-dimensional Lebesgue measure of the unit sphere in $\mathbb{R}^{d-1}$

and
$$\gamma(\alpha,p)=\int_0^1\frac{(t^p-1)(1-t^{\alpha-p-1})}{(1-t)^{1+\alpha}}dt.$$
Observe that
$$\frac{d\gamma(\alpha,p)}{dp}=\int_0^1\frac{(t^{\alpha-p-1}-t^p)|\log t|}{(1-t)^{1+\alpha}}dt$$
is positive for $p>(\alpha-1)/2$ and thus $p\mapsto\gamma(\alpha,p)$ is strictly increasing on $((\alpha-1)/2,\alpha)$. Moreover, we have

$$C(d,\alpha,\alpha-1)=C(d,\alpha,0)=0 \quad\text{and}\quad \lim_{p\uparrow\alpha}C(d,\alpha,p)=\infty. \tag{3.2.4}$$

Let $\mathcal{H}_\alpha$ be the collection of non-negative functions κ on D with the property that there exist constants $C_1,C_2\geq 0$ and $\eta\in[0,\alpha)$ such that $\kappa(x)\leq C_2$ for all $x\in D$ with $\delta_D(x)\geq 1$ and

$$|\kappa(x)-C_1\delta_D(x)^{-\alpha}|\leq C_2\delta_D(x)^{-\eta}, \tag{3.2.5}$$

for all $x\in D$ with $\delta_D(x)<1$. If $\alpha\leq 1$, then we further assume that $C_1>0$. It follows from (3.2.4) that we can find a unique $p\in[\alpha-1,\alpha)\cap(0,\alpha)$ such that $C_1=C(d,\alpha,p)$. For each $p\in[\alpha-1,\alpha)\cap(0,\alpha)$, we define

$$\mathcal{H}_\alpha(p):=\{\kappa\in\mathcal{H}_\alpha:\ \text{the constant } C_1 \text{ in (3.2.5) is } C(d,\alpha,p)\}.$$

Note that $\mathcal{H}_\alpha=\cup_{p\in[\alpha-1,\alpha)\cap(0,\alpha)}\mathcal{H}_\alpha(p)$. We fix $\kappa\in\mathcal{H}_\alpha(p)$ and let Y be a Hunt process on D corresponding to the Feynman-Kac semigroup of X^D through the multiplicative functional $e^{-\int_0^t\kappa(X_s^D)ds}$. That is,

$$\mathbb{E}^x\left[f(Y_t)\right]=\mathbb{E}^x\left[e^{-\int_0^t\kappa(X_s^D)ds}f(X_t^D)\right],\quad t\geq 0, x\in D. \tag{3.2.6}$$

Since, by Example 3.1.15, $\kappa(x)dx\in\mathbf{K}_1(D)$, it follows from Theorem 3.1.20 that Y has a transition density $q^D(t,x,y)$ with the following estimate:

$$q^D(t,x,y)\simeq\mathbb{P}^x(\zeta>t)\mathbb{P}^y(\zeta>t)\left(t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}}\right), \tag{3.2.7}$$

for $(t,x,y) \in (0,1] \times D \times D$. To get explicit estimate of $\mathbb{P}^x(\zeta > t)$, we will estimate $\mathbb{P}^x(Y_{\tau_{\mathcal{U}(x,t)}} \in D)$ and use Lemma 3.1.19.

For $f \in C_c^2(D)$, define

$$L^\alpha f(x) := \mathcal{A}_{d,\alpha} \lim_{\epsilon \downarrow 0} \int_{D, |y-x|>\epsilon} \frac{f(y)-f(x)}{|y-x|^{d+\alpha}} dy,$$
$$Lf(x) := L^\alpha f(x) - \kappa(x) f(x).$$

The above operator L coincides with the restriction to $C_c^2(D)$ of the generator of the transition semigroup of Y in $C_0(D)$.

For $q \in \mathbb{R}$, define $h_q : \mathbb{R}^d \to [0,\infty)$ by

$$h_q(x) = \delta_D(x)^q.$$

Lemma 3.2.1. *Let $p \in [\alpha - 1, \alpha) \cap (0, \alpha)$ and suppose $\kappa \in \mathcal{H}_\alpha(p)$. Then for any $q \in [p, \alpha)$, there exist constants $A_1 > 0$ and $A_2 \in (0, 1/4)$ depending only on $p, q, d, \alpha, \Lambda, C_2, \eta, R_2$ such that the following inequalities hold:*

(i) If $q > p$, then

$$A_1^{-1} \delta_D(x)^{q-\alpha} \le Lh_q(x) \le A_1 \delta_D(x)^{q-\alpha}$$

for every $x \in D$ with $0 < \delta_D(x) < A_2$.

(ii) If $q = p$, then

$$|Lh_p(x)| \le A_1 (\delta_D(x)^{p-\eta} + |\log \delta_D(x)|)$$

for every $x \in D$ with $0 < \delta_D(x) < A_2$.

Proof. Without loss of generality, we assume $R_2 = 1$. Let $x \in D$ with $\delta_D(x) < A_2$ where the constant $A_2 \in (0, 1/4)$ will be chosen later. Let $z \in \partial D$ be a point such that $\delta_D(x) = |x - z|$. Then there exist a $C^{1,1}$ function $\psi : \mathbb{R}^{d-1} \to \mathbb{R}$ such that $\psi(z) = \nabla \psi(z) = 0$ and an orthonormal coordinate system CS_z such that $z = 0$ and $x = (\widetilde{x}, x_d) = (0, x_d)$ in CS_z, and that

$$D \cap B(z,1) = \{y = (\widetilde{y}, y_d) \text{ in } CS_z : y_d > \psi(\widetilde{y})\} \cap B(z,1).$$

Observe that

$$\begin{aligned} Lh_q(x) &= L^\alpha h_q(x) - C(d,\alpha,p)x_d^{q-\alpha} - (\kappa(x) - C(d,\alpha,p)x_d^{-\alpha})h_q(x) \\ &=: I - II - III. \end{aligned}$$

By (3.2.5), it holds that $|III| \le C_2 x_d^{q-\eta}$.

For any open subset $U \subset \mathbb{R}^d$, define

$$\kappa_U(z) = \mathcal{A}_{d,\alpha} \int_{U^c} \frac{dy}{|y-z|^{d+\alpha}}, \quad z \in U.$$

Recall that $w_q(y) = (y_d)^q$ for $y \in \mathbb{R}^d_+$ and $w_q(y) = 0$ otherwise. Since $h_q(x) = w_q(x) = x_d^q$, by (3.2.3), we have

$$\begin{aligned} I &= \mathcal{A}_{d,\alpha} \lim_{\epsilon \downarrow 0} \left[\int_{\mathbb{R}^d, |y-x|>\epsilon} \frac{h_q(y) - h_q(x)}{|y-x|^{d+\alpha}} dy \right] + \kappa_D(x) h_q(x) \\ &= \mathcal{A}_{d,\alpha} \lim_{\epsilon \downarrow 0} \left[\int_{\mathbb{R}^d, |y-x|>\epsilon} \frac{h_q(y) - w_q(y)}{|y-x|^{d+\alpha}} dy + \int_{\mathbb{R}^d, |y-x|>\epsilon} \frac{w_q(y) - w_q(x)}{|y-x|^{d+\alpha}} dy \right] \\ &\quad + \kappa_D(x) w_q(x) \\ &= \mathcal{A}_{d,\alpha} \lim_{\epsilon \downarrow 0} \left[\int_{\mathbb{R}^d, |y-x|>\epsilon} \frac{h_q(y) - w_q(y)}{|y-x|^{d+\alpha}} dy \right] + C(d,\alpha,q) x_d^{q-\alpha} \\ &\quad + (\kappa_D(x) - \kappa_{\mathbb{R}^d_+}(x)) w_q(x). \end{aligned}$$

According to [16, Lemma 5.6], if $1 < \alpha < 2$, then there is $c = c(d,\alpha,\Lambda) > 0$ such that $|\kappa_D(x) - \kappa_{\mathbb{R}^d_+}(x)| \le c x_d^{1-\alpha}$. By a similar calculation as in [16, Lemma 5.6], one can show that for $\alpha \le 1$, $|\kappa_D(x) - \kappa_{\mathbb{R}^d_+}(x)| \le c(|\log x_d| \mathbf{1}_{\{\alpha=1\}} + 1)$. Thus, for any $0 < \alpha < 2$, since $q \ge p \ge (\alpha-1)_+$, we get

$$|(\kappa_D(x) - \kappa_{\mathbb{R}^d_+}(x)) w_q(x)| \le c x_d^q (x_d^{1-\alpha} + |\log x_d|) \le c. \tag{3.2.8}$$

Now, we bound $I_\epsilon := \int_{\mathbb{R}^d, |y-x|>\epsilon} \frac{h_q(y)-w_q(y)}{|y-x|^{d+\alpha}} dy$. Since D is a $C^{1,1}$ open set, there is a constant $r_0 = r_0(d,\Lambda) \in (0,1)$ such that $B_1 := B(r_0\mathbf{e}_d, r_0) \subset D$ and $B_2 := B(-r_0\mathbf{e}_d, r_0) \subset D^c$ where $\mathbf{e}_d := (\tilde{0}, 1)$. We define

$$
\begin{aligned}
E &:= \{y = (\widetilde{y}, y_d) : |\widetilde{y}| < r_0/4,\ |y_d| < r_0/2\}, \\
E_1 &:= \{y \in E : y_d > 2|\widetilde{y}|^2\}, \qquad E_2 := \{y \in E : y_d < -2|\widetilde{y}|^2\}.
\end{aligned}
$$

It is easy to check that $E_1 \subset B_1 \cap E \subset D$ and $E_2 \subset B_2 \cap E \subset D^c$. Thus, since $h_q(y) = w_q(y) = 0$ for $y \in E_2$, we get

$$
\begin{aligned}
I_\epsilon = &\int_{E^c, |y-x|>\epsilon} \frac{h_q(y) - w_q(y)}{|y-x|^{d+\alpha}} dy + \int_{E_1, |y-x|>\epsilon} \frac{h_q(y) - w_q(y)}{|y-x|^{d+\alpha}} dy \\
&+ \int_{E \setminus (E_1 \cup E_2), |y-x|>\epsilon} \frac{h_q(y) - w_q(y)}{|y-x|^{d+\alpha}} dy =: J_{1,\epsilon} + J_{2,\epsilon} + J_{3,\epsilon}.
\end{aligned}
$$

Recall that $x = (\widetilde{0}, x_d)$ and $x_d = \delta_D(x) < A_2$. We take A_2 smaller than $r_0/4$. Then we see that for every $y = (\widetilde{y}, y_d)$ with $|y_d| \ge r_0/2 > 2x_d$,

$$
|y-x|^2 \ge |\widetilde{y}|^2 + (|y_d| - |x_d|)^2 \ge |y|^2/4
$$

and for every $y = (\widetilde{y}, y_d)$ with $|\widetilde{y}| \ge r_0/4 > x_d$,

$$
|y-x|^2 - \frac{1}{4}|y|^2 = \frac{3}{4}|\widetilde{y}|^2 + \frac{3}{4}|y_d - \frac{4}{3}x_d|^2 - \frac{1}{3}x_d^2 \ge \frac{3}{4}|\widetilde{y}|^2 - \frac{1}{3}x_d^2 > 0.
$$

Therefore, for every $y \in E^c$, $|y-x| \ge |y|/2$. Since $|h_q(y) - w_q(y)| \le 2|y|^q$, it follows that for all $\epsilon \in (0,1)$,

$$
|J_{1,\epsilon}| \le 2^{1+d+\alpha} \int_{E^c} |y|^{q-d-\alpha} dy \le 2^{1+d+\alpha} \int_{|y|>r_0/4} |y|^{q-d-\alpha} dy = c.
$$

Next, for every $y \in D$, using the inequality $(a+b)^{1/2} \le a^{1/2} + b/(2a^{1/2})$ for $a, b > 0$, we get that

$$
\delta_D(y) \le \mathrm{dist}(y, B_2) = ((y_d + r_0)^2 + |\widetilde{y}|^2)^{1/2} - r_0 \le y_d + r_0 + \frac{|\widetilde{y}|^2}{2(y_d + r_0)} - r_0
$$

$$\le y_d + \frac{|\widetilde{y}|^2}{2r_0} < \left(1 + \frac{1}{4r_0}\right) y_d. \tag{3.2.9}$$

Thus, by the mean value theorem and a change of the variables, it holds that for all $\epsilon \in (0,1)$,

$$\begin{aligned}
J_{2,\epsilon} &\le \int_{E_1, |y-x|>\epsilon} \frac{(y_d + |\widetilde{y}|^2/(2r_0))^q - y_d^q}{|y-x|^{d+\alpha}} dy \\
&\le \frac{q}{2r_0} \int_{E_1, |y-x|>\epsilon} \frac{|\widetilde{y}|^2 \sup_{y_d \le s \le (1+1/(4r_0)) y_d} s^{q-1}}{|y-x|^{d+\alpha}} dy \\
&\le c \int_{E_1, |y-x|>\epsilon} \frac{|\widetilde{y}|^2 y_d^{q-1}}{|y-x|^{d+\alpha}} dy = c x_d^{q+1-\alpha} \int_{B(0,1/x_d)} \frac{|\widetilde{u}|^2 u_d^{q-1}}{|u - \mathbf{e}_d|^{d+\alpha}} du.
\end{aligned}$$

Besides, since $\delta_D(y) \ge \delta_{B_1}(y) \ge r_0 - ((r_0 - y_d)^2 + |\widetilde{y}|^2)^{1/2}$ for every $y \in E_1$, using the mean value theorem and a change of the variables again, we get that for all $\epsilon \in (0,1)$,

$$\begin{aligned}
J_{2,\epsilon} &\ge -\int_{E_1, |y-x|>\epsilon} \frac{y_d^q - (r_0 - ((r_0 - y_d)^2 + |\widetilde{y}|^2)^{1/2})^q}{|y-x|^{d+\alpha}} dy \\
&\ge -\frac{q}{2} \int_{E_1, |y-x|>\epsilon} \frac{|\widetilde{y}|^2}{|y-x|^{d+\alpha}} \Big(\sup_{\lambda \in [0, |\widetilde{y}|^2]} \frac{(r_0 - ((r_0 - y_d)^2 + \lambda)^{1/2})^{q-1}}{((r_0 - y_d)^2 + \lambda)^{1/2}} \Big) dy \\
&\ge -\frac{q}{r_0} \int_{E_1, |y-x|>\epsilon} \frac{|\widetilde{y}|^2 y_d^{q-1}}{|y-x|^{d+\alpha}} dy = -\frac{q}{r_0} x_d^{q+1-\alpha} \int_{B(0,1/x_d)} \frac{|\widetilde{u}|^2 u_d^{q-1}}{|u - \mathbf{e}_d|^{d+\alpha}} du
\end{aligned}$$

Using the inequality $|u - \mathbf{e}_d| \ge |u|/4$ for $u \in \mathbb{R}^d \setminus B(0,2)$, since $(\alpha - 1)_+ \le q < \alpha$, we see that

$$\begin{aligned}
&\int_{B(0,1/x_d)} \frac{|\widetilde{u}|^2 u_d^{q-1}}{|u - \mathbf{e}_d|^{d+\alpha}} du = \int_{B(0,2)} \frac{|\widetilde{u}|^2 u_d^{q-1}}{|u - \mathbf{e}_d|^{d+\alpha}} du + \int_{B(0,1/x_d) \setminus B(0,2)} \frac{|\widetilde{u}|^2 u_d^{q-1}}{|u - \mathbf{e}_d|^{d+\alpha}} du \\
&\le 2^{q-1} \int_{B(0,2)} |u - \mathbf{e}_d|^{2-d-\alpha} du + 4^{d+\alpha} \int_{B(0,1/x_d) \setminus B(0,2)} |u|^{q+1-d-\alpha} du \\
&\le c \Big(\int_0^2 l^{1-\alpha} dl + \int_2^{x_d^{-1}} l^{q-\alpha} dl \Big) \le c(1 + |\log x_d|).
\end{aligned}$$

Therefore, we deduce that $|J_{2,\epsilon}| \le c(1 + |\log x_d|)$ for all $\epsilon \in (0,1)$.

It remains to bound $|J_{3,\epsilon}|$. Denote by $m_{d-1}(dx)$ the $(d-1)$-dimensional

Hausdorff measure. We observe that there is $c_1 > 0$ such that

$$m_{d-1}(\{y : |\widetilde{y}| = l, -2|\widetilde{y}|^2 \le y_d \le 2|\widetilde{y}|^2\}) \le c_1 l^d \quad \text{for all } 0 < l < 1.$$

For every $y \in E \setminus (E_1 \cup E_2)$, by (3.2.9), we see that $|h_q(y)|$, $|w_q(y)| \le (y_d + |\widetilde{y}|^2/(2r_0))^q \le (2+1/(2r_0))^q |\widetilde{y}|^{2q}$. Therefore, since $q \ge (\alpha-1)_+$, it holds that

$$|J_{3,\epsilon}| \le c \int_0^1 \int_{|\widetilde{y}|=l, y\in E\setminus(E_1\cup E_2)} l^{2q-d-\alpha} m_{d-1}(dy) dl \le c \int_0^1 l^{2q-\alpha} dl \le c.$$

Combining the above estimates, we conclude that $|I_\epsilon| \le c(1+\log|x_d|)$.

We have proved that

$$|Lh_q(x)| \le (C(d,\alpha,q) - C(d,\alpha,p)) x_d^{q-\alpha} + c(1+\log|x_d|) + C_2 x_d^{q-\eta}. \quad (3.2.10)$$

When $q > p$, we note that $C(d,\alpha,q) > C(d,\alpha,p)$ and $q-\alpha < 0 \wedge (q-\eta)$. Hence, the desired result follows by taking A_2 small enough. When $q = p$, we get the result immediately from (3.2.10). $\Box$

Fix $q \in (p,\alpha)$ such that $q < p - \eta + \alpha$. Then define $A_3 := A_1(p) \vee A_1(q)$, $A_4 := A_2(p) \wedge A_2(q)$, where A_1 and A_2 are the constants in Lemma 3.2.1, and

$$v_1(x) := h_p(x) + h_q(x).$$

By Lemma 3.2.1, for any $x \in D$ with $\delta_D(x) < A_4$, we have

$$Lv_1(x) \ge A_3^{-1} \delta_D(x)^{q-\alpha} - A_3(\delta_D(x)^{p-\eta} + |\log \delta_D(x)|).$$

Thus, there exist $A_5 \in (0, A_4)$ and $A_6 > 0$ such that

$$Lv_1(x) \ge 2A_6 \delta_D(x)^{q-\alpha} \quad \text{for all } x \in D \text{ with } \delta_D(x) < A_5. \quad (3.2.11)$$

Define $v_2(x) := h_p(x) - \frac{1}{2} h_q(x)$. By the same argument, we can find $A_7 \in (0, A_4)$ and $A_8 > 0$ such that $Lv_2(x) \le -2A_8 \delta_D(x)^{q-\alpha}$ for all $x \in D$ with $\delta_D(x) < A_7$.

Now, we are ready to estimate $\mathbb{P}^x(Y_{\tau_{\mathcal{U}(x,t)}} \in D)$. We continue to assume $R_2 = 1$. Note that D is a κ-fat open set with characteristics $(1, \kappa)$. Recall that r_t is defined as $r_t = \Phi^{-1}(t)R_1/(3\Phi^{-1}(T))$ in Subsection 3.1.6. Since in the current setting $\Phi(t) = t^{1/\alpha}$, we can take $r_t = t^{1/\alpha}/3$ in the definition of $\mathcal{U}(x,t)$. Let $A_9 \in (0, A_5/2]$ be a constant which will be chosen later. Without loss of generality we assume $\kappa < A_7 \wedge A_9$.

Fix $(t,x) \in (0,1] \times D$. If $\delta_D(x) \geq \kappa t^{1/\alpha}/3$, then we have $\mathbb{P}^x(Y_{\tau_{\mathcal{U}(x,t)}} \in D) \simeq 1$ in view of Lemma 3.1.19 and (3.1.18). Recall that $z_{x,t} \in D$ is a point such that $B(z, 3\kappa r_t) \subset B(x, 3r_t) \cap D$. Assume that $\delta_D(x) < \kappa t^{1/\alpha}/3$. In this case, we have $|x - z_{x,t}| \geq \delta_D(z_{x,t}) - \delta_D(x) > \kappa t^{1/\alpha} - \kappa t^{1/\alpha}/3 > \kappa t^{1/\alpha}/2$ and hence we should choose the second definition of $\mathcal{U}(x,t)$ so that $\mathcal{U}(x,t) = B(x, \kappa t^{1/\alpha}/3) \cap D$. Let $w \in \partial D$ be the point such that $|x - w| = \delta_D(x)$. Define

$$D^{\text{bdry}}(l) := \{y \in D : |y - w| < l\}, \quad D^{\text{int}}(l) := \{y \in D^{\text{bdry}}(2) : \delta_D(y) > l\}.$$

Note that $\mathcal{U}(x,t) \subset D^{\text{bdry}}(\kappa) \subset D^{\text{bdry}}(A_9)$ by the triangle inequality and the assumption that $\kappa < A_9$.

Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ be a non-negative radial function such that $\varphi(y) = 0$ for $|y| > 1$ and $\int_{\mathbb{R}^d} \varphi(y)dy = 1$. For $k \geq 1$, we set $\varphi_k(y) := 6^{kd}\varphi(6^k y)$ and

$$f_k(y) := \varphi_k * (v_1 \mathbf{1}_{D^{\text{int}}(5^{-k})})(y) = \int_{D^{\text{int}}(5^{-k}) \cap B(y, 6^{-k})} \varphi_k(y-u)v_1(u)du, \quad y \in \mathbb{R}^d.$$

Since $f_k(y) = 0$ if $y \notin D^{\text{int}}(5^{-k} - 6^{-k})$, we see that $f_k \in C_c^\infty(D)$ and hence Lf_k is well-defined everywhere. Pick any $z \in \mathcal{U}(x,t)$ (hence $\delta_D(z) < A_9$) such that $\delta_D(z) > \max\{2^{-k/(q-p)}, 2^{-pk/(d+q)}\} =: a_k$ and observe that

$$\begin{aligned}
Lf_k(z) &= L(\varphi_k * v_1)(z) - L(\varphi_k * v_1 - f_k)(z) \\
&= L(\varphi_k * v_1)(z) + \kappa(z)(\varphi_k * v_1 - f_k)(z) \\
&\quad + \mathcal{A}_{d,\alpha} \lim_{\epsilon \downarrow 0} \int_{D, |y-z|>\epsilon} \frac{f_k(y) - (\varphi_k * v_1)(y) - f_k(z) + (\varphi_k * v_1)(z)}{|y-z|^{d+\alpha}} dy \\
&=: M_1(z) + M_2(z) + M_3(z) = M_1 + M_2 + M_3.
\end{aligned}$$

Define $D-u := \{y-u : y \in D\}$ for $u \in \mathbb{R}^d$. Using Fubini's theorem and a change of the variables, we see that

$$\begin{aligned}
M_1 &= \lim_{\epsilon\downarrow 0} \mathcal{A}_{d,\alpha} \int_{D,|y-z|>\epsilon} \int_{\mathbb{R}^d} \frac{v_1(y-u)-v_1(z-u)}{|y-z|^{d+\alpha}} \varphi_k(u)dudy - \kappa(z)(\varphi_k * v_1)(z) \\
&= \lim_{\epsilon\downarrow 0} \int_{\mathbb{R}^d} \left(\mathcal{A}_{d,\alpha} \int_{D-u,|y-(z-u)|>\epsilon} \frac{v_1(y)-v_1(z-u)}{|y-(z-u)|^{d+\alpha}} dy \right) \varphi_k(u)du \\
&\quad - \int_{\mathbb{R}^d} \kappa(z-u)v_1(z-u)\varphi_k(u)du + \int_{\mathbb{R}^d} (\kappa(z-u)-\kappa(z))v_1(z-u)\varphi_k(u)du \\
&= \lim_{\epsilon\downarrow 0} \int_{\mathbb{R}^d} I_{z,\epsilon}(u)\varphi_k(u)du + \int_{\mathbb{R}^d} (\kappa(z-u)-\kappa(z))v_1(z-u)\varphi_k(u)du, \quad (3.2.12)
\end{aligned}$$

where $I_{z,\epsilon}(u) := \mathcal{A}_{d,\alpha} \int_{D-u,|y-(z-u)|>\epsilon} \frac{v_1(y)-v_1(z-u)}{|y-(z-u)|^{d+\alpha}} dy - \kappa(z-u)v_1(z-u)$.

To bound $\int_{\mathbb{R}^d} I_{z,\epsilon}(u)\varphi_k(u)du$, we need some preparation. For $|u| < 6^{-k}$, let $w_u \in \partial D$ be a point such that $\delta_D(z-u) = |z-u-w_u|$. By the triangle inequality and the assumption that $\delta_D(z) > a_k \geq 2^{-k}$, we have

$$(3^k-1)|u| < \delta_D(z)-|u| \leq \delta_D(z-u) \leq \delta_D(z)+|u| \leq (1+3^{-k})\delta_D(z). \quad (3.2.13)$$

Let $\psi_u : \mathbb{R}^{d-1} \to \mathbb{R}$ be a $C^{1,1}$ function and CS_{w_u} an orthonormal coordinate system with origin at w_u such that $\psi_u(\widetilde{0}) = 0$, $\nabla\psi_u(\widetilde{0}) = \widetilde{0}$, $\|\nabla\psi_u\|_\infty \leq \Lambda$, the coordinate of $z-u$ in CS_{w_u} is $(\widetilde{0}, \delta_D(z-u))$ and $D \cap B(w_u,1) = \{y^u = (\widetilde{y}^u, y^u_d) \text{ in } CS_{w_u} : y^u_d > \psi_u(\widetilde{y}^u)\} \cap B(w_u,1)$. Using the coordinate system CS_{w_u}, we have that for all $q_0 \in [p,\alpha)$, $\epsilon \in (0,1)$ and $|u| < 6^{-k}$,

$$\begin{aligned}
&\left| \int_{D-u,|y-(z-u)|>\epsilon} \frac{h_{q_0}(y)-h_{q_0}(z-u)}{|y-(z-u)|^{d+\alpha}} dy \right| \\
&\leq \left| \int_{B(z-u,\epsilon)^c} \frac{h_{q_0}(y^u)-\delta_D(z-u)^{q_0}}{|y^u-(z-u)|^{d+\alpha}} dy^u \right| + \int_{(D-u)^c} \frac{|h_{q_0}(y^u)-\delta_D(z-u)^{q_0}|}{|y^u-(z-u)|^{d+\alpha}} dy^u \\
&\leq \left| \int_{B(z-u,\epsilon)^c} \frac{h_{q_0}(y^u)-(y^u_d)^{q_0}_+}{|y^u-(z-u)|^{d+\alpha}} dy^u \right| + \left| \int_{B(z-u,\epsilon)^c} \frac{(y^u_d)^{q_0}_+-\delta_D(z-u)^{q_0}}{|y^u-(z-u)|^{d+\alpha}} dy^u \right| \\
&\quad + \int_{B(z-u,\delta_D(z))^c} \frac{|y^u|^{q_0}+\delta_D(z-u)^{q_0}}{|y^u-(z-u)|^{d+\alpha}} dy^u \\
&=: N_1(z,u,\epsilon) + N_2(z,u,\epsilon) + N_3(z,u) = N_1+N_2+N_3.
\end{aligned}$$

By the proof of Lemma 3.2.1 and (3.2.13), we see that for all $|u| < 6^{-k}$ and $\epsilon \in (0,1)$, $N_1 \le c_1(1 + |\log \delta_D(z-u)|) \le c_1(2 + |\log \delta_D(z)|)$. Moreover, by [16, p.120-121] and (3.2.13), we obtain $N_2 \le c_2 \delta_D(z-u)^{q_0-\alpha} \le c_3 \delta_D(z)^{q_0-\alpha}$ uniformly in $\epsilon \in (0,1)$. Lastly, using the triangle inequality $|y^u| \le |y^u - (z-u)^u| + |(z-u)^u| = |y^u - (z-u)^u| + \delta_D(z-u)$, we also have

$$\begin{aligned} N_3 &\le c_4 \int_{\delta_D(z)}^{\infty} \big((l + \delta_D(z-u))^{q_0} + \delta_D(z-u)^{q_0}\big) l^{-\alpha-1} dl \\ &\le c_4 \int_{\delta_D(z)}^{\infty} \big((l + 2\delta_D(z))^{q_0} + 2^{q_0}\delta_D(z)^{q_0}\big) l^{-\alpha-1} dl \le c_5 \delta_D(z)^{q_0-\alpha}. \end{aligned}$$

Thus, we conclude that for all $q_0 \in [p, \alpha)$ there exists $c_6 = c_6(q_0) > 0$ such that for all $|u| < 6^{-k}$ and $\epsilon \in (0,1)$,

$$\left| \int_{D-u, |y-(z-u)|>\epsilon} \frac{h_{q_0}(y) - h_{q_0}(z-u)}{|y-(z-u)|^{d+\alpha}} dy \right| \le c_6 \delta_D(z)^{q_0-\alpha}. \tag{3.2.14}$$

Therefore, $I_{z,\epsilon}(u)$ converges as $\epsilon \downarrow 0$ uniformly in $|u| < 6^{-k}$, and by (3.2.11), it holds that for all large k such that $6^{-k} < A_5/2$ and all $|u| < 6^{-k}$,

$$\begin{aligned} \lim_{\epsilon \downarrow 0} I_{z,\epsilon}(u) &= Lv_1(z-u) - \mathcal{A}_{d,\alpha} v_1(z-u) \int_{(D-u)\setminus D} \frac{dy}{|y-(z-u)|^{d+\alpha}} \\ &\quad + \mathcal{A}_{d,\alpha} \int_{D \setminus (D-u)} \frac{v_1(z-u) - v_1(y)}{|y-(z-u)|^{d+\alpha}} dy \\ &\ge 2A_6 \delta_D(z-u)^{q-\alpha} - \mathcal{A}_{d,\alpha} v_1(z-u) \int_{(D-u)\setminus D} \frac{dy}{|y-(z-u)|^{d+\alpha}}. \end{aligned}$$

The inequality above is valid since for all $|u| < 6^{-k}$ and $y \in D \setminus (D-u)$, by (3.2.13), it holds that $\delta_D(y) \le |u| \le \delta_D(z-u)$, implying $v_1(z-u) \ge v_1(y)$. Observe that by (3.2.13), for all $|u| < 6^{-k}$,

$$\begin{aligned} &\int_{(D-u)\setminus D} \frac{dy}{|y-(z-u)|^{d+\alpha}} \\ &\le \int_{((D-u)\setminus D) \cap B(w_u,1)} \frac{dy}{|y-(z-u)|^{d+\alpha}} + \int_{B(w_u,1)^c} \frac{dy}{|y-(z-u)|^{d+\alpha}} \end{aligned}$$

$$\begin{aligned}
&\le \int_{|\widetilde{y}^u|<\delta_D(z-u)/(2\Lambda)} \int_{\psi_u(\widetilde{y}^u)-|u|}^{\psi_u(\widetilde{y}^u)} \frac{1}{(\delta_D(z-u)-y^u_d)^{d+\alpha}} dy^u_d d\widetilde{y}^u \\
&\quad + \int_{\delta_D(z-u)/(2\Lambda)\le|\widetilde{y}^u|\le 1} \int_{\psi_u(\widetilde{y}^u)-|u|}^{\psi_u(\widetilde{y}^u)} \frac{1}{|\widetilde{y}^u|^{d+\alpha}} dy^u_d d\widetilde{y}^u + 2^{d+\alpha}\int_{B(0,1)^c} \frac{dy}{|y|^{d+\alpha}} \\
&\le \frac{2^{d+\alpha}|u|}{\delta_D(z-u)^{d+\alpha}} \int_{|\widetilde{y}^u|<\delta_D(z-u)/(2\Lambda)} d\widetilde{y}^u + |u| \int_{\delta_D(z-u)/(2\Lambda)\le|\widetilde{y}^u|\le 1} \frac{d\widetilde{y}^u}{|\widetilde{y}^u|^{d+\alpha}} + c \\
&\le A_{10}(6^{-k}\delta_D(z-u)^{-\alpha-1}+1),
\end{aligned}$$

for some constant $A_{10}>0$. We used the facts that the coordinates of $z-u$ in CS_{w_u} are $(\widetilde{0},\delta_D(z-u))$, and $\delta_D(z-u)<1/2$ by (3.2.13) so that for all $y^u \in B(w_u,1)^c$, $|y^u-(z-u)| \ge |y^u-w_u| - |(z-u)-w_u| \ge |y^u| - \delta_D(z-u) \ge |y^u|/2$ in the second inequality above, and the fact that for all $|\widetilde{y}^u| < \delta_D(z-u)/(2\Lambda)$, $|\psi_u(\widetilde{y}^u)| \le \|\nabla\psi_u\|_\infty |\widetilde{y}^u| \le \delta_D(z-u)/2$ in the third inequality. Hence, since $\lim_{k\to\infty} 6^{-k}a_k^{p-q-1} = 0$, $p-q-1<0$ and $p+\alpha>\alpha>q$, by taking $A_9 < (A_6/(6\mathcal{A}_{d,\alpha}A_{10}))^{1/(p-q+\alpha)}$, we get that for all sufficiently large k and all $|u|<6^{-k}$,

$$\begin{aligned}
\lim_{\epsilon\downarrow} I_{z,\epsilon}(u) &\ge 2A_6\delta_D(z-u)^{q-\alpha} - \mathcal{A}_{d,\alpha}A_{10}\big(6^{-k}\delta_D(z-u)^{-\alpha-1}+1\big)v_1(z-u) \\
&\ge 2A_6\delta_D(z-u)^{q-\alpha} - 2\mathcal{A}_{d,\alpha}A_{10}\big(6^{-k}\delta_D(z-u)^{p-\alpha-1}+\delta_D(z-u)^p\big) \\
&\ge \big(2A_6 - 4\mathcal{A}_{d,\alpha}A_{10}\big(6^{-k}a_k^{p-q-1}+A_9^{p-q+\alpha}\big)\big)\delta_D(z-u)^{q-\alpha} \\
&\ge A_6\delta_D(z-u)^{q-\alpha} \ge A_6(\delta_D(z)+|u|)^{q-\alpha}.
\end{aligned}$$

We used the fact that $v_1(z-u) \le 2\delta_D(z-u)^p$ in the second, and (3.2.13) and the fact that $a_k < \delta_D(z) < A_9$ in the third inequality above.

Now, since the support of φ_k is contained in $B(0,6^{-k})$, using the dominated convergence theorem (which is applicable due to (3.2.14)), (3.2.5) and (3.2.13), we get from (3.2.12) that for all sufficiently large k,

$$\begin{aligned}
M_1 &\ge A_6\int_{\mathbb{R}^d}(\delta_D(z)+|u|)^{q-\alpha}\varphi_k(u)du + \int_{\mathbb{R}^d}(\kappa(z-u)-\kappa(z))v_1(z-u)\varphi_k(u)du \\
&\ge A_6\int_{\mathbb{R}^d}(\delta_D(z)+|u|)^{q-\alpha}\varphi_k(u)du
\end{aligned}$$

$$\begin{aligned}
&+ C_1 \int_{\mathbb{R}^d} (\delta_D(z-u)^{-\alpha} - \delta_D(z)^{-\alpha}) v_1(z-u)\varphi_k(u)du \\
&- C_2 \int_{\mathbb{R}^d} (\delta_D(z-u)^{-\eta} + \delta_D(z)^{-\eta}) v_1(z-u)\varphi_k(u)du \\
\geq\ & A_6(1+3^{-k})^{q-\alpha}\delta_D(z)^{q-\alpha} - C_1(1-(1+3^{-k})^{-\alpha})\delta_D(z)^{-\alpha}(\varphi_k * v_1)(z) \\
&- C_2(1+(1-3^{-k})^{-\eta})\delta_D(z)^{-\eta}(\varphi_k * v_1)(z).
\end{aligned}$$

Since $(\varphi_k * v_1)(z) \leq 2(1+3^{-k})^q \delta_D(z)^p$, $q < p - \eta + \alpha$ and $2^{-k/(q-p)} \leq a_k < \delta_D(z) < A_9$, by taking $A_9 < (A_6/(432C_2))^{1/(p-\eta+\alpha-q)}$, it follows that for all sufficiently large k,

$$\begin{aligned}
M_1 &\geq \frac{7}{8}A_6\delta_D(z)^{q-\alpha} - 3\big(\alpha C_1 3^{-k}\delta_D(z)^{p-q} + 3C_2\delta_D(z)^{p-\eta+\alpha-q}\big)\delta_D(z)^{q-\alpha} \\
&\geq \left(\frac{7}{8}A_6 - 3\alpha C_1 3^{-k} a_k^{p-q} - 9C_2 A_9^{p-\eta+\alpha-q}\right)\delta_D(z)^{q-\alpha} \geq \frac{5}{6}A_6\delta_D(z)^{q-\alpha}.
\end{aligned}$$

Next, we calculate M_2. Note that for every $k \geq 2$, $u \in B(0, 6^{-k})$ and $y \in D$ such that $\delta_D(y) > 4^{-k}$ and $|y-w| \leq 1$, we have $\delta_D(y-u) \geq 4^{-k} - 6^{-k} > 5^{-k}$ and $|y-u-w| \leq |y-w| + |u| < 2$, and therefore

$$1 - \mathbf{1}_{D^{\text{int}}(5^{-k})}(y-u) = 0. \tag{3.2.15}$$

In particular, since φ_k is supported in $B(0, 6^{-k})$, $\delta_D(z) > 2^{-pk/(d+q)} > 4^{-k}$ and $|z-w| \leq |z-x| + |x-w| < 2t^{1/\alpha}/3 < 1$, for all $k \geq 2$, we have

$$M_2 = \kappa(z)\int_{\mathbb{R}^d} \big(1 - \mathbf{1}_{D^{\text{int}}(5^{-k})}(z-u)\big) v_1(z-u)\varphi_k(u)du = 0.$$

Finally, using (3.2.15), by taking A_9 sufficiently smaller than A_6, for all k large enough, we have

$$\begin{aligned}
&|M_3| \leq \mathcal{A}_{d,\alpha} \lim_{\epsilon \downarrow 0} \int_{D, |y-z|>\epsilon} \int_{\mathbb{R}^d} \varphi_k(u) \frac{\big(1 - \mathbf{1}_{D^{\text{int}}(5^{-k})}(y-u)\big) v_1(y-u)}{|y-z|^{d+\alpha}} dudy \\
&\leq c_7 \int_{D, \delta_D(y) \leq 4^{-k}} \int_{\mathbb{R}^d} \varphi_k(u) \frac{\delta_D(y-u)^p}{|y-z|^{d+\alpha}} dudy
\end{aligned}$$

$$
\begin{aligned}
&\quad + c_7 \int_{D,|y-w|>1} \int_{\mathbb{R}^d} \varphi_k(u) \frac{(\delta_D(y-u)+1)^q}{|y-z|^{d+\alpha}} dudy \\
&\le c_7 \int_{D,\delta_D(y)\le 4^{-k}} \int_{\mathbb{R}^d} \varphi_k(u) \frac{(\delta_D(y)+|u|)^p}{|y-z|^{d+\alpha}} dudy \\
&\quad + c_7 \int_{|y-w|>1} \int_{\mathbb{R}^d} \varphi_k(u) \frac{(|y-w|+|u|+1)^q}{(3^{-1}|y-w|)^{d+\alpha}} dudy \\
&\le c_8 \left(\int_{D,\delta_D(y)\le 4^{-k}} \frac{4^{-pk}}{|y-z|^{d+\alpha}} dy + \int_{|y-w|>1} \frac{dy}{|y-w|^{d+\alpha-q}} \right) \\
&\le c_8 \left(\int_{D\cap B(z,1),\delta_D(y)\le 4^{-k}} \frac{4^{-pk}}{(\delta_D(z)-\delta_D(y))^{d+\alpha}} dy + \int_{|y-z|\ge 1} \frac{4^{-pk}}{|y-z|^{d+\alpha}} dy + c \right) \\
&\le c_9 \big(4^{-pk}\delta_D(z)^{-d-\alpha}+1\big) \le c_3\big(4^{-pk}2^{pk}+A_9^{\alpha-q}\big)\delta_D(z)^{q-\alpha} \le \frac{A_6}{2}\delta_D(z)^{q-\alpha}.
\end{aligned}
$$

In the second inequality above, we have used the facts that $\delta_D(y)^q \le \delta_D(y)^p$ for $\delta_D(y) < 1$ and $\delta_D(y)^q + 1 \ge \delta_D(y)^p$ for all y, since $q > p$. In the third inequality, we first estimate $|z-w| \le |z-x| + |x-w| < 2/3 < (2/3)|y-w|$ by using that $|y-w| > 1$, which implies that $|y-z| \ge |y-w| - |z-w| \ge (1/3)|y-w|$. The estimate $\delta_D(y-u) \le \delta_D(y) + |u| \le |y-w| + |u|$, follows by the choice of $w \in \partial D$. In the fourth inequality, we have used the facts that the support of φ_k is contained in $B(0, 6^{-k})$ and $\int_{\mathbb{R}^d} \varphi_k(u)du = 1$. The sixth and seventh inequalities are valid since $\delta_D(z) \le |z-w| < A_9$ and $\delta_D(z) > a_k \ge 2^{-pk/(d+q)}$.

Now we conclude that, for all sufficiently large k, $Lf_k(z) \ge 3^{-1}A_6\delta_D(z)^{q-\alpha}$ for all $z \in \mathcal{U}(x,t)$ such that $\delta_D(z) > a_k$. Recall that $f_k \in C_c^\infty(D)$ and hence contained in the domain of the generator of Y. Thus, by Dynkin's formula, it holds that for all sufficiently large k,

$$
\begin{aligned}
f_k(x) &= \mathbb{E}^x\big[f_k(Y_{\tau_{\mathcal{U}(x,t)\cap D^{\text{int}}(a_k)}})\big] - \mathbb{E}^x\left[\int_0^{\tau_{\mathcal{U}(x,t)\cap D^{\text{int}}(a_k)}} Lf_k(Y_t)dt\right] \\
&\le \mathbb{E}^x\big[f_k(Y_{\tau_{\mathcal{U}(x,t)\cap D^{\text{int}}(a_k)}})\big].
\end{aligned}
$$

Since $f_k = \varphi_k * (v_1 \mathbf{1}_{D^{\text{int}}(5^{-k})}) \to v_1 \mathbf{1}_{D^{\text{bdry}}(2)} \le v_1$ pointwise and $Y_{\tau_{\mathcal{U}(x,t)\cap D^{\text{int},2}(a_k)}} \to Y_{\tau_{\mathcal{U}(x,t)}}$ (using $\mathcal{U}(x,t) \subset D^{\text{bdry}}(2)$), it follows from the bounded convergence theorem,

$$\begin{aligned}\delta_D(x)^p \le v_1(x) &= \lim_{k\to\infty} f_k(x) \le \lim_{k\to\infty} \mathbb{E}^x\big[f_k(Y_{\tau_{\mathcal{U}(x,t)\cap D^{\text{int}}(a_k)}})\big] \\ &= \mathbb{E}^x\big[v_1(Y_{\tau_{\mathcal{U}(x,t)}}) : Y_{\tau_{\mathcal{U}(x,t)}} \in D^{\text{bdry}}(2)\big] \le \mathbb{E}^x\big[v_1(Y_{\tau_{\mathcal{U}(x,t)}})\big].\end{aligned}$$

Recall that we have assumed $\kappa < A_7 \wedge A_9 < A_5 \wedge A_7$. Set $r = r(t) := (A_5 \wedge A_7)t^{1/\alpha} > \kappa t^{1/\alpha}$. Note that for every $n \ge 1$ and $u \in D^{\text{bdry}}(2^n r)$, we have $v_1(u) \le (\delta_D(x) + 2^n r)^p + (\delta_D(x) + 2^n r)^q \le 2^{(n+1)p} r^p + 2^{(n+1)q} r^q \le 2^{(n+1)q+1} r^p$. Thus, we have

$$\begin{aligned}\mathbb{E}^x\big[v_1(Y_{\tau_{\mathcal{U}(x,t)}})\big] &\le \mathbb{E}^x\big[v_1(Y_{\tau_{\mathcal{U}(x,t)}}) : Y_{\tau_{\mathcal{U}(x,t)}} \in D^{\text{bdry}}(r)\big] \\ &\quad + \sum_{n=0}^{\infty} \mathbb{E}^x\big[v_1(Y_{\tau_{\mathcal{U}(x,t)}}) : Y_{\tau_{\mathcal{U}(x,t)}} \in D^{\text{bdry}}(2^{n+1}r) \setminus D^{\text{bdry}}(2^n r)\big] \\ &\le c_4 r^p \mathbb{P}^x\big(Y_{\tau_{\mathcal{U}(x,t)}} \in D^{\text{bdry}}(r)\big) \\ &\quad + c_4 \sum_{n=0}^{\infty} 2^{(n+1)q+1} r^p \mathbb{P}^x\big(Y_{\tau_{\mathcal{U}(x,t)}} \in D^{\text{bdry}}(2^{n+1}r) \setminus D^{\text{bdry}}(2^n r)\big)\end{aligned}$$

and that for every $n \ge 0$,

$$\begin{aligned}&\mathbb{P}^x\big(Y_{\tau_{\mathcal{U}(x,t)}} \in D^{\text{bdry}}(2^{n+1}r) \setminus D^{\text{bdry}}(2^n r)\big) \\ &\quad \le c_5 \mathbb{E}^x \int_0^{\tau_{\mathcal{U}(x,t)}} \int_{D^{\text{bdry}}(2^{n+1}r)\setminus D^{\text{bdry}}(2^n r))} |Y_s - z|^{-d-\alpha} dz ds \\ &\quad \le c_6 (2^{n+1} r)^d (2^n r)^{-d-\alpha} \mathbb{E}^x\big[\tau_{\mathcal{U}(x,t)}\big] = c_3 2^{-n\alpha} r^{-\alpha} \mathbb{E}^x\big[\tau_{\mathcal{U}(x,t)}\big].\end{aligned}$$

Since

$$\begin{aligned}\mathbb{P}^x\big(Y_{\tau_{\mathcal{U}(x,t)}} \in D^{\text{bdry}}(r)\big) &\ge c_7 \mathbb{E}^x\left[\int_0^{\tau_{\mathcal{U}(x,t)}} \int_{D^{\text{bdry}}(r)} |Y_s - z|^{-d-\alpha} dz ds\right] \\ &\ge c_8 r^{-\alpha} \mathbb{E}^x[\tau_{\mathcal{U}(x,t)}],\end{aligned}$$

and $q < \alpha$, we deduce that

$$\frac{\delta_D(x)^p}{r^p} \le c_9 \mathbb{P}^x\big(Y_{\tau_{\mathcal{U}(x,t)}} \in D^{\mathrm{bdry}}(r)\big)\left(1 + \sum_{n=0}^{\infty} 2^{n(q-\alpha)}\right) \le c_{10} \mathbb{P}^x\big(Y_{\tau_{\mathcal{U}(x,t)}} \in D\big).$$

By applying the similar argument to the function $g_k := \varphi_k * (v_2 \mathbf{1}_{D^{\mathrm{int}}(5^{-k})})$, we also have that

$$\begin{aligned}\delta_D(x)^p \ge v_2(x) &= \lim_{k\to\infty} g_k(x) \ge \lim_{k\to\infty} \mathbb{E}^x\big[g_k(Y_{\tau_{\mathcal{U}(x,t)\cap D^{\mathrm{int}}(a_k)}})\big] \\ &= \mathbb{E}^x[(v_2 \mathbf{1}_{D^{\mathrm{bdry}}(2)})(Y_{\tau_{\mathcal{U}(x,t)}})] \ge \frac{1}{2} r^p \mathbb{P}^x(Y_{\tau_{\mathcal{U}(x,t)}} \in \mathcal{W}(x,t)).\end{aligned}$$

The last inequality holds since $\mathcal{W}(x,t) \subset D^{\mathrm{bdry}}(2)$.

Therefore, in view of Lemma 3.1.19, we get $\mathbb{P}^x\big(\zeta > t\big) \simeq (\delta_D(x)/r)^p$. Finally, from (3.2.7) we conclude that

Theorem 3.2.2. *Suppose that D is a $C^{1,1}$ open set in $\mathbb{R}^d$, $d \ge 2$, with characteristics (R_2, Λ). For all $T > 0$, $p \in [\alpha-1, \alpha) \cap (0, \alpha)$ and $\eta \in [0, \alpha)$, there exists a constant $c = c(C_1, C_2, p, \alpha, d, \eta, T, R_2, \Lambda) \ge 1$ such that for all $\kappa \in \mathcal{H}_\alpha(p)$, the transition density $q^D(t,x,y)$ of the Hunt process Y on D corresponding to the Feynman-Kac semigroup of X^D via the multiplicative functional $e^{-\int_0^t \kappa(X_s^D)ds}$ satisfies that*

$$\begin{aligned}&c^{-1}\left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^p \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^p \left[t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right] \\ &\le q^D(t,x,y) \le c\left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^p \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^p \left[t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right]\end{aligned}$$

for all $(t,x,y) \in (0,T) \times D \times D$.

In the case $D = \mathbb{R}^d_+$ and $\kappa(x) = C(d,\alpha,p) x_d^{-\alpha}$, one can use the scaling property to get that the two-sided heat estimates in Theorem 3.2.2 is valid for all $t > 0$.

Remark 3.2.3. Theorem 3.2.2 also holds in $d = 1$. In fact, let $D \subset \mathbb{R}$ be a union of open intervals with a localization radius r_0 and $C(1,\alpha,p) =$

$\mathcal{A}(1,-\alpha)\gamma(\alpha,p)$. The first difference of the proof appears in the bound of $|III|$ in Lemma 3.2.1. We use the following calculation instead of [16, Lemma 5.6]: for every $x \in D$ with $\delta_D(x) < r_0/2$,

$$\begin{aligned}|\kappa_D(x) - \kappa_{\mathbb{R}_+}(x)| &\le |\kappa_{(-r_0,0)^c}(x) - \kappa_{(0,r_0)}(x)| \\ &= \mathcal{A}(1,-\alpha)\left(\int_{-\infty}^{-r_0} + \int_{r_0}^{\infty} \frac{dy}{|y-x|^{1+\alpha}}\right) \\ &= \frac{\mathcal{A}(1,-\alpha)}{\alpha}((r_0-x)^{-\alpha} + (r_0+x)^{-\alpha}) \le c.\end{aligned}$$

Moreover, the bound for $|I_\epsilon|$ is easy in Lemma 3.2.1: Since $h_q(y) = w_q(y)$ for $y \in (-\infty, r_0)$, $I_\epsilon \le c\int_{r_0}^\infty y^{q-1-\alpha}dy = c$.

Remark 3.2.4. It follows from [16, pp.94–95] that Z^D can be obtained from X^D via a Feynman-Kac perturbation of the form $e^{-\int_0^t \kappa_D(X_s^D)ds}$. In view of (3.2.8), κ_D satisfies condition (3.2.5) with $C_1 = \frac{\mathcal{A}_{d,\alpha}}{\alpha}\frac{\omega_{d-1}}{2}\beta(\frac{\alpha+1}{2}, \frac{d-1}{2})$. By direct calculation, we can see that $\gamma(\alpha, \alpha/2) = 1/\alpha$. This means that $C_1 = C(d,\alpha,\alpha/2)$. Thus Theorem 3.2.2 recovers (3.2.2). When $\alpha \in (1,2)$, $C_1 = 0 = C(d,\alpha,\alpha-1)$ is allowed. Thus, by taking $\kappa = 0$, Theorem 3.2.2 recovers (3.2.1) as well.

We also remark here that Theorem 3.2.2 provides examples of processes studied in [42] (see (3.2.5) and [42, Proposition 4.1(ii)]).

3.2.2 Non-local perturbation in bounded $C^{1,1}$ open set

Recall that $\mathcal{A}(d,\alpha) = \alpha 2^{\alpha-1}\pi^{-d/2}\Gamma(\frac{d+\alpha}{2})\Gamma(1-\frac{\alpha}{2})^{-1}$. We also recall that we write $y = (\widetilde{y}, y_d)$ for $y \in \mathbb{R}^d$, and for $p \in \mathbb{R}$, the function $w_p : \mathbb{R}^d \to [0,\infty)$ is defined by $w_p(y) = (y_d)_+^p$. For $u : \mathbb{R}_+^d \to [0,\infty)$, $\lambda \in (0,\infty)$ and $\beta \in (-\infty, 2)$, we define

$$L_{d,\lambda}^\beta u(x) := \lim_{\epsilon \downarrow 0} \int_{\{y\in\mathbb{R}_+^d:\, \epsilon<|y-x|<\lambda\}} (u(y)-u(x))\frac{dy}{|x-y|^{d+\beta}}, \quad x \in \mathbb{R}_+^d.$$

Lemma 3.2.5. *For all positive p, λ and $\beta \in (-\infty, 2)$, there exist $c_1 = c_1(p,d,\beta,\lambda) > 0$ and $c_2 = c_2(p,d,\beta,\lambda) \in (0, 1/4)$ such that, for every $x \in \mathbb{R}_+^d$*

with $0 < x_d < c_2$*, the following inequalities hold:*

$$|L^\beta_{d,\lambda} w_p(x)| \le c_1 \begin{cases} 1 & \text{if } p > \beta; \\ |\log x_d| & \text{if } p = \beta; \\ x_d^{p-\beta} & \text{if } p < \beta. \end{cases}$$

Proof. When $\beta \le 0$, then clearly for $x \in \mathbb{R}^d_+$,

$$\int_{\mathbb{R}^d_+} \frac{|y_d^p - x_d^p|}{|y-x|^{d+\beta}} \mathbf{1}_{\{|y-x|<\lambda\}}\, dy \le c \int_{B(0,\lambda)} |z|^{-d-\beta+p} dz \le c \int_0^\lambda s^{-1-\beta+p} ds = c\lambda^{p-\beta}.$$

We now assume $\beta > 0$. For simplicity, take $x = (\widetilde{0}, x_d)$ and denote $\mathbf{e}_d = (\tilde{0}, 1)$. Then by the change of variables $z = y/x_d$, we have

$$\begin{aligned} L^\beta_{d,\lambda} w_p(x) =& \text{p.v.} \int_{\mathbb{R}^d_+} \frac{y_d^p - x_d^p}{|y-x|^{d+\beta}} \mathbf{1}_{\{|y-x|<\lambda\}}\, dy \\ =& x_d^{p-\beta} \text{p.v.} \int_{\mathbb{R}^{d-1}} \int_0^\infty \frac{z_d^p - 1}{|z-\mathbf{e}_d|^{d+\beta}} \mathbf{1}_{\{|z-\mathbf{e}_d|<\lambda/x_d\}} dz_d d\widetilde{z} =: x_d^{p-\beta} I_1 \,. \end{aligned}$$

Using the change of variables $\widetilde{z} = |z_d - 1|\widetilde{u}$, we get

$$\begin{aligned} I_1 =& \int_{\mathbb{R}^{d-1}} \frac{1}{(|\widetilde{u}|^2+1)^{(d+\beta)/2}} \left(\text{p.v.} \int_0^\infty \frac{z_d^p - 1}{|z_d - 1|^{1+\beta}} \mathbf{1}_{\{|z_d-1|<\lambda(|\widetilde{u}|^2+1)^{-1/2}/x_d\}} dz_d \right) d\widetilde{u} \\ =:& \int_{\mathbb{R}^{d-1}} \frac{1}{(|\widetilde{u}|^2+1)^{(d+\beta)/2}} I_2 d\widetilde{u} \,. \end{aligned}$$

Fix $\widetilde{u}$ and let $M := (|\widetilde{u}|^2+1)^{1/2}$. Then

$$I_2 = \lim_{\epsilon \to 0} \left(\int_{(1-\frac{\lambda}{Mx_d})_+}^{1-\epsilon} \frac{z_d^p - 1}{|z_d-1|^{1+\beta}} dz_d + \int_{1+\epsilon}^{\frac{Mx_d+\lambda}{Mx_d}} \frac{z_d^p - 1}{|z_d-1|^{1+\beta}} dz_d \right). \quad (3.2.16)$$

By using the change of variables $w = 1/z_d$, we get that, for $\epsilon < \lambda/(Mx_d)$, the second integral in (3.2.16) is equal to

$$\int_{\frac{Mx_d}{Mx_d+\lambda}}^{1-\epsilon} \frac{w^{\beta-1-p}-w^{\beta-1}}{(1-w)^{1+\beta}}\,dw + \int_{1-\epsilon}^{\frac{1}{1+\epsilon}} \frac{w^{\beta-1-p}-w^{\beta-1}}{(1-w)^{1+\beta}}\,dw\,.$$

Note that from [16, p.121], we see that

$$\left|\int_{1-\epsilon}^{\frac{1}{1+\epsilon}} \frac{w^{\beta-1-p}-w^{\beta-1}}{(1-w)^{1+\beta}}\,dw\right| \le c\epsilon^{2-\beta}\,.$$

By writing the first integral in (3.2.16) as

$$\int_{\frac{Mx_d}{Mx_d+\lambda}}^{1-\epsilon} \frac{w^p-1}{(1-w)^{1+\beta}}dw - \int_{(1-\frac{\lambda}{Mx_d})_+}^{\frac{Mx_d}{Mx_d+\lambda}} \frac{1-w^p}{(1-w)^{1+\beta}}dw,$$

and by using

$$(w^p-1)+(w^{\beta-1-p}-w^{\beta-1}) = (1-w^p)(1-w^{p-(\beta-1)})w^{\beta-1-p}, \qquad (3.2.17)$$

we have

$$\begin{aligned} I_2 &= \lim_{\epsilon\to 0}\int_{\frac{Mx_d}{Mx_d+\lambda}}^{1-\epsilon} \frac{(1-w^p)(1-w^{p-(\beta-1)})}{(1-w)^{1+\beta}}w^{\beta-1-p}\,dw - \int_{(1-\frac{\lambda}{Mx_d})_+}^{\frac{Mx_d}{Mx_d+\lambda}} \frac{1-w^p}{(1-w)^{1+\beta}}dw \\ &=: I_{21} - I_{22}\,. \end{aligned} \qquad (3.2.18)$$

First, it is easy to see that

$$0 < I_{22} \le \int_0^{\frac{Mx_d}{Mx_d+\lambda}} \frac{1-w^p}{(1-w)^{1+\beta}}dw \le c\begin{cases} 1 & \text{if } \beta\in(0,1); \\ \log(1+Mx_d/\lambda) & \text{if } \beta=1; \\ (1+Mx_d/\lambda)^{\beta-1} & \text{if } \beta\in(1,2). \end{cases}$$

Next, since $\beta < 2$, the fraction in I_{21} is integrable near 1. Thus,

$$I_{21} = \int_{\frac{Mx_d}{Mx_d+\lambda}}^{1} \frac{(1-w^p)(1-w^{p-(\beta-1)})}{(1-w)^{1+\beta}}w^{\beta-1-p}\,dw.$$

Note that, if $\frac{Mx_d}{Mx_d+\lambda} \geq 1/4$, then clearly, $I_{21} \leq c < \infty$. If $\frac{Mx_d}{Mx_d+\lambda} < 1/4$, then

$$I_{21} \leq c + \int_{\frac{Mx_d}{Mx_d+\lambda}}^{1/2} \frac{(1-w^p)(1-w^{p-(\beta-1)})}{(1-w)^{1+\beta}} w^{\beta-1-p}\, dw \leq c + c\int_{\frac{Mx_d}{Mx_d+\lambda}}^{1} w^{\beta-1-p}\, dw.$$

Thus

$$I_{21} \leq c \begin{cases} (1+\lambda/(Mx_d))^{p-\beta} & \text{if } p > \beta; \\ \log(1+\lambda/(Mx_d)) & \text{if } p = \beta. \end{cases}$$

Therefore, if $p > \beta$, then for small x_d,

$$\begin{aligned}
&|x_d^{p-\beta} I_1| \\
&\leq c x_d^{p-\beta} \int_{\mathbb{R}^{d-1}} \frac{1}{(|\widetilde{u}|^2+1)^{(d+\beta)/2}} \times \Bigg[\left(1 + \frac{\lambda}{(|\widetilde{u}|^2+1)^{1/2} x_d}\right)^{p-\beta} \\
&\qquad + \mathbf{1}_{\beta\in[1,2)} \left(1 + \frac{(|\widetilde{u}|^2+1)^{1/2} x_d}{\lambda}\right)^{\beta-1} \log\left(e + \frac{(|\widetilde{u}|^2+1)^{1/2} x_d}{\lambda}\right) \Bigg] d\widetilde{u} \\
&\leq c \int_{\mathbb{R}^{d-1}} \frac{1}{(|\widetilde{u}|^2+1)^{(d+\beta)/2}} \times \Bigg[\left(x_d + \frac{\lambda}{(|\widetilde{u}|^2+1)^{1/2}}\right)^{p-\beta} \\
&\qquad + \mathbf{1}_{\beta\in[1,2)} \left(1 + \frac{(|\widetilde{u}|^2+1)^{1/2}}{\lambda}\right)^{\beta-1} \log\left(e + \frac{(|\widetilde{u}|^2+1)^{1/2}}{\lambda}\right) \Bigg] d\widetilde{u} \\
&\leq c(\lambda) \int_{\mathbb{R}^{d-1}} \left[\frac{1}{(|\widetilde{u}|^2+1)^{(d+\beta)/2}} + \mathbf{1}_{\beta\in[1,2)} \frac{\log\left(e + (|\widetilde{u}|^2+1)^{1/2}\right)}{(|\widetilde{u}|^2+1)^{(d+1)/2}} \right] d\widetilde{u} \\
&= c(\lambda, \beta) < \infty.
\end{aligned}$$

If $p = \beta > 0$, then for small x_d,

$$\begin{aligned}
|I_1| &\leq c \int_{\mathbb{R}^{d-1}} \frac{1}{(|\widetilde{u}|^2+1)^{(d+\beta)/2}} \times \Bigg[\log\left(1 + \frac{\lambda}{(|\widetilde{u}|^2+1)^{1/2} x_d}\right) \\
&\qquad + \mathbf{1}_{\beta\in[1,2)} \left(1 + \frac{(|\widetilde{u}|^2+1)^{1/2} x_d}{\lambda}\right)^{\beta-1} \log\left(e + \frac{(|\widetilde{u}|^2+1)^{1/2} x_d}{\lambda}\right) \Bigg] d\widetilde{u} \\
&\leq c \int_{\mathbb{R}^{d-1}} \frac{1}{(|\widetilde{u}|^2+1)^{(d+\beta)/2}} \log\left(1 + \frac{\lambda}{x_d}\right) d\widetilde{u} \\
&\quad + c(\lambda) \mathbf{1}_{\beta\in[1,2)} \int_{\mathbb{R}^{d-1}} \frac{1}{(|\widetilde{u}|^2+1)^{(d+1)/2}} \log\left(e + (|\widetilde{u}|^2+1)^{1/2}\right) d\widetilde{u}
\end{aligned}$$

$$\leq c(\lambda)(|\log x_d| + 1).$$

We now assume that $0 < p < \beta$. Note that by (3.2.18), (3.2.17) and simple algebra, the limit I_2 in (3.2.16) is equal to

$$\int_{(1-\frac{\lambda}{Mx_d})_+}^{1} \frac{(1-w^p)(1-w^{p-(\beta-1)})}{(1-w)^{1+\beta}} w^{\beta-1-p}\, dw - \int_{(1-\frac{\lambda}{Mx_d})_+}^{\frac{Mx_d}{Mx_d+\lambda}} \frac{w^{\beta-1-p} - w^{\beta-1}}{(1-w)^{1+\beta}}\, dw.$$

Since $w \mapsto w^{\beta-1-p}$ is integrable near 0,

$$I_1 \leq \int_{\mathbb{R}^{d-1}} \frac{1}{(|\widetilde{u}|^2+1)^{(d+\beta)/2}} \left(\int_0^1 \frac{(1-w^p)(1-w^{p-(\beta-1)})}{(1-w)^{1+\beta}} w^{\beta-1-p}\, dw \right) d\widetilde{u} < \infty.$$

On the other hand, $-I_1 \leq c(d) I_{1,2}$, where

$$I_{1,2} := \int_0^\infty \frac{1}{(u^2+1)^{(d+\beta)/2}} \int_{(1-\frac{\lambda}{(u^2+1)^{1/2}x_d})_+}^{\frac{(u^2+1)^{1/2}x_d}{(u^2+1)^{1/2}x_d+\lambda}} \frac{w^{\beta-1-p}(1-w^p)}{(1-w)^{1+\beta}}\, dw\, u^{d-2} du.$$

Note that

$$\begin{aligned}
&\sup_{v\geq 2\lambda} \int_{1-\lambda/v}^{v/(v+\lambda)} \frac{w^{\beta-1-p}(1-w^p)}{(1-w)^{1+\beta}}\, dw \leq c \sup_{v\geq 2\lambda} \int_{1-\lambda/v}^{v/(v+\lambda)} \frac{1}{(1-w)^{\beta}} dw \\
&= c \sup_{v\geq 2\lambda} \int_{\lambda/(v+\lambda)}^{\lambda/v} t^{-\beta} dt \ \leq c \sup_{v\geq 2\lambda} (v+\lambda)^\beta \left(\frac{1}{v} - \frac{1}{v+\lambda}\right) \leq c \sup_{v\geq 2\lambda} v^{\beta-2} < \infty,
\end{aligned}$$

and, for $x_d < \lambda$,

$$\sup_{x_d \leq v < 2\lambda} \int_0^{v/(v+\lambda)} \frac{w^{\beta-1-p}(1-w^p)}{(1-w)^{1+\beta}}\, dw \leq c \sup_{x_d\leq v<2\lambda} \int_0^{2/3} w^{\beta-1-p} dw < \infty.$$

Thus for $x_d < \lambda$,

$$0 < I_{1,2} \leq \int_0^\infty \frac{\mathbf{1}_{(u^2+1)^{1/2}x_d < 2\lambda}}{(u^2+1)^{(d+\beta)/2}} \int_0^{\frac{(u^2+1)^{1/2}x_d}{(u^2+1)^{1/2}x_d+\lambda}} \frac{w^{\beta-1-p}(1-w^p)}{(1-w)^{1+\beta}}\, dw u^{d-2} du$$

$$
\begin{aligned}
&+\int_0^\infty \frac{\mathbf{1}_{(u^2+1)^{1/2}x_d\geq 2\lambda}}{(u^2+1)^{(d+\beta)/2}}\int_{(1-\frac{\lambda}{(u^2+1)^{1/2}x_d})_+}^{\frac{(u^2+1)^{1/2}x_d}{(u^2+1)^{1/2}x_d+\lambda}} \frac{w^{\beta-1-p}(1-w^p)}{(1-w)^{1+\beta}}\, dw u^{d-2} du \\
&\leq \int_0^\infty \frac{u^{d-2}}{(u^2+1)^{(d+\beta)/2}} du < \infty.
\end{aligned}
$$

The proof is complete. □

Throughout the remainder of this subsection we assume that D is a bounded $C^{1,1}$ open subset of $\mathbb{R}^d$, $\alpha \in (0,2)$ and $\beta \in (-\infty, \alpha)$. We also assume that $b(x,y)$ is a symmetric Borel function on $D\times D$ such that $C_{b,1} := \sup_{x,y\in D}|b(x,y)| < \infty$ and the function

$$
B(x,y) := \mathcal{A}_{d,\alpha} + |x-y|^{\alpha-\beta} b(x,y), \quad x,y \in D,
$$

is bounded below by a positive constant, that is, $C_{b,2} \leq B(x,y)$ for some $C_{b,2} > 0$. Clearly, $B(x,y)$ is bounded above by $\mathcal{A}_{d,\alpha} + (\text{diam}(D))^{\alpha-\beta} C_{b,1}$. We further assume that the first partials of $B(x,y)$ are bounded on $D\times D$. Note that, β and b can be negative, as long as the condition above is satisfied.

Let $(\mathcal{E}^{(B)}, \overline{\mathcal{F}})$ be the Dirichlet form on $L^2(D, dx)$ defined by

$$
\mathcal{E}^{(B)}(u,v) := \frac{1}{2}\int_D\int_D (u(x)-u(y))(v(x)-v(y))\frac{B(x,y)}{|x-y|^{d+\alpha}} dxdy, \quad u,v \in \overline{\mathcal{F}}.
$$

By [45], $(\mathcal{E}^{(B)}, \overline{\mathcal{F}})$ is a regular Dirichlet form on $\overline{D}$ and its associated Hunt process $X^{(B)}$ is conservative and lives on $\overline{D}$. Moreover, since $B(x,y)$ is bounded on $D\times D$ between two strictly positive constants, the form $(\mathcal{E}^{(B)}, \overline{\mathcal{F}})$ satisfies the assumptions of [16, Remark 2.4], so we can freely use results of [16, Section 2]. Further, $X^{(B)}$ admits a strictly positive and jointly continuous transition density $p(t,x,y)$ with respect to the Lebesgue measure dx such that

$$
C^{-1}\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) \leq p(t,x,y) \leq C\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right),
$$

for $(t,x,y) \in (0,1)\times D\times D$.

Let $L^{(B)}$ be the generator of $X^{(B)}$ in the L^2 sense. Similar to [121, Section 4], cf. also [101], we can show that $C_c^2(D)$ is contained in the domain of $L^{(B)}$ and give an explicit expression for $L^{(B)}f$ when $f \in C_c^2(D)$. Using these, one can check that the process $X^{(B)}$ satisfies Assumptions **A** and **U**.

If $m > 0$, by taking $\beta = \alpha - 2$ and $b(x,y) = \mathcal{A}_{d,\alpha}(\varphi(m^{1/\alpha}|x-y|) - 1)|x-y|^{-2}$ with

$$\varphi(r) := 2^{-(d+\alpha)}\,\Gamma\left(\frac{d+\alpha}{2}\right)^{-1} \int_0^\infty s^{\frac{d+\alpha}{2}-1} e^{-\frac{s}{4}-\frac{r^2}{s}}\, ds,$$

we cover the reflected relativistic α-stable process X^m with weight $m > 0$ in $\overline{D}$. When $\alpha \in (1,2)$, the killed process $X^{m,D}$ is the censored relativistic α-stable process in D. When $\alpha \in (0,1]$, it follows from [16, Section 2] that, starting from inside D, the process X^m neither hits nor approaches ∂D at any finite time. Thus, the killed process $X^{m,D}$ is simply X^m restricted to D.

Recall that for $u : D \to [0,\infty)$,

$$L^\beta u(x) = \mathcal{A}_{d,\beta} \lim_{\epsilon \downarrow 0} \int_{\{y \in D :\, \epsilon < |y-x|\}} (u(y) - u(x)) \frac{dy}{|x-y|^{d+\beta}}\,, \qquad x \in D\,.$$

We define

$$L_b^\beta u(x) := \lim_{\epsilon \downarrow 0} \int_{\{y \in D :\, \epsilon < |y-x|\}} (u(y) - u(x)) \frac{b(x,y)}{|x-y|^{d+\beta}} dy\,, \qquad x \in D\,.$$

Let $p \in [\alpha - 1, \alpha) \cap (0, \alpha)$, $\kappa \in \mathcal{H}_\alpha(p)$. If $\beta \geq p$, then we always assume that, there exist $C_{b,3} > 0$ and $\beta_1 > \beta - p$ such that

$$|b(x,y) - b(x,x)| \leq C_{b,3}|x-y|^{\beta_1}, \quad x, y \in D. \tag{3.2.19}$$

Note that, under (3.2.19), for any bounded Borel function u satisfying $|u(x) - u(y)| \leq c|x-y|^p$ on D,

$$|L_b^\beta u(x)| \leq c \int_{\{y \in D :\, \epsilon < |y-x|\}} \frac{|b(x,y) - b(x,x)|}{|x-y|^{d+\beta-p}} dy + \frac{|b(x,x)|}{\mathcal{A}_{d,\beta}} |L^\beta u(x)|$$

$$\leq c_1 + c_2|L^\beta u(x)|. \tag{3.2.20}$$

Recall that for an open set D and $q \geq 0$, $h_q(x) = \delta_D(x)^q$.

Lemma 3.2.6. *Let D be a bounded $C^{1,1}$ open set with characteristics (R_2, Λ). For any $q \geq p$, there exist constants $c_1 > 0$ and $c_2 \in (0, (R_2 \wedge 1)/4)$ depending only on $p, q, d, \beta, R_2, \Lambda$, diam$(D)$, $C_{b,1}, C_{b,2}, C_{b,3}, \beta_1$ such that for every $x \in D$ with $0 < \delta_D(x) < c_2$, the following inequalities hold:*

$$|L_b^\beta h_q(x)| \leq c_1 \begin{cases} 1 & \text{if } q > \beta; \\ |\log \delta_D(x)| & \text{if } q = \beta; \\ \delta_D(x)^{q-\beta} & \text{if } q < \beta. \end{cases}$$

Proof. Without loss of generality we assume diam$(D) \leq 1$ and let $x \in D$ with $\delta_D(x) < R_2/4$. Choose a point $z \in \partial D$ such that $\delta_D(x) = |x - z|$. Then, there exists a $C^{1,1}$ function $\Gamma : \mathbb{R}^{d-1} \mapsto \mathbb{R}$ such that $\Gamma(z) = \nabla\Gamma(z) = 0$ and an orthonormal coordinate system CS_z with origin at z such that

$$D \cap B(z, R_2) = \{y = (\widetilde{y}, y_d) \text{ in } CS_z : y_d > \Gamma(\widetilde{y})\} \cap B(z, R_2),$$

and $z = 0$ and $x = (\widetilde{x}, x_d) = (\widetilde{0}, x_d)$ in CS_z. For any open subset $U \subset \mathbb{R}^d$, define $\widehat{\kappa}_U(x) := \mathcal{A}_{d,\beta} \int_{U^c \cap B(x,1)} |y - x|^{-d-\beta} dy$. Recall that $w_q(y) = (y_d)_+^q$. Since $h_q(x) = w_q(x) = x_d^q$, using (3.2.20), we have

$$\begin{aligned}
L^\beta h_q(x) &= \mathcal{A}_{d,\beta} \lim_{\epsilon \downarrow 0} \left[\int_{1>|y-x|>\epsilon} \frac{h_q(y) - h_q(x)}{|y - x|^{d+\beta}} dy + \widehat{\kappa}_D(x) h_q(x) \right] \\
&= \mathcal{A}_{d,\beta} \lim_{\epsilon \downarrow 0} \left[\int_{1>|y-x|>\epsilon} \frac{h_q(y) - w_q(y)}{|y - x|^{d+\beta}} dy \right. \\
&\qquad \left. + \int_{1>|y-x|>\epsilon} \frac{w_q(y) - w_q(x)}{|y - x|^{d+\beta}} dy + \widehat{\kappa}_D(x) w_q(x) \right] \\
&= L_{d,1}^\beta w_q(x) + \mathcal{A}_{d,\beta}(\widehat{\kappa}_D(x) - \widehat{\kappa}_{\mathbb{R}_+^d}(x)) w_q(x) \\
&\quad + \mathcal{A}_{d,\beta} \lim_{\epsilon \downarrow 0} \int_{1>|y-x|>\epsilon} \frac{h_q(y) - w_q(y)}{|y - x|^{d+\beta}} dy.
\end{aligned}$$

By a similar calculation as in [16, Lemma 5.6], since $q \geq (\alpha-1)_+ \geq (\beta-1)_+$, we get

$$|(\widehat{\kappa}_D(x) - \widehat{\kappa}_{\mathbb{R}^d_+}(x))w_q(x)| \leq c x_d^q (x_d^{1-\beta} + |\log x_d|) \leq c.$$

Next, we bound $I_\epsilon := \int_{1>|y-x|>\epsilon} \frac{h_q(y)-w_q(y)}{|y-x|^{d+\beta}} dy$ for $\epsilon \in (0,1/2)$. When $q < \beta$, by the proof of Lemma 3.2.1,

$$\sup_{\epsilon<1/2} |I_\epsilon| \leq \sup_{\epsilon<1/2} \left| \int_{\mathbb{R}^d, |y-x|>\epsilon} \frac{h_q(y) - w_q(y)}{|y-x|^{d+\beta}} dy \right| \leq c(1 + \log|x_d|).$$

When $q \geq \beta$, by [44, (3.13)], we get $\sup_{\epsilon<1/2} |I_\epsilon| \leq c$. The lemma now follows from these bounds, Lemma 3.2.5 and (3.2.20). □

Define

$$\widetilde{L} f(x) := L^\alpha f(x) + L_b^\beta f(x) - \kappa(x) f(x).$$

Combining Lemmas 3.2.1 and 3.2.6, we get the following lemma.

Lemma 3.2.7. *Let $p \in [\alpha-1,\alpha) \cap (0,\alpha)$, $\beta < \alpha$ and suppose $\kappa \in \mathcal{H}_\alpha(p)$. Then for any $q \in [p,\alpha)$, there exist $c_1 > 0$ and $c_2 \in (0,1/4)$ depending only on $p, q, d, \alpha, \beta, \Lambda, C_2, \eta, R_2, C_{b,1}, C_{b,2}, C_{b,3}, \beta_1$ such that the following inequalities hold:*

(i) If $q > p$,

$$c_1^{-1} \delta_D(x)^{q-\alpha} \leq \widetilde{L} h_q(x) \leq c_1 \delta_D(x)^{q-\alpha}$$

for every $x \in D$ with $0 < \delta_D(x) < c_2$.

(ii) If $q = p$,

$$|\widetilde{L} h_p(x)| \leq c_1 (\delta_D(x)^{p-(\beta\vee\eta)} + |\log \delta_D(x)|)$$

for every $x \in D$ with $0 < \delta_D(x) < c_2$.

Recall that $X^{(B),D}$ denote the process $X^{(B)}$ killed upon exiting D. Note that the operator $\widetilde{L}$ coincides with the restriction to $C_c^2(D)$ of the generator of of the Feynman-Kac semigroup of $X^{(B),D}$ via the multiplicative functional

$e^{-\int_0^t \kappa(X_s^{(B),D})ds}$ in $C_0(D)$. We now follow the argument of the previous subsection (choosing $q \in (p, (p-(\eta \vee \beta)+\alpha) \wedge \alpha))$ and can conclude the following.

Theorem 3.2.8. *Suppose that D is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$, $d \geq 2$, with characteristics (R_2, Λ). For all $T > 0$, $p \in [\alpha-1, \alpha) \cap (0, \alpha)$, $\beta < \alpha$ and $\eta \in [0, \alpha)$, there exists $c = c(C_1, C_2, p, \alpha, \beta, d, \eta, diam(D), T, R_2, \Lambda, C_{b,1}, C_{b,2}, C_{b,3}, \beta_1) \geq 1$ such that for all $\kappa \in \mathcal{H}_\alpha(p)$, the transition density $q^D(t,x,y)$ of the Hunt process Y on D corresponding to the Feynman-Kac semigroup of $X^{(B),D}$ via the multiplicative functional $e^{-\int_0^t \kappa(X_s^{(B),D})ds}$ satisfies that*

$$
\begin{aligned}
&c^{-1}\left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^p \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^p \left[t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right] \\
&\leq q^D(t,x,y) \leq c\left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^p \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^p \left[t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right]
\end{aligned}
$$

for $(t,x,y) \in (0,T) \times D \times D$.

We remark here that Theorem 3.2.8 recovers [42, Theorem 4.8]. Let κ_D^m be the killing function of the killed relativistic α-stable process $Z^{m,D}$ in D. It follows from [16, pp.94–95] that the killed relativistic α-stable process $Z^{m,D}$ can be obtained from $X^{m,D}$ via a Feynman-Kac perturbation of the form $e^{-\int_0^t \kappa_D^m(X_s^{m,D})ds}$. It follows [51, p. 278] that $0 \leq \kappa_D(x) - \kappa_D^m(x) \leq c\delta_D(x)^{2-\alpha}$ for all $x \in D$. Combining this with (3.2.8), we get

$$
|(\kappa_D^m(x) - \kappa_{\mathbb{R}^d_+}(x))w_q(x)| \leq cx_d^q(x_d^{1-\alpha} + |\log x_d|) \leq c.
$$

Now by the same argument as in Remark 3.2.4, we see that Theorem 3.2.8 recovers the main result of [40] for bounded $C^{1,1}$ open set D.

3.2.3 $\mathbb{R}^d \setminus \{0\}$

In this subsection we assume that $\mathfrak{X} = \mathbb{R}^d$, $d \geq 2$, X is an isotropic α-stable process on $\mathbb{R}^d$ and $D = \mathbb{R}^d \setminus \{0\}$. Obviously, D is a $(1/2)$-fat open set with characteristics $(\infty, 1/2)$ and X satisfies Assumptions **A** and **U**. Since X does not hit $\{0\}$, the killed process X^D is simply the restriction of X to D.

Recall that $\mathcal{A}_{d,\alpha} = \alpha 2^{\alpha-1}\pi^{-d/2}\Gamma((d+\alpha)/2)\Gamma(1-\alpha/2)^{-1}$. Let $p \in (0,\alpha)$ and define

$$H(s) = 2\pi \frac{\pi^{\frac{d-3}{2}}}{\Gamma(\frac{d-1}{2})} \int_0^\pi \sin^{d-2}\theta \, \frac{(\sqrt{s^2-\sin^2\theta}+\cos\theta)^{1+\alpha}}{\sqrt{s^2-\sin^2\theta}} \, d\theta, \quad s \geq 1,$$

and

$$\widetilde{C}(\alpha,d,p) := \mathcal{A}_{d,\alpha} \int_1^{+\infty} (s^p-1)(1-s^{-d+\alpha-p}) s(s^2-1)^{-1-\alpha} H(s) ds.$$

Note that $p \to \widetilde{C}(\alpha,d,p)$ is strictly increasing on $(0,\alpha)$. The function $H(s)$ is positive and continuous on $[1,+\infty)$ with $H(s) \simeq s^\alpha$ for large s and

$$s(s^2-1)^{-1-\alpha}H(s) \simeq (s-1)^{-1-\alpha}, \quad s \geq 1,$$

(see the paragraph after [65, Theorem 1.1]). Thus

$$\lim_{p\downarrow 0} \widetilde{C}(\alpha,d,p) = 0 \quad \text{and} \quad \lim_{p\uparrow\alpha} \widetilde{C}(\alpha,d,p) = \infty. \tag{3.2.21}$$

Applying [65, Theorem 1.1] to $u_p := |x|^p$, we get that

$$-(-\Delta)^{\alpha/2}u_p(x) = \widetilde{C}(\alpha,d,p)\,|x|^{p-\alpha}, \quad |x|>0, x \in \mathbb{R}^d. \tag{3.2.22}$$

Let $\mathcal{G}_\alpha$ be the collection of non-negative functions on D such that for each $\kappa \in \mathcal{G}_\alpha$ there exist constants $C_1 > 0$, $C_2 \geq 0$ and $\eta \in [0,\alpha)$ such that $\kappa(x) \leq C_2$ for all x with $|x| \geq 1$ and

$$\left|\kappa(x) - C_1|x|^{-\alpha}\right| \leq C_2|x|^{-\eta}, \tag{3.2.23}$$

for all $x \in D$ with $|x| < 1$. By (3.2.21) we can find a unique $p \in (0,\alpha)$ such that $C_1 = \widetilde{C}(\alpha,d,p)$. Define

$$\mathcal{G}_\alpha(p) := \{\kappa \in \mathcal{G}_\alpha : \text{ the constant } C_1 \text{ in (3.2.23) is } \widetilde{C}(\alpha,d,p)\}.$$

Note that $\mathcal{G}_\alpha = \cup_{0<p<\alpha}\mathcal{G}_\alpha(p)$. We fix a $\kappa \in \mathcal{G}_\alpha(p)$ and let Y be a Hunt process on D corresponding to the Feynman-Kac semigroup of X^D via the multiplicative functional $e^{-\int_0^t \kappa(X_s^D)ds}$, that is,

$$\mathbb{E}^x\left[f(Y_t)\right] = \mathbb{E}^x\left[e^{-\int_0^t \kappa(X_s^D)ds} f(X_t^D)\right], \quad t \geq 0, x \in D.$$

Since, by Example 3.1.15, $\kappa(x)dx \in \mathbf{K}_1(D)$, it follows from Theorem 3.1.20 that Y has a transition density $q^D(t,x,y)$ with the following estimate

$$q^D(t,x,y) \simeq \mathbb{P}^x(\zeta > t)\mathbb{P}^y(\zeta > t)\left[t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right], \qquad (3.2.24)$$

for $(t,x,y) \in (0,1) \times D \times D$, where ζ is the lifetime of Y. Moreover, when $C_2 = 0$, $\kappa(x)dx \in \mathbf{K}_\infty(D)$ by Example 3.1.16. Thus, by Theorem 3.1.21, (3.2.24) holds for all $t > 0$.

Define

$$Lf(x) := -(-\Delta)^{\alpha/2} f(x) - \kappa(x)f(x).$$

Fix $q \in (p,\alpha)$ such that $q < p-\eta+\alpha$ and let $A = \widetilde{C}(\alpha,d,q) - \widetilde{C}(\alpha,d,p) > 0$. Define

$$v_1(x) := u_p(x) + u_q(x), \quad v_2(x) := u_p(x) - \frac{1}{2}u_q(x).$$

Since, for $|x| < C_2^{-1}$, in view of (3.2.22) and (3.2.23),

$$Lv_1(x) \geq A|x|^{q-\alpha} - 2C_2(|x|^{p-\eta} + |x|^{q-\eta})$$

and

$$Lv_2(x) \leq -2^{-1}A|x|^{q-\alpha} + (3/2)C_2(|x|^{p-\eta} + |x|^{q-\eta}),$$

there exists $c_1 > 0$ such that $Lv_1(x) \geq 0$ and $Lv_2(x) \leq 0$ whenever $0 < |x| < c_1$. Pick any $(t,x) \in (0,1) \times D$ and set $r = r(t) = c_1 t^{1/\alpha}$ for $t < 1$. Now we can follow the argument before the statement of Theorem 3.2.2 and get

$\mathbb{P}^x(\zeta > t) \simeq (1 \wedge |x|/r)^p$ for $t < 1$.

Moreover, if $\kappa(x) = \widetilde{C}(\alpha, d, p)|x|^{-\alpha}$, we can simply take $v_1(x) = v_2(x) = u_p(x)$ and $r(t) = t^{1/\alpha}$ for all $t > 0$ and get $\mathbb{P}^x(\zeta > t) \simeq (1 \wedge |x|/r(t))^p$ for all $t > 0$.

Therefore, we conclude that

Theorem 3.2.9. *For all positive $T > 0$, $p \in (0, \alpha)$ and $\eta \in [0, \alpha)$, there exists $c = c(C_1, C_2, p, \alpha, d, \eta, T) \geq 1$ such that for all $\kappa \in \mathcal{G}_\alpha(p)$, the transition density $q(t, x, y)$ of Y, the Hunt process on $\mathbb{R}^d \setminus \{0\}$ associated with the Feynman-Kac semigroup of the isotropic α-stable process Z via the multiplicative functional $e^{-\int_0^t \kappa(Z_s)ds}$, satisfies that*

$$
\begin{aligned}
c^{-1}&\left(1 \wedge \frac{|x|}{t^{1/\alpha}}\right)^p \left(1 \wedge \frac{|y|}{t^{1/\alpha}}\right)^p \left[t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right] \\
&\leq q(t, x, y) \leq c \left(1 \wedge \frac{|x|}{t^{1/\alpha}}\right)^p \left(1 \wedge \frac{|y|}{t^{1/\alpha}}\right)^p \left[t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right],
\end{aligned}
$$

for $(t, x, y) \in (0, T) \times (\mathbb{R}^d \setminus \{0\}) \times (\mathbb{R}^d \setminus \{0\})$. Moreover, if $\kappa(x) = \widetilde{C}(\alpha, d, p)|x|^{-\alpha}$, then the above estimates holds for all $t > 0$.

The last claim in Theorem 3.2.9 can be proved using the scaling property and the finite time estimates in Theorem 3.2.9. This was proved independently in [82] using a different method.

Let $\alpha \in (1, 2)$ and g be an $\mathbb{R}^d$-valued C^1 function with $\|g\|_\infty + \|\nabla g\|_\infty < \infty$. Let $\widetilde{X}^g$ be an α-stable process with drift g, that is, a non-symmetric Hunt process with generator $-(-\Delta)^{\alpha/2} f(x) + g \cdot \nabla f(x)$, see [24]. Let X^g be the Hunt process obtained from $\widetilde{X}^g$ by killing with rate $\|\mathrm{div}\, g\|$. The generator of X^g is $-(-\Delta)^{\alpha/2} f(x) + g \cdot \nabla f(x) - \|\mathrm{div} g\|_\infty f(x)$. By [24], the transition density $p(t, x, y)$ of X^g satisfies

$$
p(t, x, y) \simeq t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}, \quad (t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d. \tag{3.2.25}
$$

The dual of $-(-\Delta)^{\alpha/2} f(x) + g \cdot \nabla f(x) - \|\mathrm{div} g\|_\infty f(x)$ is $-(-\Delta)^{\alpha/2} f(x) - g \cdot \nabla f(x) - \mathrm{div}\, g(x) f(x) - \|\mathrm{div} g\|_\infty f(x)$, which is the generator of a Hunt process $\widehat{X}^g$ which can be obtained from an α-stable process with drift via the killing potential $-\mathrm{div}\, g(x) - \|\mathrm{div} g\|_\infty$. It is easy to check that X^g and $\widehat{X}^g$ are strong duals of each other with respect to the Lebesgue measure. It is also easy to check that X^g and $\widehat{X}^g$ satisfy the sector condition, thus, by [67, Theorem 4.17], all semipolar sets are polar. Moreover, since $\alpha \in (1,2)$, Assumption **U** holds true.

Fix $\kappa \in \mathcal{G}_\alpha(p)$ and let Y^g be a Hunt process on D corresponding to the Feynman-Kac semigroup of $X^{g,D}$ defined by

$$\mathbb{E}^x\left[f(Y_t^g)\right] = \mathbb{E}^x\left[e^{-\int_0^t \kappa(X_s^{g,D})ds} f(X_t^{g,D})\right], \quad t \geq 0, x \in D.$$

Note that $\kappa(x)dx \in \mathbf{K}_1(D)$ by (3.2.25). With $u_p = |x|^p$, we get that

$$|g \cdot \nabla u_p(x)| + \|\mathrm{div} g\|_\infty |u_p(x)| \leq \widetilde{C}(\alpha, d, p)\, |x|^{p-1}, \quad 0 < |x| < 1. \tag{3.2.26}$$

From (3.2.22), (3.2.26) and the assumption $\alpha \in (1,2)$, we see that terms $g \cdot \nabla f(x) - \|\mathrm{div} g\|_\infty f(x)$ and $-g \cdot \nabla f(x) - \mathrm{div}\, g(x) f(x) - \|\mathrm{div} g\|_\infty f(x)$ can be treated as lower order terms. Thus, using (3.2.26) and the assumption $\alpha \in (1,2)$, by repeating the argument of the first part of this subsection, we can easily get the following result from (3.2.25) and Theorem 3.1.20.

Theorem 3.2.10. *Suppose that $\alpha \in (1,2)$. For all positive $T > 0$, $p \in (0,\alpha)$ and $\eta \in [0,\alpha)$, there exists $c = c(C_1, C_2, p, \|g\|_\infty, \alpha, d, \eta, T, \|\nabla g\|_\infty) \geq 1$ such that for all $\kappa \in \mathcal{G}_\alpha(p)$, the transition semigroup $q^g(t,x,y)$ of Y^g satisfies that*

$$\begin{aligned} &c^{-1}\left(1 \wedge \frac{|x|}{t^{1/\alpha}}\right)^p \left(1 \wedge \frac{|y|}{t^{1/\alpha}}\right)^p \left[t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right] \\ &\leq q^g(t,x,y) \leq c\left(1 \wedge \frac{|x|}{t^{1/\alpha}}\right)^p \left(1 \wedge \frac{|y|}{t^{1/\alpha}}\right)^p \left[t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right], \end{aligned}$$

for $(t,x,y) \in (0,T) \times (\mathbb{R}^d \setminus \{0\}) \times (\mathbb{R}^d \setminus \{0\})$.

Note that, Theorem 3.2.10 also holds for the fundamental solution to $\partial_t = -(-\Delta)^{\alpha/2} + g \cdot\ \nabla - \kappa(x)$.

3.3 Appendix: Continuous additive functionals for killed non-symmetric processes

We keep the assumptions and the notations in Sections 3.1.1–3.1.3. In this section, D is an open subset of $\mathfrak{X}$ and U is a relatively compact subset of D.

Lemma 3.3.1. *If $h \in \mathcal{D}(\widehat{\mathcal{L}})$ is nonnegative, bounded and has compact support contained in U, then for any $t \geq 0$,*

$$\limsup_{\epsilon \to 0} \frac{1}{\epsilon} \int_U \widehat{P}_t^U h(x) \mathbb{P}^x(\tau_U^X \leq \epsilon) m(dx) < \infty.$$

Proof. Noticing $h(\widehat{X}_{\tau_U^X}) = 0$, we get

$$\widehat{P}_t^U h(x) = \widehat{\mathbb{E}}^x[h(\widehat{X}_t) 1_{t < \widehat{\tau}_U^X}] = \widehat{\mathbb{E}}^x h(\widehat{X}_{t \wedge \widehat{\tau}_U^X}) = h(x) + \widehat{\mathbb{E}}^x \int_0^{t \wedge \widehat{\tau}_U^X} \widehat{\mathcal{L}} h(\widehat{X}_s) ds.$$

Using this and the duality, we have

$$\begin{aligned}
&\int_U \widehat{P}_t^U h(x) \mathbb{P}^x(\tau_U^X \leq \epsilon) m(dx) = \int_U \widehat{P}_t^U h(x)(1 - P_\epsilon^U 1(x)) m(dx) \\
&= \int_U (\widehat{P}_t^U h(x) - \widehat{P}_{t+\epsilon}^U h(x)) m(dx) = -\int_U \widehat{\mathbb{E}}^x \int_t^{t+\epsilon} \widehat{\mathcal{L}} h(\widehat{X}_s) 1_{s < \widehat{\tau}_U^X} ds\, m(dx) \\
&\leq \epsilon \left(\sup_{x \in U} |\widehat{\mathcal{L}} h(x)| \right) m(U),
\end{aligned}$$

from which the conclusion follows immediately. □

Lemma 3.3.2. *Let $\mu \in \mathbf{K}_T(D)$ for some $T > 0$. If A is the continuous additive functional of X^D associated with μ, $h \in \mathcal{D}(\widehat{\mathcal{L}})$ is nonnegative, bounded and has compact support contained in U, then for any bounded Borel function f on U and $t \geq 0$,*

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_U \widehat{P}_t^U h(x) \left(\mathbb{E}^x \int_0^\epsilon \mathbf{1}_{\tau_U^X \leq s} f(X_s^D) dA_s \right) m(dx) = 0.$$

Proof. Using the strong Markov property, we get

$$\begin{aligned}\left|\mathbb{E}^x \int_0^\epsilon 1_{\tau_U^X \le s} f(X_s^D) dA_s\right| &= \left|\mathbb{E}^x \left[\mathbb{E}^{X^D_{\tau_U^X}} \int_0^{\epsilon-\tau_U^X} f(X_s^D) dA_s : \tau_U^X < \epsilon\right]\right| \\ &\le \left(\sup_{y\in\mathfrak{X}} \mathbb{E}^y \int_0^\epsilon |f(X_s^D)| dA_s\right) \mathbb{P}^x(\tau_U^X \le \epsilon).\end{aligned}$$

The assertion now follows from Lemma 3.3.1 and condition (2) in Definition 3.1.10. □

Proposition 3.3.3. *Let $\mu \in \mathbf{K}_T(D)$ for some $T > 0$. If A is the continuous additive functional of X associated with μ, then $(A_{t\wedge\tau_U^X})$ is the continuous additive functional of X^U associated with μ_U.*

Proof. Let $A_t^U := A_{t\wedge\tau_U^X}$. Then A^U is a continuous additive functional of X^U. Let $h \in \mathcal{D}(\mathcal{L})$ be non-negative, bounded and have compact support contained in U, and let f be a bounded Borel function supported in U. Define

$$g_t := \int_U h(x)\mathbb{E}^x \int_0^t f(X_s^U) dA_s^U m(dx).$$

Since

$$\begin{aligned}g_{t+\epsilon} - g_t &= \int_U h(x)\mathbb{E}^x \int_t^{t+\epsilon} f(X_s^U) dA_s^U m(dx) \\ &= \int_U h(x) P_t^U (\mathbb{E}^\cdot \int_0^\epsilon f(X_s^U) dA_s)(x) m(dx) \\ &= \int_U \widehat{P}_t^U h(x) \mathbb{E}^x \int_0^\epsilon \mathbf{1}_{\tau_U^X > s} f(X_s^U) dA_s m(dx),\end{aligned}$$

it follows from Lemma 3.3.2 that

$$\begin{aligned}\lim_{\epsilon\to 0} \frac{g_{t+\epsilon} - g_t}{\epsilon} &= \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_U \widehat{P}_t^U h(x) \mathbb{E}^x \int_0^\epsilon f(X_s^D) dA_s m(dx) \\ &= \int_U \widehat{P}_t^U h(x) f(x) \mu(dx).\end{aligned}$$

Thus, we obtain that

$$\int_U h(x)\mathbb{E}^x \int_0^t f(X_s^U)dA_s^U m(dx) = \int_0^t \int_U \widehat{P}_s^U h(x)f(x)\mu(dx)ds$$
$$= \int_U h(x) \int_0^t P_s^U f(x)ds\mu(dx).$$

Using the dominated convergence theorem and the monotone convergence theorem, one can show that the equality above is valid for all bounded non-negative Borel functions h and f supported in U. Therefore,

$$\int_U f(x)\mu(dx) = \lim_{t\downarrow 0} \mathbb{E}^{m_U}\left[\frac{1}{t}\int_0^t f(X_s^U)dA_s^U\right].$$

□

Chapter 4

Heat kernel estimates for subordinate Markov processes

In this chapter, we study heat kernel estimates for subordinate Markov processes on spaces with boundary. The main motivation comes from [97], where it was established that the jump kernels of subordinate killed Lévy processes have an unusual form not observed before. The results of this chapter is based on [59]. We begin with the following motivating and also the simplest example covered by our results.

Let $D \subset \mathbb{R}^d$, $d \geq 1$, be a bounded $C^{1,1}$ open set. Let Y be an isotropic α-stable process in $\mathbb{R}^d$, $\alpha \in (0, 2]$ and let Y^D denote the part process of Y killed upon exiting D. When $\alpha = 2$, we further assume that D is connected. The following global two-sided estimates of the heat kernel $p_D(t, x, y)$ of Y^D were obtained in [63, 128] (for $\alpha = 2$) and [35] (for $\alpha < 2$): there exist positive constants c_i, $i = 1, \dots, 8$, such that following estimates hold true. For $(t, x, y) \in (0, 1] \times D \times D$,

$$c_1 h_\alpha(t, x, y) \Big(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \Big) \leq p_D(t, x, y) \leq c_2 h_\alpha(t, x, y) \Big(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \Big),$$

for $\alpha < 2$, and

$$c_3 h_2(t, x, y)\, t^{-d/2} e^{-c_4 |x-y|^2/t} \leq p_D(t, x, y) \leq c_5 h_2(t, x, y)\, t^{-d/2} e^{-c_6 |x-y|^2/t},$$

for $\alpha = 2$, where the boundary function $h_\alpha(t,x,y)$ is given by

$$h_\alpha(t,x,y) = \Big(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\Big)^{\frac{\alpha}{2}} \Big(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\Big)^{\frac{\alpha}{2}}.$$

For $(t,x,y) \in [1,\infty) \times D \times D$,

$$c_7 e^{-\lambda_1 t} \delta_D(x)^{\frac{\alpha}{2}} \delta_D(y)^{\frac{\alpha}{2}} \le p_D(t,x,y) \le c_8 e^{-\lambda_1 t} \delta_D(x)^{\frac{\alpha}{2}} \delta_D(y)^{\frac{\alpha}{2}},$$

where λ_1 is the smallest eigenvalue of the Dirichlet (fractional) Laplacian $(-\Delta)^{\alpha/2}\big|_D$.

Let $S = (S_t)_{t\ge 0}$ be a β-stable subordinator, $\beta \in (0,1)$, independent of Y^D, and let $X = (X_t)_{t\ge 0}$ be the subordinate process: $X_t := Y^D_{S_t}$. The generator of X is equal to (the negative of) $\big((-\Delta)^{\alpha/2}\big|_D\big)^\beta$ – the fractional power of the Dirichlet fractional Laplacian. The heat kernel $q(t,x,y)$ of the subordinate process X is given by

$$q(t,x,y) = \int_0^\infty p_D(s,x,y)\mathbb{P}(S_t \in ds), \quad t>0,\ x,y \in D.$$

With a help from the results in Chapter 2, we can obtain sharp two-sided estimates of $q(t,x,y)$. Recall that $\delta_\vee(x,y) = \delta_D(x) \vee \delta_D(y)$ and $\delta_\wedge(x,y) = \delta_D(x) \wedge \delta_D(y)$ for $x,y \in D$.

Theorem 4.0.1. *(i) For all $(t,x,y) \in (0,1] \times D \times D$,*

$$q(t,x,y) \simeq \Big(1 \wedge \frac{\delta_D(x)}{t^{1/(\alpha\beta)}}\Big)^{\frac{\alpha}{2}} \Big(1 \wedge \frac{\delta_D(y)}{t^{1/(\alpha\beta)}}\Big)^{\frac{\alpha}{2}} \Big(t^{-d/(\alpha\beta)} \wedge \frac{tB^{\alpha,\beta}(t,x,y)}{|x-y|^{d+\alpha\beta}}\Big), \quad (4.0.1)$$

where

$$B^{2,\beta}(t,x,y) := \Big(1 \wedge \frac{\delta_D(x) \vee t^{1/(2\beta)}}{|x-y|}\Big)\Big(1 \wedge \frac{\delta_D(y) \vee t^{1/(2\beta)}}{|x-y|}\Big)$$

and for $\alpha < 2$,

$B^{\alpha,\beta}(t,x,y) :=$

$$\begin{cases} \left(1 \wedge \dfrac{\delta_\wedge(x,y) \vee t^{1/(\alpha\beta)}}{|x-y|}\right)^{\alpha-\alpha\beta}, & \textit{if } \beta > \dfrac{1}{2}, \\ \left(1 \wedge \dfrac{\delta_\wedge(x,y) \vee t^{1/(\alpha\beta)}}{|x-y|}\right)^{\frac{\alpha}{2}} \left(1 \wedge \dfrac{\delta_\vee(x,y) \vee t^{1/(\alpha\beta)}}{|x-y|}\right)^{\alpha(\frac{1}{2}-\beta)}, & \textit{if } \beta < \dfrac{1}{2}, \\ \left(1 \wedge \dfrac{\delta_\wedge(x,y) \vee t^{1/(\alpha\beta)}}{|x-y|}\right)^{\frac{\alpha}{2}} \log\left(e + \dfrac{(\delta_\vee(x,y) \vee t^{1/(\alpha\beta)}) \wedge |x-y|}{(\delta_\wedge(x,y) \vee t^{1/(\alpha\beta)}) \wedge |x-y|}\right), & \textit{if } \beta = \dfrac{1}{2}. \end{cases}$$

(ii) For all $(t,x,y) \in [1,\infty) \times D \times D$,

$$q(t,x,y) \simeq e^{-t\lambda_1^\beta} \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2}.$$

Remark 4.0.2. From the forms of the heat kernel estimates (4.0.1), one can easily see the following: (1) For x, y away from the boundary (in the sense that $\delta_\wedge(x,y) \geq |x-y| \vee t^{1/(\alpha\beta)}$), and for all $\beta \in (0,1)$, it holds that

$$q(t,x,y) \simeq t^{-d/(\alpha\beta)} \wedge \frac{t}{|x-y|^{d+\alpha\beta}}. \tag{4.0.2}$$

(2) Dividing (4.0.1) by t and letting $t \to 0$, we can deduce that the jump kernel is comparable with

$$\frac{B^{\alpha,\beta}(0,x,y)}{|x-y|^{d+\alpha\beta}}.$$

Thus, in view of the definition of $B^{\alpha,\beta}(t,x,y)$, one can rewrite the estimates (4.0.1) as follows:

$$q(t,x,y) \simeq \left(1 \wedge \frac{\delta_D(x)}{t^{1/(\alpha\beta)}}\right)^{\frac{\alpha}{2}} \left(1 \wedge \frac{\delta_D(y)}{t^{1/(\alpha\beta)}}\right)^{\frac{\alpha}{2}} \left(t^{-d/(\alpha\beta)} \wedge \sup_{|x-w|,|y-z|<t^{1/(\alpha\beta)}} (tJ(w,z))\right).$$

Recall that the two-sided estimates of the form (4.0.2) are valid for the heat kernel of the isotropic $\alpha\beta$-stable process in the whole space. The novelty of the estimates for $q(t,x,y)$ is in the boundary term, which is quite unusual and involves interplays among $\delta_\vee(x,y)$, $\delta_\wedge(x,y)$ and time t itself. In this respect, the form of the boundary term is very different from the boundary function $h(t,x,y)$ for the underlying process Y^D.

In this chapter, we obtain sharp two-sided estimates on the jump kernel, heat kernel and Green function for subordinate Markov processes in a setting which is more general, in several directions, than that of the example above. We allow (i) quite general subordinators, (ii) Markov processes with state space D that is either a bounded or an unbounded subset of a locally compact separable metric space, and (iii) very general form of two-sided estimates of the heat kernel $p_D(t,x,y)$ of the underlying process. We also show that parabolic functions with respect to X satisfy Hölder regularity and the parabolic Harnack inequality in Section 4.4.

4.1 Setup and main assumptions

Let (M,ρ) be a locally compact separable metric space such that all bounded closed sets are compact, and let m a positive Radon measure on M with full support. For simplicity, we write dy instead of $m(dy)$.

Let $V(x,r) := m(B(x,r))$. We assume that there exist a localization radius $R_0 \in (0,\infty]$ and constants $d_2 \geq d_1 > 0$ such that, for every $a \geq 1$, there exists a constant $C_V = C_V(a) \geq 1$ satisfying

$$C_V^{-1}\Big(\frac{R}{r}\Big)^{d_1} \leq \frac{V(x,R)}{V(x,r)} \leq C_V\Big(\frac{R}{r}\Big)^{d_2} \quad \text{for all } x \in M \;\text{ and }\; 0 < r \leq R < aR_0. \tag{4.1.1}$$

As a consequence of (4.1.1), we see that for all $R_0, \epsilon, \eta > 0$, there exists a constant $C = C(R_0,\epsilon,\eta) > 0$ such that

$$V(x,r) \leq CV(y,\eta r) \quad \text{for all } x,y \in M \;\text{ and }\; \epsilon\rho(x,y) < r \leq R_0. \tag{4.1.2}$$

If the localization radius R_0 is infinite, then the above constant C is independent of R_0 and (4.1.2) holds for $\epsilon\rho(x,y) < r < \infty$.

Let D be a proper open subset of M, and $Y^D = (Y_t^D, \mathbb{P}^x)$ be a Hunt process in D. We assume that the semigroup of Y^D admits a density $p_D(t,x,y)$.

Thus, for any non-negative Borel function f on D,

$$\mathbb{E}^x[f(Y_t^D)] = \int_D f(y)p_D(t,x,y)\,dy.$$

Let $S = (S_t)_{t\geq 0}$ be a driftless subordinator with Laplace exponent ϕ and tail measure w, independent of Y^D. We will be interested in the subordinate process $X_t := Y^D_{S_t}$. It is well known (cf. [26, p.67, pp. 73–75] and [122]) that X is also a Hunt process and admits a heat kernel $q(t,x,y)$ which is given by the formula

$$q(t,x,y) = \mathbb{E}[p_D(S_t,x,y)] = \int_0^\infty p_D(s,x,y)\mathbb{P}(S_t \in ds).$$

On the subordinator we will impose the assumption $\mathbf{Poly}_{R_1}(\beta_1,\beta_2)$ (see Definition 2.0.2 in Chapter 2). Now we explain the assumptions we impose on $p_D(t,x,y)$. These assumptions are motivated by various examples from the literature.

We first introduce two functions $\Phi, \Psi : [0,\infty) \to [0,\infty)$, both strictly increasing and satisfying $\Psi(r) \geq \Phi(r)$ for all $r \geq 0$. Moreover, we always assume that there exist constants $\alpha_1,\alpha_2,\alpha_3,\alpha_4 > 0$ and $c_1,c_2,c_3,c_4 > 0$ such that for all $R \geq r > 0$,

$$c_1\Big(\frac{R}{r}\Big)^{\alpha_1} \leq \frac{\Phi(R)}{\Phi(r)} \leq c_2\Big(\frac{R}{r}\Big)^{\alpha_2} \quad \text{and} \quad c_3\Big(\frac{R}{r}\Big)^{\alpha_3} \leq \frac{\Psi(R)}{\Psi(r)} \leq c_4\Big(\frac{R}{r}\Big)^{\alpha_4}. \tag{4.1.3}$$

Note that for every $a \geq 1$, there exist constants $c_1(a) > 0$ and $c_2(a) > 0$ such that, for all $r, R > 0$ satisfying $0 < r \leq aR$, it holds that

$$c_1(a)\Big(\frac{R}{r}\Big)^{\alpha_1} \leq \frac{\Phi(R)}{\Phi(r)} \leq c_2(a)\Big(\frac{R}{r}\Big)^{\alpha_2}. \tag{4.1.4}$$

Using [33, Lemmas 3.1 and 3.2], we may replace Φ by a nicer function.

Lemma 4.1.1. *There exists a strictly increasing differentiable function* $\widetilde{\Phi}$ *satisfying the following two properties:*
(P1) $\Phi(r) \simeq \widetilde{\Phi}(r)$ *for* $r > 0$ *and* $\widetilde{\Phi}$ *satisfies* (4.1.4)*;*

(P2) $\widetilde{\Phi}'(r) \simeq r^{-1}\widetilde{\Phi}(r)$ *and* $(\widetilde{\Phi}^{-1})'(t) \simeq t^{-1}\widetilde{\Phi}^{-1}(t)$ *for* $r, t > 0$.

Using the fact that $\sup_{u>0} u^k e^{-u} < \infty$ for every $k > 0$, we get the following lemma from the scaling property of Φ.

Lemma 4.1.2. *Let* $f : (0,\infty) \to (0,\infty)$ *be a given function. Assume that there exist constants* $c_1, p > 0$ *such that* $s^p f(s) \le c_1 t^p f(t)$ *for all* $0 < s \le t$. *Then there exists a constant* $c_2 = c_2(c_1, p) > 0$ *such that for all* $r, \kappa > 0$,

$$\int_0^r f(s) \exp\Big(- \frac{\kappa^2}{\Phi^{-1}(s)^2} \Big) ds \le \frac{c_2 r^{p+1} f(r)}{\Phi(\kappa)^p}.$$

Definition 4.1.3. We say that a function $h : (0,\infty) \times D \times D \to [0,1]$ is a *boundary function* if it satisfies the following two properties:

(H1) For each fixed $(x,y) \in D \times D$, $s \mapsto h(s,x,y)$ is nonincreasing.

(H2) There exist constants $c_1 > 0$, $\gamma \ge 0$ such that for all $x, y \in D$,

$$s^\gamma h(s,x,y) \le c_1 t^\gamma h(t,x,y), \quad 0 < s \le t < 4\Phi(\mathrm{diam}(D)) + 1,$$

with $4\Phi(\mathrm{diam}(D)) + 1$ interpreted as ∞ when D is unbounded.

A boundary function h is said to be *regular* if there exists $c_2 > 0$ such that for any $0 < t < 4\Phi(\mathrm{diam}(D)) + 1$,

$$h(t,x,y) \ge c_2 \quad \text{for all } x, y \in D \text{ with } \delta_\wedge(x,y) \ge \Phi^{-1}(t).$$

A boundary function h is said to be of *Harnack-type* if it is regular and there exists $c_3 > 0$ such that for all $x, y \in D$ and $0 < t < \Phi(\rho(x,y))$,

$$h(t,x,y) \le c_3 h(t,z,y) \quad \text{for all } z \in D, \ 2\rho(x,z) \le \rho(x,y) \wedge \delta_D(x). \quad (4.1.5)$$

From now on, $h(t,x,y)$ *always denotes a boundary function.*

Remark 4.1.4. If h is a regular boundary function, then for every $\epsilon \in (0,1)$, there exists $c = c(\epsilon) > 0$ such that for any $0 < t < 4\Phi(\mathrm{diam}(D)) + 1$,

$$h(t,x,y) \ge c \quad \text{for all } x, y \in D \text{ with } \delta_\wedge(x,y) \ge \epsilon\Phi^{-1}(t).$$

Example 4.1.5. (a) Let $p, q \geq 0$. For $t > 0$ and $x, y \in D$, define

$$h_{p,q}(t,x,y) := \Big(1 \wedge \frac{\Phi(\delta_D(x))}{t}\Big)^p \Big(1 \wedge \frac{\Phi(\delta_D(y))}{t}\Big)^q,$$
$$h_p(t,x,y) := h_{p,p}(t,x,y). \tag{4.1.6}$$

Then $h_{p,q}(t,x,y)$ is a typical example of a regular boundary function which is also of Harnack-type. The boundary function $h_p(t,x,y)$ is very typical when D is a bounded smooth open subset of $\mathbb{R}^d$.
(b) Let $h_p(t,x,y)$ be the function defined in (4.1.6). Then $h_p(t \wedge 1, x, y)$ is also a regular boundary function of Harnack-type. This is a typical boundary function for smooth exterior open sets.
(c) A quite general example of a boundary function is obtained as follows. Suppose that Y^D admits a dual process $\widehat{Y}^D$. Let ζ and $\widehat{\zeta}$ be the lifetimes of Y^D and $\widehat{Y}^D$ respectively. Assume that the survival probabilities $\mathbb{P}^x(\zeta > t)$ and $\mathbb{P}^y(\widehat{\zeta} > t)$ satisfy the following doubling property: $\mathbb{P}^x(\zeta > t/2) \simeq \mathbb{P}^x(\zeta > t)$ and $\mathbb{P}^y(\widehat{\zeta} > t/2) \simeq \mathbb{P}^y(\widehat{\zeta} > t)$ for all $0 < t < 4\Phi(\mathrm{diam}(D)) + 1$ and $x, y \in D$. Then $h(t,x,y) := \mathbb{P}^x(\zeta > t)\mathbb{P}^y(\widehat{\zeta} > t)$ is a boundary function. Moreover, the above $h(t,x,y)$ is of Harnack-type if, in addition, (1) it is regular; (2) Y^D satisfies the (interior elliptic) Harnack inequality and (3) there is $c_1 > 0$ such that for all $x \in D$ and $\Phi(\delta_D(x)) < t < \Phi(\mathrm{diam}(D))$,

$$\mathbb{P}^x(\zeta > t) \simeq \mathbb{P}^x(\zeta > \tau_{U(x,t)}) = \mathbb{P}^x(Y^D_{\tau_{U(x,t)}} \in D),$$

where $U(x,t) := B(x, c_1\Phi^{-1}(t)) \cap D$ and $\tau_V = \inf\{t > 0 : Y^D_t \notin V\}$.

In particular, under the setting and Assumptions **A** and **U** in Section 3.1, for the Hunt process Y defined right below (3.1.12) on a κ-fat open set D with a critical killing potential $\mu \in \mathbf{K}_1(D)$, by [58, Lemma 2.21], we know that the boundary function $h(t,x,y) = \mathbb{P}^x(\zeta > t)\mathbb{P}^y(\widehat{\zeta} > t)$ is of Harnack-type. See [19, 22, 41] for related work.

For later use, we record the following consequence of (H1) and (H2): Let

$k > 1$ and $s, t > 0$ satisfy $k^{-1}s \le t \le ks \le 4\Phi(\mathrm{diam}(D))$. Then

$$c_1^{-1}k^{-\gamma}h(s,x,y) \le h(t,x,y) \le c_1k^{\gamma}h(s,x,y) \quad \text{for all } x,y \in D. \tag{4.1.7}$$

Define

$$I_a(t,x,y,C_0) := \frac{1}{V(x,\Phi^{-1}(t))} \wedge \left(\frac{C_0 t}{V(x,\rho(x,y))\Psi(\rho(x,y))} + \frac{1}{V(x,\Phi^{-1}(t))}\exp\Big(-\frac{a\rho(x,y)^2}{\Phi^{-1}(t)^2}\Big)\right).$$

Definition 4.1.6. Let $h(t,x,y)$ be a boundary function.
(a) We say that $\mathbf{HK_B^h}$ holds, if D is bounded and the following estimates hold: (i) there exist constants $C_0 \in \{0,1\}$ and $c_1, c_2, c_3, c_4 > 0$ such that for all $(t,x,y) \in (0,1] \times D \times D$,

$$c_1h(t,x,y)I_{c_2}(t,x,y,C_0) \le p_D(t,x,y) \le c_3h(t,x,y)I_{c_4}(t,x,y,C_0), \tag{4.1.8}$$

and (ii) there exists a constant $\lambda_D > 0$ such that for all $(t,x,y) \in [1,\infty) \times D \times D$,

$$p_D(t,x,y) \simeq e^{-\lambda_D t}h(1,x,y). \tag{4.1.9}$$

(b) We say that $\mathbf{HK_U^h}$ holds, if the constant R_0 in (4.1.1) is infinite and (4.1.8) holds for all $(t,x,y) \in (0,\infty) \times D \times D$.

By using the function $(1 \wedge \frac{R_1}{10\Phi(\mathrm{diam}(D))})\Phi(r)$ instead of $\Phi(r)$, we may and do assume that $\Phi(\mathrm{diam}(D)) < R_1/8$ whenever $\mathbf{Poly}_{R_1}(\beta_1,\beta_2)$ and $\mathbf{HK_B^h}$ hold.

Remark 4.1.7. One can easily see that if $\mathbf{HK_B^h}$ holds, then for every $T > 0$, there exist constants $c_1, c_2, c_3, c_4 > 0$ such that (4.1.8) holds for all $(t,x,y) \in (0,T] \times D \times D$, and (4.1.9) holds for all $(t,x,y) \in [T,\infty) \times D \times D$.

Example 4.1.8. Here are several examples of processes satisfying $\mathbf{HK_B^h}$ or $\mathbf{HK_U^h}$. We will not try to give the most general examples but the reader will see from examples below that our setup is general enough to cover almost all known cases. In all examples below, the boundary functions are of Harnack type.

(a) Suppose that D is a bounded $C^{1,1}$ open subset of $\mathbb{R}^d$.

(1) If D is connected and Y^D is the killed Brownian motion in D, then $\mathbf{HK_B^h}$ is satisfied with $C_0 = 0$, $\Phi(r) = r^2$ and boundary function $h_{1/2}$. See [56] for a more general example.

(2) If $\alpha \in (0, 2)$ and Y^D is a killed isotropic α-stable process in D, then $\mathbf{HK_B^h}$ is satisfied with $\Phi(r) = \Psi(r) = r^\alpha$ and boundary function $h_{1/2}$, cf. [35]. More generally, suppose χ is a complete Bernstein function satisfying global weak scaling conditions with indices $\beta_1, \beta_2 \in (0, 1)$, Y is a subordinate Brownian motion in $\mathbb{R}^d$ via an independent subordinator with Laplace exponent χ, Y^D is the part process of Y in D. Then $\mathbf{HK_B^h}$ is satisfied with $\Phi(r) = \Psi(r) = 1/\chi(r^{-2})$ and boundary function $h_{1/2}$, cf. [41]. See [21, 73, 85] for more general examples.

(3) If D is connected and Y is the independent sum of isotropic α-stable process and Brownian motion, then its part process Y^D in D satisfies $\mathbf{HK_B^h}$ with $\Phi(r) = r^2 \wedge r^\alpha$, $\Psi(r) = r^\alpha$ and boundary function $h_{1/2}$, cf. [37]. More generally, suppose χ is a complete Bernstein function satisfying the conditions in the paragraph above and Y is the independent sum of Brownian motion and a subordinate Brownian motion via a subordinator with Laplace exponent χ, then its part process Y^D in D satisfies $\mathbf{HK_B^h}$ with $\Phi(r) = \Phi_\chi(r) := r^2 \wedge (1/\chi(r^{-2}))$, $\Psi(r) = 1/\chi(r^{-2})$ and boundary function $h_{1/2}$, cf. [43].

(4) Suppose that χ is a complete Bernstein function such that the function $\lambda \mapsto \chi(\lambda) - \lambda\chi'(\lambda)$ satisfies weak scaling conditions for $\lambda \geq a > 0$ with upper index $\delta < 2$ and lower index $\gamma > 2^{-1}\mathbf{1}_{\{\delta \geq 1\}}$. Suppose that Y is a subordinate Brownian motion in $\mathbb{R}^d$ via an independent subordinator with Laplace exponent χ, Y^D is the part process of Y in D. Then $\mathbf{HK_B^h}$ is satisfied with

$\Phi(r) = 1/\chi(r^{-2})$, $\Psi(r) = 1/(\chi(r^{-2}) - r^{-2}\chi'(r^{-2}))$ and boundary function $h_{1/2}$, cf. [88].

(5) Let $\alpha \in (1,2)$ and Y^D be a censored α-stable process in D. Then it follows from [36] that $\mathbf{HK_B^h}$ is satisfied with $\Phi(r) = \Psi(r) = r^\alpha$ and boundary function $h_{(\alpha-1)/\alpha}$.

(6) Let $\alpha \in (0,2)$ and Z^D be the part process, in D, of a reflected isotropic α-stable process in $\overline{D}$. For any $q \in [\alpha-1,\alpha)\cap(0,\alpha)$, let Y^D be the process on D corresponding to the Feynman-Kac semigroup of Z^D via the multiplicative functional $\exp(-\int_0^t C(d,\alpha,q)\text{dist}(Z_s^D, \partial D)^{-\alpha} ds)$, where the positive constant $C(d,\alpha,q)$ is defined in section 3.2.1. It follows from Theorem 3.2.2 that the small time estimates (4.1.8) holds with $\Phi(r) = \Psi(r) = r^\alpha$ and $h_{q/\alpha}$. Using the small time estimates and the argument in [50, Section 4], one can easily show that the large time estimates in Definition 4.1.6(a)(ii) also holds. Thus $\mathbf{HK_B^h}$ holds.

(7) Suppose that D is connected, $d \geq 3$ and $\kappa \geq -\frac{1}{4}$. Let Y^D be the process corresponding to $\Delta|_D - \kappa\delta_D(x)^{-2}$, the Dirichlet Laplacian in D with critical potential $\kappa\delta_D(x)^{-2}$. It follows from [64, (6)] and [66, Corollary 1.8] that the heat kernel of Y^D satisfies $\mathbf{HK_B^h}$ with $C_0 = 0$, $\Phi(r) = r^2$ and boundary function h_p, where $p = \frac{1}{2}(\frac{1}{2} + \sqrt{\frac{1}{4} + \kappa})$.

(8) Suppose that $\alpha \in (1,2)$ and $d \geq 2$. Let $b : \mathbb{R}^d \to \mathbb{R}^d$ such that $|b|$ is in the Kato class $\mathbb{K}_{d,\alpha-1}$ (see [38, Definition 1.1] for definition). Let Y be an α-stable process with drift b in $\mathbb{R}^d$, that is, a process with generator $-(-\Delta)^{\alpha/2} + b\cdot\nabla$, and let Y^D be the part process of Y in D. By [38, Theorem 1.3], $\mathbf{HK_B^h}$ holds with $\Phi(r) = \Psi(r) = r^\alpha$ and $h_{1/2}$. See also [90].

(9) For general setups in which $\mathbf{HK_B^h}$ is satisfied, see [58, Section 2] and [78].

(b) Suppose that D is an unbounded $C^{1,1}$ open subset of $\mathbb{R}^d$.

(1) If D is the domain above the graph of a bounded Lipschitz function in $\mathbb{R}^{d-1}$, then the killed Brownian motion in D satisfies $\mathbf{HK_U^h}$ with $C_0 = 0$, $\Phi(r) = r^2$ and a boundary function defined in terms of survival probabilities like in Example 4.1.5(b), which is of Harnack type, cf. [123].

(2) Suppose that D is a half-space-like $C^{1,1}$ open set in $\mathbb{R}^d$ and $\alpha \in (0,2)$. Let Y^D be the part process in D of an isotropic α-stable process. Then by [52, Theorem 1.2], $\mathbf{HK_U^h}$ is satisfied with $\Phi(r) = \Psi(r) = r^\alpha$ and boundary function $h_{1/2}$. More generally, let Y^D be the part process in D of the independent sum of Brownian motion and an isotropic α-stable process. By [39, Theorem 1.4 and Remark 1.5(ii)], $\mathbf{HK_U^h}$ is satisfied with $\Phi(r) = r^2 \wedge r^\alpha$, $\Psi(r) = r^\alpha$ and boundary function $h_{1/2}$. When D is an exterior $C^{1,1}$ open set in $\mathbb{R}^d$ with $d > \alpha$ and Y^D is part process in D of an isotropic α-stable process, it follows from [52, Theorem 1.2] that $\mathbf{HK_U^h}$ is satisfied with $\Phi(r) = \Psi(r) = r^\alpha$ and boundary function $h_{1/2}(t \wedge 1, x, y)$. See [84] for a more general example.

(3) Suppose $D = \mathbb{R}^d_+$. Let χ be a complete Bernstein function satisfying global weak scaling conditions with indices $\alpha_1, \alpha_2 \in (0,1)$, Y be a subordinate Brownian motion in $\mathbb{R}^d$ via an independent subordinator with Laplace exponent χ, Y^D be the part process of Y in D. It follows from [95, Theorem 5.10] that $\mathbf{HK_U^h}$ is satisfied with $\Phi(r) = \Psi(r) = 1/\chi(r^{-2})$ and boundary function $h_{1/2}$. See [31] for a more general example.

(4) Suppose that $D = \mathbb{R}^d_+$ and $\alpha \in (0,2)$. Let Z^D be the part process, in D, of a reflected isotropic α-stable process in $\overline{D}$. For any $q \in [\alpha-1, \alpha) \cap (0, \alpha)$, let Y^D be the process on D corresponding to the Feynman-Kac semigroup of Z^D via the multiplicative functional $\exp(-\int_0^t C(d,\alpha,q)\delta_D(Z_s^D)^{-\alpha} ds)$, where $C(d,\alpha,q)$ is defined in subsection 3.2.1. It follows from Theorem 3.2.2 that $\mathbf{HK_U^h}$ is satisfied with $\Phi(r) = \Psi(r) = r^\alpha$ and boundary function $h_{q/\alpha}$.

(5) Suppose that $D = \mathbb{R}^d \setminus \{0\}$ and $\alpha \in (0,2)$. Let Z be an isotropic α-stable process in $\mathbb{R}^d$. For any $q \in (0,\alpha)$, let Y^D be the process on D corresponding to the Feynman-Kac semigroup of Z^D via the multiplicative functional $\exp(-\int_0^t \widetilde{C}(d,\alpha,q)|Z_s^D|^{-\alpha} ds)$, where $\widetilde{C}(d,\alpha,q)$ is defined in subsection 3.2.3. It follows from Theorem 3.2.9 and [82, Theorem 1.1] that $\mathbf{HK_U^h}$ is satisfied with $\Phi(r) = \Psi(r) = r^\alpha$ and boundary function $h_{q/\alpha}$.

(6) Suppose that $D = \mathbb{R}^d \setminus \{0\}$, $d \geq 2$ or $D = (0,\infty)$. Let Y^D be a process with generator $\Delta + (a-1)|x|^{-2}\sum_{i,j=1}^d x_i x_j \partial_{ij} + \kappa |x|^{-2} \cdot \nabla - b|x|^{-2}$ for some

$a > 0$, $\kappa, b \in \mathbb{R}$ such that

$$\Lambda := \frac{1}{2}\sqrt{\frac{b}{a} + \Big(\frac{d-1+\kappa-a}{2a}\Big)^2} \geq \frac{1}{4a}\big((d-1+\kappa-a)\vee((2a-1)d+1-\kappa-3a)\big).$$

Note that when $a = 1$ and $\kappa, b \geq 0$, the above inequality is always true. It follows from [106, Proposition 4.14, Theorem 6.2, Corollary 6.4] that $\mathbf{HK_U^h}$ is satisfied with $C_0 = 0$, $\Phi(r) = r^2$ and boundary function $h_{p,q}$ where $p = \Lambda - (d-1+\kappa-a)/(4a)$ and $q = \Lambda - ((2a-1)d+1-\kappa-3a)/(4a)$.

(7) Suppose that $\alpha \in (1,2)$ and $D = \mathbb{R}^d \setminus \{0\}$, $d \geq 3$. Let Y^D be a process with generator $-(-\Delta)^{-\alpha/2} + \kappa|x|^{-\alpha}x\cdot\nabla$ for some $\kappa \in (0,\infty)$. It follows from [98, Theorems 4 and 5] that $\mathbf{HK_U^h}$ is satisfied with $\Phi(r) = \Psi(r) = r^\alpha$ and boundary function $h = h_{0,\beta/\alpha}$ for $\beta \in (0,\alpha)$ determined by the equation at the beginning of [98, Section 3.2].

We briefly discuss the term $I_a(t,x,y,C_0)$ appearing in (4.1.8). If $C_0 = 0$, then clearly

$$I_a(t,x,y,0) = \frac{1}{V(x,\Phi^{-1}(t))}\exp\Big(-\frac{a\rho(x,y)^2}{\Phi^{-1}(t)^2}\Big). \tag{4.1.10}$$

Suppose now that $C_0 = 1$ and $a > 0$.

Lemma 4.1.9. *For any $K \geq 1$, there are comparability constants depending on K such that when $t \geq K^{-1}\Phi(\rho(x,y))$,*

$$I_a(t,x,y,1) \simeq \frac{1}{V(x,\Phi^{-1}(t))}$$

and when $t \leq K\Phi(\rho(x,y))$,

$$I_a(t,x,y,1) \simeq \frac{t}{V(x,\rho(x,y))\Psi(\rho(x,y))} + \frac{1}{V(x,\Phi^{-1}(t))}\exp\Big(-\frac{a\rho(x,y)^2}{\Phi^{-1}(t)^2}\Big).$$

In particular, if $\Psi(r) \simeq \Phi(r)$ *for* $r \in (0, \mathrm{diam}(D))$*, then for each fixed* $a > 0$*,*

$$I_a(t,x,y,1) \simeq \frac{1}{V(x,\Phi^{-1}(t))} \wedge \frac{t}{V(x,\rho(x,y))\Phi(\rho(x,y))}, \quad t>0,\ x,y \in D. \tag{4.1.11}$$

4.2 Jump kernel and heat kernel estimates

With the tail measure w of the subordinator S, for a given boundary function h, we define for $(t,x,y) \in [0,\infty) \times D \times D$,

$$\mathcal{B}_h^*(x,y) := \int_0^{\Phi(\rho(x,y))} h(s,x,y)w(s)ds \tag{4.2.1}$$

and if $\phi^{-1}(1/t)^{-1} \le \Phi(\rho(x,y))$,

$$\mathcal{B}_h(t,x,y) := \int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(\rho(x,y))} h(s,x,y)w(s)ds. \tag{4.2.2}$$

Since $\int_0^r w(s)ds < \infty$ for all $r > 0$ and $h \le 1$, the integral in (4.2.1) converges. Note that, by (H1), $\mathcal{B}_h^*(x,y) \simeq \mathcal{B}_h(0,x,y)$ for all $(x,y) \in D \times D$.

4.2.1 Jump kernel estimates

The jump kernel of the subordinate process X is given by

$$J(x,y) = \int_0^\infty p_D(s,x,y)\nu(ds)\,, \quad x,y \in D. \tag{4.2.3}$$

See [26, p.74] and also [110].

Theorem 4.2.1. *Suppose that either (1)* $\mathbf{Poly}_{R_1}(\beta_1,\beta_2)$ *and* $\mathbf{HK}_{\mathbf{B}}^{\mathbf{h}}$ *hold, or (2)* $\mathbf{Poly}_\infty(\beta_1,\beta_2)$ *and* $\mathbf{HK}_{\mathbf{U}}^{\mathbf{h}}$ *hold. Then, for* $(x,y) \in D \times D$ *with* $x \neq y$*,*

$$J(x,y) \simeq \frac{C_0\mathcal{B}_h^*(x,y)}{V(x,\rho(x,y))\Psi(\rho(x,y))} + h(\Phi(\rho(x,y)),x,y)\frac{w\big(\Phi(\rho(x,y))\big)}{V(x,\rho(x,y))}. \tag{4.2.4}$$

Proof. Since the proofs are similar, we only give the proof of the case (1), which is more complicated. Fix $x, y \in D$ with $x \neq y$ and let $r := \rho(x, y) > 0$. By Remark 4.1.7, (4.1.8) and (4.1.9) hold with $T := \Phi(2\text{diam}(D))$. Then by (4.2.3) and Lemma 4.1.9,

$$\begin{aligned}
J(x,y) \asymp \frac{C_0}{V(x,r)\Psi(r)} &\int_0^{\Phi(r)} sh(s,x,y)\nu(ds) \\
&+ \int_0^{\Phi(r)} \frac{h(s,x,y)}{V(x,\Phi^{-1}(s))} \exp\Big(-\frac{cr^2}{\Phi^{-1}(s)^2}\Big)\nu(ds) \\
&+ \int_{\Phi(r)}^{T} \frac{h(s,x,y)}{V(x,\Phi^{-1}(s))}\nu(ds) + h(1,x,y)\int_T^\infty e^{-\lambda_D s}\nu(ds) \\
=:& \, C_0 J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

By $\mathbf{Poly}_{R_1}(\beta_1, \beta_2)$, there exists $a > 1$ such that $w(s/a) \geq 2w(s)$ for all $s < R_1$. Therefore, by (4.1.7), since we assumed $\Phi(\text{diam}(D)) < R_1/8$,

$$\begin{aligned}
V(x,r)\Psi(r)J_1 &= \sum_{i\in\mathbb{N}} \int_{a^{-i}\Phi(r)}^{a^{-i+1}\Phi(r)} sh(s,x,y)\nu(ds) \\
&\simeq \sum_{i\in\mathbb{N}} a^{-i}\Phi(r)h(a^{-i}\Phi(r),x,y)\big(w(a^{-i}\Phi(r)) - w(a^{-i+1}\Phi(r))\big) \\
&\simeq \sum_{i\in\mathbb{N}} a^{-i}\Phi(r)h(a^{-i}\Phi(r),x,y)w(a^{-i}\Phi(r)) \simeq \mathcal{B}_h^*(x,y).
\end{aligned}$$

Next, by (H1), the scaling and monotonicity of Φ, we get that

$$\begin{aligned}
J_2 &\geq \frac{h(\Phi(r),x,y)}{V(x,r)} \int_{\Phi(r)/a}^{\Phi(r)} \exp\Big(-\frac{c_1 r^2}{\Phi^{-1}(s)^2}\Big)\nu(ds) \\
&\geq \frac{c_2 h(\Phi(r),x,y)}{V(x,r)} \int_{\Phi(r)/a}^{\Phi(r)} \nu(ds) \geq \frac{c_2 h(\Phi(r),x,y)w(\Phi(r))}{V(x,r)}.
\end{aligned}$$

Hence, we obtain the lower bound in (4.2.4).

Now, we prove the upper bound in (4.2.4). Let $\widetilde{\Phi}$ be the function in Lemma 4.1.1. Since $s \mapsto V(x, \widetilde{\Phi}^{-1}(s))^{-1}$ and $s \mapsto h(s, x, y)$ are nonincreasing, using the Leibniz rule for product, integration by parts and the property (P2) of $\widetilde{\Phi}^{-1}$ in Lemma 4.1.1, we obtain

$$\begin{aligned} J_2 &\le c\int_0^{\Phi(r)} \frac{h(s,x,y)}{V(x,\widetilde{\Phi}^{-1}(s))} \exp\Big(-\frac{c_3r^2}{\widetilde{\Phi}^{-1}(s)^2}\Big)\Big(-\frac{d}{ds}w(s)\Big) \\ &\le c\int_0^{\Phi(r)} \frac{h(s,x,y)w(s)}{V(x,\widetilde{\Phi}^{-1}(s))}\Big(\frac{d}{ds}\exp\Big(-\frac{c_3r^2}{\widetilde{\Phi}^{-1}(s)^2}\Big)\Big)ds \\ &\le c\int_0^{\Phi(r)} \frac{h(s,x,y)w(s)}{V(x,\widetilde{\Phi}^{-1}(s))}\frac{r^2}{s\widetilde{\Phi}^{-1}(s)^2}\exp\Big(-\frac{c_3r^2}{\widetilde{\Phi}^{-1}(s)^2}\Big)ds. \end{aligned} \tag{4.2.5}$$

In the second inequality above, we used the following: Since $h \le 1$, $e^{-x} \le k^k x^{-k}$ for all $x, k > 0$ and $\lim_{s\to 0} sw(s) = 0$ (because w is the tail of the Lévy mesure ν), by using (4.1.1) and the scaling of $\widetilde{\Phi}^{-1}$, we have that

$$\begin{aligned} &\lim_{s\to 0}\frac{h(s,x,y)w(s)}{V(x,\widetilde{\Phi}^{-1}(s))}\exp\Big(-\frac{c_3r^2}{\widetilde{\Phi}^{-1}(s)^2}\Big) \le c\lim_{s\to 0}\frac{w(s)}{V(x,\widetilde{\Phi}^{-1}(s))}\Big(\frac{\widetilde{\Phi}^{-1}(s)^2}{r^2}\Big)^{(d_2+\alpha_2)/2} \\ &\le \frac{c}{r^{d_2+\alpha_2}V(x,\widetilde{\Phi}^{-1}(1))}\lim_{s\to 0} w(s)\widetilde{\Phi}^{-1}(s)^{\alpha_2} \le \frac{c\,\widetilde{\Phi}^{-1}(1)^{\alpha_2}}{r^{d_2+\alpha_2}V(x,\widetilde{\Phi}^{-1}(1))}\lim_{s\to 0} sw(s) = 0. \end{aligned}$$

By $\mathbf{Poly}_{R_1}(\beta_1,\beta_2)$, (H2), (4.1.1), (4.1.4) and the fact that $\Phi \simeq \widetilde{\Phi}$, we can use Lemma 4.1.2 with $f(s) = h(s,x,y)w(s)\ V(x,\widetilde{\Phi}^{-1}(s))^{-1}s^{-1}\widetilde{\Phi}^{-1}(s))^{-2}$ and $p = \gamma+\beta_2+1+(d_2+2)/\alpha_1$ to deduce from (4.2.5) that

$$J_2 \le \frac{ch(\Phi(r),x,y)w(\Phi(r))}{V(x,r)}. \tag{4.2.6}$$

For J_3 and J_4, since $s \mapsto V(x,\Phi^{-1}(s))^{-1}$, $s \mapsto h(s,x,y)$ and $s \mapsto w(s)$ are nonincreasing, we have by the boundedness of D that

$$J_3 + J_4 \le \frac{h(\Phi(r),x,y)w(\Phi(r))}{V(x,r)} + h(1,x,y)w(T) \le \frac{ch(\Phi(r),x,y)w(\Phi(r))}{V(x,r)}.$$

This completes the proof. □

Suppose that $\Psi \simeq \Phi$ and $C_0 = 1$. Then the first term in (4.2.4) dominates the second in view of (1.1.2). Moreover, if $\mathbf{Poly}_{R_1}(\beta_1,\beta_2)$ holds with $\beta_2 < 1$, then according to [109, Lemma 2.6, Proposition 2.9], we get that $w(s) \simeq$

$\phi(1/s)$ for all $0 < s < R_1/2$ and hence

$$J(x,y) \simeq \frac{1}{V(x,\rho(x,y))\Phi(\rho(x,y))} \int_0^{\Phi(\rho(x,y))} h(s,x,y)\phi(1/s)\,ds.$$

In case the boundary function is equal to $h_{1/2}$, the integral above can be estimated in the same way as in [97, Lemma 8.1], cf. [97, (8.4)].

Suppose that $C_0 = 0$. Then

$$J(x,y) \simeq h(\Phi(\rho(x,y)) \frac{w(h(\Phi(\rho(x,y)),x,y)}{V(x,\rho(x,y))}. \tag{4.2.7}$$

In particular, in the context of Example 4.1.8(b-1), and assuming $\beta_2 < 1$, the above formula reduces to [96, Theorem 4.4.(1)]. Similarly, if D is an exterior $C^{1,1}$ domain in $\mathbb{R}^d$, the boundary function is equal to $h_{1/2}(t \wedge 1, x, y)$ and $\beta_2 < 1$, then (4.2.7) reduces to [96, Theorem 4.4.(2)].

4.2.2 Heat kernel estimates

Let

$$\psi(r) := \frac{1}{\phi(1/\Phi(r))}, \quad r > 0. \tag{4.2.8}$$

Since ϕ and Φ are strictly increasing, ψ is also strictly increasing. Moreover, it follows from $\mathbf{Poly}_{R_1}(\beta_1,\beta_2)$, Lemma 2.1.1 and (4.1.3) that, for every $R_0 > 0$, there exist $c_1, c_2 > 0$ such that

$$c_1\Big(\frac{R}{r}\Big)^{\alpha_1\beta_1} \le \frac{\psi(R)}{\psi(r)} \le c_2\Big(\frac{R}{r}\Big)^{\alpha_2(\beta_2\wedge 1)}, \quad 0 < r < R < R_0. \tag{4.2.9}$$

In case when $\mathbf{Poly}_\infty(\beta_1,\beta_2)$ holds, (4.2.9) is valid with $R_0 = \infty$. Note that

$$\psi^{-1}(t) = \Phi^{-1}\big(\phi^{-1}(1/t)^{-1}\big), \quad t > 0. \tag{4.2.10}$$

Recall the definition of the function $\mathcal{B}_h(t,x,y)$ from (4.2.2).

Theorem 4.2.2. *Suppose that* $\mathbf{Poly}_{R_1}(\beta_1,\beta_2)$ *and* $\mathbf{HK}_{\mathbf{B}}^{\mathbf{h}}$ *hold. Then for every* $T > 0$*, the following estimates are valid for all* $(t,x,y) \in (0,T] \times D \times D$*:*

(i) *If* $\psi(\rho(x,y)) \le t$, *then*

$$q(t,x,y) \simeq \frac{h(\phi^{-1}(1/t)^{-1},x,y)}{V(x,\psi^{-1}(t))}. \tag{4.2.11}$$

(ii) *If* $\psi(\rho(x,y)) \ge t$, *then*

$$\begin{aligned} q(t,x,y) \asymp\ & \frac{C_0}{V(x,\rho(x,y))\Psi(\rho(x,y))}\Big(t\mathcal{B}_h(t,x,y) + \frac{h(\phi^{-1}(1/t)^{-1},x,y)}{\phi^{-1}(1/t)}\Big) \\ & + \frac{h(\phi^{-1}(1/t)^{-1},x,y)}{V(x,\psi^{-1}(t))}\exp\Big(-\frac{c\,\rho(x,y)^2}{\psi^{-1}(t)^2}\Big) \\ & + h(\Phi(\rho(x,y)),x,y)\frac{tw\big(\Phi(\rho(x,y))\big)}{V(x,\rho(x,y))}. \end{aligned}$$

Proof. Take $x,y \in D$ and let $r := \rho(x,y)$. We start by establishing some relations valid for all $t \in (0,T]$. By Corollary 2.2.2, there exist constants $\delta,\epsilon \in (0,1)$ such that

$$q(t,x,y) \ge \delta \inf_{s\in[\epsilon\phi^{-1}(1/t)^{-1},\phi^{-1}(1/t)^{-1}]} p_D(s,x,y), \quad t \in (0,T]. \tag{4.2.12}$$

On the other hand, by Remark 4.1.7 (with $T = \Phi(\mathrm{diam}(D))$), (4.1.10), Lemma 4.1.9, (4.1.9) and the fact that $\exp(-cr^2/\Phi^{-1}(s)^2) \simeq 1$ when $s > \Phi(r)$, we see that

$$\begin{aligned} q(t,x,y) \asymp\ & C_0\int_0^{\Phi(r)} \frac{sh(s,x,y)}{V(x,r)\Psi(r)}\mathbb{P}(S_t \in ds) \\ & + \int_0^{\Phi(\mathrm{diam}(D))} \frac{h(s,x,y)}{V(x,\Phi^{-1}(s))}\exp\Big(-\frac{cr^2}{\Phi^{-1}(s)^2}\Big)\mathbb{P}(S_t \in ds) \\ & + h(1,x,y)\int_{\Phi(\mathrm{diam}(D))}^{\infty} e^{-\lambda_D s}\mathbb{P}(S_t \in ds) \\ =:\ & C_0 I_1 + I_2 + I_3. \end{aligned} \tag{4.2.13}$$

(i) Assume that $\psi(r) \le t$. By Remark 4.1.7, (H1), (4.1.1), the scaling of

Φ^{-1} and (4.2.10), there exists $c_1 > 0$ such that

$$\inf_{s\in[\epsilon\phi^{-1}(1/t)^{-1},\phi^{-1}(1/t)^{-1}]} p_D(s,x,y) \geq c_1 \frac{h(\phi^{-1}(1/t)^{-1},x,y)}{V(x,\psi^{-1}(t))}, \quad t \in (0,T].$$

Hence, the lower bound in (4.2.11) follows from (4.2.12).

Now, we prove the upper bound in (4.2.11). First, using Corollary 2.2.3 in the first inequality below, the assumption $\Psi \geq \Phi$ and Lemma 2.1.7 with $N = \gamma + d_2/\alpha_1$ in the second, (H2), (4.2.10) and (4.1.3) in the third, and (4.1.1) in the last, we get that

$$\begin{aligned} I_1 &\leq \frac{\Phi(r)h(\Phi(r),x,y)}{V(x,r)\Psi(r)} \exp\Big(-\frac{t}{2}(H\circ\sigma)\big(t,\Phi(r)\big)\Big) \\ &\leq c_2 \frac{h(\Phi(r),x,y)}{V(x,r)} \big(\Phi(r)\phi^{-1}(1/t)\big)^{\gamma} \big(\Phi(r)\phi^{-1}(1/t)\big)^{d_2/\alpha_1} \\ &\leq c_3 \frac{h(\phi^{-1}(1/t)^{-1},x,y)}{V(x,r)} \Big(\frac{r}{\psi^{-1}(t)}\Big)^{d_2} \leq c_4 \frac{h(\phi^{-1}(1/t)^{-1},x,y)}{V(x,\psi^{-1}(t))}. \end{aligned}$$

Next, we observe that

$$\begin{aligned} I_2 &\leq \int_0^{\phi^{-1}(1/t)^{-1}} \frac{h(s,x,y)}{V(x,\Phi^{-1}(s))}\mathbb{P}(S_t \in ds) + \int_{\phi^{-1}(1/t)^{-1}}^{\infty} \frac{h(s,x,y)}{V(x,\Phi^{-1}(s))}\mathbb{P}(S_t \in ds) \\ &=: I_{2,1} + I_{2,2}. \end{aligned}$$

By (H2), (4.1.1) and (4.1.3), we can apply Corollary 2.2.3 with $p = \gamma + d_2/\alpha_1$ to get that

$$I_{2,1} \leq c_5 \frac{h(\phi^{-1}(1/t)^{-1},x,y)}{V(x,\psi^{-1}(t))}.$$

Moreover, we see from (H1), (4.2.10) and the monotonicity of ψ^{-1} that

$$I_{2,2} \leq \frac{h(\phi^{-1}(1/t)^{-1},x,y)}{V(x,\psi^{-1}(t))}\mathbb{P}(S_t \geq \phi^{-1}(1/t)^{-1}) \leq \frac{h(\phi^{-1}(1/t)^{-1},x,y)}{V(x,\psi^{-1}(t))}.$$

Lastly, by using (H1) and (H2), since ϕ and ψ are increasing and $t \leq T$, we have that

$$I_3 \le h(1,x,y) \le c_6 \frac{h(\phi^{-1}(1/t)^{-1},x,y)}{V(x,\psi^{-1}(t))}.$$

Hence, we obtain the upper bound in (4.2.11) from (4.2.13).

(ii) Assume that $\psi(r) \ge t$. First we establish the lower bound. From (4.2.12), Remark 4.1.7, Lemma 4.1.9, (4.2.10), (H1), and the scaling and monotonicity of ψ^{-1}, we get that

$$\begin{aligned} q(t,x,y) &\ge c_7 h(\phi^{-1}(1/t)^{-1},x,y) \\ &\quad \times \Big[\frac{C_0\phi^{-1}(1/t)^{-1}}{V(x,r)\Psi(r)} + \frac{1}{V(x,\psi^{-1}(t))} \exp\Big(-\frac{c_8 r^2}{\psi^{-1}(t)^2}\Big)\Big]. \end{aligned} \quad (4.2.14)$$

We also see from Remark 4.1.7 that

$$q(t,x,y) \ge \frac{c_9 C_0}{V(x,r)\Psi(r)} \int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r)} sh(s,x,y)\mathbb{P}(S_t \in ds). \quad (4.2.15)$$

Let $K>1$ be the constant in (2.2.6). If $\Phi(r) > K\phi^{-1}(1/t)^{-1}$, then by (H1), (H2) and (2.2.6), since we assumed $\Phi(\mathrm{diam}(D)) < R_1/8$,

$$\begin{aligned} &\int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r)} sh(s,x,y)\mathbb{P}(S_t \in ds) \\ &\ge \sum_{\substack{i\in\mathbb{N} \\ K^i \le 2\Phi(r)\phi^{-1}(1/t)}} \int_{2K^{i-1}\phi^{-1}(1/t)^{-1}}^{2K^i\phi^{-1}(1/t)^{-1}} sh(s,x,y)\mathbb{P}(S_t \in ds) \\ &\ge \frac{c_{10}t}{K} \sum_{\substack{i\in\mathbb{N} \\ K^i \le 2\Phi(r)\phi^{-1}(1/t)}} w\big(2K^{i-1}\phi^{-1}(1/t)^{-1}\big) \int_{2K^{i-1}\phi^{-1}(1/t)^{-1}}^{2K^i\phi^{-1}(1/t)^{-1}} ds \\ &\ge \frac{c_{10}t}{K} \int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r)/K} h(s,x,y)w(s)ds \simeq t \int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r)} h(s,x,y)w(s)ds. \end{aligned}$$

The last comparison is valid due to (1.1.2). Hence, we deduce from (4.2.15) that

$$q(t,x,y) \ge c_{11} \frac{C_0 t\mathcal{B}_h(t,x,y)}{V(x,r)\Psi(r)}.$$

In case when $\Phi(r) \le K\phi^{-1}(1/t)^{-1}$, we see from (H1), (2.0.4) and (4.2.14) that

$$\begin{aligned}\frac{C_0 t\mathcal{B}_h(t,x,y)}{V(x,r)\Psi(r)} &\le \frac{C_0 t}{V(x,r)\Psi(r)}\int_{2\phi^{-1}(1/t)^{-1}}^{4K\phi^{-1}(1/t)^{-1}} h(s,x,y)w(s)ds\\ &\le 4KC_0\frac{t\phi^{-1}(1/t)^{-1}}{V(x,r)\Psi(r)}h(\phi^{-1}(1/t)^{-1},x,y)\,2e\phi\big(\phi^{-1}(1/t)\big)\\ &\le c_{12}\,q(t,x,y).\end{aligned}$$

By using (2.2.6), Lemma 4.1.9, (H1), (4.1.1) and the scaling of Φ, we get that, if $4\Phi(r)/K > 2\phi^{-1}(1/t)^{-1}$, then

$$q(t,x,y) \ge c_{13}\int_{4\Phi(r)/K}^{4\Phi(r)} \frac{h(s,x,y)}{V(x,\Phi^{-1}(s))}\mathbb{P}(S_t\in ds) \ge c_{14}h(\Phi(r),x,y)\frac{tw(\Phi(r))}{V(x,r)}.$$

If $4\Phi(r)/K \le 2\phi^{-1}(1/t)^{-1}$, then $t \le \psi(r) \le c_{15}t$ for some $c_{15}>0$. Moreover, by (4.2.10) and (2.0.4),

$$tw(\Phi(r)) \le tw(\phi^{-1}(1/t)^{-1}) \le 2et\phi\big(\phi^{-1}(1/t)\big) = 2e.$$

Therefore, by (H1), (4.2.14) (neglecting the first term) and (4.1.3), we obtain

$$q(t,x,y) \ge c_7\frac{h(\Phi(r),x,y)}{V(x,r)}\exp\Big(-\frac{c_8c_{15}\psi^{-1}(t)^2}{\psi^{-1}(t)^2}\Big) \ge c_{16}h(\Phi(r),x,y)\frac{tw(\Phi(r))}{V(x,r)}.$$

This completes the proof of the lower bound.

Now we prove the upper bound. Recall (4.2.13). Observe that

$$\begin{aligned}&V(x,r)\Psi(r)I_1\\ &\le \int_0^{\phi^{-1}(1/t)^{-1}} sh(s,x,y)\mathbb{P}(S_t\in ds) + \int_{\phi^{-1}(1/t)^{-1}}^{2\phi^{-1}(1/t)^{-1}} sh(s,x,y)\mathbb{P}(S_t\in ds)\\ &\quad + \int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r)} sh(s,x,y)\mathbb{P}(S_t\in ds) =: K_1+K_2+K_3.\end{aligned}$$

By Corollary 2.2.3, $K_1 \le c_{17}\phi^{-1}(1/t)^{-1}h(\phi^{-1}(1/t)^{-1},x,y)$. Moreover, by (H1), we have $K_2 \le 2\phi^{-1}(1/t)^{-1}\ h(\phi^{-1}(1/t)^{-1},x,y)$. To bound K_3, we use integration by parts and Theorem 2.2.6 to obtain

$$\begin{aligned}
K_3 &= \int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r)} sh(s,x,y)\frac{d}{ds}\big(-\mathbb{P}(S_t \geq s)\big) \\
&\leq 2\phi^{-1}(1/t)^{-1}h(\phi^{-1}(1/t)^{-1},x,y) + \int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r)} h(s,x,y)\mathbb{P}(S_t \geq s)ds \\
&\quad + \int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r)} s\mathbb{P}(S_t \geq s)\frac{dh(s,x,y)}{ds} \\
&\leq c_{18}\big(\phi^{-1}(1/t)^{-1}h(\phi^{-1}(1/t)^{-1},x,y) + t\mathcal{B}_h(t,x,y)\big).
\end{aligned}$$

In the second inequality above, we used the fact that $s \mapsto h(s,x,y)$ is non-increasing (so that $s \mapsto \frac{d}{ds}h(s,x,y) \leq 0$ a.e.).

Now, we estimate I_2. We have

$$\begin{aligned}
I_2 &\leq \int_0^{2\phi^{-1}(1/t)^{-1}} \frac{h(s,x,y)}{V(x,\Phi^{-1}(s))}\exp\Big(-\frac{cr^2}{\Phi^{-1}(s)^2}\Big)\mathbb{P}(S_t \in ds) \\
&\quad + \int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r)} \frac{h(s,x,y)}{V(x,\Phi^{-1}(s))}\exp\Big(-\frac{cr^2}{\Phi^{-1}(s)^2}\Big)\mathbb{P}(S_t \in ds) \\
&\quad + \int_{4\Phi(r)}^{\infty} \frac{h(s,x,y)}{V(x,\Phi^{-1}(s))}\mathbb{P}(S_t \in ds) \\
&=: L_1 + L_2 + L_3.
\end{aligned}$$

By applying Corollary 2.2.3, we get from (H1), (H2), (4.1.1), the scaling of Φ and (4.2.10) that

$$\begin{aligned}
L_1 &\leq \exp\Big(-\frac{cr^2}{\Phi^{-1}(2\phi^{-1}(1/t)^{-1})^2}\Big)\int_0^{2\phi^{-1}(1/t)^{-1}} \frac{h(s,x,y)}{V(x,\Phi^{-1}(s))}\mathbb{P}(S_t \in ds) \\
&\leq c_{19}\frac{h(\phi^{-1}(1/t)^{-1},x,y)}{V(x,\psi^{-1}(t))}\exp\Big(-\frac{c_{20}r^2}{\psi^{-1}(t)^2}\Big).
\end{aligned}$$

Let $\widetilde{\Phi}$ be the function in Lemma 4.1.1. By using integration by parts and similar calculations to (4.2.5), we get that

$$L_2 \leq c_{21}\int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r)} \frac{h(s,x,y)}{V(x,\widetilde{\Phi}^{-1}(s))}\exp\Big(-\frac{c_{22}r^2}{\widetilde{\Phi}^{-1}(s)^2}\Big)\frac{d}{ds}\big(-\mathbb{P}(S_t \geq s)\big)$$

$$\leq c_{23}\left[\frac{h(2\phi^{-1}(1/t)^{-1},x,y)}{V(x,\widetilde{\Phi}^{-1}(2\phi^{-1}(1/t)^{-1}))}\exp\Big(-\frac{c_{22}r^2}{\widetilde{\Phi}^{-1}(2\phi^{-1}(1/t)^{-1})^2}\Big)\right.$$
$$\left.+\int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r)}\frac{h(s,x,y)}{V(x,\widetilde{\Phi}^{-1}(s))}\mathbb{P}(S_t\geq s)\frac{r^2}{s\widetilde{\Phi}^{-1}(s)^2}\exp\Big(-\frac{c_{22}r^2}{\widetilde{\Phi}^{-1}(s)^2}\Big)ds\right]$$
$$=: c_{23}\,(L_{2,1}+L_{2,2}).$$

By (H1), (4.1.1), the scaling of Φ and (4.2.10), since $\Phi\simeq\widetilde{\Phi}$, we see that

$$L_{2,1}\leq c_{24}\frac{h(\phi^{-1}(1/t)^{-1},x,y)}{V(x,\psi^{-1}(t))}\exp\Big(-\frac{c_{25}r^2}{\psi^{-1}(t)^2}\Big).$$

Also, by using Theorem 2.2.6 and repeating the arguments for obtaining (4.2.6), we get that

$$L_{2,2}\leq c_{26}t\int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r)}\frac{h(s,x,y)w(s)}{V(x,\widetilde{\Phi}^{-1}(s))}\frac{r^2}{s\widetilde{\Phi}^{-1}(s)^2}\exp\Big(-\frac{c_{22}r^2}{\widetilde{\Phi}^{-1}(s)^2}\Big)ds$$
$$\leq\frac{c_{27}th(\Phi(r),x,y)w(\Phi(r))}{V(x,r)}.$$

By (H1) and Theorem 2.2.6, we obtain

$$L_3\leq\frac{h(\Phi(r),x,y)}{V(x,r)}\mathbb{P}(S_t\geq 4\Phi(r))\leq c_{28}h(\Phi(r),x,y)\frac{tw(\Phi(r))}{V(x,r)}.$$

Finally, we estimate I_3. Using Theorem 2.2.6, (H1) and (H2), since D is bounded, we see that

$$I_3\leq c_{29}th(1,x,y)w\big(\Phi(\mathrm{diam}(D))\big)\leq c_{30}h(\Phi(r),x,y)\frac{tw(\Phi(r))}{V(x,r)}.$$

The proof is complete. □

By following the above proof, we obtain global estimates on $q(t,x,y)$ under $\mathbf{Poly}_\infty(\beta_1,\beta_2)$ and $\mathbf{HK_U^h}$.

Theorem 4.2.3. *Suppose that* $\mathbf{Poly}_\infty(\beta_1,\beta_2)$ *and* $\mathbf{HK_U^h}$ *hold. Then the assertions in Theorem 4.2.2(i)–(ii) hold for all* $(t,x,y)\in(0,\infty)\times D\times D$.

Recall from Lemma 2.1.1(i) that if $\mathbf{Poly}_{R_1}(\beta_1, \beta_2)$ holds with $\beta_2 < 1$, then $w(s) \simeq \phi(1/s)$ for $s < R_2$. Using this fact together with (4.1.11), we obatin the following simpler form of off-diagonal estimates.

Corollary 4.2.4. *Suppose that* $\mathbf{Poly}_{R_1}(\beta_1, \beta_2)$ *holds with* $\beta_2 < 1$ *and* $\Phi(r) \simeq \Psi(r)$ *for* $r \in (0, R_1)$.
(i) If $\mathbf{HK_B^h}$ *holds, then for every* $T > 0$*, the following estimates hold for all* $(t, x, y) \in (0, T] \times D \times D$*:*
(1) If $\psi(\rho(x, y)) \le t$*, then* (4.2.11) *holds.*
(2) If $\psi(\rho(x, y)) \ge t$*, then*

$$q(t,x,y) \simeq \frac{t}{V(x,\rho(x,y))} \times \begin{cases} \dfrac{h(\Phi(\rho(x,y)),x,y)}{\psi(\rho(x,y))} & \text{when } C_0 = 0, \\ \dfrac{\mathcal{B}_h(t,x,y)}{\Phi(\rho(x,y))} & \text{when } C_0 = 1. \end{cases} \tag{4.2.16}$$

(ii) If $R_1 = \infty$ *and* $\mathbf{HK_U^h}$ *holds, then (1) and (2) above hold for all* $(t, x, y) \in (0, \infty) \times D \times D$.

In the case when D is a bounded $C^{1,1}$ domain, Y^D is a killed Brownian motion in D and S is an $(\alpha/2)$-stable subordinator, part (i) of the corollary above is equivalent to [119, Theorem 4.7]. In the case when D is an exterior $C^{1,1}$ domain, Y^D is a killed Brownian motion in D and S is an $(\alpha/2)$-stable subordinator, part (ii) of the corollary above corrects [119, Theorem 4.6].

For future use, we note the following rough upper estimates on $q(t, x, y)$.

Corollary 4.2.5. *(i) Suppose that* $\mathbf{Poly}_{R_1}(\beta_1, \beta_2)$ *and* $\mathbf{HK_B^h}$ *hold. Then for every* $T > 0$*, there exists a constant* $C > 0$ *such that for all* $(t, x, y) \in (0, T] \times D \times D$*,*

$$q(t,x,y) \le Ch(\phi^{-1}(1/t)^{-1},x,y)\left(\frac{1}{V(x,\psi^{-1}(t))} \wedge \frac{t}{V(x,\rho(x,y))\psi(\rho(x,y))}\right). \tag{4.2.17}$$

(ii) Suppose that $\mathbf{Poly}_\infty(\beta_1, \beta_2)$ *and* $\mathbf{HK_U^h}$ *hold. Then, there exists a constant* $C > 0$ *such that* (4.2.17) *holds for all* $(t, x, y) \in (0, \infty) \times D \times D$.

As another corollary to Theorems 4.2.2 and 4.2.3, we obtain the following interior estimates on $q(t,x,y)$ in case of a regular boundary function.

Corollary 4.2.6. *Suppose that $h(t,x,y)$ is a regular boundary function.*
(i) If $\mathbf{Poly}_{R_1}(\beta_1,\beta_2)$ and $\mathbf{HK}^{\mathbf{h}}_{\mathbf{B}}$ hold, then for every $T>0$, the following estimates hold for all $(t,x,y)\in(0,T]\times D\times D$ satisfying $\delta_\wedge(x,y)\ge\rho(x,y)\vee\psi^{-1}(t)$.

(1) *If $\psi(\rho(x,y))\le t$, then* $\quad q(t,x,y)\simeq\dfrac{1}{V(x,\psi^{-1}(t))}$.

(2) *If $\psi(\rho(x,y))\ge t$, then*

$$q(t,x,y)\asymp\frac{C_0}{V(x,\rho(x,y))\Psi(\rho(x,y))}\left(t\int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(\rho(x,y))}w(s)ds+\frac{1}{\phi^{-1}(1/t)}\right)+\frac{1}{V(x,\psi^{-1}(t))}\exp\Big(-\frac{c\,\rho(x,y)^2}{\psi^{-1}(t)^2}\Big)+\frac{tw\big(\Phi(\rho(x,y))\big)}{V(x,\rho(x,y))}.$$

(ii) If $\mathbf{Poly}_{\infty}(\beta_1,\beta_2)$ and $\mathbf{HK}^{\mathbf{h}}_{\mathbf{U}}$ hold, then (1) and (2) above hold for all $(t,x,y)\in(0,\infty)\times D\times D$ satisfying $\delta_\wedge(x,y)\ge\rho(x,y)\vee\psi^{-1}(t)$.

Now we give the large time estimates for $q(t,x,y)$ under $\mathbf{HK}^{\mathbf{h}}_{\mathbf{B}}$.

Theorem 4.2.7. *Suppose that $\mathbf{Poly}_{R_1}(\beta_1,\beta_2)$ and $\mathbf{HK}^{\mathbf{h}}_{\mathbf{B}}$ hold. Then for every $T>0$, there are comparison constants such that*

$$q(t,x,y)\simeq e^{-t\phi(\lambda_D)}h(1,x,y),\quad (t,x,y)\in[T,\infty)\times D\times D.$$

Proof. Fix $x,y\in D$ and $T>0$. Since $\lim_{s\to0}(H\circ\sigma)(T,s)=\infty$ and the map $t\mapsto(H\circ\sigma)(t,s)$ is nondecreasing for each fixed $s>0$, there is a constant $s_0\in(0,1)$ such that

$$(H\circ\sigma)(t,s_0)\ge2\phi(\lambda_D)+1/T\quad\text{for all }t\ge T. \tag{4.2.18}$$

By (H2), (4.1.1) and (4.1.3), we can apply Corollary 2.2.3 with $f(s)=h(s,x,y)V(x,\Phi^{-1}(s))^{-1}$. Using Remark 4.1.7 (with $T=s_0$), Corollary 2.2.3

and (4.2.18), since ϕ is the Laplace exponent of S, we get that, for all $t \geq T$,

$$\begin{aligned} q(t,x,y) &\leq c_1 \int_0^{s_0} \frac{h(s,x,y)}{V(x,\Phi^{-1}(s))} \mathbb{P}(S_t \in ds) + c_1 h(1,x,y) \int_{s_0}^{\infty} e^{-\lambda_D s} \mathbb{P}(S_t \in ds) \\ &\leq c_2 \frac{h(s_0,x,y)}{V(x,\Phi^{-1}(s_0))} \exp\Big(-\frac{t}{2}(H\circ\sigma)(t,s_0)\Big) + c_1 h(1,x,y)\mathbb{E}[e^{-\lambda_D S_t}] \\ &\leq c_3 h(1,x,y) e^{-t\phi(\lambda_D)}. \end{aligned}$$

Next, we also see from Remark 4.1.7 that

$$\begin{aligned} q(t,x,y) &\geq c_4 h(1,x,y)\Big(\int_0^{\infty} e^{-\lambda_D s}\mathbb{P}(S_t \in ds) - \int_0^{s_0} e^{-\lambda_D s}\mathbb{P}(S_t \in ds)\Big) \\ &\geq c_4 h(1,x,y)(e^{-t\phi(\lambda_D)} - \mathbb{P}(S_t \leq s_0)). \end{aligned}$$

According to Proposition 2.2.1 and (4.2.18), it holds that for all $t \geq T$,

$$\mathbb{P}(S_t \leq s_0) \leq \exp\big(-t(H\circ\sigma)(t,s_0)\big) \leq e^{-2t\phi(\lambda_D)} \leq e^{-T\phi(\lambda_D)} e^{-t\phi(\lambda_D)}.$$

Therefore, the lower bound holds true and we finish the proof. □

4.3 Green function estimates

In this section, we always assume that either (1) $\mathbf{Poly}_{R_1}(\beta_1,\beta_2)$ and $\mathbf{HK}^{\mathbf{h}}_{\mathbf{B}}$ hold, or (2) $\mathbf{Poly}_{\infty}(\beta_1,\beta_2)$ and $\mathbf{HK}^{\mathbf{h}}_{\mathbf{U}}$ hold. The Green function G_D of X is given by

$$G_D(x,y) := \int_0^{\infty} q(t,x,y)dt, \qquad x,y \in D.$$

As an application of the heat kernel estimates obtained in the previous section, we can obtain two-sided estimates on the Green function.

The following proposition provides the first and most general estimate of the Green function.

Proposition 4.3.1. *There are comparison constants such that for $x, y \in D$,*

$$G_D(x,y) \simeq \frac{C_0}{V(x,\rho(x,y))\Psi(\rho(x,y))}\int_0^{\Phi(\rho(x,y))}\frac{h(s,x,y)}{\phi(1/s)}ds + \int_{\Phi(\rho(x,y))}^{2\Phi(\mathrm{diam}(D))}\frac{h(s,x,y)}{sV(x,\Phi^{-1}(s))\phi(1/s)}ds. \tag{4.3.1}$$

Proof. Since the proofs are similar, we only give the proof when $\mathbf{HK_B^h}$ holds, which is more complicated.

Take $x, y \in D$ and let $r := \rho(x,y)$. Set $T_D := 1/\phi\big(1/(2\Phi(\mathrm{diam}(D)))\big)$. By a change of variables and Lemma 2.1.5, we have that

$$\int_0^{\psi(r)}\frac{h(\phi^{-1}(1/t)^{-1},x,y)}{\phi^{-1}(1/t)}dt = \int_0^{\Phi(r)}\frac{h(s,x,y)\phi'(1/s)}{s\phi(1/s)^2}ds \simeq \int_0^{\Phi(r)}\frac{h(s,x,y)}{\phi(1/s)}ds \tag{4.3.2}$$

and

$$\begin{aligned}\int_{\psi(r)}^{T_D}\frac{h(\phi^{-1}(1/t)^{-1},x,y)}{V(x,\psi^{-1}(t))}dt &= \int_{\Phi(r)}^{2\Phi(\mathrm{diam}(D))}\frac{h(s,x,y)}{V(x,\Phi^{-1}(s))}\frac{\phi'(1/s)}{s^2\phi(1/s)^2}ds\\ &\simeq \int_{\Phi(r)}^{2\Phi(\mathrm{diam}(D))}\frac{h(s,x,y)}{sV(x,\Phi^{-1}(s))\phi(1/s)}ds. \end{aligned}\tag{4.3.3}$$

Combining with Theorem 4.2.2 (with $T = T_D$), we arrive at the lower bound in (4.3.1).

By Theorems 4.2.2 and 4.2.7 (with $T = T_D$), we have that

$$\begin{aligned}G_D(x,y) \le{}& \frac{c_0C_0}{V(x,r)\Psi(r)}\int_0^{\psi(r)} t\mathcal{B}_h(t,x,y)dt\\ &+ \frac{c_0C_0}{V(x,r)\Psi(r)}\int_0^{\psi(r)}\frac{h(\phi^{-1}(1/t)^{-1},x,y)}{\phi^{-1}(1/t)}dt\\ &+ c_0\int_0^{\psi(r)}\frac{h(\phi^{-1}(1/t)^{-1},x,y)}{V(x,\psi^{-1}(t))}\exp\Big(-\frac{c_1r^2}{\psi^{-1}(t)^2}\Big)dt\\ &+ c_0h(\Phi(r),x,y)\frac{w(\Phi(r))}{V(x,r)}\int_0^{\psi(r)} tdt\end{aligned}$$

$$
\begin{aligned}
&+ c_0 \int_{\psi(r)}^{T_D} \frac{h(\phi^{-1}(1/t)^{-1}, x, y)}{V(x, \psi^{-1}(t))} dt + c_0 h(1, x, y) \int_{T_D}^{\infty} e^{-t\phi(\lambda_D)} dt \\
&=: c_0(C_0 G_1 + C_0 G_2 + G_3 + G_4 + G_5 + G_6).
\end{aligned}
$$

By (4.3.2) and (4.3.3), we obtain the desired upper bound for $C_0G_2 + G_5$.

Note that, thanks to (4.2.9), the result of Lemma 4.1.2 remains true even if we replace Φ by ψ. Applying Lemma 4.1.2 with $p := \gamma/\beta_1 + d_2/(\alpha_1\beta_1)$, we get from (H2), Lemma 2.1.1(i & ii), (4.1.1) and (4.2.9) that

$$
G_3 \le c_1 \frac{h(\Phi(r), x, y)}{V(x, r)} \frac{\psi(r)^{p+1}}{\psi(r)^p} = c_1 h(\Phi(r), x, y) \frac{\psi(r)}{V(x, r)}.
$$

For G_4, we see from (2.0.4) that $G_4 \le eh(\Phi(r), x, y)\psi(r)/V(x, r)$.

For G_1, we use Fubini's theorem to get that

$$
\begin{aligned}
&V(x, r)\Psi(r)G_1 \\
&= \int_0^{2\Phi(r)} h(s, x, y) w(s) \int_0^{\phi(2/s)^{-1}} t dt ds + \int_{2\Phi(r)}^{4\Phi(r)} h(s, x, y) w(s) \int_0^{\psi(r)} t dt ds \\
&=: G_{1,1} + G_{1,2}.
\end{aligned}
$$

By (2.0.4), a change of variables and (H2), we get

$$
\begin{aligned}
G_{1,1} &\le 2e \int_0^{2\Phi(r)} \frac{h(s, x, y)\phi(2/s)}{\phi(2/s)^2} ds = 4e \int_0^{\Phi(r)} \frac{h(2s, x, y)}{\phi(1/s)} ds \\
&\le c_2 \int_0^{\Phi(r)} \frac{h(s, x, y)}{\phi(1/s)} ds.
\end{aligned}
$$

Besides, we get from (2.0.4), (H1) and (1.1.2) that

$$
G_{1,2} \le 2eh(2\Phi(r), x, y)\psi(r)^3 \le c_3 \int_0^{\Phi(r)} \frac{h(s, x, y)}{\phi(1/s)} ds.
$$

Clearly, $G_6 \le \phi(\lambda_D)^{-1} h(1, x, y)$.

It follows from (1.1.2) and the upper bounds above on G_3, G_4, G_6 that G_5 dominates $G_3 + G_4 + G_6$. Since G_2 dominates G_1, the proof is complete. □

In the remainder of this section, under some additional assumptions on the boundary function, we obtain Green function estimates in simpler forms. Lemma 1.1.1 will be a useful tool in all simplifications.

We start with the following condition which is a counterpart of (H2).

(H2*) There exist constants $c_1, \gamma_* > 0$ such that for all $x, y \in D$,

$$s^{\gamma_*} h(s,x,y) \geq c_1 t^{\gamma_*} h(t,x,y), \quad \Phi(\delta_\vee(x,y)) \leq s \leq t < 2\Phi(\mathrm{diam}(D)).$$

Note that the γ_* above is less than or equal to the constant γ in (H2).

Remark 4.3.2. Suppose that the boundary function $h(t,x,y)$ satisfies (H2*). Then for every $\epsilon \in (0,1)$, there exists $c_2 = c_2(\epsilon) > 0$ such that for all $x, y \in D$ and $s, t \geq 0$ with $\epsilon\Phi(\delta_\vee(x,y)) \leq s \leq t < 2\Phi(\mathrm{diam}(D))$,

$$s^{\gamma_*} h(s,x,y) \geq c_2 t^{\gamma_*} h(t,x,y).$$

Example 4.3.3. Let $p, q \geq 0$, $p+q > 0$. Recall that the boundary function $h_{p,q}(t,x,y)$ defined in (4.1.6) satisfies (H2) with $\gamma = p+q$. We claim that $h_{p,q}(t,x,y)$ also satisfies (H2*) with $\gamma_* = \gamma = p+q$. Indeed, for all $x, y \in D$ and $\Phi(\delta_\vee(x,y)) < s < t$,

$$s^{p+q} h_{p,q}(s,x,y) = \Phi(\delta_D(x))^p \Phi(\delta_D(y))^q = t^{p+q} h_{p,q}(t,x,y).$$

In the remainder of this section, we let $d_1, d_2, \gamma, \gamma_*, \beta_1, \beta_2$ and α_1, α_2 be the constants in (4.1.1), (H2), (H2*), $\mathbf{Poly}_{R_1}(\beta_1, \beta_2)$ and the scaling indices of Φ in (4.1.3), respectively.

Let

$$\widetilde{G}_D(x,y) := \int_{\Phi(\rho(x,y))}^{2\Phi(\mathrm{diam}(D))} \frac{h(s,x,y)}{sV(x,\Phi^{-1}(s))\phi(1/s)} ds$$

denote the second term on the right-hand side of the estimate (4.3.1).

Lemma 4.3.4. *The following estimates hold for all $x, y \in D$.*

(i) If $d_1 > \alpha_2(\beta_2 \wedge 1)$, then

$$\widetilde{G}_D(x,y) \simeq h\big(\Phi(\rho(x,y)),x,y\big)\frac{\psi(\rho(x,y))}{V(x,\rho(x,y))}.$$

(ii) If $d_2 < \alpha_1(\beta_1 - \gamma)$, then

$$\widetilde{G}_D(x,y) \simeq \begin{cases} h(1,x,y), & \textit{when } \mathbf{HK_B^h} \textit{ holds,} \\ \infty, & \textit{when } \mathbf{HK_U^h} \textit{ holds.} \end{cases}$$

Below, we also assume that $h(t,x,y)$ is regular and (H2) holds.*

(iii) If $\alpha_1\beta_1 > d_2 \ge d_1 > \alpha_2((\beta_2 \wedge 1) - \gamma_*)$, then

$$\widetilde{G}_D(x,y) \simeq h\big(\Phi(\rho(x,y)),x,y\big)\frac{\psi(\delta_\vee(x,y) \vee \rho(x,y))}{V(x,\delta_\vee(x,y) \vee \rho(x,y))}.$$

(iv) If $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$ and $d_1 = d_2 = \alpha_1\beta_1$, then

$$\widetilde{G}_D(x,y) \simeq h\big(\Phi(\rho(x,y)),x,y\big)\log\Big(e + \frac{\delta_\vee(x,y) \vee \rho(x,y)}{\rho(x,y)}\Big).$$

(v) If $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\gamma = \gamma_*$ and $d_1 = d_2 = \alpha_1(\beta_1 - \gamma)$, then

$$\widetilde{G}_D(x,y) \simeq \begin{cases} h(1,x,y)\log\left(e + \dfrac{\mathrm{diam}(D)}{\delta_\vee(x,y) \vee \rho(x,y)}\right), & \textit{when } \mathbf{HK_B^h} \textit{ holds,} \\ \infty, & \textit{when } \mathbf{HK_U^h} \textit{ holds.} \end{cases}$$

Proof. Take $x, y \in D$. Let $\delta_\wedge := \delta_\wedge(x,y)$ and $\delta_\vee := \delta_\vee(x,y)$. Define

$$g(s) := \frac{h(s,x,y)}{V(x,\Phi^{-1}(s))\phi(1/s)}, \quad s > 0.$$

Then

$$\widetilde{G}_D(x,y) = \int_{\Phi(\rho(x,y))}^{2\Phi(\mathrm{diam}(D))} \frac{g(s)}{s} ds.$$

Set $p_1 := -d_2/\alpha_1 + \beta_1$ and $p_2 := -d_1/\alpha_2 + (\beta_2 \wedge 1)$. By the scaling properties of $h(\cdot, x, y)$, $V(x, \cdot)$, Φ and ϕ, there exist $c_1, c_2 > 0$ such that

$$c_1\Big(\frac{r}{s}\Big)^{p_1-\gamma} \le \frac{g(r)}{g(s)} \le c_2\Big(\frac{r}{s}\Big)^{p_2}, \quad 0 < s \le r < 2\Phi(\text{diam}(D)). \tag{4.3.4}$$

If $h(t, x, y)$ is regular, then by Remark 4.1.4, for every $a > 0$, there exists $c_3 = c_3(a) > 0$ such that

$$c_3\Big(\frac{r}{s}\Big)^{p_1} \le \frac{g(r)}{g(s)} \le c_2\Big(\frac{r}{s}\Big)^{p_2}, \quad 0 < s \le r < \Phi(a\delta_\wedge) \wedge 2\Phi(\text{diam}(D)); \tag{4.3.5}$$

if furthermore (H2*) further holds, then by Remark 4.3.2, there exists $c_4 > 0$ such that

$$c_1\Big(\frac{r}{s}\Big)^{p_1-\gamma} \le \frac{g(r)}{g(s)} \le c_4\Big(\frac{r}{s}\Big)^{p_2-\gamma_*}, \quad \Phi(\delta_\vee/2) < s \le r < 2\Phi(\text{diam}(D)). \tag{4.3.6}$$

(i) By (4.3.4), since $p_2 < 0$, the result follows from Lemma 1.1.1(ii).

(ii) If D is bounded, then by (4.3.4) and Lemma 1.1.1(i), since $p_1 - \gamma > 0$, it holds that $\widetilde{G}_D(x, y) \simeq g(\Phi(\text{diam}(D)))$. By (4.1.2), there exists $c_5 > 1$ such that $c_5^{-1} \le V(z, \text{diam}(D)) \le c_5$ for all $z \in D$. Hence, by using (H1), (H2) and the definition of g, we get that $G_D(x, y) \simeq h(1, x, y)$. If D is unbounded, then we see from (4.3.4) and Lemma 1.1.1(i) that

$$\widetilde{G}_D(x, y) \simeq \lim_{r\to\infty} \int_{\Phi(\rho(x,y))}^{r} \frac{g(s)}{s} ds \simeq \lim_{r\to\infty} g(r) \ge c_1 g(1) \lim_{r\to\infty} r^{p_1-\gamma} = \infty.$$

(iii) Suppose that $\delta_\vee \le 2\rho(x, y)$. Since $p_2 - \gamma_* < 0$, by (4.3.6) and Lemma 1.1.1(ii),

$$\widetilde{G}_D(x, y) \simeq g(\Phi(\rho(x, y))) = \frac{h(\Phi(\rho(x, y)), x, y)\psi(\rho(x, y))}{V(x, \rho(x, y))}.$$

Hence, in this case, the result follows from (4.1.1) and (4.2.9).

Suppose now that $\delta_\vee > 2\rho(x, y)$. Then $\delta_\wedge \ge \delta_\vee - \rho(x, y) > \delta_\vee/2 > \rho(x, y)$. Since h is regular, $h(\Phi(\delta_\vee), x, y) \simeq h(\Phi(\rho(x, y)), x, y) \simeq 1$. Further, since $p_1 > 0$ and $p_2 - \gamma_* < 0$, by the scaling of Φ, (4.3.5), (4.3.6) and Lemma

1.1.1(i)-(ii), we get

$$\begin{aligned}\widetilde{G}_D(x,y) &\simeq \int_{\Phi(\rho(x,y))}^{2\Phi(\delta_\wedge)} \frac{g(s)}{s}ds + \int_{\Phi(\delta_\vee)}^{2\Phi(\mathrm{diam}(D))} \frac{g(s)}{s}ds \simeq g(\Phi(\delta_\wedge)) + g(\Phi(\delta_\vee)) \\ &\simeq g(\Phi(\delta_\vee)) = \frac{h(\Phi(\delta_\vee),x,y)\psi(\delta_\vee)}{V(x,\delta_\vee)} \simeq \frac{h(\Phi(\rho(x,y)),x,y)\psi(\delta_\vee)}{V(x,\delta_\vee)}.\end{aligned}$$

This finishes the proof for (iii).

(iv) Since $d_1 = d_2$, by (4.1.1) and (4.1.2), we see that for every $a > 0$, there are comparability constants depending on a such that for all $w, z \in D$ and $0 < r < a\,\mathrm{diam}(D)$,

$$V(w,r) \simeq V(z,r) \simeq r^{d_1} V(z,1). \tag{4.3.7}$$

Moreover, since $\beta_1 = \beta_2$ and $\alpha_1 = \alpha_2$, we get that

$$\phi(1/s)^{-1} \simeq s^{\beta_1}, \quad 0 < s < 2\Phi(\mathrm{diam}(D)) \quad \text{and} \quad \Phi^{-1}(s) \simeq s^{1/\alpha_1}, \quad s > 0, \tag{4.3.8}$$

so that $g(s) \simeq h(s,x,y)$ for all $0 < s < 2\Phi(\mathrm{diam}(D))$. In particular, since h is regular, we see from Remark 4.1.4 that

$$g(s) \simeq 1, \quad 0 < s < 2\Phi(\delta_\wedge). \tag{4.3.9}$$

If $\delta_\vee \le 2\rho(x,y)$, then by (4.3.6) and Lemma 1.1.1(ii),

$$\widetilde{G}_D(x,y) \simeq g(\Phi(\rho(x,y))) \simeq h(\Phi(\rho(x,y)),x,y)\log\Big(e + \frac{\delta_\vee \vee \rho(x,y)}{\rho(x,y)}\Big).$$

If $\delta_\vee > 2\rho(x,y)$, then we get $\delta_\wedge > \delta_\vee/2 > \rho(x,y)$ as in (iii), and by (4.3.9), (4.3.6) and Lemma 1.1.1(ii),

$$\widetilde{G}_D(x,y) \simeq \int_{\Phi(\rho(x,y))}^{2\Phi(\delta_\wedge)} \frac{g(s)}{s}ds + \int_{\Phi(\delta_\vee)}^{2\Phi(\mathrm{diam}(D))} \frac{g(s)}{s}ds \simeq \int_{\Phi(\rho(x,y))}^{2\Phi(\delta_\wedge)} \frac{ds}{s} + g(\Phi(\delta_\vee)).$$

Note that since $\Phi(s) \simeq s^{\alpha_1}$ for $s > 0$, we have

$$g(\Phi(\delta_\vee)) \simeq h(\Phi(\delta_\vee), x, y) \leq 1 \leq \log\Big(e + \frac{\delta_\wedge}{\rho(x,y)}\Big) \simeq \int_{\Phi(\rho(x,y))}^{2\Phi(\delta_\wedge)} \frac{ds}{s}.$$

Therefore, since $\delta_\wedge \simeq \delta_\vee > 2\rho(x,y)$ and $h(\Phi(\rho(x,y)), x, y) \simeq 1$ in this case, we get that

$$\widetilde{G}_D(x,y) \simeq \log\Big(e + \frac{\delta_\wedge}{\rho(x,y)}\Big) \simeq h(\Phi(\rho(x,y)), x, y) \log\Big(e + \frac{\delta_\vee \vee \rho(x,y)}{\rho(x,y)}\Big).$$

(v) By (4.3.7), (4.3.8), the regularity of h, (H2), Remark 4.3.2 and (4.1.7), we have

$$g(s) \simeq s^\gamma, \quad 0 < s < \Phi(\delta_\wedge) \tag{4.3.10}$$

and

$$g(s) \simeq s^\gamma h(s,x,y) \simeq t^\gamma h(t,x,y), \qquad \Phi(\delta_\vee/2) < s \leq t < 2\Phi(\mathrm{diam}(D)) + 1. \tag{4.3.11}$$

If $\delta_\vee \leq 2\rho(x,y)$, then since $\Phi(s) \simeq s^{\alpha_1}$ for $s > 0$ in this case, we get from (4.3.11) that

$$\widetilde{G}_D(x,y) \simeq h(1,x,y) \int_{\Phi(\rho(x,y))}^{2\Phi(\mathrm{diam}(D))} \frac{ds}{s} \simeq h(1,x,y) \log\Big(e + \frac{\mathrm{diam}(D)}{\rho(x,y)}\Big).$$

If $\delta_\vee > 2\rho(x,y)$, then $\delta_\wedge > \delta_\vee/2 > \rho(x,y)$ as in (iii) and hence by (4.3.10) and (4.3.11),

$$\widetilde{G}_D(x,y) \simeq \int_{\Phi(\rho(x,y))}^{2\Phi(\delta_\wedge)} s^{\gamma-1} ds + h(1,x,y) \int_{\Phi(\delta_\vee)}^{2\Phi(\mathrm{diam}(D))} \frac{ds}{s}.$$

Since $\Phi(s) \simeq s^{\alpha_1}$ for $s > 0$, $\delta_\wedge \leq \delta_\vee \leq 2\delta_\wedge$ and h is regular, by (4.3.11),

$$\begin{aligned} h(1,x,y) \int_{\Phi(\delta_\vee)}^{2\Phi(\mathrm{diam}(D))} \frac{ds}{s} &\simeq h(1,x,y) \log\Big(e + \frac{\mathrm{diam}(D)}{\delta_\vee}\Big) \\ &\geq h(1,x,y) \simeq \Phi(\delta_\wedge)^\gamma h(\Phi(\delta_\wedge), x, y) \simeq \Phi(\delta_\wedge)^\gamma \geq \gamma^{-1} \int_{\Phi(\rho(x,y))}^{2\Phi(\delta_\wedge)} s^{\gamma-1} ds. \end{aligned}$$

This completes the proof. □

In the next lemma we show that under the additional assumption that $\gamma < \beta_1 + 1$, the first term on the right-hand side of (4.3.1) is dominated by $\widetilde{G}_D(x, y)$.

Lemma 4.3.5. *If either $C_0 = 0$ or $\gamma < \beta_1 + 1$, then*

$$G_D(x, y) \simeq \widetilde{G}_D(x, y) \text{ on } D \times D.$$

Proof. When $C_0 = 0$, the assertion follows from Proposition 4.3.1. So we now assume $C_0 = 1$ and $\gamma < \beta_1 + 1$. By the scaling of ϕ and (H2), we have

$$\frac{h(t, x, y)/\phi(1/t)}{h(s, x, y)/\phi(1/s)} \geq c\Big(\frac{t}{s}\Big)^{\beta_1 - \gamma} \quad \text{for all } 0 < s \leq t < \Phi(\mathrm{diam}(D)).$$

Thus, since $\Psi \geq \Phi$, by Lemma 1.1.1(i), we get

$$\frac{1}{V(x, r)\Psi(r)} \int_0^{\Phi(r)} \frac{h(s, x, y)}{\phi(1/s)} ds \leq c\frac{h(\Phi(r), x, y)\psi(r)\Phi(r)}{V(x, r)\Psi(r)} \leq c\frac{h(\Phi(r), x, y)\psi(r)}{V(x, r)}.$$

By (1.1.2), the second term in (4.3.1) dominates the last term above. Hence, by Proposition 4.3.1, we get the assertion. □

Define

$$g_0(x, y) = \begin{cases} \dfrac{\psi(\rho(x, y))}{V(x, \rho(x, y))}, & \text{if } d_1 > \alpha_2(\beta_2 \wedge 1), \\ \log\Big(e + \dfrac{\delta_\vee(x, y) \vee \rho(x, y)}{\rho(x, y)}\Big), & \text{if } d_1 = d_2 = \alpha_1\beta_1 = \alpha_2\beta_2, \\ \dfrac{\psi(\delta_\vee(x, y) \vee \rho(x, y))}{V(x, \delta_\vee(x, y) \vee \rho(x, y))}, & \text{if } d_2 < \alpha_1\beta_1. \end{cases} \tag{4.3.12}$$

By combining Proposition 4.3.1, Lemma 4.3.4 and Lemma 4.3.5 we arrive at the following result.

Theorem 4.3.6. *Suppose that $C_0 = 0$ or $\gamma < \beta_1 + 1$, $h(t, x, y)$ is regular and (H2*) holds.*

(i) Suppose also that one of the following holds: (1) $d_1 > \alpha_2(\beta_2 \wedge 1)$ *or (2)* $d_1 = d_2 = \alpha_1\beta_1 = \alpha_2\beta_2$ *or (3)* $d_2 < \alpha_1\beta_1$*. Then it holds that*

$$G_D(x,y) \simeq h(\Phi(\rho(x,y)),x,y)g_0(x,y).$$

(ii) If $d_2 < \alpha_1(\beta_1 - \gamma)$*, then*

$$G_D(x,y) \simeq \begin{cases} h(1,x,y), & \textit{when } \mathbf{HK_B^h} \textit{ holds,} \\ \infty, & \textit{when } \mathbf{HK_U^h} \textit{ holds.} \end{cases}$$

(iii) If $\alpha_1 = \alpha_2$*,* $\beta_1 = \beta_2$*,* $\gamma = \gamma_*$ *and* $d_1 = d_2 = \alpha_1(\beta_1 - \gamma)$*, then*

$$G_D(x,y) \simeq \begin{cases} h(1,x,y)\log\left(e + \dfrac{\mathrm{diam}(D)}{\delta_\vee(x,y) \vee \rho(x,y))}\right), & \textit{when } \mathbf{HK_B^h} \textit{ holds,} \\ \infty, & \textit{when } \mathbf{HK_U^h} \textit{ holds.} \end{cases}$$

When $C_0 = 1$, Theorem 4.3.6 only deals with the case $\gamma < \beta_1 + 1$. To cover the case when γ is large, we assume the following condition.

(H2**) There exist constants $c_1 > 0$, $\gamma_{**} \in (0, 1+\beta_1)$ such that for all $x, y \in D$,

$$s^{\gamma_{**}}h(s,x,y) \le c_1 t^{\gamma_{**}}h(t,x,y), \quad \Phi(\delta_\wedge(x,y)) \le s \le t < \Phi(\delta_\vee(x,y)).$$

Example 4.3.7. For $p, q \ge 0$, let $h_{p,q}(t,x,y)$ be the boundary function defined in (4.1.6). If $p \vee q < 1 + \beta_1$, then $h_{p,q}(t,x,y)$ satisfies (H2**). Indeed, we see that for all $x, y \in D$ and $\Phi(\delta_\wedge(x,y)) \le s \le t < \Phi(\delta_\vee(x,y))$,

$$s^{p\vee q}h_{p,q}(s,x,y) = \begin{cases} \Phi(\delta_D(x))^p s^{p\vee q - p}, & \text{if } \delta_D(x) < \delta_D(y) \\ \Phi(\delta_D(y))^q s^{p\vee q - q}, & \text{if } \delta_D(x) > \delta_D(y) \end{cases} \le t^{p\vee q}h_{p,q}(t,x,y).$$

For a given boundary function h, we define for $x, y \in D$,

$$[h](x,y) := h\Big(\Phi(\delta_\vee(x,y)\rho(x,y)), x, y\Big)\left(1 \wedge \frac{\Phi(\delta_\vee(x,y))\psi(\delta_\vee(x,y))}{\Phi(\rho(x,y))\psi(\rho(x,y))}\right).$$

In particular, one can check that for all $p, q \geq 0$,

$$[h_{p,q}](x,y) \simeq \Big(1 \wedge \frac{\Phi(\delta_D(x))}{\Phi(\rho(x,y))}\Big)^p \Big(1 \wedge \frac{\Phi(\delta_D(y))}{\Phi(\rho(x,y))}\Big)^q \times \Big(1 \wedge \frac{\Phi(\delta_\vee(x,y))}{\Phi(\rho(x,y))}\Big)^{1-p-q} \Big(1 \wedge \frac{\psi(\delta_\vee(x,y))}{\psi(\rho(x,y))}\Big). \quad (4.3.13)$$

Recall that $g_0(x,y)$ is defined in (4.3.12).

Theorem 4.3.8. *Suppose that $C_0 = 1$, $h(t,x,y)$ is regular, (H2*) holds with $\gamma_* > (\beta_2 \wedge 1) + 1$ and (H2**) holds. Suppose also that one of the following holds: (1) $d_1 > \alpha_2(\beta_2 \wedge 1)$ or (2) $d_1 = d_2 = \alpha_1\beta_1 = \alpha_2\beta_2$ or (3) $d_2 < \alpha_1\beta_1$. Then it holds that*

$$G_D(x,y) \simeq [h](x,y)\frac{\Phi(\rho(x,y))}{\Psi(\rho(x,y))}\frac{\psi(\rho(x,y))}{V(x,\rho(x,y))} + h(\Phi(\rho(x,y)),x,y)g_0(x,y). \quad (4.3.14)$$

In particular, if $\Psi \simeq \Phi$, then

$$G_D(x,y) \simeq [h](x,y)g_0(x,y). \quad (4.3.15)$$

Proof. Take $x, y \in D$ and let $r := \rho(x,y)$ and $\delta_\vee := \delta_\vee(x,y)$. Observe that by the scaling of ϕ, (H1), (H2**) and the regularity of h,

$$c_1\Big(\frac{t}{s}\Big)^{\beta_1-\gamma_{**}} \leq \frac{h(t,x,y)/\phi(1/t)}{h(s,x,y)/\phi(1/s)} \leq c_2\Big(\frac{t}{s}\Big)^{\beta_2\wedge 1}, \quad 0 < s \leq t < \Phi(\delta_\vee). \quad (4.3.16)$$

Note also that by the scaling of ϕ, (H2), (H2*) and Remark 4.3.2,

$$c_3\Big(\frac{t}{s}\Big)^{\beta_1-\gamma} \leq \frac{h(t,x,y)/\phi(1/t)}{h(s,x,y)/\phi(1/s)} \leq c_4\Big(\frac{t}{s}\Big)^{\beta_2\wedge 1-\gamma_*}, \quad \frac{\Phi(\delta_\vee)}{2} \leq s \leq t < \Phi(\mathrm{diam}(D)). \quad (4.3.17)$$

Then by using Lemma 1.1.1 several times as in the proof of Lemma 4.3.4, we can conclude that the first assertion holds true.

Now we also assume that $\Psi \simeq \Phi$. If $\delta_\vee > r$, then $[h](x,y) = h(\Phi(r),x,y)$. Hence, we see from (1.1.2) that in (4.3.14), the second term dominates the first one so that (4.3.15) holds. If $\delta_\vee \le r$, then using Lemma 1.1.1(ii), (4.3.17) and the condition that $\beta_2 \wedge 1 - \gamma_* < -1$ in the second inequality below, the scaling property of ϕ and (4.1.7) in the third, and (4.3.16) in the fourth, we get

$$\int_{\Phi(r)}^{2\Phi(\mathrm{diam}(D))} \frac{h(s,x,y)}{sV(x,\Phi^{-1}(s))\phi(1/s)}ds \le \frac{1}{V(x,r)\Phi(r)} \int_{\Phi(r)}^{2\Phi(\mathrm{diam}(D))} \frac{h(s,x,y)}{\phi(1/s)}ds$$
$$\le c_5 \frac{h(\Phi(r),x,y)\psi(r)}{V(x,r)} \le \frac{c_6}{V(x,r)\Phi(r)} \int_{\Phi(r)/2}^{\Phi(r)} \frac{h(s,x,y)}{\phi(1/s)}ds \le c_7[h](x,y)\frac{\psi(r)}{V(x,r)}.$$

Note that $g_0(x,y) \simeq \psi(r)/V(x,r)$ when $\delta_\vee \le r$. Thus by Proposition 4.3.1 and (4.3.16), we get $G_D(x,y) \simeq [h](x,y)g_0(x,y)$ when $\delta_\vee \le r$. This completes the proof for (4.3.15). □

For completeness, we record the Green function estimates when $C_0 = 1$, $\beta_1 = \beta_2$ and $\gamma_* = \gamma = \beta_1 + 1$.

Theorem 4.3.9. *Suppose that $C_0 = 1$, $\beta_1 = \beta_2$, $h(t,x,y)$ is regular and (H2*) holds with $\gamma_* = \gamma = \beta_1 + 1$ and (H2**) holds. Suppose also that one of the following holds: (1) $d_1 > \alpha_2(\beta_2 \wedge 1)$ or (2) $d_1 = d_2 = \alpha_1\beta_1 = \alpha_2\beta_2$ or (3) $d_2 < \alpha_1\beta_1$. Then it holds that*

$$G_D(x,y) \simeq h(\Phi(\rho(x,y)),x,y)$$
$$\times \left[\frac{\Phi(\rho(x,y))}{\Psi(\rho(x,y)}\frac{\Phi(\rho(x,y))^{\gamma-1}}{V(x,\rho(x,y))}\log\Big(e + \frac{\delta_\vee(x,y)\vee\rho(x,y)}{\delta_\vee(x,y)}\Big) + g_0(x,y)\right].$$

In particular, if $\Psi \simeq \Phi$, then

$$G_D(x,y) \simeq h(\Phi(\rho(x,y)),x,y)g_0(x,y)\log\Big(e + \frac{\delta_\vee(x,y)\vee\rho(x,y)}{\delta_\vee(x,y)}\Big).$$

4.4 Parabolic Harnack inequality and Hölder regularity

Throughout this section, we assume that $h(t,x,y)$ is a regular boundary function and that either (1) $\mathbf{Poly}_{R_1}(\beta_1,\beta_2)$ and $\mathbf{HK_B^h}$ hold, or (2) $\mathbf{Poly}_\infty(\beta_1,\beta_2)$ and $\mathbf{HK_U^h}$ hold.

For $x_0 \in D$ and $r > 0$, let $\tau_{B(x_0,r)} := \inf\{s > 0 : X_s \notin B(x_0,r)\}$ and $X^{B(x_0,r)}$ be the part process of X in $B(x_0,r)$. Denote by $q_{B(x_0,r)}(t,x,y)$ the heat kernel of $X^{B(x_0,r)}$. By the strong Markov property, one can see that

$$q_{B(x_0,r)}(t,x,y) = q(t,x,y) - \mathbb{E}^x\left[q(t-\tau_{B(x_0,r)}, X_{\tau_{B(x_0,r)}}, y);\, \tau_{B(x_0,r)} < t\right]. \tag{4.4.1}$$

Recall the definition of ψ in (4.2.8). Using the rough upper estimates and near diagonal interior estimates on the heat kernel that obtained in Corollary 4.2.5 and Corollary 4.2.6, respectively, we deduce the following lemma from (4.4.1).

Lemma 4.4.1. *There exist constants $C > 0$ and $\epsilon \in (0,1/4)$ such that for all $x_0 \in D$ and $r \in (0, \delta_D(x_0))$,*

$$q_{B(x_0,r)}(t,x,y) \ge \frac{C}{V(x_0,\psi^{-1}(t))} \quad \textit{for all } t \in (0,\psi(\epsilon r)],\ x,y \in B\big(x_0, \epsilon\psi^{-1}(t)\big).$$

Remark 4.4.2. By using (4.2.9) we may replace $\psi(\epsilon r)$ and $\epsilon\psi^{-1}(r)$ in the statement of Lemma 4.4.1 with $\epsilon\psi(r)$ and $\psi^{-1}(\epsilon r)$ respectively, cf. [47, p.3758].

Lemma 4.4.3. *There exists $C > 1$ such that for all $x \in D$ and $r \in (0, \delta_D(x))$,*

$$C^{-1}\psi(r) \le \mathbb{E}^x[\tau_{B(x,r)}] \le C\psi(r). \tag{4.4.2}$$

Proof. Fix $x \in D$ and $r \in (0,\delta_D(x))$. Let $\epsilon \in (0,1/4)$ be as in the statement of Lemma 4.4.1. Then by Lemma 4.4.1, we have that

$$q_{B(x,r)}(\psi(\epsilon r), x, y) \ge \frac{c_1}{V(x,\epsilon r)}, \quad y \in B(x, \epsilon^2 r).$$

By (4.1.1), this implies that

$$\mathbb{P}^x(\tau_{B(x,r)} > \psi(\epsilon r)) \geq \int_{B(x,\epsilon^2 r)} q_{B(x,r)}(\psi(\epsilon r), x, y)\, dy \geq \frac{c_1 V(x, \epsilon^2 r)}{V(x, \epsilon r)} \geq c_2.$$

Hence, by Markov inequality and (4.2.9), we get that

$$\mathbb{E}^x[\tau_{B(x,r)}] \geq \psi(\epsilon r)\mathbb{P}^x(\tau_{B(x,r)} > \psi(\epsilon r)) \geq c_2\psi(\epsilon r) \geq c_3\psi(r).$$

To obtain the upper bound in (4.4.2), we first assume that $\mathbf{HK_B^h}$ holds. We claim that there exists a constant $A > 1$ such that

$$\sup_{z\in B(x,r)} \mathbb{P}^z(\tau_{B(x,r)} > \psi(Ar)) \leq \frac{1}{2}. \tag{4.4.3}$$

Indeed, according to Corollary 4.2.5(i) and Theorem 4.2.7, since $h \leq 1$, there exists $c_4 > 1$ such that

$$q(t, z, y) \leq c_4 \left(V(z, \psi^{-1}(t))^{-1}\mathbf{1}_{\{t\leq 1\}} + e^{-\phi(\lambda_D)t}\mathbf{1}_{\{t>1\}} \right), \quad z, y \in B(x, r).$$

Further, by (4.1.1), there is $c_5 > 1$ such that for all $z \in B(x, r)$,

$$V(z, c_5 r) \geq V(x, (c_5 - 1)r) \geq 2c_4 V(x, r).$$

Let $A > c_5$ be a constant such that

$$\exp\left(\phi(\lambda_D)\psi(Ac_5^{-1}\psi^{-1}(1))\right) \geq 2m(D).$$

In case when $r \leq c_5^{-1}\psi^{-1}(1)$, we get that for all $z \in B(x, r)$,

$$\begin{aligned}\mathbb{P}^z(\tau_{B(x,r)} > \psi(Ar)) &\leq \mathbb{P}^z(\tau_{B(x,r)} > \psi(c_5 r)) \leq \mathbb{P}^z(X_{\psi(c_5 r)} \in B(x, r)) \\ &= \int_{B(x,r)} q(\psi(c_5 r), z, y)dy \leq \frac{c_4 V(x, r)}{V(z, c_5 r)} \leq \frac{1}{2}.\end{aligned}$$

On the other hand, if $r > c_5^{-1}\psi^{-1}(1)$, then since $B(x, r) \subset B(x, \delta_D(x)) \subset D$, we get that for all $z \in B(x,r)$,

$$\begin{aligned}\mathbb{P}^z(\tau_{B(x,r)} > \psi(Ar)) &\le \mathbb{P}^z(X_{\psi(Ar)} \in B(x,r)) \le V(x,r)e^{-\phi(\lambda_D)\psi(Ar)} \\ &\le m(D)e^{-2m(D)} \le \frac{1}{2}.\end{aligned}$$

Hence, (4.4.3) holds true.

Now, using (4.4.3) and the Markov's property, we get that for all $n \ge 2$,

$$\begin{aligned}&\sup_{z\in B(x,r)} \mathbb{P}^z(\tau_{B(x,r)} > n\psi(Ar)) = \sup_{z\in B(x,r)} \mathbb{P}^z\big(\tau_{B(x,r)} > n\psi(Ar),\, \tau_{B(x,r)} > \psi(Ar)\big) \\ &\le \sup_{z\in B(x,r)} \mathbb{P}^z\big(\mathbb{P}^{X_{\psi(Ar)}}(\tau_{B(x,r)} > (n-1)\psi(Ar)),\, \tau_{B(x,r)} > \psi(Ar)\big) \\ &\le \sup_{z\in B(x,r)} \mathbb{P}^z(\tau_{B(x,r)} > (n-1)\psi(Ar)) \sup_{z\in B(x,r)} \mathbb{P}^z(\tau_{B(x,r)} > \psi(Ar)) \\ &\le \cdots \le \Big(\sup_{z\in B(x,r)} \mathbb{P}^z(\tau_{B(x,r)} > \psi(Ar))\Big)^n \le 2^{-n}.\end{aligned}$$

Therefore, we conclude from (4.2.9) that

$$\begin{aligned}\mathbb{E}^x[\tau_{B(x,r)}] &\le \sum_{n=1}^{\infty} n\psi(Ar)\mathbb{P}^x\big(\tau_{B(x,r)} \in ((n-1)\psi(Ar), n\psi(Ar)]\big) \\ &\le c_6 A^{\alpha_2(\beta_2\wedge 1)}\psi(r)\sum_{n=1}^{\infty} n2^{-(n-1)} = 4c_6 A^{\alpha_2(\beta_2\wedge 1)}\psi(r).\end{aligned}$$

Similarly, by using Corollary 4.2.5(ii), we can obtain the upper bound in (4.4.2) when $\mathbf{Poly}_\infty(\beta_1,\beta_2)$ and $\mathbf{HK}^{\mathbf{h}}_{\mathbf{U}}$ hold. □

Recall that the jump kernel $J(x,y)$ is given in (4.2.3). Using Theorem 4.2.1 and the fact that $h \le 1$, we get the following result.

Lemma 4.4.4. *There exists $C > 1$ such that for all $x \in D$ and $r \in (0, \delta_D(x))$,*

$$\int_{D\setminus B(x,r)} J(x,y)dy \le \frac{C}{\psi(r)}.$$

Let $Z := (V_s, X_s)_{s\ge 0}$ be the time-space process corresponding to X, where

$V_s = V_0 - s$. The augmented filtration of Z will be denoted by $(\widetilde{\mathcal{F}}_s)_{s\geq 0}$. The law of the time-space process $s \mapsto Z_s$ starting from (t,x) will be denoted by $\mathbb{P}^{(t,x)}$. For every open subset B of $[0,\infty)\times D$, define $\tau_B^Z = \inf\{s>0:\ Z_s \notin B\}$ and $\sigma_B^Z = \tau_{B^c}^Z$.

Recall that a Borel measurable function $u:[0,\infty)\times D \to \mathbb{R}$ is parabolic on $(a,b)\times B(x_0,r)$ with respect to the process X if for every relatively compact open set $U \subset (a,b)\times B(x_0,r)$ it holds that $u(t,x) = \mathbb{E}^{(t,x)} u(Z_{\tau_U^Z})$ for all $(t,x)\in U$.

We denote by $dt\otimes m$ the product of the Lebesgue measure on $[0,\infty)$ and m on E.

Lemma 4.4.5. *Let $\epsilon \in (0,1/4)$ be the constant from Lemma 4.4.1. For every $\delta \in (0,\epsilon]$, there exists $C>0$ such that for all $x\in D$, $r\in(0,\delta_D(x))$, $t \geq \delta\psi(r)$, and any compact set $A \subset [t-\delta\psi(r), t-\delta\psi(r)/2]\times B(x,\psi^{-1}(\epsilon\delta\psi(r)/2))$,*

$$\mathbb{P}^{(t,x)}(\sigma_A^Z < \tau^Z_{[t-\delta\psi(r),t]\times B(x,r)}) \geq C\frac{dt\otimes m(A)}{V(x,r)\psi(r)}. \tag{4.4.4}$$

Proof. Write $\tau_r = \tau^Z_{[t-\delta\psi(r),t]\times B(x,r)}$ and $A_s = \{y\in D:\ (s,y)\in A\}$. For any $t,r>0$ and $x\in D$ such that $B(x,r)\subset D$,

$$\begin{aligned}\delta\psi(r)\mathbb{P}^{(t,x)}(\sigma_A<\tau_r) &\geq \int_0^{\delta\psi(r)} \mathbb{P}^{(t,x)}\left(\int_0^{\tau_r} \mathbf{1}_A(t-s,X_s)ds > 0\right)du\\ &\geq \int_0^{\delta\psi(r)} \mathbb{P}^{(t,x)}\left(\int_0^{\tau_r} \mathbf{1}_A(t-s,X_s)ds > u\right)du = \mathbb{E}^{(t,x)}\left[\int_0^{\tau_r}\mathbf{1}_A(t-s,X_s)ds\right].\end{aligned} \tag{4.4.5}$$

For any $t\geq\delta\psi(r)$,

$$\begin{aligned}\mathbb{E}^{(t,x)}\left[\int_0^{\tau_r}\mathbf{1}_A(t-s,X_s)ds\right] &\geq \int_{\delta\psi(r)/2}^{\delta\psi(r)} \mathbb{P}^{(t,x)}\left((t-s,X_s^{B(x,r)})\in A\right)ds\\ &= \int_{\delta\psi(r)/2}^{\delta\psi(r)} \mathbb{P}^x(X_s^{B(x,r)}\in A_{t-s})ds = \int_{\delta\psi(r)/2}^{\delta\psi(r)} ds\int_{A_{t-s}} q_{B(x,r)}(s,x,y)\,dy.\end{aligned}$$

Let $s\in[\delta\psi(r)/2,\delta\psi(r)]$ and $y\in B(x,\psi^{-1}(\epsilon\delta\psi(r)/2))$. Then $s\leq\epsilon\psi(r)$

and $\psi^{-1}(\epsilon\delta\psi(r)/2) \le \psi^{-1}(\epsilon s)$ so that $y \in B(x, \psi^{-1}(\epsilon s))$. Hence, by (4.1.1), (4.2.9), Lemma 4.4.1 and Remark 4.4.2,

$$q_{B(x,r)}(s, x, y) \ge c_1 V(x, \psi^{-1}(s))^{-1} \ge c_2 V(x, r)^{-1}.$$

Therefore

$$\mathbb{E}^{(t,x)}\left[\int_0^{\tau_r} \mathbf{1}_A(t-s, X_s)ds\right] \ge \frac{c_2}{V(x,r)} \int_{\delta\psi(r)/2}^{\delta\psi(r)} ds \int_{A_{t-s}} dy = c_2 \frac{dt \otimes m(A)}{V(x,r)}.$$

Combining with (4.4.5), we arrive at (4.4.4). □

Using (4.1.1), (4.2.9) and Lemmas 4.4.3, 4.4.4 and 4.4.5, by following arguments in the proof of [45, Theorem 4.14] (see also the proof of [47, Proposition 3.8]), we get the following Hölder regularity for parabolic functions.

Theorem 4.4.6. *There exists a constant $\eta \in (0, 1]$ such that for all $\delta \in (0, 1)$, there exists a constant $C = C(\delta) > 0$ so that for every $x_0 \in D$, $r \in (0, \delta_D(x_0))$, $t_0 \ge 0$, and any function u on $(0, \infty) \times D$ which is parabolic in $(t_0, t_0 + \psi(r)) \times B(x_0, r)$ and bounded in $(t_0, t_0 + \psi(r)) \times D$, we have*

$$|u(s, x) - u(t, y)| \le C\left(\frac{\psi^{-1}(|s-t|) + \rho(x, y)}{r}\right)^{\eta} \operatorname*{ess\,sup}_{[t_0, t_0+\psi(r)]\times D} |u|,$$

for every $s, t \in (t_0 + \psi(r) - \psi(\delta r), t_0 + \psi(r))$ and $x, y \in B(x_0, \delta r)$.

As a consequence of Lemma 4.4.1, we get the following lemma.

Lemma 4.4.7. *Let $\epsilon \in (0, 1/4)$ be the constant from Lemma 4.4.1 and let $\delta \in (0, \epsilon/4)$ be such that $4\delta\psi(2r) \le \epsilon\psi(r)$ for all $r > 0$. Then there exists $C > 0$ such that for all $x_0 \in D$, $R \in (0, \delta_D(x_0))$, $r \in (0, \psi^{-1}(\epsilon\delta\psi(R)/2)/2]$, $x \in B(x_0, \psi^{-1}(\epsilon\delta\psi(R)/2)/2)$, $z \in B(x_0, \psi^{-1}(\epsilon\delta\psi(R)/2))$, and $\delta\psi(R)/2 \le t - s \le 4\delta\psi(2R)$,*

$$\mathbb{P}^{(t,z)}(\sigma^Z_{\{s\}\times B(x,r)} \le \tau^Z_{[s,t]\times B(x_0,R)}) \ge C\frac{V(x,r)}{V(x,R)}.$$

In the remainder of this section, we further assume that the boundary function h is of Harnack-type and show that parabolic Harnack inequality for X holds true.

Suppose that $x, y, z \in D$ are such that $\rho(x,z) \le \rho(x,y)/2$. Then

$$\frac{2}{3}\rho(x,y) \le \rho(z,y) \le \frac{3}{2}\rho(x,y).$$

As a consequence, by the scalings of Φ and Ψ, there exists $a > 1$ such that

$$\begin{aligned} a^{-1}\Phi(\rho(x,y)) &\le \Phi(\rho(z,y)) \le a\Phi(\rho(x,y)), \\ a^{-1}\Psi(\rho(x,y)) &\le \Psi(\rho(z,y)) \le a\Psi(\rho(x,y)). \end{aligned} \tag{4.4.6}$$

Proposition 4.4.8. *Suppose that h is of Harnack-type. Then there exists $C > 0$ such that for all $x, y, z \in D$ satisfying $\rho(x,z) \le (\rho(x,y) \wedge \delta_D(x))/2$,*

$$J(x,y) \le CJ(z,y).$$

Proof. The result follows from Theorem 4.2.1, (4.1.5), $\mathbf{Poly}_{R_1}(\beta_1, \beta_2)$, (4.1.7) and (4.4.6). □

Corollary 4.4.9. *Suppose that h is of Harnack-type. Then there exists $C > 0$ such that for all $x, y \in D$ and $0 < r \le (\rho(x,y) \wedge \delta_D(x))/2$, it holds that*

$$J(x,y) \le \frac{C}{V(x,r)} \int_{B(x,r)} J(z,y)dz.$$

Proof. If $z \in B(x,r)$, then $\rho(x,z) < r \le \rho(x,y)/2$. Therefore, by Proposition 4.4.8, $J(x,y) \le c_1 J(z,y)$, whence the claim immediately follows. □

Using (4.1.1), (4.2.9), Lemmas 4.4.1, 4.4.3, 4.4.5, 4.4.7, and Corollaries 4.2.5 and 4.4.9, the following result can be proved using the same arguments as in the proofs of [32, Theorem 5.2 and Lemma 5.3] (see also the proofs of [47, Lemma 4.1 and Theorem 4.3]).

Theorem 4.4.10. *Suppose that h is of Harnack-type. Then there exist constants $\delta > 0$, $C > 1$ and $K \geq 1$ such that for all $t_0 \geq 0$, $x_0 \in D$ and $R \in (0, R_1)$ with $B(x_0, CR) \subset D$, and any non-negative function u on $(0,\infty) \times D$ which is parabolic on $Q := (t_0, t_0 + 4\delta\psi(CR)) \times B(x_0, CR)$, we have*

$$\sup_{(t_1,y_1)\in Q_-} u(t_1, y_1) \leq K \inf_{(t_2,y_2)\in Q_+} u(t_2, y_2),$$

where $Q_- = [t_0 + \delta\psi(CR), t_0 + 2\delta\psi(CR)] \times B(x_0, R)$ and $Q_+ = [t_0 + 3\delta\psi(CR), t_0 + 4\delta\psi(CR)] \times B(x_0, R)$.

4.5 Examples

Recall the definitions of $h_{p,q}(t, x, y)$ from (4.1.6), and $\mathcal{B}_h(t, x, y)$ from (4.2.2). We remind the reader that $h_{p,q}(t, x, y)$ is quite typical and it is the most important boundary function. Recall that $\psi(r) = 1/\phi(1/\Phi(r))$, $\delta_\vee(x, y) = \delta_D(x) \vee \delta_D(y)$ and $\delta_\wedge(x, y) = \delta_D(x) \wedge \delta_D(y)$. For simplicity, we will use $\delta(x)$ and $\delta(y)$ instead of $\delta_D(x)$ and $\delta_D(y)$, respectively.

We let

$$\begin{aligned}
\delta^t(x) &:= \delta_D(x) \vee \psi^{-1}(t),\\
\delta^t_\vee(x, y) &:= \delta^t(x) \vee \delta^t(y) = \delta_\vee(x, y) \vee \psi^{-1}(t),\\
\delta^t_\wedge(x, y) &:= \delta^t(x) \wedge \delta^t(y) = \delta_\wedge(x, y) \vee \psi^{-1}(t).
\end{aligned}$$

The following lemma provides a list of estimates of $\mathcal{B}_{h_{p,q}}(t, x, y)$ depending on the relationship between the parameters p, q, β_1, β_2. The list is not exhaustive, but it suffices for our purpose. The proof of the lemma is rather technical and consists of carefully estimating the integral defining $\mathcal{B}_{h_{p,q}}(t, x, y)$. The factorization (4.5.1) below is inspired by (4.2.11) and (4.2.16). See also (4.5.7) below.

Lemma 4.5.1. *Let $q \geq p \geq 0$, $p + q > 0$. Suppose that* $\mathbf{Poly}_{R_1}(\beta_1, \beta_2)$ *holds* *with* $\beta_2 < 1$. *Then*

$$\mathcal{B}_{h_{p,q}}(t,x,y) = \frac{\Phi(\rho(x,y))}{\psi(\rho(x,y))}\Big(1 \wedge \frac{\Phi(\delta_D(x))}{\phi^{-1}(1/t)^{-1}}\Big)^p \Big(1 \wedge \frac{\Phi(\delta_D(y))}{\phi^{-1}(1/t)^{-1}}\Big)^q A_{p,q}(t,x,y), \tag{4.5.1}$$

where $A_{p,q}(t,x,y)$ *satisfies the following estimates for all* $x, y \in D$ *and* $0 < t \le \psi(\rho(x,y))$ *such that* $\Phi(\rho(x,y)) < R_1/8$.

(i) If $\beta_2 < 1-p-q$, *then*

$$A_{p,q}(t,x,y) \simeq \Big(1 \wedge \frac{\Phi(\delta^t(y))}{\Phi(\rho(x,y))}\Big)^{q-p}\Big(1 \wedge \frac{\Phi(\delta^t_\wedge(x,y))}{\Phi(\rho(x,y))}\Big)^p\Big(1 \wedge \frac{\Phi(\delta^t_\vee(x,y))}{\Phi(\rho(x,y))}\Big)^p.$$

(ii) If $1-p-q < \beta_1 \le \beta_2 < 1-q$, *then*

$$\begin{aligned} A_{p,q}(t,x,y) \simeq\ & \Big(1 \wedge \frac{\Phi(\delta^t(y))}{\Phi(\rho(x,y))}\Big)^{q-p}\Big(1 \wedge \frac{\Phi(\delta^t_\wedge(x,y))}{\Phi(\rho(x,y))}\Big)^p \\ & \times \Big(1 \wedge \frac{\Phi(\delta^t_\vee(x,y))}{\Phi(\rho(x,y))}\Big)^{1-q}\Big(1 \wedge \frac{\psi(\delta^t_\vee(x,y))}{\psi(\rho(x,y))}\Big)^{-1}. \end{aligned}$$

(iii) If $1-q < \beta_1 \le \beta_2 < 1-p$, *then*

$$A_{p,q}(t,x,y) \simeq \Big(1 \wedge \frac{\Phi(\delta^t(y))}{\Phi(\rho(x,y))}\Big)^{1-p}\Big(1 \wedge \frac{\psi(\delta^t(y))}{\psi(\rho(x,y))}\Big)^{-1}\Big(1 \wedge \frac{\Phi(\delta^t_\wedge(x,y))}{\Phi(\rho(x,y))}\Big)^p.$$

(iv) If $1-p < \beta_1 \le \beta_2 < 1$, *then*

$$A_{p,q}(t,x,y) \simeq \Big(1 \wedge \frac{\Phi(\delta^t_\wedge(x,y))}{\Phi(\rho(x,y))}\Big)\Big(1 \wedge \frac{\psi(\delta^t_\wedge(x,y))}{\psi(\rho(x,y))}\Big)^{-1}.$$

(v) If $\beta_1 = \beta_2 = 1-p-q$ *and* $p > 0$, *then*

$$\begin{aligned} A_{p,q}(t,x,y) \simeq\ & \Big(1 \wedge \frac{\Phi(\delta^t(y))}{\Phi(\rho(x,y))}\Big)^{q-p}\Big(1 \wedge \frac{\Phi(\delta^t_\wedge(x,y))}{\Phi(\rho(x,y))}\Big)^p \\ & \times \Big(1 \wedge \frac{\Phi(\delta^t_\vee(x,y))}{\Phi(\rho(x,y))}\Big)^p \log\Big(e + \frac{\Phi(\rho(x,y))}{\Phi(\delta^t_\vee(x,y) \wedge \rho(x,y))}\Big). \end{aligned}$$

(vi) If $\beta_1 = \beta_2 = 1 - p - q$ *and* $p = 0$*, then*

$$A_{p,q}(t,x,y) \simeq \Big(1 \wedge \frac{\Phi(\delta^t(y))}{\Phi(\rho(x,y))}\Big)^q \log\left(e + \frac{\Phi(\rho(x,y))}{\Phi(\delta^t(y) \wedge \rho(x,y))}\right).$$

(vii) If $\beta_1 = \beta_2 = 1 - q$ *and* $q = p$*, then*

$$A_{p,q}(t,x,y) \simeq \Big(1 \wedge \frac{\Phi(\delta^t_\wedge(x,y))}{\Phi(\rho(x,y))}\Big)^p \log\left(e + \frac{\Phi(\delta^t_\vee(x,y) \wedge \rho(x,y))}{\Phi(\delta^t_\wedge(x,y) \wedge \rho(x,y))}\right).$$

(viii) If $\beta_1 = \beta_2 = 1 - q$ *and* $q > p > 0$*, then*

$$A_{p,q}(t,x,y) \simeq \Big(1 \wedge \frac{\Phi(\delta^t(y))}{\Phi(\rho(x,y))}\Big)^{q-p} \Big(1 \wedge \frac{\Phi(\delta^t_\wedge(x,y))}{\Phi(\rho(x,y))}\Big)^p \times \log\left(e + \frac{\Phi(\delta^t(x) \wedge \rho(x,y))}{\Phi(\delta^t(y) \wedge \rho(x,y))}\right).$$

(ix) If $\beta_1 = \beta_2 = 1 - p$ *and* $q > p$*, then*

$$A_{p,q}(t,x,y) \simeq \Big(1 \wedge \frac{\Phi(\delta^t_\wedge(x,y))}{\Phi(\rho(x,y))}\Big)^p \log\left(e + \frac{\Phi(\delta^t(y) \wedge \rho(x,y))}{\Phi(\delta^t(x) \wedge \rho(x,y))}\right).$$

Proof. Fix $x, y \in D$ such that $r := \rho(x,y) < \Phi^{-1}(R_1/8)$, and $t \in (0, \psi(r)]$. Write $\delta_\wedge = \delta_\wedge(x,y)$, $\delta_\vee = \delta_\vee(x,y)$, $\delta^t_\wedge = \delta^t_\wedge(x,y)$ and $\delta^t_\vee = \delta^t_\vee(x,y)$. We note that since $\beta_2 < 1$, by Lemma 2.1.1(i), $w(s) \simeq \phi(1/s)$ for $s \in (0, R_1)$ and hence

$$w(\Phi(u)) \simeq \psi(u)^{-1} \quad \text{for } u \in (0, \Phi^{-1}(R_1/2)). \tag{4.5.2}$$

Define

$$A_{p,q}(t,x,y) := \frac{\psi(r)}{\Phi(r)} \frac{\mathcal{B}_{h_{p,q}}(t,x,y)}{h_{p,q}(\phi^{-1}(1/t)^{-1}, x, y)}.$$

Then from the definition, we get that

$$A_{p,q}(t,x,y) = \frac{\psi(r)}{\Phi(r)} \int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r)} \Big(1 \wedge \frac{\Phi(\delta^t_\wedge)}{s}\Big)^p \Big(1 \wedge \frac{\Phi(\delta^t_\vee)}{s}\Big)^p \times \Big(1 \wedge \frac{\Phi(\delta^t(y))}{s}\Big)^{q-p} w(s)ds.$$

If $\delta_\vee \geq 2r$, then $\delta_\wedge \geq \delta_\vee - r \geq r$ and hence we get from Lemma 1.1.1(i), $\mathbf{Poly}_{R_1}(\beta_1, \beta_2)$ (by using $\beta_2 < 1$) and (4.5.2) that

$$A_{p,q}(t,x,y) \simeq \frac{\psi(r)}{\Phi(r)} \int_{2\phi^{-1}(1/t)^{-1}}^{4\Phi(r)} w(s)ds \simeq \psi(r) w(4\Phi(r)) \simeq 1.$$

On the other hand, we see that in every of the cases (i)-(ix), the right-hand side of comparability relation for $A_{p,q}(t,x,y)$ is comparable to 1. Hence the assertion of the lemma is valid when $\delta_\vee \geq 2r$.

Suppose now that $\delta_\vee < 2r$. Since $\psi^{-1}(t) < r$, we get $\delta^t_\vee < 2r$, hence by the scaling property (4.1.3) of Φ, we have

$$3\Phi(a) \wedge 4\Phi(r) \simeq \Phi(a) \quad \text{for } \ a \in \{\delta^t(y),\, \delta^t_\wedge,\, \delta^t_\vee\}. \tag{4.5.3}$$

(i) The desired comparability follows from Lemma 1.1.1(i), $\mathbf{Poly}_{R_1}(\beta_1, \beta_2)$ and (4.5.2).

(ii) By Lemma 1.1.1(i)-(ii), $\mathbf{Poly}_{R_1}(\beta_1, \beta_2)$ and (4.5.2), since $q - p + p + \beta_2 < 1 < p + q + \beta_1$,

$$\begin{aligned} A_{p,q}(t,x,y) &\simeq \frac{\psi(r)}{\Phi(r)} \int_{2\phi^{-1}(1/t)^{-1}}^{3\Phi(\delta^t_\vee)\wedge 4\Phi(r)} \Big(1 \wedge \frac{\Phi(\delta^t_\wedge)}{s}\Big)^p \Big(1 \wedge \frac{\Phi(\delta^t(y))}{s}\Big)^{q-p} w(s)ds \\ &\quad + \frac{\psi(r)}{\Phi(r)} \Phi(\delta^t_\wedge)^p \Phi(\delta^t_\vee)^p \Phi(\delta^t(y))^{q-p} \int_{3\Phi(\delta^t_\vee)\wedge 4\Phi(r)}^{4\Phi(r)} \frac{w(s)}{s^{p+q}} ds \\ &\simeq \frac{\psi(r)}{\Phi(r)} \Big(1 \wedge \frac{\Phi(\delta^t_\wedge)}{\Phi(\delta^t_\vee)}\Big)^p \Big(1 \wedge \frac{\Phi(\delta^t(y))}{\Phi(\delta^t_\vee)}\Big)^{q-p} \Phi(\delta^t_\vee) w(\Phi(\delta^t_\vee)) \\ &\simeq \frac{\Phi(\delta^t_\wedge)^p \Phi(\delta^t(y))^{q-p}}{\Phi(r)^q} \Big(\frac{\Phi(r)}{\Phi(\delta^t_\vee)}\Big)^q \frac{\Phi(\delta^t_\vee)}{\Phi(r)} \frac{\psi(r)}{\psi(\delta^t_\vee)}. \end{aligned} \tag{4.5.4}$$

We used (4.5.3) in the second comparability above. Since Φ and ψ are increasing and satisfy scaling properties, the desired comparability holds.

(iii) If $\delta^t_\vee = \delta^t(y)$, then (4.5.4) holds by Lemma 1.1.1(i)-(ii), $\mathbf{Poly}_{R_1}(\beta_1, \beta_2)$, (4.5.2) and (4.5.3) since $p + \beta_2 < 1 < p + q + \beta_1$. The desired comparability then follows from (4.5.4). If $\delta^t_\wedge = \delta^t(y)$, then we get, by Lemma 1.1.1(i)-(ii),

$\mathbf{Poly}_{R_1}(\beta_1,\beta_2)$, (4.5.2), (4.5.3) and the assumption $q+\beta_1>1$, that

$$\begin{aligned}A_{p,q}(t,x,y) &\simeq \frac{\psi(r)}{\Phi(r)}\int_{2\phi^{-1}(1/t)^{-1}}^{3\Phi(\delta^t(y))\wedge 4\Phi(r)} w(s)ds \\ &\quad + \frac{\psi(r)}{\Phi(r)}\Phi(\delta^t(y))^q \int_{3\Phi(\delta^t(y))\wedge 4\Phi(r)}^{4\Phi(r)} \Big(1\wedge \frac{\Phi(\delta^t_\vee)}{s}\Big)^p \frac{w(s)}{s^q}ds \\ &\simeq \frac{\Phi(\delta^t(y))}{\psi(\delta^t(y))}\frac{\psi(r)}{\Phi(r)} = \Big(\frac{\Phi(\delta^t(y))}{\Phi(r)}\Big)^{1-p}\Big(\frac{\Phi(\delta^t_\wedge)}{\Phi(r)}\Big)^p \frac{\psi(r)}{\psi(\delta^t(y))}.\end{aligned}$$

(iv) Since $p+\beta_1>1$, we get from Lemma 1.1.1(i)-(ii), $\mathbf{Poly}_{R_1}(\beta_1,\beta_2)$, (4.5.2) and (4.5.3) that

$$\begin{aligned}A_{p,q}(t,x,y) &\simeq \frac{\psi(r)}{\Phi(r)}\int_{2\phi^{-1}(1/t)^{-1}}^{3\Phi(\delta^t_\wedge)\wedge 4\Phi(r)} w(s)ds \\ &+ \frac{\psi(r)}{\Phi(r)}\Phi(\delta^t_\wedge)^p \int_{3\Phi(\delta^t_\wedge)\wedge 4\Phi(r)}^{4\Phi(r)} \Big(1\wedge \frac{\Phi(\delta^t_\vee)}{s}\Big)^p \Big(1\wedge\frac{\Phi(\delta^t(y))}{s}\Big)^{q-p}\frac{w(s)}{s^p}ds \\ &\simeq \frac{\Phi(\delta^t_\wedge)}{\psi(\delta^t_\wedge)}\frac{\psi(r)}{\Phi(r)}.\end{aligned}$$

(v) Note that $w(s)\simeq s^{p+q-1}$ and $\psi(s)\simeq\Phi(s)^{1-p-q}$ for $s\in(0,R_1)$ in this case. Thus, we get from Lemma 1.1.1(i) and (4.5.3) that

$$\begin{aligned}A_{p,q}(t,x,y) &\simeq \frac{\psi(r)}{\Phi(r)}\int_{2\phi^{-1}(1/t)^{-1}}^{3\Phi(\delta^t_\vee)\wedge 4\Phi(r)} \big(s\wedge\Phi(\delta^t_\wedge)\big)^p\big(s\wedge\Phi(\delta^t(y))\big)^{q-p}s^{p-1}ds \\ &\quad + \frac{\psi(r)}{\Phi(r)}\Phi(\delta^t_\wedge)^p\Phi(\delta^t_\vee)^p\Phi(\delta^t(y))^{q-p}\int_{3\Phi(\delta^t_\vee)\wedge 4\Phi(r)}^{4\Phi(r)}\frac{ds}{s} \\ &\simeq \frac{\Phi(\delta^t_\wedge)^p\,\Phi(\delta^t_\vee)^p\,\Phi(\delta^t(y))^{q-p}}{\Phi(r)^{p+q}}\log\big(e+\frac{\Phi(r)}{\Phi(\delta^t_\vee\wedge r)}\big).\end{aligned}$$

(vi) Note that $w(s)\simeq s^{q-1}$ and $\psi(s)\simeq\Phi(s)^{1-q}$ for $s\in(0,R_1/2)$ in this case. We get from (4.5.3) that

$$\begin{aligned}A_{p,q}(t,x,y) &\simeq \frac{\psi(r)}{\Phi(r)}\int_{2\phi^{-1}(1/t)^{-1}}^{3\Phi(\delta^t(y))\wedge 4\Phi(r)} s^{q-1}ds + \frac{\psi(r)}{\Phi(r)}\Phi(\delta^t(y))^q\int_{3\Phi(\delta^t(y))\wedge 4\Phi(r)}^{4\Phi(r)}\frac{ds}{s} \\ &\simeq \frac{\Phi(\delta^t(y))^q}{\Phi(r)^q}\log\left(e+\frac{\Phi(r)}{\Phi(\delta^t(y)\wedge r)}\right).\end{aligned}$$

(vii) Note that $w(s) \simeq s^{p-1}$ and $\psi(s) \simeq \Phi(s)^{1-p}$ for $s \in (0, R_1)$ in this case. By (4.5.3), we obtain

$$\begin{aligned} A_{p,q}(t,x,y) &\simeq \frac{\psi(r)}{\Phi(r)} \int_{2\phi^{-1}(1/t)^{-1}}^{3\Phi(\delta^t_\wedge)\wedge 4\Phi(r)} s^{p-1} ds + \frac{\psi(r)}{\Phi(r)} \Phi(\delta^t_\wedge)^p \int_{3\Phi(\delta^t_\wedge)\wedge 4\Phi(r)}^{3\Phi(\delta^t_\vee)\wedge 4\Phi(r)} \frac{ds}{s} \\ &\quad + \frac{\psi(r)}{\Phi(r)} \Phi(\delta^t_\wedge)^p \Phi(\delta^t_\vee)^p \int_{3\Phi(\delta^t_\vee)\wedge 4\Phi(r)}^{4\Phi(r)} s^{-1-p} ds \\ &\simeq \frac{\Phi(\delta^t_\wedge)^p}{\Phi(r)^p} \log\left(e + \frac{\Phi(\delta^t_\vee \wedge r)}{\Phi(\delta^t_\wedge \wedge r)}\right). \end{aligned}$$

(viii) Note that $w(s) \simeq s^{q-1}$ and $\psi(s) \simeq \Phi(s)^{1-q}$ for $s \in (0, R_1)$ in this case. If $\delta^t_\vee = \delta^t(y)$, then we get from Lemma 1.1.1(i) and (4.5.3) that

$$\begin{aligned} A_{p,q}(t,x,y) &\simeq \frac{\psi(r)}{\Phi(r)} \int_{2\phi^{-1}(1/t)^{-1}}^{3\Phi(\delta^t(y))\wedge 4\Phi(r)} \big(s \wedge \Phi(\delta^t(x))\big)^p s^{q-p-1} ds \\ &\quad + \frac{\psi(r)}{\Phi(r)} \Phi(\delta^t(x))^p \Phi(\delta^t(y))^q \int_{3\Phi(\delta^t(y))\wedge 4\Phi(r)}^{4\Phi(r)} s^{-p-1} ds \\ &\simeq \frac{\Phi(\delta^t(x))^p\, \Phi(\delta^t(y))^{q-p}}{\Phi(r)^q} = \frac{\Phi(\delta^t_\wedge)^p\, \Phi(\delta^t(y))^{q-p}}{\Phi(r)^q}. \end{aligned}$$

If $\delta^t_\wedge = \delta^t(y)$, then we get from (4.5.3) that

$$\begin{aligned} A_{p,q}(t,x,y) &\simeq \frac{\psi(r)}{\Phi(r)} \int_{2\phi^{-1}(1/t)^{-1}}^{3\Phi(\delta^t(y))\wedge 4\Phi(r)} s^{q-1} ds + \frac{\psi(r)}{\Phi(r)} \Phi(\delta^t(y))^q \int_{3\Phi(\delta^t(y))\wedge 4\Phi(r)}^{3\Phi(\delta^t(x))\wedge 4\Phi(r)} \frac{ds}{s} \\ &\quad + \frac{\psi(r)}{\Phi(r)} \Phi(\delta^t(x))^p \Phi(\delta^t(y))^q \int_{3\Phi(\delta^t(x))\wedge 4\Phi(r)}^{4\Phi(r)} s^{-1-p} ds \\ &\simeq \frac{\Phi(\delta^t_\wedge)^p\, \Phi(\delta^t(y))^{q-p}}{\Phi(r)^q} \log\left(e + \frac{\Phi(\delta^t(x) \wedge r)}{\Phi(\delta^t(y) \wedge r)}\right). \end{aligned}$$

(ix) Note that $w(s) \simeq s^{p-1}$ and $\psi(s) \simeq \Phi(s)^{1-p}$ for $s \in (0, R_1)$ in this case. If $\delta^t_\vee = \delta^t(y)$, then by (4.5.3), it holds that

$$\begin{aligned} A_{p,q}(t,x,y) &\simeq \frac{\psi(r)}{\Phi(r)} \int_{2\phi^{-1}(1/t)^{-1}}^{3\Phi(\delta^t(x))\wedge 4\Phi(r)} s^{p-1} ds + \frac{\psi(r)}{\Phi(r)} \Phi(\delta^t(x))^p \int_{3\Phi(\delta^t(x))\wedge 4\Phi(r)}^{3\Phi(\delta^t(y))\wedge 4\Phi(r)} \frac{ds}{s} \\ &\quad + \frac{\psi(r)}{\Phi(r)} \Phi(\delta^t(x))^p \Phi(\delta^t(y))^q \int_{3\Phi(\delta^t(y))\wedge 4\Phi(r)}^{4\Phi(r)} s^{-1-q} ds \end{aligned}$$

$$\simeq \frac{\Phi(\delta^t_\wedge)^p}{\Phi(r)^p} \log\Big(e + \frac{\Phi(\delta^t(y)\wedge r)}{\Phi(\delta^t(x)\wedge r)}\Big).$$

If $\delta^t_\wedge(x,y) = \delta^t(y)$, then we get from Lemma 1.1.1(ii) and (4.5.3) that

$$\begin{aligned} A_{p,q}(t,x,y) &\simeq \frac{\psi(r)}{\Phi(r)} \int_{2\phi^{-1}(1/t)^{-1}}^{3\Phi(\delta^t(y))\wedge 4\Phi(r)} s^{p-1} ds \\ &\quad + \frac{\psi(r)}{\Phi(r)} \Phi(\delta^t(y))^q \int_{3\Phi(\delta^t(y))\wedge 4\Phi(r)}^{4\Phi(r)} \Big(1 \wedge \frac{\Phi(\delta^t(x))}{s}\Big)^p s^{p-q-1} ds \\ &\simeq \frac{\Phi(\delta^t(y))^p}{\Phi(r)^p} = \frac{\Phi(\delta^t_\wedge)^p}{\Phi(r)^p}. \end{aligned}$$

□

Example 4.5.2. Let $d, \alpha > 0$, $\beta \in (0,1)$ and $p, q \geq 0$ such that $p + q > 0$. Suppose that for every $r_0 \geq 1$, there are comparability constants such that

$$V(x,r) \simeq r^d, \quad x \in D,\ 0 < r < r_0. \tag{4.5.5}$$

Let Y^D be a Hunt process in D and $S = (S_t)_{t\geq 0}$ be an independent driftless subordinator with Laplace exponent ϕ. Suppose that the tail w of the Lévy measure of S satisfies

$$w(r) \simeq r^{-\beta}, \quad 0 < r < r_1, \tag{4.5.6}$$

for some $r_1 > 0$. Suppose that the heat kernel $p_D(t,x,y)$ of Y^D satisfies either $\mathbf{HK_B^{h_{p,q}}}$ or $\mathbf{HK_U^{h_{p,q}}}$ with $\Phi(r) = \Psi(r) = r^\alpha$ where the boundary function $h_{p,q}$ is defined as (4.1.6). When $\mathbf{HK_U^{h_{p,q}}}$ is satisfied, we also assume that (4.5.5) and (4.5.6) hold for all $r > 0$. See Example 4.1.8 for concrete examples of Y^D. By switching the roles of x and y if needed, without loss of generality, we assume that $q \geq p$.

Let $q(t,x,y)$, $J(x,y)$ and $G_D(x,y)$ be the heat kernel, the jump kernel and the Green function of the subordinate process $X_t := Y^D_{S_t}$ respectively. Using our theorems in Sections 4.2 and 4.3, and Lemma 4.5.1, we get explicit estimates on $q(t,x,y)$, $J(x,y)$ and $G_D(x,y)$. We list them in terms of the range of $p+q$, similar to the format of the Green function estimates for Dirichlet forms degenerate at the boundary in [91].

In particular, by putting $p=q=1/2$, we get Theorem 4.0.1.

We define $B^0_{p,q} : (0,\infty)\times D\times D \to (0,1]$ by

$$B^0_{p,q}(t,x,y) := \Big(1\wedge \frac{\delta^t_\wedge(x,y)}{\rho(x,y)}\Big)^{\alpha p}\Big(1\wedge \frac{\delta^t_\vee(x,y)}{\rho(x,y)}\Big)^{\alpha p}\Big(1\wedge \frac{\delta^t(y)}{\rho(x,y)}\Big)^{\alpha(q-p)}.$$

We also define $B^1_{p,q} : (0,\infty)\times D\times D \to (0,1]$ as follows: if $q>1-\beta$, then

$$B^1_{p,q}(t,x,y) := \begin{cases} \Big(1\wedge \frac{\delta^t_\wedge(x,y)}{\rho(x,y)}\Big)^{\alpha(1-\beta)}, & p>1-\beta,\\ \Big(1\wedge \frac{\delta^t_\wedge(x,y)}{\rho(x,y)}\Big)^{\alpha(1-\beta)}\log\Big(e+\frac{\delta^t(y)\wedge\rho(x,y)}{\delta^t(x)\wedge\rho(x,y)}\Big), & p=1-\beta,\\ \Big(1\wedge \frac{\delta^t_\wedge(x,y)}{\rho(x,y)}\Big)^{\alpha p}\Big(1\wedge \frac{\delta^t(y)}{\rho(x,y)}\Big)^{\alpha(1-\beta-p)}, & p<1-\beta,\end{cases}$$

if $q=1-\beta$, then

$$B^1_{p,q}(t,x,y) :=$$
$$\begin{cases} \Big(1\wedge \frac{\delta^t_\wedge(x,y)}{\rho(x,y)}\Big)^{\alpha(1-\beta)}\log\Big(e+\frac{\delta^t_\vee(x,y)\wedge\rho(x,y)}{\delta^t_\wedge(x,y)\wedge\rho(x,y)}\Big), & p=1-\beta,\\ \Big(1\wedge \frac{\delta^t_\wedge(x,y)}{\rho(x,y)}\Big)^{\alpha p}\Big(1\wedge \frac{\delta^t(y)}{\rho(x,y)}\Big)^{\alpha(1-\beta-p)}\log\Big(e+\frac{\delta^t(x)\wedge\rho(x,y)}{\delta^t(y)\wedge\rho(x,y)}\Big), & 0<p<1-\beta,\\ \Big(1\wedge \frac{\delta^t(y)}{\rho(x,y)}\Big)^{\alpha q}\log\Big(e+\frac{\rho(x,y)}{\delta^t(y)\wedge\rho(x,y)}\Big), & p=0,\end{cases}$$

and if $q<1-\beta$, then

$$B^1_{p,q}(t,x,y) := \Big(1\wedge \frac{\delta^t(y)}{\rho(x,y)}\Big)^{\alpha(q-p)}\times$$

$$\begin{cases} \Big(1\wedge \dfrac{\delta^t_\wedge(x,y)}{\rho(x,y)}\Big)^{\alpha p}\Big(1\wedge \dfrac{\delta^t_\vee(x,y)}{\rho(x,y)}\Big)^{\alpha(1-\beta-q)}, & p>1-\beta-q,\\ \Big(1\wedge \dfrac{\delta^t_\wedge(x,y)}{\rho(x,y)}\Big)^{\alpha p}\Big(1\wedge \dfrac{\delta^t_\vee(x,y)}{\rho(x,y)}\Big)^{\alpha p}\log\Big(e+\dfrac{\rho(x,y)}{\delta^t_\vee(x,y)\wedge\rho(x,y)}\Big), & p=1-\beta-q,\\ \Big(1\wedge \dfrac{\delta^t_\wedge(x,y)}{\rho(x,y)}\Big)^{\alpha p}\Big(1\wedge \dfrac{\delta^t_\vee(x,y)}{\rho(x,y)}\Big)^{\alpha p}, & p<1-\beta-q. \end{cases}$$

We first give heat kernel estimates which are consequences of Lemma 4.5.1, Corollary 4.2.4 and Theorem 4.2.7.

(a) It holds that for all $(t,x,y)\in(0,1]\times D\times D$,

$$\begin{aligned} q(t,x,y) &\simeq \Big(1\wedge\frac{\delta(x)}{t^{1/(\alpha\beta)}}\Big)^{\alpha p}\Big(1\wedge\frac{\delta(y)}{t^{1/(\alpha\beta)}}\Big)^{\alpha q}B^{C_0}_{p,q}(t,x,y)\Big(t^{-d/(\alpha\beta)}\wedge\frac{t}{\rho(x,y)^{d+\alpha\beta}}\Big)\\ &\simeq \Big(1\wedge\frac{\delta(x)}{t^{1/(\alpha\beta)}}\Big)^{\alpha p}\Big(1\wedge\frac{\delta(y)}{t^{1/(\alpha\beta)}}\Big)^{\alpha q}\Big(t^{-d/(\alpha\beta)}\wedge\frac{tB^{C_0}_{p,q}(t,x,y)}{\rho(x,y)^{d+\alpha\beta}}\Big), \quad (4.5.7)\end{aligned}$$

where the function $B^{C_0}_{p,q}(t,x,y)$ is defined as above.

(b) If $\mathbf{HK_B^{h_{p,q}}}$ holds, then for all $(t,x,y)\in[1,\infty)\times D\times D$,

$$q(t,x,y)\simeq e^{-t\phi(\lambda_D)}\delta(x)^{\alpha p}\delta(y)^{\alpha q},$$

and if $\mathbf{HK_U^{h_{p,q}}}$ holds, then (4.5.7) holds for all $(t,x,y)\in(0,\infty)\times D\times D$.

Next, we give estimates on the jump kernel $J(x,y)$. By Theorem 4.2.1 and the fact that $\mathcal{B}^*_h(x,y)\simeq\mathcal{B}_h(0,x,y)$ for $x,y\in D$, we deduce from (4.5.7) that for any $q\ge p\ge 0$,

$$J(x,y)\simeq\frac{B^{C_0}_{p,q}(0,x,y)}{\rho(x,y)^{d+\alpha\beta}},\quad x,y\in D.$$

Lastly, we give the Green function estimates. Define

$$\mathfrak{g}(x,y)=\Big(1\wedge\frac{\delta(x)}{\rho(x,y)}\Big)^{\alpha p}\Big(1\wedge\frac{\delta(y)}{\rho(x,y)}\Big)^{\alpha q}\times\begin{cases}\rho(x,y)^{\alpha\beta-d}, & d>\alpha\beta,\\ \log\Big(e+\dfrac{\delta_\vee(x,y)}{\rho(x,y)}\Big), & d=\alpha\beta,\\ \big(\delta_\vee(x,y)\vee\rho(x,y)\big)^{\alpha\beta-d}, & d<\alpha\beta.\end{cases}$$

When $C_0 = 0$, by Theorem 4.3.6 and Example 4.3.3, for all $x, y \in D$,

$$G_D(x,y) \simeq \begin{cases} \mathfrak{g}(x,y), & d > \alpha(\beta - p - q), \\ \delta(x)^{\alpha p}\delta(y)^{\alpha q} \log\left(e + \dfrac{\operatorname{diam}(D)}{\delta_\vee(x,y) \vee \rho(x,y)}\right), & d = \alpha(\beta - p - q), \\ \operatorname{diam}(D)^{(\alpha\beta - \alpha p - \alpha q - d)/\alpha}\delta(x)^{\alpha p}\delta(y)^{\alpha q}, & d < \alpha(\beta - p - q). \end{cases} \tag{4.5.8}$$

In particular, when $\mathbf{HK}_{\mathbf{U}}^{\mathbf{h_{p,q}}}$ holds, if $d \le \alpha(\beta - p - q)$, then $G_D(x,y) = \infty$.

Now assume that $C_0 = 1$. If $p + q < \beta + 1$, then using Theorem 4.3.6 and Example 4.3.3 again, we see that (4.5.8) also hold. If $p + q = \beta + 1$ and $q < \beta + 1$ (so that (H2**) holds, cf. Example 4.3.7), then by Theorem 4.3.9, for all $x, y \in D$,

$$G_D(x,y) \simeq \mathfrak{g}(x,y) \log\left(e + \frac{\rho(x,y)}{\delta_\vee(x,y) \wedge \rho(x,y)}\right). \tag{4.5.9}$$

If $p + q > \beta + 1$ and $q < \beta + 1$ (so, again, (H2**) holds), then by Theorem 4.3.8 and (4.3.13), for all $x, y \in D$,

$$G_D(x,y) \simeq \left(1 \wedge \frac{\delta_\vee(x,y)}{\rho(x,y)}\right)^{-\alpha(p+q-\beta-1)} \mathfrak{g}(x,y). \tag{4.5.10}$$

The unusual form of the estimates in (4.5.9)-(4.5.10) should be compared with similar estimates of the Green function obtained in a different context in [91, Theorem 1.1 (2),(3)]. Such estimates lead to anomalous boundary behavior of the corresponding Green potential, cf. [1].

Chapter 5

Heat kernel estimates for Dirichlet forms degenerate at the boundary

In this chapter, we consider symmetric Markov processes in $\mathbb{R}^d_+$ with degenerate jump kernels and critical killing potentials. The results of this chapter is based on the ongoing project [57]. The main result of this chapter is Theorem 5.6.1.

5.1 Setup

Let $d \ge 1$ and $0 < \alpha < 2$. Recall that $\mathbf{e}_d := (\widetilde{0}, 1) \in \mathbb{R}^d$ and $\mathbb{R}^d_+ := \{(\widetilde{x}, x_d) \in \mathbb{R}^d : x_d > 0\}$. We write $\overline{\mathbb{R}}^d_+ := \{(\widetilde{x}, x_d) \in \mathbb{R}^d : x_d \ge 0\}$ for the closure of $\mathbb{R}^d_+$.

We consider the following assumptions:

(A1) $\mathcal{B}(x, y) = \mathcal{B}(y, x)$ for all $x, y \in \mathbb{R}^d_+$.

(A2) If $\alpha \ge 1$, then there exist $\theta > \alpha - 1$ and $C_1 > 0$ such that

$$|\mathcal{B}(x, x) - \mathcal{B}(x, y)| \le C_1 \left(\frac{|x - y|}{x_d \wedge y_d} \right)^{\theta}, \quad x, y \in \mathbb{R}^d_+.$$

(A3-I) There exist $C_2 \ge 1$ and parameters $\beta_1, \beta_2, \beta_3, \beta_4 \ge 0$, with $\beta_1 > 0$

if $\beta_3 > 0$, and $\beta_2 > 0$ if $\beta_4 > 0$, such that

$$C_2^{-1}\widetilde{B}_{\beta_1,\beta_2,\beta_3,\beta_4}(x,y) \le \mathcal{B}(x,y) \le C_2, \quad x,y \in \mathbb{R}^d_+,$$

where

$$\begin{aligned}\widetilde{B}_{\beta_1,\beta_2,\beta_3,\beta_4}(x,y) := &\left(\frac{x_d \wedge y_d}{|x-y|} \wedge 1\right)^{\beta_1} \left(\frac{x_d \vee y_d}{|x-y|} \wedge 1\right)^{\beta_2} \\ &\times \log^{\beta_3}\left(1 + \frac{(x_d \vee y_d) \wedge |x-y|}{(x_d \wedge y_d) \wedge |x-y|}\right) \\ &\times \log^{\beta_4}\left(1 + \frac{|x-y|}{(x_d \vee y_d) \wedge |x-y|}\right). \end{aligned} \tag{5.1.1}$$

(A3-II) There exists $C_3 > 0$ such that

$$\mathcal{B}(x,y) \le C_3\widetilde{B}_{\beta_1,\beta_2,\beta_3,\beta_4}(x,y), \quad x,y \in \mathbb{R}^d_+.$$

(A4) For all $x, y \in \mathbb{R}^d_+$ and $a > 0$, $\mathcal{B}(ax, ay) = \mathcal{B}(x,y)$. In case $d \ge 2$, for all $x,y \in \mathbb{R}^d_+$ and $\widetilde{z} \in \mathbb{R}^{d-1}$, $\mathcal{B}(x + (\widetilde{z}, 0), y + (\widetilde{z}, 0)) = \mathcal{B}(x,y)$.

Throughout the chapter, we always assume that $\mathcal{B}(x,y)$ satisfies **(A1)**, **(A3-I)** *and* **(A4)**.

The definition of the function $\widetilde{B}_{\beta_1,\beta_2,\beta_3,\beta_4}(x,y)$ is motivated by Theorem 4.2.1 and Lemma 4.5.1.

Consider the following symmetric form

$$\mathcal{E}^0(u,v) := \frac{1}{2}\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+}(u(x)-u(y))(v(x)-v(y))\frac{\mathcal{B}(x,y)}{|x-y|^{d+\alpha}}dydx.$$

where the jump kernel $\mathcal{B}(x,y)|x-y|^{-d-\alpha}$ can degenerate at $\{x \in \mathbb{R}^d : x_d = 0\}$. Since $\mathcal{B}(x,y)$ is bounded, by Fatou's lemma, $(\mathcal{E}^0, C_c^\infty(\mathbb{R}^d_+))$ and $(\mathcal{E}^0, C_c^\infty(\overline{\mathbb{R}}^d_+))$ are closable in $L^2(\mathbb{R}^d_+, dx)$. Let $\mathcal{F}^0$ and $\bar{\mathcal{F}}$ be the closures of $C_c^\infty(\mathbb{R}^d_+)$ and $C_c^\infty(\overline{\mathbb{R}}^d_+)$ in $L^2(\mathbb{R}^d_+, dx)$ under the norm $\mathcal{E}^0_1 := \mathcal{E}^0 + (\cdot,\cdot)_{L^2(\mathbb{R}^d_+,dx)}$ respectively. Then $(\mathcal{E}^0, \mathcal{F}^0)$ and $(\mathcal{E}^0, \bar{\mathcal{F}})$ are regular Dirichlet forms on $L^2(\mathbb{R}^d_+, dx)$. Let $((Y^0_t)_{t\ge0}, (\mathbb{P}^x)_{\mathbb{R}^d_+\setminus\mathcal{N}_0})$ and $((\bar{Y}_t)_{t\ge0}, (\mathbb{P}^x)_{\overline{\mathbb{R}}^d_+\setminus\mathcal{N}_0'})$ be the Hunt processes associated with $(\mathcal{E}^0, \mathcal{F}^0)$ and $(\mathcal{E}^0, \bar{\mathcal{F}})$ respectively, where $\mathcal{N}_0$ and $\mathcal{N}_0'$ are exceptional sets.

For $\kappa \in (0, \infty)$, define

$$\begin{aligned} \mathcal{E}^\kappa(u,v) &:= \mathcal{E}^0(u,v) + \kappa \int_{\mathbb{R}^d_+} u(x)v(x)x_d^{-\alpha}dx, \qquad u, v \in \mathcal{F}, \\ \mathcal{F}^\kappa &:= \widetilde{\mathcal{F}}^0 \cap L^2(\mathbb{R}^d_+, \kappa x_d^{-\alpha}dx), \end{aligned}$$

where $\widetilde{\mathcal{F}}^0$ is the family of all $\mathcal{E}^0_1$-quasi-continuous functions in $\mathcal{F}^0$. Then $(\mathcal{E}^\kappa, \mathcal{F}^\kappa)$ is also a regular Dirichlet form on $L^2(\mathbb{R}^d_+, dx)$ with $C_c^\infty(\mathbb{R}^d_+)$ as a special standard core by [71, Theorems 6.1.1 and 6.1.2]. Let $((Y^\kappa_t)_{t\ge 0}, (\mathbb{P}^x)_{x\in\mathbb{R}^d_+\setminus\mathcal{N}_\kappa})$ be the Hunt process associated with $(\mathcal{E}^\kappa, \mathcal{F}^\kappa)$, where $\mathcal{N}_\kappa$ is an exceptional set.

For $\kappa \in [0,\infty)$, we denote by ζ^κ the lifetime of Y^κ. Define $Y^\kappa_t = \partial$ for $t \ge \zeta^\kappa$, where ∂ is a cemetery point added to the state space $\mathbb{R}^d_+$. We write $(\bar{P}_t)_{t\ge0}$ and $(P^\kappa_t)_{t\ge0}$ for the semigroups of $\bar{Y}$ and Y^κ respectively.

5.2 Preliminaries

Note that for any $\epsilon > 0$,

$$\log(e+r) < (2+\epsilon^{-1})r^\epsilon \quad \text{for all } r \ge 1, \tag{5.2.1}$$

$$\frac{\log(e+ar)}{\log(e+a)} < (1+\epsilon^{-1})r^\epsilon \qquad \text{for all } r \ge 1 \text{ and } a > 0. \tag{5.2.2}$$

For any $a_1, a_2, a_3, a_4 \ge 0$, we define for $t \ge 0$ and $x, y \in \mathbb{R}^d_+$

$$\begin{aligned} A_{a_1,a_2,a_3,a_4}(t,x,y) = &\left(\frac{(x_d \wedge y_d) \vee t^{1/\alpha}}{|x-y|} \wedge 1\right)^{a_1} \left(\frac{(x_d \vee y_d) \vee t^{1/\alpha}}{|x-y|} \wedge 1\right)^{a_2} \\ &\times \log^{a_3}\left(1 + \frac{((x_d \vee y_d) \vee t^{1/\alpha}) \wedge |x-y|}{((x_d \wedge y_d) \vee t^{1/\alpha}) \wedge |x-y|}\right) \\ &\times \log^{a_4}\left(1 + \frac{|x-y|}{((x_d \vee y_d) \vee t^{1/\alpha}) \wedge |x-y|}\right). \end{aligned} \tag{5.2.3}$$

We note that $A_{\beta_1,\beta_2,\beta_3,\beta_4}(0,x,y) = \widetilde{B}_{\beta_1,\beta_2,\beta_3,\beta_4}(x,y)$. It is clear that for any

$\epsilon \in [0, a_1]$,

$$A_{a_1,a_2,a_3,a_4}(t,x,y) \le A_{a_1-\epsilon,a_2+\epsilon,a_3,a_4}(t,x,y) \quad \text{for all } t \ge 0,\ x,y \in \mathbb{R}^d_+.$$

Note that for any $a > 0$, there exists $c > 0$ such that

$$A_{a_1,a_2,a_3,a_4}(t,x,y) \ge c(a \wedge 1)^{a_1+a_2}, \tag{5.2.4}$$

for all $t > 0$ and $x, y \in \mathbb{R}^d_+$ with $(x_d \wedge y_d) + t^{1/\alpha} \ge a|x-y|$.

We give some elementary properties of $A_{a_1,a_2,a_3,a_4}(t,x,y)$.

Lemma 5.2.1. *Let $a_1, a_2, a_3, a_4 \ge 0$.*
(i) For any $\epsilon \in (0, a_1 \wedge a_3)$, there exists $c_1 > 0$ such that

$$A_{a_1,a_2,a_3,a_4}(t,x,y) \le c_1 A_{a_1-\epsilon,a_2,0,a_4}(t,x,y) \quad \text{for all } t \ge 0, x,y \in \mathbb{R}^d_+.$$

(ii) For any $\epsilon \in (0, a_2 \wedge a_4)$, there exists $c_2 > 0$ such that

$$A_{a_1,a_2,a_3,a_4}(t,x,y) \le c_2 A_{a_1,a_2-\epsilon,a_3,0}(t,x,y) \quad \text{for all } t \ge 0, x,y \in \mathbb{R}^d_+.$$

Lemma 5.2.2. *Let $a_1, a_2, a_3, a_4 \ge 0$. Suppose that $a_1 > 0$ if $a_3 > 0$.*
(i) For any $a' \in [0, a_2]$ with $a' < a_1$, there exists $C > 0$ such that

$$\begin{aligned} &A_{a_1,a_2,a_3,a_4}(t,x,y) \\ &\le C\left(\frac{x_d \vee t^{1/\alpha}}{|x-y|} \wedge 1\right)^{a_1} \left(\frac{y_d \vee t^{1/\alpha}}{|x-y|} \wedge 1\right)^{a'} \log^{a_3}\left(e + \frac{(y_d \vee t^{1/\alpha}) \wedge |x-y|}{(x_d \vee t^{1/\alpha}) \wedge |x-y|}\right) \\ &\quad \times \log^{a_4}\left(e + \frac{|x-y|}{((x_d \vee y_d) \vee t^{1/\alpha}) \wedge |x-y|}\right), \end{aligned}$$

for all $t \ge 0, x, y \in \mathbb{R}^d_+$.
(ii) Assume that $a_1 > a_2$. Then there exists $C > 0$ such that

$$A_{a_1,a_2,a_3,a_4}(t,x,y)$$

$$\leq C\left(\frac{x_d \vee t^{1/\alpha}}{|x-y|}\wedge 1\right)^{a_1}\left(\frac{y_d \vee t^{1/\alpha}}{|x-y|}\wedge 1\right)^{a_2}\log^{a_3}\left(e+\frac{(y_d\vee t^{1/\alpha})\wedge|x-y|}{(x_d\vee t^{1/\alpha})\wedge|x-y|}\right)$$
$$\times \log^{a_4}\left(e+\frac{|x-y|}{((x_d\vee y_d)\vee t^{1/\alpha})\wedge |x-y|}\right),$$

for all $t \geq 0, x, y \in \mathbb{R}^d_+$.

For any $r > 0$, define processes $\bar{Y}^{(r)}$ and $Y^{\kappa,(r)}$ by $\bar{Y}^{(r)}_t := r\bar{Y}_{r^{-\alpha}t}$ and $Y^{\kappa,(r)}_t := rY^{\kappa}_{r^{-\alpha}t}$. We recall the scaling property of Y^κ from [92, Lemma 5.1] and [93, Lemma 2.1]. By the same proof, $\bar{Y}$ also has the following scaling property.

Lemma 5.2.3. *For any* $\kappa \geq 0$, $r > 0$ *and* $x \in \mathbb{R}^d_+$, $(\bar{Y}^{(r)}, \mathbb{P}^{x/r})$ *and* $(Y^{\kappa,(r)}, \mathbb{P}^{x/r})$ *have the same laws as* $(\bar{Y}, \mathbb{P}^x)$ *and* $(Y^\kappa, \mathbb{P}^x)$ *respectively.*

By **(A4)**, we get the following horizontal translation invariance property of $\bar{Y}$ and Y^κ.

Lemma 5.2.4. *For any* $\kappa \geq 0$, $\widetilde{z} \in \mathbb{R}^{d-1}$ *and* $x \in \mathbb{R}^d_+$, $(\bar{Y} + (\widetilde{z}, 0), \mathbb{P}^{x-(\widetilde{z},0)})$ *and* $(Y^\kappa + (\widetilde{z}, 0), \mathbb{P}^{x-0(\widetilde{z},0)})$ *have the same laws as* $(\bar{Y}, \mathbb{P}^x)$ *and* $(Y^\kappa, \mathbb{P}^x)$ *respectively.*

Lemma 5.2.5. *Suppose* $\alpha \leq 1$. *Then* $\mathcal{F}^0 = \bar{\mathcal{F}}$.

Proof. Define

$$\widetilde{\mathcal{C}}(u,v) := \frac{1}{2}\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+\alpha}}dydx,$$
$$\mathcal{D}(\widetilde{\mathcal{C}}) := \text{closure of } C_c^\infty(\overline{\mathbb{R}}^d_+) \text{ in } L^2(\mathbb{R}^d_+, dx) \text{ under } \widetilde{\mathcal{C}} + (\cdot,\cdot)_{L^2(\mathbb{R}^d_+,dx)}.$$

Then $(\widetilde{\mathcal{C}}, \mathcal{D}(\widetilde{\mathcal{C}}))$ is a regular Dirichlet form associated with the *reflected* α-*stable process* in $\overline{\mathbb{R}}^d_+$ in the sense of [16]. Since $\mathcal{B}$ is bounded, there exists $c_1 > 0$ such that $\mathcal{E}^0(u,u) \leq c_1\widetilde{\mathcal{C}}(u,u)$ for all $u \in C_c^\infty(\overline{\mathbb{R}}^d_+)$ and hence $\mathcal{D}(\widetilde{\mathcal{C}}) \subset \bar{\mathcal{F}}$. By [16, Theorem 2.5(i) and Remark 2.2(1)], since $\alpha \leq 1$, $\overline{\mathbb{R}}^d_+ \setminus \mathbb{R}^d_+$ is $(\widetilde{\mathcal{C}}, \mathcal{F}_C)$-polar and hence is also $(\mathcal{E}^0, \bar{\mathcal{F}})$-polar. Therefore, $\overline{\tau}_{\mathbb{R}^d_+} := \inf\{t > 0 : \bar{Y}_t \notin \mathbb{R}^d_+\} = \infty$, $\mathbb{P}^x$-a.s. for all $x \in \mathbb{R}^d_+$ and we conclude the result from [93, Section 2]. □

5.3 Nash inequality and existence of the heat kernel

For all $\gamma \geq 0$, denote by I_γ the modified Bessel function of the first kind which is defined by

$$I_\gamma(r) = \sum_{m=0}^{\infty} \frac{1}{m!\,\Gamma(\gamma+1+m)} \left(\frac{r}{2}\right)^{2m+\gamma},$$

where $\Gamma(r) := \int_0^\infty u^{r-1}e^{-u}du$ is the Gamma function. It is known that (see, e.g. [2, (9.6.7) and (9.7.1)])

$$I_\gamma(r) \simeq (1 \wedge r)^{\gamma+1/2} r^{-1/2} e^r \quad \text{for} \quad r > 0. \tag{5.3.1}$$

Define for $t > 0$ and $x, y \in \mathbb{R}^d_+$,

$$q^\gamma(t,x,y) = \frac{\sqrt{x_d y_d}}{2t} I_\gamma\left(\frac{x_d y_d}{2t}\right) \exp\left(-\frac{x_d^2 + y_d^2}{4t}\right) \prod_{i=1}^{d-1} \left(\frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{|x_i - y_i|^2}{4t}\right)\right).$$

Note that by (5.3.1),

$$q^\gamma(t,x,y) \asymp \left(1 \wedge \frac{x_d}{\sqrt{t}}\right)^{\gamma+1/2} \left(1 \wedge \frac{y_d}{\sqrt{t}}\right)^{\gamma+1/2} t^{-d/2} \exp\left(-c\frac{|x-y|^2}{t}\right).$$

By [106, Lemma 4.1 and Theorem 4.9], $q^\gamma(t,x,y)$ is the transition density of the Feller process $W^\gamma = (W^\gamma_t)_{t\geq 0}$ on $\mathbb{R}^d_+$ associated with the following regular Dirichlet form $(\mathcal{Q}^\gamma, \mathcal{D}(\mathcal{Q}^\gamma))$:

$$\mathcal{Q}^\gamma(u,v) := \int_{\mathbb{R}^d_+} \left(\nabla u(x) \cdot \nabla v(x) + \left(\gamma^2 - \frac{1}{4}\right) u(x)v(x)x_d^{-2}\right) dx,$$
$$\mathcal{D}(\mathcal{Q}^\gamma) := \text{closure of } C_c^\infty(\mathbb{R}^d_+) \text{ in } L^2(\mathbb{R}^d_+, dx) \text{ under } \mathcal{Q}^\gamma_1 = \mathcal{Q}^\gamma + (\cdot,\cdot)_{L^2(\mathbb{R}^d_+,dx)}.$$

Let $S = (S_t)_{t\geq 0}$ be an $\alpha/2$-stable subordinator independent of W^γ, and let $X^\gamma = (X^\gamma_t)_{t\geq 0}$ be the subordinate process $X^\gamma_t := W^\gamma_{S_t}$. Then X^γ is a Hunt process with no diffusion part. The transition density $p^\gamma(t,x,y)$ of X_t^γ exists and is given by

$$p^\gamma(t,x,y) = \int_0^\infty q^\gamma(s,x,y)\frac{d}{ds}\mathbb{P}(S_t \le s).$$

Also, the jump kernel $J^\gamma(dx,dy)$ and the killing measure $\kappa^\gamma(dx)$ of X^γ have densities $J^\gamma(x,y)$ and $\kappa^\gamma(x)$ that are given by the following formulas:

$$\begin{aligned} J^\gamma(x,y) &= \int_0^\infty q^\gamma(t,x,y)\,\nu_{\alpha/2}(t)dt,\\ \kappa^\gamma(x) &= \int_0^\infty \left(1-\int_{\mathbb{R}^d_+} q^\gamma(t,x,y)dy\right)\nu_{\alpha/2}(t)dt, \end{aligned}$$

where $\nu_{\alpha/2}(t) = \frac{\alpha/2}{\Gamma(1-\alpha/2)}t^{-1-\alpha/2}$ is the Lévy density of the subordinator S.

Using the scaling property and horizontal translation invariance of the function $q^\gamma(t,x,y)$, Theorem 4.2.1 and Corollary 4.2.5, we obtain the following lemma.

Lemma 5.3.1. *(i) There exists a constant* $c_{\gamma,\alpha} > 0$ *such that* $\kappa^\gamma(x) = c_{\gamma,\alpha}x_d^{-\alpha}$ *for every* $x \in \mathbb{R}^d_+$.
(ii) It holds that

$$J^\gamma(x,y) \simeq \left(1 \wedge \frac{x_d}{|x-y|}\right)^{\gamma+1/2}\left(1 \wedge \frac{y_d}{|x-y|}\right)^{\gamma+1/2}\frac{1}{|x-y|^{d+\alpha}} \quad \textit{for } x,y \in \mathbb{R}^d_+.$$

(iii) There exists a constant $C > 0$ *such that*

$$p^\gamma(t,x,y) \le Ct^{-d/\alpha}, \quad t>0,\ x,y \in \mathbb{R}^d_+.$$

Denote by $(\mathcal{C}^\gamma, \mathcal{D}(\mathcal{C}^\gamma))$ the regular Dirichlet form associated with the subordinate process X^γ. Then since X^γ has no diffusion part, we see from Lemma 5.3.1(i) that

$$\mathcal{C}^\gamma(u,u) = \int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+}(u(x)-u(y))^2 J^\gamma(x,y)dydx + c_{\gamma,\alpha}\int_{\mathbb{R}^d_+} u(x)^2 x_d^{-\alpha}dx.$$

Also, we have $C_c^\infty(\mathbb{R}^d_+) \subset \mathcal{D}(\mathcal{C}^\gamma)$ since $\mathcal{D}(\mathcal{Q}^\gamma) \subset \mathcal{D}(\mathcal{C}^\gamma)$. (See [112].)

Lemma 5.3.2. *There exists a constant $C > 0$ such that*

$$\|u\|_{L^2(\mathbb{R}^d_+,dx)}^{2(1+\alpha/d)} \le C\mathcal{C}^\gamma(u,u) \quad \textit{for every} \;\; u \in C_c^\infty(\mathbb{R}^d_+) \;\textit{with}\; \|u\|_{L^1(\mathbb{R}^d_+,dx)} \le 1.$$

Proof. By [27, Theorem 2.1] (see also [34, Theorem 3.4] and [62, Theorem II.5]), the result follows from Lemma 5.3.1(iii). □

Proposition 5.3.3. *There exists a constant $C > 0$ such that*

$$\|u\|_{L^2(\mathbb{R}^d_+,dx)}^{2(1+\alpha/d)} \le C\mathcal{E}^0(u,u) \quad \textit{for every} \;\; u \in \bar{\mathcal{F}} \;\textit{with}\; \|u\|_{L^1(\mathbb{R}^d_+,dx)} \le 1. \qquad (5.3.2)$$

Proof. We first assume that $\alpha < 1$. Let $\gamma = \beta_1 \vee \beta_2$. Using Lemmas 5.3.2 and 5.3.1(i)–(ii), the Hardy inequality in [93, Proposition 3.2] and **(A3-I)**, we get that for any $u \in C_c^\infty(\mathbb{R}^d_+)$ be such that $\|u\|_{L^1(\mathbb{R}^d_+,dx)} \le 1$,

$$\begin{aligned}\|u\|_{L^2(\mathbb{R}^d_+,dx)}^{2(1+\alpha/d)} &\le c_1 \int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+} (u(x)-u(y))^2 \frac{\widetilde{B}_{\gamma+1/2,\gamma+1/2,0,0}(x,y)}{|x-y|^{d+\alpha}} dydx \\ &\le c_2 \mathcal{E}^0(u,u),\end{aligned}$$

where $\widetilde{B}_{\gamma+1/2,\gamma+1/2,0,0}$ is defined in (5.1.1). By Lemma 5.2.5, $\bar{\mathcal{F}}$ is the closure of $C_c^\infty(\mathbb{R}^d_+)$ under $\mathcal{E}^0_1$. Therefore, we conclude that (5.3.2) is true when $\alpha < 1$.

Now, we assume that $\alpha \ge 1$. Since (5.3.2) is valid when $\alpha < 1$, we get that for every $u \in C_c^\infty(\overline{\mathbb{R}}^d_+)$ with $\|u\|_{L^1(\mathbb{R}^d_+,dx)} \le 1$,

$$\begin{aligned}\mathcal{E}^0(u,u) &\ge \frac{1}{2}\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+} (u(x)-u(y))^2 \frac{\mathcal{B}(x,y)}{|x-y|^{d+1/2}} dydx \\ &\quad - \frac{1}{2}\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+} (u(x)-u(y))^2 \frac{\mathcal{B}(x,y)}{|x-y|^{d+1/2}} \mathbf{1}_{\{|x-y|>1\}} dydx \\ &\ge c_3 \|u\|_{L^2(\mathbb{R}^d_+,dx)}^{2(1+1/(2d))} - c_4 \|u\|_{L^2(\mathbb{R}^d_+,dx)}.\end{aligned}$$

Then since $\bar{\mathcal{F}}$ is the closure of $C_c^\infty(\overline{\mathbb{R}}^d_+)$ under $\mathcal{E}^0_1$, we get from [27, Theorem 2.1] that $\|\bar{P}_1\|_{1\to\infty} \le c_5$ for all $t > 0$. By Lemma 5.2.3, it follows that

$\|\bar{P}_t\|_{1\to\infty} = t^{-d/\alpha}\|\bar{P}_1\|_{1\to\infty} \le c_5 t^{-d/\alpha}$ for $t > 0$. Using [27, Theorem 2.1] again, we conclude that (5.3.2) holds for $\alpha \ge 1$. □

As a consequence of the Nash-type inequality (5.3.2), by following the arguments given in [34, Example 5.5] (see also [45]), we get the existence and a priori upper bounds of the heat kernels $\bar{p}(t,x,y)$ and $p^\kappa(t,x,y)$ of $\bar{Y}$ and Y^κ respectively.

Proposition 5.3.4. *The processes $\bar{Y}$ and Y^κ have heat kernels $\bar{p}(t,x,y)$ and $p^\kappa(t,x,y)$ defined on $(0,\infty) \times (\overline{\mathbb{R}}^d_+ \setminus \mathcal{N}) \times (\overline{\mathbb{R}}^d_+ \setminus \mathcal{N})$ and $(0,\infty) \times (\mathbb{R}^d_+ \setminus \mathcal{N}) \times (\mathbb{R}^d_+ \setminus \mathcal{N})$ respectively, where $\mathcal{N} \subset \overline{\mathbb{R}}^d_+$ is a properly exceptional set for for both $\bar{Y}$ and Y^κ. Moreover, there exists a constant $C > 0$ such that*

$$p^\kappa(t,x,y) \le \bar{p}(t,x,y), \quad x,y \in \mathbb{R}^d_+ \setminus \mathcal{N}$$

and

$$\bar{p}(t,x,y) \le C\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right), \quad t > 0,\ x,y \in \overline{\mathbb{R}}^d_+ \setminus \mathcal{N}.$$

By using the regularization argument given in [92, Subsection 3.1] and Theorem 3.1.13, we obtain the following interior lower bounds.

Proposition 5.3.5. *For every $a \in (0,1]$, there exists a constant $C = C(a) > 0$ such that*

$$\bar{p}(t,x,y) \ge p^\kappa(t,x,y) \ge C\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) \tag{5.3.3}$$

for all $t > 0$, $x \in \mathbb{R}^d_+ \setminus \mathcal{N}$ and a.e. $y \in \mathbb{R}^d_+ \setminus \mathcal{N}$ with $x_d \wedge y_d > a(t^{1/\alpha} \vee |x-y|)$.

5.4 Parabolic Hölder regularity and consequences

For $\kappa \ge 0$ and an open set $D \subset \mathbb{R}^d_+$, we denote by $Y^{\kappa,D}$ and $P^{\kappa,D}_t$ the part of the process Y^κ killed upon exiting D and its semigroup. Let $\bar\tau_D := \inf\{t > 0 : \bar Y_t \notin D\}$, $\tau^\kappa_D := \inf\{t > 0 : Y^\kappa_t \notin D\}$ and

$$\begin{aligned} \bar p^D(t,x,y) &:= \bar p(t,x,y) - \mathbb{E}^x\big[\bar p(t-\bar\tau_D, \bar Y_{\tau_D}, y);\ \bar\tau_D < t\big], \\ p^{\kappa,D}(t,x,y) &:= p^\kappa(t,x,y) - \mathbb{E}^x\big[p^\kappa(t-\tau^\kappa_D, Y^\kappa_{\tau_D}, y);\ \tau^\kappa_D < t\big]. \end{aligned} \tag{5.4.1}$$

By the strong Markov property, $\bar p^D(t,x,y)$ and $p^{\kappa,D}(t,x,y)$ are the transition densities of $\bar Y^D$ and $Y^{\kappa,D}$ respectively.

By standard arguments, we obtain the following two results from (5.4.1), and Propositions 5.3.4 and 5.3.5. (Cf. Lemas 4.4.1 and 4.4.3.)

Lemma 5.4.1. *There exist constants $C > 0$ and $\eta \in (0, 1/4)$ such that for all $x \in \mathbb{R}^d_+ \setminus \mathcal{N}$, $r \in (0, x_d)$, $t \in (0, (\eta r)^\alpha]$ and $z \in B(x, \eta t^{1/\alpha}) \setminus \mathcal{N}$,*

$$\bar p^{B(x,r)}(t,z,y) \ge p^{\kappa,B(x,r)}(t,z,y) \ge Ct^{-d/\alpha} \quad \text{for a.e. } y \in B(x, \eta t^{1/\alpha}) \setminus \mathcal{N}.$$

Lemma 5.4.2. *There exists a constant $C > 1$ such that for all $x \in \mathbb{R}^d_+ \setminus \mathcal{N}$ and $r \in (0, x_d)$,*

$$C^{-1} r^\alpha \le \mathbb{E}^x[\tau^\kappa_{B(x,r)}] \le \sup_{z \in B(x,r)\setminus\mathcal{N}} \mathbb{E}^z[\bar\tau_{B(x,r)}] \le Cr^\alpha.$$

Let $\bar X := (V_s, \bar Y_s)_{s\ge 0}$ and $X^\kappa := (V_s, Y^\kappa_s)_{s \ge 0}$ be time-space processes where $V_s = V_0 - s$. The law of the time-space process $s \mapsto \bar X_s$ or $s \mapsto X^\kappa_s$ starting from (t,x) will be denoted by $\mathbb{P}^{(t,x)}$. For every open subset U of $[0,\infty)\mathbb{R}^d$, define $\bar\tau_U = \inf\{s > 0 :\ \bar X_s \notin U\}$ and $\tau^\kappa_U = \inf\{s > 0 :\ X^\kappa_s \notin U\}$.

Recall that a Borel measurable function $u : [0,\infty) \times \overline{\mathbb{R}}^d_+ \to \mathbb{R}$ is said to be parabolic in $(a,b) \times B(x,r) \subset (0,\infty) \times \mathbb{R}^d_+$ with respect to $\bar Y$ (or Y^κ) if for every relatively compact open set $U \subset (a,b) \times B(x,r)$ it holds that

$u(t,z) = \mathbb{E}^{(t,z)} u(\bar{X}_{\bar{\tau}_U})$ (or $= \mathbb{E}^{(t,z)} u(X^\kappa_{\tau^\kappa_U})$) for all $(t,z) \in U$ with $z \notin \mathcal{N}$.

We denote by $|A|$ the Lebesgue measure on $\mathbb{R}^{d+1}_+$. By repeating the proof of Lemma 4.4.5, we obtain the following lemma from Lemma 5.4.1.

Lemma 5.4.3. *Let $\eta \in (0,1/4)$ be the constant from Lemma 5.4.1. For every $\delta \in (0,\eta]$, there exists a constant $C_1 > 0$ such that for all $x \in \mathbb{R}^d_+ \setminus \mathcal{N}$, $r \in (0, x_d)$, $t \geq \delta r^\alpha$, and any compact set $A \subset [t - \delta r^\alpha, t - \delta r^\alpha/2] \times B(x, (\eta\delta/2)^{1/\alpha} r)$,*

$$\mathbb{P}^{(t,x)}(\sigma^\kappa_A < \tau^\kappa_{[t-\delta r^\alpha, t] \times B(x,r)}) \geq C_1 \frac{|A|}{r^{d+\alpha}}.$$

With help from Lemma 5.4.3, by repeating the proof of Theorem 4.4.6, one can obtain the following parabolic Hölder regularity from Lemmas 5.4.1 and 5.4.2.

Theorem 5.4.4. *For any $\delta \in (0,1)$, there exists $a \in (0,1]$ and $C > 0$ such that for every $x \in \mathbb{R}^d_+ \setminus \mathcal{N}$, $r \in (0, x_d)$, $t_0 \geq 0$, and any function u on $(0,\infty) \times \mathbb{R}^d_+$ which is parabolic in $(t_0, t_0 + r^\alpha) \times B(x,r)$ with respect to $\bar{Y}$ or Y^κ, and bounded in $(t_0, t_0 + r^\alpha) \times \overline{\mathbb{R}}^d_+$, we have*

$$|u(s,y) - u(t,z)| \leq C \left(\frac{|s-t|^{1/\alpha} + |y-z|}{r} \right)^a \operatorname*{ess\,sup}_{[t_0, t_0 + r^\alpha] \times \mathbb{R}^d_+} |u|,$$

for every $s,t \in (t_0 + (1-\delta^\alpha) r^\alpha, t_0 + r^\alpha)$ and $y,z \in B(x, \delta r) \setminus \mathcal{N}$.

By Theorem 4.4.6, since heat kernels $\bar{p}(t,x,y)$ and $p^\kappa(t,x,y)$ are parabolic with respect to $\bar{Y}$ and Y^κ respectively, they can be chosen to be joint continuous in $(0,\infty) \times \mathbb{R}^d_+ \times \mathbb{R}^d_+$ by a standard argument (see, e.g. [72, Lemma 5.13]). In the remainder of this paper, we always choose the joint continuous versions of $\bar{p}(t,x,y)$ and $p^\kappa(t,x,y)$. Then we can assume that the exceptional set $\mathcal{N}$ in Proposition 5.3.4 is a subset of $\overline{\mathbb{R}}^d_+ \setminus \mathbb{R}^d_+$ and the lower bound (5.3.3) holds for all $t > 0$, $x,y \in \mathbb{R}^d_+$ with $x_d \wedge y_d > a(t^{1/\alpha} \vee |x-y|)$. Moreover, by Lemmas 5.2.3 and 5.2.4, we get that

$$\bar{p}(t,x,y) = r^{-d} \bar{p}(t/r^\alpha, x/r, y/r) = \bar{p}(t, x + (\widetilde{z}, 0), y + (\widetilde{z}, 0)),$$

$$p^\kappa(t,x,y) = r^{-d} p^\kappa(t/r^\alpha, x/r, y/r) = p^\kappa(t, x+(\widetilde{z},0), y+(\widetilde{z},0)), \qquad (5.4.2)$$

for any $(t,x,y) \in (0,\infty) \times \mathbb{R}^d_+ \times \mathbb{R}^d_+$, $r>0$ and $\widetilde{z} \in \mathbb{R}^{d-1}$.

Corollary 5.4.5. *Y^κ is a strong Feller process in $\mathbb{R}^d_+$.*

Let

$$G^\kappa(x,y) = \int_0^\infty p^\kappa(t,x,y)dt \quad \text{and} \quad \bar{G}(x,y) = \int_0^\infty \bar{p}(t,x,y)dt$$

be Green functions of Y^κ and $\bar{Y}$, respectively.

From the upper bound in Proposition 5.3.4, we obtain

Corollary 5.4.6. *If $d>\alpha$, then there exists $C>0$ such that*

$$G^\kappa(x,y) \le \bar{G}(x,y) \le \frac{C}{|x-y|^{d-\alpha}}, \quad x,y \in \mathbb{R}^d_+.$$

Remark 5.4.7. The assumption $d > (\alpha+\beta_1+\beta_2) \wedge 2$ in [91, 93] is only used to show $G^\kappa(x,y) \le c|x-y|^{-d+\alpha}$. Thus, by Corollary 5.4.6, all results in [91, 93] with the assumption $d > (\alpha+\beta_1+\beta_2) \wedge 2$ hold under the weaker assumption $d>\alpha$.

5.5 Parabolic Harnack inequality and preliminary lower bounds of heat kernels

In this section we prove that the parabolic Harnack inequality holds for $(Y^\kappa, \mathbb{P}^x)$ and get some preliminary lower bounds of heat kernels of $(Y^\kappa, \mathbb{P}^x)$. All first hotting times and first exit times are with respect to $(Y^\kappa, \mathbb{P}^x)$, and we will omit the superscript κ from the notation for these stopping times.

Using Lemma 5.4.1, by following the proof of Lemma 4.4.7, we get the next lemma.

Lemma 5.5.1. *Let $\eta \in (0,1/4)$ be the constant from Lemma 5.4.1 and let $\delta \in (0,\eta/4)$ be such that $4\delta(2r)^\alpha \le \epsilon r^\alpha$ for all $r>0$. Then there exists a constant*

$C>0$ such that for all $y\in\mathbb{R}^d_+$, $R\in(0,y_d)$, $r\in(0,(\eta\delta/2)^{1/\alpha}R/2]$, $\delta R^\alpha/2\le t-s\le 4\delta(2R)^\alpha$, $x\in B(y,(\eta\delta/2)^{1/\alpha}R/2)$, and $z\in B(x_0,(\eta\delta/2)^{1/\alpha}R)$,

$$\mathbb{P}^{(t,z)}(\sigma_{\{s\}\times B(x,r)}\le\tau_{[s,t]\times B(y,R)})\ge C(r/R)^d.$$

In the remainder of this section, we assume that $\mathcal{B}(x,y)$ satisfies the following:

(B) There exists a constant $C>0$ such that

$$\mathcal{B}(x,y)\le C\mathcal{B}(z,y)\quad\text{for all } x,y,z\in\mathbb{R}^d_+ \text{ satisfying } |x-z|\le(|x-y|\wedge x_d)/2.$$

Since we always assume **(A3-I)**, if **(A3-II)** also holds true, then one can easily check that condition **(B)** is satisfied.

Clearly, **(B)** implies that there exists a constant $c>0$ such that

$$J(x,y)\le cJ(z,y)\quad\text{for all } x,y,z\in\mathbb{R}^d_+ \text{ satisfying } |x-z|\le(|x-y|\wedge x_d)/2. \tag{5.5.1}$$

Moreover, **(B)** also implies the following **UJS** type condition: there exists a constant $c>0$ such that

$$J(x,y)\le\frac{c}{r^d}\int_{B(x,r)}J(z,y)dz\quad\text{for all } x,y\in\mathbb{R}^d_+ \text{ and } 0<r\le(|x-y|\wedge x_d)/2. \tag{5.5.2}$$

Now, using Proposition 5.3.4, (5.5.1)–(5.5.2), Lemmas 5.4.1, 5.4.2, 5.4.3, 5.5.1, we can follow the arguments in the proofs of [32, Theorem 5.2 and Lemma 5.3] (see also the proof of [47, Lemma 4.1 and Theorem 4.3]), and obtain the following (interior) parabolic Harnack inequality. (Cf. Theorem 4.4.10.)

Theorem 5.5.2. *There exist constants $\delta>0$ and $C,M\ge1$ such that for all $t_0\ge0$, $x\in\mathbb{R}^d_+$ and $R\in(0,x_d)$, and any nonnegative function u on*

$(0,\infty)\times\mathbb{R}^d_+$ *which is parabolic on* $Q:=(t_0,t_0+4\delta R^\alpha)\times B(x,R)$, *we have*

$$\sup_{(t_1,y_1)\in Q_-} u(t_1,y_1)\le C\inf_{(t_2,y_2)\in Q_+} u(t_2,y_2),$$

where $Q_-=[t_0+\delta R^\alpha,t_0+2\delta R^\alpha]\times B(x,R/M)$ *and* $Q_+=[t_0+3\delta R^\alpha,t_0+4\delta R^\alpha]\times B(x,R/M)$.

Using Lemma 5.4.1 and Theorem 4.4.10, we obtain

Lemma 5.5.3. *For any positive constants* a,b, *there exists* $c=c(a,b,\kappa)>0$ *such that for all* $z\in\mathbb{R}^d_+$ *and* $r>0$ *with* $B(z,2br)\subset\mathbb{R}^d_+$,

$$\inf_{y\in B(z,br/2)}\mathbb{P}^y\left(\tau_{B(z,br)}>ar^\alpha\right)\ \ge\ c.$$

Now, we can follow the proof of [31, Proposition 3.5] to obtain the following preliminary lower bound.

Proposition 5.5.4. *For every* $a>0$, *there exists a constant* $c=c(a,\kappa)>0$ *such that*

$$p^\kappa(t,x,y)\ge ctJ(x,y)$$

for every $(t,x,y)\in(0,\infty)\times\mathbb{R}^d_+\times\mathbb{R}^d_+$ *with* $x_d\wedge y_d\ge at^{1/\alpha}$ *and* $at^{1/\alpha}\le 4|x-y|$.

5.6 Sharp heat kernel estimates with explicit boundary decays

In this section, we additionally assume that **(A2)** and **(A3-II)** hold. Note that, since **(B)** holds true under **(A3-I)** and **(A3-II)**, all results in Section 5.5 are valid under the current setting.

For $q\in(-1,\alpha+\beta_1)$, we define a constant $C(\alpha,q,\mathcal{B})$ by

$$C(\alpha,q,\mathcal{B}):=\int_{\mathbb{R}^{d-1}}\frac{1}{(|\widetilde{u}|^2+1)^{(d+\alpha)/2}}\int_0^1\frac{(s^q-1)(1-s^{\alpha-q-1})}{(1-s)^{1+\alpha}}\mathcal{B}\big((1-s)\widetilde{u},1),s\mathbf{e}_d\big)ds\,d\widetilde{u}.$$

In case $d = 1$, $C(\alpha, q, \mathcal{B})$ is defined by

$$C(\alpha, q, \mathcal{B}) = \int_0^1 \frac{(s^q - 1)(1 - s^{\alpha - q - 1})}{(1 - s)^{1+\alpha}} \mathcal{B}\big(1, s\big) ds.$$

According to [92, Lemma 5.4 and Remark 5.5], the above constant $C(\alpha, q, \mathcal{B}) \in (-\infty, \infty)$ is well-defined for every $q \in (-1, \alpha + \beta_1)$, $C(\alpha, q, \mathcal{B}) = 0$ if and only if $q \in \{0, \alpha - 1\}$, and $\lim_{q \to -1} C(\alpha, q, \mathcal{B}) = \lim_{q \to \alpha + \beta_1} C(\alpha, q, \mathcal{B}) = \infty$. Note that for every $s \in (0, 1)$, $q \mapsto (s^q - 1)(1 - s^{\alpha - q - 1})$ is strictly decreasing on $(-1, (\alpha - 1)/2)$ and strictly increasing on $((\alpha - 1)/2, \alpha + \beta_1)$. Thus, the shape of the map $q \mapsto C(\alpha, q, \mathcal{B})$ is given as follows.

q	-1	$\cdots$	$(\alpha - 1) \wedge 0$	$\cdots$	$\frac{1}{2}(\alpha - 1)$	$\cdots$	$(\alpha - 1)_+$	$\cdots$	$\alpha + \beta_1$
$C(\alpha, q, \mathcal{B})$	∞	$\searrow$	0	$\searrow$	minimum ≤ 0	$\nearrow$	0	$\nearrow$	∞

Consequently, for every $\kappa \ge 0$, there exists a unique $p_\kappa \in [(\alpha - 1)_+, \alpha + \beta_1)$ such that

$$\kappa = C(\alpha, p_\kappa, \mathcal{B}), \tag{5.6.1}$$

In the remainder of this paper, unless explicitly mentioned otherwise, we fix $\kappa \in [0, \infty)$, assume $\alpha > 1$ if $\kappa = 0$, and omit the superscript κ from the notation, i.e., write $Y^{\kappa, D}$, P_t^κ, $P_t^{\kappa, D}$, τ_D^κ, $p^\kappa(t, x, y)$, $p^{\kappa, D}(t, x, y)$ and ζ^κ as Y^D, P_t, P_t^D, τ_D, $p(t, x, y)$, $p^D(t, x, y)$ and ζ respectively. Also, we denote by p the constant p_κ in (5.6.1).

The goal of this section is to prove the following theorem. We will prove the upper bound in Proposition 5.6.16 and the lower bound in Proposition 5.6.18 below. (Cf. Corollary 4.2.4 and Lemma 4.5.1.)

Theorem 5.6.1. *Suppose that $\beta_2 < \alpha + \beta_1$. Then it holds that for all $t > 0$ and $x, y \in \mathbb{R}^d_+$,*

$$p(t, x, y) \simeq \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^p \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^p A_{\beta_1, \widehat{\beta}_2, \beta_3, \beta_4}(t, x, y) \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}}\right).$$

5.6.1 Preliminary upper bounds of heat kernels

Lemma 5.6.2. *There exists $C > 0$ such that*

$$p(t,x,y) \le C\,\mathbb{P}^x(\zeta > t/3)\mathbb{P}^y(\zeta > t/3), \quad t \in [1,\infty),\ x,y \in \mathbb{R}^d_+.$$

Proof. By the semigroup property, the symmetry of $p(t,\cdot,\cdot)$ and Proposition 5.3.4, we obtain

$$\begin{aligned}p(t,x,y) &= \int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+} p(t/3,x,z)p(t/3,z,w)p(t/3,y,w)dzdw \\ &\le c\int_{\mathbb{R}^d_+} p(t/3,x,z)dz \int_{\mathbb{R}^d_+} p(t/3,y,w)dw = c\mathbb{P}^x(\zeta > t/3)\mathbb{P}^y(\zeta > t/3).\end{aligned}$$

□

Lemma 5.6.3. *Let V_1 and V_3 be open subsets of $\mathbb{R}^d_+$ with $\mathrm{dist}(V_1,V_3) > 0$. Set $V_2 := \mathbb{R}^d_+ \setminus (V_1 \cup V_3)$. For any $x \in V_1$, $y \in V_3$ and $t > 0$, it holds that*

$$\begin{aligned}p(t,x,y) &\le \mathbb{P}^x(\tau_{V_1} < t < \zeta) \sup_{s\le t,\, z\in V_2} p(s,z,y) \\ &\quad + \mathrm{dist}(V_1,V_3)^{-d-\alpha}\int_0^t\int_{V_3}\int_{V_1} p^{V_1}(t-s,x,u)\mathcal{B}(u,w)p(s,y,w)dudwds.\end{aligned}$$

Proof. Let $x \in V_1$ and $y \in V_3$. By the strong Markov property, the Lévy system and symmetry, we get

$$\begin{aligned}&p(t,x,y) = \mathbb{E}^x\left[p(t-\tau_{V_1}, Y_{\tau_{V_1}}, y) : \tau_{V_1} < t < \zeta\right] \\ &= \mathbb{E}^x\left[p(t-\tau_{V_1}, Y_{\tau_{V_1}}, y) : \tau_{V_1} < t < \zeta, Y_{\tau_{V_1}} \in V_2\right] \\ &\quad + \mathbb{E}^x\left[p(t-\tau_{V_1}, Y_{\tau_{V_1}}, y) : \tau_{V_1} < t < \zeta, Y_{\tau_{V_1}} \in V_3\right] \\ &\le \mathbb{P}^x(\tau_{V_1} < t < \zeta) \sup_{s\le t,\, z\in V_2} p(s,z,y) \\ &\quad + \int_0^t\int_{V_3}\int_{V_1} p^{V_1}(t-s,x,u)\frac{\mathcal{B}(u,w)}{|u-w|^{d+\alpha}}p(s,w,y)dudwds \\ &\le \mathbb{P}^x(\tau_{V_1} < t < \zeta) \sup_{s\le t,\, z\in V_2} p(s,z,y)\end{aligned}$$

$$+\,\mathrm{dist}(V_1,V_3)^{-d-\alpha}\int_0^t\int_{V_3}\int_{V_1} p^{V_1}(t-s,x,u)\mathcal{B}(u,w)p(s,y,w)dudwds.$$

□

For any $a,b>0$ and $\widetilde{w}\in\mathbb{R}^{d-1}$, we define

$$D_{\widetilde{w}}(a,b):=\{x=(\widetilde{x},x_d)\in\mathbb{R}^d:\ |\widetilde{x}-\widetilde{w}|<a,\ 0<x_d<b\}$$

and $U(r):=D_{\widetilde{0}}(r/2,r/2)$ for $r>0$. In dimension 1, we abuse notation and use $D_{\widetilde{w}}(a,b)=(0,b)$ and $U(r):=(0,r/2)$.

Lemma 5.6.4. *There exists $C>0$ such that for all $r>0$ and $x\in U(2^{-4}r)$,*

$$\mathbb{P}^x(Y_{\tau_{U(r)}}\in\mathbb{R}^d_+)\le C\left(1\wedge\frac{x_d}{r}\right)^p.$$

Proof. The result follows from [91, Lemma 3.4] if $\kappa>0$ and [93, Theorem 1.1] if $\kappa=0$. □

Lemma 5.6.5. *If $p<\alpha$, then there exists $C>0$ such that for all $r>0$ and $x\in U(2^{-4}r)$,*

$$\mathbb{E}^x[\tau_{U(r)}]\le C\left(\frac{x_d}{r}\right)^p.$$

Proof. The result follows from scaling (Lemma 5.2.3), and [92, Lemma 5.13] if $\kappa>0$ and [93, Lemma 4.5] if $\kappa=0$. □

For $t>0$ and open set $D\subset\mathbb{R}^d_+$, denote by Y^d_t and $Y^{D,d}_t$ the last coordinates of Y_t and Y^D_t respectively.

Lemma 5.6.6. *For all $\gamma,t>0$ and $x\in U(1)$, it holds that*

$$\int_{\mathbb{R}^d_+} p(t,x,z)(1\wedge z_d)^\gamma dz\le\mathbb{E}^x\left[(1\wedge Y^{U(1),d}_t)^\gamma:\tau_{U(1)}>t\right]+\mathbb{P}^x(Y_{\tau_{U(1)}}\in\mathbb{R}^d_+). \tag{5.6.2}$$

In particular, it holds that

$$\mathbb{P}^x(\zeta>t)\le t^{-1}\mathbb{E}^x[\tau_{U(1)}]+\mathbb{P}^x(Y_{\tau_{U(1)}}\in\mathbb{R}^d_+). \tag{5.6.3}$$

Proof. Since $Y_t^{U(1)} = Y_t$ for $t < \tau_{U(1)}$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d_+} p(t,x,z)(1\wedge z_d)^\gamma dz = \mathbb{E}^x\left[(1\wedge Y_t^d)^\gamma : t < \zeta\right] \\
&= \mathbb{E}^x\left[(1\wedge Y_t^d)^\gamma : \tau_{U(1)} > t\right] + \mathbb{E}^x\left[(1\wedge Y_t^d)^\gamma : \tau_{U(1)} \le t < \zeta\right] \\
&\le \mathbb{E}^x\left[(1\wedge Y_t^{U(1),d})^\gamma : \tau_{U(1)} > t\right] + \mathbb{E}^x\left[1 : \tau_{U(1)} < \zeta\right] \\
&= \mathbb{E}^x\left[(1\wedge Y_t^{U(1),d})^\gamma : \tau_{U(1)} > t\right] + \mathbb{P}^x(Y_{\tau_{U(1)}} \in \mathbb{R}^d_+).
\end{aligned}$$

By taking $\gamma = 0$ in (5.6.2) and using Markov's inequality, we get

$$\mathbb{P}^x(\zeta > t) \le \mathbb{P}^x(\tau_{U(1)} > t) + \mathbb{P}^x(Y_{\tau_{U(1)}} \in \mathbb{R}^d_+) \le t^{-1}\mathbb{E}^x[\tau_{U(1)}] + \mathbb{P}^x(Y_{\tau_{U(1)}} \in \mathbb{R}^d_+).$$

□

Using (5.4.2), (5.6.3), and Lemmas 5.6.2, 5.6.4 and 5.6.5, we get the following near diagonal upper estimates when $p < \alpha$.

Lemma 5.6.7. *If $p < \alpha$, then there exists a constant $C > 0$ such that*

$$p(t,x,y) \le C\left(1\wedge \frac{x_d}{t^{1/\alpha}}\right)^p \left(1\wedge \frac{y_d}{t^{1/\alpha}}\right)^p t^{-d/\alpha}, \quad t > 0,\ x, y \in \mathbb{R}^d_+.$$

Lemma 5.6.8. *If $p < \alpha$, then there exists a constant $C > 0$ such that*

$$p(t,x,y) \le C\left(1\wedge \frac{x_d}{t^{1/\alpha}}\right)^p \left(1\wedge \frac{y_d}{t^{1/\alpha}}\right)^p \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right), \quad t > 0,\ x, y \in \mathbb{R}^d_+.$$

Proof. We claim that there exists a constant $c_1 > 0$ such that

$$p(t,x,y) \le c_1\left(1\wedge \frac{x_d}{t^{1/\alpha}}\right)^p \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right). \tag{5.6.4}$$

By (5.4.2) we can assume $\widetilde{x} = 0$ and $t = t_0 = (1/2)^\alpha$. If $x_d \ge 2^{-4}t_0^{1/\alpha}$ or $|x-y| \le 4t_0^{1/\alpha}$, then (5.6.4) follows from Proposition 5.3.4 or Lemma 5.6.7.

Hence, we assume $x_d < 2^{-4}t_0^{1/\alpha}$ and $|x-y| > 4t_0^{1/\alpha}$, and will show that

$$p(t_0,x,y) \le c_1(t_0)\frac{x_d^p}{|x-y|^{d+\alpha}}. \tag{5.6.5}$$

Let $V_1 = U(t_0^{1/\alpha})$, $V_3 = \{w \in \mathbb{R}^d_+ : |w-y| < |x-y|/2\}$ and $V_2 = \mathbb{R}^d_+ \setminus (V_1 \cup V_3)$. By Lemma 5.6.4, we have $\mathbb{P}^x(\tau_{V_1} < t_0 < \zeta) \le \mathbb{P}^x(Y_{\tau_{V_1}} \in \mathbb{R}^d_+) \le c_2(t_0^{-1/\alpha}x_d)^p$. Also, we get from Proposition 5.3.4 that

$$\sup_{s\le t_0,\, z\in V_2} p(s,z,y) \le c_2 \sup_{s\le t_0,\, z\in\mathbb{R}^d_+, |z-y|>|x-y|/2} \frac{s}{|z-y|^{d+\alpha}} = 2^{d+\alpha}c_2\frac{t_0}{|x-y|^{d+\alpha}}.$$

Next, we note that by the triangle inequality, for any $u \in V_1$ and $w \in V_3$,

$$|u-w| \ge |x-y| - |x-u| - |y-w| \ge |x-y| - t_0^{1/\alpha} - \frac{|x-y|}{2} \ge \frac{|x-y|}{4} \ge t_0^{1/\alpha}. \tag{5.6.6}$$

In particular, we see that $\big(1+\mathbf{1}_{|w|\ge1}(\log|w|)^{\beta_3}\big)|u-w|^{-\beta_1} \le c$ for $u \in V_1$ and $w \in V_3$, so by [92, Lemma 5.2(a)], we have that for any $u \in V_1$ and $w \in V_3$,

$$\mathcal{B}(u,w) \le cu_d^{\beta_1}(|\log u_d|^{\beta_3}\vee 1)\big(1+\mathbf{1}_{|w|\ge1}(\log|w|)^{\beta_3}\big)|u-w|^{-\beta_1} \le cu_d^{\beta_1}|\log u_d|^{\beta_3}. \tag{5.6.7}$$

Thus, by [92, Lemma 5.7], **(A3-II)**, [93, Lemma 5.3], (5.6.6) and (5.6.7) we get that

$$\begin{aligned}
&\int_0^t \int_{V_3}\int_{V_1} p^{V_1}(t-s,x,u)\mathcal{B}(u,w)p(s,y,w)\,du\,dw\,ds \\
&\le c_3\int_0^t \left(\int_{V_1} p^{V_1}(t-s,x,u)u_d^{\beta_1}|\log u_d|^{\beta_3}du\right)\left(\int_{V_3} p(s,y,w)dw\right)ds \\
&\le c_3\int_0^t \left(\int_{V_1} p^{V_1}(t-s,x,u)u_d^{\beta_1}|\log u_d|^{\beta_3}du\right)ds \\
&\le c_3\int_0^\infty \left(\int_{V_1} p^{V_1}(s,x,u)u_d^{\beta_1}|\log u_d|^{\beta_3}du\right)ds \\
&= c_3\mathbb{E}^x\int^{\tau_{V_1}} (Y_s^d)^{\beta_1}|\log Y_s^d|^{\beta_3}ds \le c_4x_d^p.
\end{aligned}$$

Therefore, we conclude (5.6.5) (and so (5.6.4)) from (5.6.6) and Lemma 5.6.3.

Now, by the semigroup property, symmetry and (5.6.4), since $t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}$ is comparable to the transition density of the isotropic α-stable process in $\mathbb{R}^d$, we get

$$\begin{aligned}
&p(t,x,y) = \int_{\mathbb{R}^d_+} p(t/2,x,z)p(t/2,y,z)dz \\
&\le c_1^2 \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^p \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^p \int_{\mathbb{R}^d_+} \left(t^{-d/\alpha} \wedge \frac{t}{|x-z|^{d+\alpha}}\right) \left(t^{-d/\alpha} \wedge \frac{t}{|y-z|^{d+\alpha}}\right) dz \\
&\le c_5 \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^p \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^p \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right).
\end{aligned}$$

□

5.6.2 Sharp upper bounds of heat kernels

Lemma 5.6.9. *Let $\gamma > p - \alpha$. For any $U(R) \subset D \subset U(2R)$ and any $x = (\widetilde{0}, x_d)$ with $x_d \le R/10$, it holds that*

$$\int_0^\infty \int_D p^D(t,x,z) z_d^\gamma dz dt \le C R^{\gamma+\alpha-p} x_d^p.$$

Proof. When $d > \alpha$, the result follows from [91, Proposition 6.10] if $\kappa > 0$, and from Remark 5.4.7 and [93, Proposition 6.8] if $\kappa = 0$. When $d = 1 \le \alpha$, the result follows from [57, Section 3.2]. □

We now remove the assumption $p < \alpha$ in Lemma 5.6.7.

Lemma 5.6.10. *There exists a constant $C > 0$ such that*

$$p(t,x,y) \le C \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^p \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^p t^{-d/\alpha}, \quad t > 0,\ x,y \in \mathbb{R}^d_+. \tag{5.6.8}$$

Proof. In view of (5.4.2), it suffices to prove (5.6.8) when $t = 1$.

The lemma holds when $p < \alpha$ by Lemma 5.6.7. Now, assume that (5.6.8) holds for $p < k\alpha$ for some $k \in \mathbb{N}$. We now show (5.6.8) also holds for $p \in [k\alpha, (k+1)\alpha)$ and hence (5.6.8) always holds by induction.

Fix $\epsilon > 0$ such that $p-\alpha+\epsilon < k\alpha$. Note that $\epsilon < \alpha$. Thus, $p-\alpha+\epsilon < p$ so that $C(\alpha, p-\alpha+\epsilon, \mathcal{B}) < C(\alpha, p, \mathcal{B})$. Hence, $p(1,x,y) \le p^{C(\alpha,p-\alpha+\epsilon,\mathcal{B})}(1,x,y)$. By (5.4.2) and the induction hypothesis, it holds that for any $s, u \in [0, 1/4]$ and $z, w \in \mathbb{R}^d_+$,

$$\begin{aligned} p(1-s-u, z, w) &= (1-s-u)^{-d/\alpha} p(1, (1-s-u)^{-1/\alpha} z, (1-s-u)^{-1/\alpha} w) \\ &\le 2^{d/\alpha} p^{C(\alpha,p-\alpha+\epsilon,\mathcal{B})}(1, (1-s-u)^{-1/\alpha} z, (1-s-u)^{-1/\alpha} w) \\ &\le c_3 (1 \wedge z_d)^{p-\alpha+\epsilon} (1 \wedge w_d)^{p-\alpha+\epsilon}. \end{aligned}$$

Therefore, by the semigroup property and symmetry, we get

$$\begin{aligned} p(1,x,y) &= 16 \int_0^{\frac{1}{4}} \int_0^{\frac{1}{4}} p(1,x,y) ds du \\ &= 16 \int_0^{\frac{1}{4}} \int_0^{\frac{1}{4}} \int_{\mathbb{R}^d_+} \int_{\mathbb{R}^d_+} p(s,x,z) p(1-s-u, z, w) p(u, y, w) dz dw ds du \\ &\le 16 c_3 \Big(\int_0^{\frac{1}{4}} \int_{\mathbb{R}^d_+} p(s,x,z)(1 \wedge z_d)^{p-\alpha+\epsilon} dz ds \Big) \\ &\quad \times \Big(\int_0^{\frac{1}{4}} \int_{\mathbb{R}^d_+} p(u,y,w)(1 \wedge w_d)^{p-\alpha+\epsilon} dw du \Big). \end{aligned}$$

Thus, to conclude (5.6.8) by induction, it suffices to show that there exists a constant $c_4 > 0$ such that

$$\int_0^{\frac{1}{4}} \int_{\mathbb{R}^d_+} p(s,v,z)(1 \wedge z_d)^{p-\alpha+\epsilon} dz ds \le c_4 (1 \wedge v_d)^p, \quad v \in \mathbb{R}^d_+.$$

By (5.4.2), we can assume $\widetilde{v} = 0$. If $v \notin U(2^{-4})$, then we get

$$(1 \wedge v_d)^p \ge 2^{-5p} \ge 2^{-5p} \int_0^{\frac{1}{4}} \int_{\mathbb{R}^d_+} p(s,v,z)(1 \wedge z_d)^{p-\alpha+\epsilon} dz ds.$$

Otherwise, if $v \in U(2^{-4})$, then by (5.6.2), Fubini's theorem, Lemmas 5.6.4

and 5.6.9

$$\begin{aligned}
&\int_0^{1/4}\int_{\mathbb{R}^d_+} p(s,v,z)(1\wedge z_d)^{p-\alpha+\epsilon}dzds \\
&\le \int_0^{1/4}\mathbb{E}^v\left[(1\wedge Y_s^{U(1),d})^{p-\alpha+\epsilon} : \tau_{U(1)}>s\right]ds + \int_0^{1/4}\mathbb{P}^v(Y_{\tau_{U(1)}}\in\mathbb{R}^d_+)ds \\
&\le \mathbb{E}^v\left[\int_0^{\tau_{U(1)}}(1\wedge Y_s^{U(1),d})^{p-\alpha+\epsilon}ds\right] + \frac{1}{4}\mathbb{P}^v(Y_{\tau_{U(1)}}\in\mathbb{R}^d_+) \le c_5 v_d^p.
\end{aligned}$$

The proof is complete. □

Now using Lemma 5.6.10 instead of Lemma 5.6.7 in the proof of Lemma 5.6.8, we can remove the assumption $p<\alpha$ in Lemma 5.6.8.

Lemma 5.6.11. *There exists a constant $C>0$ such that*

$$p(t,x,y)\le C\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^p\left(1\wedge\frac{y_d}{t^{1/\alpha}}\right)^p\left(t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}}\right),\quad t>0,\ x,y\in\mathbb{R}^d_+.$$

Since $\mathbb{P}^x(\zeta>t)=\int_{\mathbb{R}^d_+}p(t,x,y)dy$ for all $x\in\mathbb{R}^d_+$ and $t>0$, as a consequence fo the above lemma, we get the following result.

Corollary 5.6.12. *There exists a constant $C>0$ such that*

$$\mathbb{P}^x(\zeta>t)\le C\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^p,\quad t>0,\ x\in\mathbb{R}^d_+.$$

Recall the definition of the function $A_{a_1,a_2,a_3,a_4}(t,x,y)$ from (5.2.3).

Lemma 5.6.13. *Let $a_1,a_3,a_4\ge 0$ be constants with $a_1>0$ if $a_3>0$. Suppose that there exists a constant $C>0$ such that for all $t>0$ and $z,y\in\mathbb{R}^d_+$,*

$$p(t,z,y)\le C\left(1\wedge\frac{z_d}{t^{1/\alpha}}\right)^p A_{a_1,0,a_3,a_4}(t,z,y)\left(t^{-d/\alpha}\wedge\frac{t}{|z-y|^{d+\alpha}}\right). \tag{5.6.9}$$

Then there exists a constant $C'>0$ such that for any $t>0$ and $x=(\widetilde{0},x_d)\in\mathbb{R}^d_+$ with $x_d\le 2^{-5}$,

$$\mathbb{P}^x(\tau_{U(1)}<t<\zeta)\le Ct\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^p(x_d\vee t^{1/\alpha})^{a_1}\log^{a_3+a_4}\left(e+\frac{1}{t^{1/\alpha}}\right). \tag{5.6.10}$$

Proof. By Proposition 5.3.4, we have

$$\begin{aligned}&\sup_{s\le t,\, y\in\mathbb{R}^d_+} \mathbb{P}^y(|Y_s-y|\ge 1/4,\, s<\zeta)\\ &\le c_1 \sup_{s\le t,\, y\in\mathbb{R}^d_+} \int_{z\in\mathbb{R}^d_+,\,|z-y|\ge 1/4} \frac{s}{|z-y|^{d+\alpha}}dz \le c_2 t. \qquad (5.6.11)\end{aligned}$$

If $c_2 t \ge 1/2$, then (5.6.10) follows from Corollary 5.6.12.

Let $c_2 t < 1/2$. By the strong Markov property and (5.6.11), it holds that

$$\begin{aligned}&\mathbb{P}^x(\tau_{U(1)} < t < \zeta,\, |Y_t - Y_{\tau_{U(1)}}| \ge 1/4)\\ &= \mathbb{E}^x\left[\mathbb{P}^{Y_{\tau_{U(1)}}}\left(|Y_{t-\tau_{U(1)}} - Y_0| \ge 1/4\right) : \tau_{U(1)} < t < \zeta\right]\\ &\le \mathbb{P}^x(\tau_{U(1)} < t < \zeta) \sup_{s\le t,\, y\in\mathbb{R}^d_+} \mathbb{P}^y(|Y_s-y|\ge 1/4,\, s<\zeta)\\ &\le \frac{1}{2}\mathbb{P}^x(\tau_{U(1)} < t < \zeta). \qquad (5.6.12)\end{aligned}$$

Note that by the triangle inequality, for any $y \in \mathbb{R}^d_+ \setminus U(1)$ and $z \in B(y, 1/4)$, we have $|z-x| \ge |y-x| - |y-z| > 7/32$. Thus by (5.6.12), we have

$$\begin{aligned}&\mathbb{P}^x(\tau_{U(1)} < t < \zeta) \le 2\mathbb{P}^x(\tau_{U(1)} < t < \zeta,\, |Y_t - Y_{\tau_{U(1)}}| < 1/4)\\ &\le 2\mathbb{P}^x(|Y_t - x| > 7/32,\, t<\zeta) = 2\int_{z\in\mathbb{R}^d_+,\,|z-x|>\frac{7}{32}} p(t,x,z)dz\\ &\le c_3 t\left(1\wedge \frac{x_d}{t^{1/\alpha}}\right)^p (x_d \vee t^{1/\alpha})^{a_1} \int_{z\in\mathbb{R}^d_+,\,|z-x|>\frac{7}{32}} \log^{a_3+a_4}\left(e + \frac{|x-z|}{t^{1/\alpha}}\right)\frac{dz}{|z-x|^{d+\alpha+a_1}}\\ &\le c_4 t\left(1\wedge \frac{x_d}{t^{1/\alpha}}\right)^p (x_d \vee t^{1/\alpha})^{a_1} \log^{a_3+a_4}\left(e+\frac{1}{t^{1/\alpha}}\right)\int_{z\in\mathbb{R}^d_+,\,|z-x|>\frac{7}{32}} \frac{dz}{|z-x|^{d+\alpha}}\\ &\le c_5 t\left(1\wedge \frac{x_d}{t^{1/\alpha}}\right)^p (x_d \vee t^{1/\alpha})^{a_1} \log^{a_3+a_4}\left(e+\frac{1}{t^{1/\alpha}}\right),\end{aligned}$$

where in the third line above we used (5.6.9) and Lemma 5.2.2(ii), and in the fourth we used (5.2.2). □

Note that for any $t, k, r > 0$,

$$\left(1 \wedge \frac{r}{t^{1/\alpha}}\right)^p (r \vee t^{1/\alpha})^k = r^k \left(1 \wedge \frac{r}{t^{1/\alpha}}\right)^{p-k}. \tag{5.6.13}$$

Lemma 5.6.14. *There exists a constant $C > 0$ such that*

$$p(t,x,y) \le C \left(1 \wedge \frac{x_d \wedge y_d}{t^{1/\alpha}}\right)^p A_{\beta_1,0,\beta_3,\beta_4}(t,x,y) \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right),$$

for all $t > 0$ and $x, y \in \mathbb{R}^d_+$.

Proof. Since $A_{0,0,a_3,a_4}$ is bounded from below by a positive constant, by Lemma 5.6.11, the lemma hold for $\beta_1 = 0$.

We assume $\beta_1 > 0$ and set $b_n = \beta_1 \wedge \frac{n\alpha}{2}$ for $n \ge 0$. Below, we prove by induction that for any $n \ge 0$, there exists a constant $C > 0$ such that for all $t > 0$ and $x, y \in \mathbb{R}^d_+$,

$$p(t,x,y) \le C \left(1 \wedge \frac{x_d \wedge y_d}{t^{1/\alpha}}\right)^p A_{b_n,0,\beta_3,\beta_4}(t,x,y) \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right). \tag{5.6.14}$$

The lemma is a direct consequence of (5.6.14).

(5.6.14) holds for $n = 0$. Suppose (5.6.14) holds for $n - 1$. By symmetry and (5.4.2), we can assume $x_d \le y_d$, $\widetilde{x} = 0$ and $|x - y| = 4$ without loss of generality. If $t > 1$ or $x_d > 2^{-5}$, then (5.6.14) follows from Lemma 5.6.11 and (5.2.4).

Let $t \le 1$ and $x_d \le 2^{-5}$. Then $y_d \le x_d + |x - y| \le 4 + 2^{-5}$ by the triangle inequality. Our goal is to show that there exists a constant $c_1 > 0$ independent of t, x, y such that

$$\begin{aligned} &p(t,x,y) \\ &\le c_1 t \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^p (x_d \vee t^{1/\alpha})^{b_n} \log^{\beta_3}\left(e + \frac{y_d \vee t^{1/\alpha}}{x_d \vee t^{1/\alpha}}\right) \log^{\beta_4}\left(e + \frac{1}{y_d \vee t^{1/\alpha}}\right). \end{aligned} \tag{5.6.15}$$

Set $V_1 = U(1)$, $V_3 = B(y, 2) \cap \mathbb{R}^d_+$ and $V_2 = \mathbb{R}^d_+ \setminus (V_1 \cup V_3)$. Using

Proposition 5.3.4 and the triangle inequality, we get that

$$\sup_{s\le t,\, z\in V_2} p(s,z,y) \le c_2 \sup_{s\le t,\, z\in \mathbb{R}^d_+, |z-y|\ge 2} \frac{s}{|z-y|^{d+\alpha}} \le 2^{-d-\alpha} c_2 t \tag{5.6.16}$$

and

$$\mathrm{dist}(V_1,V_3) \ge \sup_{u\in V_1,\, w\in V_3} (4-|x-u|-|y-w|) \ge 1. \tag{5.6.17}$$

We consider the following two cases separately.

(Case 1) $p \ge \alpha + b_n$ and $10x_d < t^{1/\alpha}$.

By Lemma 5.6.4, we have $\mathbb{P}^x(\tau_{V_1} < t < \zeta) \le \mathbb{P}^x(\tau_{V_1} < \zeta) \le c_3 x_d^p$. Pick $\epsilon > 0$ such that $0 < \epsilon < \beta_1$ and $p < \alpha + \beta_1 - \epsilon$. By **(A3-II)**, (5.6.17), and Lemmas 5.2.1(i) and 5.6.9,

$$\begin{aligned}
&\int_0^t \int_{V_3} \int_{V_1} p^{V_1}(t-s,x,u)\mathcal{B}(u,w)p(s,y,w)dudwds \\
&\le c_4 \int_0^t \int_{V_3} \int_{V_1} p^{V_1}(t-s,x,u)\widetilde{B}_{(\beta_1-\epsilon)_+,0,0,0}(u,w)p(s,y,w)dudwds \\
&\le c_5 \int_0^t \int_{V_1} p^{V_1}(t-s,x,u)u_d^{\beta_1-\epsilon}du \int_{V_3} p(s,y,w)dwds \\
&\le c_5 \int_0^\infty \int_{V_1} p^{V_1}(s,x,u)u_d^{\beta_1-\epsilon}duds \le c_6 x_d^p.
\end{aligned}$$

Therefore, since $x_d < t^{1/\alpha} \le 1$ and $1 + (b_n - p)/\alpha \le 0$, we get from Lemma 5.6.3 and (5.6.16) that

$$p(t,x,y) \le c_7(t+1)x_d^p \le 2c_7 t^{1-p/\alpha-b_n/\alpha} x_d^p = 2c_7 t (x_d/t^{1/\alpha})^p (x_d \vee t^{1/\alpha})^{b_n}.$$

(Case 2) $p < \alpha + b_n$ or $10x_d \ge t^{1/\alpha}$.

By the induction hypothesis, (5.6.9) holds with $a_1 = b_{n-1}$, $a_3 = \beta_3$ and $a_4 = \beta_4$. Thus, since $b_n - b_{n-1} \le \alpha/2$, we get from Lemma 5.6.13 and (5.6.16) that

$$\mathbb{P}^x(\tau_{V_1} < t < \zeta) \sup_{s\le t,\, z\in V_2} p(s,z,y)$$

$$\leq c_8 t \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^p (x_d \vee t^{1/\alpha})^{b_{n-1}} (t^{1/\alpha})^{\alpha/2} t^{1/2} \log^{\beta_3+\beta_4}\left(e + \frac{1}{t^{1/\alpha}}\right)$$
$$\leq c_9 t \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^p (x_d \vee t^{1/\alpha})^{b_n}. \tag{5.6.18}$$

On the other hand, using **(A3-II)**, (5.6.17) and Lemma 5.2.2(i), we get for $\beta' := (b_n/2) \wedge \beta_2 \wedge p$,

$$\begin{aligned}
&\int_0^t \int_{V_3} \int_{V_1} p^{V_1}(t-s,x,u)\mathcal{B}(u,w)p(s,y,w)dudwds \\
&\leq c_{10} \int_0^t \int_{V_3} p(s,y,w) w_d^{\beta'} \log^{\beta_4}\left(e + \frac{1}{w_d}\right) \\
&\quad \times \int_{V_1} p(t-s,x,u) u_d^{\beta_1} \log^{\beta_3}\left(e + \frac{w_d}{u_d}\right) dudwds.
\end{aligned} \tag{5.6.19}$$

By (5.2.2) and Corollary 5.6.12, since $\beta_1 > 0$ and $b_n \leq \beta_1$, we get that for any $0 < s < t$ and $w \in V_3$,

$$\begin{aligned}
&\int_{u\in V_1: u_d < x_d} p(s,x,u) u_d^{\beta_1} \log^{\beta_3}\left(e + \frac{w_d}{u_d}\right) du \\
&\leq \int_{u\in V_1: u_d < x_d \vee s^{1/\alpha}} p(s,x,u) u_d^{\beta_1} \log^{\beta_3}\left(e + \frac{w_d}{u_d}\right) du \\
&\leq c_{11} (x_d \vee s^{1/\alpha})^{\beta_1} \log^{\beta_3}\left(e + \frac{w_d}{x_d \vee s^{1/\alpha}}\right) \int_{u\in V_1: u_d < x_d \vee s^{1/\alpha}} p(s,x,u) du \\
&\leq c_{12} \left(1 \wedge \frac{x_d}{s^{1/\alpha}}\right)^p (x_d \vee s^{1/\alpha})^{b_n} \log^{\beta_3}\left(e + \frac{w_d}{x_d \vee s^{1/\alpha}}\right).
\end{aligned}$$

Next, using the induction hypothesis and Lemma 5.7.3, since $b_n \leq \beta_1$ and $b_n < \alpha + b_{n-1}$, we get that for any $0 < s < t$ and $w \in V_3$,

$$\begin{aligned}
&\int_{u\in V_1: u_d \geq x_d} p(s,x,u) u_d^{\beta_1} \log^{\beta_3}\left(e + \frac{w_d}{u_d}\right) du \\
&\leq c_{13} \left(1 \wedge \frac{x_d}{s^{1/\alpha}}\right)^p \int_{u\in V_1: u_d \geq x_d} \left(\frac{x_d \vee s^{1/\alpha}}{|x-u|} \wedge 1\right)^{b_{n-1}} \left(s^{-d/\alpha} \wedge \frac{s}{|x-u|^{d+\alpha}}\right) \\
&\qquad \times \log^{\beta_3+\beta_4}\left(1 + \frac{|x-u|}{(x_d \vee s^{1/\alpha}) \wedge |x-u|}\right) u_d^{b_n} \log^{\beta_3}\left(e + \frac{w_d}{u_d}\right) du
\end{aligned}$$

$$\leq c_{14}\left(1 \wedge \frac{x_d}{s^{1/\alpha}}\right)^p (x_d \vee s^{1/\alpha})^{b_n} \log^{\beta_3}\left(e + \frac{w_d}{x_d \vee s^{1/\alpha}}\right).$$

Similarly, again spliting the integration into two parts $w_d \geq y_d$ and $w_d < y_d$, and using the induction hypothesis and Lemma 5.7.3 again, since $\beta' \leq b_n < \alpha + b_{n-1}$, and $\beta' > 0$ if $\beta_4 > 0$, we also get that for any $0 < s < t$,

$$\begin{aligned}
&\int_{V_3} p(s,y,w) w_d^{\beta'} \log^{\beta_4}\left(e + \frac{1}{w_d}\right) \log^{\beta_3}\left(e + \frac{w_d}{x_d \vee (t-s)^{1/\alpha}}\right) dw \\
&\leq c_{15}\left(1 \wedge \frac{y_d}{s^{1/\alpha}}\right)^p (y_d \vee s^{1/\alpha})^{\beta'} \\
&\quad \times \log^{\beta_4}\left(e + \frac{1}{y_d \vee s^{1/\alpha}}\right) \log^{\beta_3}\left(e + \frac{y_d \vee s^{1/\alpha}}{x_d \vee (t-s)^{1/\alpha}}\right).
\end{aligned}$$

Therefore, by (5.6.13) and (5.6.19),

$$\begin{aligned}
&\int_0^t \int_{V_3} \int_{V_1} p^{V_1}(t-s,x,u) \mathcal{B}(u,w) p(s,y,w) \, du dw ds \\
&\leq c_{16} x_d^{b_n} y_d^{\beta'} \int_0^t \left(1 \wedge \frac{x_d}{(t-s)^{1/\alpha}}\right)^{p-b_n} \left(1 \wedge \frac{y_d}{s^{1/\alpha}}\right)^{p-\beta'} \\
&\quad \times \log^{\beta_3}\left(e + \frac{y_d \vee s^{1/\alpha}}{x_d \vee (t-s)^{1/\alpha}}\right) \log^{\beta_4}\left(e + \frac{1}{y_d \vee s^{1/\alpha}}\right) ds. \qquad (5.6.20)
\end{aligned}$$

By Lemma 5.7.1, since $\beta' \leq p$, it holds that

$$\begin{aligned}
&\int_0^{t/2} \left(1 \wedge \frac{x_d}{(t-s)^{1/\alpha}}\right)^{p-b_n} \left(1 \wedge \frac{y_d}{s^{1/\alpha}}\right)^{p-\beta'} \\
&\quad \times \log^{\beta_3}\left(e + \frac{y_d \vee s^{1/\alpha}}{x_d \vee (t-s)^{1/\alpha}}\right) \log^{\beta_4}\left(e + \frac{1}{y_d \vee s^{1/\alpha}}\right) ds \\
&\leq c_{17}\left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^{p-b_n} \log^{\beta_3}\left(e + \frac{y_d \vee t^{1/\alpha}}{x_d \vee t^{1/\alpha}}\right) \\
&\quad \times \int_0^{t/2} \left(1 \wedge \frac{y_d}{s^{1/\alpha}}\right)^{(p-\beta') \wedge (\alpha/2)} \log^{\beta_4}\left(e + \frac{1}{y_d \vee s^{1/\alpha}}\right) ds \\
&\leq c_{18} t \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^{p-b_n} \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^{(p-\beta') \wedge (\alpha/2)}
\end{aligned}$$

$$\times \log^{\beta_3}\left(e+\frac{y_d\vee t^{1/\alpha}}{x_d\vee t^{1/\alpha}}\right)\log^{\beta_4}\left(e+\frac{1}{y_d\vee t^{1/\alpha}}\right)$$
$$\le c_{19}t\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^{p-b_n}\log^{\beta_3}\left(e+\frac{y_d\vee t^{1/\alpha}}{x_d\vee t^{1/\alpha}}\right)\log^{\beta_4}\left(e+\frac{1}{y_d\vee t^{1/\alpha}}\right).$$

When $p<\alpha+b_n$, we get from Lemma 5.7.1 that

$$\int_{t/2}^{t}\left(1\wedge\frac{x_d}{(t-s)^{1/\alpha}}\right)^{p-b_n}\left(1\wedge\frac{y_d}{s^{1/\alpha}}\right)^{p-\beta'}$$
$$\times \log^{\beta_3}\left(e+\frac{y_d\vee s^{1/\alpha}}{x_d\vee (t-s)^{1/\alpha}}\right)\log^{\beta_4}\left(e+\frac{1}{y_d\vee s^{1/\alpha}}\right)ds$$
$$\le c_{20}\log^{\beta_4}\left(e+\frac{1}{y_d\vee t^{1/\alpha}}\right)\int_0^{t/2}\left(1\wedge\frac{x_d}{s^{1/\alpha}}\right)^{p-b_n}\log^{\beta_3}\left(e+\frac{y_d\vee t^{1/\alpha}}{x_d\vee s^{1/\alpha}}\right)ds$$
$$\le c_{21}t\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^{p-b_n}\log^{\beta_3}\left(e+\frac{y_d\vee t^{1/\alpha}}{x_d\vee t^{1/\alpha}}\right)\log^{\beta_4}\left(e+\frac{1}{y_d\vee t^{1/\alpha}}\right).$$

When $10x_d\ge t^{1/\alpha}$, we also get from Lemma 5.7.1 that

$$\int_{t/2}^{t}\left(1\wedge\frac{x_d}{(t-s)^{1/\alpha}}\right)^{p-b_n}\left(1\wedge\frac{y_d}{s^{1/\alpha}}\right)^{p-\beta'}$$
$$\times \log^{\beta_3}\left(e+\frac{y_d\vee s^{1/\alpha}}{x_d\vee (t-s)^{1/\alpha}}\right)\log^{\beta_4}\left(e+\frac{1}{y_d\vee s^{1/\alpha}}\right)ds$$
$$\le c_{22}\int_{t/2}^{t}\left(1\wedge\frac{y_d}{s^{1/\alpha}}\right)^{(p-\beta')\wedge\frac{\alpha}{2}}\log^{\beta_3}\left(e+\frac{y_d\vee s^{1/\alpha}}{x_d}\right)\log^{\beta_4}\left(e+\frac{1}{y_d\vee s^{1/\alpha}}\right)ds$$
$$\le c_{23}t\log^{\beta_3}\left(e+\frac{y_d\vee t^{1/\alpha}}{x_d}\right)\log^{\beta_4}\left(e+\frac{1}{y_d\vee t^{1/\alpha}}\right)$$
$$\le c_{24}t\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^{p-b_n}\log^{\beta_3}\left(e+\frac{y_d\vee t^{1/\alpha}}{x_d\vee t^{1/\alpha}}\right)\log^{\beta_4}\left(e+\frac{1}{y_d\vee t^{1/\alpha}}\right). \quad (5.6.21)$$

Now (5.6.15) follows from (5.6.13), (5.6.18), (5.6.20)-(5.6.21) and Lemma 5.6.3. The proof is complete. □

Using Lemmas 5.6.14 and 5.2.2(ii), we get the following result from Lemma 5.7.3 and (5.6.13).

Lemma 5.6.15. *Let $\eta_1,\eta_2\ge 0$ and $0\le\widehat{\beta}<\alpha+\beta_1$. Assume that $\widehat{\beta}>0$*

if $\eta_1 > 0$*. There exists a constant* $C > 0$ *such that for any* $x \in \mathbb{R}^d_+$ *and* $s, k, l > 0$*,*

$$\int_{\mathbb{R}^d_+} p(s,x,z) z_d^{\widehat{\beta}} \log^{\eta_1}\left(e + \frac{k}{z_d}\right) \log^{\eta_2}\left(e + \frac{z_d}{l}\right) dz$$
$$\leq C x_d^{\widehat{\beta}} \left(1 \wedge \frac{x_d}{s^{1/\alpha}}\right)^{p-\widehat{\beta}} \log^{\eta_1}\left(e + \frac{k}{x_d \vee s^{1/\alpha}}\right) \log^{\eta_2}\left(e + \frac{x_d \vee s^{1/\alpha}}{l}\right).$$

Proposition 5.6.16. *Let* $\epsilon \in (0, \alpha/2]$ *and set* $\widehat{\beta}_2 := \beta_2 \wedge (\alpha + \beta_1 - \epsilon)$*. There exists a constant* $C = C(\epsilon) > 0$ *such that*

$$p(t,x,y) \leq C \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^p \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^p A_{\beta_1, \widehat{\beta}_2, \beta_3, \beta_4}(t,x,y) \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right),$$

for all $t > 0$ *and* $x, y \in \mathbb{R}^d_+$*.*

Proof. As in the proof of Lemma 5.6.14, by symmetry, Lemma 5.6.11 and (5.2.4), we can assume $x_d \leq y_d \wedge 2^{-5}$ and $|x-y| = 4$ without loss of generality, and it is enough to show that there exists $c_1 > 0$ such that for any $t \leq 1$,

$$p(t,x,y) \leq c_1 t \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^p \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^p (x_d \vee t^{1/\alpha})^{\beta_1} (y_d \vee t^{1/\alpha})^{\beta_2}$$
$$\times \log^{\beta_3}\left(e + \frac{y_d \vee t^{1/\alpha}}{x_d \vee t^{1/\alpha}}\right) \log^{\beta_4}\left(e + \frac{1}{y_d \vee t^{1/\alpha}}\right). \qquad (5.6.22)$$

Let $t \leq 1$. Set $V_1 = U(1)$, $V_3 = B(y,2) \cap \mathbb{R}^d_+$ and $V_2 = \mathbb{R}^d_+ \setminus (V_1 \cup V_3)$. By Lemma 5.6.14,

$$\sup_{s \leq t,\, z \in V_2} p(s,z,y) \leq c_2 \sup_{s \leq t,\, z \in \mathbb{R}^d_+, |z-y| \geq 2} \left(1 \wedge \frac{z_d \wedge y_d}{s^{1/\alpha}}\right)^p A_{\beta_1, 0, \beta_3, \beta_4}(s,z,y) \frac{s}{|z-y|^{d+\alpha}}. \qquad (5.6.23)$$

Note that $y_d \leq x_d + |x-y| \leq 4 + 2^{-5}$ and $z_d \leq y_d + |z-y|$ for any $z \in \mathbb{R}^d_+$ by the triangle inequality. Thus, by Lemma 5.7.2(i), we have that for any

$0 < s \le t$ and $z \in \mathbb{R}^d_+$ with $|z-y| \ge 2$,

$$\begin{aligned}
&\left(1 \wedge \frac{z_d \wedge y_d}{s^{1/\alpha}}\right)^p A_{\beta_1,0,\beta_3,\beta_4}(s,z,y)\frac{s}{|z-y|^{d+\alpha}} \\
&\le \left(1 \wedge \frac{y_d}{s^{1/\alpha}}\right)^p (y_d \vee s^{1/\alpha})^{\beta_1} \log^{\beta_3+\beta_4}\left(e + \frac{|z-y|}{s^{1/\alpha}}\right)\frac{s}{|z-y|^{d+\alpha}} \\
&\le c_3 \left(1 \wedge \frac{y_d}{s^{1/\alpha}}\right)^p (y_d \vee s^{1/\alpha})^{\beta_1} s \log^{\beta_3+\beta_4}\left(e + \frac{2}{s^{1/\alpha}}\right). \qquad (5.6.24)
\end{aligned}$$

Since

$$\begin{aligned}
&\left(1 \wedge \frac{y_d}{s^{1/\alpha}}\right)^p (y_d \vee s^{1/\alpha})^{\beta_1} s \log^{\beta_3+\beta_4}\left(e + \frac{2}{s^{1/\alpha}}\right) \\
&= \begin{cases} \left(s \log^{\beta_3+\beta_4}\left(e + \frac{2}{s^{1/\alpha}}\right)\right) y_d^p (s^{1/\alpha} \vee y_d)^{\beta_1 - p} & \text{for } \beta_1 > p; \\ \left(s^{(\beta_1+\alpha-p)/\alpha} \log^{\beta_3+\beta_4}\left(e + \frac{2}{s^{1/\alpha}}\right)\right) y_d^{\beta_1} \left(s^{1/\alpha} \wedge y_d\right)^{p-\beta_1} & \text{for } \beta_1 \le p, \end{cases}
\end{aligned}$$

using the fact that $\beta_1 + \alpha - p > 0$, we get from (5.6.23)–(5.6.24) that

$$\sup_{s \le t,\, z \in V_2} p(s,z,y) \le c_4 \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^p (y_d \vee t^{1/\alpha})^{\beta_1} t \log^{\beta_3+\beta_4}\left(e + \frac{2}{t^{1/\alpha}}\right). \quad (5.6.25)$$

Note that (5.6.9) holds with $a_1 = \beta_1$, $a_3 = \beta_3$ and $a_4 = \beta_4$ by Lemma 5.6.14. Thus, by Lemma 5.6.13, (5.6.25) and (5.2.1), since $y_d \vee t^{1/\alpha} < 5$ and $(\widehat{\beta}_2 - \beta_1)_+ < \alpha$, we obtain

$$\begin{aligned}
&\mathbb{P}^x(\tau_{V_1} < t < \zeta) \sup_{s \le t,\, z \in V_2} p(s,z,y) \\
&\le c_5 t \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^p \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^p (x_d \vee t^{1/\alpha})^{\beta_1} (y_d \vee t^{1/\alpha})^{\beta_1} t \log^{2(\beta_3+\beta_4)}\left(e + \frac{2}{t^{1/\alpha}}\right) \\
&\le c_6 t \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^p \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^p (x_d \vee t^{1/\alpha})^{\beta_1} (y_d \vee t^{1/\alpha})^{\widehat{\beta}_2}. \qquad (5.6.26)
\end{aligned}$$

Next, we show that there exists $c_7 > 0$ such that

$$\int_0^t \int_{V_3} \int_{V_1} p^{V_1}(t-s,x,u)\mathcal{B}(u,w)p(s,y,w)\,dudwds$$

$$\le c_7 t x_d^{\beta_1} y_d^{\widehat{\beta}_2} \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^{p-\beta_1} \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^{p-\widehat{\beta}_2}$$
$$\times \log^{\beta_3}\left(e + \frac{y_d \vee t^{1/\alpha}}{x_d \vee t^{1/\alpha}}\right) \log^{\beta_4}\left(e + \frac{1}{y_d \vee t^{1/\alpha}}\right). \qquad (5.6.27)$$

Once we get (5.6.27), by (5.6.13) and (5.6.26), we can apply Lemma 5.6.3 to get (5.6.22) and finish the proof. We consider the cases $\beta_1 > \beta_2$ and $\beta_1 \le \beta_2$ separately.

(Case 1) $\beta_1 > \beta_2$.

Since $|u-w| \simeq 1$ and $w_d \le 4$ for $u \in V_1$ and $w \in V_3$, we have from Lemma 5.2.2(ii) and (5.6.17) that for any $0 \le s_1 < s_2 \le t$,

$$\int_{s_1}^{s_2} \int_{V_3} \int_{V_1} p^{V_1}(t-s,x,u)\mathcal{B}(u,w)p(s,y,w)\,du\,dw\,ds$$
$$\le c_8 \int_{s_1}^{s_2} \int_{V_3} p(s,y,w) w_d^{\beta_2} \log^{\beta_4}\left(e + \frac{1}{w_d}\right)$$
$$\times \left[\int_{V_1} p(t-s,x,u) u_d^{\beta_1} \log^{\beta_3}\left(e + \frac{w_d}{u_d}\right) du\right] dw\,ds.$$

Since $p < \alpha + \beta_1$, using Lemma 5.6.15 twice and Lemma 5.7.1, we get that

$$\int_{t/2}^{t} \int_{V_3} p(s,y,w) w_d^{\beta_2} \log^{\beta_4}\left(e + \frac{1}{w_d}\right)$$
$$\times \left[\int_{V_1} p(t-s,x,u) u_d^{\beta_1} \log^{\beta_3}\left(e + \frac{w_d}{u_d}\right) du\right] dw\,ds$$
$$\le c_9 x_d^{\beta_1} \int_{t/2}^{t} \left(1 \wedge \frac{x_d}{(t-s)^{1/\alpha}}\right)^{p-\beta_1}$$
$$\times \int_{V_3} p(s,y,w) w_d^{\beta_2} \log^{\beta_4}\left(e + \frac{1}{w_d}\right) \log^{\beta_3}\left(e + \frac{w_d}{x_d \vee (t-s)^{1/\alpha}}\right) dw\,ds$$
$$\le c_{10} x_d^{\beta_1} y_d^{\beta_2} \int_{t/2}^{t} \left(1 \wedge \frac{x_d}{(t-s)^{1/\alpha}}\right)^{p-\beta_1} \left(1 \wedge \frac{y_d}{s^{1/\alpha}}\right)^{p-\beta_2}$$
$$\times \log^{\beta_3}\left(e + \frac{y_d \vee s^{1/\alpha}}{x_d \vee (t-s)^{1/\alpha}}\right) \log^{\beta_4}\left(e + \frac{1}{y_d \vee s^{1/\alpha}}\right) ds$$
$$\le c_{11} x_d^{\beta_1} y_d^{\beta_2} \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^{p-\beta_2} \log^{\beta_4}\left(e + \frac{1}{y_d \vee t^{1/\alpha}}\right)$$

$$
\begin{aligned}
&\times \int_0^{t/2} \left(1 \wedge \frac{x_d}{s^{1/\alpha}}\right)^{p-\beta_1} \log^{\beta_3}\left(e + \frac{y_d \vee t^{1/\alpha}}{x_d \vee s^{1/\alpha}}\right) ds \\
&\le c_{12} t x_d^{\beta_1} y_d^{\beta_2} \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^{p-\beta_1} \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^{p-\beta_2} \\
&\quad \times \log^{\beta_3}\left(e + \frac{y_d \vee t^{1/\alpha}}{x_d \vee t^{1/\alpha}}\right) \log^{\beta_4}\left(e + \frac{1}{y_d \vee t^{1/\alpha}}\right).
\end{aligned}
$$

Similarly, when $y_d \ge t^{1/\alpha}$, we obtain from Lemma 5.6.15 that

$$
\begin{aligned}
&\int_0^{t/2} \int_{V_3} p(s,y,w) w_d^{\beta_2} \log^{\beta_4}\left(e + \frac{1}{w_d}\right) \\
&\qquad \times \left[\int_{V_1} p(t-s,x,u) u_d^{\beta_1} \log^{\beta_3}\left(e + \frac{w_d}{u_d}\right) du\right] dw ds \\
&\le c_{13} x_d^{\beta_1} y_d^{\beta_2} \int_0^{t/2} \left(1 \wedge \frac{x_d}{(t-s)^{1/\alpha}}\right)^{p-\beta_1} \left(1 \wedge \frac{y_d}{s^{1/\alpha}}\right)^{p-\beta_2} \\
&\qquad \times \log^{\beta_3}\left(e + \frac{y_d \vee s^{1/\alpha}}{x_d \vee (t-s)^{1/\alpha}}\right) \log^{\beta_4}\left(e + \frac{1}{y_d \vee s^{1/\alpha}}\right) ds \\
&\le c_{14} t x_d^{\beta_1} y_d^{\beta_2} \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^{p-\beta_1} \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^{p-\beta_2} \\
&\quad \times \log^{\beta_3}\left(e + \frac{y_d \vee t^{1/\alpha}}{x_d \vee t^{1/\alpha}}\right) \log^{\beta_4}\left(e + \frac{1}{y_d \vee t^{1/\alpha}}\right).
\end{aligned}
$$

Therefore, it remains to bound

$$
\int_0^{t/2} \int_{V_3} \int_{V_1} p^{V_1}(t-s,x,u) \mathcal{B}(u,w) p(s,y,w) du dw ds
$$

when $x_d \le y_d < t^{1/\alpha}$.

Assume that $x_d \le y_d < t^{1/\alpha}$. Using $\beta_1 > \beta_2$, we have from (5.2.1) and Lemma 5.6.15 that for any $0 < s < t/2$ and $w \in V_3$,

$$
\begin{aligned}
&\int_{u \in V_1 : u_d \le w_d} p(t-s,x,u) u_d^{\beta_1} \log^{\beta_3}\left(e + \frac{w_d}{u_d}\right) du \\
&\le c_{15} w_d^{\beta_1-\beta_2} \int_{u \in V_1 : u_d \le w_d} p(t-s,x,u) u_d^{\beta_2} du
\end{aligned}
$$

$$\leq c_{16} t^{(\beta_2-\beta_1)/\alpha} w_d^{\beta_1-\beta_2} x_d^{\beta_1} \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^{p-\beta_1}.$$

Hence, we get from Lemmas 5.6.15 and 5.7.1 that

$$\begin{aligned}
&\int_0^{t/2}\int_{V_3}\int_{V_1: u_d \leq w_d} p^{V_1}(t-s,x,u)\mathcal{B}(u,w)p(s,y,w)dudwds \\
&\leq c_{17}\int_0^{t/2}\int_{V_3} p(s,y,w) w_d^{\beta_2}\log^{\beta_4}\left(e+\frac{1}{w_d}\right) \\
&\qquad\quad \times \left[\int_{V_1:u_d\leq w_d} p(t-s,x,u)u_d^{\beta_1}\log^{\beta_3}\left(e+\frac{w_d}{u_d}\right)du\right]dwds \\
&\leq c_{18} t^{(\beta_2-\beta_1)/\alpha} x_d^{\beta_1}\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^{p-\beta_1}\int_0^{t/2}\int_{V_3} p(s,y,w)w_d^{\beta_1}\log^{\beta_4}\left(e+\frac{1}{w_d}\right)dwds \\
&\leq c_{19} t^{(\beta_2-\beta_1)/\alpha} x_d^{\beta_1} y_d^{\beta_1}\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^{p-\beta_1}\int_0^{t/2}\left(1\wedge\frac{y_d}{s^{1/\alpha}}\right)^{p-\beta_1}\log^{\beta_4}\left(e+\frac{1}{y_d\vee s^{1/\alpha}}\right)ds \\
&= c_{20} t x_d^{\beta_1} y_d^{\beta_2}\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^{p-\beta_1}\left(1\wedge\frac{y_d}{t^{1/\alpha}}\right)^{p-\beta_2}\log^{\beta_4}\left(e+\frac{1}{y_d\vee t^{1/\alpha}}\right).
\end{aligned}$$

On the other hand, we pick any $\epsilon' \in (0, \beta_1-\beta_2)$ such that $p < \alpha+\beta_1-\epsilon'$ and get from (5.2.1) and Lemmas 5.6.15 and 5.7.1 that

$$\begin{aligned}
&\int_0^{t/2}\int_{V_3}\int_{V_1: u_d \geq w_d} p^{V_1}(t-s,x,u)\mathcal{B}(u,w)p(s,y,w)dudwds \\
&\leq c_{21}\int_0^{t/2}\int_{V_3} p(s,y,w) w_d^{\beta_1-\epsilon'}\log^{\beta_4}\left(e+\frac{1}{w_d}\right) \\
&\qquad\quad \times \left[\int_{V_1:u_d\geq w_d} p(t-s,x,u)u_d^{\beta_2+\epsilon'}\log^{\beta_3}\left(e+\frac{u_d}{w_d}\right)du\right]dwds \\
&\leq c_{22} x_d^{\beta_2+\epsilon'} y_d^{\beta_1-\epsilon'}\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^{p-\beta_2-\epsilon'}\int_0^{t/2}\left(1\wedge\frac{y_d}{s^{1/\alpha}}\right)^{p-\beta_1+\epsilon'}\log^{\beta_4}\left(e+\frac{1}{y_d\vee s^{1/\alpha}}\right)ds \\
&\leq c_{23} t x_d^{\beta_1} y_d^{\beta_2}\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^{p-\beta_1}\left(1\wedge\frac{y_d}{t^{1/\alpha}}\right)^{p-\beta_2}\log^{\beta_4}\left(e+\frac{1}{y_d\vee t^{1/\alpha}}\right).
\end{aligned}$$

The proof for (Case 1) is complete.

(Case 2) $\beta_1 \leq \beta_2$.

By **(A3-II)** and (5.6.17), we have

$$\begin{aligned}
&\int_0^t \int_{V_3} \int_{V_1} p^{V_1}(t-s,x,u)\mathcal{B}(u,w)p(s,y,w)dudwds \\
&\le c_{24} \int_0^t \int_{V_3} p(s,y,w) w_d^{\widehat{\beta}_2} \log^{\beta_4}\left(e+\frac{1}{w_d}\right) \\
&\qquad \times \left[\int_{V_1} p(t-s,x,u) u_d^{\beta_1} \log^{\beta_3}\left(e+\frac{w_d}{u_d}\right) du\right] dwds \\
&\quad + c_{24} \int_0^t \int_{V_3} p(s,y,w) w_d^{\beta_1} \\
&\qquad \times \left[\int_{V_1 : w_d < u_d} p(t-s,x,u) u_d^{\widehat{\beta}_2} \log^{\beta_3}\left(e+\frac{u_d}{w_d}\right) \log^{\beta_4}\left(e+\frac{1}{u_d}\right) du\right] dwds \\
&=: I_1 + I_2.
\end{aligned}$$

Since $p-\widehat{\beta}_2 \le p-\beta_1 < \alpha$ and $\widehat{\beta}_2 < \alpha+\beta_1$, we see from Lemmas 5.6.15 and 5.7.1 that

$$\begin{aligned}
I_1 \le c_{25} t x_d^{\beta_1} y_d^{\widehat{\beta}_2} \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^{p-\beta_1} \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^{p-\widehat{\beta}_2} \\
\times \log^{\beta_3}\left(e+\frac{y_d \vee t^{1/\alpha}}{x_d \vee t^{1/\alpha}}\right) \log^{\beta_4}\left(e+\frac{1}{y_d \vee t^{1/\alpha}}\right).
\end{aligned}$$

Let $\epsilon'' > 0$ be such that $p \vee \widehat{\beta}_2 < \alpha+\beta_1-\epsilon''$. Using (5.2.2) twice and (5.6.13), since $x_d \le y_d$, we also get from Lemmas 5.6.15 and 5.7.1 that

$$\begin{aligned}
I_2 &\le c_{29} \int_0^t \int_{V_3} p(s,y,w) w_d^{\beta_1-\epsilon''} \\
&\times \left[\int_{V_1 : w_d < u_d} p(t-s,x,u) u_d^{\widehat{\beta}_2+\epsilon''} \log^{\beta_3}\left(e+\frac{u_d}{w_d}\right) \log^{\beta_4}\left(e+\frac{1}{u_d}\right) du\right] dwds \\
&\le c_{31} t \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^p \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^p (x_d \vee t^{1/\alpha})^{\widehat{\beta}_2+\epsilon''} (y_d \vee t^{1/\alpha})^{\beta_1-\epsilon''} \\
&\quad \times \log^{\beta_3}\left(e+\frac{x_d \vee t^{1/\alpha}}{y_d \vee t^{1/\alpha}}\right) \log^{\beta_4}\left(e+\frac{1}{x_d \vee t^{1/\alpha}}\right) \\
&\le c_{32} t \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^p \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^p (x_d \vee t^{1/\alpha})^{\beta_1} (y_d \vee t^{1/\alpha})^{\widehat{\beta}_2} \\
&\quad \times \log^{\beta_3}\left(e+\frac{x_d \vee t^{1/\alpha}}{y_d \vee t^{1/\alpha}}\right) \log^{\beta_4}\left(e+\frac{1}{y_d \vee t^{1/\alpha}}\right).
\end{aligned}$$

The proof is complete. □

5.6.3 Lower bound estimates

Recall that $D_{\widetilde{w}}(a,b) = \{x = (\widetilde{x}, x_d) \in \mathbb{R}^d : |\widetilde{x} - \widetilde{w}| < a, 0 < x_d < b\}$ for $a, b > 0$ and $\widetilde{w} \in \mathbb{R}^{d-1}$, and, in dimension 1, $D_{\widetilde{w}}(a,b) = (0,b)$.

For $x = (\widetilde{x}, x_d) \in \mathbb{R}^d_+$, we define

$$V_x = \begin{cases} D_{\widetilde{x}}(2,2) & \text{if } x_d < 1 \\ B(x,1/2) & \text{if } x_d \geq 1, \end{cases} \quad \text{and} \quad W_x = D_{\widetilde{x}}(2, 8 + x_d) \setminus D_{\widetilde{x}}(2, 5 + x_d).$$

Lemma 5.6.17. *There exist constants $M > 1$ and $c > 0$ such that for all $x \in \mathbb{R}^d_+$,*

$$\inf_{z \in W_x} p(M, x, z) \geq c(x_d \wedge 1)^p.$$

Proof. By Proposition 5.3.5,

$$\inf_{w,z \in W_x, 1 \leq s \leq M} p(s, w, z) \geq c_1(M) > 0 \quad \text{for all } x \in \mathbb{R}^d_+. \tag{5.6.28}$$

If $x_d \geq 1$, then the result follows from Proposition 5.3.5. Suppose $x_d < 1$. Then by the strong Markov property and (5.6.28), for all $M > 1$ and $z \in W_x$,

$$\begin{aligned} p(M, x, z) &\geq \mathbb{E}^x[p(M - \tau_{V_x}, Y_{\tau_{V_x}}, z) : \tau_{V_x} \leq M - 1, Y_{\tau_{V_x}} \in W_x] \\ &\geq \left(\inf_{w \in W_x, 1 \leq s \leq M} p(s, w, z) \right) \mathbb{P}^x(\tau_{V_x} \leq M - 1, Y_{\tau_{V_x}} \in W_x) \\ &\geq c_1(\mathbb{P}^x(Y_{\tau_{V_x}} \in W_x) - \mathbb{P}^x(\tau_{V_x} > M - 1)). \end{aligned}$$

By [92, Lemma 5.10] for $\kappa > 0$ and [93, Theorem 1.1] for $\kappa = 0$, we have

$$\mathbb{P}^x(Y^\kappa_{\tau_{V_x}} \in W_x) \geq \mathbb{P}^x(Y^\kappa_{\tau_{V_x}} \in D_{\widetilde{x}}(2,8) \setminus D_{\widetilde{x}}(2,6)) \geq 2c_2 x_d^p$$

Moreover, by Corollary 5.6.12, we also have

$$\mathbb{P}^x(\tau_{V_x} > M - 1)) \leq \mathbb{P}^x(\zeta > M - 1) \leq c_3(x_d/(M-1)^{1/\alpha})^p.$$

Thus, we can choose $M = 1 + (c_3/c_2)^{\alpha/p}$ so that $2c_2 - c_3(M-1)^{-p/\alpha} = c_2$, which implies $p(M, x, z) \geq c_2 x_d^p$. □

Recall the definition of $A_{\beta_1,\beta_2,\beta_3,\beta_4}(t, x, y)$ from (5.2.3).

Proposition 5.6.18. *There exists a constant $c > 0$ such that*

$$p(t, x, y) \geq c \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^p \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^p A_{\beta_1,\beta_2,\beta_3,\beta_4}(t, x, y) \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right),$$

for all $t > 0$ and $x, y \in \mathbb{R}^d_+$.

Proof. Without loss of generality, we assume $x_d \leq y_d$. Let $M > 1$ be the constant in Lemma 5.6.17. By the semigroup property and Lemma 5.6.17,

$$\begin{aligned} p(2M+1, x, y) &\geq \int_{W_x} \int_{W_y} p(M, x, z) p(1, z, w) p(M, w, y) dz dw \\ &\geq c_1 \left(\inf_{z \in W_x} p(M, x, z)\right) \left(\inf_{w \in W_y} p(M, y, w)\right) \left(\inf_{(z,w) \in W_x \times W_y} p(1, x, z)\right) \\ &\geq c_2 (x_d \wedge 1)^p (y_d \wedge 1)^p \left(\inf_{(z,w) \in W_x \times W_y} p(1, z, w)\right). \end{aligned} \tag{5.6.29}$$

We see from Propositions 5.3.5 and 5.5.4 that

$$\begin{aligned} &\inf_{(z,w) \in W_x \times W_y} p(1, z, w) \\ &\geq c \begin{cases} 1 & \text{if } |x-y| \leq 3, \\ |x-y|^{-d-\alpha} \left(\frac{x_d \vee 1}{|x-y|} \wedge 1\right)^{\beta_1} \left(\frac{y_d \vee 1}{|x-y|} \wedge 1\right)^{\beta_2} & \\ \quad \times \log^{\beta_3}\left(1 + \frac{(y_d \vee 1) \wedge |x-y|}{(x_d \vee 1) \wedge |x-y|}\right) \log^{\beta_4}\left(1 + \frac{|x-y|}{(y_d \vee 1) \wedge |x-y|}\right) & \text{if } |x-y| > 3. \end{cases} \end{aligned}$$

Combining the above with (5.6.29), and using scaling property (5.4.2), we arrive at the reuslt. □

5.7 Appendix: Some calculations

In this section, we give some technical lemmas which are used in the proofs of main results. The following three lemmas can be proved by using (5.2.1), (5.2.2) and Lemma 1.1.1(i).

Lemma 5.7.1. *Let $\gamma < \alpha$ and $b \ge 0$. There exists $C \ge 1$ such that for any $t, k, l > 0$,*

$$\int_0^t \left(1 \wedge \frac{k}{s^{1/\alpha}}\right)^\gamma \log^b \left(e + \frac{l}{k \vee s^{1/\alpha}}\right) ds \le Ct\left(1 \wedge \frac{k}{t^{1/\alpha}}\right)^\gamma \log^b \left(e + \frac{l}{k \vee t^{1/\alpha}}\right).$$

For $\gamma, \eta_1, \eta_2 \ge 0$ and $k, l > 0$, define

$$f_{\gamma,\eta_1,\eta_2,k,l}(r) := r^\gamma \log^{\eta_1} \left(e + \frac{k}{r}\right) \log^{\eta_2} \left(e + \frac{r}{l}\right).$$

Lemma 5.7.2. *Let $\gamma, \eta_1, \eta_2 \ge 0$ and $k, l > 0$.*
(i) For any $\epsilon > 0$, there exists $C > 0$ independent of k and l such that

$$\frac{f_{\gamma,\eta_1,\eta_2,k,l}(ar)}{f_{\gamma,\eta_1,\eta_2,k,l}(r)} \le Ca^{\gamma+\epsilon} \quad \text{for all } a \ge 1 \text{ and } r > 0.$$

(ii) Assume that $\gamma > 0$ if $\eta_1 > 0$. Then there exists $C > 0$ independent of k and l such that

$$\frac{f_{\gamma,\eta_1,\eta_2,k,l}(ar)}{f_{\gamma,\eta_1,\eta_2,k,l}(r)} \ge C \quad \text{for all } a \ge 1 \text{ and } r > 0.$$

Lemma 5.7.3. *Let $b_1, b_2, \eta_1, \eta_2 \ge 0$ and $0 \le \gamma < \alpha + b_1$. Assume that $\gamma > 0$ if $\eta_1 > 0$. There exists $C > 0$ such that for any $x \in \mathbb{R}^d_+$ and $s, k, l > 0$,*

$$\begin{aligned}
&\int_{\mathbb{R}^d_+} \left(1 \wedge \frac{x_d \vee s^{1/\alpha}}{|x-z|}\right)^{b_1} \log^{b_2} \left(1 + \frac{|x-z|}{(x_d \vee s^{1/\alpha}) \wedge |x-z|}\right) \\
&\qquad \times \left(s^{-d/\alpha} \wedge \frac{s}{|x-z|^{d+\alpha}}\right) z_d^\gamma \log^{\eta_1} \left(e + \frac{k}{z_d}\right) \log^{\eta_2} \left(e + \frac{z_d}{l}\right) dz \\
&\le C(x_d \vee s^{1/\alpha})^\gamma \log^{\eta_1} \left(e + \frac{k}{x_d \vee s^{1/\alpha}}\right) \log^{\eta_2} \left(e + \frac{x_d \vee s^{1/\alpha}}{l}\right).
\end{aligned}$$

Chapter 6

Estimates on the fundamental solution of general time fractional equation

In this chapter, we give estimates for the fundamental solution of general time fractional equation. The results in this chapter are based on [54]. By adapting the notion of boundary function introduced in Chapter 4, we generalize some results in [54].

The time fractional diffusion equation $\partial_t^\beta u = \Delta u$ $(0 < \beta < 1)$ has been used in various fields to model the diffusions on sticky and trapping environment. Here, ∂_t^β is the Caputo derivative of order β which is defined as

$$\partial_t^\beta u(t) := \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t (t-s)^{-\beta} (u(s) - u(0)) ds,$$

where $\Gamma(z) := \int_0^\infty x^{z-1} e^{-x} dx$ is the gamma function. Motivated by the above definition of the Caputo derivative, in [30], the author introduced generalized time fractional derivatives. Let w be a nonnegative function satisfying the following condition:

(Ker) w is a right continuous nonincreasing function on $(0, \infty)$ with

$$\lim_{s\to 0+} w(s) = \infty, \quad \lim_{s\to\infty} w(s) = 0 \ \text{ and } \ \int_0^\infty (1 \wedge s)(-dw(s)) < \infty.$$

Definition 6.0.1. *For a function $u : [0, \infty) \to \mathbb{R}$, the generalized time fractional derivative ∂_t^w with respect to the kernel w is given by*

$$\partial_t^w u(t) := \frac{d}{dt} \int_0^t w(t-s)(u(s) - u(0))ds, \tag{6.0.1}$$

whenever the above integral makes sense.

We note that, the kernel $w(t) = t^{-\beta}/\Gamma(1-\beta)$ for the Caputo derivative of order β $(0 < \beta < 1)$ satisfies condition **(Ker)**.

Let (M, ρ) be a locally compact separable metric space and m is a Radon measure on M. Let D be a Borel subset of M, and $(T_t)_{t\geq 0}$ be a uniformly bounded strongly continuous semigroup with infinitesimal generator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ in some Banach space $(\mathbb{B}, \|\cdot\|)$. For a given kernel w satisfying **(Ker)**, consider the following time fractional equation with Dirichlet boundary condition:

$$\begin{cases} \partial_t^w u(t,x) = \mathcal{L}u(t,x), & x \in D, \ \ t > 0, \\ u(0,x) = f(x), & x \in D, \\ u(t,x) = 0, & \text{vanishes continuously on } \partial D \ \text{ for all } \ t > 0. \end{cases} \tag{6.0.2}$$

Examples and topics related to the problem (6.0.2) can be found in [6, 49, 100, 103, 104, 105, 124]. See also [69, 70] for examples of time fractional equations with non-linear noises. In [30], the author established the probabilistic representation for the fundamental solution of time fractional equation (6.0.2) (without Dirichlet boundary condition). This procedure can be described as follows: For a given w satisfying condition **(Ker)**, let $\nu(ds)$ be a measure on $(0, \infty)$ such that $w(s) = \nu((s, \infty))$ for all $s > 0$. Define a function ϕ by

$$\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda s})\nu(ds) \quad \text{for } \ \lambda \geq 0. \tag{6.0.3}$$

Then, since $|1-e^{-\lambda s}| \le (1+\lambda)(1 \wedge s)$ for $s > 0$ and **(Ker)** holds true, ϕ is a Bernstein function. Let $S = (S_r)_{r \ge 0}$ be a subordinator whose Laplace exponent is given by ϕ, and write its inverse by $E_t := \inf\{r > 0 : S_r > t\}$. Then, if we overlook the boundary condition, it is established in [30, Theorem 2.3] that for all $f \in \mathcal{D}(\mathcal{L})$, $u(t,x) := \mathbb{E}[T_{E_t} f(x)]$ is a unique solution to (6.0.2) in the following sense:

(1) $\sup_{t>0} \|u(t,\cdot)\| < \infty$, $x \mapsto u(t,x)$ is in $\mathcal{D}(\mathcal{L})$ for each $t \ge 0$ with $\sup_{t \ge 0} \|\mathcal{L}u(t,\cdot)\| < \infty$, and both $t \mapsto u(t,\cdot)$ and $t \mapsto \mathcal{L}u(t,\cdot)$ are continuous in $(\mathbb{B}, \|\cdot\|)$;
(2) for every $t > 0$, $I^w_t[u] := \int_0^t w(t-s)(u(s,x) - f(x))ds$ is absolutely convergent in $(\mathbb{B}, \|\cdot\|)$ and

$$\lim_{\delta \to 0} \frac{1}{\delta}(I^w_{t+\delta}[u] - I^w_t[u]) = \mathcal{L}u(t,x) \quad \text{in } (\mathbb{B}, \|\cdot\|).$$

We will see that if $\{T_t, t \ge 0\}$ admits a transition density enjoying certain types of estimates, then the solution $u(t,x)$ satisfies the following boundary condition (see Corollary 6.1.8 for a precise statement).

(3) if f is bounded, then for all $t > 0$, $x \mapsto u(t,x)$ vanishes continuously on ∂D.

Conversely, for any driftless subordinator S with an infinite Lévy measure $\nu(ds)$, its tail measure $w(s) := \nu((s,\infty))$ satisfies condition **(Ker)**. Therefore, S is in one-to-one correspondence with generalized time fractional derivative ∂^w_t defined by (6.0.1).

Suppose that the semigroup $(T_t)_{t \ge 0}$ has a heat kernel $q(t,x,y)$ with respect to the reference measure m. Then for $f \in \mathcal{D}(\mathcal{L})$,

$$\begin{aligned} u(t,x) = \mathbb{E}[T_{E_t} f(x)] &= \int_0^\infty T_r f(x) d_r \mathbb{P}(E_t \le r) = \int_0^\infty T_r f(x) d_r \mathbb{P}(S_r \ge t) \\ &= \int_M f(y) \left(\int_0^\infty q(r,x,y) d_r \mathbb{P}(S_r \ge t) \right) m(dy). \end{aligned}$$

Therefore, it is natural to say that

$$p(t,x,y) := \int_0^\infty q(r,x,y)d_r\mathbb{P}(S_r \geq t) \tag{6.0.4}$$

is the fundamental solution to the equation (6.0.2).

In this chapter, using the expression (6.0.4), with helps from the results obtained in Chapter 2, we establish two-sided estimates for the fundamental solution of general time fractional equation $\partial_t^w u = \mathcal{L}u$ including the ones with the Dirichlet boundary condition. Throughout the chapter, we always assume that w satisfies condition **(Ker)**, and denote by S and E the associated subordinator (via (6.0.3)) and its inverse, respectively. We note that, since $w(0+) = \infty$, S is not a compound Poisson process. Therefore, a.s., $r \mapsto S_r$ is strictly increasing and $t \mapsto E_t$ is continuous.

6.1 Setup and main results

Let (M,ρ) be a locall compact separable metric space, and m a positive Radon measure on M with full support. As in Section 4.1, we assume that the volume function $V(x,r) := m(B(x,r))$ satisfies the uniform volume doubling condition (4.1.1), and D be a subset of M. We also let $\Phi, \Psi : [0,\infty) \to [0,\infty)$ be strictly increasing functions satisfying scaling properties (4.1.3) with constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0$, and $\Psi(r) \geq \Phi(r)$ for all $r \geq 0$.

Let $h(t,x,y)$ be a boundary function in the sense of Definition 4.1.3. Throughout the chapter, we always assume that

$$h \textit{ satisfies (H2) with } \gamma < 2. \tag{6.1.1}$$

Recall that the boundary function $h_p(t,x,y)$ defined by (4.1.6) satisfies (H2) with $\gamma = 2p$.

With the Laplace exponent ϕ of the subordinator S, we define for $(t,x,y) \in$

$(0,\infty)\times D\times D$ satisfying $4\Phi(\rho(x,y))\le\phi(t^{-1})^{-1}$,

$$\mathcal{I}_h(t,x,y):=\int_{\Phi(\rho(x,y))}^{1/(2\phi(t^{-1}))}\frac{h(r,x,y)}{V(x,\Phi^{-1}(r))}dr.$$

Recall the definitions of conditions for w, $\mathbf{Poly}_{R_1,\le}(\beta_2)$, $\mathbf{Poly}^{\infty}_{R_2,\le}(\beta_2)$, $\mathbf{Sub}^{\infty}(\gamma,\theta)$ and $\mathbf{Trun}^{\infty}_{R_2}$ from the begining of Chapter 2, and $\mathbf{HK}^{\mathbf{h}}_{\mathbf{B}}$ and $\mathbf{HK}^{\mathbf{h}}_{\mathbf{U}}$ from Definition 4.1.6. Throughout the chapter, we regard $\mathbf{HK}^{\mathbf{h}}_{\mathbf{B}}$ and $\mathbf{HK}^{\mathbf{h}}_{\mathbf{U}}$ as conditions for $q(t,x,y)$, that is, we say that $\mathbf{HK}^{\mathbf{h}}_{\mathbf{B}}$ holds if (4.1.8) and (4.1.9) hold with $q(t,x,y)$ instead of $p_D(t,x,y)$, and $\mathbf{HK}^{\mathbf{h}}_{\mathbf{U}}$ holds if (4.1.8) holds for all $(t,x,y)\in(0,\infty)\times D\times D$ with $q(t,x,y)$ instead of $p_D(t,x,y)$.

Now, we state our main results which are modifications of [54, Theorems 1.15, 1.16 and 1.18] by allowing the boundary function $h(t,x,y)$ to be more general form. The proofs will be given in Section 6.2. Let $p(t,x,y)$ be the function defined by (6.0.4).

Theorem 6.1.1. *Suppose that $\mathbf{Poly}_{R_1,\le}(\beta_2)$ and $\mathbf{HK}^{\mathbf{h}}_{\mathbf{B}}$ hold. Then the following estimates are valid for all $(t,x,y)\in(0,R_1)\times D\times D$:*

(i) Suppose that $\Phi(\rho(x,y))\phi(t^{-1})\le 1/4$. Then we have

$$p(t,x,y)\simeq\frac{h(1/\phi(t^{-1}),x,y)}{V\big(x,\Phi^{-1}(1/\phi(t^{-1}))\big)}+w(t)\mathcal{I}_h(t,x,y).\tag{6.1.2}$$

In particular, if $\mathbf{Poly}_{R_1,\le}(\beta_2)$ holds with $\beta_2<1$, then

$$p(t,x,y)\simeq\phi(t^{-1})\mathcal{I}_h(t,x,y).\tag{6.1.3}$$

(ii) If $\Phi(\rho(x,y))\phi(t^{-1})>1/4$, then

$$p(t,x,y)\asymp h(1/\phi(t^{-1}),x,y)\\ \times\bigg(\frac{C_0}{\phi(t^{-1})\Psi(\rho(x,y))V(x,\rho(x,y))}+\frac{\exp\big(-c\mathcal{N}(t,\rho(x,y))\big)}{V\big(x,\Phi^{-1}(1/\phi(t^{-1}))\big)}\bigg),$$

where $\mathcal{N}(t,l)$ *is the solution to the following equation:*

$$\frac{\mathcal{N}(t,l)/t}{\phi\big(\mathcal{N}(t,l)/t\big)} = \frac{\Phi(l/\sqrt{\mathcal{N}(t,l)})}{t}. \tag{6.1.4}$$

Remark 6.1.2. Since $\lambda \mapsto \phi(\lambda)/\lambda$ is decreasing with $\lim_{\lambda\to 0}\phi(\lambda)/\lambda = w(0+) = \infty$ and $\lim_{\lambda\to\infty}\phi(\lambda)/\lambda = 0$, and Φ is increasing with $\Phi(0) = 0$ and $\lim_{l\to\infty}\Phi(l) = \infty$, the equation (6.1.4) always have a unique solution.

Theorem 6.1.3. *Suppose that* $\mathbf{Poly}_{\infty,\le}(\beta_2)$ *and* $\mathbf{HK}^{\mathbf{h}}_{\mathbf{U}}$ *hold. Then the assertions in Theorem 6.1.1(i)-(ii) hold for all* $(t,x,y) \in (0,\infty)\times D\times D$.

Now, we give large time estimtates for $p(t,x,y)$ under $\mathbf{HK}^{\mathbf{h}}_{\mathbf{B}}$.

Theorem 6.1.4. *Suppose that* $\mathbf{Poly}^{\infty}_{R_2,\le}(\beta_2)$ *and* $\mathbf{HK}^{\mathbf{h}}_{\mathbf{B}}$ *hold. Then for every* $T > 0$*, there are comparability constants depending on* T *such that for all* $(t,x,y) \in [T,\infty)\times D\times D$,

$$p(t,x,y) \simeq w(t)\int_{\Phi(\rho(x,y))}^{2\Phi(\mathrm{diam}(D))}\frac{h(r,x,y)}{V(x,\Phi^{-1}(r))}dr.$$

When condition $\mathbf{Sub}^{\infty}(\gamma,\theta)$ holds, the bounds for fundamental solution decrease subexponentially as $t\to\infty$. Moreover, when $0<\beta<1$ and D is bounded, we obtain the sharp upper bounds that decrease with exactly the same rate as the upper bound for w as $t\to\infty$.

Theorem 6.1.5. *Suppose that* $\mathbf{Sub}^{\infty}(\gamma,\theta)$ *holds. Then for every* $T>0$*, the following estimates are valid for all* $(t,x,y)\in[T,\infty)\times D\times D$.

(i) Assume that $\mathbf{HK}^{\mathbf{h}}_{\mathbf{B}}$ *holds. Then, there exist constants* $L_1, L_2 > 0$ *independent of* θ*, and* $c>1$ *such that in the case when* $\gamma\in(0,1)$*, we have*

$$c^{-1}w(t)\int_{\Phi(\rho(x,y))}^{2\Phi(\mathrm{diam}(D))}\frac{h(r,x,y)}{V(x,\Phi^{-1}(r))}dr$$
$$\le p(t,x,y) \le c\exp\big(-\theta t^{\gamma}\big)\int_{\Phi(\rho(x,y))}^{2\Phi(\mathrm{diam}(D))}\frac{h(r,x,y)}{V(x,\Phi^{-1}(r))}dr,$$

and in the case when $\gamma = 1$, we have

$$c^{-1}\Big(w(t)\int_{\Phi(\rho(x,y))}^{2\Phi(\mathrm{diam}(D))}\frac{h(r,x,y)}{V(x,\Phi^{-1}(r))}dr + h(1,x,y)e^{-L_1 t}\Big)$$
$$\le p(t,x,y) \le c\Big(\exp\big(-\frac{\theta}{2}t\big)\int_{\Phi(\rho(x,y))}^{2\Phi(\mathrm{diam}(D))}\frac{h(r,x,y)}{V(x,\Phi^{-1}(r))}dr + h(1,x,y)e^{-L_2 t}\Big).$$

(ii) Assume that $\mathbf{HK_U^h}$ holds.

(1) If $\Phi(\rho(x,y))\phi(t^{-1}) \le 1/4$, then there exists $c > 1$ such that

$$c^{-1}\Big(\frac{h(t,x,y)}{V\big(x,\Phi^{-1}(t)\big)} + w(t)\int_{\Phi(\rho(x,y))}^{1/\phi(t^{-1})}\frac{h(r,x,y)}{V(x,\Phi^{-1}(r))}dr\Big)$$
$$\le p(t,x,y) \le c\Big(\frac{h(t,x,y)}{V\big(x,\Phi^{-1}(t)\big)} + \exp\big(-\frac{\theta}{2}t^{\gamma}\big)\int_{\Phi(\rho(x,y))}^{1/(2\phi(t^{-1}))}\frac{h(r,x,y)}{V(x,\Phi^{-1}(r))}dr\Big).$$

(2) If $\Phi(\rho(x,y))\phi(t^{-1}) > 1/4$, then

$$p(t,x,y) \asymp q(ct,x,y)$$
$$\asymp h(t,x,y)\Big(\frac{C_0 t}{V(x,\rho(x,y))\Psi(\rho(x,y))} + \frac{\exp\big(-c\rho(x,y)^2/\Phi^{-1}(t)^2\big)}{V(x,\Phi^{-1}(t))}\Big).$$

Our last main theorem deals with finitely supported w. Let $h_p(t,x,y)$ be the boundary function defined in (4.1.6).

Theorem 6.1.6. *Suppose that $\mathbf{Trun}_{R_2}^{\infty}$ holds. Then the following estimates hold for all $(t,x,y) \in [R_2/2,\infty) \times D \times D$. Let $n_t := \lfloor t/R_2 \rfloor + 1 \in \mathbb{N}$.*

(i) Assume that $\mathbf{HK_B^h}$ holds with $h = h_p$.

(1) If $t \le \lfloor d_2/\alpha_1 + 2p \rfloor R_2$, then

$$p(t,x,y) \simeq \int_{\Phi(\rho(x,y))}^{2\Phi(\mathrm{diam}(D))}\frac{r^{n_t}h_p(r,x,y)}{V(x,\Phi^{-1}(r))}dr$$
$$+ (n_t R_2 - t)^{n_t}\int_{\Phi(\rho(x,y))}^{2\Phi(\mathrm{diam}(D))}\frac{r^{n_t-1}h_p(r,x,y)}{V(x,\Phi^{-1}(r))}dr.$$

(2) If $t \geq \lfloor d_2/\alpha_1 + 2p \rfloor R_2$*, then*

$$p(t,x,y) \asymp \Phi(\delta_D(x))^p \Phi(\delta_D(y))^p e^{-ct} \asymp q(ct,x,y).$$

(ii) Assume that $\mathbf{HK_U^h}$ *holds with* $h = h_p$*.*

(1) If $\Phi(\rho(x,y)) \leq t \leq \lfloor d_2/\alpha_1 + 2p \rfloor R_2$*, then*

$$p(t,x,y) \simeq \int_{\Phi(\rho(x,y))}^{2t} \frac{r^{n_t} h_p(r,x,y)}{V(x,\Phi^{-1}(r))} dr + (n_t R_2 - t)^{n_t} \int_{\Phi(\rho(x,y))}^{2t} \frac{r^{n_t - 1} h_p(r,x,y)}{V(x,\Phi^{-1}(r))} dr.$$

(2) If $\Phi(\rho(x,y)) > t$ *or* $t > \lfloor d_2/\alpha_1 + 2p \rfloor R_2$*, then*

$$p(t,x,y) \asymp q(ct,x,y) \asymp h_p(t,x,y) \times \left[\frac{1}{V(x,\Phi^{-1}(t))} \wedge \left(\frac{C_0 t}{V(x,\rho(x,y))\Psi(\rho(x,y))} + \frac{\exp\big(-c\rho(x,y)^2/\Phi^{-1}(t)^2\big)}{V(x,\Phi^{-1}(t))} \right) \right].$$

Remark 6.1.7. Note that under settings of Theorem 6.1.6, we can apply Theorem 6.1.1 to obtain the estimates of $p(t,x,y)$ for $(t,x,y) \in (0, R_2/2] \times D \times D$. Hence, we have the global estimates for $p(t,x,y)$ under those settings.

As a consequence of the estimates for the fundamental solution, we have that the solution to the Dirichlet problem (6.0.2) vanishes continuously on the boundary of D. Indeed, under mild conditions, the solution $u(t,x)$ vanishes exactly the same rate as a transition density $q(t,x,y)$.

Corollary 6.1.8. *Suppose that* $\mathbf{Poly}_{R_1,\leq}(\beta_2)$*, and one among* $\mathbf{Poly}^{\infty}_{R_2,\leq}(\beta_2)$*,* $\mathbf{Sub}^{\infty}(\gamma,\theta)$ *and* $\mathbf{Trun}^{\infty}_{R_2}$ *hold true. We also assume that either* $\mathbf{HK_B^h}$ *or* $\mathbf{HK_U^h}$ *holds with* $h = h_p$*. Then, for all bounded measurable function* f *on* D*,* $u(t,x) := \mathbb{E}[T^D_{E_t} f(x)]$ *satisfies the following boundary condition:*

For any fixed $t > 0$*, there exists* $C > 0$ *such that for every* $x \in D$*,*

$$|u(t,x)| \leq C \|f\|_\infty \Phi(\delta_D(x))^p.$$

Recall the definition of h_p from (4.1.6). In the end of this section, we give a list of estimates for $\mathcal{I}_h(t,x,y)$ when $h = h_p$. See [54, Appendix] for the proof of the following lemma.

Lemma 6.1.9. *Let $p \in [0,1)$. The following estimates are valid for all $(t,x,y) \in (0,\infty) \times D \times D$ satisfying $\Phi(\rho(x,y))\phi(t^{-1}) \le 1/4$.*

(i) If $d_2/\alpha_1 < 1-2p$, then

$$\mathcal{I}_{h_p}(t,x,y) \simeq \left(1 \wedge \frac{\Phi(\delta_\wedge(x,y))}{1/\phi(t^{-1})}\right)^p \left(1 \wedge \frac{\Phi(\delta_\vee(x,y))}{1/\phi(t^{-1})}\right)^p \frac{1/\phi(t^{-1})}{V\big(x, \Phi^{-1}(1/\phi(t^{-1}))\big)}.$$

(ii) If $\alpha_1 = \alpha_2 = \alpha$, $d_1 = d_2 = (1-2p)\alpha$ and $p > 0$, then

$$\begin{aligned}\mathcal{I}_{h_p}(t,x,y) \simeq &\left(1 \wedge \frac{\delta_\wedge(x,y)^\alpha}{1/\phi(t^{-1})}\right)^p \left(1 \wedge \frac{\delta_\vee(x,y)^\alpha}{1/\phi(t^{-1})}\right)^p \\ &\times \phi(t^{-1})^{-2p} \log\left(e + \frac{1/\phi(t^{-1})}{(\delta_\vee(x,y) \vee \rho(x,y))^\alpha}\right).\end{aligned}$$

(iii) If $1-2p < d_1/\alpha_2 \le d_2/\alpha_1 < 1-p$, then

$$\begin{aligned}\mathcal{I}_{h_p}(t,x,y) \simeq &\left(1 \wedge \frac{\Phi(\delta_\wedge(x,y))}{1/\phi(t^{-1})}\right)^p \left(1 \wedge \frac{\Phi(\delta_\vee(x,y))}{1/\phi(t^{-1})}\right)^{1-p} \\ &\times \left(1 \wedge \frac{\Phi(\delta_\vee(x,y))^{2p-1} V(x,\delta_\vee(x,y))}{\Phi(\rho(x,y))^{2p-1} V(x,\rho(x,y))}\right) \frac{1/\phi(t^{-1})}{V\big(x, \delta_\vee(x,y) \wedge \Phi^{-1}(1/\phi(t^{-1}))\big)}.\end{aligned}$$

(iv) If $\alpha_1 = \alpha_2 = \alpha$, $d_1 = d_2 = (1-p)\alpha$ and $p > 0$, then

$$\begin{aligned}\mathcal{I}_{h_p}(t,x,y) \simeq &\left(1 \wedge \frac{\delta_\wedge(x,y)^\alpha}{1/\phi(t^{-1})}\right)^p \left(1 \wedge \frac{\delta_\vee(x,y)^\alpha}{\rho(x,y)^\alpha}\right)^p \\ &\times \phi(t^{-1})^{-p} \log\left(e + \frac{\delta_\vee(x,y) \wedge \rho(x,y)}{\delta_\wedge(x,y) \wedge \rho(x,y)}\right).\end{aligned}$$

(v) If $1-p < d_1/\alpha_2 \le d_2/\alpha_1 < 1$, then

$$\begin{aligned}\mathcal{I}_{h_p}(t,x,y) \simeq &\left(1 \wedge \frac{\Phi(\delta_\wedge(x,y))}{1/\phi(t^{-1})}\right) \left(1 \wedge \frac{\Phi(\delta_\vee(x,y))}{\Phi(\rho(x,y))}\right)^p \\ &\times \left(1 \wedge \frac{\Phi(\delta_\wedge(x,y))^{p-1} V(x,\delta_\wedge(x,y))}{\Phi(\rho(x,y))^{p-1} V(x,\rho(x,y))}\right) \frac{1/\phi(t^{-1})}{V\big(x, \delta_\wedge(x,y) \wedge \Phi^{-1}(1/\phi(t^{-1}))\big)}.\end{aligned}$$

(vi) If $\alpha_1 = \alpha_2 = d_1 = d_2 = \alpha$, then

$$\mathcal{I}_{h_p}(t,x,y) \simeq \left(1 \wedge \frac{\delta_\wedge(x,y)^\alpha}{\rho(x,y)^\alpha}\right)^p \left(1 \wedge \frac{\delta_\vee(x,y)^\alpha}{\rho(x,y)^\alpha}\right)^p \times \log\left(e + \frac{\delta_\wedge(x,y)^\alpha \wedge (1/\phi(t^{-1}))}{\delta_\wedge(x,y)^\alpha \wedge \rho(x,y)^\alpha}\right).$$

(vii) If $1 < d_1/\alpha_2$, then

$$\mathcal{I}_{h_p}(t,x,y) \simeq \left(1 \wedge \frac{\Phi(\delta_\wedge(x,y))}{\Phi(\rho(x,y))}\right)^p \left(1 \wedge \frac{\Phi(\delta_\vee(x,y))}{\Phi(\rho(x,y))}\right)^p \frac{\Phi(\rho(x,y))}{V(x,\rho(x,y))}.$$

6.2 Proofs of Main results

In this section, we give the proof for Theorems 6.1.1-6.1.6. Throughout the section, we assume that $q(t,x,y)$ satisfies either $\mathbf{HK}^{\mathbf{h}}_{\mathbf{B}}$ or $\mathbf{HK}^{\mathbf{h}}_{\mathbf{U}}$.

Recall that the fundamental solution $p(t,x,y)$ is defined by (6.0.4). Using Proposition 2.2.1 and Theorems 2.2.6-2.2.8, we get the following a prior lower bound for $p(t,x,y)$.

Lemma 6.2.1. *(i) Suppose that $\mathbf{Poly}_{R_1,\le}(\beta_2)$ holds. Then, there exist constants $N > \epsilon_1 > 0$ and $c > 0$ such that for all $t \in (0, R_1)$,*

$$p(t,x,y) \ge c \inf_{r \in (\epsilon_1/\phi(t^{-1}), N/\phi(t^{-1}))} q(r,x,y). \tag{6.2.1}$$

(ii) Suppose that one among $\mathbf{Poly}^\infty_{R_2,\le}(\beta_2)$, $\mathbf{Sub}^\infty(\gamma,\theta)$ and $\mathbf{Trun}^\infty_{R_2}$ holds true. Then for every $T > 0$, there exist constants $N > \epsilon_1 > 0$ such that (6.2.1) holds for all $t \in [T, \infty)$.

Using [33, Lemmas 3.1 and 3.2], similar to Lemma 4.1.1, we may replace the volume function V by a nicer function.

Lemma 6.2.2. *For every $a \ge 1$, there exists a strictly increasing differentiable functions $\widetilde{V}(x,\cdot)$ satisfying the following two properties:*
(P1) $V(x,r) \simeq \widetilde{V}(x,r)$ for all $x \in M$ and $0 < r < aR_0$;

(P2) $d_r\widetilde{V}(x,r) \simeq r^{-1}\widetilde{V}(x,r)$ *and* $d_t\widetilde{V}^{-1}(x,t) \simeq t^{-1}\widetilde{V}^{-1}(x,t)$ *for all* $x \in M$, $0 < r < aR_0$ *and* $0 < t < V(x, aR_0)$.

Here, we give near diagonal lower estimates for $p(t,x,y)$ when the tail measure decays in mixed polynomial orders. The next result is a modification of [54, Proposition 4.1].

Proposition 6.2.3. *(i) Suppose* $\mathbf{Poly}_{R_1,\le}(\beta_2)$ *holds. Then there exists a constant* $C > 0$ *such that for all* $(t,x,y) \in (0,R_1) \times D \times D$ *satisfying* $\Phi(\rho(x,y))\phi(t^{-1}) \le 1/4$,

$$p(t,x,y) \ge C\left(\frac{h(1/\phi(t^{-1}),x,y)}{V\big(x,\Phi^{-1}(1/\phi(t^{-1}))\big)} + w(t)\mathcal{I}_h(t,x,y)\right) \tag{6.2.2}$$

(ii) Suppose $\mathbf{Poly}^{\infty}_{R_2,\le}(\beta_2)$ *and* $\mathbf{HK_U^h}$ *hold. Then for every* $T > 0$*, there are comparability constants depending on* T *such that* (6.2.2) *holds for all* $(t,x,y) \in [T,\infty) \times D \times D$ *satisfying* $\Phi(\rho(x,y))\phi(t^{-1}) \le 1/4$.

Proof. Since the proofs are similar, we only give the proof for (i).

Fix $(t,x,y) \in (0,R_1) \times D \times D$ satisfying $\Phi(\rho(x,y))\phi(t^{-1}) \le 1/4$ and set $l := \rho(x,y)$. By Theorem 2.2.6 and (2.0.4), there exsits a constant $\epsilon_2 \in (0,1/2]$ such that for all $t \in (0,R_1)$, $\mathbb{P}(S_{\epsilon_2\Phi(l)} \ge t) \le 1/2$. Then, using the Markov property and the inequality $1-(1-x)^2 \ge 3x/2$ for $x \in (0,1/2]$, we get that

$$\begin{aligned}\mathbb{P}(S_{2\epsilon_2\Phi(l)} \ge t) &\ge \mathbb{P}(S_{2\epsilon_2\Phi(l)} - S_{\epsilon_2\Phi(l)} \ge t \quad \text{or} \quad S_{\epsilon_2\Phi(l)} \ge t)\\ &\ge 1-(1-\mathbb{P}(S_{\epsilon_2\Phi(l)} \ge t))^2 \ge \frac{3}{2}\mathbb{P}(S_{\epsilon_2\Phi(l)} \ge t).\end{aligned}$$

It follows from the scaling properties of V and Φ, and (H2) that

$$\begin{aligned}p(t,x,y) &\ge c_1\int_{\epsilon_2\Phi(l)}^{2\epsilon_2\Phi(l)} \frac{h(r,x,y)}{V(x,\Phi^{-1}(r))} d_r\mathbb{P}(S_r \ge t)\\ &\ge c_2\frac{h(\Phi(l),x,y)}{V(x,l)}\big(\mathbb{P}(S_{2\epsilon_2\Phi(l)} \ge t) - \mathbb{P}(S_{\epsilon_2\Phi(l)} \ge t)\big)\end{aligned}$$

$$\geq \frac{c_2}{2}\frac{h(\Phi(l),x,y)}{V(x,l)}\mathbb{P}(S_{\epsilon_2\Phi(l)} \geq t). \tag{6.2.3}$$

On the other hand, by Lemmas 4.1.1 and 6.2.2, using integrtion by parts in the second inequality below and the scaling properties of V and Φ, (H1) and Theorem 2.2.6 in the third, we obtain

$$\begin{aligned}
p(t,x,y) &\geq c_3\int_{\epsilon_2\Phi(l)}^{1/(2\phi(t^{-1}))}\frac{h(r,x,y)}{\widetilde{V}(x,\widetilde{\Phi}^{-1}(r))}d_r\mathbb{P}(S_r\geq t)\\
&\geq -c_3\frac{h(\epsilon_2\Phi(l),x,y)}{\widetilde{V}(x,\widetilde{\Phi}^{-1}(\epsilon_2\Phi(l)))}\mathbb{P}(S_{\epsilon_2\Phi(l)}\geq t)\\
&\quad - c_3\int_{\epsilon_2\Phi(l)}^{1/(2\phi(t^{-1}))}\mathbb{P}(S_r\geq t)d_r\left(\frac{h(r,x,y)}{\widetilde{V}(x,\widetilde{\Phi}^{-1}(r))}\right)\\
&\geq -c_4\frac{h(\Phi(l),x,y)}{V(x,l)}\mathbb{P}(S_{\epsilon_2\Phi(l)}\geq t)\\
&\quad - c_4w(t)\int_{\epsilon_2\Phi(l)}^{1/(2\phi(t^{-1}))} rh(r,x,y)\, d_r\left(\frac{1}{\widetilde{V}(x,\widetilde{\Phi}^{-1}(r))}\right)\\
&\geq -c_4\frac{h(\Phi(l),x,y)}{V(x,l)}\mathbb{P}(S_{\epsilon_2\Phi(l)}\geq t)\\
&\quad + c_5w(t)\int_{\epsilon_2\Phi(l)}^{1/(2\phi(t^{-1}))}\frac{h(r,x,y)}{V(x,\Phi^{-1}(r))}dr.
\end{aligned} \tag{6.2.4}$$

Now, we conclude from Lemma 6.2.1, (6.2.3) and (6.2.4) that

$$(1+2c_4+c_2)p(t,x,y) \geq c_6\frac{h(1/\phi(t^{-1}),x,y)}{V\big(x,\Phi^{-1}(1/\phi(t^{-1}))\big)} + c_2c_5w(t)\mathcal{I}_h(t,x,y).$$

□

Now, we prove our main theorems by following arguments in the proofs given in [54, Subsections 4.1-4.3]. Although only boundary functions $h_p(t,x,y)$ and $h_p(t\wedge 1,x,y)$ for some $p\in[0,1)$ are considered in [54], with help from (6.1.1), one can repeat their arguments and get the desired results.

Proof of Theorem 6.1.1. We fix $(t,x,y)\in(0,R_1)\times D\times D$ and then write $l:=\rho(x,y)$, $V(s):=V(x,s)$ and $\widetilde{V}(s):=\widetilde{V}(x,s)$.

Case 1. Suppose that $\Phi(l)\phi(t^{-1}) \le 1/4$. By Proposition 6.2.3, it remains to prove the upper bound. Using (4.1.10) and Lemma 4.1.9, we get that

$$\begin{aligned}
p(t,x,y) \le\, & c_1 \frac{C_0}{V(l)\Psi(l)} \int_0^{\Phi(l)} r h(r,x,y) d_r \mathbb{P}(S_r \ge t) \\
& + c_1 \int_0^{\Phi(l)} \frac{h(r,x,y)}{V(\Phi^{-1}(r))} \exp\Big(-\frac{c_2 l^2}{\Phi^{-1}(r)^2}\Big) d_r \mathbb{P}(S_r \ge t) \\
& + c_1 \int_{\Phi(l)}^{1/(2\phi(t^{-1}))} \frac{h(r,x,y)}{V(\Phi^{-1}(r))} d_r \mathbb{P}(S_r \ge t) \\
& + c_1 \int_{1/(2\phi(t^{-1}))}^{1/\phi(R_1^{-1})} \frac{h(r,x,y)}{V(\Phi^{-1}(r))} d_r \mathbb{P}(S_r \ge t) \\
& + c_1 h(1,x,y) \int_{1/\phi(R_1^{-1})}^{\infty} e^{-\lambda_D r} d_r \mathbb{P}(S_r \ge t) \\
=:\, & c_1(I_1 + I_2 + I_3 + I_4 + I_5).
\end{aligned}$$

Since (H2) holds with $\gamma < 2$, using integration by parts, Theorem 2.2.6 and the fact that $\Psi \ge \Phi$, we get that

$$\begin{aligned}
I_1 &\le \frac{C_0}{V(l)\Psi(l)} \int_0^{\Phi(l)} r^{1-\gamma} r^{\gamma} h(r,x,y) d_r \mathbb{P}(S_r \ge t) \\
&\le c_2 \frac{C_0 \Phi(l)^{\gamma} h(\Phi(l),x,y)}{V(l)\Psi(l)} \int_0^{\Phi(l)} r^{1-\gamma} d_r \mathbb{P}(S_r \ge t) \\
&\le c_2 \frac{C_0 \Phi(l)\, h(\Phi(l),x,y)}{V(l)\Psi(l)} \mathbb{P}(S_{\Phi(l)} \ge t) \\
&\quad + c_2 |\gamma - 1| \frac{C_0 \Phi(l)^{\gamma} h(\Phi(l),x,y)}{V(l)\Psi(l)} \int_0^{\Phi(l)} r^{-\gamma} \mathbb{P}(S_r \ge t) dr \\
&\le c_3 w(t) \frac{C_0 \Phi(l)^2 h(\Phi(l),x,y)}{V(l)\Psi(l)} \\
&\quad + c_3 w(t) |\gamma - 1| \frac{C_0 \Phi(l)^{\gamma} h(\Phi(l),x,y)}{V(l)\Psi(l)} \int_0^{\Phi(l)} r^{1-\gamma} dr \\
&\le c_4 w(t) \frac{C_0 \Phi(l)\, h(\Phi(l),x,y)}{V(l)}. \qquad (6.2.5)
\end{aligned}$$

Next, using (H2), the inequality $\sup_{x>0} x^{\alpha_2+d_2} e^{-x^2} < \infty$ and the scaling properties of V and Φ, we get that for all $0 \le r \le \Phi(l)$,

$$\begin{aligned}\frac{h(r,x,y)}{V(\Phi^{-1}(r))}\exp\Big(-\frac{c_2l^2}{\Phi^{-1}(r)^2}\Big) &\le c_5\frac{\Phi(l)^\gamma h(\Phi(l),x,y)}{r^\gamma V(\Phi^{-1}(r))}\Big(\frac{\Phi^{-1}(r)}{l}\Big)^{\alpha_2+d_2}\\ &\le c_6\frac{\Phi(l)^\gamma h(\Phi(l),x,y)}{r^\gamma V(l)}\frac{r}{\Phi(l)}.\end{aligned}$$

Therefore, by repeating the calculations for I_1, we obtain

$$I_2 \le c_6\frac{\Phi(l)^{\gamma-1}h(\Phi(l),x,y)}{V(l)}\int_0^{\Phi(l)} r^{1-\gamma}d_r\mathbb{P}(S_r\ge t) \le c_7w(t)\frac{\Phi(l)\,h(\Phi(l),x,y)}{V(l)}.$$

Thirdly, using the scaling properties of V,Φ, (H1), (H2) and Theorem 2.2.6, we get that

$$\begin{aligned}I_3 &\le \sum_{\substack{i\ge0\\ 2^i\Phi(l)\le1/(2\phi(t^{-1}))}}\int_{2^i\Phi(l)}^{(2^{i+1}\Phi(l))\wedge1/(2\phi(t^{-1}))}\frac{h(r,x,y)}{V(\Phi^{-1}(r))}d_r\mathbb{P}(S_r\ge t)\\ &\le c_8w(t)\sum_{\substack{i\ge0\\ 2^i\Phi(l)\le1/(2\phi(t^{-1}))}}\frac{h(2^{i+1}\Phi(l),x,y)}{V(\Phi^{-1}(2^{i+1}\Phi(l)))}2^i\Phi(l) \le c_9w(t)\mathcal{I}_h(t,x,y).\end{aligned}$$

Lastly, by the monotone properties of V, Φ and (H1), it holds that

$$I_4+I_5 \le \frac{h(1/\phi(t^{-1}),x,y)}{V\big(\Phi^{-1}(1/\phi(t^{-1}))\big)} + h(1,x,y)e^{-\lambda_D/\phi(R_1^{-1})} \le c_{10}\frac{h(1/\phi(t^{-1}),x,y)}{V\big(\Phi^{-1}(1/\phi(t^{-1}))\big)}.$$

By (1.1.2) and the upper bounds above on I_1, I_2, we see that $w(t)\mathcal{I}_h(t,x,y)$ dominates I_1+I_2. The proof for (6.1.2) is complete.

Now, we further assume that $\mathbf{Poly}_{R_1,\le}(\beta_2)$ holds with $\beta_2<1$. Then by Lemma 2.1.1(i), $w(t)\simeq\phi(t^{-1})$ for $t\in(0,R_1)$. Hence, using (1.1.2), we can deduce that the second term in (6.1.2) dominates the first term which yields that (6.1.3) holds true.

Case 2. Suppose that $\Phi(l)\phi(t^{-1})>1/4$. By (4.1.10) and Lemmas 4.1.9, 4.1.1 and 6.2.2, it holds that

$$\begin{aligned}p(t,x,y) &\le c\frac{C_0}{V(l)\Psi(l)}\int_0^{1/(2\phi(t^{-1}))} rh(r,x,y)d_r\mathbb{P}(S_r\ge t)\\ &\quad + c\frac{C_0}{V(l)\Psi(l)}\int_{1/(2\phi(t^{-1}))}^{4\Phi(l)} rh(r,x,y)\big(-d_r\mathbb{P}(S_r\le t)\big)\\ &\quad + c\int_0^{4\Phi(l)}\frac{h(r,x,y)}{\widetilde{V}(\widetilde{\Phi}^{-1}(r))}\exp\Big(-\frac{a_1 l^2}{\widetilde{\Phi}^{-1}(r)^2}\Big)\big(-d_r\mathbb{P}(S_r\le t)\big)\\ &\quad + c\int_{4\Phi(l)}^{1/\phi(R_1^{-1})}\frac{h(r,x,y)}{\widetilde{V}(\widetilde{\Phi}^{-1}(r))}\big(-d_r\mathbb{P}(S_r\le t)\big)\\ &\quad + ch(1,x,y)\int_{1/\phi(R_1^{-1})}^{\infty} e^{-\lambda_D r}\big(-d_r\mathbb{P}(S_r\ge t)\big)\\ &=: c(J_1+J_2+J_3+J_4+J_5).\end{aligned}$$

By (H1) and the monotone properties of $\widetilde{V}$ and $\widetilde{\Phi}$, we deduce that J_4 dominates J_5.

By following arguments in (6.2.5) and using (2.0.4), we see that

$$\begin{aligned}J_1 &\le \frac{C_0}{V(l)\Psi(l)}\int_0^{1/(2\phi(t^{-1}))} r^{1-\gamma}r^{\gamma}h(r,x,y)d_r\mathbb{P}(S_r\ge t)\\ &\le c_1\frac{C_0\phi(t^{-1})^{-\gamma}\,h(1/\phi(t^{-1}),x,y)}{V(l)\Psi(l)}\int_0^{1/(2\phi(t^{-1}))} r^{1-\gamma}d_r\mathbb{P}(S_r\ge t)\\ &\le c_2\frac{C_0h(1/\phi(t^{-1}),x,y)}{\phi(t^{-1})V(l)\Psi(l)}\mathbb{P}(S_{1/(2\phi(t^{-1}))}\ge t)\\ &\quad + c_2|\gamma-1|\frac{C_0\phi(t^{-1})^{-\gamma}\,h(1/\phi(t^{-1}),x,y)}{V(l)\Psi(l)}\int_0^{1/(2\phi(t^{-1}))} r^{-\gamma}\mathbb{P}(S_r\ge t)dr\\ &\le c_2\frac{C_0h(1/\phi(t^{-1}),x,y)}{\phi(t^{-1})V(l)\Psi(l)}\\ &\quad + c_3\frac{C_0w(t)\phi(t^{-1})^{-\gamma}\,h(1/\phi(t^{-1}),x,y)}{V(l)\Psi(l)}\int_0^{1/(2\phi(t^{-1}))} r^{1-\gamma}dr\\ &\le c_2\frac{C_0h(1/\phi(t^{-1}),x,y)}{\phi(t^{-1})V(l)\Psi(l)} + c_4\frac{C_0w(t)h(1/\phi(t^{-1}),x,y)}{\phi(t^{-1})^2V(l)\Psi(l)} \le c_5\frac{C_0h(1/\phi(t^{-1}),x,y)}{\phi(t^{-1})V(l)\Psi(l)}.\end{aligned}$$

Next, by (H1), (H2) and integration by parts, it holds that

$$\begin{aligned}
J_2 &\le -\frac{C_0 h(1/\phi(t^{-1}),x,y)}{V(l)\Psi(l)} \int_{1/(2\phi(t^{-1}))}^{4\Phi(l)} r d_r \mathbb{P}(S_r \le t) \\
&\le \frac{C_0 h(1/\phi(t^{-1}),x,y)}{2\phi(t^{-1})V(l)\Psi(l)} \mathbb{P}(S_{1/(2\phi(t^{-1}))} \le t) \\
&\quad + \frac{C_0 h(1/\phi(t^{-1}),x,y)}{V(l)\Psi(l)} \int_{1/(2\phi(t^{-1}))}^{4\Phi(l)} \mathbb{P}(S_r \le t) dr. \qquad (6.2.6)
\end{aligned}$$

Define $\bar{b}^{-1}(t) = \sup\{s > 0 : sb(s) < t\}$ for $t > 0$. By (2.0.7), it holds that

$$\frac{1}{\phi(t^{-1})} \le \bar{b}^{-1}(t) \le \frac{c_*}{\phi(t^{-1})}, \quad c_* := \frac{e^2 - e}{e - 2}, \quad \text{for all } t > 0. \qquad (6.2.7)$$

Using Proposition 2.2.1 in the second inequality below, the change of the variables $r = \bar{b}^{-1}(t)u$ and (6.2.7) in the third, the fact that $u \mapsto (H \circ \sigma)(u,t)$ is increasing in the fourth, (2.0.8) in the first equality and (6.2.7) in the last inequality, we obtain that

$$\begin{aligned}
&\int_{1/(2\phi(t^{-1}))}^{\infty} \mathbb{P}(S_r \le t) dr \le \int_{1/(2\phi(t^{-1}))}^{c_*/\phi(t^{-1})} dr + \int_{c_*/\phi(t^{-1})}^{\infty} \mathbb{P}(S_r \le t) dr \\
&\le \frac{c_*}{\phi(t^{-1})} + \int_{c_*/\phi(t^{-1})}^{\infty} e \exp\big(- r(H \circ \sigma)(r,t) \big) dr \\
&\le \frac{c_*}{\phi(t^{-1})} + \bar{b}^{-1}(t) \int_1^{\infty} e \exp\big(- \bar{b}^{-1}(t)u \, (H \circ \sigma)(\bar{b}^{-1}(t)u, t) \big) du \\
&\le \frac{c_*}{\phi(t^{-1})} + \bar{b}^{-1}(t) \int_1^{\infty} e \exp\big(- u\bar{b}^{-1}(t) \, (H \circ \sigma)(\bar{b}^{-1}(t), t) \big) du \\
&= \frac{c_*}{\phi(t^{-1})} + \bar{b}^{-1}(t) \int_1^{\infty} e \exp e^{-u} du = \frac{c_*}{\phi(t^{-1})} + \bar{b}^{-1}(t) \le \frac{2c_*}{\phi(t^{-1})}.
\end{aligned}$$

Hence, we get from (6.2.6) that $J_2 \le (c_6 C_0 h(1/\phi(t^{-1}),x,y))/(\phi(t^{-1})V(l)\Psi(l))$.

Define the map $g : (0, 1/\phi(R_1^{-1})) \to (0,\infty)$ by

$$g(r) = \frac{1}{r^{1+\gamma}\widetilde{V}(\widetilde{\Phi}^{-1}(r))} \exp\Big(- \frac{a_1 l^2}{\widetilde{\Phi}^{-1}(r)^2} \Big).$$

Then we see from Lemmas 4.1.1 and 6.2.2 that

$$g'(r) \geq (-c_7 + c_8 l^2 \Phi^{-1}(r)^{-2}) r^{-1} g(r).$$

Thus, there exists $c_9 \in (0,1)$ such that $g(r)$ is increasing in $(0, c_9\Phi(l))$. Hence, since $\Phi(l)\phi(t^{-1}) > 1/4$, by (H1), (H2) and the scalings of V and Φ, we obtain that

$$\begin{aligned}
&-\int_0^{c_9/(4\phi(t^{-1}))} \frac{h(r,x,y)}{\widetilde{V}(\widetilde{\Phi}^{-1}(r))} \exp\Big(-\frac{a_1 l^2}{\widetilde{\Phi}^{-1}(r)^2}\Big) d_r\mathbb{P}(S_r \leq t) \\
&= -\int_0^{c_9/(4\phi(t^{-1}))} \frac{r^\gamma h(r,x,y)}{r^\gamma \widetilde{V}(\widetilde{\Phi}^{-1}(r))} \exp\Big(-\frac{a_1 l^2}{\widetilde{\Phi}^{-1}(r)^2}\Big) d_r\mathbb{P}(S_r \leq t) \\
&\leq c_{10} \frac{h(1/\phi(t^{-1}),x,y)}{V(\Phi^{-1}(1/\phi(t^{-1})))} \exp\Big(-\frac{c_{11} l^2}{\Phi^{-1}(1/\phi(t^{-1}))^2}\Big) \int_0^{c_9/(4\phi(t^{-1}))} d_r\mathbb{P}(S_r \geq t) \\
&\leq c_{10} \frac{h(1/\phi(t^{-1}),x,y)}{V(\Phi^{-1}(1/\phi(t^{-1})))} \exp\Big(-\frac{c_{11} l^2}{\Phi^{-1}(1/\phi(t^{-1}))^2}\Big). \qquad (6.2.8)
\end{aligned}$$

Note that, with the constant c_* in (6.2.7), by (H1), (H2) and the scalings of V and Φ,

$$\begin{aligned}
&-\int_{c_9/(4\phi(t^{-1}))}^{c_*/\phi(t^{-1})} \frac{h(r,x,y)}{\widetilde{V}(\widetilde{\Phi}^{-1}(r))} \exp\Big(-\frac{a_1 l^2}{\widetilde{\Phi}^{-1}(r)^2}\Big) d_r\mathbb{P}(S_r \leq t) \\
&\leq c_{12} \frac{h(1/\phi(t^{-1}),x,y)}{V(\Phi^{-1}(1/\phi(t^{-1})))} \exp\Big(-\frac{c_{13} l^2}{\Phi^{-1}(1/\phi(t^{-1}))^2}\Big). \qquad (6.2.9)
\end{aligned}$$

Moreover, using (H1), (H2), integration by parts, Proposition 2.2.1, Lemmas 4.1.1 and 6.2.2

$$\begin{aligned}
&-\int_{c_*/\phi(t^{-1})}^{4\Phi(l)} \frac{h(r,x,y)}{\widetilde{V}(\widetilde{\Phi}^{-1}(r))} \exp\Big(-\frac{a_1 l^2}{\widetilde{\Phi}^{-1}(r)^2}\Big) d_r\mathbb{P}(S_r \leq t) \\
&\leq -c_{14} \frac{h(1/\phi(t^{-1}),x,y)}{V(\Phi^{-1}(1/\phi(t^{-1})))} \int_{c_*/\phi(t^{-1})}^{4\Phi(l)} \exp\Big(-\frac{a_1 l^2}{\widetilde{\Phi}^{-1}(r)^2}\Big) d_r\mathbb{P}(S_r \leq t) \\
&\leq c_{14} \frac{h(1/\phi(t^{-1}),x,y)}{V(\Phi^{-1}(1/\phi(t^{-1})))} \exp\Big(-\frac{c_{15} a_1 l^2}{\Phi^{-1}(1/\Phi(t^{-1}))^2}\Big)
\end{aligned}$$

$$+ c_{16}\frac{h(1/\phi(t^{-1}),x,y)}{V(\Phi^{-1}(1/\phi(t^{-1})))}\int_{c_*/\phi(t^{-1})}^{4\Phi(l)} \frac{l^2}{r\Phi^{-1}(r)^2}e^{-c_{18}\mathfrak{e}_1(r)-\mathfrak{e}_2(r)}dr, \quad (6.2.10)$$

where $\mathfrak{e}_1(r) = l^2/\Phi^{-1}(r)^2$ and $\mathfrak{e}_2(r) = r(H\circ\sigma)(r,t)$. Observe that $\mathfrak{e}_1$ is decreasing and $\mathfrak{e}_1(\Phi(l)) = 1$, while $\mathfrak{e}_2$ is increasing and $\mathfrak{e}_2(\bar{b}^{-1}(t)) = 1$. It follows that there exist unique $r_* \in (\bar{b}^{-1}(t), \Phi(l))$ such that $\mathfrak{e}_1(r_*) = \mathfrak{e}_2(r_*)$ and $\mathfrak{e}_1(r) + \mathfrak{e}_2(r) > \mathfrak{e}_1(r_*)$ for all $r \in (\bar{b}^{-1}(t), 4\Phi(l))$. Therefore, using the inequality $\sup_{x>0} x^k e^{-x} < \infty$ for all $k > 0$, the scaling of Φ and (6.2.7), we get that

$$\begin{aligned}
&\int_{c_*/\phi(t^{-1})}^{4\Phi(l)} \frac{l^2}{r\Phi^{-1}(r)^2}e^{-c_{18}\mathfrak{e}_1(r)-\mathfrak{e}_2(r)}dr \\
&\le c_{19}\int_{\bar{b}^{-1}(t)}^{4\Phi(l)} \frac{l^2}{r\Phi^{-1}(r)^2}\left(\frac{\Phi^{-1}(r)}{l}\right)^{2+1/\alpha_1} e^{-2^{-1}c_{18}\mathfrak{e}_1(r)-\mathfrak{e}_2(r)}dr \\
&\le \frac{c_{20}}{\Phi(l)}e^{-(2^{-1}c_{18}\wedge 1)\mathfrak{e}_1(r_*)}\int_{\bar{b}^{-1}(t)}^{4\Phi(l)} dr \le 4c_{20}e^{-(2^{-1}c_{18}\wedge 1)\mathfrak{e}_1(r_*)}. \qquad (6.2.11)
\end{aligned}$$

By the definition of r_*, we see that

$$\mathfrak{e}_1(r_*) \le \mathfrak{e}_1(1/\phi(t^{-1}))\wedge\mathfrak{e}_2(\Phi(l)) = \left(\frac{l^2}{\Phi^{-1}(1/\phi(t^{-1}))^2}\right)\wedge\Big(\Phi(l)(H\circ\sigma)(\Phi(l),t)\Big). \quad (6.2.12)$$

Hence, we deduce from (6.2.8), (6.2.9), (6.2.10) and (6.2.11) that

$$J_3 \le c_{19}\frac{h(1/\phi(t^{-1}),x,y)}{V(\Phi^{-1}(1/\phi(t^{-1})))}e^{-c_{20}\mathfrak{e}_1(r_*)}.$$

Also, using (H1), the scalings of V and Φ, and (6.2.12), since the subordinator S is increasing, we get that

$$\begin{aligned}
J_4 &\le c_{21}\frac{h(1/\phi(t^{-1}),x,y)}{V(\Phi^{-1}(1/\phi(t^{-1})))}\mathbb{P}(S_{4\Phi(l)} \le t) \le c_{21}\frac{h(1/\phi(t^{-1}),x,y)}{V(\Phi^{-1}(1/\phi(t^{-1})))}\mathbb{P}(S_{\Phi(l)} \le t) \\
&\le ec_{21}\frac{h(1/\phi(t^{-1}),x,y)}{V(\Phi^{-1}(1/\phi(t^{-1})))}\exp\Big(-\Phi(l)\,(H\circ\sigma)(\Phi(l),t)\Big) \\
&\le ec_{21}\frac{h(1/\phi(t^{-1}),x,y)}{V(\Phi^{-1}(1/\phi(t^{-1})))}e^{-\mathfrak{e}_1(r_*)}.
\end{aligned}$$

Now, we estimates the value of $\mathcal{M}(t,l) := \mathfrak{e}_1(r_*)$. Set $s_* = r_*/\mathfrak{e}_2(r_*) = 1/(H\circ\sigma)(r_*,t)$. Then, $b(s_*) = t/r_*$. Hence, we get from (2.0.7) that

$$\frac{1}{\phi^{-1}(c/s_*)} \asymp s_* b(s_*) = \frac{r_*}{\mathfrak{e}_2(r_*)}\frac{t}{r_*} = \frac{t}{\mathcal{M}(t,l)},$$

which yields that

$$\frac{\mathcal{M}(t,l)/t}{\phi(\mathcal{M}(t,l)/t)} \simeq \frac{s_*\mathcal{M}(t,l)}{t} = \frac{r_*}{t}.$$

On the other hand, by the definition, it holds that

$$\frac{\Phi(l/\sqrt{\mathcal{M}(t,l)})}{t} = \frac{\Phi(l/\sqrt{\mathfrak{e}_1(r_*)})}{t} = \frac{r_*}{t}.$$

Therefore, by the scaling of ϕ and Φ, we conclude that $\mathcal{M}(t,l)$ is comparable with the function $\mathcal{N}(t,l)$ defined by (6.1.4) and get the desired upper bound.

For the lower bound, since $\phi(t^{-1})^{-1} < 4\Phi(l)$, by Lemmas 4.1.9 and 6.2.1, we see that

$$p(t,x,y) \geq c_{22}\frac{C_0 h(1/\phi(t^{-1}),x,y)}{\phi(t^{-1})V(l)\Psi(l)}.$$

Besides, we observe that by Proposition 2.2.1, there exist constants $A > 1$ and $\epsilon > 0$ independent of t and l such that $\mathbb{P}(S_{r_*} \leq t) \geq (1+\epsilon)\mathbb{P}(S_{Ar_*} \leq t)$. Using (H1), (H2), the scalings of V and Φ, and Proposition 2.2.1, we get that

$$\begin{aligned} p(t,x,y) &\geq -c_{23}\int_{r_*}^{Ar_*} \frac{h(r,x,y)}{V(\Phi^{-1}(r))}\exp\Big(-\frac{a_2 l^2}{\Phi^{-1}(r)^2}\Big)d_r\mathbb{P}(S_r \leq t) \\ &\geq c_{23}\frac{h(1/\phi(t^{-1}),x,y)}{V(\Phi^{-1}(1/\phi(t^{-1})))}e^{-c_{24}\mathfrak{e}_1(r_*)-c_{25}\mathfrak{e}_2(Ar_*)} \\ &\geq c_{23}\frac{h(1/\phi(t^{-1}),x,y)}{V(\Phi^{-1}(1/\phi(t^{-1})))}e^{-c_{26}\mathfrak{e}_1(r_*)}. \end{aligned}$$

The proof is complete. □

Proof of Theorem 6.1.3. By repeating arguments in the proof of Theorem 6.1.1, since the integrals of $e^{-\lambda_D r}$ are not the dominant term in all cases, we can conclude the result. We omit details here. □

Proof of Theorem 6.1.4. By repeating arguments in the proof of Theorem 6.1.1, we can deduce that for all $(t,x,y) \in [T,\infty) \times D \times D$,

$$p(t,x,y) \asymp h(1,x,y)e^{-c\lambda_D/\phi(t^{-1})} + w(t)\int_{\Phi(\rho(x,y))}^{2\Phi(\mathrm{diam}(D))} \frac{h(r,x,y)}{V(x,\Phi^{-1}(r))}dr.$$

Since $h(r,x,y)/V(x,\Phi^{-1}(r)) \geq c$ for $r < 2\Phi(\mathrm{diam}(D))$ by (H1), (H2) and the scaling of V and Φ, and $\lim_{t\to\infty} w(t)e^{a\lambda_D/\phi(t^{-1})} = \infty$ for every $a > 0$ by $\mathbf{Poly}^{\infty}_{R_2,\leq}(\beta_2)$, we conclude that the second term in the above equation dominates the first term. Thus, we get the desired result. □

Proof of Theorem 6.1.5. Since $\int_0^\infty w(s)ds < \infty$ under $\mathbf{Sub}^{\infty}(\gamma,\theta)$, by (2.0.6), for every $T > 0$, there are comparability constants depend on T such that $\phi(t^{-1}) \simeq t^{-1}$ for $t \geq T$. Using this fact, by following arguments in the proof of Theorem 6.1.1, using Theorem 2.2.8 (when $\gamma = 1$) and Theorem 2.2.9 (when $\gamma < 1$), we arrive at the result. We omit details here. □

To get Theorem 6.1.6, we first prove large time estimates for $p(t,x,y)$ under $\mathbf{Trun}^{\infty}_{R_2}$.

Lemma 6.2.4. *Suppose that* $\mathbf{Trun}^{\infty}_{R_2}$ *holds, and either* $\mathbf{HK^h_B}$ *or* $\mathbf{HK^h_U}$ *holds with* $h = h_p$, $p \in [0,1)$*. If* $t \geq ((1/2) \vee \lfloor d_2/\alpha_1 + 2p \rfloor)R_2$*, then*

$$p(t,x,y) \asymp q(ct,x,y) \quad \textit{for } x,y \in D.$$

Proof. Since the proofs are similar, we only give the proof when $\mathbf{HK^h_U}$ holds, which is more complicated.

We fix $t \geq ((1/2) \vee \lfloor d_2/\alpha_1 + 2p \rfloor)R_2$ and $x,y \in D$, and then write $l := \rho(x,y)$. Since $\mathbf{Sub}^{\infty}(1,1)$ is satisfied under $\mathbf{Trun}^{\infty}_{R_2}$, by Theorem 6.1.5, if $\Phi(l)\phi(t^{-1}) > 1/4$, then we arrive at the result.

Now, we assume that $\Phi(l)\phi(t^{-1}) < 1/4$. Since $\int_0^\infty w(s)ds < \infty$ under $\mathbf{Trun}^{\infty}_{R_2}$, as in the proof of Theorem 6.1.5, we see that $\phi(s^{-1}) \simeq s^{-1}$ for $s \geq ((1/2) \vee \lfloor d_2/\alpha_1 + 2p \rfloor)R_2$. Therefore, we get the desired lower bound from Lemma 6.2.1. Moreover, since we assumed $\Phi(l)\phi(t^{-1}) < 1/4$, it holds that $\Phi(l) < t/c_1$. Then by $\mathbf{HK^h_U}$, we get $q(t,x,y) \simeq h_p(t,x,y)/V(x,\Phi^{-1}(t))$.

Let $L > 0$ be the constant in Theorem 2.2.10(iii). By (4.1.10) and Lemma 4.1.9, since $\Psi \geq \Phi$, we get that

$$\begin{aligned} p(t,x,y) &\leq c_2 \int_0^{(c_1\Phi(l)/L)\wedge 1} \frac{r h_p(r,x,y)}{V(x,l)\Phi(l)} d_r\mathbb{P}(S_r \geq t) \\ &\quad + c_2 \int_{(c_1\Phi(l)/L)\wedge 1}^{t/L} \frac{h_p(r,x,y)}{V(x,\Phi^{-1}(r))} d_r\mathbb{P}(S_r \geq t) \\ &\quad + c_2 \int_{t/L}^{\infty} \frac{h_p(r,x,y)}{V(x,\Phi^{-1}(r))} d_r\mathbb{P}(S_r \geq t) =: c_2(K_1 + K_2 + K_3). \end{aligned}$$

By the scalings of V and Φ, we have that $K_3 \leq h_p(t/L,x,y)/V(x,\Phi^{-1}(t/L)) \leq c_3 q(t,x,y)$. Besides, using the fact that $r \mapsto r^{2p}h(r,x,y)$ is increasing, integration by parts and Theorem 2.2.10(i)-(ii), since $\lfloor t/R_2 \rfloor \geq \lfloor d_2/\alpha_1 + 2p \rfloor$, we see that

$$\begin{aligned} K_1 &\leq c_4 \frac{\Phi(l)^{2p}\, h_p(\Phi(l),x,y)}{V(x,l)\Phi(l)} \int_0^{(c_1\Phi(l)/L)\wedge 1} r^{1-2p} d_r\mathbb{P}(S_r \geq t) \\ &\leq c_5 \frac{h_p(\Phi(l),x,y)}{V(x,l)} \mathbb{P}(S_{(c_1\Phi(l)/L)\wedge 1} \geq t) \\ &\quad + c_5 \frac{\Phi(l)^{2p}\, h_p(\Phi(l),x,y)}{V(x,l)\Phi(l)} \int_0^{(c_1\Phi(l)/L)\wedge 1} r^{-2p}\mathbb{P}(S_r \geq t) dr \\ &\leq c_6 \frac{\Phi(l)^{d_2/\alpha_1+2p}\, h_p(\Phi(l),x,y)}{V(x,l)} e^{-c_7 t\log t} \\ &\quad + c_6 \frac{\Phi(l)^{2p}\, h_p(\Phi(l),x,y)}{V(x,l)\Phi(l)} e^{-c_7 t\log t} \int_0^{(c_1\Phi(l)/L)\wedge 1} r^{-2p} r^{d_2/\alpha_1+2p} dr \\ &\leq c_8 \frac{\Phi(l)^{d_2/\alpha_1+2p}\, h_p(\Phi(l),x,y)}{V(x,l)} e^{-c_7 t\log t}. \end{aligned}$$

Using the fact that $r \mapsto r^{2p}h(r,x,y)$ is increasing again, and the scalings of V and Φ, we see that

$$\frac{\Phi(l)^{d_2/\alpha_1+2p}\, h_p(\Phi(l),x,y)}{V(x,l)} \leq t^{2p} h_p(t,x,y) \frac{\Phi(l)^{d_2/\alpha_1}}{V(x,l)}$$

$$\le c_9 t^{2p} h_p(t,x,y) \frac{(\Phi(1) l^{\alpha_1})^{d_2/\alpha_1}}{l^{d_2} V(x,1)} = c_{10} t^{2p} h_p(t,x,y).$$

By the scalings of V and Φ, we also see that for every $a > 0$,

$$\lim_{s\to\infty} s^{2p} e^{-as(1\wedge \log s)} V(x, \Phi^{-1}(s)) \le c \lim_{s\to\infty} s^{2p+d_2/\alpha_1} e^{-as(1\wedge\log s)} V(x,\Phi^{-1}(1)) = 0. \tag{6.2.13}$$

In the end, we deduce that

$$K_1 \le c_{11} h_p(t,x,y) t^{2p} e^{-c_7 t \log t} \le c_{12} \frac{h_p(t,x,y)}{V(x,\Phi^{-1}(t))} \le c_{13} q(t,x,y).$$

Lastly, using the fact that $r \mapsto r^{2p} h(r,x,y)$ is increasing, Lemmas 4.1.1 and 6.2.2, integration by parts, Theorem 2.2.10(i)-(iii), the scalings of V and Φ, Lemma 1.1.1(i) and (6.2.13), we obtain that

$$\begin{aligned}
K_2 &\le c_{14} t^{2p} h_p(t,x,y) \int_{(c_1\Phi(l)/L)\wedge 1}^{t/L} \frac{1}{r^{2p}\widetilde{V}(x,\widetilde{\Phi}^{-1}(r))} d_r \mathbb{P}(S_r \ge t) \\
&\le c_{14} \frac{L^{2p} h_p(t/L,x,y)}{\widetilde{V}(x,\widetilde{\Phi}^{-1}(t/L))} \\
&\quad + c_{15} t^{2p} h_p(t,x,y) \int_{(c_1\Phi(l)/L)\wedge 1}^{t/L} \frac{1}{r^{2p+1}\widetilde{V}(x,\widetilde{\Phi}^{-1}(r))} \mathbb{P}(S_r \ge t) dr \\
&\le c_{16} q(t,x,y) + c_{16} t^{2p} e^{-c_{17} t\log t} h_p(t,x,y) \int_{(c_1\Phi(l)/L)\wedge 1}^{1} \frac{r^{\lfloor d_2/\alpha_1 + 2p\rfloor + 1}}{r^{2p+1} V(x,\Phi^{-1}(r))} dr \\
&\quad + c_{16} t^{2p} h_p(t,x,y) \int_1^{t/L} \frac{1}{r^{2p+1}\widetilde{V}(x,\widetilde{\Phi}^{-1}(r))} e^{-c_{17} t \log(t/r)} dr \\
&\le c_{16} q(t,x,y) + c_{18} t^{2p} e^{-c_{17} t (L\wedge \log t)} h_p(t,x,y) \frac{1}{V(x,\Phi^{-1}(1))} \le c_{19} q(t,x,y).
\end{aligned}$$

The proof is complete. □

Proof of Theorem 6.1.6. By Lemma 6.2.4, it remains to prove for $t \le \lfloor d_2/\alpha_1 + 2p \rfloor R_2$. For that case, by repeating arguments in the proof of Theorem 6.1.1, using Theorem 2.2.10 instead of Theorem 2.2.6, and the fact that $\phi(t^{-1}) \simeq t^{-1}$ for $t \le \lfloor d_2/\alpha_1 + 2p \rfloor R_2$, we arrive at the result. We omit details here. □

Chapter 7

Dirichlet heat kernel estimates for Lévy processes with low intensity of small jumps

This chapter is concerned with Dirichlet heat kernel estimates for a isotropic unimodal Lévy process Y with low intensity of small jumps. Typical examples of such processes are geometric stable processes and iterated geometric stable processes. (See, e.g., [17, Page 112] for the definitions of these processes.) The results in this chapter are based on [53].

In this capter, we first derive small heat kernel estimates in $\mathbb{R}^d$ by using the results and methods from [76]. Next, we study behaviours of the process near the boundary of a $C^{1,1}$ open subset D of $\mathbb{R}^d$. Under a set of conditions that give the boundary Harnack principle (see condition **(B)** below), we obtain two-sided estimates on the survival probability in D with explicit boundary decay. Using heat kernel estimates in the whole space and boundary behaviours of the process, we establish small time two-sided Dirichlet heat kernel estimates for isotropic unimodal Lévy processes in $C^{1,1}$ open sets. In particular, we prove the following factorization formula: For every $T > 0$, there are comparability constants such that for all $(t,x,y) \in [T,\infty) \times D \times D$,

$$p_D(t,x,y) \asymp \mathbb{P}^x(\tau_D > t)\mathbb{P}^y(\tau_D > t)\, p(ct,x,y), \tag{7.0.1}$$

where $p_D(t,x,y)$ is Dirichlet heat kernel in D, $p(t,x,y)$ the heat kernel of the free process, and $\tau_D := \inf\{t > 0 : Y_t \notin D\}$ the first exit time (see Theorem 7.1.1 below). Cf. Theorem 3.1.20. Since the heat kernel $p(t,x,y)$ may not be bounded, in the proof of (7.0.1), we need different arguments from ones for the proof of Theorem 3.1.20.

When D is a bounded $C^{1,1}$ open subset of $\mathbb{R}^d$, we also obtain large time estimates for $p_D(t,x,y)$, and two-sided estimates on the Green function in D. Since the killed semigroup $(P_t^D)_{t\geq 0}$ may not be compact operators for all $t > 0$ even for bounded D, our method is different from ones for obtaining large time Dirichlet heat kernel estimates of stable processes.

7.1 Setup and main results

Let $Y = (Y_t)_{t\geq 0}$ be a Lévy process in $\mathbb{R}^d$ with the Lévy-Khintchine exponent ψ, that is,

$$\mathbb{E}\Big[\exp\big(i\langle \xi, Y_t\rangle\big)\Big] = \int_{\mathbb{R}^d} e^{i\langle \xi, x\rangle} p(t,dx) = e^{-t\psi(\xi)}, \quad \xi \in \mathbb{R}^d,$$

where $p(t,dx) = p(t,0,dx)$ is the transition probability of Y. If Y is a pure jump symmetric Lévy process with Lévy measure ν, then ψ is of the form

$$\psi(\xi) = \int_{\mathbb{R}^d} (1 - \cos\langle \xi, x\rangle)\nu(dx), \quad \xi \in \mathbb{R}^d,$$

where $\int_{\mathbb{R}^d}(1 \wedge |x|^2)\nu(dx) < \infty$.

A measure $\mu(dx)$ is isotropic unimodal if it is absolutely continuous on $\mathbb{R}^d \setminus \{0\}$ with a radial and radially nonincreasing density. A Lévy process Y is isotropic unimodal if $p(t,dx)$ is isotropic unimodal for all $t > 0$. When Y is a pure jump Lévy process, Y is isotropic unimodal if and only if the Lévy measure $\nu(dx)$ of Y is isotropic unimodal. See, [126].

Throughout the chapter, we always assume that Y is a pure jump isotropic unimodal Lévy process with the Lévy-Khintchine exponent ψ. With a slight abuse of notation, we will use the notations $\psi(|x|) = \psi(x)$, $\nu(dx) = \nu(x)dx = \nu(|x|)dx$ and $p(t, dx) = p(t, x)dx = p(t, |x|)dx$ for $x \in \mathbb{R}^d$ and $t > 0$. Then, throughout the chapter, we also assume that the following condition **(A)** holds true:

(A) $\nu(\mathbb{R}^d) = \infty$ and there exist constants $-d < \alpha_1 \le \alpha_2 < 2, c_1, c_2, \kappa_1, \kappa_2 > 0$, and a continuous function $\ell : (0, \infty) \to (0, \infty)$ satisfying

$$c_1 \left(\frac{R}{r}\right)^{\alpha_1} \le \frac{\ell(R)}{\ell(r)} \le c_2 \left(\frac{R}{r}\right)^{\alpha_2} \quad \text{for all } 1 \le r \le R \tag{7.1.1}$$

such that

$$\kappa_1 r^{-d} \ell(r^{-1}) \le \nu(r) \le \kappa_2 r^{-d} \ell(r^{-1}) \quad \text{for all } r > 0. \tag{7.1.2}$$

If $d > 1$, then we assume further that either $\alpha_1 > -1$ or $\psi(\xi) = \varphi(|\xi|^2)$ for a Bernstein function φ.

Note that, since we allow the constant α_1 to be negative, the map $r \mapsto \ell(r^{-1})$ can be increasing near zero.

Here, we enumerate other main conditions which we will assume later. We say that a given function f is almost increasing if there exists $c_1 > 0$ such that $f(x) \simeq \sup_{y \in [c_0, x]} f(y)$ for $x > c_1$, and f is almost decreasing if there exists $c_2 > 0$ such that $f(x) \simeq \inf_{y \in [c_0, x]} f(y)$ for $x > c_2$.

(B) $\nu(r)$ is absolutely continuous such that $r \mapsto -\nu'(r)/r$ is nonincreasing on $(0, \infty)$ and there exists $c > 1$ such that $\nu(r) \le c\nu(r + 1)$ for all $r \ge 1$.

(C) There exist constants $\gamma < 2$ and $c_1 > 0$ such that

$$\frac{\ell(R)}{\ell(r)} \le c_1 \left(\frac{R}{r}\right)^{\gamma} \quad \text{for all } 0 < r \le R \le 1.$$

(S-1) $\limsup_{r\to\infty} \ell(r) < \infty$;

(S-2) $\limsup_{r\to\infty} \ell(r) = \infty$ and $\ell(r)$ is almost increasing;

(L-1) $\liminf_{r\to\infty} \ell(r) = 0$ and $\ell(r)$ is almost decreasing;

(L-2) $0 < \liminf_{r\to\infty} \ell(r) \le \limsup_{r\to\infty} \ell(r) < \infty$;

(D) If $d = 1$, then $\alpha_2 < 1$ where α_2 is the constant in **(A)**.

We define for $r > 0$,

$$\begin{aligned} K(r) &:= r^{-2} \int_0^r s\ell(s^{-1})ds, \qquad L(r) := \int_r^\infty s^{-1}\ell(s^{-1})ds, \\ h(r) &:= K(r) + L(r). \end{aligned} \tag{7.1.3}$$

Then we see from **(A)** that

$$K(r) \simeq r^{-2} \int_{|y|\le r} |y|^2\, \nu(y)dy \quad \text{and} \quad L(r) \simeq \int_{|y|>r} \nu(y)dy \quad \text{for } r > 0.$$

We also define

$$\ell^*(r) := \sup_{u\in[1,r]} \ell(u) \quad \text{for } r \ge 1, \quad \ell^{-1}(t) := \inf\{r \ge 1 : \ell^*(r) > t\} \quad \text{for } t > 0$$

and for $a > 0$,

$$\theta_a(r,t) := r \vee [\ell^{-1}(a/t)]^{-1} \quad \text{for } r, t > 0.$$

Now, we state our main results. For a Borel subset D of $\mathbb{R}^d$, denote by $p_D(t, x, y)$ the Dirichlet heat kernel of Y in D.

Theorem 7.1.1. *Suppose that Y is a pure jump isotropic unimodal Lévy process satisfying* **(A)** *and* **(B)**. *Let D be a $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics (R_0, Λ). If D is unbounded, we further assume that* **(C)** *holds. Then for every $T > 0$, the following estimates are valid for all $(t, x, y) \in [T, \infty)\times D\times D$.*

(i) If **(S-1)** *holds, then*

$$
\begin{aligned}
p_D(t,x,y) &\asymp \left(1 \wedge \frac{1}{tL(\delta_D(x))}\right)^{1/2} \left(1 \wedge \frac{1}{tL(\delta_D(y))}\right)^{1/2} p(ct,x,y) \\
&\asymp \left(1 \wedge \frac{1}{tL(\delta_D(x))}\right)^{1/2} \left(1 \wedge \frac{1}{tL(\delta_D(y))}\right)^{1/2} t\nu(|x-y|)e^{-cth(|x-y|)}.
\end{aligned}
$$

(ii) If **(S-2)** *holds, then there exist constants $a_0 = a_0(d,\psi) > 0$ and $c_1, c_2 > 0$ such that*

$$
\begin{aligned}
p_D(t,x,y) \le c_1 &\left(1 \wedge \frac{1}{tL(\delta_D(x))}\right)^{1/2} \left(1 \wedge \frac{1}{tL(\delta_D(y))}\right)^{1/2} \\
&\times t\nu(\theta_{a_0}(|x-y|,t)) \exp\big(-c_2 th(\theta_{a_0}(|x-y|,t))\big).
\end{aligned}
$$

Also, for every $\eta > 0$, there exist constants $c_3, c_4 > 0$ such that

$$
\begin{aligned}
p_D(t,x,y) \ge c_3 &\left(1 \wedge \frac{1}{tL(\delta_D(x))}\right)^{1/2} \left(1 \wedge \frac{1}{tL(\delta_D(y))}\right)^{1/2} \\
&\times t\nu(\theta_{\eta}(|x-y|,t)) \exp\big(-c_4 th(\theta_{\eta}(|x-y|,t))\big).
\end{aligned}
$$

Moreover, the following factorization formula holds true:

$$
p_D(t,x,y) \asymp \left(1 \wedge \frac{1}{tL(\delta_D(x))}\right)^{1/2} \left(1 \wedge \frac{1}{tL(\delta_D(y))}\right)^{1/2} p(ct,x,y).
$$

Below, we assume that D is bounded and obtain the large time estimates for the Dirichlet heat kernel and the Green function estimates under some mild assumptions.

Definition 7.1.2. A bounded set D in $\mathbb{R}^d$ is said to be of scale (r_1, r_2) if there exist $x_1, x_2 \in \mathbb{R}^d$ such that $B(x_1, r_1) \subset D \subset B(x_2, r_2)$.

Theorem 7.1.3. *Suppose that Y is a pure jump isotropic unimodal Lévy process satisfying* **(A)** *and* **(B)**. *Let D be a bounded $C^{1,1}$ open set in $\mathbb{R}^d$ with* *characteristics* (R_0, Λ) *of scale* (r_1, r_2)*. Then, the following estimates hold:*

(i) If **(L-1)** *holds, then for every* $T > 0$*, there exist constants* $a_1, a_2 > 0$ *which only depend on the dimension* d*, constants* $c_1, c_2 > 0$ *independent of* T *and* $c_3 = c_3(T) > 1$ *such that*

$$\begin{aligned}
&\frac{c_3^{-1}}{L(\delta_D(x))^{1/2}L(\delta_D(y))^{1/2}}\Big(\nu(|x-y|)e^{-c_1 th(|x-y|)} + e^{-a_1\kappa_2 th(r_1/2)}\Big) \\
&\le p_D(t,x,y) \\
&\le \frac{c_3}{L(\delta_D(x))^{1/2}L(\delta_D(y))^{1/2}}\Big(\nu(|x-y|)e^{-c_2 th(|x-y|)} + e^{-a_2\kappa_1 th(r_2)}\Big),
\end{aligned}$$

for all $(t,x,y) \in [T,\infty) \times D \times D$*, where* κ_1 *and* κ_2 *are the constants in* **(A)**.

(ii) If **(L-2)** *holds, then there exist* $T_1 \ge 0$ *and* $\lambda_1 = \lambda_1(\psi, D) > 0$ *such that for every* $T > T_1$*, there exists* $c_4 > 1$ *such that*

$$c_4^{-1}\frac{e^{-\lambda_1 t}}{L(\delta_D(x))^{1/2}L(\delta_D(y))^{1/2}} \le p_D(t,x,y) \le c_4\frac{e^{-\lambda_1 t}}{L(\delta_D(x))^{1/2}L(\delta_D(y))^{1/2}},$$

for all $(t,x,y) \in [T,\infty) \times D \times D$*. Moreover, we have*

$$\frac{\kappa_1 C_5}{2}h(r_2) \le \lambda_1 \le \kappa_2 C_4 h(r_1/2).$$

(iii) If **(S-2)** *holds, then the estimates in (ii) holds with* $T_1 = 0$*. Moreover, the constant* $-\lambda_1 < 0$ *is the largest eigenvalue of the generator of* Y^D*.*

For a Borel subset D of $\mathbb{R}^d$, the Green function $G_D(x,y)$ of Y in D is defined by

$$G_D(x,y) := \int_0^\infty p_D(t,x,y)dt.$$

Theorem 7.1.4. *Suppose that* Y *is a pure jump isotropic unimodal Lévy process satisfying* **(A)**, **(B)** *and* **(D)**. *Let* D *be a bounded* $C^{1,1}$ *open subset in* $\mathbb{R}^d$ *with characteristics* (R_0, Λ) *of scale* (r_1, r_2)*. Then, the Green function*

$G_D(x,y)$ in D satisfies the following two-sided estimates: for $x,y \in D$,

$$G_D(x,y) \simeq \left(1 \wedge \frac{L(|x-y|)}{\sqrt{L(\delta_D(x))L(\delta_D(y))}}\right) \frac{\ell(|x-y|^{-1})}{|x-y|^d L(|x-y|)^2}, \qquad (7.1.4)$$

with comparability constants depend only on d, ψ, R_0, Λ and r_2.

Remark 7.1.5. One can obtain (7.1.4) just by integrating the estimates for $p_D(t,x,y)$ given in Theorems 7.1.1 and 7.1.3. However, to use both Theorems 7.1.1 and 7.1.3, we need conditions more than **(A)**, **(B)** and **(D)**. By adopting arguments from [87] instead of integrating the Dirichlet heat kernel, we obtained the Green function estimates in more general situations.

7.2 Heat kernel estimates in $\mathbb{R}^d$

Recall the definitions of the functions K, L and h from (7.1.3). Clearly, L and h are nonincreasing. Since $\nu(r)$ is nonincreasing, it holds that

$$K(r) \geq c\nu(r)r^{-2}\int_0^r s^{d+1}ds = c_1 r^d \nu(r) \quad \text{for all } r > 0.$$

Moreover, using Lemma 1.1.1(i), since we assumed that condition **(A)** holds true, we see that

$$K(r) \simeq r^d \nu(r) \simeq \ell(r^{-1}) \quad \text{for } 0 < r \leq 1, \qquad (7.2.1)$$

and if condition **(C)** also holds, then

$$K(r) \simeq r^d \nu(r) \simeq \ell(r^{-1}) \quad \text{for } r \geq 1. \qquad (7.2.2)$$

By applying (1.1.2) to the function L, we see from (7.2.1) that $L(r) \geq cK(r)$ for $0 < r \leq 1$. Since $h(r) = K(r) + L(r)$, it follows that

Lemma 7.2.1. *There exists a constant $c_1 > 0$ such that*

$$L(r) \leq h(r) \leq c_1 L(r) \quad \textit{for all } 0 < r \leq 1.$$

By [20, (6) and (7)], there exist positive constants C_0 and C_1 which only depend on the dimension d, and the constants κ_1, κ_2 in (7.1.2) such that

$$C_0 h(r) \le \psi(r^{-1}) \le C_1 h(r) \quad \text{for all } \ r > 0. \tag{7.2.3}$$

To make some computations easier, we define $\Phi : [0, \infty) \to [0, \infty)$ by

$$\Phi(r) := L(r^{-1}) = \int_0^r s^{-1}\ell(s)ds.$$

Then we get the following lemma from (7.1.1), (7.2.3) and Lemma 7.2.1.

Lemma 7.2.2. *(i) There exist constants $c_1, c_2 > 0$ such that*

$$c_1\left(\frac{R}{r}\right)^{\alpha_1} \le \frac{\Phi(R)}{\Phi(r)} \le c_2\left(\frac{R}{r}\right)^{\alpha_2\vee(1/2)} \quad \textit{for all } 1 \le r \le R.$$

(ii) With the constant C_0 in (7.2.3), $C_0\Phi(r) \le \psi(r)$ for all $r \ge 0$. Moreover, there exists a constant $C_2 > 0$ such that $C_2\Phi(r) \ge h(r^{-1})$ for all $r \ge 1$.

In [79], Hartman and Wintner proved sufficient conditions in terms of the Lévy exponent ψ under which the transition density $p(t,\cdot)$ of Y is in $C_0(\mathbb{R}^d)$. Then, in [99], Knopova and Schilling improve that result and they also give some necessary conditions. Using Lemma 7.2.2(ii), we can formulate these conditions in terms of Φ. Since Y is isotropic unimodal, these conditions determine whether $p(t, 0) < \infty$ or $p(t, 0) = \infty$.

Proposition 7.2.3. *Let*

$$b_1 := \liminf_{r\to\infty} \frac{\Phi(r)}{\log(1+r)} \in [0,\infty] \quad \textit{and} \quad b_2 := \limsup_{r\to\infty} \frac{\Phi(r)}{\log(1+r)} \in [0,\infty].$$

(i) If $b_1 = \infty$, then $p(t, 0) < \infty$ for all $t > 0$.
(ii) If $b_2 = 0$, then $p(t, 0) = \infty$ for all $t > 0$.
(iii) If $0 < b_1 \le b_2 < \infty$, then there exist constants $T_2 \ge T_1 > 0$ such that $p(t, 0) = \infty$ for $0 < t \le T_1$ and $p(t, 0) < \infty$ for $t > T_2$.

In particular, by l'Hospital's rule, the following are true.

(i') If $\liminf_{r\to\infty}\ell(r)=\infty$*, then* $p(t,0)<\infty$ *for all* $t>0$.
(ii') If $\limsup_{r\to\infty}\ell(r)=0$*, then* $p(t,0)=\infty$ *for all* $t>0$.
(iii') If $0<\liminf_{r\to\infty}\ell(r)\le\limsup_{r\to\infty}\ell(r)<\infty$*, then there exist* $T_2\ge T_1>0$ *such that* $p(t,0)=\infty$ *for* $0<t\le T_1$ *and* $p(t,0)<\infty$ *for* $t>T_2$.

Here, we introduce some general estimates which are established in [76]. Note that the following estimates hold no matter $p(t,0)<\infty$ or $p(t,0)=\infty$.

Proposition 7.2.4 ([76, Proposition 5.3])**.** *There are constants* $b_0, c_0>0$*, which only depend on the dimension* d *and the constant* κ_2 *in* (7.1.2) *such that for all* $t>0$ *and* $x\in\mathbb{R}^d$*,*

$$p(t,x)\ge c_0t\nu(|x|)\exp\big(-b_0th(|x|)\big).$$

Proposition 7.2.5 ([76, Theorem 5.4])**.** *There is a constant* $c_1>0$*, which only depends on the dimension* d *and* κ_2 *in* (7.1.2) *such that for all* $t>0$ *and* $x\in\mathbb{R}^d\setminus\{0\}$*,*

$$p(t,x)\le c_1t|x|^{-d}K(|x|).$$

The following lemma will be used several times to obtain heat kernel upper bounds for the whole space. (Cf. [76, Lemma 4.1 and Corollary 4.4].)

Lemma 7.2.6. *For every* $\lambda>1$*, there exists a constant* $c=c(\lambda)>0$ *such that*

$$\sup_{1<k\le\lambda}|\psi(kr)-\psi(r)|\le c\,\ell(r)\quad\text{for all } r\ge1. \tag{7.2.4}$$

Proof. Recall condition **(A)**. We first assume that either $d=1$ or $\alpha_1>-1$. For $y>0$, set $\nu_1(y)=\nu(y)$ if $d=1$, and

$$\nu_1(y):=\int_{\mathbb{R}^{d-1}}\nu\big((y^2+|z|^2)^{1/2}\big)dz\quad\text{if } d\ge2.$$

We claim that there exists a constant $c_1>0$ such that

$$\nu_1(y)\le c_1y^{-1}\ell(y^{-1})\quad\text{for all } y\in(0,1]. \tag{7.2.5}$$

If $d = 1$, then (7.2.5) follows from (7.1.2). Suppose that $\alpha_1 > -1$ and $d \geq 2$. Using (7.1.1) and a change of the variables, we get from (7.1.2) that for any $y \in (0, 1]$,

$$\frac{1}{y^{-1}\ell(y^{-1})}\int_0^1 \nu\big((y^2+k^2)^{1/2}\big)k^{d-2}dk \simeq \int_0^{1/y} \frac{k^{d-2}}{(1+k^2)^{d/2}}\frac{\ell(y^{-1}(1+k^2)^{-1/2})}{\ell(y^{-1})}dk$$
$$\leq c_2\int_0^{1/y}\frac{k^{d-2}}{(1+k^2)^{(\alpha_1+d)/2}}dk \leq c_2\int_0^1 dk + c_2\int_1^\infty k^{-2-\alpha_1}dk = \frac{c_2(2+\alpha_1)}{1+\alpha_1}.$$

Besides, since ν is nonincreasing, we also have that, for any $y \in (0, 1]$,

$$\frac{1}{y^{-1}\ell(y^{-1})}\int_1^\infty \nu\big((y^2+k^2)^{1/2}\big)k^{d-2}dk \leq \frac{1}{y^{-1}\ell(y^{-1})}\int_1^\infty \nu(k)k^{d-1}dk$$
$$= \frac{c_3}{y^{-1}\ell(y^{-1})}\int_{\xi\in\mathbb{R}^d,\,|\xi|>1}\nu(\xi)d\xi = \frac{c_4}{\ell(1)}\frac{\ell(1)}{y^{-1}\ell(y^{-1})} \leq c_5 y^{1+\alpha_1} \leq c_5.$$

Therefore, we obtain (7.2.5) with $c_1 = c_2(2+\alpha_1)/(1+\alpha_1) + c_5$.

By Fubini's theorem, it holds that for $r > 0$,

$$\psi(r) = 2\int_0^\infty \big(1-\cos(ry)\big)\nu_1(y)dy.$$

Hence, using a change of the variables, we see that for any $1 < k \leq \lambda$ and $r \geq 1$,

$$\begin{aligned}|\psi(kr)-\psi(r)| &= 2\left|\int_0^\infty \big(\cos(ry)-\cos(kry)\big)\nu_1(y)dy\right| \\ &\leq 2r^{-1}\int_0^1 \big|\cos(y)-\cos(ky)\big|\nu_1(y/r)dy \\ &\quad + 2r^{-1}\left|\int_1^\infty \cos(y)\nu_1(y/r)dy\right| + 2r^{-1}\left|\int_1^\infty \cos(ky)\nu_1(y/r)dy\right| \\ &=: I_1 + I_2 + I_3.\end{aligned}$$

By Taylor expansion of the cosine function, (7.2.5) and the assumption that (7.1.1) holds with $\alpha_2 < 2$, we get from Lemma 1.1.1(i) that

$$I_1 \le 2\lambda^2 r^{-1}\int_0^1 y^2\nu_1(y/r)dy \le 2c_6\lambda^2\ell(r)\int_0^1 y\frac{\ell(r/y)}{\ell(r)}dy \le c_7\lambda^2\ell(r).$$

Next, to bound I_2 and I_3, we use a trick from the proof of [76, Theorem 3.5]. Since $y \mapsto \nu_1(y)$ is nonincreasing, there exists a measure $-d\nu_1$ on $(0,\infty)$ such that $\nu_1(y) = \int_y^\infty(-d\nu_1(z))$ for $y > 0$. Then by Fubini theorem and (7.2.5), we obtain

$$\begin{aligned}I_2 &= 2r^{-1}\left|\int_1^\infty\int_{y/r}^\infty \cos(y)(-d\nu_1(z))dy\right| = 2r^{-1}\left|\int_{1/r}^\infty\int_1^{rz}\cos(y)dy(-d\nu_1(z))\right|\\ &\le 4r^{-1}\left|\int_{1/r}^\infty(-d\nu_1(z))\right| = 4r^{-1}\nu_1(1/r) \le 4c_8\ell(r).\end{aligned}$$

Similarly, we also have that $I_3 \le c_9\ell(r)$. Thus, we get (7.2.4) in this case.

For the case $\psi(\xi) = \varphi(|\xi|^2)$ for a Bernstein function φ, we use [76, Lemma 5.13], (7.2.1) and (7.1.1), and obtain that for any $1 < k \le \lambda$ and $r \ge 1$,

$$\begin{aligned}|\psi(kr)-\psi(r)| &= \int_{r^2}^{(kr)^2}\varphi'(u)du \le \frac{1}{r^d}\int_0^{(\lambda r)^2}u^{d/2}\varphi'(u)du\\ &\le c_7\lambda^d\ell(\lambda r) \le c_9\lambda^{d+\alpha_2}\ell(r).\end{aligned}$$

The proof is complete. □

Now, we first consider the case when **(S-2)** holds. Recall that $\ell^*(r) := \sup_{u\in[1,r]}\ell(u)$ and ℓ^{-1} is the right continuous inverse of ℓ^*. Under **(S-2)**, we see that $\lim_{r\to\infty}\ell^*(r) = \infty$ and there exists a constant $C_3 \ge 1$ such that

$$\ell(r) \le \ell^*(r) \le C_3\ell(r) \qquad \text{for all } \; r > 2.$$

Hence, in this case, by Proposition 7.2.3, $p(t,0) < \infty$ for all $t > 0$. We give the small time estimates for $p(t,0)$ under **(S-2)**.

Lemma 7.2.7. *Assume that* **(S-2)** *holds. Then, there exists* $C > 0$ *such that*

$$p(t,x) \le p(t,0) \le C\big[\ell^{-1}(a_1/t)\big]^d \exp\big(-b_1 t h(\ell^{-1}(a_1/t)^{-1})\big),$$

for all $0 < t \le t_1$ *and* $x \in \mathbb{R}^d$ *where* $a_1 := 2dC_3/C_0$, $b_1 := C_0/(4C_2C_3)$ *and* $t_1 := a_1/\ell^*(3)$.

Proof. Let $a_1 := 2dC_3/C_0$ and $t_1 := a_1/\ell^*(3)$. Then, $\ell^{-1}(a_1/t) \ge 3$ for all $t \in (0, t_1]$. By Fourier inversion theorem, (2.3.23), integration by parts and the change of variables $s = \Phi(r)$, we have that for all $t \in (0, t_1]$,

$$\begin{aligned}
&p(t,x) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-i\langle \xi, x\rangle} e^{-t\psi(\xi)} d\xi \le c_1 \int_0^\infty e^{-C_0 t\Phi(r)} r^{d-1} dr \\
&\le c_2 t \int_0^\infty r^d e^{-C_0 t \Phi(r)} \Phi'(r) dr = c_2 t \int_0^\infty \Phi^{-1}(s)^d e^{-C_0 ts} ds \\
&\le c_2 t + c_2 t \int_{\Phi(1)}^{\Phi(\ell^{-1}(a_1/t))} \Phi^{-1}(s)^d e^{-C_0 ts} ds + c_2 t \int_{\Phi(\ell^{-1}(a_1/t))}^\infty \Phi^{-1}(s)^d e^{-C_0 ts} ds \\
&=: c_2 t + I_1 + I_2.
\end{aligned}$$

Observe that for $\Phi(2) < v \le u$, we have

$$\begin{aligned}
u - v = \Phi(\Phi^{-1}(u)) - \Phi(\Phi^{-1}(v)) &= \int_{\Phi^{-1}(v)}^{\Phi^{-1}(u)} k^{-1}\ell(k) dk \\
&\ge C_3^{-1} \int_{\Phi^{-1}(v)}^{\Phi^{-1}(u)} k^{-1}\ell^*(k) dk \ge C_3^{-1}\ell^*(\Phi^{-1}(v)) \log \frac{\Phi^{-1}(u)}{\Phi^{-1}(v)}.
\end{aligned}$$

Thus, for all $\Phi(2) < v \le u$, we have that (cf. Section 3.10 in [13])

$$\frac{\Phi^{-1}(u)}{\Phi^{-1}(v)} \le \exp\left(C_3 \frac{u-v}{\ell^*(\Phi^{-1}(v))}\right). \tag{7.2.6}$$

Then, by (7.2.6) and the definition of a_1, we get

$$I_2 = c_2 t\big[\ell^{-1}(a_1/t)\big]^d \int_{\Phi(\ell^{-1}(a_1/t))}^\infty \left(\frac{\Phi^{-1}(s)}{\Phi^{-1}(\Phi(\ell^{-1}(a_1/t)))}\right)^d e^{-C_0 ts} ds$$

$$\begin{aligned}
&\le c_3\big[\ell^{-1}(a_1/t)\big]^d\int_{\Phi(\ell^{-1}(a_1/t))}^{\infty} t\exp\left(\frac{-dC_3\Phi(\ell^{-1}(a_1/t))+dC_3 s}{a_1/t}-C_0ts\right)ds\\
&\le c_4\big[\ell^{-1}(a_1/t)\big]^d\exp\big(-\frac{C_0}{2}t\Phi(\ell^{-1}(a_1/t))\big)\int_{\Phi(\ell^{-1}(a_1/t))}^{\infty} t\exp\big(-\frac{C_0}{2}ts\big)ds\\
&\le c_5\big[\ell^{-1}(a_1/t)\big]^d\exp\big(-C_0t\Phi(\ell^{-1}(a_1/t))\big).
\end{aligned}$$

To bound I_1, we define $g(r)=r^d\exp\big(-\frac{C_0}{2C_3}t\Phi(r)\big)$ for $r\ge 1$. Then

$$g'(r)=\big(d-\frac{C_0}{2C_3}t\ell(r)\big)r^{d-1}\exp\Big(-\frac{C_0}{2C_3}t\Phi(r)\Big),$$

and hence g is strictly increasing on $[1,\ell^{-1}(a_1/t))$. Hence, by the scaling of Φ, we obtain

$$\begin{aligned}
I_1 &\le c_6t\int_{\Phi(1)}^{\Phi(\ell^{-1}(a_1/t))} g(\Phi^{-1}(s))ds \le 2c_6t\int_{\Phi(\ell^{-1}(a_1/t))/2}^{\Phi(\ell^{-1}(a_1/t))} g(\Phi^{-1}(s))ds\\
&\le c_7\big[\ell^{-1}(a_1/t)\big]^d\int_{\Phi(\ell^{-1}(a_1/t))/2}^{\Phi(\ell^{-1}(a_1/t))} t\exp\Big(-\frac{C_0}{2C_3}ts\Big)ds\\
&\le c_8\big[\ell^{-1}(a_1/t)\big]^d\exp\Big(-\frac{C_0}{4C_3}t\Phi(\ell^{-1}(a_1/t))\Big).
\end{aligned}$$

We also have that

$$I_1\ge c_9t\int_{\Phi(1)}^{\Phi(3)}\Phi^{-1}(s)^d\exp(-C_0ts)ds\ge c_{10}t.$$

Finally, we deduce the result from (2.3.25). □

By a similar proof to the one for Lemma 7.2.7, we get the following lemma.

Lemma 7.2.8. *Assume that* **(S-2)** *holds. Let a_1, b_1 and t_1 be the positive constants in Lemma 7.2.7. Then, there exists $C>0$ such that*

$$p(t,x)\le Ct|x|^{-d}\ell^*(|x|^{-1})\exp\big(-b_1th(|x|)\big),$$

for all $0<t\le t_1$ and $x\in\mathbb{R}^d$ satisfying $[\ell^{-1}(a_1/t)]^{-1}\le|x|\le 1/2$.

In view of Lemmas 7.2.7 and 7.2.8, we define for $a, r, t > 0$,

$$\theta_a(r,t) := r \vee [\ell^{-1}(a/t)]^{-1}. \tag{7.2.7}$$

Note that both $r \mapsto \theta_a(r,t)$ and $t \mapsto \theta_a(r,t)$ are increasing, while $a \mapsto \theta_a(r,t)$ is decreasing.

Combining Lemmas 7.2.7 and 7.2.8, and Proposition 7.2.5, we arrive at the following result.

Proposition 7.2.9. *Assume that* **(S-2)** *holds. For every* $T > 0$*, there exists* $C > 0$ *such that for all* $(t,x) \in (0,T] \times \mathbb{R}^d$*,*

$$p(t,x) \le Ct\frac{K(\theta_{a_1}(|x|,t))}{\big[\theta_{a_1}(|x|,t)\big]^d}\exp\big(-b_1 t h(\theta_{a_1}(|x|,t))\big),$$

where a_1 *and* b_1 *are the constants in Lemma 7.2.7.*

Using the fact that $p(t,)$ is radially nonincreasing, we obtain the following two-sided heat kernel estimates under **(S-2)** from Propositions 7.2.4 and 7.2.9.

Corollary 7.2.10. *Assume that* **(S-2)** *holds. For every* $T > 0$*, there exists* $C > 1$ *such that for every fixed* $\eta > 0$*, we have that for all* $(t,x) \in (0,T] \times \mathbb{R}^d$*,*

$$\begin{aligned} &C^{-1}t\nu(\theta_\eta(|x|,t))\exp\big(-b_0 t h(\theta_\eta(|x|,t))\big) \\ &\le p(t,x) \le Ct\frac{K(\theta_{a_1}(|x|,t))}{\big[\theta_{a_1}(|x|,t)\big]^d}\exp\big(-b_1 t h(\theta_{a_1}(|x|,t))\big), \end{aligned} \tag{7.2.8}$$

where b_0 *is the constant in Proposition 7.2.4, and* a_1 *and* b_1 *are the constants in Lemma 7.2.7.*

In the rest of this section, we assume that **(S-1)** holds and obtain heat kernel estimates in analogous form to (7.2.8). Note that, under **(S-1)**, by Proposition 7.2.3, it holds that $p(t,0) = \infty$ for small t.

Proposition 7.2.11. *Assume that* **(S-1)** *holds. Then, there exist constants* $t_0, C > 0$ *such that for all* $(t,x) \in (0,t_0] \times \mathbb{R}^d$,

$$p(t,x) \le Ct|x|^{-d}K(|x|)\exp\big(-t\psi(|x|^{-1})\big). \tag{7.2.9}$$

Proof. Let $\omega(r) = K(1)\mathbf{1}_{\{0<r\le 1\}}(r) + K(r^{-1})\mathbf{1}_{\{r>1\}}(r)$ for $r > 0$. By (7.2.1), Lemma 7.2.6, **(A)** and **(S-1)**, there exists $c_1 > 0$ such that $c_1\omega(r)$ satisfies the assumptions (5.7) and (5.8) in [76]. Therefore, by [76, Proposition 5.6], there exist $t_0, C > 0$ such that for all $t \in (0,t_0]$ and $0 < |x| < 1$, the estimate (7.2.9) holds. Moreover, for $t \in (0,t_0]$ and $|x| \ge 1$, we see that $e^{-t\psi(|x|^{-1})} \simeq 1$ and get (7.2.9) from Proposition 7.2.5. □

Combining the above proposition with Proposition 7.2.4, using the semigroup property, we deduce the following result.

Corollary 7.2.12. *Assume that* **(S-1)** *holds. For every* $T > 0$*, there exist constants* $b_2 > 0$*,* $C > 1$ *such that for all* $(t,x) \in (0,T] \times \mathbb{R}^d$,

$$C^{-1}t\nu(|x|)\exp\big(-b_0th(|x|)\big) \le p(t,x) \le Ct|x|^{-d}K(|x|)\exp\big(-b_2th(|x|)\big),$$

where b_0 *is the constant in Proposition 7.2.4.*

7.3 Survival probability estimates with explicit decay

For an open subset D of $\mathbb{R}^d$, we denote $\tau_D = \inf\{t > 0 : Y_t \notin D\}$. In this section, we obtain two-sided estimates for the survival probability $\mathbb{P}^x(\tau_D > t)$ which play a crucial role in factorization of the Dirichlet heat kernel. We first state the general two-sided estimates for the survival probability in balls which are established in [76, Proposition 5.2].

Proposition 7.3.1. *There exist positive constants* c_1, c_2, C_4 *and* C_5 *which* *only depend on the dimension d such that for all $t, r > 0$,*

$$\begin{aligned} &c_1 \exp\big(- \kappa_2 C_4 t h(r)\big) \le \mathbb{P}_x(\tau_{B(x,r)} > t) \\ &\le \sup_{z \in B(x,r)} \mathbb{P}^z(\tau_{B(x,r)} > t) \le c_2 \exp\big(- \kappa_1 C_5 t h(r)\big), \end{aligned} \tag{7.3.1}$$

where κ_1 and κ_2 are the constants in **(A)**. *Consequently, it holds that*

$$\mathbb{E}^x[\tau_{B(x,r)}] = \int_0^\infty \mathbb{P}^x(\tau_{B(x,r)} > s)ds \simeq h(r)^{-1} \quad \textit{for } r > 0.$$

In the rest of this section, we assume that condition **(B)** holds true.

Let Y^d be the last coordinate of Y, $M_t = \sup_{s \le t} Y^d_s$ and $\mathscr{L}_t$ be the local time at 0 for $M_t - Y^d_t$, the last coordinate of Y reflected at the supremum. Define the ascending ladder-height process as $H_t = Y^d_{\mathscr{L}^{-1}_t} = M_{\mathscr{L}^{-1}_t}$ where $\mathscr{L}^{-1}$ is the right continuous inverse of $\mathscr{L}$. Then, the renewal function V of Y is defined by

$$V(s) = \int_0^\infty \mathbb{P}(H_t \le s)dt, \quad s \in \mathbb{R}.$$

Since the process Y is isotropic unimodal, there are several known properties for the renewal function. (See, [117, Theorem 1.2], [12, p.74] and [21, Section 1.2].)

Recall that a function $u : \mathbb{R}^d \to \mathbb{R}$ is said to be harmonic in $D \subset \mathbb{R}^d$ if for every open set B whose closure is a compact subset of D, $u(x) = \mathbb{E}^x[u(Y_{\tau_B})]$ for all $x \in B$.

Lemma 7.3.2. *(i) $V(s) = 0$ for $s < 0$ and V is strictly increasing and unbounded.*
(ii) V is subadditive; that is, $V(s + r) \le V(s) + V(r)$ for all $s, r \in \mathbb{R}$.
(iii) V is absolutely continuous and harmonic on $(0, \infty)$ for the process Y^d_t. Also, V' is a positive harmonic function for Y^d_t on $(0, \infty)$.

According to [22, Proposition 2.4], the relation (7.2.3) can be extended to include the renewal function. Precisely, there exist comparison constants which are only depend on the dimension d and the constant κ_1 and κ_2 in

(7.1.2) such that $h(r) \simeq \psi(r^{-1}) \simeq V(r)^{-2}$ for $r > 0$. Then, by Lemmas 7.2.1 and 7.2.2, we get that

$$L(r) \simeq h(r) \simeq \psi(r^{-1}) \simeq \Phi(r^{-1}) \simeq V(r)^{-2} \qquad \text{for } 0 < r \leq 1. \quad (7.3.2)$$

By (7.3.2), we get from Lemma 7.2.2 that there are $c_1, c_2, c_3, c_4 > 0$ such that

$$c_1 \left(\frac{R}{r}\right)^{\frac{\alpha_1}{2}} \leq \frac{V(R)}{V(r)} \leq c_2 \left(\frac{R}{r}\right)^{\frac{\alpha_1 \vee (1/2)}{2}} \qquad \text{for all } 0 < r \leq R \leq 1. \quad (7.3.3)$$

and

$$c_3 \left(\frac{R}{r}\right)^{\alpha_1} \leq \frac{L(r)}{L(R)} \leq c_4 \left(\frac{R}{r}\right)^{\alpha_2 \vee (1/2)} \qquad \text{for all } 0 < r \leq R \leq 1. \quad (7.3.4)$$

Proposition 7.3.3. *The renewal function V is twice-differentiable on $(0, \infty)$, and there exists $c_1 > 0$ such that*

$$|V''(r)| \leq c_1 \frac{V'(r)}{r \wedge 1} \quad \textit{and} \quad V'(r) \leq c_1 \frac{V(r)}{r \wedge 1}, \quad r > 0.$$

Proof. Since **(A)** and **(B)** hold, the scale-invariant Harnack inequality holds for Y. (See, [74, Theorem 1.9].) Then, the results follows from [102, Theorem 1.1] and Lemma 7.3.2(iii). □

Define $w(x) := V((x_d)^+)$ for $x \in \mathbb{R}^d$. Since the renewal function V is harmonic on $(0, \infty)$ for Y^d, by the strong Markov property, w is harmonic in $\mathbb{R}^d_+$ with respect to Y.

Lemma 7.3.4. *For all $\lambda > 0$, there exists $c_1 = c_1(d, \lambda) > 0$ such that for any $r > 0$,*

$$\sup_{\{x \in \mathbb{R}^d : 0 < x_d \leq \lambda r\}} \int_{B(x,r)^c} w(y) \nu(|x-y|) dy \leq c_1 V(r)^{-1}.$$

Proof. See, the proof of [73, Proposition 3.2]. □

We define an operator $\mathcal{L}_Y$ as follows: for $\epsilon > 0$ and $x \in \mathbb{R}^d$,

$$\mathcal{L}_Y^\epsilon f(x) := \int_{B(x,\varepsilon)^c} (f(y) - f(x))\nu(|x-y|)dy,$$
$$\mathcal{L}_Y f(x) := P.V. \int_{\mathbb{R}^d} (f(y) - f(x))\nu(|x-y|)dy = \lim_{\varepsilon \downarrow 0} \mathcal{L}_Y^\epsilon f(x),$$
$$\mathcal{D}(\mathcal{L}_Y) := \left\{ f \in C_0^2(\mathbb{R}^d) : P.V. \int_{\mathbb{R}^d} (f(y) - f(x))\nu(|x-y|)dy \text{ exists in } \mathbb{R} \right\}.$$

Proposition 7.3.5. *For all $x \in \mathbb{R}_+^d$, $\mathcal{L}_Y w(x) = 0$.*

Proof. By Proposition 7.3.3 and Lemma 7.3.4, using [29, Lemma 2.3 and Theorem 2.11], the proof is essentially the same as the one given in [73, Theorem 3.3]. We omit details here. □

Using Proposition 7.3.5, by following arguments in the proof of Lemma 3.2.1, we get the following lemma. See [53, Lemma 3.6] for the proof.

Lemma 7.3.6. *Let D be a $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics (R_0, Λ). For any $Q \in \partial D$ and $r > 0$, we define*

$$h_r(y) = h_{r,Q}(y) := V(\delta_D(y))\mathbf{1}_{D \cap B(Q,r)}(y).$$

Then, there exist constants $R_1 = R_1(R_0, \Lambda, \psi, d) \in (0, (R_0 \wedge 1)/2]$ and $c_1 = c_1(R_0, \Lambda, \psi, d) > 1$ independent of Q such that for every $r \in (0, R_1)$, $\mathcal{L}_Y h_r$ is well defined in $D \cap B(Q, r/4)$ and

$$|\mathcal{L}_Y h_r(x)| \le \frac{c_1}{V(r)} \quad \text{for all } x \in D \cap B(Q, r/4)\,.$$

For $l > 0$, we define $D_{int}(l) := \{y \in D : \delta_D(y) > l\}$.

Using Dynkin's formula with an approximation argument, the Lévy system, and (7.3.2), one can follow the arguments given in the below of Lemma 3.2.1 to deduce the following lemma.

Lemma 7.3.7. *Let D be a $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics (R_0, Λ) and R_1 be the constant in Lemma 7.3.6. Then, there exist constants $R_2 =$*

$R_2(R_0, \Lambda, \psi, d) \in (0, R_1/16]$ and $c_1 = c_1(R_0, \Lambda, \psi, d) > 1$ such that for every $r \in (0, R_2]$ and $x \in D$ with $\delta_D(x) < r/2$,

$$\frac{c_1^{-1}}{L(\delta_D(x))^{1/2} L(r)^{1/2}} \leq \mathbb{E}_x[\tau_{D\cap B(z,r)}] \leq \frac{c_1}{L(\delta_D(x))^{1/2} L(r)^{1/2}}. \tag{7.3.5}$$

and

$$\mathbb{P}_x\left(Y_{\tau_{D\cap B(z,r)}} \in D_{int}(r/4)\right) \geq c_1^{-1}\left(\frac{L(r)}{L(\delta_D(x))}\right)^{1/2}, \tag{7.3.6}$$

where $z \in \partial D$ is the point satisfying $\delta_D(x) = |x - z|$.

Fix $T > 0$ and D a $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics (R_0, Λ). Let R_2 be the constant in Lemma 7.3.7. For $t \in (0, T]$, we set

$$r_t = r_t(T, R_0, \Lambda, \psi, d) := \frac{L^{-1}(1/t)}{L^{-1}(1/T)} R_2.$$

For $x \in D$ with $\delta_D(x) < r_t/2$, we define an open neighborhood $U(x,t)$ of x and an open ball $W(x,t) \subset D \setminus U(x,t)$ as follows:

Find $z_x \in \partial D$ satisfying $\delta_D(x) = |x - z_x|$ and let $v_x := z_x + 2r_t(x - z_x)/|x - z_x|$. Then, we have $\delta_D(v_x) \geq r_t/\sqrt{1+\Lambda^2}$. We define

$$U(x,t) := D \cap B(z_x, r_t) \quad \text{and} \quad W(x,t) := B\big(v_x, \frac{r_t}{2\sqrt{1+\Lambda^2}}\big) \subset D. \tag{7.3.7}$$

By the construction, one can see that

$$r_t/2 \leq |u - w| \leq 4r_t \quad \text{for all } u \in U(x,t) \text{ and } w \in W(x,t). \tag{7.3.8}$$

Proposition 7.3.8. *Let D be a $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics (R_0, Λ). Let r_t and $U(x,t)$ be defined above. For every $T > 0$ and $M \geq 1$, it holds that for all $t \in (0, T]$ and $x \in D$ with $\delta_D(x) < r_t/2$,*

$$\begin{aligned}\mathbb{P}^x(\tau_D > t) \simeq \mathbb{P}^x(\tau_D > Mt) \simeq \mathbb{P}^x(Y_{\tau_{U(x,t)}} \in D) \\ \simeq t^{-1}\mathbb{E}^x[\tau_{U(x,t)}] \simeq \big(tL(\delta_D(x))\big)^{-1/2},\end{aligned}$$

where the comparability constants depend only on T, M, ψ, R_0, Λ and d.

Proof. Recall that $z_x \in \partial D$ is the point satisfying $\delta_D(x) = |x - z_x|$. Let

$$o_x = z_x + \frac{r_t(x - z_x)}{2|x - z_x|} \in D.$$

Indeed, we have $r_t/(2\sqrt{1+\Lambda^2}) \le \delta_D(o_x) \le r_t/2$. Since we assumed **(A)** and **(B)**, the assumptions in [74, Theorem 1.9] are satisfied and hence by that theorem, the (scale-invariant) boundary Harnack principle for Y holds true. Therefore, we get

$$\mathbb{P}^x(Y_{\tau_{U(x,t)}} \in D) \le \frac{\mathbb{P}^x(Y_{\tau_{U(x,t)}} \in D)}{\mathbb{P}^{o_x}(Y_{\tau_{U(x,t)}} \in D)} \le c_1 \frac{\mathbb{P}^x(Y_{\tau_{U(x,t)}} \in W(x,t))}{\mathbb{P}^{o_x}(Y_{\tau_{U(x,t)}} \in W(x,t))}, \quad (7.3.9)$$

where $W(x,t)$ is the subset of D defined as in just before the proposition. By the Lévy system, the scaling of ν, (7.3.8) and Lemma 7.3.7, we get

$$\begin{aligned}\mathbb{P}^x(Y_{\tau_{U(x,t)}} \in W(x,t)) &= \mathbb{E}^x\left[\int_0^{\tau_{U(x,t)}} \int_{W(x,t)} \nu(|Y_s - w|) dw ds\right] \\ &\simeq \mathbb{E}^x[\tau_{U(x,t)}]\nu(r_t) r_t^d \simeq L(r_t)^{-1/2} L(\delta_D(x))^{-1/2} \nu(r_t) r_t^d.\end{aligned}$$

Similarly, we also get that $\mathbb{P}^{o_x}(Y_{\tau_{U(x,t)}} \in W(x,t)) \simeq \mathbb{E}^{o_x}[\tau_{U(x,t)}]\nu(r_t) r_t^d \simeq L(r_t)^{-1}\nu(r_t) r_t^d$. Then, using the strong Markov property, Chebyshev's inequality, (7.3.9) and Lemma 7.3.7, since $L(r_t) \asymp t^{-1}$, we obtain

$$\begin{aligned}\mathbb{P}^x(\tau_D > t) &\le \mathbb{P}^x(\tau_{U(x,t)} > t) + \mathbb{P}^x(Y_{\tau_{U(x,t)}} \in D) \\ &\le t^{-1}\mathbb{E}^x[\tau_{U(x,t)}] + c_2 L(r_t)^{1/2} L(\delta_D(x))^{-1/2} \\ &\le t^{-1} L(r_t)^{-1/2} L(\delta_D(x))^{-1/2} + c_2 L(r_t)^{1/2} L(\delta_D(x))^{-1/2} \\ &\le c_3 t^{-1/2} L(\delta_D(x))^{-1/2}.\end{aligned}$$

On the other hand, for any $a > 0$, using the strong Markov property, (7.3.1), (7.3.2), Lemma 7.3.7 and Markov inequality, we get that

$$\mathbb{P}^x(\tau_D > at) \ge \mathbb{P}^x\big(\tau_{U(x,t)} < at, Y_{\tau_{U(x,t)}} \in D_{int}(r_t/4),$$

$$
\begin{aligned}
&\qquad |Y_{\tau_{U(x,t)}} - Y_{\tau_{U(x,t)}+s}| \le r_t/4 \text{ for all } 0 < s < at\big) \\
&\ge \mathbb{P}^x\big(\tau_{U(x,t)} < at, Y_{\tau_{U(x,t)}} \in D_{int}(r_t/4)\big)\mathbb{P}^0(\tau_{B(0,r_t/4)} > at) \\
&\ge c_4\big(\mathbb{P}^x(Y_{\tau_{U(x,t)}} \in D_{int}(r_t/4)) - \mathbb{P}^x(\tau_{U(x,t)} \ge at)\big) \\
&\ge c_4 t^{-1}\big(c_5\mathbb{E}^x[\tau_{U(x,t)}] - a^{-1}\mathbb{E}^x[\tau_{U(x,t)}]\big).
\end{aligned}
$$

Take $a = (2c_5^{-1}) \vee M$. Then by Lemma 7.3.7 and the third inequality in the above inequalities, we obtain

$$
\begin{aligned}
\mathbb{P}^x(\tau_D > Mt) &\ge \mathbb{P}^x(\tau_D > at) \\
&\ge \frac{c_4}{2}\mathbb{P}^x(Y_{\tau_{U(x,t)}} \in D_{int}(r_t/4)) \ge ct^{-1/2}L(\delta_D(x))^{-1/2}.
\end{aligned}
$$

The proof is complete. □

Using Proposition 7.3.8 when $\delta_D(x) < r_t/2$, and (7.3.1) and (7.3.2) when $\delta_D(x) \ge r_t/2$, we arrive at the following result.

Corollary 7.3.9. *Let D be a $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics (R_0, Λ). For all $T > 0$, there exists a constant $c_1 = c_1(d, T, \psi, R_0, \Lambda) > 1$ such that for every $t \in (0, T]$ and $x \in D$,*

$$
c_1^{-1}\left(1 \wedge \frac{1}{tL(\delta_D(x))}\right)^{1/2} \le \mathbb{P}^x(\tau_D > t) \le c_1\left(1 \wedge \frac{1}{tL(\delta_D(x))}\right)^{1/2}.
$$

Corollary 7.3.10. *Let D be a bounded $C^{1,1}$ open subset in $\mathbb{R}^d$ with characteristics (R_0, Λ) of scale (r_1, r_2). Then, there exists $c_1 = c_1(R_0, \Lambda, \psi, d) > 1$ such that for all $t > 0$ and $x \in D$,*

$$
\begin{aligned}
&c_1^{-1}\left(1 \wedge \frac{1}{(t\wedge 2)L(\delta_D(x))}\right)^{1/2} \exp\big(-\kappa_2 C_4 t h(r_1/2)\big) \\
&\qquad \le \mathbb{P}^x(\tau_D > t) \le c_1\left(1 \wedge \frac{1}{(t\wedge 2)L(\delta_D(x))}\right)^{1/2} \exp\big(-\kappa_1 C_5 t h(r_2)\big),
\end{aligned}
$$

where κ_1, κ_2 are the constants in **(A)** *and C_4, C_5 are the ones in* (7.3.1).

Proof. Fix $(t, x) \in (0, \infty) \times D$. If $t \le 2$, then the assertion follows from

Corollary 7.3.9.

Suppose that $t > 2$. Let $x_1, x_2 \in \mathbb{R}^d$ be the points satisfying $B(x_1, r_1) \subset D \subset B(x_2, r_2)$. By the semigroup property, (7.3.1) and Corollary 7.3.9, we get that

$$\begin{aligned}
\mathbb{P}^x(\tau_D > t) &= \int_D p_D(t,x,y)dy \le \int_D \int_D p_D(1,x,z) p_{B(x_2,r_2)}(t-1,z,y)dzdy \\
&\le \mathbb{P}^x(\tau_D > 1) \sup_{z\in D} \mathbb{P}^z(\tau_{B(x_2,r_2)} > t-1) \\
&\le \frac{c_1}{L(\delta_D(x))^{1/2}} \exp\big(-\kappa_1 C_5 t h(r_2)\big).
\end{aligned}$$

To prove the lower bound, we first assume that $\delta_D(x) < R_2/2$ with the constant R_2 in Lemma 7.3.7. Without loss of generality, we may assume that $R_2 \le r_1/2$. Let $z \in \partial D$ be the point satisfying $\delta_D(x) = |x-z|$ and θ be the shift operator defined as $Y_t \circ \theta_s = Y_{s+t}$. Using the strong Markov property, (7.3.6), the Lévy system and (7.3.1), we have

$$\begin{aligned}
\mathbb{P}^x(\tau_D > t) &\ge \mathbb{E}^x\Big[Y_{\tau_{D\cap B(z,R_2)}} \in D_{int}(R_2/4),\, Y_{\tau_{B(Y_0,R_2/4)}} \circ \theta_{\tau_{D\cap B(z,R_2)}} \in B(x_1, \frac{r_1}{2}), \\
&\qquad \tau_D \circ \theta_{\tau_{B(Y_0,R_2/4)}} \circ \theta_{\tau_{D\cap B(z,R_2)}} > t\Big] \\
&\ge \frac{c_2 L(R_2)^{1/2}}{L(\delta_D(x))^{1/2}} \inf_{w\in D_{int}(R_2/4)} \mathbb{P}^w\big(Y_{\tau_{B(w,R_2/4)}} \in B(x_1, \frac{r_1}{2})\big) \inf_{y\in B(x_1,r_1/2)} \mathbb{P}^y(\tau_{B(x_1,r_1)} > t) \\
&\ge \frac{c_3}{L(\delta_D(x))^{1/2}} \exp\big(-\kappa_2 C_4 t h(r_1/2)\big).
\end{aligned}$$

Similarly, if $\delta_D(x) \ge R_2/2$, then we have

$$\begin{aligned}
\mathbb{P}^x(\tau_D > t) &\ge \mathbb{E}^x[Y_{\tau_{B(x,R_2/4)}} \in B(x_1, r_1/2), \tau_D \circ \theta_{\tau_{B(x,R_2/4)}} > t] \\
&\ge c_4 \inf_{y\in B(x_1,r_1/2)} \mathbb{P}^y(\tau_{B(x_1,r_1)} > t) \ge c_5 \exp\big(-\kappa_2 C_4 t h(r_1/2)\big).
\end{aligned}$$

□

7.4 Small time Dirichlet heat kernel estimates in $C^{1,1}$ open set

In this section, we provide the proof of Theorem 7.1.1. Let $T > 0$ be a fixed constant and D be a fixed $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics (R_0, Λ). Throughout the section, we assume that condition **(B)** holds, and **(C)** further holds if D is unbounded. Then by **(A)** and **(C)**, we obtain

$$\nu(|x-y|) \simeq \nu(2|x-y|) \quad \text{for } x, y \in D. \tag{7.4.1}$$

By (7.2.1), (7.2.2), Corollary 7.2.10 and Corollary 7.2.12, for every $T > 0$, the following heat kernel estimates hold true for all $(t, x) \in (0, T] \times \mathbb{R}^d$ with the constant b_0 in Proposition 7.2.4:

(1) If **(S-1)** holds, then there exist constants $c_1 > 1$ and $b_2 > 0$ such that

$$c_1^{-1} t\nu(|x|) \exp\big(-b_0 t h(|x|)\big) \le p(t,x) \le c_1 t\nu(|x|) \exp\big(-b_2 t h(|x|)\big). \tag{7.4.2}$$

(2) If **(S-2)** holds, then there exist a constant $c_2 > 1$ such that for all $\eta > 0$,

$$\begin{aligned} &c_2^{-1} t\nu(\theta_\eta(|x|,t)) \exp\big(-b_0 t h(\theta_\eta(|x|,t))\big) \\ &\le p(t,x) \le c_2 t\nu(\theta_{a_1}(|x|,t)) \exp\big(-b_1 t h(\theta_{a_1}(|x|,t))\big), \end{aligned} \tag{7.4.3}$$

where a_1, b_1 are the constants in Lemma 7.2.7, and $\theta_a(r,t) = r \vee [\ell^{-1}(a/t)]^{-1}$ is defined by (7.2.7).

Before giving the proof of Theorem 7.1.1, we obtain a lower bound of $p_D(t,x,y)$ without **(S-1)** and **(S-2)**. This result will be used later to obtain Green function estimates.

Proposition 7.4.1. *For every $T > 0$, there exist $c_1, c_2 > 0$ depend only on d, ψ, T, R_0, Λ such that for all $(t,x,y) \in (0,T] \times D \times D$,*

$$p_D(t,x,y) \ge c_2 \left(1 \wedge \frac{1}{tL(\delta_D(x))}\right)^{1/2} \left(1 \wedge \frac{1}{tL(\delta_D(y))}\right)^{1/2}$$

$$\times\, t\nu(|x-y|)\exp\big(-c_1 th(|x-y|)\big).$$

Proof. Let R_2 be the constant in Lemma 7.3.7. Fix $(t,x,y)\in(0,T]\times D\times D$ and set

$$r_t=\frac{L^{-1}(1/t)}{L^{-1}(1/T)}R_2 \quad\text{and}\quad l_t(x,y)=r_t\wedge\frac{|x-y|}{4}. \tag{7.4.4}$$

Note that, by (7.3.2), (7.3.3) and (7.3.4), we have $V(r_t)\simeq t^{1/2}$ and $L(r_t)\simeq h(r_t)\asymp t^{-1}$.

Let $z_x, z_y\in\partial D$ be the points satisfying $\delta_D(x)=|x-z_x|$ and $\delta_D(y)=|y-z_y|$. By (7.3.3), there exists a constant $m>1$ such that

$$mV(\delta k)\geq \delta V(k)\quad\text{for all }\ 0<\delta\leq 1,\ 0<k\leq 1. \tag{7.4.5}$$

Case 1. Suppose that $|x-y|\leq R_2$. Define open neighborhoods of x and y as follows:

$$\mathcal{O}(x)=\begin{cases} B\big(x,V^{-1}[\frac{1}{8m}V(|x-y|)]\big), & \text{if}\quad 8mV(\delta_D(x))\geq V(|x-y|);\\ D\cap B(z_x,\frac{1}{3}|x-y|), & \text{if}\quad 8mV(\delta_D(x))<V(|x-y|),\end{cases}$$

and

$$\mathcal{O}(y)=\begin{cases} B\big(y,V^{-1}[\frac{1}{8m}V(|x-y|)]\big), & \text{if}\quad 8mV(\delta_D(y))\geq V(|x-y|);\\ D\cap B\big(z_y,\frac{1}{3}|x-y|\big), & \text{if}\quad 8mV(\delta_D(y))<V(|x-y|).\end{cases}$$

Then, we see that $x\in\mathcal{O}(x)\subset D$, $y\in\mathcal{O}(y)\subset D$ and $|u-w|\simeq|x-y|$ for all $u\in\mathcal{O}(x)$ and $w\in\mathcal{O}(y)$. Thus, by the strong Markov property and (7.4.1),

$$\begin{aligned} p_D(t,x,y)&\geq t\,\mathbb{P}^x(\tau_{\mathcal{O}(x)}>t)\mathbb{P}^y(\tau_{\mathcal{O}(y)}>t)\inf_{u\in\mathcal{O}(x),w\in\mathcal{O}(y)}\nu(|u-w|)\\ &\geq ct\nu(|x-y|)\,\mathbb{P}^x(\tau_{\mathcal{O}(x)}>t)\mathbb{P}^y(\tau_{\mathcal{O}(y)}>t). \end{aligned} \tag{7.4.6}$$

If $8mV(\delta_D(x))\geq V(|x-y|)$, then we see from (7.3.1) and (7.3.2) that

$$\mathbb{P}^x(\tau_{\mathcal{O}(x)}>t)\geq c\exp\big(-c_1 th(|x-y|)\big). \tag{7.4.7}$$

Next, we assume that $8mV(\delta_D(x)) < V(|x-y|)$. Then by the monotonicity of V and (7.4.5), we get $|x-y| > 8\delta_D(x)$. Let $\rho := V^{-1}(\varepsilon V(l_t(x,y)))$ where $\varepsilon \in (0, (8m)^{-1})$ will be chosen later. Then, by (7.3.2) and (7.3.3), it holds that that

$$\begin{aligned} V(\rho) &\simeq V(l_t(x,y)) \simeq t^{1/2} \wedge V(|x-y|), \\ h(\rho) &\simeq h(l_t(x,y)) \simeq t^{-1} \vee h(|x-y|). \end{aligned} \tag{7.4.8}$$

Note that we can not expect that $\rho \simeq l_t(x,y)$ in general.

If $8\delta_D(x) \geq \rho$, then by (7.3.1) and (7.4.8), we have

$$\mathbb{P}^x(\tau_{\mathcal{O}(x)} > t) \geq \mathbb{P}^x(\tau_{B(x,\rho/8)} > t) \geq c\exp\big(-c_2 t h(|x-y|)\big). \tag{7.4.9}$$

Indeed, by Lemma 7.2.2(i) and (7.3.2), we see that $h(\rho/8) \asymp h(4\rho)$. Thus, if $l_t(x,y) = |x-y|/4$, then we get (7.4.9). Otherwise, if $l_t(x,y) = r_t$, then $\mathbb{P}^x(\tau_{\mathcal{O}(x)} > t) \simeq 1 \simeq \exp\big(-c_3 t h(|x-y|)\big)$ and hence (7.4.9) holds.

If $8\delta_D(x) < \rho$, then there is a piece of annulus $\mathcal{A}(x) \subset \{w \in \mathcal{O}(x) : \rho < |w - z_x| < |x-y|/4\}$ such that $\mathrm{dist}(\mathcal{A}(x), \partial\mathcal{O}(x)) > \rho/8$. Recall that θ is shift operator. Using the strong Markov property, the Lévy system, (7.3.1), (7.3.5), (7.3.2) and (7.3.3), we obtain

$$\begin{aligned} &\mathbb{P}^x(\tau_{\mathcal{O}(x)} > t) \geq \mathbb{P}^x\big(Y_{\tau_{B(z_x,\rho/2)\cap D}} \in \mathcal{A}(x),\ \tau_{\mathcal{O}(x)} \circ \theta_{\tau_{B(z_x,\rho/2)\cap D}} > t\big) \\ &\geq \mathbb{P}^x\big(Y_{\tau_{B(z_x,\rho/2)\cap D}} \in \mathcal{A}(x)\big) \inf_{z\in\mathcal{A}(x)} \mathbb{P}^z(\tau_{\mathcal{O}(x)} > t) \\ &\geq c\mathbb{E}^x\left[\int_0^{\tau_{B(z_x,\rho/2)\cap D}} \int_{\mathcal{A}(x)} \nu(|Y_s - w|)dwds\right] \mathbb{P}^0(\tau_{B(0,\rho/8)} > t) \\ &\geq c\mathbb{E}^x\big[\tau_{B(z_x,\rho/2)\cap D}\big] \int_\rho^{|x-y|/4} (-L'(k))dk \exp\big(-c_3 t h(|x-y|)\big) \\ &\geq c\big(c_4^{-1}V(\rho)^{-2} - c_4 V(|x-y|)^{-2}\big) L(\delta_D(x))^{-1/2} V(\rho) \exp\big(-c_3 t h(|x-y|)\big), \end{aligned}$$

where the constant $c_4 > 1$ is independent of ε. Now, we choose $\varepsilon = (2c_4)^{-1} \wedge$

$(16m)^{-1}$. Then, we get from (7.4.8) that

$$\mathbb{P}^x(\tau_{\mathcal{O}(x)} > t) \geq ct^{-1/2}L(\delta_D(x))^{-1/2}\exp\big(-c_3th(|x-y|)\big).$$

Finally, by combining the above inequality with (7.4.7) and (7.4.9), we deduce that

$$\mathbb{P}^x(\tau_{\mathcal{O}(x)} > t) \geq c\left(1 \wedge \frac{1}{tL(\delta_D(x))}\right)^{1/2}\exp\big(-c_4th(|x-y|)\big).$$

By the same way, we get $\mathbb{P}^y(\tau_{\mathcal{O}(y)} > t) \geq c\big(1 \wedge \frac{1}{tL(\delta_D(y))}\big)^{1/2}\exp\big(-c_4th(|x-y|)\big)$. Therefore we get the desired lower bound from (7.4.6).

Case 2. Suppose that $|x-y| > R_2$. In this case, we let $D_x := D \cap B(x, R_2/4)$ and $D_y := D \cap B(y, R_2/4)$. By the same argument as (7.4.6), (7.4.1) and Corollary 7.3.9, we get

$$\begin{aligned}p_D(t,x,y) &\geq t\,\mathbb{P}^x(\tau_{D_x} > t)\mathbb{P}^y(\tau_{D_y} > t)\inf_{u\in D_x, w\in D_y}\nu(|u-w|)\\ &\geq c\left(1 \wedge \frac{1}{tL(\delta_D(x))}\right)^{1/2}\left(1 \wedge \frac{1}{tL(\delta_D(y))}\right)^{1/2}t\nu(|x-y|).\end{aligned}$$

The proof is complete □

Now, we are ready to prove Theorem 7.1.1.

Proof of Theorem 7.1.1. Fix $(t,x,y) \in (0,T] \times D \times D$ and continue using the notation r_t and $l_t(x,y)$ in (7.4.4).

(i) By Proposition 7.4.1, it remains to show that there exist $c_1 > 0, b_3 \in (0, b_0]$ such that

$$p_D(t,x,y) \leq c_1\left(1 \wedge \frac{1}{tL(\delta_D(x))}\right)^{1/2}t\nu(|x-y|)\exp\big(-b_3th(|x-y|)\big), \quad (7.4.10)$$

where b_0 is the constant in Proposition 7.2.4. Indeed, if (7.4.10) holds, then by the semigroup property and (7.4.2), we get

$$\begin{aligned}
p_D(t,x,y) &= \int_D p_D(t/2,x,z)p_D(t/2,y,z)dz \\
&\le c\left(1\wedge \frac{1}{tL(\delta_D(x))}\right)^{\frac12}\left(1\wedge \frac{1}{tL(\delta_D(y))}\right)^{\frac12}\int_D p\big(\frac{b_3}{2b_0}t,x,z\big)p\big(\frac{b_3}{2b_0}t,y,z\big)dz \\
&\le c\left(1\wedge \frac{1}{tL(\delta_D(x))}\right)^{\frac12}\left(1\wedge \frac{1}{tL(\delta_D(y))}\right)^{\frac12} t\nu(|x-y|)\exp\big(-\frac{b_2b_3}{b_0}th(|x-y|)\big),
\end{aligned}$$

which yields the desired upper bound.

Now, we prove (7.4.10). If $\delta_D(x)\ge r_t/2$, then (7.4.10) is a consequence of (7.4.2) and the trivial bound that $p_D(t,x,y)\le p(t,x-y)$. Hence, we assume that $\delta_D(x)<r_t/2$. By (7.3.4), there exists a constant $M>1$ such that

$$ML(16k)\ge L(k) \quad \text{for all } \ k\le 1/16. \tag{7.4.11}$$

By the semigroup property, monotonicity of $p(t,\cdot)$ and Proposition 7.3.8,

$$\begin{aligned}
&p_D(t,x,y) \\
&\le \left(\int_{\{z\in D:|y-z|>|x-y|/2\}}+\int_{\{z\in D:|x-z|>|x-y|/2\}}\right)p_D(t/2,x,z)p_D(t/2,z,y)dz \\
&\le p(t/2,|x-y|/2)\big(\mathbb{P}^x(\tau_D>t/2)+\mathbb{P}^y(\tau_D>t/2)\big) \\
&\le cp(t/2,|x-y|/2)\big(t^{-1/2}L(\delta_D(x))^{-1/2}+t^{-1/2}L(\delta_D(y))^{-1/2}\big).
\end{aligned}$$

Thus, if $ML(\delta_D(y))\ge L(\delta_D(x))$, then (7.4.10) holds true. Therefore, we assume that $ML(\delta_D(y))<L(\delta_D(x))$. Since L is decreasing, it follows from (7.4.11) that $\delta_D(y)>16\delta_D(x)$ and hence $|x-y|\ge|y-z_x|-|z_x-x|\ge\delta_D(y)-\delta_D(x)>15\delta_D(x)$ where $z_x\in\partial D$ is the point satisfying $\delta_D(x)=|x-z_x|$. Define

$$W_1:=D\cap B(z_x,l_t(x,y)),\quad W_3:=\big\{w\in D:|w-y|\le|x-y|/2\big\}$$

and $W_2:=D\setminus(W_1\cup W_3)=\{w\in D\setminus W_1:|w-y|>|x-y|/2\}$. Then, for

$u \in W_1$ and $w \in W_3$, we obtain

$$|u-w| \ge |x-y| - |z_x - x| - |u - z_x| - |y-w| \ge \frac{|x-y|}{6}. \qquad (7.4.12)$$

Using the strong Markov property, we get that

$$\begin{aligned} p_D(t,x,y) &= \mathbb{E}^x\big[\, p_D(t-\tau_{W_1}, Y_{\tau_{W_1}}, y) : \tau_{W_1} < t\big] \\ &= \mathbb{E}^x[\, p_D(t-\tau_{W_1}, Y_{\tau_{W_1}}, y) : \tau_{W_1} < t, Y_{\tau_{W_1}} \in W_3] \\ &\quad + \mathbb{E}^x[p_D(t-\tau_{W_1}, Y_{\tau_{W_1}}, y) : \tau_{W_1} \in (0, 2t/3], Y_{\tau_{W_1}} \in W_2] \\ &\quad + \mathbb{E}^x[p_D(t-\tau_{W_1}, Y_{\tau_{W_1}}, y) : \tau_{W_1} \in (2t/3, t), Y_{\tau_{W_1}} \in W_2] \\ &=: I_1 + I_2 + I_3. \end{aligned} \qquad (7.4.13)$$

First, by the Lévy system and (7.4.12), we get

$$\begin{aligned} I_1 &= \int_0^t \int_{W_3} \int_{W_1} p_{W_1}(s,x,u)\nu(|w-u|)p_D(t-s,w,y)dudwds \\ &\le \nu(|x-y|/6) \int_0^t \mathbb{P}^x(\tau_{W_1} > s) \int_{W_3} p(t-s, y-w)dwds. \end{aligned} \qquad (7.4.14)$$

By (7.4.2) and Lemma 7.2.1, for all $s \in (0,T]$ and $l \in (0, 2r_t]$, we have

$$\int_{B(y,l)} p(s, y-w)dw \le c\int_0^l -sL'(k)\exp\big(-c_2 sL(k)\big)dk \le c\exp\big(-c_3 sh(l)\big). \qquad (7.4.15)$$

Since $h(r_t) \simeq 1$, it follows that for all $s \in (0,t]$,

$$\begin{aligned} \int_{W_3} p(s,y-w)dw &\le \begin{cases} c\exp\big(-c_3 sh(|x-y|)\big), & \text{if } \ |x-y| \le 2r_t; \\ 1, & \text{if } \ |x-y| > 2r_t \end{cases} \\ &\le c\exp\big(-c_3 sh(|x-y|)\big). \end{aligned} \qquad (7.4.16)$$

Using the semigroup property and Proposition 7.3.8, we get from (7.4.15) that

$$\begin{aligned}
&\mathbb{P}^x(\tau_{W_1} > 2t/3) = \int_{W_1}\int_{W_1} p_{W_1}(t/3,x,v)p_{W_1}(t/3,v,u)dvdu \\
&\le \mathbb{P}^x(\tau_D > t/3)\int_{B(0,2l_t(x,y))} p(t/3,u)du \\
&\le ct^{-1/2}L(\delta_D(x))^{-1/2}\exp\big(-3^{-1}c_3th(2l_t(x,y))\big) \\
&\le ct^{-1/2}L(\delta_D(x))^{-1/2}\exp\big(-3^{-1}c_3th(|x-y|)\big). \qquad (7.4.17)
\end{aligned}$$

Then, using (7.4.14), (7.4.1), (7.4.15), (7.4.17) and Proposition 7.3.8, we obtain

$$\begin{aligned}
I_1 &\le c\nu(|x-y|)\int_0^t \mathbb{P}^x(\tau_{W_1} > s)\int_{W_3} p(t-s,y-w)dwds \\
&\le c\nu(|x-y|)\exp\big(-c_3th(|x-y|)/3\big)\int_0^{2t/3}\mathbb{P}^x(\tau_D > s)ds \\
&\quad + c\nu(|x-y|)\mathbb{P}^x(\tau_{W_1} > 2t/3)\int_0^{t/3}\exp\big(-c_3sh(|x-y|)\big)ds \\
&\le ct^{-1/2}L(\delta_D(x))^{-1/2}t\nu(|x-y|)\exp\big(-3^{-1}c_3th(|x-y|)\big). \qquad (7.4.18)
\end{aligned}$$

Secondly, by (7.4.2), (7.4.1), (7.3.2) and Proposition 7.3.8, since $p(t,\cdot)$ is radially nonincreasing, we get

$$\begin{aligned}
I_2 &\le c\,\mathbb{P}^x(Y_{\tau_{W_1}} \in W_2)\sup_{s\in[t/3,t)} p(s,|x-y|/2) \\
&\le c\,\mathbb{P}^x(Y_{\tau_{W_1}} \in W_2)\nu(|x-y|)\Big(\sup_{s\in[t/3,t)} s\exp\big(-b_2sh(|x-y|)\big)\Big) \\
&\le c\begin{cases} L(r_t)^{1/2}L(\delta_D(x))^{-1/2}t\nu(|x-y|)\exp\big(-3^{-1}b_2th(|x-y|)\big) \\ \qquad\qquad\qquad\qquad \text{if } |x-y| \ge 4r_t; \\ L(|x-y|)^{1/2}L(\delta_D(x))^{-1/2}t\nu(|x-y|)\exp\big(-3^{-1}b_2th(|x-y|)\big) \\ \qquad\qquad\qquad\qquad \text{if } |x-y| < 4r_t \end{cases} \\
&\le c\,t^{-1/2}L(\delta_D(x))^{-1/2}t\nu(|x-y|)\exp\big(-4^{-1}b_2th(|x-y|)\big). \qquad (7.4.19)
\end{aligned}$$

In the last inequality, we used the facts that $e^x \ge 2e\sqrt{x}$ for $x > 0$ and $h(r) \ge L(r)$ for $r > 0$.

Lastly, we note that $t \mapsto te^{-at}$ is increasing on $(0, 1/a)$ and decreasing on $(1/a, \infty)$. Thus, using similar calculation as the one given in (7.4.17), by monotonicity of $p(t, \cdot)$, (7.4.2), (7.4.1), Proposition 7.3.8 and (7.3.2), we have

$$
\begin{aligned}
I_3 &\le c\,\mathbb{P}^x(\tau_{W_1} > 2t/3)\nu(|x-y|)\Big(\sup_{s\in(0,t/3)} s\exp\big(-b_2 sh(|x-y|)\big)\Big) \\
&\le c\begin{cases} \mathbb{P}^x(\tau_{W_1} > 2t/3)\nu(|x-y|)h(|x-y|)^{-1} & \text{if } b_2th(|x-y|) \ge 3; \\ \mathbb{P}^x(\tau_D > 2t/3)\nu(|x-y|)t\exp\big(-3^{-1}b_2th(|x-y|)\big) & \\ & \text{if } b_2th(|x-y|) < 3 \end{cases} \\
&\le c\begin{cases} t^{-1/2}L(\delta_D(x))^{-1/2}t\nu(|x-y|)\exp\big(-3^{-1}c_3th(|x-y|)\big) & \\ & \text{if } b_2th(|x-y|) \ge 3; \\ t^{-1/2}L(\delta_D(x))^{-1/2}t\nu(|x-y|)\exp\big(-2^{-1}b_2th(|x-y|)\big) & \\ & \text{if } b_2th(|x-y|) < 3. \end{cases}
\end{aligned}
$$

Combining the above inequality with (7.4.18), (7.4.19) and (7.4.13), we arrive at (7.4.10).

(ii) We use the same notations as in the proof of (i) and follow that proof.

(Upper bound) By the semigroup property and (7.4.3), it suffices to show that there exist positive constants c_1 and b_4 such that

$$
\begin{aligned}
p_D(t,x,y) &\le c_1\Big(1 \wedge \frac{1}{tL(\delta_D(x))}\Big)^{1/2} \\
&\qquad \times t\nu(\theta_{3a_1}(|x-y|,t))\exp\big(-b_4th(\theta_{3a_1}(|x-y|,t))\big). \quad (7.4.20)
\end{aligned}
$$

By the similar argument to the one given in the proof of (i), we may assume $\delta_D(x) < r_t/2$ and $\delta_D(y) > 16\delta_D(x)$.

To prove (7.4.20), we first assume that $|x-y| \le [\ell^{-1}(3a_1/t)]^{-1}$. In this case, we have that $\theta_{a_1}(|x-y|, t/3) = [\ell^{-1}(3a_1/t)]^{-1}$. Then, by the semigroup property, (7.4.3) and Proposition 7.3.8, we get

$$p_D(t,x,y) = \int_D p_D(2t/3,x,z)p_D(t/3,z,y)dz \le c\,\mathbb{P}^x(\tau_D > 2t/3)p(t/3,0)$$
$$\le ct^{-1/2}L(\delta_D(x))^{-1/2}t\nu(\theta_{3a_1}(|x-y|,t))\exp\big(-3^{-1}b_1th(\theta_{3a_1}(|x-y|,t))\big).$$

Now, suppose that $|x-y| > [\ell^{-1}(3a_1/t)]^{-1}$. In this case, we use (7.4.13) and find upper bounds for I_1, I_2 and I_3. Observe that for all $s \in (0,T]$ and $l \in (0,2r_t]$, by (7.4.3) and the similar calculation to the one given in (7.4.15),

$$\int_{B(y,l)} p(s,y-w)dw \le c\begin{cases} l^d[\ell^{-1}(a_1/s)]^d\exp\big(-b_1sh([\ell^{-1}(a_1/s)]^{-1})\big), \\ \qquad\qquad\qquad\qquad \text{if } l \le [\ell^{-1}(a_1/s)]^{-1}; \\ \exp\big(-c_2sh(l)\big), \qquad \text{if } l > [\ell^{-1}(a_1/s)]^{-1} \end{cases}$$
$$\le c\exp\big(-c_3sh(\theta_{a_1}(l,s))\big). \tag{7.4.21}$$

Then, by using (7.4.21) instead of (7.4.15), we have that for all $0 < s \le T$,

$$\mathbb{P}^x(\tau_{W_1} > s) = \int_{W_1}\int_{W_1} p_{W_1}(s/3,x,u)p_{W_1}(2s/3,u,v)dudv$$
$$\le cs^{-1/2}L(\delta_D(x))^{-1/2}\exp\big(-c_4sh(\theta_{a_1}(|x-y|,2s/3))\big).$$

Hence, by the similar arguments to the ones for (7.4.16) and (7.4.18), we get

$$I_1 \le ct^{-1/2}L(\delta_D(x))^{-1/2}t\nu(|x-y|)\exp\big(-c_5th(|x-y|)\big).$$

Next, by (7.4.3), (7.4.1), monotonicity of h and the assumption that $|x-y| > [\ell^{-1}(3a_1/t)]^{-1}$, we have

$$\sup_{s\in[t/3,t)} p(s,|x-y|/2)$$
$$\le ct\sup_{s\in[t/3,t)}\Big[\nu(\theta_{a_1}(|x-y|,s))\exp\big(-3^{-1}b_1th(\theta_{a_1}(|x-y|,s))\big)\Big]$$
$$\le ct\nu(|x-y|)\exp\big(-c_7th(|x-y|)\big).$$

Therefore, by following argument in the proof for (7.4.19), we get

$$I_2 \le ct^{-1/2}L(\delta_D(x))^{-1/2}t\nu(|x-y|)\exp\big(-c_8th(|x-y|)\big).$$

Lastly, we note that since $|x-y| > [\ell^{-1}(3a_1/t)]^{-1}$,

$$\begin{aligned}&\sup_{s\in(0,t/3)}\Big[s\nu(\theta_{a_1}(|x-y|,s))\exp\big(-b_1sh(\theta_{a_1}(|x-y|,s))\big)\Big]\\&=\sup_{s\in(0,t/3)}\Big[s\nu(|x-y|)\exp\big(-b_1sh(|x-y|)\big)\Big].\end{aligned}$$

From this, by the same proof for estimating I_3 given in (i), we obtain

$$I_3 \le ct^{-1/2}L(\delta_D(x))^{-1/2}t\nu(|x-y|)\exp\big(-c_9th(|x-y|)\big).$$

The proof for the desired upper bound is complete.

(Lower bound) Fix $\eta > 0$. By Proposition 7.4.1, it remains to prove the lower bound when $|x-y| < [\ell^{-1}(\eta/t)]^{-1} \wedge R_2$, where R_2 is the constant in Lemma 7.3.7. Let $\zeta_t := [\ell^{-1}(\eta/t)]^{-1} \wedge R_2$ and define open neighborhoods of x and y as follows. Recall that $z_x, z_y \in \partial D$ are the points satisfying $\delta_D(x) = |x-z_x|$ and $\delta_D(y) = |y-z_y|$. We define

$$\mathcal{U}(x) = \begin{cases} B\big(x, V^{-1}(\frac{1}{8m}V(\zeta_t))\big) & \text{if } 8mV(\delta_D(x)) \ge V(\zeta_t);\\ B(z_x, \frac{1}{3}\zeta_t)\cap D, & \text{if } 8mV(\delta_D(x)) < V(\zeta_t)\end{cases}$$

and

$$\mathcal{U}(y) = \begin{cases} B\big(y, V^{-1}(\frac{1}{8m}V(\zeta_t))\big) & \text{if } 8mV(\delta_D(y)) \ge V(\zeta_t);\\ B(z_y, \frac{1}{3}\zeta_t)\cap D & \text{if } 8mV(\delta_D(y)) < V(\zeta_t),\end{cases}$$

where m is the constants in (7.4.5). Then $x \in \mathcal{U}(x) \subset D$ and $y \in \mathcal{U}(y) \subset D$.

By considering the cases $8mV(\delta_D(x)) \ge V(\zeta_t)$ and $8mV(\delta_D(x)) > V(\zeta_t)$ separately, one can see that there exist $c_1 > 0$ independent of η, and $c_2 = c_2(\eta) > 0$ such that

$$\mathbb{P}^x(\tau_{\mathcal{U}(x)} > t) \ge c_2\Big(1 \wedge \frac{1}{tL(\delta_D(x))}\Big)^{1/2}\exp\big(-c_1th(\zeta_t)\big). \quad (7.4.22)$$

Let $w_x := z_x + 4\zeta_t(x - z_x)/|x - z_x| \in D$ and define

$$\mathcal{W}_0 := B\big(w_x, \frac{\zeta_t}{2\sqrt{1+\Lambda^2}}\big) \quad \text{and} \quad \mathcal{W} := B\big(w_x, \frac{\zeta_t}{\sqrt{1+\Lambda^2}}\big) \subset D.$$

Then, for all $u \in \mathcal{U}(x)$ and $v \in \mathcal{W}$, we have $|u - v| \simeq \zeta_t$. Moreover, since $|x - y| < \zeta_t$, we also have $|u' - v| \simeq \zeta_t$ for all $u' \in \mathcal{U}(y)$ and $v \in \mathcal{W}$. Thus, for every $v \in \mathcal{W}_0$, by following arguments in the proofs for (7.4.6), (7.3.1) and (7.4.22), we get

$$\begin{aligned} p_D(t/2, x, v) &\geq ct\nu(\zeta_t)\mathbb{P}^x(\tau_{\mathcal{U}(x)} > t/2)\mathbb{P}^v\big(\tau_{B(v,\zeta_t/(2\sqrt{1+\Lambda^2}))} > t/2\big) \\ &\geq c\Big(1 \wedge \frac{1}{tL(\delta_D(x))}\Big)^{1/2} t\nu(\zeta_t)\exp\big(-c_2 th(\zeta_t)\big). \end{aligned}$$

Similarly, we also have that

$$p_D(t/2, v, y) \geq c\Big(1 \wedge \frac{1}{tL(\delta_D(y))}\Big)^{1/2} t\nu(\zeta_t)\exp\big(-c_2 th(\zeta_t)\big).$$

By the semigroup property and **(A)**, it follows that

$$\begin{aligned} p_D(t, x, y) &\geq \int_{\mathcal{W}} p_D(t/2, x, v)p_D(t/2, v, y)dv \\ &\geq c\Big(1 \wedge \frac{1}{tL(\delta_D(x))}\Big)^{1/2}\Big(1 \wedge \frac{1}{tL(\delta_D(y))}\Big)^{1/2} t^2|\mathcal{W}|\,\nu(\zeta_t)^2\exp\big(-2c_2 th(\zeta_t)\big) \\ &\geq c\Big(1 \wedge \frac{1}{tL(\delta_D(x))}\Big)^{1/2}\Big(1 \wedge \frac{1}{tL(\delta_D(y))}\Big)^{1/2} t^2\ell(\zeta_t^{-1})\nu(\zeta_t)\exp\big(-2c_2 th(\zeta_t)\big). \end{aligned}$$

If $\zeta_t = [\ell^{-1}(\eta/t)]^{-1}$, then since ℓ is almost increasing, we get $\ell(\zeta_t^{-1}) \simeq t^{-1}$. Hence, we are done. If $\zeta_t = R_2$, then $t \simeq 1$ so that $t^2\ell(\zeta_t^{-1})\nu(\zeta_t)\exp\big(-2c_2 th(\zeta_t)\big) \simeq t\nu([\ell^{-1}(\eta/t)]^{-1})\exp\big(-cth([\ell^{-1}(\eta/t)]^{-1})\big) \simeq 1$. The proof is complete. □

7.5 Large time estimates

In this section, we give the proof of Theorem 7.1.3. Let D be a fixed bounded $C^{1,1}$ open subset in $\mathbb{R}^d$ of scale (r_1, r_2) and $x_1, x_2 \in \mathbb{R}^d$ be the fixed points satisfying $B(x_1, r_1) \subset D \subset B(x_2, r_2)$.

We mention that under condition **(L-1)**, the semigroup $(P_t^D)_{t\geq 0}$ of Y_t^D may not be compact operators in $L^2(D)$, though D is bounded. (See, Proposition 7.2.3.) Hence, in that case, we need some lemmas to obtain the large time estimates instead of spectral theory.

Lemma 7.5.1. *There exists a constant $C > 0$ which only depend on the dimension d such that for all $(t, x, y) \in (0, \infty) \times D \times D$,*

$$p_D(t, x, y) \leq Cp(t/2, |x - y|/2) \exp\big(-2^{-1}\kappa_1 C_5 t h(r_2)\big).$$

Proof. By the semigroup property, we have

$$\begin{aligned}
&p_D(t, x, y) \\
&\leq \left(\int_{\{z\in D:|y-z|>|x-y|/2\}} + \int_{\{z\in D:|x-z|>|x-y|/2\}}\right) p_D(t/2, x, z)p_D(t/2, z, y)dz \\
&\leq p(t/2, |x - y|/2)\big(\mathbb{P}^x(\tau_{B(x_2,r_2)} > t/2) + \mathbb{P}^y(\tau_{B(x_2,r_2)} > t/2)\big).
\end{aligned}$$

Hence, we get the result from (7.3.1). □

Define for $r \geq 1$,

$$\widehat{\ell}(r) := \sup_{s\in[1,r]} \frac{1}{\ell(s)} \qquad \text{and} \qquad \widehat{\Phi}(r) := \int_1^r \frac{1}{k\widehat{\ell}(k)}dk.$$

If **(L-1)** holds, then by following the proof of Lemma 7.2.2, we see that there exist positive constants C_6 and C_7 which only depend on the dimension d, and the constants κ_1 and κ_2 in (7.1.2) such that

$$\widehat{\ell}(r)^{-1} \simeq \ell(r) \quad \text{for } r \geq 2 \tag{7.5.1}$$

and

$$C_6\widehat{\Phi}(r) \le \psi(r) \quad \text{and} \quad h(r^{-1}) \le C_7\widehat{\Phi}(r) \qquad \text{for all } r \ge 2. \tag{7.5.2}$$

Lemma 7.5.2. *Assume that* **(L-1)** *holds. Then, there exists a constant $b_5 = b_5(d, \psi, r_2) > 0$ such that for every $T > 0$, there exist $c_1, c_2 > 0$ such that for all $t \in [T, \infty)$ and $|x| \le 2r_2$,*

$$p(t, x) \le c_1 + c_2\nu(|x|)\exp\big(-b_5 t h(|x|)\big). \tag{7.5.3}$$

Proof. Fix $t \in [T, \infty)$ and $x \in \mathbb{R}^d$ satisfying $|x| \le 2r_2$, and let $r := |x|$. By [76, (5.4)], the mean value theorem, Lemma 7.2.6, (7.5.1) and (7.5.2), we have that

$$\begin{aligned}
r^d p(t, x) &\le c\int_{\mathbb{R}^d} \left(e^{-t\psi(|z|/r)} - e^{-t\psi(2|z|/r)}\right) e^{-|z|^2/4} dz \\
&\le c\int_{|z|\le 2r} dz + ct\int_{|z|>2r} \sup_{|z|\le y\le 2|z|} e^{-t\psi(y/r)}\big|\psi(2|z|/r) - \psi(|z|/r)\big|e^{-|z|^2/4}dz \\
&\le cr^d + ct\int_{2r}^{4r_2} e^{-C_6t\widehat{\Phi}(u/r)}\frac{u^{d-1}}{\widehat{\ell}(u/r)}du + ct\int_{4r_2}^{\infty} e^{-C_6t\widehat{\Phi}(u/r)}\frac{u^{d-1}}{\widehat{\ell}(u/r)}e^{-u^2/4}du \\
&=: cr^d + I_1 + I_2.
\end{aligned} \tag{7.5.4}$$

Using scaling properties of $\widehat{\ell}$ and $\widehat{\Phi}$, and (7.5.1), since $\widehat{\ell}$ and $\widehat{\Phi}$ are increasing, we get that

$$\begin{aligned}
I_2 &\le \frac{ct}{\widehat{\ell}(4r_2/r)}e^{-2^{-1}C_6t\widehat{\Phi}(4r_2/r)}e^{-2^{-1}C_6t\widehat{\Phi}(4r_2/r)}\int_{4r_2}^{\infty} u^{d-1}e^{-u^2/4}du \\
&\le \frac{ct}{\widehat{\ell}(1/r)}e^{-2^{-1}C_6t\widehat{\Phi}(4r_2/(2r_2))}e^{-c_1t\widehat{\Phi}(1/r)} \le c\ell(1/r)e^{-c_1t\widehat{\Phi}(1/r)}.
\end{aligned} \tag{7.5.5}$$

In the last inequality above, we used the fact that $\sup_{s>0} se^{-2^{-1}C_6s\widehat{\Phi}(2)} < \infty$.

Next, we set $q_{\gamma,k}(u) := u^\gamma \exp(-kt\widehat{\Phi}(u))$ for $u \ge 2$ and $\gamma, k > 0$. Then for any $\gamma, k > 0$, $\frac{d}{du}q_{\gamma,k}(u) = (\gamma - kt\widehat{\ell}(u)^{-1})q_{\gamma-1,k}(u)$. Since $\widehat{\ell}$ is increasing, it follows that there exists $u_0 \in [2, \infty)$ such that q is decreasing on $[2, u_0]$ and increasing on $[u_0, \infty)$. Thus, for any $[a, b] \subset [2, \infty)$ and $\gamma, k > 0$, it holds that

$$\sup_{u \in [a,b]} q_{\gamma,k}(u) = q_{\gamma,k}(a) \vee q_{\gamma,k}(b). \tag{7.5.6}$$

Note that $d + \alpha_1 > 0$ since we assumed **(A)**. Let

$$\rho := \frac{d+\alpha_1}{2} \in (\frac{\alpha_1}{2} \vee 0, d+\alpha_1) \quad \text{and} \quad \epsilon := \frac{d+\alpha_1-\rho}{d+\rho} \in (0,1). \tag{7.5.7}$$

If $q_{d+\rho, C_6}(2) \geq q_{d+\rho, C_6}(4r_2/r)$, then using a change of the variables, (7.5.6) and the fact that $\sup_{s>0} se^{-C_6 s \widehat{\Phi}(2)} < \infty$, since $\widehat{\ell}$ is increasing, we get

$$\begin{aligned} I_1 &= ctr^d \int_2^{4r_2/r} \frac{u^d}{u\widehat{\ell}(u)} e^{-C_6 t \widehat{\Phi}(u)} du = ctr^d \int_2^{4r_2/r} \frac{q_{d+\rho, C_6}(u)}{u^{1+\rho}\widehat{\ell}(u)} du \\ &\leq ctr^d \frac{q_{d+\rho, C_6}(2)}{\widehat{\ell}(2)} \int_2^{4r_2/r} \frac{du}{u^{1+\rho}} \leq ctr^d e^{-C_6 t \widehat{\Phi}(2)} \leq \frac{c}{C_6 \widehat{\Phi}(2)} (2r_2)^d. \end{aligned}$$

Hence, we obtain (7.5.3) from (7.5.4), (7.5.5), (7.1.2) and (7.5.2) in this case.

If $q_{d+\rho, C_6}(2) < q_{d+\rho, C_6}(4r_2/r)$, then by a change of the variables, (7.5.1), (7.5.6), (7.5.7) and scaling properties of ℓ and $\widehat{\Phi}$, since $\epsilon \in (0,1)$, we get that

$$\begin{aligned} I_1 &= ctr^d \int_2^{4r_2/r} \frac{q_{d+\alpha_1-\rho, C_6}(u)}{u^{1-\rho} u^{\alpha_1} \widehat{\ell}(u)} du \leq \frac{ctr^d}{(4r_2/r)^{\alpha_1} \widehat{\ell}(4r_2/r)} \int_2^{4r_2/r} \frac{q_{d+\alpha_1-\rho, \epsilon C_6}(u)}{u^{1-\rho}} du \\ &\leq ctr^{d+\alpha_1} \ell(1/r) \big(q_{d+\alpha_1-\rho, \epsilon C_6}(2) \vee q_{d+\alpha_1-\rho, \epsilon C_6}(4r_2/r) \big) \int_2^{4r_2/r} \frac{du}{u^{1-\rho}} \\ &\leq ctr^{d+\alpha_1-\rho} \ell(1/r) \big(q_{d+\rho, C_6}(2) \vee q_{d+\rho, C_6}(4r_2/r) \big)^{(d+\alpha_1-\rho)/(d+\rho)} \\ &= ctr^{d+\alpha_1-\rho} \ell(1/r) q_{d+\alpha_1-\rho, \epsilon C_6}(4r_2/r) = ct\ell(1/r) \exp\big(-\epsilon C_6 t \widehat{\Phi}(4r_2/r) \big) \\ &\leq c\ell(1/r) \exp\big(-2^{-1} \epsilon C_6 t \widehat{\Phi}(4r_2/r) \big) \leq c\ell(1/r) \exp\big(-c_2 t \widehat{\Phi}(1/r) \big). \end{aligned}$$

Then we get (7.5.3) by using (7.5.4), (7.5.5), (7.1.2) and (7.5.2) again. □

Proof of Theorem 7.1.3. Choose any $x, y \in D$ and denote $a(x,y) := L(\delta_D(x))^{-1/2} L(\delta_D(y))^{-1/2}$.

(i) Let $x_1 \in D$ be a point such that $B(x_1, r_1) \subset D$. Using the semigroup property, Theorem 7.1.1(i), (7.4.2) and (7.3.1), we get that for all $t \geq T$,

$$\begin{aligned} p_D(t,x,y) &\geq \int_{B(x_1,\frac{r_1}{4})}\int_{B(x_1,\frac{3r_1}{4})} p_D(\frac{T}{4},x,u)\,p_D(t-\frac{T}{2},u,v)\,p_D(\frac{T}{4},v,y)dvdu \\ &\geq c_1^2 a(x,y)\int_{B(x_1,\frac{r_1}{4})}\int_{B(x_1,\frac{3r_1}{4})} p(c_2T,2r_2)^2\,p_D(t-\frac{T}{2},u,v)dvdu \\ &\geq c_3 a(x,y)\int_{B(x_1,\frac{r_1}{4})} \mathbb{P}^u(\tau_{B(x_1,\frac{3r_1}{4})} > t-\frac{T}{2})du \\ &\geq c_4 a(x,y)\inf_{u\in B(x_1,\frac{r_1}{4})} \mathbb{P}^u(\tau_{B(x_1,\frac{3r_1}{4})} > t-\frac{T}{2}) \geq c_5 a(x,y)e^{-\kappa_2 C_4 th(r_1/2)}. \quad (7.5.8)\end{aligned}$$

Moreover, since D is a bounded set and L is decreasing, one can follow the proof of Proposition 7.4.1, after changing the definition of $l_t(x,y)$ therein from $r_t \wedge (|x-y|/4)$ to $|x-y|/4$, and see that for all $t \geq T$,

$$\begin{aligned} &p_D(t,x,y) \\ &\geq c_6\left(1\wedge\frac{L(|x-y|)}{L(\delta_D(x))}\right)^{1/2}\left(1\wedge\frac{L(|x-y|)}{L(\delta_D(x))}\right)^{1/2} t\nu(|x-y|)e^{-c_7th(|x-y|)} \\ &\geq c_6T\left(1\wedge\frac{L(2r_2)}{L(\delta_D(x))}\right)^{1/2}\left(1\wedge\frac{L(2r_2)}{L(\delta_D(x))}\right)^{1/2} \nu(|x-y|)e^{-c_7th(|x-y|)} \\ &\geq c_8 a(x,y)\nu(|x-y|)e^{-c_7th(|x-y|)}. \quad (7.5.9)\end{aligned}$$

By combining (7.5.8) with (7.5.9), we get the desired lower bound.

On the other hand, using the semigroup property, Theorem 7.1.1(i), Corollary 7.2.12, Lemma 7.5.1 and Lemma 7.5.2, we get that

$$\begin{aligned} &p_D(t,x,y) = \int_D\int_D p_D(\frac{T}{4},x,u)\,p_D(t-\frac{T}{2},u,v)\,p_D(\frac{T}{4},v,y)dudv \\ &\leq c_9 a(x,y)e^{-2^{-1}\kappa_1 C_5 th(r_2)} \\ &\quad\times\int_D\int_D p(\frac{c_{10}T}{4},\frac{|x-u|}{2})\,p(\frac{2t-T}{4},\frac{|u-v|}{2})\,p(\frac{c_{10}T}{4},\frac{|v-y|}{2})dudv \\ &\leq c_{11} a(x,y)\,p(\frac{2t-(1-2c_{10})T}{4},\frac{|x-y|}{2})\,e^{-2^{-1}\kappa_1 C_5 th(r_2)} \\ &\leq c_{12} a(x,y)\left(1+\nu(|x-y|)e^{-2^{-1}b_5 th(|x-y|)}\right)e^{-2^{-1}\kappa_1 C_5 th(r_2)}.\end{aligned}$$

The proof for (i) is complete.

(ii)-(iii) Since the proof for (iii) is similar and easier, we only give the proof for (ii).

By Proposition 7.2.3, there exist $T_0 > 0$ such that the semigroup $(P_t^D)_{t \geq T_0}$ consists of compact operators. Let $0 < \mu_1 < 1$ be the largest eigenvalue of the operator $P_{T_0}^D$ and $\phi_1 \in L^2(D)$ be the corresponding eigenfunction with unit L^2-norm. For each $n \geq 1$, we denote by $(\mu_{n,k})_{k\geq 1}$ the discrete spectrum of $P_{nT_0}^D$, arranged in decreasing order and repeated according to their multiplicity and $(\phi_{n,k})_{k\geq 1}$ be the corresponding eigenfunctions with unit L^2-norm. Then, by the semigroup property, we have $\mu_{n,1} = \mu_1^n$ and $\phi_{n,1} = \phi_1$ for all $n \geq 1$. From the eigenfunction expansion of $p_D(nT_0, u, \cdot)$, Parseval's identity and Cauchy inequality, we see that for all $n \geq 1$,

$$\begin{aligned} \int_{D\times D} p_D(nT_0, u, v)dudv &= \sum_{k=1}^{\infty} \mu_{n,k} \left(\int_D \phi_{n,k}(v)dv \right)^2 \\ &\leq \sup_k \mu_{n,k} \|\phi_{n,k}\|_{L^2(D)}^2 \|\mathbf{1}_D\|_{L^2(D)}^2 = \mu_1^n |D|. \end{aligned} \tag{7.5.10}$$

Besides, for all $s > 0$ and $u \in D$, using the fact that $p(T_0, 0) < \infty$ and Cauchy inequality, we get that

$$\begin{aligned} \phi_1(u) &\leq \int_D \int_D p_D(s, u, z) p_D(T_0, z, v) \phi_1(y) dzdv \leq c_0 \mathbb{P}^u(\tau_D > s) \int_D \phi_1(v)dv \\ &\leq c_{13} \mathbb{P}^u(\tau_D > s) \|\phi_1\|_{L^2(D)} \|\mathbf{1}_D\|_{L^2(D)} = c_{13} |D|^{1/2}\, \mathbb{P}^u(\tau_D > s). \end{aligned}$$

Thus, we obtain for all $0 < s \leq T_0$ and $n \geq 1$,

$$\begin{aligned} &\int_D \int_D \mathbb{P}^u(\tau_D > s) p_D(nT_0, u, v) \mathbb{P}^v(\tau_D > s) dudv \\ &\geq \mu_1^n \left(\int_D \mathbb{P}^z(\tau_D > s)\phi_1(z)dz \right)^2 \geq \mu_1^n \left(\int_D c_{13}^{-1} |D|^{-1/2}\, \phi_1(z)^2 dz \right)^2 \\ &\geq c_{13}^{-2} \mu_1^n |D|^{-1}. \end{aligned} \tag{7.5.11}$$

Let $t \geq 4T_0$. We set $n := \lfloor (t - 3T_0)/T_0 \rfloor \geq 1$ and $s := (t - (n+2)T_0)/2 \in$

$[T_0/2, T_0)$. Recall $a(x,y) = L(\delta_D(x))^{-1/2}L(\delta_D(y))^{-1/2}$. From the semigroup property, (7.5.10) and Corollary 7.3.9, using the fact that $p(T_0, 0) < \infty$, we deduce that

$$\begin{aligned} p_D(t,x,y) &= \int_{D\times D\times D\times D} p_D(s,x,z_1)p_D(T_0,z_1,z_2)p_D(nT_0,z_2,z_3) \\ &\qquad\qquad \times p_D(T_0,z_3,z_4)p_D(s,z_4,y)dz_1dz_2dz_3dz_4 \\ &\le c_{14}^2\Big(\int_D p_D(s,x,z_1)dz_1\Big)\Big(\int_D p_D(s,z_4,y)dz_4\Big)\int_{D\times D} p_D(nT_0,z_2,z_3)dz_2dz_3 \\ &\le c_{14}^2\,\mu_1^n|D|\,\mathbb{P}^x(\tau_D > s/2)\mathbb{P}^y(\tau_D > s/2) \le c_{15}a(x,y)e^{-\lambda_1 t}, \end{aligned}$$

where $\lambda_1 := T_0^{-1}\log(\mu_1^{-1})$. Moreover, using Theorem 7.1.1, Corollary 7.3.9 and (7.5.11), we also get that

$$\begin{aligned} p_D(t,x,y) &= \int_{D\times D} p_D(s,x,z_1)p_D((n+2)T_0,z_1,z_2)p_D(s,z_2,y)dz_1dz_2 \\ &\ge c_{16}a(x,y)\int_{D\times D} \mathbb{P}^{z_1}(\tau_D > s/2)p_D((n+2)T_0,z_1,z_2)\mathbb{P}^{z_2}(\tau_D > s/2)dz_1dz_2 \\ &\ge c_{17}a(x,y)e^{-\lambda_1 t}. \end{aligned}$$

The proof is complete. □

7.6 Green function estimates

In this section, we give the proof of Theorem 7.1.4. Throughout this section, we assume that **(D)** further holds, and D is a Borel subset of $\mathbb{R}^d$.

Using the subadditivity of the renewal function V and (7.3.2), we get the following lemma.

Lemma 7.6.1. [53, Lemma 7.1] *It holds that for all $x, y \in D$,*

$$\left(1\wedge\frac{V(\delta_D(x))}{V(|x-y|)}\right)\left(1\wedge\frac{V(\delta_D(y))}{V(|x-y|)}\right) \simeq \left(1\wedge\frac{V(\delta_D(x))V(\delta_D(y))}{V(|x-y|)^2}\right).$$

In particular, if D is bounded then for all $x, y \in D$

$$\left(1 \wedge \frac{L(|x-y|)}{L(\delta_D(x))}\right)^{1/2} \left(1 \wedge \frac{L(|x-y|)}{L(\delta_D(y))}\right)^{1/2} \simeq \left(1 \wedge \frac{L(|x-y|)}{\sqrt{L(\delta_D(x))L(\delta_D(y))}}\right).$$

Since we assumed **(A)** and **(D)** hold true, (7.1.1) holds with $\alpha_2 < 2 \wedge d$. Using this fact, we get the following lemma.

Lemma 7.6.2. [53, Lemma 7.2] *It holds that*

$$\liminf_{r\to 0} \frac{\nu(r)}{L(r)} = \liminf_{r\to 0} \frac{\nu(r)}{L(r)^2} = \infty.$$

Recall that the Green function $G_D(x, y)$ is defined by

$$G_D(x, y) := \int_0^\infty p_D(t, x, y)dt.$$

Since the process Y may be recurrent, we can not expect to obtain upper estimates for $G_{\mathbb{R}^d}(x, y)$ in general. However, when D is bounded, we can establish a prior upper estimates for $G_D(x, y)$ regardless of transience of Y using Lemma 7.5.1.

Lemma 7.6.3. [53, Lemma 7.3] *Suppose that D is bounded. Then, there exists a constant $c_1 = c_1(d, \psi, \mathrm{diam}(D)) > 0$ such that for all $x, y \in D$,*

$$G_D(x, y) \le \frac{c_1 \ell(|x-y|^{-1})}{|x-y|^d L(|x-y|)^2} \simeq \frac{\nu(|x-y|)}{L(|x-y|)^2}.$$

Now, we prove Theorem 7.1.4.

Proof of Theorem 7.1.4. By (7.1.2) and Lemma 7.2.1, it suffices to prove that for all $x, y \in D$,

$$G_D(x, y) \simeq \Big(1 \wedge \big[a(x, y)L(|x-y|)\big]\Big) \frac{\nu(|x-y|)}{h(|x-y|)^2},$$

where $a(x, y) := L(\delta_D(x))^{-1/2} L(\delta_D(y))^{-1/2}$.

(Lower bound) Using Proposition 7.4.1 in the second line below, the change of the variables $s = th(|x-y|)$ in the third line, the fact that $h(r) \geq L(r)$ for all $r > 0$ in the fourth line, and Lemma 7.6.1 and (7.3.2) in the fifth line, we get that

$$\begin{aligned}
G_D(x,y) &\geq \int_0^1 p_D(t,x,y)dt \\
&\geq c\nu(|x-y|)\int_0^1 \left(1 \wedge \frac{1}{tL(\delta_D(x))}\right)^{1/2}\left(1 \wedge \frac{1}{tL(\delta_D(y))}\right)^{1/2} te^{-c_1 th(|x-y|)}dt \\
&= c\frac{\nu(|x-y|)}{h(|x-y|)^2}\int_0^{h(|x-y|)} \left(1 \wedge \frac{h(|x-y|)}{sL(\delta_D(x))}\right)^{1/2}\left(1 \wedge \frac{h(|x-y|)}{sL(\delta_D(y))}\right)^{1/2} se^{-c_1 s}ds \\
&\geq c\frac{\nu(|x-y|)}{h(|x-y|)^2}\left(1 \wedge \frac{L(|x-y|)}{L(\delta_D(x))}\right)^{1/2}\left(1 \wedge \frac{L(|x-y|)}{L(\delta_D(y))}\right)^{1/2}\int_0^{h(2r_2)\wedge 1} se^{-c_1 s}ds \\
&\geq c\Big(1 \wedge \big[a(x,y)L(|x-y|)\big]\Big)\frac{\nu(|x-y|)}{h(|x-y|)^2}. \qquad (7.6.1)
\end{aligned}$$

(Upper bound) Using boundary Harnack principle and Lemma 7.6.3, one can prove the upper bound following the proofs of [87, Theorem 1.2 and Theorem 6.4] and [94, Theorem 4.6] line by line. Below, we give the main steps of the proof only.

By the boundary Harnack principle (which holds true under **(A)** and **(B)** by [74, Theorem 1.9]), Lemma 7.6.3 and (7.6.1), we can follow the proof of [87, Theorem 6.4] to obtain

$$G_D(x,y) \leq c\frac{g_D(x)g_D(y)}{g_D(A)^2}\frac{\nu(|x-y|)}{h(|x-y|)^2}, \qquad (7.6.2)$$

where $g_D(z) := G_D(z,z_0)\wedge c_1$ for some fixed constant $c_1 > 0$, $z_0 \in D$ is a fixed point in D and $A \in \mathcal{B}(x,y)$, where $\mathcal{B}(x,y)$ is given by [87, (6.7)]. Moreover, we can also follow the proof of [94, Theorem 4.6] to show that for all $z \in D$,

$$g_D(z) \simeq L(\delta_D(z))^{-1/2}. \qquad (7.6.3)$$

Indeed, let $R_3 := \delta_D(z_0) \wedge R_2$ where R_2 is the constant in Lemma 7.3.7. If

$\delta_D(z) \geq R_3/8$, then $L(\delta_D(z))^{-1/2} \simeq 1 \simeq g_D(z)$ by (7.6.1) and Lemma 7.6.2. Hence, (7.6.3) holds in this case.

Next, we assume that $\delta_D(z) < R_3/8$. Then $|z - z_0| \geq \delta_D(z_0) - \delta_D(z) \geq 7R_3/8$. Thus, by Lemma 7.6.2, $g_D(z) \simeq G_D(z, z_0)$. Choose $w_z \in \partial D$ satisfying $\delta_D(z) = |z-w_z|$. Let $z^* := w_z + R_3(z-w_z)/(4|z-w_z|) \in D$ and define $U(z,1)$ as (7.3.7). Then, by the boundary Harnack principle, (7.3.6), Lemma 7.6.3, (7.6.1) and Proposition 7.3.8, we get

$$\begin{aligned} g_D(z) \simeq G_D(z, z_0) &\simeq G_D(z^*, z_0) \frac{\mathbb{P}^z\big(Y_{\tau_{U(z,1)}} \in D\big)}{\mathbb{P}^{z^*}\big(Y_{\tau_{U(z,1)}} \in D\big)} \\ &\simeq \mathbb{P}^z\big(Y_{\tau_{U(z,1)}} \in D\big) \simeq L(\delta_D(z))^{-1/2}. \end{aligned}$$

Hence, (7.6.3) is valid.

We see from the definition of $\mathcal{B}(x,y)$ that $\delta_D(A) \geq c_2|x-y|$. Thus, by combining (7.6.2) and (7.6.3), we get from (7.3.4) that

$$G_D(x,y) \leq c_3 a(x,y) L(\delta_D(A)) \frac{\nu(|x-y|)}{h(|x-y|)^2} \leq c_4 a(x,y) L(|x-y|) \frac{\nu(|x-y|)}{h(|x-y|)^2}.$$

This together with Lemma 7.6.3 complete the proof. □

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국문초록

마르코프 확률과정의 추이확률밀도는 확률론과 해석학 모두에서 중요한 연구대상이다. 무한소생성자가 $\mathcal{L}$로 주어진 마르코프 확률과정의 추이확률밀도함수 $p(t,x,y)$는 편미분방정식 $\partial_t u = \mathcal{L}u$의 기본해이다. 따라서 추이확률밀도 $p(t,x,y)$는 작용소 $\mathcal{L}$의 열핵으로도 알려져있다. 열핵의 중요성에도 불구하고, 열핵에 대한 정확한 표현은 극히 드문 경우에만 알려져있다. 대신에, 열핵에 대한 추정에 대해 많은 연구가 이루어지고 있다. 본 학위논문은 마르코프 도약과정의 열핵 추정에 대한 것으로 크게 여섯 부분으로 이루어져 있다. 논문의 첫번째 부분에서는 종속자, 즉, 감소하지 않는 일차원 레비 과정을 다룬다. 두번째 부분에서는 임계 킬링이 있는 비국소적 작용소의 열핵을 다룬다. 이를 통해 세번째 부분에서는 킬링이 있는 마르코프 확률과정의 종속과정에 대한 연구를 진행한다. 네번째 부분에서는, 열핵의 안정성 이론의 관점에서 세번째 부분의 결과를 바탕으로, 디리클레 형식을 이용하여 정의된 퇴화와 임계 킬링이 있는 도약과정의 열핵에 대한 추정을 연구한다. 다섯번째 부분은 일반적인 시간분수적 디리클레 문제의 기본해에 대한 것이다. 마지막 부분에서는 작은 도약이 비교적 드물게 일어나는 등방성 단봉분포를 갖는 레비 과정의 디리클레 열핵에 대한 추정을 다룬다.

주요어휘: 마르코프 확률과정, 열핵 추정, 비국소적 작용소, 디리클레 형식
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