이학박사 학위논문

Maximal and sharp regularity bounds on averages over curves

(곡선 위에서 평균의 최적 정칙성과 극대 유계)

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이 논문을 이학박사 학위논문으로 제출함

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Maximal and sharp regularity bounds on averages over curves

A dissertation
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Abstract

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In this thesis, we study the problems of characterizing maximal and smoothing bounds on averages over curves in $\mathbb{R}^d$. Maximal and sharp smoothing estimates for integral transforms defined by averages over submanifolds are fundamental subjects in harmonic analysis, which have been extensively studied since the 1970s. Despite the simple geometric structure of curves, maximal and smoothing bounds on averages over curves have remained largely unknown except for those in low dimensions. We make breakthrough contributions to the problems in every dimension. First of all, we prove the optimal L^p Sobolev regularity estimate for averages over curves in every dimension $d \geq 3$ except for some endpoint cases. This settles the conjecture raised by Beltran, Guo, Hickman, and Seeger. Secondly, we obtain the local smoothing estimate of sharp order. As a consequence, we establish, for the first time, nontrivial L^p boundedness of the maximal averages over curves when $d \geq 4$. Lastly, we prove the maximal bound on the optimal range when $d = 3$.

Key words: Averaging operator, Maximal bound, Sobolev regularity, Local smoothing
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Contents

Chapter 1

Introduction

The maximal operators associated to averages over geometric objects are important topics in mathematical analysis. One of the most important examples is the Hardy-Littlewood maximal function:

$$\mathbb{M}f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy$$

where $B(x,r)$ is the ball of radius r centered at x in $\mathbb{R}^d$. It is well-known that $\mathbb{M}$ is bounded on L^p for $1 < p \leq \infty$ and $\mathbb{M}$ is bounded from L^1 to weak L^1. The Maximal bounds imply pointwise convergence. More precisely, L^1 to weak L^1 boundedness of $\mathbb{M}$ proves the classical Lebesgue's differentiation theorem: If $f \in L^1_{loc}(\mathbb{R}^d)$, then

$$\lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) dy = f(x), \quad \text{for a.e. } x.$$

Let $O \subset B(0,1)$ be a measurable set such that $|O| > 0$. Then the same results hold for the maximal operator $M_O f(x) := \sup_{r>0}(1/|O|) \int_O |f(x-ry)| dy$ since $M_O f \lesssim (1/|O|)\mathbb{M}f$. This gives a rise to a natural question:

> Is it possible to obtain L^p boundedness of the maximal function when $|O| = 0$?

Such kind of maximal bounds and related convergence problems have received much attention for the last half century(see [57]). For example, averaging and maximal operators associated to hypersurfaces have been most extensively studied.

Let $S \subset \mathbb{R}^d$ be a compact submanifold. Define a measure $\mathfrak{m}_t$ supported on tS by

$$\langle \mathfrak{m}_t, f\rangle := \int_S f(ty)d\mathfrak{m}_S(y)$$

where $\mathfrak{m}_S$ is the induced Lebesgue measure on S. Now, we set

$$\mathcal{A}_t f(x) := f * \mathfrak{m}_t(x)$$

and

$$Mf(x) := \sup_{t>0} |\mathcal{A}_t f(x)|.$$

The regularity property of $\mathcal{A}_t$ is a fundamental subject in harmonic analysis, which has been extensively studied since the 1970s. There is an immense body of literature devoted to the subject (see, for example, [57, 40, 55, 16] and references therein). However, numerous problems remain wide open. The regularity property is typically addressed in the frameworks of L^p improving, L^p Sobolev regularity, and local smoothing estimates, to which L^p boundedness of the maximal average is also closely related(see [57],[53]).

First, we consider regularity property of $\mathcal{A}_t$ for a fixed $t \neq 0$. For the purpose it is sufficient to consider $\mathcal{A}_1$. The followings are called L^p improving and L^p Sobolev regularity problems.

Question 1 (L^p improving)**.** *What is the optimal $q(p) \geq p$ for which*

$$\|\mathcal{A}_1 f\|_{q(p)} \lesssim \|f\|_p$$

holds for all $f \in L^p$?

Question 2 (L^p Sobolev regularity)**.** *What is the optimal $\alpha(p)$ for which*

$$\|\mathcal{A}_1 f\|_{L^p_{\alpha(p)}} \lesssim \|f\|_p \tag{1.0.1}$$

holds for all $f \in L^p$?

Secondly, we consider problems for $\mathcal{A}_t$ with varying t. The followings are known as local smoothing and maximal estimate problems.

Question 3 (Local smoothing)**.** *What is the optimal $\alpha(p)$ for which*

$$\|\chi(t)\mathcal{A}_t f\|_{L^p_{\alpha(p)}(\mathbb{R}^{d+1})} \lesssim \|f\|_p \tag{1.0.2}$$

holds for all $f \in L^p(\mathbb{R}^d)$ where χ is smooth function defined on $(1/2, 4)$?

Question 4 (Maximal estimate)**.** *What is the optimal p for which*

$$\|Mf\|_p \lesssim \|f\|_p$$

holds for all $f \in L^p$?

Stein's remarkable result [54] tells that the decay of $\widehat{\mathfrak{m}_S}$ implies optimal maximal estimates when S is the sphere in $\mathbb{R}^d$ for $d \geq 3$. After stein's result, numerous authors have studied regularity estimates when S is a hypersurface. In their result, the geometric property of S plays an important role in deciding regularity properties of averages over S. The best possible results are available when S is nondegenerate, that is to say, S has nonvanishing Gaussian curvature.

When S is nondegenerate, L^p improving, L^p Sobolev, and maximal estimates are well-understood. Littman proved L^p improving: $\|\mathcal{A}_1 f\|_q \lesssim \|f\|_p$ if and only if (p^{-1}, q^{-1}) in the closed triangle with vertices $(0,0), (1,1), (d/(d+1), 1/(d+1)))$. By the works of Miyachi, Seeger, Sogge, and Stein, it is known that: $\|\mathcal{A}_1 f\|_{L^p_\alpha} \lesssim \|f\|_p$ holds if and only if $\alpha \leq \min\{\frac{d-1}{p}, (d-1)(1-\frac{1}{p})\}$. The optimal maximal estimates were shown by Stein [54] and Bourgain [10]: M is bounded on L^p if and only if $p > d/(d-1)$. The key ingredient was the L^2 estimate. By Plancherel's Theorem, we have $\|\mathcal{A}_1 f\|_2 = \|\widehat{\mathcal{A}_1 f}\|_2$. Note that $\widehat{\mathcal{A}_1 f}(\xi) = \widehat{\mathfrak{m}_S}\widehat{f}(\xi)$. Hence, the decay of $|\widehat{\mathfrak{m}_S}|$ determines Sobolev regularity for $p = 2$. The stationary phase method gives

$$|\widehat{\mathfrak{m}_S}(\xi)| \lesssim |\xi|^{-(d-1)/2}$$

if S is nondegenerate. So $\mathcal{A}_1$ maps L^2 boundedly to $L^2_{(d-1)/2}$, and it plays an important role in proving regularity estimates and associated maximal estimates.

Local smoothing is more involved. In fact, there is no local smoothing for $p = 2$. A natural conjecture is, when S is nondegenerate hypersurface, (1.0.2) holds for $\alpha < d/p$ when $p \geq 2d/(d-1)$, which is basically equivalent to Sogge's local smoothing conjecture for the wave equation [53] is S is strictly convex. Recently, Guth, Wang, and Zhang [21] proved the sharp local smoothing estimate when $d = 2$ using the sharp square function estimate. When $d \geq 3$, it is currently known that (1.0.2) holds for $\alpha < d/p$ when $p \geq 2(d+1)/(d-1)$ due to Bourgain and Demeter's decoupling theorem [11] and Beltran, Hickman, and Sogge [5].

When S is a degenerate hypersurface, regularity problems become more complicated. The problem is better understood when S is a convex hypersurface of finite type. In this case, Bruna, Nagel, and Wainger [6] proved the sharp decay of $|\widehat{\mathfrak{m}_S}|$ using nonisotropic balls associated to S. By obtaining a variant of the decay estimate in [6], Nagel, Seeger, and Wainger [36] proved maximal estimates for convex hypersurfaces under a certain assumption. In $\mathbb{R}^3$, Iosevich, Sawyer, and Seeger [25] proved the sharp L^p improving, sharp L^p Sobolev, and optimal maximal estimates. For general degenerate hypersurfaces, using Newton polyhedra, Ikromov, Kempe, and Muller [24] and Buschenhenke, Dendrinos, Ikromov, and Muller [2] proved the sharp maximal bounds for averages over hypersurfaces of finite type in $\mathbb{R}^3$ where both principal curvatures of S do simultaneously vanish.

When the dimension of S is less than $d-1$, there is no general result unless S is a curve. When S is a nondegenerate curve (i.e. $S=\gamma$ satisfies (1.1.1)), the L^p improving property of $\mathcal{A}_1$ now has a complete characterization. However, L^p Sobolev and local smoothing estimates are far less well understood. Recently, there has been progress in low dimensions $d=3,4$ ([44, 4]), but it does not seem feasible to extend the approaches in recent works to higher dimensions.

In this thesis, we undertake the study of L^p-Sobolev, local smoothing, and L^p maximal bounds for the averaging operators defined by curves.

1.1 Main results

From now on, we assume that S is a smooth curve. Let $I=[-1,1]$ and γ be a smooth curve from I to $\mathbb{R}^d$. We have the explicit form of $\mathfrak{m}_t$ when $S=\gamma$:

$$\langle \mathfrak{m}_t, f\rangle := \int f(t\gamma(y))\psi(y)dy$$

where $\psi \in C_0^\infty((-1,1))$. And $\mathcal{A}_t f$ is given by

$$\mathcal{A}_t f(x) = \int_I f(x-t\gamma(s))\psi(s)ds.$$

We study the above-mentioned problems on $\mathcal{A}_t$ under the assumption that γ is nondegenerate, that is to say,

$$\det(\gamma'(s),\cdots,\gamma^{(d)}(s)) \gtrsim 1, \ \forall s\in I. \tag{1.1.1}$$

If γ is nondegenerate, Van der corput's lemma gives

$$|\widehat{\mathfrak{m}_1}(\xi)| \lesssim |\xi|^{-1/d}. \tag{1.1.2}$$

The bound yields the optimal L^2–$L^2_{1/d}$ estimate for $\mathcal{A}_1$. However, for the sharp L^p smoothing estimate ($p > 2$) we need to exploit finer properties of $\widehat{\mathfrak{m}}_t$.

When $d = 2$, we have rather a precise asymptotic expansion of $\widehat{\mathfrak{m}}_t$, which makes it possible to relate $\mathcal{A}_t$ to other forms of operators. In fact, one can use the estimate for the wave operator (e.g., [50, 59, 30]) to obtain local smoothing estimate. However, in higher dimensions $d \geq 3$, to compute $\widehat{\mathfrak{m}}_t$ explicitly is not a simple matter. Even worse, this becomes much more complicated as d increases since one has to take into account the derivatives $\gamma^{(k)}(s) \cdot \xi$, $k = 2, \ldots, d$. To overcome the difficulty, we prove regularity estimates under the local nondegeneracy assumption (Theorem 3.1.2 and 4.0.1) and express the averaging operator as a sum of adjoint restriction operators(Lemma 5.4.2 and 5.4.4).

The followings are the main results of this thesis.

- Optimal Sobolev regularity estimate
- Sharp local smoothing estimate
- Maximal estimate on the optimal range when $d = 3$
- Maximal estimate when $d \geq 4$.

We close the introduction by summarizing our main results.

L^p **Sobolev regularity**. By the standard duality argument, (1.0.1) for $p \geq 2$ implies Sobolev regularity estimate for $1 < p \leq 2$. Indeed, (1.0.1) for $1 < p < \infty$ implies $\|\mathcal{A}_1 f\|_{L^{p'}_{\alpha(p)}} \lesssim \|f\|_{p'}$ where p' is the conjugate exponent of p.

Let $p \geq 2$. When $d = 2$, (1.0.1) holds if and only if $\alpha \leq 1/p$ (e.g., see [14]). When $d \geq 3$, the first positive result for sharp Sobolev regularity was proved by Pramanik and Seeger. They proved for smooth nondegenerate curve γ in $\mathbb{R}^3$, $\mathcal{A}_1$ maps L^p boundedly to $L^p_{1/p}$ for large p. The order $1/p$ is sharp in that L^p Sobolev estimate fails if $\alpha > 1/p$ (see section 4.3). And $L^p \to L^p_{1/p}$ Sobolev estimate fails if $p < 2d - 2$ (see [4]). Naturally, we have the following conjecture:

Conjecture 1. *Let $d \geq 2$, $p \geq 2$, and γ is a smooth nondegenerate curve in $\mathbb{R}^d$. Then $\mathcal{A}_1$ maps L^p boundedly to $L^p_{1/p}$ if $p > 2d-2$.*

When $d = 3$, the conjecture was verified by the conditional result of Pramanik and Seeger [44] and the decoupling inequality due to Bourgain and Demeter [11] (see [39, 59] for earlier results). The case $d = 4$ was recently obtained by Beltran et al [4]. Our first result proves the conjecture for every $d \geq 5$.

Theorem 1.1.1. *Let $p \geq 2$, and γ is a smooth nondegenerate curve in $\mathbb{R}^d$. Then $\mathcal{A}_1$ maps L^p boundedly to $L^p_{1/p}$ if $p > 2d-2$.*

Interpolation with the $L^2 \to L^2_{1/d}$ estimate gives (1.0.1) for $\alpha < (p+2)/(2dp)$ when $2 < p \leq 2(d-1)$. It is also known that (1.0.1) fails if $\alpha > \alpha(p) := \min(1/p, (p+2)/(2dp))$ (see [4, Proposition 1.2]). Thus, only the estimate (1.0.1) with $\alpha = \alpha(p)$ remains open for $2 < p \leq 2(d-1)$. Those endpoint estimates seem to be a subtle problem. Our argument provides alternative proofs of the previous results for $d = 3, 4$. Theorem 1.1.1 remains valid as long as $\gamma \in \mathrm{C}^{2d}(I)$ (see Theorem 4.0.1). However, we do not try to optimize the regularity assumption.

Local smoothing estimate. Compared with the L^p Sobolev estimate, the additional integration in t is expected to yield extra smoothing. Such a phenomenon is called *local smoothing*, which has been studied for the dispersive equations to a great extent(e.g., see [52, 17]). However, the local smoothing for the averaging operators exhibits considerably different nature.

In particular, there is no local smoothing when $p = 2$. Besides, a bump function example shows $\alpha \leq 1/d$. As we shall see later, the estimate (1.0.2) fails unless $\alpha \leq 2/p$ (Proposition 3.7.1). So, it seems to be plausible to conjecture following:

Conjecture 2. *Let $p \geq 2$. Suppose γ is a smooth nendegenerate curve in $\mathbb{R}^d$. Define $\mathfrak{A}f(t,x) = \chi(t)\mathcal{A}_t(x)$ where χ is smooth function defined on $(1/2, 4)$. Then $\mathfrak{A}$ maps $L^p(\mathbb{R}^d)$ boundedly to $L^p_\alpha(\mathbb{R}^{d+1})$ if $\alpha < \min\{2/p, 1/d\}$.*

If the conjecture holds, for $p > d$ there exists $\alpha > 1/p$ such that $\mathfrak{A}$ maps $L^p(\mathbb{R}^d)$ boundedly to L^p_α. Therefore, Conjecture 2 implies that M is bounded in L^p for $p > d$. This gives maximal bound on the optimal range.

For $d = 2$, the conjecture follows by the recent result on Sogge's local smoothing conjecture for the wave operator ([53, 62, 31, 11]), which is due to

Guth, Wang, and Zhang [21]. When $d = 3$, some local smoothing estimates were utilized by Pramanik and Seeger [44] and Beltran et al. [3] to prove L^p maximal bound. Nevertheless, for $d \geq 3$, no local smoothing estimate up to the sharp order $2/p$ has been known previously. We prove following local smoothing estimate with sharp order.

Theorem 1.1.2. *Let $d \geq 3$. Suppose γ is a smooth nendegenerate curve in $\mathbb{R}^d$. Define $\mathfrak{A}f(t,x) = \chi(t)\mathcal{A}_t(x)$ where χ is smooth function defined on $(1/2, 4)$. Then $\mathfrak{A}$ maps $L^p(\mathbb{R}^d)$ boundedly to $L^p_\alpha(\mathbb{R}^{d+1})$ if $p \geq 4d-2$ and $\alpha < 2/p$.*

Theorem 1.1.2 remains valid as long as $\gamma \in \mathrm{C}^{3d+1}(I)$ (see Theorem 3.1.2).

L^p **maximal bound**. The local smoothing estimate (1.0.2) has been of particular interest in connection to L^p boundedness of the maximal operator $Mf(x) = \sup_{0<t} |\mathcal{A}_t f(x)|$ ([35, 50, 44, 3]) and problems in geometric measure theory (see, e.g., [62]). If the estimate (1.0.2) holds for some $\alpha > 1/p$, L^p boundedness of M follows by a standard argument relying on the Sobolev embedding ([44]).

The circular maximal theorem was proved by Bourgain [10] (also, see [53, 35, 48, 50, 30]). Afterwards, a natural question was whether the maximal operator M under consideration in the current paper is bounded on L^p for some $p \neq \infty$ when $d \geq 3$. In view of an interpolation argument based on L^2 estimate ([54]), proving L^p boundedness of M becomes more challenging as d increases since the decay of the Fourier transform of $\mathfrak{m}_t$ gets weaker (see (1.1.2)). Though the question was raised as early as in the late 1980s, it remained open for any $d \geq 3$ until recently. In $\mathbb{R}^3$, the first positive result was obtained by Pramanik and Seeger [44] and the range of p was further extended to $p > 4$ thanks to the decoupling inequality for the cone [11]. The maximal estimate in [44] was shown by exploiting L^p local smoothing phenomena of the averaging operator. However, being compared with the average over hypersurfaces or curves in $\mathbb{R}^2$, the L^p local smoothing property of $\mathcal{A}_t$ is not well understood.

We instead try to make use of L^p-L^q type smoothing estimate which has a close connection to the adjoint restriction estimate. Usefulness of such estimates has been manifested in the study of L^p improving property of the localized circular and spherical maximal functions [50, 30] (also see [1, 46, 7]). We prove L^p boundedness of M on the optimal range in $\mathbb{R}^3$:

Theorem 1.1.3. *Let $d = 3$ and γ is a smooth nondegenerate curve. Then M is bounded on L^p if and only if $p > 3$.*

The same result was independently obtained by Beltran et al. [3]. However, no nontrivial L^p boundedness of M has been known in higher dimensions. The following establishes existence of such L^p maximal bound for every $d \geq 4$.

Theorem 1.1.4. *Let $d \geq 4$. Suppose γ is a smooth nondegenerate curve. Then, for $p > 2(d-1)$ we have*

$$\|Mf\|_{L^p(\mathbb{R}^d)} \leq C\|f\|_{L^p(\mathbb{R}^d)}. \tag{1.1.3}$$

Theorem 1.1.4 is a consequence of Theorem 1.1.2. Since the estimate (1.0.2) with $p = 2$ and $\alpha = 1/d$ holds true, interpolation gives (1.0.2) for some $\alpha > 1/p$ when $2d - 2 < p < \infty$. So, the maximal estimate (1.1.3) follows, as mentioned before, by a standard argument. A natural conjecture is that M is bounded on L^p if and only if $p > d$. M can not be bounded on L^p if $p \leq d$ (see section 5.7).

Curves of maximal type. By the simple geometric feature of curve, we can get regularity estimates for degenerate cases. We say a smooth curve γ from I to $\mathbb{R}^d$ is of finite type if there is an ℓ such that $\operatorname{span}\{\gamma^{(1)}(s), \dots, \gamma^{(\ell)}(s)\} = \mathbb{R}^d$ for each $s \in I$. The type at s is defined to be the smallest of such ℓ and the maximal type is the supremum over all $s \in I$ of the type at s.

Assume that γ is a curve of maximal type $n \geq d$. In a small neighborhood for a fixed point s_0, we can consider γ as a perturbation of $(c_1 s^{n_1}, \dots, c_d s^{n_d})$ for some constants c_i and integers n_i. Near a degenerate point, dyadic decomposition in s and scaling make curve nondegenerate. Thus, regularity estimates for nondegenerate curves imply similar results for curves of maximal type (see section 4.4). So we have following corollaries.

Corollary 1.1.5. *Let $d \geq 3$, $\ell > d$ and $2 \leq p < \infty$. Suppose γ is a curve of maximal type ℓ. Then $\mathcal{A}_1$ maps L^p boundedly to L^p_α for $\alpha \leq \min\big(\alpha(p), 1/\ell\big)$ if $p \neq \ell$ when $\ell \geq 2d-2$, and if $p \in [2, 2\ell/(2d-\ell)) \cup (2d-2, \infty)$ when $d < \ell < 2d-2$.*

By interpolation (1.0.1) holds for $\alpha < \min\big(\alpha(p), 1/\ell\big)$ if $p = \ell$ when $\ell \geq 2d-2$, and if $2\ell/(2d-\ell) \leq p \leq 2d-2$ when $d < \ell < 2d-2$. These estimates are sharp. Since a finite type curve contains a nondegenerate subcurve and the $L^2 \to L^2_{1/\ell}$ estimate is optimal, (1.0.1) fails if $\alpha > \min\big(\alpha(p),\, 1/\ell\big)$. When $\ell \geq 2d-2$, Corollary 1.1.5 completely answers the problem of the Sobolev regularity estimate (1.0.1). In fact, the failure of $L^\ell \to L^\ell_{1/\ell}$ bound was shown in [4] using Christ's example [14]. By [51, Theorem 1.1] Corollary 1.1.5 also gives $H^1(\mathbb{R}^d) \to L^{1,\infty}(\mathbb{R}^d)$ bound on the lacunary maximal function $f \to \sup_{k\in\mathbb{Z}} |f * \mathfrak{m}_{2^k}|$ whenever γ is of finite type.

Corollary 1.1.6. *Let $d = 3$ and $\ell > d$. Suppose γ is a curve of maximal type ℓ. Then M is bounded in L^p if and only if $p > \max(\ell, 3)$.*

Corollary 1.1.7. *Let $d \geq 4$ and $\ell > d$. Suppose γ is a curve of maximal type ℓ. Then M is bounded in L^p if $p > \max(\ell, 2(d-1))$.*

This gives the optimal maximal estimate associated to averages over smooth curves of maximal type ℓ when $\ell \geq 2(d-1)$.

Chapter 2

Preliminary

2.1 Decoupling inequality

By Plancherel's theorem, if $\{f_j\}_j$ have disjoint fourier support, then

$$\|\sum_j f_j\|_2 \leq (\sum_j \|f_j\|_2^2)^{1/2}.$$

Indeed, the above inequality is equality. L^p analogue of the above inequality is a useful tool for L^p estimate. However, for L^p analogue, disjoint fourier support condition doesn't ensure that for some constant C independent of $\sharp\{j\}$,

$$\|\sum_j f_j\|_p \leq C(\sum_j \|f_j\|_p^2)^{1/2}.$$

We call this kind of inequality $\ell^2 L^p$ decoupling inequality. The Littlewood-Paley inequality and Minkowski inequality give a kind of example of decoupling inequality. In this section, we are concerned with decoupling inequality which is needed for proving regularity estimates.

Decoupling inequality has various applications. One of the typical examples is the sharp local smoothing estimates. The study in this direction was initiated by Wolff [62]. Wolff proved the sharp local smoothing estimate for the wave equation by showing sharp decoupling inequality for light cone in $\mathbb{R}^3$. Later, Bourgain and Demeter extended Wolff's result on the optimal range.

Let $\Gamma = \{(\xi_1, \xi_2, \xi_3) : 1/2 \leq \xi_n = |(\xi_1, \xi_2)| \leq 1\}$. And define $\Gamma(\delta)$ as a δ-neighborhood of Γ. Bourgain and Demeter proved the following sharp decoupling inequality:

Theorem 2.1.1 (Bourgain, Demeter). *Let $p \geq 6$ and* $\operatorname{supp} \widehat{f} \subset \Gamma(\delta^2)$. *Then for $\epsilon > 0$, there exist C such that*

$$\|f\|_p \leq C\delta^{-1+\frac{4}{p}-\epsilon}(\sum_{\Theta} \|f_\Theta\|_p^p)^{1/p}, \tag{2.1.1}$$

where $\operatorname{supp} \widehat{f_\Theta} \subset \Theta$ *and $\{\Theta\}$ is a partition of $\Gamma(\delta^2)$ satisfying angular length of each Θ is δ.*

Bourgain and Demeter also proved analogue of decoupling inequality for nondegenerate hypersurfaces and cones in all dimensions. Later, Bourgain, Demeter, and Guth obtained sharp decoupling inequality for nondegenerate curve in optimal range of p. In this paper, we use variant version of decoupling inequality in [12].

We denote $\mathbf{r}_\circ^N(s) = (s, s^2/2!, \dots s^N/N!)$, and consider a collection of curves from I to $\mathbb{R}^N$ which are small perturbations of $\mathbf{r}_\circ^N$:

$$\mathfrak{C}(\epsilon_\circ; N) := \{\mathbf{r} \in \mathrm{C}^{2N+1}(I) : \|\mathbf{r} - \mathbf{r}_\circ^N\|_{\mathrm{C}^{2N+1}(I)} < \epsilon_\circ\}.$$

For $\mathbf{r} \in \mathfrak{C}(\epsilon_\circ; N)$ and $s \in I$, we define

$$\mathcal{N}_{\mathbf{r}}(s, \delta) = \Big\{\mathbf{r}(s) + \sum_{1 \leq j \leq N} u_j \mathbf{r}^{(j)}(s) : \ |u_j| \leq \delta^j, \ j = 1, \dots, N\Big\}.$$

Let $s_1, \dots, s_l \in I$ be δ-separated points, i.e., $|s_n - s_j| \geq \delta$ if $n \neq j$, such that $\bigcup_{j=1}^l (s_j - \delta, s_j + \delta) \supset I$. Then, we set

$$\theta_j = \mathcal{N}_{\mathbf{r}}(s_j, \delta), \qquad 1 \leq j \leq l.$$

The following is due to Bourgain, Demeter, and Guth [12] (also see [19]).

Theorem 2.1.2 (Bourgain, Demeter, Guth). *Let $0 < \delta \ll 1$. Suppose $\mathbf{r} \in \mathfrak{C}(\epsilon_\circ; N)$ for a small enough $\epsilon_\circ > 0$. Then, if $2 \leq p \leq N(N+1)$, for $\epsilon > 0$ we have*

$$\Big\| \sum_{1 \leq j \leq l} f_j \Big\|_{L^p(\mathbb{R}^N)} \leq C_\epsilon \delta^{-\epsilon} \Big(\sum_{1 \leq j \leq l} \|f_j\|_{L^p(\mathbb{R}^N)}^2 \Big)^{1/2} \tag{2.1.2}$$

whenever $\operatorname{supp} \widehat{f_j} \subset \theta_j$ *for $1 \leq j \leq l$.*

The constant C_ϵ can be taken to be independent of particular choices of the δ-separated points $s_1, \dots, s_l$. One can obtain a conical extension of the inequality (2.1.2) by modifying the argument in [11] which deduces the decoupling inequality for the cone from that for the paraboloid (see [4, Proposition 7.7]). Let us consider conical sets

$$\overline{\theta}_j = \{(\eta, \rho) \in \mathbb{R}^N \times [1,2] : \eta/\rho \in \theta_j\}, \quad 1 \le j \le l.$$

Corollary 2.1.3. *Let* $0 < \delta \le 1$ *and let* $\mathbf{r} \in \mathfrak{C}(\epsilon_\circ; N)$ *with a small enough* $\epsilon_\circ > 0$*. Then, if* $2 \le p \le N(N+1)$*, for* $\epsilon > 0$ *we have*

$$\Big\| \sum_{1\le j\le l} F_j \Big\|_{L^p(\mathbb{R}^{N+1})} \le C_\epsilon \delta^{-\epsilon} \Big(\sum_{1\le j\le l} \|F_j\|^2_{L^p(\mathbb{R}^{N+1})} \Big)^{1/2} \tag{2.1.3}$$

whenever $\operatorname{supp} \widehat{F}_j \subset \overline{\theta}_j$ *for* $1 \le j \le l$.

Proof of Corollary 2.1.3. Let us set $a = \delta^{\epsilon_0}$. We first consider the decoupling inequality with a truncated conic covering of small height. For the purpose we denote

$$\overline{\mathcal{N}}^a_{\mathbf{r}}(J, \delta) = \{(\eta, \rho) \in \mathbb{R}^N \times [\rho_0, \rho_0 + a] : \eta/\rho \in \mathcal{N}_{\mathbf{r}}(J, \delta)\},$$

where J is a subinterval of I. Suppose $\operatorname{supp} \widehat{F} \subset \overline{\mathcal{N}}^a_{\mathbf{r}}(I, \delta)$, then the projection of $\operatorname{supp} \widehat{F}$ to η-plane is contained in $\rho_0 \mathbf{r} + O(a) \subset \mathcal{N}_{\rho_0 \mathbf{r}}(I, Ca^{\frac{1}{N}})$ for some $C > 0$. Let $\overline{\mathcal{C}}^a(Ca^{\frac{1}{N}})$ denote a conic $Ca^{\frac{1}{N}}$-adapted covering of $\rho_0 \mathbf{r}$ with height a. Since $\rho_0 \in [1,2]$, by (2.1.16) and rescaling we have

$$\big\| F \big\|_{L^p(\mathbb{R}^{N+1})} \le C_\epsilon a^{-\frac{\epsilon}{N}} \Big(\sum_{\overline{\theta} \in \overline{\mathcal{C}}^a(Ca^{\frac{1}{N}})} \|F_{\overline{\theta}}\|^2_{L^p(\mathbb{R}^{N+1})} \Big)^{\frac{1}{2}} \tag{2.1.4}$$

whenever $F = \sum_{\overline{\theta} \in \overline{\mathcal{C}}^a(Ca^{\frac{1}{N}})} F_{\overline{\theta}}$ and $\widehat{F_{\overline{\theta}}}$ is supported in $\overline{\theta}$. The key observation is that the set $\overline{\theta}$ can be projected into a hypersurface P so that the projection of $\overline{\theta}$ is contained in a $Ca^{\frac{2}{N}}$-(anisotropic) neighborhood of a nondegenerate curve contained in P. This allows us to decompose further $F_{\overline{\theta}}$ into functions which are Fourier supported in smaller sets belonging to a $Ca^{\frac{2}{N}}$-adapted covering with height a.

To see this, we consider the conic set

$$\overline{\mathcal{N}}^a_{\mathbf{r}}([s_0, s_0 + l], \delta)$$

with $a \le l \le Ca^{\frac{1}{N}}$ and its projection to a hyperplane. Let us denote by P_0 the plane which contains $\rho_0(\mathbf{r}(s_0), 1)$ and is perpendicular to the vector $\rho_0(\mathbf{r}(s_0), 1)$. Also, denote by Π_0 the orthogonal projection to P_0 and set

$$\overline{\mathbf{r}}_0(s) = \Pi_0 \rho_0(\mathbf{r}(s), 1) \tag{2.1.5}$$

for $s_0 \le s \le s_0 + l$. Then, it follows that $\overline{\mathbf{r}}_0(s_0) = \rho_0(\mathbf{r}(s_0), 1) = \Pi_0(0,0)$ and $\overline{\mathbf{r}}_0$ is a nondegenerate curve in P_0 if l is small enough. We claim

$$\Pi_0 \overline{\mathcal{N}}_{\mathbf{r}}^{a}([s_0, s_0 + l], \delta) \subset \mathcal{N}_{\overline{\mathbf{r}}_0}([s_0, s_0 + l], Ca^{\frac{1}{N}} l) \tag{2.1.6}$$

for some constant $C > 0$ as long as $\delta \le a^{\frac{1}{N}} l$. To see this, let $\xi \in \Pi_0 \overline{\mathcal{N}}_{\mathbf{r}}^{a}([s_0, s_0 + l], \delta)$. Since Π_0 is an affine map, we may write $\Pi_0 \mathbf{v} = \mathrm{M}\mathbf{v} + \overline{\mathbf{r}}_0(s_0)$ for a matrix M. Thus, we have

$$\xi = \Pi_0 \rho(\mathbf{r}(s) + v, 1) = \Pi_0 \rho(\mathbf{r}(s), 1) + \Pi_0 \rho(v, 0) - \overline{\mathbf{r}}_0(s_0)$$

for some $s \in [s_0, s_0 + l]$, $\rho \in [\rho_0, \rho_0 + a]$, and $v \in \mathcal{N}_{\mathbf{r}}^0(s, C\delta)$. We also note that $\Pi_0 \rho(v, 0) - \overline{\mathbf{r}}_0(s_0) \in \mathcal{N}_{\overline{\mathbf{r}}_0}^0(s, C\delta)$ because $v \in \mathcal{N}_{\mathbf{r}}^0(s, C\delta)$. So, $\xi - \overline{\mathbf{r}}_0(s) = \Pi_0 \rho(\mathbf{r}(s), 1) - \Pi_0 \rho_0(\mathbf{r}(s), 1) + u$ for some $u \in \mathcal{N}_{\overline{\mathbf{r}}_0}^0(s, C\delta)$. Using (2.1.5) and $\Pi_0 \mathbf{v} = \mathrm{M}\mathbf{v} + \overline{\mathbf{r}}_0(s_0)$, we have

$$\xi - \overline{\mathbf{r}}_0(s) = \rho_0^{-1}(\rho_0 - \rho)\big(\overline{\mathbf{r}}_0(s) - \overline{\mathbf{r}}_0(s_0)\big) + u.$$

Since $\overline{\mathbf{r}}_0(s_0) = \rho_0(\mathbf{r}(s_0), 1)$ and $\overline{\mathbf{r}}_0$ is nondegenerate, using Taylor's theorem we see $\overline{\mathbf{r}}_0(s) - \overline{\mathbf{r}}_0(s_0) \in \mathcal{N}_{\overline{\mathbf{r}}_0}^0(s, Cl)$ if $s \in [s_0, s_0 + l]$. Hence, it follows that

$$\rho_0^{-1}(\rho - \rho_0)\big(\overline{\mathbf{r}}_0(s) - \overline{\mathbf{r}}_0(s_0)\big) \in \mathcal{N}_{\overline{\mathbf{r}}_0}^0(s, Ca^{\frac{1}{N}} l)$$

since $\rho \in [\rho_0, \rho_0 + a] \subset [1, 2]$. Therefore, $\xi - \overline{\mathbf{r}}_0(s) \in \mathcal{N}_{\overline{\mathbf{r}}_0}^0(s, Ca^{\frac{1}{N}} l)$. This proves the claim (2.1.6).

In particular, if J is an interval of length $l = a^{\frac{1}{N}}$, the projection of the set $\overline{\mathcal{N}}_{\mathbf{r}}^{a}(J, a^{\frac{1}{N}})$ in an appropriate direction is contained in $\mathcal{N}_{\overline{\mathbf{r}}_0}(J, Ca^{\frac{2}{N}})$ for a nondegenerate curve $\overline{\mathbf{r}}_0$. Putting this together with (2.1.16), we obtain a decoupling inequality for each $F_{\overline{\theta}}$, $\overline{\theta} \in \overline{\mathcal{C}}^a(Ca^{\frac{1}{N}})$,* which decouples into functions Fourier supported in a $Ca^{\frac{2}{N}}$-adapted covering. This observation combined with (2.1.4) gives

$$\|F\|_{L^p(\mathbb{R}^{N+1})} \le C_\epsilon^2 a^{-\frac{1+2}{N}\epsilon} \Big(\sum_{\overline{\theta}' \in \overline{\mathcal{C}}^a(Ca^{\frac{2}{N}})} \|F_{\overline{\theta}'}\|_{L^p(\mathbb{R}^{N+1})}^2 \Big)^{\frac{1}{2}}.$$

*By finite decomposition and harmless affine transforms we may assume $\overline{\mathbf{r}}_0 \in \mathfrak{C}(\epsilon_\circ)$ for a sufficiently small $\epsilon_\circ > 0$.

Using (2.1.6) and (2.1.16), we can continue this up to n-th stage as long as $a^{\frac{n}{N}} \ge \delta > a^{\frac{n+1}{N}}$, that is to say, $\epsilon_0 n \le N < \epsilon_0(n+1)$. So, we get

$$\|F\|_{L^p(\mathbb{R}^{N+1})} \le C_\epsilon^n a^{-\frac{\epsilon}{N}(1+\cdots+n)} \Big(\sum_{\overline{\theta} \in \overline{\mathcal{C}}^a(Ca^{\frac{n}{N}})} \|F_{\overline{\theta}}\|^2_{L^p(\mathbb{R}^{N+1})} \Big)^{\frac{1}{2}},$$

where $F = \sum_{\overline{\theta} \in \overline{\mathcal{C}}^a(Ca^{\frac{n}{N}})} F_{\overline{\theta}}$ and $\operatorname{supp} \widehat{F_{\overline{\theta}}} \subset \overline{\theta}$. Since $a = \delta^{\epsilon_0}$, further decomposing $\overline{\theta} \in \overline{\mathcal{C}}^a(Ca^{\frac{n}{N}})$ into as many as $O(\delta^{-C\epsilon_0})$ slabs contained in $\overline{\mathcal{C}}^a(\delta)$, we obtain

$$\|F\|_{L^p(\mathbb{R}^{N+1})} \le C_\epsilon^{C/\epsilon_0} \delta^{-C\epsilon\epsilon_0^{-1} - C\epsilon_0} \Big(\sum_{\overline{\theta} \in \overline{\mathcal{C}}^a(\delta)} \|F_{\overline{\theta}}\|^2_{L^p(\mathbb{R}^{N+1})} \Big)^{\frac{1}{2}}.$$

Finally, let $\mathcal{C} = \{\theta_1, \dots, \theta_k\}$ be a δ-adapted covering of $\mathbf{r} \in \mathfrak{C}(\epsilon_\circ)$ and $\overline{\mathcal{C}} = \{\overline{\theta}_1, \dots, \overline{\theta}_k\}$ be the associated conic covering. Further decomposition of the conic set $\overline{\theta}_k$ into as many as $O(a^{-1})$ truncated conic coverings of height a yields the desired inequality at the expense of a $\delta^{-\epsilon_0}$ factor. This gives (2.1.3) with $\epsilon = C\epsilon_0$ if we take $\epsilon = \epsilon_0^2$. □

The decoupling inequality (2.1.3) dose not fit the symbols which appear later when we decompose $\mathfrak{a}$ (see Section 3.5.1 and Section 4.2). As to be seen later, those symbols are related to the slabs of the following form.

Definition 2.1.4. *Let $N \ge 2$ and $\tilde{\mathbf{r}} \in \mathfrak{C}(\epsilon_\circ; N+1)$. For $s \in I$, we denote by $\mathbf{s}(s, \delta, \rho; \tilde{\mathbf{r}})$ the set of $(\tau, \eta) \in \mathbb{R} \times \mathbb{R}^N$ which satisfies*

$$\begin{aligned} &\rho^{-1} \le |\langle \tilde{\mathbf{r}}^{(N+1)}(s), (\tau, \eta) \rangle| \le 2\rho, \\ &|\langle \tilde{\mathbf{r}}^{(j)}(s), (\tau, \eta) \rangle| \le \delta^{N+1-j}, \qquad j = N, \dots, 1, \end{aligned}$$

The same form of decoupling inequality continues to be valid for the slabs $\mathbf{s}(s_1, \delta, 1; \tilde{\mathbf{r}}), \dots, \mathbf{s}(s_l, \delta, 1; \tilde{\mathbf{r}})$. Beltran et al. [4, Theroem 4.4] showed, using the Frenet–Serret formulas, that those slabs can be generated as conical extensions of the slabs given by a nondegenerate curve in $\mathbb{R}^N$. Thus, the following is a consequence of Corollary 2.1.3 and a simple manipulation using decomposition and rescaling.

Corollary 2.1.5. *Let $0 < \delta \le 1$, $\rho \ge 1$, and $\tilde{\mathbf{r}} \in \mathfrak{C}(\epsilon_\circ; N+1)$ for a small enough $\epsilon_\circ > 0$. Denote $\mathbf{s}_j = \mathbf{s}(s_j, \delta, \rho; \tilde{\mathbf{r}})$ for $1 \le j \le l$. Then, if $2 \le p \le$*

$N(N+1)$, for $\epsilon > 0$ there is a constant $C_\epsilon = C_\epsilon(\rho)$ such that

$$\Big\| \sum_{1\le j\le l} F_j \Big\|_{L^p(\mathbb{R}^{N+1})} \le C_\epsilon \delta^{-\epsilon} \Big(\sum_{1\le j\le l} \|F_j\|^2_{L^p(\mathbb{R}^{N+1})} \Big)^{1/2} \tag{2.1.7}$$

whenever $\operatorname{supp} \widehat{F_j} \subset \mathbf{s}_j$ *for* $1 \le j \le l$.

Proof of Corollary 2.1.5. In order to prove Corollary 2.1.5, it suffices to show that for $\rho = 1$, $\mathbf{s}_1, \dots, \mathbf{s}_l$ form a conic δ-adapted covering of a nondegenerate curve $\mathbf{r}$ in $\mathbb{R}^N$. Then Corollary 2.1.5 immediately follows from Corollary 2.1.3.

By finite decomposition and translation, we may assume that $\widetilde{\mathbf{r}}$ is defined on the interval $[-c_\circ, c_\circ]$ for a small enough $c_\circ > 0$. Let

$$(\tau, \xi) = (\tau, \xi_1, \dots, \xi_N) \in \mathbf{s}(s, \delta, 1; \tilde{\mathbf{r}}), \quad (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^N$$

with $[s-\delta, s+\delta] \subset [-c_\circ, c_\circ]$. Since $\widetilde{\mathbf{r}} \in \mathfrak{C}(\epsilon_\circ; N+1)$, the second condition in definition of $\mathbf{s}(s, \delta, 1; \tilde{\mathbf{r}})$ guarantees $|\xi_N| \sim 1$. We need only to show that there exists a smooth nondegenerate curve $\mathbf{r}$ such that

$$\xi_N^{-1}(\tau, \xi_1, \dots, \xi_{N-1}) \in \mathbf{r}(s) + \mathcal{N}^0_{\mathbf{r}}(s, C\delta) \tag{2.1.8}$$

for $s \in [\delta - c_\circ, c_\circ - \delta]$. Let $\{\mathbf{e}^1(s), \dots \mathbf{e}^{N+1}(s)\}$ be the Frenet $(N+1)$-frame given by the Gram-Schmidt process of the vectors $\{\widetilde{\mathbf{r}}^{(1)}(s), \dots, \widetilde{\mathbf{r}}^{(N+1)}(s)\}$. Then, if $\epsilon_\circ > 0$ is small enough, by definition of $\mathbf{s}(s, \delta, 1; \tilde{\mathbf{r}})$ we have

$$|\langle \mathbf{e}^j(s), (\tau, \xi)\rangle| \le 2\delta^{N+1-j}, \; j = 1, \dots, N, \quad 2^{-2} \le |\langle \mathbf{e}^{N+1}(s), (\tau, \xi)\rangle| \le 2. \tag{2.1.9}$$

Since $\widetilde{\mathbf{r}} \in \mathfrak{C}(\epsilon_\circ; N+1)$, $\|\mathbf{e}^j(s) - e_j\|_\infty \le C(\epsilon_\circ + c_\circ)$. We set $\mathbf{e}^{N+1}(s) = (a_1(s), \dots, a_{N+1}(s))$, then we have $|a_{N+1}(s)| \sim 1$. Define a curve $\overline{\mathbf{r}} : [-c_\circ, c_\circ] \to \mathbb{R}^{N+1}$ by

$$\overline{\mathbf{r}}(s) = (a_{N+1})^{-1} \mathbf{e}^{N+1}(s). \tag{2.1.10}$$

Note that $\overline{\mathbf{r}}_{N+1}(s) = 1$ and consider $\mathbf{r} \in \mathbb{R}^N$ which is given by

$$\overline{\mathbf{r}}(s) = (\mathbf{r}(s), 1).$$

We now show (2.1.8) holds with $\mathbf{r}$ in the above. To this end we recall the Frenet-Serret formula:

$$(\mathbf{e}^j)'(s) = -\kappa_{j-1}(s)\mathbf{e}^{j-1}(s) + \kappa_j(s)\mathbf{e}^{j+1}(s), \quad 1 \le j \le N+1, \tag{2.1.11}$$

where $\kappa_0(s) = \kappa_{N+1}(s) = 0$ and $\kappa_j(s) = \langle (\mathbf{e}^j)'(s), \mathbf{e}^{j+1}(s)\rangle$, $j = 1, \dots, N$. We note $\bar{\mathbf{r}}' \in \operatorname{span}\{\mathbf{e}^N, \mathbf{e}^{N+1}\}$ and $(\mathbf{e}^\ell)' \in \operatorname{span}\{\mathbf{e}^{\ell-1}, \mathbf{e}^{\ell+1}\}$ for $2 \le \ell \le N$. Using (2.1.10), we see $\bar{\mathbf{r}}^{(j)} \in \operatorname{span}\{\mathbf{e}^{N+1-j}, \dots, \mathbf{e}^{N+1}\}$ and $\langle \bar{\mathbf{r}}^{(j)}, \mathbf{e}^{N+1-j}\rangle \neq 0$. Thus, $\bar{\mathbf{r}}^{(1)}, \dots, \bar{\mathbf{r}}^{(N)}$ are linearly independent, so $\bar{\mathbf{r}}^{(1)}, \dots, \bar{\mathbf{r}}^{(N)}$ span $\mathbb{R}^N \times \{0\}$ because $\bar{\mathbf{r}}^{(j)} = (\mathbf{r}^{(j)}, 0) \in \mathbb{R}^N \times \{0\}$. Therefore, for some $u_j \in \mathbb{R}$, $j = 1, \dots, N$, we can write

$$\xi_N^{-1}(\tau, \xi) - \bar{\mathbf{r}}(s) = \sum_{j=1}^{N} u_j\, \bar{\mathbf{r}}^{(j)}(s). \tag{2.1.12}$$

By (2.1.11), $\langle \bar{\mathbf{r}}^{(1)}, \mathbf{e}^N\rangle = -(a_{N+1})^{-1}\kappa_N$, $\langle \bar{\mathbf{r}}^{(2)}, \mathbf{e}^{N-1}\rangle = (a_{N+1})^{-1}\kappa_N\kappa_{N-1}$, and so on. Since $|a_{N+1}| \sim 1$ and $|\kappa_\ell| \sim 1$ for $1 \le \ell \le N$, we have

$$|\langle \bar{\mathbf{r}}^{(j)}, \mathbf{e}^{N+1-j}\rangle| \sim 1, \quad 0 \le j \le N. \tag{2.1.13}$$

Since $\bar{\mathbf{r}}^{(j)} \in \operatorname{span}\{\mathbf{e}^{N+1-j}, \dots, \mathbf{e}^{N+1}\}$, taking inner product against $\mathbf{e}^1(s)$ on both sides of (2.1.12), we have $|\langle \xi_N^{-1}(\tau, \xi), \mathbf{e}^1(s)\rangle| = |u_N \langle \bar{\mathbf{r}}^{(N)}, \mathbf{e}^1(s)\rangle| \sim |u_N|$ by (2.1.13). Therefore, (2.1.9) gives $|u_N| \lesssim \delta^N$. In the same manner we also get $|u_j| \lesssim \delta^j$ for $j = N-1, \dots, 1$. This therefore gives (2.1.12) with $|u_j| \lesssim \delta^j$, $j = N, \dots, 1$, which proves (2.1.8). $\square$

For our purpose, we use a modified form. If $p_* \in [2, N(N+1)]$, then we have

$$\Big\| \sum_{1 \le j \le l} F_j \Big\|_{L^p(\mathbb{R}^{N+1})} \le C_\epsilon \delta^{-1 + \frac{2+p_*}{2p} - \epsilon} \Big(\sum_{1 \le j \le l} \|F_j\|^p_{L^p(\mathbb{R}^{N+1})} \Big)^{1/p}$$

for $p \ge p_*$. The case $p = p_*$ follows by (2.1.7) and Hölder's inequality. Interpolation with the trivial $\ell^\infty L^\infty$–L^∞ estimate gives the estimate for $p \ge p_*$. We choose different p_* for the particular purposes. In fact, for the local smoothing estimate we take $p_* = 4N - 2$ to have

$$\Big\| \sum_{1 \le j \le l} F_j \Big\|_{L^p(\mathbb{R}^{N+1})} \le C_\epsilon \delta^{-1 + \frac{2N}{p} - \epsilon} \Big(\sum_{1 \le j \le l} \|F_j\|^p_{L^p(\mathbb{R}^{N+1})} \Big)^{1/p} \tag{2.1.14}$$

for $p \ge 4N - 2$ (see Section 3.5.2). For the L^p Sobolev regularity estimate, we observe

$$\Big\| \sum_{1 \le j \le l} F_j \Big\|_{L^p(\mathbb{R}^{N+1})} \le C_{\epsilon_0} \delta^{-1 + \frac{N+1}{p} + \epsilon_0} \Big(\sum_{1 \le j \le l} \|F_j\|^p_{L^p(\mathbb{R}^{N+1})} \Big)^{1/p} \tag{2.1.15}$$

holds for some $\epsilon_0 = \epsilon_0(p) > 0$ if $2N < p < \infty$. Indeed, we need only to take $p_* > 2N$ close enough to $2N$. The presence ϵ_0 in (2.1.15) is crucial for proving the optimal Sobolev regularity estimate (see Proposition 4.1.4).

The inequalities (2.1.14) and (2.1.15) trivially extend to cylindrical forms via the Minkowski inequality. For example, set $\tilde{\mathbf{s}}_j = \big\{(\xi, \eta) \in \mathbb{R}^{N+1} \times \mathbb{R}^M : \xi \in \mathbf{s}_j\big\}$ for $1 \le j \le l$. Then, using (2.1.14), we have

$$\big\| \sum_{1 \le j \le l} G_j \big\|_{L^p(\mathbb{R}^{N+M+1})} \le C_\epsilon \delta^{-1+\frac{2N}{p}-\epsilon} \big(\sum_{1 \le j \le l} \|G_j\|^p_{L^p(\mathbb{R}^{N+M+1})} \big)^{1/p} \tag{2.1.16}$$

whenever $\widehat{G}_j$ is supported in $\tilde{\mathbf{s}}_j$. Clearly, we also have a similar extension of (2.1.15).

2.2 Multilinear restriction estimate

The classical restriction conjecture is one of the most interesting problems in harmonic analysis. For understanding the restriction operator, its adjoint, extension operator has been studied intensively. Let $U \subset \mathbb{R}^{d-1}$ be a compact neighborhood of origin and $\Phi : U \to \mathbb{R}$ be smooth. Define extension operator $\mathfrak{E}$ by

$$\mathfrak{E}g(x) := \int_U g(\xi) e^{ix\cdot(\xi, -\Phi(\xi))} d\xi.$$

The classical restriction conjecture is:

Conjecture 3 (Linear Restriction)**.** *If Φ has nonvanishing determinant of Hessian, $q > \frac{2d}{d-1}$ and $p' \le \frac{d-1}{d+1}q$, then $\|\mathfrak{E}g\|_{L^q(\mathbb{R}^d)} \lesssim \|g\|_{L^p(U)}$.*

It is well-known that the restriction conjecture implies the Kakeya conjecture. Here, we describe the quantitative version of the Kakeya conjecture. For $0 < \delta \ll 1$, we call T δ-tube if T is a rectangular box of dimensions about $1 \times \delta \times \cdots \times \delta$. We say that T_1, T_2 are δ-transversal if directions of T_1, T_2 are δ-separated on $\mathbb{S}^{d-1}$. And we say that T_1, T_2 are transversal if there exist c independent of δ such that T_1, T_2 are c-transversal. Then the linear Kakeya conjecture is:

Conjecture 4 (Linear Kakeya)**.** *Let $0 < \delta \ll 1$, and $\mathbb{T}$ be a collection of δ-transversal δ-tubes. For $\frac{d}{d-1} < q \le \infty$ there is a constant C independent of $\delta, \mathbb{T}$ such that*

$$\|\sum_{T \in \mathbb{T}} \chi_T\|_q \le C\delta^{\frac{d-1}{q}} (\sharp\mathbb{T})^{\frac{1}{q}}$$

where χ_A *is indicator function(*$\chi_A(x) = 1$ *if* $x \in A$*, otherwise* $\chi_A = 0$*).*

For understanding the restriction problem, various techniques have been developed. Multilinear restriction estimate is one of them. For $1 \leq k \leq d$, let $U_k \subset \mathbb{R}^{d-1}$ be a compact subset of an open set $U_k' \subset \mathbb{B}^{d-1}(0, 2^2)$ and $\Phi_k : U_k' \to \mathbb{R}^d$ be a smooth mapping and $\mathfrak{E}_k$ is associated extension operator. The multilinear variant of linear restriction is following:

Conjecture 5 (Multilinear Restriction)**.** *Let* $d \geq 2$*,* $\theta \in (0,1]$*,* $\|\Phi_k\|_{C^2(U_k')} \leq B$ *and let* $N_k(\xi) = \frac{(\nabla\Phi_k(\xi),1)}{|(\nabla\Phi_k(\xi),1)|}$*. Suppose* $|\det(N_1(\xi_1), \ldots, N_d(\xi_d))| \geq \theta$ *for* $\xi_k \in U_k$*,* $k = 1, \ldots, d$*,* $q \geq \frac{2d}{d-1}$*, and* $p' \leq \frac{d-1}{d} q$*. Then there exist* C *depending only on* $B, \theta, d, U_1, \ldots, U_d$ *such that*

$$\|\prod_{k=1}^{d} \mathfrak{E}_k g_k\|_{L^{q/d}} \leq C \prod_{k=1}^{d} \|g_k\|_{L^p(U_k)}. \tag{2.2.1}$$

The multilinear conjecture is reduced to the endpoint case $p = 2$, $q = \frac{2d}{d-1}$. For multilinear restriction, a curvature condition is not needed. Instead of a curvature condition, the transversality condition on normal vectors N_k becomes crucial. Bennett, Carbery, and Tao [9] proved near-optimal multilinear restriction estimate:

Theorem 2.2.1 (Bennett, Carbery, Tao)**.** *Let* $d \geq 2$*,* $\theta \in (0,1]$*,* $\|\Phi_k\|_{C^2(U_k')} \leq B$ *and let* $N_k(\xi) = \frac{(\nabla\Phi_k(\xi),1)}{|(\nabla\Phi_k(\xi),1)|}$*. Suppose* $|\det(N_1(\xi_1), \ldots, N_d(\xi_d))| \geq \theta$ *for* $\xi_k \in U_k$*,* $k = 1, \ldots, d$*. Then, for any* $\varepsilon > 0$

$$\|\prod_{k=1}^{d} \mathfrak{E}_k g_k\|_{L^{\frac{2}{d-1}}(\mathbb{B}^d(0,R))} \leq C_\varepsilon(\theta) R^\varepsilon \prod_{k=1}^{d} \|g_k\|_{L^2(U_k)} \tag{2.2.2}$$

whenever $R \geq 1$*. The constant* $C_\varepsilon(\theta)$ *takes the form* $C\theta^{-C_\varepsilon}$ *for a constant* $C_\varepsilon > 0$*.*

They proved the above theorem by showing that multilinear restriction is essentially equivalent to multilinear Kakeya estimate. They obtained following near-optimal multilinear Kakeya estimate:

Theorem 2.2.2 (Bennett, Carbery, Tao)**.** *Let* $0 < \delta \ll 1$ *and* $\mathbb{T}_1, \ldots, \mathbb{T}_d$ *be transversal familes of* δ*-tubes. For* $\epsilon > 0$*, there exist* C *independent of* $\epsilon, \delta, \mathbb{T}_1, \ldots, \mathbb{T}_d$ *such that*

$$\|\prod_{j=1}^{d} \Big(\sum_{T_j \in \mathbb{T}_j} \chi_{T_j} \Big)\|_{L^{\frac{1}{d-1}}(B(0,1))} \leq C\delta^{-\epsilon} \prod_{j=1}^{d} (\delta^{d-1} \sharp \mathbb{T}_j).$$

It is not difficult to see that the argument in [9] continues to work with $\mathrm{C}^{1,\alpha}$ surface, $\alpha > 0$. Using theorem 2.2.2, we can prove the following theorem.

Theorem 2.2.3. *Let $d \geq 2$, $\theta \in (0,1]$, $\|\Phi_k\|_{\mathrm{C}^{1,\alpha}(U_k')} \leq B$ and let $N_k(\xi) = \frac{(\nabla\Phi_k(\xi),1)}{|(\nabla\Phi_k(\xi),1)|}$. Suppose $|\det(N_1(\xi_1), \dots, N_d(\xi_d))| \geq \theta$ for $\xi_k \in U_k$, $k = 1, \dots, d$. Then, for any $\varepsilon > 0$*

$$\Big\| \prod_{k=1}^{d} T_k g_k \Big\|_{L^{\frac{2}{d-1}}(\mathbb{B}^d(0,R))} \leq C_\varepsilon(\theta) R^\varepsilon \prod_{k=1}^{d} \|g_k\|_{L^2(U_k)} \tag{2.2.3}$$

whenever $R \geq 1$. The constant $C_\varepsilon(\theta)$ takes the form $C\theta^{-C_\varepsilon}$ for a constant $C_\varepsilon > 0$.

The theorem holds with C^1 curve, even with Lipschitz curve when $d = 2$ but it is unknown whether the same continues to be true in higher dimensions. Once one makes a couple of crucial observations concerning the $\mathrm{C}^{1,\alpha}$ surfaces, it is not difficult to prove Theorem 2.2.3 through routine adaptation of the arguments in [9, Proposition 2.1].

For the proof of Theorem 2.2.3, first of all, we note that

$$|\Phi_k(\xi + h) - \Phi_k(\xi) - \nabla\Phi_k(\xi) \cdot h| \leq CB|h|^{\alpha+1} \tag{2.2.4}$$

whenever $\xi + h, \xi \in U_k$. If Φ_k is assumed to be in $\mathrm{C}^{1,\alpha}(U_k)$ instead of $\mathrm{C}^{1,\alpha}(U_k')$, this can not be completely clear. In such a case we need to impose an additional condition such that U_k has a $\mathrm{C}^{1,\alpha}$ boundary (e.g. see [18, pp. 136–137]). On the other hand, if U_k is convex, (2.2.4) is a simple consequence of the mean value theorem. Since U_k is compact, there is a positive number ρ_k such that x, y are contained in a ball which is a subset of U_k' whenever $x, y \in U_k$ and $|x - y| \leq \rho_k$. Therefore we get (2.2.4) for $|h| \leq \rho_k$ and this is enough to show (2.2.4) for any $\xi, \xi + h \in U_k$ because U_k is compact and $\nabla\Phi_k$ is continuous.

Proof of Theorem 2.2.3. Let us denote $\Sigma_k = \{(\xi, -\Phi_k(\xi)) : \xi \in U_k\}$. We consider the estimate

$$\Big\| \prod_{k=1}^{d} \widehat{G_k} \Big\|_{L^{\frac{2}{d-1}}(\mathbb{B}^d(0,R))} \leq C_0 R^{-\frac{d}{2}} \prod_{k=1}^{d} \|G_k\|_{L^2(\mathbb{R}^d)} \tag{2.2.5}$$

for $R \geq 1$ when G_k is supported in $\Sigma_k(1/R) := \{(\xi, \tau) \in \mathbb{R}^{d-1} \times \mathbb{R} : \operatorname{dist}((\xi, \tau), \Sigma_k) < 1/R\}$. The estimate (2.2.3) is equivalent to (2.2.5) with

$C_0 = CR^\varepsilon$ (see [9]). Let $\mathcal{C}(R)$ be the infimum of C_0 with which (2.2.5) holds. The key part of the proof is to establish the implication

$$\mathcal{C}(R) \leq R^b \implies \mathcal{C}(R) \leq C(\theta, \varepsilon) R^{\frac{b}{1+\alpha}+\varepsilon} \tag{2.2.6}$$

for any $\varepsilon > 0$ where b is a positive constant. Via iteration the exponent of R can be suppressed to be arbitrary small and hence we get the estimate (2.2.3).

Let ϕ be a real-valued bump function adapted to $B(0, C)$, such that its Fourier transform is non-negative on the unit ball. For each $R \geq 1$ and $x \in \mathbb{R}^d$, define $\phi^x_{R,\alpha}(\xi) := e^{2\pi i x\cdot\xi} R^{d/(1+\alpha)} \phi(R^{1/(1+\alpha)}\xi)$. Then $\mathcal{C}(R) \leq R^b$ implies

$$\| \prod_{k=1}^d \widehat{\phi^x_{R,\alpha} G_k} \|_{L^{\frac{2}{d-1}}(\mathbb{B}^d(x, R^{\frac{1}{1+\alpha}}))} \lesssim R^{\frac{b}{1+\alpha} - \frac{d}{2(1+\alpha)}} \prod_{k=1}^d \|G_k * \phi^x_{R,\alpha}\|_{L^2(\mathbb{R}^d)}$$

Applying $(R^{-d/(1+\alpha)} \int_{x\in\mathbb{B}(0,R)} |\cdot|^{2/(d-1)} dx)^{(d-1)/2}$, $\| \prod_{k=1}^d \widehat{G_k} \|_{L^{\frac{2}{d-1}}(\mathbb{B}^d(0,R))}$ is bounded by

$$R^{\frac{b}{1+\alpha} - \frac{d}{2(1+\alpha)}} \big(R^{-\frac{d}{1+\alpha}} \int_{B(0,R)} (\prod_{k=1}^d \|G_k * \phi^x_{R,\alpha}\|^2_{L^2(\mathbb{R}^d)})^{\frac{1}{d-1}} dx\big)^{\frac{d-1}{2}}. \tag{2.2.7}$$

Using (2.2.4) we see that the set $\Sigma_k(1/R) \cap \mathbb{B}^d((\xi,\tau), R^{-1/(1+\alpha)})$, $(\xi,\tau) \in \Sigma_k$ is contained in a C/R neighborhood of the tangent plane to Σ_k at (ξ,τ). Thus $\Sigma_k(1/R)$ can be covered with a collection $\{\mathfrak{R}^k_j\}$ of finitely overlapping rectangles of dimensions about $R^{-1} \times R^{-1/(1+\alpha)} \times \cdots \times R^{-1/(1+\alpha)}$ which are essentially tangential to $\Sigma_k(1/R)$. These rectangles provide a decomposition of $G_k = \sum_j G^k_j$ while $\operatorname{supp} G^k_j \subset \mathfrak{R}^k_j$. Then (2.2.7) is bounded by

$$CR^{\frac{b}{1+\alpha} - \frac{d}{2(1+\alpha)}} \big(R^{-\frac{d}{1+\alpha}} \int_{B(0,R)} (\prod_{k=1}^d \sum_j \|\widehat{G^k_j}\|^2_{L^2(B(x,R^{\frac{1}{1+\alpha}}))})^{\frac{1}{d-1}} dx\big)^{\frac{d-1}{2}}.$$

And note that

$$\|\widehat{G^k_j}\|^2_{L^2(B(x,R^{\frac{1}{1+\alpha}}))} \lesssim R^{-\frac{\alpha}{1+\alpha}} |\widehat{\bar{G}^k_j}|^2 * \chi_{\mathfrak{R}^{k,*}_j}(x),$$

where $\bar{G}^k_j = G^k_j / g^k_j$, g^k_j is a Schwartz function which satisfies $g^k_j \sim 1$ on $\mathfrak{R}^k_j$ and $|\widehat{g^k_j}(x+y)| \lesssim R^{-(d+\alpha)/(1+\alpha)} \chi_{\mathfrak{R}^{k,*}_j}(x)$ for all $x, y \in \mathbb{R}^d$ with $|y| \leq R^{1/(1+\alpha)}$, and $\mathfrak{R}_j^{k,*}$ is dual of $\mathfrak{R}_j^k$. Here, $\mathfrak{R}_j^{k,*}$ is the rectangle centered at the origin of dimensions about $R^1 \times R^{1/(1+\alpha)} \times \cdots \times R^{1/(1+\alpha)}$ which are essentially normal to $\Sigma_k(1/R)$.

Using change of variables $x \mapsto Rx$ and Theorem 2.2.2, we can get

$$\|\prod_{k=1}^{d} \widehat{G_k}\|_{L^{\frac{2}{d-1}}(\mathbb{B}^d(0,R))} \lesssim R^{\frac{b}{1+\alpha}+\frac{\alpha\epsilon}{2(\alpha+1)}-\frac{d}{2}} \prod_{k=1}^{d} \big(\sum_j \|\widehat{\bar{G}_j^k}\|_2^2\big)^{\frac{1}{2}}.$$

Then Plancherel's theorem and the fact $|\bar{G}_j^k| \sim |G_j^k|$ show (2.2.6). $\square$

Chapter 3

Local smoothing estimate

In this chapter, we concern the local smoothing estimate:

$$\|\chi(t)\mathcal{A}_t f\|_{L^p_\alpha(\mathbb{R}^{d+1})} \le C\|f\|_{L^p(\mathbb{R}^d)}, \tag{3.0.1}$$

where χ is a smooth function supported in $(1/2, 4)$. To prove L^p $(p \neq 2)$ smoothing properties of $\mathcal{A}_t$, we need more than the decay of $\widehat{\mathfrak{m}}_t$, i.e., (1.1.2). The common approach in [44, 3, 4] to get around this difficulty was to use detailed decompositions (of various scales) on the Fourier side away from the conic sets where $\widehat{\mathfrak{m}}_t$ decays slowly. The consequent decompositions were then combined with the decoupling or square function estimate [39, 43, 44, 45, 3, 4]. However, this type of approach based on fine scale decomposition becomes exceedingly difficult to manage as the dimension d gets larger and, consequently, does not seem to be tractable in higher dimensions.

To overcome the difficulty, we develop a new strategy which allows us to dispense with such sophisticated decompositions. We briefly discuss the key ingredients of our approach.

• The main novelty of the paper lies in an induction argument which we build on the local nondegeneracy assumption:

$$\sum_{\ell=1}^{L} |\langle \gamma^{(\ell)}(s), \xi\rangle| \ge B^{-1}|\xi| \tag{$\mathfrak{N}(L,B)$}$$

for a constant $B \ge 1$. To prove our results, we consider the operator $\mathcal{A}_t[\gamma, a]$ (see (3.1.2) below for its definition). Clearly, $\mathfrak{N}(d, B')$ holds for a constant $B' > 0$ if γ satisfies (1.1.1). However, instead of considering the case $L = d$

alone, we prove the estimate for all $L = 2, \dots, d$ under the assumption that $\mathfrak{N}(L, B)$ holds on the support of a. See Theorem 3.1.2 and 4.0.1. A trivial (yet, important) observation is that $\mathfrak{N}(L-1, B)$ implies $\mathfrak{N}(L, B)$, so we may think of $\mathcal{A}_t[\gamma, a]$ as being more degenerate as L gets larger. Thanks to this hierarchical structure, we may use an inductive strategy along the number L. See Proposition 3.1.3 and 4.1.1 below.

• We extend the rescaling [22, 26] and iteration [44] arguments. Roughly speaking, we combine the first with the induction assumption in Proposition 3.1.3 (or 4.1.1) to handle the less degenerate parts, and use the latter to deal with the remaining part. In order to generalize the arguments, we introduce a class of symbols which are naturally adjusted to a small subcurve (Definition 3.2.1). We also use the decoupling inequalities for the nondegenerate curves obtained by Beltran et al. [4] (Corollary 2.1.5). Their inequalities were deduced from those due to Bourgain, Demeter, and Guth [12]. Instead of applying the inequalities directly, we use modified forms which are adjusted to the sharp smoothing orders of the specific estimates (see (2.1.14) and (2.1.15)). This makes it possible to obtain the sharp estimates on an extended range.

3.1 Local smoothing with localized frequency

In this section, we consider an extension of Theorem 1.1.2 via microlocalization (see Theorem 3.1.2 below) which we can prove inductively. We then reduce the matter to proving Proposition 3.4.1, which we show by applying Proposition 3.4.2. We also obtain some preparatory results.

Let $1 \le L \le d$ be a positive integer and $B \ge 1$ be a large number. For quantitative control of estimates we consider the following two conditions:

$$\max_{0 \le j \le 3d+1} |\gamma^{(j)}(s)| \le B, \qquad s \in I, \tag{3.1.1}$$

$$\mathrm{Vol}\left(\gamma^{(1)}(s), \dots, \gamma^{(L)}(s)\right) \ge 1/B, \qquad s \in I, \qquad \mathfrak{V}(L, B)$$

where $\mathrm{Vol}(v_1, \dots, v_L)$ denotes the L-dimensional volume of the parallelepiped generated by $v_1, \dots, v_L$. By finite decomposition, rescaling, and a change of variables, the constant B can be taken to be close to 1 (see Section 3.3).

Notation. For nonnegative quantities A and D, we denote $A \lesssim D$ if there exists an independent positive constant C such that $A \le CD$, but the constant

C may differ at each occurrence depending on the context, and $A \lesssim_B D$ means the inequality holds with an implicit constant depending on B. Throughout the paper, the constant C mostly depends on B. However, we do not make it explicit every time since it is clear in the context. By $A = O(D)$ we denote $|A| \lesssim D$.

Definition 3.1.1. *For $k \geq 0$, we denote $\mathbb{A}_k = \{\xi \in \mathbb{R}^d : 2^{k-1} \leq |\xi| \leq 2^{k+1}\}$. We say $a \in \mathrm{C}^{d+L+2}(\mathbb{R}^{d+2})$ is a symbol of type (k, L, B) relative to γ if* $\operatorname{supp} a \subset I \times [2^{-1}, 4] \times \mathbb{A}_k$*, $\mathfrak{N}(L, B)$ holds for γ whenever $(s, t, \xi) \in \operatorname{supp} a$ for some t, and*

$$|\partial_s^j \partial_t^l \partial_\xi^\alpha a(s, t, \xi)| \leq B|\xi|^{-|\alpha|}$$

for $(j, l, \alpha) \in \mathcal{I}_L := \{(j, l, \alpha) : 0 \leq j \leq 1,\ 0 \leq l \leq 2L,\ |\alpha| \leq d + L + 2\}$.

We define an integral operator by

$$\mathcal{A}_t[\gamma, a]f(x) = (2\pi)^{-d} \iint_{\mathbb{R}} e^{i(x - t\gamma(s))\cdot\xi}\, a(s, t, \xi) ds\, \widehat{f}(\xi)\, d\xi. \tag{3.1.2}$$

Note $\mathcal{A}_t f = \mathcal{A}_t[\gamma, \psi]f$. Theorem 1.1.2 is a consequence of the following.

Theorem 3.1.2. *Let $\gamma \in \mathrm{C}^{3d+1}(I)$ satisfy* (3.1.1) *and $\mathfrak{V}(L, B)$ for some $B \geq 1$. Suppose a is a symbol of type (k, L, B) relative to γ. Then, if $p \geq 4L - 2$, for any $\epsilon > 0$ there is a constant $C_\epsilon = C_\epsilon(B)$ such that*

$$\|\mathcal{A}_t[\gamma, a]f\|_{L^p(\mathbb{R}^{d+1})} \leq C_\epsilon 2^{(-\frac{2}{p}+\epsilon)k} \|f\|_{L^p(\mathbb{R}^d)}. \tag{3.1.3}$$

Theorem 3.1.2 is trivial when $L = 1$. Indeed, (3.1.3) follows from the estimate $|\mathcal{A}_t[\gamma, a]f(x)| \lesssim_B \int_I K * |f|(x - t\gamma(s))\, ds$ where $K(x) = 2^{(d-1)k}(1 + |2^k x|)^{-d-3}$. To show this, note $|\gamma'(s) \cdot \xi| \sim 2^k$ if $(s, t, \xi) \in \operatorname{supp} a$ for some t. By integration by parts in s, $\mathcal{A}_t[\gamma, a] = t^{-1}\mathcal{A}_t[\gamma, \tilde{a}]$ where $\tilde{a} = i(\gamma'(s) \cdot \xi\, \partial_s a - \gamma''(s) \cdot \xi\, a)/(\gamma'(s) \cdot \xi)^2$. Since $|\partial_\xi^\alpha \tilde{a}| \lesssim |\xi|^{-|\alpha|-1}$ for $|\alpha| \leq d + 3$, routine integration by parts in ξ gives the estimate (e.g., see *Proof of Lemma 3.2.4*). When $L = 2$, Theorem 3.1.2 is already known by the result in [44, Theorem 4.1] and the decoupling inequality in [11].

Once we have Theorem 3.1.2, the proof of Theorem 1.1.2 is straightforward. By Littlewood-Paley decomposition it is sufficient to show (3.1.3) for $p \geq 4d - 2$ with $a_k(s, t, \xi) = \psi(s)\chi(t)\beta(2^{-k}|\xi|)$, where $\beta \in \mathrm{C}_c^\infty((1/2, 2))$. This can be made rigorous using $\iint e^{-it(\tau + \gamma(s)\cdot\xi)}\psi(s)\chi(t) ds dt = O((1 + |\tau|)^{-N})$ for any N if $|\tau| \ge (1+\max_{s\in\operatorname{supp}\psi}|\gamma(s)|)|\xi|$. Since γ satisfies (1.1.1), a_k is of type (k,d,B) relative to γ for a large B. Therefore, Theorem 1.1.2 follows from Theorem 3.1.2.

Theorem 3.1.2 is immediate from the next proposition, which places Theorem 3.1.2 in an inductive framework.

Proposition 3.1.3. *Let $2 \le N \le d$. Suppose Theorem 3.1.2 holds for $L = N-1$. Then, Theorem 3.1.2 holds true with $L = N$.*

To prove Proposition 3.1.3, from this section to Section 3.5 we fix $N \in [2,d]$, γ satisfying $\mathfrak{V}(N,B)$, and a symbol a of type (k,N,B) relative to γ.

One of the main ideas is that by a suitable decomposition of the symbol we can separate from $\mathcal{A}_t[\gamma,a]$ the less degenerate part which corresponds to $L = N-1$. To this part we apply the assumption combined with a rescaling argument. To do this, we introduce a class of symbols which are adjusted to short subcurves of γ.

3.2 Symbols associated to subcurves

We begin with some notations. Let $N \ge 2$, and let δ and B' denote the numbers such that

$$2^{-k/N} \le \delta \le 2^{-7dN}B^{-6N}, \qquad B \le B' \le B^C$$

for a large constant $C \ge 3d+1$. We note that $\mathfrak{V}(N-1,B')$ holds for some B'. In fact, $\mathfrak{V}(N-1,B^2)$ follows by (3.1.1) and $\mathfrak{V}(N,B)$.

For $s \in I$, we define a linear map $\widetilde{\mathcal{L}}_s^\delta : \mathbb{R}^d \mapsto \mathbb{R}^d$ as follows:

$$\begin{aligned} (\widetilde{\mathcal{L}}_s^\delta)^\intercal \gamma^{(j)}(s) &= \delta^{N-j}\gamma^{(j)}(s), \qquad && j = 1,\dots,N-1, \\ (\widetilde{\mathcal{L}}_s^\delta)^\intercal v &= v, && v \in \big(\mathrm{V}_s^{\gamma,N-1}\big)^\perp, \end{aligned} \tag{3.2.1}$$

where $\mathrm{V}_s^{\gamma,\ell} = \operatorname{span}\big\{\gamma^{(j)}(s) : j = 1,\dots,\ell\big\}$. $\widetilde{\mathcal{L}}_s^\delta$ is well-defined since $\mathfrak{V}(N-1,B^2)$ holds for γ. The linear map $\widetilde{\mathcal{L}}_s^\delta$ naturally appears when we rescale a subcurve of length about δ (see the proofs of Lemma 3.2.4 and 3.3.1). We denote

$$\mathcal{L}_s^\delta(\tau,\xi) = \big(\delta^N\tau - \gamma(s)\cdot\widetilde{\mathcal{L}}_s^\delta\xi,\ \widetilde{\mathcal{L}}_s^\delta\xi\big), \qquad (\tau,\xi) \in \mathbb{R}\times\mathbb{R}^d. \tag{3.2.2}$$

We set $G(s) = (1, \gamma(s))$ and define

$$\Lambda_k(s, \delta, B') = \bigcap_{0 \le j \le N-1} \big\{(\tau, \xi) \in \mathbb{R} \times \mathbb{A}_k : |\langle G^{(j)}(s), (\tau, \xi)\rangle| \le B' 2^{k+5} \delta^{N-j}\big\}.$$

Definition 3.2.1. *Let $(s_0, \delta) \in (-1, 1) \times (0, 1)$ such that $I(s_0, \delta) := [s_0 - \delta, s_0 + \delta] \subset I$. Then, by $\mathfrak{A}_k(s_0, \delta)$ we denote the set of $\mathfrak{a} \in \mathrm{C}^{d+N+2}(\mathbb{R}^{d+3})$ such that*

$$\operatorname{supp} \mathfrak{a} \subset I(s_0, \delta) \times [2^{-1}, 2^2] \times \Lambda_k(s_0, \delta, B), \tag{3.2.3}$$
$$\big|\partial_s^j \partial_t^l \partial_{\tau,\xi}^\alpha \mathfrak{a}\big(s, t, \mathcal{L}_{s_0}^\delta(\tau, \xi)\big)\big| \le B\delta^{-j} |(\tau, \xi)|^{-|\alpha|}, \qquad (j, l, \alpha) \in \mathcal{I}_N. \tag{3.2.4}$$

We define $\operatorname{supp}_\xi \mathfrak{a} = \bigcup_{s,t,\tau} \operatorname{supp} \mathfrak{a}(s, t, \tau, \cdot)$ and $\operatorname{supp}_s \mathfrak{a}$ is defined likewise. And define $\operatorname{supp}_{s,\xi} \mathfrak{a} = \bigcup_{t,\tau} \operatorname{supp} \mathfrak{a}(\cdot, t, \tau, \cdot)$, and $\operatorname{supp}_{\tau,\xi} \mathfrak{a}$ is defined likewise. We note *a statement $S(s, \xi)$, depending on (s, ξ), holds on* $\operatorname{supp}_{s,\xi} \mathfrak{a}$ *if and only if $S(s, \xi)$ holds whenever $(s, t, \tau, \xi) \in \operatorname{supp} \mathfrak{a}$ for some t, τ.*

Denote $\mathrm{V}_s^{G,\ell} = \operatorname{span}\{(1, 0), G'(s), \dots, G^{(\ell)}(s)\}$. We take a close look at the map $\mathcal{L}_s^\delta$. By (3.2.1) and (3.2.2) we have

$$\begin{cases} (\mathcal{L}_s^\delta)^\intercal G(s) = \delta^N (1, 0), & \\ (\mathcal{L}_s^\delta)^\intercal G^{(j)}(s) = \delta^{N-j} G^{(j)}(s), & j = 1, \dots, N-1, \\ (\mathcal{L}_s^\delta)^\intercal v = v, & v \in (\mathrm{V}_s^{G,N-1})^\perp. \end{cases} \tag{3.2.5}$$

The first identity is clear since $(\mathcal{L}_s^\delta)^\intercal(\tau, \xi) = (\delta^N \tau, (\widetilde{\mathcal{L}}_s^\delta)^\intercal \xi - \tau (\widetilde{\mathcal{L}}_s^\delta)^\intercal \gamma(s))$. The second and the third follow from (3.2.1) since $G^{(j)} \in \{0\} \times \mathbb{R}^d$, $1 \le j \le N-1$, $\big(\mathrm{V}_s^{G,N-1}\big)^\perp \subset \{0\} \times \mathbb{R}^d$, and $(\mathcal{L}_s^\delta)^\intercal(0, \xi) = (0, (\widetilde{\mathcal{L}}_s^\delta)^\intercal \xi)$. Furthermore, there is a constant $C = C(B)$, independent of s and δ, such that

$$|\mathcal{L}_s^\delta(\tau, \xi)| \le C|(\tau, \xi)|. \tag{3.2.6}$$

Note that (3.2.6) is equivalent to $|(\mathcal{L}_s^\delta)^\intercal(\tau, \xi)| \le C|(\tau, \xi)|$. The inequality is clear from (3.2.1) because $\mathfrak{V}(N-1, B^2)$ holds and all the eigenvalues of $(\widetilde{\mathcal{L}}_s^\delta)^\intercal$ are contained in the interval $(0, 1]$.

Lemma 3.2.2. *Let $\mathcal{L}_s^\delta(\tau, \xi) \in \Lambda_k(s, \delta, B')$ and $\mathfrak{V}(N-1, B')$ holds for γ. Then, there exists a constant $C = C(B')$ such that*

$$C^{-1} |(\tau, \xi)| \le 2^k \le C|\xi|. \tag{3.2.7}$$

Proof. Since $\mathcal{L}_s^\delta(\tau,\xi) \in \Lambda_k(s,\delta,B')$, by (3.2.2) we have $2^{k-1} \le |\widetilde{\mathcal{L}}_s^\delta \xi| \le 2^{k+1}$. So, the second inequality in (3.2.7) is clear from (3.2.6) if we take $\tau = 0$.

To show the first inequality, from (3.2.5) we have $|\langle (1,0),(\tau,\xi)\rangle| \le B' 2^{k+5}$ and $|\langle G^{(j)}(s),(\tau,\xi)\rangle| \le B' 2^{k+5}$, $1 \le j \le N-1$, because $\mathcal{L}_s^\delta(\tau,\xi) \in \Lambda_k(s,\delta,B')$. Also, if $v \in (\mathrm{V}_s^{G,N-1})^\perp$ and $|v| = 1$, we see $|\langle v,(\tau,\xi)\rangle| = |\langle v,\mathcal{L}_s^\delta(\tau,\xi)\rangle| \le 2^{k+1}$, by (3.2.5). Since $\mathfrak{V}(N-1,B')$ holds and $\mathrm{V}_s^{G,N-1} \oplus (\mathrm{V}_s^{G,N-1})^\perp = \mathbb{R}^{d+1}$, we get $|(\tau,\xi)| \le C 2^k$ for some $C = C(B')$. $\square$

The following shows the matrices $\mathcal{L}_s^\delta$, $\mathcal{L}_{s_0}^\delta$ are close to each other if so are s, s_0.

Lemma 3.2.3. *Let $s, s_0 \in (-1,1)$ and γ satisfy $\mathfrak{V}(N-1,B')$. If $|s-s_0| \le \delta$, then there exists a constant $C = C(B') \ge 1$ such that*

$$C^{-1}|(\tau,\xi)| \le |(\mathcal{L}_{s_0}^\delta)^{-1}\mathcal{L}_s^\delta(\tau,\xi)| \le C|(\tau,\xi)|. \tag{3.2.8}$$

Proof. It suffices to prove that (3.2.8) holds if $|s-s_0| \le c\delta$ for a constant $c > 0$, independent of s and s_0. Applying this finitely many times, we can remove the additional assumption. Moreover, it is enough to show

$$\|(\mathcal{L}_s^\delta)^\intercal(\mathcal{L}_{s_0}^\delta)^{-\intercal} - \mathrm{I}\| \lesssim_{B'} c \tag{3.2.9}$$

when $|s-s_0| \le c\delta$. Here, $\|\cdot\|$ denotes a matrix norm. Taking $c > 0$ sufficiently small, we get (3.2.8).

By (3.2.5), $(\mathcal{L}_s^\delta)^\intercal(\mathcal{L}_{s_0}^\delta)^{-\intercal}G^{(j)}(s_0) = \big(\mathcal{L}_s^\delta\big)^\intercal \delta^{-(N-j)}G^{(j)}(s_0)$ for $j = 1,\dots,N-1$. Let $s_0 = s + c'\delta$, $|c'| \le c$. Expanding $G^{(j)}$ in Taylor series at s, by (3.1.1) we have

$$(\mathcal{L}_s^\delta)^\intercal(\mathcal{L}_{s_0}^\delta)^{-\intercal}G^{(j)}(s_0) = \big(\mathcal{L}_s^\delta\big)^\intercal\Big(\sum_{\ell=j}^{N-1}\delta^{-(N-j)}G^{(\ell)}(s)\frac{(c'\delta)^{\ell-j}}{(\ell-j)!} + O\big(c^{N-j}B'\big)\Big)$$

for $j = 1,\dots,N-1$. By (3.2.5) and the mean value theorem, we get

$$(\mathcal{L}_s^\delta)^\intercal(\mathcal{L}_{s_0}^\delta)^{-\intercal}G^{(j)}(s_0) = G^{(j)}(s_0) + O(cB'), \quad j = 1,\dots,N-1.$$

From (3.2.5) we also have $(\mathcal{L}_s^\delta)^\intercal(\mathcal{L}_{s_0}^\delta)^{-\intercal}(1,0) = \delta^{-N}(\mathcal{L}_s^\delta)^\intercal G(s_0)$. A similar argument also shows $(\mathcal{L}_s^\delta)^\intercal(\mathcal{L}_{s_0}^\delta)^{-\intercal}(1,0) = (1,0) + O(cB')$.

Let $\{v_N,\dots,v_d\}$ denote an orthonormal basis of $(\mathrm{V}_{s_0}^{G,N-1})^\perp$. By $\mathfrak{V}(N-1,B')$ and (3.1.1) it follows that $|\gamma^{(j)}(s_0)| \ge (B')^{-1-N}$, $j = 1,\dots,N-1$. Since $|\gamma^{(j)}(s) - \gamma^{(j)}(s_0)| \le cB'\delta$, there is an orthonormal basis $\{v_N(s),\dots,v_d(s)\}$ of

$(\mathrm{V}_s^{G,N-1})^\perp$ such that $|v_j(s) - v_j| \lesssim_{B'} c\delta$, $j = N, \dots, d$. So, we have $|(\mathcal{L}_s^\delta)^\intercal v_j - v_j| \lesssim_{B'} c\delta$ by (3.2.6). Since $(\mathcal{L}_{s_0}^\delta)^{-\intercal} v_j = v_j$, it follows that $|(\mathcal{L}_s^\delta)^\intercal(\mathcal{L}_{s_0}^\delta)^{-\intercal} v_j - v_j| \lesssim_{B'} c\delta$, $j = N, \dots, d$.

We denote by M the matrix $[(1,0), G'(s_0), \dots, G^{(N-1)}(s_0), v_N, \dots, v_d]$. Combining all together, we have $\|(\mathcal{L}_s^\delta)^\intercal(\mathcal{L}_{s_0}^\delta)^{-\intercal}\mathrm{M} - \mathrm{M}\| \lesssim_{B'} c$. Note that $\mathfrak{V}(N-1, B')$ gives $|\mathrm{M}^{-1}v| \lesssim_{B'} |v|$ for $v \in \mathbb{R}^{d+1}$. Therefore, we obtain (3.2.9). □

For a continuous function $\mathfrak{a}$ supported in $I \times [1/2, 4] \times \mathbb{R} \times \mathbb{A}_k$, we set

$$m[\mathfrak{a}](\tau, \xi) = \iint e^{-it'(\tau + \gamma(s)\cdot\xi)} \mathfrak{a}(s, t', \tau, \xi) ds dt', \tag{3.2.10}$$

$$\mathcal{T}[\mathfrak{a}]f(x,t) = (2\pi)^{-d-1} \iint e^{i(x\cdot\xi + t\tau)} m[\mathfrak{a}](\tau,\xi)\widehat{f}(\xi)\, d\xi d\tau. \tag{3.2.11}$$

Lemma 3.2.4. *Suppose $\mathfrak{a} \in C^{d+3}(\mathbb{R}^{d+3})$ satisfies (3.2.3) and (3.2.4) for $j = l = 0$ and $|\alpha| \le d+3$. Then, there is a constant $C = C(B)$ such that*

$$\|\mathcal{T}[\mathfrak{a}]f\|_{L^\infty(\mathbb{R}^{d+1})} \le C\delta \|f\|_{L^\infty(\mathbb{R}^d)}, \tag{3.2.12}$$

$$\|(1-\tilde\chi)\mathcal{T}[\mathfrak{a}]f\|_{L^p(\mathbb{R}^{d+1})} \le C 2^{-k}\delta^{1-N}\|f\|_{L^p(\mathbb{R}^d)}, \quad p > 1, \tag{3.2.13}$$

where $\tilde\chi \in \mathrm{C}_c^\infty((2^{-2}, 2^3))$ such that $\tilde\chi = 1$ on $[3^{-1}, 6]$.

Proof. We first note

$$\mathcal{T}[\mathfrak{a}]f(x,t) = \int K[\mathfrak{a}](s,t,\cdot) * f(x)\, ds, \tag{3.2.14}$$

where

$$K[\mathfrak{a}](s,t,x) = (2\pi)^{-d-1} \iiint e^{i(t-t', x-t'\gamma(s))\cdot(\tau,\xi)} \mathfrak{a}(s,t',\tau,\xi)\, d\xi d\tau dt'. \tag{3.2.15}$$

Since $\operatorname{supp}_s \mathfrak{a} \subset I(s_0, \delta)$, to prove (3.2.12) we need only to show

$$\|K[\mathfrak{a}](s,\cdot)\|_{L_t^\infty L_x^1} \le C, \qquad s \in I(s_0,\delta) \tag{3.2.16}$$

for some $C = C(B) > 0$. To this end, changing variables $(\tau,\xi) \to 2^k\mathcal{L}_s^\delta(\tau,\xi)$ in the right hand side of (3.2.15) and noting $|\det \mathcal{L}_s^\delta| = \delta^N |\det \widetilde{\mathcal{L}}_s^\delta| = \delta^{N(N+1)/2}$, we get

$$K[\mathfrak{a}](s,t,x) = C_* \iiint e^{i2^k(t-t', x-t\gamma(s))\cdot(\delta^N\tau, \widetilde{\mathcal{L}}_s^\delta\xi)} \mathfrak{a}(s,t',2^k\mathcal{L}_s^\delta(\tau,\xi))\, d\xi d\tau dt',$$

where $C_* = (2\pi)^{-d-1}\delta^{N(N+1)/2}2^{k(d+1)}$. Since $\mathfrak{a}$ satisfies (3.2.3), by (3.2.8) and Lemma 3.2.2 we have supp $\mathfrak{a}(s,t,2^k\mathcal{L}_s^\delta\cdot) \subset \{(\tau,\xi) : |(\tau,\xi)| \lesssim_B 1\}$. Besides, by (3.2.4) and (3.2.8) it follows that $|\partial_{\tau,\xi}^\alpha\big(\mathfrak{a}(s,t,2^k\mathcal{L}_s^\delta(\tau,\xi))\big)| \lesssim_B 1$ for $|\alpha| \le d+3$. Thus, repeated integration by parts in τ,ξ yields

$$|K[\mathfrak{a}](s,t,x)| \lesssim C_* \int_{1/2}^4 \Big(1 + 2^k\big|\big(\delta^N(t-t'), (\widetilde{\mathcal{L}}_s^\delta)^\intercal(x - t\gamma(s))\big)\big|\Big)^{-d-3} dt',$$

by which we obtain (3.2.16) as desired.

It is easy to show (3.2.13). The above estimate for $K[\mathfrak{a}]$ gives

$$\|(1-\tilde\chi)K[\mathfrak{a}](s,t,\cdot)\|_{L_x^1} \lesssim \delta^{-N}2^{-k}|t-1|^{-1}|1-\tilde\chi(t)|.$$

Since $\mathrm{supp}_s\, \mathfrak{a} \subset I(s_0,\delta)$, (3.2.13) for $p>1$ follows by (3.2.14), Minkowski's and Young's convolution inequalities. $\square$

3.3 Rescaling

Let $\mathfrak{a} \in \mathfrak{A}_k(s_0,\delta)$. Suppose that

$$\sum_{j=1}^{N-1} \delta^j |\langle\gamma^{(j)}(s),\xi\rangle| \ge 2^k\delta^N/B' \tag{3.3.1}$$

holds on $\mathrm{supp}_{s,\xi}\,\mathfrak{a}$ for some $B'>0$. Then, via decomposition and rescaling, we can bound the L^p norm of $\mathcal{T}[\mathfrak{a}]f$ by those of the operators given by symbols of type $(j, N-1, \tilde B)$ relative to a curve for some $\tilde B$ and j (see Lemma 3.3.1 below).

To do so, we define a rescaled curve $\gamma_{s_0}^\delta : I \to \mathbb{R}^d$ by

$$\gamma_{s_0}^\delta(s) = \delta^{-N}(\widetilde{\mathcal{L}}_{s_0}^\delta)^\intercal\big(\gamma(\delta s + s_0) - \gamma(s_0)\big). \tag{3.3.2}$$

As $\delta \to 0$, the curves $\gamma_{s_0}^\delta$ get close to a nondegenerate curve in N dimensional vector space, so the curves behave in a uniform way. In particular, (3.1.1) and $\mathfrak{V}(N,B)$ hold for some B for $\gamma_{s_0}^\delta$ if $\delta < \delta'$ for a constant $\delta' = \delta'(B)$ small enough.

Note $(\gamma_{s_0}^\delta)^{(j)}(s) = \delta^{j-N}(\widetilde{\mathcal{L}}_{s_0}^\delta)^\intercal\gamma^{(j)}(\delta s + s_0)$, $1 \le j \le N-1$, and $|(\gamma_{s_0}^\delta)^{(j)}(s)| \lesssim B\delta$, $N+1 \le j \le 3d+1$. Thus, Taylor series expansion and (3.2.1) give

$$(\gamma_{s_0}^\delta)^{(j)}(s) = \sum_{k=0}^{N-j-1} \frac{\gamma^{(j+k)}(s_0)}{k!}s^k + \frac{(\widetilde{\mathcal{L}}_{s_0}^\delta)^\intercal\gamma^{(N)}(s_0)}{(N-j)!}s^{N-j} + O(B\delta)$$

for $j = 1, \dots, N-1$. By (3.3.2), we have $(\gamma^\delta_{s_0})^{(N)}(s) = (\widetilde{\mathcal{L}}^\delta_{s_0})^\intercal \gamma^{(N)}(s_0) + O(\delta)$. We write $\gamma^{(N)}(s_0) = v + v'$ where $v \in \mathrm{V}^{\gamma,N-1}_s$ and $v' \in (\mathrm{V}^{\gamma,N-1}_s)^\perp$. Then, $(\widetilde{\mathcal{L}}^\delta_{s_0})^\intercal \gamma^{(N)}(s_0) = (\widetilde{\mathcal{L}}^\delta_{s_0})^\intercal v + v'$. Since $|(\widetilde{\mathcal{L}}^\delta_{s_0})^\intercal v| \lesssim_B \delta$ and $|v'| \le B$, $|(\widetilde{\mathcal{L}}^\delta_{s_0})^\intercal \gamma^{(N)}(s_0)| \le B + C\delta$ for some $C = C(B)$. Thus, $\gamma = \gamma^\delta_{s_0}$ satisfies (3.1.1) with B replaced by $3B$ if $\delta < \delta'$.

An elementary argument (elimination) shows

$$\mathrm{Vol}\big((\gamma^\delta_{s_0})^{(1)}(s), \dots, (\gamma^\delta_{s_0})^{(N)}(s)\big) = \mathrm{Vol}\big(\gamma^{(1)}(s_0), \dots, \gamma^{(N)}(s_0)\big) + O(\delta)$$

since $(\widetilde{\mathcal{L}}^\delta_{s_0})^\intercal \gamma^{(N)}(s_0) = (\widetilde{\mathcal{L}}^\delta_{s_0})^\intercal v + v'$ and $\gamma^{(N)}(s_0) = v + v'$. Taking δ' small enough, from $\mathfrak{V}(N, B)$ for γ we see $\mathfrak{V}(N, 3B)$ hold for $\gamma = \gamma^\delta_{s_0}$ if $0 < \delta < \delta'$.

The next lemma (cf. [26, Lemma 2.9]) plays a crucial role in what follows.

Lemma 3.3.1. *Let $2 \le N \le d$, $\mathfrak{a} \in \mathfrak{A}_k(s_0, \delta)$, and $j_* = \log(2^k \delta^N)$. Suppose (3.3.1) holds on $\mathrm{supp}_{s,\xi}\, \mathfrak{a}$. Then, there exist constants C, $\tilde{B} \ge 1$, and $\delta' > 0$ depending on B, and symbols $a_1, \dots, a_{l_*}$ of type $(j, N-1, \tilde{B})$ relative to $\gamma^\delta_{s_0}$, such that*

$$\big\|\tilde{\chi}\, \mathcal{T}[\mathfrak{a}] f\big\|_{L^p(\mathbb{R}^{d+1})} \le C\delta \sum_{1 \le l \le C} \big\|\mathcal{A}_t[\gamma^\delta_{s_0}, a_l] \tilde{f}_l\big\|_{L^p(\mathbb{R}^{d+1})},$$

$\|\tilde{f}_l\|_p = \|f\|_p$, and $j \in [j_* - C, j_* + C]$ as long as $0 < \delta < \delta'$.

Proof. We set $\mathfrak{a}_{\delta,s_0}(s, t, \tau, \xi) = \mathfrak{a}\big(\delta s + s_0, t, \tau, \xi\big)$. Combining (3.2.10) and (3.2.11), we write $\mathcal{T}[\mathfrak{a}] f$ as an integral (e.g., see (3.2.14) and (3.2.15)). Then, the change of variables $s \to \delta s + s_0$ and $(\tau, \xi) \to (\tau - \gamma(s_0) \cdot \xi,\, \xi)$ gives

$$\mathcal{T}[\mathfrak{a}] f(x, t) = (2\pi)^{-d-1}\, \delta \iint e^{i\langle x - t\gamma(s_0), \xi\rangle} \mathcal{J}(s, t, \xi) \widehat{f}(\xi)\, ds d\xi,$$

where

$$\mathcal{J}(s, t, \xi) = \iint e^{it\tau} e^{-it'(\,\tau + (\gamma(\delta s + s_0) - \gamma(s_0)) \cdot \xi\,)}\, \mathfrak{a}_{\delta,s_0}\big(s, t', \tau - \gamma(s_0) \cdot \xi,\, \xi\big)\, dt' d\tau.$$

Let $\tilde{f}$ be given by $\mathcal{F}(\tilde{f}) = |\det \delta^{-N} \widetilde{\mathcal{L}}^\delta_{s_0}|^{1-1/p} \widehat{f}(\delta^{-N} \widetilde{\mathcal{L}}^\delta_{s_0} \cdot\,)$ where $\mathcal{F}(\tilde{f}\,)$ denotes the Fourier transform of $\tilde{f}$. Then, $\|\tilde{f}\|_p = \|f\|_p$. Changing variables $\xi \to \delta^{-N} \widetilde{\mathcal{L}}^\delta_{s_0} \xi$ gives

$$\mathcal{T}[\mathfrak{a}] f(x, t) = C_d \iint e^{i\langle x - t\gamma(s_0), \delta^{-N} \widetilde{\mathcal{L}}^\delta_{s_0} \xi\rangle} \mathcal{J}(s, t, \delta^{-N} \widetilde{\mathcal{L}}^\delta_{s_0} \xi) \mathcal{F}(\tilde{f}\,)(\xi)\, ds d\xi,$$

where $C_d = (2\pi)^{-d-1}\,\delta|\det \delta^{-N}\widetilde{\mathcal{L}}^{\delta}_{s_0}|^{1/p}$. This leads us to set

$$\tilde{a}(s,t,\xi) = \frac{1}{2\pi}\iint e^{-it'(\tau+\gamma^{\delta}_{s_0}(s)\cdot\xi)}\tilde{\chi}(t)\mathfrak{a}_{\delta,s_0}\big(s,t+t',\delta^{-N}\mathcal{L}^{\delta}_{s_0}(\tau,\xi)\big)\,dt'd\tau. \tag{3.3.3}$$

It is easy to check $\tilde{a} \in C^{d+N+2}(\mathbb{R}^{d+2})$, since so is $\mathfrak{a}$ and $\gamma \in \mathrm{C}^{3d+1}$. By (3.3.2) and (3.2.2), we note $\tilde{\chi}(t)\mathcal{J}(s,t,\delta^{-N}\widetilde{\mathcal{L}}^{\delta}_{s_0}\xi) = 2\pi e^{-it\gamma^{\delta}_{s_0}(s)\cdot\xi}\,\tilde{a}(s,t,\xi)$. Therefore,

$$\tilde{\chi}(t)\mathcal{T}[\mathfrak{a}]f(x,t) = \delta|\det \delta^{-N}\widetilde{\mathcal{L}}^{\delta}_{s_0}|^{\frac{1}{p}}\,\mathcal{A}_t[\gamma^{\delta}_{s_0},\tilde{a}\,]\tilde{f}\,\big(\delta^{-N}(\widetilde{\mathcal{L}}^{\delta}_{s_0})^{\intercal}(x-t\gamma(s_0))\big),$$

and a change of variables gives

$$\big\|\tilde{\chi}\,\mathcal{T}[\mathfrak{a}]f\big\|_{L^p(\mathbb{R}^{d+1})} = \delta\big\|\mathcal{A}_t[\gamma^{\delta}_{s_0},\tilde{a}\,]\tilde{f}\big\|_{L^p(\mathbb{R}^{d+1})}. \tag{3.3.4}$$

We shall obtain symbols of type $(j,N-1,\tilde{B})$ from $\tilde{a}$ via decomposition and rescaling. To this end, we first note

$$\operatorname{supp}_{\xi}\tilde{a} \subset \big\{\xi\in\mathbb{R}^d : C^{-1}\delta^N 2^k \le |\xi| \le C\delta^N 2^k\big\} \tag{3.3.5}$$

for a constant $C = C(B) \ge 1$. This follows by Lemma 3.2.2 since there exists τ such that $\delta^{-N}\mathcal{L}^{\delta}_{s_0}(\tau,\xi) \in \Lambda_k(s_0,\delta,B)$ if $\xi \in \operatorname{supp}_{\xi}\tilde{a}$. We claim

$$|\partial_s^j\partial_t^l\partial_\xi^\alpha\tilde{a}(s,t,\xi)| \lesssim_B |\xi|^{-|\alpha|}, \qquad (j,l,\alpha)\in\mathcal{I}_{N-1}. \tag{3.3.6}$$

To show (3.3.6), let us set

$$\mathfrak{b}(s,t,t',\tau,\xi) = \tilde{\chi}(t)\mathfrak{a}_{\delta,s_0}(s,t+t',\delta^{-N}\mathcal{L}^{\delta}_{s_0}(\tau,\xi)\big).$$

Note $0\le j\le 1$. Taking derivatives on both sides of (3.3.3), we have

$$\partial_s^j\partial_t^l\partial_\xi^\alpha\tilde{a}(s,t,\xi) = \mathcal{I}[\mathfrak{b}_1] := \frac{1}{2\pi}\iint e^{-it'(\tau+\gamma^{\delta}_{s_0}(s)\cdot\xi)}\mathfrak{b}_1(s,t,t',\tau,\xi)\,dt'd\tau,$$

where

$$\mathfrak{b}_1 = \sum_{\substack{u_1+u_2=j,\\ \alpha_1+\alpha_2+\alpha_3=\alpha}} C_{\alpha,u}\big(t'\gamma^{\delta\,\prime}_{s_0}\cdot\xi\big)^{u_1-|\alpha_1|}(t'\gamma^{\delta\,\prime}_{s_0})^{\alpha_1}(t'\gamma^{\delta}_{s_0})^{\alpha_2}\,\partial_s^{u_2}\partial_t^l\partial_\xi^{\alpha_3}\mathfrak{b},$$

with $0 \le u_1 \le 1$, $0 \le |\alpha_1| \le u_1$, and constants $C_{\alpha,u}$ satisfying $|C_{\alpha,u}| = 1$. Integration by parts $u_1 + |\alpha_2|$ times in τ gives $\partial_s^j \partial_t^l \partial_\xi^\alpha \tilde a = \mathcal{I}[\mathfrak{b}_2]$, where

$$\mathfrak{b}_2 = \sum_{\substack{u_1+u_2=j,\\ \alpha_1+\alpha_2+\alpha_3=\alpha}} C'_{\alpha,u}\big({\gamma^\delta_{s_0}}' \cdot \xi\big)^{u_1-|\alpha_1|} ({\gamma^\delta_{s_0}}')^{\alpha_1} (\gamma^\delta_{s_0})^{\alpha_2}\, \partial_\tau^{u_1+|\alpha_2|} \partial_s^{u_2} \partial_t^l \partial_\xi^{\alpha_3} \mathfrak{b}$$

with constants $C'_{\alpha,u}$ satisfying $|C'_{\alpha,u}| = 1$. We decompose $\mathcal{I}[\mathfrak{b}_2] = \mathcal{I}[\chi_E \mathfrak{b}_2] + \mathcal{I}[\chi_{E^c}\mathfrak{b}_2]$ where $E = \{(\tau,\xi) : |\tau + \gamma^\delta_{s_0}(s)\cdot\xi| \le 1\}$. Then, integrating by parts in t' for $\mathcal{I}[\chi_{E^c}\mathfrak{b}_2]$, we obtain

$$|\mathcal{I}[\mathfrak{b}_2]| \lesssim \iint \chi_E |\mathfrak{b}_2| \,+ \frac{\chi_{E^c}|\partial_{t'}^2 \mathfrak{b}_2|}{|\tau + \gamma^\delta_{s_0}(s)\cdot\xi|^2}\, dt' d\tau.$$

Since $\mathfrak{a} \in \mathfrak{A}_k(s_0,\delta)$, $|\partial_s^{j'} \partial_t^{l'} \partial_{\tau,\xi}^{\alpha'} \mathfrak{b}| \lesssim_B |\xi|^{-|\alpha'|}$ for $(j',l',\alpha') \in \mathcal{I}_N$. It is also clear that $|{\gamma^\delta_{s_0}}'(s)| \lesssim 1$ if $\delta < \delta'$. Thus, $|\mathfrak{b}_2| = O(|\xi|^{-|\alpha|})$ and $|\partial_{t'}^2 \mathfrak{b}_2| = O(|\xi|^{-|\alpha|})$ if $l \le 2(N-1)$. Since $\partial_s^j \partial_t^l \partial_\xi^\alpha \tilde a = \mathcal{I}[\mathfrak{b}_2]$, we get (3.3.6).

Now, we decompose $\tilde a$. Let $\tilde\chi_1, \tilde\chi_2$, and $\tilde\chi_3 \in C_c^\infty(\mathbb{R})$ such that $\tilde\chi_1 + \tilde\chi_2 + \tilde\chi_3 = 1$ on $\operatorname{supp}\tilde\chi$ and $\operatorname{supp}\tilde\chi_\ell \subset [2^{\ell-3}, 2^\ell]$. Also, let $\beta \in \mathrm{C}_c^\infty((2^{-1},2))$ such that $\sum \beta(2^{-k}\cdot) = 1$ on $\mathbb{R}_+$. Then, we set

$$a_{\ell,j}(s,t,\xi) = \tilde\chi_\ell(t)\beta(2^{-j}|\xi|)\tilde a(s,t,\xi),$$

so $\sum_{\ell,j} a_{\ell,j} = \tilde a$. By (3.3.5), $a_{\ell,j} = 0$ if $|j - j_*| > C$ for some $C > 0$.

Denoting $(a)_\rho(s,t,\xi) = a(s,\rho t,\rho^{-1}\xi)$, via rescaling we can observe that $\mathcal{A}_{\rho t}[\gamma^\delta_{s_0}, a]g(x) = \mathcal{A}_t[\gamma^\delta_{s_0}, (a)_\rho]g(\rho\,\cdot)(x/\rho)$. Thus, changes of variables yield

$$\|\mathcal{A}_t[\gamma^\delta_{s_0}, a_{\ell,j}]\tilde f\|_{L^p(\mathbb{R}^{d+1})} = 2^{(\ell-2)/p}\|\mathcal{A}_t[\gamma^\delta_{s_0}, (a_{\ell,j})_{2^{\ell-2}}]\tilde f_\ell\|_{L^p(\mathbb{R}^{d+1})},$$

where $\tilde f_\ell = 2^{(\ell-2)d/p}\tilde f(2^{\ell-2}\cdot)$. Since $\mathcal{A}_t[\gamma^\delta_{s_0}, \tilde a] = \sum_{\ell,j}\mathcal{A}_t[\gamma^\delta_{s_0}, a_{\ell,j}]$, by (3.3.4) we get

$$\big\|\tilde\chi\, \mathcal{T}[\mathfrak{a}]f\big\|_{L^p(\mathbb{R}^{d+1})} \lesssim \delta \sum\nolimits_{\ell,j} \big\|\mathcal{A}_t[\gamma^\delta_{s_0}, (a_{\ell,j})_{2^{\ell-2}}]\tilde f_\ell\big\|_{L^p(\mathbb{R}^{d+1})}.$$

To complete the proof, we only have to relabel $(a_{\ell,j})_{2^{\ell-2}}$, $\ell = 1,2,3$, $j_* - C \le j \le j_* + C$. Indeed, since $\tilde a \in C^{d+N+2}$, $(a_{\ell,j})_{2^{\ell-2}} \in C^{d+N+2}$, which is supported in $I \times [2^{-1},4] \times \mathbb{A}_{j+\ell-2}$. Obviously, (3.3.6) holds for $\tilde a = (a_{\ell,j})_{2^{\ell-2}}$ because $\ell = 1,2,3$. Changing variables $s \to \delta s + s_0$ and $\xi \to \delta^{-N}\widetilde{\mathcal{L}}^\delta_{s_0}\xi$ in (3.3.1), by (3.3.2) we see that (3.3.1) on $\operatorname{supp}_{s,\xi}\mathfrak{a}$ is equivalent to $\sum_{j=1}^{N-1} |\langle(\gamma^\delta_{s_0})^{(j)}(s), \xi\rangle| \ge 2^k\delta^N/B'$ for $(s,\xi) \in \operatorname{supp}_{s,\xi}\mathfrak{a}_{\delta,s_0}(\,\cdot\,, \delta^{-N}\mathcal{L}^\delta_{s_0}\,\cdot)$. Note $\operatorname{supp}_{s,\xi}\mathfrak{a}_{\delta,s_0}(\,\cdot\,, \delta^{-N}\mathcal{L}^\delta_{s_0}\,\cdot) \supset \operatorname{supp}_{s,\xi}\tilde a$. So, the same holds on $\operatorname{supp}_{s,\xi}\tilde a$ and hence on $\operatorname{supp}_{s,\xi}(a_{\ell,j})_{2^{\ell-2}}$ if B' replaced by $2B'$. Therefore, $C^{-1}(a_{\ell,j})_{2^{\ell-2}}$ is of type $(j+\ell-2, N-1, \tilde B)$ relative to $\gamma^\delta_{s_0}$ for a large constant $C = C(B)$. $\square$

3.4 Localizing frequency near degenerate set

For the proof of Proposition 3.1.3, we make some reductions by decomposing the symbol a. We fix a sufficiently small positive constant

$$\delta_* \le \min\{\delta', (2^{7d}B^6)^{-N}\},$$

which is to be specified in what follows. Here δ' is the number given in Lemma 3.3.1.

We recall that γ satisfies (3.1.1), $\mathfrak{N}(N,B)$, $\mathfrak{V}(N,B)$, and a is of type (k,N,B) relative to γ. We set

$$\eta_N(s,\xi) = \prod_{1\le j\le N-1} \beta_0\Big(B2^{-k-1}\delta_*^{j-N}\langle\gamma^{(j)}(s),\,\xi\rangle\Big), \tag{3.4.1}$$

where $\beta_0 \in \mathrm{C}_c^\infty((-1,1))$ such that $\beta_0 = 1$ on $[-1/2,1/2]$. It is easy to see $|\partial_s^j\partial_t^l\partial_\xi^\alpha(a\eta_N)| \le C|\xi|^{-|\alpha|}$ for $(j,l,\alpha)\in\mathcal{I}_N$, and the same holds for $a(1-\eta_N)$.

Note $\sum_{j=1}^{N-1}|\gamma^{(j)}(s)\cdot\xi| \ge (2B)^{-1}\delta_*^N|\xi|$ on $\mathrm{supp}_{s,\xi}(a(1-\eta_N))$. So, we see $a(1-\eta_N)$ is a symbol of type $(k,N-1,B')$ for $B' = CB^2\delta_*^{-C}$ with a large C. Applying the assumption (Theorem 3.1.2 with $L=N-1$ and $B=B'$), we obtain

$$\|\mathcal{A}_t[\gamma, a(1-\eta_N)]f\|_{L^p(\mathbb{R}^{d+1})} \le C2^{(-\frac{2}{p}+\epsilon)k}\|f\|_{L^p(\mathbb{R}^d)}, \quad p \ge 4N-6.$$

Thus, it suffices to consider $\mathcal{A}_t[\gamma, a\eta_N]$. Since $\mathfrak{N}(N,B)$ holds on $\mathrm{supp}_{s,\xi}\, a$,

$$|\gamma^{(N)}(s)\cdot\xi| \ge (2B)^{-1}|\xi| \tag{3.4.2}$$

whenever $(s,t,\xi)\in \mathrm{supp}\, a\eta_N$ for some t.

Basic assumption Before we continue to prove the estimate for $\mathcal{A}_t[\gamma, a\eta_N]$, we make several assumptions which are clearly permissible by elementary decompositions.

Decomposing a, we may assume that $\mathrm{supp}_\xi\, a$ is contained in a narrow conic neighborhood and $\mathrm{supp}_s\, a \subset I(s_0,\delta_*)$ for some s_0. Let us set

$$\Gamma_k = \big\{\xi\in\mathbb{A}_k : \mathrm{dist}\left(|\xi|^{-1}\xi, |\xi'|^{-1}\xi'\right) < \delta_* \ \text{ for some } \ \xi'\in\mathrm{supp}_\xi(a\eta_N)\big\}.$$

We may also assume $\gamma^{(N-1)}(s')\cdot\xi' = 0$ for some $(s',\xi')\in I(s_0,\delta_*)\times\Gamma_k$. Otherwise, $|\gamma^{(N-1)}(s)\cdot\xi| \gtrsim |\xi|$ on $\mathrm{supp}_{s,\xi}\, a\eta_N$ and hence $a\eta_N = 0$ if we take

B large enough. By (3.4.2) and the implicit function theorem, there exists σ such that

$$\gamma^{(N-1)}(\sigma(\xi)) \cdot \xi = 0 \tag{3.4.3}$$

in a narrow conic neighborhood of ξ' where $\sigma \in C^{2d+2}$ since $\gamma \in \mathrm{C}^{3d+1}(I)$. So, decomposing a further, we may assume $\sigma \in \mathrm{C}^{2d+2}(\Gamma_k)$ and $\sigma(\xi) \in I(s_0, \delta_*)$ for $\xi \in \Gamma_k$. Furthermore, since σ is homogeneous of degree zero,

$$|\partial_\xi^\alpha \sigma(\xi)| \le C|\xi|^{-|\alpha|}, \qquad \xi \in \Gamma_k \tag{3.4.4}$$

for a constant $C = C(B)$ if $|\alpha| \le 2d + 2$. Any symbol which appears in what follows is to be given by decomposing the symbol a with appropriate cutoff functions. So, *the s, ξ-supports of the symbols are assumed to be contained in $I(s_0, \delta_*) \times \Gamma_k$.*

We break a to have further localization on the Fourier side. Let

$$\mathfrak{a}_1(s, t, \tau, \xi) = a\eta_N \, \beta_0\big(2^{-2k}\delta_*^{-2N}|\tau + \langle\gamma(s), \xi\rangle|^2\big)$$

and $\mathfrak{a}_0 = a\eta_N - \mathfrak{a}_1$. Then, by Fourier inversion

$$\mathcal{A}_t[\gamma, a\eta_N]f = \mathcal{T}[\mathfrak{a}_1]f + \mathcal{T}[\mathfrak{a}_0]f.$$

It is easy to show $\|\mathcal{T}[\mathfrak{a}_0]f\|_p \lesssim_B 2^{-2k}\|f\|_p$ for $1 \le p \le \infty$. Indeed, consider $\tilde{\mathfrak{a}}_0 = -(\tau + \gamma(s)\cdot\xi)^{-2}\partial_t^2\mathfrak{a}_0$. By (3.2.10) and integration by parts in t', $m[\mathfrak{a}_0] = m[\tilde{\mathfrak{a}}_0]$ and hence $\mathcal{T}[\mathfrak{a}_0] = \mathcal{T}[\tilde{\mathfrak{a}}_0]$. Thanks to (3.2.14), it is sufficient to show

$$\big|K[\tilde{\mathfrak{a}}_0](s, t, x)\big| \le C\, 2^{k(d-1)} \int \big(1 + 2^k|t - t'| + 2^k|x - t'\gamma(s)|\big)^{-d-3} \, dt'$$

for a constant $C = C(B, \delta_*)$. Note $|\tau + \langle\gamma(s), \xi\rangle| \gtrsim 2^k$ on $\operatorname{supp}\tilde{\mathfrak{a}}_0$, and recall (3.2.15). Rescaling and integration by parts in τ, ξ, as in the proof of Lemma 3.2.4, show the estimate.

The difficult part is to estimate $\mathcal{T}[\mathfrak{a}_1]$. Since δ_* is a fixed constant, it is obvious that $C^{-1}\mathfrak{a}_1 \in \mathfrak{A}_k(s_0, \delta_*)$ for some $C = C(B, \delta_*)$. So, the desired estimate for $\mathcal{T}[\mathfrak{a}_1]$ follows once we have the next proposition.

Proposition 3.4.1. *Let $\mathfrak{a} \in \mathfrak{A}_k(s_0, \delta_*)$ with $\operatorname{supp}_\xi \mathfrak{a} \subset \Gamma_k$. Suppose Theorem 3.1.2 holds for $L = N - 1$. Then, if $p \ge 4N - 2$, for $\epsilon > 0$ we have*

$$\big\|\mathcal{T}[\mathfrak{a}]f\big\|_{L^p(\mathbb{R}^{d+1})} \le C_\epsilon 2^{-\frac{2}{p}k + \epsilon k}\|f\|_{L^p(\mathbb{R}^d)}. \tag{3.4.5}$$

Therefore, the proof of Proposition 3.1.3 is completed if we prove Proposition 3.4.1. For the purpose, we use Proposition 3.4.2 below, which allows us to decompose $\mathcal{T}[\mathfrak{a}]$ into operators given by symbols with smaller s-supports while the consequent minor parts have acceptable bounds. This type of argument was used in [44] when $L = 2$.

3.4.1 Iterative argument

Let δ_0 and δ_1 be positive numbers such that

$$2^{7d} B^6 \delta_0^{(N+1)/N} \le \delta_1 \le \delta_0 \le \delta_*, \qquad 2^{-k/N} \le \delta_1. \tag{3.4.6}$$

Then, it is clear that

$$B^{6N} \delta_0^{j+1} \le 2^{-7dN} \delta_1^j, \qquad j = 1, \dots, N. \tag{3.4.7}$$

For $n \ge 0$, we denote $\mathfrak{J}_n^\mu = \{\nu \in \mathbb{Z} : |2^n \delta_1 \nu - \delta_0 \mu| \le \delta_0\}$.

Proposition 3.4.2. *For μ such that $\delta_0 \mu \in I(s_0, \delta_*) \cap \delta_0 \mathbb{Z}$, let $\mathfrak{a}^\mu \in \mathfrak{A}_k(\delta_0 \mu, \delta_0)$ with* $\mathrm{supp}_{s,\xi}\, \mathfrak{a}^\mu \subset I(s_0, \delta_*) \times \Gamma_k$*. Suppose Theorem 3.1.2 holds for $L = N - 1$. Then, if $p \ge 4N - 2$, for $\epsilon > 0$ there exist a constant $C_\epsilon = C_\epsilon(B) \ge 2$ and symbols $\mathfrak{a}_\nu \in \mathfrak{A}_k(\delta_1 \nu, \delta_1)$ with* $\mathrm{supp}_{s,\xi}\, \mathfrak{a}_\nu \subset I(s_0, \delta_*) \times \Gamma_k$*, $\nu \in \cup_\mu \mathfrak{J}_0^\mu$, such that*

$$\Big(\sum_\mu \|\mathcal{T}[\mathfrak{a}^\mu] f\|_p^p\Big)^{\frac1p} \le C_\epsilon \Big(\frac{\delta_1}{\delta_0}\Big)^{\frac{2N}{p} - 1 - \epsilon} \Big(\sum_\nu \|\mathcal{T}[\mathfrak{a}_\nu] f\|_p^p\Big)^{\frac1p} + C_\epsilon \delta_0^{-\frac{2N}{p} + 1 + \epsilon} 2^{-\frac{2k}{p} + 2\epsilon k} \|f\|_p.$$

Rest of this section, assuming Proposition 3.4.2, we prove Proposition 3.4.1.

Let $\mathfrak{a} \in \mathfrak{A}_k(s_0, \delta_*)$. We may assume $s_0 = \delta_* \mu$ for some $\mu \in \mathbb{Z}$. To apply Proposition 3.4.2 iteratively, we need to choose an appropriate decreasing sequence of positive numbers since the decomposition is subject to the condition (3.4.6).

Let $\delta_0 = \delta_*$, so $(2^{7d} B^6)^N \delta_0 < 1$. Let J be the largest integer such that

$$(2^{7d} B^6)^{N(\frac{N+1}{N})^{J-1} - N} \delta_0^{(\frac{N+1}{N})^{J-1}} > 2^{-\frac{k}{N}}.$$

So, $J \le C_1 \log k$ for a constant $C_1 \ge 1$. We set

$$\delta_J = 2^{-\frac{k}{N}}, \qquad \delta_j = (2^{7d} B^6)^{N(\frac{N+1}{N})^j - N} \delta_0^{(\frac{N+1}{N})^j} \tag{3.4.8}$$

for $j = 1, \dots, J-1$. Thus, it follows that

$$2^{7d} B^6 \delta_j^{(N+1)/N} \le \delta_{j+1} < \delta_j, \qquad j = 0, \dots, J-1. \tag{3.4.9}$$

For a given $\epsilon > 0$, let $\tilde\epsilon = \epsilon/4$. Since $\mathfrak{a} \in \mathfrak{A}_k(\delta_0\mu, \delta_0)$ and (3.4.6) holds for δ_0 and δ_1, applying Proposition 3.4.2 to $\mathcal{T}[\mathfrak{a}]$, we have

$$\|\mathcal{T}[\mathfrak{a}]f\|_p \le C_{\tilde\epsilon}\big(\delta_1/\delta_0\big)^{\frac{2N}{p}-1-\tilde\epsilon}\Big(\sum_{\nu_1}\|\mathcal{T}[\mathfrak{a}_{\nu_1}]f\|_p^p\Big)^{\frac1p} + C_{\tilde\epsilon}\delta_0^{-\frac{2N}{p}+1+\tilde\epsilon}2^{-\frac2p k+2\tilde\epsilon k}\|f\|_p,$$

where $\mathfrak{a}_{\nu_1} \in \mathfrak{A}_k(\delta_1\nu_1, \delta_1)$, $\nu_1 \in \mathfrak{J}_0^\mu$. Thanks to (3.4.9) we may apply again Proposition 3.4.2 to $\mathcal{T}[\mathfrak{a}_{\nu_1}]$ while δ_0, δ_1 replaced by δ_1, δ_2, respectively. Repeating this procedure up to J-th step yields symbols $\mathfrak{a}_\nu \in \mathfrak{A}_k(\delta_J\nu, \delta_J)$, $\delta_J\nu \in \delta_J\mathbb{Z} \cap I(\delta_0\mu, \delta_0)$, such that

$$\|\mathcal{T}[\mathfrak{a}]f\|_p \le C_{\tilde\epsilon}^J\delta_J^{\frac{2N}{p}-1-\tilde\epsilon}\Big(\sum_\nu\|\mathcal{T}[\mathfrak{a}_\nu]f\|_p^p\Big)^{\frac1p} + \sum_{0\le j\le J-1} C_{\tilde\epsilon}^{j+1}\delta_0^{-\frac{2N}{p}+1+\tilde\epsilon}2^{-\frac2p k+2\tilde\epsilon k}\|f\|_p$$

for $p \ge 4N-2$. Now, assuming

$$\Big(\sum_\nu\|\mathcal{T}[\mathfrak{a}_\nu]f\|_p^p\Big)^{1/p} \lesssim_B 2^{-k/N}\|f\|_p, \qquad 2 \le p \le \infty \tag{3.4.10}$$

for the moment, we can finish the proof of Proposition 3.4.1. Since $C_{\tilde\epsilon} \ge 2$, combining the above inequalities, we get

$$\|\mathcal{T}[\mathfrak{a}]f\|_p \lesssim_B C_{\tilde\epsilon}^{J+1}\big(2^{-\frac2p k+\frac{\tilde\epsilon}{N}k} + 2^{-\frac2p k+2\tilde\epsilon k}\big)\|f\|_p.$$

Note $J \le C_1\log k$, so $C_{\tilde\epsilon}^{J+1} \le C'2^{\epsilon k/2}$ for some C' if k is sufficiently large. Thus, the right hand side is bounded by $C2^{-2k/p+\epsilon k}\|f\|_p$.

It remains to show (3.4.10) for $2 \le p \le \infty$. By interpolation, it is enough to obtain (3.4.10) for $p = \infty$ and $p = 2$. The case $p = \infty$ follows by (3.2.12) since $\mathfrak{a}_\nu \in \mathfrak{A}_k(\delta_J\nu, \delta_J)$. So, we need only to prove (3.4.10) for $p = 2$. To do this, we first observe the following, which shows $\mathrm{supp}_\xi\, \mathfrak{a}_\nu$ are finitely overlapping.

Lemma 3.4.3. *For $b \ge 1$, $s \in I(s_0, \delta_*)$, and $0 < \delta \le \delta_*$, let us set*

$$\Lambda_k'(s,\delta,b) = \bigcap_{1\le j\le N-1}\big\{\xi \in \Gamma_k : |\langle\gamma^{(j)}(s), \xi\rangle| \le b2^k\delta^{N-j}\big\}. \tag{3.4.11}$$

If $\Lambda_k'(s_1,\delta,b) \cap \Lambda_k'(s_2,\delta,b) \ne \emptyset$ for some $s_1, s_2 \in I(s_0,\delta_*)$, then there is a constant $C = C(B)$ such that $|s_1 - s_2| \le Cb\delta$.

Proof. Let $\xi \in \Lambda_k'(s_1, \delta, b) \cap \Lambda_k'(s_2, \delta, b)$. Since $|\gamma^{(N-1)}(s_j) \cdot \xi| \le b2^k\delta$, $j = 1, 2$, by (3.4.3) and (3.4.2) we see $|s_j - \sigma(\xi)| \le 2^2 bB\delta$, $j = 1, 2$, using the mean value theorem. This implies $|s_1 - s_2| \le 2^3 bB\delta$. $\square$

We recall (3.2.10). Since (3.4.2) holds on $\mathrm{supp}_{s,\xi}\, \mathfrak{a}_\nu$, by van der Corput's lemma (e.g., see [55, Corollary, p. 334]) we have

$$|m[\mathfrak{a}_\nu](\tau, \xi)| \lesssim 2^{-k/N} \big(\|\mathfrak{a}_\nu(\cdot, t, \tau, \xi)\|_\infty + \|\partial_s \mathfrak{a}_\nu(\cdot, t, \tau, \xi)\|_1 \big) \lesssim_B 2^{-k/N}.$$

The second inequality is clear since $\mathfrak{a}_\nu \in \mathfrak{A}_k(\delta_J\nu, \delta_J)$. From (3.2.11) note $\mathcal{F}(\mathcal{T}[\mathfrak{a}_\nu]f) = m[\mathfrak{a}_\nu]\widehat{f}$. By (3.2.10), $\mathrm{supp}_\xi\, \mathcal{F}(\mathcal{T}[\mathfrak{a}_\nu]f) \subset S_\nu := \Lambda_k'(\delta_J\nu, \delta_J, 2^5B)$, since $\mathrm{supp}_\xi\, \mathfrak{a}_\nu \subset S_\nu$. By Lemma 3.4.3 it follows that the sets S_ν overlap at most $C = C(B)$ times. Therefore, Plancherel's theorem and the estimate above yield

$$\| \sum_\nu \mathcal{T}[\mathfrak{a}_\nu]f\|_2^2 \lesssim_B 2^{-2k/N} \sum_\nu \int_{S_\nu} \int_{\{\tau : |\tau + \gamma(\delta_J\nu)\cdot\xi| \le 2^5 B\}} d\tau\, |\widehat{f}(\xi)|^2\, d\xi$$

since $\mathrm{supp}\, \mathfrak{a}_\nu \subset \Lambda_k(\delta_J\nu, \delta_J, B)$. This gives (3.4.10) for $p = 2$.

3.5 Decoupling in a local coordinate

In this section, we prove Proposition 3.4.2 by applying the decoupling inequality. Meanwhile, the induction assumption (Theorem 3.1.2 with $L = N - 1$) plays an important role. We decompose a given symbol $\mathfrak{a}^\mu \in \mathfrak{A}_k(\delta_0\mu, \delta_0)$ into the symbols with their s-supports contained in intervals of length about δ_1 while the consequent minor contribution is controlled within an acceptable bound. To achieve it up to δ_1 satisfying (3.4.6), we approximate $\langle G(s), (\tau, \xi)\rangle$ in a local coordinate system near the set $\{(s, \xi) : \langle \gamma^{(N-1)}(s), \xi\rangle = 0\}$.

3.5.1 Decomposition of the symbol $\mathfrak{a}^\mu$

We begin by introducing some notations.

Fixing $\mu \in \mathbb{Z}$ such that $\delta_0\mu \in I(s_0, \delta_*)$, we consider linear maps

$$y_\mu^j(\tau, \xi) = \langle G^{(j)}(\delta_0\mu), (\tau, \xi)\rangle, \qquad j = 0, 1, \dots, N.$$

In particular, $y_\mu^j(\tau, \xi) = \langle \gamma^{(j)}(\delta_0\mu), \xi\rangle$ if $1 \le j \le N$. By (3.4.2) it follows that

$$|y_\mu^N(\tau, \xi)| \ge (2B)^{-1}|\xi|. \tag{3.5.1}$$

We denote

$$\omega_\mu(\xi) = \frac{y_\mu^{N-1}(\tau,\xi)}{y_\mu^N(\tau,\xi)},$$

which is close to $\delta_0\mu - \sigma(\xi)$ (see (3.5.5)). Then, we define $\mathfrak{g}_\mu^N, \mathfrak{g}_\mu^{N-1}, \dots, \mathfrak{g}_\mu^0$ recursively, by setting $\mathfrak{g}_\mu^N = y_\mu^N$, and

$$\mathfrak{g}_\mu^j(\tau,\xi) = y_\mu^j(\tau,\xi) - \sum_{\ell=j+1}^{N} \frac{\mathfrak{g}_\mu^\ell(\tau,\xi)}{(\ell-j)!}(\omega_\mu(\xi))^{\ell-j}, \quad j = N-1, \dots, 0. \tag{3.5.2}$$

Note that $\mathfrak{g}_\mu^{N-1} = 0$ and (3.5.2) can be rewritten as follows:

$$y_\mu^m(\tau,\xi) = \sum_{\ell=m}^{N} \frac{\mathfrak{g}_\mu^\ell(\tau,\xi)}{(\ell-m)!}(\omega_\mu(\xi))^{\ell-m}, \quad m = 0, \dots, N. \tag{3.5.3}$$

The identity continues to hold for $m = N$ since $\mathfrak{g}_\mu^N = y_\mu^N$. Apparently, $\mathfrak{g}_\mu^1, \dots, \mathfrak{g}_\mu^N$ are independent of τ since so are $y_\mu^1, \dots, y_\mu^N$.

For $j = 1, \dots, N$, set

$$\mathcal{E}_j(\xi) := (y_\mu^N(\tau,\xi))^{-1} \int_{\sigma(\xi)}^{\delta_0\mu} \frac{\langle \gamma^{(N+1)}(r), \xi \rangle}{j!} (\sigma(\xi) - r)^j dr. \tag{3.5.4}$$

By (3.5.4) with $j = 1$ and integration by parts, we have

$$\mathcal{E}_1(\xi) = \sigma(\xi) - \delta_0\mu + \omega_\mu(\xi). \tag{3.5.5}$$

Lemma 3.5.1. *For $0 \le j \le N-1$, we have*

$$\langle G^{(j)}(\sigma(\xi)), (\tau,\xi) \rangle = \sum_{\ell=j}^{N} \frac{\mathfrak{g}_\mu^\ell \ (\mathcal{E}_1)^{\ell-j}}{(\ell-j)!} - y_\mu^N \mathcal{E}_{N-j}. \tag{3.5.6}$$

Proof. When $j = N-1$, (3.5.6) is clear. To show (3.5.6) for $j = 0, 1, \dots, N-2$, by Taylor's theorem with integral remainder we have

$$\langle G^{(j)}(\sigma(\xi)), (\tau,\xi) \rangle = \sum_{m=j}^{N} y_\mu^m(\tau,\xi) \frac{(\sigma(\xi) - \delta_0\mu)^{m-j}}{(m-j)!} - y_\mu^N(\tau,\xi) \mathcal{E}_{N-j}(\xi).$$

Using (3.5.3) and then changing the order of the sums, we see

$$\langle G^{(j)}(\sigma(\xi)), (\tau, \xi)\rangle = \sum_{\ell=j}^{N} \mathfrak{g}_\mu^\ell \Big(\sum_{m=j}^{\ell} \frac{(\sigma(\xi) - \delta_0\mu)^{m-j}}{(\ell - m)!(m-j)!} (\omega_\mu)^{\ell-m} \Big) - y_\mu^N \, \mathcal{E}_{N-j}.$$

The sum over m equals $(\sigma(\xi) - \delta_0\mu + \omega_\mu)^{\ell-j}/(\ell - j)!$. So, (3.5.6) follows by (3.5.5). $\square$

We now decompose the symbol $\mathfrak{a}^\mu \in \mathfrak{A}_k(\delta_0\mu, \delta_0)$ by making use of $\mathfrak{g}_\mu^j$, $j = 0, \dots, N-2$. We define

$$\mathfrak{G}_N^\mu(s, \tau, \xi) = \sum_{j=0}^{N-2} \big(2^{-k}\mathfrak{g}_\mu^j(\tau, \xi)\big)^{\frac{2N!}{N-j}} + (s - \sigma(\xi))^{2N!}. \tag{3.5.7}$$

Let $\beta_N = \beta_0 - \beta_0(2^{2N!}\cdot)$, so $\sum_{\ell\in\mathbb{Z}} \beta_N(2^{2N!\ell}\cdot) = 1$. We also take $\zeta \in \mathrm{C}_c^\infty((-1, 1))$ such that $\sum_{\nu\in\mathbb{Z}} \zeta(\cdot - \nu) = 1$. For $n \ge 0$ and $\nu \in \mathfrak{J}_n^\mu$, we set

$$\mathfrak{a}_\nu^{\mu,n} = \mathfrak{a}^\mu \times \begin{cases} \beta_0\big(\delta_1^{-2N!}\,\mathfrak{G}_N^\mu\big)\,\zeta(\delta_1^{-1}s - \nu), & n = 0, \\ \beta_N\big((2^n\delta_1)^{-2N!}\,\mathfrak{G}_N^\mu\big)\,\zeta(2^{-n}\delta_1^{-1}s - \nu), & n \ge 1. \end{cases}$$

Then, it follows that

$$\mathfrak{a}^\mu = \sum_{n\ge0} \sum_{\nu\in\mathfrak{J}_n^\mu} \mathfrak{a}_\nu^{\mu,n}. \tag{3.5.8}$$

Lemma 3.5.2. *There is a constant $C = C(B)$ such that $C^{-1}\mathfrak{a}_\nu^{\mu,n}$ is in $\mathfrak{A}_k(2^n\delta_1\nu, 2^n\delta_1)$ for $n \ge 0$, μ, and ν.*

The proof of Lemma 3.5.2 is elementary though it is somewhat involved. We postpone the proof until Section 3.6.

We collect some elementary facts regarding $\mathfrak{a}_\nu^{\mu,n}$. First, we may assume

$$2^n\delta_1 \lesssim_B \delta_0 \tag{3.5.9}$$

since, otherwise, $\mathfrak{a}_\nu^{\mu,n} = 0$. Note $|\langle\gamma^{(N-1)}(\delta_0\mu), \xi\rangle| \le B2^{k+5}\delta_0$ if $\xi \in \mathrm{supp}_\xi\, \mathfrak{a}_\mu$. Then, (3.4.2), (3.4.3), and the mean value theorem show

$$|\sigma(\xi) - \delta_0\mu| \le B^2 2^7 \delta_0 \tag{3.5.10}$$

for $\xi \in \operatorname{supp}_\xi \mathfrak{a}_\mu$. If $(\tau,\xi) \in \operatorname{supp}_{\tau,\xi} \mathfrak{a}_\mu \subset \Lambda_k(\delta_0\mu, \delta_0, B)$, $|y_\mu^j(\tau,\xi)| \le B2^{k+5}\delta_0^{N-j}$ for $0 \le j \le N-1$. Since $|\omega_\mu| \lesssim B^2\delta_0$, (3.5.2) gives $|\mathfrak{g}_\mu^j(\tau,\xi)| \lesssim_B 2^{k+5}\delta_0^{N-j}$ for $0 \le j \le N-2$. Therefore, $\mathfrak{G}_N^\mu \lesssim_B \delta_0^{2N!}$ on the support of $\mathfrak{a}_\mu$. This gives (3.5.9).

Since $\mathfrak{G}_N^\mu \le (2^n\delta_1)^{2N!}$ on $\operatorname{supp} \mathfrak{a}_\nu^{\mu,n}$, the following hold on the support of $\mathfrak{a}_\nu^{\mu,n}$:

$$|s - \sigma(\xi)| \le 2^n\delta_1, \tag{3.5.11}$$

$$2^{-k}|\mathfrak{g}_\mu^j(\tau,\xi)| \le (2^n\delta_1)^{N-j}, \qquad 0 \le j \le N-1. \tag{3.5.12}$$

The inequality (3.5.12) holds true for $j = N-1$ since $\mathfrak{g}_\mu^{N-1} = 0$. We also have

$$|\mathcal{E}_j(\xi)| \le B^2(B^2 2^7\delta_0)^{j+1}, \tag{3.5.13}$$

$$|\sigma(\xi) - 2^n\delta_1\nu| \le 2^{n+1}\delta_1. \tag{3.5.14}$$

on $\operatorname{supp}_\xi \mathfrak{a}_\nu^{\mu,n}$. By using (3.5.4), (3.5.10), and (3.5.1), it is easy to show (3.5.13). Since $|s - 2^n\delta_1\nu| \le 2^n\delta_1$ on $\operatorname{supp}_s \mathfrak{a}_\nu^{\mu,n}$, (3.5.14) follows by (3.5.11).

3.5.2 Decoupling for the symbol $\mathfrak{a}^\mu$

By (3.5.8) and the Minkowski inequality we have

$$\Big(\sum_\mu \big\|\mathcal{T}[\mathfrak{a}^\mu]f\big\|_p^p\Big)^{1/p} \le \sum_{n\ge 0}\Big(\sum_\mu \Big\|\sum_{\nu\in\mathfrak{J}_n^\mu} \mathcal{T}[\mathfrak{a}_\nu^{\mu,n}]f\Big\|_p^p\Big)^{1/p}. \tag{3.5.15}$$

We apply the inequality (2.1.14) to $\sum_{\nu\in\mathfrak{J}_n^\mu} \mathcal{T}[\mathfrak{a}_\nu^{\mu,n}]f$ after a suitable linear change of variables. The symbols $\mathfrak{a}_\nu^{\mu,0}$ are to constitute the set $\{\mathfrak{a}_\nu\}$ while the operators associated to $\mathfrak{a}_\nu^{\mu,n}$, $n \ge 1$ are to be handled similarly as in Section 2.

Applying the decoupling inequality

To prove Proposition 3.4.2, we first show

$$\Big\|\sum_{\nu\in\mathfrak{J}_n^\mu} \mathcal{T}[\mathfrak{a}_\nu^{\mu,n}]f\Big\|_p \le C_\epsilon\big(2^n\delta_1/\delta_0\big)^{\frac{2N}{p}-1-\epsilon}\Big(\sum_{\nu\in\mathfrak{J}_n^\mu} \big\|\mathcal{T}[\mathfrak{a}_\nu^{\mu,n}]f\big\|_p^p\Big)^{1/p} \tag{3.5.16}$$

for $p \geq 4N-2$. To apply the inequality (2.1.14), we consider $\mathrm{supp}_{\tau,\xi}\, \mathfrak{a}_\nu^{\mu,n}$, which contains the Fourier support of $\mathcal{T}[\mathfrak{a}_\nu^{\mu,n}]f$ as is clear from (3.2.10) and (3.2.11).

We set

$$\mathbf{y}_\mu(\tau,\xi) = \big(y_\mu^0(\tau,\xi), \dots, y_\mu^N(\tau,\xi)\big).$$

Lemma 3.5.3. *Let $\mathbf{r} = \mathbf{r}_\circ^{N+1}$ and matrix $\mathcal{D}_\delta := (\delta^{-N}e_1, \delta^{1-N}e_2, \dots, \delta^0 e_{N+1})$ where e_j denotes the j-th standard unit vector in $\mathbb{R}^{N+1}$. On* $\mathrm{supp}_{\tau,\xi}\, \mathfrak{a}_\nu^{\mu,n}$, *we have*

$$\Big|\Big\langle \mathcal{D}_{\delta_0}\mathbf{y}_\mu(\tau,\xi), \mathbf{r}^{(j)}\Big(\frac{2^n\delta_1}{\delta_0}\nu - \mu\Big)\Big\rangle\Big| \lesssim 2^k\Big(\frac{2^n\delta_1}{\delta_0}\Big)^{N+1-j}, \qquad 1 \leq j \leq N, \tag{3.5.17}$$

$$(2B)^{-1}2^{k-1} \leq \big|\big\langle \mathbf{y}_\mu(\tau,\xi), \mathbf{r}^{(N+1)}\big\rangle\big| \leq B2^{k+1}. \tag{3.5.18}$$

Proof. We write $\mathbf{r} = (\mathbf{r}_1, \dots, \mathbf{r}_{N+1})$. Note $\mathbf{r}_m^{(j)}(s) = s^{m-j}/(m-j)!$ for $m \geq j$. By (3.5.3) we have

$$y_\mu^{m-1}\mathbf{r}_m^{(j)}(2^n\delta_1\nu - \delta_0\mu) = \sum_{\ell=m-1}^{N} \mathfrak{g}_\mu^\ell \frac{(2^n\delta_1\nu - \delta_0\mu)^{m-j}}{(\ell+1-m)!(m-j)!}\,\omega_\mu^{\ell+1-m}$$

for $m \geq j$. Since $\mathbf{r}_m^{(j)}(s) = 0$ for $j > m$, taking sum over m gives

$$\big\langle \mathbf{y}_\mu, \mathbf{r}^{(j)}(2^n\delta_1\nu - \delta_0\mu)\big\rangle = \sum_{\ell=j-1}^{N} \mathfrak{g}_\mu^\ell \frac{(2^n\delta_1\nu - \delta_0\mu + \omega_\mu)^{\ell+1-j}}{(\ell+1-j)!}.$$

From (3.5.5) note $2^n\delta_1\nu - \delta_0\mu + \omega_\mu = 2^n\delta_1\nu - \sigma(\xi) + \mathcal{E}_1$. Thus, (3.5.14), (3.5.13) with $j=1$, and (3.4.7) with $j=1$ show $|2^n\delta_1\nu - \delta_0\mu + \omega_\mu| \lesssim 2^n\delta_1$. Using (3.5.12), we obtain

$$\big|\big\langle \mathbf{y}_\mu(\tau,\xi), \mathbf{r}^{(j)}(2^n\delta_1\nu - \delta_0\mu)\big\rangle\big| \lesssim 2^k(2^n\delta_1)^{N+1-j}, \qquad 1 \leq j \leq N.$$

By homogeneity it follows that $\langle \eta, \mathbf{r}^{(j)}(\delta_0 s)\rangle = \delta_0^{N+1-j}\langle \mathcal{D}_{\delta_0}\eta, \mathbf{r}^{(j)}(s)\rangle$ for $\eta \in \mathbb{R}^{N+1}$. Therefore, we get (3.5.17). For (3.5.18) note $\mathbf{r}^{(N+1)} = (0, \dots, 0, 1)$, so $\langle \mathbf{y}_\mu, \mathbf{r}^{(N+1)}\rangle = y_\mu^N$ and (3.5.18) follows by (3.5.1). $\square$

Let $\mathrm{V} = \mathrm{span}\{\gamma'(\delta_0\mu), \dots, \gamma^{(N)}(\delta_0\mu)\}$ and $\{v_{N+1}, \dots, v_d\}$ be an orthonormal basis of $\mathrm{V}^\perp$. Since γ satisfies $\mathfrak{V}(N,B)$, for each $\xi \in \mathbb{R}^d$ we can write

$$\xi = \overline{\xi} + \sum_{N+1 \leq j \leq d} y_j(\xi)v_j, \tag{3.5.19}$$

where $\overline{\xi} \in \mathrm{V}$ and $y_j(\xi) \in \mathbb{R}$, $N+1 \le j \le d$. We define a linear map $\mathrm{Y}_\mu^{\delta_0}$ by

$$\mathrm{Y}_\mu^{\delta_0}(\tau,\xi) = \big(2^{-k}\mathcal{D}_{\delta_0}\mathbf{y}_\mu(\tau,\xi),\, y_{N+1}(\xi), \dots,\, y_d(\xi)\big).$$

Then, by (3.5.17) and (3.5.18) we see

$$\mathrm{Y}_\mu^{\delta_0}(\mathrm{supp}_{\tau,\xi}\, \mathfrak{a}_\nu^{\mu,n}) \subset \mathbf{s}\Big(\frac{2^n\delta_1}{\delta_0}\nu - \mu, C\frac{2^n\delta_1}{\delta_0}, 2^2B; \mathbf{r}_\circ^{N+1}\Big) \times \mathbb{R}^{d-N} \tag{3.5.20}$$

for some $C > 1$. Thus, we have the inequality (2.1.14) for $\delta = C2^n\delta_1/\delta_0$, the collection of slabs $\mathbf{s}(2^n\delta_1\nu/\delta_0 - \mu, C2^n\delta_1/\delta_0, CB; \mathbf{r}_\circ^{N+1})$, $\nu \in \mathfrak{J}_n^\mu$. Therefore, via cylindrical extension in $y_{N+1}, \dots, y_d$ (see (2.1.16)) and the change of variables $(\tau,\xi) \to \mathrm{Y}_\mu^{\delta_0}(\tau,\xi)$ we obtain (3.5.16) since the decoupling inequality is not affected by affine change of variables in the Fourier side.

Combining (3.5.15) and (3.5.16), we obtain

$$\Big(\sum_\mu \|\mathcal{T}[\mathfrak{a}^\mu]f\|_p^p\Big)^{1/p} \le \sum_{n\ge 0} \mathbf{E}_n$$

for $p \ge 4N-2$, where

$$\mathbf{E}_n = C_\epsilon\big(2^n\delta_1/\delta_0\big)^{\frac{2N}{p}-1-\epsilon}\Big(\sum_\mu \sum_{\nu\in\mathfrak{J}_n^\mu} \|\mathcal{T}[\mathfrak{a}_\nu^{\mu,n}]f\|_p^p\Big)^{1/p}.$$

Since the intervals $I(\delta_0\mu,\delta_0)$ overlap, there are at most three nonzero $\mathfrak{a}_\nu^{\mu,0}$ for each ν. We take $\mathfrak{a}_\nu = \mathfrak{a}_\nu^{\mu,0}$ which maximizes $\|\mathcal{T}[\mathfrak{a}_\nu^{\mu,0}]f\|_p$. Then, it is clear that $\mathbf{E}_0 \le 3^{1/p}C_\epsilon(\delta_1/\delta_0)^{\frac{2N}{p}-1-\epsilon}(\sum_\nu \|\mathcal{T}[\mathfrak{a}_\nu]f\|_p^p)^{1/p}$. By Lemma 3.5.2, $C^{-1}\mathfrak{a}_\nu \in \mathfrak{A}_k(\delta_1\nu,\delta_1)$ for a constant C. Thus, the proof of Proposition 3.4.2 is now reduced to showing

$$\sum_{n\ge 1} \mathbf{E}_n \lesssim_B \delta_0^{-\frac{2N}{p}+1+\epsilon} 2^{-\frac{2}{p}k+2\epsilon k}\|f\|_p, \quad p \ge 4N-2. \tag{3.5.21}$$

3.5.3 Estimates for less degenerate parts

Estimates for $\mathbf{E}_n$ when $n \ge 1$

To show (3.5.21) we decompose $\mathfrak{a}_\nu^{\mu,n}$ so that (3.5.26) or (3.5.27) (see Lemma 3.5.5 below) holds on the s,ξ-supports of the resulting symbols. If (3.5.26) holds, we use the assumption after rescaling, whereas we handle the other case using estimates for the kernels of the operators.

Let

$$\bar{\mathfrak{G}}_N^\mu(s,\xi) = \sum_{1\le j\le N-2} \big(2^{-k}\mathfrak{g}_\mu^j\big)^{\frac{2N!}{N-j}} + \big(s-\sigma(\xi)\big)^{2N!}. \tag{3.5.22}$$

The right hand side is independent of τ since so are $\mathfrak{g}_\mu^j$, $1\le j\le N-2$.

Let $C_0 = 2^{2d}B$. We set

$$\mathfrak{a}_{\nu,1}^{\mu,n} = \mathfrak{a}_\nu^{\mu,n}\,\beta_0\Big((2^{-k}\mathfrak{g}_\mu^0)^{2(N-1)!}/(C_0^{2N!}\bar{\mathfrak{G}}_N^\mu)\Big), \quad n\ge 1, \tag{3.5.23}$$

and $\mathfrak{a}_{\nu,2}^{\mu,n} = \mathfrak{a}_\nu^{\mu,n} - \mathfrak{a}_{\nu,1}^{\mu,n}$, so $\mathfrak{a}_\nu^{\mu,n} = \mathfrak{a}_{\nu,1}^{\mu,n} + \mathfrak{a}_{\nu,2}^{\mu,n}$. Similarly as before, we have the following, which we prove in Section 3.6.

Lemma 3.5.4. *There exists a constant $C = C(B)$ such that $C^{-1}\mathfrak{a}_{\nu,1}^{\mu,n}$, and $C^{-1}\mathfrak{a}_{\nu,2}^{\mu,n}$ are contained in $\mathfrak{A}_k(2^n\delta_1\nu, 2^n\delta_1)$ for $n\ge 1$.*

The estimate (3.5.21) follows if we show

$$\Big(\sum_\mu \sum_{\nu\in\mathfrak{J}_n^\mu} \|\mathcal{T}[\mathfrak{a}_{\nu,1}^{\mu,n}]f\|_p^p\Big)^{1/p} \le C_\epsilon 2^{-\frac{2}{p}k+\epsilon k}(2^n\delta_1)^{-\frac{2N}{p}+1+\epsilon}\|f\|_p, \quad p\ge 4N-6, \tag{3.5.24}$$

for any $\epsilon>0$, and

$$\Big(\sum_\mu \sum_{\nu\in\mathfrak{J}_n^\mu} \|\mathcal{T}[\mathfrak{a}_{\nu,2}^{\mu,n}]f\|_p^p\Big)^{1/p} \lesssim_B 2^{-\frac{(N+2)k}{2N}}(2^n\delta_1)^{-\frac{N}{2}}\|f\|_p, \quad 2\le p\le\infty \tag{3.5.25}$$

when $n\ge 1$. Thanks to (3.5.9), those estimates give

$$\sum_{n\ge 1}\mathbf{E}_n \le C_\epsilon \delta_0^{-\frac{2N}{p}+1+\epsilon} \sum_{1\le n\le \log_2(C\delta_0/\delta_1)} \big(2^{-\frac{2}{p}k+\epsilon k} + 2^{-\frac{(N+2)k}{2N}}(2^n\delta_1)^{\frac{2N}{p}-\frac{N+2}{2}-\epsilon}\big)\|f\|_p$$

for $p\ge 4N-2$. Note $\log_2(\delta_0/\delta_1)\le Ck$ from (3.4.6). So, (3.5.21) follows since $4N-2 > 4N/(N+2)$ and $\delta_1\ge 2^{-k/N}$.

In order to prove the estimates (3.5.24) and (3.5.25), we start with the next lemma.

Lemma 3.5.5. *Let $n\ge 1$. For a constant $C = C(B) > 0$, we have the following:*

$$\sum_{1\le j\le N-1}(2^n\delta_1)^{-(N-j)}|\langle\gamma^{(j)}(s),\xi\rangle| \ge C2^k, \qquad (s,\xi)\in \mathrm{supp}_{s,\xi}\,\mathfrak{a}_{\nu,1}^{\mu,n}, \tag{3.5.26}$$

$$(2^n\delta_1)^{-N}|\tau+\langle\gamma(s),\xi\rangle| \ge C2^k, \qquad (s,\xi)\in \mathrm{supp}_{s,\xi}\,\mathfrak{a}_{\nu,2}^{\mu,n}. \tag{3.5.27}$$

Proof. We first prove (3.5.26). Since $\mathfrak{G}_N^\mu \ge 2^{-2N!-1}(2^n\delta_1)^{2N!}$ on $\operatorname{supp}_{s,\xi} \mathfrak{a}_\nu^{\mu,n}$, one of the following holds on $\operatorname{supp} \mathfrak{a}_{\nu,1}^{\mu,n}$:

$$|s-\sigma(\xi)| \ge (2^3C_0B)^{-1}2^n\delta_1, \tag{3.5.28}$$

$$2^{-k}|\mathfrak{g}_\mu^j(\tau,\xi)| \ge (2^2C_0)^{-(N-j)}(2^n\delta_1)^{N-j} \tag{3.5.29}$$

for some $1 \le j \le N-2$, where $C_0 = 2^{2d}B$ (see (3.5.23)). If (3.5.28) holds, by (3.4.2) and (3.4.3) it follows that $(2^n\delta_1)^{-1}|\langle\gamma^{(N-1)}(s),\xi\rangle| \gtrsim 2^k$. Thus, to show (3.5.26) we may assume (3.5.28) fails, i.e., (3.5.29) holds for some $1 \le j \le N-2$. So, there is an integer $\ell \in [0, N-2]$ such that (3.5.29) fails for $\ell+1 \le j \le N-2$, whereas (3.5.29) holds for $j = \ell$. By (3.5.6) and (3.5.13), we have

$$|\langle G^{(\ell)}(\sigma(\xi)),(\tau,\xi)\rangle| \ge |\mathfrak{g}_\mu^\ell| - \sum_{j=\ell+1}^{N} |\mathfrak{g}_\mu^j| \frac{(B^6 2^{14}\delta_0^2)^{j-\ell}}{(j-\ell)!} - 2B^3(B^2 2^7\delta_0)^{N+1-\ell}|\xi|. \tag{3.5.30}$$

Thus, by (3.4.7), $|\langle G^{(\ell)}(\sigma(\xi)),(\tau,\xi)\rangle| \ge (2^3C_0)^{-(N-\ell)}2^k(2^n\delta_1)^{N-\ell}$. Also, (3.5.6) and our choice of ℓ give $|\langle G^{(j)}(\sigma(\xi)),(\tau,\xi)\rangle| \le (2C_0)^{-(N-j)}2^k(2^n\delta_1)^{N-j}$ for $\ell+1 \le j \le N-2$. Combining this with $|s-\sigma(\xi)| < (2^3C_0B)^{-1}2^n\delta_1$ and expanding $G^{(\ell)}$ in Taylor series at $\sigma(\xi)$, we see $|\langle G^{(\ell)}(s),(\tau,\xi)\rangle| \ge C2^k(2^n\delta_1)^{N-\ell}$ for some $C = C(B) > 0$. This proves (3.5.26).

We now show (3.5.27), which is easier. On $\operatorname{supp} \mathfrak{a}_{\nu,2}^{\mu,n}$, $|\mathfrak{g}_\mu^0| \ge 2^{k-N-1}(2^n\delta_1)^N$ and $2^{-k}|\mathfrak{g}_\mu^j| \le 2C_0^{-(N-j)}(2^n\delta_1)^{N-j}$ for $j = 1,\dots,N-2$. Using (3.5.30) with $\ell = 0$, by (3.4.7) and (3.4.6) we get $(2^n\delta_1)^{-N}|\tau + \langle\gamma(\sigma(\xi)),\xi\rangle| \ge 2^{-N-2}2^k$. We note that $|s-\sigma(\xi)| \le 2C_0^{-1}2^n\delta_1$ and $|\langle G^{(j)}(\sigma(\xi)),(\tau,\xi)\rangle| \le C_0^{-1}2^k(2^n\delta_1)^{N-j}$ for $1 \le j \le N-2$ on $\operatorname{supp} \mathfrak{a}_{\nu,2}^{\mu,n}$. Since $|\langle G^{(N)}(s),(\tau,\xi)\rangle| \le B2^{k+1}$, using Taylor series expansion at $\sigma(\xi)$ as above, we see (3.5.27) holds true for some $C = C(B) > 0$. $\square$

Additionally, we make use of disjointness of $\operatorname{supp}_\xi \mathfrak{a}_\nu^{\mu,n}$ by combining Lemma 3.4.3 and the next one.

Lemma 3.5.6. *There is a positive constant $C = C(B)$ such that*

$$|(\widetilde{\mathcal{L}}_s^\delta)^{-1}\xi| \le Cb2^k \tag{3.5.31}$$

whenever $\xi \in \Lambda_k'(s,\delta,b)$ (see (3.4.11)). If $\xi \in \Gamma_k$ and (3.5.31) holds with $C = 1$, then $\xi \in \Lambda_k'(s,\delta,C_1b)$ for some $C_1 = C_1(B) > 0$.

Proof. Let $\eta \in \mathbb{R}^d$ and $\{v_N, \dots, v_d\}$ be an orthonormal basis of $(\mathrm{span}\{\gamma^{(j)}(s) : 1 \le j \le N-1\})^\perp$. We write $\eta = \sum_{j=1}^{N-1} \mathbf{c}_j \gamma^{(j)}(s) + \sum_{j=N}^{d} \mathbf{c}_j v_j$. Since $\mathfrak{V}(N, B)$ holds for γ, $|\eta| \sim |(\mathbf{c}_1, \cdots, \mathbf{c}_d)|$. Let $\xi \in \Lambda_k'(s, \delta, b)$. Then, (3.2.1) gives

$$\langle \eta, (\widetilde{\mathcal{L}}_s^\delta)^{-1}\xi\rangle = \langle (\widetilde{\mathcal{L}}_s^\delta)^{-\intercal}\eta, \xi\rangle = \sum_{j=1}^{N-1} \delta^{j-N} \mathbf{c}_j \langle \gamma^{(j)}(s), \xi\rangle + \sum_{j=N}^{d} \mathbf{c}_j \langle v_j, \xi\rangle.$$

Thus, by (3.4.11) we get $|\langle \eta, (\widetilde{\mathcal{L}}_s^\delta)^{-1}\xi\rangle| \le Cb|\eta|2^k$, which shows (3.5.31).

By (3.2.1), $\langle \gamma^{(j)}(s), \xi\rangle = \delta^{N-j}\langle \gamma^{(j)}(s), (\widetilde{\mathcal{L}}_s^\delta)^{-1}\xi\rangle$ for $1 \le j \le N-1$. Therefore, (3.5.31) with $C = 1$ gives $|\langle \gamma^{(j)}(s), \xi\rangle| \le C_1 b \delta^{N-j} 2^k$ for a constant $C_1 > 0$ when $1 \le j \le N-1$. This proves the second statement. □

Now, we are ready to prove the estimates (3.5.24) and (3.5.25). We first show (3.5.24).

Proof of (3.5.24). By Lemma 3.5.4, $C^{-1}\mathfrak{a}_{\nu,1}^{\mu,n} \in \mathfrak{A}_k(2^n\delta_1\nu, 2^n\delta_1)$ for some $C > 0$, and (3.5.26) holds on $\mathrm{supp}_{s,\xi}\, \mathfrak{a}_{\nu,1}^{\mu,n}$. Thus, taking $\delta = 2^n\delta_1$ and $s_0 = 2^n\delta_1\nu$, we may use Lemma 3.3.1 for $\tilde\chi\mathcal{T}[\mathfrak{a}_{\nu,1}^{\mu,n}]f$ to get

$$\big\|\tilde\chi\mathcal{T}[\mathfrak{a}_{\nu,1}^{\mu,n}]f\big\|_{L^p(\mathbb{R}^{d+1})} \le C \sum_{1\le l\le C} \delta\big\|\mathcal{A}_t[\gamma_{s_0}^\delta,\, a_l]\tilde f_l\big\|_{L^p(\mathbb{R}^{d+1})},$$

where $\|\tilde f_l\|_p = \|f\|_p$, a_l are of type $(j, N-1, B')$ relative to $\gamma_{s_0}^\delta$ for some $B' > 0$, and $2^j \sim (2^n\delta_1)^N 2^k$. As seen before, $\gamma = \gamma_{s_0}^\delta$ satisfies $\mathfrak{V}(N, 3B)$ and (3.1.1) with B replaced by $3B$ for $\delta \le \delta_*$. So, $\gamma = \gamma_{s_0}^\delta$ satisfies $\mathfrak{V}(N-1, B')$ for a large B'.

Therefore, we may apply the assumption (Theorem 3.1.2 with $L = N-1$) to $\mathcal{A}_t[\gamma_{s_0}^\delta, a_l]$, which gives $\|\mathcal{A}_t[\gamma_{s_0}^\delta, a_l]f\|_p \le C_\epsilon\big(2^k(2^n\delta_1)^N\big)^{-\frac{2}{p}+\epsilon}\|f\|_p$ for a constant $C_\epsilon = C_\epsilon(B')$. Consequently, we obtain

$$\|\tilde\chi\mathcal{T}[\mathfrak{a}_{\nu,1}^{\mu,n}]f\|_p \le C_\epsilon 2^{-\frac{2}{p}k+\epsilon k}(2^n\delta_1)^{1-\frac{2N}{p}+\epsilon}\|f\|_p$$

for $p \ge 4(N-1)-2$. Besides, since $C^{-1}\mathfrak{a}_{\nu,1}^{\mu,n} \in \mathfrak{A}_k(2^n\delta_1\nu, 2^n\delta_1)$, by (3.2.13) we have $\|(1-\tilde\chi)\mathcal{T}[\mathfrak{a}_{\nu,1}^{\mu,n}]f\|_{L^p(\mathbb{R}^{d+1})} \lesssim_B 2^{-k}(2^n\delta_1)^{1-N}\|f\|_{L^p(\mathbb{R}^d)}$ for $p > 1$. Note $2^n\delta_1 \gtrsim 2^{-k/N}$. Combining those two estimates yields

$$\|\mathcal{T}[\mathfrak{a}_{\nu,1}^{\mu,n}]f\|_p \le C_\epsilon 2^{-\frac{2}{p}k+\epsilon k}(2^n\delta_1)^{1-\frac{2N}{p}+\epsilon}\|f\|_p. \tag{3.5.32}$$

To exploit disjointness of $\operatorname{supp}_\xi \mathfrak{a}^{\mu,n}_{\nu,1}$, we define a multiplier operator by

$$\mathcal{F}(P^\delta_s f)(\xi) = \beta_0\big(|(\widetilde{\mathcal{L}}^\delta_s)^{-1}\xi|/(C_0 2^k)\big)\widehat{f}(\xi)$$

for a constant $C_0 > 0$. Since $\operatorname{supp}_\xi \mathfrak{a}^{\mu,n}_{\nu,1} \subset \Lambda'_k(2^n\delta_1\nu,\ 2^n\delta_1, 2^5 B)$, by Lemma 3.5.6 we may choose C_0 large enough so that $\beta_0\big(|(\widetilde{\mathcal{L}}^{2^n\delta_1}_{2^n\delta_1\nu})^{-1}\cdot|/(C_0 2^k)\big) = 1$ on $\operatorname{supp}_\xi \mathfrak{a}^{\mu,n}_{\nu,1}$. Thus, $\mathcal{T}[\mathfrak{a}^{\mu,n}_{\nu,1}]f = \mathcal{T}[\mathfrak{a}^{\mu,n}_{\nu,1}]P^{2^n\delta_1}_{2^n\delta_1\nu}f$. Combining this and (3.5.32), we obtain

$$\Big(\sum_\mu \sum_{\nu\in\mathfrak{J}^\mu_n} \|\mathcal{T}[\mathfrak{a}^{\mu,n}_{\nu,1}]f\|_p^p\Big)^{1/p} \le C_\epsilon 2^{-\frac{2}{p}k+\epsilon k}(2^n\delta_1)^{1-\frac{2N}{p}+\epsilon}\Big(\sum_\mu \sum_{\nu\in\mathfrak{J}^\mu_n} \|P^{2^n\delta_1}_{2^n\delta_1\nu}f\|_p^p\Big)^{1/p}$$

for a constant $C_\epsilon = C_\epsilon(B)$ if $p \ge 4N-6$. Therefore, (3.5.24) follows if we show

$$\Big(\sum_\mu \sum_{\nu\in\mathfrak{J}^\mu_n} \|P^{2^n\delta_1}_{2^n\delta_1\nu}f\|_p^p\Big)^{1/p} \lesssim_B \|f\|_p, \qquad 2 \le p \le \infty. \tag{3.5.33}$$

By interpolation it suffices to obtain (3.5.33) for $p = 2, \infty$. The case $p = \infty$ is trivial since $\|P^{2^n\delta_1}_{2^n\delta_1\nu}f\|_\infty \lesssim \|f\|_\infty$. For $p = 2$, (3.5.33) follows by Plancherel's theorem since $\operatorname{supp}\beta_0\big(|(\widetilde{\mathcal{L}}^{2^n\delta_1}_{2^n\delta_1\nu})^{-1}\cdot|/(C_0 2^k)\big)\widehat{f}$, $\nu \in \mathfrak{J}^\mu_n$ are finitely overlapping. Indeed, by Lemma 3.5.6 we have $\operatorname{supp}\beta_0\big(|(\widetilde{\mathcal{L}}^{2^n\delta_1}_{2^n\delta_1\nu})^{-1}\cdot|/(C_0 2^k)\big)\widehat{f} \subset \Lambda'_k(2^n\delta_1\nu, 2^n\delta_1, C_1 B)$ for a constant C_1. It is clear from lemma 3.4.3 that $\Lambda'_k(2^n\delta_1\nu, 2^n\delta_1, C_1 B)$, $\nu \in \mathfrak{J}^\mu_n$ overlap at most $C = C(B)$ times. $\square$

The proof of (3.5.25) is much easier since we have a favorable estimate for the kernel of $\mathcal{T}[\mathfrak{a}^{\mu,n}_{\nu,2}]$ thanks to the lower bound (3.5.27).

Proof of (3.5.25). Let

$$\mathfrak{b}(s,t,\tau,\xi) = i^{-1}(\tau + \langle\gamma(s),\xi\rangle)^{-1}\partial_t \mathfrak{a}^{\mu,n}_{\nu,2}(s,t,\tau,\xi).$$

Then, integration by parts in t shows $m[\mathfrak{a}^{\mu,n}_{\nu,2}] = m[\mathfrak{b}]$. Note (3.5.27) holds and $C^{-1}\mathfrak{a}^{\mu,n}_{\nu,2} \in \mathfrak{A}_k(2^n\delta_1\nu, 2^n\delta_1)$ for a constant $C \ge 1$. Thus, $\mathfrak{a} := C^{-1}2^k(2^n\delta_1)^N\mathfrak{b}$ satisfies, with $\delta = 2^n\delta_1$ and $s_0 = 2^n\delta_1\nu$, (3.2.3) and (3.2.4) for $0 \le j \le 1$, $0 \le l \le 2N-1$, $|\alpha| \le d+N+2$. Applying (3.2.12), we obtain $\|\mathcal{T}[\mathfrak{a}^{\mu,n}_{\nu,2}]f\|_\infty \lesssim_B 2^{-k}(2^n\delta_1)^{1-N}\|f\|_\infty$. Since $\delta_1 \ge 2^{-k/N}$, this gives

$$\|\mathcal{T}[\mathfrak{a}^{\mu,n}_{\nu,2}]f\|_\infty \lesssim_B 2^{-\frac{(N+2)k}{2N}}(2^n\delta_1)^{-\frac{N}{2}}\|f\|_\infty. \tag{3.5.34}$$

By interpolation it is sufficient to show (3.5.25) for $p = 2$. Note that we have $\|b(\cdot,t,\tau,\xi)\|_\infty + \|\partial_s b(\cdot,t,\tau,\xi)\|_1 \lesssim 2^{-k}(2^n\delta_1)^{-N}$. Thus, (3.4.2) and van der Corput's lemma in s give $|m[\mathfrak{a}^{\mu,n}_{\nu,2}](\tau,\xi)| \lesssim 2^{-k(1+N)/N}(2^n\delta_1)^{-N}$. Since $\mathrm{supp}_\xi\, \mathfrak{a}^{\mu,n}_{\nu,2} \subset \Lambda'_k(2^n\delta_1\nu, 2^n\delta_1, 2^5B)$, as before, we have $\mathcal{T}[\mathfrak{a}^{\mu,n}_{2,\nu}]f = \mathcal{T}[\mathfrak{a}^{\mu,n}_{2,\nu}]P^{2^n\delta_1}_{2^n\delta_1\nu}f$ with $C_0 > 0$ large enough. Thus, by Plancherel's theorem

$$\|\mathcal{T}[\mathfrak{a}^{\mu,n}_{\nu,2}]f\|^2_{L^2} \lesssim_B 2^{-\frac{2(1+N)}{N}k}(2^n\delta_1)^{-2N}\iint_{\{\tau:|\mathfrak{g}^0_\mu(\tau,\xi)|\le 2^{k+1}(2^n\delta_1)^N\}} d\tau\, |\mathcal{F}(P^{2^n\delta_1}_{2^n\delta_1\nu}f)(\xi)|^2\, d\xi.$$

Combining this and (3.5.33) yields (3.5.25) for $p = 2$. □

3.6 Bounds on the symbols

Proof of Lemma 3.5.2

To simplify notations, we denote

$$\delta_* = 2^n\delta_1, \qquad s_* = 2^n\delta_1\nu$$

for the rest of this section. To prove Lemma 3.5.2, we verify (3.2.3) and (3.2.4) with $\mathfrak{a} = \mathfrak{a}^{\mu,n}_\nu$, $\delta = \delta_*$, and $s_0 = s_*$. The first is easy. In fact, since $\mathfrak{a}^\mu \in \mathfrak{A}_k(\delta_0\mu, \delta_0)$ and $\mathrm{supp}_s\, \mathfrak{a}^{\mu,n}_\nu \subset I(s_*, \delta_*)$, we only need to show

$$|\langle G^{(j)}(s_*), (\tau,\xi)\rangle| \le B2^{k+5}\delta_*^{N-j}, \quad j = 0, \dots, N-1 \tag{3.6.1}$$

on $\mathrm{supp}_{\tau,\xi}\, \mathfrak{a}^{\mu,n}_\nu$. Using (3.5.6) and (3.5.12) together with (3.4.7) and (3.5.13), one can easily obtain

$$|\langle G^{(j)}(\sigma(\xi)), (\tau,\xi)\rangle| \le 2^{k+1}\delta_*^{N-j}, \quad j = 0, \dots, N-1 \tag{3.6.2}$$

on $\mathrm{supp}_{\tau,\xi}\, \mathfrak{a}^{\mu,n}_\nu$. Expanding $\langle G^{(j)}(s), (\tau,\xi)\rangle$ in Taylor's series at $\sigma(\xi)$ gives (3.6.1) since (3.5.14) holds.

We now proceed to show (3.2.4) with $\mathfrak{a} = \mathfrak{a}^{\mu,n}_\nu$, $\delta = \delta_*$, and $s_0 = s_*$. Since $\mathfrak{a}^{\mu,n}_\nu$ consists of three factors $\mathfrak{a}^\mu$, $\beta_N(\delta_*^{-2N!}\mathfrak{G}^\mu_N)$, and $\zeta(\delta_*^{-1}s - \nu)$, by Leibniz's rule it is sufficient to consider the derivatives of each of them. The bounds on the derivatives $\zeta(\delta_*^{-1}s - \nu)$ are clear. So, it suffices to show (3.2.4) for

$$\mathfrak{a} \;=\; \mathfrak{a}^\mu,\; \beta_N(\delta_*^{-2N!}\mathfrak{G}^\mu_N)$$

with $\delta = \delta_*$ and $s_0 = s_*$ whenever $(\tau,\xi) \in \mathrm{supp}\, \mathfrak{a}^{\mu,n}_\nu(s,t,\mathcal{L}^{\delta_*}_{s_*}\cdot)$.

We handle $\mathfrak{a}^\mu$ first. That is to say, we show

$$\big|\partial_s^j\partial_t^l\partial_{\tau,\xi}^\alpha\big(\mathfrak{a}^\mu(s,t,\mathcal{L}_{s_*}^{\delta_*}(\tau,\xi))\big)\big| \lesssim_B \delta_*^{-j}|(\tau,\xi)|^{-|\alpha|}, \qquad (j,l,\alpha)\in\mathcal{I}_N, \tag{3.6.3}$$

for $(\tau,\xi)\in\operatorname{supp}\mathfrak{a}_\nu^{\mu,n}(s,t,\mathcal{L}_{s_*}^{\delta_*}\cdot)$. Since $\mathfrak{a}^\mu\in\mathfrak{A}_k(\delta_0\mu,\delta_0)$ and $|s_*-\delta_0\mu|\le\delta_0$, we have

$$|\partial_s^j\partial_t^l\partial_{\tau,\xi}^\alpha\big(\mathfrak{a}^\mu(s,t,\mathcal{L}_{s_*}^{\delta_0}(\tau,\xi))\big)| \lesssim_B \delta_0^{-j}|(\tau,\xi)|^{-|\alpha|}, \qquad (j,l,\alpha)\in\mathcal{I}_N. \tag{3.6.4}$$

One can show this using (3.2.8). We consider $\mathcal{U}:=(\mathcal{L}_{s_*}^{\delta_0})^{-1}\mathcal{L}_{s_*}^{\delta_*}$. By (3.2.5) we have $|\mathcal{U}^\intercal z|\lesssim_B|z|$ because $|\delta_0^{-1}2^n\delta_1|\lesssim_B 1$. Thus, (3.6.4) gives

$$|\partial_s^j\partial_t^l\partial_{\tau,\xi}^\alpha\big(\mathfrak{a}^\mu(s,t,\mathcal{L}_{s_*}^{\delta_0}\mathcal{U}(\tau,\xi))\big)| \lesssim_B \delta_0^{-j}|\mathcal{U}(\tau,\xi)|^{-|\alpha|}$$

for $(\tau,\xi)\in\operatorname{supp}\mathfrak{a}_\nu^{\mu,n}(s,t,\mathcal{L}_{s_*}^{\delta_*}\cdot)$.

Let $(\tau,\xi)\in\operatorname{supp}\mathfrak{a}_\nu^{\mu,n}(s,t,\mathcal{L}_{s_*}^{\delta_*}\cdot)$. Then, $\mathcal{L}_{s_*}^{\delta_0}\mathcal{U}(\tau,\xi)=\mathcal{L}_{s_*}^{\delta_*}(\tau,\xi)\in\Lambda_k(s_*,\delta_*,B)$, so $|\widetilde{\mathcal{L}}_{s_*}^{\delta_*}\xi|\sim|(\tau,\xi)|$ by Lemma 3.2.2. This and (3.2.6) give

$$|(\tau,\xi)|\sim|\widetilde{\mathcal{L}}_{s_*}^{\delta_*}\xi|\le|\mathcal{L}_{s_*}^{\delta_*}(\tau,\xi)|\le|\mathcal{U}(\tau,\xi)|$$

for $(\tau,\xi)\in\operatorname{supp}\mathfrak{a}_\nu^{\mu,n}(s,t,\mathcal{L}_{s_*}^{\delta_*}\cdot)$. So, we obtain (3.6.3) since $\delta_*\lesssim\delta_0$.

We continue to show (3.2.4) for $\mathfrak{a}=\beta_N(\delta_*^{-2N!}\mathfrak{G}_N^\mu)$. Note $\delta_*^{-2N!}\mathfrak{G}_N^\mu$ is a sum of $(\delta_*^{-1}(s-\sigma(\xi)))^{2N!}$ and $(\delta_*^{-(N-j)}2^{-k}\mathfrak{g}_\mu^j)^{2N!/(N-j)}$, $0\le j\le N-2$. Since the exponents $2N!/(N-j)$ are even integers, for the desired bounds on $\partial_{\tau,\xi}^\alpha(\beta_N(\delta_*^{-2N!}\mathfrak{G}_N^\mu))$ it suffices to show the same bounds on the derivatives of

$$\delta_*^{-1}(s-\sigma(\xi)), \qquad \delta_*^{-(N-j)}2^{-k}\mathfrak{g}_\mu^j, \quad 0\le j\le N-2.$$

The bound on $\partial_\xi^\alpha\delta_*^{-1}(s-\sigma)$ is a consequence of (3.2.7) and the following lemma. To simplify notations, we denote

$$\Xi=\mathcal{L}_{s_*}^{\delta_*}(\tau,\xi), \qquad \tilde{\Xi}=\widetilde{\mathcal{L}}_{s_*}^{\delta_*}\xi.$$

Lemma 3.6.1. *If $\Xi\in\operatorname{supp}_{\tau,\xi}\mathfrak{a}_\nu^{\mu,n}$, then we have*

$$|\delta_*^{-1}\partial_\xi^\alpha(\sigma(\tilde{\Xi}))|\lesssim_B|\xi|^{-|\alpha|}, \qquad 1\le|\alpha|\le 2d+2. \tag{3.6.5}$$

Proof. By (3.4.3), $\gamma^{(N-1)}(\sigma(\tilde{\Xi}))\cdot\tilde{\Xi}=0$. Differentiation gives

$$\gamma^{(N)}(\sigma(\tilde{\Xi}))\cdot\tilde{\Xi}\ \nabla_\xi(\sigma(\tilde{\Xi}))+(\widetilde{\mathcal{L}}_{s_*}^{\delta_*})^\intercal\gamma^{(N-1)}(\sigma(\tilde{\Xi}))=0. \tag{3.6.6}$$

Denote $s = \sigma(\tilde{\Xi})$. By (3.2.1), $(\widetilde{\mathcal{L}}^{\delta_*}_{s_*})^\intercal \gamma^{(N-1)}(s) = \delta_* (\widetilde{\mathcal{L}}^{\delta_*}_{s_*})^\intercal (\widetilde{\mathcal{L}}^{\delta_*}_{s})^{-\intercal} \gamma^{(N-1)}(s)$. Since $|s_* - s| \le \delta_*$, by Lemma 3.2.3 we have $|(\widetilde{\mathcal{L}}^{\delta_*}_{s_*})^\intercal \gamma^{(N-1)}\big(\sigma(\tilde{\Xi})\big)| \lesssim_B \delta_*$. Besides, $|\gamma^{(N)}(\sigma(\tilde{\Xi})) \cdot \tilde{\Xi}| \gtrsim |\tilde{\Xi}| \sim 2^k$ (see (3.4.2)). Thus, (3.6.6) and (3.2.7) give

$$|\nabla_\xi (\sigma(\tilde{\Xi}))| \lesssim_B \delta_* |\xi|^{-1},$$

which proves (3.6.5) with $|\alpha| = 1$.

We show the bounds on the derivatives of higher order by induction. Assume that (3.6.5) holds true for $|\alpha| \le L$. Let α' be a multi-index such that $|\alpha'| = L+1$. Then, differentiating (3.6.6) and using the induction assumption, one can easily see $\gamma^{(N)}(\sigma(\tilde{\Xi})) \cdot \tilde{\Xi}\ \partial^{\alpha'}_\xi (\sigma(\tilde{\Xi})) = O(\delta_* |\xi|^{-L})$, by which we get (3.6.5) for $|\alpha| = L + 1$. Since $\sigma \in C^{2d+2}$, one can continue this as far as $L \le 2d + 1$. $\square$

The proof of Lemma 3.5.2 is now completed if we show

$$\big|2^{-k} \partial^\alpha_{\tau,\xi} \big(\mathfrak{g}^\ell_\mu(\Xi)\big)\big| \lesssim_B \delta_*^{N-\ell} 2^{-k|\alpha|}, \qquad |\alpha| \le d + N + 2 \tag{3.6.7}$$

for $0 \le \ell \le N - 2$ whenever $\Xi \in \operatorname{supp} \mathfrak{a}^{\mu,n}_\nu(s, t, \cdot)$. To this end, we use the following.

Lemma 3.6.2. *For $j = 0, \dots, N$, we set*

$$A_j = \delta_*^{-(N-j)} 2^{-k} \big\langle G^{(j)}(\sigma(\tilde{\Xi})), \Xi \big\rangle.$$

If $(\tau, \xi) \in \operatorname{supp} \mathfrak{a}^{\mu,n}_\nu(s, t, \mathcal{L}^{\delta_*}_{s_*} \cdot)$, then for $j = 0, \dots, N$ we have

$$|\partial^\alpha_{\tau,\xi} A_j| \lesssim_B |(\tau, \xi)|^{-|\alpha|}, \qquad 1 \le |\alpha| \le 2d + 2. \tag{3.6.8}$$

Proof. When $j = N$, the estimate (3.6.8) follows by Lemma 3.6.1 and (3.2.7). So, we may assume $j \le N - 1$. Differentiating A_j, we have

$$\nabla_{\tau,\xi} A_j = B_j + D_j,$$

where

$$B_j = \delta_*^{-1} \big(0, \nabla_\xi (\sigma(\tilde{\Xi}))\big) A_{j+1}, \qquad D_j = \delta_*^{-(N-j)} 2^{-k} (\mathcal{L}^{\delta_*}_{s_*})^\intercal G^{(j)}(\sigma(\tilde{\Xi})).$$

Note $(\mathcal{L}^{\delta_*}_{s_*})^\intercal G^{(j)}(s_*) = \delta_*^{N-j} G^{(j)}(s_*)$ for $0 \le j \le N - 1$. Since $|s_* - \sigma(\tilde{\Xi})| \lesssim \delta_*$, similarly as before, Lemma 3.2.3 and (3.2.5) give

$$|(\mathcal{L}^{\delta_*}_{s_*})^\intercal G^{(j)}(\sigma(\tilde{\Xi}))| \lesssim_B \delta_*^{N-j}, \qquad 0 \le j \le N - 1. \tag{3.6.9}$$

By Lemma 3.6.1 and (3.6.2), $|B_j| \lesssim |\xi|^{-1}$. Thus, for $\Xi \in \Lambda_k(s_*, \delta_*, B)$, we have

$$|\nabla_{\tau,\xi} A_j| \lesssim_B |\xi|^{-1} + 2^{-k} \lesssim_B |(\tau,\xi)|^{-1}, \quad j = 0, \dots, N-1,$$

For the second inequality we use (3.2.7). This gives (3.6.8) when $|\alpha| = 1$.

To show (3.6.8) for $2 \le |\alpha| \le 2d+2$, we use backward induction. By (3.4.3) we note $A_{N-1} = 0$, so (3.6.8) trivially holds when $j = N-1$. We now assume that (3.6.8) holds true if $j_0 + 1 \le j \le N-1$ for some j_0. Lemma 3.6.1, (3.2.7), and the induction assumption show $\partial_{\tau,\xi}^{\alpha'} B_{j_0} = O(|(\tau,\xi)|^{-1-|\alpha'|})$ for $1 \le |\alpha'| \le 2d+1$. Concerning D_{j_0}, observe that $\partial_\xi^{\alpha'}(G^{(j_0)}(\sigma(\tilde\Xi)))$ is given by a sum of the terms

$$G^{(j)}(\sigma(\tilde\Xi)) \prod_{n=1}^{j-j_0} \partial_\xi^{\alpha'_n}(\sigma(\tilde\Xi)),$$

where $j \ge j_0$ and $\alpha'_1 + \cdots + \alpha'_{j-j_0} = \alpha'$. Hence, Lemma 3.6.1, (3.6.9), and (3.2.7) give $\partial_\xi^{\alpha'} D_{j_0} = O(|(\tau,\xi)|^{-1-|\alpha'|})$ for $1 \le |\alpha'| \le 2d+1$. Therefore, combining the estimates for B_{j_0} and D_{j_0}, we get $\partial_{\tau,\xi}^{\alpha'} \nabla_{\tau,\xi} A_{j_0} = O(|(\tau,\xi)|^{-1-|\alpha'|})$. This proves (3.6.8) for $j = j_0$. $\square$

Before proving (3.6.7), we first note

$$|\partial_\xi^\alpha(\mathcal{E}_j(\tilde\Xi))| \lesssim_B \delta_*^j |\xi|^{-|\alpha|}, \quad |\alpha| \le 2d+2 \tag{3.6.10}$$

for $j = 1, \dots, N$. This can be shown by a routine computation. Indeed, differentiating (3.5.4), and using Lemma 3.6.1 and (3.4.7), one can easily see (3.6.10) since $|\sigma(\tilde\Xi) - \delta_0\mu| \lesssim \delta_0$.

To show (3.6.7) for $0 \le \ell \le N-2$, we again use backward induction. Observe that (3.6.7) holds for $\ell = N, N-1$, and assume that (3.6.7) holds for $j+1 \le \ell \le N$ for some $j \le N-2$. By (3.5.6) we have

$$2^{-k}\mathfrak{g}_\mu^j = \delta_*^{N-j} A_j - \sum_{j+1\le \ell \le N} (2^{-k}\mathfrak{g}_\mu^\ell)(\mathcal{E}_1)^{\ell-j}/(\ell-j)! + 2^{-k} y_\mu^N \mathcal{E}_{N-j}.$$

Thus, by Lemma 3.6.2 and (3.6.10), we get (3.6.7) with $\ell = j$. This completes the proof of Lemma 3.5.2.

Proof of Lemma 3.5.4

Lemma 3.5.4 can be proved in a similar way as the previous subsection. So, we shall be brief.

By Lemma 3.5.2 we have $C^{-1}\mathfrak{a}_{\nu}^{\mu,n} \in \mathfrak{A}_k(s_*, \delta_*)$ for a constant $C \geq 1$, so it suffices to show $C^{-1}\mathfrak{a}_{\nu,1}^{\mu,n} \in \mathfrak{A}_k(s_*, \delta_*)$ for some $C \geq 1$. The support condition (3.2.3) is obvious, so we need only to show (3.2.4) with $\mathfrak{a} = \mathfrak{a}_{\nu,1}^{\mu,n}$, $\delta = \delta_*$, and $s_0 = s_*$. Moreover, recalling (3.5.23), it is enough to consider the additional factor only, i.e., to show

$$\Big|\partial_{\tau,\xi}^{\alpha}\Big(\beta_0\Big(\frac{\big(\delta_*^{-N}2^{-k}\mathfrak{g}_{\mu}^{0}(\mathcal{L}_{s_*}^{\delta_*}(\tau,\xi))\big)^{2(N-1)!}}{C_0^{2N!}\delta_*^{-2N!}\bar{\mathfrak{G}}_N^{\mu}(s,\widetilde{\mathcal{L}}_{s_*}^{\delta_*}\xi)}\Big)\Big)\Big| \lesssim |(\tau,\xi)|^{-|\alpha|}$$

for $(\tau,\xi) \in \operatorname{supp}\mathfrak{a}_{\nu,1}^{\mu,n}(s,t,\mathcal{L}_{s_*}^{\delta_*}\cdot)$. Since $\delta_*^{-2N!}\bar{\mathfrak{G}}_N^{\mu} \gtrsim 1$ on $\operatorname{supp}_{s,\xi}\mathfrak{a}_{\nu,1}^{\mu,n}$, one can obtain the estimate in the same way as in the proof of Lemma 3.5.2.

3.7 Sharpness of Theorem 1.1.2

Before closing this chapter, we show the optimality of the regularity exponent α in Theorem 1.1.2.

Proposition 3.7.1. *Suppose* (3.0.1) *holds for* $\psi(0) \neq 0$. *Then* $\alpha \leq 2/p$.

Proof. We write $\gamma = (\gamma_1, \dots, \gamma_d)$. Via an affine change of variables, we may assume $\gamma_1(0) = 0$ and $\gamma_1'(s) \neq 0$ on an interval $J = [-\delta_0, \delta_0]$ for $0 < \delta_0 \ll 1$. Since $\psi(0) \neq 0$, we may also assume $\psi \geq 1$ on J.

We choose $\zeta_0 \in \mathcal{S}(\mathbb{R})$ such that $\operatorname{supp}\widehat{\zeta_0} \subset [-1,1]$ and $\zeta_0 \geq 1$ on $[-r_1, r_1]$ where $r_1 = 1 + 2\max\{|\gamma(s)| : s \in J\}$. Denoting $\bar{x} = (x_1, \dots, x_{d-1})$ and $\bar{\gamma}(t) = (\gamma_1(t), \dots, \gamma_{d-1}(t))$, we define

$$\bar{\mathcal{A}}_t h(x) = \int e^{it\lambda\gamma_d(s)}\zeta_0(x_d - t\gamma_d(s))h(\bar{x} - t\bar{\gamma}(s))\psi(s)\,ds.$$

Let $\zeta \in \mathrm{C}_c^{\infty}((-2,2))$ be a positive function such that $\zeta = 1$ on $[-1,1]$. For a positive constant $c \ll \delta_0$, let $g_1(\bar{x}) = \sum_{\nu\in\lambda^{-1}\mathbb{Z}\cap[-c,c]}\zeta(\lambda|\bar{x} + \bar{\gamma}(\nu)|)$. We consider

$$g(\bar{x}) = e^{-i\lambda\varphi(x_1)}g_1(\bar{x}),$$

where $\varphi(s) = \gamma_d \circ (-\gamma_1)^{-1}(s)$. We claim that, if c is small enough,

$$|\bar{\mathcal{A}}_t g(x)| \gtrsim 1, \qquad (x,t) \in S_c, \tag{3.7.1}$$

where $S_c = \{(x,t) : |\bar{x}| \le c\lambda^{-1},\ |x_d| \le c,\ |t-1| \le c\lambda^{-1}\}$. To show this, note

$$\bar{\mathcal{A}}_t g(x) = \int e^{i\lambda(t\gamma_d(s)-\varphi(x_1-t\gamma_1(s)))}\zeta_0(x_d - t\gamma_d(s))g_1(\bar{x}-t\bar{\gamma}(s))\,\psi(s)\,ds.$$

Let $(x,t) \in S_c$. Then, $\operatorname{supp} g_1(\bar{x}-t\bar{\gamma}(\cdot)) \subset [-C_1c, C_1c]$ for some $C_1 > 0$. Since $\varphi(s) = \gamma_d \circ (-\gamma_1)^{-1}(s)$, by the mean value theorem we see $|\varphi(x_1 - t\gamma_1(s)) - \gamma_d(s)| \le 2r_0c\lambda^{-1}$ where $r_0 = 10r_1 \max\{|\partial_s\varphi(s)| : s \in (-\gamma_1)(J_*)\}$ and $J_* = [-(C_1+1)c, (C_1+1)c]$. Thus, we have

$$|t\gamma_d(s) - \varphi(x_1 - t\gamma_1(s))| \le 3r_0c\lambda^{-1}. \tag{3.7.2}$$

Besides, if λ is sufficiently large, $g_1(\bar{x} - t\bar{\gamma}(s)) = \sum_{\nu\in\lambda^{-1}\mathbb{Z}\cap[-c,c]}\zeta(\lambda|\bar{x} - (t-1)\bar{\gamma}(s) + \bar{\gamma}(\nu) - \bar{\gamma}(s)|) \gtrsim 1$ if $s \in [-c/2, c/2]$. Since $\operatorname{supp} g_1(\bar{x} - t\bar{\gamma}(\cdot)) \subset J$ with c small enough and $\zeta_0(x_d - t\gamma_d(s)) \ge 1$, we get $\int \zeta_0(x_d - t\gamma_d(s))g_1(\bar{x} - t\bar{\gamma}(s))\psi(s)\,ds \gtrsim 1$. Therefore, (3.7.1) follows by (3.7.2) if c is small enough, i.e., $c \ll 1/(3r_0)$.

We set $f(x) = e^{-i\lambda x_d}\zeta_0(x_d)g(\bar{x})$. Then, $\chi(t)\mathcal{A}_tf(x) = e^{-i\lambda x_d}\chi(t)\bar{\mathcal{A}}_tg(x)$. By our choice of ζ_0, $\operatorname{supp}\widehat{f} \subset \{\xi : |\xi_d + \lambda| \le 1\}$, so $\operatorname{supp}\mathcal{F}(\chi(t)\mathcal{A}_tf) \subset \{(\tau,\xi) : |\xi_d + \lambda| \le 1\}$. This gives

$$\lambda^\alpha\|\chi(t)\mathcal{A}_tf\|_{L^p(\mathbb{R}^{d+1})} \lesssim \|\chi(t)\mathcal{A}_tf\|_{L^p_\alpha(\mathbb{R}^{d+1})}. \tag{3.7.3}$$

Indeed, $\lambda^\alpha\|\chi(t)\mathcal{A}_tf\|_{L^p(\mathbb{R}^{d+1})} \lesssim \|\chi(t)\mathcal{A}_tf\|_{L^p(\mathbb{R}_{t,\bar{x}};L^p_\alpha(\mathbb{R}_{x_d}))}$ by Mihlin's multiplier theorem in x_d. Similarly, one also sees $\|F\|_{L^p(\mathbb{R}_{t,\bar{x}};L^p_\alpha(\mathbb{R}_{x_d}))} \le C\|F\|_{L^p_\alpha(\mathbb{R}^{d+1})}$ for $\alpha \ge 0$ and any F. Combining those inequalities gives (3.7.3).

From (3.7.1) we have $\|\chi(t)\mathcal{A}_tf\|_p = \|\chi(t)\bar{\mathcal{A}}_tg\|_p \ge C\lambda^{-d/p}$. Note that $\operatorname{supp} g$ is contained in a $O(\lambda^{-1})$-neighborhood of $-\bar{\gamma}$, so it follows that $\|f\|_p \lesssim \lambda^{-(d-2)/p}$. Therefore, by (3.7.3) the inequality (3.0.1) implies $\lambda^\alpha\lambda^{-d/p} \lesssim \lambda^{-(d-2)/p}$. Taking $\lambda \to \infty$ gives $\alpha \le 2/p$. □

Chapter 4

Sobolev estimate

Let $2 \le p \le \infty$. We set $\mathcal{A}f = \mathcal{A}_1 f$, and we are concerned with the L^p Sobolev regularity estimate

$$\|\mathcal{A}f\|_{L^p_\alpha(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)}. \tag{4.0.1}$$

In this chapter, we prove Theorem 1.1.1, whose proof proceeds in a similar way as that of Theorem 1.1.2. However, we provide some details to make it clear how the optimal bounds are achieved. Since there are no t, τ variables for the symbols, the proof is consequently simpler but some modifications are necessary.

For a large $B \ge 1$, we assume

$$\max_{0\le j\le 2d} |\gamma^{(j)}(s)| \le B, \qquad s \in I. \tag{4.0.2}$$

Let $2 \le L \le d$. For γ satisfying $\mathfrak{V}(L,B)$ we say $\bar{a} \in \mathrm{C}^{d+1}(\mathbb{R}^{d+1})$ is *a symbol of type* (k,L,B) *relative to* γ if $\operatorname{supp}\bar{a} \subset I \times \mathbb{A}_k$, $\mathfrak{N}(L,B)$ holds for γ on $\operatorname{supp}\bar{a}$, and

$$|\partial_s^j \partial_\xi^\alpha \bar{a}(s,\xi)| \le B|\xi|^{-|\alpha|} \tag{4.0.3}$$

for $0 \le j \le 1$ and $|\alpha| \le d+1$. As before, Theorem 1.1.1 is a straightforward consequence of the following. We denote $\mathcal{A}[\gamma,\bar{a}] = \mathcal{A}_1[\gamma,\bar{a}]$.

Theorem 4.0.1. *Suppose* $\gamma \in \mathrm{C}^{2d}(I)$ *satisfies* (4.0.2) *and* $\bar{a}$ *is a symbol of type* (k,L,B) *relative to* γ *for some* $B \ge 1$*. Then, if* $p > 2(L-1)$*, for a constant* $C = C(B)$

$$\|\mathcal{A}[\gamma,\bar{a}]f\|_{L^p(\mathbb{R}^d)} \le C2^{-k/p}\|f\|_{L^p(\mathbb{R}^d)}. \tag{4.0.4}$$

In order to prove Theorem 1.1.1, we consider $\bar{a}_k(s,\xi) := \psi(s)\beta(2^{-k}|\xi|)$, where $\beta \in \mathrm{C}_c^\infty((1/2,4))$. By (1.1.1) $\bar{a}_k$ is a symbol of type (k,d,B) relative to γ for some B, thus Theorem 4.0.1 gives (4.0.4) for $p > 2(d-1)$. The estimate (4.0.4) for each dyadic pieces can be put together by the result in [42]. So, we get (4.0.1) with $\alpha = \alpha(p)$ when $p > 2(d-1)$ (e.g., see [4]).

Interpolation with $\|\mathcal{A}[\gamma,\bar{a}_k]f\|_2 \lesssim 2^{-k/d}\|f\|_2$ which follows from (1.1.2) gives $\|\mathcal{A}[\gamma,\bar{a}_k]f\|_p \lesssim_B 2^{-\alpha k}\|f\|_p$ for $\alpha \le \alpha(p)$ with strict inequality when $p \in (2, 2(d-1)]$. Using those estimates, we can prove Corollary 1.1.5. Indeed, if γ is a curve of maximal type $\ell > d$, a typical scaling argument gives $\|\mathcal{A}[\gamma,\bar{a}_k]f\|_p \lesssim_B 2^{-\min(\alpha(p),1/\ell)k}\|f\|_p$ for $p \neq \ell$ when $\ell \ge 2d-2$, and for $p \in [2, 2\ell/(2d-\ell)) \cup (2d-2,\infty)$ when $d < \ell < 2d-2$. As above, one can combine the estimates ([42]) to get (4.0.1).

4.1 Sobolev estimate with localized frequency

The case $L = 2$ is easy. Since $\bar{a}$ is a symbol of type $(k,2,B)$ relative to γ, van der Corput's lemma and Plancherel's theorem give (4.0.4) for $p = 2$. Interpolation with L^∞ estimate shows (4.0.4) for $p \ge 2$. When $L \ge 3$, we have the following, which immediately yields Theorem 4.0.1.

Proposition 4.1.1. *Let $3 \le N \le d$. Suppose Theorem 4.0.1 holds for $L = N-1$. Then Theorem 4.0.1 holds true with $L = N$.*

To prove the proposition, we fix $N \in [3,d]$ and γ satisfying $\mathfrak{V}(N,B)$, and $\bar{a}$ of type (k,N,B) relative to γ.

For s_0 and $\delta > 0$ such that $I(s_0,\delta) \subset I$, let

$$\bar{\Lambda}_k(s_0,\delta,B) = \bigcap_{1\le j\le N-1} \big\{\xi \in \mathbb{A}_k : |\langle \gamma^{(j)}(s_0),\xi\rangle| \le B2^{k+5}\delta^{N-j}\big\}.$$

By $\bar{\mathfrak{A}}_k(s_0,\delta)$ we denote a collection of $\bar{\mathfrak{a}} \in \mathrm{C}^{d+1}(\mathbb{R}^{d+1})$ such that $\operatorname{supp}\bar{\mathfrak{a}} \subset I(s_0,\delta) \times \bar{\Lambda}_k(s_0,\delta,B)$ and $|\partial_s^j\partial_\xi^\alpha \bar{\mathfrak{a}}(s,\widetilde{\mathcal{L}}_{s_0}^\delta\xi)| \le B\delta^{-j}2^{-k|\alpha|}$ for $0 \le j \le 1$ and $|\alpha| \le d+1$.

The next lemma which plays the same role as Lemma 3.3.1 can be shown through routine adaptation of the proof of Lemma 3.3.1.

Lemma 4.1.2. *Let $\bar{\mathfrak{a}} \in \bar{\mathfrak{A}}_k(s_0,\delta)$ and $j_* = \log(2^k\delta^N)$. Suppose* (3.3.1) *holds on* $\operatorname{supp}\bar{\mathfrak{a}}$. *Then, there exist constants C, $\tilde{B} \ge 1$, and $\delta' > 0$ depending on*

B, and symbols $\bar{\mathfrak{a}}_1, \dots, \bar{\mathfrak{a}}_{l_*}$ of type $(j, N-1, \tilde{B})$ relative to $\gamma^{\delta}_{s_0}$, such that

$$\|\mathcal{A}[\gamma, \bar{\mathfrak{a}}]f\|_{L^p(\mathbb{R}^d)} \le C\delta \sum_{1\le l\le C} \big\|\mathcal{A}[\gamma^{\delta}_{s_0}, \bar{\mathfrak{a}}_l]\tilde{f}_l\big\|_{L^p(\mathbb{R}^d)},$$

$\|\tilde{f}_l\|_p = \|f\|_p$, and $j \in [j_* - C, j_* + C]$ as long as $0 < \delta < \delta'$.

The order of necessary regularity on γ is reduced since $\bar{\mathfrak{a}}$ is independent of τ, t. Actually, we may take $\tilde{a}(s,\xi) = \bar{\mathfrak{a}}(\delta s + s_0, \delta^{-N}\widetilde{\mathcal{L}}^{\delta}_{s_0}\xi)$ while following the *Proof of Lemma 3.3.1* since validity of (4.0.3) is clear for $\bar{a} = \tilde{a}$.

Using η_N (see (3.4.1)), we break

$$\mathcal{A}[\gamma, \bar{a}] = \mathcal{A}[\gamma, \bar{a}\eta_N] + \mathcal{A}[\gamma, \bar{a}(1-\eta_N)].$$

Note that $C^{-1}\bar{a}(1-\eta_N)$ is of type $(k, N-1, B')$ relative to γ for some large constants B' and C, so we may apply the assumption to $\mathcal{A}[\gamma, \bar{a}(1-\eta_N)]f$. Consequently, we have the estimate (4.0.4) for $\bar{a} = \bar{a}(1-\eta_N)$ if $p > 2N-4$.

To obtain the estimate for $\mathcal{A}[\gamma, \bar{a}\eta_N]$, as before, we may assume $\operatorname{supp} \bar{a}\eta_N \subset I(s_0, \delta_*) \times \bar{\Gamma}_k$ for some s_0 and a small δ_*. Here, $\bar{\Gamma}_k$ is defined in the same way as Γ_k by replacing $a\eta_N$ by $\bar{a}\eta_N$. Since (3.4.2) holds on $\operatorname{supp}(\bar{a}\eta_N)$, we may work under the same *Basic assumption* as in Section 3.4. That is to say, we have σ on $\bar{\Gamma}_k$ satisfying (3.4.3) and $\sigma(\xi) \in I(s_0, \delta_*)$ for $\xi \in \bar{\Gamma}_k$. Furthermore, $\sigma \in \mathrm{C}^{d+1}$ since $\gamma \in \mathrm{C}^{2d}(I)$, and (3.4.4) holds for $\xi \in \bar{\Gamma}_k$ and $|\alpha| \le d+1$. Thus, (4.0.3) remains valid for the symbols given subsequently by decomposing $\bar{a}$ with cutoff functions associated with σ, and $\bar{\mathfrak{G}}^{\mu}_N$.

Apparently, $C^{-1}\bar{a}\eta_N \in \bar{\mathfrak{A}}_k(s_0, \delta_*)$ for a constant $C = C(B, \delta_*)$, therefore the proof of Proposition 4.1.1 is completed if we show the following.

Proposition 4.1.3. *Let $3 \le N \le d$ and $\bar{\mathfrak{a}} \in \bar{\mathfrak{A}}_k(s_0, \delta_*)$ with $\operatorname{supp}_\xi \bar{\mathfrak{a}} \subset \bar{\Gamma}_k$. Suppose Theorem 4.0.1 holds for $L = N-1$. Then, if $p > 2(N-1)$, we have* (4.0.4).

We prove Proposition 4.1.3 using the next, which corresponds to Proposition 3.4.2. In what follows, we denote $\mathcal{A}[\bar{a}] = \mathcal{A}[\gamma, \bar{a}]$.

Proposition 4.1.4. *Let δ_0 and δ_1 satisfy* (3.4.6). *For μ such that $\delta_0\mu \in I(s_0, \delta_*) \cap \delta_0\mathbb{Z}$, let $\bar{\mathfrak{a}}^{\mu} \in \bar{\mathfrak{A}}_k(\delta_0\mu, \delta_0)$ with $\operatorname{supp} \bar{\mathfrak{a}}^{\mu} \subset I(s_0, \delta_*) \times \bar{\Gamma}_k$. Suppose Theorem 4.0.1 holds for $L = N-1$. Then, if $p \in (2N-2, \infty)$, there are constants $\epsilon_0 > 0$, $C_0 = C_0(\epsilon_0, B) \ge 2$, and symbols $\bar{\mathfrak{a}}_\nu \in \bar{\mathfrak{A}}_k(\delta_1\nu, \delta_1)$ with $\operatorname{supp} \bar{\mathfrak{a}}_\nu \subset I(s_0, \delta_*) \times \bar{\Gamma}_k$, $\nu \in \cup_\mu \mathfrak{J}^{\mu}_0$, such that*

$$\Big(\sum_{\mu} \|\mathcal{A}[\bar{\mathfrak{a}}^{\mu}]f\|_p^p\Big)^{\frac{1}{p}} \le C_0\big(\delta_1/\delta_0\big)^{\frac{N}{p}-1+\epsilon_0}\Big(\sum_{\nu} \|\mathcal{A}[\bar{\mathfrak{a}}_\nu]f\|_p^p\Big)^{\frac{1}{p}} + C_0\delta_0^{-\frac{N}{p}+1}2^{-\frac{k}{p}}\|f\|_p.$$

Let δ' be given as in Lemma 4.1.2, and let $\delta_\circ > 0$ be a positive constant such that

$$\delta_\circ \le \min\{\delta',\ (2^{7d}B^6)^{-N} C_0^{-2N/\epsilon_0}\}. \tag{4.1.1}$$

Proof of Proposition 4.1.3. Set $\delta_0 = \delta_\circ$, and let $\delta_1, \dots, \delta_J$ be given by (3.4.8). Then, applying Proposition 4.1.4 iteratively up to J-th step (see Section 3.4.1), we have symbols $\bar{\mathfrak{a}}_\nu \in \bar{\mathfrak{A}}_k(\delta_J\nu, \delta_J)$, $\delta_J\nu \in I(s_0, \delta_0)$, such that

$$\big\|\mathcal{A}[\bar{\mathfrak{a}}]f\big\|_p \le C_0^J \delta_J^{\frac{N}{p}-1+\epsilon_0}\big(\sum_\nu \|\mathcal{A}[\bar{\mathfrak{a}}_\nu]f\|_p^p\big)^{1/p} + 2^{-\frac{k}{p}}\delta_0^{-\frac{N}{p}+1-\epsilon_0} \sum_{0\le j\le J-1} C_0^{j+1}\delta_j^{\epsilon_0}\|f\|_p.$$

By (4.1.1) and (3.4.8), $\delta_j \le C_0^{-2((N+1)/N)^j N/\epsilon_0}$, $0 \le j \le J-1$. So, $\sum_{j=0}^{J-1} C_0^{j+1}\delta_j^{\epsilon_0}$ is bounded by a constant C_1, and $C_0^J\delta_J^{\epsilon_0} \le C_1$. Thus, the matter is now reduced to showing

$$\big(\sum_\nu \|\mathcal{A}[\bar{\mathfrak{a}}_\nu]f\|^p_{L^p(\mathbb{R}^d)}\big)^{1/p} \lesssim_B 2^{-\frac{k}{N}}\|f\|_{L^p(\mathbb{R}^d)}, \qquad 2 \le p \le \infty,$$

which corresponds to (3.4.10). The case $p = \infty$ follows from the estimate $\|\mathcal{A}[\bar{\mathfrak{a}}]f\|_{L^\infty} \le C\delta\|f\|_{L^\infty}$ when $\bar{\mathfrak{a}} \in \bar{\mathfrak{A}}_k(s_0, \delta)$ for some s_0, δ (cf. (3.2.12)). One can obtain this in the same manner as in the proof of Lemma 3.2.4. The case $p = 2$ can be handled similarly as before, using Plancherel's theorem and van der Corput's lemma combined with Lemma 3.4.3 and (3.4.2). $\square$

The proof of Proposition 4.1.4 is similar to that of Proposition 3.4.2. Instead of (2.1.14) we use the estimate (2.1.15), in which the exponent is adjusted to the sharp Sobolev regularity estimate. However, a similar approach breaks down if one tries to obtain the local smoothing estimate (3.0.1) with the optimal regularity $\alpha = 2/p$. To do so, we need the inequality (2.1.7) for $4N - 2 < p \le N(N+1)$. However, there is no such estimate available when $N = 2$.

4.2 Removing ϵ-loss in regularity

Let $\bar{\mathfrak{a}}^\mu \in \bar{\mathfrak{A}}_k(\delta_0\mu, \delta_0)$. For $\nu \in \mathfrak{J}_n^\mu$, set

$$\bar{\mathfrak{a}}_\nu^{\mu,n} = \bar{\mathfrak{a}}^\mu \times \begin{cases} \beta_0\big(\delta_1^{-2N!}\,\bar{\mathfrak{G}}_N^\mu\big)\,\zeta(\delta_1^{-1}s-\nu), & n = 0,\\ \beta_N\big((2^n\delta_1)^{-2N!}\,\bar{\mathfrak{G}}_N^\mu\big)\,\zeta(2^{-n}\delta_1^{-1}s-\nu), & n \ge 1,\end{cases}$$

(see (3.5.22)). Let $\bar{\mathbf{y}}_\mu = (y_\mu^1, \dots, y_\mu^N)$, and let $\bar{\mathcal{D}}_\delta$ denote the $N \times N$ matrix $(\delta^{1-N}\bar{e}_1,\, \delta^{2-N}\bar{e}_2, \dots,\, \delta^0\bar{e}_N)$ where $\bar{e}_j$ is the j-th standard unit vector in $\mathbb{R}^N$. Recalling (3.5.19), we consider a linear map

$$\bar{\Upsilon}_\mu^{\delta_0}(\xi) = \big(2^{-k}\bar{\mathcal{D}}_{\delta_0}\bar{\mathbf{y}}_\mu,\, y_{N+1}, \dots,\, y_d\big).$$

Let $\mathbf{r}$ denote the curve $\mathbf{r}_\circ^N$. Note that (3.5.11) and (3.5.12) hold on $\operatorname{supp}\bar{\mathfrak{a}}_\nu^{\mu,n}$. Similarly as in *Proof of Lemma 3.5.3*, we see $|\langle \bar{\mathbf{y}}_\mu, \mathbf{r}^{(j)}((2^n\delta_1/\delta_0)\nu - \mu)\rangle| \lesssim 2^k(2^n\delta_1/\delta_0)^{N-j}$ for $1 \le j \le N-1$ and $2^{k-2}/B \le \big|\langle \bar{\mathbf{y}}_\mu, \mathbf{r}^{(N)}\rangle\big| \le CB2^k$ on $\operatorname{supp}_\xi \bar{\mathfrak{a}}_\nu^{\mu,n}$. Thus, as before (cf. (3.5.20)), we have

$$\bar{\Upsilon}_\mu^{\delta_0}(\operatorname{supp}_\xi \bar{\mathfrak{a}}_\nu^{\mu,n}) \subset \mathbf{s}\Big(\frac{2^n\delta_1}{\delta_0}\nu - \mu,\, C\frac{2^n\delta_1}{\delta_0},\, CB;\, \mathbf{r}_\circ^N\Big) \times \mathbb{R}^{d-N}$$

for some $C > 0$. Note $\operatorname{supp}\mathcal{F}(\mathcal{A}[\bar{\mathfrak{a}}_\nu^{\mu,n}]f) \subset \operatorname{supp}_\xi \bar{\mathfrak{a}}_\nu^{\mu,n}$. Therefore, changing variables, by (2.1.15) with N replaced by $N-1$ and its cylindrical extension (e.g.,(2.1.16)), we get

$$\Big\| \sum_{\nu \in \mathfrak{J}_n^\mu} \mathcal{A}[\bar{\mathfrak{a}}_\nu^{\mu,n}]f \Big\|_p \le C_0\big(2^n\delta_1/\delta_0\big)^{\frac{N}{p}-1+\epsilon_0}\Big(\sum_{\nu \in \mathfrak{J}_n^\mu} \big\|\mathcal{A}[\bar{\mathfrak{a}}_\nu^{\mu,n}]f\big\|_p^p\Big)^{1/p} \tag{4.2.1}$$

for $2N-2 < p < \infty$ (cf. (3.5.16)). Since $\mathcal{A}[\bar{\mathfrak{a}}^\mu]f = \sum_n \sum_{\nu \in \mathfrak{J}_n^\mu} \mathcal{A}[\bar{\mathfrak{a}}_\nu^{\mu,n}]f$, by Minkowski's inequality and (4.2.1), we have $(\sum_\mu \|\mathcal{A}[\bar{\mathfrak{a}}^\mu]f\|_p^p)^{1/p}$ bounded by

$$\sum_{n \ge 0} \bar{\mathbf{E}}_n := C_0 \sum_{n \ge 0} \big(2^n\delta_1/\delta_0\big)^{\frac{N}{p}-1+\epsilon_0}\Big(\sum_\mu \sum_{\nu \in \mathfrak{J}_n^\mu} \|\mathcal{A}[\bar{\mathfrak{a}}_\nu^{\mu,n}]f\|_p^p\Big)^{1/p}.$$

The proof of Lemma 3.5.2 also shows $C^{-1}\bar{\mathfrak{a}}_\nu^{\mu,n} \in \bar{\mathfrak{A}}_k(2^n\delta_1\nu, 2^n\delta_1)$ for a positive constant C. Therefore, the matter is reduced to obtaining

$$\Big(\sum_\mu \sum_{\nu \in \mathfrak{J}_n^\mu} \|\mathcal{A}[\bar{\mathfrak{a}}_\nu^{\mu,n}]f\|_{L^p(\mathbb{R}^d)}^p\Big)^{1/p} \lesssim_B (2^n\delta_1)^{1-\frac{N}{p}}2^{-\frac{k}{p}}\|f\|_{L^p(\mathbb{R}^d)}, \qquad n \ge 1 \tag{4.2.2}$$

for $p > 2(N-2)$. This gives $\sum_{n \ge 1} \bar{\mathbf{E}}_n \lesssim_B \delta_0^{-N/p+1}2^{-k/p}\|f\|_p$ since $2^n\delta_1 \le C\delta_0$.

The proof of (4.2.2) is similar with that of (3.5.24). Since $C^{-1}\bar{\mathfrak{a}}_\nu^{\mu,n} \in \bar{\mathfrak{A}}_k(2^n\delta_1\nu, 2^n\delta_1)$, we have $\mathcal{A}[\bar{\mathfrak{a}}_\nu^{\mu,n}]f = \mathcal{A}[\bar{\mathfrak{a}}_\nu^{\mu,n}]P_{2^n\delta_1\nu}^{2^n\delta_1}f$. Besides, (3.5.28) or (3.5.29) for some $1 \le j \le N-2$ holds on $\operatorname{supp}\bar{\mathfrak{a}}_\nu^{\mu,n}$. Thus, (3.3.1) holds with $\delta = 2^n\delta_1$ for some B' on $\operatorname{supp}\bar{\mathfrak{a}}_\nu^{\mu,n}$ for $n \ge 1$ (see *Proof of Lemma 3.5.5*). Therefore, applying Lemma 4.1.2 to $\mathcal{A}[\bar{\mathfrak{a}}_\nu^{\mu,n}]f$ and then the assumption (Theorem 4.0.1 with $L = N-1$), we obtain

$$\|\mathcal{A}[\bar{\mathfrak{a}}_\nu^{\mu,n}]f\|_{L^p(\mathbb{R}^d)} \lesssim_B (2^n\delta_1)^{1-\frac{N}{p}} 2^{-\frac{k}{p}} \|P_{2^n\delta_1\nu}^{2^n\delta_1} f\|_p.$$

This combined with (3.5.33) gives (4.2.2) as desired.

4.3 Sharpness of Theorem 1.1.1

Before closing this chapter, we show the optimality of the regularity exponent α in Theorem 1.1.1. This is easier analogue of Proposition 3.7.1 with fixed $t = 1$.

Proposition 4.3.1. *Suppose* (4.0.1) *holds for* $\psi(0) \neq 0$. *Then* $\alpha \le 1/p$.

Proof. We write $\gamma = (\gamma_1, \dots, \gamma_d)$. Via an affine change of variables, we may assume $\gamma_1(0) = 0$ and $\gamma_1'(s) \neq 0$ on an interval $J = [-\delta_0, \delta_0]$ for $0 < \delta_0 \ll 1$. Since $\psi(0) \neq 0$, we may also assume $\psi \ge 1$ on J.

We choose $\zeta_0 \in \mathcal{S}(\mathbb{R})$ such that $\operatorname{supp} \widehat{\zeta_0} \subset [-1, 1]$ and $\zeta_0 \ge 1$ on $[-r_1, r_1]$ where $r_1 = 1 + 2\max\{|\gamma(s)| : s \in J\}$. Denoting $\bar{x} = (x_1, \dots, x_{d-1})$ and $\bar{\gamma}(t) = (\gamma_1(t), \dots, \gamma_{d-1}(t))$, we define

$$\bar{\mathcal{A}}h(x) = \int e^{i\lambda\gamma_d(s)} \zeta_0(x_d - \gamma_d(s)) h(\bar{x} - \bar{\gamma}(s)) \psi(s)\, ds.$$

Let $\zeta \in \mathrm{C}_c^\infty((-2,2))$ be a positive function such that $\zeta = 1$ on $[-1,1]$. For a positive constant $c \ll \delta_0$, let $g_1(\bar{x}) = \sum_{\nu \in \lambda^{-1}\mathbb{Z} \cap [-c,c]} \zeta(\lambda|\bar{x} + \bar{\gamma}(\nu)|)$. We consider

$$g(\bar{x}) = e^{-i\lambda\varphi(x_1)} g_1(\bar{x}),$$

where $\varphi(s) = \gamma_d \circ (-\gamma_1)^{-1}(s)$. We claim that, if c is small enough,

$$|\bar{\mathcal{A}}g(x)| \gtrsim 1, \qquad x \in S_c, \tag{4.3.1}$$

where $S_c = \{x : |\bar{x}| \le c\lambda^{-1},\ |x_d| \le c\}$. To show this, note

$$\bar{\mathcal{A}}g(x) = \int e^{i\lambda(\gamma_d(s) - \varphi(x_1 - \gamma_1(s)))} \zeta_0(x_d - \gamma_d(s)) g_1(\bar{x} - \bar{\gamma}(s))\, \psi(s)\, ds.$$

Let $x \in S_c$. Then, $\operatorname{supp} g_1(\bar{x} - \bar{\gamma}(\cdot)) \subset [-C_1c, C_1c]$ for some $C_1 > 0$. Since $\varphi(s) = \gamma_d \circ (-\gamma_1)^{-1}(s)$, by the mean value theorem we see

$$|\varphi(x_1 - \gamma_1(s)) - \gamma_d(s)| \le r_0 c\lambda^{-1} \tag{4.3.2}$$

where $r_0 = 10r_1 \max\{|\partial_s \varphi(s)| : s \in (-\gamma_1)(J_*)\}$ and $J_* = [-(C_1+1)c, (C_1+1)c]$. Besides, if λ is sufficiently large, $g_1(\bar{x} - \bar{\gamma}(s)) = \sum_{\nu \in \lambda^{-1}\mathbb{Z} \cap [-c,c]} \zeta(\lambda|\bar{x} + \bar{\gamma}(\nu) - \bar{\gamma}(s)|) \gtrsim 1$ if $s \in [-c/2, c/2]$. Since $\operatorname{supp} g_1(\bar{x} - t\bar{\gamma}(\cdot)) \subset J$ with c small enough and $\zeta_0(x_d - \gamma_d(s)) \ge 1$, we get $\int \zeta_0(x_d - \gamma_d(s)) g_1(\bar{x} - \bar{\gamma}(s))\psi(s)\, ds \gtrsim 1$. Therefore, (4.3.1) follows by (4.3.2) if c is small enough, i.e., $c \ll 1/(3r_0)$.

We set $f(x) = e^{-i\lambda x_d}\zeta_0(x_d) g(\bar{x})$. Then, $\mathcal{A}f(x) = e^{-i\lambda x_d}\bar{\mathcal{A}}g(x)$. By our choice of ζ_0, $\operatorname{supp} \widehat{f} \subset \{\xi : |\xi_d + \lambda| \le 1\}$, so $\operatorname{supp} \mathcal{F}(\mathcal{A}f) \subset \{\xi : |\xi_d + \lambda| \le 1\}$. This gives

$$\lambda^\alpha \|\mathcal{A}f\|_{L^p(\mathbb{R}^{d+1})} \lesssim \|\mathcal{A}f\|_{L^p_\alpha(\mathbb{R}^{d+1})}. \tag{4.3.3}$$

From (4.3.1) we have $\|\mathcal{A}f\|_p = \|\bar{\mathcal{A}}g\|_p \ge C\lambda^{-(d-1)/p}$. Note that $\operatorname{supp} g$ is contained in a $O(\lambda^{-1})$-neighborhood of $-\bar{\gamma}$, so it follows that $\|f\|_p \lesssim \lambda^{-(d-2)/p}$. Therefore, by (4.3.3) the inequality (4.0.1) implies $\lambda^\alpha \lambda^{-(d-1)/p} \lesssim \lambda^{-(d-2)/p}$. Taking $\lambda \to \infty$ gives $\alpha \le 1/p$. $\square$

4.4 Finite type curves

Before closing this chapter, we show the corollaries for curves of maximal type based on the argument in [44]. Let γ be a smooth curve of maximal type ℓ. For fixed point s_0, we can find integers n_i and orthonormal vectors $\{v_i\}_{i=1,\dots,d}$ satisfying $n_1 < \cdots < n_d$,

$$\langle \gamma^{(j)}(s_0), v_i \rangle = 0, \text{ for } 1 \le j < n_i, \quad \langle \gamma^{(n_i)}(s_0), v_i \rangle \ne 0.$$

After rotation, we can assume that

$$\gamma(s + s_0) = \gamma(s_0) + (s^{n_1} g_1(s), \cdots, s^{n_d} g_d(s))$$

where g_i are smooth function with $g_i(0) \ne 0$. There exist $\rho_{s_0} \in (0,1)$ such that $g_i(s) \ne 0$ for $|s| \le \rho_{s_0}$. It is enough to consider that

$$\mathcal{A}_t[\psi_{s_0}]f(x) = \int f(x - t\gamma(s))\psi_{s_0}(s) ds$$

where $\psi_{s_0} \in C_0^\infty(s_0 - \rho_{s_0}, s_0 + \rho_{s_0})$. Using dyadic decomposition in s, we have $\mathcal{A}_t[\psi_{s_0}]f = \sum_{j \ge 0} \mathcal{A}_t[\psi_{s_0}\beta(2^j|s - s_0|)]f$. Here, $\beta \in C_0^\infty((1/2, 2))$ such that $\sum_j \beta(2^j|\cdot|) \equiv 1$. After translation and scaling, we have $\mathcal{A}_t[\psi_{s_0}]f = \sum_{j\geq 0} \mathcal{A}_{t,j} f$ where

$$\mathcal{A}_{t,j} f(x) := 2^{-j} \int f(x - t\gamma(2^{-j}s + s_0))\psi_{s_0}(2^{-j}s + s_0)\beta(|s|)ds.$$

By D_j we denote the $d \times d$ matrix $(2^{jn_1}e_1, \cdots, 2^{jn_d}e_d)$ where e_i is the i-th standard unit vector in $\mathbb{R}^d$. Define $f_j(x) = f(D_j x)$ then we have

$$\mathcal{A}_{t,j} f(D_{-j}x) := 2^{-j} \int f_{-j}(x - tD_j\gamma(2^{-j}s+s_0))\psi_{s_0}(2^{-j}s+s_0)\beta(|s|)ds. \quad (4.4.1)$$

Now set

$$\gamma_j = (s^{n_1}g_1(2^{-j}s), \cdots, s^{n_d}g_d(2^{-j}s)).$$

Then $D_j\gamma(2^{-j}s+s_0) = D_j\gamma(s_0)+\gamma_j(s)$ and γ_j is nondegenerate for sufficiently large j for $1/2 \leq |s| \leq 2$.

Proof of Corollary 1.1.5,1.1.6 and 1.1.7. By (4.4.1),

$$\|\mathcal{A}_{t,j} f\|_{L^p_\alpha} \lesssim 2^{(n_d\alpha-1)j}|D_{-j}|^{1/p}\|\mathcal{A}_t[\gamma_j, \psi_j]f_{-j}\|_{L^p_\alpha}$$

where $\psi_j(s) = \psi_{s_0}(2^{-j}s + s_0)\beta(|s|)$ and $\mathcal{A}_t[\gamma, \psi]f = \int f(\cdot - t\gamma(s))\psi(s)ds$. Firstly, since γ_j is nondegenerate in the support of ψ_j, theorem 1.1.1 implies that if $p > 2d-2$ we have

$$\|\mathcal{A}_{1,j} f\|_{L^p_{1/p}} \lesssim 2^{(n_d/p-1)j}|D_{-j}|^{1/p}\|f_{-j}\|_p \lesssim 2^{(n_d/p-1)j}\|f\|_p.$$

Thus, if $p > \max\{2d-2, \ell\}$ we have $\|\mathcal{A}_1[\psi_{s_0}]f\|_{L^p_{1/p}} \lesssim \|f\|_p$ which implies corollary 1.1.5.

Secondly, theorem 1.1.2 implies that if $p > 2d-2$ there exist $\alpha > 1/p$ satisfying

$$\|\chi(t)\mathcal{A}_{t,j} f\|_{L^p_\alpha(\mathbb{R}^{d+1})} \lesssim 2^{(n_d\alpha-1)j}|D_{-j}|^{1/p}\|f_{-j}\|_p \lesssim 2^{(n_d\alpha-1)j}\|f\|_p.$$

We can conclude that if $p > \max\{2d-2, \ell\}$ there exist $1/p < \alpha < 1/n_d$ satisfying $\|\chi(t)\mathcal{A}_t[\psi_{s_0}]f\|_{L^p_\alpha(\mathbb{R}^{d+1})} \lesssim \|f\|_p$ which implies corollary 1.1.6 and 1.1.7. $\square$

Chapter 5

Maximal estimate

As mentioned before, Theorem 1.1.2 implies Theorem 1.1.4. In this chapter, we concentrate on proving Theorem 1.1.3 which is the optimal maximal estimate associated to averages over curves in $\mathbb{R}^3$. We assume that the curve γ has nonvanishing curvature and torsion, equivalently,

$$\det(\gamma'(s), \gamma''(s), \gamma'''(s)) \neq 0 \tag{5.0.1}$$

for $s \in I = [-1, 1]$. And set

$$Af(x,t) := \mathcal{A}_t f(x), \;\; Mf = \sup_{t>0} Af(x,t).$$

The condition is the natural nondegeneracy condition which is commonly used in the studies related to space curves and the most typical examples are the helix and the moment curve (s, s^2, s^3). Recall that Theorem 1.1.3 is:

Theorem 5.0.1. *Suppose that $\gamma : I \to \mathbb{R}^3$ is a smooth curve which has nonvanishing curvature and torsion, and ψ is a nontrivial, nonnegative, smooth function supported in $(-1, 1)$. Then, there is a constant C such that*

$$\|Mf\|_{L^p(\mathbb{R}^3)} \leq C\|f\|_{L^p(\mathbb{R}^3)} \tag{5.0.2}$$

for all $f \in L^p(\mathbb{R}^3)$ if and only if $p > 3$.

The assumption that ψ is smooth is not necessary and it is clear that the theorem holds for a continuous ψ. Even though γ is assumed to be smooth, there is a positive integer D such that (5.0.2) holds for $\gamma \in \mathrm{C}^D(I)$ (see Remark 1).

Our argument in this chapter is closely related to the induction strategy developed by Ham and Lee [22]. They obtained the sharp adjoint restriction estimate for the space curve in $L^p(\mu)$ when μ is an α-dimensional measure (see Section 5.1.1 for the definition). The work was in turn inspired by the multilinear approach due to Bourgain and Guth [13]. Main novelty of the current paper lies in devising an induction argument which directly works for the maximal operator. In contrast to the adjoint restriction operator a suitable form of multilinear estimate is not so obvious for the averaging operator A. In order to prove a multilinear estimate for A which enjoys better boundedness property under a certain additional assumption, we first express the operator A as a sum of adjoint restriction operators and then relate them to geometry of the curves so that the transversality condition can be reformulated in terms of the relative positions between the associated curves. Unfortunately, some of the consequent adjoint restriction operators are associated to $\mathrm{C}^{1,1/2}$ surfaces but not to C^2 surfaces, so we can not directly apply the multilinear restriction estimate which is due to Bennett, Carbery, and Tao [9]. However, it is not difficult to see that the argument in [9] continues to work for the $\mathrm{C}^{1,1/2}$ surfaces (see Theorem 2.2.3). We also make use of some of the results from [44] to strengthen the multilinear estimate and also to deal with the nondegenerate part, whereas the difficult degenerate part is to be handled by the multilinear estimate which we prove in Section 5.4, 5.5.

Structure of the chapter. In Section 5.1 we show that the maximal estimate can be deduced from a form of weighted estimates. And we formalize the induction setup to prove the weighted estimates in Section 5.2. In Section 5.4, 5.5 we obtain a weighted multilinear estimate for A under a certain separation condition. In Section 5.6 we establish the maximal bound putting the previous estimates together and show the optimality of the range of p in Section 5.7.

5.1 Connection with α-dimensional weight

In this section we reduce the proof of maximal estimate to showing a form of weighted estimates for the averaging operators.

By the argument in [10] (also see [47]) which relies on Littlewood-Paley decomposition and scaling one can obtain the maximal estimate (5.0.2) from that for $\sup_{1\le t\le 2}\big|Af(x,t)\big|$. More precisely, it is sufficient to show that there is an $\varepsilon_p > 0$ such that

$$\|Af\|_{L^p_x L^\infty_t(\mathbb{R}^3\times[1,2])} \le C\lambda^{-\varepsilon_p}\|f\|_{L^p(\mathbb{R}^3)} \tag{5.1.1}$$

for all $f \in \mathcal{S}(\mathbb{R}^3)$ whenever

$$\operatorname{supp}\widehat{f} \subset \mathbb{A}_\lambda := \{\xi \in \mathbb{R}^3 : 3\lambda/4 \le |\xi| \le 7\lambda/4\},\ \ \lambda \ge 1. \tag{5.1.2}$$

For the rest of the paper, we assume (5.1.2) unless it is mentioned otherwise.

Notation. Throughout the paper C, $C_1, \dots$ and c are supposed to be independent positive constants, and C_ε, C_δ are constants depending on ε, δ but all of these constants may vary at each appearance. In addition to the conventional notation $\widehat{\cdot}$ we use $\mathcal{F}$ and $\mathcal{F}^{-1}$ to denote the Fourier and inverse Fourier transforms, respectively. By $Q_1 = \mathcal{O}(Q_2)$ we denote $|Q_1| \le CQ_2$ for a constant C and we also use the notation $Q_1 = \mathcal{O}_s(Q_2)$ if $|Q_1| \le Q_2$.

5.1.1 Estimate with α-dimensional measure

Let $\mathbb{B}^d(z,r)$ denote the ball of radius r which is centered at $z \in \mathbb{R}^d$. Let μ be a positive Borel measure on $\mathbb{R}^4$. For $0 < \alpha \le 4$ we say μ is α-dimensional if there is a constant C such that

$$\mu(\mathbb{B}^4(z,r)) \le Cr^\alpha$$

for all $r > 0$ and $z \in \mathbb{R}^4$. For an α-dimensional measure μ we define

$$\langle\mu\rangle_\alpha = \sup_{z\in\mathbb{R}^4, r>0} r^{-\alpha}\mu(\mathbb{B}^4(z,r)).$$

Instead of directly proving the maximal estimate (5.1.1) we obtain estimates for Af with α-dimensional measures. From those estimates we can deduce the estimate (5.1.1). As far as the authors are aware, it seems that this type of argument deducing the maximal estimate from the estimates with α-dimensional measures first appeared in [37]. (See also [62, p.1283] for a related discussion.)

Theorem 5.1.1. *Let μ be 3-dimensional. Suppose that $\gamma : I \to \mathbb{R}^3$ is a smooth curve satisfying* (5.0.1)*. Then, for $p > 3$ there is an $\varepsilon_p > 0$ such that*

$$\|Af\|_{L^p(\mathbb{R}^3\times[1,2],d\mu)} \le C\langle\mu\rangle_3^{\frac{1}{p}}\lambda^{-\varepsilon_p}\|f\|_{L^p(\mathbb{R}^3)} \tag{5.1.3}$$

whenever $\widehat{f}$ is supported on $\mathbb{A}_\lambda$.

We shall work only with 3-dimensional measures even though it is possible to prove such estimates with α-dimensional measure, $\alpha \neq 3$ on a certain range of p (see Remark 2). The following shows the estimate (5.1.3) implies (5.1.1).

Lemma 5.1.2. *Suppose* (5.1.3) *holds true for any* 3*-dimensional measure* μ*. Then the estimate* (5.1.1) *holds.*

To prove this, we start with an elementary lemma.

Lemma 5.1.3. *Let* $\eta \in \mathrm{C}_0^\infty([2^{-3}, 2^3])$ *and* $\psi \in \mathrm{C}_0^\infty(I)$*. Set* $r_0 = 4\max\{|\gamma(s)| : s \in \mathrm{supp}\psi\} + 1$ *and*

$$K_\eta(x,t) = (2\pi)^{-3} \iint e^{i(x\cdot\xi - t\gamma(s)\cdot\xi)}\psi(s)\, ds\, \eta(\lambda^{-1}|\xi|)\, d\xi.$$

If $|x| \geq r_0$ *and* $|t| \leq 2$*, then* $|K_\eta(x,t)| \leq C\|\eta\|_{\mathrm{C}^{2N+3}} E_N(x)$ *for any* $N \geq 1$ *where* $E_N(x) := \lambda^{-N}(1+|x|)^{-N}$*.*

Changing variables we note $K_\eta(x,t) = \frac{\lambda^3}{(2\pi)^3}\iint e^{i\lambda(x\cdot\xi - t\gamma(s)\cdot\xi)}\psi(s)\, ds\, \eta(|\xi|)\, d\xi$. Then repeated integration by parts in ξ gives the desired estimate since $|\nabla_\xi(x\cdot\xi - t\gamma(s)\cdot\xi)| \geq 2^{-1}|x|$ if $|x| \geq r_0$ and $|t| \leq 2$.

Proof of Lemma 5.1.2. To obtain (5.1.1) it suffices to show the local estimate

$$\|Af\|_{L^p_x L^\infty_t(\mathbb{B}^3(0,1)\times[1,2])} \leq C\lambda^{-\varepsilon_p}\|f\|_{L^p(\mathbb{R}^3)}. \tag{5.1.4}$$

This is obvious if $\widehat{f}$ is not assumed to be supported in $\mathbb{A}_\lambda$. However, we may handle f as if it were supported on a ball of radius r_0. Since $\mathrm{supp}\,\widehat{f} \subset \mathbb{A}_\lambda$, $Af(\cdot,t) = K_\eta(\cdot,t) * f$ for an η such that $\eta \in \mathrm{C}_c^\infty((2^{-1},2))$ and $\eta = 1$ on $[3/4, 7/4]$. So, Lemma 5.1.3 gives $|K_\eta(x,t)| \leq CE_N(x)$ if $|x| \geq r_0$ and $|t| \leq 2$. Thus, by the typical localization argument (e.g., see the proof of Lemma 5.5.4) one can easily see that (5.1.4) implies (5.1.1).

In order to prove (5.1.4), using the Kolmogorov-Seliverstov-Plessner linearization, it is enough to show

$$\|Af(\cdot,\mathbf{t}(\cdot))\|_{L^p(\mathbb{B}^3(0,1))} \leq C\lambda^{-\varepsilon_p}\|f\|_{L^p(\mathbb{R}^3)} \tag{5.1.5}$$

for a measurable function $\mathbf{t} : \mathbb{B}^3(0,1) \to [1,2]$ with C independent of $\mathbf{t}$. Since $\widehat{f}$ is supported in $\mathbb{A}_\lambda$, Af is uniformly continuous on every compact subset. So, for (5.1.4) we may assume that $\mathbf{t}$ is continuous. With a continuous function

$\mathbf{t}$, the positive linear functional $\mathrm{C}_c(\mathbb{R}^4) \ni F \mapsto \int_{\mathbb{B}^3(0,1)} F(x, \mathbf{t}(x))dx$ defines a measure μ^* by the relation

$$\int F(x,t)\, d\mu(x,t) = \int_{\mathbb{B}^3(0,1)} F(x, \mathbf{t}(x))\, dx, \quad F \in \mathrm{C}_c(\mathbb{R}^4).$$

We now notice that μ is a 3-dimensional measure. Since $\mathbb{B}^4((x_\circ, t_\circ), r) \subset \{(x,t) \in \mathbb{R}^3 \times \mathbb{R} : |x - x_\circ| \le r\}$,

$$\mu\big(\mathbb{B}^4((x_\circ, t_\circ), r)\big) = \int_{\mathbb{B}^3(0,1)} \chi_{\mathbb{B}^4((x_\circ,t_\circ),r)}(x, \mathbf{t}(x))\, dx \le \int \chi_{\mathbb{B}^3(x_\circ,r)}(x)\, dx = \frac{4}{3}\pi r^3$$

for any $r > 0$ and $(x_\circ, t_\circ) \in \mathbb{R}^3 \times \mathbb{R}$. Thus we have $\langle \mu \rangle_3 \le 4\pi/3$. Noting $\|Af(\cdot, \mathbf{t}(\cdot))\|_{L^p(\mathbb{B}^3(0,1))} = \|Af\|_{L^p(d\mu)}$, we apply Theorem 5.1.1 and get (5.1.5) with C independent of $\mathbf{t}$. $\square$

5.1.2 Weighted estimate

For $0 < \alpha \le 4$ let us denote by Ω^α the collection of nonnegative measurable functions ω on $\mathbb{R}^4$ such that the measure $\omega\, dxdt$ is α-dimensional. For a simpler notation we denote

$$[\omega]_\alpha = \langle \omega\, dxdt \rangle_\alpha$$

for $\omega \in \Omega^\alpha$. Even though Ω^α is properly contained in the set of α-dimensional measures, the fact that $\operatorname{supp} \widehat{f} \subset \mathbb{A}_\lambda$ allows us to recover the estimate (5.1.3) from an estimate against $\omega \in \Omega^\alpha$.

Lemma 5.1.4. *Let $\tilde{I} = [2^{-1}, 2^2]$. Suppose that*

$$\|Af\|_{L^p(\mathbb{R}^3 \times \tilde{I}, \omega)} \le C[\omega]_3^{\frac{1}{p}} \lambda^{-\varepsilon_p} \|f\|_{L^p(\mathbb{R}^3)} \tag{5.1.6}$$

whenever $\omega \in \Omega^3$ and $\widehat{f}$ is supported on $\mathbb{A}_\lambda$, then (5.1.3) *holds for any* 3-*dimensional measure μ.*

The proof of the maximal estimate (5.1.1) is now reduced to showing (5.1.6). Lemma 5.1.4 of course remains valid for any $\alpha \in (0, 4]$.

To show Lemma 5.1.4 we make use of the next two lemmas: Lemma 5.1.5 and 5.1.6. The former can be shown following the standard argument (for example, see [33, pp. 47–49]), so we omit the proof.

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*In fact, μ becomes a regular Borel measure by the Riesz-Markov-Kakutani representation theorem.

Lemma 5.1.5. *Let $0 < \alpha \le 4$ and $\varphi \in \mathcal{S}(\mathbb{R}^4)$. Set $\varphi_\lambda = \lambda^4\varphi(\lambda\,\cdot)$. If μ is an α-dimensional measure, then $|\varphi|_\lambda * \mu \in \Omega^\alpha$ and $[|\varphi|_\lambda * \mu]_\alpha \le C_\varphi \langle\mu\rangle_\alpha$.*

In what follows $\widetilde{\chi}$ denotes a function in $\mathrm{C}_0^\infty(\tilde{I})$ which satisfies $\widetilde{\chi} = 1$ on $[1,2]$, and β, β_0 respectively denote the functions such that $\beta \in \mathrm{C}_0^\infty([2^{-1},2])$, $\beta = 1$ on $[3/4, 7/4]$; $\beta_0 \in \mathrm{C}_0^\infty([-2,2])$, $\beta_0 = 1$ on $[-1,1]$.

Lemma 5.1.6. *Let $r_0 = 1 + 4\max\{|\gamma(s)| : s \in \mathrm{supp}\psi\}$ and let*

$$m(\xi,\tau) = \iint \widetilde{\chi}(t) e^{-it(\tau+\gamma(s)\cdot\xi)}\psi(s)\, ds dt\, \beta(\lambda^{-1}|\xi|), \quad (\xi,\tau) \in \mathbb{R}^3 \times \mathbb{R}.$$

Then, we have $|\mathcal{F}^{-1}\big(m(\xi,\tau)(1-\beta_0((\lambda r_0)^{-1}\tau))\big)| \le C_N\|\psi\|_\infty \widetilde{E}_t^N$ for any $N > 0$ where $\widetilde{E}_t^N := (1+|t|)^{-N}E_N$.

Proof. Let $\rho_\ell(t) = (-it)^{k+\ell}\widetilde{\chi}(t)$ and note that $\partial_\xi^\alpha\partial_\tau^k m(\xi,\tau)$ is a sum of the terms $\int \widehat{\rho}_{|\alpha_1|}(\tau+\gamma(s)\cdot\xi)(\gamma(s))^{\alpha_1}\psi(s)\, ds \times \mathcal{O}(\lambda^{-|\alpha_2|})$ with $\alpha_1+\alpha_2 = \alpha$. Thus we have $|\partial_\xi^\alpha\partial_\tau^k m(\xi,\tau)| \le C_N\|\psi\|_\infty r_0^{|\alpha|}(r_0\lambda)^{-N}(1+|\tau|)^{-N}$ for any N if $|\tau| \ge r_0\lambda$, and we get the desired estimate by routine integration by parts. □

Proof of Lemma 5.1.4. We define an auxiliary operator $\widetilde{A}$ by

$$\mathcal{F}(\widetilde{A}h)(\xi,\tau) = \beta_0((\lambda r_0)^{-1}\tau)\mathcal{F}\big(\widetilde{\chi}(t)Ah\big)(\xi,\tau).$$

Since $\widehat{f}$ is supported in $\mathbb{A}_\lambda$, we have $|(\widetilde{\chi}(t)A - \widetilde{A})f| \le C\widetilde{E}_t^N * |f|$ by Lemma 5.1.6. We then note that $\int \widetilde{E}_t^N(x-y)d\mu(x,t) \le C\lambda^{-N}\langle\mu\rangle_3$ and $\int \widetilde{E}_t^N(x-y)dy \le C\lambda^{-N}$. Thus by Schur's test we get

$$\|\widetilde{E}_t^N * f\|_{L^p(\mathbb{R}^3\times\mathbb{R},d\mu)} \le C\langle\mu\rangle_3^{\frac{1}{p}}\lambda^{-N}\|f\|_{L^p(\mathbb{R}^3)} \tag{5.1.7}$$

for $1 \le p \le \infty$ and a large N. So, in order to obtain (5.1.3), it suffices to prove

$$\|\widetilde{A}f\|_{L^p(\mathbb{R}^3\times[1,2],d\mu)} \le C\langle\mu\rangle_3^{\frac{1}{p}}\lambda^{-\varepsilon_p}\|f\|_{L^p(\mathbb{R}^3)}. \tag{5.1.8}$$

Since the space time Fourier transform of $\widetilde{A}f$ is supported in $\mathbb{B}^4(0, 2^2r_0\lambda)$, $\widetilde{A}f = \widetilde{A}f * \varphi_{r_0\lambda}$ for some $\varphi \in \mathcal{S}(\mathbb{R}^4)$, which gives $|\widetilde{A}f|^p \le C|\widetilde{A}f|^p * |\varphi_{r_0\lambda}|$ via Hölder's inequality. Thus we have

$$\|\widetilde{A}f\|_{L^p(\mathbb{R}^3\times[1,2],d\mu)} \le C\|\widetilde{A}f\|_{L^p(\mathbb{R}^3\times\mathbb{R},\omega)},$$

where we set $\omega = |\varphi_{r_0\lambda}| * \mu$. Therefore, using $|(\widetilde{\chi}(t)A - \widetilde{A})f| \le C\widetilde{E}_t^N * |f|$ again, we have only to obtain the estimate for $\widetilde{\chi}(t)Af$ in $L^p(\mathbb{R}^3\times\mathbb{R},\omega)$ since the minor part can be handled as before. Since $[\omega]_3 \le C\langle\mu\rangle_3$ by Lemma 5.1.5, the estimate (5.1.8) follows from (5.1.6) because $\mathrm{supp}\,\widetilde{\chi} \subset \tilde{I}$. □

5.2 Reduction after normalization

In order to prove the estimate (5.1.6), as mentioned before, we use an induction type argument over a class of curves. For the purpose we need to normalize the curves properly so that the induction assumption applies. This step is important especially for defining the induction quantity and proving uniform estimates (cf. [22, 29]).

5.2.1 Normalization of curves and weights

Let $D \ge 2^5$ be a positive integer which is taken to be large. Let $\gamma \in \mathrm{C}^D(I)$ which satisfies (5.0.1). Then, for $s_\circ$ and $0 < \delta \ll 1$ such that $[s_\circ - \delta, s_\circ + \delta] \subset I$, we define

$$\mathrm{M}^\delta_\gamma(s_\circ) = \big(\delta\gamma'(s_\circ), \delta^2\gamma''(s_\circ), \delta^3\gamma'''(s_\circ)\big)$$

and

$$\gamma^\delta_{s_\circ}(s) = (\mathrm{M}^\delta_\gamma(s_\circ))^{-1}\big(\gamma(\delta s + s_\circ) - \gamma(s_\circ)\big). \tag{5.2.1}$$

Let $\gamma_\circ(s) = (s, s^2/2!, s^3/3!)$. We consider a class of curves which are small perturbations of the curve $\gamma_\circ$ in $\mathrm{C}^D(I)$. For $\varepsilon_\circ > 0$, we set

$$\mathfrak{C}^D(\varepsilon_\circ) = \big\{\gamma \in \mathrm{C}^D(I) : \ \|\gamma - \gamma_\circ\|_{\mathrm{C}^D(I)} \le \varepsilon_\circ\big\}.$$

Using an affine map, one can transform a small enough sub-curve of any $\gamma \in \mathrm{C}^D(I)$ satisfying (5.0.1) so as to be contained in $\mathfrak{C}^D(\varepsilon_\circ)$. The following lemma is a slight modification of [22, Lemma 2.1].

Lemma 5.2.1. *Let $s_\circ \in (-1, 1)$ and $\gamma \in \mathrm{C}^D(I)$ satisfying (5.0.1) on I. Then, for any $\varepsilon_\circ > 0$, there exists $\delta_* = \delta_*(\varepsilon_\circ, \gamma) > 0$ such that $\gamma^\delta_{s_\circ} \in \mathfrak{C}^D(\varepsilon_\circ)$ whenever $[s_\circ - \delta, s_\circ + \delta] \subset I$ and $|\delta| \le \delta_*$. Additionally, if $\gamma \in \mathfrak{C}^D(\varepsilon_\circ)$ and $\varepsilon_\circ < 2^{-5}$, there is a uniform $\delta_\circ > 0$ such that $\gamma^\delta_{s_\circ} \in \mathfrak{C}^D(\varepsilon_\circ)$ whenever $[s_\circ - \delta, s_\circ + \delta] \subset I$ with $|\delta| \le \delta_\circ$.*

For a matrix M we denote $\|\mathrm{M}\| = \sup_{|z|=1} |\mathrm{M}z|$.

Proof. By Taylor's expansion, we have

$$\begin{aligned}\gamma(\delta s + s_\circ) - \gamma(s_\circ) &= \delta\gamma'(s_\circ)s + \delta^2\gamma''(s_\circ)\frac{s^2}{2} + \delta^3\gamma'''(s_\circ)\frac{s^3}{3!} + \widetilde{R}(s_\circ, \delta, s) \\ &= \mathrm{M}^\delta_\gamma(s_\circ)\gamma_\circ(s) + \widetilde{R}(s_\circ, \delta, s)\end{aligned}$$

and $\|\widetilde{R}(s_\circ,\delta,\cdot)\|_{\mathrm{C}^D(I)} \le C\delta^4$. By (5.2.1), $\gamma^\delta_{s_\circ}(s) = \gamma_\circ(s) + (\mathrm{M}^\delta_\gamma(s_\circ))^{-1}\widetilde{R}(s_\circ,\delta,s)$. Since $\|(\mathrm{M}^\delta_\gamma(s_\circ))^{-1}\| \le C_1\delta^{-3}$ for a constant C_1, taking a positive δ_* such that $CC_1\delta_* \le \varepsilon_\circ$ we have $\|(\mathrm{M}^\delta_\gamma(s_\circ))^{-1}\widetilde{R}(s_\circ,\delta,\cdot)\|_{\mathrm{C}^D(I)} \le \varepsilon_\circ$ and, hence, $\gamma^\delta_{s_\circ} \in \mathfrak{C}^D(\varepsilon_\circ)$ for $0<\delta\le\delta_*$. The second assertion can also be shown in the same manner, so we omit the detail. □

For $\delta>0$ we denote by D_δ the diagonal matrix $(\delta e_1, \delta^2 e_2, \delta^3 e_3)$. To normalize the weights we need the next lemma which can be shown by the argument in [22].

Lemma 5.2.2. *Let $0<\alpha\le 4$, $0<\delta\ll 1$ and $\omega\in\Omega^\alpha$, and let M be a 4×4 nonsingular matrix. Set $\omega^\delta(x,t) = \omega\big(\mathrm{D}_\delta x, t\big)$ and $\omega_{\mathrm{M}}(x,t) = \omega\big(\mathrm{M}(x,t)\big)$. Then for a constant C independent of ω and δ we have*

$$[\omega^\delta]_\alpha \le C\delta^{3\alpha-12}[\omega]_\alpha, \tag{5.2.2}$$

$$[\omega_{\mathrm{M}}]_\alpha \le |\det \mathrm{M}|^{-1}\|\mathrm{M}\|^\alpha[\omega]_\alpha. \tag{5.2.3}$$

Proof. The inequality (5.2.2) is equivalent to

$$\int_{\mathbb{B}^4(y,r)} \omega(\mathrm{D}_\delta x,t)\,dxdt \le C\delta^{3\alpha-12}[\omega]_\alpha r^\alpha$$

for $y\in\mathbb{R}^4$ and $r>0$. To see this, changing variables $x\to \mathrm{D}_\delta^{-1}\mathrm{x}$, the left hand side is equal to $\delta^{-6}\int \chi_{\mathbb{B}^4(y,r)}(\mathrm{D}_\delta^{-1}x,t)\,\omega(x,t)\,dxdt$. Then we note that the set $\{(x,t) : (\mathrm{D}_\delta^{-1}x,t)\in\mathbb{B}^4(y,r)\}$ is contained in a rectangle $\mathcal{R}_\delta$ of dimensions about $\delta r\times\delta^2 r\times\delta^3 r\times r$. Since $\mathcal{R}_\delta$ is covered by at most $C\delta^{-6}$ many balls of radius $\delta^3 r$, the inequality follows.

For (5.2.3) we only have to show

$$\int_{\mathbb{B}^4(y,r)} \omega(\mathrm{M}(x,t))\,dxdt \le |\det\mathrm{M}|^{-1}\|\mathrm{M}\|^\alpha[\omega]_\alpha r^\alpha$$

for $y\in\mathbb{R}^4$ and $r>0$. Changing variables, we see that the left hand side equals $|\det\mathrm{M}|^{-1}\int\chi_{\mathbb{B}^4(y,r)}(\mathrm{M}^{-1}(x,t))\omega(x,t)dxdt$. So, we get the inequality since $(x,t)\in\mathbb{B}^4(\mathrm{M}y,\|\mathrm{M}\|r)$ if $\mathrm{M}^{-1}(x,t)\in\mathbb{B}^4(y,r)$. □

5.2.2 Induction quantity

Throughout the chapter we fix a small positive constant $c_\circ$. To show (5.1.6) for a smooth curve satisfying (5.0.1), it is sufficient to handle $\gamma\in\mathfrak{C}^D(\varepsilon_\circ)$

with a small $\varepsilon_\circ > 0$ while $\psi \in \mathrm{C}^D$ and $\operatorname{supp}\psi \subset [-c_\circ, c_\circ]$. As we shall see later, this can be shown by a finite decomposition and changing variables via affine transformation.

Definition 5.2.3. *Let $c_\circ$, $\varepsilon_\circ$ and δ be the numbers such that $0 < c_\circ \le 2^{-10}$, $0 < \varepsilon_\circ \le c_\circ^2$, and*

$$0 < \delta \le \min(c_\circ, \delta_\circ) \tag{5.2.4}$$

where $\delta_\circ$ is given in Lemma 5.2.1. The number δ is to be chosen later (see Section 5.6). We also denote $J_\circ = [-c_\circ, c_\circ]$ and

$$\mathfrak{J}(\delta) = \big\{J : J = [c_\circ\delta(k-1), c_\circ\delta(k+1)],\ k \in \mathbb{Z},\ |k| \le (c_\circ\delta)^{-1} + 1\big\},$$

so that the intervals in $\mathfrak{J}(\delta)$ cover I. For each $J \in \mathfrak{J}(\delta)$ we define $\mathfrak{N}^D(J)$ to be the set of functions such that $\psi \in \mathrm{C}_0^D(J)$ and $\|\psi(|J|\,\cdot)\|_{\mathrm{C}^D(\mathbb{R})} \le 1$. For a given interval J we denote by ψ_J a function in $\mathfrak{N}^D(J)$.

For a smooth function a on $I \times \tilde I \times \mathbb{A}_\lambda$, following [44], we define an integral operator by setting

$$A^\gamma[a]f(x,t) = (2\pi)^{-3} \iint e^{i(x - t\gamma(s))\cdot\xi} a(s,t,\xi)\, ds \widehat{f}(\xi)\, d\xi. \tag{5.2.5}$$

In particular, we note $Af = A^\gamma[\psi]f$ as is clear by Fourier inversion.

Let us take $\zeta \in \mathrm{C}_0^\infty([-1,1])$ such that $\zeta \ge 0$ and $\sum_{k\in\mathbb{Z}} \zeta(s-k) = 1$. For an interval J we denote by c_J the center of J and set $\zeta_J(s) = \zeta(2(s - c_J)/|J|)$. Consequentially, $\zeta_J \in \mathrm{C}_0^\infty(J)$ and $\sum_{J\in\mathfrak{J}(\delta)} \zeta_J(s) = 1$ for $s \in I$. As a result, we have

$$A^\gamma[\psi]f(x,t) = \sum_{J\in\mathfrak{J}(\delta)} A^\gamma[\psi\zeta_J]f(x,t) \tag{5.2.6}$$

if $\operatorname{supp}\psi \subset I_\circ$. The following is one of the key lemmas which relates the estimate for the average over a short curve to that over a larger one.

Lemma 5.2.4. *Let $I' \subset \tilde I$ be an interval, and let $\omega \in \Omega^3$, $J = [s_\circ - c_\circ\delta, s_\circ + c_\circ\delta] \in \mathfrak{J}(\delta)$ and $\psi_J \in \mathfrak{N}^D(J)$. Suppose that $\gamma \in \mathrm{C}^D(I)$ satisfies (5.0.1) and $\operatorname{supp}\widehat{f} \subset \mathbb{A}_\lambda$. Then, there are $\widetilde\omega \in \Omega^3$, $\widetilde f$ with $\|\widetilde f\|_p = \|f\|_p$, and $\psi_{J_\circ} \in \mathfrak{N}^D(J_\circ)$ which satisfy the following:*

$$\|A^\gamma[\psi_J]f\|_{L^p(\mathbb{R}^3\times I',\omega)} = \delta^{1-\frac{3}{p}} \|A^{\gamma^\delta_{s_\circ}}[\psi_{J_\circ}]\widetilde f\|_{L^p(\mathbb{R}^3\times I',\widetilde\omega)}, \tag{5.2.7}$$

$$[\widetilde\omega]_3 \le C(1 + |\gamma(s_\circ)|)^3 |\det \mathrm{M}^1_\gamma(s_\circ)|^{-1} (1 + \|\mathrm{M}^1_\gamma(s_\circ)\|)^3 [\omega]_3, \tag{5.2.8}$$

and

$$\operatorname{supp}\mathcal{F}(\widetilde{f}\,) \subset \Big\{\xi : \frac{3}{4}d_*\delta^3\lambda \le |\xi| \le \frac{7}{4}d^*\delta\lambda\Big\}, \tag{5.2.9}$$

where $1/d_* = \|(\mathrm{M}^1_\gamma(s_\circ))^{-t}\|$ *and* $1/d^* = \inf_{|z|=1}|(\mathrm{M}^1_\gamma(s_\circ))^{-t}z|$.

Proof. We denote $\psi_{J_\circ}(s) = \psi_J(\delta s + s_\circ)$. Then it is clear that $\psi_{J_\circ}\in \mathfrak{N}^D(J_\circ)$. We set

$$\widetilde{f}(x) = |\det(\mathrm{M}^\delta_\gamma(s_\circ))|^{\frac{1}{p}} f(\mathrm{M}^\delta_\gamma(s_\circ)x),$$

so $\|\widetilde{f}\|_p = \|f\|_p$ and the Fourier transform of $\widetilde{f}$ is supported in the set $S_\lambda = \{\xi : 3\lambda/4 \le |(\mathrm{M}^\delta_\gamma(s_\circ))^{-t}\xi| \le 7\lambda/4\}$ because $\operatorname{supp}\widehat{f} \subset \mathbb{A}_\lambda$. Since $\mathrm{M}^\delta_\gamma(s_\circ) = \mathrm{M}^1_\gamma(s_\circ)\mathrm{D}_\delta$, it is easy to see that $S_\lambda \subset \{\xi : 3\lambda d_*/4 \le |\mathrm{D}_\delta^{-1}\xi| \le 7\lambda d^*/4\}$, thus we get (5.2.9).

We now define $\overline{\omega}$ and $\widetilde{\omega}$ by setting $\overline{\omega}(x,t) = \omega(x + t\gamma(s_\circ), t)$ and

$$\widetilde{\omega}(x,t) = \delta^3\overline{\omega}(\mathrm{M}^\delta_\gamma(s_\circ)x, t),$$

respectively. Denoting by M the matrix such that $\mathrm{M}(x,t) = (x + t\gamma(s_\circ), t)$, we note that $\overline{\omega} = \omega_{\mathrm{M}}$, $\det\mathrm{M} = 1$, and $\|\mathrm{M}\| \le 1 + |\gamma(s_\circ)|$. Thus using (5.2.3) we have $[\overline{\omega}]_3 \le (1 + |\gamma(s_\circ)|)^3[\omega]_3$. Similarly, let M' denote the matrix such that $\mathrm{M}'(x,t) = (\mathrm{M}^1_\gamma(s_\circ)x, t)$. Then $\widetilde{\omega} = \delta^3(\overline{\omega}_{\mathrm{M}'})^\delta$ since $\mathrm{M}^\delta_\gamma(s_\circ) = \mathrm{M}^1_\gamma(s_\circ)\mathrm{D}_\delta$. Using (5.2.2) and (5.2.3) we get $[\widetilde{\omega}]_3 \le C|\det\mathrm{M}^1_\gamma(s_\circ)|^{-1}(1 + \|\mathrm{M}^1_\gamma(s_\circ)\|)^3[\overline{\omega}]_3$ since $\det\mathrm{M}' = \det\mathrm{M}^1_\gamma(s_\circ)$ and $\|\mathrm{M}'\| \le 1 + \|\mathrm{M}^1_\gamma(s_\circ)\|$. Combining these two inequalities gives (5.2.8).

To complete the proof it remains to show (5.2.7). Note that $A^\gamma[\psi_J]f(x,t) = \delta\int f\big(x - t\gamma(s_\circ) - t\mathrm{M}^\delta_\gamma(s_\circ)\gamma^\delta_{s_\circ}(s)\big)\psi_J(\delta s + s_\circ)\,ds$, changing variables $s \to \delta s + s_\circ$ and using (5.2.1). We thus have

$$A^\gamma[\psi_J]f(x,t) = \delta|\det\mathrm{M}^\delta_\gamma(s_\circ)|^{-\frac{1}{p}}\int \widetilde{f}\big((\mathrm{M}^\delta_\gamma(s_\circ))^{-1}(x - t\gamma(s_\circ)) - t\gamma^\delta_{s_\circ}(s)\big)\psi_{J_\circ}ds.$$

Therefore the change of variables $x \to \mathrm{M}^\delta_\gamma(s_\circ)x + t\gamma(s_\circ)$ yields (5.2.7). $\square$

Reduction

Let $\gamma \in \mathrm{C}^D(I)$ be a curve satisfying (5.0.1). For a given $\varepsilon_\circ > 0$ we take $\delta = \delta_*$ where δ_* is the number given in Lemma 5.2.1. We now apply (5.2.6) to $\psi \in \mathrm{C}^D_0(I)$ and then Lemma 5.2.4 to each interval J, so that we have

$$\|A^\gamma[\psi]f\|_{L^p(\mathbb{R}^3\times\tilde{I},\omega)} \le \delta^{1-\frac{3}{p}}\sum_{J\in\mathfrak{J}(\delta)}\big\|A^{\gamma^\delta_{c_J}}[\psi^J]\widetilde{f}^J\big\|_{L^p(\mathbb{R}^3\times\tilde{I},\widetilde{\omega}^J)},$$

where $\gamma^{\delta}_{c_J} \in \mathfrak{C}^D(\varepsilon_\circ)$ (by Lemma 5.2.1), $[\widetilde{\omega}^J]_3 \le C_J[\omega]_3$, $C^{-1}\psi^J \in \mathfrak{N}^D(J_\circ)$ for some constants C_J, $C > 0$, and $\widetilde{f}^J$ satisfies that $\|\widetilde{f}^J\|_p \le \|f\|_p$ and $\operatorname{supp}\mathcal{F}(\widetilde{f}^J) \subset \{\xi : (B^J)^{-1}\lambda \le |\xi| \le B^J\lambda\}$ for a constant B^J. Since there are at most $C\delta_*^{-1}$ many intervals, for the estimate (5.1.6) it is enough to obtain estimate for each $A^{\gamma^{\delta}_{c_J}}[\psi^J]\widetilde{f}^J$ against the weight $\widetilde{\omega}^J$. Hence, after decomposing $\widetilde{f}^J$ via Littlewood-Paley projection and replacing $\widetilde{\omega}^J$ with $(C_J[\omega]_3)^{-1}\widetilde{\omega}^J$, in order to show (5.1.6) we need only to consider the curve $\gamma \in \mathfrak{C}^D(\varepsilon_\circ)$ and the weight ω with $[\omega]_3 \le 1$.

Furthermore, since $A^\gamma[\psi]f(x,t) = A^\gamma[\psi]f(\frac{\cdot}{r})(rx,rt)$, by scaling after splitting $\tilde{I}$ into three intervals $[2^{-1},1],[1,2]$ and $[2,4]$, the proof of (5.1.6) now reduces to showing

$$\|A^\gamma[\psi]f\|_{L^p(\mathbb{R}^3\times[1,2],\omega)} \le C\lambda^{-\varepsilon_p}\|f\|_{L^p(\mathbb{R}^3)}$$

for $[\omega]_3 \le 1$, $\gamma \in \mathfrak{C}^D(\varepsilon_\circ)$, and $\psi \in \mathfrak{N}^D(J_\circ)$ for some D.

Definition 5.2.5. *Fixing $p, \varepsilon_\circ, D$, for $\lambda \ge 1$ we define the quantity $Q(\lambda)$ by*

$$Q(\lambda) = \sup\big\{\|A^\gamma[\psi]f\|_{L^p(\mathbb{R}^3\times[1,2],\omega)} : \gamma \in \mathfrak{C}^D(\varepsilon_\circ),\ \psi \in \mathfrak{N}^D(J_\circ),\\ [\omega]_3 \le 1,\ \operatorname{supp}\widehat{f} \subset \mathbb{A}_\lambda,\ \|f\|_{L^p(\mathbb{R}^3)} \le 1\big\}.$$

An elementary estimate gives $Q(\lambda) \le C\lambda^2$ for $1 \le p \le \infty$.

Thanks to the discussion in the above and Lemma 5.1.4, Theorem 5.1.1 now follows from the next proposition, which we prove in Section 5.6.

Proposition 5.2.6. *For $p \in (3,\infty)$, there are positive constants $\varepsilon_\circ$, D, ε_p, and C such that*

$$Q(\lambda) \le C\lambda^{-\varepsilon_p}. \tag{5.2.10}$$

In order to show (5.2.10) we need only to handle $A^\gamma[\psi]$ with $\psi \in \mathfrak{N}^D(J_\circ)$, which we decompose in the fashion of (5.2.6). Thus it suffices work with the intervals $J \cap J_\circ \ne \emptyset$. We set

$$\mathfrak{J}_\circ(\delta) = \big\{J \in \mathfrak{J}(\delta) : J \subset (1+2c_\circ)J_\circ\big\}.$$

What follows next is a consequence of Lemma 5.2.4, which plays an important role in proving (5.2.10).

Lemma 5.2.7. *Let $J \in \mathfrak{J}_\circ(\delta)$ and $\psi_J \in \mathfrak{N}^D(J)$. Suppose $\gamma \in \mathfrak{C}^D(\varepsilon_\circ)$, $[\omega]_3 \leq 1$, and* $\operatorname{supp} \widehat{f} \subset \mathbb{A}_\lambda$. *If $\delta^3\lambda \geq 2^2$ and $\varepsilon_\circ > 0$ is sufficiently small, there are constants C, independent of γ, ω, and ψ_J, such that*

$$\left\| A^\gamma[\psi_J] f \right\|_{L^p(\mathbb{R}^3\times[1,2],\omega)} \leq C\delta^{1-\frac{3}{p}} K_\delta(\lambda) \|f\|_{L^p(\mathbb{R}^3)}, \tag{5.2.11}$$

where

$$K_\delta(\lambda) = \sum_{2^{-2}\delta^3\lambda \leq 2^j \leq 2^2\delta\lambda} Q(2^j).$$

Proof. We denote $J = [s_\circ - c_\circ\delta, s_\circ + c_\circ\delta]$. Since $\gamma \in \mathfrak{C}^D(\varepsilon_\circ)$, $\gamma^\delta_{s_\circ} \in \mathfrak{C}^D(\varepsilon_\circ)$ by Lemma 5.2.1 and our choice of δ, i.e., (5.2.4). Noting that $s_\circ \in 2J_\circ$, $\gamma \in \mathfrak{C}^D(\varepsilon_\circ)$ and $\varepsilon_\circ \leq c_\circ^2$, we see that $|\gamma(s_\circ)| \leq 3c_\circ$ and $\|\mathrm{M}^1_\gamma(s_\circ) - \mathrm{I}_3\| \leq 5c_\circ$. If we use $\sum_{\ell=0}^\infty (\mathrm{I}_3 - \mathrm{M}^1_\gamma(\mathrm{s}_\circ))^\ell = (\mathrm{M}^1_\gamma(\mathrm{s}_\circ))^{-1}$, it follows $\|(\mathrm{M}^1_\gamma(s_\circ))^{-1} - \mathrm{I}_3\| \leq \frac{5c_\circ}{1-5c_\circ}$. Since $\|\mathrm{M}\| = \|\mathrm{M}^t\|$ for any matrix M, $\|(\mathrm{M}^1_\gamma(s_\circ))^{-t} - \mathrm{I}_3\| < 1/100$. So, we have

$$\frac{99}{100} \leq \inf_{|z|=1} |(\mathrm{M}^1_\gamma(s_\circ))^{-t} z|, \quad \|(\mathrm{M}^1_\gamma(s_\circ))^{-t}\|, \quad |\det \mathrm{M}^1_\gamma(s_\circ)| \leq \frac{101}{100}.$$

Therefore, by (5.2.8) and (5.2.9) we see respectively that $[\widetilde{\omega}]_3 \leq C$ with a constant C independent of γ and that $\operatorname{supp} \mathcal{F}(\widetilde{f}\,) \subset \{\xi : 2^{-1}\delta^3\lambda \leq |\xi| \leq 2\delta\lambda\}$. Let $\beta_* \in \mathrm{C}_0^\infty([3/4, 7/4])$ such that $\sum_j \beta_*(2^{-j}\cdot) = 1$. We decompose

$$\widetilde{f} = \sum_{2^{-2}\delta^3\lambda \leq 2^j \leq 2^2\delta\lambda} \widetilde{f}_j,$$

where $\widetilde{f}_j = \mathcal{F}^{-1}\big(\beta_*(2^{-j}|\cdot|)\mathcal{F}(\widetilde{f}\,)\big)$. By (5.2.7) it follows that

$$\left\| A^\gamma[\psi_J] f \right\|_{L^p(\mathbb{R}^3\times[1,2],\omega)} \leq \delta^{1-\frac{3}{p}} \sum_{2^{-2}\delta^3\lambda \leq 2^j \leq 2^2\delta\lambda} \|A^{\gamma^\delta_{s_\circ}}[\psi_{J_\circ}]\widetilde{f}_j\|_{L^p(\mathbb{R}^3\times[1,2],\widetilde{\omega})}.$$

Since $\operatorname{supp} \mathcal{F}(\widetilde{f}_j) \subset \mathbb{A}_{2^j}$ and $\|\widetilde{f}_j\|_p \leq C_{\beta_*}\|f\|_p$ and since $\gamma^\delta_{s_\circ} \in \mathfrak{C}^D(\varepsilon_\circ)$, $\psi_{J_\circ} \in \mathfrak{N}^D(J_\circ)$ and $[\widetilde{\omega}]_3 \leq C$, we have $\|A^{\gamma^\delta_{s_\circ}}[\psi_{J_\circ}]\widetilde{f}_j\|_{L^p(\mathbb{R}^3\times[1,2],\widetilde{\omega})} \leq CQ(2^j)\|f\|_p$ while C is independent of γ, ω, and ψ_J. Therefore we get (5.2.11). $\square$

5.3 Decomposition

To show the inequality (5.2.10) we need only to deal with $\gamma \in \mathfrak{C}^D(\varepsilon_\circ)$ and $\psi \in \mathfrak{N}^D(J_\circ)$, therefore it suffices to consider the curve γ over the interval $(1+2c_\circ)J_\circ$. This additional localization is helpful for simplifying the argument which follows henceforth.

5.3.1 Decomposition in Fourier side

Since $\varepsilon_\circ \le c_\circ^2$, it is clear that

$$|\gamma'(s) - e_1| \le 2c_\circ,\ |\gamma''(s) - e_2| \le 2c_\circ,\ |\gamma'''(s) - e_3| \le 2c_\circ \tag{5.3.1}$$

for $s \in (1+2c_\circ)J_\circ$ and $\gamma \in \mathfrak{C}^D(\varepsilon_\circ)$. Thus we have $|\gamma'(s)\cdot\xi| + |\gamma''(s)\cdot\xi| \ge c_\circ|\xi|$ if $|\xi_1| \ge 3c_\circ|\xi|$ or $|\xi_2| \ge 3c_\circ|\xi|$. Using Proposition 5.3.8 below we can handle the contribution from the part of frequency $|\xi_1| \ge 3c_\circ|\xi|$ or $|\xi_2| \ge 3c_\circ|\xi|$ since the condition (5.3.7) is satisfied. We shall mainly concentrate on the case where ξ is included in the set

$$\mathbb{A}_\lambda^* := \big\{\xi : 2^{-1}\lambda \le |\xi| \le 2\lambda,\ |\xi_1| \le 2^2c_\circ|\xi|,\ |\xi_2| \le 2^2c_\circ|\xi|\big\}.$$

The following is easy to see.

Lemma 5.3.1. *There exists a function $\sigma \in \mathrm{C}^{D-2}(\mathbb{A}_\lambda^*)$, homogeneous of degree 0, such that, for $\xi \in \mathbb{A}_\lambda^*$, $|\sigma(\xi)| \le 5c_\circ$ and*

$$\gamma''(\sigma(\xi))\cdot\xi = 0.$$

Indeed, we need to solve the equation $\gamma''(s)\cdot\xi = 0$ for a given ξ, equivalently, $\xi_3^{-1}\xi_2 + s + e(\xi, s) = 0$ where $e(\xi, s)$ is a function of homogeneous of degree zero and $\|e(\xi,\cdot)\|_{\mathrm{C}^{D-2}} \le 2\varepsilon_\circ$. An elementary argument shows existence of $\sigma(\xi)$ and the implicit function theorem guarantees that $\sigma \in \mathrm{C}^{D-2}(\mathbb{A}_\lambda^*)$ since $\gamma \in \mathfrak{C}^D(\varepsilon_\circ)$. It is clear that $|\sigma(\xi)| \le 5c_\circ$ because $\xi_3^{-1}\xi_2 + \sigma(\xi) + e(\xi, \sigma(\xi)) = 0$.

For $\xi \in \mathbb{A}_\lambda^*$, we denote

$$\Lambda_\gamma(\xi) = \gamma'''(\sigma(\xi))\cdot\xi,$$
$$R_\gamma(\xi) = -\frac{\gamma'(\sigma(\xi))\cdot\xi}{\Lambda_\gamma(\xi)}.$$

If $\xi \in \mathbb{A}_\lambda^*$ and $\sigma(\xi) \in (1+2c_\circ)J_\circ$, by (5.3.1) we have $2^{-2}\lambda \le |\Lambda_\gamma(\xi)| \le 2^2\lambda$, $|\gamma'(\sigma(\xi))\cdot\xi - \xi_1| \le 2^3c_\circ\lambda$, and $\big|\Lambda_\gamma(\xi) - \xi_3\big| \le 2^3c_\circ\lambda$, so $|R_\gamma(\xi)| \le 2^6c_\circ$.

Decomposition of the operator $A^\gamma[\psi_J]$

By Taylor's expansion we have

$$\gamma'(s)\cdot\xi = -\Lambda_\gamma(\xi)R_\gamma(\xi) + 2^{-1}\Lambda_\gamma(\xi)(s-\sigma(\xi))^2 + \mathcal{O}(\varepsilon_\circ\lambda|s-\sigma(\xi)|^3), \tag{5.3.2}$$

$$\gamma''(s)\cdot\xi = \Lambda_\gamma(\xi)(s-\sigma(\xi)) + \mathcal{O}(\varepsilon_\circ\lambda|s-\sigma(\xi)|^2) \tag{5.3.3}$$

for $s \in J$ and $\xi \in \mathbb{A}_\lambda^*$. Thus $\gamma'(s) \cdot \xi$ and $\gamma''(s) \cdot \xi$ have lower bounds if $\sigma(\xi)$ is distanced from J, so it is not difficult to have control over the contribution from the associated frequency. However, if $\sigma(\xi)$ is close to J for $\xi \in \operatorname{supp} \widehat{f}$, the behavior of $A^\gamma[\psi_J]f$ becomes less favorable. This leads us to define, for $K \ge 1$ and $J \in \mathfrak{J}_\circ(\delta)$,

$$\mathcal{R}_J(K) = \big\{\xi : |\gamma'(c_J)\cdot\xi| \le K c_\circ^2 \delta^2 \lambda,\ |\gamma''(c_J)\cdot\xi| \le K c_\circ \delta\lambda,\ 2^{-2}\lambda \le |\xi_3| \le 2^2\lambda\big\},$$

which contains the unfavorable frequency part of $A^\gamma[\psi_J]f$. Concerning the sets $\mathcal{R}_J(K)$ we have the next lemma which we use later.

Lemma 5.3.2. *Let $\gamma \in \mathfrak{C}^D(\varepsilon_\circ)$. If $\varepsilon_\circ > 0$ is sufficiently small, we have the following with C independent of γ and δ:*

$$\sum_{J \in \mathfrak{J}_\circ(\delta)} \chi_{\mathcal{R}_J(2^6)} \le C. \tag{5.3.4}$$

Proof. In order to show (5.3.4) it is sufficient to verify that the sets $\mathbf{r}_J := \{\xi : \lambda\xi \in \mathcal{R}_J(2^6)\}$ overlap each other at most C many times. Note that $\mathbf{r}_J$ is contained in $2^8 c_\circ\delta$ neighborhood of the line L_J passing through the origin with its direction parallel to $\gamma'(c_J) \times \gamma''(c_J)$. Since $\mathbf{r}_J \subset \{\xi : 2^{-4} \le |\xi| \le 2^4\}$, it is sufficient to show that the directions of the lines L_J are separated from each other by a distance at least $2^{-1}c_\circ\delta$. This in turn follows from the fact that

$$\frac{d}{ds}\Big(\gamma'(s) \times \gamma''(s)\Big) = \gamma'(s) \times \gamma'''(s) = -e_2 + \mathcal{O}_s(5c_\circ)$$

for $\gamma \in \mathfrak{C}^D(\varepsilon_\circ)$ because the distance between the centers c_J of J is at least $c_\circ\delta$. Since $s \in [-2c_\circ, 2c_\circ]$ and $\gamma \in \mathfrak{C}^D(\varepsilon_\circ)$, we have $|\gamma'(s) - e_1| \le 2c_\circ(1 + 2c_\circ)$ and $|\gamma'''(s) - e_3| \le c_\circ^2$. Thus the last equality is clear. $\square$

Let $\widetilde{\beta} \in \mathrm{C}_0^\infty([2^{-2}, 2^2])$ such that $\widetilde{\beta} = 1$ on $[2^{-1}, 2]$. Then we set

$$\widetilde{\chi}_{\mathcal{R}_J}(\xi) = \beta_0\Big(\frac{|\gamma'(c_J)\cdot\xi|}{2^5 c_\circ^2 \delta^2 \lambda}\Big)\beta_0\Big(\frac{|\gamma''(c_J)\cdot\xi|}{2^5 c_\circ\delta\lambda}\Big)\widetilde{\beta}\Big(\frac{|\xi_3|}{\lambda}\Big),$$

so that $\widetilde{\chi}_{\mathcal{R}_J}$ is supported in $\mathcal{R}_J(2^6)$ and $\widetilde{\chi}_{\mathcal{R}_J}(\xi) = 1$ if $\xi \in \mathcal{R}_J(2^5) \cap \mathbb{A}_\lambda^*$. We set

$$P_J f = \mathcal{F}^{-1}(\widetilde{\chi}_{\mathcal{R}_J}\widehat{f}\,).$$

The following is a consequence of (5.3.4).

Lemma 5.3.3. *If $\varepsilon_\circ$ is small enough, we have $\big(\sum_{J\in\mathfrak{J}_\circ(\delta)}\|P_Jf\|_p^p\big)^{1/p}\le C\|f\|_p$ for $2\le p\le\infty$ whenever $\gamma\in\mathfrak{C}^D(\varepsilon_\circ)$.*

The inequality follows from interpolation between the cases $p=2$ and $p=\infty$. Plancherel's theorem and (5.3.4) give $(\sum_J\|P_Jf\|_2^2)^{1/2}\le C\|f\|_2$ and the estimate $\max_J\|P_Jf\|_\infty\le C\|f\|_\infty$ is obvious.

Decomposition away from the conic surface $\mathcal{C}_\lambda$

We further decompose $A^\gamma[\psi_J]P_Jf$ in Fourier side taking into account how close ξ is to the conic set $\mathcal{C}_\lambda:=\{\xi\in\mathbb{A}_\lambda^*: R_\gamma(\xi)=0\}$. To this end we set

$$\widetilde{\chi}_{\mathbb{A}_\lambda^*}(\xi)=\beta_0\Big(\frac{\xi_1}{2c_\circ|\xi|}\Big)\beta_0\Big(\frac{\xi_2}{2c_\circ|\xi|}\Big)\beta(\lambda^{-1}|\xi|).$$

For $0<\nu\ll1$, we define the cutoff functions $\pi_{\mathbf{c}}$, $\pi_{\mathbf{e}}$, $\pi_{\mathbf{o}}^1$, and $\pi_{\mathbf{o}}^0$ by setting

$$\begin{aligned}\pi_{\mathbf{c}}(\xi)&=\widetilde{\chi}_{\mathbb{A}_\lambda^*}(\xi)\beta_0(\lambda^{\frac23-2\nu}|R_\gamma(\xi)|),\\ \pi_{\mathbf{e}}(\xi)&=\beta(\lambda^{-1}|\xi|)-\widetilde{\chi}_{\mathbb{A}_\lambda^*}(\xi)\beta_0(\delta^{-100}|R_\gamma(\xi)|),\end{aligned}$$

and, for $j=0,1$,

$$\pi_{\mathbf{o}}^j(\xi)=\widetilde{\chi}_{\mathbb{A}_\lambda^*}(\xi)\chi_{\{\xi:(-1)^{j+1}R_\gamma(\xi)>0\}}\big(\beta_0(\delta^{-100}|R_\gamma(\xi)|)-\beta_0(\lambda^{\frac23-2\nu}|R_\gamma(\xi)|)\big).$$

The support of $\widetilde{\chi}_{\mathbb{A}_\lambda^*}$ is contained in $\mathbb{A}_\lambda^*$ and $\pi_{\mathbf{c}}+\pi_{\mathbf{o}}^1+\pi_{\mathbf{o}}^0+\pi_{\mathbf{e}}=\beta(\lambda^{-1}|\cdot|)$ almost everywhere. The functions $\pi_{\mathbf{c}}$, $\pi_{\mathbf{o}}^1+\pi_{\mathbf{o}}^0$, and $\beta(\lambda^{-1}|\cdot|)-\pi_{\mathbf{e}}$ roughly split the set $\mathbb{A}_\lambda^*$ into three regions $\{\xi:|R_\gamma(\xi)|\le C\lambda^{2\nu-2/3}\}$, $\{\xi:C\lambda^{2\nu-2/3}\le|R_\gamma(\xi)|\le C_1\delta^{100}\}$, and $\{\xi:C_1\delta^{100}\le|R_\gamma(\xi)|\}$. The division between the first set and the other two reflects different asymptotic behaviors of the multiplier $A^\gamma[\psi_J](e^{i(\cdot)\cdot\xi})(0,t)$ as $|\xi|\to\infty$. The further division of the second and the third sets is necessitated to guarantee the transversality condition for the multilinear estimate, which is to be discussed in the next section.

We also define the associated multiplier operators $\mathcal{P}_{\mathbf{c}}$, $\mathcal{P}_{\mathbf{o}}^1$, $\mathcal{P}_{\mathbf{o}}^0$, and $\mathcal{P}_{\mathbf{e}}$ by

$$\widehat{\mathcal{P}_{\mathbf{c}}g}(\xi)=\pi_{\mathbf{c}}(\xi)\widehat{g}(\xi),\ \ \mathcal{F}(\mathcal{P}_{\mathbf{o}}^jg)(\xi)=\pi_{\mathbf{o}}^j(\xi)\widehat{g}(\xi),\ j=0,1,\ \ \widehat{\mathcal{P}_{\mathbf{e}}g}(\xi)=\pi_{\mathbf{e}}(\xi)\widehat{g}(\xi).$$

Besides, we set $\mathcal{P}_{\mathbf{n}}=\mathcal{P}_{\mathbf{c}}+\mathcal{P}_{\mathbf{o}}^1+\mathcal{P}_{\mathbf{o}}^0$. Then easy estimates for the kernels of the operators give

$$\|\mathcal{P}_{\mathbf{c}}\|_{p\to p}\le C_1\lambda^C,\ \ \|\mathcal{P}_{\mathbf{o}}^j\|_{p\to p}\le C_1\lambda^C,\ j=0,1,\tag{5.3.5}$$

for $1 \le p \le \infty$ and some constants $C, C_1 > 0$. It is possible to get better bounds if we use the decoupling or the square function estimate for the cone (for example, [31, 21]) but we do not attempt to do so since it is irrelevant to our purpose. Similarly, we also have

$$\|\mathcal{P}_{\mathbf{e}}\|_{p\to p} \le C_1\delta^{-C}, \quad \|\mathcal{P}_{\mathbf{n}}\|_{p\to p} \le C_1\delta^{-C} \tag{5.3.6}$$

for $1 \le p \le \infty$. For the former we need only to note that $\|\mathcal{F}^{-1}(\pi_{\mathbf{e}})\|_{L^1(\mathbb{R}^3)} \le C_1\delta^{-C}$. The latter follows from the former because the multiplier associated to the operator $\mathcal{P}_{\mathbf{n}}$ is $\beta(\lambda^{-1}|\cdot|) - \pi_{\mathbf{e}}$.

5.3.2 Nondegenerate part

Decomposition of the operator A in Fourier side gives rise to the operators of the form of (5.2.5) such as $A^\gamma[\psi_J]P_J$, $A^\gamma[\psi_J](1-P_J)$, $\dots$, $A^\gamma[\psi_J]\mathcal{P}_{\mathbf{e}}$. If $|\gamma'(s)\cdot\xi| + |\gamma''(s)\cdot\xi| \ge C|\xi|$ on the support of a, we can handle $A^\gamma[a]$ using the following theorem which is a straightforward consequence of [44, Theorem 4.1].

Theorem 5.3.4. *Let $K \ge 1$ and $[s_\circ - 2r, s_\circ + 2r] \subset I$ with $K^{-1} \le r$. Suppose that $a(s,t,\xi)$ is a smooth function supported in $[s_\circ - r, s_\circ + r] \times \tilde{I} \times \mathbb{A}_\lambda$ and $|\partial_s^{j_1}\partial_t^{j_2}\partial_\xi^\alpha a(s,t,\xi)| \le B|\xi|^{-|\alpha|}$ for $|\alpha| \le 5$ and $j_1, j_2 = 0, 1$. Also, assume that*

$$|\gamma'(s)\cdot\xi| + |\gamma''(s)\cdot\xi| \ge K^{-1}|\xi| \tag{5.3.7}$$

whenever $(s,t,\xi) \in \operatorname{supp} a$ for some $t \in \tilde{I}$. Then, if $p \ge 6$ and $\varepsilon_\circ > 0$ is small enough, for $\varepsilon > 0$

$$\|A^\gamma[a]f\|_{L^p(\mathbb{R}^3\times\tilde{I})} \le C_\varepsilon B K^C \lambda^{-\frac{2}{p}+\varepsilon}\|f\|_{L^p(\mathbb{R}^3)} \tag{5.3.8}$$

whenever $\gamma \in \mathfrak{C}^D(\varepsilon_\circ)$ and $\widehat{f}$ is supported in $\mathbb{A}_\lambda$.

The statement of Theorem 5.3.4 differs from the one in [44] in a couple of aspects. First, the range of p is enlarged to $p \ge 6$ † thanks to the ℓ^p-decoupling inequality for the cone [11]. Secondly, there is an extra factor K^C in (5.3.8). The estimate (5.3.8) can be seen following the argument in [44]. It is also possible to deduce (5.3.8) from that with $K \in [2^{-1}, 2]$ by finite

†The critical case $p = 6$ can be included by interpolation with a trivial estimate.

decomposition and making use of scaling and affine transform. Uniformity of the bound over $\gamma \in \mathfrak{C}^D(\varepsilon_\circ)$ is clear.

The estimate $|\int e^{-it\gamma(s)\cdot\xi} a(s,t,\xi)ds| \le C_1 B K^C |\xi|^{-\frac{1}{2}}$ follows by (5.3.7) and van der Corput's Lemma. Then $\|A^\gamma[a]f\|_{L^2(\mathbb{R}^3\times\tilde{I})} \le C_1 B K^C \lambda^{-\frac{1}{2}} \|f\|_{L^2(\mathbb{R}^3)}$ by Plancherel's theorem. Interpolation between the estimate and (5.3.8) with $p=6$ gives

Corollary 5.3.5. *Under the same assumption as in Theorem 5.3.4, if* $2 \le p \le 6$ *and* $\varepsilon_\circ$ *is small enough, for* $\varepsilon > 0$

$$\|A^\gamma[a]f\|_{L^p(\mathbb{R}^3\times\tilde{I})} \le C_\varepsilon B K^C \lambda^{-\frac{1}{4}-\frac{1}{2p}+\varepsilon} \|f\|_{L^p(\mathbb{R}^3)}$$

whenever $\gamma \in \mathfrak{C}^D(\varepsilon_\circ)$ *and* $\widehat{f}$ *is supported in* $\mathbb{A}_\lambda$.

We also make use of the following ([44, Theorem 1.4]).

Theorem 5.3.6. *Let* $J \subset I$ *be a compact interval of length* δ *and* $\psi_J \in \mathfrak{N}^D(J)$*. Then, if* $p \ge 6$ *and* $\varepsilon_\circ$ *is small enough, for* $\varepsilon > 0$

$$\|A^\gamma[\psi_J]f\|_{L^p(\mathbb{R}^3\times\tilde{I})} \le C_\varepsilon \delta^{-C} \lambda^{-\frac{4}{3p}+\varepsilon} \|f\|_{L^p(\mathbb{R}^3)} \tag{5.3.9}$$

whenever $\gamma \in \mathfrak{C}^D(\varepsilon_\circ)$ *and* $\widehat{f}$ *is supported in* $\mathbb{A}_\lambda$.

Compared with [44, Theorem 1.4], the range of p is extended to $p \ge 6$ by the aforementioned decoupling inequality [11]. The estimate (5.3.9) with additional factor δ^{-C} can be shown by scaling and its uniformity over $\gamma \in \mathfrak{C}^D(\varepsilon_\circ)$ is also obvious.

Estimates for $A^\gamma[\psi_J](1-P_J)$ and $A^\gamma[\psi_J]\mathcal{P}_{\mathbf{e}}$

The condition (5.3.7) is satisfied on the support of $\psi_J(s)(1-\widetilde{\chi}_{\mathcal{R}_J}(\xi))$. Thus, using Corollary 5.3.5, we can get a favorable estimate for $A^\gamma[\psi_J](1-P_J)$. We also obtain the similar estimate for $A^\gamma[\psi_J]\mathcal{P}_{\mathbf{e}}$ (see Proposition 5.3.8 below).

Proposition 5.3.7. *Let* $[\omega]_3 \le 1$*, and* $J \in \mathfrak{J}_\circ(\delta)$*. If* $2 \le p \le 6$ *and* $\varepsilon_\circ > 0$ *is small enough, for* $\varepsilon > 0$ *there are constants* C *and* C_ε *such that*

$$\|A^\gamma[\psi_J](1-P_J)f\|_{L^p(\mathbb{R}^3\times[1,2],\omega)} \le C_\varepsilon \delta^{-C} \lambda^{\frac{1}{2}(\frac{1}{p}-\frac{1}{2})+\varepsilon} \|f\|_{L^p(\mathbb{R}^3)} \tag{5.3.10}$$

whenever $\operatorname{supp}\widehat{f} \subset \mathbb{A}_\lambda$*,* $\gamma \in \mathfrak{C}^D(\varepsilon_\circ)$*, and* $\psi_J \in \mathfrak{N}^D(J)$.

Proof. We set

$$a(s,t,\xi) = \widetilde{\chi}(t)\psi_J(s)(1-\widetilde{\chi}_{\mathcal{R}_J}(\xi))\beta(\lambda^{-1}|\xi|),$$

so that $A^\gamma[a]f = \widetilde{\chi}(t)A^\gamma[\psi_J](1-P_J)f$. We claim that (5.3.7) holds on the support of a with $K = C_1\delta^{-2} > 0$, $C_1 = C_1(c_\circ)$.

To see this, it suffices to consider the case $\xi \in \mathbb{A}^*_\lambda$ because of (5.3.1). We first note that $\xi \in \mathcal{R}_J(2^5)$ if $\sigma(\xi) \in [\, c_J - |J|, c_J + |J|\,]$ and $|R_\gamma(\xi)| \le 2^3 c_\circ^2\delta^2$. Indeed, since $|\sigma(\xi) - c_J| \le 2c_\circ\delta$, by (5.3.2) we have $|\gamma'(c_J)\cdot\xi| \le 2^5 c_\circ^2\delta^2\lambda$, and we get $|\gamma''(c_J)\cdot\xi| \le 2^3 c_\circ\delta\lambda$ from (5.3.3). So, it follows $\xi \in \mathcal{R}_J(2^5)$ since $\xi \in \mathbb{A}^*_\lambda$. Hence, if $\xi \in \mathrm{supp}(1-\widetilde{\chi}_{\mathcal{R}_J})\beta(\lambda^{-1}|\cdot|) \cap \mathbb{A}^*_\lambda$, we have $\sigma(\xi) \notin [\, c_J - |J|, c_J + |J|\,]$ or $|R_\gamma(\xi)| \ge 2^3 c_\circ^2\delta^2$. In the first case we have $|\gamma''(s)\cdot\xi| \ge 2^{-2}c_\circ\delta\lambda$ by (5.3.3). Thus we may assume $|R_\gamma(\xi)| \ge 2^3 c_\circ^2\delta^2$ and $|s - \sigma(\xi)| \le 3c_\circ\delta$ and then we get $|\gamma'(s)\cdot\xi| \ge 2c_\circ^2\delta^2\lambda$ using (5.3.2). This shows the claim.

Since (5.3.7) holds on the support of a, by Corollary 5.3.5 we have the estimate

$$\|\widetilde{\chi}(t)A^\gamma[\psi_J](1-P_J)f\|_{L^p(\mathbb{R}^3\times\mathbb{R})} \le C_\varepsilon\delta^{-C}\lambda^{-\frac{1}{4}-\frac{1}{2p}+\varepsilon}\|f\|_{L^p(\mathbb{R}^3)} \tag{5.3.11}$$

for $2 \le p \le 6$. We use the estimate to obtain the weighted estimate (5.3.10) and argue similarly as in the proof of Lemma 5.1.4. So, we shall be brief.

As before, let us define an operator $\widetilde{A}_J$ by

$$\mathcal{F}(\widetilde{A}_J h)(\xi,\tau) = \beta_0((\lambda r_0)^{-1}\tau)\beta(\lambda^{-1}|\xi|)\mathcal{F}\big(\widetilde{\chi}(t)A^\gamma[\psi_J]h\big)(\xi,\tau),$$

where $r_0 = 1 + 4\max\{|\gamma(s)| : s \in \mathrm{supp}\psi_J\}$. Then we have $|(\widetilde{\chi}(t)A^\gamma[\psi_J] - \widetilde{A}_J)h| \le C\widetilde{E}_t^N * |h|$ for any N if we use Lemma 5.1.6. Putting together this (e.g., (5.1.7)), $[\omega]_3 \le 1$ and $\|(1-P_J)f\|_p \le C\|f\|_p$, we see that

$$\|\widetilde{\chi}(t)A^\gamma[\psi_J](1-P_J)f\|_{L^p(\mathbb{R}^3\times\mathbb{R},\omega)} \le \|\widetilde{A}_J(1-P_J)f\|_{L^p(\mathbb{R}^3\times\mathbb{R},\omega)} + C\lambda^{-N}\|f\|_p.$$

The Fourier transform of $\widetilde{A}_J(1-P_J)f$ is supported in $\mathbb{B}^4(0, 2^2 r_0\lambda)$. By Lemma 5.5.5 we thus get $\|\widetilde{A}_J(1-P_J)f\|_{L^p(\mathbb{R}^3\times\mathbb{R},\omega)} \le C\lambda^{1/p}\|\widetilde{A}_J(1-P_J)f\|_{L^p(\mathbb{R}^3\times\mathbb{R})}$. Disregarding the minor contribution from $(\widetilde{\chi}(t)A^\gamma[\psi_J] - \widetilde{A}_J)(1-P_J)f$, we only need to obtain the estimate for $\widetilde{\chi}(t)A^\gamma[\psi_J](1-P_J)f$ in $L^p(\mathbb{R}^3\times\mathbb{R})$. Therefore we obtain the estimate (5.3.10) by (5.3.11). $\square$

Proposition 5.3.8. *Under the same assumption as in Proposition 5.3.7, if $2 \le p \le 6$ and $\varepsilon_\circ > 0$ is small enough, for any $\varepsilon > 0$*

$$\|A^\gamma[\psi_J]\mathcal{P}_{\mathbf{e}}f\|_{L^p(\mathbb{R}^3\times[1,2],\omega)} \le C_\varepsilon\delta^{-C}\lambda^{\frac{1}{2}(\frac{1}{p}-\frac{1}{2})+\varepsilon}\|f\|_{L^p(\mathbb{R}^3)}$$

whenever $\mathrm{supp}\,\widehat{f} \subset \mathbb{A}_\lambda$, $\gamma \in \mathfrak{C}^D(\varepsilon_\circ)$ *and* $\psi_J \in \mathfrak{N}^D(J)$.

Proof. We set $\pi_{\mathbf{e}}^1(\xi) = \widetilde{\chi}_{\mathbb{A}_\lambda^*}(\xi)\big(1 - \beta_0(\delta^{-100}|R_\gamma(\xi)|)\big)$ and $\pi_{\mathbf{e}}^2(\xi) = \beta(\lambda^{-1}|\xi|) - \widetilde{\chi}_{\mathbb{A}_\lambda^*}(\xi)$, so that $\pi_{\mathbf{e}} = \pi_{\mathbf{e}}^1 + \pi_{\mathbf{e}}^2$. Then we break $\widetilde{\chi}(t)A^\gamma[\psi_J]\mathcal{P}_{\mathbf{e}}f = A^\gamma[a^1]f + A^\gamma[a^2]f$, where

$$a^j(s,t,\xi) = \widetilde{\chi}(t)\psi_J(s)\pi_{\mathbf{e}}^j(\xi), \ \ j = 1, 2.$$

We first consider $A^\gamma[a^1]f$. After decomposing ψ_J into the bump functions ψ_ℓ supported in finitely overlapping intervals J_ℓ such that $\delta^{100} \le |J_\ell| \le 2\delta^{100}$, $\psi_J = \sum \psi_\ell$, and $|\psi_\ell^{(k)}| \le C_k\delta^{-100k}$, we set $a_\ell^1(s,t,\xi) = \widetilde{\chi}(t)\psi_\ell(s)\pi_{\mathbf{e}}^1(\xi)$. By (5.3.3) $|\gamma''(s) \cdot \xi| \ge 2^{-3}\lambda\delta^{100}$ for $s \in \operatorname{supp}\psi_\ell$ if $\sigma(\xi) \notin [c_{J_\ell} - |J_\ell|, c_{J_\ell} + |J_\ell|]$. Otherwise, from (5.3.2) we have $|\gamma'(s) \cdot \xi| \ge 2^{-2}\delta^{100}\lambda$ for $s \in \operatorname{supp}\psi_\ell$ since $|R_\gamma(\xi)| \ge \delta^{100}$ on $\operatorname{supp}\pi_{\mathbf{e}}^1$. Therefore (5.3.7) holds with $K = C\delta^{-100}$ for $(s,t,\xi) \in \operatorname{supp} a_\ell^1$. An application of Corollary 5.3.5 with $a = a_\ell^1$ gives

$$\|A^\gamma[a_\ell^1]f\|_{L^p(\mathbb{R}^3\times\mathbb{R})} \le C_\varepsilon\delta^{-C}\lambda^{-\frac{1}{4}-\frac{1}{2p}+\varepsilon}\|f\|_{L^p(\mathbb{R}^3)}.$$

Arguing similarly as in the proof of Proposition 5.3.7, we get the weighted estimate $\|A^\gamma[a_\ell^1]f\|_{L^p(\mathbb{R}^3\times[1,2],\omega)} \le C_\varepsilon\delta^{-C}\lambda^{-\frac{1}{4}+\frac{1}{2p}+\varepsilon}\|f\|_{L^p(\mathbb{R}^3)}$. Summation over ℓ thus gives the desired estimate since there are at most $C\delta^{-100}$ many ℓ.

The estimate $\|A^\gamma[a^2]f\|_{L^p(\mathbb{R}^3\times[1,2],\omega)} \le C_\varepsilon\lambda^{-\frac{1}{4}+\frac{1}{2p}+\varepsilon}\|f\|_{L^p(\mathbb{R}^3)}$ can be obtained likewise but more straightforwardly since $|\gamma'(s)\cdot\xi| + |\gamma''(s)\cdot\xi| \ge c_\circ|\xi|$ on $\operatorname{supp} a^2$. $\square$

5.4 Asymptotic expansions of the multiplier

The main object of this and next sections is to prove the following weighted multilinear estimate for $A^\gamma[\psi_J]\mathcal{P}_{\mathbf{n}}f$. Throughout this and the next sections we assume $\gamma \in \mathfrak{C}^D(\varepsilon_\circ)$ with an $\varepsilon_\circ$ small enough.

Proposition 5.4.1. *Let $J_k \in \mathfrak{J}_\circ(\delta)$, $1 \le k \le 4$, and $[\omega]_3 \le 1$. Suppose that $\widehat{f_1}, \dots, \widehat{f_4}$ are supported in $\mathbb{A}_\lambda$ and dist$(J_\ell, J_k) \ge \delta$, $\ell \ne k$. If $14/5 < p \le 6$, there are constants $\varepsilon_p > 0$, D, and $C_\delta > 0$ such that*

$$\Big\| \prod_{k=1}^4 |A^\gamma[\psi_{J_k}](\mathcal{P}_{\mathbf{n}}P_{J_k}f_k)|^{\frac{1}{4}} \Big\|_{L^p(\mathbb{R}^3\times[1,2],\omega)} \le C_\delta\lambda^{-\varepsilon_p}\prod_{k=1}^4 \|f_k\|_{L^p(\mathbb{R}^3)}^{\frac{1}{4}} \tag{5.4.1}$$

whenever $\gamma \in \mathfrak{C}^D(\varepsilon_\circ)$ and $\psi_{J_k} \in \mathfrak{N}^D(J_k)$, $1 \le k \le 4$.

In order to prove Proposition 5.4.1 we first try to express $A^\gamma[\psi_{J_k}]\mathcal{P}_{\mathbf{n}}$ as a sum of adjoint restriction operators. To do so, we expand the Fourier multiplier of the operator $A^\gamma[\psi_{J_k}]\mathcal{P}_{\mathbf{n}}$ into a series of suitable form. We handle separately $A^\gamma[\psi_{J_k}]\mathcal{P}_{\mathbf{c}}$ (Lemma 5.4.2) and $A^\gamma[\psi_{J_k}]\mathcal{P}^j_{\mathbf{o}}$, $j = 1, 0$ (Lemma 5.4.4). The estimates in Lemma 5.4.2 and 5.4.4 are somewhat rough but we do not attempt to make them as efficient as possible.

Multiplier of $A^\gamma[\psi_J]\mathcal{P}_{\mathbf{c}}$

Let $J \in \mathfrak{J}_\circ(\delta)$. For $\psi_J \in \mathfrak{N}^D(J)$ we set

$$m_J(t,\xi) = (2\pi)^{-3} \int e^{-it\gamma(s)\cdot\xi} \psi_J(s)\, ds.$$

The multiplier $m_J\pi_{\mathbf{c}}$ of $A^\gamma[\psi_J]\mathcal{P}_{\mathbf{c}}$ has the worse decay in ξ as the zeros of $\gamma'(s)\cdot\xi$ and $\gamma''(s)\cdot\xi$ are close to each other. We define

$$\Phi^{\mathbf{c}}(\xi) = \gamma(\sigma(\xi))\cdot\xi, \quad \xi \in \mathbb{A}^*_\lambda,$$

and an adjoint restriction operator $\mathcal{T}^{\mathbf{c}}_\lambda$ by setting

$$\mathcal{T}^{\mathbf{c}}_\lambda g(x,t) = \int_{\mathcal{C}^{\mathbf{c}}_\lambda(\delta)} e^{i(x\cdot\xi - t\Phi^{\mathbf{c}}(\xi))} g(\xi)\, d\xi,$$

where $\mathcal{C}^{\mathbf{c}}_\lambda(\delta) = \{\xi \in \mathbb{A}^*_\lambda : |R_\gamma(\xi)| \le 2\delta^{100}\}$. We note that $\operatorname{supp}\pi_{\mathbf{c}} \subset \mathcal{C}^{\mathbf{c}}_\lambda(\delta)$.

Lemma 5.4.2. *Let $0 < \nu \ll 1$ and $J \in \mathfrak{J}_\circ(\delta)$. Suppose $\gamma \in \mathfrak{C}^D(\varepsilon_\circ)$, $\psi_J \in \mathfrak{N}^D(J)$, and $\widehat{f}$ is supported on $\mathbb{A}_\lambda$. Then we have*

$$A^\gamma[\psi_J]\mathcal{P}_{\mathbf{c}} f = \sum_{\ell\in\mathbb{Z}:\, |\ell|\le\lambda^{10\nu}} e^{it\ell}\mathcal{T}^{\mathbf{c}}_\lambda\big(c_\ell\pi_{\mathbf{c}}\widehat{f}\,\big) + \mathcal{E}_{\mathbf{c}} f, \quad t \in \tilde{I},$$

and the following hold with C, C_N, and C_δ independent of γ and ψ_J:

$$|c_\ell(\xi)| \le C_N \lambda^{\nu-\frac{1}{3}}(1+\lambda^{-3\nu}|\ell|)^{-N} \tag{5.4.2}$$

for any N and

$$\|\mathcal{E}_{\mathbf{c}} f\|_{L^q(\mathbb{R}^3\times\tilde{I})} \le C_\delta \lambda^{C-\frac{3}{2}\nu D}\|f\|_p, \quad 1\le p\le q\le\infty. \tag{5.4.3}$$

Summation over ℓ results from the Fourier series expansion in t of an amplitude function which appears after factoring out $e^{-it\Phi^{\mathbf{c}}(\xi)}$. This simplifies the amplitude function depending both on ξ and t which causes considerable loss in bound when we attempt to directly apply the multilinear restriction estimate (for example see [9, Theorem 6.2]).

For the proof of Lemma 5.4.2 and Lemma 5.4.4 below we write $m_J(t,\xi)$ in a different form. Changing of variables $s \to s + \sigma(\xi)$, we have

$$m_J(t,\xi) = (2\pi)^{-3} e^{-it\Phi^{\mathbf{c}}(\xi)} \int e^{-it\phi(s,\xi)} \psi_J(s+\sigma(\xi)) ds, \tag{5.4.4}$$

where

$$\phi(s,\xi) := \gamma(s+\sigma(\xi)) \cdot \xi - \gamma(\sigma(\xi)) \cdot \xi \, .$$

We here note that $J \subset (1+2c_\circ)J_\circ$ and $|\sigma(\xi)| \le 5c_\circ$ for $\xi \in \mathbb{A}^*_\lambda$ by Lemma 5.3.1. Thus $\phi \in \mathrm{C}^{D-2}([-1/2,1/2] \times \mathbb{A}^*_\lambda)$ and $\operatorname{supp} \psi_J(\cdot + \sigma(\xi)) \subset 2^3 J_\circ$. Since $\gamma \in \mathfrak{C}^D(\varepsilon_\circ)$ and $\gamma''(\sigma(\xi)) \cdot \xi = 0$, by Taylor's expansion it follows that

$$\phi(s,\xi) = \Lambda_\gamma(\xi)\big(- R_\gamma(\xi) s + \frac{1}{6} s^3 + \Theta(s,\xi)\big), \tag{5.4.5}$$

$$|\partial_s^k \Theta(s,\xi)| \le C_k \varepsilon_\circ |s|^{\max(4-k,0)}, \quad 0 \le k \le D. \tag{5.4.6}$$

In what follows we occasionally resort to (5.4.5) and (5.4.6) to exploit the properties of the phase function $\phi(\cdot,\xi)$.

Proof of Lemma 5.4.2. We need to consider $m_J(t,\xi)$ while $\xi \in \operatorname{supp} \pi_{\mathbf{c}}$. We break

$$\psi_J(s+\sigma(\xi)) = a_m(s,\xi) + a_e(s,\xi),$$

where $a_m(s,\xi) = \psi_J(s+\sigma(\xi))\beta_0(2^{-4}\lambda^{\frac{1}{3}-\nu} s)$. Then we put

$$\mathcal{I}_\theta(t,\xi) = (2\pi)^{-3} \int e^{-it\phi(s,\xi)} a_\theta(s,\xi) ds, \quad \theta \in \{m,e\}.$$

By (5.4.4) it follows

$$m_J(t,\xi) = e^{-it\Phi^{\mathbf{c}}(\xi)} \big(\mathcal{I}_m(t,\xi) + \mathcal{I}_e(t,\xi)\big).$$

The major term is $\mathcal{I}_m$ while $\mathcal{I}_e$ decays fast as $\lambda \to \infty$. Let $\chi_\circ \in \mathrm{C}_0^\infty([0,2\pi])$ such that $\chi_\circ = 1$ on the interval $[2^{-1}, 2^2]$. Expanding $\chi_\circ(t)\mathcal{I}_m(t,\xi)$ into Fourier series in t over the interval $[0,2\pi]$ we have

$$\chi_\circ(t)\mathcal{I}_m(t,\xi) = \sum_{\ell \in \mathbb{Z}} c_\ell(\xi) e^{it\ell}.$$

Note that $\mathcal{F}(\chi_\circ \mathcal{I}_m(\cdot,\xi))(\ell) = (2\pi)^{-3}\int \widehat{\chi_\circ}(\ell+\phi(s,\xi))a_m(s,\xi)ds$. Since $|\phi(s,\xi)| \le C\lambda^{3\nu}$ on the support of $a_m(\cdot,\xi)$ by (5.4.5), we have $|\mathcal{F}(\chi_\circ \mathcal{I}_m(\cdot,\xi))(\ell)| \le C\lambda^{\nu-\frac{1}{3}}|\ell|^{-N}$ for any N if $|\ell| \ge C_1\lambda^{3\nu}$ for a large C_1. Thus we get (5.4.2) for any $N > 0$. We also note that $|\partial_\xi^\alpha \phi| \le C$ and $|\partial_\xi^\alpha a_m| \le C_\delta$ because $|\partial_\xi^\alpha \sigma| \le C\lambda^{-|\alpha|}$ on $\mathbb{A}_\lambda^*$ for $|\alpha| \le D-2$ (see Lemma 5.3.1). By the same argument we obtain, for any $N > 0$,

$$|\partial_\xi^\alpha c_\ell(\xi)| \le C_\delta \lambda^{\nu-\frac{1}{3}}(1+\lambda^{-3\nu}|\ell|)^{-N}. \tag{5.4.7}$$

We now put

$$\mathcal{E}_{\mathbf{c}} g(x,t) = \sum_{|\ell|>\lambda^{10\nu}} e^{it\ell}\mathcal{F}_x^{-1}\big(c_\ell e^{-it\Phi^{\mathbf{c}}}\pi_{\mathbf{c}}\widehat{g}\,\big) + \mathcal{F}_x^{-1}(\mathcal{I}_e(t,\cdot)e^{-it\Phi^{\mathbf{c}}}\pi_{\mathbf{c}}\widehat{g}\,).$$

We shall show (5.4.3) to complete the proof. The terms $\mathcal{F}_x^{-1}\big(c_\ell e^{-it\Phi^{\mathbf{c}}}\pi_{\mathbf{c}}\widehat{g}\,\big)$ in the summation can be handled easily. Combining the estimate (5.4.7) and $|\partial_\xi^\alpha e^{-it\Phi^{\mathbf{c}}}| \le C$ for $|\alpha| \le 4$, we see that $\mathcal{F}_x^{-1}(c_\ell e^{-it\Phi^{\mathbf{c}}}\pi_{\mathbf{c}}\widehat{g}\,) = K_t * \mathcal{P}_{\mathbf{c}} g$ and $|K_t| \le C_\delta \lambda^C(1+\lambda^{-3\nu}|\ell|)^{-N}(1+|x|)^{-4}$. Thus, the convolution inequality gives

$$\|\mathcal{F}_x^{-1}(c_\ell e^{-it\Phi^{\mathbf{c}}}\pi_{\mathbf{c}}\widehat{g}\,)\|_{L^q(\mathbb{R}^3\times\tilde{I})} \le C_\delta\lambda^C(1+\lambda^{-3\nu}|\ell|)^{-N}\|\mathcal{P}_{\mathbf{c}} g\|_p$$

for $1 \le p \le q \le \infty$. Taking a large $N \ge D$ and using the estimate in (5.3.5), we obtain $\sum_{|\ell|\ge\lambda^{10\nu}} \|\mathcal{F}_x^{-1}(c_\ell e^{-it\Phi^{\mathbf{c}}}\pi_{\mathbf{c}}\widehat{g}\,)\|_{L^q(\mathbb{R}^3\times\tilde{I})} \le C_\delta\lambda^{C-2\nu D}\|g\|_p$.

In order to show the estimate for $\mathcal{F}_x^{-1}(\mathcal{I}_e(t,\cdot)e^{-it\Phi^{\mathbf{c}}}\pi_{\mathbf{c}}\widehat{g}\,)$ we claim

$$|\partial_\xi^\alpha \mathcal{I}_e(t,\xi)| \le C_\delta\lambda^{-\frac{3}{2}\nu(D-|\alpha|)} \tag{5.4.8}$$

for $\xi \in \operatorname{supp}\pi_{\mathbf{c}}$ and $|\alpha| \le 4$. Using (5.4.8) for $|\alpha| \le 4$, similarly as before, we see $\mathcal{F}_x^{-1}(\mathcal{I}_e(t,\cdot)e^{-it\Phi^{\mathbf{c}}}\pi_{\mathbf{c}}\widehat{g}\,) = K_t * \mathcal{P}_{\mathbf{c}} g$ with $|K_t| \le C_\delta\lambda^{C-\frac{3}{2}\nu D}(1+|x|)^{-4}$. Therefore, the convolution inequality and (5.3.5) give

$$\big\|\mathcal{F}_x^{-1}(\mathcal{I}_e(t,\cdot)e^{-it\Phi^{\mathbf{c}}}\pi_{\mathbf{c}}\widehat{g}\,)\big\|_{L^q(\mathbb{R}^3\times\tilde{I})} \le C_\delta\lambda^{C-\frac{3}{2}\nu D}\|g\|_p, \quad 1 \le p \le q \le \infty.$$

Now it remains to show (5.4.8). We recall $a_e(s,\xi) = \psi_J(s+\sigma(\xi))(1-\beta_0(2^{-4}\lambda^{\frac{1}{3}-\nu}s))$. Since $|s| \ge 2^4\lambda^{\nu-\frac{1}{3}}$ on the support of $a_e(\cdot,\xi)$ and $|R_\gamma(\xi)| \le 2\lambda^{2\nu-\frac{2}{3}}$ for $\xi \in \operatorname{supp}\pi_{\mathbf{c}}$, by (5.4.5) and (5.4.6) it follows that $C_1\lambda|s|^2 \le |\partial_s\phi(s,\xi)| \le C_2\lambda|s|^2$ and

$$\begin{aligned} C_3\lambda|s|^{3-k} \le |\partial_s^k\phi(s,\xi)| &\le C_4\lambda|s|^{3-k}, \qquad k=2,3,\\ |\partial_s^k\phi(s,\xi)| &\le C_5\varepsilon_\circ\lambda, \qquad 4 \le k \le D \end{aligned} \tag{5.4.9}$$

for some positive constants $C_1, \dots, C_5$. Thus, noting $|\partial_s^k a_e(s,\xi)| \le C_\delta \lambda^{(\frac{1}{3}-\nu)k}$ for $0 \le k \le D$, we have

$$b_{\ell+1} := \frac{|\partial_s^{\ell+1}\phi(s,\xi)|}{|\partial_s\phi(s,\xi)|^{\ell+1}} \le C_\delta \lambda^{-\frac{3}{2}\nu(\ell+1)}, \qquad b'_\ell := \frac{|\partial_s^{\ell} a_e(s,\xi)|}{|\partial_s\phi(s,\xi)|^{\ell}} \le C_\delta \lambda^{-3\nu\ell} \tag{5.4.10}$$

for $\ell \ge 1$ if $\xi \in \operatorname{supp}\pi_{\mathbf{c}}$ and $|s| \ge 2^4\lambda^{\nu-\frac{1}{3}}$. After integrating by parts $D-1$ times we see that $|\mathcal{I}_e(t,\xi)|$ is bounded by a finite sum of the terms $C\int \prod_{j=1}^m \mathcal{M}_{\ell_j} ds$ where $\mathcal{M}_\ell \in \{b_{\ell+1}, b'_\ell\}$, $\sum_{j=1}^m \ell_j = D-1$, and $\ell_j \ge 1$. Using (5.4.10) we get $|\mathcal{I}_e(t,\xi)| \le C_\delta \lambda^{-\frac{3}{2}\nu D}$ for $\xi \in \operatorname{supp}\pi_{\mathbf{c}}$. Furthermore, since $\partial_s^k \partial_\xi^\alpha \phi$, $\alpha \ne 0$ are bounded, the same argument shows (5.4.8). $\square$

Multipliers of $A^\gamma[\psi_J]\mathcal{P}_{\mathbf{o}}^1$ and $A^\gamma[\psi_J]\mathcal{P}_{\mathbf{o}}^0$

We obtain similar expansions for $m_J\pi_{\mathbf{o}}^j$, $j = 0, 1$. As we shall see, $m_J\pi_{\mathbf{o}}^0$ is a minor term decaying rapidly as $\lambda \to \infty$ (see (5.4.21)). We concentrate on the case $\xi \in \operatorname{supp}\pi_{\mathbf{o}}^1$ for the moment.

Let $\rho_1 \in \mathrm{C}_0^\infty([2^{-5}, 2^5])$, $\rho_0 \in \mathrm{C}_c^\infty([0, 2^{-4}))$ and $\rho_2 \in \mathrm{C}^\infty((2^4, \infty))$ such that $\rho_1 = 1$ on $[2^{-4}, 2^4]$ and $\rho_0 + \rho_1 + \rho_2 = 1$ on $[0, \infty)$. For $j = 0, 1, 2$, we set

$$\begin{aligned} a_j(s,\xi) &= \psi_J(s+\sigma(\xi))\rho_j\big(R_\gamma^{-1/2}(\xi)|s|\big), \\ \mathcal{I}_j(t,\xi) &= (2\pi)^{-3}\int e^{-it\phi(s,\xi)} a_j(s,\xi)\, ds, \end{aligned}$$

and then we have

$$m_J(t,\xi) = e^{-it\Phi^{\mathbf{c}}(\xi)}\big(\mathcal{I}_0(t,\xi) + \mathcal{I}_1(t,\xi) + \mathcal{I}_2(t,\xi)\big). \tag{5.4.11}$$

The main term is $\mathcal{I}_1$ while $\mathcal{I}_0$ and $\mathcal{I}_2$ are rapidly decaying as $\lambda \to \infty$ (see (5.4.22) below). The second derivative of the phase function does not vanish on $\operatorname{supp} a_1(\cdot,\xi)$, so we may apply the method of stationary phase for $\mathcal{I}_1(t,\xi)$. For the purpose we set

$$\widetilde{\phi}(s,\xi) = L^{-1}(\xi)\phi\big(R_\gamma^{1/2}(\xi)s, \xi\big), \tag{5.4.12}$$

where $L(\xi) = \Lambda_\gamma(\xi)R_\gamma(\xi)^{\frac{3}{2}}$ and set

$$\begin{aligned} a^\pm(s,\xi) &= \psi_J\big(R_\gamma^{1/2}(\xi)s + \sigma(\xi)\big)\rho_1(\pm s), \\ \mathcal{I}_1^\pm(t,\xi) &= (2\pi)^{-3}R_\gamma^{1/2}(\xi)\int e^{-itL(\xi)\widetilde{\phi}(s,\xi)} a^\pm(s,\xi)\, ds. \end{aligned}$$

By scaling $s \to R_\gamma^{1/2}(\xi)s$ we have

$$\mathcal{I}_1(t,\xi) = \mathcal{I}_1^+(t,\xi) + \mathcal{I}_1^-(t,\xi). \tag{5.4.13}$$

We try to find the stationary points of the function $\widetilde{\phi}(\cdot,\xi)$ which give rise to two different phase functions $\Phi^\pm$ (see (5.4.15) below). As we shall see later, it is important for application of the multilinear restriction estimate how smooth these phase functions are. So, we deal with the matter carefully.

Lemma 5.4.3. *There are $\tau_+, \tau_- \in \mathrm{C}^{D-4}(\mathbb{A}_\lambda^* \times [-\delta^{10}, \delta^{10}])$, homogeneous of degree zero, such that $\pm\tau_\pm(\xi,\theta) \in [2^{-1}, 2]$ and, if $R_\gamma(\xi) \ge 0$,*

$$\partial_s \widetilde{\phi}\big(\tau_\pm(\xi, R_\gamma^{1/2}(\xi)), \xi\big) = 0. \tag{5.4.14}$$

Proof. We begin by setting

$$\Theta_0(s,\xi) = s^{-3}\Theta(s,\xi),$$

which is homogeneous of degree zero in ξ. One can see $\Theta_0 \in \mathrm{C}^{D-3}([-1/2,1/2]\times \mathbb{A}_\lambda^*)$ because $\Theta_0(s,\xi) = (s/3!)\int_0^1 (1-t)^3\gamma^{(4)}(st+\sigma(\xi))\cdot\xi\Lambda_\gamma^{-1}(\xi)dt$ by Taylor's theorem with integral remainder. Then we consider the function

$$\widetilde{\phi}_0(s,\xi,\theta) = -s + \frac{s^3}{3!} + s^3\Theta_0(\theta s,\xi)$$

with $(s,\xi,\theta) \in \Omega^\pm := (\pm[2^{-5},2^5]) \times \mathbb{A}_\lambda^* \times [-\delta^{10},\delta^{10}]$. It is clear that $\widetilde{\phi}_0 \in \mathrm{C}^{D-3}(\Omega^\pm)$.

Since Θ_0, $\partial_s\Theta_0$ and $\partial_s^2\Theta_0$ are $\mathcal{O}(\varepsilon_\circ)$ as can be seen using (5.4.5) and (5.4.6), we have $\partial_s\widetilde{\phi}_0(s,\xi,\theta) = -1 + s^2/2 + \mathcal{O}(\varepsilon_\circ)$ and $\partial_s^2\widetilde{\phi}_0(s,\xi,\theta) = s + \mathcal{O}(\varepsilon_\circ)$. We now note that $\partial_s\widetilde{\phi}_0(\cdot,\xi,\theta)$ has two distinct zeros which are respectively close to $\sqrt{2}$ and $-\sqrt{2}$, thus by the implicit function theorem there are $\tau_+(\xi,\theta)$ and $\tau_-(\xi,\theta)$ such that $\partial_s\widetilde{\phi}\big(\tau_\pm(\xi,\theta),\xi,\theta\big) = 0$ and $\pm\tau_\pm(\xi,\theta) \in [2^{-1},2]$ if $\varepsilon_\circ$ is small enough. Additionally, τ_+ and τ_- are $D-4$ times continuously differentiable since so is $\partial_s\widetilde{\phi}_0$. By (5.4.5) and (5.4.12) we note that $\widetilde{\phi}_0(s,\xi,R_\gamma^{1/2}(\xi)) = \widetilde{\phi}(s,\xi)$, thus it follows that $\widetilde{\phi}_0(s,\xi,R_\gamma^{1/2}(\xi)) = \partial_s\widetilde{\phi}(s,\xi)$ when $R_\gamma(\xi) \ge 0$. Therefore we obtain (5.4.14). □

We set

$$s_\pm(\xi) = R_\gamma^{1/2}(\xi)\tau_\pm\big(\xi, R_\gamma^{1/2}(\xi)\big).$$

Then from (5.4.12) it follows $\gamma'\big(s_\pm(\xi)+\sigma(\xi)\big)\cdot\xi=0$. We define

$$\Phi^\pm(\xi)=\gamma\big(s_\pm(\xi)+\sigma(\xi)\big)\cdot\xi \tag{5.4.15}$$

for $\xi\in\mathbb{A}^*_\lambda\cap\{\xi:R_\gamma(\xi)\ge 0\}$. If $R_\gamma(\xi)=0$ for some ξ, $\nabla\Phi^\pm(\xi)$ may not exist because $R_\gamma^{1/2}$ is not differentiable at ξ. However, $\nabla\Phi^\pm$ can be defined to be a continuous function on $\mathbb{A}^*_\lambda\cap\{\xi:R_\gamma(\xi)\ge 0\}$. Indeed, differentiating (5.4.15) gives

$$\nabla\Phi^\pm(\xi)=\gamma\big(s_\pm(\xi)+\sigma(\xi)\big) \tag{5.4.16}$$

if $R_\gamma(\xi)>0$. Thus $\nabla\Phi^\pm$ becomes continuous on $\mathbb{A}^*_\lambda\cap\{\xi:R_\gamma(\xi)\ge 0\}$ if we set $\nabla\Phi^\pm(\xi)=\gamma\big(\sigma(\xi)\big)$ when $R_\gamma(\xi)=0$ since γ, σ are continuous.

We define the adjoint restriction operators $\mathcal{T}^\pm_\lambda$ by

$$\mathcal{T}^\pm_\lambda g(x,t)=\int_{\mathcal{C}^{\mathbf{o}}_\lambda(\delta)} e^{i(x\cdot\xi-t\Phi^\pm(\xi))}g(\xi)\,d\xi,$$

where $\mathcal{C}^{\mathbf{o}}_\lambda(\delta):=\{\xi\in\mathbb{A}^*_\lambda:0\le R_\gamma(\xi)\le 2\delta^{100}\}$. Putting together the discussion so far with the method of stationary phase we can obtain

Lemma 5.4.4. *Let $0<\nu\ll 1$, $M=[\frac{D-1}{3}]$, and $J\in\mathfrak{J}_\circ(\delta)$. Suppose $\gamma\in\mathfrak{C}^D(\varepsilon_\circ)$, $\psi_J\in\mathfrak{N}^D(J)$, and $\widehat f$ is supported on $\mathbb{A}_\lambda$. Then, we have*

$$A^\gamma[\psi_J](\mathcal{P}^1_{\mathbf{o}}+\mathcal{P}^0_{\mathbf{o}})f=\sum_\pm\sum_{\ell=0}^{M-1}t^{-\frac{2\ell+1}{2}}\mathcal{T}^\pm_\lambda\big(\gamma^\pm_\ell\pi^1_{\mathbf{o}}\widehat f\,\big)+\mathcal{E}_{\mathbf{o}}f,\quad t\in\tilde I, \tag{5.4.17}$$

and the following hold with C and C_δ independent of γ and ψ_J:

$$|\gamma^\pm_\ell(\xi)|\le C_\delta\lambda^{-\frac13-\frac{\nu}{2}}\lambda^{-3\ell\nu} \tag{5.4.18}$$

for $0\le\ell\le M-1$ and

$$\|\mathcal{E}_{\mathbf{o}}f\|_{L^q(\mathbb{R}^3\times\tilde I)}\le C_\delta\lambda^{C-3\nu M}\|f\|_p,\quad 1\le p\le q\le\infty. \tag{5.4.19}$$

It should be noted that the expansion in (5.4.17) is obtained only on the support of $\pi^1_{\mathbf{o}}$ but not on the larger set $\mathcal{C}^{\mathbf{o}}_\lambda(\delta)$.

We now proceed to apply to $\mathcal{I}^\pm_1$ the method of stationary phase. We first note that $\operatorname{supp}a^\pm(\cdot,\xi)\subset\pm[2^{-5},2^5]$ and, as seen in the above, the phase $\widetilde\phi(\cdot,\xi)$ has the stationary points $\tau_\pm(\xi,R_\gamma^{1/2}(\xi))$ while $\partial_s^2\widetilde\phi(\cdot,\xi)=s+\mathcal{O}_s(\varepsilon_\circ)$ for $\xi\in\mathbb{A}^*_\lambda\cap\{\xi:0\le R_\gamma(\xi)\le 2\delta^{100}\}$. We also note that $|L(\xi)|\ge 2^{-1}\lambda^{3\nu}$ for

$\xi \in \operatorname{supp} \pi_{\mathbf{o}}^1$ and that $L(\xi)\widetilde{\phi}(\tau_\pm(\xi, R_\gamma^{1/2}(\xi)), \xi) = \gamma\big(s_\pm(\xi) + \sigma(\xi)\big) \cdot \xi - \Phi^{\mathbf{c}}(\xi)$. Bring all these observations together, we now apply [23, Theorem 7.7.5] (also see [23, Theorem 7.7.6]) and obtain

$$\mathcal{I}_1^\pm(t, \xi) = e^{it(\Phi^{\mathbf{c}}(\xi) - \Phi^\pm(\xi))} R_\gamma^{1/2}(\xi) \sum_{\ell=0}^{M-1} d_\ell^\pm(\xi)(tL(\xi))^{-\frac{1}{2}-\ell} + e_M^\pm(t, \xi) \tag{5.4.20}$$

for $\xi \in \operatorname{supp} \pi_{\mathbf{o}}^1$ where $M = [\frac{D-1}{3}]$ and $e_M^\pm(t, \xi) = \mathcal{O}\big(|tL(\xi)|^{-M}\big)$. The functions $d_\ell^\pm(\xi)$ are bounded on the support of $\pi_{\mathbf{o}}^1$ since so are $\partial_s^k \widetilde{\phi}$ and $\partial_s^k a^\pm$.

Proof of Lemma 5.4.4. Recalling (5.4.11) and (5.4.13) we write

$$m_J(\pi_{\mathbf{o}}^1 + \pi_{\mathbf{o}}^0) = e^{-it\Phi^{\mathbf{c}}}(\mathcal{I}_1^+ + \mathcal{I}_1^-)\pi_{\mathbf{o}}^1 + e^{-it\Phi^{\mathbf{c}}}(\mathcal{I}_0 + \mathcal{I}_2)\pi_{\mathbf{o}}^1 + m_J \pi_{\mathbf{o}}^0.$$

Using (5.4.20), we now put

$$\mathcal{E}(t, \cdot) = e^{-it\Phi^{\mathbf{c}}}\big(e_M^+(t, \cdot) + e_M^-(t, \cdot)\big)\pi_{\mathbf{o}}^1 + e^{-it\Phi^{\mathbf{c}}}\big(\mathcal{I}_0(t, \cdot) + \mathcal{I}_2(t, \cdot)\big)\pi_{\mathbf{o}}^1 + m_J(t, \cdot)\pi_{\mathbf{o}}^0,$$

and then we set $\mathcal{E}_{\mathbf{o}} f = \mathcal{F}_\xi^{-1}(\mathcal{E}(t, \cdot)\widehat{f}\,)$ and $\gamma_\ell^\pm(\xi) = R_\gamma^{1/2}(\xi) d_\ell^\pm(\xi)(L(\xi))^{-\frac{1}{2}-\ell}$. Thus we have (5.4.17) and the inequality (5.4.18) follows because $|L(\xi)| \geq 2^{-1}\lambda^{3\nu}$ and $d_\ell^\pm$ are bounded on the support of $\pi_{\mathbf{o}}^1$.

To show (5.4.19) we use the following:

$$|\partial_\xi^\alpha m_J(t, \xi)| \leq C_\delta \lambda^{-\frac{3}{2}\nu(D-|\alpha|)}, \qquad \xi \in \operatorname{supp} \pi_{\mathbf{o}}^0, \tag{5.4.21}$$

and

$$\begin{aligned} &|\partial_\xi^\alpha \mathcal{I}_0(t, \xi)| \leq C_\delta \lambda^{-\frac{3}{2}\nu(D-|\alpha|)}, \\ &|\partial_\xi^\alpha \mathcal{I}_2(t, \xi)| \leq C_\delta \lambda^{-\frac{3}{2}\nu(D-|\alpha|)}, \end{aligned} \qquad \xi \in \operatorname{supp} \pi_{\mathbf{o}}^1. \tag{5.4.22}$$

Assuming this for the moment we obtain (5.4.19). Note that $|\partial_\xi^\alpha \Phi^{\mathbf{c}}| \leq C\lambda^{1-|\alpha|}$ and $|\partial_\xi^\alpha e_M^\pm| \leq C_1 \lambda^{C-3\nu M}$ for $|\alpha| \leq 4$. Combining this, (5.4.21) and (5.4.22) for $|\alpha| \leq 4$ and using the estimate (5.3.5), we get (5.4.19) in the same manner as before.

To complete the proof, we are left to prove (5.4.21) and (5.4.22). Let us first consider (5.4.21) which is easier. Since $R_\gamma(\xi) \leq -\lambda^{2\nu - \frac{2}{3}}$ for $\xi \in \operatorname{supp} \pi_{\mathbf{o}}^0$, by (5.4.5) we see that $|\partial_s \phi| \geq C_1 \lambda\big(-R_\gamma(\xi) + s^2(1/2 - \varepsilon_\circ |s|)\big) \geq C_2 \lambda \max(s^2, \lambda^{2\nu - \frac{2}{3}})$ for some $C_1, C_2 > 0$. Combining this with (5.4.9), we have (5.4.10) holds for $\ell \geq 1$ when a_e is replaced by $\psi_J(s + \sigma(\xi))$. Thus integration by parts gives $|m_J(t,\xi)| \le C_\delta \lambda^{-\frac{3}{2}\nu D}$ if $R_\gamma(\xi) \le -\lambda^{2\nu-\frac{2}{3}}$. The same argument also works for $\partial_\xi^\alpha m_J(t,\xi)$, so we obtain (5.4.21).

We now show (5.4.22) only with $\alpha = 0$, and the derivatives $\partial_\xi^\alpha \mathcal{I}_0$ and $\partial_\xi^\alpha \mathcal{I}_2$ can be handled likewise. We consider $\mathcal{I}_0$ first. By (5.4.5) we have $|\partial_s \phi| \ge C\lambda R_\gamma(\xi)$ for $|s| \le 2^{-4} R_\gamma^{1/2}(\xi)$. Combining this with (5.4.9), we get the first estimate in (5.4.10) for $\ell \ge 1$ when $|s| \le 2^{-4} R_\gamma^{1/2}(\xi)$ because $\lambda^{2\nu-\frac{2}{3}} \le R_\gamma(\xi)$. Note that $|\partial_s^\ell a_0(s,\xi)| \le C_\delta R_\gamma^{-\ell/2}(\xi)$, hence for $\ell \ge 1$ we have the second estimate in (5.4.10) with a_e replaced by a_0. Therefore repeated integration by parts gives the estimate for $\mathcal{I}_0$. We can handle $\mathcal{I}_2$ in the same manner. Since $|s| \ge 2^4 R_\gamma^{1/2}(\xi)$, by (5.4.5) we have $C_1\lambda|s|^2 \le |\partial_s \phi(s,\xi)| \le C_2 \lambda |s|^2$ and obviously $|\partial_s^\ell a_2(s,\xi)| \le C_\delta R_\gamma^{-\ell/2}(\xi)$. So, we get the estimate (5.4.10) for $|s| \ge 2^4 R_\gamma^{1/2}(\xi)$ and $\ell \ge 1$ while a_e is replaced by a_2. Thus integration by parts gives the estimate for $\mathcal{I}_2$. $\square$

In contrast to $\Phi^{\mathbf{c}}$ the 2nd derivatives of $\Phi^\pm$ are no longer bounded. However, a computation with $\gamma = \gamma_\circ$ ‡ leads us to expect that $\Phi^\pm \in \mathrm{C}^{1,1/2}$. What follows shows this holds true for $\gamma \in \mathfrak{C}^D(\varepsilon_\circ)$.

Lemma 5.4.5. *For $\xi_1, \xi_2 \in \mathcal{C}_1^{\mathbf{o}}(\delta)$, there is a constant C independent of γ such that*

$$|\nabla \Phi^\pm(\xi_1) - \nabla \Phi^\pm(\xi_2)| \le C|\xi_1 - \xi_2|^{\frac{1}{2}}. \tag{5.4.23}$$

Proof. Let us set $\tau_0^\pm(\xi) = \tau_\pm(\xi, R_\gamma^{1/2}(\xi))$, so $s_\pm(\xi) = R_\gamma^{1/2}(\xi)\tau_0^\pm(\xi)$. Using (5.4.16) and applying the mean value inequality to γ, it is easy to see

$$|\nabla \Phi^\pm(\xi_1) - \nabla \Phi^\pm(\xi_2)| \le C|s_\pm(\xi_1) - s_\pm(\xi_2)| + C|\sigma(\xi_1) - \sigma(\xi_2)|.$$

Since $\sigma \in \mathrm{C}^{D-2}(\mathbb{A}_\lambda^*)$ from Lemma 5.3.1, we only have to consider the first term on the right hand side, which is in turn bounded by

$$|R_\gamma^{1/2}(\xi_1) - R_\gamma^{1/2}(\xi_2)||\tau_0^\pm(\xi_1)| + R_\gamma^{1/2}(\xi_2)|\tau_0^\pm(\xi_1) - \tau_0^\pm(\xi_2)|.$$

It is easy to see that $|R_\gamma^{1/2}(\xi_1) - R_\gamma^{1/2}(\xi_2)| \le C|\xi_1 - \xi_2|^{\frac{1}{2}}$. Since $\tau^\pm$ is $D-4$ times continuously differentiable in a region containing $\mathcal{C}_1^{\mathbf{o}}(\delta)$ (Lemma 5.4.3) and $\tau_0^\pm(\xi) = \tau_\pm(\xi, R_\gamma^{1/2}(\xi))$, by the mean value inequality it follows that $|\tau_0^\pm(\xi_1) - \tau_0^\pm(\xi_2)| \le C|R_\gamma^{1/2}(\xi_1) - R_\gamma^{1/2}(\xi_2)| + C|\xi_1 - \xi_2|$. Consequently, we get the inequality (5.4.23). $\square$

‡If $\gamma = \gamma_\circ$, $\Phi^{\mathbf{c}}(\xi) = -\xi_1\xi_2/\xi_3 + \xi_2^3/(3\xi_3^2)$ and $\Phi^\pm(\xi) = \Phi^{\mathbf{c}}(\xi) \mp 3^{-1}\xi_3\big(\xi_2^2/\xi_3^2 - 2\xi_1/\xi_3\big)^{3/2}$.

5.5 Multilinear restriction estimate

In this section we obtain a form of multilinear restriction estimate which we need to prove (5.4.1). The surfaces associated to $\Phi^{\mathbf{c}}$ and $\Phi^{\pm}$ have a certain curvature property, so it is possible to get an L^2-L^q smoothing estimate using the typical TT^* argument. However, the consequent estimate is not so strong enough as to be useful for controlling the maximal operator. Instead, we utilize 4-linear estimates which are to be deduced from the multilinear restriction estimate under transversality assumption ([9]).

Multilinear restriction estimate for $\mathrm{C}^{1,\alpha}$ hypersurfaces

For the adjoint restriction estimate, the surfaces are typically assumed to be compact and twice continuously differentiable. The same assumption was also made for the multilinear restriction estimate in [9, Theorem 1.16] but the phase functions $\Phi^{\pm}$ no longer have bounded second derivatives. Nevertheless, it is not difficult to see that the argument in [9] continues to work with $\mathrm{C}^{1,\alpha}$ surface, $\alpha > 0$(see Theorem 2.2.3).

Making use of Theorem 2.2.3 we obtain the following.

Proposition 5.5.1. *Let $\theta_1, \ldots, \theta_4 \in \{\mathbf{c}, +, -\}$ and let $J_k \in \mathfrak{J}_\circ(\delta)$, $1 \le k \le 4$. Suppose that $\gamma \in \mathfrak{C}^D(\varepsilon_\circ)$ and dist$(J_\ell, J_k) \ge \delta$, $\ell \ne k$. Then, for any $\varepsilon > 0$ and $R \ge 1$ there is a constant C_ε such that*

$$\Big\| \prod_{k=1}^{4} |\mathcal{T}_1^{\theta_k}\big(\widetilde{\chi}_{\mathcal{R}_{J_k}}(\lambda\cdot) g_k\big)|^{\frac14} \Big\|_{L^{\frac83}(\mathbb{B}^4(0,R))} \le C\delta^{-C_\varepsilon} R^{\varepsilon} \prod_{k=1}^{4} \|g_k\|_2^{\frac14}. \tag{5.5.1}$$

Proof. We begin with recalling that $\widetilde{\chi}_{\mathcal{R}_{J_k}}(\lambda\cdot)$ is supported in $\lambda^{-1}\mathcal{R}_{J_k}(2^6)$ and that $|R_\gamma(\xi)| \le 2\delta^{100}$ if $\xi \in \mathcal{C}_1^{\mathbf{c}}(\delta)$ or $\mathcal{C}_1^{\mathbf{o}}(\delta)$. Since $\nabla_\xi \Phi^{\mathbf{c}}(\xi) = \gamma(\sigma(\xi)) + \gamma'(\sigma(\xi)) \cdot \xi \nabla\sigma(\xi)$, we have $\nabla_\xi \Phi^{\mathbf{c}}(\xi) = \gamma(\sigma(\xi)) + \mathcal{O}(\delta^{100})$ for $\xi \in \mathcal{C}_1^{\mathbf{c}}(\delta)$. If $\xi \in \mathcal{C}_1^{\mathbf{o}}(\delta)$, by (5.4.16) we have $\nabla_\xi \Phi^{\pm}(\xi) = \gamma(\sigma(\xi)) + \mathcal{O}_s(2^2\delta^{50})$ because $|R_\gamma(\xi)| \le 2\delta^{100}$. Thus

$$\mathrm{N}_k(\xi) := |(\nabla\Phi^{\theta_k}(\xi), 1)|^{-1}(\nabla\Phi^{\theta_k}(\xi), 1)$$

which is normal to the surface $(\xi, -\Phi^{\theta_k}(\xi))$ satisfies

$$\mathrm{N}_k(\xi) = \frac{(\gamma(\sigma(\xi)), 1)}{\sqrt{|\gamma(\sigma(\xi))|^2 + 1}} + \mathcal{O}_s(2^3\delta^{50}), \quad \xi \in \mathcal{C}_1^{\theta_k}(\delta), \quad k = 1, \ldots, 4,$$

where we denote $\mathcal{C}_1^{\pm}(\delta) = \mathcal{C}_1^{\mathbf{o}}(\delta)$.

Let $\xi_k \in \lambda^{-1}\mathcal{R}_{J_k}(2^6) \cap \mathcal{C}_1^{\theta_k}(\delta)$, $k = 1, \dots, 4$. Then we have $\sigma(\xi_k) \in [-3c_\circ, 3c_\circ]$ since $J_k \subset (1+2c_\circ)J_\circ$. Let Γ denote the matrix whose k-th column is the vector $\big(\gamma(\sigma(\xi_k)), 1\big)$, $k = 1, \dots, 4$. By the generalized mean value theorem (see for example [41, Part V, Ch.1, 95]) there exists $u_k \in [-3c_\circ, 3c_\circ]$ such that

$$\det \Gamma = \det \begin{pmatrix} \gamma(u_1) & \gamma'(u_2) & \gamma''(u_3) & \gamma'''(u_4) \\ 1 & 0 & 0 & 0 \end{pmatrix} \prod_{1 \le \ell < k \le 4} |\sigma(\xi_\ell) - \sigma(\xi_k)|.$$

Since $\gamma \in \mathfrak{C}^D(\varepsilon_\circ)$ and $u_1, \dots, u_4 \in [-3c_\circ, 3c_\circ]$, the determinant on the right hand side has its absolute value $1 + \mathcal{O}_s(\varepsilon_\circ)$ regardless of γ (for example see (5.3.1)). On the other hand, using (5.3.3) with $s = c_{J_k}$, for $\xi_k \in \lambda^{-1}\mathcal{R}_{J_k}(2^6) \cap \mathcal{C}_1^{\theta_k}(\delta)$ we have $|c_{J_k} - \sigma(\xi_k)| \le 2^{-2}\delta$ with a small enough $\varepsilon_\circ$, and we also have $|c_{J_\ell} - c_{J_k}| \ge (1+2c_\circ)\delta$, $\ell \ne k$ because $\operatorname{dist}(J_\ell, J_k) \ge \delta$. So, $|\sigma(\xi_\ell) - \sigma(\xi_k)| > 2^{-1}\delta$ if $\ell \ne k$, thus we have $\prod_{1 \le \ell < k \le 4} |\sigma(\xi_\ell) - \sigma(\xi_k)| > 2^{-6}\delta^6$. Consequently, we obtain

$$|\det(\mathrm{N}_1(\xi_1), \dots, \mathrm{N}_4(\xi_4))| > 2^{-7}\delta^6$$

provided that $\xi_k \in \lambda^{-1}\mathcal{R}_{J_k}(2^6) \cap \mathcal{C}_1^{\theta_k}(\delta)$ for $k = 1, \dots, 4$. That is to say, the transversality condition holds uniformly regardless of the choice of $\theta_1, \dots, \theta_4 \in \{\mathbf{c}, +, -\}$.

We now note that $\Phi^{\mathbf{c}}$ is continuously differentiable at least twice in a region containing $\mathcal{C}_1^{\mathbf{c}}(\delta)$ and that $\|\Phi^\pm\|_{\mathrm{C}^{1,1/2}(\mathcal{C}_1^{\mathbf{o}}(\delta))} \le C$ by Lemma 5.4.5. To apply Theorem 2.2.3 we need only to make it sure that $\Phi^\pm$ extends as a $\mathrm{C}^{1,1/2}$ function to an open set containing $\mathcal{C}_1^{\mathbf{o}}(\delta)$. The only part of the boundary which can be problematic is $S := \{\xi : R_\gamma(\xi) = 0\} \cap \mathcal{C}_1^{\mathbf{o}}(\delta)$ since $\Phi^\pm$ is homogenous and $D-4$ times continuously differentiable on $\{\xi : R_\gamma(\xi) = 2\delta^{100}\} \cap \mathcal{C}_1^{\mathbf{o}}(\delta)$ (see Lemma 5.3.1 and 5.4.3). We note that $R_\gamma(\xi) = 0$ if and only if $g(\xi) := \gamma'(\sigma(\xi)) \cdot \xi = 0$. Since $\nabla g(\xi) = \gamma''(\sigma(\xi)) = e_2 + \mathcal{O}_s(6c_\circ)$ for $\xi \in \mathbb{A}_1^*$ by Lemma 5.3.1 and since $g \in \mathrm{C}^{D-2}(\mathbb{A}_1^*)$, by the implicit function theorem it follows that S is a part of a C^{D-2} boundary. Thus we can extend $\Phi^\pm$ to be a $\mathrm{C}^{1,1/2}$ function across S (e.g., [18, pp. 136–137]). Therefore we may apply Theorem 2.2.3 and get the estimate (5.5.1). $\square$

As $\Phi^{\mathbf{c}}$, $\Phi^\pm$ are homogeneous of degree 1, the following is an immediate consequence of Proposition 5.5.1 by means of scaling and Plancherel's theorem.

Corollary 5.5.2. *Under the same assumption as in Proposition 5.5.1, for any $\varepsilon > 0$ there is a $C_\varepsilon = C_\varepsilon(\delta) > 0$ such that*

$$\Big\| \prod_{k=1}^{4} |\mathcal{T}_\lambda^{\theta_k}(\widetilde{\chi}_{\mathcal{R}_{J_k}} \widehat{f}_k)|^{\frac{1}{4}} \Big\|_{L^{\frac{8}{3}}(\mathbb{B}^4(0,2^3))} \le C_\varepsilon \lambda^\varepsilon \prod_{k=1}^{4} \|f_k\|_2^{\frac{1}{4}}. \tag{5.5.2}$$

5.5.1 Multilinear estimate for $A^\gamma[\psi_{J_k}]\mathcal{P}_{\mathbf{n}} P_{J_k}$

We are ready to prove Proposition 5.4.1. We first show the multilinear estimate in $L^q(\mathbb{R}^3 \times \tilde{I})$ from which we deduce the weighted estimate.

Proposition 5.5.3. *Let $J_k \in \mathfrak{J}_\circ(\delta)$, $1 \le k \le 4$. Suppose that dist$(J_\ell, J_k) \ge \delta$, $\ell \neq k$. If $1/q = 5/(8p) + 1/16$ and $2 \le p \le 6$, for $\varepsilon > 0$ there are constants $C_\varepsilon = C_\varepsilon(\delta)$ and $D = D(\varepsilon)$ such that*

$$\Big\| \prod_{k=1}^{4} |A^\gamma[\psi_{J_k}]\mathcal{P}_{\mathbf{n}} P_{J_k} f_k|^{\frac{1}{4}} \Big\|_{L^q(\mathbb{R}^3 \times \tilde{I})} \le C_\varepsilon \lambda^{-\frac{1}{3p} - \frac{1}{6} + \varepsilon} \prod_{k=1}^{4} \|f_k\|_{L^p(\mathbb{R}^3)}^{\frac{1}{4}} \tag{5.5.3}$$

whenever $\gamma \in \mathfrak{C}^D(\varepsilon_\circ)$, $\psi_{J_k} \in \mathfrak{N}^D(J_k)$, and $\widehat{f}_k$ is supported on $\mathbb{A}_\lambda$.

By the localization argument it is sufficient for the estimate (5.5.3) to show its local counterpart. In fact, we have

Lemma 5.5.4. *Let $1 \le p \le q \le \infty$ and $b \in \mathbb{R}$, and let $I' \subset \tilde{I}$ be an interval. Let $\gamma \in \mathfrak{C}^D(\varepsilon_\circ)$, $\omega \in \Omega^\alpha$, $0 < \alpha \le 4$, and $\psi_{J_k} \in \mathfrak{N}^D(J_k)$, $J_k \in \mathfrak{J}_\circ(\delta)$, $1 \le k \le 4$. If*

$$\Big\| \prod_{k=1}^{4} |A^\gamma[\psi_{J_k}]\mathcal{P}_{\mathbf{n}} P_{J_k} f_k|^{\frac{1}{4}} \Big\|_{L^q(\mathbb{B}^3(0,1) \times I', \omega)} \le B \lambda^b [\omega]_\alpha^{\frac{1}{q}} \prod_{k=1}^{4} \|f_k\|_{L^p(\mathbb{R}^3)}^{\frac{1}{4}} \tag{5.5.4}$$

holds for a large enough $D = D(b)$, then we have

$$\Big\| \prod_{k=1}^{4} |A^\gamma[\psi_{J_k}]\mathcal{P}_{\mathbf{n}} P_{J_k} f_k|^{\frac{1}{4}} \Big\|_{L^q(\mathbb{R}^3 \times I', \omega)} \le C_\delta B \lambda^b [\omega]_\alpha^{\frac{1}{q}} \prod_{k=1}^{4} \|f_k\|_{L^p(\mathbb{R}^3)}^{\frac{1}{4}}. \tag{5.5.5}$$

Proof. Let $K_k(\cdot, t)$ denote the kernel of the operator $A^\gamma[\psi_{J_k}]\mathcal{P}_{\mathbf{n}} P_{J_k}$. We note that the multiplier of $\mathcal{P}_{\mathbf{n}} P_{J_k}$ is given by $m(\xi) = \widetilde{\chi}_{\mathbb{A}_\lambda^*}(\xi)\beta_0(\delta^{-100}|R_\gamma(\xi)|)\widetilde{\chi}_{\mathcal{R}_{J_k}}(\xi)$ and $\|m(\lambda\cdot)\|_{\mathrm{C}^M} \le C\delta^{-CM}$ for $M \le D-2$. Since $|\gamma(s)| \le 2(c_\circ + \varepsilon_\circ)$ for $s \in J_k$, by Lemma 5.1.3 we have $|K_k(x,t)| \le C_\delta E_M(x)$ for $M \le (D-5)/2$ if $|x| \ge 2$ and $t \in \tilde I$. For $\mathbf{k} \in \mathbb{Z}^3$ set $B_{\mathbf{k}} = \mathbb{B}^3(\mathbf{k}, 1)$ and $B'_{\mathbf{k}} = \mathbb{B}^3(\mathbf{k}, 3)$. Then we have

$$|A^\gamma[\psi_{J_k}]\mathcal{P}_{\mathbf{n}} P_{J_k} f| \le \sum_{\mathbf{k}\in\mathbb{Z}^3} \chi_{B_{\mathbf{k}}} |A^\gamma[\psi_{J_k}]\mathcal{P}_{\mathbf{n}} P_{J_k}(\chi_{B'_{\mathbf{k}}} f)| + C_\delta E_M * |f|.$$

Taking $M = 4N+9$ and combining this with $|A^\gamma[\psi_{J_k}]\mathcal{P}_{\mathbf{n}} P_{J_k} g| \le C_\delta \lambda^3 (1+|\cdot|)^{-N} * |g|$, we see that $\prod_{k=1}^4 |A^\gamma[\psi_{J_k}]\mathcal{P}_{\mathbf{n}} P_{J_k} f_k|$ is bounded by

$$\sum_{\mathbf{k}\in\mathbb{Z}^3} \chi_{B_{\mathbf{k}}} \prod_{k=1}^4 |A^\gamma[\psi_{J_k}]\mathcal{P}_{\mathbf{n}} P_{J_k}(\chi_{B'_{\mathbf{k}}} f_k)| + C_\delta \prod_{k=1}^4 (E_N * |f_k|).$$

Since $\|E_N * |f|\|_{L^q(\mathbb{R}^3\times I',\omega)} \le C[\omega]_\alpha^{1/q} \lambda^{-N} \|f\|_p$ for $1 \le p \le q$, taking a large $N \ge -b$, we may disregard the second term. We now use (5.5.4) to get

$$\Big\| \prod_{k=1}^4 |A^\gamma[\psi_{J_k}]\mathcal{P}_{\mathbf{n}} P_{J_k}(\chi_{B'_{\mathbf{k}}} f_k)|^{\frac14} \Big\|_{L^q(B_{\mathbf{k}}\times I',\omega)} \le B\lambda^b [\omega]_\alpha^{\frac1q} \prod_{k=1}^4 \|\chi_{B'_{\mathbf{k}}} f_k\|_{L^p(\mathbb{R}^3)}^{\frac14}.$$

Thus the desired estimate (5.5.5) follows by taking summation over $\mathbf{k}$ and Hölder's inequality since $B'_{\mathbf{k}}$ overlap each other at most 6^2 times. □

Thanks to Lemma 5.5.4, the proof of Proposition 5.5.3 is reduced to showing

$$\Big\| \prod_{k=1}^4 |A^\gamma[\psi_{J_k}]\mathcal{P}_{\mathbf{n}} P_{J_k} f_k|^{\frac14} \Big\|_{L^q(\mathbb{B}^3(0,1)\times\tilde I)} \le C_\varepsilon \lambda^{-\frac{1}{3p}-\frac16+\varepsilon} \prod_{k=1}^4 \|f_k\|_{L^p(\mathbb{R}^3)}^{\frac14} \tag{5.5.6}$$

for p, q satisfying $1/q = 5/(8p) + 1/16$ and $2 \le p \le 6$. Since $\|\mathcal{P}_{\mathbf{n}} P_{J_k} g\|_p \le C_\delta \|g\|_p$ by (5.3.6), using the estimate (5.3.9) with $p = 6$ after Hölder's inequality, we get the estimate (5.5.6) with $p = 6$. Thus in view of interpolation we only have to obtain

$$\Big\| \prod_{k=1}^4 |A^\gamma[\psi_{J_k}]\mathcal{P}_{\mathbf{n}} P_{J_k} f_k|^{\frac14} \Big\|_{L^{\frac83}(\mathbb{B}^3(0,1)\times\tilde I)} \le C_\varepsilon \lambda^{-\frac13+\varepsilon} \prod_{k=1}^4 \|f_k\|_{L^2(\mathbb{R}^3)}^{\frac14}. \tag{5.5.7}$$

Proof of (5.5.7). For a given $\varepsilon > 0$ we fix ν such that $10\nu = 2^{-1}\varepsilon$ and then take an integer D such that $D \ge C_1/\nu$ with a large constant C_1. For simplicity let us set

$$F_k = A^\gamma[\psi_{J_k}]\mathcal{P}_{\mathbf{n}} P_{J_k} f_k, \quad k = 1, \dots, 4. \tag{5.5.8}$$

By Lemma 5.4.2 and 5.4.4, we have

$$F_k = F_k^{\mathbf{c}} + F_k^{+} + F_k^{-} + \mathcal{E} f_k, \quad k = 1, \dots, 4,$$

where $\mathcal{E}$ satisfies $\|\mathcal{E} f_k\|_q \le C_\delta \lambda^{C-\nu D} \|f_k\|_p$ for $1 \le p \le q \le \infty$, and

$$F_k^{\mathbf{c}} = \sum_{|\ell| \le \lambda^{10\nu}} e^{it\ell} \mathcal{T}_\lambda^{\mathbf{c}}(c_\ell \pi_{\mathbf{c}} \widetilde{\chi}_{\mathcal{R}_{J_k}} \widehat{f}_k), \qquad F_k^{\pm} = \sum_{0 \le m \le M-1} t^{-\frac{2m+1}{2}} \mathcal{T}_\lambda^{\pm}(\gamma_m^{\pm} \pi_{\mathbf{o}}^1 \widetilde{\chi}_{\mathcal{R}_{J_k}} \widehat{f}_k).$$

We thus need to handle the terms $\Pi_{k=1}^4 h_k$ where $h_k \in \{F_k^{\mathbf{c}}, F_k^{\pm}, \mathcal{E} f_k\}$, $1 \le k \le 4$. Any product which has $\mathcal{E} f_k$ as one of its factors is easily handled by taking C_1 large enough if one uses Hölder's inequality and the trivial estimates $\|\mathcal{T}_\lambda^{\mathbf{c}}(\pi_{\mathbf{c}} \widehat{g})\|_q \le C_\delta \lambda^C \|g\|_p$ and $\|\mathcal{T}_\lambda^{\pm}(\pi_{\mathbf{o}}^1 \widehat{g})\|_q \le C_\delta \lambda^C \|g\|_p$, which hold for $1 \le p \le q \le \infty$. So, it suffices to obtain the estimates for the products which consist only of the terms $F_k^{\mathbf{c}}$, $F_k^{\pm}$. By (5.4.2) and (5.4.18) we have $\sum_{|\ell| \le \lambda^{10\nu}} \lambda^{\frac{1}{3}-\nu} \|c_\ell\|_\infty \le C\lambda^{3\nu}$ and $\sum_{\ell=0}^{M-1} \|\gamma_\ell^{\pm}\|_\infty \le C_\delta \lambda^{-\frac{1}{3}-\frac{\nu}{2}}$. Thus, using the estimate (5.5.2) and Plancherel's theorem, we obtain

$$\Big\| \prod_{k=1}^4 |F_k^{\theta_k}|^{\frac14} \Big\|_{L^{\frac{8}{3}}(\mathbb{B}^4(0,2^3))} \le C_\varepsilon \lambda^{-\frac13 + 10\nu + \frac{\varepsilon}{2}} \prod_{k=1}^4 \|f_k\|_2^{\frac14},$$

where $\theta_k \in \{\mathbf{c}, +, -\}$ for $1 \le k \le 4$. Therefore we get (5.5.7). □

5.5.2 Putting altogether

We are in a position to prove Proposition 5.4.1. By Lemma 5.5.4, it suffices to show that

$$\Big\| \prod_{k=1}^4 |A^\gamma[\psi_{J_k}] \mathcal{P}_{\mathbf{n}} P_{J_k} f_k|^{\frac14} \Big\|_{L^p(\mathbb{B}^3(0,1)\times[1,2],\omega)} \le C_\delta \lambda^{-\varepsilon_p} \prod_{k=1}^4 \|f_k\|_{L^p(\mathbb{R}^3)}^{\frac14} \tag{5.5.9}$$

for $14/5 < p \le 6$. The exponent $p/4$ here is not necessarily bigger than or equal to 1, so we can not use Hölder's inequality to exploit the fact that the Fourier transform of $\mathcal{P}_{\mathbf{n}} P_{J_k} f_k$ is supported in $\mathbb{B}^3(0, 2\lambda)$. Nevertheless, the following lemma enables us to deal with the case $p/4 < 1$.

Lemma 5.5.5. *Let $0 < p \le \infty$, $0 < \alpha \le 4$ and $\omega \in \Omega^\alpha$. Suppose that $F \in L^p(\mathbb{R}^4, \omega)$ and $\widehat{F}$ is supported on $\mathbb{B}^4(0, \lambda)$. Then we have*

$$\|F\|_{L^p(\mathbb{R}^4,\omega)} \le C[\omega]_\alpha^{\frac1p} \lambda^{\frac{4-\alpha}{p}} \|F\|_{L^p(\mathbb{R}^4)}. \tag{5.5.10}$$

Proof. Note that $wdxdt \ll dxdt$ since $\omega \in \Omega^\alpha$. Thus $\|F\|_{L^\infty(\mathbb{R}^4,\omega)} \le \|F\|_{L^\infty(\mathbb{R}^4)}$. When $1 \le p < \infty$, as already seen, (5.5.10) is a simple consequence of Hölder's inequality, so we only need to consider $p \in (0,1)$.

Let us take $\varphi \in \mathcal{S}(\mathbb{R}^4)$ such that $\widehat{\varphi} = 1$ on $\mathbb{B}^4(0,1)$ and $\widehat{\varphi}$ is supported on $\mathbb{B}^4(0,2)$. Then we have $F = F * \varphi_\lambda$ since $\widehat{F}$ is supported on $\mathbb{B}^4(0,\lambda)$. We first claim that

$$|F|^p \le C|F|^p * |\varphi|_\lambda^p, \tag{5.5.11}$$

where we denote $|\varphi|_\lambda^p = (|\varphi|^p)_\lambda$. Once we have (5.5.11) the proof of (5.5.10) is rather straightforward. Indeed, by (5.5.11) it follows that $\|F\|^p_{L^p(\mathbb{R}^4,\omega)} \le C\int |F(x)|^p |\varphi|_\lambda^p * \omega(x)dx \le C\|F\|^p_{L^p(\mathbb{R}^4)} \||\varphi|^p_\lambda * \omega\|_\infty$. Since $\||\varphi|^p_\lambda * \omega\|_\infty \le C\lambda^{4-\alpha}[\omega]_\alpha$, this gives (5.5.10).

We now turn to the proof of (5.5.11). By scaling we may assume $\lambda = 1$, otherwise one may replace F with $F(\cdot/\lambda)$. To show (5.5.11) when $\lambda = 1$, we first notice that

$$|F| * |\varphi|(x) = \int |F(y)\varphi(x-y)|dy \le \|F\varphi(x-\cdot)\|_\infty^{1-p} \, |F|^p * |\varphi|^p(x).$$

Since the Fourier transform of $F\varphi(x-\cdot)$ is supported in $\mathbb{B}^4(0,5)$, $F\varphi(x-\cdot) = \big(F\varphi(x-\cdot)\big) * 5^{-1}\varphi(5^{-1}\cdot)$ and $\|F\varphi(x-\cdot)\|_\infty \le C\|F\varphi(x-\cdot)\|_1 = C|F|*|\varphi|(x)$. Combining this and the inequality above gives $(|F| * |\varphi|)^p \le C|F|^p * |\varphi|^p$. Since $|F| \le |F| * |\varphi|$, we get (5.5.11) with $\lambda = 1$. $\square$

Proof of (5.5.9)

Here we keep using the simpler notation (5.5.8). The estimate (5.5.9) follows if we show

$$\Big\| \prod_{k=1}^4 |\widetilde{\chi} F_k|^{\frac14} \Big\|_{L^p(\mathbb{B}^3(0,1)\times \tilde{I},\omega)} \le C_\delta \lambda^{-\varepsilon_p} \prod_{k=1}^4 \|f_k\|^{\frac14}_{L^p(\mathbb{R}^3)}. \tag{5.5.12}$$

We deduce the weighted estimate from Proposition 5.5.3 in the same way as in the proof of Proposition 5.3.7. The difference is that we are dealing with a multilinear estimate and the exponent $p/4$ can be less than 1. To apply Lemma 5.5.5 we break $\widetilde{\chi} F_k = \widetilde{A}_k f_k + \mathcal{E}_k f_k$ where

$$\mathcal{F}(\widetilde{A}_k f_k)(\xi,\tau) = \beta_0((\lambda r_0)^{-1}\tau)\mathcal{F}(\widetilde{\chi} F_k)(\xi,\tau)$$

and $r_0 = 1 + 4\max\{|\gamma(s)| : s \in \mathrm{supp}\psi_{J_k}, k = 1, \dots, 4\}$. By Lemma 5.1.6 we have $|\mathcal{E}_k f_k(x,t)| \le C\widetilde{E}_t^M * |f_k|(x)$ for any M. Thus, taking a large M and using the trivial estimate $|\widetilde{\chi} F_k| \le C\lambda^3(1 + |\cdot|)^{-M} * |f_k|$, one can easily see

$$\Big\| \prod_{k=1}^4 |\widetilde{\chi} F_k|^{\frac14} \Big\|_{L^q(\mathbb{B}^3(0,1)\times\tilde I,\omega)} \le C \Big\| \prod_{k=1}^4 |\widetilde{A}_k f_k|^{\frac14} \Big\|_{L^q(\mathbb{R}^4,\omega)} + C\lambda^{-N} \prod_{k=1}^4 \|f_k\|_{L^p(\mathbb{R}^3)}^{\frac14}$$

for a large N. Since $[\omega]_3 \le 1$ and the support of $\mathcal{F}(\prod_{k=1}^4 \widetilde{A}_k f_k)$ is contained in a ball of radius $2^4 r_0\lambda$, the inequality (5.5.10) gives $\| \prod_{k=1}^4 |\widetilde{A}_k f_k|^{\frac14} \|_{L^q(\mathbb{R}^4,\omega)} \le C\lambda^{1/q} \| \prod_{k=1}^4 |\widetilde{A}_k f_k|^{\frac14} \|_{L^q(\mathbb{R}^4)}$. To estimate the last one in $L^q(\mathbb{R}^4)$, using the estimate $|\mathcal{E}_k f_k(x,t)| \le C\widetilde{E}_t^M * |f_k|$ again and disregarding the minor contributions, it suffices to obtain the bound on $\| \prod_{k=1}^4 |\widetilde{\chi} F_k|^{\frac14} \|_{L^q(\mathbb{R}^4)}$. Since $\mathrm{supp}\,\widetilde{\chi} \subset \tilde I$, by the estimate (5.5.3) we get

$$\Big\| \prod_{k=1}^4 |\widetilde{\chi} F_k|^{\frac14} \Big\|_{L^q(\mathbb{B}^3(0,1)\times\tilde I,\omega)} \le C_\varepsilon(\delta)\lambda^{\frac{7}{24}(\frac1p - \frac{5}{14}) + \varepsilon} \prod_{k=1}^4 \|f_k\|_{L^p(\mathbb{R}^3)}^{\frac14}$$

for $1/q = 5/(8p) + 1/16$ and $2 \le p \le 6$. Finally, we obtain (5.5.12) for $14/5 < p \le 6$ by Hölder's inequality since $\|\omega\|_{L^1(\mathbb{B}^3(0,1)\times\tilde I)} \le C[\omega]_3$ and $q \ge p$. $\square$

Remark 1. *In the above we try to obtain the estimate* (5.4.1) *on a range of* p *as large as possible by suppressing* ν *arbitrary small (*Proof of (5.5.7)*). This forces us to take a large* $D \ge C_1/\nu$*. However, to obtain the maximal estimate it is enough to have the estimate* (5.4.1) *on the smaller range* $3 < p \le 6$ *instead of* $14/5 < p \le 6$*. For* $3 < p \le 6$*, we can prove* (5.4.1) *with a fixed* ν *and* D*. For example, optimizing the estimates at various places, we can take* $\nu = 1/397$ *and* $D = 720$*. In other words, Theorem 5.0.1 holds true for* $\gamma \in \mathrm{C}^{720}(I)$*.*

5.6 Closing induction argument

In this section we complete the proof of Theorem 5.0.1. We prove the sufficiency in this section. By the reduction in Section 5.2.2, Lemma 5.1.4) and Lemma 5.1.2, it suffices to prove Proposition 5.2.6, which also proves Theorem 5.1.1.

Decomposition

We first decompose the averaging operator $A^\gamma[\psi]$ in such a way that we can use the multilinear estimate obtained in Section 5.4, 5.5. The following Lemma 5.6.1 is a slight modification of [22, Lemma 2.8]. Let us set

$$\mathfrak{J}_*^4(\delta) = \big\{(J_1, \dots, J_4) : J_1, \dots, J_4 \in \mathfrak{J}_\circ(\delta), \quad \min_{\ell \neq k} \operatorname{dist}(J_\ell, J_k) \geq \delta\big\}.$$

Lemma 5.6.1. *Let $\psi \in \mathfrak{N}^D(J_\circ)$ and $\gamma \in \mathfrak{C}^D(\varepsilon_\circ)$. There is a constant $C = C(D)$ independent of $z = (x,t)$, γ, and δ such that*

$$|A^\gamma[\psi]f(z)| \leq C \max_{J \in \mathfrak{J}_\circ(\delta)} |A^\gamma[\psi_J]f(z)| + C\delta^{-1} \sum_{(J_1,\dots,J_4) \in \mathfrak{J}_*^4(\delta)} \prod_{k=1}^4 \big|A^\gamma[\psi_{J_k}]f(z)\big|^{\frac{1}{4}}, \tag{5.6.1}$$

where $\psi_J \in \mathfrak{N}^D(J)$ and $\psi_{J_k} \in \mathfrak{N}^D(J_k)$.

Proof. Let us recall (5.2.6). It is clear that there is a constant $C_D > 0$ such that $C_D^{-1}\psi\zeta_J \in \mathfrak{N}^D(J)$ for $J \in \mathfrak{J}_\circ(\delta)$. Setting $\psi_J = C_D^{-1}\psi\zeta_J$ we have

$$A^\gamma[\psi]f(z) = C_D \sum_{J \in \mathfrak{J}_\circ(\delta)} A^\gamma[\psi_J]f(z).$$

Let us set $\mathfrak{J}_1 = \mathfrak{J}_\circ(\delta)$. For a fixed z, define J_1^* to be an interval in $\mathfrak{J}_1$ such that $|A^\gamma[\psi_{J_1^*}]f(z)| = \max_{J \in \mathfrak{J}_1} |A^\gamma[\psi_J]f(z)|$. For $k = 2,3,4$, we recursively define $\mathfrak{J}_k$ and $J_k^* \in \mathfrak{J}_k$. Let $\mathfrak{J}_k = \{J \in \mathfrak{J}_{k-1} : \operatorname{dist}(J, J_{k-1}^*) \geq \delta\}$ and let $J_k^* \in \mathfrak{J}_k$ denote an interval such that $|A^\gamma[\psi_{J_k^*}]f(z)| = \max_{J \in \mathfrak{J}_k} |A^\gamma[\psi_J]f(z)|$. Thus, if $\operatorname{dist}(J, J_k^*) \geq \delta$ for all $1 \leq k \leq 3$, we have $|A^\gamma[\psi_J]f| \leq |A^\gamma[\psi_{J_k^*}]f|$ for $1 \leq k \leq 4$.

Let us denote $\mathcal{J} = \bigcup_{k=1}^3 \{J \in \mathfrak{J}_\circ(\delta) : \operatorname{dist}(J, J_k^*) < \delta\}$. Splitting the sum into the cases $J \in \mathcal{J}$ and $J \notin \mathcal{J}$, we have

$$C_D^{-1}|A^\gamma[\psi]f(z)| \leq \sum_{J \in \mathcal{J}} |A^\gamma[\psi_J]f(z)| + \sum_{J \notin \mathcal{J}} |A^\gamma[\psi_J]f(z)|.$$

The first on the right hand side is bounded by $C \max_{J \in \mathfrak{J}_\circ(\delta)} |A^\gamma[\psi_J]f(z)|$ and the second by $C\delta^{-1} \prod_{k=1}^4 |A^\gamma[\psi_{J_k^*}]f(z)|^{\frac{1}{4}}$. (5.6.1) follows since $\operatorname{dist}(J_k^*, J_\ell^*) \geq \delta$ if $k \neq \ell$. $\square$

The next lemma gives control over the first term on the right hand side of (5.6.1).

Lemma 5.6.2. *Let $2 < p \le 6$, and let $[\omega]_3 \le 1$ and $\psi_J \in \mathfrak{N}^D(J)$ for each $J \in \mathfrak{J}_\circ(\delta)$. If $\delta^3\lambda \ge 2^2$ and $\varepsilon_\circ > 0$ is sufficiently small, there is an $\varepsilon_p > 0$ such that*

$$\| \max_{J\in\mathfrak{J}_\circ(\delta)} |A^\gamma[\psi_J]f|\|_{L^p(\mathbb{R}^3\times[1,2],\omega)} \le C\big(\delta^{1-\frac{3}{p}}K_\delta(\lambda) + C_\delta\lambda^{-\varepsilon_p}\big)\|f\|_{L^p(\mathbb{R}^3)}$$

whenever $\gamma \in \mathfrak{C}^D(\varepsilon_\circ)$ and $\widehat{f}$ is supported on $\mathbb{A}_\lambda$.

Proof of Lemma 5.6.2. By the embedding $\ell^p \subset \ell^\infty$ and Minkowski's inequality,

$$\| \max_{J\in\mathfrak{J}_\circ(\delta)} |A^\gamma[\psi_J]f|\|^p_{L^p(\mathbb{R}^3\times[1,2],\omega)} \le 2^p(\mathrm{I} + \mathrm{II}),$$

where

$$\mathrm{I} = \sum_{J\in\mathfrak{J}_\circ(\delta)} \|A^\gamma[\psi_J]P_Jf\|^p_{L^p(\mathbb{R}^3\times[1,2],\omega)},\ \mathrm{II} = \sum_{J\in\mathfrak{J}_\circ(\delta)} \|A^\gamma[\psi_J](f-P_Jf)\|^p_{L^p(\mathbb{R}^3\times[1,2],\omega)}.$$

For II we apply Proposition 5.3.7. Taking $\varepsilon_p = \frac{1}{4}(\frac{1}{2}-\frac{1}{p})$ and using the estimate (5.3.10) with $\varepsilon = \varepsilon_p/2$, we have $\mathrm{II}^{\frac{1}{p}} \le C_\delta\lambda^{-\varepsilon_p}\|f\|_{L^p(\mathbb{R}^3)}$ since there are at most $C\delta^{-1}$ many J. To handle I, we invoke Lemma 5.2.7 and then use Lemma 5.3.3 to obtain

$$\mathrm{I} \le C\delta^{p-3}K_\delta(\lambda)^p \sum_{J\in\mathfrak{J}_\circ(\delta)} \|P_Jf\|_p^p \le C\delta^{p-3}K_\delta(\lambda)^p\|f\|_p^p.$$

Therefore the desired bound follows. □

Now we consider the product terms appearing in (5.6.1).

Lemma 5.6.3. *Let $\frac{14}{5} < p \le 6$, $[\omega]_3 \le 1$, and $(J_1,\dots,J_4) \in \mathfrak{J}^4_*(\delta)$. If $\delta^3\lambda \ge 2^2$ and $\varepsilon_\circ > 0$ is small enough, there are positive constants ε_p, c, D such that*

$$\Big\|\prod_{k=1}^4 |A^\gamma[\psi_{J_k}]f|^{\frac{1}{4}}\Big\|_{L^p(\mathbb{R}^3\times[1,2],\omega)} \le C_\delta\Big(\lambda^{-\varepsilon_p} + \lambda^{-c}K_\delta(\lambda)\Big)\|f\|_{L^p(\mathbb{R}^3)} \qquad (5.6.2)$$

whenever $\gamma \in \mathfrak{C}^D(\varepsilon_\circ)$, $\psi_{J_k} \in \mathfrak{N}^D(J_k)$, $k = 1,\dots,4$, and $\widehat{f}$ is supported in $\mathbb{A}_\lambda$.

Proof. For each $1 \le k \le 4$ we split $f = b_k + g_k$, where

$$b_k = \mathcal{P}_{\mathbf{n}} P_{J_k} f, \quad g_k = \mathcal{P}_{\mathbf{n}}(1 - P_{J_k})f + \mathcal{P}_{\mathbf{e}} f.$$

We here use $f = \mathcal{P}_{\mathbf{n}} f + \mathcal{P}_{\mathbf{e}} f$ because $\widehat{f}$ is supported on $\mathbb{A}_\lambda$. Thus, the left hand side of (5.6.2) is bounded by a constant times

$$\mathfrak{M} = \sum_{h_k \in \{b_k, g_k\}} \Big\| \prod_{k=1}^{4} |A^\gamma[\psi_{J_k}] h_k|^{\frac{1}{4}} \Big\|_{L^p(\mathbb{R}^3 \times [1,2], \omega)}.$$

We consider the cases $(h_1, \dots, h_4) = (b_1, \dots, b_4)$ and $(h_1, \dots, h_4) \ne (b_1, \dots, b_4)$. For the former case we use Proposition 5.4.1 and the estimate (5.3.6). Since $14/5 < p \le 6$, there is an $\varepsilon_p > 0$ such that

$$\Big\| \prod_{k=1}^{4} |A^\gamma[\psi_{J_k}] b_k|^{\frac{1}{4}} \Big\|_{L^p(\mathbb{R}^3 \times [1,2], \omega)} \le C_\delta \lambda^{-\varepsilon_p} \|f\|_{L^p(\mathbb{R}^3)}.$$

For the other case we combine Proposition 5.3.7, 5.3.8, and Lemma 5.2.7. In fact, Proposition 5.3.7 and 5.3.8 followed by (5.3.6) yield

$$\|A^\gamma[\psi_{J_k}] g_k\|_{L^p(\mathbb{R}^3 \times [1,2], \omega)} \le C_\varepsilon \delta^{-C} \lambda^{\frac{1}{2}(\frac{1}{p} - \frac{1}{2}) + \varepsilon} \|f\|_{L^p(\mathbb{R}^3)}$$

for $2 \le p \le 6$. If we consider a particular case $(h_1, \dots, h_4) = (b_1, b_2, b_3, g_4)$, by Hölder's inequality and the above estimate we have

$$\Big\| \prod_{k=1}^{4} |A^\gamma[\psi_{J_k}] h_k|^{\frac{1}{4}} \Big\|_{L^p(\mathbb{R}^3 \times [1,2], \omega)} \le C_\delta \lambda^{-c} \|f\|^{\frac{1}{4}}_{L^p(\mathbb{R}^3)} \prod_{k=1}^{3} \big\|A^\gamma[\psi_{J_k}] b_k\big\|^{\frac{1}{4}}_{L^p(\mathbb{R}^3 \times [1,2], \omega)}$$

for a constant $c > 0$ because $p > 14/5$. We apply Lemma 5.2.7 to handle the last three factors. Since $\|b_k\| \le C_1 \delta^{-C} \|f\|_{L^p(\mathbb{R}^3)}$ from (5.3.6), the inequality (5.2.11) gives

$$\Big\| \prod_{k=1}^{4} |A^\gamma[\psi_{J_k}] h_k|^{\frac{1}{4}} \Big\|_{L^p(\mathbb{R}^3 \times [1,2], \omega)} \le C_\delta \lambda^{-c} K_\delta(\lambda)^{\frac{3}{4}} \|f\|_{L^p(\mathbb{R}^3)}.$$

We can deal with the remaining products similarly. As a consequence, we obtain

$$\mathfrak{M} \le C_\delta \Big(\lambda^{-\varepsilon_p} + \sum_{\ell=1}^{3} \lambda^{-(4-\ell)c} K_\delta(\lambda)^{\frac{\ell}{4}} \Big) \|f\|_{L^p(\mathbb{R}^3)}$$

and therefore the bound (5.6.2) after a simple manipulation since we may assume $\varepsilon_p \le c$ taking a smaller ε_p if necessary. $\square$

We now conclude the proof of (5.2.10) putting together the previous estimates.

Proof of (5.2.10)

Since $\|Af\|_{L^\infty(\mathbb{R}^3\times[1,2],\omega)} \le C\|f\|_{L^\infty(\mathbb{R}^3)}$, by interpolation it is sufficient to show (5.2.10) for $3 < p < 6$. Let $p \in (3,6)$ and take an $\varepsilon_\circ > 0$ small enough and a large D such that the estimates in Lemma 5.6.2 and 5.6.3 hold whenever $\gamma \in \mathfrak{C}^D(\varepsilon_\circ)$ and $\psi_J \in \mathfrak{N}^D(J)$, $J \in \mathfrak{J}_\circ(\delta)$.

Let $\gamma \in \mathfrak{C}^D(\varepsilon_\circ)$, $\omega \in \Omega^3$ with $[\omega]_3 \le 1$ and $\psi \in \mathfrak{N}^D(J_\circ)$, and let f be a function such that $\operatorname{supp}\widehat{f} \subset \mathbb{A}_\lambda$ and $\|f\|_p \le 1$. By (5.6.1) and Minkowski's inequality we see that $\|A^\gamma[\psi]f\|_{L^p(\mathbb{R}^3\times[1,2],\omega)}$ is bounded by

$$C\Big\|\max_{J\in\mathfrak{J}_\circ(\delta)} |A^\gamma[\psi_J]f|\Big\|_{L^p(\mathbb{R}^3\times[1,2],\omega)} + C_\delta \sum_{(J_1,\dots,J_4)\in\mathfrak{J}_*^4(\delta)} \Big\|\prod_{k=1}^4 |A^\gamma[\psi_{J_k}]f|^{\frac14}\Big\|_{L^p(\mathbb{R}^3\times[1,2],\omega)}.$$

Then Lemma 5.6.2 and 5.6.3 gives

$$\|A^\gamma[\psi]f\|_{L^p(\mathbb{R}^3\times[1,2],\omega)} \le C\big(\delta^{1-\frac3p} + \lambda^{-c}\big)K_\delta(\lambda) + C_\delta\lambda^{-\varepsilon_p}$$

if $2^2\delta^{-3} \le \lambda$. Taking supremum over f, ω, ψ, and γ, we obtain

$$Q(\lambda) \le C\big(\delta^{1-\frac3p} + \lambda^{-c}\big)K_\delta(\lambda) + C_\delta\lambda^{-\varepsilon_p} \tag{5.6.3}$$

for $2^2\delta^{-3} \le \lambda$. In order to close the induction we need to modify $Q(\lambda)$ slightly. Fix $0 < b$ which is to be chosen later. We define

$$\overline{Q}_b(\lambda) = \sup_{1\le r\le\lambda} r^b Q(r).$$

We observe $\lambda^b K_\delta(\lambda) \le 2^{2b}\delta^{-3b}\sum_{2^{-2}\delta^3\lambda\le 2^j\le 2^2\delta\lambda} 2^{jb}Q(2^j)$, and hence we have $\lambda^b K_\delta(\lambda) \le C|\log\delta|\delta^{-3b}\overline{Q}_b(2^2\delta\lambda)$. Multiplying λ^b to both sides of (5.6.3), we thus get

$$\lambda^b Q(\lambda) \le C\big(\delta^{1-\frac3p} + \lambda^{-c}\big)|\log\delta|\delta^{-3b}\overline{Q}_b(2^2\delta\lambda) + C_\delta\lambda^{b-\varepsilon_p}$$

for $2^2\delta^{-3} \le \lambda$. We now choose a small enough b such that $1 - \frac3p - 3b > 0$ and $b - \varepsilon_p < 0$, then fix a small enough $\delta > 0$ such that $C\delta^{1-\frac3p}|\log\delta|\delta^{-3b} \le 2^{-2}$ and $2^2\delta \le 1$. Such a choice is clearly possible because $p > 3$. Let $\lambda_\circ$ be a large number such that $\delta^{1-\frac{3}{p}} \geq \lambda_\circ^{-c}$ and $2^2\delta^{-3} \leq \lambda_\circ$. Then we have the inequality $\lambda^b Q(\lambda) \leq 2^{-1}\overline{Q}_b(\lambda) + C_\delta$ for $\lambda \geq \lambda_\circ$ since $\overline{Q}_b$ is increasing. This obviously implies

$$\lambda^b Q(\lambda) \leq 2^{-1}\overline{Q}_b(r) + C_\delta$$

for $\lambda_\circ \leq \lambda \leq r$. Note that $\overline{Q}_b(\lambda_\circ) \leq \lambda_\circ^b C_2$ for a certain constant C_2 (because of the trivial estimate $Q(\lambda) \leq C\lambda^2$). Taking supremum over $\lambda \in [1, r]$ we get $\overline{Q}_b(r) \leq 2^{-1}\overline{Q}_b(r) + \lambda_\circ^b C_2 + C_\delta$. Therefore we have $\overline{Q}_b(r) \leq C_3$ for a constant C_3 and conclude $Q(\lambda) \leq C_3\lambda^{-b}$ for $\lambda \geq 1$. □

Remark 2. *Routine adaptation of our argument proves L^p improving property of the localized maximal operator $\overline{M}f(x) := \sup_{1\leq t\leq 2}\big|Af(x,t)\big|$. In fact, if γ is smooth, the estimate $\|\overline{M}f\|_{L^q(\mathbb{R}^3)} \leq C\|f\|_{L^p(\mathbb{R}^3)}$ holds provided that $(1/p, 1/q)$ is contained in the interior of the triangle with vertices $(0,0)$, $(1/3, 1/3)$, and $(19/66, 8/33)$. It is possible to extend the range slightly making use of the estimate (5.3.9) with $p > 6$. Furthermore, one can show that $\overline{M}$ is bounded from L^p to $L^p(d\mu)$ for $p > 9 - 2\alpha$ when μ is an α dimensional measure and $3 > \alpha > \frac{65-\sqrt{865}}{12} = 2.9657\ldots$*

5.7 Sharpness of Theorem 5.0.1

To prove that L^p boundedness of M fails for $p \leq 3$, it is sufficient to show the next proposition. Our construction below is a modification of Stein's example in [54].

Proposition 5.7.1. *Let $p \leq d$ and $\psi \not\equiv 0$ be a nonnegative continuous function supported in I. Suppose $\gamma : I \to \mathbb{R}^d$ be a smooth nondegenerate curve. Then there is an $h \in L^p(\mathbb{R}^d)$ such that $Mh = \infty$ on a nonempty open set.*

Proof. Since $\psi \geq 0$ and $\psi \not\equiv 0$, we may assume that $\psi(s) \geq c$ on an interval $J \subset I$ for some $c > 0$. By (1.1.1) we may additionally assume that $|\gamma(s)| \geq c$ on J taking a subinterval of J if necessary because the condition (1.1.1) can not be satisfied if there is no such a subinterval.

Since $\gamma'(s), \cdots, \gamma^{(d)}(s)$ are linearly independent, we can write

$$\gamma(s) = \sum_{i=1}^{d} c_i(s)\gamma^{(i)}(s), \quad s \in J \tag{5.7.1}$$

for some smooth functions $c_1, \cdots, c_d$. We claim that there is an $s_\circ \in J$ such that $c_d(s_\circ) \neq 0$. Suppose that there is no such $s_\circ \in J$, that is to say, $c_d(s) \equiv 0$ for all $s \in J$. Differentiating both side of (5.7.1), we have $(c_1'(s) - 1)\gamma'(s) + \sum_{i=2}^{d-1}[c_{i-1}(s) + c_i'(s)]\gamma^{(i)}(s) + c_{d-1}(s)\gamma^{(d)}(s) = 0$, which implies $c_{d-1}(s) \equiv 0$, $c_{i-1}(s) + c_i'(s) \equiv 0$, and $c_1'(s) \equiv 1$ for $s \in J$. This leads to a contradiction and proves the claim. Therefore there are $s_\circ \in J$ and $\delta > 0$ such that

$$|c_d(s)| \geq c, \quad s \in [s_\circ - \delta, s_\circ + \delta] \subset J$$

for some $c > 0$. We only consider the case $c_d(s) \geq c$ since the other case can be handled similarly.

For $x \in \mathbb{R}^d$ let $y = (y_1, \cdots, y_d)$ denote the coordinate of x with respect to the basis $\{\gamma'(s_\circ), \cdots, \gamma^{(d)}(s_\circ)\}$, i.e., $x = y_1\gamma'(s_\circ) + \cdots + y_d\gamma^{(d)}(s_\circ)$, and we set $\overline{y} = (y_1, \cdots, y_{d-1})$. For some $\varepsilon > 0$ we take $g(t) = \chi_{[0,2^{-1}]}(t)|t|^{-\frac{1}{d}}|\log|t||^{-\frac{1}{d}-\varepsilon}$ and then we consider

$$h(x) = \chi_0(|\overline{y}|)g(y_d),$$

where $\chi_0 \in \mathrm{C}_0^\infty([-2, 2])$ is a nonnegative function such that $\chi_0 = 1$ on $[-1, 1]$. It is easy to see $h \in L^p(\mathbb{R}^d)$ for $p \leq d$ because $g \in L^d(\mathbb{R})$ for $p \leq d$. Thus we only have to show that $\sup_{0<t} Ah = \infty$ on a nonempty open set.

Let us write $\gamma(s) = \sum_{i=1}^d a_i(s)\gamma^{(i)}(s_\circ)$ and $\overline{a}(s) = (a_1(s), \cdots, a_{d-1}(s))$. Since $c_j(s_\circ) = a_j(s_\circ)$, $j = 1, \ldots, d$, by Taylor's expansion we have

$$\gamma(s) = \sum_{i=1}^d \big(c_i(s_\circ) + (s - s_\circ)^i/i!\big)\gamma^{(i)}(s_\circ) + \mathcal{O}((s - s_\circ)^{d+1}).$$

So, $y_d - ta_d(s) = y_d - tc_d(s_\circ) - t\big((s - s_\circ)^d/d! + \mathcal{O}((s - s_\circ)^{d+1})\big)$. For $y_d > 0$ we take $t = y_d/c_d(s_\circ) > 0$. Then it follows that $C_1 y_d|s - s_\circ|^d \leq |y_d - ta_d(s)| \leq C_2 y_d|s - s_\circ|^d$ for some $C_1, C_2 > 0$, so $|g(y_d - ta_d(s))| \geq C y_d^{-\frac{1}{d}}|s - s_\circ|^{-1}|\log(y_d|s - s_\circ|^d)|^{-\frac{1}{d}-\varepsilon}$ provided that $|s - s_\circ| < c'$ for a small $c' > 0$ and $0 < y_d \leq 1$. Thus by our choice of δ and $s_\circ$ we have

$$Ah\Big(x, \frac{y_d}{c_d(s_\circ)}\Big) \geq C y_d^{-\frac{1}{d}} \int_{|s-s_\circ|\leq\delta'} \widetilde{\chi}_0(y, s)|s - s_\circ|^{-1}|\log(y_d|s - s_\circ|^d)|^{-\frac{1}{d}-\varepsilon}\, ds$$

for $0 < y_d \leq 1$ where $\delta' = \min(\delta, c')$ and $\widetilde{\chi}_0(y, s) = \chi_0(|\overline{y} - \frac{y_d}{c_d(s_\circ)}\overline{a}(s)|)$. Since $\widetilde{\chi}_0(y, s) \geq 1$ if $|y| \leq r_\circ$ for a small enough $r_\circ > 0$, we have

$$Ah\Big(x, \frac{y_d}{c_d(s_\circ)}\Big) \geq C y_d^{-\frac{1}{d}} \int_{|s-s_\circ|\leq\min(\delta', y_d^{\frac{1}{d}}/10)} |s - s_\circ|^{-1}|\log|s - s_\circ||^{-\frac{1}{d}-\varepsilon}\, ds = \infty$$

for $y \in \mathbb{B}^d(0, r_\circ) \cap \{y : 0 < y_d < 1\}$ as desired. $\square$

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국문초록

이 학위 논문에서는 $\mathbb{R}^d$ 곡선으로 정의되는 평균에 대한 극대 유계와 정칙성을 규명하는 연구를 진행한다. 다양체 위에서 평균으로 정의되는 적분 변환의 극대 유계와 최적 정칙성 문제는 조화 해석학의 기본 주제로, 1970년대부터 널리 연구되어 왔다. 곡선의 간단한 기하학적 구조에도 불구하고, 곡선 위에서 평균의 극대 유계와 최적 정칙성은 낮은 차원 일부를 제외하고 거의 알려지지 않았다. 이 논문의 결과는 모든 차원에서 이들 문제들에 돌파구를 마련하는 획기적인 기여를 한다. 첫째, 삼차원 이상에서 곡선 위의 평균 연산자에 대한 최적 소볼레프 정칙 유계를 부분적 끝점 문제를 제외하고 모두 증명한다. 이는 벨트란, 구오, 히크만, 시거에 의해 제기된 추측을 완전히 해결한 것이다. 둘째, 삼차원 이상 공간안의 곡선위에서 평균값 연산자의 국소 평활 유계를 최적 차수까지 증명한다. 그 결과로, 사차원 이상에서 자명하지 않은 극대 유계를 최초로 보인다. 마지막으로, 삼차원 공간안의 곡선위에서 극대 유계를 최적 범위에서 증명한다.

주요어휘: 평균값 연산자, 극대 유계, 소볼레프 정칙성, 국소적 평활화
학번: 2016-20240

감사의 글

먼저, 저에게 늘 가르침을 주시고 저를 올바르게 지도해주신 이상혁 교수님께 감사를 드립니다. 교수님께서 늘 모범을 보여주시고 배울 기회를 주셔서 올바르게 성장할 수 있었습니다. 교수님께서 보여주신 열정은 늘 제게 큰 자극이 되었습니다. 앞으로도 교수님께서 주신 가르침 잊지 않고 더욱 정진하겠습니다. 그리고 이 자리를 빌어, 바쁘신 와중에도 학위 논문 심사를 맡아주신 김준일 교수님, 김판기 교수님, 정인지 교수님, 조용근 교수님께 감사하다는 말씀을 드립니다.

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