이학박사 학위논문

Nonlocal elliptic equations with Orlicz growth

(오리츠 성장조건을 가지는 비국소 타원형 편미분방정식)

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Nonlocal elliptic equations with Orlicz growth

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Abstract

Nonlocal elliptic equations with Orlicz growth

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This thesis involves various regularity results of nonlocal elliptic equations with Orlicz growth.

First, we prove the existence and uniqueness of a weak solution to a nonlocal Dirichlet problem with Orlicz growth by using variational methods.

Next, we show local Hölder continuity of a weak solution to such a nonlocal elliptic equation by obtaining a suitable Sobolev-Poincaré type inequality and a logarithmic estimate.

Finally, we derive Harnack inequality by finding a precise tail estimate.

Key words: Nonlocal operator, Orlicz growth, N-function, Local boundedness, Hölder continuity, Harnack inequality
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Contents

Chapter 1

Introduction

This thesis is concerned with various regularity results for the following nonlocal equation

$$\mathcal{L}u = 0 \quad \text{in} \quad \Omega, \tag{1.1}$$

defined on a bounded domain Ω in $\mathbb{R}^n$, $n \geq 2$, by

$$\mathcal{L}v(x) := \text{p.v.} \int_{\mathbb{R}^n} g\left(\frac{|v(x) - v(y)|}{|x - y|^s}\right) \frac{v(x) - v(y)}{|v(x) - v(y)|} K(x, y) \frac{dy}{|x - y|^s}, \tag{1.2}$$

where $0 < s < 1$ and $g : [0, \infty) \to [0, \infty)$ is a strictly increasing, continuous function that satisfies $g(0) = 0$, $\lim\limits_{t \to \infty} g(t) = \infty$ and

$$1 < p \leq \frac{tg(t)}{G(t)} \leq q < \infty \quad \text{for some} \quad 1 < p \leq q, \quad \text{where} \quad G(t) := \int_0^t g(s)\, ds. \tag{1.3}$$

Note that this inequality means that the growth of the function G varies between p and q, which is a natural outgrowth of the p-th power function. This condition covers the power case $g(t) = t^{p-1}$, the borderline case $g(t) = t^{p-1} \log(e + t)$, the mixed case $g(t) = t^{p-1} + t^{q-1}$, and so on. For more examples of g and applications of problems with such growth, we refer to [4, 36, 56].

$K : \mathbb{R}^n \times \mathbb{R}^n \to (0, \infty]$ is a symmetric, i.e., $K(x, y) = K(y, x)$, and measurable kernel that satisfies

$$\frac{\lambda}{|x-y|^n} \le K(x,y) \le \frac{\Lambda}{|x-y|^n}, \qquad x,y \in \mathbb{R}^n, \tag{1.4}$$

for some constants $0 < \lambda \le \Lambda$. Note that the symmetry condition of K is not necessary. However, by considering the kernel $\tilde{K}(x,y) = \frac{K(x,y)+K(y,x)}{2}$, we shall always assume this symmetry, see [44, Section 1.5] for more details.

A main point in this thesis is that the function G is an N-function satisfying the Δ_2 and ∇_2 conditions (see Chapter 2) and that a simple example of the kernel $K(x,y)$ is $a(x,y)|x-y|^{-n}$ with $\lambda \le a(x,y) \le \Lambda$. In particular, we point out that in the case when $K(x,y) = |x-y|^{-n}$, $\mathcal{L}$ becomes the so-called (s-)fractional G-Laplace operator and we denote it by $\mathcal{L} = (-\Delta)^s_G$.

The goal of this thesis is to establish local Hölder continuity and Harnack inequality for the nonlocal problem (1.1). In addition, we will discuss the existence and uniqueness of weak solutions to (1.1) with Dirichlet boundary condition.

An obvious example is a particular situation that $g(t) = t$ and $K(x,y) = |x-y|^{-n}$ in which case it reduces to the s-fractional Laplace operator $(-\Delta)^s$. There have been many developments in the regularity theory for nonlocal elliptic and parabolic equations of fractional Laplace type. We refer to [11, 12, 13, 39, 40, 45, 49, 57] for various regularity results including Hölder continuity, Harnack inequality, self improving property, L^p-regularity and so on. On the other hand, for the fractional p-Laplacian type equations, i.e., $g(t) = t^{p-1}$ with $1 < p < \infty$, Di Castro, Kuusi and Palatucci in [21, 22] established nonlocal De Giorgi-Nash-Moser theory. They proved local Hölder regularity along with Harnack inequality by employing the so-called tail(see Chapter 2). We also mention that Cozzi [17] proved similar regularity results by using a notion of fractional De Giorgi class. We further refer to [5, 6, 17, 19, 21, 27, 32, 33, 34, 41, 42, 43, 44, 47, 51, 52] and references therein for further discussions on the nonlocal nonlinear equations of the fractional p-Laplacian type.

A general non-autonomous fractional nonlocal operator can be written as

$$\mathcal{L}v(x) \coloneqq \text{p.v.} \int_{\mathbb{R}^n} h\left(x, y, \frac{|v(x)-v(y)|}{|x-y|^s}\right) \frac{v(x)-v(y)}{|v(x)-v(y)|} K(x,y) \frac{dy}{|x-y|^s}.$$

If $h(x,y,t) = t^{p-1}$, then we say that the operator or equation satisfies the p-growth condition. On the other hand, if $h(x,y,t)$ has a more general structure, then we say that the operator or equation satisfies a non-standard growth condition. Typical examples of non-standard growth conditions include the variable growth condition: $h(x,y,t) = t^{p(x,y)-1}$, the double phase condition: $h(x,y,t) = t^{p-1} + a(x,y)t^{q-1}$, and the general growth condition: $h(x,y,t) = g(t)$. There has been a great deal of studies concerning fractional nonlocal equations with nonstandard growth conditions, in particular, for Hölder regularity in [55, 14] with the variable growth condition and in [10, 20, 28] with the double phase condition, respectively.

We are mainly focusing on the general growth condition. The local one corresponding to the nonlocal equation (1.1) is the so called G-Laplace equation:

$$\operatorname{div}\left(g(|Du|)\frac{Du}{|Du|}\right) = 0 \quad \text{in} \quad \Omega, \quad \text{where} \quad g(t) = G'(t),$$

for which Lieberman [46] proved Hölder regularity and Harnack inequality of weak solutions under the condition (1.3). We also refer to [3, 7, 18, 26, 37, 38, 48, 53, 58] and references therein for the regularity results for equations of the G-Laplacian type. In particular, the papers [18, 48] deal with problems modeled by the G-Laplace equation with G not necessarily satisfying the Δ_2 and ∇_2-conditions.

According to the local regularity results for the G-Laplace equation, obtaining analogous regularity results to the corresponding nonlocal equation (1.1) has been a naturally interesting issue. Especially, Hölder regularity alongside Harnack inequality has been studied in [15, 30, 31] and [16, 28], respectively. However, Hölder regularity results in [15, 30, 31] are established with a strong Dirichlet boundary condition or the boundedness of a weak solution or a restrictive condition on g like $q < p^*$ in (1.3). Meanwhile, in [28], Harnack inequality for (1.1) has been proved under additional assumptions to (1.3), which are

$$G(t_1 t_2) \leq cG(t_1)G(t_2), \qquad t_1, t_2 \geq 0 \tag{1.5}$$

and

$$\min\{t^p, t^q\} \leq G(t) \leq c \max\{t^p, t^q\}, \qquad t \geq 0 \tag{1.6}$$

for some constant c independent of t_1 and t_2.

In this thesis, on the other hand, we prove local Hölder regularity and Harnack inequality for the equation (1.1) with the assumption (1.3) only and without boundary data. We do not require the a priori assumption of the boundedness of a solution. In particular, we need neither (1.5) nor (1.6). Therefore, we first obtain local boundedness of a weak solution with a suitable estimate (1.7). To this end, we focus on finding inequalities and embeddings on fractional Orlicz-Sobolev spaces $W^{s,G}$ (see Champer 2). Especially, we proved an integral version of a fractional Sobolev-Poincaré inequality in $W^{s,G}$ which plays a major role in the proof of the main results in this thesis. We also give a more simplified proof to obtain a natural form of Harnack inequality. It is worthwhile to mention that Chaker, Kim and Weidner [16] proved Harnack inequality for (1.1) by using a different approach.

With the definition of a weak solution, the related function spaces and the tail to be introduced in details in the next chapter, we now state our main results.

Theorem 1.0.1 (Hölder continuity). *Let $0 < s < 1$. Suppose that $u \in \mathbb{W}^{s,G}(\Omega) \cap L^g_s(\mathbb{R}^n)$ is a weak solution to (1.1) with (1.2), (1.3) and (1.4). Then $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha \equiv \alpha(n,s,p,q,\lambda,\Lambda) \in (0,1)$. Moreover, there exist positive constants c_b and c_h depending on n,s,p,q,λ and Λ such that for any $B_r(x_0) \Subset \Omega$,*

$$\|u\|_{L^\infty(B_{r/2}(x_0))} \le c_b r^s G^{-1}\left(\fint_{B_r(x_0)} G\left(\frac{|u|}{r^s}\right)dx\right) + r^s g^{-1}(r^s \mathrm{Tail}(u;x_0,r/2)) \tag{1.7}$$

and

$$\begin{aligned}[u]_{C^{0,\alpha}(B_{r/2}(x_0))} \le \frac{c_h}{r^\alpha}\Bigg[& r^s G^{-1}\left(\fint_{B_r(x_0)} G\left(\frac{|u|}{r^s}\right)dx\right) \\ & + r^s g^{-1}(r^s \mathrm{Tail}(u;x_0,r/2))\Bigg].\end{aligned} \tag{1.8}$$

Theorem 1.0.2 (Harnack inequality). *Let $0 < s < 1$. Under assumptions (1.2), (1.3) and (1.4), let $u \in \mathbb{W}^{s,G}(\Omega) \cap L^g_s(\mathbb{R}^n)$ be a weak solution to (1.1) such that $u \ge 0$ in a ball $B_R \equiv B_R(x_0) \subset \Omega$. Then the inequality*

$$\sup_{B_r} u \le c \inf_{B_r} u + c r^s g^{-1}(r^s\,\mathrm{Tail}(u_-;x_0,R))$$

holds for any concentric ball $B_r(x_0) \subset B_{R/2}(x_0)$, where $c \equiv c(n,s,p,q,\lambda,\Lambda)$.

Remark 1.0.3. We can get the same results in Theorem 1.0.1 and Theorem 1.0.2 under weaker conditions on g that $g(0) = 0$, $\lim_{t\to\infty} g(t) = \infty$, and

$$\frac{g(s)}{s^{p-1}} \le L\frac{g(t)}{t^{p-1}} \quad \text{and} \quad \frac{g(t)}{t^{q-1}} \le L\frac{g(s)}{s^{q-1}} \quad \text{for all} \quad 0 < s \le t,$$

for some $1 < p \le q$ and $L \ge 1$. Note that the above inequality implies Δ_2 and ∇_2-conditions of the function $t \mapsto tg(t)$. Under this condition, there exists an increasing continuous function $\tilde{g}$ with $\tilde{g}(0) = 0$ and $\lim_{t\to\infty} \tilde{g}(t) = \infty$ such that $\tilde{g}$ satisfies (1.3) and $\tilde{g} \approx g$ (i.e., there exists a positive constant $c \ge 1$ such that $c^{-1}g \le \tilde{g} \le cg$). See [36, Chapter 2] for details. Then we get the results in Theorem 1.0.1 and Theorem 1.0.2 with respect to $\tilde{g}$. Therefore, by this equivalence, we can obtain the same estimates for g. Nevertheless, we adopt the condition (1.3) instead of the above one, as the proof of the equivalence is rather technical and the condition (1.3) is simpler and widely used.

Our approach here is based on the De Giorgi approach established in [21, 22], in particular, for the fractional p-Laplacian type equations in the setting of fractional Sobolev spaces $W^{s,p}$. On the other hand, to the fractional G-Laplacian type equations, this approach can not be directly applied, as $G(st) \not\approx G(s)G(t)$ (this equivalence is true when $G(t) = t^p$). Indeed, we are forced to face a more complicated and delicate situation under which we need to make a very careful systematic analysis to overcome the complexity and difficulty coming from such a G-Laplacian type nonlocal problem. Moreover, an integral version of Sobolev-Poincaré type inequality plays an essential role in the process of De Giorgi iteration, which is not known in the fractional Orlicz-Sobolev space as of today, as far as we are concerned.

This thesis is organized as follows. In Chapter 2 we introduce notations, function spaces, weak solutions and fundamental inequalities that will be used throughout this thesis. In Chapter 3 we prove the existence and uniqueness of weak solutions to (1.1) with Dirichlet boundary conditions. In Chapter 4 we first derive two essential estimates for weak solutions to (1.1) in Section 4.1. One is a Caccioppoli type inequality and the other is a logarithmic estimate. Then we prove local boundedness and Hölder continuity of a weak solution. Chapter 5 is devoted to the proof of Harnack inequality. We notice that Chapter 3 and Chapter 4 are based on the joint work with Sun-Sig Byun and Jihoon Ok [8]. Chapter 5 is based on the paper [9] jointed with Sun-Sig Byun and Kyeong Song.

Chapter 2

Preliminaries

In this chapter we introduce notations and preliminaries, which will be used throughout this thesis. $B_r(x_0)$ denotes the ball in $\mathbb{R}^n$ with radius $r > 0$ centered at $x_0 \in \mathbb{R}^n$. When the center is clear in the context, we write it by B_r for the sake of simplicity. The average of an integrable function f on B_r is defined as

$$(f)_{B_r} = \fint_{B_r} f\,dx = \frac{1}{|B_r|}\int_{B_r} f\,dx.$$

We denote by c to mean a universal constant that can be computed by given quantities such as $n, s, p, q, \lambda, \Lambda$. This generic constant can vary from line to line. We write $A \approx B$ if there exists some constant $c \geq 1$ such that $\frac{1}{c}A \leq B \leq cA$.

2.1 Properties for function G

Throughout this thesis we always assume that $G \in C^1([0,\infty))$ satisfies (1.3). Then $G : [0,\infty) \to [0,\infty)$ is an *N-function*(nice Young function), i.e., it is increasing and convex, and satisfies $\lim\limits_{t\to 0+} \frac{G(t)}{t} = 0$ and $\lim\limits_{t\to\infty} \frac{G(t)}{t} = \infty$. We always assume $G(1) = 1$.

The conjugate function $G^* : [0,\infty) \to [0,\infty)$ is defined by

$$G^*(t) \coloneqq \sup_{s\geq 0}\,(st - G(s)), \quad t \geq 0.$$

Then we have from (1.3) that for every $t \in [0, \infty)$,

$$\begin{cases} a^q G(t) \leq G(at) \leq a^p G(t) & \text{if } 0 < a < 1 \\ a^p G(t) \leq G(at) \leq a^q G(t) & \text{if } a > 1, \end{cases} \tag{2.1}$$

and that

$$\begin{cases} a^{p'} G^*(t) \leq G^*(at) \leq a^{q'} G^*(t) & \text{if } 0 < a < 1 \\ a^{q'} G^*(t) \leq G^*(at) \leq a^{p'} G^*(t) & \text{if } a > 1. \end{cases} \tag{2.2}$$

where p' and q' are the Hölder conjugates of p and q, respectively. Also we see that G satisfies the following Δ_2- and ∇_2-conditions (see [50, Proposition 2.3]):

(Δ_2) there exists a constant $\kappa > 1$ such that

$$G(2t) \leq \kappa G(t) \quad \text{for all} \quad t \geq 0; \tag{2.3}$$

(∇_2) there exists a constant $l > 1$ such that

$$G(t) \leq \frac{1}{2l} G(lt) \quad \text{for all} \quad t \geq 0, \tag{2.4}$$

where the constants κ and l are to be determined by q and p. Note that G satisfies the ∇_2-condition if and only if G^* does the Δ_2-condition. In addition, from the definition of the conjugate function, we have

$$ts \leq G(t) + G^*(s), \qquad t, s \geq 0. \tag{2.5}$$

From (2.1), we deduce that for every $\epsilon \in (0, 1)$

$$ts \leq \epsilon^{1-q} G(t) + \epsilon G^*(s), \qquad t, s \geq 0, \tag{2.6}$$

which is Young's inequality with ϵ. We further have from (1.3) that

$$G^*(g(t)) = tg(t) - G(t) \leq (q-1) G(t), \quad t \geq 0. \tag{2.7}$$

Also the convexity and (2.1) imply

$$2^{-1}(G(t) + G(s)) \leq G(t+s) \leq 2^{q-1}(G(t) + G(s)),$$

which will be used often later.

2.2 Fractional Orlicz-Sobolev spaces

For an open subset U in $\mathbb{R}^n$, we denote by $\mathcal{M}(U)$ to mean the class of all real-valued measurable functions on U. For an N-function G satisfying the Δ_2 and ∇_2 conditions, we define the *Orlicz space* $L^G(U)$ as

$$L^G(U) := \left\{ v \in \mathcal{M}(U) \;\Big|\; \int_U G(|v(x)|)\,dx < \infty \right\},$$

which is a Banach space with the Luxemburg norm defined as

$$\|v\|_{L^G(U)} := \inf \left\{ \lambda > 0 \;\Big|\; \int_U G\left(\frac{|v(x)|}{\lambda}\right) dx \leq 1 \right\}.$$

Then note that

$$\|v\|_{L^G(U)} \leq \int_U G\left(|v|\right) dx + 1. \tag{2.8}$$

We next let $0 < s < 1$ and define the *fractional Orlicz-Sobolev space* $W^{s,G}(U)$ as

$$W^{s,G}(U) := \left\{ v \in L^G(U) \;\Big|\; \int_U \int_U G\left(\frac{|v(x) - v(y)|}{|x-y|^s}\right) \frac{dxdy}{|x-y|^n} < \infty \right\},$$

which is also a Banach space with the norm

$$\|v\|_{W^{s,G}(U)} := \|v\|_{L^G(U)} + [v]_{s,G,U},$$

where $[v]_{s,G,U}$ is the Gagliardo semi-norm defined by

$$[v]_{s,G,U} := \inf \left\{ \lambda > 0 \;\Big|\; \int_U \int_U G\left(\frac{|v(x) - v(y)|}{\lambda |x-y|^s}\right) \frac{dxdy}{|x-y|^n} \leq 1 \right\}.$$

Thus we have

$$[v]_{s,G,U} \leq \int_U \int_U G\left(\frac{|v(x) - v(y)|}{|x-y|^s}\right) \frac{dxdy}{|x-y|^n} + 1. \tag{2.9}$$

We introduce the function space to which weak solutions of (1.2) belong, see the next subsection for the concept of a weak solution. We write

$$C_\Omega := (\Omega \times \mathbb{R}^n) \cup (\mathbb{R}^n \times \Omega). \tag{2.10}$$

Then the space $\mathbb{W}^{s,G}(\Omega)$ consists of all functions $v \in \mathcal{M}(\mathbb{R}^n)$ with $v|_\Omega \in L^G(\Omega)$ and

$$\iint_{C_\Omega} G\left(\frac{|v(x)-v(y)|}{|x-y|^s}\right)\frac{dxdy}{|x-y|^n} < \infty.$$

Note that if $v \in \mathbb{W}^{s,G}(\Omega)$, then $v|_\Omega \in W^{s,G}(\Omega)$.

2.3 Weak solution and tail

We first recall g with (1.3) and K with (1.4) to define a weak solution to (1.1).

Definition 2.3.1. *$u \in \mathbb{W}^{s,G}(\Omega)$ is a weak solution (resp. subsolution or supersolution) to (1.1) if*

$$\iint_{C_\Omega} g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|u(x)-u(y)|}(\eta(x)-\eta(y))K(x,y)\,dxdy = 0$$
$$(\textit{resp.} \le 0 \textit{ or} \ge 0)$$

for any $\eta \in \mathbb{W}^{s,G}(\Omega)$ (resp. nonnegative $\eta \in \mathbb{W}^{s,G}(\Omega)$) such that $\eta = 0$ in $\mathbb{R}^n \setminus \Omega$.

We next write

$$L^g_s(\mathbb{R}^n) := \left\{u \in \mathcal{M}(\mathbb{R}^n) : \int_{\mathbb{R}^n} g\left(\frac{|u(x)|}{(1+|x|)^s}\right)\frac{dx}{(1+|x|)^{n+s}} < \infty\right\},$$

and the tail of $u \in L^g_s(\mathbb{R}^n)$ for the ball $B_R(x_0)$ is denoted by

$$\mathrm{Tail}(u; x_0, R) := \int_{\mathbb{R}^n \setminus B_R(x_0)} g\left(\frac{|u(x)|}{|x-x_0|^s}\right)\frac{dx}{|x-x_0|^{n+s}}. \tag{2.11}$$

We notice that $u \in L^g_s(\mathbb{R}^n)$ if and only if $\mathrm{Tail}(u; x_0, R) < \infty$ for all $x_0 \in \mathbb{R}^n$

and $R > 0$. Indeed, for $x \in \mathbb{R}^n \setminus B_R(x_0)$, a direct computation leads to

$$\frac{1+|x|}{|x-x_0|} \le 1 + \frac{1+|x_0|}{R}.$$

Then it follows from (2.1) that

$$\begin{aligned}
&\mathrm{Tail}(u; x_0, R) \\
&\quad = \int_{\mathbb{R}^n \setminus B_R(x_0)} g\left(\frac{|u(x)|}{(1+|x|)^s}\left(\frac{1+|x|}{|x-x_0|}\right)^s\right)\left(\frac{1+|x|}{|x-x_0|}\right)^{n+s}\frac{dx}{(1+|x|)^{n+s}} \\
&\quad \le \left(1+\frac{1+|x_0|}{R}\right)^{n+sq}\int_{\mathbb{R}^n \setminus B_R(x_0)} g\left(\frac{|u(x)|}{(1+|x|)^s}\right)\frac{dx}{(1+|x|)^{n+s}} < \infty.
\end{aligned}$$

To show the converse relation, choose two different points x_1, x_2 with $|x_1| > 1$, $|x_2| > 1$, and let $0 < R \le \frac{|x_1-x_2|}{4}$. Then we find that for $x \in \mathbb{R}^n$

$$\frac{|x-x_i|}{1+|x|} \le 1 + \frac{|x_i|-1}{1+|x|} \le |x_i|, \quad i = 1, 2.$$

Therefore we can estimate as above that

$$\begin{aligned}
&\int_{\mathbb{R}^n} g\left(\frac{|u(x)|}{(1+|x|)^s}\right)\frac{dx}{(1+|x|)^{n+s}} \\
&\quad \le \int_{\mathbb{R}^n \setminus B_R(x_1)} g\left(\frac{|u(x)|}{(1+|x|)^s}\right)\frac{dx}{(1+|x|)^{n+s}} \\
&\qquad + \int_{\mathbb{R}^n \setminus B_R(x_2)} g\left(\frac{|u(x)|}{(1+|x|)^s}\right)\frac{dx}{(1+|x|)^{n+s}} \\
&\quad \le |x_1|^{n+sq}\int_{\mathbb{R}^n \setminus B_R(x_1)} g\left(\frac{|u(x)|}{|x-x_1|^s}\right)\frac{dx}{|x-x_1|^{n+s}} \\
&\qquad + |x_2|^{n+sq}\int_{\mathbb{R}^n \setminus B_R(x_2)} g\left(\frac{|u(x)|}{|x-x_2|^s}\right)\frac{dx}{|x-x_2|^{n+s}} < \infty.
\end{aligned}$$

Remark 2.3.2. Observe that

$$\begin{aligned}
&R^s g^{-1}\left(R^s \mathrm{Tail}(u; x_0, R)\right) \\
&\quad = R^s g^{-1}\left(R^s \int_{\mathbb{R}^n \setminus B_R(x_0)} g\left(\frac{|u(x)|}{|x-x_0|^s}\right)\frac{dx}{|x-x_0|^{n+s}}\right).
\end{aligned} \tag{2.12}$$

In particular, if $g(t) = t^{p-1}$, (2.12) is reduced to

$$\left[R^{sp} \int_{\mathbb{R}^n \setminus B_R(x_0)} \frac{|u(x)|^{p-1}}{|x - x_0|^{n+sp}} \, dx \right]^{\frac{1}{p-1}},$$

which is the tail used in [22]. In this thesis we use (2.11) instead of (2.12) for simplicity.

Remark 2.3.3.

1. Note that $W^{s,G}(\mathbb{R}^n) \subset \mathbb{W}^{s,G}(\Omega) \cap L^g_s(\mathbb{R}^n)$.

2. Let ψ be an N-function satisfying $g(t) \leq c\psi(t)$ for $t \geq t_0$, where c and t_0 are some positive constants. If $u \in L^\psi(\mathbb{R}^n)$ or $u \in L^\psi(B_R) \cap L^\infty(\mathbb{R}^n \setminus B_R)$, then $u \in L^g_s(\mathbb{R}^n)$.

Chapter 3

Existence and uniqueness

In this chapter we prove the existence and uniqueness of a weak solution to (1.1) with Dirichlet boundary conditions. The proof is based on a direct method in the calculus of variations. We refer the reader to [2, 22, 29, 35] for the details. Before introducing the compact embedding in $W^{s,G}(\Omega)$, we need the following definition.

Definition 3.0.1. *Let A and B be two Young functions. We say B grows essentially more slowly near infinity than A if*

$$\lim_{t\to\infty}\frac{B(\lambda t)}{A(t)}=0 \tag{3.1}$$

for every $\lambda > 0$.

Note that the condition (3.1) is equivalent to

$$\lim_{t\to\infty}\frac{A^{-1}(t)}{B^{-1}(t)}=0.$$

Let $s \in (0,1)$ and let A be a Young function such that

$$\int_0^1\left(\frac{t}{A(t)}\right)^{\frac{s}{n-s}}dt<\infty \quad \text{and} \quad \int_1^\infty\left(\frac{t}{A(t)}\right)^{\frac{s}{n-s}}dt=\infty. \tag{3.2}$$

Then $A_{\frac{n}{s}}$ is given by

$$A_{\frac{n}{s}}(t) := A(H^{-1}(t)) \quad \text{for} \quad t \geq 0, \tag{3.3}$$

where the function $H : [0,\infty) \to [0,\infty)$ obeys

$$H(t) := \left(\int_0^t \left(\frac{\tau}{A(\tau)} \right)^{\frac{s}{n-s}} d\tau \right)^{\frac{n-s}{n}} \quad \text{for } \ t \geq 0. \tag{3.4}$$

The following lemma is related to compact embedding of $W^{s,G}$.

Lemma 3.0.2. *[1, Theorem 3.5] Let $s \in (0,1)$ and let A be a Young function fulfilling* (3.2)*. Let $A_{\frac{n}{s}}$ be the Young function defined as in* (3.3)*. Assume that B is a Young function. Then the following properties are equivalent.*

1. *B grows essentially more slowly near infinity than $A_{\frac{n}{s}}$.*
2. *The embedding*

$$W^{s,A}(U) \to L^B(U)$$

is compact for every bounded domain $U \subset \mathbb{R}^n$ with Lipschitz boundary.

From the above lemma, we see the following compact embedding result.

Lemma 3.0.3. *Suppose an N-function G satisfies* (1.3) *and $0 < s < 1$. The embedding*

$$W^{s,G}(U) \to L^G(U)$$

is compact for any bounded domain $U \subset \mathbb{R}^n$ with Lipschitz boundary.

Proof. We first consider the case $0 < s < \frac{n}{q}$. Note that the condition (1.3) implies

$$G(t) \geq t^q \ \text{ for } \ 0 < t < 1 \quad \text{and} \quad G(t) \leq t^q \ \text{ for } \ t \geq 1.$$

Then recalling the assumption $\frac{(q-1)s}{n-s} < 1$, we get

$$\int_0^1 \left(\frac{t}{G(t)} \right)^{\frac{s}{n-s}} dt \leq c \int_0^1 t^{-\frac{(q-1)s}{n-s}} dt < \infty \tag{3.5}$$

and

$$\int_1^\infty \left(\frac{t}{G(t)} \right)^{\frac{s}{n-s}} dt \geq c \int_1^\infty t^{-\frac{(q-1)s}{n-s}} dt = \infty.$$

Thus G satisfies (3.2) with $A = G$.

Next, we will check that G grows essentially more slowly near infinity than $G_{\frac{n}{s}}$. For $t > G(1) = 1$, we recall the definition of $G_{\frac{n}{s}}$ to see

$$\begin{aligned}\frac{G_{\frac{n}{s}}^{-1}(t)}{G^{-1}(t)} &= \frac{H(G^{-1}(t))}{G^{-1}(t)}\\ &= \frac{1}{G^{-1}(t)}\left(\int_0^1 \left(\frac{\tau}{G(\tau)}\right)^{\frac{s}{n-s}} d\tau\right)^{\frac{n-s}{n}} \\ &\quad + \frac{1}{G^{-1}(t)}\left(\int_1^{G^{-1}(t)} \left(\frac{\tau}{G(\tau)}\right)^{\frac{s}{n-s}} d\tau\right)^{\frac{n-s}{n}},\end{aligned} \tag{3.6}$$

where H is as in (3.4) with $A = G$. Let us consider the second term in the right-hand side. Since $\frac{\tau}{G(\tau)}$ is non-increasing,

$$\begin{aligned}\frac{1}{G^{-1}(t)}\left(\int_1^{G^{-1}(t)} \left(\frac{\tau}{G(\tau)}\right)^{\frac{s}{n-s}} d\tau\right)^{\frac{n-s}{n}} &\le \frac{1}{G^{-1}(t)}\left(\frac{1}{G(1)}\right)^{\frac{s}{n}} (G^{-1}(t)-1)^{\frac{n-s}{n}}\\ &\le \frac{(G^{-1}(t))^{\frac{n-s}{n}}}{G^{-1}(t)} = (G^{-1}(t))^{-\frac{s}{n}}.\end{aligned} \tag{3.7}$$

Gathering together (3.5), (3.6) and (3.7) gives

$$\frac{G_{\frac{n}{s}}^{-1}(t)}{G^{-1}(t)} \le \frac{c}{G^{-1}(t)} + (G^{-1}(t))^{-\frac{s}{n}},$$

hence $\lim_{t\to\infty} \frac{G_{\frac{n}{s}}^{-1}(t)}{G^{-1}(t)} = 0$. Therefore, Lemma 3.0.2 directly implies the compact embedding from $W^{s,G}(U)$ to $L^G(U)$ when $s < \frac{n}{q}$.

On the other hand, for the case $s \ge \frac{n}{q}$, a simple modification to [23, Proposition 2.1] shows that the embedding $W^{s,G}(U) \to W^{\tilde{s},G}(U)$ is continuous, i.e., $\|u\|_{W^{\tilde{s},G}(U)} \le c\|u\|_{W^{s,G}(U)}$, for every $\tilde{s} \in (0, s)$. Now take any number $\tilde{s} \in (0, \frac{n}{q})$. Since $W^{s,G}(U) \subset W^{\tilde{s},G}(U)$ and $W^{\tilde{s},G}(U) \to L^G(U)$ is compact, the embedding $W^{s,G}(U) \to L^G(U)$ is also compact when $s \ge \frac{n}{q}$. □

We next recall the following Poincaré type inequality.

Lemma 3.0.4. *[29, Corollary 6.2] Let $U \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Suppose G is an N-function satisfying* (1.3). *Then there exists a constant $c > 0$ depending on n, s, p, q and U such that*

$$\int_U G(|u|)dx \le c \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G\left(\frac{|u(x) - u(y)|}{|x - y|^s}\right) \frac{dxdy}{|x - y|^n}$$

for every $s \in (0, 1)$ and $u \in W^{s,G}(U)$.

To prove the existence and uniqueness of weak solutions to (1.1), we consider the following energy functional

$$\mathcal{I}[v] \coloneqq \iint_{C_\Omega} G\left(\frac{|v(x) - v(y)|}{|x - y|^s}\right) K(x, y)\, dxdy, \tag{3.8}$$

where C_Ω is from (2.10) and

$$v \in \mathcal{A}_f(\Omega) \coloneqq \{v \in \mathbb{W}^{s,G}(\Omega) : \ v = f \quad \text{in} \ \ \mathbb{R}^n \setminus \Omega\}.$$

We say $u \in \mathcal{A}_f(\Omega)$ is a minimizer of $\mathcal{I}$ over $\mathcal{A}_f(\Omega)$ if $\mathcal{I}[u] \le \mathcal{I}[v]$ for all $v \in \mathcal{A}_f(\Omega)$.

Theorem 3.0.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, the operator $\mathcal{L}$ and an N-function G be given as in Chapter 1, and $f \in \mathbb{W}^{s,G}(\Omega)$. Then there exists a unique minimizer u of $\mathcal{I}$ over $\mathcal{A}_f(\Omega)$. Moreover, a function $u \in \mathcal{A}_f(\Omega)$ is the minimizer of $\mathcal{I}$ over $\mathcal{A}_f(\Omega)$ if and only if it is the weak solution to*

$$\begin{cases} \mathcal{L}u = 0 & \text{in} \ \ \Omega, \\ u = f & \text{in} \ \ \mathbb{R}^n \setminus \Omega. \end{cases} \tag{3.9}$$

Proof. **Step 1.** We first prove the existence and uniqueness of minimizer of $\mathcal{I}$. Since $f \in \mathcal{A}_f(\Omega)$, $\mathcal{A}_f(\Omega)$ is nonempty. We now choose a minimizing sequence $\{u_m\}_{m\ge1}$ in $\mathcal{A}_f(\Omega)$ so that $\mathcal{I}[u_m]$ is non-increasing and $\lim_{m\to\infty} \mathcal{I}[u_m] = \inf_{w\in\mathcal{A}_f(\Omega)} \mathcal{I}[w]$. Set $v_m \coloneqq u_m - f$. Then $\{v_m\}_{m\ge1} \subset \mathbb{W}^{s,G}(\Omega)$ and $v_m = 0$ in $\mathbb{R}^n \setminus \Omega$. Choose a ball $B \equiv B_R(0)$ such that $B \supset \Omega$. In order to use the compactness argument, we need to show that $\|v_m\|_{W^{s,G}(B)}$ is bounded for m. Using (2.8), (2.9) and Lemma 3.0.4, and the fact that $v_m = 0$ in $\mathbb{R}^n \setminus \Omega$, we find

$$
\begin{aligned}
\|v_m\|_{W^{s,G}(B)} &\le \int_B G\left(|v_m|\right) dx + \int_B \int_B G\left(\frac{|v_m(x)-v_m(y)|}{|x-y|^s}\right) \frac{dxdy}{|x-y|^n} + 2 \\
&\le c\left[\iint_{C_\Omega} G\left(\frac{|v_m(x)-v_m(y)|}{|x-y|^s}\right) K(x,y)\, dxdy + 2\right] \\
&\le c(\mathcal{I}[u_m] + \mathcal{I}[f] + 2) \le c.
\end{aligned}
$$

Since $\mathcal{I}[u_m]$ is bounded, so is $\|v_m\|_{W^{s,G}(B)}$. By the compactness result in Lemma 3.0.3, there exist a subsequence $\{v_{m_j}\}_{j\ge1}$ and $v \in W^{s,G}(B)$ such that

$$
\begin{cases}
v_{m_j} \rightharpoonup v & \text{weakly in } \; W^{s,G}(B), \\
v_{m_j} \to v & \text{in } \; L^G(B), \qquad\qquad \text{as} \quad j \to \infty. \\
v_{m_j} \to v & \text{a.e. in } \; B,
\end{cases}
$$

Now extend v by zero outside B and set $u = v + f$. Then we see that $u \in \mathcal{A}_f(\Omega)$ and $\lim\limits_{j\to\infty} u_{m_j} = u$ a.e. in $\mathbb{R}^n$. Therefore $v \in \mathbb{W}^{s,G}(\Omega)$ such that $v = 0$ in $\mathbb{R}^n \setminus \Omega$ and so $v + f \in \mathcal{A}_f(\Omega)$. Then Fatou's lemma implies

$$
\mathcal{I}[u] \le \liminf_{j\to\infty} \mathcal{I}[u_{m_j}] = \inf_{w\in\mathcal{A}_f} \mathcal{I}[w].
$$

The uniqueness directly follows from the convexity of G. Indeed, to prove this, we first suppose that $u, v \in \mathcal{A}_f(\Omega)$ are two different minimizers of $\mathcal{I}$. Then $\mathcal{I}[u] = \mathcal{I}[v]$. Since G is strictly convex, we have

$$
\mathcal{I}[u] \le \mathcal{I}\left[\frac{u+v}{2}\right] < \frac{\mathcal{I}[u] + \mathcal{I}[v]}{2} = \mathcal{I}[u],
$$

which is a contradiction.

Step 2. We next show the equivalence between the minimizer of (3.8) and a weak solution to (3.9). Suppose u is the minimizer of (3.8). Then for any $\eta \in \mathbb{W}^{s,G}(\Omega)$ with $\eta = 0$ in $\mathbb{R}^n \setminus \Omega$, $\mathcal{I}[u + \tau\eta]$ has a critical point at $\tau = 0$.

Thus

$$\begin{aligned}
0 &= \frac{d}{d\tau}\mathcal{I}[u+\tau\eta]\Big|_{\tau=0} \\
&= \iint_{C_\Omega} \frac{d}{d\tau} G\left(\frac{|u(x)-u(y)+\tau(\eta(x)-\eta(y))|}{|x-y|^s}\right)\Big|_{\tau=0} K(x,y)\,dxdy \\
&= \iint_{C_\Omega} g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|u(x)-u(y)|}(\eta(x)-\eta(y))K(x,y)\,\frac{dxdy}{|x-y|^s}.
\end{aligned}$$

Therefore u is a weak solution to (3.9).

On the other hand, suppose u is a weak solution to (3.9). Then for any $v \in \mathcal{A}_f(\Omega)$, we see that $u - v \in \mathbb{W}^{s,G}(\Omega)$ and that $u - v = 0$ in $\mathbb{R}^n \setminus \Omega$. We then test $\eta \coloneqq u - v$ in the weak formulation of (3.9) to discover

$$\begin{aligned}
0 = &\iint_{C_\Omega} g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)|u(x)-u(y)|K(x,y)\frac{dxdy}{|x-y|^s} \\
&- \iint_{C_\Omega} g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|u(x)-u(y)|}(v(x)-v(y))K(x,y)\frac{dxdy}{|x-y|^s}.
\end{aligned} \tag{3.10}$$

Let us look at the integrand of the second term with respect to the measure $K(x,y)\,dxdy$ on the right-hand side. From (2.5) and (2.7), we see

$$\begin{aligned}
g&\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|u(x)-u(y)|}\frac{v(x)-v(y)}{|x-y|^s} \\
&\le g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{|v(x)-v(y)|}{|x-y|^s} \\
&\le G^*\left(g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\right) + G\left(\frac{|v(x)-v(y)|}{|x-y|^s}\right) \\
&= \frac{|u(x)-u(y)|}{|x-y|^s} g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right) \\
&\quad - G\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right) + G\left(\frac{|v(x)-v(y)|}{|x-y|^s}\right).
\end{aligned} \tag{3.11}$$

We combine (3.10) and (3.11) to conclude that $\mathcal{I}[v] \geq \mathcal{I}[u]$. Therefore u is the minimizer of $\mathcal{I}$. □

Chapter 4

Hölder regularity

4.1 Auxiliary estimates

In this section we derive two estimates for the weak solutions to (1.1) that play essential roles in the proof of the main theorems. The first one is a Caccioppoli type estimate. A similar Caccioppoli type estimate in the Orlicz setting can be also found in [15].

Proposition 4.1.1 (Caccioppoli type estimate)**.** *Let* $u \in \mathbb{W}^{s,G}(\Omega) \cap L^g_s(\mathbb{R}^n)$ *be a weak solution to* (1.1)*. Then for any* $k \geq 0$*,* $B_r \equiv B_r(x_0) \Subset \Omega$ *and* $\phi \in C_0^\infty(B_r)$ *with* $0 \leq \phi \leq 1$*, we have*

$$\begin{aligned}
&\int_{B_r}\int_{B_r} G\left(\frac{|w_\pm(x)-w_\pm(y)|}{|x-y|^s}\right)\min\{\phi^q(x),\phi^q(y)\}\frac{dxdy}{|x-y|^n} \\
&\quad \leq c\int_{B_r}\int_{B_r} G\left(\frac{|\phi(x)-\phi(y)|}{|x-y|^s}\max\{w_\pm(x),w_\pm(y)\}\right)\frac{dxdy}{|x-y|^n} \\
&\qquad + c\int_{B_r} w_\pm(x)\phi^q(x)\,dx\left(\sup_{y\,\in\,\mathrm{supp}\,\phi}\int_{\mathbb{R}^n\setminus B_r} g\left(\frac{w_\pm(x)}{|x-y|^s}\right)\frac{dx}{|x-y|^{n+s}}\right),
\end{aligned} \tag{4.1}$$

where $w_\pm := (u-k)_\pm = \max\{\pm(u-k),0\}$ *and* $c > 0$ *depends on* n, s, p, q, λ *and* Λ*.*

Proof. We only consider w_+, as the same argument can apply to w_-. Take

$\eta \coloneqq w_+\phi^q \in \mathbb{W}^{s,p}(\Omega)$ as a test function to find

$$
\begin{aligned}
0 &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|u(x)-u(y)|}(\eta(x)-\eta(y))K(x,y)\frac{dxdy}{|x-y|^s}\\
&= \int_{B_r}\int_{B_r} g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|u(x)-u(y)|}(\eta(x)-\eta(y))K(x,y)\frac{dxdy}{|x-y|^s}\\
&\quad + 2\int_{\mathbb{R}^n\setminus B_r}\int_{B_r} g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|u(x)-u(y)|}\eta(x)K(x,y)\frac{dxdy}{|x-y|^s}\\
&=: I + II.
\end{aligned}
\tag{4.2}
$$

Note that $\eta(x) = 0$ for $x \in B_r \cap \{u(x) < k\}$. We divide the latter part of the proof into two steps.

Step 1. In this step we derive an estimate in terms of w_+ from (4.2). We first consider the integrand of I with respect to the measure, $K(x,y)\frac{dxdy}{|x-y|^s}$. In the case when $u(x) \geq u(y)$ for $x, y \in B_r$, we have

$$
\begin{aligned}
& g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|u(x)-u(y)|}(\eta(x)-\eta(y))\\
&= g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)(\eta(x)-\eta(y))\\
&= \begin{cases} g\left(\frac{w_+(x)-w_+(y)}{|x-y|^s}\right)(\eta(x)-\eta(y)) & \text{if } \ u(x)\geq u(y)\geq k,\\ g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\eta(x) & \text{if } \ u(x)\geq k > u(y),\\ 0 & \text{if } \ k > u(x) \geq u(y)\end{cases} \qquad (4.3)\\
&\geq g\left(\frac{w_+(x)-w_+(y)}{|x-y|^s}\right)(\eta(x)-\eta(y))\\
&= g\left(\frac{|w_+(x)-w_+(y)|}{|x-y|^s}\right)\frac{w_+(x)-w_+(y)}{|w_+(x)-w_+(y)|}(\eta(x)-\eta(y)).
\end{aligned}
$$

On the other hand, in the case when $u(x) < u(y)$ for $x, y \in B_r$, we exchange the roles of x and y in (4.3) to obtain the same result. Then we recall the assumption (1.4) to get

$$I \geq \lambda \int_{B_r}\int_{B_r} g\left(\frac{|w_+(x)-w_+(y)|}{|x-y|^s}\right)\frac{w_+(x)-w_+(y)}{|w_+(x)-w_+(y)|}(\eta(x)-\eta(y))\frac{dxdy}{|x-y|^{n+s}}. \tag{4.4}$$

Next, let us consider II. Note that

$$\begin{aligned} &g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|u(x)-u(y)|}\eta(x) \\ &\qquad \geq \begin{cases} -g\left(\frac{w_+(y)}{|x-y|^s}\right)\eta(x) & \text{if } u(y)>u(x)\geq k, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Inserting this inequality into II, we deduce

$$II \geq -2\Lambda \int_{\mathbb{R}^n\setminus B_r}\int_{B_r} g\left(\frac{w_+(y)}{|x-y|^s}\right)\eta(x)\frac{dxdy}{|x-y|^{n+s}}. \tag{4.5}$$

We then combine (4.2), (4.4), and (4.5) to discover

$$\begin{aligned} &\int_{B_r}\int_{B_r} g\left(\frac{|w_+(x)-w_+(y)|}{|x-y|^s}\right)\frac{w_+(x)-w_+(y)}{|w_+(x)-w_+(y)|}(\eta(x)-\eta(y))\frac{dxdy}{|x-y|^{n+s}} \\ &\quad \leq \frac{2\Lambda}{\lambda}\int_{\mathbb{R}^n\setminus B_r}\int_{B_r} g\left(\frac{w_+(y)}{|x-y|^s}\right)\eta(x)\frac{dxdy}{|x-y|^{n+s}}. \end{aligned} \tag{4.6}$$

Step 2. Set

$$III \coloneqq \int_{B_r}\int_{B_r} g\left(\frac{|w_+(x)-w_+(y)|}{|x-y|^s}\right)\frac{w_+(x)-w_+(y)}{|w_+(x)-w_+(y)|}(\eta(x)-\eta(y))\frac{dxdy}{|x-y|^{n+s}}$$

and

$$IV \coloneqq \int_{\mathbb{R}^n\setminus B_r}\int_{B_r} g\left(\frac{w_+(y)}{|x-y|^s}\right)\eta(x)\frac{dxdy}{|x-y|^{n+s}}.$$

Then we see from (4.6) that $III \leq \frac{2\Lambda}{\lambda}IV$.

To estimate III, we first look at the integrand of III with respect to the measure $\frac{dxdy}{|x-y|^n}$. Consider the following three cases:

(1) $w_+(x) > w_+(y)$ and $\phi(x) \le \phi(y)$,

(2) $w_+(x) > w_+(y)$ and $\phi(x) > \phi(y)$,

(3) $w_+(x) \le w_+(y)$.

In the case (1), we have

$$\begin{aligned}
&g\left(\frac{|w_+(x)-w_+(y)|}{|x-y|^s}\right)\frac{w_+(x)-w_+(y)}{|w_+(x)-w_+(y)|}\frac{w_+(x)\phi^q(x)-w_+(y)\phi^q(y)}{|x-y|^s}\\
&\quad = g\left(\frac{w_+(x)-w_+(y)}{|x-y|^s}\right)\frac{w_+(x)-w_+(y)}{|x-y|^s}\phi^q(y)\\
&\qquad - g\left(\frac{w_+(x)-w_+(y)}{|x-y|^s}\right)\frac{\phi^q(y)-\phi^q(x)}{|x-y|^s}w_+(x)\\
&\quad \ge pG\left(\frac{w_+(x)-w_+(y)}{|x-y|^s}\right)\phi^q(y)\\
&\qquad - qg\left(\frac{w_+(x)-w_+(y)}{|x-y|^s}\right)\phi^{q-1}(y)\frac{\phi(y)-\phi(x)}{|x-y|^s}w_+(x),
\end{aligned} \tag{4.7}$$

where we have used (1.3) and the following elementary inequality

$$\phi^q(y)-\phi^q(x) \le q\phi^{q-1}(y)(\phi(y)-\phi(x)).$$

We further estimate the second term on the right-hand side of (4.7). By using (2.6) and (2.7), we have that for $\epsilon \in (0,1)$,

$$\begin{aligned}
&g\left(\frac{w_+(x)-w_+(y)}{|x-y|^s}\right)\phi^{q-1}(y)\frac{\phi(y)-\phi(x)}{|x-y|^s}w_+(x)\\
&\quad \le \epsilon G^*\left(g\left(\frac{w_+(x)-w_+(y)}{|x-y|^s}\right)\phi^{q-1}(y)\right)+c(\epsilon)G\left(\frac{\phi(y)-\phi(x)}{|x-y|^s}w_+(x)\right)\\
&\quad \le \epsilon(q-1)G\left(\frac{w_+(x)-w_+(y)}{|x-y|^s}\right)\phi^q(y)+c(\epsilon)G\left(\frac{\phi(y)-\phi(x)}{|x-y|^s}w_+(x)\right).
\end{aligned} \tag{4.8}$$

For the last inequality, we have used (2.2) with $a=\phi^{q-1}(y)\le 1$. Choosing

$\epsilon = \min\left\{\frac{p}{2(q-1)}, \frac{1}{2}\right\}$ and plugging (4.8) into (4.7), we discover

$$
\begin{aligned}
&g\left(\frac{|w_+(x)-w_+(y)|}{|x-y|^s}\right)\frac{w_+(x)-w_+(y)}{|w_+(x)-w_+(y)|}\frac{w_+(x)\phi^q(x)-w_+(y)\phi^q(y)}{|x-y|^s}\\
&\quad\geq \frac{p}{2}G\left(\frac{|w_+(x)-w_+(y)|}{|x-y|^s}\right)\min\{\phi^q(x),\phi^q(y)\} \qquad (4.9)\\
&\quad - cG\left(\frac{|\phi(x)-\phi(y)|}{|x-y|^s}\max\{w_+(x),w_+(y)\}\right).
\end{aligned}
$$

In the case (2), we use (1.3) to have

$$
\begin{aligned}
&g\left(\frac{|w_+(x)-w_+(y)|}{|x-y|^s}\right)\frac{w_+(x)-w_+(y)}{|w_+(x)-w_+(y)|}\frac{w_+(x)\phi^q(x)-w_+(y)\phi^q(y)}{|x-y|^s}\\
&\quad\geq g\left(\frac{w_+(x)-w_+(y)}{|x-y|^s}\right)\frac{w_+(x)-w_+(y)}{|x-y|^s}\phi^q(x)\\
&\quad\geq pG\left(\frac{|w_+(x)-w_+(y)|}{|x-y|^s}\right)\min\{\phi^q(x),\phi^q(y)\}.
\end{aligned}
$$

Therefore, we also obtain the estimate (4.9) in this case. Moreover, since the integrand is invariant with the exchanging of x and y, we again have the estimate (4.9) in the case (3). Consequently, we obtain

$$
\begin{aligned}
III \geq\ & c\int_{B_r}\int_{B_r} G\left(\frac{|w_+(x)-w_+(y)|}{|x-y|^s}\right)\min\{\phi^q(x),\phi^q(y)\}\frac{dxdy}{|x-y|^n}\\
&- c\int_{B_r}\int_{B_r} G\left(\frac{|\phi(x)-\phi(y)|}{|x-y|^s}\max\{w_+(x),w_+(y)\}\right)\frac{dxdy}{|x-y|^n}.
\end{aligned}
$$

To estimate IV, we first use Fubini's theorem to find

$$
\begin{aligned}
IV &= \int_{\mathbb{R}^n\setminus B_r}\int_{B_r} g\left(\frac{w_+(y)}{|x-y|^s}\right)\eta(x)\frac{dxdy}{|x-y|^{n+s}}\\
&\leq \int_{B_r} w_+(x)\phi^q(x)\,dx\left(\sup_{x\,\in\,\mathrm{supp}\,\phi}\int_{\mathbb{R}^n\setminus B_r} g\left(\frac{w_+(y)}{|x-y|^s}\right)\frac{dy}{|x-y|^{n+s}}\right).
\end{aligned}
$$

Hence we obtain (4.1), as $III \leq cIV$. □

Remark 4.1.2. In Proposition 4.1.1, the estimate (4.1) for w_+ (resp. w_-) still holds true when u is a weak subsolution (resp. supersolution) to (1.1).

The second one to be derived is a logarithmic estimate. This will be used in the proof of the decay estimate for the oscillation of weak solutions, Lemma 4.4.1. We need the following elementary inequality.

Lemma 4.1.3. *[22, Lemma 3.1] Let $q \geq 1$ and $\epsilon \in (0,1]$. Then*

$$|a|^q \leq (1+c_q\epsilon)|b|^q + (1+c_q\epsilon)\epsilon^{1-q}|a-b|^q$$

for every $a, b \in \mathbb{R}^n$. Here $c_q > 0$ depends on n and q.

Proposition 4.1.4 (Logarithmic estimate)**.** *Let $u \in \mathbb{W}^{s,G}(\Omega) \cap L^g_s(\mathbb{R}^n)$ be a weak supersolution to* (1.1) *with $u \geq 0$ in $B_R \equiv B_R(x_0) \subset \Omega$. Then for any $d > 0$ and $0 < r < \frac{R}{2}$, we have*

$$\begin{aligned}\int_{B_r}\int_{B_r} &|\log(u(x)+d) - \log(u(y)+d)|\frac{dxdy}{|x-y|^n} \\ &\leq cr^n + c\frac{r^{n+s}}{g(d/r^s)}\mathrm{Tail}(u_-; x_0, R)\end{aligned} \tag{4.10}$$

for some $c = c(n, s, p, q, \lambda, \Lambda) > 0$. In addition, we have the estimate

$$\int_{B_r} |h - (h)_{B_r}|\, dx \leq cr^n\left[1 + \frac{r^s}{g(d/r^s)}\mathrm{Tail}(u_-; x_0, R)\right], \tag{4.11}$$

where

$$h := \min\left\{(\log(a+d) - \log(u+d))_+, \log b\right\}, \quad a > 0 \quad \text{and} \quad b > 1.$$

Proof. Write $v(x) := u(x) + d$ and fix a cut-off function $\phi \in C_0^\infty(B_{3r/2})$ such that $0 \leq \phi \leq 1$, $|D\phi| \leq 4/r$ and $\phi \equiv 1$ in B_r. Since $\frac{v}{G(v/r^s)}$ is nonnegative in B_R and belongs to $\mathbb{W}^{s,G}(\Omega)$, we can take $\eta = \frac{v\phi^q}{G(v/r^s)}$ as a test function to find

$$\begin{aligned}0 \leq &\int_{B_{2r}}\int_{B_{2r}} g\left(\frac{|v(x)-v(y)|}{|x-y|^s}\right)\frac{v(x)-v(y)}{|v(x)-v(y)|}(\eta(x)-\eta(y))K(x,y)\frac{dxdy}{|x-y|^s} \\ &+ 2\int_{\mathbb{R}^n\setminus B_{2r}}\int_{B_{2r}} g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|u(x)-u(y)|}\eta(x)K(x,y)\frac{dxdy}{|x-y|^s} \\ =: &\ I + II.\end{aligned} \tag{4.12}$$

We define

$$F = F(x,y) := g\left(\frac{|v(x)-v(y)|}{|x-y|^s}\right)\frac{v(x)-v(y)}{|v(x)-v(y)|}\frac{\eta(x)-\eta(y)}{|x-y|^s}, \quad x,y\in B_{2r}.$$

Note that $F(x,y) = F(y,x)$. We divide the remaining proof into five steps.

Step 1. We first assume that $v(y) \le v(x) \le 2v(y)$ for $x, y \in B_{2r}$ to assert that

$$F(x,y) \le -\tilde{c}(\log v(x) - \log v(y))\phi(x)^q + c\left(\frac{|x-y|}{r}\right)^s + c\left(\frac{|x-y|}{r}\right)^{(1-s)p} \tag{4.13}$$

for some small constant $\tilde{c} > 0$ and large constant $c > 0$ depending on n, p and q. To prove this, let us suppose $\phi(x) \ge \phi(y)$. By the definition of η, we get

$$\begin{aligned} F(x,y) &= g\left(\frac{v(x)-v(y)}{|x-y|^s}\right)\left(\frac{v(x)}{G(v(x)/r^s)} - \frac{v(y)}{G(v(y)/r^s)}\right)\frac{\phi^q(x)}{|x-y|^s} \\ &\quad + g\left(\frac{v(x)-v(y)}{|x-y|^s}\right)\frac{v(y)}{G(v(y)/r^s)}\frac{\phi^q(x)-\phi^q(y)}{|x-y|^s} \\ &=: F_1(x,y) + F_2(x,y). \end{aligned} \tag{4.14}$$

Before estimating F_1 and F_2, we apply Mean Value Theorem to the mapping $t \mapsto \frac{t}{G(t/r^s)}$ for $v(y) \le t \le v(x)$ and use the inequality

$$\left(\frac{t}{G(t/r^s)}\right)' = \frac{G(t/r^s) - (t/r^s)g(t/r^s)}{G^2(t/r^s)} \le -\frac{p-1}{G(t/r^s)} \qquad \text{(by (1.3))},$$

to find

$$\frac{v(x)}{G(v(x)/r^s)} - \frac{v(y)}{G(v(y)/r^s)} \le -(p-1)\frac{v(x)-v(y)}{G(v(x)/r^s).} \tag{4.15}$$

We again apply Mean value theorem to the mapping $t \mapsto t^q$ for $\phi(y) \le t \le \phi(x)$ to have

$$\phi^q(x) - \phi^q(y) \le q\phi^{q-1}(x)(\phi(x)-\phi(y)). \tag{4.16}$$

Putting (4.15) into F_1 and using (1.3) and the fact that $v(x) \leq 2v(y)$, we have

$$\begin{aligned} F_1 &\leq -(p-1)g\left(\frac{v(x)-v(y)}{|x-y|^s}\right)\frac{v(x)-v(y)}{|x-y|^s}\frac{\phi^q(x)}{G(v(x)/r^s)} \\ &\leq -c_1 G\left(\frac{v(x)-v(y)}{|x-y|^s}\right)\frac{\phi^q(x)}{G(v(y)/r^s)} \end{aligned} \tag{4.17}$$

for some small constant $c_1 = c_1(p,q) > 0$.

We use (4.16) and recall (2.6) with $\epsilon = \min\left\{\frac{c_1}{2q(q-1)}, \frac{1}{2}\right\}$ and (2.7), to discover

$$\begin{aligned} F_2 &\leq qg\left(\frac{v(x)-v(y)}{|x-y|^s}\right)\phi^{q-1}(x)\frac{\phi(x)-\phi(y)}{|x-y|^s}\frac{v(y)}{G(v(y)/r^s)} \\ &\leq q\Bigg[\epsilon(q-1)G\left(\frac{v(x)-v(y)}{|x-y|^s}\right)\phi^q(x) \\ &\qquad + \epsilon^{1-q}G\left(\frac{\phi(x)-\phi(y)}{|x-y|^s}v(y)\right)\Bigg]\frac{1}{G(v(y)/r^s)} \\ &\leq \left[\frac{c_1}{2}G\left(\frac{v(x)-v(y)}{|x-y|^s}\right)\phi^q(x) + cG\left(\frac{\phi(x)-\phi(y)}{|x-y|^s}v(y)\right)\right]\frac{1}{G(v(y)/r^s)}. \end{aligned} \tag{4.18}$$

We then combine (4.14), (4.17), and (4.18) and use the fact that $|D\phi| \leq 4/r$ and $|x-y| \leq 4r$ for $x, y \in B_{2r}$, to obtain (4.13).

We next suppose $\phi(x) < \phi(y)$. Using (4.15) and (1.3), we have

$$\begin{aligned} F(x,y) &= g\left(\frac{v(x)-v(y)}{|x-y|^s}\right)\left(\frac{\phi^q(x)v(x)}{G(v(x)/r^s)} - \frac{\phi^q(y)v(y)}{G(v(y)/r^s)}\right)\frac{1}{|x-y|^s} \\ &\leq g\left(\frac{v(x)-v(y)}{|x-y|^s}\right)\left(\frac{v(x)}{G(v(x)/r^s)} - \frac{v(y)}{G(v(y)/r^s)}\right)\frac{\phi^q(y)}{|x-y|^s} \\ &\leq -cg\left(\frac{v(x)-v(y)}{|x-y|^s}\right)\frac{v(x)-v(y)}{|x-y|^s}\frac{\phi^q(y)}{G(v(x)/r^s)} \\ &\leq -cG\left(\frac{v(x)-v(y)}{|x-y|^s}\right)\frac{\phi^q(x)}{G(v(y)/r^s)}. \end{aligned}$$

Therefore we also have

$$F(x,y) \le -cG\left(\frac{v(x)-v(y)}{|x-y|^s}\right)\frac{\phi^q(x)}{G(v(y)/r^s)} + c\left(\frac{|x-y|}{r}\right)^{(1-s)p}. \tag{4.19}$$

In addition, by Mean Value Theorem,

$$\begin{aligned}\log v(x) - \log v(y) &\le \frac{v(x)-v(y)}{v(y)} = \frac{(v(x)-v(y))/|x-y|^s}{v(y)/r^s}\frac{|x-y|^s}{r^s} \\ &\le \left\{G\left(\frac{v(x)-v(y)}{|x-y|^s}\right)\frac{1}{G(v(y)/r^s)} + 1\right\}\frac{|x-y|^s}{r^s} \\ &\le cG\left(\frac{v(x)-v(y)}{|x-y|^s}\right)\frac{1}{G(v(y)/r^s)} + \frac{|x-y|^s}{r^s}.\end{aligned}$$

For the second inequality we have used the fact that $\frac{G(t)}{t}$ is increasing for t. This estimate and (4.19) imply finally (4.13).

Step 2. We now assume that $v(x) > 2v(y)$ for $x, y \in B_{2r}$ to claim that

$$\begin{aligned}F(x,y) &\le -\tilde{c}(\log v(x) - \log v(y))\phi^q(y) \\ &\quad + c\left(\frac{|x-y|}{r}\right)^{s(p-1)} + c\left(\frac{|x-y|}{r}\right)^{(1-s)q}\end{aligned} \tag{4.20}$$

for some small constant $\tilde{c} > 0$ and large constant $c > 0$ depending on n, p and q. To this end, we recall the definition of η to see that

$$\begin{aligned}F(x,y) &= g\left(\frac{v(x)-v(y)}{|x-y|^s}\right)\left(\frac{v(x)}{G(v(x)/r^s)} - \frac{v(y)}{G(v(y)/r^s)}\right)\frac{\phi^q(y)}{|x-y|^s} \\ &\quad + g\left(\frac{v(x)-v(y)}{|x-y|^s}\right)\frac{v(x)}{G(v(x)/r^s)}\frac{\phi^q(x)-\phi^q(y)}{|x-y|^s} \\ &=: F_3(x,y) + F_4(x,y).\end{aligned}$$

Since $\frac{t}{G(t)}$ is decreasing for t, we have

$$\begin{aligned}
F_3 &\leq g\left(\frac{v(x)-v(y)}{|x-y|^s}\right)\left(\frac{2v(y)}{G(2v(y)/r^s)}-\frac{v(y)}{G(v(y)/r^s)}\right)\frac{\phi^q(y)}{|x-y|^s}\\
&= g\left(\frac{v(x)-v(y)}{|x-y|^s}\right)\frac{v(y)}{G(v(y)/r^s)}\left(2\frac{G(v(y)/r^s)}{G(2v(y)/r^s)}-1\right)\frac{\phi^q(y)}{|x-y|^s}\\
&\leq -\left(1-\frac{1}{2^{p-1}}\right)g\left(\frac{v(x)-v(y)}{|x-y|^s}\right)\frac{v(y)}{G(v(y)/r^s)}\frac{\phi^q(y)}{|x-y|^s}.
\end{aligned}$$

On the other hand, in light of Lemma 4.1.3, we find that for $\epsilon \in (0,1)$,

$$\begin{aligned}
F_4 \leq c_q\epsilon g&\left(\frac{v(x)-v(y)}{|x-y|^s}\right)\frac{v(x)}{G(v(x)/r^s)}\frac{\phi^q(y)}{|x-y|^s}\\
&+c\epsilon^{1-q}g\left(\frac{v(x)-v(y)}{|x-y|^s}\right)\frac{v(x)}{G(v(x)/r^s)}\frac{|\phi(x)-\phi(y)|^q}{|x-y|^s}.
\end{aligned}$$

In addition, using the fact that $\frac{t}{G(t)}$ is decreasing for t, $2v(y) < v(x)$, $|D\phi| \leq c/r$, $|x-y| \leq 4r$ for $x, y \in B_{2r}$ and $tg(t) \leq qG(t)$, we discover

$$\begin{aligned}
F_4 &\leq c_q\epsilon g\left(\frac{v(x)-v(y)}{|x-y|^s}\right)\frac{v(y)}{G(v(y)/r^s)}\frac{\phi^q(y)}{|x-y|^s}\\
&\quad + c\epsilon^{1-q}r^s\left(\frac{v(x)-v(y)}{v(x)}\right)^{p-1}\left(\frac{r}{|x-y|}\right)^{s(q-1)}\frac{|\phi(x)-\phi(y)|^q}{|x-y|^s}\\
&\leq c_q\epsilon g\left(\frac{v(x)-v(y)}{|x-y|^s}\right)\frac{v(y)}{G(v(y)/r^s)}\frac{\phi^q(y)}{|x-y|^s}+c\epsilon^{1-q}\left(\frac{|x-y|}{r}\right)^{(1-s)q}.
\end{aligned}$$

We then choose $\epsilon = \min\left\{\frac{1}{2c_q}\left(1-\frac{1}{2^{p-1}}\right), \frac{1}{2}\right\}$, and combine the above estimates to discover

$$F \leq -cg\left(\frac{v(x)-v(y)}{|x-y|^s}\right)\frac{v(y)}{G(v(y)/r^s)}\frac{\phi^q(y)}{|x-y|^s}+c\left(\frac{|x-y|}{r}\right)^{(1-s)q}.$$

Note that

$$\frac{v(y)}{G(v(y)/r^s)}\frac{1}{|x-y|^s} \geq \frac{1}{4^s}\frac{v(y)/r^s}{G(v(y)/r^s)} \geq \frac{p}{4^s}\frac{1}{g(v(y)/r^s)}$$

to have

$$F(x,y) \le -cg\left(\frac{v(x)-v(y)}{|x-y|^s}\right)\frac{\phi^q(y)}{g(v(y)/r^s)} + c\left(\frac{|x-y|}{r}\right)^{(1-s)q}. \tag{4.21}$$

Moreover, since $v(x) > 2v(y)$,

$$\log v(x) - \log v(y) \le \log 2(v(x)-v(y)) - \log v(y) \le c\left(\frac{2(v(x)-v(y))}{v(y)}\right)^{p-1},$$

where we have used the fact that $\log t < \frac{t^{p-1}}{p-1}$. Note that

$$\frac{g(s)}{s^{p-1}} \le q\frac{G(s)}{s^p} \le q\frac{G(t)}{t^p} \le \frac{q}{p}\frac{g(t)}{t^{p-1}} \quad \text{for any} \quad t \ge s > 0,$$

to discover

$$\begin{aligned}\log v(x) - \log v(y) &\le c\left(\frac{(v(x)-v(y))/|x-y|^s}{v(y)/r^s}\frac{|x-y|^s}{r^s}\right)^{p-1}\\ &\le cg\left(\frac{v(x)-v(y)}{|x-y|^s}\right)\frac{1}{g(v(y)/r^s)} + c\left(\frac{|x-y|}{r}\right)^{s(p-1)}.\end{aligned}$$

This and (4.21) imply the estimate (4.20).

Step 3. We next estimate I in (4.12). We recall (4.13) when $v(y) \le v(x) \le 2v(y)$, and (4.20) when $v(x) > 2v(y)$, and use the fact $F(x,y) = F(y,x)$, to discover that for every $x, y \in B_{2r}$,

$$\begin{aligned}F(x,y) \le& -\tilde{c}\,|\log v(x) - \log v(y)|\min\{\phi(x),\phi(y)\}^q\\ &+ c\left(\frac{|x-y|}{r}\right)^s + c\left(\frac{|x-y|}{r}\right)^{(1-s)p} + c\left(\frac{|x-y|}{r}\right)^{s(p-1)}.\end{aligned}$$

Then since $\phi \equiv 1$ in B_r and $K(x,y)$ satisfies (1.4), we have

$$\begin{aligned} I &\le -\frac{\tilde{c}}{\lambda}\int_{B_r}\int_{B_r} |\log v(x) - \log v(y)| \frac{dxdy}{|x-y|^n} \\ &\quad + c\int_{B_{2r}}\int_{B_{2r}} \left[\left(\frac{|x-y|}{r}\right)^s + \left(\frac{|x-y|}{r}\right)^{(1-s)p} + \left(\frac{|x-y|}{r}\right)^{s(p-1)}\right]\frac{dxdy}{|x-y|^n} \\ &\le -\frac{\tilde{c}}{\lambda}\int_{B_r}\int_{B_r} |\log v(x) - \log v(y)| \frac{dxdy}{|x-y|^n} + cr^n. \end{aligned} \tag{4.22}$$

Step 4. We next estimate II. Observe that for $x \in B_R$ and $y \in \mathbb{R}^n$,

$$\begin{aligned} g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|u(x)-u(y)|} &\le g\left(\frac{(u(x)-u(y))_+}{|x-y|^s}\right) \\ &\le c\left[g\left(\frac{u(x)}{|x-y|^s}\right) + g\left(\frac{u(y)_-}{|x-y|^s}\right)\right]. \end{aligned}$$

Recalling $\operatorname{supp}\phi \subset B_{3r/2}$, we have

$$\begin{aligned} II &\le c\int_{\mathbb{R}^n\setminus B_{2r}}\int_{B_{3r/2}} g\left(\frac{(u(x)-u(y))_+}{|x-y|^s}\right)\frac{r^s}{g(v(x)/r^s)}\frac{dxdy}{|x-y|^{n+s}} \\ &\le c\int_{B_R\setminus B_{2r}}\int_{B_{3r/2}} g\left(\frac{(u(x)-u(y))_+}{|x-y|^s}\right)\frac{r^s}{g(v(x)/r^s)}\frac{dxdy}{|x-y|^{n+s}} \\ &\quad + c\int_{\mathbb{R}^n\setminus B_R}\int_{B_{3r/2}} g\left(\frac{u(x)}{|x-y|^s}\right)\frac{r^s}{g(v(x)/r^s)}\frac{dxdy}{|x-y|^{n+s}} \\ &\quad + c\int_{\mathbb{R}^n\setminus B_R}\int_{B_{3r/2}} g\left(\frac{u(y)_-}{|x-y|^s}\right)\frac{r^s}{g(v(x)/r^s)}\frac{dxdy}{|x-y|^{n+s}} \\ &=: II_1 + II_2 + II_3. \end{aligned}$$

Since $u \ge 0$ in B_R and $v = u + d$, we see that

$$(u(x)-u(y))_+ \le v(x) \quad \text{and} \quad u(x) \le v(x), \qquad x, y \in B_R.$$

Thus

$$
\begin{aligned}
II_1 &\le c\int_{B_R\setminus B_{2r}}\int_{B_{3r/2}} g\left(\frac{(u(x)-u(y))_+}{r^s}\right)\frac{r^s}{g(v(x)/r^s)}\frac{dxdy}{|x-y|^{n+s}}\\
&\le cr^s\int_{B_R\setminus B_{2r}}\int_{B_{3r/2}}\frac{dxdy}{|x-y|^{n+s}}\\
&\le cr^s\int_{B_{3r/2}}\int_{\mathbb{R}^n\setminus B_{r/2}(x)}\frac{dydx}{|x-y|^{n+s}}\le cr^n
\end{aligned}
$$

and

$$
\begin{aligned}
II_2 &\le c\int_{\mathbb{R}^n\setminus B_R}\int_{B_{3r/2}} g\left(\frac{u(x)}{r^s}\right)\frac{r^s}{g(v(x)/r^s)}\frac{dxdy}{|x-y|^{n+s}}\\
&\le cr^s\int_{\mathbb{R}^n\setminus B_{2r}}\int_{B_{3r/2}}\frac{dxdy}{|x-y|^{n+s}}\le cr^n.
\end{aligned}
$$

Observing that for any $x\in B_{3r/2}$ and $y\in\mathbb{R}^n\setminus B_{2r}$

$$
\frac{|y-x_0|}{|x-y|}\le 1+\frac{|x-x_0|}{|x-y|}\le 1+\frac{3r/2}{2r-(3r/2)}=4,
$$

we find

$$
\begin{aligned}
II_3 &\le c\int_{\mathbb{R}^n\setminus B_R}\int_{B_{3r/2}} g\left(\frac{u(y)_-}{|y-x_0|^s}\right)\frac{r^s}{g(d/r^s)}\frac{dxdy}{|y-x_0|^{n+s}}\\
&\le c\frac{r^{n+s}}{g(d/r^s)}\mathrm{Tail}(u_-;x_0,R).
\end{aligned}
$$

Consequently, we have

$$
II\le cr^n+c\frac{r^{n+s}}{g(d/r^s)}\mathrm{Tail}(u_-;x_0,R).
$$

Inserting this estimate and (4.22) into (4.12), we get (4.10).

Step 5. Now we are ready to prove the estimate (4.11). Observe that

$$\int_{B_r} |h-(h)_{B_r}|dx \leq c\int_{B_r}\int_{B_r} |h(x)-h(y)|\frac{dxdy}{|x-y|^n}.$$

Since $h(x)$ is a truncation of $\log v(x)$,

$$\int_{B_r}\int_{B_r} |h(x)-h(y)|\frac{dxdy}{|x-y|^n} \leq c\int_{B_r}\int_{B_r} |\log v(x)-\log v(y)|\frac{dxdy}{|x-y|^n}.$$

Combining (4.10) and the above inequalities, we finally obtain (4.11). □

4.2 Sobolev-Poincaré inequality

We notice that the Sobolev inequality and the Sobolev-Poincaré inequality for the fractional Orlicz-Sobolev space $W^{s,G}(B_r)$ are well known in terms of the Luxemburg norms. However, it does not directly imply a certain integral version of the Sobolev-Poincaré inequality. For the sake of completeness, we need to prove the following Sobolev-Poincaré inequality for functions in $W^{s,G}(B_r)$.

Lemma 4.2.1 (Sobolev-Poincaré inequality)**.** *Let $s \in (0,1)$. Then there exists $\theta = \theta(n,s) > 1$ such that if G is an N-function satisfying the Δ_2 condition* (2.3) *and the ∇_2 condition* (2.4) *with constants κ and l, and $f \in W^{s,G}(B_r)$, then*

$$\left(\fint_{B_r} G\left(\frac{|f-(f)_{B_r}|}{r^s}\right)^{\theta} dx\right)^{\frac{1}{\theta}} \leq c\fint_{B_r}\int_{B_r} G\left(\frac{|f(x)-f(y)|}{|x-y|^s}\right)\frac{dy\,dx}{|x-y|^n}, \tag{4.23}$$

where $c = c(n,s,\kappa,l) > 0$.

Proof. We first show that

$$|f(x)-(f)_{B_r}| \leq c\int_{B_r}\left[\int_{B_r}\frac{|f(y)-f(z)|}{|y-z|^{n+s}}\,dz\right]\frac{dy}{|x-y|^{n-s}}, \quad \text{a.e. } x \in B_r, \tag{4.24}$$

by using a standard chain argument (see for instance [25] and references therein). Fix any Lebesgue's point $x \in B_r$ for f. For each $i \in \mathbb{N}_0$, set $r_i =$

$2^{-i}r$. Then there exists a sequence $\{B^i\}_{i=0}^{\infty}$ of balls in B_r such that $x \in B^i$, $B^i \subset B_{2r_i}(x) \cap B_r$, $B^{i+1} \subset B^i$, $r_i \le$ (the radius of B^i) $\le 2r_i$. In particular, we can choose $B^0 = B_r$ and $B^i = B_{r_i}(x)$ for large i with $r_i \le \operatorname{dist}(x, \partial B_r)$. Then,

$$\begin{aligned}
|f(x) - (f)_{B_r}| &\le \sum_{i=0}^{\infty} |(f)_{B^{i+1}} - (f)_{B^i}| \le \sum_{i=0}^{\infty} \fint_{B^{i+1}} |f(y) - (f)_{B^i}|\,dy \\
&\le c \sum_{i=0}^{\infty} r_i^{-n} \int_{B^i} |f(y) - (f)_{B^i}|\,dy \\
&\le c \sum_{i=0}^{\infty} r_i^{-n+s} \int_{B^i}\int_{B^i} \frac{|f(y) - f(z)|}{|y-z|^{n+s}}\,dz\,dy.
\end{aligned}$$

Set

$$h(y) \coloneqq \int_{B_r} \frac{|f(y) - f(z)|}{|y-z|^{n+s}}\,dz, \quad y \in B_r.$$

Then

$$\begin{aligned}
|f(x) - (f)_{B_r}| &\le c \sum_{i=0}^{\infty} r_i^{-n+s} \int_{B_{2r_i}(x)\cap B_r} h(y)\,dy \\
&\le c \sum_{i=0}^{\infty}\sum_{j=i}^{\infty} 2^{(n-s)i} r^{-n+s} \int_{(B_{2r_j}(x)\setminus B_{2r_{j+1}}(x))\cap B_r} h(y)\,dy \\
&= c \sum_{j=0}^{\infty} \left(\sum_{i=0}^{j} 2^{(n-s)(i-j)}\right) \int_{(B_{2r_j}(x)\setminus B_{2r_{j+1}}(x))\cap B_r} r^{-n+s} h(y)\,dy \\
&\le c \sum_{j=0}^{\infty} \int_{(B_{2r_j}(x)\setminus B_{2r_{j+1}}(x))\cap B_r} \frac{h(y)}{|x-y|^{n-s}}\,dy \\
&= c \int_{B_r} \frac{h(y)}{|x-y|^{n-s}}\,dy,
\end{aligned}$$

and this is (4.24).

We next prove the desired estimate following the argument in [24, Theorem 7]. To do this, note that for $s > 0$ there exists $c(n,s) \geq 1$ such that

$$\frac{1}{c(n,s)} \leq r^{-s} \int_{B_r} \frac{1}{|x-y|^{n-s}}\, dy \leq c(n,s), \quad \text{for every } \ x \in B_r. \tag{4.25}$$

Using (4.24) with s replaced by $\frac{s}{2}$ and Jensen's inequality, we have

$$\begin{aligned}
&\fint_{B_r} G\left(\frac{|f(x)-(f)_{B_r}|}{r^s}\right)^{\theta} dx \\
&\leq c \fint_{B_r} G\left(r^{-s} \int_{B_r} \left[\int_{B_r} \frac{|f(y)-f(z)|}{|y-z|^{n+s/2}}\, dz\right] \frac{dy}{|x-y|^{n-s/2}}\right)^{\theta} dx \\
&= c \fint_{B_r} G\left(r^{-s} \int_{B_r} \left[\int_{B_r} \frac{|f(y)-f(z)|}{|y-z|^{s}} \frac{dz}{|y-z|^{n-s/2}}\right] \frac{dy}{|x-y|^{n-s/2}}\right)^{\theta} dx \\
&\leq c \fint_{B_r} \left[r^{-s} \int_{B_r} \int_{B_r} G\left(\frac{|f(y)-f(z)|}{|y-z|^{s}}\right) \frac{dz}{|y-z|^{n-s/2}} \frac{dy}{|x-y|^{n-s/2}}\right]^{\theta} dx.
\end{aligned}$$

Denote

$$L := \int_{B_r} \int_{B_r} G\left(\frac{|f(y)-f(z)|}{|y-z|^{s}}\right) \frac{dz\, dy}{|y-z|^{n}}.$$

Then recall the fact that $|y-z| \leq 2r$ and use Jensen's inequality and Fubini's theorem, to discover

$$\begin{aligned}
&\fint_{B_r} G\left(\frac{|f(x)-(f)_{B_r}|}{r^s}\right)^{\theta} dx \\
&\leq cL^{\theta} \fint_{B_r} \left[L^{-1} \int_{B_r} \frac{r^{-s/2}}{|x-y|^{n-s/2}} \left(\int_{B_r} G\left(\frac{|f(y)-f(z)|}{|y-z|^{s}}\right) \frac{dz}{|y-z|^{n}}\right) dy\right]^{\theta} dx \\
&\leq cL^{\theta-1} \fint_{B_r} \int_{B_r} \left(\frac{r^{-s/2}}{|x-y|^{n-s/2}}\right)^{\theta} \left(\int_{B_r} G\left(\frac{|f(y)-f(z)|}{|y-z|^{s}}\right) \frac{dz}{|y-z|^{n}}\right) dy\, dx \\
&= cL^{\theta-1} \fint_{B_r} \int_{B_r} \left[\int_{B_r} \left(\frac{r^{-s/2}}{|x-y|^{n-s/2}}\right)^{\theta} dx\right] G\left(\frac{|f(y)-f(z)|}{|y-z|^{s}}\right) \frac{dz\, dy}{|y-z|^{n}}.
\end{aligned}$$

We now choose $\theta = \theta(n,s)$ such that

$$1 < \theta < \frac{n}{n-s/2}.$$

From this choice and (4.25) with s replaced by $s\theta/2 - n(\theta-1)$, we discover

$$\frac{1}{c} \le r^{n(\theta-1)-s\theta/2} \int_{B_r} \frac{1}{|x-y|^{(n-s/2)\theta}}\, dx \le c \quad \text{for every} \quad x \in B_r.$$

Consequently,

$$\begin{aligned}
&\fint_{B_r} G\left(\frac{|f(x)-(f)_{B_r}|}{r^s}\right)^{\theta} dx \\
&\qquad \le c(|B_r|^{-1}L)^{\theta-1} \fint_{B_r}\int_{B_r} G\left(\frac{|f(y)-f(z)|}{|y-z|^s}\right)\frac{dz\,dy}{|y-z|^n} \\
&\qquad \le c\left(\fint_{B_r}\int_{B_r} G\left(\frac{|f(y)-f(z)|}{|y-z|^s}\right)\frac{dz\,dy}{|y-z|^n}\right)^{\theta}.
\end{aligned}$$

This finishes the proof. □

Remark 4.2.2. In Lemma 4.2.1, we selected $\theta > 1$ such that $\theta \in (1, \frac{n}{n-s/2})$. This selection is not optimal and it is possible to consider a larger value θ. However the condition $\theta > 1$ is enough in the proof of Theorem 4.3.2 below.

4.3 Local boundedness

This section is devoted to the proof of the local boundedness of weak solutions to (1.1) with the estimate (1.7) in Theorem 1.0.1. Key ingredients of the proof are the Caccioppoli type estimate, Proposition 4.1.1, and the Sobolev-Poincaré type inequality in Lemma 4.2.1.

The following lemma will be used in the De Giorgi iteration.

Lemma 4.3.1. *[35, Lemma 7.1] Let $\beta > 0$ and A_i be a sequence of real positive numbers such that*

$$A_{i+1} \le C B^i A_i^{1+\beta}$$

with $C > 0$ and $B > 1$. If $A_0 \le C^{-\frac{1}{\beta}} B^{-\frac{1}{\beta^2}}$, then we have

$$A_i \le B^{-\frac{i}{\beta}} A_0 \quad \textit{hence, in particular,} \quad \lim_{i\to\infty} A_i = 0.$$

Now, we are ready to prove the local boundedness of weak solutions to (1.1).

Theorem 4.3.2. *Let $u \in \mathbb{W}^{s,G}(\Omega) \cap L^g_s(\mathbb{R}^n)$ be a weak subsolution to* (1.1) *and $B_r \Subset \Omega$. Then we have*

$$\sup_{B_{r/2}} u_+ \leq c_b r^s G^{-1}\left(\fint_{B_r} G\left(\frac{u_+}{r^s}\right)dx\right) + r^s g^{-1}(r^s \, Tail(u_+; x_0, r/2)), \tag{4.26}$$

where $c_b = c_b(n, s, p, q, \lambda, \Lambda) > 0$. Moreover, if u is a weak solution to (1.1), *then $u \in L^\infty_{\mathrm{loc}}(\Omega)$ and we have the estimate* (1.7).

Proof. Suppose that u is a weak subsolution. Fix $B_r = B_r(x_0) \Subset \Omega$. For any $j \in \mathbb{N}_0$, write

$$\begin{aligned} &r_j = (1 + 2^{-j})\frac{r}{2}, \quad \tilde{r}_j = \frac{r_j + r_{j+1}}{2}, \quad B_j = B_{r_j}(x_0), \quad \tilde{B}_j = B_{\tilde{r}_j}(x_0), \\ &k_j = (1 - 2^{-j})k, \quad \tilde{k}_j = \frac{k_j + k_{j+1}}{2}, \quad w_j = (u - k_j)_+ \text{ and } \tilde{w}_j = (u - \tilde{k}_j)_+. \end{aligned}$$

Note from the above setting that

$$B_{j+1} \subset \tilde{B}_j \subset B_j, \quad k_j \leq \tilde{k}_j \leq k_{j+1} \quad \text{and} \quad w_{j+1} \leq \tilde{w}_j \leq w_j. \tag{4.27}$$

We take any cut-off functions $\phi_j \in C^\infty_0(\tilde{B}_j)$ such that $0 \leq \phi_j \leq 1$, $\phi_j \equiv 1$ in B_{j+1} and $|D\phi_j| \leq 2^{j+4}/r$. Putting ϕ_j into the Caccioppoli inequality (4.1) with $w_+ = \tilde{w}_j$ (see Remark 4.1.2) and dividing the inequality by $|B_{j+1}|$, we get

$$\begin{aligned} &\fint_{B_{j+1}} \int_{B_{j+1}} G\left(\frac{|\tilde{w}_j(x) - \tilde{w}_j(y)|}{|x-y|^s}\right)\frac{dxdy}{|x-y|^n} \\ &\leq c \fint_{B_j} \int_{B_j} G\left(\frac{|\phi_j(x) - \phi_j(y)|}{|x-y|^s} \max\{\tilde{w}_j(x), \tilde{w}_j(y)\}\right)\frac{dxdy}{|x-y|^n} \\ &\quad + c \fint_{B_j} \tilde{w}_j(x)\phi_j^q(x)dx \left(\sup_{y \in \operatorname{supp}\phi_j} \int_{\mathbb{R}^n \setminus B_j} g\left(\frac{\tilde{w}_j(x)}{|x-y|^s}\right)\frac{dx}{|x-y|^{n+s}}\right) \\ &=: I + II. \end{aligned} \tag{4.28}$$

We first look at the first term I in the right-hand side of the above inequality.

Since $|\phi_j(x) - \phi_j(y)| \le \|D\phi_j\|_{L^\infty}|x-y| \le c2^j|x-y|/r$, we find

$$
\begin{aligned}
I &\le c\fint_{B_j}\int_{B_j} G\left(2^j r^{-1}|x-y|^{1-s}\max\{\tilde{w}_j(x),\tilde{w}_j(y)\}\right)\frac{dxdy}{|x-y|^n} \\
&\le c2^{qj}\fint_{B_j}\int_{B_j} G\left(\frac{\max\{\tilde{w}_j(x),\tilde{w}_j(y)\}}{r^s}\right)\left(\frac{|x-y|}{r}\right)^{(1-s)p}\frac{dxdy}{|x-y|^n} \\
&\le c2^{qj}r^{-(1-s)p}\fint_{B_j} G\left(\frac{\tilde{w}_j(x)}{r^s}\right)\left(\int_{B_j}\frac{dy}{|x-y|^{n-(1-s)p}}\right)dx \\
&\le c2^{qj}\fint_{B_j} G\left(\frac{w_j(x)}{r^s}\right)dx.
\end{aligned}
\tag{4.29}
$$

To estimate II, we write

$$
II_1 = \fint_{B_j}\tilde{w}_j(x)\phi_j^q(x)dx \quad\text{and}\quad II_2 = \sup_{y\,\in\,\operatorname{supp}\phi_j}\int_{\mathbb{R}^n\setminus B_j} g\left(\frac{\tilde{w}_j(x)}{|x-y|^s}\right)\frac{dx}{|x-y|^{n+s}}.
$$

Since g is increasing and $w_j \ge \tilde{k}_j - k_j$ in $\{u_j \ge \tilde{k}_j\}$, we have

$$
G\left(\frac{w_j}{r^s}\right) \ge \frac{1}{q}\frac{w_j}{r^s}g\left(\frac{w_j}{r^s}\right) \ge \frac{1}{q}\frac{\tilde{w}_j}{r^s}g\left(\frac{\tilde{k}_j-k_j}{r^s}\right) \ge c2^{-(q-1)j}\frac{\tilde{w}_j}{r^s}g\left(\frac{k}{r^s}\right).
$$

Thus

$$
II_1 \le c2^{(q-1)j}\frac{r^s}{g\left(k/r^s\right)}\fint_{B_j} G\left(\frac{w_j}{r^s}\right)dx. \tag{4.30}
$$

In order to estimate II_2, we notice that for $x \in \mathbb{R}^n \setminus B_j$ and $y \in \tilde{B}_j$,

$$
\frac{|x-x_0|}{|x-y|} \le \frac{|x-y|+|y-x_0|}{|x-y|} \le 1 + \frac{\tilde{r}_j}{r_j-\tilde{r}_j} \le 2^{j+4}.
$$

This and (4.27) imply

$$\begin{aligned} II_2 &\leq \sup_{y\in\tilde{B}_j}\int_{\mathbb{R}^n\setminus B_{r/2}} g\left(\frac{w_0}{|x-y|^s}\right)\frac{dx}{|x-y|^{n+s}} \\ &\leq c2^{(n+sq)j}\int_{\mathbb{R}^n\setminus B_{r/2}} g\left(\frac{u_+}{|x-x_0|^s}\right)\frac{dx}{|x-x_0|^{n+s}} \\ &= c2^{(n+sq)j}\mathrm{Tail}(u_+;x_0,r/2). \end{aligned} \tag{4.31}$$

In light of (4.30) and (4.31), we deduce

$$II \leq c2^{(n+sq+q)j}\frac{r^s}{g\left(k/r^s\right)}\left(\fint_{B_j} G\left(\frac{w_j}{r^s}\right)dx\right)\mathrm{Tail}(u_+;x_0,r/2). \tag{4.32}$$

Combining (4.28), (4.29), and (4.32), and applying the Sobolev-Poincaré inequality (4.23) to the left-hand side of (4.28), we have

$$\begin{aligned} &\left(\fint_{B_{j+1}} G^{\theta}\left(\frac{|\tilde{w}_j-(\tilde{w})_{B_{j+1}}|}{r_{j+1}^s}\right)dx\right)^{\frac{1}{\theta}} \\ &\leq c2^{(n+sq+q)j} \\ &\quad\times\left[\fint_{B_j} G\left(\frac{w_j}{r^s}\right)dx+\frac{r^s}{g\left(k/r^s\right)}\left(\fint_{B_j} G\left(\frac{w_j}{r^s}\right)dx\right)\mathrm{Tail}(u_+;x_0,r/2)\right] \end{aligned} \tag{4.33}$$

for some $\theta=\theta(n,s)>1$. On the other hand, recalling the definition of r_{j+1} and using Jensen's inequality and (4.27), we discover

$$\begin{aligned} &\left(\fint_{B_{j+1}} G^{\theta}\left(\frac{\tilde{w}_j}{r^s}\right)dx\right)^{\frac{1}{\theta}} \\ &\leq c\left(\fint_{B_{j+1}} G^{\theta}\left(\frac{|\tilde{w}_j-(\tilde{w}_j)_{B_{j+1}}|}{r^s}\right)dx\right)^{\frac{1}{\theta}}+cG\left(\frac{(\tilde{w}_j)_{B_{j+1}}}{r^s}\right) \\ &\leq c\left(\fint_{B_{j+1}} G^{\theta}\left(\frac{|\tilde{w}_j-(\tilde{w}_j)_{B_{j+1}}|}{r_{j+1}^s}\right)dx\right)^{\frac{1}{\theta}}+c\fint_{B_j} G\left(\frac{w_j}{r^s}\right)dx. \end{aligned} \tag{4.34}$$

Let us estimate the left-hand side of (4.34). Notice that the relations in (4.27) yield

$$G^{\theta}\left(\frac{\tilde{w}_j}{r^s}\right) \geq G^{\theta-1}\left(\frac{\tilde{w}_j}{r^s}\right) G\left(\frac{w_{j+1}}{r^s}\right) \geq G^{\theta-1}\left(\frac{k_{j+1}-\tilde{k}_j}{r^s}\right) G\left(\frac{w_{j+1}}{r^s}\right).$$

Therefore it follows that

$$\begin{aligned} &G^{\frac{\theta-1}{\theta}}\left(\frac{k}{r^s}\right)\left(\fint_{B_{j+1}} G\left(\frac{w_{j+1}}{r^s}\right) dx\right)^{\frac{1}{\theta}} \\ &\leq c2^{qj} G^{\frac{\theta-1}{\theta}}\left(\frac{k_{j+1}-\tilde{k}_j}{r^s}\right)\left(\fint_{B_{j+1}} G\left(\frac{w_{j+1}}{r^s}\right) dx\right)^{\frac{1}{\theta}} \\ &\leq c2^{qj}\left(\fint_{B_{j+1}} G^{\theta}\left(\frac{\tilde{w}_j}{r^s}\right) dx\right)^{\frac{1}{\theta}}. \end{aligned} \tag{4.35}$$

Taking into account (4.33), (4.34) and (4.35), we deduce that

$$\begin{aligned} &G^{\frac{\theta-1}{\theta}}\left(\frac{k}{r^s}\right)\left(\fint_{B_{j+1}} G\left(\frac{w_{j+1}}{r^s}\right) dx\right)^{\frac{1}{\theta}} \\ &\leq c2^{(n+sq+2q)j}\left[1+\frac{r^s}{g\left(k/r^s\right)}\mathrm{Tail}(u_+;x_0,r/2)\right]\left(\fint_{B_j} G\left(\frac{w_j}{r^s}\right) dx\right). \end{aligned} \tag{4.36}$$

Denote

$$a_j := \frac{1}{G(k/r^s)}\fint_{B_j} G\left(\frac{w_j}{r^s}\right) dx. \tag{4.37}$$

Then (4.36) is identical to

$$a_{j+1} \leq c_2 2^{(n+sq+2q)\theta j}\left[1+\frac{r^s}{g\left(k/r^s\right)}\mathrm{Tail}(u_+;x_0,r/2)\right]^{\theta} a_j^{\theta}$$

for some $c_2 > 0$ depending on n, s, p, q, λ and Λ. At this stage, choose

$$k = r^s G^{-1}\left(c_3 \fint_{B_r} G\left(\frac{u_+}{r^s}\right) dx\right) + r^s g^{-1}(r^s \mathrm{Tail}(u_+; x_0, r/2)),$$

where $c_3 = (c_2 2^\theta)^{\frac{1}{\theta-1}} 2^{\frac{(n+sq+2q)\theta}{(\theta-1)^2}}$. Then we see that

$$a_{j+1} \le (c_2 2^\theta) 2^{(n+sq+2q)\theta j} a_j^\theta \quad \text{and} \quad a_0 \le c_3^{-1} = (c_2 2^\theta)^{-\frac{1}{\theta-1}} 2^{-\frac{(n+sq+2q)\theta}{(\theta-1)^2}}.$$

Set $c_b = \max\left\{c_3^{1/p}, c_3^{1/q}\right\}$. Since Lemma 4.3.1 implies $a_j \to 0$ as $j \to \infty$, we discover

$$\sup_{B_{r/2}} u_+ \le k \le c_b r^s G^{-1}\left(\fint_{B_r} G\left(\frac{u_+}{r^s}\right) dx\right) + r^s g^{-1}(r^s \mathrm{Tail}(u_+; x_0, r/2)),$$

and this is (4.26).

If u is a weak solution, then $-u$ is a weak subsolution. Then we have the estimate (4.26) with u_+ replaced by $(-u)_+ = u_-$. This completes the proof. □

4.4 The proof of Theorem 1.0.1

We complete the proof of Theorem 1.0.1 by obtaining (1.8). Let $u \in \mathbb{W}^{s,G}(\Omega) \cap L^g_s(\mathbb{R}^n)$ be a weak solution to (1.1). Let $B_r \equiv B_r(x_0) \Subset \Omega$. For $\alpha \in (0,1)$, $\sigma \in (0,1)$ and $i \in \mathbb{N}_0$, we write

$$r_i := \sigma^i \frac{r}{2} \quad \text{and} \quad B_i = B_{r_i}(x_0) \tag{4.38}$$

and define

$$\nu_i := \left(\frac{r_i}{r_0}\right)^\alpha \nu_0 = \sigma^{\alpha i} \nu_0 \tag{4.39}$$

with

$$\nu_0 := 2\left(c_b r^s G^{-1}\left(\fint_{B_r} G\left(\frac{|u|}{r^s}\right) dx\right) + r^s g^{-1}(r^s \mathrm{Tail}(u; x_0, r/2))\right), \tag{4.40}$$

where c_b is as in (1.7).

For the proof of (1.8), it is enough to show the following oscillation decay estimate.

Lemma 4.4.1. *Under the above setting, there exist small $\alpha, \sigma \in (0,1)$ depending on n, s, p, q, λ and Λ such that for every $i \in \mathbb{N}_0$,*

$$\operatorname*{osc}_{B_i} u := \sup_{B_i} u - \inf_{B_i} u \le \nu_i. \tag{4.41}$$

Proof. First of all, we assume that

$$\alpha \le \frac{sp}{2(p-1)} \quad \text{and} \quad \sigma < \frac{1}{4}. \tag{4.42}$$

We prove this lemma by induction. Obviously, (4.41) holds true for $i = 0$ from (1.7) and the definition of ν_0. Suppose that for some $j \ge 0$,

$$\operatorname*{osc}_{B_i} u \le \nu_i \quad \text{for all} \quad i \in \{0, 1, 2, \cdots, j\}, \tag{4.43}$$

and then we will prove (4.41) for $i = j+1$. We define u_j by

$$u_j := \begin{cases} u - \inf\limits_{B_j} u, & \text{if} \ \ |2B_{j+1} \cap \{u \ge \inf\limits_{B_j} u + \nu_j/2\}| \ge \frac{1}{2}|2B_{j+1}|, \\ \nu_j - (u - \inf\limits_{B_j} u), & \text{if} \ \ |2B_{j+1} \cap \{u \le \inf\limits_{B_j} u + \nu_j/2\}| \ge \frac{1}{2}|2B_{j+1}|, \end{cases}$$

where $2B_{j+1} := B_{2r_{j+1}}(x_0)$. Then $u_j \ge 0$ in B_j and

$$\frac{|2B_{j+1} \cap \{u_j \ge \nu_j/2\}|}{|2B_{j+1}|} \ge \frac{1}{2}. \tag{4.44}$$

We divide the remaining part of the proof into three steps.

Step 1. We first estimate $\mathrm{Tail}(u_j; x_0, r_j)$. Define T_1 and T_2 as follows:

$$\begin{aligned} \mathrm{Tail}(u_j; x_0, r_j) &= \sum_{i=1}^{j} \int_{B_{i-1}\setminus B_i} g\left(\frac{|u_j(x)|}{|x-x_0|^s}\right) \frac{dx}{|x-x_0|^{n+s}} \\ &\quad + \int_{\mathbb{R}^n \setminus B_0} g\left(\frac{|u_j(x)|}{|x-x_0|^s}\right) \frac{dx}{|x-x_0|^{n+s}} \\ &=: T_1 + T_2. \end{aligned} \tag{4.45}$$

Before estimating T_1 and T_2, observe that the definition of u_j and the induction hypothesis (4.43) imply

$$\sup_{B_i} |u_j| \leq 2\nu_i \quad \text{for all} \quad i \leq j. \tag{4.46}$$

Moreover, the local boundedness of u implies

$$|u_j| \leq |u| + \nu_j + \sup_{B_j} |u| \leq |u| + 2\nu_0. \tag{4.47}$$

We now estimate T_1. Recall (4.46) to find

$$\begin{aligned} T_1 &\leq \sum_{i=1}^{j} \int_{B_{i-1}\setminus B_i} g\left(\frac{\sup_{B_{i-1}} |u_j|}{|x-x_0|^s}\right) \frac{dx}{|x-x_0|^{n+s}} \\ &\leq c \sum_{i=1}^{j} \int_{B_{i-1}\setminus B_i} g\left(\frac{\nu_{i-1}}{r_i^s}\right) \left(\frac{r_i^s}{|x-x_0|^s}\right)^{p-1} \frac{dx}{|x-x_0|^{n+s}} \\ &= c \sum_{i=1}^{j} r_i^{s(p-1)} g\left(\frac{\nu_{i-1}}{r_i^s}\right) \int_{B_{i-1}\setminus B_i} \frac{dx}{|x-x_0|^{n+sp}} \leq c \sum_{i=1}^{j} \frac{1}{r_i^s} g\left(\frac{\nu_{i-1}}{r_i^s}\right). \end{aligned} \tag{4.48}$$

In order to estimate T_2, we write $\tilde{g}(t) \coloneqq G(t)/t$. Note that (1.3) implies $p\tilde{g}(t) \leq g(t) \leq q\tilde{g}(t)$ and

$$(p-1)\frac{G(t)}{t^2} \leq \tilde{g}'(t) = \frac{tg(t) - G(t)}{t^2} \leq (q-1)\frac{G(t)}{t^2}. \tag{4.49}$$

Now set $h(t) \coloneqq \tilde{g}(t^{1/(q-1)})$. Using (4.49), we get

$$0 \leq \frac{p-1}{q-1} \frac{G\left(t^{\frac{1}{q-1}}\right)}{t^{\frac{1}{q-1}+1}} \leq h'(t) = \frac{1}{q-1}\tilde{g}'\left(t^{\frac{1}{q-1}}\right) t^{\frac{1}{q-1}-1} \leq \frac{G\left(t^{\frac{1}{q-1}}\right)}{t^{\frac{1}{q-1}+1}} = \frac{h(t)}{t}$$

and so

$$\left(\frac{h(t)}{t}\right)' = \frac{th'(t) - h(t)}{t^2} \leq 0.$$

Therefore $h(t)$ is non-decreasing and $h(t)/t$ is non-increasing. We then set ψ

be the concave envelope of h to conclude that $\frac{\psi}{2} \leq h \leq \psi$, see [54, Lemma 2.2] for details. Additionally, considering (4.47) and the inequalities $p\tilde{g}(t) \leq g(t) \leq q\tilde{g}(t)$, we find

$$\begin{aligned}
T_2 &\leq c \int_{\mathbb{R}^n \setminus B_0} \tilde{g}\left(\frac{\nu_0}{|x-x_0|^s}\right) \frac{dx}{|x-x_0|^{n+s}} + c \int_{\mathbb{R}^n \setminus B_0} g\left(\frac{|u(x)|}{|x-x_0|^s}\right) \frac{dx}{|x-x_0|^{n+s}} \\
&= c \int_{\mathbb{R}^n \setminus B_0} h\left(\left(\frac{\nu_0}{|x-x_0|^s}\right)^{q-1}\right) \frac{dx}{|x-x_0|^{n+s}} + c\mathrm{Tail}(u; x_0, r_0) \\
&\leq c \int_{\mathbb{R}^n \setminus B_0} \psi\left(\left(\frac{\nu_0}{|x-x_0|^s}\right)^{q-1}\right) \frac{dx}{|x-x_0|^{n+s}} + c\mathrm{Tail}(u; x_0, r_0).
\end{aligned}$$

Now we use Jensen's inequality with respect to the measure $\frac{dx}{|x-x_0|^{n+s}}$. Then

$$\begin{aligned}
T_2 &\leq \frac{c}{r_0^s} \psi\left(r_0^s \int_{\mathbb{R}^n \setminus B_0} \left(\frac{\nu_0}{|x-x_0|^s}\right)^{q-1} \frac{dx}{|x-x_0|^{n+s}}\right) + c\mathrm{Tail}(u; x_0, r_0) \\
&\leq \frac{c}{r_0^s} h\left(\left(\frac{\nu_0}{r_0^s}\right)^{q-1}\right) + c\mathrm{Tail}(u; x_0, r_0) \\
&\leq \frac{c}{r_0^s} \tilde{g}\left(\frac{\nu_0}{r_0^s}\right) + c\mathrm{Tail}(u; x_0, r_0) \leq \frac{c}{r_0^s} g\left(\frac{\nu_0}{r_0^s}\right) + c\mathrm{Tail}(u; x_0, r_0).
\end{aligned}$$

We recall (4.40) to discover

$$T_2 \leq \frac{c}{r_0^s} g\left(\frac{\nu_0}{r_0^s}\right) \leq \frac{c}{r_1^s} g\left(\frac{\nu_0}{r_1^s}\right). \tag{4.50}$$

We combine (4.45), (4.48), and (4.50), and recall (4.38) and (4.39) to have

$$
\begin{aligned}
\mathrm{Tail}(u_j; x_0, r_j) &\leq \sum_{i=1}^{j} \frac{c}{r_i^s} g\left(\frac{\nu_{i-1}}{r_i^s}\right) = \sum_{i=1}^{j} \frac{c}{r_i^s} g\left(\frac{\nu_j}{r_{j+1}^s}\sigma^{(s-\alpha)(j-i+1)}\right) \\
&\leq \frac{c}{r_{j+1}^s} g\left(\frac{\nu_j}{r_{j+1}^s}\right) \sum_{i=1}^{j} \sigma^{(sp-\alpha(p-1))(j-i+1)} \\
&\leq \frac{c}{r_{j+1}^s} g\left(\frac{\nu_j}{r_{j+1}^s}\right) \frac{\sigma^{sp-\alpha(p-1)}}{1-\sigma^{sp-\alpha(p-1)}} \\
&\leq \frac{c}{r_{j+1}^s} g\left(\frac{\nu_j}{r_{j+1}^s}\right) \sigma^{sp-\alpha(p-1)},
\end{aligned}
\tag{4.51}
$$

by taking $\sigma > 0$ sufficiently small so that

$$
\sigma^{sp-\alpha(p-1)} \leq \sigma^{\frac{sp}{2}} \leq \frac{1}{2}. \tag{4.52}
$$

Step 2. In this step, we look at

$$
\frac{|2B_{j+1} \cap \{u_j \leq 2\epsilon\nu_j\}|}{|2B_{j+1}|}, \quad \text{where } \ \epsilon := \sigma^{\frac{sp-\alpha(p-1)}{q-1}} \leq \sigma^{\frac{sp}{2(q-1)}} < 1. \tag{4.53}
$$

For $k > 0$ to be determined later, we write

$$
v := \min\left\{\left[\log\left(\frac{\nu_j/2 + \epsilon\nu_j}{u_j + \epsilon\nu_j}\right)\right]_+, k\right\}.
$$

Applying Proposition 4.1.4 with $u = u_j$, $r = 2r_{j+1}$, $R = r_j$, $a \equiv \nu_j/2$, $b \equiv \exp(k)$ and $d = \epsilon\nu_j$ and using (4.51), we find

$$
\begin{aligned}
⨍_{2B_{j+1}} |v - (v)_{2B_{j+1}}|\, dx &\leq c\left[1 + \frac{g(\nu_j/r_{j+1}^s)}{g(\epsilon\nu_j/r_{j+1}^s)}\sigma^{sp-\alpha(p-1)}\right] \\
&\leq c(1 + \epsilon^{1-q}\sigma^{sp-\alpha(p-1)}) \leq c.
\end{aligned}
\tag{4.54}
$$

On the other hand, using the fact $\{v=0\}=\{u_j\geq \nu_j/2\}$ and (4.44), we see

$$\begin{aligned} k &= \frac{1}{|2B_{j+1}\cap\{u_j\geq \nu_j/2\}|}\int_{2B_{j+1}\cap\{v=0\}} k\,dx \\ &\leq \frac{2}{|2B_{j+1}|}\int_{2B_{j+1}}(k-v)\,dx = 2[k-(v)_{2B_{j+1}}]. \end{aligned}$$

Integrating the above inequality over $2B_{j+1}\cap\{v=k\}$ and using (4.54), we get

$$\begin{aligned} \frac{|2B_{j+1}\cap\{v=k\}|}{|2B_{j+1}|}k &\leq \frac{2}{|2B_{j+1}|}\int_{2B_{j+1}\cap\{v=k\}}(k-(v)_{2B_{j+1}})\,dx \\ &\leq \frac{2}{|2B_{j+1}|}\int_{2B_{j+1}}|v-(v)_{2B_{j+1}}|\,dx\leq c. \end{aligned}$$

Here we assume $\sigma>0$ is sufficiently small so that

$$\sqrt{\epsilon}=\sigma^{\frac{sp-\alpha(p-1)}{2(q-1)}}\leq \sigma^{\frac{sp}{4(q-1)}}\leq \frac{1}{6}, \tag{4.55}$$

and take

$$k=\log\left(\frac{\nu_j/2+\epsilon\nu_j}{3\epsilon\nu_j}\right)\geq \log\left(\frac{1}{6\epsilon}\right)\geq \frac{1}{2}\log\left(\frac{1}{\epsilon}\right),$$

from which, together with (4.53), we discover

$$\frac{|2B_{j+1}\cap\{u_j\leq 2\epsilon\nu_j\}|}{|2B_{j+1}|}\leq \frac{c}{k}\leq \frac{c_4}{\log(1/\sigma)}$$

for some $c_4>0$ depending on n,p,q,λ and Λ.

Step 3. Finally, we prove (4.41) for $i=j+1$. For any $m\in\mathbb{N}_0$, we write

$$\begin{aligned} &\rho_m=(1+2^{-m})r_{j+1},\quad \tilde\rho_m=\frac{\rho_m+\rho_{m+1}}{2},\quad B^m=B_{\rho_m},\quad \tilde B^m=B_{\tilde\rho_m}, \\ &k_m=(1+2^{-m})\epsilon\nu_j \quad\text{and}\quad w_m=(k_m-u_j)_+=(u_j-k_m)_-. \end{aligned}$$

Note that $r_{j+1}<\rho_m\leq 2r_{j+1}$ and $\epsilon\nu_j<k_m\leq 2\epsilon\nu_j$. Take cut-off functions $\phi_m\in C_0^\infty(\tilde B^m)$ such that $0\leq\phi_m\leq 1$, $\phi_m\equiv 1$ in B^{m+1} and $|D\phi_m|<2^{m+4}/r_{j+1}$. Applying the Caccioppoli inequality (4.1) to $w_-=w_m$, $\phi=\phi_m$

and $B_r = B^m$, we have

$$\begin{aligned}
&\int_{B^{m+1}} \int_{B^{m+1}} G\left(\frac{|w_m(x) - w_m(y)|}{|x-y|^s}\right) \frac{dxdy}{|x-y|^n} \\
&\quad \leq c \int_{B^m} \int_{B^m} G\left(\frac{|\phi_m(x) - \phi_m(y)|}{|x-y|^s} \max\{w_m(x), w_m(y)\}\right) \frac{dxdy}{|x-y|^n} \\
&\qquad + c \int_{B^m} w_m(x)\phi_m^q(x)dx \left(\sup_{y \in \tilde{B}^m} \int_{\mathbb{R}^n \setminus B^m} g\left(\frac{w_m(x)}{|x-y|^s}\right) \frac{dx}{|x-y|^{n+s}}\right).
\end{aligned} \tag{4.56}$$

As in the proof of the local boundedness, we use the Sobolev-Poincaré inequality (4.23), Jensen's inequality and (4.56), to find

$$\begin{aligned}
I &:= \left(\fint_{B^{m+1}} G^\theta\left(\frac{w_m}{\rho_{m+1}^s}\right) dx\right)^{\frac{1}{\theta}} \\
&\leq c\left(\fint_{B^{m+1}} G^\theta\left(\frac{w_m - (w_m)_{B^{m+1}}}{\rho_{m+1}^s}\right) dx\right)^{\frac{1}{\theta}} \\
&\quad + c\left(\fint_{B^{m+1}} G^\theta\left(\frac{(w_m)_{B^{m+1}}}{\rho_{m+1}^s}\right) dx\right)^{\frac{1}{\theta}} \\
&\leq c\fint_{B^{m+1}} \int_{B^{m+1}} G\left(\frac{|w_m(x) - w_m(y)|}{|x-y|^s}\right) \frac{dxdy}{|x-y|^n} + cG\left(\frac{(w_m)_{B^{m+1}}}{\rho_{m+1}^s}\right) \\
&\leq c\fint_{B^m} \int_{B^m} G\left(\frac{|\phi_m(x) - \phi_m(y)|}{|x-y|^s} \max\{w_m(x), w_m(y)\}\right) \frac{dxdy}{|x-y|^n} \\
&\quad + c\fint_{B^m} w_m(x)\phi_m^q(x)\, dx \left(\sup_{y \in \tilde{B}^m} \int_{\mathbb{R}^n \setminus B^m} g\left(\frac{w_m(x)}{|x-y|^s}\right) \frac{dx}{|x-y|^{n+s}}\right) \\
&\quad + c\fint_{B^{m+1}} G\left(\frac{w_m}{\rho_{m+1}^s}\right) dx =: II + III + IV.
\end{aligned} \tag{4.57}$$

We write

$$A_m := \frac{|B^m \cap \{u_j \leq k_m\}|}{|B^m|}.$$

From the definition of u_j, k_m and A_m, we estimate I as follows:

$$\begin{aligned} I &\geq \frac{1}{|B^{m+1}|^{\frac{1}{\theta}}} \left(\int_{B^{m+1}\cap\{u_j \leq k_{m+1}\}} G^\theta \left(\frac{k_m - k_{m+1}}{\rho_{m+1}^s} \right) dx \right)^{\frac{1}{\theta}} \\ &= A_{m+1}^{\frac{1}{\theta}} G \left(\frac{k_m - k_{m+1}}{\rho_{m+1}^s} \right) \geq c 2^{-qm} A_{m+1}^{\frac{1}{\theta}} G \left(\frac{\epsilon \nu_j}{r_{j+1}^s} \right). \end{aligned} \tag{4.58}$$

Since $|\phi_m(x) - \phi_m(y)| \leq c 2^m \frac{|x-y|}{r_{j+1}}$, we find

$$\begin{aligned} II &\leq \frac{c 2^{qm}}{|B^m|} \int_{B^m} \int_{B^m} G \left(\frac{\max\{w_m(x), w_m(y)\}}{r_{j+1}^s} \right) \left(\frac{|x-y|}{r_{j+1}} \right)^{(1-s)p} \frac{dxdy}{|x-y|^n} \\ &\leq \frac{c 2^{qm}}{|B^m|} r_{j+1}^{-(1-s)p} \int_{B^m \cap \{u_j \leq k_m\}} \int_{B^m} G \left(\frac{k_m}{r_{j+1}^s} \right) \frac{dydx}{|x-y|^{n-(1-s)p}} \\ &\leq \frac{c 2^{qm}}{|B^m|} G \left(\frac{k_m}{r_{j+1}^s} \right) |B^m \cap \{u_j \leq k_m\}| \leq c 2^{qm} G \left(\frac{\epsilon \nu_j}{r_{j+1}^s} \right) A_m. \end{aligned} \tag{4.59}$$

As for III, set

$$III_1 = \fint_{B^m} w_m(x) \phi^q(x)\, dx$$

and

$$III_2 = \sup_{y \in \tilde{B}^m} \int_{\mathbb{R}^n \setminus B^m} g \left(\frac{w_m(x)}{|x-y|^s} \right) \frac{dx}{|x-y|^{n+s}}.$$

Then we have

$$III_1 \leq |B^m|^{-1} \int_{B^m \cap \{u_j \leq k_m\}} k_m dx \leq c \epsilon \nu_j A_m.$$

Using the fact that $\frac{|x-x_0|}{|x-y|} \leq 1 + \frac{|y-x_0|}{|x-y|} \leq 1 + \frac{\tilde{\rho}_m}{\rho_m - \tilde{\rho}_m} \leq c 2^m$ for $x \in \mathbb{R}^n \setminus B^m$

and $y \in \tilde{B}^m$, we have

$$\begin{aligned} III_2 &\le c \int_{\mathbb{R}^n \setminus B_{j+1}} g\left(\frac{w_m(x)}{|x-x_0|^s} 2^{sm} \right) 2^{(n+s)m} \frac{dx}{|x-x_0|^{n+s}} \\ &\le c 2^{(n+sq)m} \mathrm{Tail}(w_m; x_0, r_{j+1}). \end{aligned}$$

Moreover, since $u_j \ge 0$ in B_j, we have $w_m \le k_m \le 2\epsilon\nu_j$ in B_j and $w_m \le k_m + |u_j| \le 2\epsilon\nu_j + |u_j|$ in $\mathbb{R}^n \setminus B_j$. Then from (4.51), we see

$$\begin{aligned} \mathrm{Tail}(w_m; x_0, r_{j+1}) &\le c \int_{B_j \setminus B_{j+1}} g\left(\frac{\epsilon\nu_j}{|x-x_0|^s} \right) \frac{dx}{|x-x_0|^{n+s}} + c\mathrm{Tail}(u_j; x_0, r_j) \\ &\le c \int_{\mathbb{R}^n \setminus B_{j+1}} g\left(\frac{\epsilon\nu_j}{r_{j+1}^s} \right) \left(\frac{r_{j+1}}{|x-x_0|} \right)^{(p-1)s} \frac{dx}{|x-x_0|^{n+s}} \\ &\quad + \frac{c}{r_{j+1}^s} g\left(\frac{\nu_j}{r_{j+1}^s} \right) \sigma^{sp-\alpha(p-1)} \\ &\le \frac{c}{r_{j+1}^s} g\left(\frac{\epsilon\nu_j}{r_{j+1}^s} \right). \end{aligned}$$

Therefore we obtain

$$III \le c 2^{(n+sq)m} \frac{\epsilon\nu_j}{r_{j+1}^s} g\left(\frac{\epsilon\nu_j}{r_{j+1}^s} \right) A_m \le c 2^{(n+sq)m} G\left(\frac{\epsilon\nu_j}{r_{j+1}^s} \right) A_m. \tag{4.60}$$

We recall the notation for IV to find

$$IV \le cG\left(\frac{\epsilon\nu_j}{r_{j+1}^s} \right) A_m. \tag{4.61}$$

We finally combine (4.57), (4.58), (4.59), (4.60), and (4.61), to discover

$$A_{m+1} \le c 2^{(n+sq+2q)\theta m} A_m^{1+\beta}, \quad \text{where } \beta = \theta - 1.$$

Recall that

$$A_0 = \frac{|2B_{j+1} \cap \{u_j \le 2\epsilon\nu_j\}|}{|2B_{j+1}|} \le \frac{c_4}{\log(1/\sigma)}$$

and choose $\sigma > 0$ sufficiently small such that

$$\frac{c_4}{\log(1/\sigma)} \leq c^{-1/\beta} 2^{-[n+sq+2q]\theta/\beta^2}. \tag{4.62}$$

Here, we notice that the constant σ is determined from (4.42), (4.52), and (4.62), hence depends only on n, s, p, q, λ and Λ. Then we apply Lemma 4.3.1 to see that $\lim\limits_{m\to\infty} A_m = 0$, which implies

$$u_j > \epsilon\nu_j \quad \text{in} \quad B_{j+1}. \tag{4.63}$$

If $u_j = u - \inf\limits_{B_j} u$, then (4.63) implies $\inf\limits_{B_{j+1}} u \geq \epsilon\nu_j + \inf\limits_{B_j} u$ and therefore

$$\operatorname*{osc}_{B_{j+1}} u \leq \sup_{B_j} u - \inf_{B_{j+1}} u \leq \sup_{B_j} u - (\epsilon\nu_j + \inf_{B_j} u) = \operatorname*{osc}_{B_j} u - \epsilon\nu_j \leq (1-\epsilon)\nu_j.$$

On the other hand, if $u_j = \nu_j - (u - \inf\limits_{B_j} u)$, then we have $\sup\limits_{B_{j+1}} u \leq (1-\epsilon)\nu_j + \inf\limits_{B_j} u$ from (4.63). Thus

$$\operatorname*{osc}_{B_{j+1}} u \leq \sup_{B_{j+1}} u - \inf_{B_j} u \leq (1-\epsilon)\nu_j.$$

Considering both cases, we obtain

$$\operatorname*{osc}_{B_{j+1}} u \leq (1-\epsilon)\nu_j = \frac{1-\sigma^{\frac{sp-\alpha(p-1)}{q-1}}}{\sigma^\alpha}\nu_{j+1} \leq \frac{1-\sigma^{\frac{sp}{q-1}}}{\sigma^\alpha}\nu_{j+1} \leq \nu_{j+1},$$

by taking $\alpha = \alpha(n, s, p, q, \lambda, \Lambda) > 0$ sufficiently small so that

$$\sigma^\alpha \geq 1 - \sigma^{\frac{sp}{q-1}}.$$

This completes the proof. $\square$

Chapter 5

Harnack inequality

5.1 Density lemma

In this section, we prove a density lemma and weak Harnack inequality which will play a crucial role in the proof of Harnack inequality. The proof is based on [21, Section 3]. We first state a modified version of Theorem 4.3.2 as follows.

Lemma 5.1.1. *Let* $u \in \mathbb{W}^{s,G}(\Omega) \cap L^g_s(\mathbb{R}^n)$ *be a weak subsolution to* (1.1) *and* $B_r \Subset \Omega$. *Set* $\gamma = \frac{\theta(q-1)}{(\theta-1)p}$. *Then for any ball* $B_r \equiv B_r(x_0) \Subset \Omega$ *and* $\delta \in (0,1)$, *we have*

$$\sup_{B_{r/2}} u_+ \leq c\delta^{-\gamma} r^s G^{-1}\left(\fint_{B_r} G\left(\frac{u_+}{r^s}\right) dx\right) + \delta r^s g^{-1}(r^s \operatorname{Tail}(u_+; x_0, r/2)) \quad (5.1)$$

for some $c \equiv c(n, s, p, q, \lambda, \Lambda) > 0$.

Proof. Let $k > 0$ be a number to de determined later. In the proof of Theorem 4.3.2, for each $i \in \mathbb{N}_0$, we have

$$a_{i+1} \leq c_1 2^{(n+sq+2q)\theta i}\left[1 + \frac{r^s}{g\left(k/r^s\right)} \operatorname{Tail}(u_+; x_0, r/2)\right]^\theta a_i^\theta$$

for some $c_1 \equiv c_1(n, s, p, q, \lambda, \Lambda) > 0$. Here a_i is as in (4.37) and $\theta \equiv \theta(n, s) > 1$ is the constant in Lemma 4.2.1. Now we set

$$k = r^s G^{-1}\left(c_2 \delta^{\frac{\theta}{\theta-1}(1-q)} \fint_{B_r} G\left(\frac{u_+}{r^s}\right) dx\right) + \delta r^s g^{-1}(r^s \operatorname{Tail}(u_+; x_0, r/2)),$$

where $c_2 = (c_1 2^\theta)^{\frac{1}{\theta-1}} 2^{\frac{(n+sq+2q)\theta}{(\theta-1)^2}}$, in order to have

$$a_{i+1} \leq (c_1 2^\theta) 2^{(n+sq+2q)\theta i} a_i^\theta \quad \text{and} \quad a_0 \leq c_2^{-1} = (c_1 2^\theta)^{-\frac{1}{\theta-1}} 2^{-\frac{(n+sq+2q)\theta}{(\theta-1)^2}}.$$

Hence Lemma 4.3.1 implies $\lim_{i\to\infty} a_i = 0$, which in turn gives $\sup_{B_{r/2}} u_+ \leq k$. Then (5.1) follows. □

Lemma 5.1.2. *Let $u \in \mathbb{W}^{s,G}(\Omega) \cap L^g_s(\mathbb{R}^n)$ be a weak supersolution to (1.1) such that $u \geq 0$ in $B_R \equiv B_R(x_0) \Subset \Omega$. Let $k \geq 0$. Suppose that there exists $\sigma \in (0,1]$ such that*

$$|B_r \cap \{u \geq k\}| \geq \sigma |B_r|, \tag{5.2}$$

for some $0 < r < R/16$. Then there exists a constant $\delta \equiv \delta(n,s,p,q,\lambda,\Lambda,\sigma) \in (0,1/4)$ such that

$$\inf_{B_{4r}} u \geq \delta k - r^s g^{-1}(r^s \operatorname{Tail}(u_-; x_0, R)). \tag{5.3}$$

Proof. **Step 1: A preliminary estimate.** We first show that

$$\left| B_{6r} \cap \left\{ u \leq 2\delta k - \frac{1}{2} r^s g^{-1}(r^s \operatorname{Tail}(u_-; x_0, R)) \right\} \right| \leq \frac{\bar{c}}{\sigma \log \frac{1}{2\delta}} |B_{6r}|. \tag{5.4}$$

for every $\delta \in (0,1/4)$, where $\bar{c}$ depends only on n, s, p, q, λ and Λ.

Using Lemma 4.1.4 in B_{6r} with

$$h = \min\left\{ (\log(k+d) - \log(u+d))_+, \log \frac{1}{2\delta} \right\}$$

for

$$d := \frac{1}{2} r^s g^{-1}(r^s \operatorname{Tail}(u_-; x_0, R)),$$

we have

$$\fint_{B_{6r}} |h - (h)_{B_{6r}}|\, dx \leq c. \tag{5.5}$$

Since $\{h = 0\} = \{u \geq k\}$, the assumption (5.2) implies

$$|B_{6r} \cap \{h = 0\}| \geq |B_r \cap \{h = 0\}| \geq \sigma |B_r| = \frac{\sigma}{6^n} |B_{6r}|.$$

Thus we have

$$\begin{aligned}\log\frac{1}{2\delta} &= \fint_{B_{6r}\cap\{h=0\}}\left(\log\frac{1}{2\delta}-h\right)dx\\ &\le \frac{c}{\sigma}\fint_{B_{6r}}\left(\log\frac{1}{2\delta}-h\right)dx = \frac{c}{\sigma}\left(\log\frac{1}{2\delta}-(h)_{B_{6r}}\right).\end{aligned}$$

Integrating the above inequality over $B_{6r}\cap\left\{h=\log\frac{1}{2\delta}\right\}$ and using (5.5), we get

$$\begin{aligned}\left|B_{6r}\cap\left\{h=\log\frac{1}{2\delta}\right\}\right|\log\frac{1}{2\delta} &\le \frac{c}{\sigma}\int_{B_{6r}\cap\left\{h=\log\frac{1}{2\delta}\right\}}|h-(h)_{B_{6r}}|\,dx\\ &\le \frac{c}{\sigma}\int_{B_{6r}}|h-(h)_{B_{6r}}|\,dx \le \frac{\bar{c}}{\sigma}|B_{6r}|\end{aligned}$$

for some $\bar{c}\equiv\bar{c}(n,s,p,q,\lambda,\Lambda)$. Then, using the relation

$$\left\{h=\log\frac{1}{2\delta}\right\}=\{u\le 2\delta(k+d)-d\}\supset\{u\le 2\delta k-d\},$$

we obtain (5.4):

$$|B_{6r}\cap\{u\le 2\delta k-d\}|\le|B_{6r}\cap\{u\le 2\delta(k+d)-d\}|\le\frac{\bar{c}}{\sigma\log\frac{1}{2\delta}}|B_{6r}|.$$

Step 2: Expansion of positivity. We now determine the constant δ to prove (5.3). Here we may assume $2d\le\delta k$, otherwise there is nothing to prove. For each $i\in\mathbb{N}\cup\{0\}$, we set

$$\begin{aligned}&\rho_i=\left(4+\frac{1}{2^{i-1}}\right)r,\quad \tilde{\rho}_i=\frac{\rho_i+\rho_{i+1}}{2},\quad B_i=B_{\rho_i},\quad \tilde{B}_i=B_{\tilde{\rho}_i},\\ &l_i=\left(1+\frac{1}{2^{i+1}}\right)\delta k\quad\text{and}\quad w_i=(l_i-u)_+.\end{aligned}$$

We notice that the above settings give $4r\le\rho_i\le 6r$ and $B_{i+1}\subset\tilde{B}_i\subset B_i$. Take cut-off functions $\phi_i\in C_0^\infty(\tilde{B}_i)$ satisfying $\phi_i\equiv 1$ in B_{i+1}, $0\le\phi_i\le 1$ and

$|D\phi_i| \le 2^{i+3}/r$. Applying Lemma 4.1.1 with w_i, $\tilde{B}_i$ and ϕ_i, we obtain

$$
\begin{aligned}
&\fint_{B_{i+1}} \int_{B_{i+1}} G\left(\frac{|w_i(x)-w_i(y)|}{|x-y|^s}\right)\frac{dxdy}{|x-y|^n} \\
&\quad \le c\fint_{B_i}\int_{B_i} G\left(\frac{|\phi_i(x)-\phi_i(y)|}{|x-y|^s}\max\{w_i(x),w_i(y)\}\right)\frac{dxdy}{|x-y|^n} \\
&\qquad + c\fint_{B_i} w_i(x)\phi_i^q(x)\,dx\left(\sup_{y\in\tilde{B}_i}\int_{\mathbb{R}^n\setminus B_i} g\left(\frac{w_i(x)}{|x-y|^s}\right)\frac{dx}{|x-y|^{n+s}}\right). \qquad (5.6)
\end{aligned}
$$

On the other hand, applying Lemma 4.2.1 and Jensen's inequality, we have

$$
\begin{aligned}
\left(\fint_{B_{i+1}} G^\theta\left(\frac{w_i}{r^s}\right)dx\right)^{\frac{1}{\theta}} &\le c\left(\fint_{B_{i+1}} G^\theta\left(\frac{|w_i-(w_i)_{B_{i+1}}|}{\rho_{i+1}^s}\right)dx\right)^{\frac{1}{\theta}} \\
&\quad + c\fint_{B_{i+1}} G\left(\frac{w_i}{r^s}\right)dx \\
&\le c\fint_{B_{i+1}}\int_{B_{i+1}} G\left(\frac{|w_i(x)-w_i(y)|}{|x-y|^s}\right)\frac{dxdy}{|x-y|^n} \\
&\quad + c\fint_{B_i} G\left(\frac{w_i}{r^s}\right)dx.
\end{aligned} \qquad (5.7)
$$

where we have also used the fact $\rho_i \approx r$. Combining (5.6) and (5.7), we deduce that

$$
\begin{aligned}
I &:= \left(\fint_{B_{i+1}} G^\theta\left(\frac{w_i}{r^s}\right)dx\right)^{\frac{1}{\theta}} \\
&\le c\fint_{B_i}\int_{B_i} G\left(\frac{|\phi_i(x)-\phi_i(y)|}{|x-y|^s}\max\{w_i(x),w_i(y)\}\right)\frac{dxdy}{|x-y|^n} \\
&\quad + c\fint_{B_i} w_i(x)\phi_i^q(x)\,dx\left(\sup_{y\in\tilde{B}_i}\int_{\mathbb{R}^n\setminus B_i} g\left(\frac{w_i(x)}{|x-y|^s}\right)\frac{dx}{|x-y|^{n+s}}\right) \qquad (5.8)\\
&\quad + c\fint_{B_i} G\left(\frac{w_i}{r^s}\right)dx \\
&=: II + III + IV.
\end{aligned}
$$

We first estimate I from below. We denote

$$A_i = \frac{|B_i \cap \{u < l_i\}|}{|B_i|}.$$

Since $w_i \geq l_i - l_{i+1}$ in $\{u < l_{i+1}\}$, we get

$$A_{i+1}^{\frac{1}{\theta}} G\left(\frac{l_i - l_{i+1}}{r^s}\right) = \left(\frac{1}{|B_{i+1}|}\int_{B_{i+1}\cap\{u<l_{i+1}\}} G^\theta\left(\frac{l_i - l_{i+1}}{r^s}\right) dx\right)^{\frac{1}{\theta}} \leq I.$$

In order to estimate II, we use $|\phi_i(x) - \phi_i(y)| \leq c2^i|x-y|/r$ and Fubini's theorem as follows:

$$\begin{aligned}
II &\leq c\fint_{B_i}\int_{B_i} G\left(\frac{|\phi_i(x)-\phi_i(y)|}{|x-y|^s} w_i(x)\right)\frac{dxdy}{|x-y|^n} \\
&\leq c\fint_{B_i}\int_{B_i\cap\{u<l_i\}} G\left(\frac{|\phi_i(x)-\phi_i(y)|}{|x-y|^s} l_i\right)\frac{dxdy}{|x-y|^n} \\
&\leq \frac{c2^{qi}}{\rho_i^{(1-s)p}|B_i|}\int_{B_i\cap\{u<l_i\}}\int_{B_i} G\left(\frac{l_i}{\rho_i^s}\right)\frac{dydx}{|x-y|^{n-(1-s)p}} \leq c2^{qi} G\left(\frac{l_i}{r^s}\right) A_i.
\end{aligned}$$

We next estimate III. we note that for any $x \in \mathbb{R}^n \setminus B_i$ and $y \in \tilde{B}_i$,

$$\frac{|x-x_0|}{|x-y|} \leq 1 + \frac{|y-x_0|}{|x-y|} \leq 1 + \frac{\tilde{\rho}_i}{\rho_i - \tilde{\rho}_i} \leq 2^{i+4}.$$

Then

$$\begin{aligned}
III &\leq \frac{c}{|B_i|}\int_{B_i\cap\{u<l_i\}} l_i dx \times 2^{(n+sq)i}\int_{\mathbb{R}^n\setminus B_i} g\left(\frac{w_i(x)}{|x-x_0|^s}\right)\frac{dx}{|x-x_0|^{n+s}} \\
&\leq cl_iA_i \times 2^{(n+sq)i}\,\mathrm{Tail}(w_i; x_0, \rho_i) \\
&\leq cl_iA_i \times 2^{(n+sq)i}(\mathrm{Tail}(l_i; x_0, \rho_i) + \mathrm{Tail}(u_-; x_0, \rho_i)). \qquad (5.9)
\end{aligned}$$

We further estimate the right-hand side. A direct calculation gives

$$\mathrm{Tail}(l_i; x_0, \rho_i) \leq c\rho_i^{s(q-1)}\int_{\mathbb{R}^n\setminus B_i} g\left(\frac{l_i}{\rho_i^s}\right)\frac{dx}{|x-x_0|^{n+sq}} \leq \frac{c}{r^s} g\left(\frac{l_i}{r^s}\right) \qquad (5.10)$$

We recall the assumptions $u \geq 0$ in B_R and $2d \leq \delta k$ to estimate $\mathrm{Tail}(u_-; x_0, \rho_i)$

as

$$\text{Tail}(u_-; x_0, \rho_i) = \text{Tail}(u_-; x_0, R) \leq \frac{1}{r^s} g\left(\frac{\delta k}{r^s}\right) \leq \frac{1}{r^s} g\left(\frac{l_i}{r^s}\right). \tag{5.11}$$

Plugging (5.10) and (5.11) into (5.9), we have

$$III \leq c 2^{(n+sq)i} G\left(\frac{l_i}{r^s}\right) A_i.$$

Finally, It is straightforward to check that

$$IV \leq \frac{c}{|B_i|} \int_{B_i \cap \{u < l_i\}} G\left(\frac{l_i}{r^s}\right) dx = cG\left(\frac{l_i}{r^s}\right) A_i.$$

Connecting the estimates found for I, II, III and IV to (5.8), and then using the fact that

$$\frac{G(l_i/r^s)}{G((l_i - l_{i+1})/r^s)} \leq \left(\frac{l_i}{l_i - l_{i+1}}\right)^q \leq 2^{q(i+3)},$$

we have

$$A_{i+1} \leq c_0 2^{(n+sq+q)\theta i} A_i^{1+\beta},$$

where $\beta = \theta - 1$. Now it remains to choose a proper $\delta \in (0, 1/4)$ in order to get $\lim\limits_{i\to\infty} A_i = 0$. From (5.4), there exists $\bar{c} > 0$ such that

$$A_0 = \frac{|B_{6r} \cap \{u < \frac{3}{2}\delta k\}|}{|B_{6r}|} \leq \frac{|B_{6r} \cap \{u < 2\delta k - d\}|}{|B_{6r}|} \leq \frac{\bar{c}}{\sigma \log \frac{1}{2\delta}}.$$

Let $\nu := c_0^{-1/\beta} 2^{-(n+sq+q)\theta/\beta^2}$ and choose $\delta := \frac{1}{4} \exp\left(-\frac{\bar{c}}{\sigma\nu}\right) < \frac{1}{4}$. Then

$$A_0 \leq \frac{\bar{c}}{\sigma \log \frac{1}{2\delta}} \leq c_0^{-1/\beta} 2^{-(n+sq+q)\theta/\beta^2}.$$

Therefore Lemma 4.3.1 gives $\lim\limits_{i\to\infty} A_i = 0$, which implies $\inf\limits_{B_{4r}} u \geq \delta k$. This finishes the proof. □

Once we have proved the above lemma, we can proceed with exactly the same arguments as in [21, Lemma 4.1] to obtain the following lemma.

Lemma 5.1.3 (Weak Harnack Inequality)**.** *Let $u \in \mathbb{W}^{s,G}(\Omega) \cap L^g_s(\mathbb{R}^n)$ be a weak supersolution to* (1.1) *such that $u \geq 0$ in $B_R \equiv B_R(x_0) \Subset \Omega$. Then there exist constants $t \in (0,1)$ and $c \geq 1$, both depending only on n, s, p, q, λ, and Λ, such that for $B_r \subset B_R$,*

$$\left(\fint_{B_r} u^t dx \right)^{\frac{1}{t}} \leq c \inf_{B_r} u + c r^s g^{-1}(r^s \operatorname{Tail}(u_-; x_0, R)).$$

5.2 The proof of Theorem 1.0.2

We first prove the following lemma, which gives the control of tail contribution for the Harnack inequality.

Lemma 5.2.1. *Let $u \in \mathbb{W}^{s,G}(\Omega) \cap L^g_s(\mathbb{R}^n)$ be a weak solution to* (1.1) *such that $u \geq 0$ in $B_R \equiv B_R(x_0) \Subset \Omega$. Then, for any $0 < r < R$,*

$$\operatorname{Tail}(u_+; x_0, r) \leq c \left[r^{-s} g \left(r^{-s} \sup_{B_r} u \right) + \operatorname{Tail}(u_-; x_0, R) \right]$$

holds for some $c \equiv c(n, s, p, q, \lambda, \Lambda)$.

Proof. Let $k := \sup_{B_r} u$. We may assume $k > 0$ because if $k = 0$, then $u \equiv 0$ in B_r so u satisfies Harnack inequality. Take a cut-off function $\phi \in C_0^\infty(B_{3r/4})$ satisfying $\phi \equiv 1$ in $B_{r/2}$, $0 \leq \phi \leq 1$ and $|D\phi| \leq 8/r$. We test (1.1) with $\eta \equiv (u - 2k)\phi^q$ to have

$$\begin{aligned} 0 &= \int_{B_r} \int_{B_r} g\left(\frac{|u(x) - u(y)|}{|x-y|^s} \right) \frac{u(x) - u(y)}{|u(x) - u(y)|} (\eta(x) - \eta(y)) K(x,y) \frac{dxdy}{|x-y|^s} \\ &\quad + 2 \int_{\mathbb{R}^n \setminus B_r} \int_{B_r} g\left(\frac{|u(x) - u(y)|}{|x-y|^s} \right) \frac{u(x) - u(y)}{|u(x) - u(y)|} \eta(x) K(x,y) \frac{dxdy}{|x-y|^s} \\ &=: I + II. \end{aligned} \tag{5.12}$$

Step 1: Estimate of I. We first assume $\phi(x) \geq \phi(y)$. We observe the following inequalities:

$$
\begin{aligned}
&g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|u(x)-u(y)|}\frac{u(x)-u(y)}{|x-y|^s} \geq p\,G\left(\frac{(u(x)-u(y))_+}{|x-y|^s}\right),\\
&g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|u(x)-u(y)|} \leq g\left(\frac{(u(x)-u(y))_+}{|x-y|^s}\right).
\end{aligned}
$$

Using the fact $-2k \leq u-2k \leq -k$ in B_r and putting the above inequalities into I, we discover

$$
\begin{aligned}
I &= \int_{B_r}\int_{B_r} g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|u(x)-u(y)|}\frac{u(x)-u(y)}{|x-y|^s}\phi^q(x)K(x,y)\,dxdy\\
&\quad+\int_{B_r}\int_{B_r} g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|u(x)-u(y)|} \qquad (5.13)\\
&\qquad\qquad\times\frac{\phi^q(x)-\phi^q(y)}{|x-y|^s}(u(y)-2k)K(x,y)\,dxdy\\
&\geq p\int_{B_r}\int_{B_r} G\left(\frac{(u(x)-u(y))_+}{|x-y|^s}\right)\phi^q(x)K(x,y)\,dxdy\\
&\quad-2k\int_{B_r}\int_{B_r} g\left(\frac{(u(x)-u(y))_+}{|x-y|^s}\right)\frac{\phi^q(x)-\phi^q(y)}{|x-y|^s}K(x,y)\,dxdy. \qquad (5.14)
\end{aligned}
$$

We further estimate the last term in the above display. Using the inequality $\phi^q(x)-\phi^q(y) \leq q\phi^{q-1}(x)(\phi(x)-\phi(y))$ and applying Young's inequality with $\varepsilon = \min\{\frac{p}{4}, \frac{1}{2}\}$, we get

$$
\begin{aligned}
&2k\int_{B_r}\int_{B_r} g\left(\frac{(u(x)-u(y))_+}{|x-y|^s}\right)\frac{\phi^q(x)-\phi^q(y)}{|x-y|^s}K(x,y)\,dxdy\\
&\quad\leq \frac{p}{2}\int_{B_r}\int_{B_r} G\left(\frac{(u(x)-u(y))_+}{|x-y|^s}\right)\phi^q(x)K(x,y)\,dxdy\\
&\qquad+c\int_{B_r}\int_{B_r} G\left(\frac{\phi(x)-\phi(y)}{|x-y|^s}k\right)K(x,y)\,dxdy \qquad (5.15)
\end{aligned}
$$

Then we combine (5.13) and (5.15), and then use (1.4) to get

$$\begin{aligned} I \geq& \frac{p}{2} \int_{B_r} \int_{B_r} G\left(\frac{(u(x)-u(y))_+}{|x-y|^s}\right) \phi^q(x) K(x,y)\, dxdy \\ &- c \int_{B_r} \int_{B_r} G\left(\frac{\phi(x)-\phi(y)}{|x-y|^s} k\right) K(x,y)\, dxdy \\ \geq& -\frac{c}{r^{(1-s)p}} \int_{B_r} \int_{B_r} G\left(\frac{k}{r^s}\right) \frac{dxdy}{|x-y|^{n-(1-s)p}} \geq -cr^n G\left(\frac{k}{r^s}\right). \end{aligned} \tag{5.16}$$

We note that, by interchanging the roles of x and y, we can also obtain (5.16) when $\phi(x) < \phi(y)$.

Step 2: Estimate of II. We start by estimating

$$\begin{aligned} II \geq& \int_{\mathbb{R}^n \setminus B_r} \int_{B_r} g\left(\frac{(u(y)-k)_+}{|x-y|^s}\right) k\phi^q(x) K(x,y) \frac{dxdy}{|x-y|^s} \\ &- \int_{\mathbb{R}^n \setminus B_r} \int_{B_r} g\left(\frac{(u(x)-u(y))_+}{|x-y|^s}\right) 2k\phi^q(x) K(x,y) \frac{dxdy}{|x-y|^s} \\ =:& \, II_1 - II_2. \end{aligned} \tag{5.17}$$

We observe that

$$\frac{r}{4} \leq \frac{1}{4}|y-x_0| \leq |x-y| \leq \frac{7}{4}|y-x_0| \quad \text{for} \quad x \in \operatorname{supp}\phi \subseteq B_{3r/4},\ y \in \mathbb{R}^n \setminus B_r, \tag{5.18}$$

in order to estimate II_1 as

$$\begin{aligned} II_1 \geq& \frac{k}{c} \int_{\mathbb{R}^n \setminus B_r} \int_{B_{r/2}} g\left(\frac{u_+(y)}{|y-x_0|^s}\right) \frac{dxdy}{|y-x_0|^{n+s}} \\ &- ck \int_{B_r} \int_{\mathbb{R}^n \setminus B_{r/4}(x)} g\left(\frac{k}{r^s}\right) \frac{dydx}{|x-y|^{n+s}} \\ \geq& \frac{kr^n}{c} \operatorname{Tail}(u_+; x_0, r) - ckr^{n-s} g\left(\frac{k}{r^s}\right). \end{aligned} \tag{5.19}$$

Next, using the inequality $(u(x)-u(y))_+ \leq u(x) + u_-(y)$ for $x \in B_r$ and

$y \in \mathbb{R}^n$, along with (1.4) and (5.18), we estimate II_2 as

$$\begin{aligned} II_2 &\leq \int_{\mathbb{R}^n \setminus B_r} \int_{B_r} g\left(\frac{k+u_-(y)}{|x-y|^s}\right) 2k\phi^q(x) K(x,y) \frac{dxdy}{|x-y|^s} \\ &\leq ck \int_{B_r} \int_{\mathbb{R}^n \setminus B_{r/4}(x)} g\left(\frac{k}{r^s}\right) \frac{dydx}{|x-y|^{n+s}} \\ &\quad + ck \int_{\mathbb{R}^n \setminus B_R} \int_{B_r} g\left(\frac{u_-(y)}{|y-x_0|^s}\right) \frac{dxdy}{|y-x_0|^{n+s}} \\ &\leq ckr^{n-s} g\left(\frac{k}{r^s}\right) + ckr^n \operatorname{Tail}(u_-; x_0, R). \end{aligned} \tag{5.20}$$

Combining (5.17), (5.19) and (5.20), we have

$$II \geq \frac{kr^n}{c} \operatorname{Tail}(u_+; x_0, r) - ckr^{n-s} g\left(\frac{k}{r^s}\right) - ckr^n \operatorname{Tail}(u_-; x_0, R)). \tag{5.21}$$

In turn, connecting (5.16) and (5.21) to (5.12) and then dividing both sides by kr^n, the desired estimate follows. □

Before proving Theorem 1.0.2, we introduce the following technical lemma.

Lemma 5.2.2. *[35, Lemma 6.1] Let $F(t)$ be a nonnegative bounded function defined for $0 \leq T_0 \leq t \leq T_1$. Suppose that for $T_0 \leq \sigma < \tau \leq T_1$,*

$$F(\sigma) \leq c_1(\tau - \sigma)^{-\gamma} + c_2 + \zeta F(\tau),$$

where $c_1, c_2, \gamma \geq 0$ and $\zeta \in (0,1)$ are constants. Then there exists a constant $c \equiv c(\gamma, \zeta)$ such that for every $T_0 \leq \rho < R \leq T_1$, we have

$$F(\rho) \leq c[c_1(R-\rho)^{-\gamma} + c_2].$$

We are now ready to prove our main result.

Proof of Theorem 1.0.1. Let us fix a ball $B_R \equiv B_R(x_0)$ as in the statement. By Lemma 5.1.1 and Lemma 5.2.1, we have for any ball $B_\rho \equiv B_\rho(x_0) \Subset \Omega$

$$\sup_{B_{\rho/2}} u \leq c\delta^{-\gamma} \rho^s G^{-1}\left(\fint_{B_\rho} G\left(\frac{u}{\rho^s}\right) dx\right) + c\delta \sup_{B_\rho} u + c\delta \rho^s g^{-1}(\rho^s \operatorname{Tail}(u_-; x_0, R)).$$

By using [54, Lemma 2.2], we can find a concave function $\tilde{G}(t)$ such that $\tilde{G}(t) \approx G(t^{1/q})$. Then applying Jensen's inequality with $\tilde{G}^{-1}(t) \approx (G^{-1}(t))^q$ implies

$$\sup_{B_{\rho/2}} u \le c\delta^{-\gamma} \left(\fint_{B_\rho} u^q \, dx \right)^{\frac{1}{q}} + c\delta \sup_{B_\rho} u + c\delta \rho^s g^{-1}(\rho^s \operatorname{Tail}(u_-; x_0, R)).$$

We now choose $\rho = (\tau - \sigma)r$ with $\frac{1}{2} \le \sigma \le \tau \le 1$. A standard covering argument gives

$$\sup_{B_{\sigma r}} u \le \frac{c\delta^{-\gamma}}{(\tau - \sigma)^{\frac{n}{q}}} \left(\fint_{B_{\tau r}} u^q \, dx \right)^{\frac{1}{q}} + c\delta \sup_{B_{\tau r}} u + c\delta r^s g^{-1}(r^s \operatorname{Tail}(u_-; x_0, R)).$$

Choosing $\delta = \frac{1}{4c}$ and using Young's inequality, we have

$$\begin{aligned} \sup_{B_{\sigma r}} u &\le \frac{c}{(\tau - \sigma)^{\frac{n}{q}}} \left(\sup_{B_{\tau r}} u \right)^{\frac{q-t}{q}} \left(\fint_{B_{\tau r}} u^t \, dx \right)^{\frac{1}{q}} + \frac{1}{4} \sup_{B_{\tau r}} u \\ &\quad + c r^s g^{-1}(r^s \operatorname{Tail}(u_-; x_0, R)) \\ &\le \frac{1}{2} \sup_{B_{\tau r}} u + \frac{c}{(\tau - \sigma)^{\frac{n}{t}}} \left(\fint_{B_r} u^t \, dx \right)^{\frac{1}{t}} + c r^s g^{-1}(r^s \operatorname{Tail}(u_-; x_0, R)), \end{aligned}$$

where $t \equiv t(n, s, p, q, \lambda, \Lambda) \in (0, 1)$ is the constant chosen in Lemma 5.1.3. Then Lemma 5.2.2 implies

$$\sup_{B_r} u \le c \left(\fint_{B_r} u^t \, dx \right)^{\frac{1}{t}} + c r^s g^{-1}(r^s \operatorname{Tail}(u_-; x_0, R)).$$

Finally, combining this estimate with Lemma 5.1.3, we finish the proof. □

Bibliography

[1] Angela Alberico, Andrea Cianchi, Luboš Pick, and Lenka Slavíková, *On fractional Orlicz-Sobolev spaces*, Anal. Math. Phys. **11** (2021), no. 2, Paper No. 84, 21.

[2] Anna Kh. Balci, Andrea Cianchi, Lars Diening, and Vladimir Maz'ya, *A pointwise differential inequality and second-order regularity for nonlinear elliptic systems*, Math. Ann. **383** (2022), no. 3-4, 1775–1824.

[3] Paolo Baroni, *Riesz potential estimates for a general class of quasilinear equations*, Calc. Var. Partial Differential Equations **53** (2015), no. 3-4, 803–846.

[4] Paolo Baroni and Casimir Lindfors, *The Cauchy-Dirichlet problem for a general class of parabolic equations*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **34** (2017), no. 3, 593–624.

[5] Lorenzo Brasco and Erik Lindgren, *Higher Sobolev regularity for the fractional p-Laplace equation in the superquadratic case*, Adv. Math. **304** (2017), 300–354.

[6] Lorenzo Brasco, Erik Lindgren, and Armin Schikorra, *Higher Hölder regularity for the fractional p-Laplacian in the superquadratic case*, Adv. Math. **338** (2018), 782–846.

[7] Sun-Sig Byun and Yumi Cho, *Nonlinear gradient estimates for generalized elliptic equations with nonstandard growth in nonsmooth domains*, Nonlinear Anal. **140** (2016), 145–165.

[8] Sun-Sig Byun, Hyojin Kim, and Jihoon Ok, *Local Hölder continuity for fractional nonlocal equations with general growth*, Math. Ann. (2022), to appear.

[9] Sun-Sig Byun, Hyojin Kim, and Kyeong Song, *Nonlocal Harnack inequality for fractional elliptic equations with Orlicz growth*, preprint.

[10] Sun-Sig Byun, Jihoon Ok, and Kyeong Song, *Hölder regularity for weak solutions to nonlocal double phase problems*, J. Math. Pures Appl. (2022), to appear.

[11] Luis Caffarelli, Chi Hin Chan, and Alexis Vasseur, *Regularity theory for parabolic nonlinear integral operators*, J. Amer. Math. Soc. **24** (2011), no. 3, 849–869.

[12] Luis Caffarelli and Luis Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations **32** (2007), no. 7-9, 1245–1260.

[13] ———, *Regularity theory for fully nonlinear integro-differential equations*, Comm. Pure Appl. Math. **62** (2009), no. 5, 597–638.

[14] Jamil Chaker and Minhyun Kim, *Local regularity for nonlocal equations with variable exponents*, arXiv:2107.06043 (2021).

[15] Jamil Chaker, Minhyun Kim, and Marvin Weidner, *Regularity for nonlocal problems with non-standard growth*, Calc. Var. Partial Differential Equations **61** (2022), no. 6, Paper No. 227.

[16] ———, *Harnack inequality for nonlocal problems with non-standard growth*, Math. Ann. (2022), to appear.

[17] Matteo Cozzi, *Regularity results and Harnack inequalities for minimizers and solutions of nonlocal problems: a unified approach via fractional De Giorgi classes*, J. Funct. Anal. **272** (2017), no. 11, 4762–4837.

[18] Cristiana De Filippis and Giuseppe Mingione, *Lipschitz bounds and nonautonomous integrals*, Arch. Ration. Mech. Anal. **242** (2021), no. 2, 973–1057.

[19] ———, *Gradient regularity in mixed local and nonlocal problems*, Math. Ann., to appear.

[20] Cristiana De Filippis and Giampiero Palatucci, *Hölder regularity for nonlocal double phase equations*, J. Differential Equations **267** (2019), no. 1, 547–586.

[21] Agnese Di Castro, Tuomo Kuusi, and Giampiero Palatucci, *Nonlocal Harnack inequalities*, J. Funct. Anal. **267** (2014), no. 6, 1807–1836.

[22] ______, *Local behavior of fractional p-minimizers*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **33** (2016), no. 5, 1279–1299.

[23] Eleonora Di Nezza, Giampiero Palatucci, and Enrico Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), no. 5, 521–573.

[24] Lars Diening and Frank Ettwein, *Fractional estimates for non-differentiable elliptic systems with general growth*, Forum Math. **20** (2008), no. 3, 523–556.

[25] Lars Diening, Mikyoung Lee, and Jihoon Ok, *Parabolic weighted Sobolev-Poincaré type inequalities*, Nonlinear Anal. **218** (2022), Paper No. 112772, 13.

[26] Lars Diening, Bianca Stroffolini, and Anna Verde, *Everywhere regularity of functionals with ϕ-growth*, Manuscripta Math. **129** (2009), no. 4, 449–481.

[27] Mengyao Ding, Chao Zhang, and Shulin Zhou, *Local boundedness and Hölder continuity for the parabolic fractional p-Laplace equations*, Calc. Var. Partial Differential Equations **60** (2021), no. 1, Paper No. 38, 45.

[28] Yuzhou Fang and Chao Zhang, *On weak and viscosity solutions of nonlocal double phase equations*, Int. Math. Res. Not. (2021).

[29] Julián Fernández Bonder and Ariel M. Salort, *Fractional order Orlicz-Sobolev spaces*, J. Funct. Anal. **277** (2019), no. 2, 333–367.

[30] Julián Fernández Bonder, Ariel M. Salort, and Hernán Vivas, *Interior and up to the boundary regularity for the fractional g-Laplacian: the convex case*, Nonlinear Anal. **223** (2022), Paper No. 113060, 31.

[31] ______, *Global Hölder regularity for eigenfunctions of the fractional g-Laplacian*, arXiv:2112.00830 (2021).

[32] Giovanni Franzina and Giampiero Palatucci, *Fractional p-eigenvalues*, Riv. Math. Univ. Parma (N.S.) **5** (2014), no. 2, 373–386.

[33] Prashanta Garain and Juha Kinnunen, *On the regularity theory for mixed local and nonlocal quasilinear elliptic equations*, Trans. Amer. Math. Soc. **375** (2022), no. 8, 5393–5423.

[34] Jacques Giacomoni, Deepak Kumar, and Konijeti Sreenadh, *Interior and boundary regularity results for strongly nonhomogeneous p,q-fractional problems*, Adv. Calc. Var. (2021).

[35] Enrico Giusti, *Direct methods in the calculus of variations*, World Scientific Publishing Co., Inc., River Edge, NJ (2003).

[36] Petteri Harjulehto and Peter Hästö, *Orlicz spaces and generalized Orlicz spaces*, Lecture Notes in Mathematics, vol. 2236, Springer, Cham (2019).

[37] Petteri Harjulehto, Peter Hästö, and Mikyoung Lee, *Hölder continuity of ω-minimizers of functionals with generalized Orlicz growth*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **22** (2021), no. 2, 549–582.

[38] Peter Hästö and Jihoon Ok, *Maximal regularity for local minimizers of non-autonomous functionals*, J. Eur. Math. Soc. (JEMS) **24** (2022), no. 4, 1285–1334.

[39] Moritz Kassmann, *The theory of De Giorgi for non-local operators*, C. R. Math. Acad. Sci. Paris **345** (2007), no. 11, 621–624.

[40] ———, *A priori estimates for integro-differential operators with measurable kernels*, Calc. Var. Partial Differential Equations **34** (2009), no. 1, 1–21.

[41] Janne Korvenpää, Tuomo Kuusi, and Erik Lindgren, *Equivalence of solutions to fractional p-Laplace type equations*, J. Math. Pures Appl. (9) **132** (2019), 1–26.

[42] Janne Korvenpää, Tuomo Kuusi, and Giampiero Palatucci, *The obstacle problem for nonlinear integro-differential operators*, Calc. Var. Partial Differential Equations **55** (2016), no. 3, Art. 63, 29.

[43] ———, *Fractional superharmonic functions and the Perron method for nonlinear integro-differential equations*, Math. Ann. **369** (2017), no. 3-4, 1443–1489.

[44] Tuomo Kuusi, Giuseppe Mingione, and Yannick Sire, *Nonlocal equations with measure data*, Comm. Math. Phys. **337** (2015), no. 3, 1317–1368.

[45] ———, *Nonlocal self-improving properties*, Anal. PDE **8** (2015), no. 1, 57–114.

[46] Gary M. Lieberman, *The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations*, Comm. Partial Differential Equations **16** (1991), no. 2-3, 311–361.

[47] Erik Lindgren, *Hölder estimates for viscosity solutions of equations of fractional p-Laplace type*, NoDEA Nonlinear Differential Equations Appl. **23** (2016), no. 5, Art. 55, 18.

[48] Paolo Marcellini, *Everywhere regularity for a class of elliptic systems without growth conditions*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **23** (1996), no. 1, 1–25.

[49] Tadele Mengesha, Armin Schikorra, and Sasikarn Yeepo, *Calderon-Zygmund type estimates for nonlocal PDE with Hölder continuous kernel*, Adv. Math. **383** (2021), Paper No. 107692, 64.

[50] Mihai Mihăilescu and Vicenţiu Rădulescu, *Neumann problems associated to nonhomogeneous differential operators in Orlicz-Sobolev spaces*, Ann. Inst. Fourier (Grenoble) **58** (2008), no. 6, 2087–2111.

[51] Simon Nowak, *Higher Hölder regularity for nonlocal equations with irregular kernel*, Calc. Var. Partial Differential Equations **60** (2021), no. 1, Paper No. 24, 37.

[52] ———, *Regularity theory for nonlocal equations with VMO coefficients*, Ann. Inst. H. Poincaré Anal. Non Linéaire (2022), to appear.

[53] Jihoon Ok, *Regularity of ω-minimizers for a class of functionals with non-standard growth*, Calc. Var. Partial Differential Equations **56** (2017), no. 2, Paper No. 48, 31.

[54] ———, *Partial Hölder regularity for elliptic systems with non-standard growth*, J. Funct. Anal. **274** (2018), no. 3, 723–768.

[55] ______, *Local Hölder regularity for nonlocal equations with variable powers*, Calc. Var. Partial Differential Equations **62** (2023), no. 1, Paper No. 32, to appear.

[56] M. M. Rao and Z. D. Ren, *Applications of Orlicz spaces*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 250, Marcel Dekker, Inc., New York (2002).

[57] Luis Silvestre, *Hölder estimates for solutions of integro-differential equations like the fractional Laplace*, Indiana Univ. Math. J. **55** (2006), no. 3, 1155–1174.

[58] Fengping Yao and Shulin Zhou, *Calderón-Zygmund estimates for a class of quasilinear elliptic equations*, J. Funct. Anal. **272** (2017), no. 4, 1524–1552.

국문초록

이 학위논문에서는 오리츠 성장조건을 가진 비국소 타원형 편미분방정식의 다양한 정칙성 결과들을 다룬다. 우선 변분법을 이용하여 오리츠 성장조건을 가진 비국소 디리클레 문제의 약해의 존재성과 유일성을 증명한다. 그 다음, 주어진 비국소 타원형 방정식에 적합한 형태의 소볼레프 - 푸앵카레 부등식과 로그가늠을 유도한 뒤 이를 통해 약해의 국소적 횔더 연속성을 보인다. 마지막으로, 정밀한 꼬리가늠을 통해 하르나크 부등식을 유도한다.

주요어휘: 비국소 작용소, 오리츠 성장조건, N-함수, 국소적 유계, 횔더 연속성, 하르나크 부등식
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