이학 박사 학위논문

Nonlinear partial differential equations on irregular domains

(비정칙 영역에서의 비선형 편미분 방정식)

2022년 8월

서울대학교 대학원

수리과학부

이세찬

Nonlinear partial differential equations on irregular domains

(비정칙 영역에서의 비선형 편미분 방정식)

지도교수 이기암

이 논문을 이학 박사 학위논문으로 제출함

2022년 4월

서울대학교 대학원

수리과학부

이세찬

이세찬의 이학 박사 학위논문을 인준함

2022년 6월

위 원 장	변 순 식	(인)
부위원장	이 기 암	(인)
위 원	김 성 훈	(인)
위 원	박 진 완	(인)
위 원	정 인 지	(인)

Nonlinear partial differential equations on irregular domains

A dissertation
submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
to the faculty of the Graduate School of
Seoul National University

by

Sechan Lee

Dissertation Director : Professor Ki-Ahm Lee

Department of Mathematical Sciences
Seoul National University

August 2022

Abstract

Nonlinear partial differential equations on irregular domains

Sechan Lee
Department of Mathematical Sciences
The Graduate School
Seoul National University

This thesis consists of three papers concerning nonlinear elliptic equations on irregular domains. In the first paper, we establish the Wiener criterion, which characterizes a regular boundary point via nonlinear potential theory, for fully nonlinear equations in non-divergence form. Our approach is based on the investigation of non-variational capacity, and the construction of barrier functions using a homogeneous solution. The second and third papers discuss the random homogenization of an obstacle problem for elliptic operators with Orlicz growth and fully nonlinear operators, respectively. In both cases, the limit profile satisfies a homogenized equation without obstacles, if we assume the stationary ergodicity on the perforating holes with critical size. The heart of analysis lies in capturing the asymptotic behavior of oscillating solutions, by means of energy and viscosity method, respectively.

Key words: Wiener criterion, random homogenization, fully nonlinear operator, Orlicz space
Student Number: 2016-20241

Contents

Chapter 1

Introduction

The analysis on solutions of the partial differential equations becomes more complicated when the irregularity on domains are assumed. For example, the non-smoothness on the boundary of domains or the randomly perforating holes on the interior of domains induce such difficulties. Nevertheless, we are still able to describe several regular properties of solutions by employing an energy method for operators in divergence form and a viscosity method for operators in non-divergence form, respectively. Roughly speaking, we will concentrate on capturing the asymptotic behaviors of solutions near the singular point, in terms of capacity or homogeneous solution.

In the first part of this thesis, we are concerned with the irregularity on the boundary of domains. To illustrate the issues, let Ω be an open and bounded subset in $\mathbb{R}^n$, f be a boundary data on $\partial\Omega$, and $\mathcal{M}$ be an elliptic operator. For the existence of a solution u (in a suitable sense) to the Dirichlet problem

$$\begin{cases} \mathcal{M}[u] = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases}$$

one may apply Perron's method. If the solvability of the Dirichlet problem on any balls is known and $\mathcal{M}$ allows a comparison principle, it is rather straightforward to prove that the upper Perron solution $\overline{H}_f$ satisfies $\mathcal{M}[\overline{H}_f] = 0$ in Ω. Nevertheless, we cannot ensure that the boundary condition $u = f$ on $\partial\Omega$

is satisfied by the upper Perron solution, in general. Instead, we are forced to discover an additional condition for the boundary $\partial\Omega$, which enables us to capture the boundary behavior of $\overline{H}_f$.

To be precise, we say a boundary point $x_0 \in \partial\Omega$ is *regular* with respect to Ω, if

$$\lim_{\Omega \ni y \to x_0} \overline{H}_f(y) = f(x_0).$$

whenever $f \in C(\partial\Omega)$. One simple characterization of a regular boundary point is to find a *barrier function*. As a consequence, by constructing proper barrier functions, geometric criteria on $\partial\Omega$ such as an exterior sphere condition or an exterior cone condition have been invoked to guarantee the boundary continuity at $x_0 \in \partial\Omega$ for a variety of elliptic operators. Here note that aforementioned conditions only serve as sufficient conditions for a boundary point to be regular. In other words, these conditions do not reflect the individual character of each operator, and so they are not sharp enough to be a necessary condition for a regular boundary point.

On the other hand, in the pioneering works [79, 80], Wiener provided an alternative criterion for a regular boundary point, based on potential theory. Namely, for the Laplacian operator ($\mathcal{M} = \Delta$), $x_0 \in \partial\Omega$ is regular if and only if the Wiener integral diverges, i.e.

$$\int_0^1 \frac{\mathrm{cap}_2(\overline{B_t(x_0)} \setminus \Omega, B_{2t}(x_0))}{\mathrm{cap}_2(\overline{B_t(x_0)}, B_{2t}(x_0))} \frac{\mathrm{d}t}{t} = \infty,$$

where $\mathrm{cap}_2(K, \Omega)$ is defined by the variational capacity of the Laplacian operator. Surprisingly, the Wiener criterion becomes both a sufficient and necessary condition for the regularity of a boundary point. Here the notion of capacity is used to measure the 'size' of sets in view of given differential equations. Roughly speaking, $x_0 \in \partial\Omega$ is regular if and only if Ω^c is 'thick' enough at x_0 in the potential theoretic sense.

Both linear and nonlinear potential theory have been extensively studied in literature; see [11, 37, 38, 53, 66, 78] and references therein. Since the main ingredient of potential theory comes from the integration by parts, the theory and corresponding Wiener criterion have been developed mostly for operators in divergence form. Littman, Stampacchia and Weinburger [65] demonstrated the coincidence between the regular points for uniformly elliptic operators $\mathcal{M} = D_j(a_{ij}D_i)$, where a_{ij} is bounded and measurable, and for the Laplacian operator. For the p-Laplacian operator ($\mathcal{M} = \Delta_p, p > 1$), Maz'ya [68] verified the sufficiency of the p-Wiener criterion, i.e. $x_0 \in \partial\Omega$ is regular for Δ_p if

$$\int_0^1 \left(\frac{\mathrm{cap}_p(\overline{B_t(x_0)} \setminus \Omega, B_{2t}(x_0))}{\mathrm{cap}_p(\overline{B_t(x_0)}, B_{2t}(x_0))} \right)^{1/(p-1)} \frac{\mathrm{d}t}{t} = \infty.$$

For the converse direction, Lindqvist and Martio [64] proved the necessity of the Wiener criterion under the assumption $p > n-1$. Later, Kilpeläinen and Malý [46] extended this result to any $p > 1$, via the Wolff potential estimate. For the other available results on the Wiener criterion, we refer to [2] for $p(x)$-Laplacian operators, [59] for operators with Orlicz growth, and [47] for nonlocal operators. Note that all of these results consider elliptic operators in divergence form.

For elliptic operators in non-divergence form, relatively small amounts of results for the Wiener criterion are known. While the equivalence was obtained for $\mathcal{M} = D_j(a_{ij}D_i)$ with merely measurable coefficients in [65], Miller [71, 72] discovered the non-equivalence with respect to $\mathcal{M} = a_{ij}D_{ij}u$, even if the coefficients a_{ij} are continuous. More precisely, he presented examples of linear operators $\mathcal{M}$ in non-divergence form and domains Ω such that $x_0 \in \partial\Omega$ is regular for $\mathcal{M}$, but x_0 is irregular for Δ, and vice-versa. We also refer [50, 55]. On the other hand, Bauman [7] developed the Wiener test for $\mathcal{M} = a_{ij}D_{ij}u$ with continuous coefficients a_{ij}. He proved that $x_0 \in \partial\Omega$ is regular if and only if

(i) $\mathrm{cap}_{\mathcal{M}}(\{x_0\}) > 0$, or

(ii) $\sum_{j=1}^{\infty} \widetilde{g}(x_0, x_0 + 2^{-j}e) \cdot \mathrm{cap}_{\mathcal{M}}(\Omega^c \cap (\overline{B_{2^{-j}}(x_0)} \setminus B_{2^{-j-1}}(x_0))) = \infty$.

Here $\widetilde{g}$ is the normalized Green function and e is a unit vector in $\mathbb{R}^n$.

In Chapter 2, we formulate the Wiener criterion for fully nonlinear elliptic equations in non-divergence form, which is the main result of [60]. Unlike the cases of operators in divergence form, we cannot define the variational capacity by minimizing the corresponding energy. Instead, under the assumption that the operator is positively homogeneous of degree one, we explain the non-variational capacity based on the growth rate of homogeneous solutions. One can see that the non-variational capacity plays a crucial role in investigating the boundary regularity of solutions.

In the second part of this thesis, we present the random homogenization result for elliptic equations with highly oscillating obstacles. Indeed, a variety of physical and biological phenomena can be modeled by partial differential equations on the media with periodic structure (or oscillating obstacles). Then the solutions, u_ε, of these equations are expected to possess periodic oscillation in microscopic scale (often denoted by ε), which is much smaller than the size of the domain with macroscopic scale. The homogenization process is interested in describing the asymptotic behavior of u_ε when $\varepsilon \to 0$ and determining the effective model which is satisfied by the limit solution $u = \lim_{\varepsilon\to 0} u_\varepsilon$.

There has been a large body of literature on the periodic homogenization of linear and nonlinear PDEs; for classical results, see [8, 13, 22, 26, 42] and references therein. Here we concentrate on summarizing the homogenization results which are closely related to our circumstances.

Cioranescu and Murat employed an energy method to analyze the asymptotic behavior of u_ε in their paper [17], entitled "A strange term coming from nowhere". To be precise, they proved that the solution u_ε of Laplace equation ($-\Delta u_\varepsilon = f$) in a perforated domain with critical hole size, converges to the solution u of Laplace equation with an additional term depending on the capacity of holes ($-\Delta u + \mu u = f$). The proof relies on the construction of appropriate correctors with desired properties under abstract framework. Note that in their periodic setting, all holes have the identical size and $u_\varepsilon = 0$

on T_ε rather than $u_\varepsilon \geq 0$ on T_ε.

The homogenization result in [17] was extended to the stationary ergodic setting for the Laplace equations with obstacles by Caffarelli and Mellet [15]. Here the hypothesis of stationary ergodicity is an extension of the notion of periodicity or almost periodicity, and it requires a random variable to have self-averaging behavior. They overcame the difficulty coming from randomness by exploiting the subadditive ergodic theorem: we refer to [1, 16, 21] for details. Tang [77] generalized this result for p-Laplacian operator $(1 < p < n)$ in the stationary ergodic setting.

Furthermore, Γ-convergence methods can be applied in homogenization; see two books [10, 20]. Ansini and Braides [3] described the asymptotic behavior of p-energy type Dirichlet problem in periodically perforated domain. Focardi extended the results for fractional obstacle problems in stationary ergodic setting [28] and in aperiodic setting [29].

Caffarelli and Lee [12] developed a viscosity method for periodic homogenization of Laplacian and fully nonlinear operator with highly oscillating obstacles. They considered a viscosity solution satisfying a uniformly elliptic equation with non-divergence structure, and established a viscosity method to find an effective equation satisfied by the limit function. See also [48] and [61] for an application of a viscosity method for periodic homogenization of nonlinear parabolic equations and semilinear equations, respectively.

In Chapter 3 and 4, we consider elliptic equations with Orlicz growth [57] and fully nonlinear elliptic equations [58], respectively. Let us briefly explain the common main ingredient in Chapter 3 and 4: the correctors. As usual in the homogenization process, the correctors are essential tools to estimate the difference between ε-solutions and the limit solution. In the abstract framework, we first find out the desired properties for correctors to explain the limit profile in the homogenization. Then we construct such correctors and prove that they possess such properties. Note that these correctors are also can be employed to the homogenization result of non-critical hole sizes. In short, when the perforating hole sizes are not critical, we can obtain rather trivial effective equations without additional terms.

Since we consider an obstacle problem with highly oscillating obstacles in microscopic scale, we require oscillating correctors with prescribed values on each hole with random size. To be precise, the corrector must behave like a fundamental solution near each perforated hole, to explain the oscillatory behavior of u_ε. Thus, we will adopt the Dirac-delta measure δ (energy) for operators in divergence form and the homogeneous solution Φ (viscosity) for operators in non-divergence form, respectively. In particular, for non-divergence form operators, we will modify the homogeneous solution to approximate the Dirac-delta measure in sense of 'shape'.

Moreover, another important ingredient in the random homogenization process is the subadditive ergodic theorem. This theorem enables to describe the self-averaging behavior of given random process, which satisfies the stationary ergodic property. To determine the critical value or the critical function which appears in the limit equation, we first study the measure of a contact set. Indeed, we define an auxiliary function which solves an obstacle problem with randomness, and then investigate the coincidence set. Since the measure of a contact sets satisfy the subadditivity, we will conclude that there exists a critical value which separate behaviors of aforementioned auxiliary functions.

Chapter 2

The Wiener Criterion for Fully Nonlinear Elliptic Equations

2.1 Introduction

The goal of this chapter is to establish the Wiener criterion for fully nonlinear elliptic operators, by implementing potential theoretic tools. To illustrate the issues, we consider an *Issacs operator*, i.e. an operator F with the following two properties:

(F1) F is *uniformly elliptic*: there exist positive constants $0 < \lambda \le \Lambda$ such that for any $M \in \mathcal{S}^n$,

$$\lambda \|N\| \le F(M+N) - F(M) \le \Lambda \|N\|, \quad \forall N \ge 0.$$

Here we write $N \ge 0$ whenever N is a non-negative definite symmetric matrix.

(F2) F is *positively homogeneous of degree one*: $F(tM) = tF(M)$ for any $t > 0$ and $M \in \mathcal{S}^n$.

Throughout this chapter, we suppose that F satisfies (F1) and (F2), unless otherwise stated. Typical examples of operators satisfying (F1) and (F2) are the Pucci extremal operators $\mathcal{P}^+_{\lambda,\Lambda}$ and $\mathcal{P}^-_{\lambda,\Lambda}$, defined by

$$\mathcal{P}^+_{\lambda,\Lambda}(M) = \Lambda \sum_{e_i>0} e_i + \lambda \sum_{e_i<0} e_i, \quad \mathcal{P}^-_{\lambda,\Lambda}(M) = \Lambda \sum_{e_i<0} e_i + \lambda \sum_{e_i>0} e_i,$$

where $e_i = e_i(M)$ are the eigenvalues of M. For a fully nonlinear operator F satisfying (F1) and (F2), we define a *dual operator*

$$\widetilde{F}(M) := -F(-M), \quad \text{for } M \in \mathcal{S}^n.$$

Then it is obvious that $\widetilde{F}$ also satisfies (F1) and (F2). One important property that F satisfying (F1) and (F2) possesses is the existence of a homogeneous solution V:

Lemma 2.1.1 (A homogeneous solution; [4, 12]). *There exists a non-constant solution of $F(D^2u) = 0$ in $\mathbb{R}^n \setminus \{0\}$ that is bounded below in B_1 and bounded above in $\mathbb{R}^n \setminus B_1$. Moreover, the set of all such solutions is of the form $\{aV + b \,|\, a > 0, b \in \mathbb{R}\}$, where $V \in C^{1,\gamma}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ can be chosen to satisfy one of the following homogeneity relations: for all $t > 0$*

$$V(x) = V(tx) + \log t \quad \text{in } \mathbb{R}^n \setminus \{0\} \quad \text{where } \alpha^* = 0,$$

or

$$V(x) = t^{\alpha^*} V(tx), \ \alpha^* V > 0 \quad \text{in } \mathbb{R}^n \setminus \{0\},$$

for some number $\alpha^* \in (-1, \infty) \setminus \{0\}$ that depends only on F and n. We call the number $\alpha^* = \alpha^*(F)$ the scaling exponent of F.

We are ready to state our first main theorem, namely, the sufficiency of the Wiener criterion:

Theorem 2.1.2 (The sufficiency of the Wiener crietrion). *If*

$$\int_0^1 \mathrm{cap}_F(\overline{B_t(x_0)} \setminus \Omega, B_{2t}(x_0)) \frac{\mathrm{d}t}{t} = \infty$$

and

$$\int_0^1 \mathrm{cap}_{\widetilde{F}}(\overline{B_t(x_0)} \setminus \Omega, B_{2t}(x_0)) \frac{\mathrm{d}t}{t} = \infty,$$

then the boundary point $x_0 \in \partial\Omega$ *is (*F*-)regular.*

We remark that the Wiener integral is again defined in terms of a capacity, but the definition of a F-capacity is quite different from the variational capacity for the Laplacian case; see Section 2.3 for details. Furthermore, as a corollary of Theorem 2.1.2, we will derive the quantitative estimate for a modulus of continuity at a regular boundary point (Lemma 2.4.7), and suggest another geometric condition, called an exterior corkscrew condition (Corollary 2.4.9).

Our second main theorem is concerned with the necessity of the Wiener criterion. We propose a partial result on the necessary condition, i.e. exploiting the additional structure of F, we show that the Wiener integral at $x_0 \in \partial\Omega$ must diverge whenever x_0 is a regular boundary point.

Theorem 2.1.3 (The necessity of the Wiener criterion)**.** *Suppose that* F *is concave and* $\alpha^*(F) < 1$*. If a boundary point* $x_0 \in \partial\Omega$ *is regular, then*

$$\int_0^1 \mathrm{cap}_F(\overline{B_t(x_0)} \setminus \Omega, B_{2t}(x_0)) \frac{\mathrm{d}t}{t} = \infty.$$

Note that the assumption $\alpha^*(F) < 1$ in the fully nonlinear case corresponds to the assumption $p > n-1$ in the p-Laplacian case, [64]. The underlying idea for both cases is to utilize the non-zero capacity of a line segment (or a set of Hausdorff dimension 1). Further comments on this assumption can be found in Section 2.5.

In this chapter, the main difficulty arises from the inherent lack of divergence structure; we cannot define a variational capacity by means of an energy minimizer, and moreover, we cannot employ integral estimates involving Sobolev inequality and Poincaré inequality. Instead, we will develop potential theory with non-divergence structure by the construction of appropriate barrier functions using the homogeneous solution, and by the application of the comparison principle and Harnack inequality. In short, our strategy is to capture the local boundary behavior of the upper Perron solution $\overline{H}_f$ in terms of newly defined capacity $\mathrm{cap}_F(K,B)$ and the capacity potential (or the balayage) $\hat{R}^1_K(B)$, using prescribed tools. Heuristically, the non-variational capacity measures the 'height' of the F-solution with the boundary value 0 on ∂B and 1 on ∂K, while the variational capacity measures the 'energy' of such function. We emphasize that although our notion of capacity does not satisfy the subadditive property in general, it was still able to recover certain properties of the variational capacity.

Finally, we would like to point out that the dual operator $\widetilde{F}$ is different from F, for general F. Thus, even though u is an F-supersolution, we cannot guarantee $-u$ is an F-subsolution. Moreover, a similar feature is found in the growth rate of the homogeneous solution for F; two growth rates of an upward-pointing homogeneous solution and a downward-pointing one can be different. This phenomenon naturally leads us

(i) to describe the local behavior of both the upper Perron solution $\overline{H}_f$ and the lower Perron solution $\underline{H}_f$ for regularity at $x_0 \in \partial\Omega$;

(ii) to construct two (upper/lower) barrier functions when characterizing a regular boundary point;

(iii) to display two different Wiener integrals in our main theorem,

which differ from the previous results that appeared in [7, 46, 79].

This chapter is organized as follows. In Section 2.2, we summarize the terminology and preliminary results for our main theorems. In short, we introduce F-superharmonic functions and Poisson modification and then perform Perron's method. In Section 2.3, we first define a balayage and a capacity for uniformly elliptic operators in non-divergence form. Then we prove several capacitary estimates by constructing auxiliary functions and provide the characterization of a regular boundary point via balayage. Section 2.4 consists of potential theoretic estimates for the capacity potential. Then we prove the sufficiency of the Wiener criterion and several corollaries. Finally, Section 2.5 is devoted to the proof of the (partial) necessity of the Wiener criterion.

2.2 Perron's Method

2.2.1 F-Supersolutions and F-Superharmonic Functions

In this subsection, we only require the condition (F1) for an operator F. To illustrate Perron's method precisely, we start with two different notions of solutions for a uniformly elliptic operator F: F-solutions and F-harmonic functions. Indeed, we will prove that these two notions coincide.

Definition 2.2.1 (F-supersolution)**.** A lower semi-continuous [resp. upper semi-continuous] function u in Ω is a *(viscosity) F-supersolution* [resp. *(viscosity) F-subsolution*] in Ω, when the following condition holds:

if $x_0 \in \Omega$, $\varphi \in C^2(\Omega)$ and $u - \varphi$ has a local minimum at x_0, then

$$F(D^2\varphi(x_0)) \leq 0.$$

[resp. if $u - \varphi$ has a local maximum at x_0, then $F(D^2\varphi(x_0)) \geq 0$.]

We say that $u \in C(\Omega)$ a *(viscosity) F-solution* if u is both an F-subsolution and an F-supersolution.

Lemma 2.2.2. *Suppose that a lower semi-continuous function u is an F-supersolution in Ω. Then*

$$u(x) = \liminf_{\Omega \ni y \to x} u(y) \quad \textit{for any } x \in \Omega.$$

Proof. We argue by contradiction: suppose that

$$u(x_0) < \liminf_{\Omega \ni y \to x} u(y) \quad \text{for some } x_0 \in \Omega.$$

Then for any $\varphi \in C^2(\Omega)$, it follows that $u - \varphi$ has a local minimum at x_0 and so we can test this function. Therefore,

$$F(D^2\varphi(x_0)) \leq 0 \quad \text{for any } \varphi \in C^2(\Omega),$$

which is impossible. □

Theorem 2.2.3. *(i) (Stability) Let $\{u_k\}_{k\geq 1} \subset C(\Omega)$ be a sequence of F-solutions in Ω. Assume that u_k converges uniformly in every compact set of Ω to u. Then u is an F-solution in Ω.*

(ii) (Compactness) Suppose that $\{u_k\}_{k\geq 1} \subset C(\Omega)$ is a locally uniformly bounded sequence of F-solutions in Ω. Then it has a subsequence that converges locally uniformly in Ω to an F-solution.

Theorem 2.2.4 (Harnack convergence theorem)**.** *Let $\{u_k\}_{k\geq 1} \subset C(\Omega)$ be an increasing sequence of F-solutions in Ω. Then the function $u = \lim_{k\to\infty} u_k$ is either an F-solution or identically $+\infty$ in Ω.*

Proof. If $u(x) < \infty$ for some $x \in \Omega$, it follows from Harnack inequality that u is locally bounded in Ω. The interior C^α-estimate yields that the sequence u_k is equicontinuous in every compact subset of Ω. Thus, applying Arzela-Ascoli theorem and Theorem 2.2.3 (i), we finish the proof. □

We demonstrate two essential tools for Perron's method, namely, the comparison principle and the solvability of the Dirichlet problem in a ball.

Theorem 2.2.5 (Comparison principle for F-super/subsolutions, [43, 44])**.** *Let Ω be a bounded open subset of $\mathbb{R}^n$. Let $v \in \mathrm{USC}(\overline{\Omega})$ [resp.$u \in \mathrm{LSC}(\overline{\Omega})$] be an F-subsolution [resp. F-supersolution] in Ω and $v \leq u$ on $\partial\Omega$. Then $v \leq u$ in $\overline{\Omega}$.*

In the previous theorem, $\mathrm{USC}(\overline{\Omega})$ denotes the set of all upper semicontinuous functions from $\overline{\Omega}$ to $\mathbb{R}$. Moreover, note that for a lower semicontinuous function f, there exists an increasing sequence of continuous functions $\{f_n\}$ such that $f_n \to f$ pointwise as $n \to \infty$.

Theorem 2.2.6 (The solvability of the Dirichlet problem)**.** *Let Ω satisfy a uniform exterior cone condition and $f \in C(\partial\Omega)$. Then there exists a unique F-solution $u \in C(\overline{\Omega})$ of the Dirichlet problem*

$$\begin{cases} F(D^2u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

Proof. The existence depends on the construction of global barriers achieving given boundary data and the standard Perron's method; see [19, 69] and [18, 40]. Then the uniqueness comes from the comparison principle, Theorem 2.2.5. □

Definition 2.2.7 (F-superharmonic function)**.** A function $u : \Omega \to (-\infty, \infty]$ is called *F-superharmonic* if

(i) u is lower semi-continuous;

(ii) $u \not\equiv \infty$ in each component of Ω;

(iii) u satisfies the comparison principle in each open $D \subset\subset \Omega$: If $h \in C(\overline{D})$ is an F-solution in D, and if $h \leq u$ on ∂D, then $h \leq u$ in D.

Analagously, a function $u : \Omega \to [-\infty, \infty)$ is called *F-subharmonic* if

(i) u is upper semi-continuous;

(ii) $u \not\equiv -\infty$ in each component of Ω;

(iii) u satisfies the comparsion principle in each open $D \subset\subset \Omega$: If $h \in C(\overline{D})$ is an F-solution in D, and if $h \geq u$ on ∂D, then $h \geq u$ in D.

We say that $u \in C(\Omega)$ is *F-harmonic* if u is both F-subharmonic and F-superharmonic.

Lemma 2.2.8. *(i) If u is F-superharmonic, then $au+b$ is F-superharmonic whenever a and b are real numbers and $a \geq 0$.*

(ii) *If u and v are F-superhmaronic, then the function* $\min\{u,v\}$ *is F-superharmonic.*

(iii) *Suppose that u_i, $i = 1, 2, \cdots$, are F-superharmonic in Ω. If the sequence u_i is increasing or converges uniformly on compact subsets of Ω, then in each component of Ω, the limit function $u = \lim_{i\to\infty} u_i$ is F-superharmonic unless $u \equiv \infty$.*

Theorem 2.2.9 (Comparison principle for F-super/subharmonic functions)**.** *Suppose that u is F-superharmonic and that v is F-subharmonic in Ω. If*

$$\limsup_{y\to x} v(y) \le \liminf_{y\to x} u(y)$$

for all $x \in \partial\Omega$, then $v \le u$ in Ω.

Proof. Fix $\varepsilon > 0$ and let

$$K_\varepsilon := \{x \in \Omega : v(x) \ge u(x) + \varepsilon\}.$$

Then K_ε is a compact subset of Ω and so there exists an open cover D_ε such that $K_\varepsilon \subset D_\varepsilon \subset \Omega$ where D_ε is a union of finitely many balls B_i, and $\partial D_\varepsilon \subset \Omega \setminus K_\varepsilon$. Since u is lower semi-continuous, v is upper semi-continuous and ∂D_ε is compact, we can choose a continuous function θ on ∂D_ε such that $v \le \theta \le u + \varepsilon$ on ∂D_ε. Moreover, since D_ε satisfies a uniform exterior cone condition, there exists $h \in C(\overline{D})$ which is the unique F-solution in D_ε that coincides with θ on ∂D_ε by applying Theorem 2.2.6. Now the definition of F-super/subharmonic functions yields that

$$v \le h \le u + \varepsilon \quad \text{in } D_\varepsilon.$$

Hence, $v \le u + \varepsilon$ in Ω and the desired result follows by letting $\varepsilon \to 0$. $\square$

Now we describe the equivalence of F-supersolution and F-superharmonic function; see also [39, 49, 52].

Theorem 2.2.10. *u is an F-supersolution in Ω if and only if u is F-superharmonic in Ω.*

Proof. Assume first that u is an F-supersolution in Ω. To show that u is F-superharmonic, we only need to verify the property (iii) in the definition of F-superharmonic functions. Let $D \subset\subset \Omega$ be an open set and take $h \in C(\overline{D})$ to be an F-solution in D such that $h \leq u$ on ∂D. Thus, applying the comparison principle for F-super/subsolutions (Theorem 2.2.5) for u and h, we conclude that $h \leq u$ in $\overline{D}$.

Assume now that u is F-superharmonic in Ω. For any $x_0 \in \Omega$, take $\varphi \in C^2(B_r(x_0))$ such that $u - \varphi$ has a local minimum 0 at x_0. We need to prove that

$$F(D^2\varphi(x_0)) \leq 0. \tag{2.2.1}$$

We argue by contradiction; suppose that (2.2.1) fails. By the continuity of the operator F, there exist $\tau > 0$ and $\rho \in (0, r)$ such that

$$F(D^2\varphi(x)) > \tau \quad \text{in } B_\rho(x_0).$$

Consider a cut-off function $\eta \in C_0^2(B_\rho(x_0))$ with $\operatorname{supp}\eta \subset B_{\rho/2}(x_0)$ and $\eta(x_0) = 1$. Since the uniform ellipticity gives

$$F(D^2(\varphi + \varepsilon\eta)) \geq F(D^2\varphi) + \varepsilon\, \mathcal{P}^-_{\lambda,\Lambda}(D^2\eta) \quad \text{for any } \varepsilon > 0,$$

we can choose a sufficiently small $\varepsilon_0 > 0$ so that

$$F(D^2(\varphi + \varepsilon_0\eta)) \geq 0 \quad \text{in } B_\rho(x_0).$$

In other words, since $\varphi + \varepsilon_0\eta \in C^2(B_\rho(x_0))$, $\varphi + \varepsilon_0\eta$ is an F-subsolution in $B_\rho(x_0)$. Furthermore, by a similar argument as in the first part, we have $\varphi + \varepsilon_0\eta$ is F-subharmonic in $B_\rho(x_0)$. On the other hand, on $\partial B_{\rho/2}(x_0)$, we have

$$\varphi(x) + \varepsilon_0\eta(x) = \varphi(x) \le u(x).$$

Thus, by the comparison principle for F-super/subharmonic functions (Theorem 2.2.9) for u and $\varphi + \varepsilon_0\eta$, we conclude that $\varphi + \varepsilon_0\eta \le u$ in $B_{\rho/2}(x_0)$. In particular, letting $x = x_0$, we have $\varphi(x_0) + \varepsilon_0 \le u(x_0)$, which contradicts to the fact that $u(x_0) = \varphi(x_0)$. $\square$

The result for F-subsolution and F-subharmonic function can be derived in the same manner and consequently, a function u is an F-solution if and only if it is F-harmonic.

2.2.2 Perron's Method

Lemma 2.2.11 (Pasting lemma). *Let $D \subset \Omega$ be open. Also let u and v be F-superharmonic in Ω and D, resepctively. If the function*

$$s := \begin{cases} \min\{u, v\} & \text{in } D, \\ u & \text{in } \Omega \setminus D, \end{cases}$$

is lower semi-continuous, then s is F-superharmonic in Ω.

Proof. Let $G \subset\subset \Omega$ be open and $h \in C(\overline{G})$ be F-harmonic such that $h \le s$ on ∂G. Then $h \le u$ in $\overline{G}$. In particular, since s is lower semi-continuous,

$$\lim_{D\cap G \ni y \to x} h(y) \le u(x) = s(x) \le \liminf_{D\cap G \ni y \to x} v(y)$$

for all $x \in \partial D \cap G$. Thus,

$$\lim_{D\cap G \ni y \to x} h(y) \le s(x) \le \liminf_{D\cap G \ni y \to x} v(y)$$

for all $x \in \partial(D \cap G)$, and Theorem 2.2.9 implies $h \le v$ in $D \cap G$. Therefore, $h \le s$ in G and the lemma is proved. $\square$

Suppose that u is F-superharmonic in Ω and that $B \subset\subset \Omega$ is an open ball. Let

$$u_B := \inf \Big\{ v : v \text{ is } F\text{-superharmonic in } B, \\ \liminf_{y \to x} v(y) \geq u(x) \text{ for each } x \in \partial B \Big\}.$$

Then define the *Poisson modification* $P(u, B)$ of u in B to be the function

$$P(u,B) := \begin{cases} u_B & \text{in } B, \\ u & \text{in } \Omega \setminus B. \end{cases}$$

Lemma 2.2.12 (Poisson modification)**.** *The Poisson modification $P(u,B)$ is F-superharmonic in Ω, F-harmonic in B, and $P(u,B) \leq u$ in Ω.*

Proof. By definition, it is clear that $P(u,B) \leq u$ in Ω. To show $P(u,B)$ is F-harmonic in B, choose an increasing sequence of continuous functions $\{\theta_j\}_{j \geq 1}$ on ∂B such that $u = \lim_{j\to\infty} \theta_j$. (recall that this is possible since u is lower semi-continuous.) Then let $h_j \in C(\overline{B})$ be the F-solution of the Dirichlet problem $F(D^2 h_j) = 0$ in B and $h_j = \theta_j$ on ∂B by Theorem 2.2.6. The comparison principle yields that h_j is also an increasing sequence. Thus, applying Harnack convergence theorem (Theorem 2.2.4), we have the limit function $h = \lim_{j\to\infty} h_j$ is an F-solution in B. Since

$$\liminf_{y\to x} h(y) \geq \lim_{j\to\infty} \liminf_{y \to x} h_j(y) = \lim_{j\to\infty} h_j(x) = \lim_{j\to\infty} \theta_j(x) = u(x), \tag{2.2.2}$$

for any $x \in \partial B$, we have $h \geq P(u,B)$ in B by the definition of u_B. On the other hand, since $h_j(x) \leq \liminf_{y\to x} v(y)$ where $x \in \partial B$ and v is an admissible function for u_B, we have $h \leq P(u,B)$ in B by applying the comparison principle, letting $j \to \infty$ and taking the infimum over v. Therefore, $P(u,B) = h$ is F-harmonic in B.

Finally, if we show that $P(u,B)$ is lower semi-continuous, then it immediately follows from the pasting lemma that $P(u,B)$ is F-superharmonic in Ω. Indeed, it is enough to show that $P(u,B)$ is lower semi-continuous at each point $x \in \partial B$; recall (2.2.2). $\square$

Remark 2.2.13 (Perron's method)**.** Let Ω be an open, bounded subset of $\mathbb{R}^n$ and f be a bounded function on $\partial\Omega$. The *upper class* $\mathcal{U}_f = \mathcal{U}_f(\Omega)$ consists of all functions u in Ω such that

(i) u is F-superharmonic in Ω;

(ii) u is bounded below;

(iii) $\liminf_{\Omega \ni y \to x} u(y) \geq f(x)$ for each $x \in \partial\Omega$.

Then we define the *upper Perron solution* of f by

$$\overline{H}_f = \overline{H}_f(\Omega) := \inf_{u \in \mathcal{U}_f} u.$$

Similarly, let the *lower class* $\mathcal{L}_f = \mathcal{L}_f(\Omega)$ be the set of all F-subharmonic functions v in Ω which are bounded above and such that

$$\limsup_{\Omega \ni y \to x} v(y) \leq f(x) \quad \text{for each } x \in \partial\Omega,$$

and define the *lower Perron solution* of f by

$$\underline{H}_f = \underline{H}_f(\Omega) := \sup_{v \in \mathcal{L}_f} v.$$

Then the comparison principle yields that $\underline{H}_f \leq \overline{H}_f$.

Lemma 2.2.14. *The Perron solutions $\overline{H}_f$ and $\underline{H}_f$ are F-solutions in Ω.*

Proof. This proof is based on the argument used in [45]. Fix an open ball B with $B \subset\subset \Omega$. Next, choose a countable, dense subset $X = \{x_1, x_2, ...\}$ of B and then for each $j = 1, 2, ...$, choose $u_{i,j} \in \mathcal{U}_f$ such that

$$\lim_{i \to \infty} u_{i,j}(x_j) = \overline{H}_f(x_j).$$

Moreover, replacing $u_{i,j+1}$ by $\min\{u_{i,j}, u_{i,j+1}\}$ if necessary, we have

$$\lim_{i\to\infty} u_{i,j}(x_k) = \overline{H}_f(x_k),$$

for each $k = 1, 2..., j$ and each j. Now, let $U_{i,j} := P(u_{i,j}, B)$ be the Poisson modification of $u_{i,j}$ in B. Then we observe that $\overline{H}_f \le U_{i,j} \le u_{i,j}$ and $U_{i,j}$ is F-harmonic in B. By compactness (Theorem 2.2.3 (ii)), $U_{i,j}$ converges locally uniformly to F-harmonic v_j in B (passing to a subsequence, if necessary). Again by compactness, v_j converges locally uniformly to F-harmonic h in B.

By the construction of h, it follows immediately that

$$\overline{H}_f \le h$$

in B and $\overline{H}_f = h$ on X. For any $u \in \mathcal{U}_f$ and its Poisson modification $U = P(u, B)$, we have $u \ge U \ge \overline{H}_f$. Since $U \ge h$ on X (which is dense in B) and U, h are continuous in B, it follows that $U \ge h$ in B. Thus, $u \ge h$ in B which implies that

$$\overline{H}_f \ge h$$

in B. Hence, $\overline{H}_f = h$ is F-harmonic in Ω and a similar argument for $\underline{H}_f$ completes the proof. $\square$

We emphasize that although we proved that $F(D^2\overline{H}_f) = 0$ in Ω, we cannot guarantee that $\overline{H}_f$ enjoys the boundary condition of the Dirichlet problem, $\overline{H}_f = f$ on $\partial\Omega$. To investigate the boundary behavior of the Perron solutions and ensure the solvability of the Dirichlet problem, we need to introduce further concepts, namely, a *regular point* and a *barrier function.*

Definition 2.2.15 (A regular point)**.** A boundary point $x_0 \in \partial\Omega$ is (F-)*regular* with respect to Ω, if

$$\lim_{\Omega\ni y\to x_0} \overline{H}_f(y) = f(x_0) \quad \text{and} \quad \lim_{\Omega\ni y\to x_0} \underline{H}_f(y) = f(x_0)$$

whenever $f \in C(\partial\Omega)$. An open and bounded set Ω is called *regular* if each $x_0 \in \partial\Omega$ is a regular boundary point.

Remark 2.2.16. Suppose that an operator $\mathcal{M}$ satisfies $\mathcal{M}[-u] = -\mathcal{M}[u]$; for example, any linear operator L and p-Laplcian operators Δ_p possess this property. Then we have

$$\overline{H}_f = -\underline{H}_{-f},$$

and so in this case, we can equivalently call $x_0 \in \partial\Omega$ is regular if

$$\lim_{\Omega \ni y \to x_0} \overline{H}_f(y) = f(x_0)$$

whenever $f \in C(\partial\Omega)$. Nevertheless, for the general fully nonlinear operator F, we do not have this property. Therefore, it seems that we have to require both conditions simultaneously, when we define a regular point for F. To the best of our knowledge, it is unknown whether the two conditions in the definition are redundant. One possible approach to show that only one condition is essential is to prove that f is resolutive whenever f is continuous on $\partial\Omega$; see Definition 2.2.17 for the definition of resolutivity.

Before we define a barrier function, which characterizes a regular boundary point, we shortly deal with the resolutivity of boundary data:

Definition 2.2.17 (Resolutivity)**.** We say that a bounded function f on $\partial\Omega$ is (F-)*resolutive* if the upper and the lower Perron solutions $\overline{H}_f$ and $\underline{H}_f$ coincide in Ω. When f is resolutive, we write $H_f := \overline{H}_f = \underline{H}_f$.

Lemma 2.2.18. *Let Ω be a bounded open set of $\mathbb{R}^n$, let f and g be bounded functions on $\partial\Omega$, and let c be any real number.*

(i) If $f = c$ on $\partial\Omega$, then f is resolutive and $H_f = c$ in Ω.

(ii) $\overline{H}_{f+c} = \overline{H}_f + c$ and $\underline{H}_{f+c} = \underline{H}_f + c$. If f is resolutive, then $f + c$ is resolutive and $H_{f+c} = H_f + c$.

(iii) *If $c > 0$, then $\overline{H}_{cf} = c\overline{H}_f$ and $\underline{H}_{cf} = c\underline{H}_f$. If f is resolutive, then cf is resolutive and $H_{cf} = cH_f$ for $c \geq 0$.*

(iv) *If $f \leq g$, then $\overline{H}_f \leq \overline{H}_g$ and $\underline{H}_f \leq \underline{H}_g$.*

Note that the resolutivity of f does not imply

$$\lim_{y \to x} H_f(y) = f(x)$$

for $x \in \partial\Omega$. However, the converse is true in some sense:

Lemma 2.2.19. *Let Ω be an open and bounded subset of $\mathbb{R}^n$ and f be a bounded function on $\partial\Omega$. Suppose that there exists F-harmonic h in Ω such that*

$$\lim_{\Omega \ni y \to x} h(y) = f(x)$$

for any $x \in \partial\Omega$. Then $\overline{H}_f = h = \underline{H}_f$. In particular, f is resolutive.

Proof. Since $h \in \mathcal{U}_f \cap \mathcal{L}_f$, we have $\overline{H}_f \leq h \leq \underline{H}_f$. □

Lemma 2.2.20. *If u is a bounded F-superharmonic (or F-subharmonic) function on the bounded open set Ω such that $f(x) = \lim_{\Omega \ni y \to x} u(y)$ exists for all $x \in \partial\Omega$, then f is a resolutive boundary function.*

Proof. Obviously, we have $u \in \mathcal{U}_f$ and so $\overline{H}_f \leq u$ in Ω. Then since $\overline{H}_f$ is F-harmonic in Ω and

$$\limsup_{\Omega \ni y \to x} \overline{H}_f(y) \leq \lim_{\Omega \ni y \to x} u(y) = f(x),$$

we have $\overline{H}_f \in \mathcal{L}_f$, which implies that $\overline{H}_f \leq \underline{H}_f$. Because $\underline{H}_f \leq \overline{H}_f$ always holds, we conclude that f is resolutive. An analogous argument works for the F-subharmonic case. □

2.2.3 Characterization of a Regular Point

Definition 2.2.21 (Barrier)**.** Let $x_0 \in \partial\Omega$. A function $w^+ : \Omega \to \mathbb{R}$ [resp. w^-] is an *upper barrier* [resp. *lower barrier*] in Ω at the point x_0 if

(i) w^+ [resp. w^-] is F-superharmonic [resp. F-subharmonic] in Ω;

(ii) $\liminf_{\Omega \ni y \to x} w^+(y) > 0$ [resp. $\limsup_{\Omega \ni y \to x} w^-(y) < 0$] for each $x \in \partial\Omega \setminus \{x_0\}$;

(iii) $\lim_{\Omega \ni y \to x_0} w^+(y) = 0$. [resp. $\lim_{\Omega \ni y \to x_0} w^-(y) = 0$.]

Observe that the maximum principle indicates that an upper barrier w^+ is positive in Ω and a lower barrier w^- is negative in Ω. Moreover, under the condition (F2), cw^+ is still an upper barrier for any constant $c > 0$ and an upper barrier w^+. See also [74].

Now we can deduce that a regular boundary point is characterized by the existence of upper and lower barriers.

Theorem 2.2.22. *Let $x_0 \in \partial\Omega$. Then the following are equivalent:*

(i) x_0 is regular;

(ii) there exist an upper barrier and a lower barrier at x_0.

Proof. (ii) $\implies$ (i) For $f \in C(\partial\Omega)$ and $\varepsilon > 0$, there is $\delta > 0$ such that $|x - x_0| \le \delta$ with $x \in \partial\Omega$ implies $|f(x) - f(x_0)| < \varepsilon$. Moreover, for $M := \sup_{\partial\Omega} |f|$, there exists a large number $K > 0$ such that

$$K \cdot \liminf_{\Omega \ni y \to x} w^+(y) \ge 2M \text{ for all } x \in \partial\Omega \text{ with } |x - x_0| \ge \delta.$$

Here we used that $x \mapsto \liminf_{\Omega \ni y \to x} w^+(y)$ is lower semi-continuous on $\partial\Omega$. Then since $Kw^+ + f(x_0) + \varepsilon \in \mathcal{U}_f$, we have

$$\overline{H}_f(y) \le Kw^+(y) + f(x_0) + \varepsilon,$$

which implies that

$$\limsup_{\Omega \ni y \to x_0} \overline{H}_f(y) \leq f(x_0).$$

An analogous argument leads to

$$\liminf_{\Omega \ni y \to x_0} \underline{H}_f(y) \geq f(x_0).$$

Since $\underline{H}_f \leq \overline{H}_f$, we conclude that

$$\lim_{\Omega \ni y \to x_0} \overline{H}_f(y) = f(x_0) = \lim_{\Omega \ni y \to x_0} \underline{H}_f(y),$$

i.e. x_0 is a regular boundary point.

(i) $\implies$ (ii) Define a distance function d by

$$d(y) := |y - x_0|^2$$

so that d is continuous, non-negative and $d(y) = 0$ if and only if $y = x_0$. Moreover, since $D^2 d = 2I$, we have $F(D^2 d) = 2F(I) > 0$, i.e. d is F-subharmonic.

Then letting $w^+ := \underline{H}_d$, we have w^+ is F-harmonic in Ω and it follows from $d \in \mathcal{L}_d$ that $w^+ \geq d$ in Ω. Thus, for any $x \in \partial\Omega \setminus \{x_0\}$,

$$\liminf_{\Omega \ni y \to x} w^+(y) \geq d(x) = |x - x_0|^2 > 0.$$

Furthermore, since x_0 is regular, we have

$$\lim_{\Omega \ni y \to x_0} w^+(y) = d(x_0) = 0,$$

and so w^+ is a desired upper barrier. The existence of a lower barrier is guaranteed by considering $\widetilde{d}(y) := -d(y) = -|y - x_0|^2$ and $w^- := \overline{H}_{\widetilde{d}}$.

$\square$

Indeed, the barrier characterization is a *local* property:

Lemma 2.2.23. *Let $x_0 \in \partial\Omega$ and $G \subset \Omega$ be open with $x_0 \in \partial G$. If x_0 is regular with respect to Ω, then x_0 is regular with respect to G.*

Proof. By Theorem 2.2.22, there exist an upper barrier w^+ and a lower barrier w^- with respect to Ω at x_0. Then $w^+|_G$ and $w^-|_G$ become the desired barriers with respect to G at x_0. Again by Theorem 2.2.22, x_0 is regular with respect to G. $\square$

Lemma 2.2.24. *Let $x_0 \in \partial\Omega$ and B be a ball containing x_0. Then x_0 is regular with respect to Ω if and only if x_0 is regular with respect to $B \cap \Omega$.*

Proof. By Lemma 2.2.23, one direction is immediate. For the opposite direction, suppose that x_0 is regular with respect to $B \cap \Omega$. Then there exist an upper barrier w^+ and a lower barrier w^- with respect to $B \cap \Omega$. If we let $m := \min_{\partial B \cap \Omega} w^+ > 0$ (the minimum exists because w^+ is lower semicontinuous), then the pasting lemma, Lemma 2.2.11, shows that

$$s^+ := \begin{cases} \min\{w^+, m\} & \text{in } B \cap \Omega, \\ m & \text{in } \Omega \setminus B, \end{cases}$$

is F-superharmonic in Ω. One can easily verify that s^+ is an upper barrier with respect to Ω at x_0. Similarly, a lower barrier s^- can be constructed. $\square$

The barrier characterization leads to another useful corollary, which enables us to write x_0 is regular instead of F-regular, without ambiguity.

Corollary 2.2.25. *A boundary point $x_0 \in \partial\Omega$ is F-regular if and only if x_0 is $\widetilde{F}$-regular.*

Proof. Suppose that x_0 is F-regular. By Theorem 2.2.22, there exists an upper barrier w_F^+ and a lower barrier w_F^-. If we let $w_{\widetilde{F}}^+ := -w_F^-$ and $w_{\widetilde{F}}^- := -w_F^+$, then $w_{\widetilde{F}}^+$ and $w_{\widetilde{F}}^-$ become an upper barrier and a lower barrier for $\widetilde{F}$, respectively. Therefore, again by Theorem 2.2.22, x_0 is $\widetilde{F}$-regular. $\square$

Now we present one sufficient condition that guarantees a regular boundary point, namely the exterior cone condition. In Section 2.4, we suggest another sufficient condition, namely the Wiener criterion, which contains this exterior cone condition as a special case.

Theorem 2.2.26 (Exterior cone condition)**.** *Suppose that* Ω *satisfies an exterior cone condition at* $x_0 \in \partial\Omega$. *Then* x_0 *is a regular boundary point.*

Proof. The proof relies on the construction of a local barrier at x_0. See [19, 70, 73] for details. □

Corollary 2.2.27. *All polyhedra and all balls are regular. Furthermore, every open set can be exhausted by regular open sets. Here a bounded open set* Ω *is called a polyhedron if* $\partial\Omega = \partial\overline{\Omega}$ *and if* $\partial\Omega$ *is contained in a finite union of* $(n-1)$*-hyperplanes.*

Proof. Since polyhedra and balls satisfy the uniform exterior cone condition, the first assertion follows from Theorem 2.2.26. For the second assertion, exhaust Ω by domains $D_1 \subset\subset D_2 \subset\subset \cdots \subset\subset \Omega$. Then since $\overline{D_j}$ is compact, there exists a finite union of open cubes $Q_{j_i} (\subset D_{j+1})$ that covers $\overline{D_j}$. Letting $P_j := \bigcup_i \operatorname{int} \overline{Q_{j_i}}$ which is a polyhedron by the construction, we obtain the desired exhaustion. □

2.3 Balayage and Capacity

2.3.1 Balayage and Capacity Potential

We define the *lower semi-continuous regularization* $\hat{u}$ of any function $u : E \to [-\infty, \infty]$ by

$$\hat{u}(x) := \lim_{r \to 0} \inf_{E \cap B_r(x)} u.$$

Lemma 2.3.1. *Suppose that $\mathcal{F}$ is a family of F-superharmonic functions in Ω, locally uniformly bounded below. Then the lower semi-continuous regularization s of* $\inf \mathcal{F}$*,*

$$s(x) = \lim_{r\to 0} \inf_{B_r(x)} (\inf \mathcal{F}),$$

is F-superharmonic in Ω.

Proof. Since $\mathcal{F}$ is locally uniformly bounded below, s is lower semi-continuous. Fix an open $D \subset\subset \Omega$ and let $h \in C(\overline{D})$ be an F-harmonic function satisfying $h \leq s$ on ∂D. Then $h \leq u$ in D whenever $u \in \mathcal{F}$. It follows from the continuity of h that $h \leq s$ in D. □

Definition 2.3.2 (Balayage and capacity potential)**.**

(i) For $\psi : \Omega \to (-\infty, \infty]$ which is locally bounded below, let

$$\Phi^\psi = \Phi^\psi(\Omega) := \{u : u \text{ is } F\text{-superharmonic in } \Omega \text{ and } u \geq \psi \text{ in } \Omega\}.$$

Then the function

$$R^\psi = R^\psi(\Omega) := \inf \Phi^\psi$$

is called the *reduced function* and its lower semi-continuous regularization

$$\hat{R}^\psi = \hat{R}^\psi(\Omega)$$

is called the *balayage* of ψ in Ω. By Lemma 2.3.1, $\hat{R}^\psi$ is F-superharmonic in Ω.

(ii) If u is a non-negative function on a set $E \subset \Omega$, we write

$$\Phi^u_E = \Phi^\psi, \quad R^u_E = R^\psi, \quad \hat{R}^u_E = \hat{R}^\psi,$$

where

$$\psi = \begin{cases} u & \text{in } E, \\ 0 & \text{in } \Omega \setminus E. \end{cases}$$

The function $\hat{R}_E^u$ is called the *balayage of* u *relative to* E.

(iii) In particular, we call the function $\hat{R}_E^1$ the (F-)*capacity potential* of E in Ω.

Remark 2.3.3. For an operator in divergence form, there exists an alternative method to define the capacity potential. For simplicity, suppose that the operator is given by the p-Laplacian. Let Ω be bounded and $K \subset \Omega$ be a compact set. For $\psi \in C_0^\infty(\Omega)$ with $\psi \equiv 1$ on K, the p-harmonic function u in $\Omega \setminus K$ with $u - \psi \in W_0^{1,p}(\Omega \setminus K)$ is called the *capacity potential* of K in Ω and denoted by $\mathcal{R}(K, \Omega)$. Here note that $\mathcal{R}(K, \Omega)$ is independent of the particular choice of ψ and the existence of the capacity potential is guaranteed by the variational method. Indeed, both definitions of capacity potentials coincide; see [37, Chapter 9] for details.

Lemma 2.3.4. *The balayage $\hat{R}_E^u$ is F-harmonic in $\Omega \setminus \overline{E}$ and coincides with R_E^u there. If, in addition, u is F-superharmonic in Ω, then $\hat{R}_E^u = u$ in the interior of E.*

Proof. Observe first that if v_1 and v_2 are in Φ_E^u, then so is $\min\{v_1, v_2\}$. Hence, the family Φ_E^u is downward directed and we may invoke Choquet's topological lemma (see [37, Lemma 8.3]): there is a decreasing sequence of functions $v_i \in \Phi_E^u$ with the limit v such that

$$\hat{v}(x) = \hat{R}_E^u(x)$$

for all $x \in \Omega$.

Next, we choose a ball $B \subset\subset \Omega \setminus \overline{E}$ and consider a Poisson modification $s_i = P(v_i, B)$. Then it follows that $s_i \in \Phi_E^u$ and $s_{i+1} \leq s_i \leq v_i$. Thus, we have

$$R_E^u \le s := \lim_{i\to\infty} s_i \le v,$$

which implies that $\hat{R}_E^u = \hat{v} = \hat{s}$. Moreover, since s is F-harmonic in B (Harnack convergence theorem, Theorem 2.2.4), we know that $\hat{s} = s$. Therefore, we conclude that the balayage $\hat{R}_E^u$ is F-harmonic in $\Omega \setminus \overline{E}$. The second assertion of the lemma is rather immediate since $u \in \Phi_E^u$ if u is F-superharmonic in Ω. □

Lemma 2.3.5. *Let K be a compact subset of Ω and consider $R_K^1 = R_K^1(\Omega)$ and $\hat{R}_K^1 = \hat{R}_K^1(\Omega)$.*

(i) $0 \le \hat{R}_K^1 \le R_K^1 \le 1$ in Ω.

(ii) $R_K^1 = 1$ in K.

(iii) $R_K^1 = \hat{R}_K^1$ in $(\partial K)^c$.

(iv) $\hat{R}_K^1$ is F-superharmonic in Ω and F-harmonic in $\Omega \setminus K$.

Proof. (i) It immediately follows from the definition of R_K^1 and the comparison principle.

(ii) Since $1 \in \Phi^\psi(\Omega)$, we have $R_K^1 \le 1$ in Ω. On the other hand, for any $u \in \Phi^\psi(\Omega)$, we have $u \ge \psi = 1$ in K and so $R_K^1 \ge 1$ in K.

(iii), (iv) It immediately follows from Lemma 2.3.4 and part (ii).

□

The following theorem shows that the capacity potential can be understood as the upper Perron solution:

Theorem 2.3.6. *Suppose that K is a compact subset of a bounded, open set Ω and that $u = \hat{R}_K^1(\Omega)$ is the capacity potential of K in Ω. Moreover, let f*

be a function such that

$$f = \begin{cases} 1 & on\ \partial K, \\ 0 & in\ \partial\Omega. \end{cases}$$

Then

$$\hat{R}^1_K(\Omega) = \overline{H}_f(\Omega \setminus K)$$

in $\Omega \setminus K$.

Proof. Lemma 2.3.4 shows that $\hat{R}^1_K = R^1_K$ in $\Omega \setminus K$. Then recall that

$$R^1_K(\Omega) = \inf \Phi^1_K = \inf\{v : v \text{ is } F\text{-superharmonic in } \Omega \text{ and } v \geq \psi \text{ in } \Omega\},$$

where

$$\psi = \begin{cases} 1 & \text{in } K, \\ 0 & \text{in } \Omega \setminus K, \end{cases}$$

and

$$\begin{aligned} \overline{H}_f(\Omega \setminus K) = \inf \mathcal{U}_f = \inf\{v :& v \text{ is } F\text{-superharmonic in } \Omega \setminus K, \\ & \liminf_{\Omega\setminus K \ni y \to x} v(y) \geq f(x) \text{ for each } x \in \partial(\Omega \setminus K)\}, \end{aligned}$$

where

$$f = \begin{cases} 1 & \text{on } \partial K, \\ 0 & \text{in } \partial\Omega. \end{cases}$$

(i) Suppose that $v \in \Phi^1_K$. Since $v \geq 0$ in Ω, we have $\liminf_{\Omega\setminus K \ni y \to x} v(y) \geq 0 = f(x)$ for $x \in \partial\Omega$. Moreover, since v is lower semi-continuous, we have

$$\liminf_{\Omega\setminus K \ni y \to x} v(y) \geq \liminf_{\Omega \ni y \to x} v(y) \geq v(x) \geq 1 = f(x),$$

for $x \in \partial K$. Therefore, we conclude $v \in \mathcal{U}_f$, which implies that $\overline{H}_f(\Omega \setminus K) \leq R^1_K(\Omega)$ in $\Omega \setminus K$.

(ii) Suppose that $v \in \mathcal{U}_f$. We consider $\overline{v} := \min\{1, v\} \in \mathcal{U}_f$ so that $0 \leq \overline{v} \leq 1$ in $\Omega \setminus K$. Then since $u \equiv 1$ is F-superharmonic in Ω, the function

$$s = \begin{cases} \min\{1, \overline{v}\} = \overline{v} & \text{in } \Omega \setminus K, \\ 1 & \text{in } K \end{cases}$$

is F-superharmonic in Ω by pasting lemma, Lemma 2.2.11. Obviously, $s \in \Phi^1_K$ and so $R^1_K(\Omega) \leq \overline{v} \leq v$ in $\Omega \setminus K$.

□

2.3.2 Capacity

In general, for an operator in divergence form, we consider a *variational capacity*, which comes from minimizing the energy among admissible functions. On the other hand, for an operator in non-divergence form, we cannot consider the corresponding energy, and so we require an alternative approach to attain a proper notion of capacity. Our definition of a capacity is in the same context with [7] for linear operators in non-divergence form and [52] for the Pucci extremal operators.

Definition 2.3.7 (Non-variational capacity)**.** For a ball $B = B_{2r}(x_0)$, we fix a ball $B' = B_{7/5r}(x_0) \subset B$ and a point $y_0 = x_0 + \frac{3}{2}re_1$. Then we define a *capacity* for fully nonlinear operator F by

$$\begin{aligned} \operatorname{cap}(K, B) = \operatorname{cap}_F(K, B) := \inf\{u(y_0) : u \text{ is } F\text{-superharmonic in } B, \\ u \geq 0 \text{ in } B, \text{ and } u \geq 1 \text{ in } K\} \end{aligned} \tag{2.3.1}$$

whenever K is a compact subset of B'.

Comparing the definitions of capacity and capacity potential, we immediately notice that

$$\operatorname{cap}(K, B) = \hat{R}^1_K(B)(y_0).$$

Moreover, appealing to Theorem 2.3.6, we further have

$$\operatorname{cap}(K, B) = \overline{H}_f(B \setminus K)(y_0),$$

where the boundary data f on $\partial(B \setminus K)$ is given by

$$f = \begin{cases} 1 & \text{on } \partial K, \\ 0 & \text{in } \partial B. \end{cases}$$

Finally, considering Harnack inequality for $\hat{R}^1_K(B)$ on the sphere $\partial B_{3r/2}(x_0)$, we notice that capacities defined for different choices of $y_0 \in \partial B_{3r/2}(x_0)$ are comparable.

Lemma 2.3.8 (Properties of capacity)**.** *Fix a ball $B = B_{2r}(x_0)$. Then the set function $K \mapsto \operatorname{cap}(K, B)$, where K is a compact subset of $B' = B_{7/5r}(x_0)$, enjoys the following properties:*

(i) $0 \leq \operatorname{cap}(K, B) \leq 1$.

(ii) If $K_1 \subset K_2 \subset B'$, then

$$\operatorname{cap}(K_1, B) \leq \operatorname{cap}(K_2, B).$$

(iii) If a monotone sequence of compact sets $\{K_j\}_{j=1}^{\infty}$ satisfies $B' \supset K_1 \supset K_2 \supset \cdots$, then

$$\operatorname{cap}(K, B) = \lim_{j \to \infty} \operatorname{cap}(K_j, B), \quad \textit{for } K := \bigcap_{j=1}^{\infty} K_j.$$

(iv) (Subadditivity) We further suppose that F is convex. If K_1 and K_2 are

compact subsets of B', then

$$\operatorname{cap}(K_1 \cup K_2, B) \leq \operatorname{cap}(K_1, B) + \operatorname{cap}(K_2, B).$$

Proof. (i) Recalling Lemma 2.3.5, we have $0 \leq \operatorname{cap}(K, B) \leq 1$.

(ii) If $K_1 \subset K_2$, then $\Phi^1_{K_2} \subset \Phi^1_{K_1}$ and so $\operatorname{cap}(K_1, B) \leq \operatorname{cap}(K_2, B)$.

(iii) Since $\operatorname{cap}(K_j, B) \geq \operatorname{cap}(K, B)$ by (ii), it is immediate that

$$\operatorname{cap}(K, B) \leq \lim_{j\to\infty} \operatorname{cap}(K_j, B).$$

For the reversed inequality, fix small $\varepsilon > 0$ and $u \in \Phi^1_K(B)$. If j is large enough, then $K_j \subset \{u \geq 1 - \varepsilon\}$ and so

$$\lim_{j\to\infty} \operatorname{cap}(K_j, B) \leq \operatorname{cap}(\{u \geq 1 - \varepsilon\}, B) \leq \frac{1}{1-\varepsilon} u(y_0).$$

Letting $\varepsilon \to 0^+$ and taking infimum for $u \in \Phi^1_K(B)$, we conclude that

$$\lim_{j\to\infty} \operatorname{cap}(K_j, B) \leq \operatorname{cap}(K, B).$$

(iv) Let $v_1 \in \Phi^1_{K_1}(B)$ and $v_2 \in \Phi^1_{K_2}(B)$. Since F is convex, we can apply [14, Theorem 5.8] to obtain $\frac{1}{2}(v_1 + v_2)$ is F-superharmonic in B. Moreover, it follows from the assumption (F2) that $v_1 + v_2 \in \Phi^1_{K_1\cup K_2}(B)$ and so $R^1_{K_1\cup K_2}(B) \leq v_1 + v_2$. Putting the infimum on this inequality and evaluating at y_0, we conclude that

$$\operatorname{cap}(K_1 \cup K_2, B) \leq \operatorname{cap}(K_1, B) + \operatorname{cap}(K_2, B).$$

□

We would like to remove the restriction of compact sets when defining a capacity. For this purpose, when $U \subset B'$ is open, we set the *inner capacity*

$$\operatorname{cap}_*(U, B) := \sup_{K \subset U,\, K \text{ compact}} \operatorname{cap}(K, B).$$

Then for an arbitrary set $E \subset B'$, we set the *outer capacity*

$$\operatorname{cap}^*(E, B) := \inf_{E \subset U \subset B',\, U \text{ open}} \operatorname{cap}_*(U, B).$$

Lemma 2.3.9. *Fix a ball $B = B_{2r}(x_0)$. For a compact subset K of $B' = B_{7r/5}(x_0)$, we have*

$$\operatorname{cap}(K, B) = \operatorname{cap}^*(K, B).$$

In other words, there is no ambiguity in having two different definitions for the capacity of compact sets.

Proof. (i) For any open set U satisfying $K \subset U \subset B'$, the definition of the inner capacity yields that

$$\operatorname{cap}(K, B) \leq \operatorname{cap}_*(U, B).$$

By taking the infimum over such U, we conclude that

$$\operatorname{cap}(K, B) \leq \operatorname{cap}^*(K, B).$$

(ii) Define a sequence of compact sets $\{K_j\}_{j=1}^{\infty}$ by

$$K_j := \{x \in \mathbb{R}^n : \operatorname{dist}(x, K) \leq 1/j\},$$

and a sequence of open sets $\{U_j\}_{j=1}^{\infty}$ by

$$U_j := \{x \in \mathbb{R}^n : \operatorname{dist}(x, K) < 1/j\}.$$

We may assume $K_1 \subset B'$. Then we have

$$B' \supset K_1 \supset U_1 \supset K_2 \supset U_2 \supset \cdots \supset K, \quad \text{and} \quad K = \bigcap_j K_j.$$

Applying Lemma 2.3.8 (ii), it follows that

$$\operatorname{cap}_*(U_j, B) \leq \operatorname{cap}(K_j, B).$$

By the definition of outer capacity,

$$\operatorname{cap}^*(K, B) \leq \operatorname{cap}_*(U_j, B) \leq \operatorname{cap}(K_j, B), \quad \text{for any } j \in \mathbb{N}.$$

Now letting $j \to \infty$, Lemma 2.3.8 (iii) leads to

$$\operatorname{cap}^*(K, B) \leq \operatorname{cap}(K, B).$$

□

Roughly speaking, we have the following correspondance:

$$\begin{aligned} \text{the variational capacity} &\longleftrightarrow \text{divergence operator,} \\ \text{the height capacity} &\longleftrightarrow \text{non-divergence operator.} \end{aligned}$$

In the following lemma, we explain why the definition of height capacity is reasonable in some sense. In other words, we claim that for the Laplacian operator Δ, two definitions of capacity are comparable.

Lemma 2.3.10 (The variational capacity and the height capacity)**.** *Suppose $n \geq 3$ and fix two balls $B = B_{2r}(x_0)$, $B' = B_{7r/5}(x_0)$ and a point $y_0 = \frac{3}{2}re_1 + x_0 \in \partial B_{3r/2}(x_0)$. Then for any compact set $K \subset B'$, we have*

$$\operatorname{cap}_{\Delta,\mathrm{var}}(K, B) \sim \operatorname{cap}_{\Delta,\mathrm{height}}(K, B)\, r^{n-2},$$

where the comparable constant depends only on n.

Proof. We may assume $x_0 = 0$. We denote by u the capacity potential with respect to K in B. Note that u is harmonic in $B \setminus K$.

We begin with the variational capacity:

$$\mathrm{cap}_{\Delta,\mathrm{var}}(K, B) = \int_{B\setminus K} |\nabla u|^2 \,\mathrm{d}x = \int_{\partial K} \frac{\partial u}{\partial \mathbf{n}} \,\mathrm{d}s = -\int_{\partial B} \frac{\partial u}{\partial \mathbf{n}} \,\mathrm{d}s.$$

Here we applied the divergence theorem and used the behavior of u on the boundary.

On the other hand, recalling the definition of height capacity, we have

$$\mathrm{cap}_{\Delta,\mathrm{height}}(K, B) = u(y_0).$$

By Harnack inequality, there exist constants $c_1, c_2 > 0$ which only depend on n such that

$$c_1 u(y_0) \le u(x) \le c_2 u(y_0) \quad \text{for any } x \in \partial B_{3r/2}.$$

Thus, if we set $m_- := \min_{\partial B_{3r/2}} u$ and $m_+ := \max_{\partial B_{3r/2}} u$, then we have

$$c_1 \mathrm{cap}_{\Delta,\mathrm{height}}(K, B) \le m_- \le m_+ \le c_2 \mathrm{cap}_{\Delta,\mathrm{height}}(K, B).$$

Moreover, we consider two barriers $h^\pm$ which solve the Dirichlet problem in $B_{2r} \setminus B_{3r/2}$:

$$\begin{cases} \Delta h^\pm = 0 & \text{in } B_{2r} \setminus B_{3r/2}, \\ h^\pm = m_\pm & \text{on } \partial B_{3r/2}, \\ h^\pm = 0 & \text{on } \partial B_{2r}. \end{cases}$$

Indeed, using the homogeneous solution $V(x) = |x|^{2-n}$, one can compute $h^\pm$ explicitly:

$$h^\pm(x) = m_\pm \cdot \frac{|x|^{2-n} - (2r)^{2-n}}{(3r/2)^{2-n} - (2r)^{2-n}}.$$

Then the comparison principle between u and $h^{\pm}$ leads to

$$h^{-} \leq u \leq h^{+} \quad \text{in } B_{2r} \setminus B_{3r/2},$$

and so

$$\frac{c(n)\, m_{-}}{r} = -\frac{\partial h^{-}}{\partial \mathbf{n}} \leq -\frac{\partial u}{\partial \mathbf{n}} \leq -\frac{\partial h^{+}}{\partial \mathbf{n}} = \frac{c(n)\, m_{+}}{r} \quad \text{on } \partial B.$$

Therefore, we conclude that

$$c_1(n) r^{n-2} \mathrm{cap}_{\Delta,\mathrm{height}}(K, B) \leq \mathrm{cap}_{\Delta,\mathrm{var}}(K, B) \leq c_2(n) r^{n-2} \mathrm{cap}_{\Delta,\mathrm{height}}(K, B).$$

□

Next, we estimate the capacity of a ball B_ρ with respect to the larger ball B_{2r}. Indeed, the capacity of a ball can capture the growth rate of the homogeneous solution V of F.

Lemma 2.3.11 (Capacitary estimate for balls)**.** *Let $B = B_{2r}(x_0)$, $B' = B_{\frac{7}{5}r}(x_0)$ and $y_0 = x_0 + \frac{3}{2} r e_1$. Then for any $0 < \rho < \frac{7}{5} r$, there exists a constant $c = c(n, \lambda, \Lambda) > 0$ which is independent of r and ρ such that*

(i) ($\alpha^* > 0$)

$$\frac{1}{c} \frac{r^{-\alpha^*}}{\rho^{-\alpha^*} - (2r)^{-\alpha^*}} \leq \mathrm{cap}_F(\overline{B_\rho(x_0)}, B_{2r}(x_0)) \leq \frac{c r^{-\alpha^*}}{\rho^{-\alpha^*} - (2r)^{-\alpha^*}}.$$

(ii) ($\alpha^* < 0$)

$$\frac{1}{c} \frac{r^{-\alpha^*}}{(2r)^{-\alpha^*} - \rho^{-\alpha^*}} \leq \mathrm{cap}_F(\overline{B_\rho(x_0)}, B_{2r}(x_0)) \leq \frac{c r^{-\alpha^*}}{(2r)^{-\alpha^*} - \rho^{-\alpha^*}}.$$

(iii) ($\alpha^* = 0$)

$$\frac{1}{c} \frac{1}{\log(2r) - \log \rho} \leq \mathrm{cap}_F(\overline{B_\rho(x_0)}, B_{2r}(x_0)) \leq \frac{c}{\log(2r) - \log \rho}.$$

Proof. We may assume $x_0 = 0$. Applying the argument after the definition of a capacity, we have

$$\mathrm{cap}_F(\overline{B_\rho}, B_{2r}) = \hat{R}_{\overline{B_\rho}}(B_{2r})(y_0) = \overline{H}_f(B_{2r} \setminus \overline{B_\rho})(y_0),$$

where the boundary data f is given by

$$f = \begin{cases} 1 & \text{on } \partial B_\rho, \\ 0 & \text{in } \partial B_{2r}. \end{cases}$$

Moreover, since a ball is a regular domain, we can write $\overline{H}_f(B_{2r} \setminus \overline{B_\rho}) = v$ where v is the unique solution of the Dirichlet problem

$$\begin{cases} F(D^2 v) = 0 & \text{in } B_{2r} \setminus \overline{B_\rho}, \\ v = 1 & \text{on } \partial B_\rho, \\ v = 0 & \text{in } \partial B_{2r}. \end{cases}$$

Note that $\overline{H}_f(B_{2r} \setminus \overline{B_\rho})$ is continuous upto the boundary. We now split three cases according to the sign of $\alpha^*(F)$.

(i) ($\alpha^* > 0$) In this case, for the homogeneous solution $V(x) = |x|^{-\alpha^*} V\left(\frac{x}{|x|}\right)$, denote

$$V_+ := \max_{|x|=1} V(x) \quad \text{and} \quad V_- := \min_{|x|=1} V(x)$$

and choose two points x_+, x_- with $|x_+| = 1 = |x_-|$ so that

$$V(x_+) = V_+ \quad \text{and} \quad V(x_-) = V_-.$$

We define two functions

$$v^+(x) := \frac{V(x) - (2r)^{-\alpha^*} V_-}{[\rho^{-\alpha^*} - (2r)^{-\alpha^*}] V_-} \quad \text{and} \quad v^-(x) := \frac{V(x) - (2r)^{-\alpha^*} V_+}{[\rho^{-\alpha^*} - (2r)^{-\alpha^*}] V_+}.$$

Then we have

$$\begin{aligned}
F(D^2v^+) = 0 = F(D^2v^-) &\text{ in } B_{2r} \setminus \overline{B_\rho}, \\
v^+ \geq 1 \text{ on } \partial B_\rho &\text{ and } v^+ \geq 0 \text{ on } \partial B_{2r}, \\
v^- \leq 1 \text{ on } \partial B_\rho &\text{ and } v^- \leq 0 \text{ on } \partial B_{2r}.
\end{aligned}$$

Thus, the comparison principle yields that

$$v^- \leq v = \overline{H}_f(B_{2r} \setminus \overline{B_\rho}) = \hat{R}_{\overline{B_\rho}}(B_{2r}) \leq v^+ \quad \text{in } B_{2r} \setminus \overline{B_\rho}.$$

Finally, applying Harnack inequality for v on $\partial B_{3r/2}$, there exists a constant $c_1 > 0$ which is independent of $r > 0$ such that

$$\frac{1}{c_1} v\left(\frac{3rx_+}{2}\right) \leq v(y_0) \leq c_1 v\left(\frac{3rx_-}{2}\right).$$

Therefore, we have the desired upper bound:

$$\operatorname{cap}_F(\overline{B_\rho}, B_{2r}) \leq c_1 v\left(\frac{3rx_-}{2}\right) \leq c_1 v^+\left(\frac{3rx_-}{2}\right) = \frac{c\, r^{-\alpha^*}}{\rho^{-\alpha^*} - (2r)^{-\alpha^*}}.$$

Similarly, we derive the lower bound:

$$\operatorname{cap}_F(\overline{B_\rho}, B_{2r}) \geq \frac{1}{c_1} v\left(\frac{3rx_+}{2}\right) \geq \frac{1}{c_1} v^-\left(\frac{3rx_+}{2}\right) = \frac{1}{c}\frac{r^{-\alpha^*}}{\rho^{-\alpha^*} - (2r)^{-\alpha^*}}.$$

(ii) ($\alpha^* < 0$) For simplicity, we assume that the upward-pointing homogeneous solution is given by

$$V(x) = -|x|^{-\alpha^*}.$$

Then we can explicitly write the capacity potential:

$$v(x) = \frac{(2r)^{-\alpha^*} - |x|^{-\alpha^*}}{(2r)^{-\alpha^*} - \rho^{-\alpha^*}}.$$

Thus,

$$\operatorname{cap}_F(\overline{B_\rho}, B_{2r}) = v(y_0) \sim \frac{r^{-\alpha^*}}{(2r)^{-\alpha^*} - \rho^{-\alpha^*}}.$$

For general V, we can compute by a similar argument as in part (i). For example, if $V(x) = -|x|^{-\alpha^*} V\left(\frac{x}{|x|}\right)$, then define

$$v^+(x) := \frac{(2r)^{-\alpha^*} V_+ + V(x)}{[(2r)^{-\alpha^*} - \rho^{-\alpha^*}] V_+} \quad \text{and} \quad v^-(x) := \frac{(2r)^{-\alpha^*} V_- + V(x)}{[(2r)^{-\alpha^*} - \rho^{-\alpha^*}] V_-}.$$

(iii) ($\alpha^* = 0$) Again for simplicity, we may assume the upward-pointing homogeneous solution is given by

$$V(x) = -\log|x|.$$

Similarly, we can explicitly write the capacity potential:

$$v(x) = \frac{\log(2r) - \log|x|}{\log(2r) - \log\rho}.$$

Thus,

$$\operatorname{cap}_F(\overline{B_\rho}, B_{2r}) = v(y_0) \sim \frac{1}{\log(2r) - \log\rho}.$$

For general V, we can compute by a similar argument as in part (i). For example, if $V(x) = V\left(\frac{x}{|x|}\right) - \log|x|$, then define

$$v^+(x) := \frac{\log(2r) - V_- + V(x)}{\log(2r) - \log\rho} \quad \text{and} \quad v^-(x) := \frac{\log(2r) - V_+ + V(x)}{\log(2r) - \log\rho}.$$

$\square$

We can observe that the capacity of a single point is determined according to the sign of the scaling exponent $\alpha^*(F)$. In fact, one can expect the results of the following lemma taking $\rho \to 0^+$ in the capacitary estimate, Lemma 2.3.11.

Lemma 2.3.12. *For $z_0 \in \mathbb{R}^n$, choose a ball $B = B_{2r}(x_0)$ so that $z_0 \in B' = B_{7r/5}(x_0)$.*

(i) If $\alpha^*(F) \geq 0$, then $\operatorname{cap}_F(\{z_0\}, B) = 0$.

(ii) If $\alpha^*(F) < 0$, then $\operatorname{cap}_F(\{z_0\}, B) > 0$.

Proof. (i) Let

$$V(x) = \begin{cases} |x|^{-\alpha^*} V\left(\frac{x}{|x|}\right) & \text{if } \alpha^* > 0, \\ -\log|x| + V\left(\frac{x}{|x|}\right) & \text{if } \alpha^* = 0. \end{cases}$$

be the homogeneous solution of F. Then for $m := \min_{x \in \partial B} V(x - z_0)$ and any $\varepsilon > 0$, we have

$$\varepsilon \cdot [V(x - z_0) - m] \in \Phi^1_{\{z_0\}}$$

due to the minimum principle and $\lim_{x \to z_0} V(x - z_0) = \infty$. Thus,

$$\operatorname{cap}(\{z_0\}, B) = \hat{R}^1_{\{z_0\}}(y_0) = R^1_{\{z_0\}}(y_0) \leq \varepsilon \cdot [V(y_0 - z_0) - m].$$

Since $\varepsilon > 0$ is arbitrary, we finish the first part of proof.

(ii) Let $V(x) = -|x|^{-\alpha^*} V\left(\frac{x}{|x|}\right)$ be the homogeneous solution of F. Then for $\max_{x \in \partial B} V(x - z_0) =: -M < 0$, we consider

$$u(x) := 1 + \frac{V(x - z_0)}{M}.$$

Since $\sup_{\partial B} u = 0$ and V is a homogeneous function, we have $\sup_{\partial B_{7/5r}} u > 0$. On the other hand, recalling Theorem 2.3.6,

$$\hat{R}^1_{\{z_0\}} = \overline{H}_f(\Omega \setminus \{z_0\}) \geq \underline{H}_f(\Omega \setminus \{z_0\}),$$

where the boundary data f is given by

$$f(x) = \begin{cases} 1 & \text{if } x = z_0, \\ 0 & \text{if } x \in \partial B. \end{cases}$$

Then $u \in \mathcal{L}_f$ and so $\underline{H}_f(\Omega \setminus \{z_0\}) \geq u$. Therefore, we conclude that

$$\sup_{\partial B_{7/5r}} \hat{R}^1_{\{z_0\}} > 0$$

and by Harnack inequality, $\operatorname{cap}(\{z_0\}, B) > 0$ as desired.

□

2.3.3 Capacity Zero Sets

Definition 2.3.13. A set E in $\mathbb{R}^n$ is said to be of (F-)*capacity zero*, or to have (F-)*capacity zero* if

$$\operatorname{cap}_F(E, B) = 0$$

whenever $E \subset B' \subset B$. In this case, we write $\operatorname{cap}_F E = 0$.

According to Lemma 2.3.12 (i), we immediately notice that every single point is of F-capacity zero if $\alpha^*(F) \geq 0$. Indeed, we are going to show that: to check whether a compact set K is of capacity zero or not, it is enough to test with respect to one ball B (Corollary 2.3.15). For this purpose, we require the following version of a capacitary estimate, called "comparable lemma".

Lemma 2.3.14 (Comparable lemma)**.** *If $K \subset B' = B_{7r/5}$ and $0 < r \leq s \leq 2r$, then there exists a universal constant $c > 0$ such that*

$$\frac{1}{c} \operatorname{cap}_F(K, B_{2r}) \leq \operatorname{cap}_F(K, B_{2s}) \leq c \operatorname{cap}_F(K, B_{2r}).$$

Proof. We may assume $x_0 = 0$. We claim that for $0 < r \le s \le \frac{21}{20}r$, we have

$$\frac{1}{c}\operatorname{cap}_F(K, B_{2r}) \le \operatorname{cap}_F(K, B_{2s}) \le c\operatorname{cap}_F(K, B_{2r}).$$

Indeed, we may iterate this inequality finitely many times to conclude the desired inequality for $0 < r \le s \le 2r$. Moreover, let $y_r = \frac{3}{2}re_1$, $y_s = \frac{3}{2}se_1$ and denote $u_r := \hat{R}^1_K(B_{2r})$, $u_s := \hat{R}^1_K(B_{2s})$. By the definition of the capacity potential, it is immediate that $u_r \le u_s$ in B_{2r}. In particular, we have

$$\operatorname{cap}_F(K, B_{2r}) = u_r(y_r) \le u_s(y_r).$$

On the other hand, an application of Harnack inequality for u_s (in a small neighborhood of $B_{3s/2} \setminus B_{10s/7}$) yields that there exists a constant $c > 0$ which is independent of the choice of r and s such that

$$u_s(y_r) \le cu_s(y_s) = c\operatorname{cap}_F(K, B_{2s}).$$

Here note that $|y_r| = \frac{3}{2}r \ge \frac{10}{7}s > \frac{7}{5}s$ and $R^1_K(B_{2s})$ is F-harmonic in $B_{2s} \setminus B_{7s/5}$ and $B_{3s/2} \setminus B_{10s/7} \subset B_{2s} \setminus B_{7s/5}$. Therefore, it finishes the proof for the first inequality.

Next, for the second inequality, we first assume that $\alpha^*(F) > 0$ and the homogeneous solution is given by $V(x) = |x|^{-\alpha^*}$ (for computational simplicity) and let

$$M := \max_{\partial B_{2r}} u_s \in [0, 1).$$

Then recalling Theorem 2.3.6, the comparison principle yields that

$$(1 - M)u_r + M \ge u_s \quad \text{in } B_{2r} \setminus K. \tag{2.3.2}$$

Now choose $z \in \partial B_{3r/2}$ so that $u_s(z) = \max_{\partial B_{3r/2}} u_s =: M_1$. Then it can be easily checked that the function

$$w(x) := M_1 \cdot \frac{|x|^{-\alpha^*} - (2s)^{-\alpha^*}}{(3r/2)^{-\alpha^*} - (2s)^{-\alpha^*}}$$

is F-harmonic in $B_{2s} \setminus B_{3r/2}$ and by the comparison principle, $w \geq u_s$ in $B_{2s} \setminus B_{3r/2}$. (here again note that $\frac{7}{5}s < \frac{3}{2}r$.) In particular,

$$\begin{aligned} M_1 \cdot \frac{(2r)^{-\alpha^*} - (2s)^{-\alpha^*}}{(3r/2)^{-\alpha^*} - (2s)^{-\alpha^*}} &\geq M, \\ M_1 \cdot \frac{(3s/2)^{-\alpha^*} - (2s)^{-\alpha^*}}{(3r/2)^{-\alpha^*} - (2s)^{-\alpha^*}} &\geq u_s \left(\frac{3}{2} s e_1 \right) = \mathrm{cap}_F(K, B_{2s}). \end{aligned}$$

Since $(3r/2)^{-\alpha^*} - (2r)^{-\alpha^*} \geq (3s/2)^{-\alpha^*} - (2s)^{-\alpha^*}$ or equivalently,

$$(3r/2)^{-\alpha^*} - (2s)^{-\alpha^*} \geq [(3s/2)^{-\alpha^*} - (2s)^{-\alpha^*}] + [(2r)^{-\alpha^*} - (2s)^{-\alpha^*}],$$

we obtain

$$u_s(z) = M_1 \geq M + \mathrm{cap}_F(K, B_{2s}). \tag{2.3.3}$$

Moreover, by (2.3.2) and (2.3.3), we have $u_r(z) \geq (1-M)u_r(z) \geq \mathrm{cap}_F(K, B_{2s})$ and then Harnack inequality leads to

$$\mathrm{cap}_F(K, B_{2s}) \leq c \, \mathrm{cap}_F(K, B_{2r}),$$

for constant $c > 0$ which is independent of r and s. Finally, for the general homogeneous solution or the case of $\alpha^*(F) \leq 0$, one can follow the idea of Lemma 2.3.11. □

Corollary 2.3.15. *Suppose that* $\mathrm{cap}(K, B) = 0$ *for* $K \subset B' \subset B$. *Then*

(i) *for any ball* B_1 *such that* $K \subset B_1'$ *and* $B_1 \subset B'$, *we have*

$$\mathrm{cap}(K, B_1) = 0;$$

(ii) for any ball B_2 such that $B_2' \supset B$, we have

$$\mathrm{cap}(K, B_2) = 0;$$

(iii) K is of F-capacity zero.

Proof. (i) Apply the first inequality of Lemma 2.3.14 finitely many times.

(ii) Apply the second inequality of Lemma 2.3.14 finitely many times.

(iii) It is an immediate consequence of (i) and (ii).

□

2.3.4 Another Characterization of a Regular Point

The definitions of a reduced function and a balayage depend on the choice of an operator F. In this subsection, we need to distinguish an operator and its dual operator, so we will specify the dependence by denoting $\hat{R}_K^{1,F}(\Omega)$ or $\hat{R}_K^{1,\widetilde{F}}(\Omega)$. We now provide a key lemma for our first main theorem, the sufficiency of the Wiener criterion:

Lemma 2.3.16. *A boundary point $x_0 \in \partial\Omega$ is regular if*

$$\hat{R}^{1,\widetilde{F}}_{\overline{B}\setminus\Omega}(2B)(x_0) = 1 = \hat{R}^{1,F}_{\overline{B}\setminus\Omega}(2B)(x_0)$$

whenever B is a ball centered at x_0.

Proof. For $f \in C(\partial\Omega)$, consider the upper Perron solution $\overline{H}_f = \overline{H}_f(\Omega)$. We may assume $f(x_0) = 0$ and $\max_{\partial\Omega} |f| \leq 1$. For $\varepsilon > 0$, we can choose a ball B with center x_0 such that $\partial(2B) \cap \Omega \neq \varnothing$ and $|f| < \varepsilon$ in $2B \cap \partial\Omega$. Then we define

$$u = \begin{cases} 1 + \varepsilon - \hat{R}^{1,\widetilde{F}}_{\overline{B}\setminus\Omega}(2B) & \text{in } \Omega \cap 2B, \\ 1 + \varepsilon & \text{in } \Omega \setminus 2B. \end{cases}$$

Since $\hat{R}^{1,\widetilde{F}}_{\overline{B}\setminus\Omega}(2B)$ is a $\widetilde{F}$-solution in $\Omega\cap 2B$, $1+\varepsilon-\hat{R}^{1,\widetilde{F}}_{\overline{B}\setminus\Omega}(2B)$ is F-harmonic in $\Omega\cap 2B$. On the other hand, by Theorem 2.3.6, $\hat{R}^{1,\widetilde{F}}_{\overline{B}\setminus\Omega}(2B)$ can be considered as the upper Perron solution for the operator $\widetilde{F}$. Then since a ball is regular, we have

$$\lim_{y\to x}\hat{R}^{1,\widetilde{F}}_{\overline{B}\setminus\Omega}(2B)(y)=0 \quad \text{for all } x\in\partial(2B).$$

Thus, u is continuous in Ω and by the pasting lemma, u is F-superharmonic in Ω. Moreover, it can be easily checked that

$$\liminf_{y\to x} u(y)\geq f(x) \quad \text{for any } x\in\partial\Omega.$$

Therefore, $u\in\mathcal{U}_f$ and so $\overline{H}_f\leq u$. In particular,

$$\begin{aligned}\limsup_{\Omega\ni y\to x_0}\overline{H}_f(y)\leq \limsup_{\Omega\ni y\to x_0}u(y)&=1+\varepsilon-\liminf_{\Omega\ni y\to x_0}\hat{R}^{1,\widetilde{F}}_{\overline{B}\setminus\Omega}(2B)(y)\\ &\leq 1+\varepsilon-\hat{R}^{1,\widetilde{F}}_{\overline{B}\setminus\Omega}(2B)(x_0)=\varepsilon.\end{aligned}$$

For the converse inequality, we define

$$v=\begin{cases}-1-\varepsilon+\hat{R}^{1,F}_{\overline{B}\setminus\Omega}(2B) & \text{in } \Omega\cap 2B,\\ -1-\varepsilon & \text{in } \Omega\setminus 2B.\end{cases}$$

Then by a similar argument, $v\in\mathcal{L}_f$ and so,

$$\liminf_{\Omega\ni y\to x_0}\underline{H}_f(y)\geq -\varepsilon.$$

Consequently, since $\varepsilon>0$ is arbitrary, we conclude that

$$\lim_{\Omega\ni y\to x_0}\overline{H}_f(y)=\lim_{\Omega\ni y\to x_0}\underline{H}_f(y)=0=f(x_0),$$

i.e. x_0 is regular. □

Next, we provide a converse direction of the above lemma: i.e. a characterization of an irregular boundary point. We expect that this lemma may be employed to prove the necessity of the Wiener criterion for the general case.

Lemma 2.3.17 (Characterization of an irregular boundary point). *If there exists a constant $\rho > 0$ such that the capacity potential $u = u_\rho$ of $\overline{B_\rho(x_0)} \setminus \Omega$ with respect to $B_{2\rho}(x_0)$ satisfies the inequality*

$$u(x_0) = \hat{R}^1_{\overline{B_\rho(x_0)}\setminus\Omega}(B_{2\rho}(x_0)) < 1,$$

then the boundary point $x_0 \in \partial\Omega$ is irregular.

Proof. Since the capacity potential u is the lower semi-continuous regularization, we have

$$u(x_0) = \liminf_{\Omega \ni x \to x_0} u(x) < 1. \tag{2.3.4}$$

Moreover, by definition, we have $u_{\rho'} \leq u_\rho$ when $0 < \rho' < \rho$. Thus, we can choose a sufficiently small $\rho > 0$ such that (2.3.4) holds and $\Omega \cap \partial B_{2\rho}(x_0) \neq \varnothing$.

We now define a smooth boundary data f on $\partial(\Omega \cap B_{2\rho}(x_0))$ such that $f(x) = 3/2$ if $x \in \partial\Omega \cap B_{\rho/2}(x_0)$, $0 \leq f(x) \leq 3/2$ if $x \in \partial\Omega \cap (B_\rho(x_0) \setminus B_{\rho/2}(x_0))$ and $f(x) = 0$ on the remaining part of $\partial(\Omega \cap B_{2\rho}(x_0))$. Then we consider the lower Perron solution $\underline{H}_f(\Omega \cap B_{2\rho}(x_0))$. We claim that the following inequality holds:

$$\underline{H}_f(x) \leq \frac{1}{2} + u(x), \quad x \in \Omega \cap B_{2\rho}(x_0). \tag{2.3.5}$$

Recalling the comparison principle, it is enough to check the above inequality on the boundary of the domain $\Omega \cap B_{2\rho}(x_0)$. For this purpose, let $v \in \mathcal{L}_f(\Omega \cap B_{2\rho}(x_0))$ and $w \in \mathcal{U}_g(B_{2\rho}(x_0) \setminus (\overline{B_\rho(x_0)} \setminus \Omega))$ where g is given by (recall Theorem 2.3.6)

$$g = \begin{cases} 1 & \text{on } \partial(\overline{B_\rho(x_0)} \setminus \Omega), \\ 0 & \text{in } \partial B_{2\rho}(x_0). \end{cases}$$

(i) (on $\partial\Omega \cap B_{2\rho}(x_0)$) First, for $x \in \partial\Omega \cap B_\rho(x_0)$, we have

$$\limsup_{y\to x} v(y) \le f(x) \le \frac{3}{2} = \frac{1}{2} + g(x) \le \frac{1}{2} + \liminf_{y\to x} w(y).$$

Next, for $x \in \partial\Omega \cap (B_{2\rho}(x_0) \setminus B_\rho(x_0))$, we have

$$\limsup_{y\to x} v(y) \le f(x) = 0 \le \frac{1}{2} + g(x) \le \frac{1}{2} + \liminf_{y\to x} w(y).$$

(ii) (on $\Omega \cap \partial B_{2\rho}(x_0)$) Similarly, we obtain

$$\limsup_{y\to x} v(y) \le f(x) = 0 \le \frac{1}{2} + g(x) \le \frac{1}{2} + \liminf_{y\to x} w(y).$$

Since v and w are F-subharmonic and F-superharmonic, respectively, we derive that

$$v \le \frac{1}{2} + w, \quad \text{in } \Omega \cap B_{2\rho}(x_0).$$

Taking the supremum on v and the infimum on w, we conclude (2.3.5) which implies that

$$\liminf_{\Omega\cap B_{2\rho}(x_0)\ni x\to x_0} \underline{H}_f(x) \le \frac{1}{2} + \liminf_{\Omega\cap B_{2\rho}(x_0)\ni x\to x_0} u(x) < \frac{3}{2} = f(x_0).$$

Therefore, x_0 is irregular with respect to $\Omega\cap B_{2\rho}(x_0)$. Recalling Lemma 2.2.24, we deduce that x_0 is irregular with respect to Ω. $\square$

2.4 A Sufficient Condition for the Regularity of a Boundary Point

In this section, we prove the sufficiency of the Wiener criterion and its sequential corollaries, via the potential estimates. More precisely, we first develop quantitative estimates for the capacity potential $\hat{R}^1_K(B)$ by employing capacitary estimates obtained in Section 2.3. Then we adopt the characterization of a regular boundary point in terms of the capacity potential to deduce the desired conclusion.

Definition 2.4.1. We say that a set E is F*-thick* at z if the Wiener integral diverges, i.e.

$$\int_0^1 \mathrm{cap}_F(E \cap \overline{B_t(z)}, B_{2t}(z)) \frac{\mathrm{d}t}{t} = \infty. \tag{2.4.1}$$

For simplicity, we write

$$\varphi_F(z, E, t) := \mathrm{cap}_F(E \cap \overline{B_t(z)}, B_{2t}(z)),$$

for the capacity density function in (2.4.1).

Remark 2.4.2. Recalling Lemma 2.3.11, there exists a constant $c > 0$ which is independent of $t > 0$ such that

$$1/c \leq \mathrm{cap}_F(\overline{B_t}, B_{2t}) \leq c.$$

Thus, one may write an equivalent form of (2.4.1):

$$\int_0^1 \frac{\mathrm{cap}_F(E \cap \overline{B_t(z)}, B_{2t}(z))}{\mathrm{cap}_F(\overline{B_t(z)}, B_{2t}(z))} \frac{\mathrm{d}t}{t} = \infty,$$

which is a similar form to the Wiener integral appearing in [46, 79].

We now state an equivalent form of our main theorem, Theorem 2.1.2:

If Ω^c is both F-thick and $\widetilde{F}$-thick at $x_0 \in \partial\Omega$, then x_0 is regular.

To prove this statement, we need several auxiliary lemmas regarding the capacity potential.

Lemma 2.4.3. *Fix a ball B. Suppose that $K \subset B'$ is compact and $v =$*

$\hat{R}^1_K(B)$. *If* $0 < \gamma < 1$ *and* $K_\gamma := \{x \in B : v(x) \geq \gamma\} \subset B'$, *then*

$$\mathrm{cap}(K_\gamma, B) = \frac{1}{\gamma}\mathrm{cap}(K, B).$$

Proof. We write $v_\gamma := \hat{R}^1_{K_\gamma}(B)$. Then by Lemma 2.3.4 and the definition of a reduced function,

$$v_\gamma = R^1_{K_\gamma}(B) = \inf \Phi^1_{K_\gamma} \quad \text{in } B \setminus K_\gamma.$$

(i) Clearly, $v = \hat{R}^1_K(B)$ is F-superharmonic in B and so is v/γ due to (F2). Since $v \geq \gamma$ in K_γ, we have $v/\gamma \geq 1$ in K_γ. Thus, $v/\gamma \in \Phi^1_{K_\gamma}$ and so

$$\frac{v}{\gamma} \geq v_\gamma \quad \text{in } B \setminus K_\gamma.$$

(ii) Recalling Theorem 2.3.6, $v_\gamma = \overline{H}_{f_\gamma}(B \setminus K_\gamma)$ in $B \setminus K_\gamma$ where

$$f_\gamma = \begin{cases} 1 & \text{on } \partial K_\gamma, \\ 0 & \text{on } \partial B. \end{cases}$$

Then for $u \in \mathcal{U}_{f_\gamma}(B \setminus K_\gamma)$, we have

$$\liminf_{B\setminus K_\gamma \ni y \to x} u(y) \geq f_\gamma(x) = 1 = \frac{v(x)}{\gamma},$$

for any $x \in \partial K_\gamma$. Since u is F-superharmonic and v/γ is F-harmonic in $B \setminus K_\gamma$, the comparison principle leads to $u \geq v/\gamma$ in $B \setminus K_\gamma$ and so

$$v_\gamma \geq \frac{v}{\gamma} \quad \text{in } B \setminus K_\gamma.$$

Consequently, we conclude that

$$\mathrm{cap}(K_\gamma, B) = v_\gamma(y_0) = \frac{1}{\gamma}v(y_0) = \frac{1}{\gamma}\mathrm{cap}(K, B).$$

□

Lemma 2.4.4. *Fix a ball $B = B_{2r}(x_0)$. Let $K \subset B_r = B_r(x_0)$ be a compact set and $v = \hat{R}^1_K(B)$. Then there exists a constant $c > 0$ which is independent of K and r such that*

$$v(x) \geq c \,\mathrm{cap}(K, B),$$

for any $x \in B_r$.

Proof. Denote

$$M := \sup_{\partial B_{6r/5}} v, \quad m := \inf_{\partial B_{6r/5}} v.$$

Since v is a non-negative F-solution in $B \setminus K$, Harnack inequality yields that there exists a constant $c_1 > 0$ independent of $r > 0$ such that

$$c_1 M \leq m. \tag{2.4.2}$$

Morevoer, the strong maximum principle in $B \setminus B_{6r/5}$ implies that

$$K_M := \{v \in B : v(x) \geq M\} \subset \overline{B_{6r/5}},$$

and so

$$\mathrm{cap}(K_M, B) \leq \mathrm{cap}(\overline{B_{6r/5}}, B) \sim 1. \tag{2.4.3}$$

Here we applied Lemma 2.3.11 and the comparable constant does not depend on K and r.

Now since $K_M \subset B'$, we can apply Lemma 2.4.3:

$$\mathrm{cap}(K_M, B) = \frac{1}{M}\mathrm{cap}(K, B). \tag{2.4.4}$$

Finally, combining (2.4.2), (2.4.3) and (2.4.4), we conclude that

$$m \geq c_1 M = c_1 \cdot \frac{\operatorname{cap}(K, B)}{\operatorname{cap}(K_M, B)} \geq c_2 \operatorname{cap}(K, B),$$

and the minimum principle leads to the desired result. $\square$

We may rewrite the previous lemma as

$$\hat{R}^1_K(B_{2r})(x) \geq c\,\varphi_F(x_0, K, r), \quad \text{for any } x \in B_r. \tag{2.4.5}$$

Lemma 2.4.5. *Let $x_0 \in \partial\Omega$, $\rho > 0$ and*

$$w = 1 - \hat{R}^1_{\overline{B_\rho(x_0)}\setminus\Omega}(B_{2\rho}(x_0)).$$

Then for all $0 < r \leq \rho$, there exists a constant $c > 0$ such that

$$w(x) \leq \exp\left(-c\int_r^\rho \varphi_F(x_0, \Omega^c, t)\frac{\mathrm{d}t}{t}\right),$$

for any $x \in B_r(x_0)$.

Proof. Denote $B_i = B_{2^{1-i}\rho}(x_0)$. Fix $0 < r \leq \rho$ and let k be the integer with $2^{-k}\rho < r \leq 2^{1-k}\rho$. Then write for $i = 0, 1, 2, ...$

$$v_i := \hat{R}^1_{\overline{B_{i+1}}\setminus\Omega}(B_i)$$

and

$$a_i := \varphi_F(x_0, \Omega^c, 2^{-i}\rho).$$

Since $e^t \geq 1 + t$, estimate (2.4.5) yields that

$$v_i \geq ca_i \geq 1 - \exp(-ca_i) \quad \text{in } B_{i+1}.$$

Thus, denoting $m_0 := \inf_{B_1} v_0$, we have $1 - m_0 \leq \exp(-ca_0)$. Next, let

$D_1 := B_1 \setminus (\overline{B_2} \cap \Omega^c)$ and

$$\psi_1 := \begin{cases} 1 & \text{in } \overline{B_2} \cap \Omega^c, \\ m_0 & \text{in } D_1. \end{cases}$$

Then we write $u_1 := \hat{R}^{\psi_1}(B_1)$ be the balayage with respect to the ψ_1 in B_1. It immediately follows from the definition of balayage that

$$\frac{u_1 - m_0}{1 - m_0} = \hat{R}^1_{\overline{B_2} \cap \Omega^c}(B_1) = v_1.$$

Again, denoting $m_1 := \inf_{B_2} u_1$, we obtain

$$1 - m_1 \le (1 - m_0) \exp(-ca_1) \le \exp(-c(a_1 + a_0)).$$

Now iterate this step: let $D_i := B_i \setminus (\overline{B_{i+1}} \cap \Omega^c)$ and

$$\psi_i = \begin{cases} 1 & \text{in } \overline{B_{i+1}} \cap \Omega^c, \\ m_{i-1} & \text{in } D_i. \end{cases}$$

Denoting $u_i := \hat{R}^{\psi_i}(B_i)$ and $m_i := \inf_{B_{i+1}} u_i$, we have

$$\frac{u_i - m_{i-1}}{1 - m_{i-1}} = v_i$$

and so

$$1 - m_i \le (1 - m_{i-1}) \exp(-ca_i) \le \exp\left(-c \sum_{j=0}^{i} a_j\right).$$

Furthermore, we claim that $u_i \ge u_{i+1}$ in B_{i+1}. Indeed, by Theorem 2.3.6, $u_i = \overline{H}_{f_i}(D_i)$ in D_i where $f_i \in C(\partial D_i)$ is given by

$$f_i = \begin{cases} 1 & \text{in } \partial(\overline{B_{i+1}} \cap \Omega^c), \\ m_{i-1} & \text{in } \partial B_i. \end{cases}$$

Thus, for $u \in \mathcal{U}_{f_i}(D_i)$, we have

$$\begin{aligned}
&\liminf_{D_{i+1}\ni y\to x} u(y) \geq 1 \geq \limsup_{D_{i+1}\ni y\to x} u_{i+1}(y) \quad \text{for any } x \in \partial(\overline{B_{i+2}} \cap \Omega^c),\\
&\liminf_{D_{i+1}\ni y\to x} u(y) \geq m_i = \limsup_{D_{i+1}\ni y\to x} u_{i+1}(y) \quad \text{for any } x \in \partial B_{i+1}.
\end{aligned}$$

Therefore, by the comparison principle, $u \geq u_{i+1}$ in D_{i+1} and so $u_i = \overline{H}_{f_i}(D_i) \geq u_{i+1}$ in B_{i+1}.

Repeating the argument above, we conclude that $v_0 \geq u_1 \geq \cdots \geq u_k$ in B_k, which implies that

$$w = 1 - v_0 \leq 1 - u_k \leq 1 - m_k \leq \exp\left(-c\sum_{j=0}^{k} a_j\right) \quad \text{in } B_{k+1}.$$

Finally, the result follows from

$$\int_r^\rho \varphi_F(x_0, \Omega^c, t)\frac{\mathrm{d}t}{t} \leq c\sum_{i=1}^{k} a_i,$$

which can be easily checked from the dyadic decomposition. Indeed, we can deduce from Lemma 2.3.11 and Lemma 2.3.14 that if $t \leq s \leq 2t$, then

$$\mathrm{cap}_F(\overline{B_t} \setminus K, B_{2t}) \sim \mathrm{cap}_F(\overline{B_t} \setminus K, B_{2s}),$$

where the comparable constant only depends on n, λ, Λ and these results also hold for $\mathrm{cap}_{\widetilde{F}}(\cdot)$. $\square$

We are ready to prove the sufficiency of the Wiener criterion.

Proof of Theorem 2.1.2. Let $x_0 \in \partial\Omega$, $\rho > 0$ and define

$$w_{F,\rho} := 1 - \hat{R}^{1,F}_{B_\rho(x_0)\setminus\Omega}(B_{2\rho}(x_0)) \quad \text{and} \quad w_{\widetilde{F},\rho} := 1 - \hat{R}^{1,\widetilde{F}}_{B_\rho(x_0)\setminus\Omega}(B_{2\rho}(x_0)).$$

Then applying Lemma 2.4.5 for both functions, we have that for all $0 < r \leq \rho$, there exist a constant $c_1, c_2 > 0$ such that

$$w_{F,\rho}(x) \leq \exp\left(-c_1 \int_r^\rho \varphi_F(x_0, \Omega^c, t)\frac{\mathrm{d}t}{t}\right),$$
$$w_{\widetilde{F},\rho}(x) \leq \exp\left(-c_2 \int_r^\rho \varphi_{\widetilde{F}}(x_0, \Omega^c, t)\frac{\mathrm{d}t}{t}\right),$$

for any $x \in B_r(x_0)$. Letting $r \to 0^+$, we conclude that

$$\hat{R}^{1,F}_{\overline{B_\rho(x_0)}\setminus\Omega}(B_{2\rho}(x_0))(x_0) = 1 = \hat{R}^{1,\widetilde{F}}_{\overline{B_\rho(x_0)}\setminus\Omega}(B_{2\rho}(x_0))(x_0).$$

Since $\rho > 0$ can be arbitrarily chosen, an application of Lemma 2.3.16 yields that $x_0 \in \partial\Omega$ is a regular boundary point. (Note that a boundary point x_0 is F-regular if and only if it is $\widetilde{F}$-regular; Corollary 2.2.25.) $\square$

On the other hand, if additional information is imposed on the boundary data f, i.e. the boundary data f has its maximum (or minimum) at $x_0 \in \partial\Omega$, then we can deduce the continuity of the Perron solution at x_0 under a relaxed condition:

Corollary 2.4.6. *Suppose that $f \in C(\partial\Omega)$ attains its maximum [resp. minimum] at $x_0 \in \partial\Omega$. If Ω^c is F-thick [resp. $\widetilde{F}$-thick] at $x_0 \in \partial\Omega$, then*

$$\lim_{\Omega\ni y\to x_0} \overline{H}_f(y) = f(x_0) = \lim_{\Omega\ni y\to x_0} \underline{H}_f(y).$$

Proof. Similarly as in the proof of the previous theorem, this corollary is the consequence of Lemma 2.3.16 and Lemma 2.4.5. $\square$

Furthermore, if the given boundary data $f \in C(\partial\Omega)$ is resolutive, then we are able to obtain a quantitative estimate for the modulus of continuity.

Lemma 2.4.7 (The modulus of continuity)**.** *Suppose that Ω is an open and bounded subset of $\mathbb{R}^n$. Let $f \in C(\partial\Omega)$.*

If $x_0 \in \partial\Omega$ with $f(x_0) = 0$, then for $0 < r \leq \rho$, we have

$$\sup_{\Omega_r} \underline{H}^F_f \leq \max_{\partial\Omega_{2\rho}} f + \max_{\partial\Omega} f \cdot \exp\left(-c \int_r^\rho \varphi_{\widetilde{F}}(x_0, \Omega^c, t)\frac{\mathrm{d}t}{t}\right)$$

and

$$\inf_{\Omega_r} \overline{H}_f^F \geq \min_{\partial\Omega_{2\rho}} f + \min_{\partial\Omega} f \cdot \exp\left(-c\int_r^\rho \varphi_F(x_0, \Omega^c, t)\frac{\mathrm{d}t}{t}\right)$$

where $\Omega_r := \Omega \cap B_r(x_0)$ *and* $\partial\Omega_{2\rho} := \partial\Omega \cap B_{2\rho}(x_0)$.

Furthermore, if f *is resolutive, then we have the quantitative estimate:*

$$\begin{aligned}\min_{\partial\Omega_{2\rho}} f + \min_{\partial\Omega} f \cdot \exp\left(-c\int_r^\rho \varphi_F(x_0, \Omega^c, t)\frac{\mathrm{d}t}{t}\right) \leq \inf_{\Omega_r} H_f^F \\ \leq \sup_{\Omega_r} H_f^F \leq \max_{\partial\Omega_{2\rho}} f + \max_{\partial\Omega} f \cdot \exp\left(-c\int_r^\rho \varphi_{\widetilde{F}}(x_0, \Omega^c, t)\frac{\mathrm{d}t}{t}\right),\end{aligned}$$

where $H_f^F := \overline{H}_f^F = \underline{H}_f^F$.

Proof. Let $v = \hat{R}^{1,\widetilde{F}}_{\overline{B_\rho(x_0)}\setminus\Omega}(B_{2\rho}(x_0))$ be the capacity potential of $\overline{B_\rho} \setminus \Omega$ with respect to $B_{2\rho}$. Then let $w := 1 - v$ and write

$$s := w \cdot \max_{\partial\Omega} f + \max_{\partial\Omega_{2\rho}} f.$$

Note that since we assumed $f(x_0) = 0$, we have $\max_{\partial\Omega} f \geq 0$ and $\max_{\partial\Omega_{2\rho}} f \geq 0$. For $u \in \mathcal{L}_f^F$, u is F-subharmonic and s is F-harmonic in $\Omega_{2\rho}$. Moreover,

$$\liminf_{y\to x} s(y) \geq \max_{\partial\Omega_{2\rho}} f \geq \limsup_{y\to x} u(y) \quad \text{for any } x \in \partial\Omega \cap B_{2\rho}$$

and

$$\liminf_{y\to x} s(y) \geq \max_{\partial\Omega} f \geq \limsup_{y\to x} u(y) \quad \text{for any } x \in \Omega \cap \partial B_{2\rho}.$$

Thus, the comparison principle yields that $s \geq u$ in $\Omega_{2\rho}$ and so $s \geq \underline{H}_f^F$ in $\Omega_{2\rho}$. On the other hand, let

$$\widetilde{s} := \left(1 - \hat{R}^{1,F}_{\overline{B_\rho(x_0)}\setminus\Omega}(B_{2\rho}(x_0))\right) \cdot \max_{\partial\Omega}(-f) + \max_{\partial\Omega_{2\rho}}(-f).$$

By the same argument, we derive $\widetilde{s} \geq \underline{H}^{\widetilde{F}}_{-f} = -\overline{H}_f^F$ in $\partial\Omega_{2\rho}$.

An application of Lemma 2.4.5 for w (and $\widetilde{w}$) finishes the proof. $\square$

Now we present a new geometric condition for a regular boundary point, namely the exterior corkscrew condition; see also [41, 62].

Definition 2.4.8. We say that Ω satisfies the *exterior corkscrew condition* at $x_0 \in \partial\Omega$ if there exists $0 < \delta < 1/4$ and $R > 0$ such that for any $0 < r < R$, there exists $y \in B_r(x_0)$ such that $\overline{B_{\delta r}(y)} \subset \Omega^c \cap B_r(x_0)$.

Note that if Ω satisfies an exterior cone condition at $x_0 \in \partial\Omega$, then Ω satisfies an exterior corkscrew condition at x_0. Thus, the following corollary obtained from the (potential theoretic) Wiener criterion is a generalized result of Theorem 2.2.26.

Corollary 2.4.9 (Exterior corkscrew condition)**.** *Suppose that Ω satisfies an exterior corkscrew condition at $x_0 \in \partial\Omega$. Then x_0 is a regular boundary point. Moreover, if f is Hölder continuous at x_0 and is resolutive, then H_f is Hölder continuous at x_0.*

Proof. A small modification of Lemma 2.3.11 and its proof, we have

$$\mathrm{cap}(\overline{B_{\delta r}(y)}, B_{2r}(x_0)) \sim 1, \quad \text{for } \delta \in (0, 1/4) \text{ and } \overline{B_{\delta r}(y)} \subset B_{2r}(x_0),$$

where the comparable constant depends only on n, λ, Λ and δ. Thus, if x_0 satisfies an exterior corkscrew condition, then we have

$$\begin{aligned}
\int_0^1 \mathrm{cap}_F(\overline{B_t(x_0)} \setminus \Omega, B_{2t}(x_0)) \frac{\mathrm{d}t}{t} &\geq \int_0^1 \mathrm{cap}_F(\overline{B_{\delta t}(y)}, B_{2t}(x_0)) \frac{\mathrm{d}t}{t} \geq \infty, \\
\int_0^1 \mathrm{cap}_{\widetilde{F}}(\overline{B_t(x_0)} \setminus \Omega, B_{2t}(x_0)) \frac{\mathrm{d}t}{t} &\geq \int_0^1 \mathrm{cap}_{\widetilde{F}}(\overline{B_{\delta t}(y)}, B_{2t}(x_0)) \frac{\mathrm{d}t}{t} \geq \infty,
\end{aligned}$$

and so x_0 is a regular boundary point by the Wiener criterion.

Next, for the second statement, we may assume $f(x_0) = 0$ by adding a constant for f, if necessary. Since f is resolutive, we can apply the quantitative estimate obtained in Lemma 2.4.7:

$$\begin{aligned}\min_{\partial\Omega_{2\rho}} f + \min_{\partial\Omega} f \cdot \exp\left(-c\int_r^\rho \varphi_F(x_0, \Omega^c, t)\frac{\mathrm{d}t}{t}\right) &\le \inf_{\Omega_r} H_f^F \\ \le \sup_{\Omega_r} H_f^F \le \max_{\partial\Omega_{2\rho}} f + \max_{\partial\Omega} f \cdot \exp&\left(-c\int_r^\rho \varphi_{\widetilde{F}}(x_0, \Omega^c, t)\frac{\mathrm{d}t}{t}\right),\end{aligned}$$

Here

(i) f is Hölder continuous at x_0: there exists a constant $C > 0$ such that $|f(x)| = |f(x) - f(x_0)| \le C|x - x_0|^\gamma \le C\rho^\gamma$ for $x \in \partial\Omega_{2\rho}$.

(ii) Ω satisfies an exterior corkscrew condition at x_0:

$$\exp\left(-c\int_r^\rho \varphi_{\widetilde{F}}(x_0, \Omega^c, t)\frac{\mathrm{d}t}{t}\right) \le \exp\left(-c_1\int_r^\rho \frac{\mathrm{d}t}{t}\right) = \left(\frac{r}{\rho}\right)^{c_1}.$$

Thus, choosing $\rho = r^{1/2}$, we conclude that the Perron solution H_f is Hölder continuous at x_0. □

Remark 2.4.10 (Example)**.** In this example, we suppose $n = 2$, $F = \mathcal{P}^+_{\lambda,\Lambda}$ with ellipticity constants $0 < \lambda < \Lambda$. Then it immediately follows that

$$\widetilde{F} = \mathcal{P}^-_{\lambda,\Lambda}, \quad \alpha^*(F) = (n-1)\frac{\lambda}{\Lambda} - 1 < 0, \quad \alpha^*(\widetilde{F}) = (n-1)\frac{\Lambda}{\lambda} - 1 > 0.$$

We consider a domain $\Omega = B_1(0) \setminus \{0\} \subset \mathbb{R}^2$ and its boundary point $0 \in \partial\Omega$.

(i) Since $\alpha^*(F) < 0$, we know that a single point has non-zero capacity. More precisely, recalling the homogeneous solution for F is given by

$$V(x) = -|x|^{1-\frac{\lambda}{\Lambda}},$$

there exists a constant $c = c(\lambda, \Lambda) > 0$ such that

$$\mathrm{cap}_F(\{0\}, B_{2t}(0)) = c.$$

Therefore, we have

$$\int_0^\rho \operatorname{cap}_F(\{0\}, B_{2t}(0))\frac{\mathrm{d}t}{t} = c\int_0^\rho \frac{\mathrm{d}t}{t} = \infty.$$

In other words, Ω^c is F-thick at 0.

(ii) On the other hand, since $\alpha^*(\widetilde{F}) > 0$, we know that a single point is of capacity zero. Therefore, we have

$$\int_0^\rho \frac{\operatorname{cap}_{\widetilde{F}}(\{0\}, B_{2t}(0))}{\operatorname{cap}_{\widetilde{F}}(\overline{B_t(0)}, B_{2t}(0))}\frac{\mathrm{d}t}{t} = 0.$$

In other words, Ω^c is not $\widetilde{F}$-thick at 0 and we cannot apply our Wiener's criterion.

(iii) Let $f_1 \in C(\partial\Omega)$ is a boundary data given by

$$f_1(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } |x| = 1. \end{cases}$$

Then clearly the function $u(x) = 1 - |x|^{1-\frac{\lambda}{\Lambda}} = 1 - V(x)$ is the solution for this Dirichlet problem. In particular, in this case, we have $\overline{H}_{f_1} = \underline{H}_{f_1}$ (i.e. f_1 is resolutive) and

$$\lim_{\Omega \ni x \to 0} H_{f_1}(x) = 1 = f_1(0).$$

Alternatively, one can apply Corollary 2.4.6 to reach the same conclusion, since f_1 attains its maximum at 0 and Ω^c is F-thick at 0.

(iv) Let $f_2 \in C(\partial\Omega)$ is a boundary data given by

$$f_2(x) = \begin{cases} -1 & \text{if } x = 0, \\ 0 & \text{if } |x| = 1. \end{cases}$$

Then since the zero function belongs to $\mathcal{U}_{f_2}$, we have $\overline{H}_{f_2} \leq 0$. Moreover, since $\varepsilon(1-|x|^{-(\frac{\Lambda}{\lambda}-1)}) \in \mathcal{L}_{f_2}$ for any $\varepsilon > 0$, we have $\underline{H}_{f_2} \geq \varepsilon(1-|x|^{-(\frac{\Lambda}{\lambda}-1)})$. Letting $\varepsilon \to 0$, we conclude $\underline{H}_{f_2} \geq 0$.

Therefore, we deduce that $\overline{H}_{f_2} = \underline{H}_{f_2} = 0$. Furthermore, it follows that

$$\lim_{\Omega \ni x \to 0} H_{f_2}(x) = 0 \neq -1 = f_2(0),$$

which implies that 0 is an irregular boundary point for Ω.

2.5 A Necessary Condition for the Regularity of a Boundary Point

In this section, we provide the necessity of the Wiener criterion, under additional structure on the operator F. Indeed, our strategy is to employ the argument made in [64] which proved the necessity of the p-Wiener criterion for p-Laplacian operator with $p > n-1$. Since the assumption $p > n-1$ was essentially imposed to ensure the capacity of a line segment is non-zero in [64], we begin with finding the corresponding assumptions in the fully nonlinear case.

Lemma 2.5.1. *Suppose that F is convex and $\alpha^*(F) > s$ for some $s > 0$. Let K be a compact subset in $B_r(\subset \mathbb{R}^n)$ such that $\mathcal{H}^s(K) < \infty$, where $\mathcal{H}^s$ is the s-dimensional Hausdorff measure. Then*

$$\operatorname{cap}_F(K) = 0.$$

Proof. For any $\delta > 0$, define

$$\mathcal{H}^s_\delta(K) := \inf \sum_i r_i^s,$$

where the infimum is taken over all countable covers of K by balls B_i with diameter r_i not exceeding δ. Then since $\sup_{\delta>0} \mathcal{H}^s_\delta(K) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(K) = \mathcal{H}^s(K) < \infty$ and K is compact, for each $\delta \in (0, r)$, there exist finitely many open balls $\{B_i = B_{r_i}(x_i)\}_{i=1}^N$ such that $r_i < \delta$, $\bigcup_{i=1}^N B_i \supset K$, and

$$\sum_{i=1}^{N} r_i^s \leq \mathcal{H}^s(K) + 1 < \infty. \tag{2.5.1}$$

Now we consider the homogeneous solution $V(x) = |x|^{-\alpha^*} V\left(\frac{x}{|x|}\right)$ of F. Here we may assume $\min_{|x|=1} V(x) = 1$ by normalizing V. If we let $W_i(x) := r_i^{\alpha^*} V(x - x_i)$, then it immediately follows that W_i is non-negative and F-superharmonic in $\mathbb{R}^n$, and $W_i(x) \geq 1$ on B_i.

Finally, we let $W := \sum_{i=1}^N W_i (\geq 0)$. Since F is convex, W is F-superharmonic in $\mathbb{R}^n$. Moreover, $W \geq 1$ on $\bigcup_{i=1}^N B_i$ and in particular, $W \geq 1$ on K. Therefore, $W \in \Phi_K^1(B_{4r})$ and so

$$\begin{aligned} \operatorname{cap}_F(K, B_{4r}) \leq W(y_0) &\leq r^{-\alpha^*} \max_{|x|=1} V(x) \cdot \sum_{i=1}^{N} r_i^{\alpha^*} \\ &\leq r^{-\alpha^*} \max_{|x|=1} V(x) \cdot \left(\mathcal{H}^s(K) + 1\right) \delta^{\alpha^* - s}, \end{aligned}$$

where we used (2.5.1) and $\alpha^* > s$. Letting $\delta \to 0$, we finish the proof. $\square$

Now we prove the partial converse statement of Lemma 2.5.1. Indeed, here we only consider the compact set K is given by a line segment L, whose Hausdorff dimension is exactly 1.

Lemma 2.5.2. *Suppose that F is concave and $\alpha^*(F) < 1$. Let $L = \{x_0 + se : ar \leq s \leq br\}$ be a line segment in $B_r(x_0)$, where e is an unit vector in $\mathbb{R}^n$ and $0 < a < b < 1$ are constants satisfying $b - a < \frac{1}{2}$. Then*

$$\operatorname{cap}_F(L, B_{2r}) > 0.$$

Proof. Note that since L is a line segment, for any $\delta > 0$, one can cover L by open balls $B_i = B_{3\delta}(x_i)$, $1 \leq i \leq N(\delta)$ where $x_i \in L$, $|x_i - x_j| \geq 2\delta$ whenever $i \neq j$, and $N(\delta) \sim \frac{(b-a)r}{\delta}$. We write such cover by $K_\delta := \bigcup_{i=1}^{N(\delta)} \overline{B_i}$. Recalling Lemma 2.3.9 and its proof, for any $\varepsilon > 0$, there exist a sufficiently small $\delta > 0$ and corresponding cover K_δ such that

$$\mathrm{cap}_F(K_\delta, B_{2r}) \le \mathrm{cap}_F(L, B_{2r}) + \varepsilon.$$

If we denote $\widetilde{B}_i := B_\delta(x_i)$ and $\widetilde{K}_\delta = \bigcup_{i=1}^{N(\delta)} \overline{\widetilde{B}}_i$, then we have $\widetilde{B}_i$ are pairwise disjoint and

$$\mathrm{cap}_F(\widetilde{K}_\delta, B_{2r}) \le \mathrm{cap}_F(L, B_{2r}) + \varepsilon.$$

On the other hand, for simplicity, we suppose that the homogeneous solution V is given by $V(x) = |x|^{-\alpha^*}$ and $\alpha^*(F) \in (0,1)$. Note that if $\alpha^* < 0$, then a single point has a positive capacity (Lemma 2.3.12) and the result immediately follows. Other cases can be shown by similar argument as in Lemma 2.5.1. For each $i = 1, 2, \cdots, N(\delta)$, write

$$W_i(x) := \left(\frac{|x - x_i|}{\delta}\right)^{-\alpha^*} \quad \text{and} \quad W(x) = \sum_{i=1}^{N(\delta)} W_i(x).$$

Since F is concave, W is F-subharmonic in $\mathbb{R}^n \setminus \bigcup_{i=1}^{N(\delta)} \{x_i\}$.

(i) (On $\partial\widetilde{K}_\delta$) For $y \in \partial\widetilde{K}_\delta$, let $y \in \partial B_i$ for some i. Then for $j \neq i$, we have

$$|y - x_j| \ge |x_i - x_j| - |y - x_i| = |x_i - x_j| - \delta,$$

and so

$$\begin{aligned} W(y) &\le 2(1 + 2^{-\alpha^*} + \cdots + N(\delta)^{-\alpha^*}) \\ &\le 2\left(1 + \int_2^{N(\delta)} \frac{1}{s^{\alpha^*}}\,\mathrm{d}s\right) \le c\,N(\delta)^{1-\alpha^*}. \end{aligned}$$

Here we used the condition $\alpha^* < 1$.

(ii) (On ∂B_{2r}) For $z \in \partial B_{2r}$, $|z - x_i| \geq 2r - br = (2-b)r$, and so

$$W(z) \leq \left(\frac{(2-b)r}{\delta}\right)^{-\alpha^*} \cdot N(\delta).$$

Therefore, for

$$\widetilde{W}(x) := \frac{W(x) - \left(\frac{(2-b)r}{\delta}\right)^{-\alpha^*} \cdot N(\delta)}{c\, N(\delta)^{1-\alpha^*}},$$

we have

$$\widetilde{W} \text{ is } F\text{-subharmonic in } B \setminus \widetilde{K}_\delta, \quad \widetilde{W} \leq 0 \text{ on } \partial B_{2r}, \quad \text{and} \quad \widetilde{W} \leq 1 \text{ on } \partial \widetilde{K}_\delta.$$

Note that since $\widetilde{K}_\delta$ and B_{2r} is regular domains, the capacity potential $\hat{R}^1_{\widetilde{K}_\delta}(B_{2r})$ satisfies:

$$\hat{R}^1_{\widetilde{K}_\delta}(B_{2r}) = 0 \text{ on } \partial B_{2r}, \quad \text{and} \quad \hat{R}^1_{\widetilde{K}_\delta}(B_{2r}) = 1 \text{ on } \partial \widetilde{K}_\delta.$$

Hence, the comparison principle yields that

$$\hat{R}^1_{\widetilde{K}_\delta}(B_{2r}) \geq \widetilde{W} \quad \text{in } B_{2r} \setminus \widetilde{K}_\delta.$$

In particular, putting $x = x_0 + \frac{3}{2}re$, we conclude that

$$|x - x_i| \leq 3r/2 - ar = \left(\frac{3}{2} - a\right) r,$$

and so

$$
\begin{aligned}
\hat{R}^1_{\widetilde{K}_\delta}(B_{2r})\left(x_0+\frac{3}{2}re\right) &\geq \widetilde{W}\left(x_0+\frac{3}{2}re\right) \\
&\geq \frac{\left[\left(\frac{(3/2-a)r}{\delta}\right)^{-\alpha^*}-\left(\frac{(2-b)r}{\delta}\right)^{-\alpha^*}\right]\cdot N(\delta)}{c\,N(\delta)^{1-\alpha^*}} \\
&\geq c_1(b-a)^{\alpha^*}\left[\left(\frac{3}{2}-a\right)^{-\alpha^*}-(2-b)^{-\alpha^{-*}}\right].
\end{aligned}
$$

Finally, by applying Harnack inequality for $\hat{R}^1_{\widetilde{K}_\delta}(B_{2r})$ on $\partial B_{3r/2}$, we have

$$
\begin{aligned}
\varepsilon+\mathrm{cap}_F(L,B_{2r}) &\geq \mathrm{cap}_F(\widetilde{K}_\delta,B_{2r}) \\
&\geq c_2\,(b-a)^{\alpha^*}\left[\left(\frac{3}{2}-a\right)^{-\alpha^*}-(2-b)^{-\alpha^{-*}}\right]>0.
\end{aligned}
$$

Since $\varepsilon>0$ is arbitrary, we finish the proof. □

The idea of the previous lemma can be modified to derive the 'spherical symmetrization' result:

Lemma 2.5.3 (Spherical symmetrization)**.** *Suppose that F is concave and $\alpha^*(F)<1$. Let K be a compact subset in $B_r(x_0)$ such that K meets $S(t):=\{x\in\mathbb{R}^n:|x-x_0|=t\}$ for all $t\in(ar,br)$, where $0<a<b<1$ are constants satisfying $b<\frac{1}{4}$. Then there exists a constant $c=c(n,F,a,b)$ such that*

$$\mathrm{cap}_F(K,B_{2r})\geq c(n,F,a,b)>0.$$

Proof. The proof is similar to the one of Lemma 2.5.2. By assumption, we can choose $x_{(t)}\in K\cap S(t)$ for all $t\in(ar,br)$. In particular, for small $\delta>0$, we define $x_i:=x_{(ar+2\delta i)}$ for $i=1,2,\cdots,N(\delta)$ so that

$$ar+2\delta\cdot N(\delta)<br\leq ar+2\delta\cdot(N(\delta)+1).$$

Note that $N(\delta) \sim \frac{(b-a)r}{\delta}$. Moreover, for $\delta > 0$, we define a set K_δ by

$$K_\delta = \bigcup_{i=1}^{N(\delta)} \overline{B_i},$$

where $B_i = B_{x_i}(\delta)$. Again recalling Lemma 2.3.9 and its proof, for any $\varepsilon > 0$, there exists a sufficiently small $\delta > 0$ such that

$$\mathrm{cap}_F(K_\delta, B_{2r}) \leq \mathrm{cap}_F(K, B_{2r}) + \varepsilon.$$

On the other hand, for simplicity, we suppose that the homogeneous solution V is given by $V(x) = |x|^{-\alpha^*}$ and $\alpha^*(F) \in (0,1)$. For each $i = 1, 2, \cdots, N(\delta)$, write

$$W_i(x) := \left(\frac{|x - x_i|}{\delta}\right)^{-\alpha^*} \quad \text{and} \quad W(x) = \sum_{i=1}^{N(\delta)} W_i(x).$$

Since F is concave, W is F-subharmonic in $\mathbb{R}^n \setminus \bigcup_{i=1}^{N(\delta)} \{x_i\}$.

(i) (On ∂K_δ) For $y \in \partial K_\delta$, let $y \in \partial B_i$ for some i. Then for $j \neq i$, we have

$$|y - x_j| \geq |x_i - x_j| - |y - x_i| = |x_i - x_j| - \delta \geq 2|i-j|\delta - \delta,$$

and so

$$\begin{aligned} W(y) &\leq 2(1 + 2^{-\alpha^*} + \cdots + N(\delta)^{-\alpha^*}) \\ &\leq 2\left(1 + \int_2^{N(\delta)} \frac{1}{s^{\alpha^*}}\,\mathrm{d}s\right) \leq c\,N(\delta)^{1-\alpha^*}. \end{aligned}$$

Here we used the condition $\alpha^* < 1$.

(ii) (On ∂B_{2r}) For $z \in \partial B_{2r}$, $|z - x_i| \geq 2r - br = (2-b)r$, and so

$$W(z) \leq \left(\frac{(2-b)r}{\delta}\right)^{-\alpha^*} \cdot N(\delta).$$

Therefore, for

$$\widetilde{W}(x) := \frac{W(x) - \left(\frac{(2-b)r}{\delta}\right)^{-\alpha^*} \cdot N(\delta)}{c\, N(\delta)^{1-\alpha^*}},$$

we have

$$\widetilde{W} \text{ is } F\text{-subharmonic in } B \setminus K_\delta, \quad \widetilde{W} \le 0 \text{ on } \partial B_{2r}, \quad \text{and} \quad \widetilde{W} \le 1 \text{ on } \partial K_\delta.$$

Note that since K_δ and B_{2r} are regular domains, the capacity potential $\hat{R}^1_{K_\delta}(B_{2r})$ satisfies:

$$\hat{R}^1_{K_\delta}(B_{2r}) = 0 \text{ on } \partial B_{2r}, \quad \text{and} \quad \hat{R}^1_{K_\delta}(B_{2r}) = 1 \text{ on } \partial K_\delta.$$

Hence, the comparison principle yields that

$$\hat{R}^1_{K_\delta}(B_{2r}) \ge \widetilde{W} \quad \text{in } B_{2r} \setminus K_\delta.$$

In particular, putting $x = x_0 + \frac{3}{2} r e_1$, we conclude that

$$|x - x_i| \le 3r/2 + br = \left(\frac{3}{2} + b\right) r,$$

and so

$$\begin{aligned} \hat{R}^1_{K_\delta}(B_{2r}) \left(x_0 + \frac{3}{2} r e_1\right) &\ge \widetilde{W}\left(x_0 + \frac{3}{2} r e_1\right) \\ &\ge \frac{\left[\left(\frac{(3/2+b)r}{\delta}\right)^{-\alpha^*} - \left(\frac{(2-b)r}{\delta}\right)^{-\alpha^*}\right] \cdot N(\delta)}{c\, N(\delta)^{1-\alpha^*}} \\ &\ge c_1 (b-a)^{\alpha^*} \left[\left(\frac{3}{2} + b\right)^{-\alpha^*} - (2-b)^{-\alpha^{-*}}\right]. \end{aligned}$$

Hence,

$$\begin{aligned}\varepsilon + \mathrm{cap}_F(K, B_{2r}) &\geq \mathrm{cap}_F(K_\delta, B_{2r}) \\ &\geq c_2\,(b-a)^{\alpha^*}\left[\left(\frac{3}{2}+b\right)^{-\alpha^*} - (2-b)^{-\alpha^*}\right] > 0.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we finish the proof. □

Let E be a regular set in a ball B_{2r}. Let $u = \hat{R}^1_E(B_{2r})$ be the capacity potential. For $\gamma \in (0,1)$, let $A_\gamma = \{x \in B_{2r} : u(x) < \gamma\}$.

Lemma 2.5.4. *Suppose that F is concave and $\alpha^*(F) < 1$. Then there exists a constant $c_1 > 0$ depending only on n, λ, Λ such that: if*

$$\gamma \geq c_1 \mathrm{cap}_F(E, B_{2r}),$$

then the set A_γ contains a sphere $S(t) := \{x \in \mathbb{R}^n : |x - x_0| = t\}$ for some $t \in (r/10, r/5)$.

Proof. For $0 < \gamma < 1$, let $E_\gamma := \{x \in B_{2r} : u(x) \geq \gamma\}$. We argue by contradiction: suppose that A_γ does not contain any $S(t)$ for $t \in (r/10, r/5)$. Then the set E_γ meets $S(t)$ for all $t \in (r/10, r/5)$ and we have

$$\mathrm{cap}_F(E_\gamma, B_{2r}) \geq c(n, F) > 0,$$

by employing Lemma 2.5.3 for $a = 1/10$ and $b = 1/5$.

On the other hand, by Lemma 2.4.3, we have

$$\mathrm{cap}_F(E_\gamma, B_{2r}) = \frac{1}{\gamma}\mathrm{cap}_F(E, B_{2r}).$$

Combining two estimates above, we obtain

$$\gamma \leq \frac{1}{c(n,F)}\,\mathrm{cap}_F(E, B_{2r}).$$

Therefore, by choosing $c_1 = \frac{1}{c(n,F)} + 1$, we arrive at a contradiction. □

Now we are ready to prove the necessity of the Wiener criterion.

Proof of Theorem 2.1.3. For simplicity, we write $B_r = B_r(x_0)$. Suppose that Ω^c is not F-thick at $x_0 \in \partial\Omega$, i.e.

$$\int_0^1 \mathrm{cap}_F(\overline{B_t} \setminus \Omega, B_{2t}) \frac{\mathrm{d}t}{t} < \infty.$$

For $\varepsilon > 0$ to be determined, choose $r_1 > 0$ small enough so that

$$\int_0^{r_1} \mathrm{cap}_F(\overline{B_t} \setminus \Omega, B_{2t}) \frac{\mathrm{d}t}{t} < \varepsilon.$$

Set $r_{i+1} = r_i/2$ and

$$a_i = \mathrm{cap}_F(\overline{B_{r_i}} \setminus \Omega, B_{2r_i}).$$

Applying Lemma 2.3.14,

$$\sum_{i=2}^{\infty} a_i \leq c_0(n, \lambda, \Lambda)\, \varepsilon.$$

Next, by Corollary 2.2.27 and Lemma 2.3.9, for each i, choose a regular domain E_i such that $\overline{B_{r_i}} \setminus \Omega \subset E_i$ and

$$b_i := \mathrm{cap}_F(E_i, B_{2r_i}) < a_i + \varepsilon \cdot 2^{-i}.$$

Then we have

$$\sum_{i=2}^{\infty} b_i \leq (c_0 + 1)\, \varepsilon$$

and so $b_i \leq (c_0 + 1)\, \varepsilon$ for $i = 2, 3, \cdots$. Moreover, let $u_i := \hat{R}^1_{E_i}(B_{2r_i})$ be the capacity potential. By Lemma 2.5.4, for $\gamma_i = c_1 \cdot b_i$, the set

$$A_i = \{x \in B_{2r_i} : u_i(x) < \gamma_i\}$$

contains $S(t_i)$ for some $t_i \in (r_i/10, r_i/5)$. Now by selecting $\varepsilon = \frac{1}{2(c_0+1)c_1} > 0$, we have $\gamma_i < 1$. In particular, since $u_2 = 1$ on E_2 and $S(t_2) \subset A_2$, we conclude that $S(t_2) \subset \Omega$.

Next, let $f \in C(\partial\Omega)$ be the boundary function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in B_{t_2} \cap \partial\Omega, \\ 0 & \text{if } x \in \partial\Omega \setminus B_{t_2}. \end{cases}$$

Then we have the following results for the lower Perron solution $\underline{H}_f = \underline{H}_f(\Omega)$:

(i) $\underline{H}_f \not\equiv 1$: Choose $r > 0$ large enough so that $\Omega \subset B_r$. Moreover, set a domain $\Omega_0 := B_r \setminus (B_{t_2} \cap \Omega)$ and a boundary function $f_0 \in C(\partial\Omega_0)$ by

$$f_0(x) = \begin{cases} 1 & \text{if } x \in B_{t_2} \cap \partial\Omega, \\ 0 & \text{if } x \in \partial B_r. \end{cases}$$

Then since B_r is regular, we have $\overline{H}_{f_0}(\Omega_0) < 1$ in $B_r \setminus B_{t_2}$. On the other hand, for any $v \in \mathcal{L}_f(\Omega)$ and $w \in \mathcal{U}_{f_0}(\Omega_0)$, one can check that $v \leq w$ in Ω using the comparison principle. Therefore, we conclude that $\underline{H}_f(\Omega) \leq \overline{H}_{f_0}(\Omega_0)$ and so $\underline{H}_f(\Omega) \not\equiv 1$.

(ii) $\max_{S(t_2)} \underline{H}_f =: M < 1$: This is an immediate consequence of the strong maximum principle for $\underline{H}_f$ and part (i).

For $u := \frac{\underline{H}_f - M}{1-M}$ which is F-harmonic in Ω and $u \leq 0$ in $S(t_2)$, we claim that

$$\liminf_{\Omega \ni x \to x_0} u(x) < \frac{1}{2}. \tag{2.5.2}$$

Indeed, since $S(t_2) \subset B_{r_3}$ and E_3 is a regular domain, we have

$$u(x) \leq 0 \leq \liminf_{y \to x} u_3(y) \quad \text{for any } x \in \partial B_{t_2} = S(t_2),$$
$$\limsup_{y \to x} u(y) \leq 1 = \liminf_{y \to x} u_3(y) \quad \text{for any } x \in \partial E_3.$$

Thus, the comparison principle yields that $u \leq u_3$ in $B_{t_2} \setminus E_3$. In particular, since $S(t_3) \subset A_3$, we observe that

$$u \le u_3 < \gamma_3 \quad \text{on } S(t_3).$$

Iterating this argument (for example, consider $u - \gamma_3$ instead of u), we conclude that

$$u \le \sum_{k=3}^{i} \gamma_k \le \sum_{k=3}^{\infty} \gamma_k = c_1 \cdot \sum_{k=3}^{\infty} b_i \le c_1 \, (c_0 + 1) \, \varepsilon = \frac{1}{2} \quad \text{on each } S(t_i),$$

which leads to (2.5.2).

Finally, recalling the definition of u, the estimate (2.5.2) is equivalent to

$$\liminf_{\Omega \ni x \to x_0} \underline{H}_f(x) < 1 = f(x_0),$$

which implies that $x_0 \in \partial\Omega$ is an irregular boundary point. $\square$

Chapter 3

Random Homogenization of φ-Laplace Equations with Highly Oscillating Obstacles

3.1 Introduction

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a given probability space. For every $\omega \in \Omega$ and every $\varepsilon > 0$, we consider a domain $D_\varepsilon(\omega)$ obtained by perforating holes from an open, bounded domain D of $\mathbb{R}^n$. We denote by $T_\varepsilon(\omega)$ the set of holes (we then have $D_\varepsilon(\omega) = D \setminus T_\varepsilon(\omega)$). The goal of this chapter is to study the asymptotic behavior of the minimizer, u_ε, of the φ-Laplacian functional as $\varepsilon \to 0$.

More precisely, let u_ε be the solution of the following obstacle problem:

$$\min\Big\{\int_D \varphi(|\nabla u|)\,\mathrm{d}x - \int_D fu\,\mathrm{d}x; u \in W_0^{1,\varphi}(D), u \geq 0 \text{ a.e. in } T_\varepsilon(\omega)\Big\}.$$

Here $\varphi : [0,\infty) \to [0,\infty)$ is an N-function satisfying the $\Delta_2 \cap \nabla_2$-condition and $f \in L^{\varphi^*}(D)$, which will be defined in Section 3.2. In particular, when we set $\varphi(t) = \frac{1}{p}t^p$, for $p > 1$, then it becomes a p-Laplacian obstacle problem and so φ-Laplacian operator is a natural generalization of p-Laplacian operator. A typical example for an N-function is $\varphi(t) = t^p \log^q(1+t)$, where $p > 1$ and

$q > 1 - p$.

Now suppose that the hole is a union of balls centered at each lattice point, i.e.

$$T_\varepsilon(\omega) = \cup_{k\in\mathbb{Z}^n} B_{a^\varepsilon(k,\omega)}(\varepsilon k),$$

where a radius of ball, $a^\varepsilon(k,\omega)$, will be determined randomly. Then we will prove that there exists a critical radius $a^\varepsilon(k,\omega) \ll \varepsilon$ such that the homogenized problem is no longer an obstacle problem, but an elliptic boundary value problem with a new term that comes from the influence of the obstacles on the holes. If a^ε is a critical radius which is determined by capacity condition, then there exists a function $g : [0,\infty) \to [0,\infty)$ such that $u = \lim_{\varepsilon\to 0} u^\varepsilon$ solves

$$\min\Big\{ \int_D \varphi(|\nabla u|)\,\mathrm{d}x + \int_D g(u_-)\,\mathrm{d}x - \int_D fu\,\mathrm{d}x; u \in W_0^{1,\varphi}(D) \Big\}.$$

In this chapter, we will concentrate on the nontrivial case with critical radius a^ε, where the limit solution satisfies an equation with an additional term. In fact, the behavior of limit solution u can be different (but trivial) if the radius of holes a^ε is not critical. First if the order of the decay rate of a^ε is higher than the critical one, then the obstacles rarely restrict the behavior of limit solution. Thus, the limit solution will be a solution of the following variational problem without obstacles:

$$\min\Big\{ \int_D \varphi(|\nabla u|)\,\mathrm{d}x - \int_D fu\,\mathrm{d}x; u \in W_0^{1,\varphi}(D) \Big\}.$$

Second, on the contrary, if the order of the decay rate is lower than the critical one, then the obtacles completely enforce the behavior of limit solution. Thus, the limit solution will be a solution of the following obstacle problem:

$$\min\Big\{ \int_D \varphi(|\nabla u|)\,\mathrm{d}x - \int_D fu\,\mathrm{d}x; u \in W_0^{1,\varphi}(D), u \geq 0 \text{ a.e. in } D \Big\}.$$

The main difficulty for extending the homogenization theorems of p-

Laplacian operator to those of φ-Laplacian operator is that the growth exponent of an N-function $\varphi(t)$ may vary with respect to t. As a result, we cannot expect the homogeneity property for an N-function φ, i.e. there is no constant $C > 0$ such that $\varphi(xy) \neq C\varphi(x)\varphi(y)$ in general. In fact, if a submultiplicative function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is differentiable at $x = 1$ and $f(1) = 1$, then $f(x) = x^p$ for all $x \in \mathbb{R}^+$ and some $p \in \mathbb{R}$. Here f is said to be submultiplicative if the inequality $f(xy) \leq f(x)f(y)$ holds for all $x, y \in \mathbb{R}^+$, and see [27] for details. Therefore, we cannot find out an explicit formula for the critical function g or the critical hole size a^ε (or the capacity of holes) in φ-Laplacian case. Moreover, since the norm in Orlicz space is defined by Luxemburg sense, there is a restriction when we replace the norm of functions to the modular of them, involving Hölder's inequality and Poincaré inequality.

The main idea for the construction of corrector w^ε is that for the critical value β_0, w^ε will behave like the fundamental solution of φ-Laplacian, near the holes. To capture this property, we also construct several intermediate functions between corrector w^ε and the fundamental solution h^ε, such as $v^\varepsilon_{\beta_0,D}$ and w^ε_τ, which will be defined precisely in Section 3.4. Note that these auxiliary functions will be defined by the solution of different obstacle problems and the existence of these functions is guaranteed by Perron's method. Finally, exploiting the similarity between w^ε and h^ε, we can obtain a strong convergence of w^ε in $L^\varphi(D)$.

The plan for this chapter is as follows. In Section 3.2, we introduce preliminaries which include definitions and well-known results about an N-function and Orlicz space. In Section 3.3, we first state two assumptions on the holes: capacity condition and condition on stationary ergodicity. Moreover, Section 3.3 contains the statement and proof of our main theorem, and several lemmas for correctors. In Section 3.4, we first find the critical value β_0 by studying a measure of contact set. Then we construct a corrector from solving an obstacle problem which depends on the critical value β_0, and show the desired properties for this corrector. Finally, in Section 3.5, we prove Lemma 3.3.5 and Lemma 3.3.12.

3.2 Preliminaries

The study for an N-function and its related Orlicz space was initiated by extending theory of classical L^p space. For an overview of Orlicz space theory, we refer two books [23, 32] and references therein; both books contain generalization of theorems in L^p space, such as Sobolev embedding theorem, density theorem and Poincaré inequality. Liebermann [63] proved a Harnack inequality for a solution of φ-Laplace equations by obtaining local boundedness and weak Harnack inequality. See also [6]. Moreover, in [24, 25], Diening, Stroffolini and Verde studied the regularity of φ-harmonic maps. In fact, they showed that the minimizer of φ-Laplacian energy has a Hölder continuous gradient by using Lipschitz truncation method as a main tool.

We first introduce some definitions and facts about an N-function and Orlicz space. Here we always denote D by an open, bounded subset in $\mathbb{R}^n$. Also note that in this chapter, we denote $f \sim g$ for two functions f, g when there exist two constants $c_1, c_2 > 0$ such that $c_1 f(t) \le g(t) \le c_2 f(t)$.

Definition 3.2.1 (N-function). $\varphi : [0, \infty) \to [0, \infty)$ is called an *N-function* if

(i) $\varphi(0) = 0$,

(ii) φ is strictly increasing and convex,

(iii) $\lim_{x\to 0^+} \frac{\varphi(x)}{x} = 0, \lim_{x\to\infty} \frac{\varphi(x)}{x} = \infty$.

Definition 3.2.2 (Δ_2-condition). An N-function φ is said to satisfy the Δ_2-*condition* if there exists $c > 0$ such that for all $t \ge 0$, we have

$$\varphi(2t) \le c\varphi(t).$$

We denote the smallest possible constant by $\Delta_2(\varphi)$. Since $\varphi(t) \le \varphi(2t)$ holds for an N-function φ, the Δ_2-condition is equivalent to the relation $\varphi(2t) \sim \varphi(t)$.

Definition 3.2.3 (∇_2-condition)**.** An N-function φ is said to satisfy the ∇_2-*condition* if there exists $a > 1$ such that for all $t \geq 0$ we have

$$\varphi(t) \leq \frac{\varphi(at)}{2a}.$$

Definition 3.2.4 (Conjugate function)**.** For an N-function φ, we define the *Young conjugate* φ^* of φ by

$$\varphi^*(t) = \sup_{s \geq 0} \{ts - \varphi(s)\}.$$

Here φ^* is again an N-function and $(\varphi^*)^* = \varphi$. Moreover, we may equivalently define

$$\varphi^*(t) := \int_0^t (\varphi')^{-1}(s)\, \mathrm{d}s,$$

which implies that $(\varphi^*)'(t) = (\varphi')^{-1}(t)$ for $t > 0$.

Lemma 3.2.5. *The following statements are equivalent:*

(i) φ satisfies the ∇_2-condition.

(ii) φ^* satisfies the Δ_2-condition.

Remark 3.2.6. By the definition of an N-function and the Δ_2-condition, we can easily check that if φ is an N-function satisfying the Δ_2-condition, then

$$\varphi(t) \sim t\varphi'(t),$$

uniformly in $t > 0$. Moreover, we can check that uniformly in $t > 0$,

$$\varphi^*(\varphi'(t)) \sim \varphi(t),$$

whenever φ satisfies the $\Delta_2 \cap \nabla_2$-condition.

Lemma 3.2.7 ([5, Lemma 2.1])**.** *An N-function φ satisfies the Δ_2-condition if and only if*

$$\sup_{t>0} \frac{t\varphi'(t)}{\varphi(t)} < +\infty.$$

Moreover, its conjugate function φ^* satisfies the Δ_2-condition if and only if

$$\inf_{t>0} \frac{t\varphi'(t)}{\varphi(t)} > 1.$$

For a constant $p_\varphi > 0$, by differentiating the function $\frac{\varphi(t)}{t^{p_\varphi}}$, we have

$$t \mapsto \frac{\varphi(t)}{t^{p_\varphi}} \text{ is non-decreasing if and only if } t\varphi'(t) \geq p_\varphi \varphi(t) \text{ for any } t \geq 0.$$

Similarly, for $q_\varphi > 0$, we have

$$t \mapsto \frac{\varphi(t)}{t^{q_\varphi}} \text{ is non-increasing if and only if } t\varphi'(t) \leq q_\varphi \varphi(t) \text{ for any } t \geq 0.$$

Lemma 3.2.8 ([35])**.** *Suppose that for an N-function φ, there exist $1 < p_\varphi \leq q_\varphi < \infty$ such that*

$$t \mapsto \frac{\varphi'(t)}{t^{p_\varphi - 1}} \textit{ is non-decreasing and } t \mapsto \frac{\varphi'(t)}{t^{q_\varphi - 1}} \textit{ is non-increasing.}$$

Then φ satisfies the following properties for any $s, t \geq 0$:

(i) $\min\{s^{p_\varphi - 1}, s^{q_\varphi - 1}\}\varphi'(t) \leq \varphi'(st) \leq \max\{s^{p_\varphi - 1}, s^{q_\varphi - 1}\}\varphi'(t)$.

(ii) $t \mapsto \frac{\varphi(t)}{t^{p_\varphi}}$ *is non-decreasing and* $t \mapsto \frac{\varphi(t)}{t^{q_\varphi}}$ *is non-increasing.*

(iii) $\min\{s^{p_\varphi}, s^{q_\varphi}\}\varphi(t) \leq \varphi(st) \leq \max\{s^{p_\varphi}, s^{q_\varphi}\}\varphi(t)$.

(iv) $c_1 \min\{s^{\frac{p_\varphi}{p_\varphi - 1}}, s^{\frac{q_\varphi}{q_\varphi - 1}}\}\varphi^*(t) \leq \varphi^*(st) \leq c_2 \max\{s^{\frac{p_\varphi}{p_\varphi - 1}}, s^{\frac{q_\varphi}{q_\varphi - 1}}\}\varphi^*(t)$.

We now state assumptions for φ, which are necessary for our main theorem:

Assumption 3.2.9. Let φ be an N-function, which is C^1 on $(0, \infty)$. We also suppose that there exist constants $1 < p_\varphi \leq q_\varphi < n$ such that

$$t \mapsto \frac{\varphi'(t)}{t^{p_\varphi - 1}} \text{ is non-decreasing and } t \mapsto \frac{\varphi'(t)}{t^{q_\varphi - 1}} \text{ is non-increasing.}$$

We remark that under these assumptions, φ satisfies $\Delta_2 \cap \nabla_2$-condition and φ' satisfies Δ_2-condition.

Definition 3.2.10 (Orlicz space)**.** For an N-function φ, *Orlicz space* $L^\varphi(D)$ consists of all Lebesgue measurable functions defined in D, satisfying

$$\int_D \varphi(\lambda f(x))\,\mathrm{d}x < \infty \quad \text{for some } \lambda > 0.$$

Here $L^\varphi(D)$ is a Banach space with the Luxemburg norm

$$\|f\|_{L^\varphi(D)} := \inf\left\{\kappa > 0 : \int_D \varphi\left(\frac{|f(x)|}{\kappa}\right)\,\mathrm{d}x \le 1\right\}.$$

Moreover, we define the Orlicz-Sobolev space $W^k L^\varphi(D)$: the set of measurable functions f on D with weak derivatives $D^\alpha f \in L^\varphi(D)$ for all $|\alpha| \le k$.

Lemma 3.2.11 (Young's inequality and Hölder's inequality)**.** *Let φ be an N-function. Then the following Young's inequality holds:*

$$ab \le \varphi(a) + \varphi^*(b) \quad \textit{for all } a, b \ge 0.$$

Assume that $u \in L^\varphi(D)$ and $v \in L^{\varphi^*}(D)$; then the following Hölder's inequality holds:

$$\int_D uv\,\mathrm{d}x \le 2\|u\|_\varphi \|v\|_{\varphi^*}.$$

Remark 3.2.12 (Norm-modular relation; [23])**.**

(i) Let φ be an N-function. Then we define the norm and the modular of a function $f \in L^\varphi(D)$ as following:

$$\|f\|_\varphi := \inf\left\{\lambda > 0 : \int_D \varphi\left(\frac{|f|}{\lambda}\right)\,\mathrm{d}x \le 1\right\},$$
$$\rho_\varphi(f) = \int_D \varphi(|f|)\,\mathrm{d}x.$$

(ii) For an N-function φ and a function $f \in L^\varphi(D)$, we have the following norm-modular relation by their definitions:

$$\|f\|_\varphi \leq 1 \implies \rho_\varphi(f) \leq \|f\|_\varphi,$$
$$\|f\|_\varphi > 1 \implies \rho_\varphi(f) \geq \|f\|_\varphi.$$

In particular, we obtain $\|f\|_\varphi \leq \rho_\varphi(f) + 1$.

(iii) Let φ be an N-function and let f_k, f be functions in $L^\varphi(D)$. We say that f_k converges strongly in norm to f if $\|f_k - f\|_\varphi \to 0$. Note that $\|f_k - f\|_\varphi \to 0$ if and only if $\rho_\varphi(\lambda(f_k - f)) \to 0$ for all $\lambda > 0$.

On the other hand, we say that f_k converges modularly to f if there exists $\lambda > 0$ such that $\rho_\varphi(\lambda(f_k - f)) \to 0$.

(iv) Let φ be an N-function satisfying the Δ_2-condition. Then the modular convergence is equivalent to the norm convergence in $L^\varphi(D)$.

Definition 3.2.13 (φ-capacity)**.** (i) For $A \subset \mathbb{R}^n$,

$$\mathrm{cap}_\varphi(A) := \inf\left\{\int_{\mathbb{R}^n} |\nabla u|\varphi'(|\nabla u|)\,\mathrm{d}x : \ u \in C_0^\infty(\mathbb{R}^n), \ u = 1 \text{ on } \partial A\right\}.$$

(ii) For any open set $D \subset \mathbb{R}^n$ and compact set $K \subset D$,

$$\mathrm{cap}_\varphi(K, D) := \inf\left\{\int_D |\nabla u|\varphi'(|\nabla u|)\,\mathrm{d}x : u \in C_c^\infty(D), u \equiv 1 \text{ on } K\right\}.$$

Theorem 3.2.14 (Compact embedding; [32, Theorem 6.3.7])**.** *Let φ be an N-function satisfying the $\Delta_2 \cap \nabla_2$-condition. Then $W_0^{1,\varphi}(D)$ is compactly embedded in $L^\varphi(D)$.*

Theorem 3.2.15 (Poincaré inequality; [32, Theorem 6.2.8])**.** *Let φ be an N-function satisfying the $\Delta_2 \cap \nabla_2$-condition. Then for every $u \in W_0^{1,\varphi}(D)$, we have*

$$\|u\|_{L^\varphi(D)} \leq c \,\mathrm{diam}(D)\|\nabla u\|_{L^\varphi(D)}.$$

Theorem 3.2.16 (Comparison principle)**.** *Let φ be an N-function and suppose that $\Delta_\varphi u \geq \Delta_\varphi v$ holds in a bounded domain D. If the inequality*

$$\limsup_{x\to\zeta} u(x) \leq \liminf_{x\to\zeta} v(x)$$

holds for any $\zeta \in \partial D$, then $u \leq v$ in D a.e.

Lemma 3.2.17 (Harnack inequality; [6])**.** *Let u be a locally bounded and non-negative solution of equation*

$$-\nabla\cdot\left(\frac{\varphi'(|\nabla u|)}{|\nabla u|}\nabla u\right) = \mathcal{B}(\cdot, u)$$

in Ω, where φ is an N-function satisfying the $\Delta_2 \cap \nabla_2$-condition and

$$|\mathcal{B}(x,u)| \leq \alpha\varphi'(|u(x)|) + \beta,$$

for α, β are non-negative numbers.

Let $B_R \subset D$ be a ball of radius $0 < R \leq 1$. There exists a positive constant $\mathcal{N} = \mathcal{N}(\alpha, p_\varphi, q_\varphi, n)$ such that

$$\sup_{B_{R/2}} u \leq \mathcal{N}\left(\inf_{B_{R/2}} u + LR\right),$$

where L is any non-negative constant such that $\beta \leq \varphi'(L)$ and $B_{R/2}$ is the ball of radius $R/2$ concentric with B_R.

3.3 Main Theorem

Before we state the main theorem, let us first make precise the assumptions on the holes $T_\varepsilon(\omega) = \left(\cup_{k\in\mathbb{Z}^n} B_{a^\varepsilon(k,\omega)}(\varepsilon k)\right) \cap D$.

Assumption 3.3.1. For all $k \in \mathbb{Z}^n$ and a.e. $\omega \in \Omega$, there exists $\gamma(k,\omega)$ (independent of ε) such that

$$\mathrm{cap}_\varphi(B_{a^\varepsilon(k,\omega)}(\varepsilon k)) = \varepsilon^n \gamma(k,\omega),$$

where $\operatorname{cap}(A)$ denotes the capacity of a subset A of $\mathbb{R}^n$. Moreover, we assume that there exists a constant $\overline{\gamma} > 0$:

$$\gamma(k,\omega) \leq \overline{\gamma} \quad \text{for all } k \in \mathbb{Z}^n \text{ and a.e. } \omega \in \Omega.$$

Assumption 3.3.2. The process $\gamma : \mathbb{Z}^n \times \Omega \mapsto [0,\infty)$ is *stationary ergodic*: there exists a family of measure-preserving transformations $\tau_k : \Omega \to \Omega$ such that

(i) (stationary) $\gamma(k+k',\omega) = \gamma(k,\tau_{k'}\omega)$ for all $k,k' \in \mathbb{Z}^n$ and $\omega \in \Omega$;

(ii) (ergodic) if $A \subset \Omega$ and $\tau_k A = A$ for all $k \in \mathbb{Z}^n$, then $P(A) = 0$ or $P(A) = 1$. (in other words, the only invariant set of positive measure is the whole set.)

Theorem 3.3.3. *Assume that $T_\varepsilon(\omega)$ satisfies Assumption 3.3.1 and Assumption 3.3.2. Also let φ be an N-function satisfying Assumption 3.2.9. Then there exists a function $g : [0,\infty) \to [0,\infty)$ such that when ε goes to zero, the solution $u^\varepsilon(x,\omega)$ of*

$$\min\Big\{ \int_D \varphi(|\nabla u|)\,\mathrm{d}x - \int_D fu\,\mathrm{d}x;\ u \in W_0^{1,\varphi}(D), u \geq 0 \ \text{a.e. in } T_\varepsilon(\omega)\Big\}$$

converges weakly in $W^{1,\varphi}(D)$ and almost surely $\omega \in \Omega$ to the solution $\bar{u}(x)$ of the following minimization problem:

$$\min\Big\{ \int_D \varphi(|\nabla u|) + g(u_-)\,\mathrm{d}x - \int_D fu\,\mathrm{d}x;\ u \in W_0^{1,\varphi}(D)\Big\}.$$

Moreover, $g(\cdot)$ is an N-function satisfying the $\Delta_2 \cap \nabla_2$-condition; in particular, if $\varphi(t) = \frac{1}{p}t^p$, then $g(t) = \beta_0\varphi(t)$.

Remark 3.3.4. The Euler-Lagrange equations for the minimization problem yield

$$\begin{cases} -\Delta_\varphi u^\varepsilon = f & \text{for } x \in D_\varepsilon, \\ u^\varepsilon(x) \geq 0 & \text{for } x \in T_\varepsilon, \\ u^\varepsilon(x) = 0 & \text{for } \partial D \setminus T_\varepsilon, \end{cases}$$

where

$$\Delta_\varphi v := -\nabla \cdot \left(\frac{\varphi'(|\nabla v|)}{|\nabla v|} \nabla v \right).$$

3.3.1 Key Lemmas

In this subsection, we introduce several lemmas which are essential for proving our main theorem. We will prove Lemma 3.3.5 and Lemma 3.3.12, which describe the behavior of the corrector w^ε, in Section 3.4 and Section 3.5.

Lemma 3.3.5. *There exist a non-negative constant β_0 and a function $w^\varepsilon(x,\omega)$ such that*

$$\begin{cases} \Delta_\varphi w^\varepsilon = \beta_0 & \text{in } D_\varepsilon(\omega), \\ w^\varepsilon(x,\omega) = 1 & \text{for } x \in T_\varepsilon(\omega), \\ w^\varepsilon(x,\omega) = 0 & \text{for } x \in \partial D \setminus T_\varepsilon(\omega), \\ w^\varepsilon(\cdot,\omega) \rightharpoonup 0 & \text{in } W^{1,\varphi}(D), \end{cases}$$

for almost all $\omega \in \Omega$, and w^ε also satisfies the following properties:

(i) Let ψ_1 be a function and ψ_2 be an N-function such that $\psi_2(\psi_1(t)) \sim \varphi(t)$ for uniformly in $t > 0$. Then for any $\eta \in \mathcal{D}(D)$,

$$\lim_{\varepsilon \to 0} \int_D \psi_1(|\nabla w^\varepsilon|)\eta \, \mathrm{d}x = 0.$$

(ii) For any $\eta \in \mathcal{D}(D)$,

$$\lim_{\varepsilon \to 0} \int_D \varphi'(|\nabla w^\varepsilon|)|\nabla w^\varepsilon|\eta \, \mathrm{d}x = \int_D \beta_0 \eta \, \mathrm{d}x.$$

(iii) *For any sequence* $\{v^\varepsilon\} \subset W_0^{1,\varphi}(D)$ *with the property:* $v^\varepsilon \to v$ *weakly in* $W_0^{1,\varphi}(D)$ *as* $\varepsilon \to 0$ *and* $v^\varepsilon = 0$ *on* T_ε *and any* $\eta \in \mathcal{D}(D)$*, we have that*

$$\lim_{\varepsilon\to 0} \int_D \frac{\varphi'(|\nabla w^\varepsilon|)}{|\nabla w^\varepsilon|} \nabla w^\varepsilon \cdot \nabla v^\varepsilon \eta \, \mathrm{d}x = -\int_D \beta_0 v \eta \, \mathrm{d}x.$$

Remark 3.3.6. In the first part of Lemma 3.3.5, the assumption $\psi_2(\psi_1(t)) \sim \varphi(t)$ implies that, roughly speaking, the growth rate of ψ_1 is smaller than that of φ. For example, for the simplest case, if $\psi_1(t) = t$, we can choose $\psi_2(t) = \varphi(t)$. Moreover, if $\psi_1(t) = \varphi'(t)$, then we can choose $\psi_2(t) = \varphi^*(t)$.

Remark 3.3.7. We introduce the initial and limit energies:

$$I[u] := \int_D \varphi(|\nabla u|) \, \mathrm{d}x - \int_D f u \, \mathrm{d}x$$

and

$$I_0[u] := \int_D \varphi(|\nabla u|) + g(u_-) \, \mathrm{d}x - \int_D f u \, \mathrm{d}x.$$

With these notations, we have that $u^\varepsilon(x,\omega)$ satisfies

$$I[u^\varepsilon] = \inf_{v\in K_\varepsilon} I[v]$$

with $K_\varepsilon = \{v \in W_0^{1,\varphi}(D); v \geq 0 \text{ a.e. in } T_\varepsilon\}$.

Since $\{u^\varepsilon\}$ is bounded in $W^{1,\varphi}(D)$, there exists a function $\bar{u}(x,\omega)$ such that

$$u^\varepsilon(\cdot,\omega) \rightharpoonup \bar{u}(\cdot,\omega) \text{ in } W_0^{1,\varphi}(D) \text{ -weak.}$$

Now to prove the main theorem, it is enough to show that for almost every ω, $\bar{u}(\cdot,\omega)$ satisfies:

$$I_0[\bar{u}] = \inf_{v\in W_0^{1,\varphi}(D)} I_0[v].$$

Lemma 3.3.8. *Let* w^ε *be a corrector function defined in Lemma 3.3.5. Then* *there exists a function* $g : [0, \infty) \to [0, \infty)$ *such that*

$$\lim_{\varepsilon \to 0} \int_D \varphi(|\nabla w^\varepsilon|\eta) \, \mathrm{d}x = \int_D g(\eta) \, \mathrm{d}x,$$

for any $\eta \in \mathcal{D}(D)$*, with* $\eta \geq 0$*. Moreover,* g *is an increasing function on* $[0, \infty)$*.*

Proof. Let η be a simple function, i.e.

$$\eta := \sum_{i=1}^{N} a_i \chi_{A_i},$$

where $a_i > 0$ and $A_i \subset D$ are mutually disjoint. Then

$$\int_D \varphi(|\nabla w^\varepsilon|\eta) \, \mathrm{d}x = \sum_{i=1}^{N} \int_D \varphi(|\nabla w^\varepsilon|a_i) \chi_{A_i} \, \mathrm{d}x.$$

Now define

$$\mu_a^\varepsilon(A, \omega) := \int_D \varphi(a|\nabla w^\varepsilon(x, \omega)|) \chi_A \, \mathrm{d}x,$$

for $\omega \in \Omega$, $a \geq 0$ and $A \subset D$. First to check the subadditive property of the random variable μ_a^ε, let $(A_i)_{i \in I}$ be a finite family of sets such that

$$\begin{aligned} &A_i \subset A \text{ for all } i \in I, \\ &A_i \cap A_j = \varnothing \text{ for all } i \neq j, \\ &|A - \cup_{i \in I} A_i| = 0. \end{aligned}$$

Then we have

$$\mu_a^\varepsilon(A, \omega) \leq \sum_{i \in I} \mu_a^\varepsilon(A_i, \omega),$$

which yields the subadditive property. Moreover, we have

$$\begin{aligned}\mu_a^\varepsilon(A,\omega) &= \int_D \varphi(a|\nabla w^\varepsilon(x,\omega)|)\chi_A \,\mathrm{d}x \\ &\le c\int_D |\nabla w^\varepsilon|\varphi'(|\nabla w^\varepsilon|)\chi_A \,\mathrm{d}x \le (c\beta_0+1)|A|,\end{aligned}$$

by part (ii) of Lemma 3.3.5. Finally, thanks to the ergodicity of the transformations τ_k, it follows from the subadditive ergodic theorem (see [16]) that for each a there exists a constant $g(a) \ge 0$ such that

$$\lim_{\varepsilon\to 0} \frac{\mu_a^\varepsilon(A,\omega)}{|A|} = g(a),$$

or equivalently,

$$\lim_{\varepsilon\to 0}\int_D \varphi(a|\nabla w^\varepsilon(x,\omega)|)\chi_A \,\mathrm{d}x = \int_D g(a)\chi_A \,\mathrm{d}x.$$

Note that this limit $g(a)$ increases when a increases by the definition of $\mu_a^\varepsilon(A,\omega)$. This construction of g finishes the proof. □

Lemma 3.3.9. *Let g be the function constructed in Lemma 3.3.8. Then g is an N-function satisfying the $\Delta_2 \cap \nabla_2$-condition.*

Proof. Recall the construction of g in the proof of Lemma 3.3.8:

$$g(a) = \lim_{\varepsilon\to 0}\frac{\mu_a^\varepsilon(A,\omega)}{|A|} = \lim_{\varepsilon\to 0}\frac{\int_D \varphi(a|\nabla w^\varepsilon(x,\omega)|)\chi_A \,\mathrm{d}x}{|A|}.$$

It is clear that $g(0)=0$, g is increasing and convex.

For $0<s<1$, using Lemma 3.2.8, we have $g(s) \le g(1)\cdot s^{p_\varphi}$, which implies that

$$0 \le \lim_{s\to 0}\frac{g(s)}{s} \le \lim_{s\to 0}[g(1)\cdot s^{p_\varphi-1}] = 0,$$

since $p_\varphi > 1$. Similarly, we can show that $\lim_{s\to\infty}\frac{g(s)}{s} = \infty$.

Moreover, since φ satisfies the Δ_2-condition,

$$g(2a) \leq \Delta_2(\varphi) g(a).$$

Thus, g satisfies the Δ_2-condition and $\Delta_2(g) \leq \Delta_2(\varphi)$. Using a similar argument, we conclude that g satisfies the $\Delta_2 \cap \nabla_2$-condition. $\square$

Remark 3.3.10. Note that if $\varphi(t) = \frac{1}{p} t^p$, then we have

$$\begin{aligned}
\lim_{\varepsilon \to 0} \int_D \varphi(|\nabla w^\varepsilon| \eta) \, \mathrm{d}x &= \lim_{\varepsilon \to 0} \int_D |\nabla w^\varepsilon|^p \varphi(\eta) \, \mathrm{d}x \\
&= \lim_{\varepsilon \to 0} \int_D |\nabla w^\varepsilon| \varphi'(|\nabla w^\varepsilon|) \varphi(\eta) \, \mathrm{d}x \\
&= \int_D \beta_0 \varphi(\eta) \, \mathrm{d}x,
\end{aligned}$$

by applying part (ii) of Lemma 3.3.5.

Thus, in this case, we can calculate $g(t) = \beta_0 \varphi(t)$ explicitly and recover the result for p-Laplacian in [77].

Remark 3.3.11. In [15, 77] and this chapter, the existence of the critical value β_0 is guaranteed by the subadditive ergodic theorem, and then the corresponding homogenized operator is implicitly defined. However, following the arguments in [3] and [28], we are able to express the critical value β_0 explicitly, in terms of the expectation of the stationary ergodic process $\gamma(k, \omega)$, for the p-Laplacian operators. Indeed, they showed that β_0 (or g) can be determined explictly when φ satisfies a growth condition of order p $(1 < p < n)$, i.e. there exist two constants $c_1, c_2 > 0$ such that

$$c_1(x^p - 1) \leq \varphi(x) \leq c_2(x^p + 1) \quad \text{for any } x \geq 0.$$

In particular, [28, Section 6] discussed how to recover the homogenization results in [15, 77] and to compute β_0, using the Γ-convergence method. Note that in our general setting, φ has varying growth order (roughly, from p_φ to q_φ), which is not adequate for applying those results.

Lemma 3.3.12. *If Lemma 3.3.5 holds and u^ε is the solution of*

$$\min\Big\{\int_D \varphi(|\nabla v|) - fv\,\mathrm{d}x : v \in W_0^{1,\varphi}(D), u \geq 0 \text{ a.e. in } T_\varepsilon(\omega)\Big\},$$

then

$$\liminf_{\varepsilon\to 0} I[u^\varepsilon] \geq I_0[u^0],$$

where u^0 is the weak limit of $\{u^\varepsilon\}$ in $W^{1,\varphi}(D)$.

3.3.2 Proof of Theorem 3.3.3

Proof. For any $\eta \in \mathcal{D}(D)$, the function $\eta+\eta_- w^\varepsilon$ is non-negative on $T_\varepsilon(\omega)$ and is thus admissible for the initial obstacle problem. In particular, by definition of u^ε, we have

$$I[u^\varepsilon] \leq I[\eta + \eta_- w^\varepsilon],$$

where

$$I[\eta+\eta_- w^\varepsilon] = \int_D \varphi(|\nabla\eta + \nabla\eta_- w^\varepsilon + \eta_-\nabla w^\varepsilon|)\,\mathrm{d}x - \int_D f(\eta+\eta_- w^\varepsilon)\,\mathrm{d}x.$$

Note that since φ is increasing,

$$\int_D \varphi(|\nabla\eta + \nabla\eta_- w^\varepsilon + \eta_-\nabla w^\varepsilon|)\,\mathrm{d}x \leq \int_D \varphi(|\nabla\eta + \nabla\eta_- w^\varepsilon| + |\eta_-\nabla w^\varepsilon|)\,\mathrm{d}x.$$

For convenience, let $r^\varepsilon := |\nabla\eta+\nabla\eta_- w^\varepsilon|$ and $s^\varepsilon := |\eta_-\nabla w^\varepsilon|$. Then the integral in the right-hand side can be written as

$$\int_D \varphi(r^\varepsilon + s^\varepsilon)\,\mathrm{d}x.$$

Now we will show that

$$\lim_{\varepsilon\to 0}\int_D \varphi(r^\varepsilon+s^\varepsilon)\,\mathrm{d}x = \lim_{\varepsilon\to 0}\int_D \varphi(r^\varepsilon)\,\mathrm{d}x + \lim_{\varepsilon\to 0}\int_D \varphi(s^\varepsilon)\,\mathrm{d}x. \tag{3.3.1}$$

To prove this equation (3.3.1), we need a technical lemma:

Lemma 3.3.13. *Let φ be an N-function satisfying Assumption 3.2.9. Then for $x, y > 0$, we have*

$$|\varphi(x+y) - \varphi(x) - \varphi(y)| \leq c(x\varphi'(y) + y\varphi'(x)),$$

where the constant c depends only on $\Delta_2(\varphi')$.

Proof. (i) $(x \geq y)$ Let $f(y) := \varphi(x+y) - \varphi(x) - \varphi(y)$ and then $f(0) = 0$. By mean value theorem, we have

$$f(y) - f(0) = yf'(t), \quad \text{for } t \in (0, y).$$

Here direct calculation yields that

$$|f'(y)| = |\varphi'(x+y) - \varphi'(y)| \leq 2\varphi'(x+y).$$

Recalling φ is convex and applying Assumption 3.2.9,

$$|f'(t)| \leq 2\varphi'(x+y) \leq 2\varphi'(2x) \leq c\varphi'(x).$$

Thus, we conclude that

$$|\varphi(x+y) - \varphi(x) - \varphi(y)| = |f(y) - f(0)| \leq cy\varphi'(x).$$

(ii) $(x < y)$ By symmetry, we have $|\varphi(x+y) - \varphi(x) - \varphi(y)| \leq cx\varphi'(y)$.

□

Applying this technical lemma for $x = r^\varepsilon, y = s^\varepsilon$, we obtain that

$$\left| \int_D [\varphi(r^\varepsilon + s^\varepsilon) - \varphi(r^\varepsilon) - \varphi(s^\varepsilon)] \, \mathrm{d}x \right| \leq c \int_D (\underbrace{r^\varepsilon \varphi'(s^\varepsilon)}_{=I} + \underbrace{s^\varepsilon \varphi'(r^\varepsilon)}_{=II}) \, \mathrm{d}x.$$

(I) For $r^\varepsilon \varphi'(s^\varepsilon)$, note that

$$r^\varepsilon \varphi'(s^\varepsilon) \le |\nabla \eta| \cdot \varphi'(|\nabla w^\varepsilon|) + C|w^\varepsilon| \cdot \varphi'(|\nabla w^\varepsilon|).$$

First by Lemma 3.3.5 (i), (for $\psi_1(t) = \varphi'(t)$ and $\psi_2(t) = \varphi^*(t)$) we have

$$\lim_{\varepsilon \to 0} \int_D |\nabla \eta| \cdot \varphi'(|\nabla w^\varepsilon|) \, \mathrm{d}x = 0.$$

Next by Hölder's inequality, we have

$$\begin{aligned} \int_D |w^\varepsilon| \cdot \varphi'(|\nabla w^\varepsilon|) \, \mathrm{d}x &\le 2\|w^\varepsilon\|_\varphi \|\varphi'(\nabla w^\varepsilon)\|_{\varphi^*} \\ &\le c\|w^\varepsilon\|_\varphi \left(1 + \int_D \varphi(|\nabla w^\varepsilon|) \, \mathrm{d}x\right), \end{aligned}$$

where we used the norm-modular relation and $\varphi^*(\varphi'(t)) \sim \varphi(t)$ for the last inequality. Since $w^\varepsilon \to 0$ strongly in $L^\varphi(D)$, we obtain

$$\lim_{\varepsilon \to 0} \int_D |w^\varepsilon| \cdot \varphi'(|\nabla w^\varepsilon|) \, \mathrm{d}x = 0.$$

(II) For $s^\varepsilon \varphi'(r^\varepsilon)$, note that

$$s^\varepsilon \varphi'(r^\varepsilon) \le C|\nabla w^\varepsilon| \cdot (\varphi'(|\nabla \eta|) + \varphi'(|w^\varepsilon|)).$$

Again by Lemma 3.3.5 (i), (for $\psi_1(t) = t$ and $\psi_2(t) = \varphi(t)$) we have

$$\lim_{\varepsilon \to 0} \int_D |\nabla w^\varepsilon| \cdot \varphi'(|\nabla \eta|) \, \mathrm{d}x = 0.$$

Similarly as in the first case, we obtain

$$\lim_{\varepsilon \to 0} \int_D |\nabla w^\varepsilon| \cdot \varphi'(|w^\varepsilon|) \, \mathrm{d}x = 0.$$

Combining two cases above, we conclude that

$$\lim_{\varepsilon\to 0}\int_D (r^\varepsilon\varphi'(s^\varepsilon)+s^\varepsilon\varphi'(r^\varepsilon))\,\mathrm{d}x=0,$$

which proves the equation (3.3.1).

Note that applying Lemma 3.3.8 with $w^\varepsilon \to 0$ strongly in $L^\varphi(D)$,

$$\lim_{\varepsilon\to 0}\int_D \varphi(|\nabla\eta+\nabla\eta_- w^\varepsilon|)\,\mathrm{d}x=\int_D\varphi(|\nabla\eta|)\,\mathrm{d}x,$$
$$\lim_{\varepsilon\to 0}\int_D \varphi(|\eta_-\nabla w^\varepsilon|)\,\mathrm{d}x=\int_D g(\eta_-)\,\mathrm{d}x.$$

Therefore, we have :

$$\limsup_{\varepsilon\to 0}\int_D \varphi(|\nabla\eta+\nabla\eta_- w^\varepsilon+\eta_-\nabla w^\varepsilon|)\,\mathrm{d}x\le\int_D\varphi(|\nabla\eta|)\,\mathrm{d}x+\int_D g(\eta_-)\,\mathrm{d}x,$$

which implies that

$$I_0[\eta]\ge\limsup_{\varepsilon\to 0} I[\eta+\eta_- w^\varepsilon]\ge\liminf_{\varepsilon\to 0} I[u^\varepsilon].$$

By Lemma 3.3.12, we have $I_0[\eta]\ge I_0[\bar{u}]$. Since the set $\{\eta\in\mathcal{D}(D):\eta_-\in\mathcal{D}(D)\}$ is dense in $W_0^{1,\varphi}(D)$, we obtain the desired result. □

3.4 Critical Value β_0 and Corrector w^ε

3.4.1 Find the Critical Value β_0

We introduce the following obstacle problem: for every open set $A\subset\mathbb{R}^n$ and $\beta\in\mathbb{R}$,

$$v^\varepsilon_{\beta,A}(x,\omega)=\inf\Big\{v(x);\Delta_\varphi v\le\beta-\sum_{k\in\mathbb{Z}^n\cap\varepsilon^{-1}A}\gamma(k,\omega)\varepsilon^n\delta(x-\varepsilon k)\text{ in }A,$$
$$v\ge 0\text{ in }A,\ v=0\text{ on }\partial A\Big\}.$$

Moreover, we set

$$m^{\varepsilon}_{\beta}(A,\omega) = |\{x \in A; v^{\varepsilon}_{\beta,A}(x,\omega) = 0\}|.$$

Then from the subadditive ergodic theorem, [1] and [21], for any $\beta \in \mathbb{R}$, there is a constant $l(\beta) \geq 0$ such that

$$\lim_{\varepsilon \to 0} \frac{m^{\varepsilon}_{\beta}(B_1(x_0),\omega)}{|B_1(x_0)|} = l(\beta),$$

for any $B_1(x_0) \subset \mathbb{R}^n$.

Lemma 3.4.1. *(i) $l(\beta)$ is a non-decreasing funcion of β.*

(ii) $l(\beta) = 0$ for $\beta < 0$.

(iii) $l(\beta) > 0$ for β is large enough.

Proof. (i) The proof follows immediately from the inequality

$$v_{\beta,A} \leq v_{\beta',A} \text{ for any } \beta, \beta' \text{ such that } \beta' \leq \beta.$$

(ii) If $\beta < 0$, let

$$u_{\beta}(x) := \frac{n}{|\beta|} \cdot \left[\varphi^{*}\left(\frac{|\beta|}{n}\right) - \varphi^{*}\left(\frac{|\beta|}{n} r\right)\right],$$

for $r = |x - x_0|$. Then $\Delta_{\varphi} u_{\beta} = \beta$ in $B_1(x_0)$ (see Remark 3.4.2 and Remark 3.4.3 below.) Moreover, $u_{\beta} > 0$ in $B_1(x_0)$ and $u_{\beta} = 0$ on $\partial B_1(x_0)$.

Therefore, by comparison principle, we deduce that:

$$v^{\varepsilon}_{\beta,B_1(x_0)} \geq u_{\beta} > 0 \quad \text{in } B_1(x_0).$$

Therefore, $m^{\varepsilon}_{\beta}(B_1(x_0),\omega) = 0$ for $\beta < 0$, which implies that $l(\beta) = 0$ for $\beta < 0$.

(iii) Let

$$a = a(k,\omega) = \left(\frac{nc\gamma(k,\omega)}{\beta}\right)^{\frac{1}{n}},$$

where the constant $c = c(n) = \frac{1}{nw_n}$. We define a rotationally symmetric function $g^{\varepsilon}_{\beta,k}(x,\omega)$ for any $k \in \mathbb{Z}^n$ as follows:

$$g^{\varepsilon}_{\beta,k}(x,\omega) := \begin{cases} \int_r^{a\varepsilon} (\varphi')^{-1}\left(c\gamma(k,\omega)\varepsilon^n s^{1-n} - \frac{\beta}{n}s\right) \mathrm{d}s, & \text{if } 0 \le r \le a\varepsilon, \\ 0, & \text{if } r \ge a\varepsilon, \end{cases}$$

where $r = |x - \varepsilon k|$. Note that by the definition of $g^{\varepsilon}_{\beta,k}$, we know that $g^{\varepsilon}_{\beta,k}$ and $\nabla g^{\varepsilon}_{\beta,k}$ vanish along $\partial B_{a_\varepsilon}$. Thus, by Remark 3.4.2 below, we obtain that

$$\Delta_\varphi g^{\varepsilon}_{\beta,k}(x,\omega) \le \beta - \gamma(k,\omega)\varepsilon^n \delta(x - \varepsilon k) \quad \text{in } \mathbb{R}^n,$$

and $g^{\varepsilon}_{\beta,k}(x,\omega) = 0$ if $x \notin B_{a\varepsilon}(\varepsilon k)$.

On the other hand, if we choose β large so that $\beta \ge 2^n nc\gamma(k,\omega)$, i.e.

$$\frac{1}{2} \ge a = \left(\frac{nc\gamma(k,\omega)}{\beta}\right)^{\frac{1}{n}},$$

then the support of function $g^{\varepsilon}_{\beta,k}(x,\omega)$ is contained in the cell ball $B_{\frac{\varepsilon}{2}}(\varepsilon k)$. Now we consider the sum of all $g^{\varepsilon}_{\beta,k}$:

$$g^{\varepsilon}_{\beta}(x,\omega) := \sum_{k\in\varepsilon^{-1}B_1\cap\mathbb{Z}^n} g^{\varepsilon}_{\beta,k}(x,\omega).$$

By the definition, we know that for any two different $k, k' \in \varepsilon^{-1}B_1 \cap \mathbb{Z}^n$, $g^{\varepsilon}_{\beta,k}$ and $g^{\varepsilon}_{\beta,k'}$ have disjoint support. Then

$$\Delta_\varphi g^{\varepsilon}_{\beta}(x,\omega) \le \beta - \sum_{k\in\varepsilon^{-1}B_1\cap\mathbb{Z}^n} \gamma(k,\omega)\varepsilon^n \delta(x - \varepsilon k),$$

and $g^{\varepsilon}_{\beta}(x,\omega) \ge 0$ for $x \in B_1$ and $g^{\varepsilon}_{\beta}(x,\omega) = 0$ on ∂B_1.

Therefore, for almost surely $\omega \in \Omega$,

$$0 \leq v^{\varepsilon}_{\beta,B_1}(x,\omega) \leq g^{\varepsilon}_{\beta}(x,\omega), \quad \text{for a.e. } x \in B_1.$$

Then by the definition of g^{ε}_{β}, we deduce that

$$\bigcup_{k\in\varepsilon^{-1}B_1\cap\mathbb{Z}^n} (B_1 \setminus B_{a\varepsilon}(\varepsilon k)) \subset \{x \in B_1 : v^{\varepsilon}_{\beta,B_1} = 0\},$$

which implies

$$m^{\varepsilon}_{\beta}(B_1,\omega) \geq |B_1| - C\varepsilon^{-n}(a\varepsilon)^n = |B_1| - Ca^n.$$

Thus, we have $l(\beta) > 0$ if a is small enough, i.e. β is large enough.

$\square$

Remark 3.4.2 (Idea of construction for $g^{\varepsilon}_{\beta,k}$)**.** First note that a function

$$u^{\varepsilon}_{\beta,k}(x) = \frac{\beta}{2n}|x|^2 + c\gamma(k,\omega)\varepsilon^n|x|^{2-n}$$

solves

$$\Delta u^{\varepsilon}_{\beta,k} = \beta - \gamma(k,\omega)\varepsilon^n\delta(x-\varepsilon k).$$

Here a direct calculation yields

$$\nabla u^{\varepsilon}_{\beta,k} = \left(\frac{\beta}{n}r - c_n\gamma(k,\omega)\varepsilon^n r^{1-n}\right)\frac{x}{r},$$

where $r = |x|$.

Since we require

$$\Delta_{\varphi} g^{\varepsilon}_{\beta,k} = \beta - \gamma(k,\omega)\varepsilon^n\delta(x-\varepsilon k) = \Delta u^{\varepsilon}_{\beta,k},$$

we have

$$\frac{\varphi'(|\nabla g^{\varepsilon}_{\beta,k}|)}{|\nabla g^{\varepsilon}_{\beta,k}|}\nabla g^{\varepsilon}_{\beta,k} = \nabla u^{\varepsilon}_{\beta,k}.$$

Moreover, we know that $g^\varepsilon_{\beta,k}$ should be rotationally symmetric and so we let $g^\varepsilon_{\beta,k}(x) = f^\varepsilon_{\beta,k}(r)$ for $r = |x|$. Then

$$\nabla g^\varepsilon_{\beta,k}(x) = (f^\varepsilon_{\beta,k})'(r) \cdot \frac{x}{r}.$$

Finally, combining these results, we conclude that

$$(f^\varepsilon_{\beta,k})'(r) = -(\varphi')^{-1}\left(c_n\gamma(k,\omega)\varepsilon^n r^{1-n} - \frac{\beta}{n}r\right),$$

when r is small.

Remark 3.4.3 (Estimate for fundamental solution)**.** From the above idea, we can estimate the fundamental solution f and the solution for $\Delta_\varphi u_\beta = \beta$. More precisely, we can obtain an explicit formula for these functions:

(i) $-\Delta_\varphi f = \delta_0$ and $\lim_{|x|\to\infty} f(x) = 0$ hold when we define

$$f(x) = \int_r^\infty (\varphi')^{-1}(cs^{1-n})\,\mathrm{d}s = \int_r^\infty (\varphi^*)'(cs^{1-n})\,\mathrm{d}s, \quad \text{where } r = |x|.$$

(ii) For $\beta \geq 0$, (with the condition $u_\beta(0) = 0$)

$$u_\beta(x) = \int_0^r (\varphi')^{-1}\left(\frac{\beta}{n}s\right)\mathrm{d}s = \int_0^r (\varphi^*)'\left(\frac{\beta}{n}s\right)\mathrm{d}s = \frac{n}{\beta}\cdot\varphi^*\left(\frac{\beta}{n}r\right).$$

Similarly for $\beta < 0$, (with the condition $u_\beta(x) = 0$ on $|x| = 1$)

$$\begin{aligned} u_\beta(x) = \int_r^1 (\varphi^*)'\left(-\frac{\beta}{n}s\right)\mathrm{d}s &= \int_r^1 (\varphi^*)'\left(\frac{|\beta|}{n}s\right)\mathrm{d}s \\ &= \frac{n}{|\beta|}\cdot\left[\varphi^*\left(\frac{|\beta|}{n}\right) - \varphi^*\left(\frac{|\beta|}{n}r\right)\right]. \end{aligned}$$

According to Lemma 3.4.1, $\beta_0 := \sup\{\beta;\, l(\beta) = 0\}$ is well-defined, finite and non-negative. Thus, we can define the corrector function w^ε as follows:

$$w^\varepsilon(x,\omega) := \inf\Big\{v(x); \Delta_\varphi v \leq \beta_0 \text{ in } D\setminus T_\varepsilon, v \geq 1 \text{ on } T_\varepsilon,\ v = 0 \text{ on } \partial D\setminus T_\varepsilon\Big\}.$$

Then for almost surely $\omega \in \Omega$, we have that

$$\begin{cases} \Delta_\varphi w^\varepsilon(x,\omega) = \beta_0 & \text{for } x \in D \setminus T_\varepsilon, \\ w^\varepsilon(x,\omega) = 1 & \text{for } x \in T_\varepsilon, \\ w^\varepsilon(x,\omega) = 0 & \text{for } x \in \partial D \setminus T_\varepsilon. \end{cases}$$

3.4.2 $W^{1,\varphi}$ boundedness of $\{w^\varepsilon\}$

To show that $\{w^\varepsilon\}$ is uniformly bounded in $W^{1,\varphi}(D)$, we split the proof into two parts: $\{w^\varepsilon\}$ is uniformly bounded in $L^\varphi(D)$ and $\{\nabla w^\varepsilon\}$ is also uniformly bounded in $L^\varphi(D)$.

Proof. To prove the first part, we need to introduce an auxiliary function $v(x)$: let v be the solution to the following problem

$$\begin{cases} \Delta_\varphi v = \beta_0 & \text{in } D, \\ v = 0 & \text{on } \partial D. \end{cases}$$

By comparison principle, for almost surely $\omega \in \Omega$,

$$v(x) \le w^\varepsilon(x,\omega) \le 1 \text{ for a.e. } x \in D.$$

Hence, $\int_D \varphi(|w^\varepsilon|)\,\mathrm{d}x \le C$, which implies that $\{w^\varepsilon\}$ is uniformly bounded in $L^\varphi(D)$.

To show that $\{\nabla w^\varepsilon\}$ is also uniformly bounded in $L^\varphi(D)$, we define the function h^ε as follows: first for each $k \in \mathbb{Z}^n$, define h_k^ε be the φ-capacity function of $B_{a^\varepsilon}(\varepsilon k)$ with respect to $B_{\varepsilon/2}(\varepsilon k)$. Then define $h^\varepsilon = \sum_{k\in\mathbb{Z}^n} h_k^\varepsilon$.

Obviously, $w^\varepsilon - h^\varepsilon = 0$ on T_ε and ∂D. Hence, from integration by parts,

$$\begin{aligned} \int_{D_\varepsilon} \beta_0(h^\varepsilon - w^\varepsilon)\,\mathrm{d}x &= \int_{D_\varepsilon} \Delta_\varphi w^\varepsilon (h^\varepsilon - w^\varepsilon)\,\mathrm{d}x \\ &= \int_{D_\varepsilon} \varphi'(|\nabla w^\varepsilon|)|\nabla w^\varepsilon|\,\mathrm{d}x - \int_{D_\varepsilon} \frac{\varphi'(|\nabla w^\varepsilon|)}{|\nabla w^\varepsilon|}\nabla w^\varepsilon \cdot \nabla h^\varepsilon\,\mathrm{d}x. \end{aligned}$$

Then by Young's inequality and the relation $\varphi^*(\varphi'(t)) \sim \varphi(t)$, we obtain that:

$$\begin{aligned}\left|\int_{D_\varepsilon} \frac{\varphi'(|\nabla w^\varepsilon|)}{|\nabla w^\varepsilon|}\nabla w^\varepsilon \cdot \nabla h^\varepsilon \,\mathrm{d}x\right| &\leq \delta \int_{D_\varepsilon} \varphi^*(\varphi'(|\nabla w^\varepsilon|))\,\mathrm{d}x + C(\delta)\int_{D_\varepsilon} \varphi(|\nabla h^\varepsilon|)\,\mathrm{d}x \\ &\leq c\delta \int_{D_\varepsilon} \varphi(|\nabla w^\varepsilon|)\,\mathrm{d}x + C(\delta)\int_{D_\varepsilon} \varphi(|\nabla h^\varepsilon|)\,\mathrm{d}x.\end{aligned}$$

Now using the relation $\varphi'(t)t \sim \varphi(t)$ and choosing $\delta > 0$ small enough, we have:

$$\int_{D_\varepsilon} \varphi(|\nabla w^\varepsilon|)\,\mathrm{d}x \leq C\left(\underbrace{\int_{D_\varepsilon} \varphi(|\nabla h^\varepsilon|)\,\mathrm{d}x}_{=I} + \underbrace{\int_{D_\varepsilon} \beta_0|h^\varepsilon - w^\varepsilon|\,\mathrm{d}x}_{=II}\right).$$

(I) First integral: we need a uniform bound for $\int_{D_\varepsilon} \varphi(|\nabla h^\varepsilon|)\,\mathrm{d}x$ which is independent of ε. Note that since we have defined h^ε in terms of φ-capacity function,

$$\int_{D_\varepsilon} \varphi(|\nabla h^\varepsilon|)\,\mathrm{d}x \sim \sum_{k\in\mathbb{Z}^n\cap\varepsilon^{-1}D} \mathrm{cap}_\varphi(\overline{B_{a^\varepsilon}(\varepsilon k)}, B_{\varepsilon/2}(\varepsilon k)).$$

We chose $a^\varepsilon(k,\omega)$ so that $\mathrm{cap}_\varphi(\overline{B_{a^\varepsilon(k,\omega)}(\varepsilon k)}) = \gamma(k,\omega)\varepsilon^n \leq \bar{\gamma}\varepsilon^n$. Also the number of summands (i.e. capacity terms) is proportional to ε^{-n}. Thus, we conclude that

$$\int_{D_\varepsilon} \varphi(|\nabla h^\varepsilon|)\,\mathrm{d}x \leq C,$$

where C is a uniform constant.

(II) Second integral: Since $|h^\varepsilon| \leq 1$ (recall that h^ε is a φ-capacity function) and $v \leq w^\varepsilon \leq 1$, we obtain the uniform bound for the second integral. This completes the proof.

□

3.4.3 $w^\varepsilon \to 0$ in $L^\varphi(D)$ as $\varepsilon \to 0$

To show that w^ε converges to zero strongly in $L^\varphi(D)$, we need to compare the corrector w^ε with the auxiliary function $v^\varepsilon_{\beta_0,D}$. Indeed, we will compare the function $v^\varepsilon_{\beta_0,D}$ with the fundamental solution h^ε_k, which will be defined at Lemma 3.4.4, and then investigate the limiting behavior of $v^\varepsilon_{\beta_0,D}$ as $\varepsilon \to 0$.

Lemma 3.4.4. *(i) $v^\varepsilon_{\beta_0,D}(x,\omega) \geq h^\varepsilon_k(x,\omega) - o(1)$ for a.e. $x \in B_{\frac{\varepsilon}{2}}(\varepsilon k)$ and a.s. $\omega \in \Omega$, where h^ε_k denotes the fundamental solution with singularity at εk. More precisely,*

$$h^\varepsilon_k(x,\omega) := \int_r^\infty (\varphi^*)'(c\gamma(k,\omega)\varepsilon^n s^{1-n})\,\mathrm{d}s, \quad \text{where } r = |x - \varepsilon k|.$$

(ii) For any $\tau > 0$, $\bar{v}^\varepsilon_{\beta_0+\tau,D}$ converges to 0 in $L^\varphi(D)$ as ε goes to 0 for a.s. $\omega \in \Omega$, where $\bar{v}^\varepsilon_{\beta_0+\tau,D}$ is defined as follows:

$$v^\varepsilon_{\beta_0+\tau,D} := \inf\Big\{v(x); \Delta_\varphi v \leq \beta_0 + \tau - \sum_{k\in\mathbb{Z}^n} \gamma(k,\omega)\varepsilon^n\delta(x-\varepsilon k) \text{ in } D, \\ v \geq 0 \text{ in } D,\ v = 0 \text{ on } \partial D\Big\},$$

and let $\bar{v}^\varepsilon_{\beta_0+\tau,D} = \min\{v^\varepsilon_{\beta_0+\tau,D}, 1\}$.

Proof. (i) Let

$$b(k,\omega) = \left(\frac{nc\gamma(k,\omega)}{\beta_0}\right)^{\frac{1}{n}},$$

where the constant c is the same as in Lemma 3.4.1 (iii). Then we define the function $h^\varepsilon_{\beta_0,k}(x,\omega)$ as follows: if $b \geq \frac{1}{2}$, then

$$h^\varepsilon_{\beta_0,k}(x,\omega) := \begin{cases} \int_r^{\frac{\varepsilon}{2}} (\varphi^*)'\left(c\gamma(k,\omega)\varepsilon^n s^{1-n} - \frac{\beta_0}{n}s\right)\,\mathrm{d}s, & \text{if } 0 \leq r \leq \frac{\varepsilon}{2}, \\ 0, & \text{if } r \geq \frac{\varepsilon}{2}, \end{cases}$$

and if $b \le \frac{1}{2}$, then

$$h^\varepsilon_{\beta_0,k}(x,\omega) := \begin{cases} \int_r^{b\varepsilon} (\varphi^*)' \left(c\gamma(k,\omega)\varepsilon^n s^{1-n} - \frac{\beta_0}{n} s\right) \mathrm{d}s, & \text{if } 0 \le r \le b\varepsilon, \\ 0, & \text{if } r \ge b\varepsilon, \end{cases}$$

where $r = |x - \varepsilon k|$ and $x \in B_{\varepsilon/2}(\varepsilon k)$.

If $b \ge \frac{1}{2}$, then for a.e. $x \in B_{\frac{\varepsilon}{2}}(\varepsilon k)$ and almost surely $\omega \in \Omega$, we have that

$$\Delta_\varphi h^\varepsilon_{\beta_0,k}(x,\omega) = \beta_0 - \gamma(k,\omega)\varepsilon^n \delta(x - \varepsilon k),$$

and $h^\varepsilon_{\beta_0,k}(x,\omega) = 0$ if $|x - \varepsilon k| = \frac{\varepsilon}{2}$. Thus, we can apply comparison principle and as a result, for almost surely $\omega \in \Omega$,

$$h^\varepsilon_{\beta_0,k}(x,\omega) \le v^\varepsilon_{\beta_0,D}(x,\omega) \quad \text{a.e. } x \in B_{\frac{\varepsilon}{2}}(\varepsilon k).$$

We can prove the same result for the latter case ($b \le \frac{1}{2}$) using a similar argument. Combining these two cases, for almost surely $\omega \in \Omega$, we have

$$h^\varepsilon_{\beta_0,k}(x,\omega) \le v^\varepsilon_{\beta_0,D}(x,\omega) \quad \text{a.e. } x \in B_{\frac{\varepsilon}{2}}(\varepsilon k).$$

Now by direct computation, we obtain that for almost surely $\omega \in \Omega$,

$$h^\varepsilon_{\beta_0,k}(x,\omega) \ge h^\varepsilon_k(x,\omega) - o(1) \quad \text{a.e. } x \in B_{\frac{\varepsilon}{2}}(\varepsilon k).$$

Therefore, for almost surely $\omega \in \Omega$, we conclude that

$$v^\varepsilon_{\beta_0,D}(x,\omega) \ge h^\varepsilon_k(x,\omega) - o(1) \quad \text{a.e. } x \in B_{\frac{\varepsilon}{2}}(\varepsilon k).$$

(ii) From the definition of $\{v^\varepsilon_{\beta_0+\tau,D}\}$, we know that for almost surely $\omega \in \Omega$,

$$\Delta_\varphi v^\varepsilon_{\beta_0+\tau,D} \ge - \sum_{k \in \mathbb{Z}^n \cap \varepsilon^{-1} D} \gamma(k,\omega)\varepsilon^n \delta(\cdot - \varepsilon k).$$

Hence, by testing $\bar{v}^{\varepsilon}_{\beta_0+\tau,D}$, we have

$$\int_{B_1} \Delta_{\varphi} v^{\varepsilon}_{\beta_0+\tau,D} \cdot \bar{v}^{\varepsilon}_{\beta_0+\tau,D} \,\mathrm{d}x \geq - \sum_{k\in\mathbb{Z}^n\cap\varepsilon^{-1}D} \gamma(k,\omega)\varepsilon^n \bar{v}^{\varepsilon}_{\beta_0+\tau,D}(\varepsilon k).$$

By part (i), we can easily check that $\bar{v}^{\varepsilon}_{\beta_0+\tau,D}(\varepsilon k) = 1$ for $k \in \mathbb{Z}^n \cap \varepsilon^{-1}D$. Since $\varphi'(t)t \sim \varphi(t)$ and $|\{k \in \mathbb{Z}^n \cap \varepsilon^{-1}D\}| \sim \varepsilon^{-n}$, we obtain

$$\int_D \varphi(|\nabla \bar{v}^{\varepsilon}_{\beta_0+\tau,D}|) \,\mathrm{d}x \leq C,$$

where C is a universal constant. Therefore, $\{\bar{v}^{\varepsilon}_{\beta_0+\tau,D}\}$ is uniformly bounded in $W^{1,\varphi}_0(D)$.

For almost surely $\omega \in \Omega$,

$$\lim_{\varepsilon\to 0} \frac{|\{\bar{v}^{\varepsilon}_{\beta_0+\tau,D} = 0\} \cap B_r(x_0)|}{|B_r(x_0)|} = l(\beta_0+\tau) > 0,$$

for any $B_r(x_0) \subset D$. Thus, by a version of Poincaré inequality (see [31, Lemma 4.8]), there exists a constant $C = C(\beta_0+\tau, n)$ such that

$$\|\bar{v}^{\varepsilon}_{\beta_0+\tau,D}\|_{L^{\varphi}(B_r(x_0))} \leq Cr\|\nabla \bar{v}^{\varepsilon}_{\beta_0+\tau,D}\|_{L^{\varphi}(B_r(x_0))}.$$

Since x_0 can be arbitrarily chosen, by summing above inequality and applying norm-modular relation, we obtain that:

$$\int_D \varphi(|\bar{v}^{\varepsilon}_{\beta_0+\tau,D}|) \,\mathrm{d}x \leq Cr.$$

Here we may choose $r = \sqrt{\varepsilon}$ and follow the above argument. By letting $\varepsilon \to 0$, and then we conclude that

$$\lim_{\varepsilon\to 0} \int_D \varphi(|\bar{v}^{\varepsilon}_{\beta_0+\tau,D}|) \,\mathrm{d}x = 0.$$

□

Remark 3.4.5 (Capacity function)**.** Here we will prove that, in fact, h^{ε}_k

defined in Lemma 3.4.4 (i) is the capacity function for $B_{a^\varepsilon(k,\omega)}(\varepsilon k)$. In other words, we will show that $\lim_{|x|\to\infty} h_k^\varepsilon(x) = 0$, $h_k^\varepsilon = 1$ on $\partial B_{a^\varepsilon}$, and

$$\Delta_\varphi h_k^\varepsilon = 0, \quad \text{in } \mathbb{R}^n \setminus B_{a^\varepsilon}.$$

Note that in Assumption 3.2.9 (i), we chose a^ε so that

$$\operatorname{cap}_\varphi(B_{a^\varepsilon(k,\omega)}(\varepsilon k)) = \varepsilon^n \gamma(k,\omega). \tag{3.4.1}$$

Proof. Without loss of generality, we let $k = 0$ and $h^\varepsilon := h_0^\varepsilon$. First as we checked in Remark 3.4.2, we have that

$$\Delta_\varphi h^\varepsilon = -\gamma \varepsilon^n \delta_0, \quad \text{in } \mathbb{R}^n.$$

In particular, we have $\Delta_\varphi h^\varepsilon = 0$ for $x \in \mathbb{R}^n \setminus \{0\}$. Moreover,

$$\lim_{|x|\to\infty} h^\varepsilon(x) = 0$$

follows directly from the definition of h^ε in Lemma 3.4.4 (i).

Finally, to prove that $h_k^\varepsilon = 1$ on $\partial B_{a^\varepsilon}$, choose $b^\varepsilon > 0$ so that

$$1 = \int_{b^\varepsilon}^\infty (\varphi')^{-1}(c\gamma(k,\omega)\varepsilon^n s^{1-n})\, \mathrm{d}s. \tag{3.4.2}$$

Then h^ε is the capacity function for a set B_{b^ε} and so we have

$$\begin{aligned}
\operatorname{cap}_\varphi(B_{b^\varepsilon}) &= \int_{\mathbb{R}^n \setminus B_{b^\varepsilon}} |\nabla h^\varepsilon| \varphi'(|\nabla h^\varepsilon|)\, \mathrm{d}x \\
&= \int_{b^\varepsilon}^\infty (c\gamma\varepsilon^n r^{1-n}) \cdot (\varphi')^{-1}(c\gamma\varepsilon^n r^{1-n}) n\omega_n r^{n-1}\, \mathrm{d}r \\
&= \gamma\varepsilon^n \int_{b^\varepsilon}^\infty (\varphi')^{-1}(c\gamma\varepsilon^n r^{1-n})\, \mathrm{d}r = \gamma\varepsilon^n \\
&= \operatorname{cap}_\varphi(B_{a^\varepsilon}).
\end{aligned}$$

Here we used a change of variables, $c = \frac{1}{n\omega_n}$ (Lemma 3.4.1 (iii)), (3.4.1), (3.4.2) and

$$|\nabla h^\varepsilon(x)| = (\varphi')^{-1}(c\gamma\varepsilon^n r^{1-n}), \quad \text{where } r = |x|.$$

Therefore, we conclude that $a^\varepsilon = b^\varepsilon$ and we finish the proof.

□

To finish the proof of $w^\varepsilon \to 0$ in $L^\varphi(D)$, it remains to show that the corrector w^ε has the same limiting property as $\bar{v}^\varepsilon_{\beta_0+\tau,D}$. For this purpose, we introduce a new auxiliary function w^ε_τ as follows: for any $\tau > 0$,

$$w^\varepsilon_\tau(x,\omega) := \inf\Big\{v(x) : \Delta_\varphi v \le \beta_0 + \tau \text{ in } D_\varepsilon, v \ge 1 \text{ on } T_\varepsilon \text{ and } v = 0 \text{ on } \partial D\Big\}.$$

Obviously, for almost surely $\omega \in \Omega$, $w^\varepsilon(x,\omega) \ge w^\varepsilon_\tau(x,\omega)$ for a.e. $x \in D$ and $\{w^\varepsilon_\tau\}$ is also bounded in $W^{1,\varphi}(D)$ by previous result. More precisely, $\{w^\varepsilon_\tau\}$ satisfies the following property:

Proposition 3.4.6. *(i) For almost surely $\omega \in \Omega$, we have*

$$\|w^\varepsilon - w^\varepsilon_\tau\|_{W^{1,\varphi}(D)} \to 0,$$

as $\tau \to 0^+$.

(ii) For almost surely $\omega \in \Omega$, we have

$$\lim_{\varepsilon\to 0} \int_D \varphi(|w^\varepsilon_\tau|)\,\mathrm{d}x = 0.$$

Proof. For simplicity, we write $S : \mathbb{R}^n \to \mathbb{R}^n$ by

$$S(Q) := \frac{\varphi'(|Q|)}{|Q|}Q, \quad \text{where } Q \in \mathbb{R}^n.$$

(i) First by repeating the proof for $W^{1,\varphi}(D)$ boundedness of w^ε (see Section 3.4.2), we know that w^ε_τ is uniformly bounded in $W^{1,\varphi}(D)$ when $\tau \in (0,1)$. Moreover, since $w^\varepsilon_\tau - w^\varepsilon \in W^{1,\varphi}_0(D)$, we can use it as a test function:

$$
\begin{aligned}
-\tau \int_D (w_\tau^\varepsilon - w^\varepsilon)\,\mathrm{d}x &= -\int_D (\Delta_\varphi w_\tau^\varepsilon - \Delta_\varphi w^\varepsilon)\cdot(w_\tau^\varepsilon - w^\varepsilon)\,\mathrm{d}x \\
&= \int_D (S(\nabla w_\tau^\varepsilon) - S(\nabla w^\varepsilon))\cdot(\nabla w_\tau^\varepsilon - \nabla w^\varepsilon)\,\mathrm{d}x.
\end{aligned}
$$

Recalling the proof in Section 3.4.2, we know that $\{w_\tau^\varepsilon - w^\varepsilon\}$ is bounded in $L^\infty(D)$ when $\tau \in (0,1)$. This yields that

$$
\int_D (S(\nabla w_\tau^\varepsilon) - S(\nabla w^\varepsilon))\cdot(\nabla w_\tau^\varepsilon - \nabla w^\varepsilon)\,\mathrm{d}x \to 0,
$$

when $\tau \to 0^+$. Therefore, by [54, Theorem 4], we conclude that

$$
w_\tau^\varepsilon \to w^\varepsilon \quad \text{in } W^{1,\varphi}(D),
$$

when $\tau \to 0^+$.

(ii) We follow the proof of Proposition 3.3. in [77]: decompose the function w_τ^ε into

$$
w_\tau^\varepsilon = (w_\tau^\varepsilon)_+ - (w_\tau^\varepsilon)_-,
$$

and then estimate each part. First by Lemma 3.4.4 (i), Remark 3.4.5, and comparison principle (between $v^\varepsilon_{\beta_0+\tau,D} + o(1)$ and w_τ^ε in D_ε), we know

$$
0 \le (w_\tau^\varepsilon)_+ \le \bar{v}^\varepsilon_{\beta_0+\tau,D} + o(1).
$$

Therefore, applying Lemma 3.4.4 (ii), we obtain

$$
\lim_{\varepsilon\to 0} \int_D \varphi((w_\tau^\varepsilon)_+)\,\mathrm{d}x = 0.
$$

Next, to estimate the negative part $(w_\tau^\varepsilon)_-$, we may assume that

$$
\sup_{B_{\varepsilon/2}(\varepsilon k)} (w_\tau^\varepsilon)_- > 0.
$$

Since $\Delta_\varphi w^\varepsilon_\tau = \beta_0 + \tau$ in D_ε, then w^ε_τ is continuous in D and so is $(w^\varepsilon_\tau)_-$. Thus, for small $\varepsilon > 0$, we can apply Harnack inequality, Lemma 3.2.17, to $(w^\varepsilon_\tau)_-$, in a ball with a radius $R = \varepsilon \to 0$. However, note that the constants $\mathcal{N}$ and L in Lemma 3.2.17 do not depend on $\varepsilon > 0$. Therefore, we have that for a.s. $\omega \in \Omega$,

$$\sup_{B_{\varepsilon/2}(\varepsilon k)} (w^\varepsilon_\tau)_- = o(1),$$

which implies that

$$\lim_{\varepsilon\to 0} \int_D \varphi((w^\varepsilon_\tau)_-)\, \mathrm{d}x = 0.$$

Therefore, combining two results above, we conclude that

$$\lim_{\varepsilon\to 0} \int_D \varphi(|w^\varepsilon_\tau|)\, \mathrm{d}x = 0.$$

□

Hence, by Proposition 3.4.6, we have that

$$\lim_{\varepsilon\to 0} \int_D \varphi(|w^\varepsilon|)\, \mathrm{d}x = 0.$$

Thereofore, we can select a subsequence from $\{w^\varepsilon\}$ which converges to zero weakly in $W^{1,\varphi}(D)$.

3.5 Proof of Lemma 3.3.5 and Lemma 3.3.12

3.5.1 Proof of Lemma 3.3.5

Proof. (i) Without loss of generality, we assume that $\eta \in \mathcal{D}(D)$ and $\eta \geq 0$ on D. Let θ be in $(0, 1)$. To prove property (i), we need to prove the two facts:

$$\limsup_{\varepsilon\to 0} \int_{D\cap\{w^\varepsilon\leq\theta\}} \psi_1(|\nabla w^\varepsilon|)\eta\, \mathrm{d}x \leq C(\beta_0, \eta)\theta$$

and

$$\limsup_{\varepsilon\to 0}\int_{D\cap\{w^\varepsilon>\theta\}}\psi_1(|\nabla w^\varepsilon|)\eta\,\mathrm{d}x = 0$$

In fact, if we let $w_\theta^\varepsilon = (\theta - w^\varepsilon)_+$, then $w_\theta^\varepsilon\eta \in W_0^{1,\varphi}(D)$ and $w_\theta^\varepsilon\eta$ converges to $\theta\eta$ weakly in $W_0^{1,\varphi}(D)$. Moreover, since $\theta < 1$, $w_\theta^\varepsilon = 0$ on the holes T_ε. Hence, from integration by parts,

$$\lim_{\varepsilon\to 0}\int_D \frac{\varphi'(|\nabla w^\varepsilon|)}{|\nabla w^\varepsilon|}\nabla w^\varepsilon\cdot\nabla(w_\theta^\varepsilon\eta)\,\mathrm{d}x = -\beta_0\theta\int_D \eta\,\mathrm{d}x.$$

which implies

$$\lim_{\varepsilon\to 0}\Big\{\int_{D\cap\{w^\varepsilon\le\theta\}}|\nabla w^\varepsilon|\varphi'(|\nabla w^\varepsilon|)\eta\,\mathrm{d}x - \int_{D\cap\{w^\varepsilon\le\theta\}}\frac{\varphi'(|\nabla w^\varepsilon|)}{|\nabla w^\varepsilon|}\nabla w^\varepsilon\cdot\nabla\eta\; w_\theta^\varepsilon\,\mathrm{d}x\Big\}$$
$$= \beta_0\theta\int_D\eta\,\mathrm{d}x$$

Applying Hölder's inequality, we have

$$\begin{aligned}\Big|\int_{D\cap\{w^\varepsilon\le\theta\}}\frac{\varphi'(|\nabla w^\varepsilon|)}{|\nabla w^\varepsilon|}\nabla w^\varepsilon\cdot\nabla\eta\; w_\theta^\varepsilon\,\mathrm{d}x\Big| &\le \int_{D\cap\{w^\varepsilon\le\theta\}}\varphi'(|\nabla w^\varepsilon|)\cdot|\nabla\eta|w_\theta^\varepsilon\,\mathrm{d}x\\ &\le C(\eta)\|\varphi'(|\nabla w^\varepsilon|)\|_{\varphi^*}\|w_\theta^\varepsilon\|_\varphi.\end{aligned}$$

Recalling the norm-modular relation and the relation $\varphi^*(\varphi'(t)) \sim \varphi(t)$, we obtain

$$\|\varphi'(|\nabla w^\varepsilon|)\|_{\varphi^*} \le \int_D \varphi^*(\varphi'(|\nabla w^\varepsilon|))\,\mathrm{d}x + 1 \le C\int_D\varphi(|\nabla w^\varepsilon|)\,\mathrm{d}x + 1 \le C.$$

Then using $w_\theta^\varepsilon \rightharpoonup \theta$ weakly in $W^{1,\varphi}(D)$ and the definition of the Orlicz space norm,

$$\lim_{\varepsilon\to 0}\|w_\theta^\varepsilon\|_\varphi = \|\theta\|_\varphi \le C\theta.$$

Thus,

$$\limsup_{\varepsilon\to 0}\int_{D\cap\{w^\varepsilon\le\theta\}}|\nabla w^\varepsilon|\varphi'(|\nabla w^\varepsilon|)\eta\,\mathrm{d}x \le C(\beta_0,\eta)\theta,$$

which implies

$$\limsup_{\varepsilon\to 0}\int_{D\cap\{w^\varepsilon\le\theta\}}\varphi(|\nabla w^\varepsilon|)\eta\,\mathrm{d}x \le C(\beta_0,\eta)\theta,$$

by $t\varphi'(t)\sim\varphi(t)$.

Now let ψ_2 be an N-function such that $\psi_2(\psi_1(t))\sim\varphi(t)$, for uniformly in $t>0$. Then by Hölder's inequality, we have that

$$\begin{aligned}\int_{D\cap\{w^\varepsilon\le\theta\}}\psi_1(|\nabla w^\varepsilon|)\eta\,\mathrm{d}x &\le 2\|\sqrt{\eta}\psi_1(|\nabla w^\varepsilon|)\chi_{D\cap\{w^\varepsilon\le\theta\}}\|_{\psi_2}\cdot\|\sqrt{\eta}\|_{\psi_2^*}\\ &\le 2\left(\int_{D\cap\{w^\varepsilon\le\theta\}}\psi_2(\sqrt{\eta}\psi_1(|\nabla w^\varepsilon|))\,\mathrm{d}x\right)^{1/p_{\psi_2}}\cdot\|\sqrt{\eta}\|_{\psi_2^*}\\ &\le c(\eta,\psi_2)\left(\int_{D\cap\{w^\varepsilon\le\theta\}}\varphi(|\nabla w^\varepsilon|)\widetilde{\eta}\,\mathrm{d}x\right)^{1/p_{\psi_2}}.\end{aligned}$$

Thus, we conclude that

$$\limsup_{\varepsilon\to 0}\int_{D\cap\{w^\varepsilon\le\theta\}}\psi_1(|\nabla w^\varepsilon|)\eta\,\mathrm{d}x \le C(\eta,\psi_2,\varphi,\beta_0)\theta^{\frac{1}{p_{\psi_2}}}.$$

Similarly for the integral $\int_{D\cap\{w^\varepsilon>\theta\}}\psi_1(|\nabla w^\varepsilon|)\eta\,\mathrm{d}x$, we again apply Hölder's inequality, then

$$\int_{D\cap\{w^\varepsilon>\theta\}}\psi_1(|\nabla w^\varepsilon|)\eta\,\mathrm{d}x \le 2\|\sqrt{\eta}\psi_1(|\nabla w^\varepsilon|)\|_{\psi_2}\cdot\|\sqrt{\eta}\chi_{D\cap\{w^\varepsilon>\theta\}}\|_{\psi_2^*}$$

Here the sequence $\{w^\varepsilon\}$ is uniformly bounded in $W^{1,\varphi}(D)$ and

$$\lim_{\varepsilon\to 0}\int_{D\cap\{w^\varepsilon>\theta\}}\widetilde{\eta}\,\mathrm{d}x = 0.$$

This implies

$$\limsup_{\varepsilon\to 0}\int_{D\cap\{w^\varepsilon>\theta\}}\psi_1(|\nabla w^\varepsilon|)\eta\,\mathrm{d}x = 0.$$

Therefore,

$$\limsup_{\varepsilon\to 0}\int_D \psi_1(|\nabla w^\varepsilon|)\eta\,\mathrm{d}x \le C(\eta,\psi_2,\varphi,\beta_0)\theta^{\frac{1}{p_{\psi_2}}}.$$

Since θ is an arbitrarily small positive number, we conclude that

$$\lim_{\varepsilon\to 0}\int_D \psi_1(|\nabla w^\varepsilon|)\eta\,\mathrm{d}x = 0.$$

(ii) Let $\eta \in \mathcal{D}(D)$. Then since $\eta(1-w^\varepsilon)\in\mathcal{D}(D)$ and from integration by parts,

$$\begin{aligned}\int_D \beta_0\eta(1-w^\varepsilon)\,\mathrm{d}x &= \int_D \nabla\cdot S(\nabla w^\varepsilon)\ \eta(1-w^\varepsilon)\,\mathrm{d}x\\ &= \int_D \nabla\eta\cdot S(\nabla w^\varepsilon)(w^\varepsilon-1)\,\mathrm{d}x + \int_D \eta S(\nabla w^\varepsilon)\cdot\nabla w^\varepsilon\,\mathrm{d}x.\end{aligned}$$

Since w^ε goes to 0 weakly in $W^{1,\varphi}(D)$,

$$\lim_{\varepsilon\to 0}\beta_0\int_D \eta(1-w^\varepsilon)\,\mathrm{d}x = \int_D \beta_0\eta\,\mathrm{d}x.$$

Recall that w^ε converges to 0 strongly in $L^\varphi(D)$ and $\{\nabla w^\varepsilon\}$ is bounded in $L^\varphi(D)$. Hence, by Hölder's inequality, we have that

$$\lim_{\varepsilon\to 0}\left|\int_D \nabla\eta\cdot S(\nabla w^\varepsilon)w^\varepsilon\,\mathrm{d}x\right| \le c(\eta)\int_D \varphi'(|\nabla w^\varepsilon|)|w^\varepsilon|\,\mathrm{d}x = 0.$$

Finally, by part (i), we know that (let $\psi_1(t)=\varphi'(t)$ and $\psi_2(t)=\varphi^*(t)$.)

$$\lim_{\varepsilon\to 0}\left|\int_D \nabla\eta\cdot S(\nabla w^\varepsilon)\,\mathrm{d}x\right| \le \int_D |\nabla\eta|\varphi'(|\nabla w^\varepsilon|)\,\mathrm{d}x = 0.$$

Therefore,

$$\lim_{\varepsilon\to 0}\int_D \eta S(\nabla w^\varepsilon)\cdot\nabla w^\varepsilon\,\mathrm{d}x = \int_D \beta_0\eta\,\mathrm{d}x.$$

(iii) From integration by parts, we have that

$$
\begin{aligned}
-\int_D \beta_0 v^\varepsilon \eta &= -\int_D \nabla \cdot S(\nabla w^\varepsilon)\ v^\varepsilon \eta \\
&= \int_D \eta S(\nabla w^\varepsilon) \cdot \nabla v^\varepsilon + \int_D v^\varepsilon S(\nabla w^\varepsilon) \cdot \nabla \eta.
\end{aligned}
$$

Since $\{v^\varepsilon\}$ is bounded in $W_0^{1,\varphi}(D)$ with $q_\varphi < n$, by Sobolev embedding theorem (see [32, Corollary 6.3.4]), $\{v^\varepsilon\}$ is bounded in $L^\psi(D)$ for an N-function ψ which satisfies

$$
t^{-\frac{1}{n}} \varphi^{-1}(t) \sim \psi^{-1}(t). \tag{3.5.1}
$$

Hence, by Hölder's inequality,

$$
\left| \int_D v^\varepsilon S(\nabla w^\varepsilon) \cdot \nabla \eta \right| \leq 2 \|v^\varepsilon\|_\psi \|\varphi'(\nabla w^\varepsilon) \nabla \eta\|_{\psi^*}.
$$

Then by applying part (i), (let $\psi_1 = \psi^* \circ \varphi'$ and $\psi_2 = \varphi^* \circ (\psi^*)^{-1}$.)

$$
\lim_{\varepsilon \to 0} \|\varphi'(\nabla w^\varepsilon) \nabla \eta\|_{\psi^*} = 0.
$$

Note that (3.5.1) ensures that ψ_2 is an N-function. (see [32, Theorem 2.4.10].) Therefore,

$$
\lim_{\varepsilon \to 0} \int_D S(\nabla w^\varepsilon) \cdot \nabla v^\varepsilon \eta \, \mathrm{d}x = -\int_D \beta_0 v \eta \, \mathrm{d}x.
$$

□

3.5.2 Proof of Lemma 3.3.12

For a general N-function φ, we cannot expect that φ has a multiplicative property. Here a function f is said to have a multiplicative property if there exists a constant $C > 0$ such that $f(xy) = Cf(x)f(y)$, for any $x, y > 0$. Note that for p-Laplacian case ($\varphi(t) = t^p$), we have a multiplicative property with $C = 1$. Thus, we can separate the solution u^ε and the test function η within the function φ and apply Lemma 3.3.5 directly. However, to prove

Lemma 3.3.12 for a general N-function φ, we need some revised version of part (ii), (iii) in Lemma 3.3.5:

Lemma 3.5.1. *Let w^ε be a corrector function defined in Lemma 3.3.5.*

(i) There exists a function $h : [0,\infty) \to [0,\infty)$ such that

$$\lim_{\varepsilon\to 0} \nabla \cdot \left(\frac{\varphi'(|\nabla w^\varepsilon|\eta)}{|\nabla w^\varepsilon|} \nabla w^\varepsilon \right) = h(\eta), \ (\textit{in distribution sense})$$

for any $\eta \in \mathcal{D}(D)$, with $\eta \geq 0$.

(ii) For any $\eta \in \mathcal{D}(D)$ with $\eta \geq 0$,

$$\lim_{\varepsilon\to 0} \int_D \varphi'(|\nabla w^\varepsilon|\eta)|\nabla w^\varepsilon|\eta \,\mathrm{d}x = \int_D \eta h(\eta)\,\mathrm{d}x.$$

(iii) For any sequence $\{v^\varepsilon\} \subset W_0^{1,\varphi}(D)$ with the property: $v^\varepsilon \to v$ weakly in $W_0^{1,\varphi}(D)$ as $\varepsilon \to 0$ and $v^\varepsilon = 0$ on T_ε and any $\eta \in \mathcal{D}(D)$ with $\eta \geq 0$, we have that

$$\lim_{\varepsilon\to 0} \int_D \frac{\varphi'(|\nabla w^\varepsilon|\eta)}{|\nabla w^\varepsilon|} \nabla w^\varepsilon \cdot \nabla v^\varepsilon \,\mathrm{d}x = -\int_D v h(\eta)\,\mathrm{d}x.$$

Proof. (i) Follow the proof of Lemma 3.3.8 and recall that $w^\varepsilon \to 0$ strongly in $L^\varphi(D)$.

(ii), (iii) Follow the proof of part (ii), (iii) in Lemma 3.3.5. The only difference is using

$$\lim_{\varepsilon\to 0} \nabla \cdot \left(\frac{\varphi'(|\nabla w^\varepsilon|\eta)}{|\nabla w^\varepsilon|} \nabla w^\varepsilon \right) = h(\eta),$$

instead of

$$\nabla \cdot \left(\frac{\varphi'(|\nabla w^\varepsilon|)}{|\nabla w^\varepsilon|} \nabla w^\varepsilon \right) = \beta_0.$$

□

Lemma 3.5.2. *For an N-function φ, we have*

$$\varphi'(t)t - \varphi^*(\varphi'(t)) = \varphi(t),$$

for all $t \geq 0$.

Proof. Recall the definition of φ^*:

$$\varphi^*(t) := \sup_{s\geq 0}(st - \varphi(s)).$$

In particular, it yields

$$\varphi^*(\varphi'(t)) := \sup_{s\geq 0}(s\varphi'(t) - \varphi(s)).$$

If we denote $f(s) := s\varphi'(t) - \varphi(s)$, then $f'(s) = \varphi'(t) - \varphi'(s)$. Since φ' is increasing function, we know that f attains its maximum at $s = t$. It finishes the proof. □

Proof of Lemma 3.3.12. Let us decompose $u^\varepsilon = u_+^\varepsilon - u_-^\varepsilon$. Obviously, we have (up to subsequence, if necessary)

$$\liminf_{\varepsilon\to 0} I[u^\varepsilon] = \lim_{\varepsilon\to 0} I[u^\varepsilon],$$

and $u_\pm^\varepsilon \rightharpoonup u_\pm^0$ weakly in $W^{1,\varphi}(D)$, respectively. Here note that

$$\int_D \varphi(|\nabla u^\varepsilon|)\,\mathrm{d}x = \int_D \varphi(|\nabla u_+^\varepsilon|)\,\mathrm{d}x + \int_D \varphi(|\nabla u_-^\varepsilon|)\,\mathrm{d}x,$$

and

$$\int_D \varphi(|\nabla u^0|)\,\mathrm{d}x = \int_D \varphi(|\nabla u_+^0|)\,\mathrm{d}x + \int_D \varphi(|\nabla u_-^0|)\,\mathrm{d}x.$$

For u_+^ε, we apply the classical lower semicontinuity property ([23]):

$$\liminf_{\varepsilon\to 0} \int_D \varphi(|\nabla u_+^\varepsilon|)\,\mathrm{d}x \geq \int_D \varphi(|\nabla u_+^0|)\,\mathrm{d}x.$$

In order to prove Lemma 3.3.12, we need to show the following revised lower semicontinuity property:

$$\liminf_{\varepsilon\to 0}\int_D \varphi(|\nabla u_-^\varepsilon|)\,\mathrm{d}x \geq \int_D \varphi(|\nabla u_-^0|)\,\mathrm{d}x + \int_D g(u_-^0)\,\mathrm{d}x.$$

Let $\theta > 0$ be any small positive number and η is a test function in $\mathcal{D}(D)$.

Firstly, we claim that

$$\liminf_{\varepsilon\to 0}\int_{w^\varepsilon\leq\theta} \varphi(|\nabla u_-^\varepsilon|) \geq \int_D \frac{\varphi'(|\nabla\eta|)}{|\nabla\eta|}\nabla\eta\cdot\nabla u_-^0 - \int_D \varphi^*(\varphi'(|\nabla\eta|)). \quad (3.5.2)$$

In fact, from Young's inequality, we have

$$\int_{w^\varepsilon\leq\theta} \frac{\varphi'(|\nabla\eta|)}{|\nabla\eta|}\nabla\eta\cdot\nabla u_-^\varepsilon \leq \int_{w^\varepsilon\leq\theta}\varphi(|\nabla u_-^\varepsilon|) + \int_{w^\varepsilon\leq\theta}\varphi^*(\varphi'(|\nabla\eta|)).$$

Since w^ε converges to 0 weakly in $W^{1,\varphi}(D)$, then $|\{w^\varepsilon > \theta\}| \to 0$ as ε goes to 0. Hence,

$$\lim_{\varepsilon\to 0}\int_{w^\varepsilon>\theta}\varphi(|\nabla\eta|)\,\mathrm{d}x = 0,$$

which implies that (by Hölder's inequality and equivalence of norm-modular convergence)

$$\lim_{\varepsilon\to 0}\int_{w^\varepsilon>\theta}\frac{\varphi'(|\nabla\eta|)}{|\nabla\eta|}\nabla\eta\cdot\nabla u_-^\varepsilon\,\mathrm{d}x = 0.$$

Since u_-^ε converges to u_-^0 weakly in $W^{1,\varphi}(D)$, we obtain the estimate (3.5.2).

Next, we will prove that for a test function η with $\eta \geq 0$,

$$\begin{aligned}\int_{w^\varepsilon>\theta}\varphi(|\nabla u_-^\varepsilon|)\,\mathrm{d}x \geq &-\int_D \varphi^*(\varphi'(|\nabla w^\varepsilon|\eta))\,\mathrm{d}x \\ &- \int_D \frac{\varphi'(|\nabla w^\varepsilon|\eta)}{|\nabla w^\varepsilon|}\nabla w^\varepsilon\cdot\nabla u_-^\varepsilon\,\mathrm{d}x - C\theta - f(\theta).\end{aligned} \quad (3.5.3)$$

Indeed, by Young's inequality, we have

$$-\int_{w^\varepsilon>\theta}\frac{\varphi'(|\nabla w^\varepsilon|\eta)}{|\nabla w^\varepsilon|}\nabla w^\varepsilon\cdot\nabla u_-^\varepsilon \leq \int_{w^\varepsilon>\theta}\varphi^*(\varphi'(|\nabla w^\varepsilon|\eta)) + \int_{w^\varepsilon>\theta}\varphi(|\nabla u_-^\varepsilon|).$$

Then by the proof of part (i) of Lemma 3.3.5,

$$\int_{w^\varepsilon<\theta} \varphi^*(\varphi'(|\nabla w^\varepsilon|\eta))\,\mathrm{d}x \le C\theta,$$

and by Hölder's inequality,

$$\left|\int_{w^\varepsilon<\theta} \frac{\varphi'(|\nabla w^\varepsilon|\eta)}{|\nabla w^\varepsilon|}\nabla w^\varepsilon\cdot\nabla u_-^\varepsilon\,\mathrm{d}x\right| \le 2\|\varphi'(|\nabla w^\varepsilon|\eta)\chi_{\{w^\varepsilon<\theta\}}\|_{\varphi^*}\|\nabla u_-^\varepsilon\|_{\varphi} =: f(\theta).$$

Here $f(\theta)\to 0$ when $\theta\to 0$, since

(i) $\{u_-^\varepsilon\}$ is uniformly bounded in $W^{1,\varphi}(D)$,

(ii) $\int_{w^\varepsilon<\theta}\varphi^*(\varphi'(|\nabla w^\varepsilon|\eta))\,\mathrm{d}x \le c\int_{w^\varepsilon<\theta}\varphi(|\nabla w^\varepsilon|)\widetilde{\eta}\,\mathrm{d}x \le C\theta$, and so the modular converges to 0 when $\theta\to 0$. By the equivalence of norm-modular convergence, we know $\|\varphi'(|\nabla w^\varepsilon|\eta)\chi_{\{w^\varepsilon<\theta\}}\|_{\varphi^*}\to 0$ when $\theta\to 0$.

Thus, the estimate (3.5.3) follows. Letting $\varepsilon\to 0$ in (3.5.3), we have:

$$\begin{aligned}\liminf_{\varepsilon\to 0}\int_{w^\varepsilon>\theta}\varphi(|\nabla u_-^\varepsilon|)\,\mathrm{d}x \ge &-\limsup_{\varepsilon\to 0}\int_D \varphi^*(\varphi'(|\nabla w^\varepsilon|\eta))\,\mathrm{d}x\\ &-\limsup_{\varepsilon\to 0}\int_D \frac{\varphi'(|\nabla w^\varepsilon|\eta)}{|\nabla w^\varepsilon|}\nabla w^\varepsilon\cdot\nabla u_-^\varepsilon\,\mathrm{d}x - C\theta - f(\theta).\end{aligned}$$

Here by applying part (iii) of Lemma 3.5.1, we obtain

$$\limsup_{\varepsilon\to 0}\int_D \frac{\varphi'(|\nabla w^\varepsilon|\eta)}{|\nabla w^\varepsilon|}\nabla w^\varepsilon\cdot\nabla u_-^\varepsilon\,\mathrm{d}x = -\int_D u_-^0 h(\eta)\,\mathrm{d}x.$$

Moreover, using Lemma 3.5.2, we have

$$\varphi^*(\varphi'(|\nabla w^\varepsilon|\eta)) = \varphi'(|\nabla w^\varepsilon|\eta)|\nabla w^\varepsilon|\eta - \varphi(|\nabla w^\varepsilon|\eta),$$

and thus applying Lemma 3.3.8 and part (ii) of Lemma 3.5.1, we conclude that

$$\limsup_{\varepsilon\to 0}\int_D \varphi^*(\varphi'(|\nabla w^\varepsilon|\eta))\,\mathrm{d}x = \int_D \eta h(\eta)\,\mathrm{d}x - \int_D g(\eta)\,\mathrm{d}x.$$

Therefore, we obtain the following estimate:

$$\liminf_{\varepsilon\to 0}\int_{w^\varepsilon>\theta}\varphi(|\nabla u_-^\varepsilon|)\,\mathrm{d}x \geq \int_D (u_-^0-\eta)h(\eta)\,\mathrm{d}x + \int_D g(\eta)\,\mathrm{d}x - C\theta - f(\theta). \tag{3.5.4}$$

We now combine two estimates (3.5.2) and (3.5.4) to derive

$$\begin{aligned}\liminf_{\varepsilon\to 0}\int_D\varphi(|\nabla u_-^\varepsilon|)\,\mathrm{d}x \geq &\int_D \frac{\varphi'(|\nabla\eta|)}{|\nabla\eta|}\nabla\eta\cdot\nabla u_-^0\,\mathrm{d}x - \int_D\varphi^*(\varphi'(|\nabla\eta|))\,\mathrm{d}x\\ &+\int_D (u_-^0-\eta)h(\eta)\,\mathrm{d}x + \int_D g(\eta)\,\mathrm{d}x.\end{aligned}$$

In particular, by setting $\eta = u_-^0$ (since the test functions are dense in $W_0^{1,\varphi}(D)$), we conclude (applying Lemma 3.5.2)

$$\liminf_{\varepsilon\to 0}\int_D\varphi(|\nabla u_-^\varepsilon|)\,\mathrm{d}x \geq \int_D\varphi(|\nabla u_-^0|)\,\mathrm{d}x + \int_D g(u_-^0)\,\mathrm{d}x,$$

which finishes the proof.

□

Chapter 4

Random Homogenization of Fully Nonlinear Elliptic Equations with Highly Oscillating Obstacles

4.1 Introduction

This chapter is devoted to the random homogenization of fully nonlinear equations with highly oscillating random obstacles, via a viscosity method. To state our main theorem, let $(\Omega, \mathcal{F}, \mathcal{P})$ be a given probability space. For every $\omega \in \Omega$ and every $\varepsilon > 0$, we consider a domain $D_\varepsilon(\omega)$ obtained by perforating holes from an open, bounded domain D of $\mathbb{R}^n$. We denote by $T_\varepsilon(\omega)$, the set of holes (i.e. $D_\varepsilon(\omega) = D \setminus T_\varepsilon(\omega)$) and impose two assumptions on $T_\varepsilon(\omega)$, namely Assumption 4.2.1 and Assumption 4.2.2, which will be stated later. Moreover, let us consider a special smooth function $\varphi(x)$ in D such that $\varphi \leq 0$ on ∂D and $\varphi > 0$ in some region of D. Then we are going to consider highly oscillating obstacles $\varphi_\varepsilon(x)$ which are zero in $D_\varepsilon(\omega)$ and $\varphi(x)$

on holes $T_\varepsilon(\omega)$, i.e.

$$\varphi_\varepsilon(x) := \begin{cases} \varphi(x) & \text{if } x \in T_\varepsilon(\omega), \\ 0 & \text{otherwise.} \end{cases}$$

Now we consider the standard obstacle problem asking the least viscosity supersolution of Laplacian operator above the given oscillating obstacle:

$$\begin{cases} \Delta u_\varepsilon \le 0 & \text{in } D, \\ u_\varepsilon(x) = 0 & \text{on } \partial D, \\ u_\varepsilon(x) \ge \varphi_\varepsilon(x) & \text{in } D. \end{cases} \tag{L_ε}$$

The concept of viscosity solution and its regularity can be found in [14]. Then our main theorem concerning the Laplacian operator is the following:

Theorem 4.1.1. *Let u_ε be the least viscosity supersolution of (L_ε).*

(i) There is a continuous function u such that $u_\varepsilon \rightharpoonup u$ in D with respect to L^p-norm, for $p > 0$, and for any $\delta > 0$, there is a subset $D_\delta \subset D$ and ε_0 such that for $0 < \varepsilon < \varepsilon_0$, $u_\varepsilon \to u$ uniformly in D_δ as $\varepsilon \to 0$ and $|D \setminus D_\delta| < \delta$.

(ii) There exists a critical value $\beta_0 > 0$ such that u is a viscosity solution of

$$\begin{cases} \Delta u + \beta_0(\varphi - u)_+ = 0 & \textit{in } D, \\ u = 0 & \textit{on } \partial D. \end{cases} \tag{$\overline{L}$}$$

Here the critical value β_0 can be interpreted as a capacity-like quantity; see [12] for details. Moreover, a viscosity method for the Laplacian case can be extended to a general class of fully nonlinear operators. More precisely, we will consider a fully nonlinear operator F, which satisfies two assumptions (F1) and (F2) stated in Chapter 2. Then we will deal with the fully nonlinear version of equation (L_ε): find the least viscosity supersolution u_ε such that

$$\begin{cases} F(D^2 u_\varepsilon(x)) \le 0 & \text{in } D, \\ u_\varepsilon = 0 & \text{on } \partial D, \\ u_\varepsilon(x) \ge \varphi_\varepsilon(x) & \text{in } D. \end{cases} \tag{F_ε}$$

Then our main theorem concerning the fully nonlinear operator is the following:

Theorem 4.1.2. *Let u_ε be the least viscosity supersolution of (F_ε).*

(i) There is a continuous function u such that $u_\varepsilon \rightharpoonup u$ in D with respect to L^p-norm, for $p > 0$, and for any $\delta > 0$, there is a subset $D_\delta \subset D$ and ε_0 such that for $0 < \varepsilon < \varepsilon_0$, $u_\varepsilon \to u$ uniformly in D_δ as $\varepsilon \to 0$ and $|D \setminus D_\delta| < \delta$.

(ii) There exists a fully nonlinear, uniformly elliptic operator $\overline{F}$ such that u is a viscosity solution of

$$\begin{cases} \overline{F}(D^2 u, (\varphi - u)_+) = 0 & \textit{in } D, \\ u = 0 & \textit{on } \partial D. \end{cases} \tag{$\overline{F}$}$$

We summarize the main steps of this chapter and explain related key features briefly.

(i) (The critical value β_0) In the stationary ergodic environment, the determination of the critical value β_0 is performed by an application of the subadditive ergodic theorem. For this purpose, we define a proper contact set (often with zero obstacle) of some auxiliary functions so that the measure of a contact set has a subadditive property. For the equations with divergence structure [15, 57, 77], this process has been done by considering the Dirac-delta distribution δ_0. Unfortunately, the inherent lack of divergence structure (i.e. integration by parts) prevents us from employing similar techniques. To overcome this obstruction, our idea is to approximate the homogeneous solution in the sense of "shape", which enables us to define auxiliary functions without the notion of δ_0. We now denote two auxiliary functions, namely, free solutions $w_{\beta,\sigma,A}$ and obstacle solutions $v_{\beta,\sigma,A}$ (see Section 4.4 for precise definitions). To find the critical value β_0, we further have to check the convergence of these functions when $\sigma \to 0$. Unlike the linear case, there is no monotone property for the fully nonlinear case; however, such difficulty could be overcome by the isolated singularity theorem, Theorem 4.4.11. In short, this theorem guarantees that a singular solution must behave like the corresponding homogeneous solution, near an isolated singularity. With the help of Theorem 4.4.11 and Arzela-Ascoli theorem, we derive the existence and uniqueness of such limit function.

(ii) (Properties of a corrector w^ε) After determining the critical value β_0, we define a corrector w^ε (see Section 4.5 for precise definitions). Here we require two properties for w^ε to finish the proof of our main theorem:

(P1) $\lim_{\varepsilon\to 0} w^\varepsilon = 0$ away from each hole;

(P2) $w^\varepsilon = 1$ (or $w^\varepsilon \approx 1$, see Section 4.5) on the boundary of each hole.

Note that (P2) is trivial by the definition of w^ε. Our strategy is to check these properties for the auxiliary functions $w^\varepsilon_{\beta,A} := \lim_{\varepsilon\to 0} w^\varepsilon_{\beta,\sigma,A}$ and $v^\varepsilon_{\beta,A} := \lim_{\varepsilon\to 0} v^\varepsilon_{\beta,\sigma,A}$ first, and then transport the convergence property (P1) to the corrector w^ε via the comparison principle. Indeed, we show that the auxiliary functions satisfy (P1) and (P2) by studying the theory for obstacle problems and singular solutions, together with the criticality of β_0. More precisely, we discover the "spreading effect" of obstacle solutions using the quadratic growth of obstacle problems and construct appropriate barriers using the behavior of (approximated) homogeneous solutions. Again, although the linear case is fairly straightforward, an additional challenge occurs for the fully nonlinear case; the Alexandrov-Backelman-Pucci estimate (for viscosity solutions) and the stability of coincidence sets (for obstacle problems) will help us.

This chapter is organized as follows. In Section 4.2, we investigate the behavior of u_ε away from holes, and as a consequence, we derive the convergence of u_ε to the homogenized solution u. Section 4.3 is devoted to the explanation of a homogeneous solution and its $C^{1,1}$-approximation in the sense of "shape". In Section 4.4, we define free solutions $w_{\beta,\sigma,A}$ and obstacle solutions $v_{\beta,\sigma,A}$, and prove the convergence of these auxiliary functions when $\sigma \to 0$. Then we conclude that the critical value β_0 is well-defined by the subadditive ergodic theorem. In Section 4.5, we justify two properties of $w_{\beta,\sigma,A}$, and transport the information to the corrector w^ε, which enables us to finish the proof for our main theorem. Note that to clarify the difficulties coming from nonlinearity, we deal with the Laplacian case and the fully nonlinear case in consecutive order within each section.

4.2 Estimates and Convergence

Let us make precise assumptions on the holes

$$T_\varepsilon(\omega) = \left(\cup_{k\in\mathbb{Z}^n} B_{a^\varepsilon(r(k,\omega))}(\varepsilon k)\right) \cap D,$$

where the size of hole is determined randomly, but the center of hole is periodically distributed.

Assumption 4.2.1. For all $k \in \mathbb{Z}^n$ and a.e. $\omega \in \Omega$, there exists $\gamma(k,\omega)$ (independent of ε) such that

$$a^\varepsilon(r(k,\omega))^{\alpha^*} = \varepsilon^{\alpha^*+2}\gamma(k,\omega),$$

where α^* denotes the scaling exponent of F and $a^\varepsilon(r) = r\varepsilon^{\frac{\alpha^*+2}{\alpha^*}}$. Moreover, we assume that there exists a constant $\overline{\gamma} > 0$:

$$\gamma(k,\omega) \le \overline{\gamma} \quad \text{for all } k \in \mathbb{Z}^n \text{ and a.e. } \omega \in \Omega.$$

Assumption 4.2.2. The process $\gamma : \mathbb{Z}^n \times \Omega \mapsto [0, \infty)$ is *stationary ergodic.* Recall Assumption 3.3.2.

Remark 4.2.3. In this chapter, we will concentrate on the non-trivial case with critical hole size $a^\varepsilon \approx \varepsilon^{\frac{\alpha^*+2}{\alpha^*}}$ so that the limit solution satisfies an effective equation without obstacles. In fact, the behavior of limit solution u can be different (but trivial) if the radius of holes a^ε is not critical. See [12] for details.

We now derive the estimate for the oscillation of u_ε on $\partial B_{b_\varepsilon}(k)$ where b_ε is chosen to have an intermediate growth rate between ε and a^ε. We first consider the Laplacian case.

Lemma 4.2.4. *Set $b_\varepsilon(k, \omega) = (\varepsilon a^\varepsilon(k, \omega))^{1/2}$ where $a^\varepsilon(k, \omega) \approx \varepsilon^{\frac{n}{n-2}}$ is the critical rate. Then*

$$\operatorname*{osc}_{\partial B_{b_\varepsilon}(k)} u_\varepsilon = o(\varepsilon^\gamma)$$

for $k \in \varepsilon\mathbb{Z}^n \cap \operatorname{supp} \varphi$ and for some $0 < \gamma < 1$.

Proof. See [12, Lemma 3.4] or [48, Lemma 2.9] for proof. □

Next, we control the behavior of u_ε in $D \setminus (\cup_{k\in\mathbb{Z}^n} B_{b_\varepsilon}(k))$ by constructing appropriate barrier functions $h_\varepsilon^\pm$ and applying the comparison principle with u_ε. This kind of idea was also employed in [36], which do not require the size of perforating holes to be identical. Note that in the periodic setting, similar results follows from the discrete gradient estimate [12, 48]; if we define

$$\Delta_{e_i} u_\varepsilon := \frac{u_\varepsilon(x + \varepsilon e_i) - u_\varepsilon(x)}{\varepsilon} \quad \text{for unit vector } e_i \in \mathbb{R}^n,$$

then there exists a uniform constant $C > 0$ such that

$$|\Delta_e u_\varepsilon| < C. \tag{4.2.1}$$

Lemma 4.2.5. *For $\varepsilon \in (0,1)$, let $h_\varepsilon^\pm$ be the solutions of the Dirichlet problem*

$$\begin{cases} \Delta h_\varepsilon^\pm = 0 & \text{in } D \setminus (\cup_{k\in\mathbb{Z}^n} B_{b_\varepsilon}(k)), \\ h_\varepsilon^\pm = 0 & \text{on } \partial D, \\ h_\varepsilon^+(x) = \sup_{\partial B_{b_\varepsilon}(k)} u_\varepsilon & \\ h_\varepsilon^-(x) = \inf_{\partial B_{b_\varepsilon}(k)} u_\varepsilon & \text{for } x \in \partial B_{b_\varepsilon}(k) \text{ where } k \in \mathbb{Z}^n. \end{cases}$$

Then $h_\varepsilon^\pm$ have the following properties:

(i) $0 \le h_\varepsilon^- \le u_\varepsilon \le h_\varepsilon^+$.

(ii) $h_\varepsilon^\pm \in C^{2,\alpha}$*; in particular, we have*

$$|h_\varepsilon^\pm(x) - h_\varepsilon^\pm(y)| \le C|x-y|^\alpha,$$

for any $\alpha \in (0,1)$ and any $x, y \in D \setminus (\cup_{k\in\mathbb{Z}^n} B_{b_\varepsilon}(k))$.

(iii) $h_\varepsilon^+ - h_\varepsilon^- \le \max_{k\in\mathbb{Z}^n} \operatorname{osc}_{\partial B_{b_\varepsilon}(k)} u_\varepsilon$.

Proof. (i) It follows directly from the construction of $h_\varepsilon^\pm$ and the comparison principle with u_ε.

(ii) Since the boundary data for $h_\varepsilon^\pm$ are clearly in C^α for any $\alpha \in (0,1)$, the desired result follows from the boundary $C^{2,\alpha}$-estimate; for example, see [30].

(iii) The maximum principle for $h_\varepsilon^+ - h_\varepsilon^-$ yields the inequality.

□

We also need the following version of Arzela-Ascoli theorem, whose proof is a simple modification of the original one. In short, the equicontinuous assumption in Arzela-Ascoli theorem can be relaxed to "almost equicontinuity".

Lemma 4.2.6 (Arzela-Ascoli theorem)**.** *Let A be a compact subset of $\mathbb{R}^n$. Suppose that a sequence of functions $\{f_l\}_{l\in\mathbb{N}}$ defined on A satisfies* *(i) (Uniformly bounded) There exists a constant $M > 0$ such that*

$$|f_l(x)| \leq M,$$

for any $l \in \mathbb{N}$ and $x \in A$.

(ii) (Almost equicontinuous) There exists a constant $\alpha \in (0,1)$, $C > 0$ and a function $g : \mathbb{N} \to \mathbb{R}_{\geq 0}$ such that $\lim_{l\to\infty} g(l) = 0$ and

$$|f_l(x) - f_l(y)| \leq C|x-y|^\alpha + g(l),$$

for any $x, y \in A$.

Then there exists a subsequence $\{f_{l_k}\}_{k\in\mathbb{N}}$ which converges uniformly on A. Moreover, if we denote the limit function by f, then $f \in C^\alpha(A)$.

Theorem 4.2.7 (Uniform convergence)**.** *There is a continuous function u such that $u_\varepsilon \to u$ weakly in D with respect to L^p-norm for any $p > 0$. Also for any $\delta > 0$, there is a subset $D_\delta \subset D$ and a sequence $\{\varepsilon_l\}_{l\in\mathbb{N}}$ such that $\varepsilon_l > 0$, $\lim_{l\to\infty} \varepsilon_l = 0$ and $u_{\varepsilon_l} \to u$ uniformly in D_δ as $l \to \infty$ and $|D \setminus D_\delta| < \delta$.*

Proof. See [12] and [48] for detailed proof. Here the only difference occurs when applying the discrete gradient estimate in the references. More precisely, the absence of periodicity in stationary ergodic setting prevents us from achieving the discrete gradient estimate (4.2.1). Nevertheless, Lemma 4.2.4 and Lemma 4.2.5 provides the "almost equicontinuity" of u_ε, i.e.

$$\begin{aligned}
|u_\varepsilon(x) - u_\varepsilon(y)| = u_\varepsilon(x) - u_\varepsilon(y) &\leq h_\varepsilon^+(x) - h_\varepsilon^-(y) \\
&\leq h_\varepsilon^+(x) - h_\varepsilon^+(y) + h_\varepsilon^+(y) - h_\varepsilon^-(y) \\
&\leq C|x-y|^\alpha + \max_{k\in\mathbb{Z}^n} \operatorname*{osc}_{\partial B_{b_\varepsilon}(k)} u_\varepsilon \\
&\leq C|x-y|^\alpha + o(\varepsilon^\gamma),
\end{aligned}$$

for any $x, y \in D \setminus (\cup_{k\in\mathbb{Z}^n} B_{b_\varepsilon}(k))$ where we assumed $u_\varepsilon(x) \geq u_\varepsilon(y)$ without loss of generality. Then we can apply the modified Arzela-Ascoli theorem, Lemma 4.2.6, for u_ε and finish the proof following the previous references. $\square$

Remark 4.2.8. Note that the argument for the Laplacian operator in this section can be repeated for the uniformly elliptic, fully nonlinear operator F. Indeed, we only used the comparison principle, boundary C^α-estimate, Harnack inequality, and oscillation lemma which still hold for F; for example, see [18] and [14].

In conclusion, we presented the proof for the first part of Theorem 4.1.1 and Theorem 4.1.2, which concerns the convergence of u_ε to the limit function u. In the remaining of this chapter, we will concentrate on the second part of our main theorems by constructing a proper corrector and investigating its properties.

4.3 The Approximation of a Homogeneous Solution

To determine the critical value β_0, the essential step is to define the corresponding subadditive quantity since the size of hole is not identical, but random. In the papers [15] (for Laplacian case), [77] (for p-Laplacian case) and [57] (for φ-Laplacian case), they described the subadditive quantity in terms of Dirac-delta distribution δ_0 and proved the properties of correctors using an energy method. However, this approach is not suitable for our case, because the operators that we consider do not have the divergence structure. Hence, we concentrate on the non-divergence structure of F; in particular, we will capture its "shape" and employ a viscosity method to verify the properties of correctors.

4.3.1 A Homogeneous Solution

The starting point is a homogeneous solution for F, a uniformly elliptic fully nonlinear operator being homogeneous of degree one. We recall that

Lemma 2.1.1 explains the existence, uniqueness, and behaviors of a homogeneous solution for F.

Remark 4.3.1. (i) Throughout this chapter, we denote a homogeneous solution by Φ, instead of V in Chapter 2.

(ii) In the remaining of this chapter, we concentrate on the case $\alpha^* > 0$ (which corresponds to $n \geq 3$ in the Laplacian case) to simplify the statement. Indeed, the same argument can be applied to $\alpha^* = 0$ (which corresponds to $n = 2$ in the Laplacian case) and $\alpha^* < 0$ (which corresponds to $n = 1$ in the Laplacian case).

(iii) Since F is positively homogeneous of degree one, we have $a\Phi + b$ is again a homogeneous solution for $a > 0$, $b \in \mathbb{R}$ and a homogeneous solution Φ. In the remaining of this chapter, we fix the 'normalized' homogeneous solution Φ by

$$\Phi(x) = |x|^{-\alpha^*}\Phi\left(\frac{x}{|x|}\right) =: |x|^{-\alpha^*}\phi(\theta), \quad \text{for } \theta = \frac{x}{|x|} \in S^{n-1},$$

where ϕ is chosen so that $\min_{\theta \in S^{n-1}} \phi(\theta) = 1$. Note that here we normalize a homogeneous solution in the sense of the 'height' (at $|x| = 1$) while in the Laplacian case, we typically normalize in the sense of 'mass' (i.e. measure): $-\Delta\Phi = \delta_0$.

4.3.2 Approximation of a Homogeneous Solution

In divergence case, it is natural to approximate the Dirac-delta measure δ_0 by measurable functions $\{f_\sigma\}$. More precisely, we define

$$f_\sigma(x) = \frac{1}{|B_\sigma|}\chi_{B_\sigma}(x),$$

for any $\sigma > 0$ and let a regularized homogeneous solution Φ_σ by the solution of $T\Phi_\sigma = f_\sigma$ in $\mathbb{R}^n$, where T is a uniformly elliptic operator with divergence structure. Then $\Phi_\sigma \to \Phi$ in $L^1(\mathbb{R}^n)$ and $f_\sigma \to \delta_0$ in distribution sense as $\sigma \to 0^+$.

In non-divergence case, we do not have the corresponding measure such as the Dirac-delta δ_0. In other words, it is difficult to define $F(D^2\Phi)$ in the whole space $\mathbb{R}^n$, while we know that $F(D^2\Phi) = 0$ in $\mathbb{R}^n \setminus \{0\}$; see [51]. Thus, instead of measure-sense, we focus on the 'shape' of Φ; we define an approximated homogeneous solution Φ_σ for $\sigma > 0$ by

$$\Phi_\sigma = \begin{cases} \Phi & \text{in } \mathbb{R}^n \setminus B_{\overline{a}_\sigma}, \\ W_\sigma & \text{in } B_{\overline{a}_\sigma}, \end{cases}$$

where Φ is the normalized homogeneous solution and W_σ, $\overline{a}_\sigma$ will be determined later. (note that a radius $\overline{a}_\sigma$ must converge to zero when $\sigma \to 0$.) Then we define a corresponding function ν_σ by

$$\nu_\sigma := -F(D^2\Phi_\sigma) \quad \text{in } \mathbb{R}^n.$$

Since $\Phi_\sigma = \Phi$ in $B^c_{\overline{a}_\sigma}$ and Φ is a homogeneous solution, we immediately have that $\nu_\sigma \equiv 0$ in $B^c_{\overline{a}_\sigma}$ and so $\operatorname{supp} \nu_\sigma \subset B_{\overline{a}_\sigma}$.

Laplacian case

Continuing to the argument above, we can define a radius $\overline{a}_\sigma$ and an approximated homogeneous solution W_σ. Note that for the Laplacian case, we have $\alpha^* = n-2$ and so the normalized homogeneous solution is given by $\Phi(x) = |x|^{2-n}$. On the other hand, we see that the radius of hole a^ε is assumed to comparable to $\varepsilon^{n/(n-2)}$. Since the corrector w^ε will be constructed so that $w^\varepsilon \approx 1$ near ∂T_ε (see Section 4.5), we require the homogeneous solution $\Phi(x) \approx \varepsilon^{-2}$ near $|x| = \overline{a}^\varepsilon = a^\varepsilon/\varepsilon$. Here we need to distinguish the scale ε and the scale 1.

Therefore, we let $\overline{a}_\sigma := \sigma^{\frac{2}{n-2}}$ and determine a quadratic polynomial W_σ

which is rotationally symmetric and satisifes

$$W_\sigma(x) = \Phi(x) \quad \text{and} \quad \nabla W_\sigma(x) = \nabla\Phi(x),$$

if $|x| = \overline{a}_\sigma$. Indeed, for $\sigma > 0$, we set

$$\Phi_\sigma(x) := \begin{cases} \Phi(x) = |x|^{2-n} & |x| \geq \overline{a}_\sigma, \\ W_\sigma(x) = -m_\sigma|x|^2 + k_\sigma & |x| < \overline{a}_\sigma, \end{cases}$$

where $m_\sigma = \frac{n-2}{2}\sigma^{\frac{-2n}{n-2}}$ and $k_\sigma = \frac{n}{2}\sigma^{-2}$. Then $\Phi_\sigma \in C^{1,1}(\mathbb{R}^n)$ and it follows that

$$\Delta\Phi_\sigma(x) = -\nu_\sigma(x) := \begin{cases} 0 & |x| \geq \overline{a}_\sigma, \\ -2nm_\sigma & |x| < \overline{a}_\sigma. \end{cases}$$

On the other hand, by its construction, we immediately have that

$$\Phi_\sigma \to \Phi \quad \text{as } \sigma \to 0,$$

locally uniformly on $\mathbb{R}^n \setminus \{0\}$. Moreover, for $0 < \sigma_1 \leq \sigma_2$,

$$\Phi_{\sigma_1} = \Phi_{\sigma_2} \quad \text{in } B^c_{\overline{a}_{\sigma_2}}, \tag{4.3.1}$$

and

$$\Phi_{\sigma_1} \geq \Phi_{\sigma_2} \quad \text{in } \mathbb{R}^n. \tag{4.3.2}$$

Note that this approximation is related to the Dirac-delta measure:

$$-\nu_\sigma \rightharpoonup -n(n-2)\omega_n\delta_0 \quad \text{as } \sigma \to 0,$$

in distribution sense, i.e.

$$-\int_{B_{\overline{a}_\sigma}} \nu_\sigma(x)\eta(x)\,\mathrm{d}x \to -n(n-2)\omega_n\eta(0) \quad \text{as } \sigma \to 0,$$

for any $\eta \in C_c^\infty(\mathbb{R}^n)$.

Fully nonlinear case

In this case, the radius of hole a^ε is comparable to $\varepsilon^{\frac{\alpha^*+2}{\alpha^*}}$. Since the normalized homogeneous solution is given by

$$\Phi(x) = r^{-\alpha^*}\phi(\theta),$$

in spherical coordinates, we let $\overline{a}_\sigma = \sigma^{\frac{2}{\alpha^*}}$. Moreover, we consider a (strict) superlevel set of Φ:

$$C_{\overline{a}_\sigma} := \{x = (r,\theta) \in \mathbb{R}^n : r^{-\alpha^*}\phi(\theta) > \overline{a}_\sigma^{-\alpha^*}\}.$$

Note that we have $C_{\overline{a}_\sigma} = B_{\overline{a}_\sigma}$ as before, if we let $\phi(\theta) \equiv 1$, i.e. F is rotationally symmetric. Then for $\sigma > 0$, we set

$$\Phi_\sigma(x) := \begin{cases} \Phi(x) = r^{-\alpha^*}\phi(\theta) & x \in C_{\overline{a}_\sigma}^c, \\ W_\sigma(x) = -m_\sigma(r^{\alpha^*}\phi(\theta)^{-1})^s + k_\sigma & x \in C_{\overline{a}_\sigma}, \end{cases}$$

where s (which is independent of $\sigma > 0$), m_σ and k_σ will be determined.

(i) ($\Phi_\sigma \in C^{1,1}$) We only need to check this property on $\partial C_{\overline{a}_\sigma}$. In fact, for $(r,\theta) \in \partial C_{\overline{a}_\sigma}$, we have

$$\Phi(r,\theta) = \overline{a}_\sigma^{-\alpha^*}, \quad W_\sigma(r,\theta) = -m_\sigma \overline{a}_\sigma^{\alpha^* s} + k_\sigma,$$
$$\partial_r\Phi(r,\theta) = -\frac{\alpha^*}{r}\overline{a}_\sigma^{-\alpha^*}, \quad \partial_r W_\sigma(r,\theta) = -\frac{m_\sigma \alpha^* s}{r}\overline{a}_\sigma^{\alpha^* s},$$

and

$$\nabla_\theta \Phi(r,\theta) = r^{-\alpha^*}\nabla_\theta\phi,$$
$$\nabla_\theta W_\sigma(r,\theta) = m_\sigma s r^{\alpha^* s}\phi(\theta)^{-s-1}\nabla_\theta\phi = m_\sigma s r^{-\alpha^*}\overline{a}_\sigma^{\alpha^*(s+1)}\nabla_\theta\phi.$$

Therefore, we conclude $W_\sigma \in C^{1,1}$ provided that

$$m_\sigma = \frac{1}{s}\sigma^{-2(s+1)} \quad \text{and} \quad k_\sigma = \left(1+\frac{1}{s}\right)\sigma^{-2},$$

for some constant $s > 0$.

(ii) ($F(D^2 W_\sigma) =: -\nu_\sigma \leq 0$) To verify this property, it is enough to show that there exists a sufficiently large s such that

$$\mathcal{P}^-(-D^2 W_\sigma) \geq 0.$$

For this purpose, we claim that for sufficiently large $s = s(\lambda, \Lambda, f) > 0$, we have

$$\mathcal{P}^-(D^2 w) \geq 0,$$

where $w(r,\theta) := r^s f(\theta)$ for a positive function $f \in C^2(S^{n-1})$. Indeed, one can calculate the Hessian of w as follows:

$$\text{Hess}(w) \sim (a_{ij}(s,\theta)) r^{s-2},$$

where

$$a_{ij}(s,\theta) = \begin{cases} s(s-1)f(\theta) & \text{if } (i,j) = (1,1), \\ o(s^2) g_{ij}(\theta) & \text{otherwise}, \end{cases}$$

for $g_{ij} \in C(S^{n-1}), 1 \leq i,j \leq n$. See appendix in [67] for the computation of the Hessian matrix in spherical coordinates. In short, we have the dominant s^2-order only in $(1,1)$-component of $\text{Hess}(w)$, since the power of s is added if and only if we take a radial derivative with respect to w.

Moreover, since $\det(tI - A) = (t - \lambda_1(A)) \cdots (t - \lambda_n(A))$ and the determinant function is smooth, one can easily check that the eigenvalues of

Hess(w) is given by

$$(s^2 f(\theta) + o(s^2) b_1(\theta), o(s^2) b_2(\theta), ..., o(s^2) b_n(\theta)) \cdot r^{s-2},$$

for some functions $b_i \in C(S^{n-1})$. Hence, for sufficiently large $s = s(\lambda, \Lambda, f) > 0$, we have

$$\mathcal{P}^-(D^2 w) \geq [\lambda s^2 f(\theta) - \Lambda b(\theta) o(s^2)] \cdot r^{s-2} \geq 0,$$

as claimed.

After choosing s, m_σ and k_σ in this way, we immediately have that

$$\Phi_\sigma \to \Phi \quad \text{as } \sigma \to 0,$$

locally uniformly on $\mathbb{R}^n \setminus \{0\}$. Moreover, we have $\Phi_{\sigma_1} = \Phi_{\sigma_2}$ in $C^c_{\bar{a}_{\sigma_2}}$, and $\Phi_{\sigma_1} \geq \Phi_{\sigma_2}$ in $\mathbb{R}^n$ whenever $0 < \sigma_1 \leq \sigma_2$.

4.4 The Convergence of Free Solutions and Obstacle Solutions

To apply the subadditive ergodic theorem (see [16, 21]) and determine the critical value β_0, we first consider an obstacle problem and its solution as an auxiliary function for a corrector w^ε. In view of [15, 57, 77], the forcing term of an obstacle problem was presented by the Dirac-delta measure. In contrast to those operators of divergence form, we cannot exploit this energy-type method in fully nonlinear operator of non-divergence form. Instead, to capture the behavior of a corrector w^ε, we are going to adopt the approximation of a homogeneous solution which was obtained in the previous section. Moreover, to connect the properties between an obstacle solution and a corrector, we need one more auxiliary function, namely a “free” solution.

While the argument concerning these auxiliary functions is relatively straightforward in the Laplacian case, there arises several challenges in the fully nonlinear case. Hence, we will first investigate nice properties of obstacle solutions and free solutions in the Laplacian case, and then justify the validity of those properties in the fully nonlinear case.

4.4.1 Laplacian Operator

We start with the definition of obstacle solutions and free solutions in the Laplacian operator:

Definition 4.4.1. Let A be an open and bounded subset of $\mathbb{R}^n$.

(i) For $\beta \in \mathbb{R}$, we define an "obstacle" solution

$$v_{\beta,\sigma,A}(x,\omega) := \inf\Big\{v(x) : \Delta v \le \beta - \sum_{k\in\mathbb{Z}^n\cap A} \gamma(k,\omega)\nu_\sigma(x-k) \text{ in } A,\\ v \ge 0 \text{ in } A,\ v = 0 \text{ on } \partial A\Big\}.$$

and its rescaled function $\overline{v}^\varepsilon_{\beta,\sigma}(y,\omega) := \varepsilon^2 v_{\beta,\sigma,\varepsilon^{-1}A}(y/\varepsilon,\omega)$.

(ii) For $\beta \in \mathbb{R}$, we define a "free" solution

$$w_{\beta,\sigma,A}(x,\omega) := \inf\Big\{w(x) : \Delta w \le \beta - \sum_{k\in\mathbb{Z}^n\cap A} \gamma(k,\omega)\nu_\sigma(x-k) \text{ in } A,\\ w = 0 \text{ on } \partial A\Big\}.$$

and its rescaled function $\overline{w}^\varepsilon_{\beta,\sigma}(y,\omega) := \varepsilon^2 w_{\beta,\sigma,\varepsilon^{-1}A}(y/\varepsilon,\omega)$.

Lemma 4.4.2 (Multiple sources; Δ.). *Let $0 < \sigma_1 \le \sigma_2 \ll 1$. For a non-negative function $\gamma : \mathbb{Z}^n \to \mathbb{R}_{\ge 0}$, we consider the solutions w_i, defined by*

$$\begin{cases} \Delta w_i(x) = -\sum_{k\in\mathbb{Z}^n} \gamma(k)\nu_{\sigma_i}(x-k) & \text{in } A,\\ w_i(x) = 0 & \text{on } \partial A. \end{cases}$$

Then we have $w_1 \ge w_2$ in A.

Proof. First, we let $\widetilde{w}_i(x) = \sum_{k\in\mathbb{Z}^n} \gamma(k)\Phi_{\sigma_i}(x-k)$. Then we have $\Delta\widetilde{w}_i(x) = -\sum_{k\in\mathbb{Z}^n} \gamma(k)\nu_{\sigma_i}(x-k)$, and $\widetilde{w}_1 \ge \widetilde{w}_2$ in A by (4.3.2). Recalling (4.3.1), we also have that $\widetilde{w}_1 = \widetilde{w}_2$ on ∂A. Thus, if we let g be the solution of the Dirichlet problem

$$\begin{cases} \Delta g = 0 & \text{in } A, \\ g = -\widetilde{w}_1 (= -\widetilde{w}_2) & \text{on } \partial A, \end{cases}$$

then we conclude $w_i = \widetilde{w}_i + g$, which completes the proof. □

Lemma 4.4.3 (Additional source; Δ.)**.** *Suppose that a function w_i defined by the Dirichlet problem*

$$\begin{cases} \Delta w_i = f_i & \text{in } A, \\ w_i = 0 & \text{on } \partial A, \end{cases}$$

satisfy $w_1 \geq w_2$ in A. Moreover, for a constant $\beta \geq 0$, we define a function $w_{\beta,i}$ by

$$\begin{cases} \Delta w_{\beta,i} = f_i + \beta & \text{in } A, \\ w_{\beta,i} = 0 & \text{on } \partial A. \end{cases}$$

Then we have $w_{\beta,1} \geq w_{\beta,2}$ in A.

Proof. Let g_β be the solution of the Dirichlet problem

$$\begin{cases} \Delta g_\beta = \beta & \text{in } A, \\ g_\beta = 0 & \text{on } \partial A. \end{cases}$$

Then the result follows immediately from $w_{\beta,i} = w_i + g_\beta$. □

Remark 4.4.4 (Existence of limit free solutions; Δ.)**.** For $0 < \sigma_1 \leq \sigma_2 \ll 1$, applying Lemma 4.4.2, Lemma 4.4.3 and their proofs, we have

$$w_{\beta,\sigma_1,A} \geq w_{\beta,\sigma_2,A} \quad \text{in } A, \tag{4.4.1}$$

and furthermore,

$$w_{\beta,\sigma_1,A}(x) = w_{\beta,\sigma_2,A}(x) \quad \text{if } x \in \cup_{k\in\mathbb{Z}^n}(B_{\overline{a}_{\sigma_2}}(k))^c \ . \tag{4.4.2}$$

In particular, (4.4.1), the monotonicity of $\{w_{\beta,\sigma,A}\}_{\sigma>0}$ yields the convergence of free solutions $w_{\beta,\sigma,A}$ when $\sigma \to 0^+$. We denote the limit function by $w_{\beta,A}$.

Lemma 4.4.5 (Obstacle solution; Δ.). *Let $0 < \sigma_1 \leq \sigma_2 \ll 1$. For a constant $\beta \geq 0$ and a non-negative function $\gamma : \mathbb{Z}^n \to \mathbb{R}_{\geq 0}$, we define (an obstacle solution) $v_{\beta,i}$ by*

$$\begin{aligned} v_{\beta,i}(x) := \inf \Big\{ v(x) : \Delta v \leq \beta - \sum_{k\in\mathbb{Z}^n\cap A} \gamma(k)\nu_{\sigma_i}(x-k) \ \textit{in } A, \\ v \geq 0 \ \textit{in } A,\ v = 0 \ \textit{on } \partial A \Big\}. \end{aligned} \tag{4.4.3}$$

Then we have $v_{\beta,1} \geq v_{\beta,2}$ in A.

Proof. We have the equivalent definition of an obstacle problem (4.4.3):

$$\begin{aligned} &\Delta v_{\beta,i} \leq \beta - \sum_{k\in\mathbb{Z}^n\cap A} \gamma(k)\nu_{\sigma_i}(x-k), \quad v_{\beta,i} \geq 0 \quad \text{in } A \text{ and} \\ &\Delta v_{\beta,i} = \beta - \sum_{k\in\mathbb{Z}^n\cap A} \gamma(k)\nu_{\sigma_i}(x-k) \quad \text{if } v_{\beta,i} > 0. \end{aligned}$$

Moreover, for sufficiently small $\sigma_i > 0$ and $k \in \mathbb{Z}^n$ with $\gamma(k) \neq 0$, we have $\beta - \gamma(k)\nu_{\sigma_i}(x-k) < 0$ for $|x-k| < \overline{a}_{\sigma_i}$. Since $\beta - \gamma(k)\nu_{\sigma_i}(x-k) < 0$ in $B_{\overline{a}_{\sigma_i}}(k)$ for any $k \in \mathbb{Z}^n \cap A$ with $\gamma(k) \neq 0$, we have

$$v_{\beta,i} > 0 \quad \text{in } \cup_{k\in\mathbb{Z}^n\cap A} B_{\overline{a}_{\sigma_i}}(k). \tag{4.4.4}$$

Now, if we let $\widetilde{v}(x) := v_{\beta,2}(x) + \sum_{k\in\mathbb{Z}^n\cap A} \gamma(k)(\Phi_{\sigma_1}(x-k) - \Phi_{\sigma_2}(x-k))$, then by (4.3.2), we have $\widetilde{v} \geq v_{\beta,2}$ in A. Thus, the proof will be completed if we prove $v_{\beta,1} = \widetilde{v}$. Indeed, $\widetilde{v} \geq 0$ in A. We split two cases:

(i) ($\widetilde{v}(x) = 0$) By the definition of $\widetilde{v}$, for each $k \in \mathbb{Z}^n$, we have either $\gamma(k) = 0$ or $\Phi_{\sigma_1}(x-k) = \Phi_{\sigma_2}(x-k)$. In the latter case, the construction of Φ_σ yields $|x-k| > \overline{a}_{\sigma_2} \geq \overline{a}_{\sigma_1}$ and so $\nu_{\sigma_i}(x-k) = 0$. Thus, in both cases, we have $\gamma(k)\nu_{\sigma_i}(x-k) = 0$ and so

$$0 = \Delta\widetilde{v}(x) \leq \beta - \sum_{k\in\mathbb{Z}^n\cap A} \gamma(k)\nu_{\sigma_i}(x-k) = \beta$$

(ii) ($\widetilde{v}(x) > 0$) By the definition of $\widetilde{v}$, we have either $v_{\beta,2}(x) > 0$ or $|x-k| \leq \overline{a}_{\sigma_2}$ for some $k \in \mathbb{Z}^n$ with $\gamma(k) \neq 0$. Recalling (4.4.4), in both cases, we have $v_{\beta,2}(x) > 0$ which yields that

$$\Delta v_{\beta,2}(x) = \beta - \sum_{k\in\mathbb{Z}^n\cap A} \gamma(k)\nu_{\sigma_2}(x-k).$$

Therefore, we conclude that

$$\begin{aligned}\Delta\widetilde{v}(x) =& \Delta\left(v_{\beta,2}(x) + \sum_{k\in\mathbb{Z}^n\cap A} \gamma(k)(\Phi_{\sigma_1}(x-k) - \Phi_{\sigma_2}(x-k))\right)\\ =& \beta - \sum_{k\in\mathbb{Z}^n\cap A} \gamma(k)\nu_{\sigma_1}(x-k).\end{aligned}$$

□

Remark 4.4.6 (Existence of limit obstacle solutions; Δ.)**.** We know that the assumption in Lemma 4.4.5 holds by Lemma 4.4.2 and Lemma 4.4.3, i.e. we have (4.4.1) and further (4.4.2). Thus, by applying Lemma 4.4.5 and its proof, we conclude that

$$v_{\beta,\sigma_1,A} \geq v_{\beta,\sigma_2,A} \quad \text{in } A, \tag{4.4.5}$$

and furthermore,

$$v_{\beta,\sigma_1,A}(x) = v_{\beta,\sigma_2,A}(x) \quad \text{if } x \in \cup_{k\in\mathbb{Z}^n}(B_{\overline{a}_{\sigma_2}}(k))^c\ . \tag{4.4.6}$$

In particular, (4.4.5), the monotonicity of $\{v_{\beta,\sigma,A}\}_{\sigma>0}$ yields the convergence of obstacle solutions $v_{\beta,\sigma,A}$ when $\sigma \to 0^+$.

We now define the measure of contact set for an obstacle problem and determine the critical value β_0. Indeed, we define a random variable $m_{\beta,A}$ by

$$m_{\beta,A} := |\{x \in A : v_{\beta,A} = 0\}|.$$

Lemma 4.4.7 (A subadditive quantity). *(i) The random variable $m_{\beta,A}$ is subadditive: in other words, for the finite family of sets $(A_i)_{i\in I}$ such that*

$$\begin{aligned} &A_i \subset A \quad \text{for all } i \in I, \\ &A_i \cap A_j = \varnothing \quad \text{for all } i \neq j, \\ &|A - \cup_{i\in I} A_i| = 0, \end{aligned}$$

then $m_{\beta,A} \leq \sum_{i\in I} m_{\beta,A_i}$.

(ii) The process $T_k m_{\beta,A} := m_{\beta,k+A}$ has the same distribution for all $k \in \mathbb{Z}^n$.

Proof. (i) Since $v_{\beta,A}$ is admissible for v_{β,A_i} for each i, we have

$$v_{\beta,A_i} \leq v_{\beta,A} \quad \text{in } A_i.$$

Thus, we have the desired result.

(ii) It follows immediately from our assumptions on $\gamma(k,\omega)$.

□

Due to the previous lemma, we can apply a subadditive ergodic theorem (see [16, 21]). More precisely, we have

$$l(\beta) = \lim_{t\to\infty} \frac{m_{\beta,B_t}}{|B_t|}.$$

and a scaled version:

$$l(\beta) = \lim_{\varepsilon\to 0} \frac{|\{y : \overline{v}^{\varepsilon}_{\beta}(y,\omega) = 0\}|}{|B_1|} \quad \text{a.s.},$$

where $\overline{v}^{\varepsilon}_{\beta}(y,\omega) := \varepsilon^2 v_{\beta,\varepsilon^{-1}B}(y/\varepsilon,\omega)$.

Lemma 4.4.8 (Properties of $l(\beta)$). *(i) $l(\beta)$ is non-decreasing function with respect to β.*

(ii) If $\beta < 0$, then $l(\beta) = 0$.

(iii) If $\beta > 0$ is large enough, then $l(\beta) > 0$.

Proof. (i) For $\beta_1 \le \beta_2$, we have $v_{\beta_2,A} \le v_{\beta_1,A}$ which implies that $m_{\beta_1,A} \le m_{\beta_2,A}$.

(ii) Since v_{β,σ,B_t} is a solution of an obstacle problem, we have

$$\Delta v_{\beta,\sigma,B_t} \le \beta - \sum_{k\in\mathbb{Z}^n\cap A} \gamma(k,\omega)\nu_\sigma(k,\omega) \le \beta,$$

in B_t. Thus, by the comparison principle,

$$v_{\beta,\sigma,A} \ge \frac{\beta}{2n}(|x|^2 - t^2) > 0$$

in B_t. Letting $\sigma \to 0$, we have $v_{\beta,A} > 0$ in A which implies that $l(\beta) = 0$ for $\beta < 0$.

(iii) For $k \in \mathbb{Z}^n$, we define

$$h_k(x) = \frac{\beta}{2n}|x-k|^2 + \gamma(k,\omega)\Phi_\sigma(x-k).$$

Then we have $h_k \in C^{1,1}$ and $\Delta h_k = \beta - \gamma(k,\omega)\nu_\sigma(x-k)$. Moreover, a direct calculation yields that a rotationally symmetric function h_k attains its minimum at

$$|x-k| = r_k = \left(\frac{n(n-2)\gamma(k,\omega)}{\beta}\right)^{1/n}.$$

Since $\gamma(k,\omega) \le \overline{\gamma}$, we can choose $\beta > 0$ large enough so that $r_k < 1/2$ for any $k \in \mathbb{Z}^n$. Moreover, we can choose a constant D_k so that the minimum of $h_k(x) - D_k$ is exactly 0. Now if we define

$$\widetilde{h}_k(x) = \begin{cases} h_k(x) - D_k & \text{if } |x| < r_k, \\ 0 & \text{if } |x| \geq r_k, \end{cases}$$

then $\widetilde{h}_k$ is well-defined and it belongs to $C^{1,1}$. Moreover, since $\Delta \widetilde{h}_k = \beta - \gamma(k,\omega)\nu_\sigma(x-k)$ in B_{r_k}, $\sum_{k \in \mathbb{Z}^n \cap tB} \widetilde{h}_k$ is admissible for $v_{\beta,\sigma,tB}$. Therefore, we conclude that

$$l_\sigma(\beta) := \lim_{t\to\infty} \frac{|\{x \in tB : v_{\beta,\sigma,tB} = 0\}|}{|tB|} \geq \frac{|C_1| - |B_{1/2}|}{|C_1|} \geq 1 - \frac{\omega_n}{2^n} > 0,$$

which ensures that $l(\beta) > 0$ for large enough β.

□

Finally, we let the (non-negative) critical value

$$\beta_0 := \sup\{\beta : l(\beta) = 0\},$$

which is well-defined by the previous lemma.

4.4.2 Fully Nonlinear Operator

Definition 4.4.9. (i) For $M \in \mathcal{S}^n$ and $\beta \in \mathbb{R}$, we define an "obstacle" solution

$$\begin{aligned} v_{\beta,\sigma,A;M}(x,\omega) := \inf \Big\{ & v(x) : v \geq 0 \text{ in } A,\ v = 0 \text{ on } \partial A, \\ & F(M + D^2 v(x)) \leq \beta + F(M) - \sum_{k \in \mathbb{Z}^n \cap A} \gamma(k,\omega)\nu_\sigma(x-k) \text{ in } A \Big\}, \end{aligned}$$

and its rescaled function $\overline{v}^\varepsilon_{\beta,\sigma;M}(y,\omega) := \varepsilon^2 v_{\beta,\sigma,\varepsilon^{-1}A;M}(y/\varepsilon,\omega)$.

(ii) For $M \in \mathcal{S}^n$ and $\beta \in \mathbb{R}$, we define a "free" solution

$$w_{\beta,\sigma,A;M}(x,\omega) := \inf \Big\{ w(x) : w = 0 \text{ on } \partial A, \\ F(M + D^2 w(x)) \le \beta + F(M) - \sum_{k \in \mathbb{Z}^n \cap A} \gamma(k,\omega)\nu_\sigma(x-k) \text{ in } A \Big\}.$$

and its rescaled function $\overline{w}^{\varepsilon}_{\beta,\sigma;M}(y,\omega) := \varepsilon^2 w_{\beta,\sigma,\varepsilon^{-1}A;M}(y/\varepsilon,\omega)$.

Before showing the convergence of these functions when $\sigma \to 0^+$, we first describe the local behavior of a singular solution with an isolated singularity at some point x_0. Roughly speaking, we will demonstrate that the growth rate of a singular solution near the singularity point is the same as the growth rate of the corresponding homogeneous solution Φ. This type of result was first proved by M. Bôcher [9] for the Laplacian operator in 1903. Similar results can be found in [75, 76] for quasilinear divergence-type equations, [51] for Pucci operators, [12, 4] for fully nonlinear operators with homogeneous degree one and [34, 33] for a class of subequations. Note that they considered the local behavior of solutions for equations with zero-forcing term; in the following lemmas, we present generalized results by choosing a general forcing term.

Lemma 4.4.10. *Let $u \in C(B_1 \setminus \{0\})$ be a viscosity solution of*

$$F(D^2u) = g(x) \quad \text{in } B_1 \setminus \{0\},$$

where $g \in L^\infty(B_1)$, u is bounded on ∂B_1 and $\lim_{|x|\to 0} u(x) = \infty$. Then there exist positive constants a_0 and C_0 such that

$$a_0\Phi(x) - C_0 \le u(x) \le \frac{1}{a_0}\Phi(x) + C_0.$$

Proof. We may assume u is positive in $B_1 \setminus \{0\}$ by adding a constant on u, if necessary. To show the lower bound, suppose that there exist sequences

$a_i \to 0$, $\varepsilon_i \to 0$ and $x_i \in B_1 \setminus \{0\}$ such that

$$u(x_i) \le a_i \Phi(x_i) \quad \text{for } |x_i| = \varepsilon_i. \tag{4.4.7}$$

Note that from [14], we have the Harnack inequality

$$\sup_{\partial B_{1/2}} u \le C \left(\inf_{\partial B_{1/2}} u + \|g\|_{L^n(B_1)} \right).$$

Recalling that F is positively homogeneous of degree one and considering the scaled function $u_r(x) := u(rx)$ for small $r > 0$, we deduce that

$$\sup_{\partial B_{r/2}} u \le C \left(\inf_{\partial B_{r/2}} u + r\|g\|_{L^n(B_1)} \right). \tag{4.4.8}$$

Thus, (4.4.7), (4.4.8) and the homogeneity of Φ imply that

$$\begin{aligned} u(x) &\le C(u(x_i) + \varepsilon_i \|g\|_{L^n(B_1)}) \le C(a_i \Phi(x_i) + \varepsilon_i \|g\|_{L^n(B_1)}) \\ &\le \widetilde{C} a_i \Phi(x) + C\varepsilon_i \|g\|_{L^n(B_1)}, \end{aligned}$$

for $|x| = \varepsilon_i$. Since $F(D^2(\Phi - c|x|^2)) = F(D^2\Phi - 2cI) \le F(D^2\Phi) - 2c\lambda \le g(x)$ for sufficiently large $c > 0$, the comparison principle yields that

$$u(x) \le \widetilde{C} a_i \Phi(x) + c_1 - c_2|x|^2 \quad \text{for } \varepsilon_i \le |x| \le 1.$$

for some $c_1, c_2 > 0$. Letting $i \to \infty$, we have u is bounded above in $B_1 \setminus \{0\}$ which contradicts to the assumption $\lim_{|x|\to 0} u(x) = \infty$. Therefore, we obtain the lower bound and from the similar argument, we finish the proof. □

Theorem 4.4.11 (An isolated singularity)**.** *Let $u \in C(B_1 \setminus \{0\})$ be a solution of*

$$F(D^2u) = g(x) \quad \text{in } B_1 \setminus \{0\},$$

where $g \in L^\infty(B_1)$, u is bounded on ∂B_1 and $\lim_{|x|\to 0} u(x) = \infty$. Then there *exists a positive constant a such that*

$$\lim_{x\to 0}\frac{u(x)}{\Phi(x)} = a. \tag{4.4.9}$$

Proof. Set $\widetilde{u}_\varepsilon(x) := \varepsilon^{\alpha^*}u(\varepsilon x)$. Then the homogeneity of the homogeneous solution Φ gives

$$\frac{u(\varepsilon x)}{\Phi(\varepsilon x)} = \frac{\varepsilon^{\alpha^*}u(\varepsilon x)}{\varepsilon^{\alpha^*}\Phi(\varepsilon x)} = \frac{\widetilde{u}_\varepsilon(x)}{\Phi(x)}.$$

For a compact set $K \subset \mathbb{R}^n \setminus \{0\}$, an application of Lemma 4.4.10 leads to

$$\sup_{x\in K}\frac{\widetilde{u}_\varepsilon(x)}{\Phi(x)} = \sup_{x\in\varepsilon K}\frac{u(\varepsilon x)}{\Phi(\varepsilon x)} \le \frac{1}{a_0} + C_1,$$

for some constant $C_1 > 0$ which is independent of $\varepsilon > 0$. Employing a similar argument for the lower bound, we conclude that

$$\sup_{0<\varepsilon<\varepsilon_0}\|\widetilde{u}_\varepsilon\|_{L^\infty(K)} \le C_K.$$

Since $F(D^2\widetilde{u}_\varepsilon(x)) = \varepsilon^{\alpha^*+2}g(\varepsilon x)$ holds for any $x \in B_1\setminus\{0\}$, we also obtain the uniform Hölder estimates for the sequence $\{\widetilde{u}_\varepsilon\}_{\varepsilon>0}$ in K. Therefore, Arzela-Ascoli theorem implies that there exist a subsequence $\varepsilon_j \to 0$ and a function $v \in C(\mathbb{R}^n \setminus \{0\})$ such that $\widetilde{u}_{\varepsilon_j} \to v$ locally uniformly in $\mathbb{R}^n \setminus \{0\}$. Then for any $x \in \mathbb{R}^n \setminus \{0\}$, the homogeneity of Φ yields

$$\frac{v(x)}{\Phi(x)} = \lim_{\varepsilon\to 0}\frac{\widetilde{u}_\varepsilon(x)}{\Phi(x)} = \lim_{\varepsilon\to 0}\frac{u(\varepsilon x)}{\Phi(\varepsilon x)} \in [\underline{a}, \overline{a}],$$

where

$$\underline{a} := \liminf_{\varepsilon\to 0}\inf_{|x|=\varepsilon}\frac{u(x)}{\Phi(x)}, \quad \overline{a} := \limsup_{\varepsilon\to 0}\sup_{|x|=\varepsilon}\frac{u(x)}{\Phi(x)}.$$

Here $\underline{a}, \overline{a} \in (0,\infty)$ by Lemma 4.4.10. Moreover, since $\widetilde{u}_\varepsilon \to v$ and $\varepsilon^{\alpha^*+2}g(\varepsilon x) \to 0$ uniformly on every compact subset K, we have $F(D^2v) = 0$ in $\mathbb{R}^n \setminus \{0\}$.

Finally, choose $x_\varepsilon \in \partial B_\varepsilon$ so that $u(x_\varepsilon) = \inf_{|x|=\varepsilon} \frac{u(x)}{\Phi(x)} \Phi(x_\varepsilon)$. Then there exist a (further) subsequence $\varepsilon_j \to 0$ and $y \in \partial B_1$ such that $\frac{x_{\varepsilon_j}}{\varepsilon_j} \to y$. Since

$$\begin{aligned} v(y) = \lim_{j\to\infty} \widetilde{u}_{\varepsilon_j}(\varepsilon_j^{-1} x_{\varepsilon_j}) &= \lim_{j\to\infty} \varepsilon_j^{\alpha^*} u(x_{\varepsilon_j}) \\ &= \lim_{j\to\infty} \left(\inf_{|x|=\varepsilon_j} \frac{u(x)}{\Phi(x)} \Phi\left(\frac{x_{\varepsilon_j}}{\varepsilon_j}\right)\right) = \underline{a}\Phi(y), \end{aligned}$$

we conclude that $v \equiv \underline{a}\Phi$ in $\mathbb{R}^n \setminus \{0\}$ by the strong maximum principle. Hence, we have

$$\limsup_{x\to 0} \frac{u(x)}{\Phi(x)} = \limsup_{\varepsilon\to 0} \max_{x\in\partial B_1} \frac{\widetilde{u}_\varepsilon(x)}{\Phi(x)} = \max_{x\in\partial B_1} \frac{v(x)}{\Phi(x)} = \underline{a},$$

which implies the desired result

$$\lim_{x\to 0} \frac{u(x)}{\Phi(x)} = a (:= \underline{a} = \overline{a}).$$

□

Free solutions

For notational simplicity, we write a free solution

$$w_\sigma(x,\omega) := w_{\beta,\sigma,A;M}(x,\omega),$$

where $\beta \geq 0, A \subset \mathbb{R}^n, M \in \mathcal{S}^n$ are fixed and $\sigma > 0$. Moreover, we denote

$$z_\sigma := \sum_{k\in\mathbb{Z}^n\cap A} \gamma(k,\omega)\Phi_\sigma(x-k).$$

For $0 < \sigma_1 \leq \sigma_2 \ll 1$, we have $z_{\sigma_1}(x) = z_{\sigma_2}(x)$ whenever $|x-k| \geq \overline{a}_{\sigma_2}(k)$ for any $k \in \mathbb{Z}^n$.

We will estimate $F(M + D^2 z_\sigma)$: we may expect

$$F(M + D^2 z_\sigma) \approx F(M) - \sum_{k\in\mathbb{Z}^n\cap A} \gamma(k,\omega)\nu_\sigma(x-k)$$

in A, by heuristic computation. Indeed, recall that F is positively homogeneous of degree one, $F(D^2\Phi_\sigma) = -\nu_\sigma$ by the construction of approximated homogeneous solution Φ_σ and $\Phi_\sigma(x-k)$ is "flat" away from $k \in \mathbb{Z}^n$ (i.e. $D^2\Phi_\sigma(x-k) \approx 0$, away from $k \in \mathbb{Z}^n$.) We prove this observation rigorously in the following lemma:

Lemma 4.4.12. *(i) There exists a constant $C > 0$ which is independent of $\sigma > 0$ such that*

$$\left| F(M + D^2 z_\sigma(x)) - \left(- \sum_{k \in \mathbb{Z}^n \cap A} \gamma(k,\omega)\nu_\sigma(x-k) \right) \right| \leq C. \tag{4.4.10}$$

(ii) There exists a constant $C > 0$ which is independent of $\sigma > 0$ such that

$$|F(M + D^2 z_\sigma) - F(M + D^2 w_\sigma)| \leq C.$$

(iii) There exists a constant $C > 0$ which is independent of $\sigma > 0$ such that

$$\|z_\sigma - w_\sigma\|_{L^\infty(A)} \leq C. \tag{4.4.11}$$

(iv) There exists a subsequence $\{w_{\sigma_m}\}_{m=1}^\infty$ and a limit function $w \in C(A \setminus \cup_{k \in \mathbb{Z}^n}\{k\})$ such that $w_{\sigma_m} \to w$ when $\sigma \to 0^+$ uniformly on every compact subset of $A \setminus \cup_{k\in\mathbb{Z}^n}\{k\}$.

Proof. (i) Since F is uniformly elliptic and positively homogeneous of degree one, we have

$$\begin{aligned} F(M + D^2 z_\sigma) \leq -\gamma(k_0)\nu_\sigma(x-k) + \mathcal{P}^+(M) \\ + \sum_{k \neq k_0} \gamma(k)\mathcal{P}^+(D^2\Phi(x-k)) \end{aligned}$$

where $|x - k_0| \leq 1/3$ for $k_0 \in \mathbb{Z}^n \cap A$. Since there exists a positive constant $\overline{\gamma} > 0$ such that $\gamma(k) \leq \overline{\gamma}$, there exists a constant

$C = C(F, A, M, \overline{\gamma})$ such that

$$\mathcal{P}^+(M) + \sum_{k \neq k_0} \gamma(k)\mathcal{P}^+(D^2\Phi(x-k)) \leq C,$$

where $|x - k_0| \leq 1/3$ for $k_0 \in \mathbb{Z}^n \cap A$. Similarly, we also have

$$F(M + D^2 z_\sigma) \leq \mathcal{P}^+(M) + \sum_{k \in \mathbb{Z}^n} \gamma(k)\mathcal{P}^+(D^2\Phi(x-k)) \leq C,$$

where $|x-k| > 1/3$ for any $k \in \mathbb{Z}^n \cap A$. We can apply the same argument for finding the lower bound of $F(M + D^2 z_\sigma)$ and thus, we conclude the desired result (4.4.10).

(ii) It follows directly from the part (i) and the definition of w_σ.

(iii) We may assume $x_0 = 0 \in A$ and $A \subset B_l(x_0)$ for some $l > 0$. Note that

$$\begin{aligned} F\left(D^2\left(z_\sigma + \frac{x^T M x}{2} + \frac{C(|x|^2 - l^2)}{2n\lambda}\right)\right) &\geq F(M + D^2 z_\sigma) + \mathcal{P}^-\left(\frac{C}{n\lambda} I\right) \\ &= F(M + D^2 z_\sigma) + C \\ &\geq F(M + D^2 w_\sigma) \\ &= F\left(D^2\left(w_\sigma + \frac{1}{2}x^T M x\right)\right). \end{aligned}$$

Moreover, by the construction of Φ_σ, $z_{\sigma_1} = z_{\sigma_2}$ on ∂A for any $\sigma_1, \sigma_2 > 0$. Thus, there exists a constant C independent of σ such that $|z_\sigma| \leq C$ on ∂A. Therefore, the comparison principle leads to

$$z_\sigma - C + \frac{1}{2}x^T M x + \frac{C}{2n\lambda}(|x|^2 - l^2) \leq w_\sigma + \frac{1}{2}x^T M x,$$

which implies that $z_\sigma - w_\sigma \leq C$. From the same argument, we derive (4.4.11).

(iv) Let K be a compact subset of $A \backslash \cup_{k \in \mathbb{Z}^n}\{k\}$. Again by the construction of Φ_σ, we have $z_\sigma(x) = \sum_{k \in \mathbb{Z}^n} \gamma(k)\Phi(x-k)$ in K for any sufficiently small

$\sigma > 0$. In other words, the function z_σ in K is independent of $\sigma > 0$ (if it is sufficiently small). Due to part (iii), we have a uniform L^∞-bound for w_σ: $\|w_\sigma\|_{L^\infty(K)} \leq C$, and $F(D^2 w_\sigma) = \beta$ in K. Hence, application of Arzela-Ascoli theorem together with the interior C^α-estimate and standard diagonal process ensures the existence of a convergent subsequence and corresponding limit function.

□

Theorem 4.4.13 (Convergence of free solutions). *There exists a unique limit function $w \in C(A \setminus \cup_{k\in\mathbb{Z}^n}\{k\})$ such that $w_\sigma \to w$ when $\sigma \to 0^+$ uniformly on every compact subset of $A \setminus \cup_{k\in\mathbb{Z}^n}\{k\}$. Moreover, w satisfies*

$$\begin{cases} F(M + D^2 w) = \beta + F(M) & \textit{in } A \setminus \cup_{k\in\mathbb{Z}^n}\{k\}, \\ w = 0, & \textit{on } \partial A, \end{cases}$$

and

$$\lim_{x\to k} \frac{w(x,\omega)}{\Phi(x-k)} = \gamma(k,\omega),$$

for any $k \in \mathbb{Z}^n \cap A$.

Proof. According to Lemma 4.4.12 (iv), there exists a limit function $w \in C(A \setminus \cup_{k\in\mathbb{Z}^n}\{k\})$ such that $w_{\sigma_m} \to w$ when $m \to \infty$ uniformly on every compact subset of $A \setminus \cup_{k\in\mathbb{Z}^n}\{k\}$. Recalling Proposition 2.9. in [14] (the stability of viscosity solutions), we deduce that w is a viscosity solution of

$$\begin{cases} F(M + D^2 w) = \beta + F(M) & \text{in } A \setminus \cup_{k\in\mathbb{Z}^n}\{k\}, \\ w = 0, & \text{on } \partial A. \end{cases}$$

Moreover, since $z_\sigma(k) \to \infty$ when $\sigma \to 0^+$ for $k \in \mathbb{Z}^n \cap A$ with $\gamma(k) > 0$, Lemma 4.4.12 (iii) yields that w has an isolated singularity at eath $k \in \mathbb{Z}^n \cap A$ whenever $\gamma(k) > 0$. Thus, applying Theorem 4.4.11 for an isolated singularity

$k \in \mathbb{Z}^n \cap A$, we have

$$\lim_{x \to k} \frac{w(x)}{\Phi(x-k)} = a, \tag{4.4.12}$$

for some positive constant $a > 0$.

Now we claim that $a = \gamma(k)$. Indeed, for $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x-k| < \delta \implies (a-\varepsilon)\Phi(x-k) \le w(x) \le (a+\varepsilon)\Phi(x-k),$$

by (4.4.12). Let $\{w_{\sigma_m}\}_{m=1}^{\infty}$ be a subsequence such that $w_{\sigma_m} \to w$ uniformly on every compact subset of $A \setminus \cup_{k\in\mathbb{Z}^n}\{k\}$. Then for any $m \in \mathbb{N}$ large enough, we have

$$|w(x) - w_{\sigma_m}(x)| \le 1, \quad |w_{\sigma_m}(x) - z_{\sigma_m}(x)| \le C,$$

for $\min\{\delta/2, \varepsilon\} < |x-k| < \delta$. Note that C is independent of m. (see Lemma 4.4.12 (iii).) Recalling the definition of z_σ, for m large enough, we have

$$\gamma(k)\Phi(x-k) \le z_{\sigma_m}(x) \le \gamma(k)\Phi(x-k) + C,$$

for $\min\{\delta/2, \varepsilon\} < |x-k| < \delta$ and $C > 0$ which is independent of m. Combining these estimates together, we conclude that

$$(a-\varepsilon)\Phi(x-k) - C \le \gamma(k)\Phi(x-k) \le (a+\varepsilon)\Phi(x-k) + C.$$

Dividing by $\Phi(x-k)$ and letting $\varepsilon \to 0$ leads to $a = \gamma(k)$, as desired.

Finally, it only remains to prove the uniqueness of limit functions. For this purpose, let $\overline{w}, \underline{w}$ be two limit functions of $\{w_\sigma\}_{\sigma>0}$. Then both $\overline{w}$ and

$\underline{w}$ are viscosity solutions of

$$\begin{cases} F(M + D^2 w) = \beta + F(M) & \text{in } A \setminus \cup_{k\in\mathbb{Z}^n}\{k\}, \\ w = 0, & \text{on } \partial A. \end{cases}$$

Moreover, the argument above allows us to capture the behavior of limit functions near an isolated singularity, namely:

$$\lim_{x\to k} \frac{\overline{w}(x)}{\Phi(x-k)} = \gamma(k) = \lim_{x\to k} \frac{\underline{w}(x)}{\Phi(x-k)}.$$

Fix $\varepsilon > 0$. Then there exists small enough $\delta > 0$ such that

$$(1+\varepsilon)\overline{w} \geq \underline{w},$$

for $|x-k| = \delta$ with $k \in \mathbb{Z}^n \cap A$. Employing a similar argument as in the proof of Lemma 4.4.10 and Theorem 4.4.11, there exist constants $c_1, c_2 > 0$ such that

$$F(M + D^2((1+\varepsilon)\overline{w} + \beta\varepsilon(c_1 - c_2|x|^2)) \leq \beta + F(M) = F(M + D^2\underline{w})$$

in $A \setminus \cup_k B_\delta(k)$, and

$$(1+\varepsilon)\overline{w} + \beta\varepsilon(c_1 - c_2|x|^2) \geq \underline{w} \quad \text{on } \partial(A \setminus \cup_k B_\delta(k)).$$

Applying the comparison principle and letting $\varepsilon \to 0$, we have $\overline{w} \geq \underline{w}$ and by the symmetry, we conclude that $\overline{w} = \underline{w}$. □

Obstacle solutions

For notational simplicity, we write an obstacle solution

$$v_\sigma(x,\omega) := v_{\beta,\sigma,A;M}(x,\omega),$$

where $\beta \geq 0, A \subset \mathbb{R}^n, M \in \mathcal{S}^n$ are fixed and $\sigma > 0$. The convergence of obstacle solutions $\{v_\sigma\}_{\sigma>0}$ can be achieved if we exploit the result for free solutions.

Lemma 4.4.14. *(i)* $0 \leq |F(M + D^2 w_\sigma) - F(M + D^2 v_\sigma)| \leq \beta$ *in* A.

(ii) There exists a constant $C > 0$ *which is independent of* $\sigma > 0$ *such that*

$$\|z_\sigma - v_\sigma\|_{L^\infty(A)} \leq C.$$

(iii) There exists a subsequence $\{v_{\sigma_m}\}_{m=1}^\infty$ *and a limit function* $v \in C(A \setminus \cup_{k\in\mathbb{Z}^n}\{k\})$ *such that* $v_{\sigma_m} \to v$ *when* $\sigma \to 0^+$ *uniformly on every compact subset of* $A \setminus \cup_{k\in\mathbb{Z}^n}\{k\}$.

Proof. (i) Since

$$F(M + D^2 v_\sigma) = F(M) + \left(\beta - \sum_{k\in\mathbb{Z}^n\cap A} \gamma(k,\omega)\nu_\sigma(x-k)\right)\chi_{\{v_\sigma>0\}},$$

we have

$$\begin{aligned} &F(M + D^2 w_\sigma) - F(M + D^2 v_\sigma) \\ &\qquad = \left(\beta - \sum_{k\in\mathbb{Z}^n\cap A} \gamma(k,\omega)\nu_\sigma(x-k)\right)\chi_{\{v_\sigma=0\}} = \beta\chi_{\{v_\sigma=0\}}. \end{aligned}$$

Here we used that $\nu_\sigma > 0$ near $k \in \mathbb{Z}^n$ with $\gamma(k,\omega) > 0$, recalling the proof for Lemma 4.4.5.

(ii) It follows from the comparison principle (similarly as in the proof of Lemma 4.4.12 (iii)) and (4.4.11).

(iii) See the proof of Lemma 4.4.12 (iv).

□

Theorem 4.4.15 (Convergence of obstacle solutions)**.** *There exists a unique limit function $v \in C(A \setminus \cup_{k\in\mathbb{Z}^n}\{k\})$ such that $v_\sigma \to v$ when $\sigma \to 0^+$ uniformly on every compact subset of $A \setminus \cup_{k\in\mathbb{Z}^n}\{k\}$. Moreover, v satisfies*

$$\begin{cases} F(M + D^2 v) = F(M) + \beta\chi_{\{v>0\}} & \textit{in } A \setminus \cup_{k\in\mathbb{Z}^n}\{k\}, \\ v \geq 0 & \textit{in } A \setminus \cup_{k\in\mathbb{Z}^n}\{k\}, \\ v = 0, & \textit{on } \partial A, \end{cases} \tag{4.4.13}$$

and

$$\lim_{x\to k} \frac{v(x,\omega)}{\Phi(x-k)} = \gamma(k,\omega),$$

for any $k \in \mathbb{Z}^n \cap A$.

Proof. The most part of the proof is the same as the proof of Theorem 4.4.13, which is an application of the uniform convergence obtained in the previous lemma, the stability of obstacle problems and the isolated singularity theorem, Theorem 4.4.11.

Again it only remains to prove the uniqueness part. Let $\overline{v}, \underline{v}$ be two limit functions of $\{v_\sigma\}_{\sigma>0}$. Since v_σ behaves like z_σ (or Φ_σ) near $k \in \mathbb{Z}^n$, Theorem 4.4.11 implies that

$$\lim_{x\to k} \frac{\overline{v}(x)}{\Phi(x-k)} = \gamma(k) = \lim_{x\to k} \frac{\underline{v}(x)}{\Phi(x-k)}.$$

Fix $\varepsilon > 0$. Then there exists small enough $\delta > 0$ such that

$$(1+\varepsilon)\overline{v} \geq \underline{v},$$

for $|x-k| = \delta$ with $k \in \mathbb{Z}^n \cap A$. Similarly as in the proof of Theorem 4.4.13, there exist constants $c_1, c_2 > 0$ such that

$$F(M + D^2((1+\varepsilon)\overline{v} + \beta\varepsilon(c_1 - c_2|x|^2))) \leq \beta + F(M) \quad \text{in } A \setminus \cup_k B_\delta(k),$$

$$(1+\varepsilon)\overline{v} + \beta\varepsilon(c_1 - c_2|x|^2) \geq \underline{v} \quad \text{on } \partial(A \setminus \cup_k B_\delta(k)),$$

and

$$(1+\varepsilon)\overline{v} + \beta\varepsilon(c_1 - c_2|x|^2) \geq 0 \quad \text{in } A \setminus \cup_k B_\delta(k).$$

Note that $\underline{v}$ can be written as the unique solution of the following obstacle problem

$$\begin{aligned}\inf \Big\{ v(x) : & F(M + D^2 v(x)) \leq \beta + F(M) \text{ in } A \setminus \cup_k B_\delta(k), \\ & v \geq 0 \text{ in } A \setminus \cup_k B_\delta(k),\ v \geq \underline{v} \text{ on } \cup_k \partial B_\delta(k),\ v \geq 0 \text{ on } \partial A. \Big\}\end{aligned}$$

Since the function $(1+\varepsilon)\overline{v} + \beta\varepsilon(c_1 - c_2|x|^2)$ is admissible for the obstacle problem above, we have $(1+\varepsilon)\overline{v} + \beta\varepsilon(c_1 - c_2|x|^2) \geq \underline{v}$. Letting $\varepsilon \to 0$, we have $\overline{v} \geq \underline{v}$ and by the symmetry, we conclude that $\overline{v} = \underline{v}$.

□

Now as we have done in the Laplacian case, we are able to define the measure of contact set for an obstacle problem and to determine the critical value β_0. Indeed, for any $M \in \mathcal{S}^n$, we define a random variable $m_{\beta,A;M}$ by

$$m_{\beta,A;M} := |\{x \in A : v_{\beta,A;M} = 0\}|.$$

Lemma 4.4.16 (A subadditive quantity). *(i) The random variable $m_{\beta,A;M}$ is subadditive.*

(ii) The process $T_k m_{\beta,A;M} := m_{\beta,k+A;M}$ has the same distribution for all $k \in \mathbb{Z}^n$.

Proof. See the proof of Lemma 4.4.7. □

Due to the previous lemma, an application of subadditive ergodic theorem yields

$$l(\beta; M) = \lim_{t\to\infty} \frac{m_{\beta,B_t;M}}{|B_t|} \quad \text{a.s.}$$

and a scaled version:

$$l(\beta; M) = \lim_{\varepsilon\to 0} \frac{|\{y : \overline{v}^{\varepsilon}_{\beta;M}(y,\omega) = 0\}|}{|B_1|} \quad \text{a.s.},$$

where $\overline{v}^{\varepsilon}_{\beta;M}(y,\omega) := \varepsilon^2 v_{\beta,\varepsilon^{-1}B;M}(y/\varepsilon,\omega)$.

Lemma 4.4.17 (Properties of $l(\beta; M)$)**.** *Let $M \in \mathcal{S}^n$.*

(i) $l(\beta; M)$ is non-decreasing function with respect to β.

(ii) If $\beta \leq 0$, then $l(\beta; M) = 0$.

(iii) If $\beta > 0$ is large enough, then $l(\beta; M) > 0$.

Proof. See the proof of Lemma 4.4.8. □

Finally, we let the (non-negative) critical value

$$\beta_0(M) := \sup\{\beta : l(\beta; M) = 0\},$$

which is well-defined by the previous lemma.

4.5 The Properties of Free Solutions and Obstacle Solutions

In short, the argument in the previous section enables us to show the convergence of obstacle functions $\{v_{\beta,\sigma,A}\}_{\sigma>0}$ and free functions $\{w_{\beta,\sigma,A}\}_{\sigma>0}$ when $\sigma \to 0$; so we could define the critical value β_0. Note that, in the Laplacian case, we have a further information such that $w_{\beta,\sigma_1,A} = w_{\beta,\sigma_2,A}$ and $v_{\beta,\sigma_1,A} = v_{\beta,\sigma_2,A}$ if $x \notin \cup_{k\in\mathbb{Z}^n}(B_{\overline{a}_{\sigma_2}}(k))^c$ and $0 < \sigma_1 \leq \sigma_2$.

In this section, we first extract useful properties (namely, (P1) and (P2) in Section 4.1) of limit obstacle solutions and free solutions by investigating the behaviors of approximated solutions (whose parameter is given by σ). Here we should check whether the auxiliary functions are rescaled or not carefully. Then we define the corrector in terms of the critical value β_0 and transport the desired properties for correctors by comparing to the auxiliary functions. Finally, we end up with our main homogenization result employing the correctors.

Again we justify each step above with respect to the Laplacian operator Δ first, and then to the general fully nonlinear operator F.

4.5.1 Laplacian Operator

We begin with the step which illustrates the behavior of an obstacle solution and a free solution away from perforated holes, when $\varepsilon \to 0$. In other words, we are going to show that for the critical value β_0, we have

$$\lim_{\varepsilon\to 0} \overline{w}^{\varepsilon}_{\beta_0} = 0,$$

away from holes (which will be precisely stated later). Note that

(i) we split two cases depending on the value $l(\beta)$, more precisely,

- if $l(\beta) = 0$, i.e. $\overline{v}^{\varepsilon}_{\beta}$ never meet the (zero) obstacle, then we expect that $\overline{v}^{\varepsilon}_{\beta} > 0$ in D;
- if $l(\beta) > 0$, i.e. $\overline{v}^{\varepsilon}_{\beta}$ meets the (zero) obstacle in some region, then we expect that $\overline{v}^{\varepsilon}_{\beta} = 0$ occurs throughout the whole domain (we will prove the "spreading effect" of contact point);

(ii) we first prove for an obstacle solution $\overline{v}^{\varepsilon}_{\beta}$ and transport this information to a free solution $\overline{w}^{\varepsilon}_{\beta}$.

Lemma 4.5.1. *If $l(\beta) = 0$, then $\liminf_{\varepsilon\to 0} \overline{w}^{\varepsilon}_{\beta} \geq 0$ in D.*

Proof. We may assume $D = B_1$ and write $\overline{v}^{\varepsilon}_{\beta,\sigma}(x,\omega) = \varepsilon^2 v_{\beta,\sigma,\varepsilon^{-1}B_1}(x/\varepsilon,\omega)$ and $\overline{w}^{\varepsilon}_{\beta,\sigma}(x,\omega) = \varepsilon^2 w_{\beta,\sigma,\varepsilon^{-1}B_1}(x/\varepsilon,\omega)$. First recall that for each fixed $\varepsilon > 0$, we have $v_{\beta,\sigma,\varepsilon^{-1}D} \nearrow v_{\beta,\varepsilon^{-1}B_1}$ when $\sigma \searrow 0^+$. Moreover, $v_{\beta,\sigma,\varepsilon^{-1}B_1} \neq v_{\beta,\varepsilon^{-1}B_1}$ only can occur in $\cup_{k\in\mathbb{Z}^n} B_{\overline{a}_\sigma}(k)$, where $\overline{a}_\sigma = \sigma^{\frac{2}{n-2}}$. Thus, for any sufficiently small $\sigma > 0$, the coincidence sets are identical;

$$\{\overline{v}^{\varepsilon}_{\beta} = 0\} = \{\overline{v}^{\varepsilon}_{\beta,\sigma} = 0\}. \tag{4.5.1}$$

On the other hand, recalling (4.4.4), we obtain

$$v_{\beta,\sigma,\varepsilon^{-1}B_1}(x) > 0 \quad \text{if } x \in \cup_{k\in\mathbb{Z}^n} B_{\overline{a}_\sigma}(k),$$

which yields

$$\begin{aligned}\Delta(w_{\beta,\sigma,\varepsilon^{-1}B_1} - v_{\beta,\sigma,\varepsilon^{-1}B_1}) &= \left(\beta - \sum_{k\in\mathbb{Z}^n} \gamma(k)\nu_\sigma(x-k)\right)\chi_{\{v_{\beta,\sigma,\varepsilon^{-1}B_1}=0\}} \\ &= \beta\chi_{\{v_{\beta,\sigma,\varepsilon^{-1}B_1}=0\}}.\end{aligned}$$

Applying the Alexandrov-Backelman-Pucci estimate (for exmaple, [14]), we obtain that

$$\begin{aligned}\sup_{B_{\varepsilon^{-1}}}(v_{\beta,\sigma,\varepsilon^{-1}B_1} - w_{\beta,\sigma,\varepsilon^{-1}B_1}) &\le C\varepsilon^{-1}\left(\int_{B_{\varepsilon^{-1}}} (\beta\chi_{\{v_{\beta,\sigma,\varepsilon^{-1}B_1}=0\}})^n\right)^{1/n} \\ &= C\beta\varepsilon^{-1}|\{v_{\beta,\sigma,\varepsilon^{-1}B_1} = 0\} \cap B_{\varepsilon^{-1}}|^{1/n}.\end{aligned}$$

By rescaling, we have

$$\sup_{B_1}(\overline{v}^{\varepsilon}_{\beta,\sigma} - \overline{w}^{\varepsilon}_{\beta,\sigma}) \le C\beta|\{\overline{v}^{\varepsilon}_{\beta,\sigma} = 0\} \cap B_1|^{1/n},$$

or equivalently,

$$\overline{w}^{\varepsilon}_{\beta,\sigma} \ge \overline{v}^{\varepsilon}_{\beta,\sigma} - C\beta|\{\overline{v}^{\varepsilon}_{\beta,\sigma} = 0\}|^{\frac{1}{n}} \ge -C\beta|\{\overline{v}^{\varepsilon}_{\beta,\sigma} = 0\}|^{\frac{1}{n}},$$

in B_1. Letting $\sigma \to 0^+$ and applying (4.5.1), we have $\overline{w}^\varepsilon_\beta \geq -C\beta|\{\overline{v}^\varepsilon_\beta = 0\}|^{\frac{1}{n}}$ in B_1. Finally, since $0 = l(\beta) = \lim_{\varepsilon\to 0} \frac{|\{\overline{v}^\varepsilon_\beta = 0\}|}{|B_1|}$, we conclude that $\liminf_{\varepsilon\to 0} \overline{w}^\varepsilon_\beta \geq 0$ in B_1. $\square$

Next, to estimate the upper bound for $\overline{w}^\varepsilon_\beta$ when $l(\beta) > 0$, we study the quadratic growth property of an obstacle problem. For this purpose, let $u \in L^\infty(D)$ be a non-negative solution of

$$\Delta u = f(x)\chi_{\{u>0\}} \quad \text{in } D, \tag{4.5.2}$$

for $f \in L^\infty(D)$. For an open set $D(u) = \{u > 0\}$, we define the free boundary

$$\Gamma(u) := \partial D(u) \cap D.$$

The following lemma explains the quadratic growth of the solution for an obstacle problem near the free boundary. In other words, the solution has the optimal $C^{1,1}$-regularity.

Lemma 4.5.2 (Quadratic growth)**.** *Let $u \in L^\infty(D)$, $u \geq 0$, satisfy* (4.5.2)*, $x_0 \in \Gamma(u)$, and $B_{2r}(x_0) \subset D$. Then there exists a constant $C = C(n) > 0$ such that*

$$\sup_{B_r(x_0)} u \leq C\|f\|_{L^\infty(D)} r^2.$$

Proof. See the proof of Lemma 4.5.14 which deals with the same result for the fully nonlinear operator. $\square$

Lemma 4.5.3. *If $l(\beta) > 0$, then we have*

$$\lim_{\varepsilon\to 0} \overline{v}^\varepsilon_{\beta,\sigma} = 0 \quad \text{in } D,$$

for each sufficiently small $\sigma > 0$.

Proof. We may assume $D = Q_1$, where $Q_r := [-r/2, r/2]^n$ is a cube of width

$r > 0$. By (4.5.1), we have

$$l_\sigma(\beta) := \lim_{\varepsilon \to 0} \frac{|\{x \in Q_1 : \overline{v}^\varepsilon_{\beta,\sigma} = 0\}|}{|Q_1|} = l(\beta) > 0 \tag{4.5.3}$$

for any sufficiently small $\sigma > 0$. Here we fix a sufficiently small $\sigma > 0$, and for any $m \in \mathbb{N}$, we split Q_1 into 2^{mn} smaller cubes of equal size, whose width is exactly $1/2^m$. For Q being any of these cubes, we have $\overline{v}^\varepsilon_{\beta,\sigma} \geq 0$ on ∂Q. Thus, applying the comparison principle in Q, we have

$$\lim_{\varepsilon \to 0} \frac{|\{x \in Q : \overline{v}^\varepsilon_{\beta,\sigma} = 0\}|}{|Q|} \leq l_\sigma(\beta). \tag{4.5.4}$$

Then from (4.5.3) and (4.5.4), we deduce that for sufficiently small $\varepsilon > 0$, the set $\{x \in Q : \overline{v}^\varepsilon_{\beta,\sigma} = 0\}$ is non-empty for any smaller cubes Q. In other words, we have shown the contact set $\{\overline{v}^\varepsilon_{\beta,\sigma} = 0\}$ "spreads" all over the Q_1.

For a smaller cube Q (with width $1/2^m$) such that $Q \cap \partial Q_1 = \varnothing$, the above result yields that there exists a point $x_0 \in Q$ such that $\overline{v}^\varepsilon_{\beta,\sigma}(x_0) = 0$ for any sufficiently small $\varepsilon > 0$. Recalling the definition of the obstacle solution $\overline{v}^\varepsilon_{\beta,\sigma}$, we have

$$\Delta \overline{v}^\varepsilon_{\beta,\sigma} = \left(\beta - \sum_{k \in \mathbb{Z}^n} \gamma(k) \nu_\sigma(x/\varepsilon - k) \right) \chi_{\{\overline{v}^\varepsilon_{\beta,\sigma} > 0\}} \text{ in } B_1.$$

Thus, by applying Lemma 4.5.2, we have

$$\sup_Q \overline{v}^\varepsilon_{\beta,\sigma} \leq \sup_{B_{\frac{\sqrt{n}}{2^m}}(x_0)} \overline{v}^\varepsilon_{\beta,\sigma} \leq C_n(\beta + c\overline{\gamma}\sigma^{\frac{-2n}{n-2}}) \frac{n}{2^{2m}},$$

for any smaller cubes Q such that $Q \cap \partial Q_1 = \varnothing$. Thus, letting $m \to \infty$ and then choosing sufficiently small $\varepsilon = \varepsilon(m) > 0$, we conclude that $\lim_{\varepsilon \to 0} \overline{v}^\varepsilon_{\beta,\sigma} = 0$ in Q_1. □

Since $\overline{w}^\varepsilon_\beta \leq \overline{v}^\varepsilon_\beta$ by their definitions and $\overline{v}^\varepsilon_{\beta,\sigma} = \overline{v}^\varepsilon_\beta$ in $D \setminus \cup_{k \in \mathbb{Z}^n} B_{\varepsilon \overline{a}_\sigma}(\varepsilon k)$, we deduce the following corollary:

Corollary 4.5.4. *Let $q \in (0, 1/2)$. If $l(\beta) > 0$, then*

$$\lim_{\varepsilon \to 0} \sup_{D \setminus \cup_{k \in \mathbb{Z}^n} B_{q\varepsilon}(\varepsilon k)} \overline{w}^{\varepsilon}_{\beta} \leq 0.$$

Applying Lemma 4.5.1, Corollary 4.5.4 and the fact that β_0 is the critical value, we have:

Corollary 4.5.5. *Let $B_\eta(x_0) \subset D$. Then there exists a sequence $\{\varepsilon_j\}_{j \in \mathbb{N}}$ such that $\varepsilon_j \to 0^+$ and*

$$\lim_{j \to \infty} \overline{w}^{\varepsilon_j}_{\beta_0}(x) = 0,$$

for any $x \in \partial B_\eta(x_0) \cup \{x_0\}$.

Next, we study the asymptotic behavior of auxiliary functions near the boundary of holes ∂T_ε. More precisely, we need to show that

$$\overline{w}^{\varepsilon}_{\beta_0} = 1 + o(1), \quad \text{on } \partial T_\varepsilon.$$

In short, this can be done by comparing the auxiliary functions with the normalized homogeneous solution.

Lemma 4.5.6. *For $k \in \mathbb{Z}^n$, we denote*

$$h_{\beta,\sigma,k}(x) := \frac{\beta}{2n}|x-k|^2 + \gamma(k,\omega)\Phi_\sigma(x-k).$$

(i) For every β and for every $k \in \mathbb{Z}^n$, we have

$$v_{\beta,\sigma,\varepsilon^{-1}D}(x) \geq h_{\beta,\sigma,k}(x) - \frac{\beta}{2n} - r(k,\omega)^{n-2},$$

for all $x \in B_1(k)$ and almost every $\omega \in \Omega$.

(ii) For every $\beta > \beta_0$, we have

$$v_{\beta,\sigma,\varepsilon^{-1}D}(x) \leq h_{\beta,\sigma,k}(x) + o(\varepsilon^{-2}),$$

for all $x \in B_{1/2}(k)$ and almost every $\omega \in \Omega$.

Proof. We refer to [15, Lemma 4.3]. The only difference arises from that we are considering functions with σ- dependence, which does not change the proof. Otherwise, it can also be shown by applying the comparison principle and Lemma 4.5.3; indeed, see the proof of Lemma 4.5.18 later in the fully nonlinear case. $\square$

We denote

$$h_{\beta,k}(x) := \lim_{\sigma \to 0} h_{\beta,\sigma,k}(x) = \frac{\beta}{2n}|x-k|^2 + \gamma(k,\omega)\Phi(x-k),$$

and $h^{\varepsilon}_{\beta,k}(x) := \varepsilon^2 h_{\beta,k}(x/\varepsilon)$. Then since $h_{\beta,k}|_{\partial B_{\overline{a}_\varepsilon(r(k,\omega))}(k)} = \varepsilon^{-2} + \frac{\beta}{2n}|\overline{a}_\varepsilon(r(k,\omega))|^2$, we deduce the following corollary by letting $\sigma \to 0^+$ in Lemma 4.5.6:

Corollary 4.5.7. *(i) For every β and $k \in \mathbb{Z}^n$ such that $r(k,\omega) > 0$, we have*

$$v_{\beta,\varepsilon^{-1}D}(x) \geq \varepsilon^{-2} + O(1) \quad \text{on } \partial B_{\overline{a}^\varepsilon(r(k,\omega))}(k) \text{ a.e. } \omega \in \Omega,$$

and so

$$\overline{v}^{\varepsilon}_{\beta}(x) \geq 1 + o(1) \quad \text{on } \partial T_\varepsilon(\omega) \text{ a.e. } \omega \in \Omega,$$

for all β.

(ii) For every $\beta > \beta_0$ and every $k \in \mathbb{Z}^n$, we have

$$v_{\beta,\varepsilon^{-1}D}(x) \leq \varepsilon^{-2} + o(\varepsilon^{-2}) \quad \text{on } \partial B_{\overline{a}^\varepsilon(r(k,\omega))}(k) \text{ a.e. } \omega \in \Omega,$$

and so

$$\overline{v}^{\varepsilon}_{\beta}(x) \leq 1 + o(1) \quad \text{on } \partial T_\varepsilon(\omega) \text{ a.e. } \omega \in \Omega,$$

for all $\beta > \beta_0$.

Lemma 4.5.8. *For every $k \in \mathbb{Z}^n$, $\overline{w}^\varepsilon_{\beta_0}$ satisfies*

$$h^\varepsilon_{\beta_0,k}(x) - o(1) \le \overline{w}^\varepsilon_{\beta_0}(x) \le h^\varepsilon_{\beta_0,k}(x) + o(1) \quad \forall x \in B_{\varepsilon/2}(\varepsilon k) \cap D \text{ a.e. } \omega \in \Omega.$$

In particular,

$$\overline{w}^\varepsilon_{\beta_0} = 1 + o(1) \quad \text{on } \partial T_\varepsilon \cap D.$$

Proof. Recall that for every β, we denote $\overline{v}^\varepsilon_\beta(x) = \varepsilon^2 v_{\beta,\varepsilon^{-1}D}(x/\varepsilon)$, which is defined in D and $\overline{v}^\varepsilon_\beta = 0$ on ∂D.

(i) Let $\beta > \beta_0$. Note that

$$\Delta w_{\beta_0,\sigma,\varepsilon^{-1}D} = \beta_0 - \sum_{k \in \mathbb{Z}^n \cap \varepsilon^{-1}D} \gamma(k,\omega)\nu_\sigma(x-k),$$

and

$$\Delta v_{\beta,\sigma,\varepsilon^{-1}D} = \left(\beta - \sum_{k \in \mathbb{Z}^n \cap \varepsilon^{-1}D} \gamma(k,\omega)\nu_\sigma(x-k)\right)\chi_{\{v_{\beta,\sigma,\varepsilon^{-1}D}>0\}},$$

in $\varepsilon^{-1}D$. Thus, we have

$$\Delta(w_{\beta_0,\sigma,\varepsilon^{-1}D} - v_{\beta,\sigma,\varepsilon^{-1}D}) \ge \beta_0 - \beta$$

and $w_{\beta_0,\sigma,\varepsilon^{-1}D} - v_{\beta,\sigma,\varepsilon^{-1}D} = 0$ on $\partial(\varepsilon^{-1}D)$. By rescaling, we obtain

$$\Delta(\overline{w}^\varepsilon_{\beta_0,\sigma} - \overline{v}^\varepsilon_{\beta,\sigma}) \ge \beta_0 - \beta$$

and $\overline{w}^\varepsilon_{\beta_0,\sigma} - \overline{v}^\varepsilon_{\beta,\sigma} = 0$ on ∂D. The Green representation formula yields that

$$\begin{aligned}\overline{w}^\varepsilon_{\beta_0,\sigma}(x_0) - \overline{v}^\varepsilon_{\beta,\sigma}(x_0) &\le \int_D G(x,x_0)(\beta_0-\beta)\,\mathrm{d}x \\ &\le (\beta-\beta_0)\int_D \Phi(x-x_0)\,\mathrm{d}x \le O(\beta-\beta_0),\end{aligned}$$

where $G(\cdot,\cdot)$ is the Green function on D. ($\Delta G(\cdot, x_0) = \delta_{x_0}$ and $G = 0$ on ∂D.) Applying Lemma 4.5.6 (ii), we conclude that

$$\overline{w}^{\varepsilon}_{\beta_0,\sigma}(x) \leq \varepsilon^2 h_{\beta,\sigma,k}(x/\varepsilon) + o(1) + O(\beta - \beta_0)$$

for all $x \in B_{\varepsilon/2}(\varepsilon k)$. Letting $\sigma \to 0$ and then $\beta \to \beta_0$, we obtain the desired upper bound.

(ii) Arguing as in (i), for every $\beta \leq \beta_0$, we have

$$\Delta(v_{\beta,\sigma,\varepsilon^{-1}D} - w_{\beta_0,\sigma,\varepsilon^{-1}D}) = \beta - \beta_0 - \beta\chi_{\{v_{\beta,\sigma,\varepsilon^{-1}D}=0\}}.$$

By rescaling, we obtain that

$$\Delta(\overline{v}^{\varepsilon}_{\beta,\sigma} - \overline{w}^{\varepsilon}_{\beta_0,\sigma}) = \beta - \beta_0 - \beta\chi_{\{\overline{v}^{\varepsilon}_{\beta,\sigma}=0\}}$$

Again, the Green representation formula leads to

$$\overline{v}^{\varepsilon}_{\beta,\sigma} - \overline{w}^{\varepsilon}_{\beta_0,\sigma} \leq O(\beta_0 - \beta) + C\beta|\{\overline{v}^{\varepsilon}_{\beta,\sigma} = 0\}|^{1/(n-1)} \quad \text{in } D.$$

Applying Lemma 4.5.6 (i), we conclude that

$$\overline{w}^{\varepsilon}_{\beta_0,\sigma}(x) \geq \varepsilon^2 h_{\beta,\sigma,k}(x/\varepsilon) - o(1) - O(\beta_0 - \beta) - C\beta|\{\overline{v}^{\varepsilon}_{\beta,\sigma} = 0\}|^{1/(n-1)}.$$

Letting $\sigma \to 0$, (see (4.5.1))

$$\overline{w}^{\varepsilon}_{\beta_0}(x) \geq h^{\varepsilon}_{\beta,k}(x) - o(1) - O(\beta_0 - \beta) - C\beta|\{\overline{v}^{\varepsilon}_{\beta} = 0\}|^{1/(n-1)}.$$

Since $\lim_{\varepsilon\to 0} |\{\overline{v}^{\varepsilon}_{\beta} = 0\}| = l(\beta) \cdot |D| = 0$ for any $\beta \leq \beta_0$, we obtain the desired lower bound.

$\square$

Now we define a corrector:

$$\begin{cases} \Delta w^{\varepsilon}(x,\omega) = \beta_0 & \text{for } x \in D \setminus T_{\varepsilon}, \\ w^{\varepsilon}(x,\omega) = 1 & \text{for } x \in \partial T_{\varepsilon}, \\ w^{\varepsilon}(x,\omega) = 0 & \text{for } x \in \partial D \setminus T_{\varepsilon}. \end{cases} \tag{4.5.5}$$

Lemma 4.5.9. *Let $B_\eta(x_0) \subset D$. Then there exists a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ such that $\varepsilon_j \to 0^+$ and*

$$\lim_{j\to\infty} w^{\varepsilon_j}(x) = 0,$$

for any $x \in \partial B_\eta(x_0) \cup \{x_0\}$.

Proof. Since $\Delta \overline{w}^{\varepsilon}_{\beta_0,\sigma} = \beta_0$ in $D \setminus \cup_k B_{\varepsilon \overline{a}_\sigma}(\varepsilon k)$, letting $\sigma = \varepsilon$ yields that

$$\Delta \overline{w}^{\varepsilon}_{\beta_0} = \Delta \overline{w}^{\varepsilon}_{\beta_0,\varepsilon} = \beta_0 \quad \text{in } D \setminus \cup_k B_{a^\varepsilon}(\varepsilon k) = D \setminus T_\varepsilon,$$

Recalling Lemma 4.5.8 and applying the comparison principle for w^ε and $\overline{w}^{\varepsilon}_{\beta_0}$ in $D \setminus T_\varepsilon$, we have

$$\overline{w}^{\varepsilon}_{\beta_0}(x) - o(1) \le w^\varepsilon(x) \le \overline{w}^{\varepsilon}_{\beta_0}(x) + o(1) \quad \text{in } D \setminus T_\varepsilon.$$

Therefore, the desired result follows from Corollary 4.5.5. $\square$

Finally, we are ready to finish the proof of our main theorem for the Laplacian case:

Proof of Theorem 4.1.1 (ii). We are going to show that u is a subsolution. Let us assume there is a parabola P touching u from above at x_0 and

$$\Delta P + \beta_0(\varphi(x_0) - P(x_0))_+ < -2\mu_0 < 0.$$

In a small neighborhood of x_0, $B_\eta(x_0)$, there exists another parabola Q such that

$$\begin{cases} D^2P < D^2Q & \text{in } B_\eta(x_0), \\ P(x_0) > Q(x_0) + \delta_0, & \\ P(x) < Q(x) & \text{on } \partial B_\eta(x_0). \end{cases}$$

In addition, we can choose a sufficinetly small $\varepsilon_0 > 0$ so that Q satisfies

$$\Delta Q + \beta_0(\varphi(x_0) - u(x_0) + 3\varepsilon_0) =: \Delta Q + \beta_0\xi_0 < -\mu_0 < 0,$$

and

$$|Q(x) - Q(x_0)| + |\varphi(x) - \varphi(x_0)| < \varepsilon_0$$

for $x \in B_\eta(x_0)$. Let us consider $Q_\varepsilon(x) := Q(x) + w^\varepsilon(x)\xi_0$. Then by the definition of w^ε, we have $\Delta Q_\varepsilon < -\mu_0 < 0$ and

$$Q_\varepsilon(x) = Q(x) + \xi_0 = Q(x) + \varphi(x_0) - u(x_0) + 3\varepsilon_0 > \varphi(x),$$

on $T_{a^\varepsilon} \cap B_\eta(x_0)$. Therefore, the maximum principle yields that $Q_\varepsilon \geq \varphi_\varepsilon$ in $B_\eta(x_0)$.

Now we define the function

$$v_\varepsilon := \begin{cases} \min\{u_\varepsilon, Q_\varepsilon\} & \text{in } B_\eta(x_0), \\ u_\varepsilon & \text{in } D \setminus B_\eta(x_0). \end{cases}$$

Applying Lemma 4.5.9, for sufficiently small $\varepsilon > 0$ (at least for a subsequence $\{\varepsilon_j\}$), we have $Q_\varepsilon > u_\varepsilon$ on $\partial B_\eta(x_0)$. Thus, the function v_ε is well-defined and will be a viscosity supersolution of (L_ε). Since u_ε is the least viscosity supersolution, we have $u_\varepsilon \leq v_\varepsilon \leq Q_\varepsilon$ in $B_\eta(x_0)$. Letting $\varepsilon \to 0$, we have

$$u(x_0) \leq Q(x_0) < P(x_0) = u(x_0),$$

which is a contradiction. By an argument similar to the proof of Lemma 4.1

in [12], we can show that u is also a viscosity supersolution of $(\overline{L})$. (or see the proof of Theorem 4.1.2 (ii) which deals with the argument for a viscosity supersolution of $(\overline{F})$.)

□

4.5.2 Fully Nonlinear Operator

In the Laplacian case, the strong properties of the obstacle solutions (see Lemma 4.4.5 and its proof) immediately led to the equality between coincidence sets, (4.5.1): for any sufficiently small $\sigma > 0$,

$$\{\overline{v}^{\varepsilon}_{\beta,\sigma} = 0\} = \{\overline{v}^{\varepsilon}_{\beta} = 0\}.$$

On the other hand, in the fully nonlinear case, we only have the uniform convergence of the obstacle solutions and we require some auxiliary lemmas to derive the stability of coincidence sets, which is a weaker consequence compared to (4.5.1).

We begin with a simple lemma:

Lemma 4.5.10. *For an open set $D \subset \mathbb{R}^n$, let $\{u_m\}_{m=1}^{\infty}$ and u_0 be continuous functions on D. If $u_m \to u_0$ uniformly on every compact subset of D as $m \to \infty$, then*

$$\limsup_{m\to\infty} |\{u_m = 0\}| \leq |\{u_0 = 0\}|.$$

Proof. Suppose that

$$\limsup_{m\to\infty} |\{u_m = 0\}| > |\{u_0 = 0\}|.$$

Then there exist an open neighborhood A of $\{u_0 = 0\}$ and a compact set $K \subset\subset D$ such that $(\{u_m = 0\} \cap K) \setminus A$ is non-empty (upto subsequence, if necessary). In other words, there exists a sequence of points $x_m \in K$ satisfying $x_m \in \{u_m = 0\} \setminus A$. Again, upto subsequence, there exists a point

$x_0 \in K \setminus A$ such that $x_m \to x_0$. Then for any $\varepsilon > 0$, we have

$$|u_m(x_m) - u_0(x_0)| \leq |u_m(x_m) - u_0(x_m)| + |u_0(x_m) - u_0(x_0)| < \varepsilon,$$

for sufficiently large m, which yields that

$$u_0(x_0) = \lim_{m\to\infty} u_m(x_m) = 0.$$

This contradicts to $x_0 \notin \{u_0 = 0\}$ and so we have the desired inequality. $\square$

Next, for the other direction of the inequality obtained from Lemma 4.5.10, we need an additional work in terms of obstacle problem theory. Let u be a non-negative solution of an obstacle problem $F(D^2u) = f(x)\chi_{\{u>0\}}$ in D, for $f \in L^\infty(D)$. By Theorem 1.2.1 in [56], we have $u \in C^{1,1}_{\text{loc}}(D) \cap C(\overline{D})$. We set $D(u) := \{x \in D : u(x) > 0\}$, $C(u) := \{x \in D : u(x) = 0\}$, and $\Gamma(u) := \partial D(u) \cap D$.

Lemma 4.5.11 (Non-degeneracy; [56, Lemma 3.4])**.** *Suppose that $f \geq M > 0$ in D and let x_0 be any point in $\overline{D(u)}$. Then for any ball $B_r(x_0) \subset D$,*

$$\sup_{B_r(x_0)} [u(x) - u(x_0)] \geq \frac{Mr^2}{2n\Lambda}.$$

Lemma 4.5.12. *Suppose that $f \geq M > 0$ in D and u_0, u_m, $m \in \mathbb{N}$ are non-negative solutions of $F(D^2v) = f(x)\chi_{\{v>0\}}$ in D. If $u_m \to u_0$ uniformly in every compact subset of D, then we have*

(i) $\limsup_{m\to\infty} D(u_m) \subset \overline{D(u_0)}$;

(ii) $\limsup_{m\to\infty} |D(u_m)| \leq |\overline{D(u_0)}|$;

(iii) $\liminf_{m\to\infty} |C(u_m)| \geq |C(u_0)|$.

where $\limsup_{m\to\infty} A_m$ means the set of all limit points of sequences $\{x_m\}$, $x_m \in A_m$.

Proof. (i) It is a consequence of Lemma 4.5.11; see Corollary 3.5 in [56].

(ii) Its proof is similar to the one of Lemma 4.5.10. Indeed, we suppose that

$$\limsup_{m\to\infty} |D(u_m)| > |\overline{D(u_0)}|.$$

Then there exists an open neighborhood A of $\overline{D(u_0)}$ such that there is a sequence of points $x_m \in D$ (upto subsequence, if necessary) satisfying $x_m \in D(u_m) \setminus A$. Then there is a point $x_0 \in D$ such that $x_m \to x_0$, which implies that

$$x_0 \in \limsup_{m\to\infty} D(u_m) \setminus A \subset \limsup_{m\to\infty} D(u_m) \setminus \overline{D(u_0)}.$$

This contradicts to (i).

(iii) Since $D(u_m) = D \setminus C(u_m)$ and $\overline{D(u_0)} = (D \setminus C(u_0)) \cup \Gamma(u_0)$,

$$\liminf_{m\to\infty} |C(u_m)| \geq |C(u_0)| - |\Gamma(u_0)|$$

follows from (ii). Note that when F is positively homogeneous of degree one, the free boundary $\Gamma(u_0)$ is a $C^{1,\alpha}$-graph; see [56, Theorem 3.3]. In particular, $|\Gamma(u_0)| = 0$ and thus, the desired inequality follows.

□

Combining the results from Lemma 4.5.10 and Lemma 4.5.12 (iii), we have the *stability of coincidence sets*; i.e. since $v^\varepsilon_{\beta,\sigma}$ converges to v^ε_β uniformly on every compact subset of D as $\sigma \to 0$, we have

$$\lim_{\sigma\to 0} |\{v^\varepsilon_{\beta,\sigma} = 0\}| = |\{v^\varepsilon_\beta = 0\}|. \tag{4.5.6}$$

Now, we proceed similarly as in the Laplacian case:

Lemma 4.5.13. *If* $l(\beta) = 0$, *then* $\liminf_{\varepsilon\to 0} \overline{w}^\varepsilon_\beta \geq 0$ *in* D.

Proof. We may assume $D = B_1$ and write $\overline{v}^\varepsilon_{\beta,\sigma}(x,\omega) = \varepsilon^2 v_{\beta,\sigma,\varepsilon^{-1}B_1}(x/\varepsilon,\omega)$ and $\overline{w}^\varepsilon_{\beta,\sigma}(x,\omega) = \varepsilon^2 w_{\beta,\sigma,\varepsilon^{-1}B_1}(x/\varepsilon,\omega)$. Recalling the proof for Lemma 4.4.5, we obtain $v_{\beta,\sigma,\varepsilon^{-1}B_1}(x) > 0$ if $x \in \cup_{k\in\mathbb{Z}^n} B_{\overline{a}_\sigma}(k)$, which yields

$$\begin{aligned} &F(M + D^2 w_{\beta,\sigma,\varepsilon^{-1}B_1}) - F(M + D^2 v_{\beta,\sigma,\varepsilon^{-1}B_1}) \\ &\qquad = \left(\beta - \sum_{k\in\mathbb{Z}^n} \gamma(k)\nu_\sigma(x-k)\right) \chi_{\{v_{\beta,\sigma,\varepsilon^{-1}B_1}=0\}} = \beta\chi_{\{v_{\beta,\sigma,\varepsilon^{-1}B_1}=0\}}. \end{aligned}$$

Now let h be the solution of

$$\begin{cases} \mathcal{P}^+(D^2 h) = -\beta\chi_{\{v_{\beta,\sigma,\varepsilon^{-1}B_1}=0\}} & \text{in } \varepsilon^{-1}B_1, \\ h = 0 & \text{on } \partial(\varepsilon^{-1}B_1). \end{cases}$$

Then the uniformly ellipticity of F yields that

$$\begin{aligned} F(M + D^2(w_{\beta,\sigma,\varepsilon^{-1}B_1} + h)) &\le F(M + D^2 w_{\beta,\sigma,\varepsilon^{-1}B_1}) + \mathcal{P}^+(D^2 h) \\ &= F(M + D^2 v_{\beta,\sigma,\varepsilon^{-1}B_1}), \end{aligned}$$

and thus, we have $w_{\beta,\sigma,\varepsilon^{-1}B_1} + h \ge v_{\beta,\sigma,\varepsilon^{-1}B_1}$ in $\varepsilon^{-1}B_1$ by the comparison principle. (for example, see [18]) Applying the Alexandrov-Backelman-Pucci estimate for h, we obtain that

$$\begin{aligned} \sup_{B_{\varepsilon^{-1}}} h &\le C\varepsilon^{-1} \left(\int_{B_{\varepsilon^{-1}}} (\beta\chi_{\{v_{\beta,\sigma,\varepsilon^{-1}B_1}=0\}})^n \right)^{1/n} \\ &= C\beta\varepsilon^{-1} |\{v_{\beta,\sigma,\varepsilon^{-1}B_1} = 0\} \cap B_{\varepsilon^{-1}}|^{1/n}. \end{aligned}$$

Now the remaining part is the same as in the proof of Lemma 4.5.1. The only difference arises when applying the stability of coincidence sets, (4.5.6). □

Similarly as in the Laplacian case (Lemma 4.5.2), one can prove the quadratic growth property in the fully nonlinear case:

Lemma 4.5.14 (Quadratic growth; fully nonlinear operator)**.** *Let u be a non-negative solution of an obstacle problem, i.e. u satisfies*

$$F(D^2 u) = f(x)\chi_{\{u>0\}} \quad \textit{in } D,$$

for $f \in L^\infty(D)$*. If* $x_0 \in \Gamma(u)$*, and* $B_{2r}(x_0) \subset D$*, then*

$$\sup_{B_r(x_0)} u \le C(n, \lambda, \Lambda)\|f\|_{L^\infty(D)} r^2.$$

Proof. For simplicity, we may assume $x_0 = 0$ and write $B_R = B_R(0)$ for any $R > 0$. Then we split u into the sum $u_1 + u_2$ in B_{2r}, where

$$\begin{aligned} &F(D^2 u_1) = F(D^2 u), \quad \mathcal{P}^+(D^2 u_2) = 0 \quad \text{in } B_{2r}; \\ &u_1 = 0, \quad u_2 = u \quad \text{on } \partial B_{2r}. \end{aligned}$$

We estimate these functions u_1 and u_2 separately.

(i) To estimate u_1, we consider a barrier function

$$g_-(x) := \frac{1}{2n}(4r^2 - |x|^2).$$

Then we immediately obtain

$$F(D^2 g_-) = F\left(-\frac{1}{n} I\right) \le -\lambda$$

in B_{2r} and $g_- = 0$ on ∂B_{2r}. Thus, the comparison principle yields that for $x \in B_{2r}$,

$$u_1(x) \le \frac{M}{\lambda} g_-(x) \le C(n, \lambda) M r^2,$$

where $M := \|f\|_{L^\infty(D)}$. Considering $g_+(x) := \frac{1}{2n}(|x|^2 - 4r^2)(= -g_-(x))$, we can conclude that

$$|u_1(x)| \le C(n, \lambda) M r^2.$$

(ii) To estimate u_2, note that $u_2 \ge 0$ in B_{2r} since $F(D^2 u_2) = 0$ in B_{2r} and

$u_2 = u \geq 0$ on ∂B_{2r}. Moreover, since $0 \in \Gamma(u)$, we have

$$u_2(0) = -u_1(0) \leq C(n, \lambda) M r^2,$$

by the previous result. Thus, applying Harnack inequality to a non-negative function u_2 in B_{2r}, we conclude that for $x \in B_r$,

$$u_2(x) \leq C(n, \lambda, \Lambda) u_2(0) \leq C(n, \lambda, \Lambda) M r^2.$$

Finally, combining the estimates for u_1 and u_2, we obtain the desired estiamtes for u.

□

Lemma 4.5.15. *If $l(\beta) > 0$, then we have*

$$\lim_{\varepsilon \to 0} \overline{v}^{\varepsilon}_{\beta,\sigma} = 0 \quad \textit{in } D,$$

for each sufficiently small $\sigma > 0$.

Proof. Its proof can be done by following the proof of Lemma 4.5.3. Note that here we use the stability of coincidence sets and the quadratic separation occurred at the contact point. □

Since $\overline{w}^{\varepsilon}_{\beta} \leq \overline{v}^{\varepsilon}_{\beta}$ by their definitions and $\overline{v}^{\varepsilon}_{\beta,\sigma}$ uniformly converges to $\overline{v}^{\varepsilon}_{\beta}$ on every compact subset of $D \setminus \cup_{k \in \mathbb{Z}^n} (\varepsilon k)$, we deduce the following corollary:

Corollary 4.5.16. *Let $q \in (0, 1/2)$. If $l(\beta) > 0$, then*

$$\lim_{\varepsilon \to 0} \sup_{D \setminus \cup_{k \in \mathbb{Z}^n} B_{q\varepsilon}(\varepsilon k)} \overline{w}^{\varepsilon}_{\beta} \leq 0.$$

Applying Lemma 4.5.13, Corollary 4.5.16 and the fact that β_0 is the critical value, we have:

Corollary 4.5.17. *Let $B_\eta(x_0) \subset D$. Then there exists a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ such that $\varepsilon_j \to 0^+$ and*

$$\lim_{j\to\infty} \overline{w}^{\varepsilon_j}_{\beta_0}(x) = 0,$$

for any $x \in \partial B_\eta(x_0) \cup \{x_0\}$.

Lemma 4.5.18. *For $k \in \mathbb{Z}^n$ and $M \in \mathcal{S}^n$, we denote*

$$h^-_{\beta,\sigma,k;M}(x) := \frac{\beta + F(M)}{2n\lambda}|x-k|^2 + \gamma(k,\omega)\Phi_\sigma(x-k) - \frac{1}{2}(x-k)^T M(x-k),$$

and

$$h^+_{\sigma,k;M}(x) := \frac{F(M)}{2n\Lambda}|x-k|^2 + \gamma(k,\omega)\Phi_\sigma(x-k) - \frac{1}{2}(x-k)^T M(x-k).$$

(i) For every β, we have

$$v_{\beta,\sigma,\varepsilon^{-1}D;M}(x) \geq h^-_{\beta,\sigma,k;M}(x) - \frac{\beta}{8n\lambda} - (2r(k,\omega))^{\alpha^*} - \frac{1}{8}\|M\|,$$

for all $x \in B_{1/2}(k)$ and almost every $\omega \in \Omega$.

(ii) For every $\beta > \beta_0$, we have

$$v_{\beta,\sigma,\varepsilon^{-1}D;M}(x) \leq h^+_{\sigma,k;M}(x) + o(\varepsilon^{-2}),$$

for all $x \in B_{1/2}(k)$ and almost every $\omega \in \Omega$.

Proof. (i) By a direct calculation, for $x \in B_{1/2}(k)$, we have

$$\begin{aligned} F(M + D^2 h^-_{\beta,\sigma,k;M}) &= F\left(\frac{\beta + F(M)}{n\lambda} I + \gamma(k,\omega) D^2\Phi_\sigma(x-k)\right) \\ &\geq \beta + F(M) - \gamma(k,\omega)\nu_\sigma(x-k) \\ &\geq F(M + D^2 v_{\beta,\sigma,\varepsilon^{-1}D;M}). \end{aligned}$$

Thus, the comparison principle yields the desired inequality.

(ii) Combining Theorem 4.4.15 (uniform convergence on every compact subset of $D \setminus \cup_{k\in\mathbb{Z}^n}\{\varepsilon k\}$) and Lemma 4.5.15, we have (after rescaling)

$$v_{\beta,\sigma,\varepsilon^{-1}D;M} = o(\varepsilon^{-2}),$$

if $x \in \varepsilon^{-1}D \setminus \cup_{k\in\mathbb{Z}^n} B_{1/2}(k)$. On the other hand, in $B_{1/2}(k)$,

$$\begin{aligned} F(M + D^2 h^+_{\sigma,k;M}) &= F\left(\frac{F(M)}{n\Lambda} I + \gamma(k,\omega) D^2\Phi_\sigma(x-k)\right) \\ &\le F(M) - \gamma(k,\omega)\nu_\sigma(x-k) \le F(M + D^2 v_{\beta,\sigma,\varepsilon^{-1}D;M}). \end{aligned}$$

Moreover, there exists a constant $L = L(n,F,M) \ge 0$ (independent of k) such that $h^+_{\beta,\sigma,k;M}+L \ge 0$ on $\partial B_{1/2}(k)$. Thus, again by the comparison principle, we conclude the desired inequality.

□

We denote

$$h^-_{\beta,k;M}(x) := \lim_{\sigma\to 0} h^-_{\beta,\sigma,k;M}(x), \quad h^+_{k;M}(x) := \lim_{\sigma\to 0} h^+_{\sigma,k;M}(x),$$

and

$$h^{-,\varepsilon}_{\beta,k;M}(x) := \varepsilon^2 h^-_{\beta,k;M}(x/\varepsilon) \quad h^{+,\varepsilon}_{k;M}(x) := \varepsilon^2 h^+_{k;M}(x/\varepsilon).$$

Recalling Assumption 4.2.1, we have

$$\gamma(k,\omega)(\overline{a}^\varepsilon)^{-\alpha^*} = \varepsilon^{-2},$$

where $\overline{a}^\varepsilon = a^\varepsilon(r(k,\omega))/\varepsilon$. Thus, we deduce

$$h^-_{\beta,k;M}|_{\partial B_{\overline{a}_\varepsilon(r(k,\omega))}(k)} = \varepsilon^{-2}\phi(\theta) + O(1), \quad h^+_{k;M}|_{\partial B_{\overline{a}_\varepsilon(r(k,\omega))}(k)} = \varepsilon^{-2}\phi(\theta) + O(1)$$

which yields the following corollary after letting $\sigma \to 0^+$ in Lemma 4.5.18:

Corollary 4.5.19. *(i) For every β and $k \in \mathbb{Z}^n$ such that $r(k,\omega) > 0$, we* *have*

$$v_{\beta,\varepsilon^{-1}D;M}(x) \geq \varepsilon^{-2}\phi(\theta) + O(1) \quad on\ \partial B_{\overline{a}^{\varepsilon}(r(k,\omega))}(k)\ a.e.\ \omega \in \Omega,$$

and so

$$\overline{v}^{\varepsilon}_{\beta;M}(x) \geq \phi(\theta) + o(1) \quad on\ \partial T_{\varepsilon}(\omega)\ a.e.\ \omega \in \Omega,$$

for all β*.*

(ii) For every $\beta > \beta_0$ *and every* $k \in \mathbb{Z}^n$*, we have*

$$v_{\beta,\varepsilon^{-1}D;M}(x) \leq \varepsilon^{-2}\phi(\theta) + o(\varepsilon^{-2}) \quad on\ \partial B_{\overline{a}^{\varepsilon}(r(k,\omega))}(k)\ a.e.\ \omega \in \Omega,$$

and so

$$\overline{v}^{\varepsilon}_{\beta;M}(x) \leq \phi(\theta) + o(1) \quad on\ \partial T_{\varepsilon}(\omega)\ a.e.\ \omega \in \Omega,$$

for all $\beta > \beta_0$*.*

Lemma 4.5.20. *For every* $k \in \mathbb{Z}^n$*,* $\overline{w}^{\varepsilon}_{\beta_0}$ *satisfies*

$$h^{-,\varepsilon}_{\beta_0,k;M}(x) - o(1) \leq \overline{w}^{\varepsilon}_{\beta_0;M}(x) \leq h^{+,\varepsilon}_{k;M}(x) + o(1) \quad \forall x \in B_{\varepsilon/2}(\varepsilon k) \cap D.$$

In particular,

$$\overline{w}^{\varepsilon}_{\beta_0;M} = \phi(\theta) + o(1) \quad on\ \partial T_{\varepsilon} \cap D.$$

Proof. For simplicity, we drop the subscript M in this proof. Recall that for every β, we denote

$$\overline{v}^{\varepsilon}_{\beta}(x) = \varepsilon^2 v_{\beta,\varepsilon^{-1}D}(x/\varepsilon),$$

which is defined in D and $\overline{v}^{\varepsilon}_{\beta} = 0$ on ∂D. Compared to the Laplacian case (Lemma 4.5.8), we cannot exploit the Green representation formula when estimating an auxiliary function. Instead, we will use the Alexandrov-Backelman-Pucci estimate.

(i) Let $\beta > \beta_0$. Note that

$$F(M + D^2 w_{\beta_0,\sigma,\varepsilon^{-1}D}) = F(M) + \beta_0 - \sum_{k\in\mathbb{Z}^n\cap\varepsilon^{-1}D} \gamma(k,\omega)\nu_\sigma(x-k),$$

and

$$\begin{aligned}
&F(M + D^2 v_{\beta,\sigma,\varepsilon^{-1}D}) \\
&\quad = F(M) + \left(\beta - \sum_{k\in\mathbb{Z}^n\cap\varepsilon^{-1}D} \gamma(k,\omega)\nu_\sigma(x-k)\right)\chi_{\{v_{\beta,\sigma,\varepsilon^{-1}D}>0\}} \\
&\quad = F(M) + \beta\chi_{\{v_{\beta,\sigma,\varepsilon^{-1}D}>0\}} - \sum_{k\in\mathbb{Z}^n\cap\varepsilon^{-1}D} \gamma(k,\omega)\nu_\sigma(x-k) \\
&\quad \le F(M) + \beta - \sum_{k\in\mathbb{Z}^n\cap\varepsilon^{-1}D} \gamma(k,\omega)\nu_\sigma(x-k).
\end{aligned}$$

Now let h be the solution of

$$\begin{cases} \mathcal{P}^+(D^2 h) = \beta_0 - \beta & \text{in } \varepsilon^{-1}D, \\ h = 0 & \text{on } \partial(\varepsilon^{-1}D). \end{cases}$$

Since

$$\begin{aligned}
F(M + D^2 v_{\beta,\sigma,\varepsilon^{-1}D} + D^2 h) &\le F(M + D^2 v_{\beta,\sigma,\varepsilon^{-1}D}) + \mathcal{P}^+(D^2 h) \\
&\le F(M + D^2 w_{\beta_0,\sigma,\varepsilon^{-1}D}),
\end{aligned}$$

the comparison principle leads to $v_{\beta,\sigma,\varepsilon^{-1}D} + h \ge w_{\beta_0,\sigma,\varepsilon^{-1}D}$. Then an application of Alexandrov-Backelman-Pucci estimate for h indicates that

$$\sup_{\varepsilon^{-1}D} h \le C\varepsilon^{-1}\mathrm{diam}D\|\beta-\beta_0\|_{L^n(\varepsilon^{-1}D)} \le C(\varepsilon^{-1}\mathrm{diam}D)^2(\beta-\beta_0).$$

Thus, by rescaling, we conclude that

$$\overline{w}^{\varepsilon}_{\beta_0,\sigma} \leq \overline{v}^{\varepsilon}_{\beta,\sigma} + C(\mathrm{diam}D)^2(\beta - \beta_0).$$

Applying Lemma 4.5.18 (ii), we conclude that

$$\overline{w}^{\varepsilon}_{\beta_0,\sigma}(x) \leq \varepsilon^2 h^{+}_{\sigma,k}(x/\varepsilon) + o(1) + C(\mathrm{diam}D)^2(\beta - \beta_0)$$

for all $x \in B_{\varepsilon/2}(\varepsilon k)$. Letting $\sigma \to 0$ and then $\beta \to \beta_0$, we obtain the desired upper bound.

(ii) Arguing as in (i), for every $\beta \leq \beta_0$, let h be the solution of

$$\begin{cases} \mathcal{P}^+(D^2 h) = \beta - \beta_0 - \beta\chi_{\{v_{\beta,\sigma,\varepsilon^{-1}D}=0\}} & \text{in } \varepsilon^{-1}D, \\ h = 0 & \text{on } \partial(\varepsilon^{-1}D). \end{cases}$$

Then since

$$F(M + D^2 w_{\beta_0,\sigma,\varepsilon^{-1}D} + D^2 h) \leq F(M + D^2 v_{\beta,\sigma,\varepsilon^{-1}D}),$$

we have $w_{\beta_0,\sigma,\varepsilon^{-1}D} + h \geq v_{\beta,\sigma,\varepsilon^{-1}D}$. Again, the Alexandrov-Backelman-Pucci estimate yields that

$$\sup_{\varepsilon^{-1}D} h \leq C(\varepsilon^{-1}\mathrm{diam}D)^2 \left[(\beta_0 - \beta) + \left(\frac{|\{v_{\beta,\sigma,\varepsilon^{-1}D} = 0\}|}{|\varepsilon^{-1}D|}\right)^{1/n}\right].$$

By rescaling, we obtain that

$$\overline{w}^{\varepsilon}_{\beta_0,\sigma} \geq \overline{v}^{\varepsilon}_{\beta,\sigma} - C(\mathrm{diam}D)^2 \left[(\beta_0 - \beta) + \left(\frac{|\{\overline{v}^{\varepsilon}_{\beta,\sigma} = 0\}|}{|D|}\right)^{1/n}\right].$$

Applying Lemma 4.5.18 (i) and letting $\sigma \to 0$,

$$\begin{aligned}\overline{w}^{\varepsilon}_{\beta_0}(x) &\geq \varepsilon^2 h^{-}_{\beta,k}(x/\varepsilon) - o(1)\\ &\qquad - C(\mathrm{diam} D)^2 \left[(\beta_0-\beta) + \left(\frac{|\{\overline{v}^{\varepsilon}_{\beta}=0\}|}{|D|}\right)^{1/n}\right].\end{aligned}$$

Since $\lim_{\varepsilon\to 0}\frac{|\{\overline{v}^{\varepsilon}_{\beta}=0\}|}{|D|} = l(\beta) = 0$ for any $\beta \leq \beta_0$, we obtain the desired lower bound.

□

For each symmetric matrix $M \in \mathcal{S}^n$, we define a corrector w^{ε}_M by

$$\begin{cases} F(M + D^2 w^{\varepsilon}_M) = \beta_0(M) + F(M) & \text{in } D_{\varepsilon},\\ w^{\varepsilon}_M(x) = \phi(\theta) & \text{on } \partial T_{\varepsilon},\\ w^{\varepsilon}_M = 0 & \text{on } \partial D,\end{cases}$$

where $\beta_0(M)$ is the critical value. Note that we impose the boundary condition $w^{\varepsilon} = \phi(\theta)$ on ∂T_{ε} instead of $w^{\varepsilon} = 1$, which we wrote $w^{\varepsilon} \approx 1$ in the Introduction.

Lemma 4.5.21. *Let $B_\eta(x_0) \subset D$. Then there exists a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ such that $\varepsilon_j \to 0^+$ and*

$$\lim_{j\to\infty} w^{\varepsilon_j}(x) = 0,$$

for any $x \in \partial B_\eta(x_0) \cup \{x_0\}$.

Proof. First of all, by Theorem 4.4.13, the free solution $\overline{w}^{\varepsilon}_{\beta_0;M}$ satisfies

$$F(M + D^2\overline{w}^{\varepsilon}_{\beta_0;M}) = \beta_0(M) + F(M) \quad \text{in } D \setminus \cup_{k\in\mathbb{Z}^n}\{\varepsilon k\},$$

while the corrector w^{ε}_M satisfies

$$F(M + D^2 w^{\varepsilon}_M) = \beta_0(M) + F(M).$$

Applying Lemma 4.5.20 together with the comparison principle for w^ε_M and $\overline{w}^\varepsilon_{\beta_0;M}$ in $D \setminus T_\varepsilon$, it holds that

$$\overline{w}^\varepsilon_{\beta_0;M}(x) - o(1) \leq w^\varepsilon(x) \leq \overline{w}^\varepsilon_{\beta_0;M}(x) + o(1) \quad \text{in } D \setminus T_\varepsilon.$$

Therefore, the desired result follows from Corollary 4.5.17. □

Before finding the effective equation satisfied by the limit profile u, we show the uniform ellipticity of the homogenized operator.

Lemma 4.5.22. *For $M \in \mathcal{S}^n$ and $c \geq 0$, set*

$$\begin{aligned}\overline{F}(M,c) &:= F(M + cD^2 w^\varepsilon_M) \\ &= \begin{cases} cF(M/c + D^2 w^\varepsilon_M) = c\beta_0(M/c) + F(M) & \text{if } c > 0, \\ F(M) & \text{if } c = 0. \end{cases}\end{aligned}$$

Then we have $\lambda\|N\| \leq \overline{F}(M+N,c) - \overline{F}(M,c) \leq \Lambda\|N\|$ for any $N \geq 0$.

Proof. If $c = 0$, then the result follows from the uniform ellipticity of F. For $c = 1$, we have

$$\overline{F}(M+N,1) = F(M+N+D^2 w^\varepsilon_{M+N}) \leq F(M + D^2 w^\varepsilon_{M+N}) + \Lambda\|N\|$$

We denote $\widetilde{w}$ be the solution of

$$\begin{cases} F(M + D^2\widetilde{w}) = \overline{F}(M+N,1) - \Lambda\|N\| & \text{in } D_\varepsilon, \\ \widetilde{w}(x) = \phi(\theta) & \text{on } \partial T_\varepsilon, \\ \widetilde{w} = 0 & \text{on } \partial D. \end{cases}$$

Then the comparison principle yields that $\widetilde{w} \geq w^\varepsilon_{M+N}$ and so $\widetilde{w} \geq 0$ letting $\varepsilon \to 0$. Recalling the definition of w^ε_M, we have

$$F(M + D^2\widetilde{w}) \leq F(M + D^2 w^\varepsilon_M) = \beta_0(M) + F(M) = \overline{F}(M,1),$$

and so

$$\overline{F}(M+N,1) \le \overline{F}(M,1) + \Lambda\|N\|.$$

The lower bound can be proved similarly. Moreover, considering M/c and N/c instead of M and N, we can finish the proof for general $c > 0$. □

Finally, we are ready to finish the proof of our main theorem for the fully nonlinear case:

Proof of Theorem 4.1.2 (ii). Recalling Lemma 4.5.22, $\overline{F}$ is uniformly elliptic. We are going to show that u is a supersolution. Let us assume that there is a parabola P touching u from below at x_0 and

$$\overline{F}(D^2P, (\varphi(x_0) - P(x_0))_+) > 2\mu_0 > 0.$$

In a small neighborhood of x_0, $B_\eta(x_0)$, there exists another parabola Q such that

$$\begin{cases} D^2P > D^2Q & \text{in } B_\eta(x_0), \\ P(x_0) + \delta_0 < Q(x_0), & \\ P(x) > Q(x) & \text{on } \partial B_\eta(x_0). \end{cases}$$

In addition, for $\xi_0 := (\varphi(x_0) - u(x_0))_+$, $\overline{F}(D^2Q, \xi_0) > \mu_0 > 0$. Then for a corrector w_M^ε with $M = D^2Q$, we have

$$F(D^2Q_\varepsilon) = F(D^2Q + \xi_0 D^2 w_M^\varepsilon) = \overline{F}(D^2Q, \xi_0) > \mu_0 > 0,$$

where $Q_\varepsilon(x) = Q(x) + \xi_0 w_M^\varepsilon(x)$. Since $Q_\varepsilon < u_\varepsilon$ on $\partial B_\rho(x_0)$ for sufficiently small $\varepsilon > 0$, the comparison principle yields that $Q(x_0) \le u(x_0)$. It contradicts to the fact that $Q(x_0) > P(x_0) + \delta_0 = u(x_0) + \delta_0$. A similar argument tells us u is also a subsolution. □

Bibliography

[1] M. A. Akcoglu and U. Krengel. Ergodic theorems for superadditive processes. *J. Reine Angew. Math.*, 323:53–67, 1981.

[2] Y. A. Alkhutov and O. V. Krasheninnikova. Continuity at boundary points of solutions of quasilinear elliptic equations with a nonstandard growth condition. *Izv. Ross. Akad. Nauk Ser. Mat.*, 68(6):3–60, 2004.

[3] N. Ansini and A. Braides. Asymptotic analysis of periodically-perforated nonlinear media. *J. Math. Pures Appl. (9)*, 81(5):439–451, 2002.

[4] S. N. Armstrong, B. Sirakov, and C. K. Smart. Fundamental solutions of homogeneous fully nonlinear elliptic equations. *Comm. Pure Appl. Math.*, 64(6):737–777, 2011.

[5] W. Arriagada and J. Huentutripay. Characterization of a homogeneous Orlicz space. *Electron. J. Differential Equations*, pages Paper No. 49, 17, 2017.

[6] W. Arriagada and J. Huentutripay. A Harnack inequality in Orlicz-Sobolev spaces. *Studia Math.*, 243(2):117–137, 2018.

[7] P. Bauman. A Wiener test for nondivergence structure, second-order elliptic equations. *Indiana Univ. Math. J.*, 34(4):825–844, 1985.

[8] A. Bensoussan, J.-L. Lions, and G. Papanicolaou. *Asymptotic analysis for periodic structures*, volume 5 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam-New York, 1978.

[9] M. Bôcher. Singular points of functions which satisfy partial differential equations of the elliptic type. *Bulletin of the American Mathematical Society*, 9(9):455–465, 1903.

[10] A. Braides. *Γ-convergence for beginners*, volume 22 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2002.

[11] M. Brelot. *Lectures on potential theory*, volume 19 of *Lectures on Mathematics*. Tata Institute of Fundamental Research, Bombay, 1960. Notes by K. N. Gowrisankaran and M. K. Venkatesha Murthy.

[12] L. Caffarelli and K.-A. Lee. Viscosity method for homogenization of highly oscillating obstacles. *Indiana Univ. Math. J.*, 57(4):1715–1741, 2008.

[13] L. A. Caffarelli. A note on nonlinear homogenization. *Comm. Pure Appl. Math.*, 52(7):829–838, 1999.

[14] L. A. Caffarelli and X. Cabré. *Fully nonlinear elliptic equations*, volume 43 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1995.

[15] L. A. Caffarelli and A. Mellet. Random homogenization of an obstacle problem. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(2):375–395, 2009.

[16] L. A. Caffarelli, P. E. Souganidis, and L. Wang. Homogenization of fully nonlinear, uniformly elliptic and parabolic partial differential equations in stationary ergodic media. *Comm. Pure Appl. Math.*, 58(3):319–361, 2005.

[17] D. Cioranescu and F. Murat. Un terme étrange venu d'ailleurs. II. In *Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. III (Paris, 1980/1981)*, volume 70 of *Res. Notes in Math.*, pages 154–178, 425–426. Pitman, Boston, Mass.-London, 1982.

[18] M. G. Crandall, H. Ishii, and P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)*, 27(1):1–67, 1992.

[19] M. G. Crandall, M. Kocan, P. L. Lions, and A. Swiech. Existence results for boundary problems for uniformly elliptic and parabolic fully nonlinear equations. *Electron. J. Differential Equations*, pages No. 24, 22 pp, 1999.

[20] G. Dal Maso. *An introduction to* Γ*-convergence*, volume 8 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston, Inc., Boston, MA, 1993.

[21] G. Dal Maso and L. Modica. Nonlinear stochastic homogenization and ergodic theory. *J. Reine Angew. Math.*, 368:28–42, 1986.

[22] G. Dal Maso and F. Murat. Asymptotic behaviour and correctors for Dirichlet problems in perforated domains with homogeneous monotone operators. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 24(2):239–290, 1997.

[23] L. Diening, P. Harjulehto, P. Hästö, and M. Ružička. *Lebesgue and Sobolev spaces with variable exponents*, volume 2017 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2011.

[24] L. Diening, B. Stroffolini, and A. Verde. Everywhere regularity of functionals with ϕ-growth. *Manuscripta Math.*, 129(4):449–481, 2009.

[25] L. Diening, B. Stroffolini, and A. Verde. The ϕ-harmonic approximation and the regularity of ϕ-harmonic maps. *J. Differential Equations*, 253(7):1943–1958, 2012.

[26] L. C. Evans. Periodic homogenisation of certain fully nonlinear partial differential equations. *Proc. Roy. Soc. Edinburgh Sect. A*, 120(3-4):245–265, 1992.

[27] C. E. Finol and M. Wójtowicz. Multiplicative properties of real functions with applications to classical functions. *Aequationes Math.*, 59(1-2):134–149, 2000.

[28] M. Focardi. Homogenization of random fractional obstacle problems via Γ-convergence. *Comm. Partial Differential Equations*, 34(10-12):1607–1631, 2009.

[29] M. Focardi. Aperiodic fractional obstacle problems. *Adv. Math.*, 225(6):3502–3544, 2010.

[30] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*, volume 224 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1983.

[31] Q. Han and F. Lin. *Elliptic partial differential equations*, volume 1 of *Courant Lecture Notes in Mathematics*. Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, second edition, 2011.

[32] P. Harjulehto and P. Hästö. *Orlicz spaces and generalized Orlicz spaces*, volume 2236 of *Lecture Notes in Mathematics*. Springer, Cham, 2019.

[33] F. R. Harvey and H. B. Lawson. Tangents to subsolutions: existence and uniqueness, Part I. *Ann. Fac. Sci. Toulouse Math. (6)*, 27(4):777–848, 2018.

[34] F. R. Harvey and H. B. Lawson, Jr. The Dirichlet problem with prescribed interior singularities. *Adv. Math.*, 303:1319–1357, 2016.

[35] P. Hästö and J. Ok. Maximal regularity for local minimizers of non-autonomous functionals. *J. Eur. Math. Soc. (JEMS)*, 24(4):1285–1334, 2022.

[36] D. Hauer and K.-A. Lee. A robust method for uniform convergence in domain perturbation with shrinking holes. *preprint*.

[37] J. Heinonen, T. Kilpeläinen, and O. Martio. *Nonlinear potential theory of degenerate elliptic equations*. Dover Publications, Inc., Mineola, NY, 2006. Unabridged republication of the 1993 original.

[38] L. L. Helms. *Potential theory*. Universitext. Springer-Verlag London, Ltd., London, 2009.

[39] L. Hörmander. *Notions of convexity*, volume 127 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1994.

[40] H. Ishii and P.-L. Lions. Viscosity solutions of fully nonlinear second-order elliptic partial differential equations. *J. Differential Equations*, 83(1):26–78, 1990.

[41] D. S. Jerison and C. E. Kenig. Boundary behavior of harmonic functions in nontangentially accessible domains. *Adv. in Math.*, 46(1):80–147, 1982.

[42] V. V. Jikov, S. M. Kozlov, and O. A. Oleĭnik. *Homogenization of differential operators and integral functionals*. Springer-Verlag, Berlin, 1994. Translated from the Russian by G. A. Yosifian.

[43] B. Kawohl and N. Kutev. Comparison principle and Lipschitz regularity for viscosity solutions of some classes of nonlinear partial differential equations. *Funkcial. Ekvac.*, 43(2):241–253, 2000.

[44] B. Kawohl and N. Kutev. Comparison principle for viscosity solutions of fully nonlinear, degenerate elliptic equations. *Comm. Partial Differential Equations*, 32(7-9):1209–1224, 2007.

[45] T. Kilpeläinen and P. Lindqvist. On the Dirichlet boundary value problem for a degenerate parabolic equation. *SIAM J. Math. Anal.*, 27(3):661–683, 1996.

[46] T. Kilpeläinen and J. Malý. The Wiener test and potential estimates for quasilinear elliptic equations. *Acta Math.*, 172(1):137–161, 1994.

[47] M. Kim, K.-A. Lee, and S.-C. Lee. The Wiener criterion for nonlocal Dirichlet problems. *arXiv preprint arXiv:2203.16815*, 2022.

[48] S. Kim and K.-A. Lee. Viscosity method for homogenization of parabolic nonlinear equations in perforated domains. *J. Differential Equations*, 251(8):2296–2326, 2011.

[49] J. Korvenpää, T. Kuusi, and E. Lindgren. Equivalence of solutions to fractional p-Laplace type equations. *J. Math. Pures Appl. (9)*, 132:1–26, 2019.

[50] N. V. Krylov and T. Yastrzhembskiy. On nonequivalence of regular boundary points for second-order elliptic operators. *Comm. Partial Differential Equations*, 42(3):366–387, 2017.

[51] D. A. Labutin. Isolated singularities for fully nonlinear elliptic equations. *J. Differential Equations*, 177(1):49–76, 2001.

[52] D. A. Labutin. Singularities of viscosity solutions of fully nonlinear elliptic equations. Number 1287, pages 45–57. 2002. Viscosity solutions of differential equations and related topics (Japanese) (Kyoto, 2001).

[53] N. S. Landkof. *Foundations of modern potential theory.* Die Grundlehren der mathematischen Wissenschaften, Band 180. Springer-Verlag, New York-Heidelberg, 1972. Translated from the Russian by A. P. Doohovskoy.

[54] V. K. Le. A global bifurcation result for quasilinear elliptic equations in Orlicz-Sobolev spaces. *Topol. Methods Nonlinear Anal.*, 15(2):301–327, 2000.

[55] H. Lebesgue. Sur des cas d'impossibilité du problème de dirichlet ordinaire. *CR des Séances de la Société Mathématique de France*, 17(2), 1913.

[56] K.-A. Lee. *Obstacle problems for the fully nonlinear elliptic operators.* ProQuest LLC, Ann Arbor, MI, 1998. Thesis (Ph.D.)–New York University.

[57] K.-A. Lee and S.-C. Lee. Random homogenization of φ-Laplace equations with highly oscillating obstacles. *preprint, available at https://www.iumj.indiana.edu/IUMJ/forthcoming.php*, 2021.

[58] K.-A. Lee and S.-C. Lee. Viscosity method for random homogenization of fully nonlinear elliptic equations with highly oscillating obstacles. *preprint*, 2021.

[59] K.-A. Lee and S.-C. Lee. The Wiener criterion for elliptic equations with Orlicz growth. *J. Differential Equations*, 292:132–175, 2021.

[60] K.-A. Lee and S.-C. Lee. The Wiener criterion for fully nonlinear elliptic equations. *preprint*, 2021.

[61] K.-A. Lee and M. Yoo. The viscosity method for the homogenization of soft inclusions. *Arch. Ration. Mech. Anal.*, 206(1):297–332, 2012.

[62] Y. Lian, K. Zhang, D. Li, and G. Hong. Boundary Hölder regularity for elliptic equations. *J. Math. Pures Appl. (9)*, 143:311–333, 2020.

[63] G. M. Lieberman. The natural generalization of the natural conditions of Ladyzhenskaya and Uraltseva for elliptic equations. *Comm. Partial Differential Equations*, 16(2-3):311–361, 1991.

[64] P. Lindqvist and O. Martio. Two theorems of N. Wiener for solutions of quasilinear elliptic equations. *Acta Math.*, 155(3-4):153–171, 1985.

[65] W. Littman, G. Stampacchia, and H. F. Weinberger. Regular points for elliptic equations with discontinuous coefficients. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 17:43–77, 1963.

[66] J. Malý and W. P. Ziemer. *Fine regularity of solutions of elliptic partial differential equations*, volume 51 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997.

[67] M. Masi. On compressive radial tidal forces. *American Journal of Physics*, 75(2):116–124, 2007.

[68] V. G. Maz'ya. On the continuity at a boundary point of solutions of quasi-linear elliptic equations. *Vestnik Leningrad. Univ. Mat. Mekh. Astronom.*, 25:42–55, 1970.

[69] J. H. Michael. Barriers for uniformly elliptic equations and the exterior cone condition. *J. Math. Anal. Appl.*, 79(1):203–217, 1981.

[70] K. Miller. Barriers on cones for uniformly elliptic operators. *Ann. Mat. Pura Appl. (4)*, 76:93–105, 1967.

[71] K. Miller. Exceptional boundary points for the nondivergence equation which are regular for the Laplace equation and vice-versa. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 22:315–330, 1968.

[72] K. Miller. Nonequivalence of regular boundary points for the Laplace and nondivergence equations, even with continuous coefficients. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 24:159–163, 1970.

[73] K. Miller. Extremal barriers on cones with Phragmén-Lindelöf theorems and other applications. *Ann. Mat. Pura Appl. (4)*, 90:297–329, 1971.

[74] M. Ramaswamy and S. Ramaswamy. Perron's method and barrier functions for the viscosity solutions of the Dirichlet problem for some nonlinear partial differential equations. *Z. Anal. Anwendungen*, 13(2):199–207, 1994.

[75] J. Serrin. Local behavior of solutions of quasi-linear equations. *Acta Math.*, 111:247–302, 1964.

[76] J. Serrin. Isolated singularities of solutions of quasi-linear equations. *Acta Math.*, 113:219–240, 1965.

[77] L. Tang. Random homogenization of p-Laplacian with obstacles in perforated domain. *Comm. Partial Differential Equations*, 37(3):538–559, 2012.

[78] N. S. Trudinger and X.-J. Wang. On the weak continuity of elliptic operators and applications to potential theory. *Amer. J. Math.*, 124(2):369–410, 2002.

[79] N. Wiener. Certain notions in potential theory. *Journal of Mathematics and Physics*, 3(1):24–51, 1924.

[80] N. Wiener. The Dirichlet problem. *Journal of Mathematics and Physics*, 3(3):127–146, 1924.

국문초록

본 학위 논문은 비정칙 영역에서의 비선형 타원형 방정식을 다룬 세 편의 연구논문으로 구성된다. 첫 번째 논문에서, 우리는 비선형 퍼텐셜 이론을 통해 완전 비선형 방정식에 대한 정칙 경계점을 특징짓는 위너 판정법을 확립한다. 우리의 접근 방식은 비변분 용량의 분석과 동차해를 사용한 장벽 함수의 구성을 기반으로 한다. 두 번째 및 세 번째 논문은 각각 올리츠 증가성을 가진 타원형 작용소와 완전 비선형 작용소에 대한 장애물 문제의 임의 균질화에 대해 논의한다. 임계 크기를 가진 구멍에 대해 정상 에르고딕 성질을 가정하면, 두 경우 모두 극한 함수는 장애물이 없는 균질화된 방정식을 만족한다. 분석의 핵심은 각각 에너지와 점성 방법을 통해, 진동하는 해의 점근적 행동을 포착하는 데 있다.

주요어휘: 위너 판정법, 임의 균질화, 완전 비선형 작용소, 올리츠 공간
학번: 2016-20241

감사의 글

크고 작은 어려움이 있었던 석박통합과정을 무사히 마치는 데에 정말 많은 분들의 도움이 있었습니다. 먼저 학부 때부터 박사과정까지 저를 지도해 주신 이기암 교수님께 감사드립니다. 제가 다양한 분야를 경험하고 관련 연구를 수행할 때 교수님의 뛰어난 통찰력과 끝없는 열정을 본받기 위해 항상 노력했습니다. 뿐만 아니라 교수님께서는 따뜻한 조언과 격려로 박사과정 동안 성장할 수 있도록 긍정적인 힘을 꾸준히 주셨습니다.

바쁘신 가운데에도 박사학위논문 심사를 맡아주신 변순식 교수님, 김성훈 교수님, 박진완 교수님, 정인지 교수님께도 감사의 말씀을 드립니다. 교수님들의 세심한 조언과 색다른 안목은 박사논문을 완성하는 데 큰 도움이 되었습니다.

또한 오랜 시간 동안 연구실을 함께 사용한 동료들인 효석이형, 상필이형, 형성이형, 성하형, 성한이형, 민현이형, 태훈이형, 탁원이형, 종명이형, 진제형, 성은씨, 애솔씨에게도 감사의 인사를 전합니다. 함께 수학을 공부하고 이야기하며 즐거운 연구실 생활을 할 수 있었고, 행복한 추억을 많이 남겼습니다.

마지막으로 박사과정 기간 동안 항상 격려하고 응원해 주신 동생과 부모님께 깊은 감사의 마음을 전합니다. 가족들이 언제나 저의 선택과 꿈을 지지하고 믿어주셔서, 흔들리지 않고 묵묵히 학위과정을 마무리할 수 있었습니다.