이학박사 학위논문

On the emergent behaviors of Cucker-Smale type flocks

(쿠커-스메일 유형 모델의 창발 현상에 관하여)

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변준혁

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이 논문을 이학박사 학위논문으로 제출함

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On the emergent behaviors of Cucker-Smale type flocks

A dissertation
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Abstract

In this thesis, we investigate *Cucker-Smale type* (in short, CS-type) models, mainly focusing on a case of a singular kernel. The CS-type model introduces an activation function to the CS model to describe various group phenomena, and the theory of relativity can be reflected as an example.

To motivate the CS-type model, we first introduce the relativistic Cucker-Smale (in short, RCS) model with a singular kernel. More precisely, we study collision avoidance and flocking dynamics for the RCS model with a singular communication weight. For a bounded and regular communication weight, RCS particles can exhibit collisions in finite time depending on the geometry of the initial configuration. In contrast, for a singular communication weight, when particles collide, the associated Cucker-Smale vector field becomes unbounded, and the standard Cauchy-Lipschitz theory cannot be applied, so the existence theory after collisions is problematic. Thus, the collision avoidance problem is directly linked to the global solvability of the singular RCS model and asymptotic flocking dynamics.

We then propose the CS-type model, which is a general nonlinear consensus model incorporating the RCS model. Depending on the regularity and singularity of communication weight at the origin and far-field, we provide diverse clustering patterns for collective behaviors on the real line. The singularity of the kernel induces collision avoidance or sticking property, depending on the integrability of the kernel near the origin. We study the regularity of sticking solutions of the proposed model on the real line. On the other hand, we provide a sufficient framework beyond collision avoidance property to guarantee a strict lower bound between agents in the Euclidean space. We then introduce a kinetic analog of the proposed model and study its well-posedness. We also show the structural stability in both particle and kinetic levels.

Key words: activation function, collision avoidance, emergent behavior, kinetic model, relativistic Cucker-Smale model, structural stability

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Contents

Chapter 1

Introduction

Collective behaviors of complex systems are ubiquitous in nature. In [34], the authors presented a unified equation for the first-order modeling on collective dynamics:

$$\dot{q}_i = \nu_i + \frac{\kappa}{N}\sum_{k=1}^{N}\Psi(q_k - q_i), \tag{1.0.1}$$

where ν_i is the natural velocity of the i-th agent, which describes the innate rate of changes. By suitable choice of coupling rule Ψ and ambient space $\mathcal{M}$, basic types of collective behaviors can be modeled from (1.0.1). For example, flocking of birds [25] or schooling of fish [27, 49] on real line, synchronization of fireflies [7] etc. Then it is natural to consider the inverse problem i.e., design of appropriate kernel Ψ for a given consensus behavior. Such reverse engineering problem is common in theory of deep learning, and typical approach is to approximate target function by iterated composition of functions of the form

$$x \mapsto g(Ax+b), \quad x \in \mathbb{R}^{d_1}, \quad b \in \mathbb{R}^{d_2}.$$

Above, $A : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ is a linear function, and g is a nonlinear function so-called the activation function, where the nonlinearity of g is essential to describe nontrivial phenomena [50]. In the authors' previous work [10], motivated by the theory of deep learning, they employed nonlinear *activation* *function* G and proposed the system

$$\dot{q}_i = G(\nu_i + \frac{\kappa}{N}\sum_{k=1}^{N}\Psi(q_k - q_i)), \tag{1.0.2}$$

where G is a continuously differentiable odd function, whose derivative is strictly positive. Typical example is the hyperbolic tangent function tanh, which is an activation function frequently used in deep learning. As an another example, the authors in [29] discussed that relativistic effect can be considered by choosing suitable activation function involving the Lorentz factor. For the other examples, we refer to [41].

Regarding (1.0.1) as the basic building block of variant consensus behaviors, its generalization have been studied in various manner, e.g. flocking [26], synchronization [38, 53], aggregation via Hessian communication [40] etc, where the methodology of generalization depends on the physical context. In context of flocking, Ψ is typically assumed to be an odd function, which is differentiable, concave, and increasing on $\mathbb{R}_+$. In this case, (1.0.2) is extended to the following Cucker-Smale (in short, CS) type model [10]:

$$\begin{cases} \dot{q}_i = G(p_i), \quad t > 0, \quad i \in [N] := \{1, 2, \cdots, N\}, \\ \dot{p}_i = \dfrac{\kappa}{N}\displaystyle\sum_{k=1}^{N}\psi(|q_k - q_i|)(G(p_k) - G(p_i)), \\ (q_i, p_i)\big|_{t=0+} = (q_i^0, p_i^0), \quad p_i, q_i \in \mathbb{R}^d, \end{cases} \tag{1.0.3}$$

where the kernels Ψ and ψ in (1.0.2) and (1.0.3) are coupled as $\Psi' = \psi$. If we pose $G = \mathrm{Id}$ in (1.0.3), then we have the standard CS model;

$$\begin{cases} \dot{x}_i = v_i, \quad t > 0, \quad i \in [N] := \{1, 2, \cdots, N\}, \\ \dot{v}_i = \dfrac{\kappa}{N}\displaystyle\sum_{k=1}^{N}\psi(|x_k - x_i|)(v_k - v_i), \\ (x_i, v_i)\big|_{t=0+} = (x_i^0, v_i^0), \quad x_i, v_i \in \mathbb{R}^d, \end{cases} \tag{1.0.4}$$

where x_i and v_i are regarded as a position and velocity of the i-th agent, respectively. The CS model describes the dynamics of flocking behaviors of self-propelled particles, and have been studied extensively as it unites seeming unrelated phenomena [6, 47, 51, 52]. For the mathematical analysis, kinetic description, and hydrodynamic description of CS model, we refer to [6, 15, 19, 33, 36, 43], [14] and [39], respectively.

As mentioned above, (1.0.3) incorporates *relativistic* Cucker-Smale model (in short, RCS) model. Therefore, to motivate (1.0.3), we begin with the discussion by introducing RCS model. Let $(x_i, v_i) \in \mathbb{R}^{2d}$ be the position and the relativistic velocity of the i-th CS particle satisfying $|v_i| < c$. Here, $|\cdot|$ denotes the standard ℓ_2-norm in $\mathbb{R}^d$. For a velocity vector v in $B_c(0)$ (the open ball centered at the origin with radius c), we set the Lorentz factor Γ and the quantity F as

$$\Gamma(v) := \left(1 - \frac{|v|^2}{c^2}\right)^{-1/2}, \quad F(v) := \Gamma\left(1 + \frac{\Gamma}{c^2}\right),$$

and define $\hat{w} : B_c(0) \to \mathbb{R}^d$ as

$$\hat{w}(v) := F(v)v = \frac{v}{\sqrt{1 - \frac{|v|^2}{c^2}}} + \frac{v}{c^2 - |v|^2}.$$

Then, one can show that $\hat{w}$ is bijective, and there exists an inverse function $\hat{v} := \hat{w}^{-1} : \mathbb{R}^d \to B_c(0)$ satisfying the relation:

$$v = \hat{v}(w) = \frac{w}{F(v)}. \tag{1.0.5}$$

For further details, we refer to Chapter 2. Now, we are ready to introduce the RCS model formulated in terms of state variables $(x_i, w_i := F(v_i)v_i)$:

$$\begin{cases} \dfrac{dx_i}{dt} = \hat{v}(w_i), \quad t > 0, \quad i \in [N] := \{1, \dots, N\}, \\ \dfrac{dw_i}{dt} = \dfrac{1}{N}\displaystyle\sum_{k=1}^{N} \phi(|x_i - x_k|)\left(\hat{v}(w_k) - \hat{v}(w_i)\right), \\ (x_i(0), w_i(0)) = (x_i^{in}, w_i^{in}) \in \mathbb{R}^{2d}, \end{cases} \tag{1.0.6}$$

where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a communication weight, encoding the degree of interactions between particles in terms of their relative distances. Usually, the communication weight is taken to be monotone decreasing, which means that the interactions between particles become weaker, as their relative distances become larger.

The RCS model (1.0.6) and its simplified variants have been extensively studied from various point of views, e.g., emergent behaviors on the Euclidean space [4, 29], dynamics on the Riemannian manifold [5], kinetic description and mean-field limit [4, 30], hydrodynamic description [30], etc. On the other hand, when we consider real-world applications, one of the main concerns for the multi-agent system is to guarantee collision avoidance between particles (agents). To keep particles away from mutual collisions, one may consider a singular communication weight [12, 20] or additional control inputs [22, 23, 24]. For the CS model, a singular communication weight is mainly considered due to its simple structure and investigated extensively in literature [13, 18, 31, 42, 44, 46, 45, 48]. In this work, we are mainly interested in the following specific singular communication weight ϕ:

$$\phi(r) = r^{-\alpha}, \quad \alpha > 0. \tag{1.0.7}$$

Note that the explicit form of the communication weight is not important, but the singularity at $r = 0$ will be essential in what follows.

Now, we briefly discuss a formal derivation of the general CS-type consensus model (1.0.2) from the RCS model on the real line, and review related previous results on the CS model on the real line. To set the stage, we begin with a brief description of the RCS model on the real line. Let x_i, p_i and $\hat{v}(p_i)$ be the scalar position, momentum and velocity variables of the i-th RCS particle on the real line. Then, the RCS model reads as

$$\begin{cases} \dot{x}_i = \hat{v}(p_i), \quad t > 0, \quad i \in [N], \\ \dot{p}_i = \dfrac{\kappa}{N} \displaystyle\sum_{k=1}^{N} \psi(x_k - x_i)(\hat{v}(p_k) - \hat{v}(p_i)), \\ (x_i, p_i)\big|_{t=0+} = (x_i^0, p_i^0), \end{cases} \tag{1.0.8}$$

where $\psi : \mathbb{R} \to \mathbb{R}_+$ is a Lipschitz continuous communication weight function.

To recast (1.0.8) as an abstract consensus model (1.0.2), we set $\Psi(\cdot)$ to be the antiderivative of ψ:

$$\Psi(x) := \int_0^x \psi(y)\,dy, \quad x \in \mathbb{R},$$

as long as ψ is locally integrable around $x = 0$. Note that

$$\frac{d}{dt}\Psi(x_k(t) - x_i(t)) = \frac{d}{dt}\int_0^{x_k(t)-x_i(t)} \psi(y)dy = \psi(x_k(t) - x_i(t))(v_k(t) - v_i(t)). \tag{1.0.9}$$

Hence, it follows from $(1.0.8)_2$ and (1.0.9) that

$$\frac{d}{dt}\left(p_i - \frac{\kappa}{N}\sum_{k=1}^N \Psi(x_k(t) - x_i(t))\right) = 0, \quad t > 0, \quad i \in [N]. \tag{1.0.10}$$

Now, we integrate (1.0.10) with respect to t to get

$$\begin{aligned} p_i(t) &= p_i^0 - \frac{\kappa}{N}\sum_{k=1}^N \Psi(x_k^0 - x_i^0) + \frac{\kappa}{N}\sum_{k=1}^N \Psi(x_k(t) - x_i(t)) \\ &=: \nu_i + \frac{\kappa}{N}\sum_{k=1}^N \Psi(x_k(t) - x_i(t)). \end{aligned} \tag{1.0.11}$$

Note that ν_i is a natural velocity depending only on initial data and coupling strength κ. Finally, we combine $(1.0.8)_1$ and (1.0.11) to get the first-order abstract consensus model:

$$\hat{p}(\dot{x}_i) = \nu_i + \frac{\kappa}{N}\sum_{k=1}^N \Psi(x_k - x_i), \quad i \in [N],$$

or equivalently,

$$\dot{x}_i = \hat{v}\left(\nu_i + \frac{\kappa}{N}\sum_{k=1}^N \Psi(x_k - x_i)\right), \quad i \in [N]. \tag{1.0.12}$$

Hence, if we set $q_i = x_i$ and $G = \hat{v}$, then system (1.0.12) corresponds to the special case of the general abstract consensus model (1.0.2). Another special case of (1.0.2) can be also derived from the relativistic Kuramoto model (see [41]). Other examples of activation function G, or equivalently G^{-1}, are

$$\text{Proper velocity: } G^{-1}(v) = v\left(1 - \frac{|v|^2}{c^2}\right)^{-1},$$
$$\text{Rapidity: } G^{-1}(v) = c\tanh^{-1}\left(\frac{v}{c}\right).$$

For the detailed derivation, we refer to [41]. Note that the hyperbolic tangent function, which describes the rapidity, is an activation function frequently used in machine learning. Indeed, besides the physical semantics, we may employ activation functions in deep learning satisfying (1.0.13). For example, we can consider the symmetrized sigmoid function

$$G^{-1}(v) = cS^{-1}\left(\frac{v}{c}\right), \quad S(x) := \frac{1}{1+e^{-x}} - \frac{1}{2}.$$

We also assume that for any closed interval I, there exist positive constants $m_{G'}$ and $M_{G'}$ such that

$$0 < m_{G'} \le G'(q) \le M_{G'} < \infty, \quad G(-q) = -G(q), \quad \forall\, q \in I, \tag{1.0.13}$$

because it turns out that (1.0.13) are essential properties of activation functions inducing the flocking phenomena (see Section 4.1). The most simplest and motivating example for G will be the identity mapping $G(q) = q$. In this case, system (1.0.2) reduces to system (1.0.1). We provide emergent dynamics of (1.0.2) depending on the behaviors of the communication weight function $\psi := \Psi'$ at $q = 0$ and $q = \infty$:

$$\begin{aligned}
&\text{Type I}: \int_0^\infty \psi(q)\,dq = \infty \;:\; \text{Regular, long-ranged communication weight},\\
&\text{Type II}: \int_0^\infty \psi(q)\,dq < \infty \;:\; \text{Regular, short-ranged communication wight},\\
&\text{Type III}: \psi(q) = \frac{1}{|q|^\alpha}, \quad \alpha > 0, \quad q \neq 0 \;:\; \text{Singular communication weight}.
\end{aligned} \tag{1.0.14}$$

In the thesis, we are mainly interested in the study of asymptotic consensus behavior (flocking) of (1.0.2) and (1.0.3) for each types of kernels in

(1.0.14). Analysis on singular kernel (Type III) is in particular interesting due to its extraordinary behavior. In view of (1.0.2), if $\alpha \in (0,1)$, where α represents the singularity of kernel as in (1.0.14), then two particles *stick* after some finite time if and only if they share the same natural velocity. Regarding (1.0.2) as a training model, we may interpret it as a flow classifying the agents, where each group, labeled by ν_i, shares some value q_i, where q_i depends on time, initial data and coupling Ψ. On the other hand, for $\alpha \geq 1$, two distinct states q_i and q_j never coincide through time evolution, which is usually referred as *collision avoidance*. If each agents lies on the real line, (1.0.3) might be converted into the first-order consensus model (1.0.2) and in this case, aforementioned priorities can be rigorously stated and proven. However, lifting those results into second-order model may not be trivial (see Remark 4.2.1), even if $d = 1$. The non-triviality follows from the loss of regularity from collision; as a kernel is singular, Ψ is not differentiable at the origin (i.e. at the instance two particles collide), and therefore standard calculus cannot be applied directly. The one of goals of this thesis is to overcome such difficulty and rigorously connect (1.0.2) and (1.0.3) on the real line, which may hint us for the description of a solution to (1.0.2) on general ambient space. Indeed, we provide some results for (1.0.3) on $\mathbb{R}^d$, which is parallel to the results of the real line case.

Another content of this thisis is to study structural stability of (3.1.5). More precisely, we consider the classic CS model (1.0.4) as a reference model and consider the following problem:

$$\text{If } (p_i^0, q_i^0) = (x_i^0, v_i^0), \text{ then would } G \to \mathrm{Id} \text{ implies} \\ (p_i(t), q_i(t)) \to (x_i(t), v_i(t))?$$

We provide the structural stability in both microscopic and mesoscopic level. To be more precise for the mesoscopic description, we first introduce and study the well-posedness of the following Vlasov-type equation:

$$\begin{aligned}
&\partial_t f + G(p)\cdot\nabla_q f + \nabla_p \cdot (F[f]f) = 0, \quad p, q \in \mathbb{R}^d,\\
&F[f](p,q,t) := \int_{\mathbb{R}^{2d}} \psi(|q^* - q|)(G(p^*) - G(p)) f(p^*, q^*, t)\,dp^* dq^*,
\end{aligned}$$

which is formally obtained by taking limit $N \to \infty$ to (1.0.3). We then study the convergence of solution toward a solution of

$$\partial_t f + p \cdot \nabla_q f + \nabla_p \cdot (F[f]f) = 0, \quad p, q \in \mathbb{R}^d,$$
$$F[f](p,q,t) := \int_{\mathbb{R}^{2d}} \psi(|q^* - q|)(p^* - p) f(p^*, q^*, t) dp^* dq^*,$$

in a suitable sense. In this thesis, in the same vein as the previous paragraph, we limit ourselves to the singular kernel. For a previous work regarding a regular kernel, we refer to [4].

In summary, we address the following questions throughout the thesis:

- (Q1): What are the essential properties of the RCS model (1.0.6) that cause flocking as the standard CS model (1.0.4) does? Under those properties, can we expect the CS-type consensus model (1.0.3) to show flocking behavior?

- (Q2): When the kernel ψ is singular, can we guarantee the well-posedness of the solution? If so, can we expect some special interaction between agents?

- (Q3): Can the solution to (1.0.4) converge to the solution to (1.0.3) in suitable sense, as G converges to the identity map Id?

- (Q4): Can we consider the kinetic description of (1.0.3)? If so, what will be the kinetic analog of answers to (Q1),(Q2), and (Q3)?

The rest of this thesis is organized as follows. In Chapter 2, we present the relativistic Cucker-Smale model with a singular kernel and collision avoidance property. In Chapter 3, the study first-order CS-type consensus model on the real line and analysis of the effect of kernel depending on the regularity or singularity at the origin or infinity. In Chapter 4, we discuss the asymptotic dynamics of the CS-type consensus model and consider a further analysis of results in the previous chapters. Finally, Chapter 5 is devoted to a brief summary of our main results and some discussion on the remaining issues for a future work.

Chapter 2

The relativistic Cucker-Smale model with a singular kernel

In this chapter, we first present basic properties and preliminary lemmas for the following RCS model

$$\begin{cases} \dfrac{dx_i}{dt} = \hat{v}(w_i), \quad t > 0, \quad i \in [N] := \{1, \dots, N\}, \\ \dfrac{dw_i}{dt} = \dfrac{1}{N} \displaystyle\sum_{k=1}^{N} \phi(|x_i - x_k|) \left(\hat{v}(w_k) - \hat{v}(w_i) \right), \\ (x_i(0), w_i(0)) = (x_i^{in}, w_i^{in}) \in \mathbb{R}^{2d}, \end{cases} \tag{2.0.1}$$

with a singular communication weight and recall previous results for a regular communication weight. Then, we consider a strongly singular communication weight

$$\phi(s) = s^{-\alpha}, \quad \alpha > 0, \tag{2.0.2}$$

with $\alpha \in [1, \infty)$. In this case, we provide the nonexistence of collisions for non-collisional initial data and derive a global existence of solutions to (1.0.6) with asymptotic flocking dynamics under suitable conditions on initial data and communication weight function. On the other hand, weakly singular communication weight with $\alpha \in (0, 1)$ is also considered. In this setting, we present an explicit example leading to the finite-time collision for the two-particle system, and then we derive a sufficient framework for the nonexistence of collisions, so that system (1.0.6) yields a global solution satisfying asymptotic flocking. We note that this chapter is based on the joint work [11].

Before we proceed further, we first clarify the the concept of asymptotic flocking for (2.0.1). We define a concise notation for the spatial and velocity configurations as

$$X := \{x_i\}, \quad V := \{v_i\}, \quad W := \{w_i\}, \quad Z := \{(x_i, w_i)\}.$$

Definition 2.0.1. *Let Z be a global solution to the RCS model* (1.0.6)*. Then the RCS model exhibits asymptotic flocking if the following estimates hold:*

$$\sup_{0 \le t < \infty} |x_i(t) - x_j(t)| < \infty, \quad \lim_{t \to \infty} |w_i(t) - w_j(t)| = 0, \quad i, j \in [N].$$

2.1 Introduction to the relativistic Cucker-Smale model

2.1.1 The RCS model

We briefly discuss a derivation of (1.0.6) from the relativistic thermodynamic CS model. Details can be found in [29].

Let (x_i, v_i, T_i) be the thermomechanical state consisting of position, velocity and temperature of the i-th relativistic thermodynamic CS particle. Then, the state dynamics is governed by the following coupled system of first-order ODEs:

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \quad t > 0, \quad i \in [N], \\ \dfrac{d}{dt}\left(\Gamma_i v_i \left(1 + \dfrac{\Gamma_i}{c^2}\right)\right) = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} \phi_{ij} \left(\dfrac{v_j \Gamma_j}{T_j} - \dfrac{v_i \Gamma_i}{T_i}\right), \\ \dfrac{d}{dt}\left(\Gamma_i T_i + c^2(\Gamma_i - 1)\right) = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} \zeta_{ij} \left(\dfrac{\Gamma_i}{T_i} - \dfrac{\Gamma_j}{T_j}\right), \end{cases} \tag{2.1.1}$$

where $\Gamma_i := \Gamma(v_i)$ and we used handy notation for communication weights:

$$\phi_{ij} := \phi(|x_j - x_i|) \quad \text{and} \quad \zeta_{ij} := \zeta(|x_j - x_i|), \quad i, j \in [N].$$

Note that in the relativistic regime with $c \gg |v_i|$, we have the following Taylor expansions for quantities related with Γ_i:

$$\Gamma_i = 1 + \frac{|v_i|^2}{2c^2} + \frac{3}{8}\frac{|v_i|^4}{c^4} + \cdots, \quad \Gamma_i^2 = 1 + \frac{|v_i|^2}{c^2} + \frac{|v_i|^4}{c^4} + \cdots,$$
$$c^2\Gamma_i(\Gamma_i - 1) = \frac{|v_i|^2}{2} + \frac{5}{8}\frac{|v_i|^4}{c^2} + \cdots.$$

Then, these yield

$$\lim_{c\to\infty} \Gamma_i = 1, \quad \lim_{c\to\infty} c^2(\Gamma_i - 1) = \frac{|v_i|^2}{2}.$$

Thus, in a classical limit $c \to \infty$, the relativistic system (2.1.1) reduces to the classical TCS model [35]:

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \quad t > 0, \quad i \in [N], \\ \dfrac{dv_i}{dt} = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \phi_{ij}\left(\frac{v_j}{T_j} - \frac{v_i}{T_i}\right), \\ \dfrac{d}{dt}\left(T_i + \dfrac{1}{2}|v_i|^2\right) = \dfrac{1}{N}\displaystyle\sum_{j=1}^{n} \zeta_{ij}\left(\frac{1}{T_i} - \frac{1}{T_j}\right). \end{cases}$$

Now, we return to system (2.1.1). Following the principle of system in [29], we set

$$\frac{T_i}{\Gamma_i} = T^* = 1, \quad i \in [N],$$

and ignore the third equation $(2.1.1)_3$ to derive the RCS model:

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \quad t > 0, \ i \in [N], \\ \dfrac{d}{dt}\left(\Gamma_i v_i\left(1 + \dfrac{\Gamma_i}{c^2}\right)\right) = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \phi_{ij}\left(v_j - v_i\right). \end{cases}$$

Since v_i can be represented in terms of w_i, $F(v_i)$ itself can be expressed by w_i. However, since the explicit representation of the map $\hat{v}$ is extremely complicated, we use the notation $F_i := F(v_i)$. We introduce $\Psi : [0, c) \to [0, \infty)$ defined as

$$\Psi(r) := \frac{cr}{\sqrt{c^2 - r^2}} + \frac{r}{c^2 - r^2}.$$

Then, it is easy to see

$$\hat{w}(v_i) = F_i|v_i|\frac{v_i}{|v_i|} = \left(\frac{c|v_i|}{\sqrt{c^2 - |v_i|^2}} + \frac{|v_i|}{c^2 - |v_i|^2}\right)\frac{v_i}{|v_i|} = \Psi(|v_i|)\frac{v_i}{|v_i|}.$$

Note that the map Ψ is bijective, and both Ψ and Ψ^{-1} are strictly increasing in their arguments.

2.1.2 Preliminary lemmas

In this subsection, we study several estimates to be used in later sections.

Lemma 2.1.1. *Let $Z = Z(t)$ be a solution to system* (1.0.6). *Then, the following assertions hold:*

1. *The total sum of w_i is conserved:*

$$\sum_{i=1}^N w_i(t) = \sum_{i=1}^N w_i^{in}, \quad t > 0.$$

2. *The maximal moduli of w_i and v_i decrease monotonically in time:*

$$\max_{1\le i\le N} |w_i(t)| \le \max_{1\le i\le N} |w_i(s)|, \quad \max_{1\le i\le N} |v_i(t)| \le \max_{1\le i\le N} |v_i(s)|, \quad 0 \le s \le t.$$

In particular, if the initial speed is less than c, then the speed of particles cannot exceed c:

$$|v_M^{in}| := \max_{1\le i\le N} |v_i^{in}| < c \quad \Longrightarrow \quad \max_{1\le i\le N} |v_i(t)| \le |v_M^{in}| < c.$$

Proof. (1) Since ϕ is radially symmetric, we have $\phi_{ij} = \phi_{ji}$. Then, it follows from $(1.0.6)_2$ that

$$\frac{d}{dt}\sum_{i=1}^N w_i = \frac{1}{N}\sum_{i,j=1}^N \phi_{ij}\left(\hat{v}(w_j) - \hat{v}(w_i)\right) = \frac{1}{N}\sum_{i,j=1}^N \phi_{ij}\left(\hat{v}(w_i) - \hat{v}(w_j)\right) = 0.$$

(2) Let M be an index satisfying

$$|w_M| := \max_{1\le i\le N} |w_i|.$$

Again, we use $(1.0.6)_2$ to derive

$$\frac{d|w_M|^2}{dt} = \frac{2}{N}\sum_{j=1}^{N} \phi_{Mj} w_M \cdot (v_j - v_M).$$

Since Ψ^{-1} is an increasing function, the map

$$|w| \mapsto \frac{|w|}{F} = \Psi^{-1}(|w|)$$

is an increasing function of $|w|$. Therefore, by the maximality of w_M and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} w_M \cdot (v_j - v_M) &= w_M \cdot \frac{w_j}{F_j} - \frac{|w_M|^2}{F_M} \\ &\le |w_M|\left(\frac{|w_j|}{F_j} - \frac{|w_M|}{F_M}\right) = |w_M|\big(\Psi^{-1}(|w_j|) - \Psi^{-1}(|w_M|)\big) \le 0, \end{aligned}$$

for all $j = 1, 2, \ldots, N$. Therefore, we obtain

$$\frac{d|w_M|^2}{dt} \le 0 \quad \text{for } t > 0.$$

This implies the monotonicity of $|w_M|$. Since the map $\Psi^{-1} : |w| \to |v|$ is an increasing function, we get the monotonicity of $|v_M|$.

□

Remark 2.1.1. *Thanks to Lemma 3.1.3, without loss of generality, we may assume*

$$\sum_{i=1}^{N} w_i(t) = 0, \quad t \ge 0.$$

Lemma 2.1.2. *Let $Z = Z(t)$ be a solution to system* (1.0.6) *with initial data:*

$$\max_{1\le i\le N} |w_i^{in}| \le U_w < \infty.$$

Then, there exists a positive constant $C_L = C_L(U_w)$ such that

$$\begin{aligned}&(1)\ C_L|w_i - w_j| \le |\hat{v}(w_i) - \hat{v}(w_j)| \le |w_i - w_j|,\\ &(2)\ C_L^2|w_i - w_j|^2 \le (w_i - w_j)\cdot(\hat{v}(w_i) - \hat{v}(w_j)) \le |w_i - w_j|^2.\end{aligned} \tag{2.1.2}$$

Proof. (1) • (Derivation of the left inequality): We fix indices i and j and denote them by 1 and 2. Without loss of generality, we assume $|w_1| \ge |w_2|$. We use the mean value theorem and monotonicity of the maximal modulus of velocity to find

$$\begin{aligned}|w_1 - w_2| &\le \sup_{|v|\le\Psi^{-1}(|w_1|)} \|\hat{w}'(v)\|_{op}|v_1 - v_2|\\ &\le \sup_{|v|\le\Psi^{-1}(U_w)} \|\hat{w}'(v)\|_{op}|v_1 - v_2|,\end{aligned}$$

where $\|\hat{w}'(v)\|_{op}$ stands for the operator norm of the Jacobian of $\hat{w}$ at v. On the other hand, the Jacobian $\hat{w}'(v)$ can be explicitly computed as

$$\hat{w}' = \left(\frac{|v|^2}{(c^2-|v|^2)^{\frac{3}{2}}} + \frac{2|v|^2}{(c^2-|v|^2)^2}\right)\frac{v\otimes v}{|v|^2} + \left(\frac{c}{\sqrt{c^2-|v|^2}} + \frac{1}{c^2-|v|^2}\right)I_d,$$

for $|v| \le \Psi^{-1}(U_w)$. Since the eigenvalues of $\frac{v\otimes v}{|v|^2}$ are 0 and 1 up to multiplicity, eigenvalues of the symmetric matrix $\hat{w}'$ are given by

$$\begin{aligned}\lambda_1(v) &= \frac{c}{\sqrt{c^2-|v|^2}} + \frac{1}{c^2-|v|^2} \quad \text{and}\\ \lambda_2(v) &= \frac{c}{\sqrt{c^2-|v|^2}} + \frac{1}{c^2-|v|^2} + \frac{|v|^2}{(c^2-|v|^2)^{\frac{3}{2}}} + \frac{2|v|^2}{(c^2-|v|^2)^2}.\end{aligned}$$

Therefore, the operator norm of the Jacobian $\hat{w}'(v)$, the largest singular value, is $\lambda_2(v)$. We use $\max_{k=1,2}|v_k(t)| \le |\Psi^{-1}(U_w)|$ to achieve

$$\sup_{|v|\le\Psi^{-1}(|w_1|)} \|\hat{w}'(v)\|_{op} = \sup_{|v|\le\Psi^{-1}(U_w)} \lambda_2(v) = \lambda_2(\Psi^{-1}(U_w)) =: C(U_w).$$

Hence, defining $C_L(U_w) = \frac{1}{C(U_w)}$, we have the left inequality of (1).

• (Derivation of the right inequality): To show the right inequality of (1), we note that

$$|v_1(t) - v_2(t)| \le \sup_{|w|\le U_w} \|\hat{v}'(w)\|_{op}|w_1(t) - w_2(t)|,$$

where $\hat{v}'(w)$ is now the Jacobian of $\hat{v}$ at w, which is an inverse of $\hat{w}(v)$. Thus, the eigenvalue of $\hat{v}'$ is also the inverse of λ_1 and λ_2, and in particular, they are smaller than 1. Therefore, the operator norm of $\hat{v}'(w)$ is also less than 1, which implies the desired right inequality of (1).

(2) • (Derivation of the left inequality): Again, without loss of generality, we assume $|w_1(t)| \geq |w_2(t)|$. Then, it is easy to observe

$$|v_1(t)| \geq |v_2(t)| \quad \text{and} \quad F_1 \geq F_2 \geq 1.$$

We use the above inequalities to obtain

$$\begin{aligned}
(w_1 - w_2)\cdot&(\hat{v}(w_1) - \hat{v}(w_2)) \\
&= (F_1 v_1 - F_2 v_1 + F_2 v_1 - F_2 v_2) \cdot (v_1 - v_2) \\
&= F_2 |v_1 - v_2|^2 + (F_1 - F_2)(v_1 - v_2) \cdot v_1 \\
&\geq F_2 |v_1 - v_2|^2 \geq |v_1 - v_2|^2 \geq C_L^2 |w_1 - w_2|^2.
\end{aligned}$$

• (Derivation of the right inequality): Similarly, under the assumption $|w_1(t)| \leq |w_2(t)|$,

$$\begin{aligned}
(w_1 - w_2)\cdot&(\hat{v}(w_1) - \hat{v}(w_2)) \\
&= \frac{1}{F_2}|w_1 - w_2|^2 + \left(\frac{1}{F_1} - \frac{1}{F_2}\right)(w_1 - w_2) \cdot w_1 \\
&\leq \frac{1}{F_2}|w_1 - w_2|^2 \leq \frac{c^2}{1 + c^2}|w_1 - w_2|^2 \leq |w_1 - w_2|^2.
\end{aligned}$$

This proves the right inequality. □

Remark 2.1.2. *(1) Since the Jacobian $\hat{w}'(v)$ converges asymptotically to the identity matrix as $c \to \infty$, one has $\lim_{c\to\infty} \|\hat{w}'(v)\|_{op} = 1$, therefore* $\lim_{c\to\infty} C_L = 1$. *Consequently, the estimate* (2.1.2) *implies the coincidence of v and w, when relativistic effect is not taken into account via $c \to \infty$. This also agrees with our heuristic intuition. The constant C_L depends on both U_w and c, although the explicit formula for the dependency is extremely complicated.*

(2) In [29, Lemma 6.6], the authors also obtained the following estimate:

$$(w_1 - w_2) \cdot (\hat{v}(w_1) - \hat{v}(w_2)) \geq \gamma(U_w)|w_1 - w_2|^2,$$

and $\lim_{c\to\infty} \gamma(U_w) = \frac{1}{2}$*. Therefore, Lemma 2.1.2 improves the estimate in [29].*

2.1.3 Previous results

In this subsection, we briefly review a global existence and asymptotic flocking dynamics of the RCS model with bounded and Lipschitz communication weights. Consider the RCS model with a bounded communication weight:

$$\begin{cases} \dfrac{dx_i}{dt} = \hat{v}(w_i), \quad t > 0, \quad i \in [N], \\ \dfrac{dw_i}{dt} = \dfrac{1}{N} \displaystyle\sum_{k=1}^{N} \phi_b(|x_i - x_k|) \left(\hat{v}(w_k) - \hat{v}(w_i)\right), \\ (x_i(0), w_i(0)) = (x_i^{in}, w_i^{in}) \in \mathbb{R}^{2d}, \end{cases} \tag{2.1.3}$$

where $\phi_b : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+$ is non-increasing function satisfying

$$\|\phi_b\|_{L^\infty} \leq \phi_b^* \quad \text{and} \quad [\phi_b]_{\mathrm{Lip}} := \sup_{r \neq s \in \mathbb{R}_+ \cup \{0\}} \frac{|\phi_b(r) - \phi_b(s)|}{|r - s|} < \infty.$$

Assume that there exists a positive constant $v^0 < c$ such that $\max_{1\leq i\leq N} |v_i^{in}| < v^0$. Then, the monotonicity of the radius of velocity (which still holds for the regular communication weight) implies $\max_{1\leq i\leq N} |v_i(t)| < v^0$ for all $t \geq 0$.

For asymptotic flocking estimate, we also introduce several Lyapunov functionals:

$$\mathcal{D}_x := \max_{1\leq i,j\leq N} |x_i - x_j|, \quad \mathcal{D}_w := \max_{1\leq i,j\leq N} |w_i - w_j|, \quad \mathcal{L}_w := \frac{1}{2N} \sum_{i,j=1}^{N} |w_i - w_j|^2.$$

For notational simplicity, we set

$$\phi_b^{ij} := \phi_b(|x_i - x_j|), \quad i, j \in [N].$$

In the following two theorems, we summarize state-of-the-art results for the RCS model with a regular and bounded communication weight.

Theorem 2.1.1. [29] *Let Z be a solution to* (2.1.3) *with the initial data Z^{in}, and let R be a positive constant such that*

$$\sup_{0\le t<\infty}\max_{1\le i\le N}|w_i(t)|\le R.$$

Then the following assertions hold.

1. *If $\phi_b^{ij}\equiv 1$, then there exists a positive constant $\Lambda(R)$ such that*

$$\mathcal{D}_w(t)\le \mathcal{D}_w(0)e^{-\Lambda(R)t},\quad t\ge 0.$$

2. *If ϕ_b^{ij} is constant and satisfies $\phi_m := \min_{i,j}\phi_b^{ij}>0$, then, $\mathcal{L}_w$ decays exponentially fast:*

$$\mathcal{L}_w(t)\le \mathcal{L}_w(0)e^{-2\Lambda(R)\phi_m t},\quad t\ge 0.$$

Theorem 2.1.2. [4] *Let Z be a solution to* (2.1.3) *with initial data Z^{in}. Suppose that initial data Z^{in} satisfy the following assumptions: for given positive constants α,β and C, there exists a positive constant $\mathcal{D}_x^\infty$ such that*

$$\beta<\phi(\mathcal{D}_x^\infty)\alpha,\quad \mathcal{D}_x(0)+\frac{\mathcal{D}_w(0)}{C(\alpha\phi(\mathcal{D}_x^\infty)-\beta)}<\mathcal{D}_x^\infty.$$

Then, there exists a positive constant Λ such that the following asymptotic flocking emerges:

$$\mathcal{D}_x(t)<\mathcal{D}_x^\infty,\quad \mathcal{D}_w(t)\le\mathcal{D}_w(0)e^{-\Lambda t},\quad t\ge 0.$$

2.2 Strongly singular communication weight

2.2.1 The collision avoidance

We study collision avoidance due to singular interactions for system (1.0.6) to derive a global well-posedness. Since the communication weight ϕ is singular at $r=0$, local well-posedness problem will surface up at the instant in which $x_i(t)=x_j(t)$, i.e., two particles x_i and x_j collide. Thus, to obtain a global well-posedness, it suffices to show that there will be no collisions in any finite time interval. Our first main result is concerned with the nonexistence of finite-time collisions.

Theorem 2.2.1. *Suppose that the communication weight ϕ is sufficiently singular and non-collisional:*

$$\alpha \geq 1, \qquad \min_{1 \leq i \neq j \leq N} |x_i^{in} - x_j^{in}| > 0.$$

Then, system (1.0.6) *admits a unique non-collisional global-in-time solution Z satisfying*

$$\min_{1 \leq i \neq j \leq N} |x_i(t) - x_j(t)| > 0, \quad t \geq 0.$$

Since the map $\hat{v}$ in (3.1.4) is bounded and Lipschitz continuous by Lemma 2.1.2 and the communication weight $\phi(|x_i - x_j|)$ is regular unless $x_i = x_j$, we may use the standard Cauchy-Lipschitz theory to guarantee the existence and uniqueness of the local solution to (1.0.6), before collisions happen. Therefore, it suffices to show that there are no collisions at any finite time for a global well-posedness. This will be verified step by step.

In the sequel, collision avoidance will be verified via a contradiction argument. Suppose that the first collision occurs at time $t_0 \in (0, \infty)$, and let particle x_l be the one of the particles colliding with other particles at time $t = t_0$. Moreover, we define the index set $[l]$ as the set of indices of particles colliding with x_l at $t = t_0$:

$$\begin{aligned} &|x_l - x_j| \to 0 \quad \text{as} \quad t \nearrow t_0 \quad \text{for all} \quad j \in [l], \\ \text{for some } \delta > 0, \quad &|x_l - x_k| \geq \delta > 0 \quad \text{in} \quad t \in [0, t_0) \quad \text{for all} \quad k \notin [l]. \end{aligned}$$

Now, we consider ℓ_2-norms of the system $\{(x_i, w_i)\}_{i \in [l]}$ by

$$\begin{aligned} \|W\|_{[l]} := \sqrt{\sum_{i,j \in [l]} |w_i - w_j|^2}, \quad \|V\|_{[l]} &:= \sqrt{\sum_{i,j \in [l]} |v_i - v_j|^2}, \\ \|X\|_{[l]} &:= \sqrt{\sum_{i,j \in [l]} |x_i - x_j|^2}. \end{aligned} \tag{2.2.1}$$

Then, it is straightforward to see that

$$\left| \frac{d}{dt} \|X\|_{[l]}^2 \right| = 2 \left| \sum_{i,j \in [l]} (x_i - x_j) \cdot (\hat{v}(w_i) - \hat{v}(w_j)) \right| \leq 2 \|X\|_{[l]} \|V\|_{[l]}.$$

This and (2.1.2) yield

$$\left|\frac{d}{dt}\|X\|_{[l]}\right| \le \|V\|_{[l]} \le \|W\|_{[l]}. \tag{2.2.2}$$

Next, we estimate the term $\|W\|_{[l]}$ as

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|W\|_{[l]}^2 &= \sum_{i,j\in[l]}(w_i - w_j)\cdot\left(\frac{dw_i}{dt} - \frac{dw_j}{dt}\right)\\
&= \frac{1}{N}\sum_{i,j\in[l]}\sum_{k=1}^{N}(w_i - w_j)\cdot(\phi(|x_k - x_i|)(\hat{v}(w_k) - \hat{v}(w_i))\\
&\qquad - \phi(|x_k - x_j|)(\hat{v}(w_k) - \hat{v}(w_j)))\\
&=: \frac{1}{N}\sum_{i,j\in[l]}\sum_{k=1}^{N}\mathcal{A}_{ijk} = \frac{1}{N}\sum_{i,j,k\in[l]}\mathcal{A}_{ijk} + \frac{1}{N}\sum_{i,j\in[l],k\notin[l]}\mathcal{A}_{ijk}\\
&=: \mathcal{I}_{11} + \mathcal{I}_{12}.
\end{aligned}$$

Note that $\mathcal{I}_{11}$ and $\mathcal{I}_{12}$ involve with colliding particles with x_l and non-colliding particles with x_l, respectively.

In the following two lemmas, we provide estimates for $\mathcal{I}_{1i}$ one by one.

Lemma 2.2.1. *The term $\mathcal{I}_{11}$ satisfies*

$$\mathcal{I}_{11} \le -\frac{C_L|[l]|}{2N}\phi(\|X\|_{[l]})\|W\|_{[l]}^2 =: -C_1\phi(\|X\|_{[l]})\|W\|_{[l]}^2,$$

where $|[l]|$ denotes the cardinality of the set $[l]$.

Proof. Since i, j and k are in the same index set $[l]$, we may use index switching trick ($i \leftrightarrow k$), Lemma 2.1.2 and definition of $\|X\|_{[l]}$ and $\|W\|_{[l]}$ to estimate $\mathcal{I}_{11}$:

$$\begin{aligned}
\mathcal{I}_{11} &= \frac{2}{N}\sum_{i,j,k\in[l]}(w_i - w_j)\cdot(\phi_{ki}(\hat{v}(w_k) - \hat{v}(w_i)))\\
&= \frac{1}{N}\sum_{i,j,k\in[l]}(w_i - w_j)\cdot(\phi_{ki}(\hat{v}(w_k) - \hat{v}(w_i)))
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{N}\sum_{i,j,k\in[l]}(w_k-w_j)\cdot(\phi_{ki}(\hat{v}(w_i)-\hat{v}(w_k)))\\
&=\frac{1}{N}\sum_{i,j,k\in[l]}\phi_{ki}(w_i-w_k)\cdot(\hat{v}(w_k)-\hat{v}(w_i))\\
&\le -\frac{C_L|[l]|}{N}\sum_{i,j\in[l]}\phi_{ij}|w_i-w_j|^2\\
&\le -\frac{C_L|[l]|}{N}\phi(\|X\|_{[l]})\|W\|_{[l]}^2 =: -C_1\phi(\|X\|_{[l]})\|W\|_{[l]}^2.
\end{aligned}
$$

□

Lemma 2.2.2. *The term $\mathcal{I}_{12}$ satisfies*

$$
\mathcal{I}_{12}\le \frac{2U_vL_\delta(N-|[l]|)}{N}\|W\|_{[l]}\|X\|_{[l]} =: C_2\|X\|_{[l]}\|W\|_{[l]},
$$

when L_δ is the Lipschitz constant of ϕ in (δ,∞).

Proof. By construction, we have

$$
|x_i-x_k|>\delta \quad \text{and} \quad |x_j-x_k|>\delta \quad \text{for } i,j\in[l] \text{ and } k\notin[l].
$$

Then, one has

$$
\begin{aligned}
\mathcal{I}_{12}&=\frac{1}{N}\sum_{i,j\in[l],k\notin[l]}(w_i-w_j)\cdot(\phi_{ki}(\hat{v}(w_k)-\hat{v}(w_i))-\phi_{kj}(\hat{v}(w_k)-\hat{v}(w_j)))\\
&=\frac{1}{N}\sum_{i,j\in[l],k\notin[l]}(w_i-w_j)\cdot[\phi_{ki}(\hat{v}(w_j)-\hat{v}(w_i))\\
&\qquad\qquad\qquad\qquad\qquad+(\phi_{ki}-\phi_{kj})(\hat{v}(w_k)-\hat{v}(w_j))]\\
&\le\frac{1}{N}\sum_{i,j\in[l],k\notin[l]}(\phi_{ki}-\phi_{kj})(w_i-w_j)\cdot(\hat{v}(w_k)-\hat{v}(w_j))\\
&\le\frac{L_\delta}{N}\sum_{i,j\in[l],k\notin[l]}|x_i-x_j||w_i-w_j||\hat{v}(w_k)-\hat{v}(w_j)|.
\end{aligned}
$$

The monotonicity of $\hat{v}$ in Lemma 3.1.3 implies

$$
|\hat{v}(w_k)-\hat{v}(w_j)|=|v_k-v_j|\le|v_k|+|v_j|\le 2U_v,
$$

where U_v is a constant satisfying $\max_{1\le i\le N}|v_i^{in}| \le U_v$. Therefore, we further estimate $\mathcal{I}_{12}$ as

$$\begin{aligned}\mathcal{I}_{12} &\le \frac{2U_vL_\delta}{N}\sum_{i,j\in[l],k\notin[l]}|x_i-x_j||w_i-w_j|\\ &= \frac{2U_vL_\delta(N-|[l]|)}{N}\sum_{i,j\in[l]}|x_i-x_j||w_i-w_j|\\ &\le \frac{2U_vL_\delta(N-|[l]|)}{N}\|X\|_{[l]}\|W\|_{[l]} =: C_2\|X\|_{[l]}\|W\|_{[l]}.\end{aligned}$$

□

Now, we are ready to provide a proof of Theorem 3.1.

Proof of Theorem 2.2.1: It follows from Lemma 3.2.1 and Lemma 2.2.2 to get

$$\frac{1}{2}\frac{d}{dt}\|W\|_{[l]}^2 \le -C_1\phi(\|X\|_{[l]})\|W\|_{[l]}^2 + C_2\|X\|_{[l]}\|W\|_{[l]}.$$

This yields

$$\frac{d}{dt}\|W\|_{[l]} \le -C_1\phi(\|X\|_{[l]})\|W\|_{[l]} + C_2\|X\|_{[l]}.$$

By Grönwall's lemma, we have

$$\begin{aligned}&\|W\|_{[l]}(t)\\ &\le \left[C_2\int_s^t\|X\|_{[l]}(\tau)e^{C_1\int_s^\tau\phi(\|X\|_{[l]}(\sigma))d\sigma}d\tau + \|W\|_{[l]}(s)\right]e^{-C_1\int_s^t\phi(\|X\|_{[l]}(\tau))d\tau},\end{aligned} \tag{2.2.3}$$

for any s, t with $0\le s\le t<t_0$. Now, let Φ be the primitive of ϕ:

$$\Phi(r) := \int_{r_0}^r\phi(s)\,ds = \begin{cases}\log\dfrac{r}{r_0}, & \text{if} \quad \alpha=1,\\ \dfrac{1}{1-\alpha}\left(r^{1-\alpha}-r_0^{1-\alpha}\right), & \text{if} \quad \alpha>1.\end{cases}$$

Then, one has

$$\begin{aligned}|\Phi(\|X\|_{[l]}(t))| &= \left|\int_s^t \frac{d}{dt}\Phi(\|X\|_{[l]}(\tau))d\tau + \Phi(\|X\|_{[l]}(s))\right| \\ &= \left|\int_s^t \phi(\|X\|_{[l]}(\tau))\left(\frac{d}{dt}\|X\|_{[l]}(\tau)\right)d\tau + \Phi(\|X\|_{[l]}(s))\right| \\ &\le \underbrace{\int_s^t \phi(\|X\|_{[l]}(\tau))\|W\|_{[l]}(\tau)d\tau}_{=:\mathcal{J}} + |\Phi(\|X\|_{[l]}(s))|,\end{aligned} \tag{2.2.4}$$

where we used (2.2.2) in the last inequality.

On the other hand, we use (2.2.3) to further estimate $\mathcal{J}$ as

$$\begin{aligned}\mathcal{J} &\le \int_s^t \phi(\|X\|_{[l]}(\tau)) \\ &\qquad \times \left[C_2\int_s^\tau \|X\|_{[l]}(\sigma)e^{C_1\int_s^\sigma \phi(\|X\|_{[l]}(\rho))d\rho}d\sigma + \|W\|_{[l]}(s)\right] \\ &\qquad \times e^{-C_1\int_s^\tau \phi(\|X\|_{[l]}(\sigma))d\sigma}d\tau \\ &= \int_s^t \phi(\|X\|_{[l]}(\tau)) \\ &\qquad \times \left[C_2\int_s^\tau \|X\|_{[l]}(\sigma)e^{C_1\int_s^\sigma \phi(\|X\|_{[l]}(\rho))d\rho}d\sigma\right]e^{-C_1\int_s^\tau \phi(\|X\|_{[l]}(\sigma))d\sigma}d\tau \\ &\quad + \|W\|_{[l]}(s)\int_s^t \phi(\|X\|_{[l]}(\tau))e^{-C_1\int_s^\tau \phi(\|X\|_{[l]}(\sigma))d\sigma}d\tau \\ &=: \mathcal{I}_{21} + \mathcal{I}_{22}.\end{aligned} \tag{2.2.5}$$

In the sequel, we estimate $\mathcal{I}_{2i},\ i=1,2$ one by one.

• (Estimate of $\mathcal{I}_{21}$): It follows from Lemma 3.1.3 that there exists a positive constant $C_3 = C_3(t_0)$ such that

$$\max\left\{C_2\|X\|_{[l]}, \|W\|_{[l]}\right\} < C_3, \quad t \in [0, t_0).$$

Therefore, we have

$$\mathcal{I}_{21} \le C_3\int_s^t \left(\int_s^\tau e^{C_1\int_s^\sigma \phi(\|X\|_{[l]}(\rho))d\rho}d\sigma\right)\phi(\|X\|_{[l]}(\tau))e^{-C_1\int_s^\tau \phi(\|X\|_{[l]}(\sigma))d\sigma}d\tau.$$

Now, we use the following relation

$$\phi(\|X\|_{[l]}(\tau))e^{-C_1\int_s^\tau \phi(\|X\|_{[l]}(\sigma))d\sigma} = -\frac{d}{d\tau}\frac{1}{C_1}e^{-C_1\int_s^\tau \phi(\|X\|_{[l]}(\sigma))d\sigma},$$

and integration by parts to obtain

$$\begin{aligned}\mathcal{I}_{21} \le \frac{C_3}{C_1}\bigg[&-\int_s^t e^{C_1\int_s^\tau \phi(\|X\|_{[l]}(\sigma))d\sigma}d\tau\ e^{-C_1\int_s^t \phi(\|X\|_{[l]}(\tau))d\tau} \\ &+\int_s^t e^{C_1\int_s^\tau \phi(\|X\|_{[l]}(\sigma))d\sigma}e^{-C_1\int_s^\tau \phi(\|X\|_{[l]}(\sigma))d\sigma}d\tau\bigg] \le \frac{C_3}{C_1}t_0.\end{aligned} \tag{2.2.6}$$

• (Estimate of $\mathcal{I}_{22}$): Similar to $\mathcal{I}_{21}$, one has

$$\mathcal{I}_{22} \le \frac{C_3}{C_1}\left(1 - e^{-C_1\int_s^t \phi(\|X\|_{[l]}(\tau))d\tau}\right) \le \frac{C_3}{C_1}. \tag{2.2.7}$$

In (2.2.5), we combine (2.2.6) and (2.2.7) to find

$$\mathcal{J} = \int_s^t \phi(\|X\|_{[l]}(\tau))\|W\|_{[l]}(\tau)d\tau \le \frac{C_3}{C_1}(t_0+1). \tag{2.2.8}$$

We substitute (2.2.8) into (2.2.4) to get

$$\big|\Phi(\|X\|_{[l]}(t))\big| < \frac{C_3}{C_1}(t_0+1) + \Big|\Phi(\|X\|_{[l]}(s))\Big|, \quad 0 \le s \le t < t_0.$$

In particular, since the initial data are non-collisional, we have the boundedness of $\Phi(\|X\|_{[l]}(t))$:

$$\big|\Phi(\|X\|_{[l]}(t))\big| < \frac{C_3}{C_1}(t_0+1) + \Big|\Phi(\|X\|_{[l]}(0))\Big| < +\infty, \quad 0 \le t < t_0. \tag{2.2.9}$$

However, since the index set $[l]$ is the collisional set at time t_0, we have

$$\lim_{t\nearrow t_0} \|X\|_{[l]}(t) = 0.$$

This implies

$$\lim_{t\nearrow t_0} |\Phi(\|X\|_{[l]}(t))| = \infty,$$

which is contradictory to (2.2.9). Therefore, we conclude that particles do not collide at any finite time t_0, and we have a global solution to (1.0.6). □

2.2.2 Asymptotic flocking dynamics

In this subsection, we provide asymptotic flocking dynamics of a global solution whose existence is guaranteed by the previous subsection under the conditions that the initial configuration is non-collisional and $\alpha \geq 1$. Similar to (2.2.1), we set:

$$\|X\| := \sqrt{\sum_{i=1}^N |x_i|^2}, \quad \|V\| := \sqrt{\sum_{i=1}^N |v_i|^2}, \quad \|W\| := \sqrt{\sum_{i=1}^N |w_i|^2}. \tag{2.2.10}$$

Lemma 2.2.3. *Suppose the communication weight* (3.2.2) *is sufficiently singular and initial configuration is non-collisional:*

$$\alpha \geq 1, \quad \textit{and} \quad \min_{1\leq i\neq j\leq N} |x_i^{in} - x_j^{in}| > 0.$$

Then, for a global solution Z to (1.0.6)*, the functionals of* (2.2.10) *satisfy:*

$$\left|\frac{d}{dt}\|X\|\right| \leq \|W\|, \quad \frac{d}{dt}\|W\| \leq -C_L^2\phi(\sqrt{2}\|X\|)\|W\|, \quad t > 0. \tag{2.2.11}$$

Proof. In the sequel, we derive the estimates (2.2.11) one by one.

- (First estimate in (2.2.11)): by definition of $\|X\|$, one has

$$\left|\frac{d}{dt}\|X\|^2\right| = 2\left|\sum_{i=1}^N x_i \cdot \hat{v}(w_i)\right| \leq 2\|X\|\|V\|.$$

This and Lemma 2.1.2 yield

$$\left|\frac{d}{dt}\|X\|\right| \leq \|V\| \leq \|W\|.$$

- (Second estimate in (2.2.11)): we use (1.0.6) to find

$$\frac{d}{dt}\|W\|^2 = \frac{2}{N}\sum_{i,j=1}^N \phi_{ij} w_i \cdot (\hat{v}(w_j) - \hat{v}(w_i))$$

$$
\begin{aligned}
&= \frac{1}{N}\sum_{i,j=1}^{N} \phi_{ij}(w_j - w_i)\cdot(\hat{v}(w_i) - \hat{v}(w_j)) \\
&\le -\frac{C_L^2}{N}\sum_{i,j=1}^{N}\phi_{ij}|w_i - w_j|^2 \le -C_L^2\phi(\sqrt{2}\|X\|)\frac{1}{N}\sum_{i,j=1}^{N}|w_i - w_j|^2 \\
&= -2C_L^2\phi(\sqrt{2}\|X\|)\left(\sum_{i=1}^{N}|w_i|^2\right) = -2C_L^2\phi(\sqrt{2}\|X\|)\|W\|^2.
\end{aligned}
$$

This yields the desired estimate. □

Finally, we employ the Lyapunov functional approach in [33] and Lemma 2.2.3 to derive the following flocking estimate.

Theorem 2.2.2. *Suppose the communication weight* (3.2.2) *and initial data satisfy:*

$$
\alpha \ge 1, \quad \min_{1\le i\ne j\le N}|x_i^{in} - x_j^{in}| > 0, \quad \|W(0)\| < \frac{C_L^2}{\sqrt{2}}\int_{\sqrt{2}\|X(0)\|}^{+\infty}\phi(s)\,ds, \tag{2.2.12}
$$

and let Z be a global solution to (1.0.6)*. Then, there exists a positive constant $x^\infty < +\infty$ such that*

$$
\sup_{0\le t<\infty}\|X(t)\| \le \frac{x^\infty}{\sqrt{2}}, \quad \|W(t)\| \le \|W(0)\|e^{-C_L\phi(x^\infty)t}, \quad t \ge 0.
$$

Proof. • Step A (Uniform bound for $\|X\|$): First, we use $(2.2.12)_3$ to see that there exists a positive constant $x^\infty < +\infty$ such that

$$
\|W(0)\| = \frac{C_L^2}{\sqrt{2}}\int_{\sqrt{2}\|X(0)\|}^{x^\infty}\phi(s)\,ds.
$$

Now, we introduce the Lyapunov functional $\mathcal{L}$:

$$
\mathcal{L}(t) := \frac{C_L^2}{\sqrt{2}}\int_{\sqrt{2}\|X(0)\|}^{\sqrt{2}\|X(t)\|}\phi(s)\,ds + \|W(t)\|. \tag{2.2.13}
$$

Then, it follows from Lemma 2.2.3 that

$$
\begin{aligned}
\frac{d\mathcal{L}}{dt} &= C_L^2\phi(\sqrt{2}\|X(t)\|)\frac{d\|X(t)\|}{dt} + \frac{d\|W(t)\|}{dt} \\
&\le C_L^2\phi(\sqrt{2}\|X(t)\|)\|W(t)\| - C_L^2\phi(\sqrt{2}\|X(t)\|)\|W(t)\| = 0.
\end{aligned} \tag{2.2.14}
$$

By (2.2.13) and (2.2.14), one has

$$\frac{C_L^2}{\sqrt{2}}\int_{\sqrt{2}\|X(0)\|}^{\sqrt{2}\|X(t)\|}\phi(s)\,ds+\|W(t)\|=\mathcal{L}(t)$$
$$\leq \mathcal{L}(0)=\|W(0)\|=\frac{C_L^2}{\sqrt{2}}\int_{\sqrt{2}\|X(0)\|}^{x^\infty}\phi(s)\,ds.$$

This implies the uniform boundedness of X:

$$\sup_{0\leq t<\infty}\|X(t)\|\leq\frac{x^\infty}{\sqrt{2}}. \tag{2.2.15}$$

• Step B (Exponential decay of $\|W\|$): We use $(2.2.11)_2$ and (2.2.15) to get

$$\frac{d}{dt}\|W\|\leq -C_L^2\phi(\sqrt{2}\|X\|)\|W\|\leq -C_L^2\phi(x^\infty)\|W\|.$$

This implies the desired result:

$$\|W(t)\|\leq\|W(0)\|e^{-C_L^2\phi(x^\infty)t},\quad t\geq 0.$$

□

Remark 2.2.1. *The role of condition $\alpha\geq 1$ is to guarantee the global well-posedness of (1.0.6). If a solution of (1.0.6) is globally well-posed, then results of Section 2.2 can be applied, whether the kernel is strongly singular or weakly singular. We will revisit this aspect in the proof of Proposition 2.3.1.*

2.3 Weakly singular communication weight

In this section, we study existence and non-existence of finite-time collisions for system (1.0.6) with weakly singular communication weight (e.g., long-ranged interactions):

$$\phi(r):=r^{-\alpha},\quad 0<\alpha<1.$$

First, we present a simple example of initial data leading to the finite-time collision for a two-particle system, and then provide sufficient conditions on initial data so that there are no collisions, both in finite time and asymptotically.

2.3.1 Existence of finite-time collisions

In this subsection, we provide an example of finite-time collisions using the two-particle system on a real line:

$$\begin{cases} \dfrac{dx_1}{dt} = \hat{v}(w_1), \quad \dfrac{dx_2}{dt} = \hat{v}(w_2), \quad t > 0, \\ \dfrac{dw_1}{dt} = \dfrac{\hat{v}(w_2) - \hat{v}(w_1)}{2|x_1 - x_2|^\alpha}, \quad \dfrac{dw_2}{dt} = \dfrac{\hat{v}(w_1) - \hat{v}(w_2)}{2|x_1 - x_2|^\alpha}, \quad \alpha \in (0,1), \\ (x_1(0), x_2(0), w_1(0), w_2(0)) = (x_1^{in}, x_2^{in}, w_1^{in}, w_2^{in}) \in \mathbb{R}^4. \end{cases} \tag{2.3.1}$$

In the following proposition, we construct special initial data leading to a finite-time collision.

Proposition 2.3.1. *There exist noncollisional initial data* $\{(x_i^{in}, w_i^{in})\}_{i=1}^2$ *such that a solution to* (2.3.1) *subject to it has a finite-time collision, i.e., there exists* $t_c \in (0, \infty)$ *such that*

$$x_1(t_c) = x_2(t_c).$$

Proof. Consider the initial data $\{(x_i^{in}, w_i^{in})\}$ satisfying

$$x_1^{in} < x_2^{in}, \quad w_2^{in} - w_1^{in} + \frac{(x_2^{in} - x_1^{in})^{1-\alpha}}{1-\alpha} = 0. \tag{2.3.2}$$

Suppose there is no finite-time collision. Then, we have

$$x_1(t) < x_2(t) \quad \text{for all } t > 0.$$

Then it follows from (2.3.1) that

$$\frac{d}{dt}(w_2 - w_1) = -\frac{v_2 - v_1}{|x_2 - x_1|^\alpha} = -\frac{1}{(x_2 - x_1)^\alpha}\frac{d}{dt}(x_2 - x_1) = -\frac{d}{dt}\frac{(x_2 - x_1)^{1-\alpha}}{1-\alpha}. \tag{2.3.3}$$

We integrate (2.3.3) over $(0, t)$ to obtain

$$(w_2(t) - w_1(t)) - (w_2^{in} - w_1^{in}) = -\frac{(x_2(t) - x_1(t))^{1-\alpha} - (x_2^{in} - x_1^{in})^{1-\alpha}}{1-\alpha}. \tag{2.3.4}$$

From the choice of initial configuration (2.3.2), the relation (2.3.4) reduces to

$$w_2(t) - w_1(t) = -\frac{(x_2(t) - x_1(t))^{1-\alpha}}{1-\alpha}. \tag{2.3.5}$$

On the other hand, it follows from Lemma 2.1.2 that

$$C_L(w_2 - w_1)^2 \le (w_2 - w_1)(v_2 - v_1), \tag{2.3.6}$$

and since the right-hand side of (2.3.5) is negative, we get

$$w_2 - w_1 < 0.$$

Therefore, we may divide $w_2 - w_1 (< 0)$ on both sides of (2.3.6) to obtain

$$\frac{d}{dt}(x_2 - x_1) = v_2 - v_1 \le C_L(w_2 - w_1) = -\frac{C_L(x_2 - x_1)^{1-\alpha}}{1-\alpha}, \tag{2.3.7}$$

where C_L is a constant obtained in Lemma 2.1.2. Then, Grönwall's inequality yields

$$x_2(t) - x_1(t) \le (x_2^{in} - x_1^{in})e^{-\int_0^t \frac{C_L}{(1-\alpha)(x_2(s)-x_1(s))^\alpha}ds}. \tag{2.3.8}$$

On the other hand, since we assume that there is no finite-time collision, system (1.0.6) is globally well-posed and therefore, we may use the previous result in Theorem 2.2.2. However, since we assume $0 < \alpha < 1$, the communication weight ϕ is non-integrable at the infinity, and therefore, the condition (2.2.12) always holds. Thus, there exists a uniform upper bound x_M of $|x_1|$ and $|x_2|$:

$$\max\{|x_1(t)|, |x_2(t)|\} \le x_M < \infty, \quad t > 0.$$

We use an upper bound of $|x_i(t)|$ to observe that the exponent of (2.3.8) tends to $-\infty$ as $t \to \infty$:

$$-\int_0^t \frac{C_L}{(1-\alpha)(x_2 - x_1)^\alpha} ds \le -\frac{C_L t}{(1-\alpha)(2x_M)^\alpha} \underset{t\to\infty}{\longrightarrow} -\infty.$$

This implies

$$\lim_{t\to\infty}(x_2(t) - x_1(t)) = 0.$$

Therefore, we can find an increasing sequence of times $\{t_n\}_{n=1}^{\infty}$ by

$$t_n := \inf\left\{t \ : \ x_2(s) - x_1(s) \le 2^{-n} \quad \text{whenever} \quad s \ge t\right\}, \quad n \in \mathbb{N} \cup \{0\}.$$

By definition, we have

$$x_2(t_n) - x_1(t_n) = 2^{-n}, \quad \text{and} \quad x_2(t) - x_1(t) \le 2^{-n} \quad \text{for} \quad t \ge t_n.$$

Again, we apply Grönwall's inequality to (2.3.7) to derive

$$x_2(t_{n+1}) - x_1(t_{n+1}) \le (x_2(t_n) - x_1(t_n)) e^{-\int_{t_n}^{t_{n+1}} \frac{C_L}{(1-\alpha)(x_2(s)-x_1(s))^\alpha} ds}.$$

Therefore, one has

$$\log 2 \ge \int_{t_n}^{t_{n+1}} \frac{C_L}{(1-\alpha)(x_2(s)-x_1(s))^\alpha} ds \ge (t_{n+1} - t_n) \frac{C_L 2^{\alpha n}}{1-\alpha}. \tag{2.3.9}$$

However, for each $n \in \mathbb{N}$, the relation (2.3.9) implies

$$t_n = t_0 + \sum_{k=0}^{n-1} (t_{k+1} - t_k) \le t_0 + \frac{(1-\alpha)\log 2}{C_L} \sum_{k=0}^{n-1} 2^{-\alpha k} < t_0 + \frac{(1-\alpha)\log 2}{C_L(1-2^{-\alpha})}.$$

Hence, there exists a limit $\lim_{n\to\infty} t_n = t_\infty < +\infty$ satisfying

$$x_2(t_\infty) - x_1(t_\infty) = \lim_{n\to\infty} (x_2(t_n) - x_1(t_n)) \le \lim_{n\to\infty} 2^{-n} = 0,$$

which is contradictory to the absence of finite-time collision. Therefore, the collision will occur at some finite time $t_c < +\infty$. □

Thus, for a weakly singular communication weight satisfying (2.3.2), we can obtain a finite-time collision depending on the geometry of initial configuration. In next subsection, we provide sufficient conditions which guarantee the nonexistence of collisions.

2.3.2 Sufficient conditions for collision avoidance

In this subsection, we present sufficient conditions for collision avoidance when the singularity is weak. As we observe in the previous subsection, we cannot guarantee the collision-free property of system (1.0.6), regardless of initial data, when $0 < \alpha < 1$. Therefore, if one wants to get a collision-free property for $0 < \alpha < 1$, an extra condition on the geometry of the initial configuration is needed. In this part, we present a sufficient condition on initial data that guarantee collision avoidance. We use the flocking estimate in Section 3.2 to obtain the absence of finite-time collisions.

Theorem 2.3.1. *Suppose the communication weight* (3.2.2) *and initial data satisfy*

$$\begin{aligned} &0 < \alpha < 1, \quad \min_{1 \le i \ne j \le N} |x_i^{in} - x_j^{in}| > 0, \\ &\|W(0)\| < \frac{C_L^2}{\sqrt{2}} \min\left\{ \int_{\sqrt{2}\|X(0)\|}^{x^\infty} \phi(s)\,ds,\ \phi(e^\infty) \min_{1 \le i,j \le N} |x_i^{in} - x_j^{in}| \right\}, \end{aligned} \tag{2.3.10}$$

for some positive constant $x^\infty < +\infty$. Then, the following assertions hold.

1. *There exists a global-in-time solution $\{(x_i^{in}, w_i^{in})\}_{i=1}^N$ and a constant $\delta_0 > 0$ such that*

$$\inf_{0 \le t < \infty} \min_{1 \le i,j \le N} |x_i(t) - x_j(t)| \ge \delta_0.$$

2. *Asymptotic flocking emerges:*

$$\|X(t)\| \le \frac{x^\infty}{\sqrt{2}}, \quad \|W(t)\| \le \|W(0)\| e^{-C_L^2 \phi(x^\infty) t}, \quad t \ge 0.$$

Proof. (i) Since the initial data is non-collisional, there exists at least local-in-time solution $\{(x_i, w_i)\}_{i=1}^N$ to (1.0.6). We now assume that there exists a critical time t_* such that the first collision occurs at time t_*. We denote two particles colliding at time t_* by x_i and x_j. In particular, one has

$$\lim_{t \to t_*} |x_i(t) - x_j(t)| = 0. \tag{2.3.11}$$

Since initial data satisfy (2.3.10), Theorem 2.2.2 implies

$$\|X(t)\| \le \frac{x^\infty}{\sqrt{2}} \quad \text{and} \quad \|W(t)\| \le \|W(0)\| e^{-C_L^2 \phi(x^\infty) t}, \quad 0 \le t < t_*. \tag{2.3.12}$$

Now, we estimate $|x_i(t) - x_j(t)|$ as

$$|x_i(t) - x_j(t)| \geq |x_i^{in} - x_j^{in}| - \int_0^t |v_i(s) - v_j(s)|\, ds. \tag{2.3.13}$$

On the other hand, it follows from (2.3.12) that

$$\begin{aligned}|v_i(s) - v_j(s)| &\leq |w_i(s) - w_j(s)| \leq |w_i(s)| + |w_j(s)| \\ &\leq \sqrt{2(|w_i(s)|^2 + |w_j(s)|^2)} \\ &\leq \sqrt{2}\|W(s)\| \leq \sqrt{2}\|W(0)\|e^{-C_L^2\phi(x^\infty)s}.\end{aligned}$$

Therefore, one has

$$\int_0^t |v_i(s) - v_j(s)|\, ds \leq \sqrt{2}\|W(0)\| \int_0^t e^{-C_L^2\phi(x^\infty)s}\, ds \leq \frac{\sqrt{2}\|W(0)\|}{C_L^2\phi(x^\infty)}.$$

We substitute the above estimate into (2.3.13) to obtain

$$|x_i(t) - x_j(t)| \geq |x_i^{in} - x_j^{in}| - \frac{\sqrt{2}\|W(0)\|}{C_L^2\phi(x^\infty)} > 0, \quad 0 \leq t < t_*,$$

where the last inequality comes from the initial condition (2.3.10). This contradicts to (2.3.11). Therefore, there is no finite-time collision, and the solution $\{(x_i, w_i)\}_{i=1}^N$ can be extended globally in time and thus, the flocking estimate (2.3.12) holds for the whole time $t \geq 0$. Moreover, we choose

$$\delta_0 := \min_{1\leq i,j\leq N} |x_i^{in} - x_j^{in}| - \frac{\sqrt{2}\|W(0)\|}{C_L^2\phi(x^\infty)}.$$

This yields the desired uniform boundedness for relative distances. □

Remark 2.3.1. *Since Theorem 3.3.1 does not depend on the singularity of the communication weight, the same result holds for the case when $\alpha \geq 1$ as well, which can guarantee the existence of the uniform lower bound of the distance between particles.*

Chapter 3

The first-order CS-type consensus model on the real line

In this chapter, we are interested in a general nonlinear first-order consensus model motivated by the RCS model and study its emergent dynamics. We first present a heuristic derivation of the general first-order consensus model (3.1.4) and recall previous results on the collective behaviors of the first-order consensus model (3.1.1), and then we show that ordering principle holds for system (3.1.5), when the communication weight is regular. We then present the detailed analysis on the asymptotic clustering behaviors of (3.1.5), when the regular communication weight is long-ranged and short-ranged, respectively, and we study collective behaviors for system (3.1.5) with singular coupling function. Finally, we provide structural stability from the general consensus model (3.1.5) to the standard one (3.1.1). We note that this chapter is based on the joint work [10].

Notation: Throughout the Chapter 3 and Chapter 4, for state configuration $\{q_i\}$ and natural velocity $\{\nu_i\}$, we set a natural velocity vector, state vector and a derivative of state vector by $\mathcal{N}$, Q and P, respectively:

$$Q(t) := (q_1(t), \dots, q_N(t)), \quad P(t) := (\dot{q}_1(t), \dots, \dot{q}_N(t)),$$
$$Q^0 := Q(0), \quad P^0 := P(0), \quad \mathcal{N} := (\nu_1, \nu_2, \cdots, \nu_N).$$

3.1 Consensus model on the real line

3.1.1 The CS-type consensus model on the real line

To set the stage, we begin with the first-order nonlinear consensus model introduced in [34] which combines the Kuramoto model and one-dimensional Cucker-Smale model as special examples. More precisely, let $q_i = q_i(t)$ be the real-valued quantifiable state of the i-th agent lying on the one-dimensional manifold $\mathcal{M}$ such as $\mathbb{S}^1$ and $\mathbb{R}^1$. In [34], the following consensus model was proposed:

$$\dot{q}_i = \nu_i + \frac{\kappa}{N}\sum_{k=1}^{N} \Psi(q_k - q_i), \quad i \in [N] := \{1, \dots, N\}, \tag{3.1.1}$$

where ν_i is the natural rate of changes (natural velocity for simplicity) of the i-th agent. In the context of flocking, coupling function Ψ is typically assumed to be odd, differentiable and monotonically increasing:

$$\Psi(-q) = -\Psi(q), \quad \Psi'(q) \geq 0, \quad \forall\, q \in \mathbb{R}. \tag{3.1.2}$$

The following choices:

$$(\mathcal{M}, \Psi(q)): \ (\mathbb{S}^1, \sin q), \ \left(\mathbb{R}^1, \int_0^q \psi(\eta)d\eta\right),$$

correspond to the Kuramoto model and the Cucker-Smale model on the real line with a nonnegative communication weight ψ, respectively (see a general consensus model [34]). Note that assumptions on a communication function depends on the realization of the consensus behavior; for example, choice of $(\mathcal{M}, \Psi(q)) = (\mathbb{S}^1, \sin q)$ in the Kuramoto model is based on the context of synchronization, and does not obey (3.1.2). The emergent dynamics of (3.1.1) were extensively studied in [8, 21, 31, 28, 34, **?**, 39, 40]. Throughout the thesis, in addition to (3.1.2), we suppose

$$\psi := \Psi' \quad \text{is decreasing on } [0, \infty), \tag{3.1.3}$$

which is a typical assumption on the communication weight ψ for the flocking models. In this thesis, we are interested in the clustering dynamics of generalized model for (3.1.1)–(3.1.2).

We first propose an abstract consensus model by replacing the time-derivative $\dot{q}_i$ by a suitable increasing function of $\dot{q}_i$:

$$F(\dot{q}_i) = \nu_i + \frac{\kappa}{N}\sum_{k=1}^{N}\Psi(q_k - q_i), \quad q_i(0) = q_i^0, \quad i \in [N], \tag{3.1.4}$$

where $F : \mathbb{R} \to \mathbb{R}$ is a odd function which is strictly increasing and differentiable. Equivalently system (3.1.4) can be rewritten as a more convenient form:

$$\dot{q}_i = G\left(\nu_i + \frac{\kappa}{N}\sum_{k=1}^{N}\Psi(q_k - q_i)\right), \quad q_i(0) = q_i^0 \quad i \in [N], \tag{3.1.5}$$

where we call $G = F^{-1}$ as an "*activation function*" borrowing terminology from deep learning, and we also assume that there exist positive constants $m_{G'}$ and $M_{G'}$ such that

$$0 < m_{G'} \le G'(q) \le M_{G'} < \infty, \quad \forall\, q \in \mathbb{R}. \tag{3.1.6}$$

Note that $G(-q) = -G(q)$ since F is odd. The most simplest and motivating example for G will be the identity mapping $G(q) = q$. In this case, system (3.1.4) reduces to system (3.1.1). If we set $q_i = x_i$ and $G = \hat{v}$, then system (3.1.5) corresponds to the RCS model (1.0.6). See Chapter 1 for other nontrivial examples of an activation function.

In this chapter, we will provide emergent dynamics of (3.1.5) depending on the behaviors of the communication weight function $\psi := \Psi'$ at $q = 0$ and $q = \infty$:

$$\begin{aligned}
&\text{Type I}: \int_0^\infty \psi(q)\,dq = \infty \ : \ \text{Regular, long-ranged communication weight},\\
&\text{Type II}: \int_0^\infty \psi(q)\,dq < \infty \ : \ \text{Regular, short-ranged communication wight},\\
&\text{Type III}: \psi(q) = \frac{1}{|q|^\alpha}, \quad \alpha > 0, \quad q \neq 0 \ : \ \text{Singular communication weight}.
\end{aligned} \tag{3.1.7}$$

Due to the singularity at the origin, the coupling kernel of Type III should be treated in a different manner. For details, we refer to Section 4.4.

3.1.2 Previous results

We briefly summarize previous results on the abstract consensus model (3.1.5) with $G(x) = x$. First, we recall several concepts of clustering as follows.

Definition 3.1.1. *Let $\{q_i\}$ be a solution to* (3.1.5)*. Then, the following assertions hold.*

1. *The i-th and j-th particles belong to the same cluster, if the relative state is uniformly bounded in time:*
$$\sup_{t\geq 0} |q_i(t) - q_j(t)| < \infty.$$
2. *The i-th and j-th particles segregates if the relative state satisfies*
$$\liminf_{t\to\infty} |q_i(t) - q_j(t)| = \infty.$$
3. *The configuration is asymptotically state-locked if the relative states satisfy*
$$\limsup_{t\to\infty} \max_{i\neq j} |q_i(t) - q_j(t)| < \infty.$$

In the sequel, we recall previous clustering results for system (3.1.5) with the identity map for G:
$$G(q) = q, \quad q \in \mathbb{R}.$$
In this case, system (3.1.5) becomes
$$\begin{cases} \dot{q}_i = \nu_i + \dfrac{\kappa}{N} \displaystyle\sum_{k=1}^{N} \Psi(q_k - q_i), \quad i \in [N], \\ \Psi(-q) = -\Psi(q), \quad q_i(0) = q_i^0, \quad q \in \mathbb{R}. \end{cases} \tag{3.1.8}$$
Unlike to the abstract model (3.1.4), system (3.1.8) has a conserved quantity. For a given configuration $\{(q_i, \nu_i)\}$, we set
$$C[t] := \frac{1}{N}\sum_{i=1}^{N} q_i - \frac{t}{N}\sum_{i=1}^{N} \nu_i, \quad t \geq 0.$$

Lemma 3.1.1. *Let $\{q_i\}$ be a global solution to* (3.1.8). *Then, the functional $C[t]$ is time-invariant.*

$$C[t] = C[0], \quad t \geq 0.$$

Proof. We sum up $(3.1.8)_1$ over all i to find

$$\frac{d}{dt}\sum_{i=1}^{N} q_i = \sum_{i=1}^{N} \nu_i + \frac{\kappa}{N}\sum_{i,k=1}^{N} \Psi(q_k - q_i) = \sum_{i=1}^{N} \nu_i.$$

This yields the desired estimate. □

Next, we recall clustering dynamics of (3.1.8).

Proposition 3.1.1. [31, 34] *Suppose the coupling function Ψ is short-ranged in the sense that*

$$0 < \Psi^{\infty} := \lim_{q\to\infty} \Psi(q) < \infty,$$

and let $\{q_i\}$ be a solution to (3.1.4). *If $i < j$, the initial states q_i^0 and q_j^0 satisfy*

$$q_i^0 < q_j^0,$$

then the following trichotomy holds.

1. *If $\nu_i < \nu_j$, then q_i and q_j will never collide in finite time:*

   $$|\{t_* \in (0,\infty) \ : \ q_i(t_*) = q_j(t_*)\}| = 0,$$

   where $|A|$ is the cardinality of the set A.

2. *If $\nu_i > \nu_j$, q_i and q_j will collide once in finite time:*

   $$|\{t_* \in (0,\infty) \ : \ q_i(t_*) = q_j(t_*)\}| = 1.$$

3. *If $\nu_i = \nu_j$, then the relative distance $|q_i - q_j|$ decays to zero exponentially fast:*

   $$\begin{aligned} |q_i^0 - q_j^0| \exp\left(-\kappa\psi_M t\right) &\leq |q_i(t) - q_j(t)| \\ &\leq |q_i^0 - q_j^0| \exp\left(-\frac{\kappa}{N}\psi(|q_i^0 - q_j^0|)t\right), \quad t \geq 0, \end{aligned}$$

   where $\psi_M := \max_{-\infty < r < \infty} \psi(r)$.

Remark 3.1.1. *By Lemma 3.1.1 and Proposition 3.1.1, we may assume the ordering and mean-zero properties of initial states without loss of generality:*

$$q_1^0 \le \cdots \le q_N^0, \quad \frac{1}{N}\sum_{i=1}^N q_i^0 = 0, \quad \sum_{i=1}^N \nu_i = 0.$$

Moreover, this imply

$$q_1(t) \le \cdots \le q_N(t), \quad \frac{1}{N}\sum_{i=1}^N q_i(t) = 0, \quad \forall\ t \ge 0.$$

Next, we recall asymptotic clustering of (3.1.1).

Proposition 3.1.2. [31, 34] *Suppose the natural velocity ν_i is well-ordered and has mean zero:*

$$\nu_1 < \nu_2 \cdots < \nu_N \quad \text{and} \quad \sum_{i=1}^N \nu_i = 0,$$

and let $\{q_i\}$ be a solution to (3.1.1) *with initial data $\{q_i^0\}$. Then, the following assertions hold:*

1. *The state configuration is completely segregated:*

$$\limsup_{t\to+\infty} q_1(t) = -\infty, \quad \liminf_{t\to+\infty} q_N(t) = \infty,$$
$$\liminf_{t\to+\infty} |q_{i+1}(t) - q_i(t)| = \infty, \quad i \in [N-1]$$

if and only if the coupling strength κ is sufficiently small such that

$$\kappa < \min\Big\{ -\frac{N}{N-1}\cdot\frac{\nu_1}{\Psi^\infty}, \frac{N}{2}\cdot\frac{(\nu_2-\nu_1)}{\Psi^\infty}, \ldots, \frac{N}{2}\cdot\frac{(\nu_N-\nu_{N-1})}{\Psi^\infty}, \frac{N}{N-1}\cdot\frac{\nu_N}{\Psi^\infty}\Big\}.$$

2. *The state configuration is asymptotically state-locked:*

$$\exists\ q_{ij}^\infty := \lim_{t\to\infty} |q_i(t) - q_j(t)|, \quad i,j \in [N].$$

if and only if the coupling strength κ is sufficiently large such that

$$\kappa > \max_{1\le \ell\le N-1}\left(\frac{-\frac{1}{l}\sum_{i=1}^l \nu_i}{\frac{(N-l)}{N}\Psi^\infty}\right),$$

Remark 3.1.2. *In [31], the authors provided a criterion to estimate the number of asymptotic clusters and asymptotic group velocity of each cluster in terms of initial data and coupling strength. For the singular coupling function, we refer to the recent work [55].*

3.1.3 Ordering principle for state configuration

We present the ordering principle of system (3.1.5)–(3.1.6) with the regular and long-ranged communication weight which is parallel to Proposition 3.1.1. For convenience, we assume

$$\psi_0 := \psi(0) = 1.$$

In what follows, we show that the positions of the particles are aligned according to the size of their natural velocities.

Theorem 3.1.1. *Let $\{q_i\}$ be a solution to* (3.1.5)–(3.1.6) *with initial data $\{q_i^0\}$. For fixed indices i and $j (i \neq j)$, we assume*

$$q_i^0 > q_j^0.$$

Then the following trichotomy holds.

1. *If $\nu_i > \nu_j$, then q_i and q_j will not collide in finite time:*

$$q_i(t) > q_j(t) \quad \text{for all} \quad t \geq 0.$$

2. *If $\nu_i < \nu_j$, then q_i and q_j will collide exactly once, i.e., there exists a time $t^* \in [0, \infty)$ such that*

$$q_i(t) > q_j(t) \quad \text{for} \quad 0 \leq t \leq t^*, \quad q_i(t^*) = q_j(t^*) \\ \text{and} \quad q_i(t) < q_j(t) \text{ for } t > t^*.$$

3. *If $\nu_i = \nu_j$, then q_i and q_j will not collide in finite time, and the relative distance $|q_i - q_j|$ satisfies*

$$0 < (q_i^0 - q_j^0) e^{-M_{G'}\kappa t} \leq q_i(t) - q_j(t) \leq (q_i^0 - q_j^0) e^{-\frac{2\psi\left(q_i^0 - q_j^0\right) m_{G'}\kappa}{N} t}, \quad t \geq 0.$$

Proof. (1) Suppose that there exists a finite-time collision, and let t^* be the first collision time such that

$$q_i(t) > q_j(t) \quad 0 \le t < t^* \quad \text{and} \quad q_i(t^*) = q_j(t^*). \tag{3.1.9}$$

Then, we use (3.1.5), $\nu_i > \nu_j$, (3.1.9) and the mean-value theorem to get

$$\begin{aligned}
&\frac{d}{dt}(q_i - q_j)\Big|_{t=t^*} \\
&\quad = G\left(\nu_i + \frac{\kappa}{N}\sum_{k=1}^{N}\Psi(q_k(t^*) - q_i(t^*))\right) \\
&\qquad\qquad - G\left(\nu_j + \frac{\kappa}{N}\sum_{k=1}^{N}\Psi(q_k(t^*) - q_j(t^*))\right) \\
&\quad = G'(y_{ij})\left(\underbrace{\nu_i - \nu_j}_{>0} + \frac{\kappa}{N}\sum_{k=1}^{N}\underbrace{(\Psi(q_k(t^*) - q_i(t^*)) - \Psi(q_k(t^*) - q_j(t^*)))}_{=0}\right) \\
&\quad \ge m_{G'}(\nu_i - \nu_j) > 0.
\end{aligned} \tag{3.1.10}$$

Therefore, we conclude that at time $t = t^*$, we have

$$q_i(t^*) = q_j(t^*) \quad \text{and} \quad \frac{d}{dt}\Big|_{t=t^*}(q_i - q_j) > 0.$$

By the continuity of the solution, there exists a sufficiently small $\delta > 0$ such that

$$q_i(t) < q_j(t) \quad \text{for } t^* - \delta < t < t^*.$$

This contradicts to the definition of t^* in (3.1.9). Therefore, there is no finite-time collision between q_i and q_j.

(2) We split the proof of the second assertion into two steps.

- Step A: We claim that

a finite-time collision cannot happen more than once.

Suppose that q_i and q_j collide and define the first collision time t^* as in (3.1.9). Then by the same argument as in (3.1.10), we have

$$\frac{d}{dt}(q_i - q_j)\bigg|_{t=t^*} \le m_{G'}(\nu_i - \nu_j) < 0.$$

Therefore, there exists a positive constant $\delta > 0$, such that

$$q_i(t) < q_j(t) \quad \text{for } t \in [t^*, t^* + \delta].$$

By the result in (1), with i and j reversed, we conclude that the collision does not occur afterward.

• Step B: Now, we show that finite-time collision must happen. Suppose that there is no collision:

$$q_i(t) > q_j(t) \quad \text{for all } t \ge 0,$$

which implies

$$q_k - q_i < q_k - q_j, \quad k \in [N].$$

On the other hand, since Ψ is increasing, one has

$$\Psi(q_k - q_i) - \Psi(q_k - q_j) < 0. \tag{3.1.11}$$

Then, for $t > 0$, one has

$$\begin{aligned}
&\frac{d}{dt}(q_i - q_j) \\
&\quad = G\left(\nu_i + \frac{\kappa}{N}\sum_{k=1}^{N}\Psi(q_k(t^*) - q_i(t^*))\right) - G\left(\nu_j + \frac{\kappa}{N}\sum_{k=1}^{N}\Psi(q_k(t^*) - q_j(t^*))\right) \\
&\quad = G'(y_{ij})\left(\nu_i - \nu_j + \frac{\kappa}{N}\sum_{k=1}^{N}\underbrace{\big(\Psi(q_k(t) - q_i(t)) - \Psi(q_k(t) - q_j(t))\big)}_{<0 \text{ by } (3.1.11)}\right) \\
&\quad \le m_{G'}(\nu_i - \nu_j) < 0.
\end{aligned}$$

This yields

$$q_i(t) - q_j(t) \le q_i^0 - q_j^0 + m_{G'}(\nu_i - \nu_j)t, \quad t \ge 0.$$

Therefore, there exists $t_* \le \frac{q_i^0 - q_j^0}{m_{G'}(\nu_j - \nu_i)}$ such that

$$q_i(t_*) - q_j(t_*) = 0.$$

This contradicts to the absence of collision.

(3) Again, we use the mean-value theorem twice to obtain

$$\begin{aligned}
\frac{d}{dt}(q_i - q_j) &= G\left(\nu_i + \frac{\kappa}{N}\sum_{k=1}^N \Psi(q_k - q_i)\right) - G\left(\nu_j + \frac{\kappa}{N}\sum_{k=1}^N \Psi(q_k - q_j)\right) \\
&= \frac{\kappa G'(y_{ij})}{N}\left(\sum_{k=1}^N \Psi(q_k - q_i) - \Psi(q_k - q_j)\right) \\
&= \frac{\kappa G'(y_{ij})}{N}\left(\sum_{k=1}^N \psi(z_{ijk})(q_j - q_i)\right) \\
&= -\left(\frac{\kappa G'(y_{ij})}{N}\sum_{k=1}^N \psi(z_{ijk})\right)(q_i - q_j),
\end{aligned}$$

where z_{ijk} is located between $q_k - q_i$ and $q_k - q_j$.

Since

$$0 < \frac{\kappa G'(y_{ij})}{N}\sum_{k=1}^N \psi(z_{ijk}) \le \kappa M_{G'}, \tag{3.1.12}$$

we deduce

$$0 < q_i(t) - q_j(t) < q_i^0 - q_j^0 \quad \text{for all } t > 0.$$

Hence, we have

$$-(q_i^0 - q_j^0) < q_j - q_i < z_{ijj} < 0 < z_{iji} < q_i - q_j < q_i^0 - q_j^0,$$

and therefore, we refine the lower bound of (3.1.12) as

$$\frac{2m_{G'}\kappa\psi(q_i^0 - q_j^0)}{N} < \frac{m_{G'}\kappa(\psi(z_{iji}) + \psi(z_{ijj}))}{N} < \frac{\kappa G'(y_{ij})}{N}\sum_{k=1}^N \psi(z_{ijk}) \le \kappa M_{G'}.$$

Hence, we obtain the desired upper and lower exponential decays for $q_i - q_j$:

$$0 < (q_i^0 - q_j^0)e^{-\kappa M_{G'} t} \le q_i(t) - q_j(t) \le (q_i^0 - q_j^0)e^{-\frac{2\kappa m_{G'}\psi\left(q_i^0 - q_j^0\right)}{N}t}, \quad t \ge 0.$$

This completes the proof.

□

Therefore, after a finite time depending on initial data and natural velocity, all particles will be aligned according to the ordering of their natural velocities ν_i. More precisely, the following corollary holds.

Corollary 3.1.1. *Let* $\{q_i\}$ *be a solution to* (3.1.5)–(3.1.6) *with the initial data* $\{q_i^0\}$. *Then, the following assertions hold.*

1. *Suppose that the natural velocities are distinct and increasing in the indices by reordering:*

$$\nu_1 < \nu_2 < \cdots < \nu_N.$$

Then there exists a positive time T_* *depending on initial data and the natural velocities such that*

$$q_1(t) < q_2(t) < \cdots < q_N(t), \quad \forall\ t > T_*.$$

2. *If* $q_i^0 - q_j^0 = \nu_i - \nu_j = 0$, *then* q_i *and* q_j *stick together:*

$$q_i(t) = q_j(t), \quad t \geq 0.$$

3. *If* $q_i^0 \neq q_j^0$ *and* $\nu_i = \nu_j$, *the relative distance* $|q_i - q_j|$ *decays to zero exponentially fast, and they will not collide in finite time.*

From now on, throughout Section 3.2 and Section 3.3, we assume that the initial states and the natural velocities satisfy

$$q_1^0 < q_2^0 < \cdots < q_N^0, \quad \text{and} \quad \nu_1 < \nu_2 < \cdots < \nu_N, \tag{3.1.13}$$

and collisions never happen. In the following three sections, we consider three communication weights displayed in (3.1.7) one by one.

3.2 Regular long-ranged communication weight: Type I

In this section, we study the clustering dynamics of system (3.1.4) with the long-ranged communication weight:

$$\lim_{q\to\infty} \Psi(q) = \int_0^\infty \psi(\eta)d\eta = \infty, \quad \Psi(0) = 0, \quad \text{where } \psi = \Psi'.$$

3.2.1 Asymptotic state-locking

We provide estimate on the relative distances. When the communication weight is long-ranged, we always attain a uniform lower and upper bounds for the relative distances between particles.

Theorem 3.2.1. (Asymptotic state-locking) *Suppose that initial data and natural velocities satisfy*

$$q_1^0 < q_2^0 < \cdots < q_N^0 \quad \textit{and} \quad \nu_1 < \nu_2 < \cdots < \nu_N, \tag{3.2.1}$$

and let $\{q_i\}$ *be a solution to* (3.1.5)-(3.1.6) *with the initial data* $\{q_i^0\}$. *Then, the following assertions for relative distances hold:*

1. *(Existence of a positive minimal distance): for* $i > j$, *there exists a positive constant* $\ell_1^{ij} > 0$ *such that*

$$\inf_{t\geq 0} |q_i(t) - q_j(t)| \geq \ell_1^{ij} > 0.$$

   In particular, for consecutive indices $i = j+1$ *and* j $(1 \leq j < N)$, ℓ_1^{ij} *is explicitly given as*

$$\ell_1^{ij} := \min\left\{q_i^0 - q_j^0,\ \Psi^{-1}\left(\frac{m_{G'}}{M_{G'}} \cdot \frac{\nu_i - \nu_j}{\kappa}\right)\right\} > 0.$$

2. *(Existence of a positive maximal distance): there exists a positive constant* $L_1^\infty := \max\left\{q_N^0 - q_1^0,\ \Psi^{-1}\left(\frac{M_{G'}}{m_{G'}} \cdot \frac{\nu_N - \nu_1}{\kappa}\right)\right\} < +\infty$ *such that*

$$\sup_{t\geq 0} \max_{i,j} |q_i(t) - q_j(t)| \leq L_1^\infty < \infty.$$

Proof. (1) Note that for $q, M, m \geq 0$ with $M > m$, decreasing property of ψ (3.1.3) yields

$$\Psi(q+M) - \Psi(M) = \int_M^{q+M} \psi(r)dr \leq \int_m^{q+m} \psi(r)dr = \Psi(q+m) - \Psi(m). \tag{3.2.2}$$

We fix consecutive indices i and j such that

$$j < N \quad \text{and} \quad i := j+1.$$

Then, we use the mean-value theorem, (3.2.1), (3.2.2) and $\Psi(0) = 0$ to see that for $t > 0$,

$$\begin{aligned}
\frac{d}{dt}(q_i - q_j) &= G\left(\nu_i + \frac{\kappa}{N}\sum_{k=1}^N \Psi(q_k - q_i)\right) - G\left(\nu_j + \frac{\kappa}{N}\sum_{k=1}^N \Psi(q_k - q_j)\right) \\
&= G'(y_{ij})\left(\nu_i - \nu_j + \frac{\kappa}{N}\sum_{k=1}^N \Big[\Psi(q_k - q_i) - \Psi(q_k - q_j)\Big]\right) \\
&= G'(y_{ij})\Bigg(\nu_i - \nu_j + \frac{\kappa}{N}\Bigg[-2\Psi(q_i - q_j) - \sum_{k=i+1}^N (\Psi(q_k - q_j) - \Psi(q_k - q_i)) \\
&\qquad\qquad - \sum_{k=1}^{j-1}(\Psi(q_i - q_k) - \Psi(q_j - q_k))\Bigg]\Bigg).
\end{aligned} \tag{3.2.3}$$

On the other hand, for $k > i$, we apply (3.2.2) with

$$M = q_k - q_i > 0, \quad q = q_i - q_j > 0, \quad m = 0$$

to get

$$\Psi(q_k - q_j) - \Psi(q_k - q_i) \leq \Psi(q_i - q_j) - \Psi(0) = \Psi(q_i - q_j).$$

This yields

$$-\sum_{k=i+1}^N (\Psi(q_k - q_j) - \Psi(q_k - q_i)) \geq -\sum_{k=i+1}^N \Psi(q_i - q_j) = -(N-i)\Psi(q_i - q_j). \tag{3.2.4}$$

Similarly, one has

$$-\sum_{k=1}^{j-1}(\Psi(q_i-q_k)-\Psi(q_j-q_k)) \geq -\sum_{k=1}^{j-1}\Psi(q_i-q_j) = -(j-1)\Psi(q_i-q_j). \quad (3.2.5)$$

Now we combine (3.2.3), (3.2.4) and (3.2.5) to get

$$\frac{d}{dt}(q_i-q_j) \geq G'(y_{ij})\Big(\nu_j-\nu_i-\kappa\Psi(q_i-q_j)\Big) \geq m_{G'}(\nu_i-\nu_j)-\kappa M_{G'}\Psi(q_i-q_j). \quad (3.2.6)$$

Next, we consider the differential equation:

$$\begin{cases} \dot{y} = m_{G'}(\nu_i-\nu_j)-\kappa M_{G'}\Psi(y), \quad t>0, \\ y(0) = q_i^0-q_j^0 > 0. \end{cases} \quad (3.2.7)$$

By the comparison principle of ordinary differential equation, it suffices to show that y has a uniform-in-time positive lower bound. However, since $\lim_{r\to+\infty}\Psi(r) = +\infty$, the map $\Psi:\mathbb{R}\to\mathbb{R}$ is a strictly increasing bijective function. Therefore, the differential equation for y obtains its equilibrium at $y = \Psi^{-1}\left(\frac{m_{G'}}{M_{G'}}\cdot\frac{\nu_i-\nu_j}{\kappa}\right)$. Moreover, since Ψ is a strictly increasing function, we have

$$\begin{cases} \dot{y}>0 \quad \text{if} \quad y<\Psi^{-1}\left(\frac{m_{G'}}{M_{G'}}\cdot\frac{\nu_i-\nu_j}{\kappa}\right), \\ \dot{y}<0 \quad \text{if} \quad y>\Psi^{-1}\left(\frac{m_{G'}}{M_{G'}}\cdot\frac{\nu_i-\nu_j}{\kappa}\right). \end{cases}$$

Hence, we obtain

$$y(t) \geq \min\left\{q_i^0-q_j^0,\ \Psi^{-1}\left(\frac{m_{G'}}{M_{G'}}\cdot\frac{\nu_i-\nu_j}{\kappa}\right)\right\} =: \ell_1^{ij}, \quad t\geq 0.$$

Therefore, the relative state between q_j and $q_i = q_{j+1}$ is also bounded below by ℓ_1^{ij}. Now, for general indices $1\leq i<j\leq N$ represented by $i = j+K$ $(K>0)$, we set

$$\ell_1^{ij} := \sum_{k=0}^{K-1}\ell_1^{j_{k+1}j_k}, \quad j_k := j+k,$$

to obtain the desired positive lower bound of $|q_i-q_j|$.

(2) Similar to (3.2.3), we have

$$\begin{aligned}\frac{d}{dt}(q_N - q_1) &= G'(y_{1N})\left(\nu_N - \nu_1 - \frac{\kappa}{N}\sum_{k=1}^{N}\underbrace{(\Psi(q_k - q_1) + \Psi(q_N - q_k))}_{\geq \Psi(q_N - q_1)}\right) \\ &\leq -\kappa m_{G'}\Psi(q_N - q_1) + M_{G'}(\nu_N - \nu_1).\end{aligned}$$

Since the right-hand side of the estimate is negative if and only if

$$\frac{M_{G'}}{m_{G'}}\cdot\frac{\nu_N - \nu_1}{\kappa} < \Psi(q_N - q_1),$$

by the same argument as (1), we show that

$$\begin{aligned}\sup_{t\geq 0}\max_{i\neq j}|q_i(t) - q_j(t)| &\leq \sup_{t\geq 0}|q_N(t) - q_1(t)| \\ &\leq \max\left\{q_N^0 - q_1^0, \Psi^{-1}\left(\frac{M_{G'}}{m_{G'}}\cdot\frac{\nu_N - \nu_1}{\kappa}\right)\right\}.\end{aligned}$$

Therefore, we choose

$$L_1^\infty := \max\left\{q_N^0 - q_1^0,\ \Psi^{-1}\left(\frac{M_{G'}}{m_{G'}}\cdot\frac{\nu_N - \nu_1}{\kappa}\right)\right\} > 0$$

to obtain the desired upper bound. □

3.2.2 Asymptotic momentum consensus

We provide the exponential decay of relative momentum to (3.1.5)–(3.1.6). For this, we begin with the second-order formulation of (3.1.5). Let $\{q_i\}$ be a state configuration. Then, we define

$$p_i := \nu_i + \frac{\kappa}{N}\sum_{k=1}^{N}\Psi(q_k - q_i), \quad D_p := \max_{i,j}|p_i - p_j|.$$

Then, it follows from (3.1.5) and $\Psi(q) = \int_0^q \psi(\eta)d\eta$ that

$$\frac{dp_i}{dt} = \frac{\kappa}{N}\sum_{k=1}^{N}\frac{d}{dt}\Psi(q_k - q_i) = \frac{\kappa}{N}\sum_{k=1}^{N}\psi(q_k - q_i)(G(p_k) - G(p_i)).$$

Hence, the first-order system (3.1.5) can be lifted as the second-order model:

$$\begin{cases} \dfrac{dq_i}{dt} = G(p_i), \quad t > 0, \quad i \in [N], \\ \dfrac{dp_i}{dt} = \dfrac{\kappa}{N} \displaystyle\sum_{k=1}^{N} \psi(q_k - q_i)(G(p_k) - G(p_i)). \end{cases} \tag{3.2.8}$$

It is easy to see that the total sum of p_i is preserved:

$$\sum_{i=1}^{N} p_i(t) = \sum_{i=1}^{N} p_i(0), \quad t \geq 0.$$

In particular, when the sum of natural velocities is zero, then so is the sum of p_i:

$$\sum_{i=1}^{N} p_i(t) = \sum_{i=1}^{N} \nu_i + \frac{\kappa}{N} \sum_{i,k=1}^{N} \Psi(q_k - q_i) = \sum_{i=1}^{N} \nu_i = 0, \quad t \geq 0. \tag{3.2.9}$$

Next, we study the exponential decay of D_p under suitable conditions. Throughout the current subsection, we assume that $p_k \neq 0$ for $k \in [N]$. If there exists a particle $p_k = 0$, the proof is still valid with a suitably modified estimation. For notational simplicity, we set

$$\psi_{ik} := \psi(q_k - q_i) \quad \text{and} \quad \widetilde{\psi}_{ij} := \frac{\psi_{ij}}{N} + \left(1 - \frac{\sum_{k=1}^{N} \psi_{ik}}{N} \right) \delta_{ij},$$

where δ_{ij} is the Kronecker delta. Then, it is easy to see that

$$\begin{aligned} &\widetilde{\psi}_{ij} = \widetilde{\psi}_{ji}, \quad \widetilde{\psi}_{ij} \geq \frac{\psi_{ij}}{N}, \quad \sum_{k=1}^{N} \widetilde{\psi}_{ik} = 1, \\ &\sum_{k=1}^{N} \widetilde{\psi}_{ik}(G(p_k) - G(p_i)) = \frac{1}{N} \sum_{k=1}^{N} \psi_{ik}(G(p_k) - G(p_i)). \end{aligned}$$

Now, we choose $i = N$ and $j = 1$ so that

$$p_i - p_j = D_p. \tag{3.2.10}$$

Then, for such i and j, one has

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}|p_i - p_j|^2 &= (p_i - p_j)\left(\frac{dp_i}{dt} - \frac{dp_j}{dt}\right) \\
&= (p_i - p_j)\left(\frac{\kappa}{N}\sum_{k=1}^{N}\psi_{ik}(G(p_k) - G(p_i)) - \frac{\kappa}{N}\sum_{k=1}^{N}\psi_{jk}(G(p_k) - G(p_j))\right) \\
&= \kappa(p_i - p_j)\left(\sum_{k=1}^{N}\widetilde{\psi}_{ki}(G(p_k) - G(p_i)) - \sum_{k=1}^{N}\widetilde{\psi}_{kj}(G(p_k) - G(p_j))\right) \\
&= -\kappa(p_i - p_j)(G(p_i) - G(p_j)) + \kappa(p_i - p_j)\left(\sum_{k=1}^{N}(\widetilde{\psi}_{ki} - \widetilde{\psi}_{kj})G(p_k)\right) \\
&= -\kappa(p_i - p_j)(G(p_i) - G(p_j)) \\
&\quad + \kappa(p_i - p_j)\left[\sum_{k=1}^{N}\left(\widetilde{\psi}_{ki} - \min\{\widetilde{\psi}_{ki}, \widetilde{\psi}_{kj}\} + \min\{\widetilde{\psi}_{ki}, \widetilde{\psi}_{kj}\} - \widetilde{\psi}_{kj}\right)G(p_k)\right] \\
&=: -\kappa(p_i - p_j)(G(p_i) - G(p_j)) + \mathcal{I}_1.
\end{aligned}
\tag{3.2.11}
$$

Next, we proceed to estimate the term $\mathcal{I}_1$ as follows.

$$
\begin{aligned}
\mathcal{I}_1 =&\kappa(p_i - p_j)\left(\sum_{k=1}^{N}\left(\widetilde{\psi}_{ki} - \min\{\widetilde{\psi}_{ki}, \widetilde{\psi}_{kj}\}\right)G(p_k)\right) \\
&- \kappa(p_i - p_j)\left(\sum_{k=1}^{N}\left(\widetilde{\psi}_{kj} - \min\{\widetilde{\psi}_{ki}, \widetilde{\psi}_{kj}\}\right)G(p_k)\right) \\
=&\kappa(p_i - p_j)\left(\sum_{k=1}^{N}\frac{G(p_k)}{p_k}\left(\widetilde{\psi}_{ki} - \min\{\widetilde{\psi}_{ki}, \widetilde{\psi}_{kj}\}\right)p_k\right) \\
&- \kappa(p_i - p_j)\left(\sum_{k=1}^{N}\frac{G(p_k)}{p_k}\left(\widetilde{\psi}_{kj} - \min\{\widetilde{\psi}_{ki}, \widetilde{\psi}_{kj}\}\right)p_k\right) \\
\le&\kappa(p_i - p_j)\left(\sum_{k=1}^{N}\frac{G(p_k)}{p_k}\left(\widetilde{\psi}_{ki} - \min\{\widetilde{\psi}_{ki}, \widetilde{\psi}_{kj}\}\right)p_i\right) \\
&- \kappa(p_i - p_j)\left(\sum_{k=1}^{N}\frac{G(p_k)}{p_k}\left(\widetilde{\psi}_{kj} - \min\{\widetilde{\psi}_{ki}, \widetilde{\psi}_{kj}\}\right)p_j\right),
\end{aligned}
\tag{3.2.12}
$$

where we used the relation (3.2.10) to find

$$(p_i - p_j)p_j \le (p_i - p_j)p_k \le (p_i - p_j)p_i.$$

Now, we combine (3.2.11) and (3.2.12) to find

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|p_i - p_j|^2 \\
&\quad = -\kappa(p_i - p_j)(G(p_i) - G(p_j)) \\
&\qquad + \kappa(p_i - p_j)p_i \sum_{k=1}^{N} \frac{G(p_k)}{p_k}\left(\widetilde{\psi}_{ki} - \min\{\widetilde{\psi}_{ki}, \widetilde{\psi}_{kj}\}\right) \\
&\qquad - \kappa(p_i - p_j)p_j \sum_{k=1}^{N} \frac{G(p_k)}{p_k}\left(\widetilde{\psi}_{kj} - \min\{\widetilde{\psi}_{ki}, \widetilde{\psi}_{kj}\}\right) \\
&\quad = -\kappa|p_i - p_j|^2 \sum_{k=1}^{N} \frac{G(p_k)\min\{\widetilde{\psi}_{ki}, \widetilde{\psi}_{kj}\}}{p_k} \\
&\qquad + \kappa(p_i - p_j)\sum_{k=1}^{N} \widetilde{\psi}_{ki}\left(\frac{G(p_k)p_i}{p_k} - G(p_i)\right) \\
&\qquad + \kappa(p_i - p_j)\sum_{k=1}^{N} \widetilde{\psi}_{kj}\left(G(p_j) - \frac{G(p_k)p_j}{p_k}\right) \\
&\quad =: \mathcal{I}_{11} + \mathcal{I}_{12} + \mathcal{I}_{13}.
\end{aligned} \tag{3.2.13}$$

Lemma 3.2.1. *Suppose that natural velocity set $\{\nu_i\}$ satisfies*

$$\sum_{i=1}^{N} \nu_i = 0,$$

and let $\{q_i\}$ be a solution to (3.1.5)–(3.1.6) with initial data $\{q_i^0\}$. Then, one has the following estimates:

(i) $\mathcal{I}_{11} \le -\kappa m_{G'} \psi(L_1^\infty)|p_i - p_j|^2.$

(ii) $\mathcal{I}_{12} \le \kappa(M_{G'} - m_{G'})(p_i - p_j)p_i.$

(iii) $\mathcal{I}_{13} \le -\kappa(M_{G'} - m_{G'})(p_i - p_j)p_j,$

where L_1^∞ is defined as a maximal relative distance defined in Theorem 3.2.1.

Proof. Below, we provide estimates on $\mathcal{I}_{1i}$ one by one.

• (Estimate of $\mathcal{I}_{11}$): We use the mean-value theorem to obtain

$$m_{G'} < \frac{G(p_k)}{p_k} = \frac{G(p_k) - G(0)}{p_k - 0} = G'(r_k) < M_{G'}, \quad \text{where} \quad |r_k| < |p_k|.$$

Moreover, since we have already shown that the relative states are uniformly bounded, we have

$$\widetilde{\psi}_{ki} \geq \frac{\psi_{ki}}{N} \geq \frac{\psi(L_1^\infty)}{N}.$$

Hence, we estimate $\mathcal{I}_{11}$ as

$$\mathcal{I}_{11} = -\kappa |p_i - p_j|^2 \sum_{k=1}^N \frac{G(p_k)\min\{\widetilde{\psi}_{ki}, \widetilde{\psi}_{kj}\}}{p_k} \leq -\kappa m_{G'} \psi(L_1^\infty)|p_i - p_j|^2.$$

• (Estimate of $\mathcal{I}_{12}$): Note that the sum-zero condition $\sum_{k=1}^N p_k = 0$ implies

$$p_i > 0 \quad \text{and} \quad p_j < 0.$$

Therefore, we estimate $\mathcal{I}_{12}$ as

$$\begin{aligned}\mathcal{I}_{12} &= \kappa(p_i - p_j)p_i \sum_{k=1}^N \widetilde{\psi}_{ki} \left(\frac{G(p_k)}{p_k} - \frac{G(p_i)}{p_i} \right) \\ &\leq \kappa(p_i - p_j)p_i \sum_{k=1}^N \widetilde{\psi}_{ki} \left| \frac{G(p_k)}{p_k} - \frac{G(p_i)}{p_i} \right|.\end{aligned}$$

However, we have

$$\left| \frac{G(p_k)}{p_k} - \frac{G(p_i)}{p_i} \right| = |G'(r_k) - G'(r_i)| \leq M_{G'} - m_{G'}.$$

Therefore, we further estimate $\mathcal{I}_{12}$ as

$$\mathcal{I}_{12} \leq \kappa(M_{G'} - m_{G'})(p_i - p_j)p_i.$$

• (Estimate of $\mathcal{I}_{13}$): Similarly, we estimate $\mathcal{I}_{13}$ as

$$\mathcal{I}_{13} = -\kappa(p_i - p_j)p_j \sum_{k=1}^N \widetilde{\psi}_{jk} \left(\frac{G(p_k)}{p_k} - \frac{G(p_j)}{p_j} \right) \leq -\kappa(M_{G'} - m_{G'})(p_i - p_j)p_j.$$

□

Proposition 3.2.1. *Suppose that natural velocity set $\{\nu_i\}$ satisfies*

$$\sum_{i=1}^{N} \nu_i = 0,$$

and let $\{q_i\}$ be a solution to (3.1.4) with initial data $\{q_i^0\}$. Then, we have

$$D_p(t) \le D_p(0) \exp\Big(-\kappa\Big(m_{G'}\psi(L_1^\infty) - (M_{G'} - m_{G'})\Big)t\Big), \quad t \ge 0,$$

where L_1^∞ is defined as a maximal relative distance defined in Theorem 3.2.1. Therefore, if $m_{G'}$ and $M_{G'}$ in (3.2.7) satisfy

$$\psi(L_1^\infty) > \frac{M_{G'} - m_{G'}}{m_{G'}},$$

then D_p decays to zero exponentially fast.

Proof. In (3.2.13), we use Lemma 3.2.1 to find

$$\frac{1}{2}\frac{dD_p^2}{dt} \le -\kappa m_{G'}\psi(L_1^\infty)D_p^2 + \kappa(M_{G'} - m_{G'})D_p^2,$$

or equivalently,

$$\frac{dD_p}{dt} \le -\kappa\Big(m_{G'}\psi(L_1^\infty) - (M_{G'} - m_{G'})\Big)D_p.$$

Now we use the assumption:

$$m_{G'}\psi(L_1^\infty) > M_{G'} - m_{G'}$$

to obtain the desired exponential decay of D_p. □

Remark 3.2.1. *Note that if*

$$m_{G'} \approx M_{G'} \approx \mathcal{O}(1),$$

the condition

$$\psi(L_1^\infty) > \frac{M_{G'} - m_{G'}}{m_{G'}} \tag{3.2.14}$$

holds for almost all initial data since the right-hand side of (3.2.14) can be small, and in particular, for the case of standard consensus model $G(q) = q$ where $m_{G'} = M_{G'} = 1$, the assumption (3.2.14) always holds.

As a consequence of the exponential decay of p-diameter, the position has its asymptotic limit, if the sum of natural velocities is zero.

Corollary 3.2.1. *Suppose* $\{\nu_i\}$, $m_{G'}$ *and* $M_{G'}$ *in* (3.2.7) *satisfy*

$$\sum_{i=1}^N \nu_i = 0, \quad \psi(L_1^\infty) > \frac{M_{G'} - m_{G'}}{m_{G'}},$$

where L_1^∞ *is defined as a maximal relative distance defined in Theorem 3.2.1, and let* $\{q_i\}$ *be a solution to* (3.1.5)–(3.1.6) *with initial data* $\{q_i^0\}$*. Then, there exists an asymptotic state configuration* $\{q_i^\infty\}$ *such that*

$$\lim_{t\to\infty} q_i(t) = q_i^\infty, \quad i \in [N].$$

Proof. We use (3.2.9) to derive

$$\sum_{i=1}^N p_i(t) = 0.$$

Hence, one has

$$|p_i(t)| = \left| \frac{1}{N} \sum_{k=1}^N (p_i(t) - p_k(t)) \right| \le D_p(t) \le D_p(0)\exp(-\delta t),$$

where δ is a positive constant defined by

$$\delta := \kappa\Big(m_{G'}\psi(L_1^\infty) - (M_{G'} - m_{G'})\Big).$$

Then, we use $G(0) = 0$ to find

$$q_i(t) = q_i(0) + \int_0^t G(p_i(s))\,ds \quad \text{and}$$
$$|G(p_i(s))| = |G(p_i(s)) - G(0)| \le |G'(r_i(s))||p_i(s)| \le M_{G'}|p_i(s)| \le C\exp(-\delta s).$$

This implies the existence of the limit $q_i^\infty := \lim_{t\to\infty} q_i(t)$. □

We study the orbital stability of asymptotic state. For given asymptotic configurations $\{q_i^\infty\}$ and $\{\tilde{q}_i^\infty\}$, we set

$$q_c^\infty := \frac{1}{N}\sum_{i=1}^N q_i^\infty, \qquad \tilde{q}_c^\infty := \frac{1}{N}\sum_{i=1}^N \tilde{q}_i^\infty.$$

It is well known from [34] that for system (3.1.5) with $G(x) = x$, the asymptotic state:

$$q_i^\infty := \lim_{t\to\infty} q_i(t)$$

is uniquely determined, when the averaged state is fixed. However, if G is not an identity mapping, there will no conservation law for (3.1.5). Thus, we can only expect the uniqueness of the asymptotic position up to a translation.

Proposition 3.2.2. (Orbital stability) *Let* $\{q_i\}$ *and* $\{\tilde{q}_i\}$ *be solutions to* (3.1.4) *with initial data* $\{q_i^0\}$ *and* $\{\tilde{q}_i^0\}$*, respectively. Suppose that there exist asymptotic limits* $\{q_i^\infty\}$ *and* $\{\tilde{q}_i^\infty\}$ *such that*

$$\lim_{t\to\infty} q_i(t) = q_i^\infty, \quad \lim_{t\to\infty} \tilde{q}_i(t) = \tilde{q}_i^\infty, \quad \forall\ i \in [N].$$

Then, there exists a constant shift α *independent of* i *such that*

$$q_i^\infty = \tilde{q}_i^\infty + \alpha, \quad i \in [N].$$

Proof. Since $\{q_i^\infty\}$ and $\{\tilde{q}_i^\infty\}$ are asymptotic states, they satisfy

$$G\left(\nu_i + \frac{\kappa}{N}\sum_{k=1}^N \Psi(q_k^\infty - q_i^\infty)\right) = 0, \quad G\left(\nu_i + \frac{\kappa}{N}\sum_{k=1}^N \Psi(\tilde{q}_k^\infty - \tilde{q}_i^\infty)\right) = 0.$$

Since G is bijective and $G(0) = 0$, we can conclude that

$$\nu_i = \frac{\kappa}{N}\sum_{k=1}^N \Psi(q_i^\infty - q_k^\infty) = \frac{\kappa}{N}\sum_{k=1}^N \Psi(\tilde{q}_i^\infty - \tilde{q}_k^\infty).$$

We now define $\hat{q}_i^\infty$ and $\hat{\tilde{q}}_i^\infty$ as

$$\hat{q}_i^\infty := q_i^\infty - q_c^\infty, \quad \hat{\tilde{q}}_i^\infty := \tilde{q}^\infty - \tilde{q}_c^\infty, \quad i \in [N].$$

Then, since $\hat{q}_i^\infty$ and $\hat{\tilde{q}}_i^\infty$ are translations of q_i^∞ and $\tilde{q}_i^\infty$, they also satisfy

$$\nu_i = \frac{\kappa}{N}\sum_{k=1}^N \Psi(\hat{q}_i^\infty - \hat{q}_k^\infty) = \frac{\kappa}{N}\sum_{k=1}^N \Psi(\hat{\tilde{q}}_i^\infty - \hat{\tilde{q}}_k^\infty), \quad \sum_{k=1}^N \hat{q}_k^\infty = \sum_{k=1}^N \hat{\tilde{q}}_k^\infty = 0.$$

Therefore, it follows from [34, Theorem 4.1] that

$$\hat{q}_i^\infty = \hat{\tilde{q}}_i^\infty.$$

Hence, we have

$$q_i^\infty = \hat{q}_i^\infty + q_c^\infty = \hat{\tilde{q}}_i^\infty + q_c^\infty = \tilde{q}_i^\infty + (q_c^\infty - \tilde{q}_c^\infty).$$

We set $\alpha = q_c^\infty - \tilde{q}_c^\infty$ to get the desired result. □

3.3 Regular short-ranged communication weight: Type II

We now study the clustering dynamics of system (3.1.5) with the bounded short-ranged communication weight:

$$\lim_{r\to+\infty}\int_0^r \psi(x)dx =: \Psi^\infty < \infty.$$

As in the case of long-ranged communication weight, we first provide estimates for the lower and upper bounds for relative states. Unlike the case of long-ranged communication weight, the relative states are bounded only for the large value of coupling strength. Moreover, when the coupling strength is sufficiently small, all the particles are segregated with each other. The following proposition is the counterpart of Theorem 3.2.1 for a bounded long-ranged communication weight.

Theorem 3.3.1. (Complete consensus and segregation) *Suppose that natural velocities and initial state satisfy the ordering* (3.1.13)*, and let* $\{q_i\}$ *be a solution to* (3.1.5)–(3.1.6)*. Then, the following assertions hold.*

1. *There exists a positive lower bound* $\ell_1^{ij} > 0$ *such that*

$$\inf_{t\geq 0}|q_i(t)-q_j(t)| \geq \ell_1^{ij} > 0,$$

   where ℓ_1^{ij} *is defined in Theorem 3.2.1.*

2. *Suppose that the coupling strength* κ *is sufficiently large such that*

$$\kappa > \frac{M_{G'}}{m_{G'}}\cdot\frac{N}{|i-j|+1}\cdot\frac{|\nu_i-\nu_j|}{\Psi^\infty}.$$

   Then, one has

$$\sup_{t\geq 0}|q_i(t)-q_j(t)| \max\left\{q_i^0-q_j^0, \Psi^{-1}\left(\frac{M_{G'}}{m_{G'}}\cdot\frac{N(\nu_i-\nu_j)}{\kappa(i-j+1)}\right)\right\} < \infty.$$

   In particular, if the coupling strength κ *satisfies*

$$\kappa > \frac{M_{G'}}{m_{G'}}\cdot\frac{(\nu_N-\nu_1)}{\Psi^\infty},$$

then one has

$$\sup_{t\geq 0}\max_{i\neq j}|q_i(t)-q_j(t)|<L_1^\infty<\infty,$$

where L_1^∞ *is defined in Theorem 3.2.1.*

3. *Suppose that the coupling strength* κ *is sufficiently small such that*

$$\kappa<\frac{m_{G'}}{M_{G'}}\cdot\frac{N}{(N-1+|i-j|)}\cdot\frac{|\nu_i-\nu_j|}{\Psi^\infty}. \tag{3.3.1}$$

Then, one has

$$\liminf_{t\to+\infty}|q_i(t)-q_j(t)|=\infty.$$

Proof. (1) For indices $j<N$ and $i=j+1$, we use the same estimate in (3.2.3) to obtain

$$\frac{d(q_i-q_j)}{dt}\geq m_{G'}(\nu_i-\nu_j)-M_{G'}\kappa\Psi(q_i-q_j).$$

Again, we consider the differential equation:

$$\begin{cases}\dot{y}=m_{G'}(\nu_i-\nu_j)-\kappa M_{G'}\Psi(y) & t>0,\\ y(0)=q_i^0-q_j^0>0.\end{cases}$$

Next, we consider two separate cases.

• Case A ($\Psi^\infty>\frac{m_{G'}}{M_{G'}}\cdot\frac{\nu_i-\nu_j}{\kappa}$): In this case, since Ψ is monotonically increasing, there exists a unique equilibrium $y=\Psi^{-1}\left(\frac{m_{G'}}{M_{G'}}\cdot\frac{\nu_i-\nu_j}{\kappa})\right)$ of the differential equation for y. Then, the same argument in the proof of Theorem 3.2.1 (1) holds and therefore

$$y(t)\geq\min\left\{y(0),\ \Psi^{-1}\left(\frac{m_{G'}}{M_{G'}}\cdot\frac{\nu_i-\nu_j}{\kappa})\right)\right\}.$$

• Case B ($\Psi^\infty\leq\frac{m_{G'}}{M_{G'}}\cdot\frac{\nu_i-\nu_j}{\kappa}$): In this case, one has

$$y'=m_{G'}(\nu_i-\nu_j)-\kappa M_{G'}\Psi(y)\geq m_{G'}(\nu_i-\nu_j)-\kappa M_{G'}\Psi^\infty\geq 0.$$

Therefore, we obtain

$$y(t)\geq y(0).$$

Finally, we combine Case A and Case B to derive

$$q_i(t) - q_j(t) \geq \min \left\{ q_i^0 - q_j^0,\ \Psi^{-1}\left(\frac{m_{G'}}{M_{G'}} \cdot \frac{\nu_i - \nu_j}{\kappa}) \right) \right\}.$$

(2) We fix pair of indices (i, j) such that $i > j$. We then use the mean-value theorem to obtain

$$\begin{aligned}
\frac{d(q_i - q_j)}{dt} &= G\left(\nu_i + \frac{\kappa}{N}\sum_{k=1}^{N}\Psi(q_k - q_i)\right) - G\left(\nu_j + \frac{\kappa}{N}\sum_{k=1}^{N}\Psi(q_k - q_j)\right) \\
&= G'(y_{ij})\left(\nu_i - \nu_j + \frac{\kappa}{N}\sum_{k=1}^{N}(\Psi(q_k - q_i) - \Psi(q_k - q_j))\right) \\
&\leq G'(y_{ij})\left(\nu_i - \nu_j + \frac{\kappa}{N}\sum_{j \leq k \leq i}(\Psi(q_k - q_i) - \Psi(q_k - q_j))\right) \\
&\leq G'(y_{ij})\left(\nu_i - \nu_j - \frac{\kappa}{N}\sum_{j \leq k \leq i}\Psi(q_i - q_j)\right) \\
&= G'(y_{ij})\left(\nu_i - \nu_j - \frac{(i-j+1)\kappa}{N}\Psi(q_i - q_j)\right) \\
&\leq M_{G'}(\nu_i - \nu_j) - \kappa m_{G'}\frac{(i-j+1)}{N}\Psi(q_i - q_j).
\end{aligned} \tag{3.3.2}$$

By the same argument as in the proof of Theorem 3.2.1, the unique equilibrium of the differential equation:

$$\begin{cases} \dot{z} = M_{G'}(\nu_i - \nu_j) - \kappa m_{G'}\dfrac{(i-j+1)}{N}\Psi(z), \quad t > 0, \\ z(0) = q_i^0 - q_j^0, \end{cases}$$

is given by $z = \Psi^{-1}\left(\frac{M_{G'}}{m_{G'}} \cdot \frac{N(\nu_i - \nu_j)}{\kappa(i-j+1)}\right)$. Note that an existence of the equilibrium is guaranteed from (3.3.1). Now, we use the comparison principle and similar argument as in (1) to deduce

$$0 < q_i(t) - q_j(t) \leq \max\left\{ q_i^0 - q_j^0, \Psi^{-1}\left(\frac{M_{G'}}{m_{G'}} \cdot \frac{N(\nu_i - \nu_j)}{\kappa(i-j+1)}\right)\right\}.$$

Finally, we choose $i = N$ and $j = 1$ to obtain that if

$$\kappa > \frac{M_{G'}}{m_{G'}} \cdot \frac{(\nu_N - \nu_1)}{\Psi^\infty},$$

then one has

$$\sup_{t\geq 0} |q_N(t) - q_1(t)| \leq \max\left\{ q_N^0 - q_1^0, \Psi^{-1}\left(\frac{M_{G'}}{m_{G'}} \cdot \frac{\nu_N - \nu_1}{\kappa}\right)\right\} = L_1^\infty < \infty.$$

This implies the desired boundedness.

(3) By the same estimate as in (3.3.2), we have

$$\begin{aligned}\frac{d}{dt}(q_i - q_j) &= G'(y_{ij}) \times \left(\nu_i - \nu_j - \frac{\kappa}{N}\sum_{k=1}^N (\Psi(q_k - q_j) - \Psi(q_k - q_i))\right)\\ &\geq m_{G'}(\nu_i - \nu_j) - \kappa M_{G'}\frac{(N + i - j - 1)}{N}\Psi^\infty > 0.\end{aligned}$$

Note that the first inequality comes from the the following observation:

$$\begin{aligned}0 < \Psi(q_k - q_j) - \Psi(q_k - q_i) &= \int_{q_k - q_i}^{q_k - q_j} \psi(q)dq\\ &< \begin{cases} 2\Psi^\infty & \text{for} \quad k \in \{j+1, \cdots, i-1\},\\ \Psi^\infty & \text{for} \quad k \in \{1, \cdots, j\} \cup \{i, \cdots, N\},\end{cases}\end{aligned}$$

and the second inequality is valid from the condition (3.3.1). This completes the proof of Theorem 3.3.1.

□

Theorem 3.3.2. (Improved complete segregation) *Suppose that natural velocities and initial state satisfy the ordering* (3.1.13) *and*

$$\sum_{i=1}^N \nu_i = 0,$$

and let $\{q_i\}$ *be a solution to* (3.1.5)–(3.1.6). *Then, all the particles are completely segregated in the sense that*

$$\limsup_{t\to\infty} q_1(t) = -\infty, \quad \liminf_{t\to\infty} q_N(t) = \infty,$$

$$\liminf_{t\to\infty} |q_{i+1}(t) - q_i(t)| = \infty, \quad \forall\ i \in [N-1]$$

if the coupling strength κ is sufficiently small in the following sense:

$$\kappa < \min\Big\{ -\frac{N}{N-1}\cdot\frac{\nu_1}{\Psi^\infty},\ \frac{m_{G'}}{M_{G'}}\cdot\frac{N}{2}\cdot\frac{(\nu_2-\nu_1)}{\Psi^\infty}, \\ \dots,\ \frac{m_{G'}}{M_{G'}}\cdot\frac{N}{2}\cdot\frac{(\nu_N-\nu_{N-1})}{\Psi^\infty},\ \frac{N}{N-1}\cdot\frac{\nu_N}{\Psi^\infty}\Big\}. \tag{3.3.3}$$

Proof. It follows from the zero sum condition for $\{\nu_i\}$ that

$$\nu_1 < 0 < \nu_N.$$

Then, we estimate q_1 and q_N as

$$\dot{q}_1 = G\left(\nu_1 + \frac{\kappa}{N}\sum_{k=1}^N \Psi(q_k - q_1)\right) \le G\left(\nu_1 + \frac{\kappa(N-1)}{N}\Psi^\infty\right) < 0,$$
$$\dot{q}_N = G\left(\nu_N + \frac{\kappa}{N}\sum_{k=1}^N \Psi(q_k - q_N)\right) \ge G\left(\nu_N - \frac{\kappa(N-1)}{N}\Psi^\infty\right) > 0,$$

where we used the following relations:

$$\Psi(0) = 0 \quad \text{and} \quad \Psi(r) \le \Psi^\infty \quad \text{for } r \ge 0.$$

Therefore we have

$$\limsup_{t\to\infty} q_1(t) = -\infty \quad \text{and} \quad \liminf_{t\to\infty} q_N(t) = \infty. \tag{3.3.4}$$

Now we verify that segregation occurs between two particles. Suppose on the contrary that complete segregation does not occur. Then, there exists index L and R with $L < R$ satisfying

$$\liminf_{t\to\infty}(q_R(t) - q_L(t)) < \infty,$$
$$\text{but} \quad \liminf_{t\to\infty}(q_L(t) - q_j(t)) = \liminf_{t\to\infty}(q_i(t) - q_R(t)) = \infty,$$

whenever $1 \leq j < L < R < i \leq N$. Then, the relative state between q_R and q_L can be estimated as

$$\begin{aligned} \frac{d}{dt}(q_R - q_L) &= G\left(\nu_R + \frac{\kappa}{N}\sum_{k=1}^{N}\Psi(q_k - q_R)\right) - G\left(\nu_L + \frac{\kappa}{N}\sum_{k=1}^{N}\Psi(q_k - q_L)\right) \\ &= G'(y_{RL})\left(\nu_R - \nu_L + \frac{\kappa}{N}\sum_{k=1}^{N}(\Psi(q_k - q_R) - \Psi(q_k - q_L))\right) \\ &\geq \frac{\kappa M_{G'}}{N}\sum_{k=1}^{N}(\Psi(q_k - q_R) - \Psi(q_k - q_L)) + m_{G'}(\nu_R - \nu_L). \end{aligned} \tag{3.3.5}$$

On the other hand, the conditions (3.3.4) assert that $(L, R) \neq (1, N)$. Therefore, if we assume $R \neq N$, then there exists k satisfying $R < k \leq N$ and we have

$$0 < \Psi(q_k - q_L) - \Psi(q_k - q_R) = \int_{q_k - q_R}^{q_k - q_L} \psi(r) dr \leq \int_{q_k - q_R}^{\infty} \psi(r)\, dr.$$

Since q_k segregate from q_R,

$$\lim_{t\to+\infty}(q_k(t) - q_R(t)) = +\infty \quad \Longrightarrow \quad \lim_{t\to+\infty}\int_{q_k - q_R}^{\infty}\psi(r)\, dr = 0,$$

and similar estimate holds for $L \neq 1$ case. Therefore, for any $\varepsilon > 0$, there exists a time $T(\varepsilon)$ such that

$$\begin{aligned} &0 < \Psi(q_k - q_L)(t) - \Psi(q_k - q_R)(t) < \varepsilon, \\ &\text{for} \quad k < L \quad \text{or} \quad k > R, \quad \text{and} \quad t > T(\varepsilon). \end{aligned}$$

Therefore, for a small positive constant $\delta > 0$, we can choose sufficiently large time $T^* = T^*(\delta)$ and continue estimation on (3.3.5) under $t > T^*$ as follows:

$$\begin{aligned} \frac{d}{dt}(q_R - q_L) &\geq m_{G'}(\nu_R - \nu_L) + \frac{\kappa M_{G'}}{N}\sum_{k=L}^{R}(\Psi(q_k - q_R) - \Psi(q_k - q_L)) - \delta \\ &> m_{G'}(\nu_R - \nu_L) - \frac{2\kappa M_{G'}(R - L)\Psi^{\infty}}{N} - \delta, \end{aligned} \tag{3.3.6}$$

where we used the following estimate for the second inequality.

$$
\begin{aligned}
0 < \Psi(q_k - q_L) - \Psi(q_k - q_R) &= \int_{q_k - q_R}^{q_k - q_L} \psi(q)dq \\
&< \begin{cases} 2\Psi^\infty & \text{for} \quad k \in \{L+1, \cdots, R-1\}, \\ \Psi^\infty & \text{for} \quad k \in \{L, R\}. \end{cases}
\end{aligned}
$$

Now, since κ is sufficiently small as in (3.3.3), we have

$$
\nu_R - \nu_L = \sum_{k=L}^{R-1} (\nu_{k+1} - \nu_k) > \sum_{k=L}^{R-1} \frac{M_{G'}}{m_{G'}} \cdot \frac{2\kappa \Psi^\infty}{N} = \frac{M_{G'}}{m_{G'}} \cdot \frac{2\kappa (R-L)\Psi^\infty}{N}. \tag{3.3.7}
$$

Then, it follows from (3.3.6) and (3.3.7) that we can choose $T^* \gg 1$ which makes $\frac{d}{dt}(q_R - q_L)$ strictly positive for $t \geq T^*$. This implies that q_R and q_L segregate asymptotically, which yields a contradiction. □

3.4 Singular communication weight: Type III

Finally, we consider the collective dynamics of (3.1.5)–(3.1.6) for the case of singular communication weight. To fix an idea, we consider the power-law type singular communication weight:

$$
\psi(q) = \frac{1}{|q|^\alpha}, \quad \alpha > 0, \quad q \neq 0.
$$

Note that for the first-order model (3.1.5)–(3.1.6), we need to define its antiderivative Ψ appropriately according to the singularity of the communication weight at $q = 0$.

If $0 < \alpha < 1$, then ψ is integrable at the origin. Thus, we can define

$$
\Psi(q) = \int_0^q \psi(r)dr = \operatorname{sgn}(q) \frac{|q|^{1-\alpha}}{1-\alpha}.
$$

In contrast, for $\alpha \geq 1$, ψ is not integrable at the origin. Therefore, we change the definition of Ψ by the integration from $x = 1$:

$$
\Psi(q) := \operatorname{sgn}(q) \int_1^{|q|} \psi(r)dr = \begin{cases} \operatorname{sgn}(q) \log |q| & q \neq 0, \quad \alpha = 1, \\ \operatorname{sgn}(q) \frac{1}{\alpha - 1}\left(1 - \frac{1}{|q|^{\alpha-1}}\right), & q \neq 0, \quad \alpha > 1. \end{cases}
$$

For the consistency with a regular communication weight, we define $\Psi(0) = 0$.

3.4.1 Weak singularity

We consider the dynamics of (3.1.5) when the singular communication weight has a weak singularity, i.e., $\alpha \in (0, 1)$. Even in this case, we will see that system (3.1.4) is again aligned according to the natural velocity again, just as in the regular communication weight case in Proposition 3.1.1 and Proposition 3.2.1. This will be documented in the following proposition.

Proposition 3.4.1. *Suppose the communication weight has weak singularity:*

$$\psi(q) = \frac{1}{|q|^\alpha}, \quad 0 < \alpha < 1, \quad q \neq 0,$$

and let $\{q_i\}$ be a solution to (3.1.5)–(3.1.6) *with initial data $\{q_i^0\}$. For fixed indices i and j ($i \neq j$), without loss of generality, we assume*

$$q_i^0 > q_j^0.$$

Then the following trichotomy holds.

1. *If $\nu_i > \nu_j$, then q_i and q_j will not collide in finite time:*

$$q_i(t) > q_j(t) \quad \textit{for all} \;\; t \geq 0.$$

2. *If $\nu_i < \nu_j$, then q_i and q_j will collide exactly once, i.e., there exists a time t^* such that*

$$\begin{aligned} &q_i(t) > q_j(t) \quad \textit{for} \;\; 0 \leq t < t^*, \\ &q_i(t^*) = q_j(t^*) \quad \textit{and} \quad q_i(t) < q_j(t) \quad \textit{for}\; t > t^*. \end{aligned}$$

3. *If $\nu_i = \nu_j$, then q_i and q_j will collide in finite time, and two particles will stick together after their first collision.*

Proof. Since the proof of the first two statements of Proposition 3.1.1 does not depend on the regularity of ψ, the proofs for (1) and (2) are the same as in the proof of Proposition 3.1.1. Therefore, it suffices to prove (3). For this, we separate the proof of (3) in two steps. Moreover, we assume that ψ belongs to the general class of communication weight as in Remark 3.4.1 below.

• (Collision appearance): Suppose on the contrary that collision does not occur, i.e.,

$$q_i(t) > q_j(t) \quad \text{for all } t \geq 0.$$

By the same calculation as in (3.3.2), we have

$$\frac{d}{dt}(q_i - q_j) \leq -\frac{(|i-j|+1)\kappa m_{G'}}{N}\Psi(q_i - q_j). \tag{3.4.1}$$

We now define $f : [0, \infty) \to [0, \infty)$ as

$$f(r) := \int_0^r \frac{1}{\Psi(s)} ds,$$

which is well-defined since $\frac{1}{\Psi} \in L^1_{\text{loc}}(\mathbb{R})$. Then, f satisfies

$$f(0) = 0, \quad \text{and} \quad f(r) > 0, \quad f'(r) = \frac{1}{\Psi(r)} > 0, \quad \text{for} \quad r > 0.$$

Then, the differential inequality (3.4.1) gives

$$\frac{d}{dt}f(q_i - q_j) = f'(q_i - q_j)\frac{d(q_i - q_j)}{dt} \leq -\frac{(|i-j|+1)\kappa m_{G'}}{N} < 0.$$

Therefore $f(q_i(t) - q_j(t))$ is initially positive and its derivative is strictly less than some negative constant, and therefore, there exists $T > 0$ such that

$$f(q_i(T) - q_j(T)) = 0.$$

Hence we conclude $q_i(T) = q_j(T)$, which gives a contradiction. Therefore, two particles q_i and q_j must collide.

• (Post-collision behavior): Suppose that q_i and q_j collide at time $t = T$. Below, we will show that two particles stick together afterwards:

$$q_i(t) = q_j(t), \quad \text{for } t \geq T.$$

Again, we suppose the contrary, i.e., there exists a separating time $T_1 \geq T$ such that

$$q_i(T_1) = q_j(T_1) \quad \text{but} \quad |q_i(t) - q_j(t)| > 0,$$

for t in some non-empty interval $(T_1, T_1 + \varepsilon)$. Since we have already shown that the two particles should collide after the time $T_1 + \varepsilon$, there exists $T_2 > T_1$ such that

$$|q_i(t) - q_j(t)| > 0 \quad \text{for } t \in (T_1, T_2) \ \text{ and } \ q_i(T_2) = q_j(T_2).$$

Without loss of generality, we may assume

$$q_i > q_j \quad \text{in } (T_1, T_2).$$

Then, one has

$$\begin{aligned}
0 &= (q_i(T_2) - q_j(T_2)) - (q_i(T_1) - q_j(T_1)) = \int_{T_1}^{T_2} \frac{d(q_i - q_j)}{ds}\, ds \\
&= \int_{T_1}^{T_2} \frac{\kappa G'(y_{ij}(s))}{N} \sum_{k=1}^{N} \Big(\Psi(q_k(s) - q_i(s)) - \Psi(q_k(s) - q_j(s)) \Big)\, ds \\
&\leq \frac{\kappa m_{G'}}{N} \sum_{k=1}^{N} \int_{T_1}^{T_2} \Big(\Psi(q_k(s) - q_i(s)) - \Psi(q_k(s) - q_j(s)) \Big)\, ds < 0,
\end{aligned}$$

where we used

$$\Psi(q_k - q_i) < \Psi(q_k - q_j) \quad \text{on } (T_1, T_2),$$

since we have $q_i > q_j$ on that time interval. This yields a contradiction. Therefore, we conclude that two particles stick together after the first collision. □

Remark 3.4.1. *The third assertion of Proposition 3.4.1(3) holds for a more general class of kernels satisfying the relations:*

$$\psi \in L^1_{loc}(\mathbb{R}) \quad \textit{and} \quad \frac{1}{\Psi} \in L^1_{loc}(\mathbb{R}).$$

Now, we are ready to state the state-locking as follows.

Theorem 3.4.1. *Suppose the communication weight has weak singularity:*

$$\psi(q) = \frac{1}{|q|^\alpha}, \quad 0 < \alpha < 1, \quad q \neq 0,$$

and let $\{q_i\}$ *be a solution to* (3.1.4) *with initial data* $\{q_i^0\}$*. If each natural velocity is distinct to each other, the following assertions hold.*

1. *There exists a positive lower bound* $\ell_1^{ij} > 0$ *of distance between* q_i *and* q_j *as* $t \to \infty$*:*
$$\liminf_{t\to+\infty} |q_i(t) - q_j(t)| \geq \ell_1^{ij} > 0.$$
where ℓ_1^{ij} *is defined in Theorem 3.2.1.*

2. *The relative distances* $|q_i - q_j|$ *is uniformly bounded by* $L_1^\infty < +\infty$*:*
$$\sup_{t\geq 0} \max_{i,j} |q_i(t) - q_j(t)| \leq L_1^\infty < \infty.$$
where L_1^{ij} *is defined in Theorem 3.2.1.*

Proof. When the singularity of the communication weight is weak, thanks to Proposition 3.4.1, the position and natural velocities are again aligned as in the regular communication weight. Therefore, we still may assume the condition (3.1.13). Furthermore, since the communication weight is long-ranged for $0 < \alpha < 1$, the same results hold as in Theorem 3.2.1. Since the proof does not depend on the regularity of ψ, we omit the proof. □

3.4.2 Strong singularity

We consider the case in which the singularity exponent α is large, i.e., $\alpha \geq 1$. We first begin with the preservation of collisionless property.

Proposition 3.4.2. *Suppose the communication weight has a strong singularity:*

$$\psi(q) = \frac{1}{|q|^\alpha}, \quad \alpha \geq 1, \quad q \neq 0,$$

and let $\{q_i\}$ *be a solution to* (3.1.4) *with the collisionless initial data* $\{q_i^0\}$*:*

$$q_i^0 \neq q_j^0, \quad i \neq j.$$

Then, the solution $\{q_i\}$ is also collisionless:

$$q_i(t) \neq q_j(t), \quad \textit{for all} \quad t \geq 0.$$

Proof. In this case, the communication weight is asymptotically integrable, and therefore, there exists a limit $\Psi^\infty := \lim_{x\to\infty} \Psi(x)$. Furthermore, there is no collision between particles, due to the high-singularity. Considering the second-order model (3.2.8), the proof of collision-free property is almost the same as in the proof of collision-free property of the relativistic CS model in [11]. Therefore, we omit the detailed proof for collision-free property. □

Note that for the strong singularity case, the relative states are not zero, and therefore, system (3.1.4) is globally well-posed. Therefore, unlike the regular or weakly singular communication weight, the state and natural velocity cannot be aligned as in (3.1.13). Nevertheless, we still attain the lower and upper bounds for relative states.

Theorem 3.4.2. *Suppose the communication weight has a strong singularity:*

$$\psi(q) = \frac{1}{|q|^\alpha}, \quad \alpha \geq 1, \quad q \neq 0,$$

and let $\{q_i\}$ be a solution to (3.1.5) *with the well-prepared initial data $\{q_i^0\}$:*

$$q_1^0 < q_2^0 < \cdots < q_N^0.$$

Then, the following assertions hold.

1. (Existence of a positive minimal relative state)*: There exists a positive uniform-in-time lower bound $\ell_2^\infty > 0$ of relative distances:*

$$\inf_{t\geq 0} \min_{i\neq j} |q_i(t) - q_j(t)| \geq \ell_2^\infty > 0.$$

2. (Emergence of state-locking)*: Suppose that coupling strength κ is sufficiently strong in a sense that*

$$\kappa \geq \max_{1\leq j<N} \left\{ \frac{M_{G'}}{m_{G'}} \cdot \frac{N(\nu_{j+1} - \nu_j)(\alpha - 1)}{2} \right\}.$$

Then magnitudes of the relative states $|q_i - q_j|$ are uniformly bounded above:

$$\sup_{0 \le t < \infty} \max_{i,j} |q_i(t) - q_j(t)| < L_2^\infty < \infty.$$

Proof. (1) Let $J < N$ be a time-dependent index defined by the minimizer of relative distances:

$$q_{J+1}(t) - q_J(t) = \min_{1 \le j < N} (q_{j+1}(t) - q_j(t)).$$

Then, we can divide the time interval by $\mathbb{R}_{\ge 0} = \cup_{m=0}^\infty [t_m, t_{m+1})$ and choose J, if there is more than one minimizer of $q_{j+1}(t) - q_j(t)$, in a way that

$$J = J(t) \text{ is a constant in each interval } [t_m, t_{m+1}).$$

We will use an induction argument on m to obtain a uniform positive lower bound of $q_{J+1}(t) - q_j(t)$ in $[t_m, t_{m+1})$. For simplicity, we define

$$\Delta(t) := q_{J+1}(t) - q_J(t).$$

• Step A ($m = 0$): For $q \ge 0$ and $M \ge m > 0$, we recall that (3.1.3) holds except the origin, which yields:

$$\begin{aligned} 0 < \Psi(q+M) - \Psi(M) &= \int_M^{q+M} \psi(y)dy \\ &\le \int_m^{q+m} \psi(y)dy = \Psi(q+m) - \Psi(m). \end{aligned} \tag{3.4.2}$$

In contrast to (3.2.2), we assumed the positiveness of m since ψ is not integrable at the origin. As Δ is a minimum for $q_{j+1} - q_J$, we use (3.4.2) with

$$q = \Delta, \quad M = q_k - q_{J+1} \quad \text{and} \quad m = (k - J - 1)\Delta \quad \text{for} \quad k \ge J+2$$

to obtain

$$\Psi(q_k - q_J) - \Psi(q_k - q_{J+1}) \le \Psi((k-J)\Delta) - \Psi((k-J-1)\Delta).$$

Similarly for $k \le J - 1$, we obtain

$$\Psi(q_{J+1} - q_k) - \Psi(q_J - q_k) \le \Psi((J-k+1)\Delta) - \Psi((J-k)\Delta).$$

Consequently,

$$\sum_{k=J+2}^{N}\big(\Psi(q_k-q_J)-\Psi(q_k-q_{J+1})\big)\le\Psi((N-J)\Delta)-\Psi(\Delta),$$
$$\sum_{k=1}^{J-1}\big(\Psi(q_{J+1}-q_k)-\Psi(q_J-q_k)\big)\le\Psi(J\Delta)-\Psi(\Delta).$$

Therefore, it follows from (3.4.2) that

$$\begin{aligned}&\sum_{k=1}^{J-1}\big(\Psi(q_{J+1}-q_k)-\Psi(q_J-q_k)\big)+\sum_{k=J+2}^{N}\big(\Psi(q_k-q_J)-\Psi(q_k-q_{J+1})\big)\\&\quad\le\Psi((N-J)\Delta)+\Psi(J\Delta)-2\Psi(\Delta)\le 2\left(\Psi\left(\frac{N}{2}\Delta\right)-\Psi(\Delta)\right).\end{aligned}$$

This implies

$$\sum_{k=1}^{N}\big(\Psi(q_k-q_J)-\Psi(q_k-q_{J+1})\big)\le 2\Psi\left(\frac{N}{2}\Delta\right).$$

Similar to the estimate in (3.2.6), for some $y_J(t)\in(q_J(t),q_{J+1}(t))$, we have

$$\begin{aligned}\frac{d\Delta}{dt}&=\frac{d(q_{J+1}-q_J)}{dt}\\&=G'(y_J)\left(\nu_{J+1}-\nu_J+\frac{\kappa}{N}\sum_{k=1}^{N}\big(\Psi(q_k-q_{J+1})-\Psi(q_k-q_J)\big)\right)\\&\ge G'(y_J)\left(\nu_{J+1}-\nu_J-\frac{2\kappa}{N}\Psi\left(\frac{N}{2}\Delta\right)\right).\end{aligned}$$

Since G' has positive lower and upper bounds, the term Δ satisfies

$$\frac{d\Delta}{dt}\ge\begin{cases}m_{G'}(\nu_{J+1}-\nu_J)-\mathcal{M}, & \text{if } \nu_{J+1}>\nu_J,\\ M_{G'}(\nu_{J+1}-\nu_J)-\mathcal{M}, & \text{if } \nu_{J+1}\le\nu_J,\end{cases}$$

where

$$\mathcal{M}:=\frac{\kappa}{N}\left((M_{G'}-m_{G'})\left|\Psi\left(\frac{N}{2}\Delta\right)\right|+(m_{G'}+M_{G'})\Psi\left(\frac{N}{2}\Delta\right)\right).$$

For simplicity, we define $\tilde{\Psi} : \mathbb{R}_+ \to \mathbb{R}$ by

$$\tilde{\Psi}(q) := (M_{G'} - m_{G'}) \left| \Psi\left(\frac{N}{2}q\right) \right| + (m_{G'} + M_{G'})\Psi\left(\frac{N}{2}q\right).$$

Then $\tilde{\Psi}$ is a strictly increasing function for $q > 0$ and $\lim_{q\to\infty} \tilde{\Psi}(q) = M_{G'}\Psi^\infty$.

We set

$$u(x) := \begin{cases} m_{G'}(\nu_{J+1} - \nu_J) - \frac{\kappa}{N}\tilde{\Psi}(x), & \text{if } \nu_{J+1} \geq \nu_J, \\ M_{G'}(\nu_{J+1} - \nu_J) - \frac{\kappa}{N}\tilde{\Psi}(x), & \text{if } \nu_{J+1} \leq \nu_J. \end{cases}$$

Now, we consider a differential equation $y' = u(y)$ emanating from $t_0 = 0$ with positive initial data $y(t_0) = q^0_{J+1} - q^0_J$. Then, by the comparison principle, a lower bound of y becomes a lower bound of Δ in (t_0, t_1).

$\diamond$ (Case 1: $\nu_{J+1} > \nu_J$): We define

$$y^\infty := \begin{cases} \tilde{\Psi}^{-1}\left(\frac{m_{G'}N(\nu_{J+1}-\nu_J)}{\kappa}\right), & \text{if} \quad \frac{m_{G'}N(\nu_{J+1}-\nu_J)}{\kappa} \leq M_{G'}\Psi^\infty, \\ \infty, & \text{otherwise.} \end{cases}$$

By the definition of y^∞, we have $y^\infty > 0$ and $u(y) > 0$ holds for $y \in (0, y^\infty)$. Since the initial data $y(t_0)$ is positive, if $y(t_0) < y^\infty$, then y will increase until it achieve its equilibrium at $y = y^\infty$. In particular, if $y^\infty = \infty$, then y will never decrease. If $y(t_0) \geq y^\infty$, y will decrease until it achieve its equilibrium at y^∞. Hence we conclude

$$\inf_{t\in(t_0,t_1)} \Delta(t) \geq \inf_{t\in(t_0,t_1)} y(t) \geq \min\{\Delta(t_0), y^\infty\}.$$

$\diamond$ (Case 2: $\nu_{J+1} \leq \nu_J$): By the same argument as above, we have the same result except $m_{G'}$ is replaced by $M_{G'}$:

$$\inf_{t\in(t_0,t_1)} \Delta(t) \geq \min\{\Delta(t_0), z^\infty\},$$

$$\text{where} \quad z^\infty := \begin{cases} \tilde{\Psi}^{-1}\left(\frac{M_{G'}N(\nu_{J+1}-\nu_J)}{\kappa}\right), & \text{if} \quad \frac{m_{G'}N(\nu_{J+1}-\nu_J)}{\kappa} \leq M_{G'}\Psi^\infty, \\ \infty, & \text{otherwise.} \end{cases}$$

Finally, we combine all the estimates in Case 1 and Case 2 to obtain

$$\inf_{t\in(t_0,t_1)} \Delta(t) \geq \min\{\Delta(t_0), y^\infty, z^\infty\}.$$

• Step B (Inductive step): Since y^∞ and z^∞ can be characterized in terms of ν_j and ν_{j+1} ($j = 1, 2, \cdots, N-1$), they only depend on the index j. Thus, we define

$$Y^\infty := \min_{1\leq j<N} y^\infty(j), \quad Z^\infty := \min_{1\leq j<N} z^\infty(j).$$

Our we claim is that the following lower bound for $\Delta(t)$ on the time interval (t_0, t_{k+1}) holds:

$$\inf_{t\in(t_0,t_{k+1})} \Delta(t) \geq \min\{\Delta(0), Y^\infty, Z^\infty\} > 0 \tag{3.4.3}$$

for arbitrary non-negative integer k. Since the case for $k = 0$ is already proven in Step A, we only need to show that (3.4.3) holds for $k = m + 1$ under the assumption that (3.4.3) holds for $k = m$. By using the same argument as in Step A, we have

$$\inf_{t\in(t_{m+1},t_{m+2})} \Delta(t) \geq \min\{\Delta(t_{m+1})\ ,\ Y^\infty\ ,\ Z^\infty\} > 0.$$

for any non-negative integer m. However, since (3.4.3) holds for $k = m$, we have

$$\Delta(t_{m+1}) \geq \min\{\Delta(0), Y^\infty, Z^\infty\}.$$

This implies

$$\inf_{t\in(t_{m+1},t_{m+2})} \Delta(t) \geq \min\{\Delta(0), Y^\infty, Z^\infty\}.$$

Therefore, we obtain

$$\begin{aligned}\inf_{t\in(t_0,t_{m+2})} \Delta(t) &\geq \min\left\{\inf_{t\in(t_{m+1},t_{m+2})} \Delta(t), \inf_{t\in(t_0,t_{m+1})} \Delta(t)\right\}\\ &\geq \min\{\Delta(0), Y^\infty, Z^\infty\} =: \ell_2^\infty.\end{aligned}$$

Hence, we complete the induction argument, and obtain the desired lower bound of the relative distances.

(2) Since the proof uses a similar argument as in (1), we just provide a sketch of the arguments. Suppose now that $J(t) < N$ is an index that maximizing the relative distances $q_{J+1} - q_J$ and redefine $\Delta(t) := q_{J+1} - q_J$. We only consider the time interval $[0, t_1]$, on which the index J is constant. Then again by (3.4.2), similar estimate as in (1) yields

$$\frac{d\Delta}{dt} \leq G'(y_J)\left(\nu_{J+1} - \nu_J - \frac{\kappa}{N}\Psi\left((N-1)\Delta\right)\right).$$

and, we obtain a similar upper bound for Δ as

$$\frac{d\Delta}{dt} \leq \begin{cases} M_{G'}(\nu_{J+1} - \nu_J) - \mathcal{M}', & \text{if } \nu_{J+1} > \nu_J, \\ m_{G'}(\nu_{J+1} - \nu_J) - \mathcal{M}', & \text{if } \nu_{J+1} \leq \nu_J, \end{cases}$$

where

$$\mathcal{M}' := \frac{\kappa}{N}\left(|\Psi((N-1)\Delta)|\frac{m_{G'} - M_{G'}}{2} + \Psi((N-1)\Delta)\frac{m_{G'} + M_{G'}}{2}\right).$$

• Case A ($\nu_{J+1} > \nu_J$): For simplicity, we define

$$\tilde{\Psi}(q) := |\Psi((N-1)q)|\frac{m_{G'} - M_{G'}}{2} + \Psi((N-1)q)\frac{m_{G'} + M_{G'}}{2},$$

which is a strictly increasing function with $\lim_{x\to\infty} \tilde{\Psi}(x) = m_{G'}\Psi^{\infty}$.

Then, a simple comparison principle implies

$$\Delta(t) \leq \max\left\{\Delta(0), \tilde{\Psi}^{-1}\left(\frac{M_{G'}N(\nu_{J+1} - \nu_J)}{\kappa}\right)\right\}, \quad 0 \leq t \leq t_1$$

if $\kappa \geq \frac{M_{G'}N(\nu_{J+1}-\nu_J)}{2m_{G'}\Psi^{\infty}}$.

• Case B ($\nu_{J+1} \leq \nu_J$): It follows from the governing dynamics that

$$\Delta(t) \leq \max\left\{\Delta(0), \tilde{\Psi}^{-1}\left(\frac{m_{G'}N(\nu_{J+1} - \nu_J)}{\kappa}\right)\right\}, \quad 0 \leq t \leq t_1.$$

Note that $\tilde{\Psi}^{-1}\left(\frac{m_{G'}N(\nu_{J+1}-\nu_J)}{\kappa}\right)$ is well-defined if $\nu_{J+1} \leq \nu_J$.

Therefore, if $\kappa \geq \max_{1\leq j<N} \frac{M_{G'}N(\nu_{J+1}-\nu_J)}{2m_{G'}\Psi^\infty}$, a similar induction argument as in (1) can be applied to prove the uniform-in-time upper bound for Δ:

$$\begin{aligned}\sup_{t\geq 0}\Delta(t) &\leq \max_{1\leq J<n}\left\{\Delta(0), \tilde{\Psi}^{-1}\left(\frac{M_{G'}N(\nu_{J+1}-\nu_J)}{\kappa}\right), \tilde{\Psi}^{-1}\left(\frac{m_{G'}N(\nu_{J+1}-\nu_J)}{\kappa}\right)\right\}\\ &=: L_2^\infty < \infty.\end{aligned}$$

Finally, since Ψ^∞ can be explicitly calculated as

$$\Psi^\infty = \int_1^\infty \frac{dq}{q^\alpha} = \begin{cases} \frac{1}{\alpha-1}, & \text{if } \alpha > 1, \\ \infty, & \text{if } \alpha = 1, \end{cases}$$

we have the desired result. This completes the proof of Theorem 3.4.2.

□

Remark 3.4.2. *Below, we comments on the results of Theorem 3.4.2 as follows:*

1. *In the proof of Theorem 3.4.2 for the first statement, the only required property for the kernel ψ is non-integrability at the origin. Hence, the first statement can be generalized as following; suppose that the kernel is not integrable only at the origin as*

$$\int_{-\varepsilon}^{\varepsilon}\psi(q)\,dq = \infty \quad \text{and} \quad \psi\mathbb{1}_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \in L^1_{\text{loc}}(\mathbb{R}), \quad \text{for each } \varepsilon > 0.$$

   Then, for any two particles q_j and q_{j+1} $(j < N)$, there exists a uniform-in-time lower bound of relative distances: there exists a positive constant ℓ_2^∞ such that

$$\inf_{t\geq 0}\min_{i,j}|q_i(t) - q_j(t)| \geq \ell_2^\infty > 0,$$

   where ℓ_2^∞ is defined in the proof of Theorem 3.4.2. In particular, collision does not occur.

2. *Similarly, the second statement can be generalized as follows; for $\Psi^\infty := \int_1^\infty \psi(q)dq$, if coupling strength κ is sufficiently strong in a sense that*

$$\kappa \geq \max_{1\leq j<N}\frac{M_{G'}}{m_{G'}}\cdot\frac{N}{2}\cdot\frac{(\nu_{j+1}-\nu_j)}{\Psi^\infty},$$

then relative distance $|q_i - q_j|$ is uniformly bounded above: there exists a positive constant L_2^∞ such that

$$\sup_{t\in\mathbb{R}^+} \max_{i,j} |q_i(t) - q_j(t)| \le L_2^\infty < \infty,$$

where L_2^∞ is defined in the proof of Theorem 3.4.2. In particular, long-ranged communication kernel exhibits uniform-in-time upper bound of relative states. Also note that, contrast to the previous cases, we do not assume ordering (3.1.13)*: ν_i need not be arranged in an increasing order of indices.*

3.5 Structural stability

In this section, we present the structural stability of (3.1.5), when the activation function G converges to the identity map. More precisely, we consider the following one-parameter family of system (3.1.5):

$$\dot{q}_i^\varepsilon = G_\varepsilon \left(\nu_i + \frac{\kappa}{N} \sum_{k=1}^N \Psi(q_k^\varepsilon - q_i^\varepsilon) \right), \quad i \in [N], \tag{3.5.1}$$

where $\varepsilon > 0$ is a positive constant, and $G_\varepsilon : \mathbb{R} \to \mathbb{R}$ satisfies the following properties: there exist one-parameter family of sequences $\{m_{G'_\varepsilon}\}$ and $\{M_{G'_\varepsilon}\}$ such that

$$\begin{cases} G_\varepsilon(-q) = -G_\varepsilon(q), \quad 0 < m_{G'_\varepsilon} \le G'_\varepsilon(q) \le M_{G'_\varepsilon}, \quad q \in \text{domain of } D_\varepsilon, \\ \lim\limits_{\varepsilon\to 0} m_{G'_\varepsilon} = \lim\limits_{\varepsilon\to 0} M_{G'_\varepsilon} = 1. \end{cases} \tag{3.5.2}$$

As $\varepsilon \to 0+$, one can expect $\lim_{\varepsilon\to 0} G_\varepsilon(q) = q$ pointwise, and thus (3.5.1) converges to the following system:

$$\dot{q}_i = \nu_i + \frac{\kappa}{N} \sum_{k=1}^N \Psi(q_k - q_i), \quad t > 0, \quad i \in [N]. \tag{3.5.3}$$

Then, a sequence of solution $\{q_i^\varepsilon\}$ to (3.5.1) may converges to the solution to (3.5.3). For simplicity, we only consider the case, when the communication weight is regular.

Proposition 3.5.1. *For $T \in (0, \infty)$, let $\{q_i^\varepsilon\}$ and $\{q_i\}$ be solutions to* (3.5.1) *and* (3.5.2) *with the common initial data $\{q_i^0\}$ in the time interval $[0, T]$, respectively. Suppose initial data is ordered as* (3.1.13). *If natural velocity satisfies zero-zum condition $\sum_{i=1}^N \nu_i = 0$, then one has the following assertions:*

1. (Finite-in-time convergence)*: $\{q_i^e\}$ converges to $\{q_i\}$ in any finite-time interval $[0, T]$:*

$$\lim_{\varepsilon \to 0} \sup_{0 \le t \le T} \max_{1 \le i \le N} |q_i^\varepsilon(t) - q_i(t)| = 0.$$

2. (Uniform-in-time convergence)*: If Ψ is bounded and long-ranged, convergence can be made uniformly in time:*

$$\lim_{\varepsilon \to 0} \sup_{t \ge 0} \max_{1 \le i \le N} |q_i^\varepsilon(t) - q_i(t)| = 0.$$

Proof. Throughout the proof, we denote C by a positive generic constant, independent of ε and t.

(1) It follows from (3.5.1) and (3.5.3) that the difference $q_i^\varepsilon - q_i$ satisfies

$$\begin{aligned}
\frac{d}{dt}(q_i^\varepsilon - q_i) &= G_\varepsilon\left(\nu_i + \frac{\kappa}{N}\sum_{k=1}^N \Psi(q_k^\varepsilon - q_i^\varepsilon)\right) - \left(\nu_i + \frac{\kappa}{N}\sum_{k=1}^N \Psi(q_k - q_i)\right) \\
&= G_\varepsilon\left(\nu_i + \frac{\kappa}{N}\sum_{k=1}^N \Psi(q_k^\varepsilon - q_i^\varepsilon)\right) - \left(\nu_i + \frac{\kappa}{N}\sum_{k=1}^N \Psi(q_k^\varepsilon - q_i^\varepsilon)\right) \\
&+ \left(\nu_i + \frac{\kappa}{N}\sum_{k=1}^N \Psi(q_k^\varepsilon - q_i^\varepsilon)\right) - \left(\nu_i + \frac{\kappa}{N}\sum_{k=1}^N \Psi(q_k - q_i)\right) \\
&=: \mathcal{I}_{21} + \mathcal{I}_{22}.
\end{aligned}$$

Below, we provide estimates for $\mathcal{I}_{2i},\ i = 1, 2$ one by one.

• (Estimate of $\mathcal{I}_{21}$): First note that for $q \in \mathbb{R}$,

$$|G_\varepsilon(q) - q| \le \int_0^{|q|} |G'_\varepsilon(y) - 1|\, dy \le \max\{|m_{G'_\varepsilon} - 1|, |M_{G'_\varepsilon} - 1|\}|q|.$$

Therefore, we estimate $\mathcal{I}_{21}$ as

$$\mathcal{I}_{21} \leq \max\{|m_{G'_\varepsilon} - 1|, |M_{G'_\varepsilon} - 1|\} \left| \nu_i + \frac{\kappa}{N} \sum_{k=1}^{N} \Psi(q_k^\varepsilon - q_i^\varepsilon) \right|.$$

We split the analysis for $\mathcal{I}_{21}$ according to the types of communication weight.

$\diamond$ (Bounded and long-ranged Ψ): It follows from Theorem 3.2.1 that the relative states are bounded by

$$|q_k^\varepsilon - q_l^\varepsilon| \leq \max\left\{ q_N^0 - q_1^0,\ \Psi^{-1}\left(\frac{M_{G',\varepsilon}}{m_{G',\varepsilon}} \cdot \frac{\nu_N - \nu_1}{\kappa} \right) \right\}.$$

However, since

$$\lim_{\varepsilon \to 0} m_{G'_\varepsilon} = \lim_{\varepsilon \to 0} M_{G'_\varepsilon} = 1,$$

there exists a constant q^∞, independent to ε, such that

$$\sup_{0 \leq t < \infty} \max_{1 \leq k,l \leq N} |q_k^\varepsilon(t) - q_l^\varepsilon(t)| < q^\infty, \quad \text{for all} \quad \varepsilon \ll 1.$$

Therefore, in this case,

$$\sup_{0 \leq t < \infty} \left| \nu_i + \frac{\kappa}{N} \sum_{i=1}^{N} \Psi(q_k^\varepsilon - q_i^\varepsilon) \right| \leq |\nu_i| + \kappa \Psi(q^\infty),$$

and we have

$$\mathcal{I}_{21} \leq C \max\{|m_{G'_\varepsilon} - 1|, |M_{G'_\varepsilon} - 1|\}.$$

$\diamond$ (Bounded and short-ranged Ψ): In this case, there exists an upper bound Ψ^∞ for Ψ, and thus,

$$\mathcal{I}_{21} \leq (|\nu_i| + \kappa \Psi^\infty) \max\{|m_{G'_\varepsilon} - 1|, |M_{G'_\varepsilon} - 1|\} \leq C \max\{|m_{G'_\varepsilon} - 1|, |M_{G'_\varepsilon} - 1|\}.$$

Therefore, in any case, we can bound $\mathcal{I}_{21}$ as

$$\mathcal{I}_{21} \leq C \max\{|m_{G'_\varepsilon} - 1|, |M_{G'_\varepsilon} - 1|\},$$

where C does not depend on ε.

• (Estimate of $\mathcal{I}_{22}$): We directly estimate $\mathcal{I}_{22}$ as

$$\begin{aligned}\mathcal{I}_{22} &= \frac{\kappa}{N}\sum_{k=1}^{N}(\Psi(q_k^\varepsilon - q_i^\varepsilon) - \Psi(q_k - q_i)) \le \frac{\kappa\|\Psi'\|_{L^\infty}}{N}\sum_{k=1}^{N}|(q_k^\varepsilon - q_k) - (q_i^\varepsilon - q_i)| \\ &\le 2\kappa\|\psi\|_{L^\infty}\max_{1\le i\le N}|q_i^\varepsilon(t) - q_i(t)|.\end{aligned}$$

Now, we set

$$\mathcal{Q}^\varepsilon(t) := \max_{1\le i\le N}|q_i^\varepsilon(t) - q_i(t)|.$$

Then, we combine all the estimates for $\mathcal{I}_{21}$ and $\mathcal{I}_{22}$ to obtain

$$\frac{d\mathcal{Q}^\varepsilon}{dt} \le C\max\{|m_{G'_\varepsilon} - 1|, |M_{G'_\varepsilon} - 1|\} + C\mathcal{Q}^\varepsilon, \quad \text{a.e. } t > 0.$$

Since $\mathcal{Q}^\varepsilon(0) = 0$, we have

$$\mathcal{Q}^\varepsilon(t) \le \max\{|m_{G'_\varepsilon} - 1|, |M_{G'_\varepsilon} - 1|\}\left(e^{Ct} - 1\right).$$

Therefore, we obtain

$$\lim_{\varepsilon\to 0}\mathcal{Q}^\varepsilon(t) = 0 \quad \text{for any } 0 \le t \le T.$$

(2) We now prove the uniform-in-time convergence, when ψ is bounded and long-ranged. First of all, we consider the second-order models:

$$\begin{cases}\dfrac{dq_i^\varepsilon}{dt} = G_\varepsilon(p_i^\varepsilon), \quad t > 0, \\ \dfrac{dp_i^\varepsilon}{dt} = \dfrac{\kappa}{N}\displaystyle\sum_{k=1}^{N}\psi(q_k^\varepsilon - q_i^\varepsilon)(G_\varepsilon(p_k^\varepsilon) - G_\varepsilon(p_i^\varepsilon)).\end{cases}$$

For sufficiently small ε, since

$$\lim_{\varepsilon\to 0} m_{G'_\varepsilon} = \lim_{\varepsilon\to 0} M_{G'_\varepsilon} = 1,$$

we may assume that L_1^∞ in Theorem 3.2.1 satisfies

$$\psi(L_1^\infty) > \frac{M_{G'_\varepsilon} - m_{G'_\varepsilon}}{m_{G'_\varepsilon}}.$$

Then it follows from Lemma 3.2.1 that

$$\begin{aligned}\max_{i,j} |p_i^\varepsilon(t) - p_j^\varepsilon(t)| &\le \max_{i,j} |p_i^0 - p_j^0| \exp\left(-(m_{G'}\psi(L_1^\infty) - (M_{G'} - m_{G'}))t\right)\\ &\le C\exp(-\delta t)\end{aligned}$$

for some $\delta > 0$. Therefore, for arbitrary time s, t $(0 < s < t)$, one has

$$\begin{aligned}|q_i^\varepsilon(t) - q_i(t)| &\le |q_i^\varepsilon(s) - q_i(s)| + \int_s^t |G_\varepsilon(p_i^\varepsilon)(\tau) - p_i(\tau)| d\tau\\ &\le |q_i^\varepsilon(s) - q_i(s)| + C\int_s^t \exp(-\delta\tau)d\tau\\ &\le |q_i^\varepsilon(s) - q_i(s)| + C\exp(-\delta s).\end{aligned}$$

Taking the maximum over indices, we obtain

$$\mathcal{Q}^\varepsilon(t) \le \mathcal{Q}^\varepsilon(T) + \exp(-\delta T), \quad \forall\ 0 < T < t.$$

Now we fix $\delta > 0$ and take a $T = T(\delta)$ satisfying $\exp(-\delta T) < \delta$ to deduce

$$\mathcal{Q}^\varepsilon(t) \le \mathcal{Q}^\varepsilon(T) + \delta.$$

Therefore, we take a supremum over $t > T$ to find

$$\sup_{t>T} \mathcal{Q}^\varepsilon(t) \le \mathcal{Q}^\varepsilon(T) + \delta,$$

Then we utilize finite-time convergence estimate in (1) to conclude

$$\lim_{\varepsilon\to 0} \sup_{t>0} \mathcal{Q}^\varepsilon(t) \le \delta.$$

Since δ is arbitrary small, we achieve the desired result.

□

Remark 3.5.1. *For the choice $G_\varepsilon := \hat{v}$ with the parameter $\varepsilon := c^{-1}$, the result in Proposition 3.5.1 yields the nonrelativistic limit of the RCS model [4] on the real line.*

Chapter 4

Asymptotic dynamics of the CS-type consensus model

4.1 Regular communication weight

In this chapter, we introduce and study asymptotic flocking behaviors of CS-type consensus model, which involves an activation function G. We use the following handy notation throughout the chapter:

$$\psi_{ij} := \psi(|q_i - q_j|), \quad \inf_{t\geq 0} := \inf_{0\leq t<\infty}, \quad \sup_{t\geq 0} := \sup_{0\leq t<\infty},$$
$$[n] := \{1, 2, \cdots, n\}, \ n \in \mathbb{N}, \quad \mathbb{R}_{\geq 0} := \{x \mid x \geq 0\}, \quad \mathbb{R}_+ := \{x \mid x > 0\}.$$

For configuration vectors $q_i \in \mathbb{R}^d$ and $p_i \in \mathbb{R}^d$, we denote

$$Q(t) := (q_1(t), \ldots, q_N(t)), \quad P(t) := (p_1(t), \ldots, p_N(t)),$$
$$Q^0 := Q^0, \quad P^0 := P^0, \quad \mathcal{N} := (\nu_1, \nu_2, \cdots, \nu_N).$$

For $S \subset [N]$, we define the norms on a (sub)system of $\{p_i - p_j\}_{i,j\in[N]}$ and $\{q_i - q_j\}_{i,j\in[N]}$ as

$$\|Q\|_S := \sqrt{\sum_{i,j\in S} |q_i - q_j|^2}, \quad \|P\|_S := \sqrt{\sum_{i,j\in S} |p_i - p_j|^2},$$
$$\|Q\| := \|Q\|_{[N]} \quad \|P\| := \|P\|_{[N]},$$

$$D_{P,S} := \max_{i,j\in S} |p_i - p_j|, \quad D_{Q,S} := \max_{i,j\in S} |p_i - p_j|$$
$$D_P := D_{P,[N]}, \quad D_Q := D_{Q,[N]}.$$

We note that this chapter is based on the work [9].

4.1.1 The CS-type consensus model.

In this subsection, we impose several conditions on an activation function G and study its implication to the dynamics. We recall the CS-type model:

$$\begin{cases} \dot{q}_i = G(p_i), \quad t > 0, \quad i \in [N], \\ \dot{p}_i = \dfrac{\kappa}{N} \displaystyle\sum_{k=1}^{N} \psi(|q_k - q_i|)(G(p_k) - G(p_i)), \\ (q_i, p_i)\big|_{t=0+} = (q_i^0, p_i^0), \quad p_i, q_i \in \mathbb{R}^d, \end{cases} \tag{4.1.1}$$

where $\kappa > 0$. Throughout Chapter 4, we assume

$$\psi \in (L^\infty \cap C^{0,1})(\mathbb{R}_+; \mathbb{R}_+), \quad (\psi(r) - \psi(s))(r - s) \le 0, \quad \forall r, s \in \mathbb{R}_+.$$

In (4.1.1), G is assumed to be radially symmetric. More precisely, we assume:

$$\begin{aligned} &G(p) = \begin{cases} g(|p|)\dfrac{p}{|p|} & , \quad \text{if} \quad p \neq 0, \\ 0 & , \quad \text{if} \quad p = 0, \end{cases} \qquad g \in C^1(\mathbb{R}_{\ge 0}), \quad g(0) = 0, \\ &0 < m_{g'} \le g' \le M_{g'} \text{ on any compact interval,} \\ &g \text{ is convex or concave on } \mathbb{R}_+, \end{aligned} \tag{4.1.2}$$

where $m_{g'}$ and $M_{g'}$ may depends on a compact interval.

Example 4.1.1. *We address some possible examples of activation function G.*

1. ***The CS model.*** *The simplest and most motivating example for G is the identity mapping $G(p) = p$. In this case, system* (4.1.1) *represents the standard CS model [25, 26].*

2. ***Speed limit model.*** *Suppose that* G *is bounded, say* $g(\mathbb{R}) = [0, M),\ M < \infty$. *Then we have*
$$|\dot{q}_i| = |G(p_i)| < M,$$
and the maximum speed of agents is always bounded by M. *This may not feature out for the model* (4.1.1) *since maximal speed always decrease in this case (see Proposition 4.1.1). However, despite the presence of extra force, which may increase the maximal speed(e.g., random noise [3] or bonding force [2]), we can still guarantee the speed limitation.*

3. ***Physical models.*** *Several physical effects can be reflected by the suitable choice of* G. *For example, if we involve the Lorentz factor* Γ *as follows:*
$$g^{-1} : [0, c) \mapsto \mathbb{R}, \quad g(v) := \Gamma\left(1 + \frac{\Gamma}{c^2}\right)v, \quad \Gamma := \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}},$$
then the model (4.1.1) *becomes tha relativistic Cucker-Smale (RCS) model, which is introduced as the relativistic correction of the CS model. For the derivation and emergent dynamics of the RCS model, we refer to [4, 29]. Other than relativistic effects, physical semantics like proper velocity or rapidity can be reflected [10, 41].*

4. ***Almost unit speed model.*** *In literature, several Vicsek-type models with a unit speed constraint have been studied in terms of the heading angle. For the CS model with unit speed, refer to [16]. In terms of* (4.1.1), *this might be represented by choice of* $g_0 \equiv 1$ *on* $\mathbb{R}_+$. *This does not fulfill* (4.1.2), *but can be approximated by functions satisfying* (4.1.2). *For example, for* $g_\varepsilon(p) := \tanh(p/\varepsilon)$, *we expect*
$$g_\varepsilon \xrightarrow{\varepsilon \searrow 0} g_0 = 1 \text{ on } \mathbb{R}_+,$$
and we formally have a close-to-unit speed model under $\varepsilon \ll 1$. *Compared to the model in [16], the above model does not strictly have unit speed but has the advantage of applying methodology consistent with the standard CS model.*

Technical reason for conditions (4.1.2) will naturally rise up in the following proposition.

Proposition 4.1.1. *Let* (P, Q) *be a global solution to* (4.1.1) *with initial data* (P^0, Q^0)*. Then the following holds.*

1. $\sum_{k=1}^N p_j$ *is conserved:*
$$\frac{d}{dt}\sum_{k=1}^N p_j(t) = 0.$$

2. *Maximum modulus of* p_i *decrease in time:*
$$\max_{i\in[N]}|p_i(t)| \le \max_{i\in[N]}|p_i(s)|, \quad 0 \le s \le t.$$
In particular,
$$\sup_{t\ge 0}\max_{i\in[N]}|p_i(t)| = \max_{i\in[N]}|p_i^0| =: P_M^0. \tag{4.1.3}$$

3. *For any* $t \in \mathbb{R}_+$ *and* $i, j \in [N]$,
$$m_{G'}|p_i(t) - p_j(t)| \le |G(p_i(t)) - G(p_j(t))| \le M_{G'}|p_i(t) - p_j(t)|,$$
where
$$M_{G'} := \max\{g'(|p|) : |p| \le P_M^0\}, \quad m_{G'} := \min\{g'(|p|) : |p| \le P_M^0\}.$$

4. *There exists a positive constant* $\mathcal{M} = \mathcal{M}(P^0) > 0$ *satisfying*
$$\mathcal{M}|p_i(t) - p_j(t)|^2 \le (p_i(t) - p_j(t)) \cdot (G(p_i(t)) - G(p_j(t)))$$
for any $t \in \mathbb{R}_+$ *and* $i, j \in [N]$.

Proof. (1) We sum $(4.1.1)_2$ over $i \in [N]$ and utilize the index symmetry to see
$$\frac{d}{dt}\sum_{k=1}^N p_j = \frac{\kappa}{N}\sum_{i,k=1}^N \psi(|q_k - q_i|)(G(p_k) - G(p_i))$$

$$= \frac{\kappa}{N} \sum_{i,k=1}^{N} \psi(|q_i - q_k|)(G(p_i) - G(p_k)) = 0.$$

(2) Define $M(t) \in \arg\max_{i\in[N]} |p_i(t)|$. Let time t and index $M(t)$ be fixed, and set $\ell = M(t)$. Then we have

$$\frac{d}{dt}|p_\ell|^2 = \frac{\kappa}{N} \sum_{k=1}^{N} \psi(|q_k - q_\ell|) p_\ell \cdot (G(p_k) - G(p_\ell)) \le 0,$$

where the inequality holds from the maximality of M. This proves (4.1.3).
(3) The Jacobian of G at p is

$$G'(p) = \begin{cases} \dfrac{g(|p|)}{|p|}\mathrm{Id} + \left(g'(|p|) - \dfrac{g(|p|)}{|p|}\right)\dfrac{p\otimes p}{|p|^2} & , \quad \text{if} \quad p \neq 0, \\ g'(0)\mathrm{Id} & , \quad \text{if} \quad p = 0, \end{cases}$$

Since the eigenvalues of $p \otimes p$ are 0 and $|p|^2$ up to multiplicity, eigenvalues of G' are

$$\lambda_1 = \begin{cases} \dfrac{g(|p|)}{|p|}, & \text{if} \quad p \neq 0, \\ g'(0), & \text{if} \quad p = 0, \end{cases} \qquad \lambda_2 = g'(|p|)$$

Due to symmetry of G', the largest eigenvalue is the operator norm of G', and this is bounded by $M_{G'}$ from (2). Therefore the mean value theorem implies

$$|G(p_i) - G(p_j)| \le M_{G'}|p_i - p_j|.$$

On the other hand, operator norm of inverse Jacobian $(G^{-1})'$ is $\frac{1}{\min\{\lambda_1,\lambda_2\}}$, which is less or equal to $\frac{1}{m_{G'}}$. Therefore we have

$$m_{G'}|p_i - p_j| \le |G(p_i) - G(p_j)| \le M_{G'}|p_i - p_j|.$$

(4) Throughout the proof, without the loss of generality, we assume $|p_i| \ge |p_j|$. First suppose that g is convex on $\mathbb{R}_+$, so that $|\cdot| \mapsto \frac{g(|\cdot|)}{|\cdot|}$ is an increasing function. Then,

$$(p_i - p_j)\cdot(G(p_i) - G(p_j))$$

$$
\begin{aligned}
&= \frac{g(|p_j|)}{|p_j|}|p_i - p_j|^2 + \left(\frac{g(|p_i|)}{|p_i|} - \frac{g(|p_j|)}{|p_j|}\right)(p_i - p_j)\cdot p_i \\
&\geq m_{G'}|p_i - p_j|^2.
\end{aligned}
$$

Now suppose that g is concave on $\mathbb{R}_+$. Since g is increasing, g^{-1} is convex and this yields

$$
\begin{aligned}
&(p_i - p_j)\cdot(G(p_i) - G(p_j)) \\
&\quad = \frac{g^{-1}(|G(p_j)|)}{|G(p_j)|}|G(p_i) - G(p_j)|^2 \\
&\quad\quad + \left(\frac{g^{-1}(|G(p_i)|)}{|G(p_i)|} - \frac{g^{-1}(|G(p_j)|)}{|G(p_j)|}\right)(G(p_i) - G(p_j))\cdot G(p_i) \\
&\quad \geq \frac{m_{G'}^2}{M_{G'}}|p_i - p_j|^2.
\end{aligned}
$$

We pose $\mathcal{M} := \min\{m_{G'}, \frac{m_{G'}^2}{M_{G'}}\}$ to complete the proof. Since $m_{G'}$ and $M_{G'}$ depends only on P_0, so is $\mathcal{M}$. □

Remark 4.1.1.

1. *In* (4.1.2), *$m_{g'}$ and $M_{g'}$ depend on the interval. However, thanks to the uniform-in-time boundedness of $|p_i|$, we can fix the interval by $[0, P_M^0]$, and this enables us to fix $m_{g'}$ and $M_{g'}$ according to the initial data, namely $M_{G'}$ and $m_{G'}$.*

2. *In Proposition 4.1.1, if an ambient space of* (4.1.1) *is one-dimensional* $(d = 1)$*, then the inner product is merely a scalar multiplication, and (3) proves (4) without convexity or concavity assumption on g.*

4.1.2 Emergence of asymptotic flocking.

The CS model is one of the most successful models designing the flocking behavior, and we can still expect the emergent flocking of (4.1.1) as well. We recall the definition of (asymptotic) flocking for a model (4.1.1).

Definition 4.1.1. *Let* (P, Q) *be a global solution to* (4.1.1).

1. *We say that (P,Q) exhibits a (mono-cluster) flocking if*

$$\sup_{t\geq 0} D_Q(t) < \infty, \quad \lim_{t\to\infty} D_P(t) = 0.$$

2. *We say that (P,Q) exhibits a bi-cluster flocking if there exists a nonempty proper subset S of $[N]$ satisfying*

$$\sup_{t\geq 0} \max\{D_{Q,S}(t), D_{Q,[N]-S}(t)\} < \infty, \quad \sup_{t\geq 0} \min_{\substack{i\in S\\ j\notin S}} |q_i(t) - q_j(t)| = \infty,$$
$$\lim_{t\to\infty} D_{P,S}(t) = \lim_{t\to\infty} D_{P,[N]-S}(t) = 0.$$

In the following lemma, we estimate the relative distance for an arbitrary collection of agents, which will be used repeatedly through Section 4.1, Section 4.2, and Section 4.3.

Lemma 4.1.1 (Subsystem estimation)**.** *Let (P,Q) be a solution to* (4.1.1)*. For any $[l] \subset [N]$, we have the following differential inequalities.*

1.

$$\frac{d}{dt}\|P\|_{[l]} \leq -\frac{\kappa \mathcal{M} l}{N}\psi(\|Q\|_{[l]})\|P\|_{[l]} + \frac{2\kappa M_{G'}(N-l)P_M^0 L_{\psi,[l]}}{N}\|Q\|_{[l]},$$

$$L_{\psi,[l]}(t) := \sup_{\substack{r,s\geq q_{[l]}(t),\\ r\neq s}} \left|\frac{\psi(r)-\psi(s)}{r-s}\right| < \infty, \quad q_{[l]}(t) := \min_{\substack{i'\in[l]\\ j'\notin[l]}} |q_{i'}(t) - q_{j'}(t)|. \tag{4.1.4}$$

2.

$$\frac{d}{dt}\|P\|_{[l]} \leq -\frac{\kappa \mathcal{M} l}{N}\psi(\|Q\|_{[l]})\|P\|_{[l]} + \frac{4\kappa P_M^0 M_{G'} l(N-l)}{N} \max_{\substack{i'\in[l]\\ j'\notin[l]}} \psi_{i'j'}.$$

Proof. (1) For simplicity, let $p_{ij} := p_i - p_j$. We expand $\|P\|_{[l]}$ as

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|P\|_{[l]}^2 &= \sum_{i,j\in[l]} p_{ij}\cdot\left(\frac{dp_i}{dt} - \frac{dp_j}{dt}\right)\\ &= \frac{\kappa}{N}\sum_{i,j\in[l]}\sum_{k=1}^{N}\underbrace{p_{ij}\cdot\left(\psi_{ki}(G(p_k)-G(p_i)) - \psi_{kj}(G(p_k)-G(p_j))\right)}_{=:\mathcal{A}_{ijk}}\\ &= \frac{\kappa}{N}\sum_{i,j,k\in[l]}\mathcal{A}_{ijk} + \frac{\kappa}{N}\sum_{i,j\in[l],k\notin[l]}\mathcal{A}_{ijk}\\ &=: \frac{\kappa\mathcal{I}_1}{N} + \frac{\kappa\mathcal{I}_2}{N}.\end{aligned}$$

For the estimate of $\mathcal{I}_1$, we utilize the symmetry of indices to use the index switching trick $(i,j,k)\to(j,k,i)$ to obtain

$$\begin{aligned}\mathcal{I}_1 &= \sum_{i,j,k\in[l]}\psi_{ki}p_{ij}\cdot(G(p_k)-G(p_i)) - \sum_{i,j,k\in[l]}\psi_{kj}p_{ij}\cdot(G(p_k)-G(p_j))\\ &= \sum_{i,j,k\in[l]}\psi_{ki}p_{ij}\cdot(G(p_k)-G(p_i)) + \sum_{i,j,k\in[l]}\psi_{ki}p_{jk}\cdot(G(p_k)-G(p_i))\\ &= \sum_{i,j,k\in[l]}\psi_{ki}p_{ik}\cdot(G(p_k)-G(p_i))\\ &\le -\mathcal{M}\sum_{i,j,k\in[l]}\psi_{ki}|p_{ik}|^2 \le -\mathcal{M}l\psi(\|Q\|_{[l]})\|P\|_{[l]}^2.\end{aligned}$$

For the estimate of $\mathcal{I}_2$, we use the Lipschitz continuity of ψ to get

$$|\psi_{ki}-\psi_{kj}| \le L_{\psi,[l]}(t)\big||q_k-q_i| - |q_k-q_j|\big| \le L_{\psi,[l]}(t)|q_i-q_j|,$$

where $L_{\psi,[l]}$ is a nonnegative function defined as (4.1.4). Then a direct computation yields

$$\begin{aligned}\mathcal{I}_2 &= \sum_{i,j\in[l],k\notin[l]} p_{ij}\cdot\left(\psi_{ki}(G(p_k)-G(p_i)) - \psi_{kj}(G(p_k)-G(p_j))\right)\\ &= \sum_{i,j\in[l],k\notin[l]} p_{ij}\cdot\left[\psi_{ki}(G(p_j)-G(p_i)) + (\psi_{ki}-\psi_{kj})(G(p_k)-G(p_j))\right]\\ &\le \sum_{i,j\in[l],k\notin[l]}(\psi_{ki}-\psi_{kj})p_{ij}\cdot(G(p_k)-G(p_j))\end{aligned}$$

$$
\begin{aligned}
&\le 2M_{G'}P_M^0 L_{\psi,[l]} \sum_{i,j\in[l],k\notin[l]} |q_i - q_j||p_i - p_j|. \\
&\le 2M_{G'}(N-l)P_M^0 L_{\psi,[l]}\|P\|_{[l]}\|Q\|_{[l]},
\end{aligned}
$$

where we used

$$0 \le \mathcal{M}|p_i - p_j|^2 \le p_{ij}(G(p_i) - G(p_j))$$

for the first inequality. Combining the estimates altogether, we obtain

$$\frac{d}{dt}\|P\|_{[l]} \le -\frac{\kappa\mathcal{M}l}{N}\psi(\|Q\|_{[l]})\|P\|_{[l]} + \frac{2\kappa M_{G'}(N-l)P_M^0 L_{\psi,[l]}}{N}\|Q\|_{[l]}.$$

(2) We estimate $\mathcal{I}_2$ as following.

$$
\begin{aligned}
\mathcal{I}_2 &\le \sum_{i,j\in[l],k\notin[l]} (\psi_{ki} - \psi_{kj})p_{ij} \cdot (G(p_k) - G(p_j)) \\
&\le 2\max_{\substack{i'\in[l]\\ j'\notin[l]}} \psi_{i'j'} \sum_{i,j\in[l],k\notin[l]} |p_i - p_j||G(p_k) - G(p_j)| \\
&\le 4P_M^0 M_{G'} l(N-l) \max_{\substack{i'\in[l]\\ j'\notin[l]}} \psi_{i'j'}\|P\|_{[l]}.
\end{aligned}
$$

Together with the estimate of $\mathcal{I}_1$, we have

$$\frac{d}{dt}\|P\|_{[l]} \le -\frac{\kappa\mathcal{M}l}{N}\psi(\|Q\|_{[l]})\|P\|_{[l]} + \frac{4\kappa P_M^0 M_{G'} l(N-l)}{N}\max_{\substack{i'\in[l]\\ j'\notin[l]}} \psi_{i'j'}.$$

□

In particular, choice of $[l] = [N]$ *leads to the following result on the emergence of flocking.*

Theorem 4.1.1 (Emergence of asymptotic flocking)**.** *Let* (P, Q) *be a solution to* (4.1.1)*.*

1. *The following three statements are equivalent.*
   - *(a)* (P, Q) *exhibits flocking;* $\sup_{t\ge 0} D_Q(t) < \infty,\ \lim_{t\to 0} D_P(t) = 0$*.*
   - *(b)* D_P *decays exponentially;* $D_P(t) \le Be^{-Ct},\ B, C > 0$*.*

(c) Agents are spatially bounded; $\sup_{t\geq 0} D_Q(t) < \infty$.

2. *Suppose that*
$$\|P^0\| < \frac{\mathcal{M}\kappa}{M_{G'}} \int_{\|Q^0\|}^{+\infty} \psi(s)\,ds.$$
Then (P,Q) *exhibits flocking. In particular, if* $\|\psi\|_{L^1(\mathbb{R}_+)} = \infty$*, then a flocking happens unconditionally.*

Proof. Define the function $\mathcal{L}$ as
$$\mathcal{L}(t) := \frac{\mathcal{M}\kappa}{M_{G'}} \int_{\|Q^0\|}^{\|Q(t)\|} \psi(s)\,ds + \|P(t)\|.$$
We claim that $\dot{\mathcal{L}} \leq 0$. We first observe that
$$\left|\frac{d}{dt}\|Q\|_{[l]}^2\right| = 2\left|\sum_{i,j\in[l]} (q_i - q_j)\cdot(G(p_i) - G(p_j))\right| \leq 2M_{G'}\|P\|_{[l]}\|Q\|_{[l]}. \quad (4.1.5)$$
We then choose $[l] = [N]$ and apply Lemma 4.1.1 to obtain
$$\frac{d}{dt}\|P\| \leq -\kappa\mathcal{M}\psi(\|Q\|)\|P\|. \quad (4.1.6)$$
Then this proves the claim:
$$\begin{aligned}\frac{d\mathcal{L}}{dt} &= \frac{\mathcal{M}\kappa}{M_{G'}}\psi(\|Q(t)\|)\frac{d\|Q(t)\|}{dt} + \frac{d\|P(t)\|}{dt}\\ &\leq \mathcal{M}\kappa\psi(\|Q(t)\|)\|P(t)\| - \mathcal{M}\kappa\psi(\|Q(t)\|)\|P(t)\| = 0.\end{aligned}$$
(• Proof of (1)) Implications from (a) to (c) and (b) to (a) are clear. Suppose that (c) holds. Since any norms are equivalent in a finite dimensional space, we prove the statement for the norm $\|\cdot\|$. We have
$$\frac{d}{dt}\|P\| \leq -\kappa\mathcal{M}\psi(\|Q\|)\|P\| \leq -\kappa\mathcal{M}\psi\left(\sup_{t\geq 0}\|Q(t)\|\right)\|P\| \leq -C\|P\|$$
for some positive constant C independent of time. This yields
$$\|P(t)\| \leq e^{-tC}\|P^0\|.$$

(• Proof of (2)) Since $\mathcal{L}$ decrease in time, we have

$$\frac{\mathcal{M}\kappa}{M_{G'}}\int_{\|Q^0\|}^{\|Q(t)\|}\psi(s)\,ds+\|P(t)\|=\mathcal{L}(t)\le\mathcal{L}(0)=\|P^0\|<\frac{\mathcal{M}\kappa}{M_{G'}}\int_{\|Q^0\|}^{+\infty}\psi(s)\,ds. \tag{4.1.7}$$

This proves $\sup_{t\ge0}\|Q(t)\|<\infty$, and hence $\sup_{t\ge0}D_Q(t)<\infty$. Therefore the flocking emerges. □

In particular, Theorem 4.1.1 states that if flocking happens, then relative states converge at an exponential rate. Therefore, if agents are far enough away, they will not collide.

Corollary 4.1.1. *Suppose that there exists a positive constant* $0<M<\infty$ *satisfying*

$$\frac{M_{G'}\|P^0\|}{\kappa\mathcal{M}}<\min\left\{\int_{\|Q^0\|}^{M}\psi(r)dr,\psi(M)\min_{i,j\in[N]}|q_i^0-q_j^0|\right\}. \tag{4.1.8}$$

Then we have

$$\inf_{t\ge0}\min_{i,j\in[N]}|q_i(t)-q_j(t)|>0.$$

Proof. From (4.1.7) and (4.1.8), we have $\sup_{t\ge0}\|Q(t)\|<M<\infty$. This yields

$$\begin{aligned}|q_i(t)-q_j(t)|&\ge|q_i^0-q_j^0|-\int_0^t|G(p_i(s))-G(p_j(s))|ds\\&\ge|q_i^0-q_j^0|-M_{G'}\int_0^t D_P(s)ds\\&\ge|q_i^0-q_j^0|-M_{G'}\int_0^t\|P(s)\|ds\\&\ge|q_i^0-q_j^0|-M_{G'}\|P^0\|\int_0^t\exp\left(-\kappa\mathcal{M}\int_0^s\psi(\|Q(\tau)\|)d\tau\right)ds\\&\ge|q_i^0-q_j^0|-M_{G'}\|P^0\|\int_0^t\exp\left(-\kappa\mathcal{M}\int_0^s\psi(M)d\tau\right)ds\\&\ge|q_i^0-q_j^0|-\frac{M_{G'}\|P^0\|}{\kappa\mathcal{M}\psi(M)}>0.\end{aligned}$$

□

Remark 4.1.2. *Suppose that kernel is of the form*

$$\psi(|q|) = |q|^{-\alpha}, \quad \alpha > 1.$$

In this case, even if $\psi \notin (L^\infty \cap C^{0,1})(\mathbb{R}_+;\mathbb{R}_+)$, *the result in Corollary 4.1.1 still holds. In fact, a priori condition relaxes to*

$$\|P^0\| < \frac{\mathcal{M}\kappa}{M_{G'}} \int_{\|Q^0\|}^{+\infty} \psi(s)\, ds.$$

and the proof will be provided in Theorem 4.3.1.

4.1.3 Application to bi-cluster flocking

Theorem 4.1.1 demonstrates a close relationship between spatial boundedness and the emergence of flocking. Likewise, spacial boundedness plays an essential role in the bi-cluster flocking.

Proposition 4.1.2. *Let* (P, Q) *be a solution to* (4.1.1)*. Then the following two statements are equivalent.*

1. (P, Q) *exhibits the bi-cluster flocking.*
2. *There exists a partition* $\{A, B\}$ *of* $[N]$ *satisfying*

$$\sup_{t\geq 0} \max\{D_{Q,A}(t), D_{Q,B}(t)\} < \infty, \quad \sup_{t\geq 0} \min_{\substack{i\in A \\ j\in B}} |q_i(t) - q_j(t)| = \infty.$$

To prove Proposition 4.1.2, we introduce a preliminary lemma.

Lemma 4.1.2. *Let* (P, Q) *be a solution to* (4.1.1)*. Suppose that there exists a partition* $\{A, B\}$ *of* $[N]$ *satisfying*

$$\sup_{t\geq 0} \max\{D_{Q,A}(t), D_{Q,B}(t)\} < \infty.$$

Then whenever two groups generate different cluster, two groups segregate:

$$\sup_{t\in\mathbb{R}_+} \min_{i\in A, j\in B} |q_i(t) - q_j(t)| = \infty \quad \Longrightarrow \quad \lim_{t\to\infty} \min_{i\in A, j\in B} |q_i(t) - q_j(t)| = \infty.$$

Proof. It suffices to prove

$$\liminf_{t\to\infty} \min_{i\in A, j\in B} |q_i(t) - q_j(t)| = \infty.$$

Suppose on the contrary that

$$\liminf_{t\to\infty} \min_{i\in A, j\in B} |q_i(t) - q_j(t)| = M_{AB} < \infty.$$

Then there exists a time sequence $\{t_n\}_{n\in\mathbb{N}}$ satisfying

$$t_1 < t_2 < \cdots, \quad \lim_{n\to\infty} t_n = \infty, \quad \sup_{n\in\mathbb{N}} \min_{i\in A, j\in B} |q_i(t_n) - q_j(t_n)| < 1 + M_{AB}.$$

As each groups are spatially bounded, we have

$$\begin{aligned}
&\sup_{n\in\mathbb{N}} \max_{i,j\in[N]} |q_i(t_n) - q_j(t_n)| \\
&\qquad < 1 + M_{AB} + \sup_{t\in\mathbb{R}_+} \|Q(t)\|_A + \sup_{t\in\mathbb{R}_+} \|Q(t)\|_B =: M'_{AB} < \infty.
\end{aligned}$$

Then for any $T > 0$,

$$\sup_{n\in\mathbb{N}} \sup_{t\in(t_n, t_n+T)} D_Q(t) \le M'_{AB} + 2TM_{G'}P_M^0 =: M'_{AB,T} < \infty,$$

since $p_i(t)$ (resp. $\dot{q}_i(t) = (G(p_i(t)))$ is bounded above by P_M^0 (resp. $M_{G'}P_M^0$) from Proposition 4.1.1. Therefore if $(t_1, \infty) \subset \cup_{n\in\mathbb{N}}(t_n, t_n+T)$ for some finite T, the flocking must emerge from Theorem 4.1.1. Since A and B generates different cluster, this cannot happen, which yields

$$\limsup_{n\to\infty} (t_{n+1} - t_n) = \infty.$$

Passing to a subsequence, we may assume $\lim_{n\to\infty}(t_{n+1} - t_n) = \infty$. Let ℓ be an index maximizing $|p_i|$. We recall that

$$\frac{d}{dt}|p_\ell|^2 = \frac{2\kappa}{N}\sum_{k=1}^N \psi_{k\ell} p_\ell \cdot (G(p_k) - G(p_\ell)) \le \frac{2\kappa}{N}\psi(D_Q)\sum_{k=1}^N p_\ell \cdot (G(p_k) - G(p_\ell)). \tag{4.1.9}$$

As a maximum of R.H.S. in (4.1.9) is achieved when each p_i have same direction (i.e. $\cos(p_i, p_j) = 1$), we further estimate

$$\begin{aligned}
\frac{d}{dt}|p_\ell|^2 &\le \frac{2\kappa}{N}\psi(D_Q)\sum_{k=1}^N p_\ell \cdot \left(\frac{g(|p_k|)}{|p_\ell|}p_\ell - \frac{g(|p_\ell|)}{|p_\ell|}p_\ell\right) \\
&\le \frac{2\kappa m_{G'}}{N}\psi(D_Q)\sum_{k=1}^N |p_\ell|\left(|p_k| - |p_\ell|\right) \\
&= -\frac{2\kappa m_{G'}}{N}\psi(D_Q)|p_\ell|\left(N|p_\ell| - \sum_{k=1}^N |p_k|\right) \\
&= -2\kappa m_{G'}\psi(D_Q)|p_\ell|\left(|p_\ell| - \frac{1}{N}\left|\sum_{k=1}^N p_k\right|\right) \qquad (\because \cos(p_i,p_j)=1) \\
&=: -2\kappa m_{G'}\psi(D_Q)|p_\ell|\left(|p_\ell| - |p^0_{\text{ave}}|\right).
\end{aligned}$$

Note that p^0_{ave} is a constant vector, since the derivative of $\sum_{k=1}^N p_k$ is zero. Thus we have

$$\frac{d}{dt}\left(|p_\ell| - |p^0_{\text{ave}}|\right) = \frac{d}{dt}|p_\ell| \le -\kappa m_{G'}\psi(D_Q)\left(|p_\ell| - |p^0_{\text{ave}}|\right),$$

which leads to

$$0 \le |p_\ell(t)| - |p^0_{\text{ave}}| \le \exp\left(-\kappa m_{G'}\int_s^t \psi(D_Q(u))du\right)(|p_\ell(s)| - |p^0_{\text{ave}}|), \ s \le t,$$

where the first inequality comes from the maximality of ℓ. Now fix T and take $N \gg 1$, so that each interval $(t_n, t_n + T)$ is disjoint for each $n \ge N$. Then

$$0 \le |p_\ell(t_n + T)| - |p^0_{\text{ave}}| \le \exp\left(-\kappa m_{G'}T\psi(M'_{AB,T})\right)(|p_\ell(t_n)| - |p^0_{\text{ave}}|), \ n \ge N.$$

On the other hand, $|p_\ell|$ is unconditionally decreasing from Proposition 4.1.1. Thus,

$$\begin{aligned}
0 \le |p_\ell(t_{n+1} + T)| - |p^0_{\text{ave}}| &\le \exp\left(-\kappa m_{G'}T\psi(M'_{AB,T})\right)(|p_\ell(t_{n+1})| - |p^0_{\text{ave}}|) \\
&\le \exp\left(-\kappa m_{G'}T\psi(M'_{AB,T})\right)(|p_\ell(t_n + T)| - |p^0_{\text{ave}}|) \\
&\le \exp\left(-\kappa m_{G'}2T\psi(M'_{AB,T})\right)(|p_\ell(t_n)| - |p^0_{\text{ave}}|).
\end{aligned}$$

Then a straightforward induction yields

$$\begin{aligned} 0 &\le |p_\ell(t^*)| - |p^0_{\text{ave}}| \\ &\le \exp\left(-\kappa m_{G'}(m+1)T\psi(M'_{AB,T})\right)(|p_\ell(t_n)| - |p^0_{\text{ave}}|), \quad t^* \ge t_{n+m} + T, \end{aligned}$$

and therefore

$$\lim_{t\to\infty} |p_\ell(t)| = |p^0_{\text{ave}}|.$$

We again use the maximality of ℓ and apply the squeeze theorem to find

$$N|p^0_{\text{ave}}| = \left|\sum_{k=1}^N p_k(t)\right| \le \sum_{k=1}^N |p_k(t)| \le N\,|p_\ell(t)|\,,$$

so that

$$\lim_{t\to\infty}\sum_{k=1}^N \frac{1}{N}\,|p_k(t)| = |p^0_{\text{ave}}|.$$

Now we claim

$$\lim_{t\to\infty} p_k(t) = p^0_{\text{ave}}, \quad k \in [N]. \tag{4.1.10}$$

It suffices to show $\lim_{t\to\infty}|p_k(t)| = |p^0_{\text{ave}}|$ for each k. Suppose the contrary. Then since

$$\limsup_{t\to\infty}|p_k(t)| \le \limsup_{t\to\infty}|p_\ell(t)| = |p^0_{\text{ave}}|, \quad k \in [N],$$

there exists a constant P_m satisfying

$$\liminf_{t\to\infty}\min_{i\in[N]}|p_i(t)| < P_m < |p^0_{\text{ave}}|,$$

and there exists a time sequence $\{s_n\}_{n\in\mathbb{N}}$ such that

$$0 < s_1 < s_2 < \cdots, \quad \lim_{n\to\infty} s_n = \infty, \quad \min_{i\in[N]}|p_i(s_n)| < P_m.$$

This leads to

$$|p^0_{\text{ave}}| = \frac{1}{N}\sum_{i=1}^N \lim_{n\to\infty}|p_i(s_n)| \le \frac{1}{N}P_m + \lim_{n\to\infty}\frac{N-1}{N}|p_\ell(s_n)| < |p^0_{\text{ave}}|$$

which yields a contradiction, verifying the claim (4.1.10). Finally, since flocking does not happen, from Theorem 4.1.1 we have

$$\|P(t_n)\| \geq \frac{\mathcal{M}\kappa}{M_{G'}} \int_{\|Q(t_n)\|}^{+\infty} \psi(s)\, ds$$

for any t_n. As $\|Q(t_n)\|$ is uniformly bounded in n from the definition of t_n, let $Q_M < \infty$ be its upper bound. Since $\|P(t_n)\|$ converges to zero from (4.1.10), we obtain

$$0 < \frac{\mathcal{M}\kappa}{M_{G'}} \int_{Q_M}^{+\infty} \psi(s)\, ds \leq \frac{\mathcal{M}\kappa}{M_{G'}} \int_{\|Q(t_n)\|}^{+\infty} \psi(s)\, ds \leq \|P(t_n)\| \xrightarrow{n\to\infty} 0,$$

and this completes the proof by contradiction. □

Proof of Proposition 4.1.2. Clearly, (1) implies (2). Suppose that (2) holds. From Lemma 4.1.1, we have

$$\frac{d}{dt}\|P\|_A \leq -\frac{\kappa\mathcal{M}l}{N}\psi(\|Q\|_A)\|P\|_A + \frac{4\kappa P_M^0 M_{G'} l(N-l)}{N} \max_{\substack{i'\in A\\ j'\in B}} \psi_{i'j'}.$$

From the assumptions on (2) and Lemma 4.1.2, we have

$$\inf_{t\in\mathbb{R}_+} \left(\frac{\kappa\mathcal{M}l}{N}\psi(\|Q(t)\|_A)\right) \geq C_1 > 0, \quad \lim_{t\to\infty} \max_{\substack{i'\in A\\ j'\in B}} \psi_{i'j'} = 0,$$

for some positive constant C_1. Therefore for any $\varepsilon > 0$, there exists a time T satisfying

$$\frac{d}{dt}\|P(t)\|_A \leq -C_1\|P(t)\|_A + \varepsilon, \quad \forall t > T(\varepsilon) > 0.$$

Then by the comparison principle, we have

$$0 \leq \limsup_{t\to\infty} \|P\|_A \leq \frac{\varepsilon}{C_1}.$$

Since the choice of $\varepsilon > 0$ is arbitrary, we conclude $\|P(t)\|_A \xrightarrow{t\to\infty} 0$. The proof of $\|P(t)\|_B \xrightarrow{t\to\infty} 0$ is similar. □

Example 4.1.2. *In this example, we briefly sketch an example that achieves a bi-cluster flocking. For convenience, let* $G = \mathrm{Id}$ *so that* $m_{G'} = M_{G'} = \mathcal{M} = 1$. *Suppose that* $D_Q^0 \neq 0$. *Then from [17, Theorem 5.1], under some well-prepared initial configuration, there exists a set of indices* $[l]$ *(after re-ordering), which is a nonempty proper subset of* $[N]$*, and a positive constant* C *such that*

$$\min_{i\in[l]\ \ j\notin[l]} |q_i(t) - q_j(t)| \geq Ct.$$

This leads to

$$\int_0^t \max_{i\in[l], j\notin[l]} \psi(q_i(s) - q_j(s))ds \leq \int_0^t \psi(Cs)ds \leq \frac{\|\psi\|_{L^1(\mathbb{R}_+)}}{C}.$$

Then, integrating the second estimate in Lemma 4.1.1 leads to

$$\|P(t)\|_{[l]} + \frac{\kappa l}{N}\int_{\|Q^0\|_{[l]}}^{\|Q(t)\|_{[l]}} \psi(r)dr \leq \|P^0\|_{[l]} + \frac{4\kappa P_M^0 l(N-l)\|\psi\|_{L^1(\mathbb{R}_+)}}{CN}.$$

Therefore, if a velocity deviation in a group $[l]$ *is small and* C *is sufficiently large in the sense that*

$$\frac{N}{\kappa l}\|P^0\|_{[l]} + \frac{4P_M^0(N-l)\|\psi\|_{L^1(\mathbb{R}_+)}}{C} < \int_{\|Q^0\|_{[l]}}^{\infty} \psi(r)dr,$$

then we have $\sup_{t\in\mathbb{R}_+} \|Q(t)\|_{[l]} < \infty$. *Similarly, we have* $\sup_{t\in\mathbb{R}_+} \|Q(t)\|_{[N]-[l]} < \infty$ *for large* $C > 0$*, and this implies the bi-cluster flocking. Note that* C *can be chosen sufficiently large for a suitable choice of initial data. For the detail, we refer to [17].*

4.2 Analysis under weakly singular communications

In this section, we consider the kernel of form

$$\psi(x) = \frac{1}{|x|^\alpha}, \quad \alpha \in (0,1).$$

In this case, particles may collide (for the colliding example, see [11]) and the vector field blows up. The description of such a solution is not straightforward, as provided in the following Definition and Theorem [45, 46].

Definition 4.2.1. *Let B_i and ψ_n be defined as*

$$B_i(t) := \{k \in [N] : x_k(t) \neq x_i(t) \text{ or } v_k(t) \neq v_i(t)\},$$

$$\psi_n(s) := \begin{cases} \psi(s) & \text{if} \quad s \geq (n-1)^{-\frac{1}{\alpha}} \\ \text{smooth and monotone} & \text{if} \quad n^{-\frac{1}{\alpha}} \leq s \leq (n-1)^{-\frac{1}{\alpha}} \\ n & \text{if} \quad s \leq n^{-\frac{1}{\alpha}} \end{cases}$$

and let $0 = T_0 \leq T_1 \leq T_{N_s}$ be the set of all times of sticking (i.e. $x_i(t) - x_j(t) = v_i(t) - v_j(t)$ for some i) and $T_{N_s+1} := T$ be a given positive number. For $n \in \{0, \cdots, N_s\}$, on each interval $[T_n, T_{n+1}]$, consider the problem

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = \frac{1}{N} \sum_{k \in B_i(T_n)} (v_k - v_i) \psi_n(|x_k - x_i), \\ x_i \equiv x_j \quad \text{if} \quad j \notin B_i(T_n), \end{cases} \tag{4.2.1}$$

for $t \in [T_n, T_{n+1}]$, with initial data $x(T_n), v(T_n)$. We say that (x, v) solve (4.2.1) on the time interval $[0, T]$ with weight $\psi(s) = s^{-\alpha}$ if and only if for all $n = 0, \cdots, N_s$ and arbitrary small $\varepsilon > 0$, the function $x \in (C^1([0, T]))^{Nd}$ is a weak in $(W^{2,1}([T_n, T_{n+1} - \varepsilon]))^{Nd}$ solution of (4.2.1).

Proposition 4.2.1.

1. *Let $\alpha \in (0, \frac{1}{2})$ be given. Then for all $T > 0$ and arbitrary initial data, there exists a unique $x \in W^{2,1}([0, T]) \subset C^1([0, T])$ that solves (4.1.1) with communication weight $\psi(s) = \frac{1}{|x|^\alpha}$ weakly in $W^{2,1}([0, T])$.*

2. *Let $\alpha \in (\frac{1}{2}, 1)$ be given. Then there exists a unique solution in the sense of Definition 4.2.1.*

Although the description of a collisional solution under a singular kernel is somewhat non-trivial, if we restrict (4.1.1) on the real line, we may convert it into the first-order model, and its analysis may hint at the property of a solution in the second-order model as well. In this section, we are interested in the model (4.1.1) on the *real line*, equipped with a weakly singular kernel:

$$\begin{cases} \dot{q}_i = G(p_i), \quad t > 0, \quad i \in [N], \\ \dot{p}_i = \dfrac{\kappa}{N}\displaystyle\sum_{k=1}^{N} \psi(q_k - q_i)(G(p_k) - G(p_i)), \qquad \psi(s) = \dfrac{1}{|s|^\alpha}, \quad \alpha \in (0,1). \\ (q_i, p_i)\big|_{t=0+} = (q_i^0, p_i^0), \quad p_i, q_i \in \mathbb{R}, \end{cases} \tag{4.2.2}$$

If ψ is regular and (P, Q) is a classical solution of (4.2.2), then we have a following relation:

$$\int_0^t \psi(q_k(s) - q_i(s))(G(p_k(s)) - G(p_i(s)))ds = \int_0^{q_k(t)-q_i(t)} \psi(s)ds.$$

Therefore P is a solution of

$$\dot{q}_i = G(\nu_i + \frac{\kappa}{N}\sum_{k=1}^{N} \Psi(q_k - q_i)), \tag{4.2.3}$$

provided that two systems are coupled by the following relationship:

$$\Psi(r) := \int_0^r \psi(x)dx, \quad \nu_i := p_i^0 - \frac{\kappa}{N}\sum_{k=1}^{N} \Psi(q_k^0 - q_i^0). \tag{4.2.4}$$

On the other hand, the converse holds; if Ψ is differentiable, a solution of (4.2.3) is also a solution of (4.2.2) under (4.2.4), and therefore two models are equivalent. What if ψ weakly singular? In this case, we have

$$\psi(q) = \frac{1}{|q|^\alpha} \ (\alpha \in (0,1)) \quad \Longrightarrow \quad \Psi(q) = \int_0^q \psi(r)dr = \mathrm{sgn}(q)\frac{|q|^{1-\alpha}}{1-\alpha}. \tag{4.2.5}$$

As Ψ is continuous, Peano's theorem guarantees a classical solution of (4.2.3). However, this may *not* be a classical solution of (4.2.2), since a solution of (4.2.2) requires more regularity of q_i than (4.2.3), but the regularity of Ψ (and hence regularity of $\dot{q}_i$) breaks down at the origin.

Example 4.2.1. *Consider a two-particle system with*

$$G = \mathrm{Id}, \quad \psi(q) = \frac{1}{\sqrt{q}}, \quad \Psi(q) = 2\mathrm{sgn}(q)\sqrt{|q|}.$$

Then, (4.2.3) *is of the form*

$$\dot{q_1} = \nu_1 + \frac{\kappa}{2}\Psi(q_2 - q_1), \quad \dot{q_2} = \nu_2 + \frac{\kappa}{2}\Psi(q_1 - q_2).$$

If we pose $\nu_1 = \nu_2 = 0$ *and* $q_1^0 \geq q_2^0$*, we have a following classical solution of* (4.2.3).

$$q_i = \begin{cases} \frac{1}{2}\left((q_1^0 + q_2^0) + \kappa^2\big(t - \frac{1}{\kappa}\sqrt{|q_1^0 - q_2^0|}\big)^2\right), & \text{if } i = 1,\ t < \frac{1}{\kappa}\sqrt{|q_1^0 - q_2^0|}, \\ \frac{1}{2}\left((q_1^0 + q_2^0) - \kappa^2\big(t - \frac{1}{\kappa}\sqrt{|q_1^0 - q_2^0|}\big)^2\right), & \text{if } i = 2,\ t < \frac{1}{\kappa}\sqrt{|q_1^0 - q_2^0|}, \\ \frac{q_1^0 + q_2^0}{2}, & \text{otherwise.} \end{cases}$$

However, since $\dot{q}_i$ *is not differentiable, we cannot recover a classical solution of* (4.2.2).

The above example illustrates that two models are not equivalent under the classical regime if ψ is singular; a solution of (4.2.3) need not be twice differentiable. Therefore, if one attempts to make two models equivalent, one needs to enlarge the concept of solution of (4.2.2). It turns out that the Sobolev space $W^{2,1}$ is an appropriate function space, as described in the following theorem.

Theorem 4.2.1. *Let* (P, Q) *be a solution to* (4.2.2)*. Then the following assertions holds.*

1. *The model* (4.2.2) *has a unique global (weak) solution where*

$$q_i \in W^{2,\gamma}([0, T]),$$

for each $i \in [N]$, $T \in \mathbb{R}_+$*, and*

$$\gamma \in \left[1, \frac{1}{\max\{1 - K, \alpha\}}\right), \quad K := \frac{m_{G'} 2^{1-2\alpha}(1 - \alpha)}{N M_{G'} \alpha}.$$

2. *Flocking emerges unconditionally:*

$$\sup_{t\geq 0}\max_{i,j\in[N]}|q_i(t)-q_j(t)|<\infty,\quad \max_{i,j\in[N]}|p_i(t)-p_j(t)|\lesssim e^{-Ct},\quad C>0.$$

Remark 4.2.1. *Below, we list some comments about Theorem 4.2.1*

1. *Theorem 4.2.1 states that q_i is always continuously differentiable and p_i is always differentiable almost everywhere, and the regularity of p_i improves as α decreases. Roughly speaking, when α is close to 1, then p_i is close to an absolutely continuous function, and when α is close to 0, then p_i is close a to Lipschitz continuous function. In fact, if only sticking happens and collision does not occur (see Definition 4.2.2), then p_i can be indeed Lipschitz, as described in Example 4.2.1.*

2. *Equivalence between* (4.2.2) *and* (4.2.3) *is not trivial for a singular kernel. When the kernel ψ is regular, equivalence is essentially based on the following change of variable formula:*

$$\int_0^t \psi(q_k(s)-q_i(s))(G(p_k(s))-G(p_i(s)))ds=\int_0^{q_k(t)-q_i(t)}\psi(s)ds.$$

The above formula holds if ψ is continuous and $t\mapsto q_k(t)-q_i(t)$ is continuously differentiable. However, if ψ is merely a nonnegative measurable function, even if $t\mapsto q_k(t)-q_i(t)$ is absolutely continuous, change of variable formula requires either monotonicity of q_k-q_i or integrability of ψ and $\psi(q_k(s)-q_i(s))(G(p_k(s))-G(p_i(s)))$ (see Lemma 4.2.2). Therefore, due to the possibility of pathological behavior near a collision time, a change of variable formula cannot be applied directly.

3. *In [44], the authors provided a framework to rigorously derive a kinetic description of the model* (4.1.1) *(under $G=\mathrm{Id}$) with a weakly singular communication in a weak-atomic sense. For this, the solution should have a regularity of $W^{2,1}$ and therefore the derivation was limited to the case where $\alpha\in(0,1/2)$ (see Proposition 4.2.1). Theorem 4.2.1 states that a weak-atomic solution can be derived for any $\alpha\in(0,1)$ on the real line.*

Before we establish the equivalence between (4.2.2) and (4.2.3), we first review the dynamics of (4.2.3). Recall that

$$\mathcal{N} := (\nu_1, \nu_2, \cdots, \nu_N), \quad \Psi(r) := \int_0^r \psi(x)dx, \quad \nu_i := p_i^0 - \frac{\kappa}{N}\sum_{k=1}^N \Psi(q_k^0 - q_i^0).$$

Proposition 4.2.2. *[11] Suppose that communication weight has weak singularity of the following form:*

$$\psi(q) = \frac{1}{|q|^\alpha}, \quad 0 < \alpha < 1, \quad q \neq 0,$$

and let Q be a solution to (4.2.3) *with initial data $(Q^0, \mathcal{N})$. For fixed indices i and j $(i \neq j)$, suppose that*

$$q_i^0 > q_j^0.$$

Then the following trichotomy holds.

1. *If $\nu_i > \nu_j$, then q_i and q_j will not collide in finite time:*

$$q_i(t) > q_j(t) \quad \textit{for all} \quad t \geq 0.$$

2. *If $\nu_i < \nu_j$, then q_i and q_j will collide exactly once, i.e., there exists a time t^* such that*

$$q_i(t) > q_j(t) \quad \textit{for} \quad 0 \leq t < t^*,$$
$$q_i(t^*) = q_j(t^*) \quad \textit{and} \quad q_i(t) < q_j(t) \quad \textit{for } t > t^*.$$

3. *If $\nu_i = \nu_j$, then q_i and q_j will collide in finite time, and two particles will stick together after their first collision.*

4. *If $\nu_i \neq \nu_j$, then we have*

$$\liminf_{t\to\infty} |q_i(t) - q_j(t)| > 0.$$

Therefore, two particles p_i and p_j will overlap in some time unless $(p_i^0 - p_j^0)(\nu_i - \nu_j) > 0$, and will eventually 'stick' if and only if $\nu_i = \nu_j$. It is easy to see that

$$\nu_i = \nu_j \quad \Longleftrightarrow \quad p_i(t) = p_j(t) \quad \text{implies} \quad \dot{p}_i(t) = \dot{p}_j(t),$$

as illustrated in Example 4.2.1. If a kernel is regular, then (1),(2), and (4) in Proposition 4.2.2 still hold, but (3) does not happen; agents never stick unless they are sticking at the initial state [11, 31]. Thus such 'finite-in-time sticking' characterizes a singular kernel [45, 46]. In what follows, we clarify the definition of sticking and related concepts:

Definition 4.2.2. *Let Q be a C^1 solution of* (4.2.3)*. Consider two agents q_i and q_j.*

1. *We say q_i and q_j collide at time t if*
$$q_i(t) = q_j(t) \quad \text{but} \quad \dot{q}_i(t) \neq \dot{q}_j(t).$$
2. *We say q_i and q_j stick at time t if*
$$q_i(t) = q_j(t) \quad \text{and} \quad \dot{q}_i(t) = \dot{q}_j(t).$$

From Proposition 4.2.2, if particles stick at some instance, they stick afterwards. Therefore, if some agents stick at time t among N agents, the system immediately changes into a system of weighted $N'(< N)$ agents. To describe this phenomena, we define sets of collisional indices, sticking indices and their time set as follows:

$$\begin{aligned}
&C_i(t) := \{j \in [N] \mid q_i \text{ and } q_j \text{ collide at time } t\},\\
&S_i(t) := \{j \in [N] \mid q_i \text{ and } q_j \text{ stick at time } t\},\\
&\mathcal{T} := \bigcup_{i \in [N]} \left(\{t \in \mathbb{R}_+ : |C_i(t)| \neq 0\} \cup \{t \in \mathbb{R}_+ : |S_i| \text{ is discontinuous at } t\}\right).
\end{aligned} \tag{4.2.6}$$

Note that C_i,S_i and $\mathcal{T}$ depends on solution of (4.2.3), and S_i is discontinuous at the instance when two particles start to stick.

Proposition 4.2.3. *Suppose that an initial data $(Q^0, \mathcal{N})$ of* (4.2.3) *is given.*

1. *System* (4.2.3) *has a unique global classical (i.e. $q_i(t) \in C^1(\mathbb{R}_+; \mathbb{R})$ for each i) solution. In particular, $C_i(t)$,$S_i(t)$ and $\mathcal{T}$ are well defined for each $i \in [N]$ and $t \in \mathbb{R}_+$.*

2. *For each* $i \in [N]$, $|C_i(t)|$ *is zero for all but finitely many* $t \in \mathbb{R}_+$.
3. *For each* $i \in [N]$ *and* $0 \le s \le t < \infty$, *we have* $S_i(s) \subset S_i(t)$. *In particular,* $|S_i|$ *is a right-continuous increasing step function.*
4. $\mathcal{T}$ *has a finite cardinality.*

Proof. If we prove (1), then the other statements are direct consequences of Proposition 4.2.2 and (1). Therefore we focus on the proof of (1). The existence of a global classical solution is guaranteed by Peano's Theorem. Therefore it suffices to verify the uniqueness. Suppose that there exists two solutions $Q = (q_1, \cdots, q_N)$ and $Q' = (q_1', \cdots, q_2')$ with same initial data $(Q^0, \mathcal{N}) = (q_1^0, \cdots, q_N^0, \nu_1, \cdots, \nu_N)$, which is neither collisional nor sticking (i.e. $\prod_{\substack{i,j\in[N]\\ i\neq j}}(q_i^0 - q_j^0) \neq 0$). Let $(C_i, S_i, \mathcal{T})$ and $(C_i', S_i', \mathcal{T}')$ be defined as (4.2.6) with respect to Q and Q', respectively. From Proposition 4.2.2, there exists finite number of times $\{t_i\}_{i\in[M]}$ sayisfting

$$[0,\infty) = \bigcup_{c=0}^{M}[t_c, t_{c+1}), \quad 0 = t_0 < t_1 < \cdots < t_M = +\infty, \quad M < \infty,$$

such that $\{t_1, \cdots, t_M\} = \mathcal{T}$. Similarly, we set $[0,\infty) = \bigcup_{c=0}^{M'}[t_c', t_{c+1}')$ with respect to Q'. Now we use an induction argument to prove $t_c = t_c'$ and $Q = Q'$ on $[0, t_c)$ for each $c = 1, 2, \cdots, M$.

• (c=1) Since Ψ is locally Lipschitz except the origin and Ψ is not evaluated at 0 in $t \in [0, \min\{t_1, t_1'\})$, the standard theory of ODE guarantees that $Q(t) = Q'(t)$ in $[0, \min\{t_1, t_1'\})$. Without loss of generality, suppose that $t_1 \le t_1'$. Since we have global existence of a classical solution, both of Q and Q' uniquely extends to $[0, t_1]$ and they are same. In particular, if q_i and q_j collide or starts to stick at t_1, then so are q_i' and q_j'. Therefore we have $t_1 = t_1'$ and $Q = Q'$ in $[0, t_1] = [0, t_1']$.

• (Inductive step) Suppose $t_n = t_n'$ for $n = 1, 2, \cdots, c$ and assume that solution is unique in $[0, t_c)$, so that $S_k(t)$ and $C_k(t)$ are well defined for each $k \in [N]$ in $t \in [0, t_c)$. We claim that for any $t^* \in [t_c, \min\{t_{c+1}, t'_{c+1}\})$, we have $Q = Q'$ in $[0, t^*]$.

To prove this by contradiction, suppose that

$$Q(T) \neq Q'(T) \quad \text{for some } T \in (t_c, t^*]. \tag{4.2.7}$$

Let $D(t) := \max_{i\in[N]} |q_i(t) - q'_i(t)|$ and $M = M(t)$ be a time-dependent index satisfying $D(t) = |q_M(t) - q'_M(t)|$. Let time $t \in (t_c, t^*]$ and index $M(t) = \ell$ be fixed. If we assume, without loss of generality, that $q_\ell(t) \geq q'_\ell(t)$, then we have

$$\begin{aligned}
&\dot{q}_\ell(t) - \dot{q}'_\ell(t) \\
&\quad = G(\nu_i + \frac{\kappa}{N}\sum_{k=1}^{N} \Psi(q_k(t) - q_\ell(t))) - G(\nu_i + \frac{\kappa}{N}\sum_{k=1}^{N} \Psi(q'_k(t) - q'_\ell(t))) \\
&\quad = \frac{\kappa G'(\tilde{q}_i)}{N}\sum_{k=1}^{N} \left(\Psi(q_k(t) - q_\ell(t)) - \Psi(q'_k(t) - q'_\ell(t))\right), \quad t \in (t_c + \varepsilon, t^*],
\end{aligned}$$

where we used the mean value theorem for the last equality. Then definition of M yields

$$q_k(t) - q'_k(t) \leq q_\ell(t) - q'_\ell(t) \iff q_k(t) - q_\ell(t) \leq q'_k(t) - q'_\ell(t).$$

As Ψ is increasing, we have $\dot{q}_\ell(t) - \dot{q}'_\ell(t) \leq 0$. From the existence of a global solution, each $\dot{q}_i$ and $\dot{q}'_i$ are uniformly bounded in $[t_c, t^*)$. Therefore we have $\dot{D}(t) \leq 0$ for almost every $t \in [t_c, t^*)$, and

$$0 \leq D(t^*) \leq D(t_c) = 0,$$

where the equality comes from the induction hypothesis. Therefore, by the same argument as in the $c = 1$ case, we have $t_{c+1} = t'_{c+1}$ and $D(t) \equiv 0$ on $(t_c, t_{c+1}]$. By induction, we conclude $D \equiv 0$ on $(t_c, t^*]$. This contradicts (4.2.7), which completes the proof for not overlapping initial data. The proof for a collisional or sticking initial data follows by letting $t_1 = 0$. □

Lemma 4.2.1. *Let* $(P, \mathcal{N})$ *be a solution to* (4.2.3) *with a communication of the form* (4.2.5)*. For sufficiently small* $\varepsilon > 0$*, we have the following assertions.*

1. *If p_i and p_j collide at $T > 0$, then*
$$C\varepsilon \le |q_i(T \pm \varepsilon) - q_j(T \pm \varepsilon)|,$$
for some positive constant $C > 0$.

2. *If p_i and p_j stick at $T > 0$,*
$$D_1 \varepsilon^{\frac{1}{\alpha}} \le |q_i(T - \varepsilon) - q_j(T - \varepsilon)| \le D_2 \varepsilon^{\frac{1}{\alpha}},$$
where
$$D_1 = \left(\frac{2\kappa m_{G'} \alpha}{N(1-\alpha)} \right)^{\frac{1}{\alpha}}, \quad D_2 = \left(\frac{\kappa M_{G'} 2^\alpha \alpha}{1-\alpha} \right)^{\frac{1}{\alpha}}.$$

Proof. Throughout the proof, we set
$$\mathcal{T} = \{t_1, t_2, \cdots, t_c\}, \quad t_1 < t_2 < \cdots < t_c, \quad t_0 = 0,$$
where $\mathcal{T}$ is defined as Proposition 4.2.3.

(• Proof of (1)). Without loss of generality, set $\nu_i > \nu_j$ and $q_j^0 > q_i^0$. Suppose that q_i collide with q_j at $t_C \in \mathcal{T}$. First, we use the mean value theorem to observe

$$\begin{aligned} &\frac{d}{dt}(q_i - q_j)|_{t=t_C} \\ &\quad = G\left(\nu_i + \frac{\kappa}{N} \sum_{k=1}^N \Psi(q_k - q_i) \right)\Big|_{t=t_C} - G\left(\nu_j + \frac{\kappa}{N} \sum_{k=1}^N \Psi(q_k - q_j) \right)\Big|_{t=t_C} \\ &\quad = G'(y_{ij})(\nu_i - \nu_j) \ge m_{G'}(\nu_i - \nu_j) =: v_{ij} > 0. \end{aligned} \tag{4.2.8}$$

Then from the continuity of the solution, for some $\delta > 0$ we have
$$\frac{v_{ij}}{2} \le G(p_i(t)) - G(p_j(t)), \quad t \in [t_C - \delta, t_C + \delta].$$
Therefore, as $q_i(t_C) - q_j(t_C) = 0$, for any $0 < \varepsilon \le \delta$, we obtain
$$|q_i(t_C \pm \varepsilon) - q_j(t_C \pm \varepsilon)| \ge \left| \int_0^\varepsilon G(p_i(t_C \pm s)) - G(p_j(t_C \pm s)) ds \right| \ge \frac{\varepsilon v_{ij}}{2}.$$

(• Proof of (2)). Suppose that, $q_1, q_2, \cdots, q_{l-1}, q_l$ *starts* to stick at time $t_S \in \mathcal{T}$ ($S \in [c]$) simultaneously with suitable reordering of indices, and set

$$q_1(t) < q_2(t) < \cdots < q_{l-1}(t) < q_l(t), \quad \text{for any } t \in (t_{S-1}, t_S).$$

Let $i := j+1 (i, j \in [l])$. We use the mean-value theorem twice to obtain

$$\begin{aligned}
\frac{d}{dt}(q_i - q_j) &= G\left(\nu_i + \frac{\kappa}{N}\sum_{k=1}^{N}\Psi(q_k - q_i)\right) - G\left(\nu_j + \frac{\kappa}{N}\sum_{k=1}^{N}\Psi(q_k - q_j)\right) \\
&= \frac{\kappa G'(y_{ij})}{N}\left(\sum_{k=1}^{N}\big(\Psi(q_k - q_i) - \Psi(q_k - q_j)\big)\right) \\
&= \frac{\kappa G'(y_{ij})}{N}\left(\sum_{k=1}^{N}\psi(z_{ijk})(q_j - q_i)\right) \\
&= -\left(\frac{\kappa G'(y_{ij})}{N}\sum_{k=1}^{N}\psi(z_{ijk})\right)(q_i - q_j),
\end{aligned}$$

where z_{ijk} is located between $q_k - q_i$ and $q_k - q_j$. Note that the second mean value theorem is valid since i and j are consecutive, so that Ψ is differentiable in the interval $(q_k - q_i, q_k - q_j)$. In particular, for $k \in \{i, j\}$, z_{ijk} is specifically

$$\psi(z_{ijj}) = \psi(z_{iji}) = \frac{\Psi(q_i - q_j)}{q_i - q_j} = \frac{1}{(1-\alpha)(q_i - q_j)^{\alpha}}.$$

Therefore, we have

$$\begin{aligned}
&\frac{d}{dt}(q_i - q_j) \\
&\quad \le -\frac{\kappa m_{G'}}{N}(\psi(z_{ijj}) + \psi(z_{iji}))(q_i - q_j) \\
&\quad = -C_1(q_i - q_j)^{1-\alpha}, \quad C_1 := \frac{2\kappa m_{G'}}{N(1-\alpha)},
\end{aligned}$$

and $C_1 > 0$ is independent of initial data. Now we recall the following ODE:

$$\begin{aligned}
&\dot{x} = -C_1 x^{1-\alpha}, \quad x(0) = x^0 > 0 \\
&\implies \quad x(t) = (C_1\alpha)^{\frac{1}{\alpha}}\left(\frac{(x^0)^{\alpha}}{C_1\alpha} - t\right)^{\frac{1}{\alpha}}, \quad t \in \left(0, \frac{(x^0)^{\alpha}}{C_1\alpha}\right).
\end{aligned}$$

Then, by the comparison principle, for any sufficiently small $\delta > 0$, we have

$$
\begin{aligned}
&q_i(t_S-\varepsilon) - q_j(t_S-\varepsilon) \\
&\qquad \le (C_1\alpha)^{\frac{1}{\alpha}} \left(\frac{(q_i(t_S-\delta) - q_j(t_S-\delta))^\alpha}{C_1\alpha} - (\delta-\varepsilon) \right)^{\frac{1}{\alpha}}, \quad \varepsilon \in [0,\delta].
\end{aligned} \tag{4.2.9}
$$

Now we claim that

$$
q_i(t_S-\varepsilon) - q_j(t_S-\varepsilon) \ge (C_1\alpha\varepsilon)^{\frac{1}{\alpha}}, \quad \varepsilon \in [0,\delta]. \tag{4.2.10}
$$

Suppose that (4.2.10) does not hold. Then for some $\varepsilon^* \in [0,\delta]$, we have

$$
q_i(t_S-\varepsilon^*) - q_j(t_S-\varepsilon^*) < (C_1\alpha\varepsilon^*)^{\frac{1}{\alpha}}.
$$

Then (4.2.9) under $\delta = \varepsilon^*$ yields

$$
q_i(t_S-\varepsilon) - q_j(t_S-\varepsilon) < (C_1\alpha\varepsilon)^{\frac{1}{\alpha}}, \quad \varepsilon \in [0,\varepsilon^*].
$$

Then, as the inequality is strict and $(C_1\alpha\varepsilon)^{\frac{1}{\alpha}}|_{\varepsilon=0} = 0$, there exists $\varepsilon^{**} \in (0,\varepsilon^*]$ satisfying

$$
q_i(t_S-\varepsilon^{**}) - q_j(t_S-\varepsilon^{**}) = 0.
$$

However, since t_S is a time that q_i and q_j *starts* to stick, this is awkward, verifying (4.2.10). Since choice of i and $j = i-1$ was arbitrary and C_1 is independent of indices, we have

$$
\min_{i\neq j, i,j\in[l]} |q_i(t_S-\varepsilon) - q_j(t-\varepsilon)| \ge (C_1\alpha\varepsilon)^{\frac{1}{\alpha}} =: D_1\varepsilon^{\frac{1}{\alpha}}.
$$

Now take any $i', j' \in [l]$ with $i' > j'$. Since $\psi(|r|)$ decreasing in $|r|$, we have

$$
\begin{aligned}
\Psi(q_k - q_{j'}) - \Psi(q_k - q_{i'}) &= \int_{q_k-q_{i'}}^{q_k-q_{j'}} \psi(r)dr \\
&\le \int_{-\frac{q_{i'}-q_{j'}}{2}}^{\frac{q_{i'}-q_{j'}}{2}} \psi(r)dr = \frac{2^\alpha}{1-\alpha}(q_{i'}-q_{j'})^{1-\alpha}.
\end{aligned}
$$

We apply the mean value theorem to get

$$\begin{aligned}\frac{d}{dt}(q_{i'} - q_{j'}) &= \frac{\kappa G'(y_{i'\ell})}{N}\left(\sum_{k=1}^{N}\big(\Psi(q_k - q_{i'}) - \Psi(q_k - q_{j'})\big)\right) \\ &\geq -\frac{\kappa M_{G'} 2^{\alpha}}{1-\alpha}(q_{i'} - q_{j'})^{1-\alpha} =: -C_2(q_{i'} - q_{j'})^{1-\alpha}, \quad t \in (t_S - \varepsilon, t_S),\end{aligned}$$

for a positive constant $C_2 > 0$ independent of i' and j'. We then apply similar technique to derive (4.2.10) to yield

$$q_{i'}(t_S - \varepsilon) - q_{j'}(t_S - \varepsilon) \leq (C_2 \alpha \varepsilon)^{\frac{1}{\alpha}}, \quad \varepsilon \in [0, \delta],$$

and therefore

$$\max_{i \neq j, i,j \in [l]} |q_i(t_S - \varepsilon) - q_j(t - \varepsilon)| \leq (C_2 \alpha \varepsilon)^{\frac{1}{\alpha}} =: D_2 \varepsilon^{\frac{1}{\alpha}}.$$

□

Lemma 4.2.2. *Suppose that* $0 \leq f \in L^1_{\text{loc}}(\mathbb{R})$ *and* u *is absolutely continuous on* $[a, b]$. *If* $(f \circ u) \times u' \in L^1([a, b])$, *then*

$$\int_{u(a)}^{u(b)} f(x)dx = \int_a^b f(u(t))u'(t)dt.$$

Proof. First suppose that $0 \leq f$ is bounded and measurable. For some constant c, define

$$F(x) := \int_c^x f(t)dt,$$

Then from boundedness of f, we have $F \in C^{0,1}(\mathbb{R})$. Thus $F \circ u$ is absolutely continuous and

$$(F \circ u)'(t) = f'(u(t))u'(t),$$

for almost every $t \in [a, b]$. Therefore we have

$$\begin{aligned}\int_a^b f(u(t))u'(t)dt &= \int_a^b (F \circ u)'(t)dt = (F \circ u)(b) - (F \circ u)(b) \\ &= F(u(b)) - F(u(a)) = \int_{u(a)}^{u(b)} F'(x)dx = \int_{u(a)}^{u(b)} f(x)dx.\end{aligned}$$

Now suppose that $0 \le f \in L^1_{\text{loc}}(\mathbb{R})$. Define an approximating function f_n as

$$f_n(x) := \begin{cases} f(x), & \text{if } 0 \le f(x) \le n, \\ 0, & \text{if } f(x) > n. \end{cases}$$

Then since f_n is bounded, we have

$$\int_a^b f_n(u(t))u'(t)dt = \int_{u(a)}^{u(b)} f_n(x)dx.$$

From the integrability of f and $f(u(t))|u'(t)|$, we have a desired result from the dominated convergence theorem. □

As a direct consequence, we can establish the equivalence between (4.2.2) and (4.2.3) whenever $\psi(q_i - q_j)(G(p_i) - G(p_j))$ is locally integrable for each $i, j \in [N]$. More precisely, let $\Psi(\cdot)$ be an antiderivative of ψ:

$$\Psi(x) := \int_0^x \psi(y)dy, \quad x \in \mathbb{R},$$

as long as ψ is locally integrable. Let (P, Q) be a solution to (4.2.2), where $p_i \in W^{2,1}([0,T])$ for any $T > 0$. Then from Lemma 4.2.2, it follows that

$$\begin{aligned} \frac{d}{dt}\Psi(q_k(t) - q_i(t)) &= \frac{d}{dt}\int_{q_k(0)-q_i(t)}^{q_k(t)-q_i(t)} \psi(y)dy \\ &= \frac{d}{dt}\int_0^t \psi(q_k(t) - q_i(t))(G(q_k(t)) - G(q_i(t)))dt \\ &= \psi(q_k(t) - q_i(t))(G(p_k(t)) - G(p_i(t))), \end{aligned}$$

for almost every t. Hence, it follows from (4.2.2) that

$$\frac{d}{dt}\left(p_i - \frac{\kappa}{N}\sum_{k=1}^N \Psi(q_k(t) - q_i(t))\right) = 0, \quad i \in [N],$$

for almost every t. Now, we integrate above with respect to t to get

$$p_i(t) = p_i^0 - \frac{\kappa}{N}\sum_{k=1}^N \Psi(q_k^0 - q_i^0) + \frac{\kappa}{N}\sum_{k=1}^N \Psi(q_k(t) - q_i(t))$$

$$=: \nu_i + \frac{\kappa}{N}\sum_{k=1}^{N}\Psi(q_k(t)-q_i(t)).$$

Conversely, if each q_i is continuously differentiable and $\dot{q}_i$ is absolutely continuous in any finite time interval, we recover (4.2.2) from (4.2.3) for almost every t by direct differentiation.

Now we are ready to prove Theorem 4.2.1.

Proof of Theorem 4.2.1. Let $\mathcal{T} = (t_1, t_2, \cdots, t_c)$ and $t_0 = 0, t_{c+1} = \infty$, where t_i is increasing with respect to indices. Let $[l]$ be a set of indices sticking at $t_S \in \mathcal{T}$ as in the proof of Lemma 4.2.1. For any $\varepsilon > 0, k \in [c]$ and $i, j \in [N]$, we have either

$$q_i(t) \equiv q_j(t) \text{ for } t \geq t_k, \quad \text{or} \quad \inf_{t\in(t_k+\varepsilon, t_{k+1}-\varepsilon)} |q_i(t)-q_j(t)| > C > 0, \quad (4.2.11)$$

for some constant $C > 0$ from Proposition 4.2.2. Thus $\Psi(q_i(s) - q_j(s))$ is continuously differentiable for s where $|s - t_k| > \varepsilon,\ t_k \in \mathcal{T}$. Therefore by Lemma 4.2.2, (4.2.2) and (4.2.3) are equivalent in time $T \in (t_k+\varepsilon, t_{k+1}-\varepsilon)$. Now consider a bounded regular communcation $\tilde{\psi}$ satisfying

$$\psi(x) = \tilde{\psi}(x), \quad x \in (C, \infty),$$

and its antiderivative $\tilde{\Psi}(x) := \int_0^x \tilde{\psi}(r)dr$. Let $T \in (t_k + \varepsilon, t_{k+1} - \varepsilon)$. For the former case of (4.2.11), we have

$$\begin{aligned}\Psi(q_i(T) - q_j(T)) &- \Psi(q_i(t_k+\varepsilon) - q_j(t_k+\varepsilon)) \\ &= 0 = \tilde{\Psi}(q_i(T) - q_j(T)) - \tilde{\Psi}(q_i(t_k+\varepsilon) - q_j(t_k+\varepsilon)).\end{aligned}$$

For the latter case of (4.2.11), since ψ and $\tilde{\psi}$ are same in (C, ∞), we have

$$\begin{aligned}\Psi(q_i(T) - q_j(T)) &- \Psi(q_i(t_k+\varepsilon) - q_j(t_k+\varepsilon)) \\ &= \int_{q_i(t_k+\varepsilon)-q_j(t_k+\varepsilon)}^{q_i(T)-q_j(T)} \psi(r)dr \\ &= \int_{q_i(t_k+\varepsilon)-q_j(t_k+\varepsilon)}^{q_i(T)-q_j(T)} \tilde{\psi}(r)dr\end{aligned}$$

$$= \tilde{\Psi}(q_i(T) - q_j(T)) - \tilde{\Psi}(q_i(t_k + \varepsilon) - q_j(t_k + \varepsilon)).$$

This yields

$$\begin{aligned}
p_i(T) &= \nu_i + \frac{\kappa}{N}\sum_{k=1}^{N} \Psi(q_k(T) - q_i(T)) \\
&= p_i(t_k + \varepsilon) - \frac{\kappa}{N}\sum_{k=1}^{N} \Psi(q_k(t_k + \varepsilon) - q_i(t_k + \varepsilon)) \\
&\qquad\qquad + \frac{\kappa}{N}\sum_{k=1}^{N} \Psi(q_k(T) - q_i(T)) \\
&= p_i(t_k + \varepsilon) - \frac{\kappa}{N}\sum_{k=1}^{N} \tilde{\Psi}(q_k(t_k + \varepsilon) - q_i(t_k + \varepsilon)) \\
&\qquad\qquad + \frac{\kappa}{N}\sum_{k=1}^{N} \tilde{\Psi}(q_k(T) - q_i(T)) \\
&= p_i(t_k + \varepsilon) + \frac{\kappa}{N}\sum_{k=1}^{N} \int_{t_k+\varepsilon}^{T} \tilde{\psi}(q_k(s) - q_i(s))(G(p_k(s)) - G(p_i(s)))ds.
\end{aligned}$$

Therefore, even if we change the kernel of (4.2.2) from ψ to $\tilde{\psi}$ at time $t_S + \varepsilon$, p_i still remains as a solution of a differential equation in time $(t_k + \varepsilon, t_{k+1} - \varepsilon)$. Now consider a differential equation

$$\begin{aligned}
\dot{\tilde{q}}_i &= G(\tilde{\nu}_i + \frac{\kappa}{N}\sum_{k=1}^{N} \tilde{\Psi}(\tilde{q}_k(t) - \tilde{q}_i(t))), \\
\tilde{\nu}_i &:= \tilde{p}_i^0 - \frac{\kappa}{N}\sum_{k=1}^{N} (\tilde{\Psi}(\tilde{q}_k^0 - \tilde{q}_i^0)), \quad \tilde{q}_i^0 = \tilde{q}_i(0),
\end{aligned}$$

where $\tilde{q}_i^0 = q_i(t_k + \varepsilon)$, $\tilde{p}_i^0 = p_i(t_k + \varepsilon)$. Since (4.2.11) also holds for $\tilde{q}_i$, by the same argument, we can replace $\tilde{\psi}$ to ψ as well. In other word, for $t \in (t_k + \varepsilon, t_{k+1} - \varepsilon)$, a value of solution is independent of the value of ψ near the origin. Therefore we may assume that ψ is regular and use Lemma 4.1.1 for any $t \notin \mathcal{T}$. As a choice of $\varepsilon > 0$ is arbitrary, we have

$$\frac{d}{dt}\|P\|_{[l]} \le -\frac{\kappa m_{G'} l}{N}\psi(\|Q\|_{[l]})\|P\|_{[l]} + \frac{2\kappa M_{G'}(N-l)P_M^0 L_{\psi,[l]}}{N}\|Q\|_{[l]},$$

$$L_{\psi,[l]}(t) := \sup_{\substack{r,s\geq q_{[l]}(t),\\ r\neq s}} \left|\frac{\psi(r)-\psi(s)}{r-s}\right| < \infty, \quad q_{[l]}(t) := \min_{\substack{i'\in[l]\\ j'\notin[l]}} |q_{i'}(t)-q_{j'}(t)|,$$

for $t \notin \mathcal{T}$. As p_i and p_j stick at t_S, Lemma 4.2.1 yields

$$D_1\varepsilon^{\frac{1}{\alpha}} \leq |q_i(t_S-\varepsilon)-q_j(t_S-\varepsilon)| \leq D_2\varepsilon^{\frac{1}{\alpha}},$$

where

$$D_1 = \left(\frac{2\kappa m_{G'}\alpha}{N(1-\alpha)}\right)^{\frac{1}{\alpha}}, \quad D_2 = \left(\frac{\kappa M_{G'}2^\alpha\alpha}{1-\alpha}\right)^{\frac{1}{\alpha}}.$$

Since $[l]$ is a set of sticking particles, particles p_i and p_j with indices $i\in[l]$ and $j\notin[l]$ are either collisional or separated at time t_S. Therefore for $0<\delta\ll 1$, Lemma 4.2.1 yields

$$\begin{aligned} L_{\psi,[l]}(t_S-\delta) &\leq \left|\frac{d}{dx}\frac{1}{x^\alpha}\right|\Bigg|_{x=C\delta} = \alpha C^{-\alpha-1}\delta^{-\alpha-1} \\ \|Q(t_S-\delta)\|_{[l]} &\leq \sqrt{(l^2-l)\times(D_2\delta^{\frac{1}{\alpha}})^2} \leq lD_2\delta^{\frac{1}{\alpha}}. \end{aligned}$$

Therefore for $0<\delta\ll 1$,

$$\begin{aligned} \frac{d}{dt}\|P(t_S-\delta)\|_{[l]} &\leq -\frac{\kappa m_{G'}l^{1-\alpha}D_2^{-\alpha}}{N}\delta^{-1}\|P(t-\delta)\|_{[l]} \\ &\quad + \frac{2\kappa M_{G'}(N-l)P_M^0}{N}C^{-1-\alpha}\delta^{-1-\alpha}lD_2\delta^{\frac{1}{\alpha}} \\ &=: -K_1\delta^{-1}\|P(t-\delta)\|_{[l]} + K_2\delta^{-1-\alpha+\frac{1}{\alpha}}. \end{aligned}$$

Then for fixed $\varepsilon>0$ and $t_*<t-\varepsilon$, the Grönwall inequality yields

$$\begin{aligned} &\|P(t-\varepsilon)\|_{[l]} \leq \\ &\quad \exp\left(-K_1\int_{t^*}^{t_S-\varepsilon}\frac{1}{t_S-s}ds\right) \\ &\quad \times\left[\|P(t^*)\|_{[l]} + K_2\int_{t^*}^{t_S-\varepsilon}\exp\left(K_1\int_{t^*}^{s}\frac{1}{t_S-u}du\right)(t_S-s)^{-1-\alpha+\frac{1}{\alpha}}ds\right] \\ &\quad = \frac{\varepsilon^{K_1}}{(t_s-t^*)^{K_1}}\times\left[\|P(t^*)\| + K_2(t_s-t^*)^{K_1}\int_{t^*}^{t_S-\varepsilon}(t_S-s)^{-1-\alpha+\frac{1}{\alpha}-K_1}ds\right] \end{aligned}$$

$$\lesssim \begin{cases} \varepsilon^{K_1}(\log(t_S - t^*) - \log\varepsilon), & \text{if } -\alpha + \frac{1}{\alpha} - K_1 = 0, \\ \varepsilon^{K_1}((t_s - t^*)^{-\alpha+\frac{1}{\alpha}-K_1} - \varepsilon^{-\alpha+\frac{1}{\alpha}-K_1}), & \text{if } -\alpha + \frac{1}{\alpha} - K_1 \neq 0. \end{cases}$$

Therefore, if q_i and q_j starts to stick at time t_S, then for $\varepsilon \ll 1$,

$$\begin{aligned} \psi(q_i - q_j)(G(p_i) - G(p_j))|_{t=t_S-\varepsilon} & \\ &\leq M_{G'}\psi((D_1\varepsilon^{\frac{1}{\alpha}}))D_{P,[l]}(t-\varepsilon) \\ &\leq M_{G'}\psi((D_1\varepsilon^{\frac{1}{\alpha}}))\|P(t-\varepsilon)\|_{[l]} \\ &\lesssim \begin{cases} \varepsilon^{K_1-1}(1-\log\varepsilon), & \text{if } -\alpha + \frac{1}{\alpha} - K_1 = 0, \\ |\varepsilon^{K_1-1} - \varepsilon^{-1-\alpha+\frac{1}{\alpha}}|, & \text{if } -\alpha + \frac{1}{\alpha} - K_1 \neq 0. \end{cases} \end{aligned}$$

As $G(p_i(t)) - G(p_j(t)) \equiv 0$ for $t \geq t_S$, we have

$$\begin{aligned} &\psi(q_i - q_j)(G(p_i) - G(p_j))|_{[t_S-\varepsilon, t_S+\varepsilon]} \in L^p, \\ &\text{where} \quad p \in \left[1, \frac{1}{\max\{0, 1-K_1, 1+\alpha-1/\alpha\}}\right). \end{aligned}$$

Furthermore, if $1+\alpha-1/\alpha \leq 0$ and $1 \leq K_1$, then we can choose $p = \infty$. Note that K_1 is increasing in l, the number of simultaneously sticking particles, so that

$$K_1 = \frac{\kappa m_{G'} l^{1-\alpha} D_2^{-\alpha}}{N} = \frac{m_{G'} l^{1-\alpha}(1-\alpha)}{N M_{G'} 2^\alpha \alpha} \geq \frac{m_{G'} 2^{1-2\alpha}(1-\alpha)}{N M_{G'} \alpha} =: K.$$

If q_i and q_j collide at time t_S, where $\nu_i > \nu_j$, by similar calculation in (4.2.8) we have

$$m_{G'}(\nu_i - \nu_j) \leq G(p_i(t_S)) - G(p_j(t_S)) \leq M_{G'}(\nu_i - \nu_j),$$

and $G(p_i(t_S)) - G(p_j(t_S))$ is nonzero bounded in a neighborhood of t_S. Together with Lemma 4.2.1, this yields

$$\psi(q_i - q_j)(G(p_i) - G(p_j))|_{t=t_S\pm\varepsilon} \lesssim \varepsilon^{-\alpha}.$$

In this case, we have

$$\psi(q_i - q_j)(G(p_i) - G(p_j))|_{[t_S-\varepsilon, t_S+\varepsilon]} \in L^p, \quad \text{where} \quad p \in \left[1, \frac{1}{\alpha}\right).$$

If p_i and p_j are neither collisional nor sticking at t_S, then $\psi(q_i - q_j)(G(p_i) - G(p_j))$ is bounded near t_S. On the other hand, $\psi(q_i(s) - q_j(s))(G(p_i(s)) - G(p_j(s)))$ is bounded for s such that $|s - t_c| > \varepsilon,\ t_c \in \mathcal{T}$. Putting the results altogether, we conclude

$$\sum_{i=1}^{N} \psi(q_i - q_j)(G(p_i) - G(p_j)) \in L^p_{\text{loc}}(\mathbb{R}_+),$$
$$\text{where} \quad p \in \left[1, \frac{1}{\max\{1-K, \alpha\}}\right).$$

This proves the first assertion of the Theorem 4.2.1.

To prove the second assertion, we consider the solution to (4.2.2) emanating from an initial data $(P(T), Q(T))$, where $T > t_c$. Then from Proposition 4.2.2, we have either

$$q_i(t) \equiv q_j(t) \text{ for } t > T, \quad \text{or} \quad \inf_{t \geq T} |q_i(t) - q_j(t)| > C > 0,$$

for some constant $C > 0$. Therefore, as we did in the beginning of the proof, a value of solution for $t > T$ is independent of the value of ψ near the origin. Therefore we may assume that ψ is regular and apply Theorem 4.1.1. Since $\int_r^\infty \psi(x)dx = \infty$ for any $r > 0$, we conclude that the flocking emerges unconditionally. $\square$

4.3 Analysis under strongly singular communications

In this section, we consider strongly singular communications, which are not integrable near the origin. A typical example is

$$\psi(q) = \frac{1}{|q|^\alpha}, \quad \text{where} \quad \alpha \geq 1. \tag{4.3.1}$$

As in the previous section, the well-posedness of a solution is directly related to singularity arising from a collision. For a strongly singular kernel case, this issue can be treated by the so-called 'collision avoidance property' of the strongly singular kernel.

Proposition 4.3.1. *Suppose that*

$$\psi \in (C^{0,1}_{\mathrm{loc}} \cap L^1_{\mathrm{loc}})(\mathbb{R}_+;\mathbb{R}_+), \quad (\psi(r)-\psi(s))(r-s) \le 0, \quad \forall r,s \in \mathbb{R}_+,$$

and let Q be a solution of (4.1.1) *with noncollisional initial data (P^0,Q^0). If $\int_0^\varepsilon \psi(r)dr = \infty$ for any $\varepsilon > 0$, then there exists a unique global classical solution with the collision avoidance property:*

$$\inf_{t\in[0,T]} \min_{\substack{i,j\in[N]\\ i\neq j}} |q_i(t)-q_j(t)| > 0, \quad \forall T \in \mathbb{R}_+.$$

Furthermore, if the ambient space is one-dimensional($d=1$), then we have

$$\inf_{t\ge 0} \min_{\substack{i,j\in[N]\\ i\neq j}} |q_i(t)-q_j(t)| > 0.$$

Proof. Although the Proposition can proved by direct modification of [10, Theorem 5.2] and [11, Theorem 3.1], we provide a more simple proof here. Since Q^0 is non-collisional, from the standard Cauchy-Lipschitz theory, a solution is well-posed before the first collision time τ (i.e. the smallest $\tau > 0$ satisfying $q_i(\tau) = q_j(\tau)$ for some $i,j,i \neq j$). To establish the global well-posedness, we first observe that the collision does not happen in any finite time. Suppose that the first collision time $\tau \in \mathbb{R}_+$ exists and let q_i be a colliding particle. By the rearrangement of indices suppose define a set of indices $[l] \subset [N]$ as

$$[l] := \{j \in [N] \mid q_i(\tau) = q_j(\tau)\} \neq \emptyset.$$

Then, for any $\varepsilon > 0$, we can apply the second estimate in Lemma 4.1.1 in time $t \in [0, \tau-\varepsilon)$. From the definition of $[l]$, there exists two positive constants $C_1, C_2 > 0$ satisfying

$$\frac{d}{dt}\|P\|_{[l]} \le -C_1\psi(\|Q\|_{[l]})\|P\|_{[l]} + C_2\|Q\|_{[l]}, \quad t \in [0,\tau-\varepsilon). \tag{4.3.2}$$

Now define the functional

$$\tilde{\mathcal{L}}(t) := \frac{C_1}{M_{G'}} \int_{\|Q^0\|_{[l]}}^{\|Q(t)\|_{[l]}} \psi(s)\, ds.$$

Then $|\tilde{\mathcal{L}}(t)| + \|P(t)\|_{[l]}$ have a linear or sub-linear growth; there exists a positive constant C satisfying

$$\begin{aligned}\frac{d}{dt}|\tilde{\mathcal{L}}(t)| + \frac{d}{dt}\|P(t)\|_{[l]} &\le \left|\frac{d}{dt}\tilde{\mathcal{L}}(t)\right| + \frac{d}{dt}\|P(t)\|_{[l]}\\ &= \left|\frac{C_1}{M_{G'}}\psi(\|Q(t)\|_{[l]})\frac{d}{dt}\|Q(t)\|_{[l]}\right| + \frac{d}{dt}\|P(t)\|_{[l]} \le C_2\|Q\|_{[l]} < C < \infty,\end{aligned}$$

where we used (4.1.5) and (4.3.2) for the second inequality. Therefore if collision happen, there exists a constant C satisfying

$$\infty = \lim_{t\nearrow\tau}|\tilde{\mathcal{L}}(t)| \lesssim C(1+\tau) < \infty,$$

which yields a contradiction. Therefore collision cannot happen in any finite time, and this proves the existence and uniqueness of a global classical solution.

Now suppose $d = 1$. From (4.1.1), we can deduce an integral equation

$$G^{-1}(\dot{q}_i(t)) = G^{-1}(\dot{q}_i(0)) + \frac{\kappa}{N}\int_0^t \sum_{k=1}^N \alpha_{ki}(s)(G^{-1}(\dot{q}_k(s)) - G^{-1}(\dot{q}_i(s)))ds,$$

for $t \in \mathbb{R}_+$, where the modified kernel α_{ki} is defined as

$$\alpha_{ki}(t) = \begin{cases} 0 & \text{if } \dot{q}_k(t) = \dot{q}_i(t),\\ \phi(q_k(t) - q_i(s)) \times \dfrac{\dot{q}_k(t) - \dot{q}_i(t)}{G^{-1}(\dot{q}_k(t)) - G^{-1}(\dot{q}_i(t))} & \text{if } \dot{q}_k(t) \ne \dot{q}_i(t).\end{cases}$$

so that α_{ki} is nonnegative, measurable and symmetric with respect to indices. Since a finite-in-time collision never happens, a solution is well defined globally, and [37, Theorem 1] guarantees the existence the following uniform-in-t bound U for each $i, j \in [N] (i \ne j)$:

$$\begin{aligned}\left|\int_{q_i^0 - q_j^0}^{q_i(t)-q_j(t)} \psi(r)dr\right| &= \left|\int_0^t \psi(q_i(s) - q_j(s))(\dot{q}_i(s) - \dot{q}_j(s))ds\right|\\ &= \left|\int_0^t \alpha_{ij}(s)(G^{-1}(\dot{q}_i(s)) - G^{-1}(\dot{q}_j(s)))ds\right|\end{aligned}$$

$$\leq \int_0^\infty \alpha_{ij}(s)|(G^{-1}(\dot{q}_i(s)) - G^{-1}(\dot{q}_j(s))|ds =: U < \infty.$$

As ψ is not integrable near the origin, we conclude

$$\sup_{t\geq 0} \max_{\substack{i,j\in[N] \\ i\neq j}} \left| \int_{q_i^0 - q_j^0}^{q_i(t)-q_j(t)} \psi(r)dr \right| \leq U \quad \Longrightarrow \quad \inf_{t\geq 0} \min_{\substack{i,j\in[N] \\ i\neq j}} |q_i(t) - q_j(t)| > 0.$$

□

Remark 4.3.1.

1. *From Proposition 4.3.1, the origin of the kernel is not referred to in any finite time. Therefore, under the same assumption in Proposition 4.3.1, although ψ is not Lipschitz, the results of Theorem 4.1.1 still hold.*

2. *Although the proof of Proposition 4.3.1 is rather simple, explicit lower bounds between agents cannot be deduced. For the explicit expression for a lower bound, refer to the proof of [10, Theorem 5.2].*

3. *If the kernel is weakly singular at the origin (i.e., $\int_0^\varepsilon \psi(r)dr < \infty$ for some $\varepsilon > 0$), then a collision might happen, as described in the previous section.*

For the Euclidean space of arbitrary dimension, the authors of [54] derived an existence of strict positive lower bound for relative distances under $\alpha > 2$ and $G = \mathrm{Id}$:

$$\min_{i,j\in[N]} |q_i^0 - q_j^0| > 0 \quad \Rightarrow \quad \inf_{t\geq 0} \min_{i,j\in[N]} |q_i(t) - q_j(t)| \geq L_\infty > 0,$$

by employing a suitable potential energy with a dissipative structure. Unfortunately, the dissipation of potential energy heavily depends on the Galilean invariance, which (4.1.1) lacks due to the presence of an activation function. Instead, we provide an alternative characterization for the existence of L_∞.

Theorem 4.3.1. *Let (P, Q) be a solution of* (4.1.1) *with a kernel of the form* (4.3.1). *Suppose that Q is non-collisional and further assume that*

1. *$\alpha \neq 1$, and*

2. (P, Q) exhibits flocking.

Then there exists a strictly positive lower bound of distance between agents:

$$\inf_{t\geq 0}\min_{\substack{i,j\in[N]\\ i\neq j}}|q_i(t)-q_j(t)|>0.$$

Proof. From Proposition 4.3.1, the result of Theorem 4.1.1 holds for kernel of the form (4.3.1) as well. Therefore there exists a constant $C>0$ satisfying

$$\max_{i,j}|p_i(t)-p_j(t)|\lesssim e^{-tC},\quad t\in\mathbb{R}_+,$$

and the limit $\lim_{t\to\infty}|q_i(t)-q_j(t)|$ always exists for any indices. In particular if $\lim_{t\to\infty}|q_i(t)-q_j(t)|=0$, then we have

$$\begin{aligned}|q_i(t)-q_j(t)|&=\left|\int_\infty^t(\dot{q}_i(t)-\dot{q}_j(t))dt\right|=\left|\int_t^\infty(\dot{q}_j(t)-\dot{q}_i(t))dt\right|\\&\leq M_{G'}\int_t^\infty|p_i(s)-p_j(s)|ds\lesssim\int_t^\infty e^{-sC}ds\lesssim e^{-tC}.\end{aligned}\tag{4.3.3}$$

Now for some index i, suppose that there exists a set $[l]\subset[N]$ defined as

$$[l]:=\{j\in[N]\mid\lim_{t\to\infty}(q_i(t)-q_j(t))=0\}\neq\emptyset.$$

From (4.3.3), there positive constants $B,C>0$ satisfying $\|Q(t)\|_{[l]}\leq Be^{-tC}$. Therefore for sufficiently large $t\gg 1$, we obtain

$$\begin{aligned}\left|\int_{\|Q^0\|_{[l]}}^{\|Q(t)\|_{[l]}}\psi(s)ds\right|&=\int_{\|Q(t)\|_{[l]}}^{\|Q^0\|_{[l]}}\psi(s)ds\\&\geq\int_{Be^{-tC}}^{\|Q^0\|_{[l]}}\psi(s)ds=\frac{B^{1-\alpha}e^{tC(\alpha-1)}-\|Q^0\|_{[l]}^{1-\alpha}}{\alpha-1}.\end{aligned}$$

Since the limit of $|p_i(t)-p_j(t)|$ always exists and finite-in-time collision never happens, there exists a positive constant U satisfying

$$\sup_{t\geq 0}\sup_{\substack{i\in[l]\\ j\notin[l]}}\psi(q_i-q_j)<U<\infty.$$

Therefore $|\tilde{\mathcal{L}}|$ defined in the proof of Proposition 4.3.1 have a linear or sub-linear growth, which leads to the contradiction:

$$e^{tC(\alpha-1)} - 1 \lesssim \left| \int_{\|Q^0\|_{[l]}}^{\|Q(t)\|_{[l]}} \psi(s)ds \right| \lesssim 1 + t.$$

Therefore we conclude $[l] = \emptyset$, as desired. □

Remark 4.3.2. *1. Let (P, Q) be a solution to (4.1.1) with non-collisional initial data (P^0, Q^0). Corollary 4.1.1 states a high value of κ leads to not only flocking but also the strict spacing between the agents;*

$$\min_{\substack{i,j\in[N]\\ i\neq j}} \inf_{t\geq 0} |q_i(t) - q_j(t)| > 0 \tag{4.3.4}$$

Moreover, when communication is of form $\psi(x) = |x|^{-\alpha}$, the theorem in [54] states that (4.3.4) can be achieved for arbitrary $\kappa > 0$ under $\alpha > 2$ and $G = \mathrm{Id}$. However, to the author's knowledge, a further result to have (4.3.4) under $\alpha \in [1, 2]$ is missing. Meanwhile, Theorem 4.1.1 and 4.3.1 states that κ can be arbitrarily small when α is close to 1. Therefore Theorem 4.1 may complement the previous result.

2. *If a strictly positive lower bound of the relative state is guaranteed, as we did in the proof of Theorem 4.2.1, we may regularize the kernel. As an application, for example, we may apply results of stability estimates in [32] for singular kernels as well, even though proof of the theorem requires the Lipschitz continuity of the kernel. On the other hand, we may relax a priori condition for stability estimate in [1], since some of the conditions are devoted to ensuring strict lower bound between relative states.*

3. *In many cases concerning a many-body system equipped with a singular kernel, it is often desirable to guarantee a strictly positive lower bound for a relative distance between agents. This, for example, guarantees the well-definedness of a ω-limit set, which enables us to apply the dynamical system theory like LaSalle's invariance principle.*

4.4 The kinetic description

We recall the the CS-type model (4.1.1):

$$\begin{cases} \dot{q}_i = G(p_i), \quad t > 0, \quad i \in [N] := \{1, 2, \cdots, N\}, \\ \dot{p}_i = \dfrac{\kappa}{N} \displaystyle\sum_{k=1}^{N} \phi(q_k - q_i)(G(p_k) - G(p_i)), \\ (q_i, p_i)\big|_{t=0+} = (q_i^0, p_i^0), \quad p_i, q_i \in \mathbb{R}^d, \quad \phi(q) = \frac{1}{|q|^\alpha}, \end{cases} \tag{4.4.1}$$

where we used $\phi : q \mapsto \frac{1}{|q|^\alpha}$ instead of $\psi : |q| \mapsto \frac{1}{|q|^\alpha}$ for the convenience of further analysis. In the current section, we are interested with a kinetic description of CS-type model (4.4.1), which is of the form:

$$\begin{cases} \partial_t f + G(p) \cdot \nabla_q f + \nabla_p \cdot (L[f]f) = 0, \quad (t, q, p) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d, \\ L[f](t, q, p) := \int_{\mathbb{R}^{2d}} \phi\big(q_* - q\big)(G(p_*) - G(p))f(t, q_*, p_*)dq_* dp_*, \\ f(0, q, p) = f^0(q, p), \quad \phi(q) = \dfrac{1}{|q|^\alpha}. \end{cases} \tag{4.4.2}$$

For the formal derivation of (4.4.2), we use the standard BBGKY hierarchy to derive the following kinetic equation of the probability density function $f = f(t, q, p)$.

Note that (4.4.2) can be regarded as a generalization of (4.4.1). To see this, Let $\{q_i(t), p_i(t)\}_{i=1}^N$ be a solution of (4.4.1). If we regard f as a distribution(generalized function), then $f(t, q, p) = \frac{1}{N}\sum_{i=1}^N \delta_{(q_i(t), p_i(t))}$ is a distributional weak solution of (4.4.2) for Direc-delta distribution δ and vice versa (for more details, see [44]). Therefore, it might be subtle to consider a kinetic analog of (4.4.1) with non-discrete support, since Section 4.3 features out the collision avoidance property of a singular kernel. Nevertheless, one can interpret the Theorem 3.2.1 as an allowance of collision or sticking of characteristics in terms of (4.4.2), as far as singularity is not too strong. On the other hand, sticking of characteristics might be interpreted as a blow-up of one-particle distribution function, which violates the regularity of a solution.

Indeed, we will prove the *local-in-time* well-posedness of (4.4.2) under weak singularity.

4.4.1 Preliminaries in optimal transport theory

In this subsection, we present the basic definitions. Let $\mathcal{M}(\mathbb{R}^{2d})$ be the set of regular Borel measures on $\mathbb{R}^{2d}$. We denote the duality paring between measure and function by

$$\langle \mu, f \rangle := \int_{\mathbb{R}^{2d}} f(q,p) d\mu(q,p), \quad \mu \in \mathcal{M}(\mathbb{R}^{2d}).$$

To employ the language of optimal transport theory, we introduce several definitions and related properties. Definitions for a generic $p \in [1, \infty)$ will be provided, and we mainly deal with the case of $p = 1$.

Definition 4.4.1. *Let $(X, \|\cdot\|)$ be a normed vector space, $\mathcal{P}(X) \subset \mathcal{M}(X)$ be a space of probability measures on X, and and $p \in [1, \infty)$.*

1. *The Wasserstein space of order p on X is defined as a collection of probability measures with a finite p-th moment:*

$$\mathcal{P}_p(X) := \left\{ \mu \in \mathcal{P}(X) : \langle \mu, \|y\|^p \rangle = \int_X \|y\|^p d\mu(y) < \infty \right\}.$$

2. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $T : \mathbb{R}^d \to \mathbb{R}^d$ be a measurable mapping. Then the push-forward of μ by T is the measure $T\#\mu \in \mathcal{M}(\mathbb{R}^d)$ defined by*

$$T\#\mu(B) := \mu(T^{-1}(B)) \quad \text{for any Borel set } B \subset \mathbb{R}^d.$$

   Such T is called a transport map *from μ to $T\#\mu$ in the context of optimal transport.*

3. *Let μ and ν in $\mathcal{P}_p(X)$ be two measures. Then, the Wasserstein metric W_p of order p between μ and ν is given by*

$$W_p(\mu, \nu) := \left(\inf_{\gamma \in \Pi(\mu,\nu)} \int_{X \times X} \|y - \tilde{y}\|^p d\gamma(y, \tilde{y}) \right)^{\frac{1}{p}},$$

where $\Pi(\mu,\nu)$ is the collection of probability measures on $X \times X$ with marginals μ and ν:

$$\Pi(\mu,\nu) = \{\gamma \in \mathcal{M}(X \times X) \ : \ \pi_1\#\gamma = \mu, \ \pi_2\#\gamma = \nu\},$$

where $\pi_i(x_1,x_2) = x_i$. Such $\gamma \in \Pi(\mu,\nu)$ are called the transport plans, and those achieving the infimum, if exist, are called the optimal transport plans.

We recall some classic properties in Definition 4.4.1.

Remark 4.4.1.

1. *(Kantorovich-Rubinstein Duality for $p = 1$) For a compact set $X \subset \mathbb{R}^d$, Wasserstein-1 distance on $\mathcal{P}_1(X)$ coincide with the bounded Lipschitz distance, which is also known as* Monge-Kantorovich-Rubinstein distance d_{Lip}*:*

$$W_1(\mu_1,\mu_2) = d_{Lip}(\mu_1,\mu_2) := \sup_{\substack{\phi \in \mathrm{Lip}(X) \\ \mathrm{Lip}[\phi] \le 1}} \int_X \phi(x) d(\mu_1 - \mu_2)(x).$$

2. *If $\mu,\nu \in \mathcal{P}_p(X)$ $(1 \le p < \infty)$ are atomless, there exists an optimal transport map from μ to ν. That is, there exists a transport map $\mathcal{T}$ from μ to ν achieving the following infimum:*

$$\inf_T \int_X \|x - T(x)\|^p d\mu(x) = \int_X \|x - \mathcal{T}(x)\|^p d\mu(x),$$

where the infimum is taken over the set of transport maps from μ to ν. Furthermore, for a compact subset X of $\mathbb{R}^d$, it turns out that $(\mathrm{Id}\times\mathcal{T})\#\mu$ is an optimal transport. In particular, we have

$$W_p^p(\mu,\nu) = \int_X \|x - \mathcal{T}(x)\|^p d\mu(x).$$

3. *The definition of push-forward measure is equivalent to adjunction formula:*

$$\langle F\#\mu, \phi\rangle = \langle \mu, \phi \circ F\rangle \quad \textit{for any } \phi \in C_b(\mathbb{R}^d).$$

Therefore, for $\mu \in \mathcal{P}_p(\mathbb{R}^d)$ and Borel mapping $f_1, f_2 : \mathbb{R}^d \to \mathbb{R}^d$, the following inequality holds:

$$W_p^p(f_1\#\mu, f_2\#\mu) \le \Big\langle \big((f_1 \times f_2)\#\mu\big)(x,y), |x-y|^p \Big\rangle = \langle \mu, |f_1 - f_2|^p\rangle.$$

4.4.2 Local well-posedness

We start by rewriting (4.4.1) in a view of measure. Let $C([0,T];\mathcal{P}_1(X))$ be the space of all continuous probability measure-valued function from $[0,T]$ to $\mathcal{P}_1(X)$.

Definition 4.4.2 (Weak Solution)**.** *For $T \in [0,\infty]$, f is a weak solution of* (4.4.1) *on the time interval $[0,T)$ if and only if*

1. *$f \in L^\infty(0,T;L^p(\mathbb{R}^{2d}))$ and $fdqdp \in C([0,T];\mathcal{P}_1(\mathbb{R}^{2d}))$.*
2. *f satisfies the following equation for all the test functions $g \in C_c^\infty([0,T]\times\mathbb{R}^{2d})$:*

$$\int_{\mathbb{R}^{2d}} f(t,q,p)g(t,q,p)dqdp - \int_{\mathbb{R}^{2d}} f^0(q,p)g^0(q,p)dqdp$$
$$= \int_0^T \int_{\mathbb{R}^{2d}} f(\partial_t g + G(p)\cdot\nabla_x g + L[f]\cdot\nabla_p g)dqdpdt.$$

If f is a weak solution of (4.4.1), then $f(t,q,p)dqdp = \mu^t(dq,dp)$ is a measure-valued solution of (4.4.2). Conversely, if a measure valued solution μ^t of (4.4.2) is absolutely continuous with respect to Lebesgue measure with Radon-Nikodym derivative $f(t,q,p)dqdp = \mu^t(dq,dp)$, then f is a weak soluton of (4.4.1). In particular, if f is a function with sufficient regularity, μ can be translated into classical solution f. Therefore, we use $f(t,q,p)$ and $\mu^t(dq,dp)$ interchangeably: for example, when $f_i(t,q,p)dqdp = \mu_i^t(dq,dp)$ for $i=1,2$, we may abuse notation and denote

$$W_p(f_1,f_2) := W_p(\mu_1^t,\mu_2^t).$$

Similarly, we may write $f_2 = F\# f_1$ in place of $\mu_2^t = F\#\mu_1^t$.

Since the solution of (4.4.2) is globally well-posed under regular kernel, our strategy is to approximate the solution of (4.4.2) via its regularized system. We introduce a Dirac sequence of radially symmetric mollifiers in $\mathbb{R}^d$:

$$\zeta(x) = \bar{\zeta}(|x|) \geq 0, \quad \zeta \in C_c^\infty(\mathbb{R}^d), \quad \mathrm{supp}(\zeta) \subset B_1(0),$$

$$\int_{\mathbb{R}^d} \zeta dx = 1, \quad \zeta_\varepsilon(x) := \frac{1}{\varepsilon^d}\zeta\left(\frac{x}{\varepsilon}\right).$$

Using regularized communication weight, we introduce regularized system for (4.4.2):

$$\begin{cases} \partial_t f_\varepsilon + G(p)\cdot\nabla_q f_\varepsilon + \nabla_p\cdot(L_\varepsilon[f_\varepsilon]f_\varepsilon) = 0, \quad (t,q,p)\in\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}^d, \\ L_\varepsilon[f_\varepsilon](t,q,p) := \int_{\mathbb{R}^{2d}}\phi_\varepsilon\big(q_*-q\big)(G(p_*)-G(p))f_\varepsilon(t,q_*,p_*)dq_*dp_*, \\ f_\varepsilon(0,q,p) = f^0(q,p), \quad \phi_\varepsilon(q) = \dfrac{1}{|q|^\alpha}*\zeta_\varepsilon. \end{cases} \tag{4.4.3}$$

NOTATION. From now on, C will denote a generic positive constant which may vary even in the same line. Given a constant $\beta\in(1,\infty]$, Hölder conjugate of β will be denoted by γ so that $\frac{1}{\beta}+\frac{1}{\gamma}=1$. We denote the norms by

$$\|f\|_{L^\beta} := \|f\|_{L^\beta(U)} \quad \text{where } U=\mathbb{R}^d \text{ or } \mathbb{R}^{2d}, \quad \|f\| := \|f\|_{L^\infty(0,T;L^\beta)},$$

and define the momentum support and its maximum modulus by $R_\varepsilon(t) := \max_{p\in\Omega_\varepsilon(t)}|p|$ and

$$\Omega_\varepsilon(t) := \mathrm{cl}(\{\ p\in\mathbb{R}^d \mid f_\varepsilon(t,q,p)\neq 0 \text{ for some } (q,p)\in\mathbb{R}^{2d}\ \}).$$

When notational simplicity is required, we abbreviate variables in the following way.

$$(q,p) = z, \quad (q_*,p_*) = z_*, \quad dqdp = dz, \quad dq_*dp_* = dz_*.$$

Analogues to Proposition 4.1.1, we first observe the decrement in the maximal momentum.

Proposition 4.4.1. *Let $Z_\varepsilon(t)$ be a solution to the particle trajectory* (4.4.3) *emanating from an initial point in the support of classicial solution $f_{\varepsilon,0}$. If initial velocity support is bounded, then we have*

$$\frac{d}{dt}R_\varepsilon(t)\le 0, \quad t>0.$$

Proof. We denote Z by bi-characteristics $Z_\varepsilon(s) = (Q_\varepsilon(s;0,q,p), P_\varepsilon(s;0,q,p))$ satisfying

$$\begin{cases} \dfrac{dQ_\varepsilon(s)}{ds} = G(P_\varepsilon) \\ \dfrac{dQ_\varepsilon(s)}{ds} = L_\varepsilon[f_\varepsilon](s, P_\varepsilon(s)) \\ Z_\varepsilon(0) = (q,p) \end{cases} \tag{4.4.4}$$

For each t, we choose an initial data in support of $f_{\varepsilon,0}$ which generates characteristic curve (q,p) satisfying $R_\varepsilon(t) = |P_\varepsilon(t)|$. Then we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}(R_\varepsilon(t))^2 &= \frac{1}{2}\frac{d}{dt}|P_\varepsilon(t)|^2 = P_\varepsilon(t)\cdot\frac{d}{dt}P_\varepsilon(t) \\ &= \int_{\mathbb{R}^{2d}} \phi_\varepsilon(q_* - Q_\varepsilon(t))(G(p_*) - G(P_\varepsilon(t)))\cdot P_\varepsilon(t) f_\varepsilon(t,q_*,p_*)dq_*dp_*. \end{aligned}$$

Since $|p| \mapsto g(|p|)$ is an increasing function, maximality of P_ε implies

$$(G(p_*) - G(P_\varepsilon)(t))\cdot P(t) = (g(|p_*|)p_* - g(|P_\varepsilon|)P_\varepsilon)\cdot P_\varepsilon \le 0,$$

and therefore R_ε is a decreasing function. □

Proposition 4.4.2. *Let f_ε be a classical solution to* (4.4.3) *which vanish at infinity sufficiently fast with $\|f_\varepsilon^0\|_{L^\beta} < \infty$ and satisfies $R_\varepsilon(0) =: R_p < \infty$. If $\alpha\gamma < d$, then there exists a $T > 0$, independent in ε, such that the uniform $L^1 \cap L^\beta$-estimate of f_ε holds:*

$$\sup_{t\in[0,T]} \|f_\varepsilon\|_{L^1\cap L^\beta} \le C, \quad \textit{where} \quad \|\cdot\|_{L^1\cap L^\beta} := \|\cdot\|_{L^1} + \|\cdot\|_{L^\beta},$$

holds for a positive constant C independent in and ε.

Proof. Since f_ε vanish at infinity, we have

$$\frac{d}{dt}\int_{\mathbb{R}^{2d}} f_\varepsilon dz = \frac{d}{dt}\int_{\mathbb{R}^{2d}} 1\times f_\varepsilon dz = 0, \quad \text{therefore} \quad \|f_\varepsilon\|_{L^1} \equiv \|f_\varepsilon^0\|_{L^1}.$$

Now we estimate $\|f_\varepsilon\|_{L^\beta}$ by

$$\frac{d}{dt}\int_{\mathbb{R}^{2d}} f_\varepsilon^\beta dz = \int_{\mathbb{R}^{2d}} \partial_t f_\varepsilon^\beta dz = (1-\beta)\int_{\mathbb{R}^{2d}} f_\varepsilon^\beta \nabla_p \cdot L_\varepsilon[f_\varepsilon]dz. \tag{4.4.5}$$

To estimate the last term, we consider a cut-off function $\chi : \mathbb{R}^d \to \mathbb{R}$

$$\chi(x) := \begin{cases} 1 & |x| \le 1, \\ 0 & |x| > 1, \end{cases}$$

and apply Young's convolution inequality to see

$$\begin{aligned} &\|(\phi\chi) * \zeta_\varepsilon\|_{L^\gamma} \le \|\phi\chi\|_{L^\gamma} < +\infty, \\ &\|(\phi(1-\chi)) * \zeta_\varepsilon\|_{L^\infty} \le \|\phi(1-\chi)\|_{L^\infty} \le 1, \end{aligned} \tag{4.4.6}$$

where we used $\alpha\gamma < d$ to guarantee that $\|\phi\chi\|_{L^\gamma}$ is finite. Note that

$$\nabla_p \cdot G(p) = \text{trace}(G') = \text{sum of eigenvalues of } G' \le d \max_{|p| \le R_\varepsilon(0)} g'(|p|) =: G_p,$$

from Proposition 4.4.1, and G_p is independent of ε, since it depends only on the initial data. We then have

$$\begin{aligned} |\nabla_p \cdot L_\varepsilon[f_\varepsilon]| &\le G_p \int_{\mathbb{R}^{2d}} |(\phi\chi) * \zeta_\varepsilon||f_\varepsilon| dz_* + G_p \int_{\mathbb{R}^{2d}} |(\phi(1-\chi)) * \zeta_\varepsilon||f_\varepsilon| dz_* \\ &\le G_p R_p^{\frac{1}{\gamma}} \|\phi\chi\|_{L^\gamma} \|f_\varepsilon\|_{L^\beta} + \|\phi(1-\chi)\|_{L^\infty} \|f_\varepsilon\|_{L^1} \le C\|f_\varepsilon\|_{L^1 \cap L^\beta}. \end{aligned}$$

Therefore from (4.4.5) we obtain

$$\frac{d}{dt}\|f_\varepsilon\|_{L^\beta}^\beta \le C\|f_\varepsilon\|_{L^\beta}^\beta \|f_\varepsilon\|_{L^1 \cap L^\beta}.$$

Using $\frac{d}{dt}\|f_\varepsilon\|_{L^1} = 0$, we achive

$$\frac{d}{dt}\|f_\varepsilon\|_{L^1 \cap L^\beta} \le C\|f_\varepsilon\|_{L^1 \cap L^\beta}^2.$$

Hence, by the comparison principle, there exist $T > 0$, independent of ε, c, such that $\|f_\varepsilon\|_{L^1 \cap L^\beta}$ does not blow up in $[0, T]$, which is the desired result. □

Remark 4.4.2. *From Proposition 4.4.1, if an initial data f_0 of* (4.4.3) *has compact (q,p)-support, then $f(t,\cdot,\cdot)$ also have compact (q,p)-support for any finite t. In this case, L^1 norm is uniformly controlled by L^β. In particular, we can replace estimates on $\|f(t)\|_{L^1 \cap L^\beta}$ by the one using $\|f(t)\|_{L^\beta}$.*

Proposition 4.4.3. *Let f_ε and $f_{\varepsilon'}$ be two classical solutions of the system* (4.4.3) *satisfying*

$$\mathrm{supp}(f_0) \subset\subset \mathbb{R}^{2d}, \quad \|f^0\|_{L^\beta} < \infty, \quad 0 < \alpha < \frac{d}{\gamma} - 1.$$

Then there exist a constant C independent in $\varepsilon, \varepsilon'$ such that

$$\frac{d}{dt} W_1(f_\varepsilon(t), f_{\varepsilon'}(t)) \le C(W_1(f_\varepsilon(t), f_{\varepsilon'}(t)) + \varepsilon + \varepsilon'). \tag{4.4.7}$$

Proof. Throughout the proof, we denote $d\mu_\varepsilon^t := f_\varepsilon(t,q,p)dqdp$ and $d\mu_{\varepsilon'}^t := f_{\varepsilon'}(t,q,p)dqdp$. For each (t,q,p), we define characteristic curve

$$Z_\varepsilon(s) = (Q_\varepsilon(s;t,q,p), P_\varepsilon(s;t,q,p))$$

as a solution to the following ODEs:

$$\begin{cases} \frac{d}{ds} Q_\varepsilon(s) = G(P_\varepsilon(s;t,q,p)), \\ \frac{d}{ds} P_\varepsilon(s) = L_\varepsilon[f_\varepsilon]\big(s, Z_\varepsilon(s)\big), \\ (Q_\varepsilon(t;t,q,p), P_\varepsilon(t;t,q,p)) = (q,p), \end{cases}$$

so that $f_\varepsilon(t) = Z_\varepsilon(t;t_0,\cdot,\cdot)\# f_\varepsilon(t_0)$, and define $Z_{\varepsilon'}(s)$ in similar way. To emphasize a role as a transport map, $Z_\varepsilon(t_2;t_1,\cdot,\cdot)$ will be occasionally denoted by $\mathcal{T}_\varepsilon^{t_1\to t_2}$. As $\mu_\varepsilon^{t_0}$ and $\mu_{\varepsilon'}^{t_0}$ are absolutely continuous with respect to Lebesgue measure, they are atomless and we can introduce an *optimal* transport map $\mathcal{T}_{\varepsilon\to\varepsilon'}^{t_0}$ from $f_\varepsilon(t_0)$ to $f_{\varepsilon'}(t_0)$ so that

$$\mu_{\varepsilon'}^{t_0} = \mathcal{T}_{\varepsilon\to\varepsilon'}^{t_0} \# \mu_\varepsilon^{t_0}, \quad \mathcal{T}_{\varepsilon\to\varepsilon'}^{t_0}(q,p) =: (q_{\varepsilon'}, p_{\varepsilon'}),$$

and we define the transport map from $f_\varepsilon(t)$ to $f_{\varepsilon'}(t)$:

$$\mathcal{T}_{\varepsilon\to\varepsilon'}^{t} \# \mu_\varepsilon^t = \mu_{\varepsilon'}^t, \quad \text{where} \quad \mathcal{T}_{\varepsilon\to\varepsilon'}^{t} := \mathcal{T}_{\varepsilon'}^{t_0\to t} \circ \mathcal{T}_{\varepsilon\to\varepsilon'}^{t_0} \circ \mathcal{T}_\varepsilon^{t\to t_0},$$

and regard $\mathcal{T}_\varepsilon^{t_0}$ as the identity map Id.

From Remark 4.4.1, we have

$$\begin{aligned} W_1(\mu_\varepsilon^t, \mu_{\varepsilon'}^t) &= W_1(\mathcal{T}_\varepsilon^{t_0\to t} \# \mu_\varepsilon^{t_0}, (\mathcal{T}_{\varepsilon'}^{t_0\to t} \circ \mathcal{T}_{\varepsilon\to\varepsilon'}^{t_0}) \# \mu_\varepsilon^{t_0}) \\ &\le \big\langle \mu_\varepsilon^{t_0}, \big| \mathcal{T}_\varepsilon^{t_0\to t} - (\mathcal{T}_{\varepsilon'}^{t_0\to t} \circ \mathcal{T}_{\varepsilon\to\varepsilon'}^{t_0})) \big| \big\rangle \\ &= \big\langle \mu_\varepsilon^{t_0}, \big| Z_\varepsilon(t;t_0,\cdot,\cdot) - Z_{\varepsilon'}\big(t;t_0, \mathcal{T}_{\varepsilon\to\varepsilon'}^{t_0}(\cdot,\cdot)\big) \big| \big\rangle =: \mathcal{Q}_{\varepsilon,\varepsilon'}(t). \end{aligned} \tag{4.4.8}$$

Differentiating with respect to t and evaluating $t = t_0^+$, we get

$$\begin{aligned}
&\frac{d}{dt}\mathcal{Q}_{\varepsilon,\varepsilon'}(t)\Big|_{t=t_0^+} \\
&\quad \le \left\langle \mu_\varepsilon^{t_0}, \left|G(P_\varepsilon(t;t_0,\cdot,\cdot)) - G(P_{\varepsilon'}\big(t;t_0,\mathcal{T}^{t_0}_{\varepsilon\to\varepsilon'}(\cdot,\cdot))\big)\right|\right\rangle\Big|_{t=t_0^+} \\
&\qquad + \left\langle \mu_\varepsilon^{t_0}, \left|L_\varepsilon[f_\varepsilon]\big(t, Z_\varepsilon(t;t_0,\cdot,\cdot)\big) - L_{\varepsilon'}[f_{\varepsilon'}](t, Z_{\varepsilon'}\big(t;t_0,\mathcal{T}^{t_0}_{\varepsilon\to\varepsilon'}(\cdot,\cdot)\big)\right|\right\rangle\Big|_{t=t_0^+} \\
&\quad = \left\langle \mu_\varepsilon^{t_0}, \left|G(p_\varepsilon^{t_0}) - G(p_{\varepsilon'}^{t_0}))\right|\right\rangle + \left\langle \mu_\varepsilon^{t_0}, \left|L_\varepsilon[f_\varepsilon](t_0, \mathcal{T}^{t_0}_\varepsilon) - L_{\varepsilon'}[f_{\varepsilon'}](t, \mathcal{T}^{t_0}_{\varepsilon\to\varepsilon'})\right|\right\rangle \\
&\quad =: \mathcal{I}_1 + \langle \mu_\varepsilon^{t_0}, \mathcal{I}_2\rangle.
\end{aligned}$$

During the estimation of $\frac{d}{dt}\mathcal{Q}_{\varepsilon,\varepsilon'}(t)\Big|_{t=t_0^+}$, as the time t is fixed, we will suppress the upper index which represents the time configuration. For the estimate of $\mathcal{I}_1$, we notice that

$$\begin{aligned}
\mathcal{I}_1 &= \langle \mu_\varepsilon, |G(p_\varepsilon) - G(p_{\varepsilon'})|\rangle \lesssim \langle \mu_\varepsilon, |p_\varepsilon - p_{\varepsilon'}|\rangle \\
&\le \langle \mu_\varepsilon, |\mathcal{T}_\varepsilon - \mathcal{T}_{\varepsilon\to\varepsilon'}|\rangle = \langle \mu_\varepsilon, |\mathrm{Id} - \mathcal{T}_{\varepsilon\to\varepsilon'}|\rangle = W_1(\mu_\varepsilon, \mu_{\varepsilon'}),
\end{aligned} \tag{4.4.9}$$

where the last equality holds because $\mathcal{T}^{t_0}_{\varepsilon\to\varepsilon'}$ is optimal (Remark 4.4.1). Now we rewrite $\mathcal{I}_2$ as

$$\begin{aligned}
\mathcal{I}_2 &= |\langle d\mu_\varepsilon(z_*), \phi_\varepsilon(q_* - q)(G(p_*) - G(p))\rangle \\
&\qquad\qquad - \langle d\mu_{\varepsilon'}(z_*), \phi_{\varepsilon'}(q_* - q_{\varepsilon'})(G(p_*) - G(p_{\varepsilon'})\rangle| \\
&= |\langle d\mu_\varepsilon(z_*), \phi_\varepsilon(q_* - q)(G(p_*) - G(p))\rangle \\
&\qquad\qquad - \langle d\mu_\varepsilon(z_*), \phi_{\varepsilon'}(q_{*\varepsilon'} - q_{\varepsilon'})(G(p_{*\varepsilon'}) - G(p_{\varepsilon'})\rangle| \\
&= \Big|\langle d\mu_\varepsilon(z_*), \left(\phi_\varepsilon(q_* - q) - \phi_{\varepsilon'}(q_{*\varepsilon'} - q_{\varepsilon'})\right)(G(p_*) - G(p))\rangle \\
&\qquad - \langle d\mu_\varepsilon(z_*), \phi_{\varepsilon'}(q_{*\varepsilon'} - q_{\varepsilon'})(G(p_{*\varepsilon'}) - G(p_{\varepsilon'})) - (G(p_*) - G(p)))\rangle\Big| \\
&=: |\mathcal{I}_{21} + \mathcal{I}_{22}|.
\end{aligned}$$

- (Estimate of $\mathcal{I}_{21}$): We find that

$$\mathcal{I}_{21} = \langle d\mu_\varepsilon(z_*), \big(\phi_\varepsilon(q_* - q) - \phi_{\varepsilon'}(q_{*\varepsilon'} - q_{\varepsilon'})\big)(G(p_*) - G(p))\rangle$$

$$
\begin{aligned}
&\leq \left\langle d\mu_\varepsilon(z_*), \left|\left((\phi_\varepsilon - \phi_{\varepsilon'})(q_* - q)(G(p_*) - G(p)\right)\right|\right\rangle \\
&\quad + \left\langle d\mu_\varepsilon(z_*), \left|\left(\phi_{\varepsilon'}(q_* - q) - \phi_{\varepsilon'}(q_{*\varepsilon'} - q_{\varepsilon'})\right)(G(p_*) - G(p))\right|\right\rangle \\
&=: \mathcal{I}_{211} + \mathcal{I}_{212}.
\end{aligned}
$$

Since support of ζ_ε is contained in $B_\varepsilon(0)$, for any $x \neq 0$ we use the mean value theorem to have

$$
\begin{aligned}
|\phi_\varepsilon(x) - \phi(x)| &\leq \int_{\mathbb{R}^d} |\phi(x-y) - \phi(x)| \zeta_\varepsilon(y)dy \\
&= \int_{\mathbb{R}^d} \left| \frac{|x|^{-\alpha} - |x-y|^{-\alpha}}{|x| - |x-y|} \right| \big||x| - |x-y|\big| \zeta_\varepsilon(y)dy \\
&\leq \alpha \int_{\mathbb{R}^d} \left(\frac{1}{|x|^{1+\alpha}} + \frac{1}{|x-y|^{1+\alpha}} \right) |y| \zeta_\varepsilon(y)dy \\
&\leq \alpha\varepsilon \int_{\{y\,:\,\varepsilon \geq |y|\}} \left(\frac{1}{|x|^{1+\alpha}} + \frac{1}{|x-y|^{1+\alpha}} \right) \zeta_\varepsilon(y)dy \leq \frac{C\varepsilon}{|x|^{1+\alpha}}.
\end{aligned}
$$

Therefore, from a priori regularity condition $(\alpha+1)\gamma < d$, we can apply Hölder inequality near the origin and its complement to obtain

$$
\begin{aligned}
&\left\langle d\mu_\varepsilon(z_*), \left|\left((\phi_\varepsilon - \phi)(q_* - q)(G(p_*) - G(p)\right)\right|\right\rangle \\
&\leq \varepsilon R_\varepsilon(0) C \int_{\mathbb{R}^d \times \Omega_\varepsilon(0)} \frac{1}{|x|^{1+\alpha}} f_\varepsilon dz_* \leq \varepsilon C \|f_\varepsilon\|_{L^\beta},
\end{aligned}
\tag{4.4.10}
$$

and the same inequality holds for ε'. Hence we have

$$
\langle \mu_\varepsilon(dz), \mathcal{I}_{211} \rangle \leq C \|f_\varepsilon\|_{L^\beta}^2 (\varepsilon + \varepsilon'), \tag{4.4.11}
$$

where C is independent in $\varepsilon, \varepsilon', t$. Now we turn to estimate of $\mathcal{I}_{212}$. First note that

$$
\begin{aligned}
\phi_\varepsilon(x) &= \int_{\mathbb{R}^d} \frac{1}{|x-y|^\alpha} \zeta_\varepsilon(y)dy \\
&\leq \int_{\{y:2|y|<|x|\}} \frac{\zeta_\varepsilon(y)}{|x-y|^\alpha} dy + \mathbb{1}_{\{|x|\leq 2\varepsilon\}} \int_{\{y:|y|\leq\varepsilon\}} \frac{\zeta_\varepsilon(y)}{|x-y|^\alpha} dy \leq \frac{C}{|x|^\alpha},
\end{aligned}
$$

which leads to

$$
|\phi_\varepsilon(x) - \phi_\varepsilon(y)| \leq \frac{C|x-y|}{\min(|x|,|y|)^{1+\alpha}},
$$

for C independent in ε. Therefore,

$$\begin{aligned}
&\langle d\mu_\varepsilon(z), \mathcal{I}_{212}\rangle \\
&= \Big\langle \mu_\varepsilon(dz) \left\langle d\mu_\varepsilon(z_*), \left|\big(\phi_{\varepsilon'}(q_* - q) - \phi_{\varepsilon'}(q_{*\varepsilon'} - q_{\varepsilon'})\big)(G(p_*) - G(p))\right|\right\rangle \Big\rangle \\
&\le 2CR_\varepsilon(0) \int_{\mathbb{R}^{2d}\times\Omega_\varepsilon(0)^2} \left(\frac{|q_{\varepsilon'} - q|}{|x - q_*|^{1+\alpha}} + \frac{|q_{\varepsilon'} - q|}{|q_{\varepsilon'} - q_{*\varepsilon'}|^{1+\alpha}} \right) f_\varepsilon(z_*) f_\varepsilon(z) dz_* dz \\
&=: C(\mathcal{I}_{2121} + \mathcal{I}_{2122}).
\end{aligned} \tag{4.4.12}$$

We used change of variable $z \leftrightarrow z_*$ for the inequality. Now we compute $\mathcal{I}_{2122}$ to find

$$\mathcal{I}_{2121} \le C\|f_\varepsilon\|_{L^\beta} \langle \mu_\varepsilon, |q_{\varepsilon'} - q|\rangle \le C\|f_\varepsilon\|_{L^\beta} W_1\big(\mu_\varepsilon, \mu_{\varepsilon'}\big),$$

where we followed a similar procedure as in (4.4.10) for the first inequality. Then we obtain

$$\begin{aligned}
\mathcal{I}_{2122} &\le C \left\langle \mu_\varepsilon(dz), \left\langle d\mu_\varepsilon(z_*), \frac{|q_{\varepsilon'} - q|}{|q_{\varepsilon'} - q_{*\varepsilon'}|^{1+\alpha}} \right\rangle \right\rangle \\
&= C \left\langle \mu_\varepsilon(dz), |q_{\varepsilon'} - q| \left\langle d\mu_{\varepsilon'}(z_*), \frac{1}{|q_{\varepsilon'} - q_{*\varepsilon}|^{1+\alpha}} \right\rangle \right\rangle \\
&\le C\|f_{\varepsilon'}\|_{L^\beta} W_1(\mu_\varepsilon, \mu_{\varepsilon'}).
\end{aligned}$$

Then from (4.4.12), we achieve

$$\langle \mu_\varepsilon(dz), \mathcal{I}_{212}\rangle \le C \max(\|f_\varepsilon\|_{L^\beta}, \|f_{\varepsilon'}\|_{L^\beta}) W_1(\mu_\varepsilon, \mu_{\varepsilon'}). \tag{4.4.13}$$

Combining (4.4.11) and (4.4.13), we deduce the following estimation:

$$\langle \mu_\varepsilon, \mathcal{I}_{21}\rangle \le C \max(\|f_\varepsilon\|_{L^\beta}, \|f_{\varepsilon'}\|_{L^\beta}) W_1(\mu_\varepsilon, \mu_{\varepsilon'}) + C\|f_\varepsilon\|^2_{L^\beta}(\varepsilon + \varepsilon'), \tag{4.4.14}$$

for C is independent in $\varepsilon, \varepsilon'$ and t.

- (Estimate of $\mathcal{I}_{22}$): From a direct computation,

$$\begin{aligned}
\langle \mu_\varepsilon(dz), \mathcal{I}_{22}\rangle \le &\int_{\mathbb{R}^{4d}} |\phi_{\varepsilon'}(q_{*\varepsilon'} - q_{\varepsilon'})|\,|G(p_*) - G(p_{*\varepsilon'})| f_\varepsilon(t_0, z) f_\varepsilon(t_0, z_*) dz dz_* \\
&+ \int_{\mathbb{R}^{4d}} |\phi_{\varepsilon'}(q_{*\varepsilon'} - q_{\varepsilon'})|\,|G(p) - G(p_{\varepsilon'})| f_\varepsilon(t_0, z) f_\varepsilon(t_0, z_*) dz dz_*.
\end{aligned}$$

Again using same computation as (4.4.10), we have

$$\langle \mu_\varepsilon(dz), \mathcal{I}_{22} \rangle \le C \| f_{\varepsilon'} \|_{L^\beta} W_1(\mu_\varepsilon, \mu'_\varepsilon). \tag{4.4.15}$$

Finally, putting (4.4.9),(4.4.14) and (4.4.15) altogether, we achieve

$$\frac{d}{dt} \mathcal{Q}_{\varepsilon,\varepsilon'}(t) \Big|_{t=t_0^+} \le C(W_1(\mu_\varepsilon^{t_0}, \mu_{\varepsilon'}^{t_0}) + \varepsilon + \varepsilon').$$

On the other hand, since $\mathcal{T}^{t_0}_{\varepsilon \to \varepsilon'}$ is an optimal map, inequality in (4.4.8) turns to be a equality if $t = t_0$. Hence, by subtracting $W_1(\mu_\varepsilon^{t_0}, \mu_{\varepsilon'}^{t_0}) = \mathcal{Q}_{\varepsilon,\varepsilon'}(t_0)$ and then dividing $t - t_0$ from (4.4.8), the limit $t \to t_0$ leads to

$$\frac{d}{dt} W_1(f_\varepsilon(t), f_{\varepsilon'}(t)) \Big|_{t=t_0^+} \le C(W_1(\mu_\varepsilon^{t_0}, \mu_{\varepsilon'}^{t_0}) + \varepsilon + \varepsilon').$$

Since t_0 can be chosen arbitrary, we have the desired inequality

$$\frac{d}{dt} W_1(f_\varepsilon(t), f_{\varepsilon'}(t)) \le C(W_1(\mu_\varepsilon^{t}, \mu_{\varepsilon'}^{t}) + \varepsilon + \varepsilon'),$$

for C independent in $\varepsilon, \varepsilon'$ and t. □

Remark 4.4.3. *Applying Grönwall lemma to* (4.4.7) *gives*

$$W_1(f_\varepsilon(t), f_{\varepsilon'}(t)) \le e^{Ct} W_1(f_\varepsilon^0, f_{\varepsilon'}^0) + C(\varepsilon + \varepsilon') \int_0^t e^{C(t-s)} ds$$

for a positive constant C independent in $\varepsilon, \varepsilon'$ and t. Therefore, if f_ε and $f_{\varepsilon'}$ defined in finite time interval [0,T] have a same initial data, then f_ε is uniformly Cauchy in $C(0,T;\mathcal{P}_1(\mathbb{R}^{2d}))$, where $\mathcal{P}(\mathbb{R}^{2d})$ is equipped with W_1 distance, provided that ε is sufficiently small.

Now, by the limiting process $\varepsilon \to 0$, we prove local-in-time existence and uniqueness of the weak solution.

Theorem 4.4.1. *Suppose that initial data $f^0 \in (L^\beta \cap \mathcal{P}_1)(\mathbb{R}^{2d})$ of* (4.4.2) *has compact support and p, α satisfies the following relation:*

$$0 < \alpha < \frac{d}{\gamma} - 1,$$

where γ is a Hölder conjugate of β. Then there exist a unique weak solution of (4.4.2) *in a time interval* $[0,T]$*. Furthermore, local-in-time solutions satisfies uniform stability with respect to W_1 distance: for two local-in-time solutions f_1 and f_2 defined in* $[0,T]$*, we have*

$$\sup_{0\le t\le T} W_1(f_1(t), f_2(t)) \le CW_1(f_1^0, f_2^0),$$

where C is a positive constant independent of the time T.

Proof. •(Existence of solution) Again, we use f and μ interchangeably. Since a family of regular function $\{f_\varepsilon\}$ is a Cauchy sequence in $C(0,T;\mathcal{P}_1(\mathbb{R}^{2d}))$ as $\varepsilon\to 0$, there exist the limit function f. From the definition 4.4.2, we have to show

$$\langle \mu^t, g(t,\cdot,\cdot)\rangle - \langle \mu^0, g(0,\cdot,\cdot)\rangle = \int_0^t \langle \mu^s, \partial_t g + G(p)\cdot\nabla_q g + L[\mu]\cdot\nabla_p g\rangle ds.$$

for arbitrary $g\in C_c^\infty([0,T]\times\mathbb{R}^{2d})$. From standard method of characteristics, there exists a global-in-time solution f_ε of regularized solution (4.4.3). Then we have

$$\langle \mu_\varepsilon^t, g(t,\cdot,\cdot)\rangle - \langle \mu_\varepsilon^0, g(0,\cdot,\cdot)\rangle = \int_0^t \langle \mu_\varepsilon^s, \partial_t g + G(p)\cdot\nabla_q g + L_\varepsilon[\mu_\varepsilon]\cdot\nabla_p g\rangle ds.$$

As $\varepsilon\to 0$, we have

$$\begin{aligned}\langle \mu_\varepsilon^t, g(t,\cdot,\cdot)\rangle - \langle \mu_\varepsilon^0, g(0,\cdot,\cdot)\rangle &\to \langle \mu^t, g(t,\cdot,\cdot)\rangle - \langle \mu^0, g(0,\cdot,\cdot)\rangle,\\ \int_0^t \langle \mu_\varepsilon^s, \partial_t g + G(p)\cdot\nabla_q g\rangle ds &\to \int_0^t \langle \mu^s, \partial_t g + G(p)\cdot\nabla_q g\rangle ds.\end{aligned}$$

Hence, it suffices to show

$$\int_0^t \left(\langle \mu_\varepsilon^s, L_\varepsilon[\mu_\varepsilon]\cdot\nabla_p g\rangle - \langle \mu^s, L[\mu]\cdot\nabla_p g\rangle\right) ds \to 0.$$

We split the integral into three terms as

$$\left|\int_0^t\int_{\mathbb{R}^{2d}} (f_\varepsilon L_\varepsilon[f_\varepsilon] - fL[f])\cdot\nabla_p g dz dt\right|$$

$$
\begin{aligned}
&\le \left| \int_0^t \int_{\mathbb{R}^{2d}} f_\varepsilon (L_\varepsilon[f_\varepsilon] - L[f_\varepsilon]) \cdot \nabla_p g dz dt \right| \\
&+ \left| \int_0^t \int_{\mathbb{R}^{2d}} f_\varepsilon (L[f_\varepsilon] - L[f]) \cdot \nabla_p g dz dt \right| \\
&+ \left| \int_0^t \int_{\mathbb{R}^{2d}} (f_\varepsilon - f) L[f] \cdot \nabla_p g dz dt \right| \\
=: &\ \mathcal{I}_{31} + \mathcal{I}_{32} + \mathcal{I}_{33}.
\end{aligned}
$$

By the same calculation as in (4.4.10), one has

$$
\mathcal{I}_{31} \le C\varepsilon \|f_\varepsilon\|^2 \le C\varepsilon \to 0.
$$

On the other hand, we denote

$$
\begin{aligned}
\mathcal{I}_{32} &= \left| \int_0^t \int_{\mathbb{R}^{2d}} \mathcal{I}'_{32} d\big(\mu_\varepsilon^t(z_*) - \mu^t(z_*)\big) dt \right|, \\
\text{where} \quad \mathcal{I}'_{32} &= \int_{\mathbb{R}^{2d}} f_\varepsilon(t,z) \phi(q_* - q)(G(p_*) - G(p)) \cdot \nabla_p g(z) dz.
\end{aligned}
$$

Then, from a priori regularity condition we have

$$
\int_{\mathbb{R}^d} \int_{|q-q_*|<\delta} f_\varepsilon(t,z) \phi(q_* - q)(G(p_*) - G(p)) \cdot \nabla_p g(z) dq dp < C\delta,
$$

where C is independent in ε. Therefore we may neglect the singularity near the origin and regard $\mathcal{I}'_{32}$ as a bounded Lipschitz function. Therefore, interpreting $\mathcal{I}_{32}$ in terms of bounded Lipchitz distance(Remark 4.4.1), we have

$$
\mathcal{I}_{32} \le C \sup_{0 \le t \le T} W_1(f_\varepsilon(t), f(t)) \to 0, \quad \text{as} \quad \varepsilon \to 0.
$$

Similarly, one can also achieve

$$
\mathcal{I}_{33} \le C \sup_{0 \le t \le T} W_1(f_\varepsilon(t), f(t)) \to 0, \quad \text{as} \quad \varepsilon \to 0,
$$

and this proves the existence.

• (Uniqueness and stability of solution) Let f_1 and f_2 be two weak solutions of equation (4.4.2). Then, Proposition (4.4.3) implies

$$\frac{d}{dt}W_1(f_1(t), f_2(t)) \le CW_1(f_1(t), f_2(t)), \quad t \in [0, T].$$

Therefore, from the Grönwall lemma, both of uniqueness and stability are verified. □

4.4.3 Structural stability in a finite-time interval.

We consider a structural stability of kinetic CS type model with singular kernel. Then, the following two kinetic equations with the same initial data are given:

$$\begin{cases} \partial_t f_g + G(p) \cdot \nabla_q f_g + \nabla_p \cdot (L_g[f_g]f_g) = 0, \quad (t, q, p) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d, \\ L_g[f](t, q, p) := \int_{\mathbb{R}^{2d}} \phi\big(q_* - q\big)(G(p_*) - G(p))f(t, q_*, p_*)dq_*dp_*, \\ f_g(0, q, p) = f^0, \quad \phi(q) = \dfrac{1}{|q|^\alpha}, \end{cases} \tag{4.4.16}$$

$$\begin{cases} \partial_t f_\infty + w \cdot \nabla_q f_\infty + \nabla_p \cdot (L_\infty[f_\infty]f_\infty) = 0, \quad (t, q, p) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d, \\ L_\infty[f](t, q, p) := \int_{\mathbb{R}^{2d}} \phi\big(q_* - q\big)(p_* - p)f(t, q_*, p_*)dq_*dp_*, \\ f_\infty(0, q, p) = f^0, \quad \phi(q) = \dfrac{1}{|q|^\alpha}. \end{cases} \tag{4.4.17}$$

The equation (4.4.17) can be obtained by posing $G = \text{id}$ to (4.4.16). Therefore, regarding (4.4.17) as a reference model, we show that the CS type kinetic model (4.4.2) converges to the standard kinetic CS model.

Definition 4.4.3. *Let $F(p) = f(|p|)\frac{p}{|p|}$ and $G(p) = g(|p|)\frac{p}{|p|}$ be activation functions (i.e. (F, f) and (G, g) satisfies (4.1.2)). We say that F converges to G as an activation function (notation: $F \xrightarrow{\text{act}} G$) if and only if there is a collection of functions $\{F_c\}_{c\in[1,\infty)}$, parametrized by c, such that*

1. *each* $F_c = f_c(|p|)\frac{p}{|p|}$ *is an activation function,*
2. $F_1 = F,\ F_\infty = G,$
3. $f_c' \to f_\infty'$ in L^∞_{loc} *as* $c \to \infty$.

Theorem 4.4.2. *Let* f_g *and* f_∞ *be local-in-time solutions of* (4.4.16) *and* (4.4.17) *respectively, defined in* $[0,T]$. *Suppose that* α, β *and* f_0 *satisfy the following relation:*

$$\mathrm{supp}(f^0) \subset\subset \mathbb{R}^{2d}, \quad f^0 \in (L^\beta \cap \mathcal{P}_1)(\mathbb{R}^{2d}), \quad 0 < \alpha < \frac{d}{\gamma} - 1.$$

Then we have a following finite-in-time structural stability:

$$G \xrightarrow{\text{act}} \mathrm{Id} \quad \Longrightarrow \quad \sup_{t\in[0,T]} W_1(f_g(t), f_\infty(t)) \to 0,$$

where Id is the identity map.

Proof. We take a similar procedure as in Proposition (4.4.7). Let $0 < \varepsilon \ll 1$ be *fixed*. Consider two regularized systems:

$$\begin{cases} \partial_t f_g + G(p)\cdot\nabla_q f_g + \nabla_p \cdot (L_g[f_g]f_g) = 0, \quad (t,q,p) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d, \\ L_g[f](t,q,p) := \int_{\mathbb{R}^{2d}} \phi_\varepsilon\big(q_* - q\big)(G(p_*) - G(p)) f(t,q_*,p_*) dq_* dp_*, \\ f_g(0,q,p) = f^0, \quad \phi_\varepsilon(x) = \dfrac{1}{|x|^\alpha} * \zeta_\varepsilon, \end{cases} \tag{4.4.18}$$

$$\begin{cases} \partial_t f_\infty + w\cdot\nabla_q f_\infty + \nabla_p \cdot (L_\infty[f_\infty]f_\infty) = 0, \quad (t,q,p) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d, \\ L_\infty[f](t,q,p) := \int_{\mathbb{R}^{2d}} \phi_\varepsilon\big(q_* - q\big)(p_* - p) f(t,q_*,p_*) dq_* dp_*, \\ f_\infty(0,q,p) = f^0, \quad \phi_\varepsilon(x) = \dfrac{1}{|x|^\alpha} * \zeta_\varepsilon. \end{cases} \tag{4.4.19}$$

Let $\mu_g^t = f_g(t,q,p)dqdp$ and $\mu_\infty^t = f_\infty(t,q,p)dqdp$ be solutions of (4.4.18) and (4.4.19), respectively. We definte characteristic curves $Z_g = (Q_g, P_g)$ and $Z_\infty = (Q_\infty, P_\infty)$ by

$$\begin{cases} \dfrac{d}{ds}Q_g(s) = G(P_g(s;t,q,p)), \\ \dfrac{d}{ds}P_g(s) = L_g[f_g]\big(s, Z_g(s)\big), \\ (Q_g(t;t,q,p), P_g(t;t,q,p)) = (q,p), \end{cases} \qquad \begin{cases} \dfrac{d}{ds}Q_\infty(s) = P_\infty(s;t,q,p), \\ \dfrac{d}{ds}P_\infty(s) = L_\infty[f_\infty]\big(s, Z_\infty(s)\big), \\ (Q_\infty(t;t,q,p), P_\infty(t;t,q,p)) = (q,p), \end{cases}$$

respectively, so that $\mathcal{T}_g^{t\to t_0} = Z_g(t;t_0,q,p)$ and $\mathcal{T}_\infty^{t\to t_0} = Z_\infty(t;t_0,q,p)$ can be served as transport maps shifting the time configuration. Let $\mathcal{T}_{g\to\infty}^{t_0}$ be an *optimal* transport map from $\mu_g(t_0)$ to $\mu_\infty(t_0)$. We adopt same convention as in the proof of Proposition (4.4.3), but we place $g-$configuration in place of $\varepsilon-$configuration.

Following the similar calculations as in the proof of proposition (4.4.7), we have

$$W_1(\mu_g^t, \mu_\infty^t) \le \left\langle \mu_g^{t_0}, \left| Z_g(t;t_0,\cdot,\cdot) - Z_\infty\big(t;t_0,\mathcal{T}_{g\to\infty}^{t_0}(\cdot,\cdot)\big)\right| \right\rangle =: \mathcal{Q}_{g,\infty}(t),$$

which leads to

$$\begin{aligned} &\frac{d}{dt}\mathcal{Q}_{g,\infty}(t)\Big|_{t=t_0^+} \\ &\quad \le \left\langle \mu_g^{t_0}, \left| G(p_c^{t_0}) - w_\infty^{t_0}\right| \right\rangle + \left\langle \mu_g^{t_0}, \left| L_g[f_g](t_0, \mathcal{T}_g^{t_0}) - L_\infty[f_\infty](t_0, \mathcal{T}_{g\to\infty}^{t_0})\right| \right\rangle \\ &\quad \le \left\langle \mu_g^{t_0}, \left| G(p_c^{t_0}) - G(p_\infty^{t_0})\right| \right\rangle + \left\langle \mu_g^{t_0}, \left| G(p_\infty^{t_0}) - w_\infty^{t_0}\right| \right\rangle \\ &\qquad\qquad + \left\langle \mu_g^{t_0}, \left| L_g[f_g](t_0, \mathcal{T}_g^{t_0}) - L_\infty[f_\infty](t_0, \mathcal{T}_{g\to\infty}^{t_0})\right| \right\rangle \\ &\quad \le W_1(\mu_g^{t_0}, \mu_\infty^{t_0}) + \left\langle \mu_g^{t_0}, \left| G(p_\infty^{t_0}) - w_\infty^{t_0}\right| \right\rangle \\ &\qquad\qquad + \left\langle \mu_g^{t_0}, \left| L_g[f_g](t_0, \mathcal{T}_g^{t_0}) - L_\infty[f_\infty](t_0, \mathcal{T}_{g\to\infty}^{t_0})\right| \right\rangle \\ &\quad =: W_1(\mu_g^{t_0}, \mu_\infty^{t_0}) + \langle \mu_g^{t_0}, \mathcal{J}_1\rangle + \langle \mu_g^{t_0}, \mathcal{J}_2\rangle, \end{aligned}$$

where we used the same methodology as in (4.4.9) for the last inequality. From now on, during the estimation of $\frac{d}{dt}\mathcal{Q}_{g,\infty}(t)\Big|_{t=t_0^+}$, we suppress the upper index again. To estimate $\mathcal{J}_1$, we recall the decrement of maximal momentum

(Proposition 4.4.1) to obtain an uniform upper bound U_p of $|p|$, independent in g, ε and t. On the other hand, Proposition 4.1.1 implies

$$|p - G(p)| \le \|\mathrm{Id} - G\|_{op}|p| = U_p \max_{|p|\in[0,U_p]} |1 - g'(|p|)| =: U_{g'}, \tag{4.4.20}$$

from the mean value theorem, where $\|\cdot\|_{op}$ stands for the operator norm. Then, since supports of $f_g(t,\cdot,\cdot)$ and $f_\infty(t,\cdot,\cdot)$ are bounded uniformly in t, and ε (Remark 4.4.2), Hölder's inequality deduces

$$\langle \mu_g(dz), \mathcal{J}_1\rangle \le U_{g'} C \|f_g(t_0)\|_{L^\beta},$$

for C independent in ε, and t. For further estimate, we split $\mathcal{J}_2$ as

$$\begin{aligned}
\mathcal{J}_2 &= |L_g[f_g](t_0, \mathcal{T}_g) - L_\infty[f_\infty](t_0, \mathcal{T}_{g\to\infty})| \\
&= \|\langle \mu_g(dz_*), \phi_\varepsilon(q_* - q)(G(p_*) - G(p))\rangle \\
&\qquad\qquad - \langle \mu_\infty(dz_*), \phi_\varepsilon(q_* - q_\infty)(p_* - p_\infty)\rangle\| \\
&= \|\langle \mu_g(dz_*), \phi_\varepsilon(q_* - q)(G(p_*) - G(p))\rangle \\
&\qquad\qquad - \langle \mu_g(dz_*), \phi_\varepsilon(q_{*\infty} - q_\infty)(p_{*\infty} - p_\infty)\rangle\| \\
&\le \left|\langle \mu_g(dz_*), (\phi_\varepsilon(q_* - q) - \phi_\varepsilon(q_{*\infty} - q_\infty))(G(p_*) - p)\rangle\right| \\
&\quad + \left|\langle \mu_g(dz_*), \phi_\varepsilon(q_{*\infty} - q_\infty)((G(p_*) - p) - (G(p_{*\infty}) - p_\infty))\rangle\right| \\
&\quad + \left|\langle \mu_g(dz_*)\phi_\varepsilon(q_* - q)(p - G(p)) - \phi_\varepsilon(q_{*\infty} - q_\infty)(p_{*\infty} - G(p_{*\infty}))\rangle\right| \\
&=: \mathcal{J}_{21} + \mathcal{J}_{22} + \mathcal{J}_{23}.
\end{aligned}$$

From the similar calculation as in (4.4.12) – (4.4.13), we have

$$\langle \mu_g(dz), \mathcal{J}_{21}\rangle \le C W_1(\mu_g, \mu_\infty).$$

For the estimate of $\mathcal{J}_{22}$, note that

$$\begin{aligned}
\mathcal{J}_{22} &= \left|\langle \mu_g(dz_*), \phi_\varepsilon(q_{*\infty} - q_\infty)((G(p_*) - p) - (G(p_{*\infty}) - p_\infty))\rangle\right| \\
&\le \langle \mu_g(dz_*), \phi_\varepsilon(q_{*\infty} - q_\infty)|p_* - p_{*\infty}|\rangle \\
&\quad + \langle \mu_g(dz_*), \phi_\varepsilon(q_{*\infty} - q_\infty)|w - p_\infty|\rangle \\
&=: \mathcal{J}_{221} + \mathcal{J}_{222},
\end{aligned}$$

where

$$\langle \mu_g(dz), \mathcal{J}_{221}\rangle = \left\langle \mu_g(dz_*), |p_* - p_{*\infty}|\langle \mu_g(dz), \phi_\varepsilon(q_{*\infty} - q_\infty)\rangle\right\rangle$$

$$\begin{aligned}&\le C\langle \mu_g(dz_*), |p_* - p_{*\infty}|\rangle\\&\le C\langle \mu_g(dz_*), |z_* - z_{*\infty}|\rangle = CW_1(\mu_g, \mu_\infty),\end{aligned}$$

and similarly we have $\langle \mu_g(dz), \mathcal{J}_{221}\rangle \le CW_1(\mu_g, \mu_\infty)$.
Finally, using (4.4.20), we can deduce

$$\langle \mu_g(dz), \mathcal{J}_{23}\rangle \le CU_{g'}.$$

Summing the estimates altogether, one has

$$\frac{d}{dt}W_1(\mu_g^t, \mu_\infty^t)\Big|_{t=t_0^+} \le \frac{d}{dt}\mathcal{Q}_{g,\infty}(t)\Big|_{t=t_0^+} \le C\left(W_1(\mu_g^{t_0}, \mu_\infty^{t_0}) + U_{g'}\right),$$

where t_0 is arbitrary in $[0, T]$. Therefore, from the Grönwall ineqaulty, we have

$$W_1(\mu_g^t, \mu_\infty^t) \le e^{Ct}W_1(\mu_g^0, \mu_\infty^0) + U_{g'}\int_0^t e^{C(t-s)}ds, \tag{4.4.21}$$

for a positive constant C, independent in ε, and t. Now, we recall that the solution of (4.4.16) is the W_1 limit of regularized equation (4.4.3)(Proposition 4.4.3) and the convergence is uniform in time t from the Grönwall lemma (Remark (4.4.3)). Since a proof of Proposition 4.4.3 is valid for classical model (i.e. (4.4.2) with $G = \mathrm{Id}$) and (4.4.21) is an $\varepsilon-$independent estimation, by approximation, we have the desired result. □

Corollary 4.4.1. *Suppose that $G = g(|p|)\frac{p}{|p|}$ and $H = h(|p|)\frac{p}{|p|}$ are activation functions satisfying* (4.1.2). *Let f_g and f_h be local-in-time solutions of* (4.4.2) *corresponding to G and H defined on time interval $[0, T]$, respectively. If f_g and f_h has a same initial data, then we obtain*

$$G \xrightarrow{\text{act}} H \quad \Longrightarrow \quad \sup_{t\in[0,T]} W_1(f_g(t), f_h(t)) \to 0.$$

Proof. This follows from the proof of Theorem 4.4.2; we replace Id to h, and the proof is still valid. □

Chapter 5

Conclusion and future work

In this thesis, we have studied an emergent behavior of the CS-type consensus model, especially for the singular kernel.

First, we have provided a global flocking dynamics of the relativistic Cucker-Smale model with a singular communication weight. When the singularity is sufficiently strong, near the singular point, there is no finite time collision between particles, if the initial data is non-collisional. Thus, nonexistence of finite-time collisions guarantees the global existence of smooth solutions. Once we obtain a global existence of solution, we can use standard Lyapunov functional approach to find sufficient conditions in terms of initial data and communication weight function. On the other hand, when the singularity is weak at single point, one cannot guarantee that there is no finite-time collision between particles, and indeed, we can provide a simple example for the existence of the finite time collision in the one-dimensional two-particle system. Therefore, to guarantee the collision avoidance in the weak singularity regime, we need to impose an extra condition on initial data, under which one can obtain a global lower bound for relative distances. Besides the singular model, there are still a lot of topics to be studied even for the regular RCS model, such as finding a sufficient framework for bi- or multi-cluster flocking.

Second, we have proposed the asymptotic dynamics of the first-order consensus model on the real line, which was obtained from the relativistic Cucker-Smale flocking model on the real line. We provided a detailed analysis on the large-time behaviors of the proposed nonlinear consensus model, according to the regularity and singularity of the communication weights at zero and infinity. When the communication weight is regular and long-ranged, the nonlinear consensus model exhibits a complete consensus behaviors, under the mild assumptions on system parameters. On the other hand, when the communication weight is still regular but short-ranged, asymptotic clustering behavior becomes completely different, and system may present complete consensus or segregation, depending on the size of a coupling strength κ. We present sufficient conditions for the coupling strength under which either system aggregates or segregates asymptotically. We also consider the case of singular communication weight, and present similar results on the asymptotic behaviors. Finally, we also studied the structural stability of the activation function. The one-dimensional flocking model can be lifted in the kinetic and hydrodynamic levels. In particular, the singular communication weight case is more interesting, not only because it is mathematically challenging, but it is also related to the fractional diffusion. Therefore, it would be interesting to investigate the corresponding kinetic and hydrodynamic counterparts of the proposed generalized consensus model.

Third, we have presented the CS-type consensus model and studied its asymptotic behavior. In particular, when the ambient space is one-dimensional ($d = 1$), the proposed model can be transformed into the first-order consensus model, and its study provides a deeper understanding of the second model as well. The CS model for a singular kernel has several interesting properties not found in a regular kernel, depending on the integrability of the kernel near the origin. If the kernel is weakly singular, the particles can stick or collide in finite time, which leads to loss of regularity. We have studied the regularity of such solutions on the real line. On the other hand, when the kernel is strongly singular, the particles never collide in a finite time if the initial data is non-collisional. However, the existence of a strictly lower bound for relative distance for the general initial data was left as a remaining issue. We proved that a prototypical kernel with strong singularity ($\psi(|q|) = |q|^{-\alpha},\ \alpha \geq 1$)

have a strictly positive lower bound between agents, provided that singularity is not critical($\alpha \neq 1$) and flocking is guaranteed. We also provide a well-posedness and structural stability for a kinetic analog of the proposed model. Several related problems remain as future perspectives. For example, relaxing a priori condition for the existence of a strictly positive lower bound is one of the remaining problems. On the other hand, the property of sticking was not featured in the kinetic model in our thesis, and it would be interesting to investigate the realization of such special behavior in the kinetic model

Bibliography

[1] H. Ahn: *Uniform stability of the Cucker–Smale and thermodynamic Cucker–Smale ensembles with singular kernels.* Netw. Heterog. Media. **17**(5) (2022), 753–782.

[2] H. Ahn, J. Byeon, S.-Y. Ha and J. Yoon: *On the relativistic flocks over the unit sphere and the hyperboloid in a bonding force field.* J. Math. Phys. **64** (2023), Paper No. 012705.

[3] S. Ahn and and S.-Y. Ha: *Stochastic flocking dynamics of the Cucker–Smale model with multiplicative white noises.* J. Math. Phys. **51**(2010), Paper No. 103301

[4] H. Ahn, S.-Y. Ha, and J. Kim: *Nonrelativistic limits of the relativistic Cucker-Smale model and its kinetic counterparts.* J. Math. Phys. **63** (2022), 082701.

[5] H. Ahn, S.-Y. Ha, M. Kang, and W. Shim: *Emergent behaviors of relativistic flocks on Riemannian manifolds.* Phys. D: Nonlinear Phenom. **427** (2021), 133011.

[6] G. Albi, N. Bellomo, L. Fermo, S.-Y. Ha, J. Kim, L. Pareschi, D. Poyato and J. Soler: *Vehicular traffic, crowds, and swarms. From kinetic theory and multiscale methods to applications and research perspectives.* Math. Models Methods Appl. Sci. **29** (2019), 1901–2005.

[7] J. Buck and E. Buck: *Biology of synchronous flashing of fireflies.* Nature **211** (1966), 562–564.

[8] J. C. Bronski, T. E. Carty and L. Deville: *Synchronisation conditions in the Kuramoto model and their relationship to seminorms.* Nonlinearity **34** (2021), 5399–5433.

[9] J. Byeon: *Asymptotic dynamics for the Cucker-Smale model with velocity control.* Submitted.

[10] J. Byeon, S.-Y. Ha, and J. Kim: *Emergence of state-locking for the first-order nonlinear consensus model on the real line*. Kinet. Relat. Models. **16**(3) (2023), 423-457.

[11] J. Byeon, S.-Y. Ha, and J. Kim: *Asymptotic flocking dynamics of a relativistic Cucker-Smale flock under singular communications*. J. Math. Phys. **63** (2022), 012702.

[12] J. A. Carrillo, Y.-P. Choi, and M. Hauray, *Local well-posedness of the generalized Cucker-Smale model with singular kernels*. Mathematical modelling of complex systems, 17–35, ESAIM Proc. Surveys **47**, EDP Sci., Les Ulis, 2014.

[13] J. A. Carrillo, Y.-P. Choi, P. B. Mucha, and J. Peszek, *Sharp conditions to avoid collisions in singular Cucker-Smale interactions*. Nonlinear Anal. Real World Appl. **37** (2017), 317–328.

[14] J. A. Carrillo, Y.-P. Choi, E. Tadmor, and C. Tan: *Critical thresholds in 1D Euler equations with non-local forces*. Math. Models Methods Appl. Sci. **26** (2016), 185–206.

[15] J. A. Carrillo, M. Fornasier, J. Rosado, and G. Toscani: *Asymptotic flocking dynamics for the kinetic Cucker-Smale model.* SIAM J. Math. Anal. **42** (2010), 218–236.

[16] S. Choi and S.-Y. Ha: *Emergence of flocking for a multi-agent system moving with constant speed.* Commun. Math. Sci. **14** (2016), 953–972.

[17] J. Cho, S.-Y. Ha, F. Huang, C. Jin and D. Ko: *Emergence of bi-cluster flocking for the Cucker–Smale model.* Mathematical Models and Methods in Applied Sciences, **26**(06) (2016), 1191-1218.

[18] Y.-P. Choi, S.-Y. Ha, and J. Kim, *Propagation of regularity and finite-time collisions for the thermomechanical Cucker-Smale model with a singular communication*. Netw. Heterog. Media **13** (2018), 379–407.

[19] Y.-P. Choi, S.-Y. Ha, and Z. Li: *Emergent dynamics of the Cucker-Smale flocking model and its variants.* Active particles. Vol. 1. Advances in theory, models and applications, 299–331, Model. Simul. Sci. Eng. Technol., Birkhauser/Springer, Cham, 2017.

[20] Y.-P. Choi, D. Kalise, J. Peszek, and A. A. Peters, *A collisionless singular Cucker-Smale model with decentralized formation control*. SIAM J. Appl. Dyn. Syst. **18** (2019), 1954–1981.

[21] Y. P. Choi and X. Zhang: *One dimensional singular Cucker-Smale model: uniform-in-time mean-field limit and contractivity.* J. Differential Equations **287** (2021), 428–459.

[22] F. Cucker and J.-G. Dong, *A conditional, collision-avoiding, model for swarming*. Discrete Contin. Dyn. Syst. **34** (2014), 1009–1020.

[23] F. Cucker and J.-G. Dong, *A general collision-avoiding flocking framework*. IEEE Trans. Automat. Control **56** (2011), 1124–1129.

[24] F. Cucker and J.-G. Dong, *Avoiding collisions in flock*. IEEE Trans. Automat. Control **55** (2010), 1238–1243.

[25] F. Cucker and S. Smale: *Emergent behavior in flocks.* IEEE Trans. Autom. Control **52** (2007), 852–862.

[26] F. Cucker and S. Smale: *On the mathematics of emergence*. Japan J. Math. **2** (2007), 197–227.

[27] P. Degond and S. Motsch.: *Large-scale dynamics of the persistent turning walker model of fish behavior.* J. Stat. Phys. **131** (2008), 989–1022.

[28] S.-Y. Ha, S. Jin, D. Kim and D. Ko: *Convergence toward equilibrium of the first-order consensus model with random batch interactions.* J. of Differential Equations **302** (2021), 585–616.

[29] S.-Y. Ha, J. Kim, and T. Ruggeri: *From the relativistic mixture of gases to the relativistic Cucker-Smale flocking.* Arch. Rational Mech. Anal. **235** (2020), 1661–1706.

[30] S.-Y. Ha, J. Kim, and T. Ruggeri, *Kinetic and hydrodynamic models for the relativistic Cucker-Smale ensemble and emergent behaviors.* Commun. Math. Sci. **19** (2021), 1945–1990.

[31] S.-Y. Ha, J. Kim, J. Park, and X. Zhang: *Complete cluster predictability of the Cucker-Smale flocking model on the real line*. Arch. Ration. Mech. Anal. **231** (2019), 319–365.

[32] S.-Y. Ha, J. Kim and X. Zhang: *Uniform stability of the Cucker-Smale model and its application to the mean-field limit.* Kinet. Relat. Models, **11**(5) (2018), 1157–1181.

[33] S.-Y. Ha and J.-G. Liu: *A simple proof of Cucker-Smale flocking dynamics and mean field limit*. Commun. Math. Sci. **7** (2009), 297–325.

[34] S.-Y. Ha, J. Park, and X. Zhang: *A first-order reduction of the Cucker-Smale model on the real line and its clustering dynamics*. Commun. Math. Sci. **16** (2018), 1907–1931.

[35] S.-Y. Ha and T. Ruggeri, *Emergent dynamics of a thermomechanically consistent particle model*. Arch. Rational Mech. Anal. **233** (2017), 1397–1425.

[36] S.-Y. Ha and E. Tadmor: *From particle to kinetic and hydrodynamic description of flocking*. Kinet. Relat. Models **1** (2008), 415–435.

[37] J. M. Hendrickx and J.N. Tsitsiklis: *Convergence of type-symmetric and cut-balanced consensus seeking systems.* IEEE Trans. Automat. Contr. **58**(1) (2012), 214–218.

[38] Y. Kuramoto: *Chemical Oscillations, waves and turbulence.* Springer-Verlag, Berlin (1984).

[39] J. Kim: *First-order reduction and emergent behavior of the one-dimensional kinetic Cucker-Smale equation.* J. Differential Equations **302** (2021), 496–532.

[40] J. Kim: *A Cucker-Smale flocking model with the Hessian communication weight and its first-order reduction.* J. Nonlinear Sci. **32** (2022), Article No. 20.

[41] C. H. Min, H. Ahn, S.-Y. Ha and M. Kang: *Sufficient conditions for asymptotic phase-locking to the generalized Kuramoto model.* **16** Kinet. Relat. Models (2023), 97–132.

[42] P. Minakowski, P. B. Mucha, J. Peszek, and E. Zatorska, *Singular Cucker-Smale dynamics.* Active Particles, Vol. 2., 201–243, Model. Simul. Sci. Eng. Technol., Birkhäuser/Springer, Cham, 2019.

[43] S. Motsch and E. Tadmor: *Heterophilious dynamics enhances consensus.* SIAM Rev. **56** (2014), 577–621.

[44] P. B. Mucha and J. Peszek, *The Cucker-Smale equation: singular communication weight, measure-valued solutions and weak-atomic uniqueness.* Arch. Rational Mech. Anal. **227** (2018), 273–308.

[45] J. Peszek, *Discrete Cucker–Smale flocking model with a weakly singular weight.* SIAM J. Math. Anal. **47** (2015), 3671–3686.

[46] J. Peszek, *Existence of piecewise weak solutions of a discrete Cucker–Smale's flocking model with a singular communication weight*, J. Differ. Equ. (2014), **257**, 2900–2925.

[47] A. Pikovsky, M. Rosenblum and J. Kurths: *Synchronization: A universal concept in nonlinear sciences.* Cambridge University Press, Cambridge, 2001.

[48] D. Poyato and J. Soler, *Euler-type equations and commutators in singular and hyperbolic limits of kinetic Cucker-Smale models.* Math. Models Methods Appl. Sci. **27** (2017), 1089–1152.

[49] C. M. Topaz and A. L. Bertozzi.: *Swarming patterns in a two-dimensional kinematic model for biological groups.* SIAM J. Appl. Math. **65** (2004), 152–174.

[50] T. Chen and H Chen: *Universal approximation to nonlinear operators by neural networks with arbitrary activation functions and its application to dynamical systems.* IEEE Trans. Neural Netw. Learn. Syst., **6**(4) (1995), 911–917.

[51] J. Toner and Y. Tu: *Flocks, herds, and schools: A quantitative theory of flocking.* Phys. Rev. E **58** (1998), 4828–4858.

[52] T. Vicsek and A. Zefeiris.: *Collective motion.* Phys. Rep. **517** (2012), 71–140.

[53] A. T. Winfree, *Biological rhythms and the behavior of populations of coupled oscillators.* J. Theor. Biol. **16** (1967), 15–42.

[54] X. Yin, D. Yue and Z. Chen: *Asymptotic behavior and collision avoidance in the Cucker–Smale model.* IEEE Trans. Automat. Control, **65**(7) (2019), 3112–3119.

[55] X. Zhang and T. Zhu: *Complete classification of the asymptotical behavior for singular C-S model on the real line.* J. Differ. Equ. **269** (2020), 201–256.

국문초록

본 학위 논문에서는, 특이적 핵을 통해 상호작용하는 쿠커-스메일 유형의 모델들을 연구한다. 쿠커-스메일 유형 모델은 기계학습의 이론에 창안하여 쿠커-스메일 모델에 활성화 함수를 도입한 것으로서, 이를 통해 다양한 집단 행동의 묘사를 기대할 수 있다. 예를 들어, 적절한 활성화 함수를 도입하여 상대성 이론을 반영할 수 있다.

쿠커-스메일 유형 모델에 대한 동기 부여를 위해, 본 학위 논문에서는 먼저 상대론적 쿠커-스메일 모델(이하 RCS)을 소개한다. 구체적으로, 플로킹 및 핵의 특이성에 기인한 충돌방지가 일어날 조건에 대해서 연구한다. 정규성을 지닌 유계인 핵에 대하여, RCS의 입자들은 초기 상태의 기하적 구조에 따라 충돌할 수 있다. 다른 한편으로, 특이적인 핵에 대해서는 입자들이 충돌할 때 쿠커-스메일 벡터장이 유계가 아니게 되므로 표준적인 코시-립시츠 이론을 적용할 수 없고, 따라서 이 경우 해의 존재성을 논하기 어려워진다. 따라서, 충돌방지에 대한 연구는 RCS의 해의 타당성 및 플로킹 현상과 직결된다.

이후 우리는 쿠커-스메일 유형 모델을 도입한다. 해당 모델은 RCS를 포괄하는 일반화된 모델이며, 창발 현상을 기술한다. 우리는 영점 근처와 무한점 근처에서 핵의 정규성 혹은 특이성에 대응하여 실직선 위에서 발생하는 다양한 군집 유형을 연구한다. 이후 해당 모델에서 접착성을 가진 해의 정칙성을 분석한다. 다른 한편으로, 충돌 회피를 넘어 입자 간의 상대거리에 대한 하한을 보장하는 충분조건을 제공한다. 또한, 제안된 모델에 대응하는 기체 운동방정식을 도입하여 이에 대한 타당성 및 구조적 안정성을 논증한다.

주요어휘: 구조적 안정성, 기체 분자 운동론, 상대론적 쿠커-스메일 모델, 창발 현상, 충돌 방지, 활성화 함수
학번: 2019-28728

감사의 글

학위를 마무리 하면서 정말 많은 분들의 도움을 받았습니다. 우선, 저를 지도해 주시고 학업과 인생 전반에 대한 아낌없는 조언들을 해주신 하승열 교수님께 무한한 감사를 드립니다.

또한, 귀중한 시간을 내어 저의 디펜스 심사위원으로 참여해주신 강명주 교수님, 배형옥 교수님, 김도헌 교수님, 김정호 교수님께 깊은 감사를 드립니다. 그리고, 힘든 대학원 생활을 이겨내도록 든든한 버팀목이 되어준 제 대학원 동기들 재형이, 제웅이형, 강산이형, 민희 누나, 성현이, 형기형, 찬규, 성제 등등에게도 모두 고맙다는 인사를 전하고 싶습니다.

또한, 연구 선배이자 형으로서 많은 도움을 준 현진이 형, 많은 조언과 격려를 해준 명주와 한솔이, 고된 대학원 생활동안 버팀목이 되어준 연구실 동료들인 규영이, 재영이, 은택이 형, 항준이 형, 배기찬 박사님, 그리고 선배이자 롤모델로써 나아갈 길을 제시해 주신 고동남 교수님, 김도현 교수님, 노세은 교수님, 민찬호 교수님, 그리고 지금은 연구실을 떠났지만 좋은 친구이자 동료인 소영이와 Tao 등등 많은 구성원분들께 감사하고 모두 행복하시길 염원합니다.

이 모든 분들 덕분에, 매일 열심히 연구할 수 있는 원동력을 얻으며 대학원 생활을 할 수 있었습니다. 다시 한번, 감사드리며 깊은 존경을 표합니다. 앞으로, 더 나은 연구자가 되도록 열심히 공부하겠습니다.

마지막으로, 학위를 무사히 마치는데 응원을 해주신 제 사랑하는 아버지, 어머니, 형, 친척들 모두에게 감사하다는 말을 전하고 싶습니다.