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Regularity theory for local and nonlocal measure data problems

(국소 및 비국소 측도 데이터 문제의 정칙성 이론)

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수리과학부

송경

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Regularity theory for local and nonlocal measure data problems

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Kyeong Song

Dissertation Director : Professor Sun-Sig Byun

Department of Mathematical Sciences

Seoul National University

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Abstract

Regularity theory for local and nonlocal measure data problems

Kyeong Song

Department of Mathematical Sciences
The Graduate School
Seoul National University

In this thesis, we establish various regularity results for nonlinear measure data problems. The results obtained are part of a program devoted to nonlinear Calderón-Zygmund theory and nonlinear potential theory.

Firstly, we obtain maximal integrability and fractional differentiability results for elliptic measure data problems with Orlicz growth and borderline double phase growth, respectively. We also obtain fractional differentiability results for parabolic measure data problems under a minimal assumption on the coefficients.

Secondly, we obtain gradient potential estimates and fractional differentiability results for elliptic obstacle problems with measure data, by using linearization techniques. In particular, we develop a new method to obtain potential estimates for irregular obstacle problems. For the case of single obstacle problems with L^1-data, we further obtain uniqueness results and comparison principles in order to improve such regularity results.

Lastly, we show existence, regularity and potential estimates for mixed local and nonlocal equations with measure data. Also, as a first step to the regularity theory for anisotropic nonlocal problems with nonstandard growth, we establish Hölder regularity for nonlocal double phase problems by identifying sharp assumptions analogous to those for local double phase problems.

Key words: Measure data, Calderón-Zygmund theory, Potential theory, Nonstandard growth, Obstacle problem, Nonlocal operator
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Contents

Chapter 1

Introduction

This thesis is concerned with regularity theory for measure data problems. Emphasis is on nonlinear Calderón-Zygmund theory and nonlinear potential theory. Closely linked to each other, they aim at reproducing the classical Calderón-Zygmund theory and potential theory for nonlinear problems. Their several main results are based on the De Giorgi-Nash-Moser theory.

1.1 Measure data problems

We first outline the existence and regularity results for elliptic equations with measure data. One of the main features in measure data problems is that they do not in general have weak solutions. Thus, several notions of solutions have been suggested. Here we recall the notion of *SOLA (Solution Obtained as Limits of Approximations)* introduced by Boccardo and Gallouët [28, 29]. Consider the Dirichlet problem defined on a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$:

$$\begin{cases} -\mathrm{div}\,(|Du|^{p-2}Du) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\mu \in \mathcal{M}_b(\Omega)$, the space of all Borel measures with finite total mass on Ω, and $p > 2 - 1/n$. A function $u \in W_0^{1,1}(\Omega)$ is called a SOLA to (1.1) if it is a distributional solution, and moreover it is obtained as a ($W^{1,1}$- and a.e.) limit of approximating solutions $\{u_k\} \subset W_0^{1,p}(\Omega)$ to the regularized problems

$$\begin{cases} -\mathrm{div}\,(|Du_k|^{p-2}Du_k) = \mu_k & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega, \end{cases}$$

where the sequence $\{\mu_k\} \subset W^{-1,p'}(\Omega) \cap L^1(\Omega)$ converges to μ weakly* in the sense of measures and satisfies

$$\limsup_{k\to\infty} |\mu_k|(B) \le |\mu|(\overline{B}) \quad \text{for every ball } B \subset \mathbb{R}^n.$$

In [28], the authors proved the existence of a SOLA u to (1.1) satisfying

$$u \in W^{1,q}(\Omega) \qquad \forall\, q < \min\left\{ \frac{n(p-1)}{n-1}, p \right\}. \tag{1.2}$$

The result in (1.2) is optimal in the sense that we cannot in general take $q = n(p-1)/(n-1)$. This can be shown by the fundamental solution

$$G_p(x) = c(n,p) \begin{cases} |x|^{\frac{p-n}{p-1}} - 1 & \text{if } p \neq n, \\ \log |x| & \text{if } p = n, \end{cases} \tag{1.3}$$

which is the unique SOLA to the Dirichlet problem

$$\begin{cases} -\operatorname{div}\,(|Du|^{p-2}Du) = \delta_0 & \text{in } B_1(0), \\ u = 0 & \text{on } \partial B_1(0), \end{cases} \tag{1.4}$$

with δ_0 being the Dirac measure charging the origin; see [84, 165, 197]. Also, the lower bound $p > 2 - 1/n$, equivalent to $n(p-1)/(n-1) > 1$, is not avoidable in order to ensure $u \in W^{1,1}(\Omega)$. This is a common phenomenon in the theory of measure data problems, and one needs other notions of solutions when $1 < p \le 2 - 1/n$. We refer to [23, 30, 84, 160] for various notions of solutions to measure data problems; see also [73, 77, 120] for problems with nonstandard growth. All such definitions are essentially equivalent in the case of nonnegative measures [128].

The uniqueness of SOLA to general measure data problems remains open except for $p = 2$ [37, 123, 197] and $p = n$ [104, 119]. However, when the data is an L^1-function, it is possible to show the uniqueness, see [23, 85].

1.1.1 Nonlinear Calderón-Zygmund theory

Calderón-Zygmund theory is concerned with integrability and differentiability of solutions. We first consider integrability results, recalling (1.2). In fact, as can be seen by (1.3), Marcinkiewicz spaces (see Section 2.2.3 below) are the correct ones for a sharp integrability for measure data problems, see [23, 103, 104]. For (1.1), it holds that

$$\mu \in \mathcal{M}_b(\Omega) \implies |Du|^{p-1} \in \mathcal{M}^{\frac{n}{n-1}}_{\mathrm{loc}}(\Omega).$$

Since the sharpness of (1.2) is shown by counterexamples like (1.4) involving Dirac measures, one may expect better regularity results when considering "diffusive" measures. Namely, we consider the case when

$$|\mu|(B_R) \lesssim R^{n-\theta} \tag{1.5}$$

for every ball $B_R \subset \Omega$, where $\theta \in [0, n]$. In this case we say that μ belongs to the Morrey space $L^{1,\theta}(\Omega)$, and accordingly define

$$\|\mu\|_{L^{1,\theta}(\Omega)} := \sup_{B_R \subset \Omega} \frac{|\mu|(B_R)}{R^{n-\theta}}.$$

Note that a measure satisfying (1.5) cannot concentrate on sets with Hausdorff dimension less than $n-\theta$, see [6, Theorem 5.1.12]. Indeed, in [164, 168], it was identified that condition (1.5) leads to the improvement in maximal integrability, along with Morrey type regularity, of the gradient. That is,

$$\mu \in L^{1,\theta}(\Omega) \implies |Du|^{p-1} \in \mathcal{M}^{\frac{\theta}{\theta-1},\theta}_{\mathrm{loc}}(\Omega) \tag{1.6}$$

holds for $\theta \in [p, n]$, with $2 - 1/n < p \leq n$. Note that in the case $\theta < p$, we have $\mu \in W^{-1,p'}(\Omega)$ from Sobolev's embedding theorem (when $p > n$) and Adams' trace theorem [4] (when $\theta < p \leq n$). In turn, we are in the realm of weak solutions in $W^{1,p}(\Omega)$, where (1.6) is trivial since $\theta/(\theta-1) < p/(p-1)$. The parabolic analogs of (1.6) were obtained in [11, 14].

We next examine differentiability results. For the Poisson equation

$$-\triangle u = \mu \quad \text{in } \Omega, \tag{1.7}$$

the classical Calderón-Zygmund theory [67, 68] asserts that

$$\mu \in L^q_{\mathrm{loc}}(\Omega) \implies Du \in W^{1,q}_{\mathrm{loc}}(\Omega)$$

holds whenever $q \in (1, \infty)$. This result fails when $q = 1$. Nevertheless, it holds that

$$\mu \in L^1_{\mathrm{loc}}(\Omega) \implies Du \in W^{\sigma,1}_{\mathrm{loc}}(\Omega) \quad \forall\, \sigma \in (0,1),$$

which still holds when μ is merely a Borel measure. For the definition of fractional Sobolev spaces, see Section 2.2.2 below.

For nonlinear elliptic problems modeled on (1.1), higher differentiability results in the scale of fractional Sobolev spaces were first proved in [164, 168] via the variational difference quotients argument originally introduced in [140, 141]. For the case of parabolic equations with $p = 2$, see [19].

Later, differentiability results were eventually upgraded to a completely linearized form in [7]. For example, in the model case (1.1) it holds that

$$\mu \in \mathcal{M}_b(\Omega) \implies |Du|^{p-2}Du \in W^{\sigma,1}_{\mathrm{loc}}(\Omega) \quad \forall\, \sigma \in (0,1). \tag{1.8}$$

We also refer to [55] for the case of equations with coefficients.

In [164, 168], it was also observed that the Morrey condition (1.5) with $\theta \in [p, n]$ leads to fractional Sobolev-Morrey regularity of the gradient. We further note that in the case $\theta < p$, different kinds of differentiability results as the one in [164, Theorem 1.10] can be obtained.

We also mention the paper [9], where linearized Calderón-Zygmund type estimates in the scale of Besov and Triebel-Lizorkin spaces were proved for the p-Laplace equation with data in divergence form:

$$-\operatorname{div}(|Du|^{p-2}Du) = -\operatorname{div} F \quad \text{in } \Omega, \quad \text{with } F \in L^{p'}(\Omega). \tag{1.9}$$

Specifically, when $p > 2$ and $n = 2$, it holds that

$$F \in W^{\sigma,1}_{\mathrm{loc}}(\Omega) \implies |Du|^{p-2}Du \in W^{\sigma,1}_{\mathrm{loc}}(\Omega) \quad \text{whenever } \frac{2}{p} < \sigma < 1. \tag{1.10}$$

For more on nonlinear Calderón-Zygmund theory, see [165, 169].

1.1.2 Nonlinear potential theory

Potential theory is concerned with pointwise estimates and fine properties of solutions. Pointwise potential estimates for solutions to nonlinear problems modeled on (1.1) were first obtained by Kilpeläinen and Malý [129, 130], where they actually considered nonnegative p-superharmonic functions and corresponding nonnegative measures. These results were later revisited by Trudinger and Wang [196] with a different method that can be extended to subelliptic equations; see also [136]. Another approach was developed by Duzaar and Mingione [111], which applies to equations with signed measures.

We summarize those results in [111, 129, 130, 136, 196] as follows: if u is a SOLA to (1.1), then there holds

$$|u(x_0)| \leq c\mathbf{W}^{\mu}_{1,p}(x_0, R) + c\left(\fint_{B_R(x_0)} |u|^{q_0}\, dx\right)^{\frac{1}{q_0}} \tag{1.11}$$

whenever $B_R(x_0) \subset \Omega$ is a ball and the right-hand side is finite with $q_0 \coloneqq \max\{p-1, 1\}$, where

$$\mathbf{W}^{\mu}_{\beta,p}(x_0, R) \coloneqq \int_0^R \left[\frac{|\mu|(B_\rho(x_0))}{\rho^{n-\beta p}}\right]^{\frac{1}{p-1}} \frac{d\rho}{\rho}, \qquad \beta > 0,$$

is the nonlinear Wolff potential of μ. Moreover, the estimate (1.11) is sharp in the sense that the potential $\mathbf{W}^{\mu}_{1,p}$ cannot be replaced by any other smaller potential. This comes from the following lower bound, which holds when both the measure μ and the solution u are nonnegative:

$$\mathbf{W}^{\mu}_{1,p}(x_0, R) \leq cu(x_0). \tag{1.12}$$

The estimate (1.11) was extended to the p-Laplace system in [151], where no analog of (1.12) is available due to the absence of maximum principle.

Later, potential estimates for nonlinear equations were upgraded to the gradient level. The first result was obtained by Mingione [167] for the non-degenerate case $p = 2$:

$$|Du(x_0)| \leq c\mathbf{I}^{\mu}_{1}(x_0, R) + c\fint_{B_R(x_0)} |Du|\, dx \tag{1.13}$$

for a.e. $x_0 \in \Omega$ whenever $B_R(x_0) \subset \Omega$, where

$$\mathbf{I}^{\mu}_{\beta}(x_0, R) \coloneqq \int_0^R \frac{|\mu|(B_\rho(x_0))}{\rho^{n-\beta}} \frac{d\rho}{\rho}, \qquad \beta > 0,$$

is the truncated Riesz potential of μ. Note that (1.13) is the same as the one available for the Poisson equation (1.7) via representation formulas.

Subsequently, in [111] concerning the case $p \geq 2$, it was proved that

$$|Du(x_0)| \leq c\mathbf{W}^{\mu}_{\frac{1}{p},p}(x_0, R) + c\fint_{B_R(x_0)} |Du|\, dx. \tag{1.14}$$

Note that (1.14) reduces to (1.13) when $p = 2$, since $\mathbf{W}^{\mu}_{1/2,2} = \mathbf{I}^{\mu}_{1}$.

Surprisingly, in contrast to (1.11) and (1.12), it was shown in [110, 145] that gradient estimates via Riesz potentials continue to hold for nonlinear, possibly degenerate, equations with $p > 2 - 1/n$. To be precise, we have the following estimate in a completely linearized form:

$$|Du(x_0)| \le c\,[\mathbf{I}_1^{\mu}(x_0,R)]^{\frac{1}{p-1}} + c \fint_{B_R(x_0)} |Du|\,dx \tag{1.15}$$

whenever $B_R(x_0) \subset \Omega$ and the right-hand side is finite; see also [105, 173, 174, 175] for the range $1 < p \le 2 - 1/n$. In particular, when $p \ge 2$, (1.15) improves (1.14) in light of the inequality

$$[\mathbf{I}_1^{\mu}(x_0,R)]^{\frac{1}{p-1}} \le c(p)\mathbf{W}^{\mu}_{\frac{1}{p},p}(x_0,2R).$$

We note that (1.15) can be considered as a counterpart of (1.8) for potential estimates. The underlying linearization phenomena also appear in several elliptic equations with nonstandard growth [13, 38, 61, 62]; see also [47, 111, 144, 148, 149] for the corresponding results for parabolic equations. Later in [151], the estimate (1.15) was extended to the p-Laplace system with measure data. Moreover, when the data μ is regular enough to ensure the existence of weak solutions, it is possible to obtain potential estimates for elliptic systems without quasi-diagonal structure in the setting of partial regularity [63, 150].

Such Riesz potential estimates provide a universal approach to a sharp borderline regularity, such as Lipschitz regularity, for nonlinear problems [22, 88]. They also imply Calderón-Zygmund type estimates in various function spaces via the mapping properties of Riesz potentials [5, 76].

We also mention the papers [35, 78] concerning sharp maximal function estimates and potential estimates for the p-Laplace system (1.9) with data in divergence form. Specifically, in [35] it was proved that

$$\begin{aligned} |Du(x_0)|^{p-1} \le c \int_0^R \left(\fint_{B_r(x_0)} |F - (F)_{B_r(x_0)}|^{p'}\,dx \right)^{\frac{1}{p'}} \frac{dr}{r} \\ + c \fint_{B_R(x_0)} |Du|^{p-1}\,dx \end{aligned} \tag{1.16}$$

whenever $B_R(x_0) \Subset \Omega$ and the right-hand side is finite. For more on nonlinear potential theory, see [146, 170].

1.2 Elliptic measure data problems with nonstandard growth

Let us consider the following equation

$$-\mathrm{div}\left(\frac{\partial_t\Phi(x,|Du|)}{|Du|}Du\right)=\mu \quad \text{in } \Omega,$$

where $\Phi:\Omega\times[0,\infty)\to[0,\infty)$ is a generalized N-function, see Section 2.2.1 for details. The following are typical examples of nonstandard growth:

- Orlicz growth:
$$\Phi(x,t)=G(t). \tag{1.17}$$

- Variable exponents:
$$\Phi(x,t)=(t^2+s^2)^{\frac{p(x)-2}{2}}t^2, \quad p:\Omega\to(1,\infty),\ s\in[0,1]. \tag{1.18}$$

- Double phase:
$$\Phi(x,t)=t^p+a(x)t^q, \quad 1<p\le q<\infty,\ a:\Omega\to[0,\infty). \tag{1.19}$$

- Double phase in the borderline case:
$$\Phi(x,t)=t^p+a(x)t^p\log(e+t), \quad p\in(1,\infty),\ a:\Omega\to[0,\infty). \tag{1.20}$$

The case (1.17) was introduced in [157] as a natural generalization of the p-Laplacian. The nonautonomous cases (1.18)-(1.20) are typical examples of nonuniformly elliptic problems, which were first introduced in [198, 199, 200, 201]. In particular, (1.18) and (1.20) have several similarities [17]. We refer to [163, 171] for a comprehensive overview of nonuniformly elliptic problems.

There are several regularity results for measure data problems with (1.17), (1.18) and (1.20), such as global Calderón-Zygmund estimates [40, 41, 51] and potential estimates [13, 20, 32, 61, 62].

In Chapter 3, we first prove Marcinkiewicz-Morrey regularity for the case (1.17), which is announced in [56]. We then prove fractional differentiability for the case (1.20), which is announced in [54]. We also prove fractional differentiability for parabolic measure data problems with coefficients, which is announced in [39].

1.3 Elliptic obstacle problems with measure data

We next consider elliptic obstacle problems related to (1.1). As in the case of equations, usual variational inequalities are not available for such problems. In this case, Scheven [188, 189] introduced *limits of approximating solutions*, analogous to SOLA, see Chapter 4 for the precise definition. For other notions of solutions, see for instance [189] and related references.

Existence and regularity of limits of approximating solutions were first treated in [188, 189]. Specifically, in [189] gradient estimates via Wolff potentials (when $p > 2$) and Riesz potentials (when $2 - 1/n < p \leq 2$) were proved under an additional assumption that the obstacle $\psi \in W^{1,p}(\Omega) \cap W^{2,1}(\Omega)$ satisfies $\mathcal{D}\Psi := |D\psi|^{p-2}|D^2\psi| \in L^1(\Omega)$. In fact, such a higher differentiability assumption was used to treat the measures μ and $\mathcal{D}\Psi$ in the same manner, thereby obtaining estimates via potentials involving both μ and $\mathcal{D}\Psi$.

In Chapter 4, we first provide a natural approach to gradient potential estimates for obstacle problems; in particular, this leads us to remove the extra differentiability assumption on the obstacle. Moreover, Wolff potentials of μ appearing in [189] are replaced by Riesz potentials. The main outcome is a linearization phenomenon with respect to the pointwise estimates for the gradient, which allows to obtain new borderline regularity results. The main difficulty arises from the interplay between the measure and the obstacle. To overcome this, we apply an intrinsic linearization argument motivated from those developed in [7, 35]. The result is announced in [58].

By applying several ideas in the proof of potential estimates, we next prove fractional differentiability for double obstacle problems with measure data. More precisely, we prove an analog of (1.8) under a suitable differentiability assumption on the obstacles (see (4.103) below). Unlike the case of potential estimates, such a differentiability assumption is rather natural and almost sharp for the maximal differentiability result in Theorem 4.2.2 below, in view of the results in (1.8) and (1.10). Moreover, we are able to apply the linearization techniques developed in [7]. The result is announced in [59].

We also establish a comparison principle for obstacle problems with L^1-data. As a consequence, we show that the solution to a given obstacle problem with zero Dirichlet boundary condition is indeed affected by only the positive part of the obstacle, instead of the whole obstacle. This in turn improves several regularity results for such problems. The result is announced in [195].

1.4 Nonlocal equations, mixed local and nonlocal equations

The regularity theory for nonlocal problems with fractional orders has been extensively studied for the last two decades. Caffarelli and Silvestre [65] proved Harnack inequality for the fractional Laplace equation, $(-\triangle)^s u = 0$, by using an extension argument. Caffarelli, Chan and Vasseur [64] applied De Giorgi's approach to linear parabolic equations involving general kernels, and proved Hölder continuity of weak solutions. We refer to [66, 125, 126, 153, 162, 176, 177, 178, 194] for regularity results for nonlocal linear equations.

Later, for nonlocal nonlinear equations of fractional p-Laplacian type, Di Castro, Kuusi and Palatucci [93, 94] employed De Giorgi's approach to prove Hölder regularity and Harnack inequality for weak solutions. Cozzi [82] extended these results to inhomogeneous problems with lower order terms, by using fractional De Giorgi classes. We further refer to [33, 34, 115, 137, 138, 139, 159, 190] and references therein for various results for nonlocal problems of fractional p-Laplacian type. For a general overview of the history and related topics, see [36, 183].

We note that such a nonlocal operator is associated with a purely jump process. On the other hand, the generator of a general Lévy process is given by the sum of local and nonlocal operators, whose prototype is

$$-\triangle + (-\triangle)^s.$$

For elliptic and parabolic equations involving the above operator, Hölder continuity and Harnack inequality were proved in [10, 114] by combining probabilistic and analytic techniques. See also [24, 25, 26] for further results including regularity, maximum principles and other qualitative behaviors.

Subsequently, there have been also recent results for nonlinear operators modeled on

$$-\triangle_p + (-\triangle)_p^s. \tag{1.21}$$

Garain and Kinnunen [116] employed purely analytic methods based on [93, 94] to obtain regularity results including Hölder continuity and Harnack inequality. We refer to [26, 27, 117] for various results and relevant function spaces; see also [87] for a connection between a class of mixed functionals and local functionals with (p, q)-growth. We also mention the paper [89], in which the maximal regularity was established for a more general class of mixed local and nonlocal problems.

1.5 Nonlocal operators and measure data

Consider the following nonlocal nonlinear equation with measure data:

$$(-\triangle)_p^s u = \mu \quad \text{in } \Omega.$$

Existence and regularity results for such equations were obtained by Kuusi, Mingione and Sire [152, 154]. They first defined SOLA and proved its existence. Next, they proved the nonlocal analogs of the pointwise estimates (1.11) and (1.12) via perturbation arguments and excess decay estimates as in [111, 146]. More precisely, they proved the following: for any SOLA u, there holds

$$\begin{aligned} |u(x_0)| &\le c\mathbf{W}^{\mu}_{s,p}(x_0,R) + c\left(\fint_{B_R(x_0)} |u|^{q_0}\,dx\right)^{\frac{1}{q_0}} \\ &\quad + c\left(R^{sp}\int_{\mathbb{R}^n\setminus B_R(x_0)} \frac{|u(x)|^{p-1}}{|x-x_0|^{n+sp}}\,dx\right)^{\frac{1}{p-1}} \end{aligned} \tag{1.22}$$

whenever $B_R(x_0)\subset\Omega$ is a ball and the right-hand side is finite with $q_0 := \max\{p-1,1\}$. Moreover, if both μ and u are nonnegative in $B_R(x_0)$, then

$$\mathbf{W}^{\mu}_{s,p}(x_0,R/8) \le cu(x_0) + c\left(R^{sp}\int_{\mathbb{R}^n\setminus B_{R/2}(x_0)} \frac{(u_-(x))^{p-1}}{|x-x_0|^{n+sp}}\,dx\right)^{\frac{1}{p-1}}. \tag{1.23}$$

In [131], the approach in [129, 130] was extended to fractional p-superharmonic functions for a potential upper bound similar to (1.22), which in turn shows a nonlocal counterpart of the Wiener criterion. We also note that, for the case $p=2$, gradient potential estimates for SOLA were proved in [155].

The last term in each of (1.22) and (1.23) is called a nonlocal tail, which already appears in [93, 94]. Nonlocal tails play a crucial role in the local regularity theory for nonlocal fractional equations, accounting for long-range interactions of solutions.

In Chapter 5, we prove existence and potential estimates for measure data problems involving mixed local and nonlocal operators modeled on (1.21). The main results reflect both local and nonlocal characters of the problems. On one hand, since the local term has higher order, we obtain estimates via $\mathbf{W}^{\mu}_{1,p}$. On the other hand, due to the presence of the nonlocal term, we need to involve nonlocal tails in the estimates. The result is announced in [57].

1.6 Nonlocal operators with nonstandard growth

Nonlocal problems with nonstandard growth have been the object of recent investigation. As in the local case described in Section 1.2, we can consider typical examples of nonlocal operators with nonstandard growth conditions:

- Orlicz growth:

$$\mathrm{P.V.}\int_{\mathbb{R}^n} g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|u(x)-u(y)|}\frac{dy}{|x-y|^{n+s}}. \tag{1.24}$$

- Variable powers:

$$\mathrm{P.V.}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{n+s(x,y)p(x,y)}}\,dy, \tag{1.25}$$

  $s:\mathbb{R}^n\times\mathbb{R}^n\to(0,1)$, $p:\mathbb{R}^n\times\mathbb{R}^n\to(1,\infty)$.

- Double phase:

$$\begin{aligned}&\mathrm{P.V.}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}}\,dy\\&\qquad+\mathrm{P.V.}\int_{\mathbb{R}^n}a(x,y)\frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))}{|x-y|^{n+tq}}\,dy,\end{aligned} \tag{1.26}$$

  $s,t\in(0,1)$, $p,q>1$.

The techniques and results in [93, 94] were extended to nonlocal equations with Orlicz growth (1.24) and variable powers (1.25) in [45, 46] and [182], respectively. See also [70, 71, 72], where the techniques in [82] were extended, with more restrictive assumptions, to (1.24) and (1.25).

Nonlocal equations of double phase type were first treated in [91], where the authors proved Hölder regularity for viscosity solutions. A nonlocal self-improving property for bounded weak solutions was proved in [191]. We also mention the paper [113] concerning Hölder regularity for bounded weak solutions and a relationship between weak and viscosity solutions. Here, we point out that the papers [91, 113, 191] are restricted to solutions which are bounded in $\mathbb{R}^n$ and are under the assumption that $t\le s$. This means that the second term in (1.26) is a lower order term, which allows to consider a bounded, possibly discontinuous modulating coefficient $a(\cdot,\cdot)$.

In Chapter 6, we prove the local boundedness and Hölder continuity of weak solutions to nonlocal equations with double phase growth condition (1.26), which is announced in [52]. We emphasize that we deal with the case $s \le t$, which is a main difference from the papers [91, 113, 191]. This case is more delicate than the other case, since the second term in (1.26) has a higher order in the sense that $t \ge s$, $q \ge p$. To the best of our knowledge, the results presented in [52] are the first regularity results in this case. When we prove Hölder continuity in this case, we assume that the modulating coefficient $a(\cdot,\cdot)$ is Hölder continuous, which together with a restriction on the ranges of t and q allows us to replace $a(\cdot,\cdot)$ with a constant. Note that this argument is exactly the same as the one for the local double phase problem. Therefore, we are able to make the assumptions on t, q and $a(\cdot,\cdot)$ that are analogous to those for local double phase problems. Moreover, we assume that weak solutions are not bounded in $\mathbb{R}^n$, but only locally bounded in Ω. Therefore, we need to handle the nonlocal tails. The main difficulty arises in deriving the logarithmic type estimate (see Lemma 6.5.1 below). An analogous estimate was obtained for fractional p-Laplacian type equations in [93, Lemma 1.3], but we could not apply the same approach directly to our problem with nonstandard growth (1.26). In order to obtain such an estimate, we first assume that the weak solution is locally bounded, and then take advantage of the Hölder continuity of $a(\cdot,\cdot)$ in order to modify and develop the techniques used in the proof of [93, Lemma 1.3].

We also mention the paper [48], where Hölder regularity and Harnack inequality for mixed local and nonlocal double phase problems related to

$$(-\triangle)_p^s u - \operatorname{div}\left(a(x)|Du|^{q-2}Du\right) = 0 \quad \text{in } \Omega,$$

with $0 < s < 1 < p \le q$ and $a(\cdot) \ge 0$, were proved.

Chapter 2

Preliminaries

2.1 General notations

Throughout this thesis, we use the following notations.

1. Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, and $\partial\Omega$ is its boundary.

2. $B_r(x_0) \equiv B(x_0, r) \coloneqq \{x \in \mathbb{R}^n : |x - x_0| < r\}$ is the open ball in $\mathbb{R}^n$ with center $x_0 \in \mathbb{R}^n$ and radius $r > 0$. If there is no confusion, we simply write $B_r(x_0) \equiv B_r$. Moreover, given a ball B, we denote by γB the concentric ball with radius magnified by a factor $\gamma > 0$.

3. For a set $S \subset \mathbb{R}^n$, $\operatorname{diam}(S) \coloneqq \sup\{|x-y| : x, y \in S\}$ is the diameter of S. For two sets $S_1, S_2 \subset \mathbb{R}^n$, $\operatorname{dist}(S_1, S_2) \coloneqq \inf\{|x-y| : x \in S_1, y \in S_2\}$ is the distance between S_1 and S_2.

4. For each Lebesgue measurable set $S \subset \mathbb{R}^n$, $|S|$ is the (n-dimensional) Lebesgue measure of S.

5. If $f : S \to \mathbb{R}^k$ ($k \in \mathbb{N}$) is an integrable map and $0 < |S| < \infty$, we denote the integral average of f over S by

$$(f)_S \coloneqq \fint_S f\,dx \coloneqq \frac{1}{|S|}\int_S f\,dx.$$

6. The oscillation of a map f on S is defined by

$$\operatorname*{osc}_S f \coloneqq \sup_{x,y\in S} |f(x) - f(y)|.$$

7. For a real-valued function f, we denote $f_\pm := \max\{\pm f, 0\}$.

8. For each $k > 0$, we define the truncation operators $T_k, \mathfrak{T}_k : \mathbb{R} \to \mathbb{R}$ by

$$T_k(t) := \min\{k, \max\{-k, t\}\}, \quad \mathfrak{T}_k(t) := T_1(t - T_k(t)).$$

9. We denote by c a generic constant greater than or equal to one, whose value may possibly vary from line to line. Special occurrences will be denoted by c_*, c_0, c_1 or the like. We denote relevant dependencies on parameters by using parentheses, and use the abbreviation `data`. The meaning of `data` will be specified in each chapter.

10. We write $a \lesssim b$ to mean that there exists $c \equiv c(\texttt{data}) \geq 1$ such that $a \leq cb$. We write $a \approx b$ if $a \lesssim b$ and $b \lesssim a$. If the constant c depends also on χ other than `data`, we write $a \lesssim_\chi b$ and $a \approx_\chi b$.

11. For any $p \geq 1$, we denote its Hölder and Sobolev conjugate exponents by

$$p' := \begin{cases} \dfrac{p}{p-1} & \text{if } p > 1, \\ \infty & \text{if } p = 1 \end{cases}$$

and

$$p^* := \begin{cases} \dfrac{np}{n-p} & \text{if } 1 \leq p < n, \\ \text{any number in } (p, \infty) & \text{if } p \geq n, \end{cases}$$

respectively. Moreover, given $s \in (0,1)$, we denote the (s-)fractional Sobolev conjugate exponent of p by

$$p_s^* := \begin{cases} \dfrac{np}{n-sp} & \text{if } sp < n, \\ \text{any number in } (p, \infty) & \text{if } sp \geq n. \end{cases}$$

12. For a function space $X(U;\mathbb{R}^k)$ of possibly vector valued measurable maps defined on an open set $U \subseteq \mathbb{R}^n$, we define the local variant $X_{\mathrm{loc}}(U;\mathbb{R}^k)$ as that space of maps $f : U \to \mathbb{R}^k$ such that $f \in X(U';\mathbb{R}^k)$ for every $U' \Subset U$. Moreover, also in the case f is vector valued, that is $k > 1$, we simply denote $X(U;\mathbb{R}^k) \equiv X(U)$, or even $X(U;\mathbb{R}^k) \equiv X$.

13. For an open set $U \subseteq \mathbb{R}^n$, we denote by $\mathcal{M}_b(U)$ the set of all signed Borel measures on U having finite total mass. Moreover, we identify $L^1(U)$ with a subset of $\mathcal{M}_b(U)$ in the following way: for each function $\mu \in L^1(U)$ and each measurable set $S \subseteq U$, we denote

$$|\mu|(S) = \int_S |\mu| \, dx.$$

We also identify each element of $\mathcal{M}_b(U)$ with its zero extension to $\mathbb{R}^n$.

2.2 Function spaces

2.2.1 Musielak-Orlicz spaces

Let $U \subseteq \mathbb{R}^n$ be an open set. A function $\Phi : U \times [0, \infty) \to [0, \infty)$ is called a generalized Young function if $\Phi(x, \cdot)$ is a convex function that satisfies

$$\Phi(x, 0) = 0, \quad \lim_{t\to\infty} \Phi(x, t) = \infty \qquad \text{for a.e. } x \in U$$

and $\Phi(\cdot, t)$ is measurable for every $t > 0$.

We say that $\Phi : U \times [0, \infty) \to [0, \infty)$ is a generalized N-function if it is a generalized Young function satisfying $\Phi(x, t) > 0$ for all $t > 0$,

$$\lim_{t\to 0} \frac{\Phi(x, t)}{t} = 0, \quad \lim_{t\to\infty} \frac{\Phi(x, t)}{t} = \infty \qquad \text{for a.e. } x \in U$$

and $\Phi(\cdot, t)$ is positive and measurable for every $t > 0$. Additionally, we assume that $\Phi(x, \cdot) \in C^2(0, \infty)$ for a.e. $x \in U$ and there holds

$$p_1 \le \frac{t\partial_t^2 \Phi(x, t)}{\partial_t \Phi(x, t)} \le p_2 \tag{2.1}$$

for some positive constants $p_1 \le p_2$, for $t > 0$ and a.e. $x \in U$. Then we have

$$p_1 + 1 \le \frac{t\partial_t \Phi(x, t)}{\Phi(x, t)} \le p_2 + 1$$

and

$$\min\{\alpha^{p_1+1}, \alpha^{p_2+1}\}\Phi(x, t) \le \Phi(x, \alpha t) \le \max\{\alpha^{p_1+1}, \alpha^{p_2+1}\}\Phi(x, t)$$

for $t, \alpha \geq 0$ and a.e. $x \in U$. The last inequalities imply the so called Δ_2 and ∇_2 conditions; we refer to [122] for more details. Hereafter, we always assume that a generalized N-function Φ satisfies (2.1).

Given a generalized Young function Φ, we define its Young's conjugate $\Phi^* : U \times [0,\infty) \to [0,\infty)$ by

$$\Phi^*(x,t) \coloneqq \sup_{s\geq 0} \left(st - \Phi(x,s)\right), \qquad x \in U,\, t \geq 0.$$

This definition directly implies the Young's inequality

$$st \leq \Phi(x,t) + \Phi^*(x,s)$$

whenever $s, t \geq 0$ and $x \in U$.

We also note that

$$\Phi^*\left(x, \frac{\Phi(x,t)}{t}\right) \leq \Phi(x,t) \leq \Phi^*\left(x, \frac{\Phi(x,2t)}{t}\right).$$

Given a generalized N-function Φ, the Musielak-Orlicz space $L^\Phi(U;\mathbb{R}^k) \equiv L^\Phi(U)$ is the set of all Lebesgue measurable maps $f : U \to \mathbb{R}^k$ satisfying

$$\int_U \Phi(x, |f|)\, dx < \infty,$$

equipped with the following Luxemburg norm

$$\|f\|_{L^\Phi(U)} \coloneqq \inf\left\{\lambda > 0 : \int_U \Phi\left(x, \frac{|f|}{\lambda}\right)\, dx \leq 1\right\},$$

and the Musielak-Orlicz-Sobolev space $W^{1,\Phi}(U;\mathbb{R}^k) \equiv W^{1,\Phi}(U)$ is the set of all maps $f \in L^\Phi(U) \cap W^{1,1}(U)$ satisfying $|Df| \in L^\Phi(U)$, equipped with the norm

$$\|f\|_{W^{1,\Phi}(U)} \coloneqq \|f\|_{L^\Phi(U)} + \|Df\|_{L^\Phi(U)}.$$

Also, we define $W^{1,\Phi}_0(U)$ as the closure of $C^\infty_0(U)$ in $W^{1,\Phi}(U)$. By (2.1) and the fact that $\inf_{x\in U} \Phi(x,1) > 0$, they are Banach spaces. When $\Phi(x,t) = t^{p(x)}$, the spaces L^Φ and $W^{1,\Phi}$ are called variable exponent Lebesgue and Sobolev spaces, respectively. When $\Phi(\cdot)$ is independent of x, we call it an N-function. In this case the spaces L^Φ and $W^{1,\Phi}$ are called Orlicz and Orlicz-Sobolev spaces, respectively. We refer to [99, 122] for a comprehensive introduction.

We recall the following embedding theorem and Sobolev-Poincaré type inequality for Orlicz-Sobolev spaces.

Lemma 2.2.1 ([75]). *Let $U \subset \mathbb{R}^n, n \geq 2$, be a bounded open set and $\Phi(\cdot)$ an N-function satisfying*

$$\int_0 \left(\frac{s}{\Phi(s)}\right)^{\frac{1}{n-1}} ds < \infty \quad \textit{and} \quad \int^\infty \left(\frac{s}{\Phi(s)}\right)^{\frac{1}{n-1}} ds = \infty.$$

Then there exists a constant $c_S(n)$, depending only on n, such that for every $u \in W_0^{1,\Phi}(U)$ there holds

$$\int_U \Phi_n\left(\frac{|u|}{c_S(n)\left(\int_U \Phi(|Du|)\,dx\right)^{1/n}}\right) dx \leq \int_U \Phi(|Du|)\,dx,$$

where

$$\Phi_n(t) := (\Phi \circ H_n^{-1})(t) \quad \textit{for} \quad H_n(t) := \left(\int_0^t \left[\frac{s}{\Phi(s)}\right]^{\frac{1}{n-1}} ds\right)^{\frac{n-1}{n}}. \tag{2.2}$$

Lemma 2.2.2 ([63]). *Let $\Phi \in C^1[0,\infty)$ be an N-function satisfying*

$$p_1 \leq \frac{t\Phi'(t)}{\Phi(t)} \leq p_2, \qquad t \geq 0,$$

for some $1 < p_1 \leq p_2 < \infty$. Then there exist constants $\vartheta \in (0,1)$ and $c \geq 1$, both depending only on n, p_1 and p_2, such that

$$\fint_{B_R} \Phi\left(\left|\frac{f-(f)_{B_R}}{R}\right|\right) dx \leq c\left(\fint_{B_R} [\Phi(|Df|)]^\vartheta\,dx\right)^{\frac{1}{\vartheta}}$$

for any ball $B_R \subset \mathbb{R}^n$ and $f \in W^{1,\Phi}(B_R)$

We also recall the following estimate in the Zygmund space $L\log^\beta L$ (see [2, 124]): for any $q, \beta > 1$ and $f \in L^q(U)$, we have

$$\fint_U |f|\log^\beta\left(e + \frac{|f|}{(|f|)_U}\right) dx \leq c(q,\beta)\left(\fint_U |f|^q\,dx\right)^{\frac{1}{q}}. \tag{2.3}$$

2.2.2 Fractional Sobolev spaces

For any open set $U \subseteq \mathbb{R}^n$, $\alpha \in (0,1)$ and $q \geq 1$, the fractional Sobolev space $W^{\alpha,q}(U;\mathbb{R}^k) \equiv W^{\alpha,q}(U)$ is the set of all functions $f : U \to \mathbb{R}^k$ satifying

$$\begin{aligned}\|f\|_{W^{\alpha,q}(U)} &:= \|f\|_{L^q(U)} + [f]_{\alpha,q;U} \\ &:= \left(\int_U |f|^q \, dx\right)^{\frac{1}{q}} + \left(\int_U \int_U \frac{|f(x)-f(y)|^q}{|x-y|^{n+\alpha q}} \, dxdy\right)^{\frac{1}{q}}.\end{aligned}$$

We also define $W_0^{\alpha,q}(U)$ as the closure of $C_0^\infty(U)$ in $W^{\alpha,q}(U)$.

From the above definition, one can see that the strict inclusions

$$W^{1,q}_{\mathrm{loc}}(U) \subsetneq W^{t,q}_{\mathrm{loc}}(U) \subsetneq W^{s,q}_{\mathrm{loc}}(U) \subsetneq L^q_{\mathrm{loc}}(U)$$

hold whenever $0 < s < t < 1 \leq q$. For the first inclusion, we recall the following result from [95, Proposition 2.2]: If U is a bounded Lipschitz domain, then

$$\|f\|_{W^{\alpha,q}(U)} \leq c(n,\alpha,q,U)\|f\|_{W^{1,q}(U)} \qquad \forall\ f \in W^{1,p}(U).$$

The second inclusion is a special case of the following lemma. Note that it fails to hold when $s = t$, see [172].

Lemma 2.2.3. *Let $1 \leq p \leq q$ and $0 < s < t < 1$. Let $U \subset \mathbb{R}^n$ be a bounded open set. Then, for any $f \in W^{t,q}(U)$ we have*

$$[f]_{s,p;U} \leq c|U|^{\frac{q-p}{pq}}(\mathrm{diam}(U))^{t-s}[f]_{t,q;U}$$

for a constant $c \equiv c(n,s,t,p,q)$.

Proof. When $p < q$, the result follows from [82, Lemma 4.6]. When $p = q$, we directly have

$$\begin{aligned}&\left(\int_U \int_U \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}} \, dxdy\right)^{\frac{1}{p}} \\ &= \left(\int_U \int_U \frac{|f(x)-f(y)|^p}{|x-y|^{n+tp}}|x-y|^{(t-s)p} \, dxdy\right)^{\frac{1}{p}} \\ &\leq (\mathrm{diam}(U))^{t-s}\left(\int_U \int_U \frac{|f(x)-f(y)|^p}{|x-y|^{n+tp}} \, dxdy\right)^{\frac{1}{p}}.\end{aligned}$$

□

Fractional Sobolev spaces have their own embedding properties together with Poincaré type inequalities, see [95, 106, 182].

Lemma 2.2.4. *Let $\alpha \in (0,1]$ and $q \geq 1$ be such that $\alpha q < n$. Let $U \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then for any $f \in W^{\alpha,q}(U;\mathbb{R}^k)$, we have*

$$\|f\|_{L^{q^*_\alpha}(U)} \leq c\|f\|_{W^{\alpha,q}(U)}$$

for a constant $c \equiv c(n, \alpha, q, [\partial U]_{0,1}, \mathrm{diam}(U))$.

Lemma 2.2.5. *Let $\alpha \in (0,1)$ and $q \geq 1$. Let $B_R \subset \mathbb{R}^n$ be a ball and $f \in W^{\alpha,q}(B_R)$.*

(1) We have

$$\int_{B_R} |f - (f)_{B_R}|^q \, dx \leq cR^{\alpha q} \int_{B_R} \int_{B_R} \frac{|f(x) - f(y)|^q}{|x-y|^{n+\alpha q}} \, dxdy \tag{2.4}$$

for a constant $c \equiv c(n, \alpha, q)$.

(2) If $\alpha q \leq n$, then we have

$$\left(\fint_{B_R} |f - (f)_{B_R}|^{q^*_\alpha} \, dx \right)^{\frac{q}{q^*_\alpha}} \leq cR^{\alpha q} \fint_{B_R} \int_{B_R} \frac{|f(x) - f(y)|^q}{|x-y|^{n+\alpha q}} \, dxdy \tag{2.5}$$

for a constant $c \equiv c(n, \alpha, q)$.

We also recall another fractional Poincaré inequality, see [82, Lemma 4.7].

Lemma 2.2.6. *Let $\alpha \in (0,1)$ and $q \geq 1$. Let $f \in W^{\alpha,q}(B_R)$ be such that $f = 0$ a.e. on a set $\Omega_0 \subseteq B_R$, with $|\Omega_0| \geq \gamma |B_R|$ for some $\gamma \in (0,1]$. Then,*

$$\int_{B_R} |f|^q \, dx \leq cR^{\alpha q} \int_{B_R} \int_{B_R} \frac{|f(x) - f(y)|^q}{|x-y|^{n+\alpha q}} \, dxdy$$

for a constant $c \equiv c(n, \alpha, q, \gamma)$.

For a vector $h \in \mathbb{R}^n$, we denote $U_{|h|} := \{x \in U : \mathrm{dist}(x, \partial U) > |h|\}$, and define the finite difference operator $\tau_h : L^1(U) \to L^1(U_{|h|})$ by letting

$$\tau_h f(x) \equiv \tau_h(f)(x) := f(x+h) - f(x), \quad x \in U_{|h|}$$

for each possibly vector-valued function $f \in L^1(U)$, whenever $U_{|h|}$ is nonempty.

The Nikol'skii space $N^{\alpha,q}(U;\mathbb{R}^k) \equiv N^{\alpha,q}(U)$ is the set of all functions $f : U \to \mathbb{R}^k$ for which

$$\|f\|_{N^{\alpha,q}(U)} := \left(\int_U |f|^q \, dx\right)^{\frac{1}{q}} + \left(\sup_{h\neq 0} \int_{U_{|h|}} \frac{|\tau_h f|^q}{|h|^{\alpha q}} \, dx\right)^{\frac{1}{q}} < \infty.$$

If we define $N^{1,q}(U)$ in the same way, then the standard difference quotient characterization of Sobolev spaces implies $N^{1,q}(U) \subset W^{1,q}_{\mathrm{loc}}(U)$ and $W^{1,q}_{\mathrm{loc}}(U) = N^{1,q}_{\mathrm{loc}}(U)$ for $q > 1$, along with the inequality

$$\int_{B_r} |\tau_h f|^q \, dx \le c(n,q)|h|^q \int_{B_R} |Df|^q \, dx \tag{2.6}$$

for any concentric balls $B_r \Subset B_R \Subset U$, vector $h \in \mathbb{R}^n$ with $|h| \le R - r$ and $f \in W^{1,q}(B_R)$.

Its fractional counterpart is the following strict inclusions

$$W^{\alpha,q}(U) \subsetneq N^{\alpha,q}(U) \subsetneq W^{\alpha-\varepsilon,q}(U) \qquad \forall\, \varepsilon \in (0,\alpha)$$

which hold for instance for bounded Lipschitz domain U. In particular, the local version of the second inclusion is quantified as follows, see [87, Lemma 1].

Lemma 2.2.7. *Let $f \in L^q(\Omega)$, $q \ge 1$, and assume that for $\bar\alpha \in (0,1]$, $S \ge 0$ and a bounded open set $\tilde\Omega \Subset \Omega$ we have*

$$\|\tau_h f\|_{L^q(\tilde\Omega)} \le S|h|^{\bar\alpha}$$

for every $h \in \mathbb{R}^n$ satisfying $0 < |h| \le d$, where $0 < d \le \operatorname{dist}(\tilde\Omega, \partial\Omega)$. Then $f \in W^{\alpha,q}(\tilde\Omega)$ for every $\alpha \in (0,\bar\alpha)$. Moreover, for every $\alpha \in (0,\bar\alpha)$ we have

$$\|f\|_{W^{\alpha,q}(\tilde\Omega)} \le c(n,q)\left(\frac{d^{\bar\alpha-\alpha} S}{[(\bar\alpha-\alpha)q]^{1/q}} + \frac{\|f\|_{L^q(\tilde\Omega)}}{\min\{d^{n/q+\alpha},1\}}\right).$$

We then recall the definition of fractional Sobolev-Morrey spaces. Let $U \subseteq \mathbb{R}^n$ be an open set. For $\alpha \in (0,1)$, $q \ge 1$ and $\theta \in [0,n]$, we say that a map $f \in W^{\alpha,q}(U;\mathbb{R}^k)$ belongs to $W^{\alpha,q,\theta}(U;\mathbb{R}^k)$ if and only if

$$[f]^q_{\alpha,q,\theta;U} := \sup_{B_R \subset U} R^{\theta-n}[f]^q_{\alpha,q;B_R} < \infty.$$

We accordingly define $\|f\|_{W^{\alpha,q,\theta}(U)} := \|f\|_{W^{\alpha,q}(U)} + [f]_{\alpha,q,\theta;U}$.

2.2.3 Lorentz spaces, Marcinkiewicz spaces

For $\gamma \in [1, \infty)$ and $q \in (0, \infty)$, the Lorentz space $L(\gamma, q)(U)$ is defined as the set of all measurable maps $f : U \to \mathbb{R}^k$ satisfying

$$\|f\|^q_{L(\gamma,q)(U)} \coloneqq \gamma \int_0^\infty (\lambda^\gamma |\{x \in U : |f(x)| > \lambda\}|)^{\frac{q}{\gamma}} \frac{d\lambda}{\lambda} < \infty.$$

In the case $q = \infty$, $L(\gamma, \infty)(U) \equiv \mathcal{M}^\gamma(U)$ is called the Marcinkiewicz space. It is defined as the set of all measurable maps $f : U \to \mathbb{R}^k$ satisfying

$$\|f\|^\gamma_{\mathcal{M}^\gamma(U)} \coloneqq \sup_{\lambda \geq 0} \lambda^\gamma |\{x \in U : |f(x)| > \lambda\}| < \infty. \tag{2.7}$$

Marcinkiewicz spaces are often called weak Lebesgue spaces, due to the strict inclusions

$$L^\gamma(U) \subsetneq \mathcal{M}^\gamma(U) \subsetneq L^{\gamma-\varepsilon}(U) \qquad \forall\, \varepsilon \in (0, \gamma).$$

In particular, for the second inclusion we have the following Hölder type inequality:

$$\|f\|_{L^q(U)} \leq \left(\frac{\gamma}{\gamma - q}\right)^{\frac{1}{q}} |U|^{\frac{1}{q}-\frac{1}{\gamma}} \|f\|_{\mathcal{M}^\gamma(U)} \qquad \forall\, f \in \mathcal{M}^\gamma(U),$$

whenever $1 \leq q < \gamma$, see [164, Lemma 2.8]. We also have the Riesz potential embedding $I_\beta : L^1 \to \mathcal{M}^{n/(n-\beta)}$ for $\beta \in (0, n)$, see [4].

With $\theta \in [0, n]$, we define the Marcinkiewicz-Morrey space $\mathcal{M}^{\gamma,\theta}(U)$ as the set of all $f \in \mathcal{M}^\gamma(U)$ satisfying

$$\sup_{B_R \subset U} R^{\theta-n} \|f\|^\gamma_{\mathcal{M}^\gamma(B_R)} < \infty.$$

Accordingly, we define

$$\|f\|_{\mathcal{M}^{\gamma,\theta}(U)} \coloneqq \|f\|_{\mathcal{M}^\gamma(U)} + \left[\sup_{B_R \subset U} R^{\theta-n} \|f\|^\gamma_{\mathcal{M}^\gamma(B_R)}\right]^{\frac{1}{\gamma}}.$$

It is obvious that $\mathcal{M}^{\gamma,n}(U) \equiv \mathcal{M}^\gamma(U)$. We also note that $\|\cdot\|_{\mathcal{M}^{\gamma,\theta}(U)}$ is lower semicontinuous with respect to the a.e. convergence, see [166, Remark 3].

2.3 Auxiliary results

2.3.1 Basic properties of the vector fields $V(\cdot)$ and $A(\cdot)$

We consider a vector field $A : \mathbb{R}^n \to \mathbb{R}^n$ which is C^1-regular on $\mathbb{R}^n$ for $p \geq 2$ and on $\mathbb{R}^n \setminus \{0\}$ for $p < 2$. It also satisfies the following growth and ellipticity assumptions:

$$\begin{cases} |A(z)| + |\partial A(z)|(|z|^2 + s^2)^{\frac{1}{2}} \leq L(|z|^2 + s^2)^{\frac{p-1}{2}}, \\ \nu(|z|^2 + s^2)^{\frac{p-2}{2}}|\xi|^2 \leq \partial A(z)\xi \cdot \xi, \end{cases} \tag{2.8}$$

for every $z, \xi \in \mathbb{R}^n$, where $p > 1$, $0 < \nu \leq L$ and $s \geq 0$ are fixed constants. Here we denote $\texttt{data} = (n, p, \nu, L)$.

Observe that the ellipticity assumption in (2.8) implies the following monotonicity property:

$$(A(z_1) - A(z_2)) \cdot (z_1 - z_2) \approx (|z_1|^2 + |z_2|^2 + s^2)^{\frac{p-2}{2}}|z_1 - z_2|^2$$

for any $z_1, z_2 \in \mathbb{R}^n$.

We now define the auxiliary vector field $V = V_s : \mathbb{R}^n \to \mathbb{R}^n$ by

$$V(z) = V_s(z) := (|z|^2 + s^2)^{\frac{p-2}{4}} z, \qquad z \in \mathbb{R}^n,$$

which is a locally bi-Lipschitz bijection of $\mathbb{R}^n$. Moreover, for any $z_1, z_2 \in \mathbb{R}^n$,

$$|V(z_1) - V(z_2)| \approx (|z_1|^2 + |z_2|^2 + s^2)^{\frac{p-2}{4}}|z_1 - z_2|, \tag{2.9}$$

holds with the implicit constant depending only on p. In particular, we have

$$|z_1 - z_2|^p \lesssim |V(z_1) - V(z_2)|^2 \quad \text{when } p \geq 2. \tag{2.10}$$

Consequently, taking (2.9) into account, the monotonicity of $A(\cdot)$ can be written simply in terms of $V(\cdot)$. Namely, for any $z_1, z_2 \in \mathbb{R}^n$ there holds

$$(A(z_1) - A(z_2)) \cdot (z_1 - z_2) \approx |V(z_1) - V(z_2)|^2. \tag{2.11}$$

We also notice that, by the definition of $V(\cdot)$,

$$|V(z)|^2 + s^p \approx |z|^p + s^p \approx (|z| + s)^p.$$

We then recall some properties of the vector field $A(\cdot)$, see [7, 187].

Lemma 2.3.1. *The following inequalities hold for every $z, z_1, z_2 \in \mathbb{R}^n$:*

$$\begin{aligned} |A(z)| + s^{p-1} &\approx |z|^{p-1} + s^{p-1} \approx (|z| + s)^{p-1}, \\ |A(z_1) - A(z_2)| &\approx (|z_1|^2 + |z_2|^2 + s^2)^{\frac{p-2}{2}} |z_1 - z_2|. \end{aligned} \tag{2.12}$$

In particular, $A(\cdot)$ is a locally bi-Lipschitz bijection, and it holds that

$$\begin{aligned} &|A(z_1) - A(z_2)| \lesssim (|z_1|^2 + s^2)^{\frac{p-2}{2}} |z_1 - z_2| + |z_1 - z_2|^{p-1} \quad \textit{when } p \geq 2, \\ &|A(z_1) - A(z_2)| \lesssim |z_1 - z_2|^{p-1} \quad \textit{when } 1 < p \leq 2. \end{aligned}$$

We now recall some inequalities for integrals. If $S \subset \mathbb{R}^n$ is a measurable set with $0 < |S| < \infty$ and $f \in L^q(S; \mathbb{R}^k)$ for some $q \in [1, \infty]$, then we have

$$\|f - (f)_S\|_{L^q(S)} \leq 2\|f - z_0\|_{L^q(S)} \qquad \forall\, z_0 \in \mathbb{R}^k. \tag{2.13}$$

In particular, using this inequality and recalling that $A(\cdot)$ is bijective, we can prove the following lemma, see [100, Lemma A.2].

Lemma 2.3.2. *Let $p > 1$, and let B be a ball in $\mathbb{R}^n$. Given any map $f \in L^p(B; \mathbb{R}^n)$, denote by $f_A \in \mathbb{R}^n$ the vector satisfying $A(f_A) = (A(f))_B$. Then we have*

$$⨍_B |V(f) - (V(f))_B|^2\,dx \approx ⨍_B |V(f) - V((f)_B)|^2\,dx \approx ⨍_B |V(f) - V(f_A)|^2\,dx. \tag{2.14}$$

We next introduce a class of shifted N-functions and their properties. Shifted N-functions play a crucial role in various regularity estimates for problems with general growth, and they are especially effective when dealing with both super- and sub-quadratic growth simultaneously. For more on the definitions and properties of general shifted N-functions, we refer to [97, 98, 101].

For the N-function $\varphi(t) \equiv \varphi_0(t) = (s + t)^{p-2} t^2$, $t \geq 0$, we define the shifted N-function

$$\varphi_a(t) := (a + s + t)^{p-2} t^2, \qquad t \geq 0,$$

for each $a \geq 0$. Then $\varphi_a(\cdot)$ is an N-function. Moreover, a direct calculation shows that

$$\min\{p-1,1\} \le \frac{t\varphi_a''(t)}{\varphi_a'(t)} \le \max\{p-1,1\},$$
$$\min\{p,2\} \le \frac{t\varphi_a'(t)}{\varphi_a(t)} \le \max\{p,2\}$$

hold for every $t \ge 0$. Note that the second inequality implies that the family $\{\varphi_a\}_{a\ge 0}$ satisfies the Δ_2 and ∇_2 conditions uniformly in a, i.e., $\varphi_a(2t) \approx \varphi_a(t)$ uniformly in $a, t \ge 0$. In turn, the following versions of Young's inequality

$$\begin{aligned} t_1 t_2 &\le \varepsilon\varphi_a(t_1) + c\varepsilon^{1-\max\{p',2\}}(\varphi_a)^*(t_2), \\ t_1 t_2 &\le c\varepsilon^{1-\max\{p,2\}}\varphi_a(t_1) + \varepsilon(\varphi_a)^*(t_2) \end{aligned} \tag{2.15}$$

hold for all $t_1, t_2 \ge 0$ and $\varepsilon \in (0,1]$, where $c \equiv c(p)$. Moreover, we have

$$(\varphi_a)^*(t) \approx ((a+s)^{p-1} + t)^{p'-2} t^2, \qquad t \ge 0. \tag{2.16}$$

Using shifted N-functions, the monotonicity property of $A(\cdot)$ can be rephrased as follows:

$$\begin{aligned} &(A(z_1) - A(z_2)) \cdot (z_1 - z_2) \\ &\approx |V(z_1) - V(z_2)|^2 \approx \varphi_{|z_1|}(|z_1 - z_2|) \approx (\varphi_{|z_1|})^*(|A(z_1) - A(z_2)|). \end{aligned} \tag{2.17}$$

We also note the "shift change formula": for any $z_1, z_2 \in \mathbb{R}^n$, $\varepsilon \in (0,1]$ and $t \ge 0$, it holds that

$$\begin{aligned} \varphi_{|z_1|}(t) &\le c\varepsilon^{1-\max\{p',2\}}\varphi_{|z_2|}(t) + \varepsilon|V(z_1) - V(z_2)|^2, \\ (\varphi_{|z_1|})^*(t) &\le c\varepsilon^{1-\max\{p,2\}}(\varphi_{|z_2|})^*(t) + \varepsilon|V(z_1) - V(z_2)|^2. \end{aligned} \tag{2.18}$$

2.3.2 Regularity for homogeneous equations

In this section, we examine various regularity estimates for the homogeneous equation

$$-\operatorname{div} A(Dv) = 0 \quad \text{in } \Omega. \tag{2.19}$$

Let us start with the reverse Hölder's inequalities. The first estimate was proved in [164, 168]; the second estimate follows from the first one and [100, Corollary 3.4].

Lemma 2.3.3. *Let $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a weak solution to* (2.19) *under assumptions* (2.8) *with $p > 1$. Then for any $\sigma \in (0,1)$ there exists a constant $c \equiv c(\texttt{data}, \sigma)$ such that*

$$\fint_B |V(Dv) - V(z_0)|^2 \, dx \le c \left(\fint_{2B} |V(Dv) - V(z_0)|^{2\sigma} \, dx \right)^{\frac{1}{\sigma}} \tag{2.20}$$

holds for every $z_0 \in \mathbb{R}^n$, whenever $2B \Subset \Omega$. Moreover, there exists a constant $c \equiv c(\texttt{data})$ such that

$$\fint_B \varphi_{|z_0|}(|Dv - z_0|) \, dx \le c \varphi_{|z_0|} \left(\fint_{2B} |Dv - z_0| \, dx \right) \tag{2.21}$$

holds for every $z_0 \in \mathbb{R}^n$, whenever $2B \Subset \Omega$.

We now recall the following result concerning the maximal regularity for (2.19), see [7, 96, 118, 161]. Moreover, we also recall an excess decay estimate below the natural growth exponent, see [111, Section 3.2].

Lemma 2.3.4. *Let $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a weak solution to* (2.19) *under assumptions* (2.8) *with $p > 1$. Then $v \in C^{1,\beta}_{\mathrm{loc}}(\Omega)$ for some $\beta \equiv \beta(\texttt{data}) \in (0,1)$. Moreover, we have the following:*

(1) For every $t > 0$, there exists a constant $c_b \equiv c_b(\texttt{data}, t)$ such that

$$\sup_{\varepsilon B}(|Dv| + s) \le \frac{c_b}{(1-\varepsilon)^{n/t}} \left(\fint_B (|Dv|^2 + s^2)^{\frac{t}{2}} \, dx \right)^{\frac{1}{t}} \tag{2.22}$$

holds for every ball $B \Subset \Omega$ and $\varepsilon \in (0,1)$.

(2) There exists a constant $c_h \equiv c_h(\texttt{data})$ such that

$$|Dv(x_1) - Dv(x_2)| \le c_h \varepsilon^\beta \fint_B |Dv - (Dv)_B| \, dx \tag{2.23}$$

holds for every ball $B \Subset \Omega$ and $x_1, x_2 \in \varepsilon B$ with $\varepsilon \in (0, 1/2]$.

(3) There exists a constant $c \equiv c(\texttt{data})$ such that

$$\fint_{\varepsilon B} |Dv - (Dv)_{\varepsilon B}| \, dx \le c \varepsilon^\beta \fint_B |Dv - (Dv)_B| \, dx \tag{2.24}$$

holds for every ball $B \Subset \Omega$ and $\varepsilon \in (0,1)$.

Since the vector fields $V(\cdot)$ and $A(\cdot)$ are locally bi-Lipschitz, the above lemma implies that both $V(Dv)$ and $A(Dv)$ are also locally Hölder continuous. However, in order to obtain potential estimates for $A(Dv)$, we further need corresponding excess decay estimates for $A(Dv)$. For this we first need excess decay estimates for $V(Dv)$, see [9, 100]. To the best of our knowledge, such results available in the literature are only concerned with equations with additional structural assumptions [1, 110, 121] and the p-Laplace (or φ-Laplace) system [101, 102, 107], respectively. Here we give a new proof for general equations (2.19), assuming only (2.8). The proof will be divided into two parts, treating the cases $p \geq 2$ and $1 < p < 2$ in different manners.

Theorem 2.3.5. *Let $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a weak solution to* (2.19) *under assumptions* (2.8) *with $p > 1$. Then there exist constants $\alpha_V \in (0,1]$ and $c \geq 1$, both depending only on* `data`*, such that*

$$\fint_{B_\rho} |V(Dv) - (V(Dv))_{B_\rho}|^2 \, dx \leq c \left(\frac{\rho}{R}\right)^{2\alpha_V} \fint_{B_R} |V(Dv) - (V(Dv))_{B_R}|^2 \, dx$$

holds whenever $B_\rho \subset B_R \Subset \Omega$ are concentric balls.

Proof of Theorem 2.3.5 in the case $p \geq 2$

Considering only the case $\rho \leq R/4$ as usual, we first prove an L^1-decay estimate. We start with

$$\begin{aligned}
&\fint_{B_{2\rho}} |V(Dv) - V((Dv)_{B_{2\rho}})| \, dx \\
&\overset{(2.9)}{\leq} c \fint_{B_{2\rho}} (|Dv| + |(Dv)_{B_{2\rho}}| + s)^{\frac{p-2}{2}} |Dv - (Dv)_{B_{2\rho}}| \, dx \\
&\overset{p\geq 2}{\leq} c \left[\sup_{B_{2\rho}}(|Dv| + s)\right]^{\frac{p-2}{2}} \fint_{B_{2\rho}} |Dv - (Dv)_{B_{2\rho}}| \, dx \\
&\overset{(2.24)}{\leq} c \left(\frac{\rho}{R}\right)^{\beta} \left[\sup_{B_{R/2}}(|Dv| + s)\right]^{\frac{p-2}{2}} \fint_{B_{R/2}} |Dv - (Dv)_{B_{R/2}}| \, dx.
\end{aligned}$$

Here, we estimate the last integral as

$$
\begin{aligned}
&\left(\frac{\rho}{R}\right)^{\beta}\left[\sup_{B_{R/2}}(|Dv|+s)\right]^{\frac{p-2}{2}}\fint_{B_{R/2}}|Dv-(Dv)_{B_{R/2}}|\,dx\\
&\le c\left(\frac{\rho}{R}\right)^{\beta}\left[\inf_{B_{R/2}}(|Dv|+s)\right]^{\frac{p-2}{2}}\fint_{B_{R/2}}|Dv-(Dv)_{B_{R/2}}|\,dx\\
&\quad+c\left(\frac{\rho}{R}\right)^{\beta}\left[\operatorname*{osc}_{B_{R/2}}|Dv|\right]^{\frac{p-2}{2}}\fint_{B_{R/2}}|Dv-(Dv)_{B_{R/2}}|\,dx\\
&\overset{(2.23)}{\le} c\left(\frac{\rho}{R}\right)^{\beta}\fint_{B_{R/2}}(|Dv|+|(Dv)_{B_R}|+s)^{\frac{p-2}{2}}|Dv-(Dv)_{B_R}|\,dx\\
&\quad+c\left(\frac{\rho}{R}\right)^{\beta}\left(\fint_{B_R}|Dv-(Dv)_{B_R}|\,dx\right)^{\frac{p}{2}}\\
&\overset{(2.9)}{\le} c\left(\frac{\rho}{R}\right)^{\beta}\fint_{B_R}|V(Dv)-V((Dv)_{B_R})|\,dx.
\end{aligned}
$$

Combining the above two estimates yields the L^1-decay estimate

$$
\fint_{B_{2\rho}}|V(Dv)-V((Dv)_{B_{2\rho}})|\,dx\le c\left(\frac{\rho}{R}\right)^{\beta}\fint_{B_R}|V(Dv)-V((Dv)_{B_R})|\,dx,
$$

which in turn gives the desired L^2-decay estimate

$$
\begin{aligned}
\fint_{B_\rho}|V(Dv)-(V(Dv))_{B_\rho}|^2\,dx &\overset{(2.13)}{\le} c\fint_{B_\rho}|V(Dv)-V((Dv)_{B_{2\rho}})|^2\,dx\\
&\overset{(2.20)}{\le} c\left(\fint_{B_{2\rho}}|V(Dv)-V((Dv)_{B_{2\rho}})|\,dx\right)^2\\
&\overset{(2.14)}{\le} c\left(\frac{\rho}{R}\right)^{2\beta}\fint_{B_R}|V(Dv)-(V(Dv))_{B_R}|^2\,dx.
\end{aligned}
$$

Proof of Theorem 2.3.5 in the case $1<p<2$

In the following, $B_R\equiv B_R(x_0)$ is a fixed ball as in the statement, and all the balls considered will have the same center x_0 and r will denote a positive radius with $r \le R$. We accordingly denote

$$E(r) := \fint_{B_r} |V(Dv) - (V(Dv))_{B_r}|^2\,dx, \qquad \xi_r := (Dv)_{B_r}.$$

With $B_r \subset B_R$ and $\theta_0 \in (0,1)$, we consider the following alternatives:

$$\fint_{B_r} |V(Dv) - (V(Dv))_{B_r}|^2\,dx \le \theta_0 \fint_{B_r} |V(Dv)|^2\,dx \tag{2.25}$$

and

$$\fint_{B_r} |V(Dv) - (V(Dv))_{B_r}|^2\,dx > \theta_0 \fint_{B_r} |V(Dv)|^2\,dx. \tag{2.26}$$

The parameter θ_0 will be determined as a universal constant later in the proof.

Step 1: The non-degenerate case. In this case we assume that (2.25) holds. Then

$$\begin{aligned}
&\fint_{B_r} |V(Dv) - (V(Dv))_{B_r}|^2\,dx \le \theta_0 \fint_{B_r} |V(Dv)|^2\,dx \\
&\le 2\theta_0 \fint_{B_r} |V(Dv) - V((Dv)_{B_r})|^2\,dx + 2\theta_0 |V((Dv)_{B_r})|^2 \\
&\overset{(2.14)}{\le} 2c_0\theta_0 \fint_{B_r} |V(Dv) - (V(Dv))_{B_r}|^2\,dx + 2\theta_0 |V((Dv)_{B_r})|^2
\end{aligned}$$

holds for some $c_0 \equiv c_0(\mathtt{data})$. Thus, if

$$\theta_0 \le \frac{1}{4c_0}, \tag{2.27}$$

then we have

$$\begin{aligned}
\fint_{B_r} |V(Dv) - (V(Dv))_{B_r}|^2\,dx &\le \frac{2\theta_0}{1-2c_0\theta_0} |V((Dv)_{B_r})|^2 \\
&\overset{(2.17)}{\le} \frac{c\theta_0}{1-c_0\theta_0} \varphi(|\xi_r|).
\end{aligned} \tag{2.28}$$

Next, for any $\delta \in (0, 1/4]$, we observe that

$$
\begin{aligned}
|\xi_{\delta r} - \xi_r| &\le \fint_{B_{\delta r}} |Dv - \xi_r|\, dx \le \delta^{-n} \fint_{B_r} |Dv - \xi_r|\, dx \\
&\le \delta^{-n} (\varphi_{|\xi_r|})^{-1} \left(\fint_{B_r} \varphi_{|\xi_r|}(|Dv - \xi_r|)\, dx \right) \\
&\overset{(2.17),(2.14)}{\le} c\delta^{-n} (\varphi_{|\xi_r|})^{-1} \left(\fint_{B_r} |V(Dv) - (V(Dv))_{B_r}|^2\, dx \right) \\
&\overset{(2.28)}{\le} c\delta^{-n} \left(\frac{\theta_0}{1 - 2c_0\theta_0} \right)^{\frac{1}{2}} (\varphi_{|\xi_r|})^{-1} (\varphi(|\xi_r|)) \\
&\le c_1 \delta^{-n} \left(\frac{\theta_0}{1 - 2c_0\theta_0} \right)^{\frac{1}{2}} |\xi_r|
\end{aligned}
$$

holds for a constant $c_1 \equiv c_1(\mathtt{data})$, where for the last inequality we have used

$$
\varphi(|\xi_r|) = \frac{|\xi_r|^2}{(|\xi_r| + s)^{2-p}} \le 2^{2-p} \varphi_{|\xi_r|}(|\xi_r|) \le \varphi_{|\xi_r|}\left(2^{\frac{2-p}{p}} |\xi_r| \right).
$$

Now, if θ_0 satisfies

$$
c_1 \delta^{-n} \left(\frac{\theta_0}{1 - 2c_0\theta_0} \right)^{\frac{1}{2}} \le \frac{1}{2}, \tag{2.29}
$$

then it holds that

$$
\frac{1}{2}|\xi_r| \le |\xi_{\delta r}| \le \frac{3}{2}|\xi_r|, \tag{2.30}
$$

which in turn implies

$$
\begin{aligned}
\fint_{B_{\delta r}} |V(Dv) - (V(Dv))_{B_{\delta r}}|^2\, dx &\overset{(2.14),(2.17)}{\le} c \fint_{B_{\delta r}} \varphi_{|\xi_{\delta r}|}(|Dv - \xi_{\delta r}|)\, dx \\
&\overset{(2.21)}{\le} c\varphi_{|\xi_{\delta r}|} \left(\fint_{B_{2\delta r}} |Dv - \xi_{\delta r}|\, dx \right) \\
&\le c\varphi_{|\xi_{\delta r}|} \left(\fint_{B_{2\delta r}} |Dv - \xi_{2\delta r}|\, dx \right) \\
&\overset{(2.24)}{\le} c\varphi_{|\xi_{\delta r}|} \left(\delta^{\beta} \fint_{B_r} |Dv - \xi_r|\, dx \right)
\end{aligned}
$$

$$
\begin{aligned}
&\le c\delta^{\beta p} \fint_{B_r} \varphi_{|\xi_{\delta r}|}(|Dv-\xi_r|)\,dx \\
&\overset{(2.30)}{\le} c\delta^{\beta p} \fint_{B_r} \varphi_{|\xi_r|}(|Dv-\xi_r|)\,dx \\
&\overset{(2.17),(2.14)}{\le} c_2\delta^{\beta p} \fint_{B_r} |V(Dv)-(V(Dv))_{B_r}|^2\,dx.
\end{aligned}
$$

Summarizing, if $\theta_0 \equiv \theta_0(\mathtt{data},\delta) \in (0,1)$ is so small that (2.27) and (2.29) hold, then we have

$$E(\delta r) \le c_2\delta^{\beta p}E(r)$$

for a constant $c_2 \equiv c_2(\mathtt{data})$.

Step 2: The degenerate case. In this case we assume that (2.26) holds. Then for any $N \in \mathbb{N}$ we have

$$
\begin{aligned}
\fint_{B_{\delta^N r}} |V(Dv)-(V(Dv))_{B_{\delta^N r}}|^2\,dx &\overset{(2.14)}{\le} c\fint_{B_{\delta^N r}} |V(Dv)-V(\xi_{\delta^N r})|^2\,dx \\
&\overset{(2.17),(2.21)}{\le} c\varphi_{|\xi_{\delta^N r}|}\left(\fint_{B_{2\delta^N r}} |Dv-\xi_{\delta^N r}|\,dx\right) \\
&\overset{(2.24)}{\le} c\delta^{\beta N p}\varphi_{|\xi_{\delta^N r}|}\left(\fint_{B_r} |Dv-\xi_r|\,dx\right).
\end{aligned}
$$

Since $p<2$, we have $\varphi_a(t) \le \varphi(t)$ for every $a,t \ge 0$. This and (2.13) imply

$$
\begin{aligned}
\fint_{B_{\delta^N r}} |V(Dv)-(V(Dv))_{B_{\delta^N r}}|^2\,dx &\le c\delta^{\beta N p}\varphi\left(\fint_{B_r} |Dv|\,dx\right) \\
&\le c\delta^{\beta N p}\fint_{B_r} \varphi(|Dv|)\,dx \\
&\overset{(2.17)}{\le} c\delta^{\beta N p}\fint_{B_r} |V(Dv)|^2\,dx \\
&\overset{(2.26)}{\le} \frac{c_3}{\theta_0}\delta^{\beta N p}\fint_{B_r} |V(Dv)-(V(Dv))_{B_r}|^2\,dx
\end{aligned}
$$

for a constant $c_3 \equiv c_3(\mathtt{data})$. We recall that the constant $\theta_0 \equiv \theta_0(\mathtt{data},\delta) \in (0,1)$ is assumed to be small enough to satisfy (2.27) and (2.29). Now, if

$N_0 \equiv N_0(\texttt{data}, \delta) \in \mathbb{N}$ is so large that

$$\frac{c_3}{\theta_0}\delta^{\beta N_0 p} < \frac{1}{2}, \tag{2.31}$$

then it follows that

$$E(\delta^N r) \le \frac{1}{2}E(r) \qquad \forall\, N \ge N_0.$$

Step 3: Choice of the constants. We first choose $\delta \equiv \delta(\texttt{data}) \in (0, 1/2]$ so small that

$$c_2\delta^{\beta p} \le \frac{1}{2}.$$

We then choose the constants, the small one $\theta_0 \equiv \theta_0(\texttt{data}) \in (0,1)$ and the large one $N_0 \equiv N_0(\texttt{data}) \in \mathbb{N}$, in order to satisfy (2.27), (2.29) and (2.31), respectively. All in all, as a consequence of *Step 1* and *Step 2*, we have that one of the following inequalities must hold:

$$E(\delta r) \le \frac{1}{2}E(r) \tag{2.32}$$

or

$$E(\delta^N r) \le \frac{1}{2}E(r) \qquad \forall\, N \ge N_0. \tag{2.33}$$

Step 4: Conclusion. We now consider an arbitrary radius $\tilde{r}$ and set

$$N_\tau := \lceil 2N_0 n \log_2(1/\delta)\rceil + N_0,$$

where $\lceil t\rceil$ denotes the least integer greater than or equal to t. We examine the two cases.

(i) If (2.32) holds with $r = \delta^i\tilde{r}$ for every $i \in \{0, \dots, \lceil 2N_0 n \log_2(1/\delta)\rceil\}$, then we have

$$\begin{aligned} E(\delta^{N_\tau}\tilde{r}) &\le 2\delta^{-N_0 n} E(\delta^{\lceil 2N_0 n \log_2(1/\delta)\rceil}\tilde{r}) \\ &\le 2\delta^{-N_0 n}\left(\frac{1}{2}\right)^{2N_0 n \log_2(1/\delta)} E(\tilde{r}) \le \frac{1}{2}E(\tilde{r}). \end{aligned}$$

(ii) If (2.33) holds with $r = \delta^i\tilde{r}$ for at least one $i \in \{0, \dots, \lceil 2N_0 n \log_2(1/\delta)\rceil\}$, then let k be the smallest such number. Then we have

$$E(\delta^{N_\tau}\tilde{r}) \le \frac{1}{2}E(\delta^k\tilde{r}) \le \left(\frac{1}{2}\right)^{k+1} E(\tilde{r}) \le \frac{1}{2}E(\tilde{r}).$$

Therefore, in any case we obtain

$$E(\delta^{N_\tau}\tilde{r}) \le \frac{1}{2}E(\tilde{r}) \qquad \forall\, \tilde{r} \in (0, R].$$

Iterating this estimate in a standard way, we complete the proof.

Linearized excess decay estimates

Once we have Lemma 2.3.3 and Theorem 2.3.5, then we can prove the desired excess decay estimate for $A(Dv)$ by following [100, Section 5].

Theorem 2.3.6. *Let $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a weak solution to (2.19) under assumptions (2.8) with $p > 1$. Then there exist constants $\alpha_A \in (0, 1]$ and $c \ge 1$, both depending only on* `data`*, such that the estimate*

$$\fint_{B_\rho} |A(Dv) - (A(Dv))_{B_\rho}|\, dx \le c\left(\frac{\rho}{R}\right)^{\alpha_A} \fint_{B_R} |A(Dv) - (A(Dv))_{B_R}|\, dx \tag{2.34}$$

holds whenever $B_\rho \subset B_R \Subset \Omega$ are concentric balls.

By Campanato's characterization of Hölder spaces (see for instance [118, Theorem 2.9]), we also have local oscillation estimates for $V(Dv)$ and $A(Dv)$.

Corollary 2.3.7. *Let $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be as in the above theorem. Then there exist constants $c_V, c_A \ge 1$, both depending only on* `data`*, such that*

$$\sup_{x_1,x_2\in\varepsilon B} |V(Dv(x_1)) - V(Dv(x_2))|^2 \le c_V\varepsilon^{2\alpha_V} \fint_B |V(Dv) - (V(Dv))_B|^2\, dx$$

and

$$\sup_{x_1,x_2\in\varepsilon B} |A(Dv(x_1)) - A(Dv(x_2))| \le c_A\varepsilon^{\alpha_A} \fint_B |A(Dv) - (A(Dv))_B|\, dx$$

hold for every $\varepsilon \in (0, 1/2]$ and every ball $B \Subset \Omega$, where $\alpha_V \in (0, 1]$ and $\alpha_A \in (0, 1]$ are as in Theorems 2.3.5 and 2.3.6, respectively.

The next lemmas are concerned with higher differentiability for nonlinear functions of the gradient of solutions. The classical result is concerned with $V(Dv)$; we state it in the following form, by combining the two estimates in [164, Lemma 3.2].

Lemma 2.3.8. *Let $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a weak solution to (2.19) under assumptions (2.8) for $p > 1$. Then $V(Dv) \in W^{1,2}_{\mathrm{loc}}(\Omega)$. Moreover, for any $t > 0$ there exists a constant $c \equiv c(\mathtt{data}, t)$ such that*

$$\left(\fint_{B_{3R/4}} |D(V(Dv))|^2 \, dx \right)^{\frac{1}{2}} \leq \frac{c}{R} \left(\fint_{B_R} |V(Dv) - z_0|^t \, dx \right)^{\frac{1}{t}}$$

holds for every $z_0 \in \mathbb{R}^n$, whenever $B_R \Subset \Omega$.

Using this lemma, we can also obtain differentiability of $A(Dv)$. We remark that, while it is stated that $A(Dv) \in W^{1,1}$ in [7, Lemma 4.1], its proof actually gives $A(Dv) \in W^{1,2}$, as pointed out in [55, Lemma 2.3].

Lemma 2.3.9. *Let $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a weak solution to (2.19) under assumptions (2.8) for $p > 1$, and let $B_R \Subset \Omega$ be a ball. In the case $1 < p < 2$, assume further that*

$$\inf_{B_{3R/4}} (|Dv| + s) > 0.$$

Then $A(Dv) \in W^{1,2}(B_{R/2})$. Moreover, if $p \geq 2$, then

$$\left(\fint_{B_{R/2}} |D(A(Dv))|^2 \, dx \right)^{\frac{1}{2}} \leq \frac{c}{R} \left[\sup_{B_{3R/4}} (|Dv| + s) \right]^{\frac{p-2}{2}} \fint_{B_R} |V(Dv) - z_0| \, dx$$

holds for every $z_0 \in \mathbb{R}^n$, where $c \equiv c(\mathtt{data})$. Finally, if $1 < p < 2$, then

$$\left(\fint_{B_{R/2}} |D(A(Dv))|^2 \, dx \right)^{\frac{1}{2}} \leq \frac{c}{R} \left[\inf_{B_{3R/4}} (|Dv| + s) \right]^{\frac{p-2}{2}} \fint_{B_R} |V(Dv) - z_0| \, dx$$

holds for every $z_0 \in \mathbb{R}^n$, where $c \equiv c(\mathtt{data})$.

Under the additional symmetry condition on $\partial A(\cdot)$, another differentiability result is also available, see [7, Theorem 4.2] for the proof.

Lemma 2.3.10. *Let* $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ *be a weak solution to* (2.19) *under assumptions* (2.8) *for* $p > 1$, *and assume also that* $\partial A(\cdot)$ *is symmetric. Then*

$$A(Dv) \in W^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^n).$$

Moreover, for every ball $B_R \Subset \Omega$, *the inequality*

$$\left(\fint_{B_{R/2}} |D(A(Dv))|^2 \, dx \right)^{\frac{1}{2}} \leq \frac{c}{R} \fint_{B_R} |A(Dv) - z_0| \, dx$$

holds for every choice of $z_0 \in \mathbb{R}^n$, *where* $c \equiv c(\mathtt{data})$.

2.3.3 Technical lemmas

We end this chapter with two technical lemmas, see for instance [118, Chapters 6-7] and [21, Appendix B] for the proofs.

Lemma 2.3.11. *Let* $\mathcal{Z} : [0, \bar{R}] \to [0, \infty)$ *be a nondecreasing function such that*

$$\mathcal{Z}(\rho) \leq c_0 \left[\left(\frac{\rho}{R} \right)^{\delta_0} + \varepsilon \right] \mathcal{Z}(R) + \mathcal{B} R^{\gamma} \qquad \text{for every } \ 0 \leq \rho < R \leq \bar{R},$$

where $c_0, \mathcal{B}, \varepsilon \geq 0$ *and* $\gamma \in (0, \delta_0)$ *are given constants. Then there exist constants* ε_0 *and* c, *both depending only on* c_0, δ_0 *and* γ, *such that if* $\varepsilon \leq \varepsilon_0$, *then it holds that*

$$\mathcal{Z}(\rho) \leq c \left(\frac{\rho}{R} \right)^{\gamma} \mathcal{Z}(R) + c\mathcal{B} \rho^{\gamma} \qquad \text{for every } \ 0 \leq \rho \leq R \leq \bar{R}.$$

Lemma 2.3.12. *Let* $\mathcal{Z} : [\rho_0, \rho_1] \to [0, \infty)$ *be a bounded function such that*

$$\mathcal{Z}(t) \leq \varepsilon_0 \mathcal{Z}(s) + \frac{\mathcal{B}_1}{(s-t)^{\gamma_1}} + \frac{\mathcal{B}_2}{(s-t)^{\gamma_2}} \qquad \text{for every } \ \rho_0 \leq t < s \leq \rho_1,$$

where $\varepsilon_0 \in (0,1)$ *and* $\mathcal{B}_1, \mathcal{B}_2, \gamma_1, \gamma_2 \geq 0$ *are given constants. Then there exists a constant* $c \equiv c(\varepsilon_0, \gamma_1, \gamma_2)$ *such that*

$$\mathcal{Z}(\rho_0) \leq \frac{c\mathcal{B}_1}{(\rho_1 - \rho_0)^{\gamma_1}} + \frac{c\mathcal{B}_2}{(\rho_1 - \rho_0)^{\gamma_2}}.$$

Chapter 3

Elliptic and parabolic equations with measure data

3.1 Maximal integrability for elliptic measure data problems with Orlicz growth

3.1.1 Main results

We consider the following Dirichlet problem

$$\begin{cases} -\operatorname{div} A(x, Du) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\mu \in L^{1,\theta}(\Omega)$. The Carathéodory vector field $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following growth and monotonicity assumptions:

$$\begin{cases} |A(x, z)| \leq Lg(|z|), \\ \nu \dfrac{g(|z_1| + |z_2|)}{|z_1| + |z_2|} |z_1 - z_2|^2 \leq (A(x, z_1) - A(x, z_2)) \cdot (z_1 - z_2), \end{cases} \tag{3.2}$$

for every $z, z_1, z_2 \in \mathbb{R}^n$ with $|z_1| + |z_2| \neq 0$ and $x \in \Omega$, where $0 < \nu \leq L < \infty$ are fixed constants. Note in particular that the map $x \mapsto A(x, z)$ is only measurable. The function $g : [0, \infty) \to [0, \infty)$ is the derivative of an N-

function $G \in C^2(0,\infty)$ satisfying

$$1 \le g_0 \le \frac{tg'(t)}{g(t)} \le g_1 \le n-1, \qquad t>0, \tag{3.3}$$

for some positive constants g_0, g_1.

Definition 3.1.1. *A function $u \in W_0^{1,1}(\Omega)$ is a SOLA to (3.1) under assumptions (3.2) if $A(\cdot, Du) \in L^1(\Omega;\mathbb{R}^n)$,*

$$\int_\Omega A(x,Du)\cdot D\varphi\,dx = \int_\Omega \varphi\,d\mu \qquad \forall\,\varphi \in C_0^\infty(\Omega),$$

and moreover there exists a sequence of weak solutions $\{u_k\} \subset W_0^{1,G}(\Omega)$ to the problems

$$\begin{cases} -\operatorname{div} A(x,Du_k) = \mu_k & \text{in } \Omega,\\ u_k = 0 & \text{on } \partial\Omega \end{cases} \tag{3.4}$$

such that $u_k \to u$ in $W_0^{1,g}(\Omega)$, where the sequence $\{\mu_k\} \subset L^\infty(\Omega)$ converges to μ weakly in the sense of measures and satisfies*

$$\limsup_{k\to\infty} |\mu_k|(B) \le |\mu|(\bar{B})$$

for every ball $B \subset \mathbb{R}^n$.

Throughout this section, we use the abbreviation $\texttt{data} := (n, g_0, g_1, \nu, L)$. We now state our main result:

Theorem 3.1.2. *Let $u \in W_0^{1,g}(\Omega)$ be a SOLA to (3.1) under assumptions (3.2) and (3.3). Assume that (1.5) holds with $g_1 + 1 \le \theta \le n$. Then*

$$g(|Du|) \in \mathcal{M}_{\mathrm{loc}}^{\frac{\theta}{\theta-1},\theta}(\Omega).$$

Moreover, for any ball $B_R \Subset \Omega$ of radius $R \le 1$ we have

$$\begin{aligned} \|g(|Du|)\|_{\mathcal{M}^{\theta/(\theta-1)}(B_R)}^{\theta/(\theta-1)} &= \sup_{\lambda\ge 0} \lambda^{\frac{\theta}{\theta-1}} |\{x \in B_R : g(|Du(x)|) > \lambda\}| \\ &\le c[|\mu|(\Omega) + \|\mu\|_{L^{1,\theta}(\Omega)}]^{\frac{\theta}{\theta-1}} R^{n-\theta}, \end{aligned} \tag{3.5}$$

where $c \equiv c(\texttt{data}, \operatorname{dist}(B_R, \partial\Omega))$.

Remark 3.1.3. *When $g(t) = t^{p-1}$, Theorem 3.1.2 reduces to [164, Theorem 1.8]. Moreover, in the case of general measures ($\theta = n$), Theorem 3.1.2 gives a result similar to [77, Theorem 3.2]. One can also consider the case of generalized Morrey space as in [15], by assuming that*

$$\|\mu\|_{L^{1,\phi}(\Omega)} := \sup_{B_R \subset \Omega} \phi(R) \frac{|\mu|(B_R)}{R^n} < \infty.$$

Then we expect that the approach in this section can be extended to regularity results in generalized Marcinkiewicz spaces.

Remark 3.1.4. *The Morrey condition* (1.5) *also implies fractional Sobolev-Morrey regularity results, that for the sake of simplicity we state for the model case*

$$-\mathrm{div} \left(\frac{g(|Du|)}{|Du|} Du \right) = \mu.$$

Namely, combining [38, Theorem 1.1] and Lemma 3.1.10 below, one can prove

$$\mu \in L^{1,\theta} \implies \frac{g(|Du|)}{|Du|} Du \in W^{\sigma,1,\theta}_{\mathrm{loc}} \quad \forall\, \sigma \in (0,1).$$

3.1.2 Some technical results

We first recall the reverse Hölder's inequality and higher integrability results for the homogeneous equation

$$-\operatorname{div} A(x, Dv) = 0 \quad \text{in } \Omega. \tag{3.6}$$

Lemma 3.1.5 ([13, 180])**.** *Let $v \in W^{1,G}_{\mathrm{loc}}(\Omega)$ be a weak solution to* (3.6) *under assumptions* (3.2) *and* (3.3)*. Then for every ball $B_R \Subset \Omega$, there holds*

$$\fint_{B_{R/2}} G(|Dv|)\, dx \le c\, G \left(\fint_{B_R} |Dv|\, dx \right)$$

for some $c \equiv c(\mathtt{data})$. Moreover, there exists $\chi \equiv \chi(\mathtt{data}) > 1$ such that

$$\fint_{B_{R/2}} [G(|Dv|)]^{\chi}\, dx \le c \left[G \left(\fint_{B_R} |Dv|\, dx \right) \right]^{\chi}$$

holds whenever $B_R \Subset \Omega$, where $c \equiv c(\mathtt{data})$.

Using this lemma, we obtain a decay estimate for (3.6) below the natural growth.

Lemma 3.1.6. *Let $v \in W^{1,G}_{\mathrm{loc}}(\Omega)$ be a weak solution to* (3.6) *under assumptions* (3.2) *and* (3.3). *Then there exists an exponent $\alpha \equiv \alpha(\texttt{data}) \in (0,1]$ such that for every*

$$\xi \in \left[1, \frac{n}{n-1}\right), \tag{3.7}$$

there holds

$$\fint_{B_\rho} [g(|Dv|)]^\xi \, dx \le c \left(\frac{\rho}{R}\right)^{\xi g_1(\alpha-1)} \fint_{B_R} [g(|Dv|)]^\xi \, dx$$

for a constant $c \equiv c(\texttt{data}, \xi)$, whenever $B_\rho \subset B_R \Subset \Omega$ are concentric balls.

Proof. As usual we only consider the case $\rho \le R/2$, otherwise the lemma follows trivially. A standard decay estimate for (3.6) is

$$\int_{B_\rho} G(|Dv|) \, dx \le c \left(\frac{\rho}{R}\right)^{n+(g_0+1)(\alpha-1)} \int_{B_{R/2}} G(|Dv|) \, dx,$$

see [180, Lemma 3.4] for the proof. Now, for ξ as in (3.7) we consider the auxiliary function

$$f_\xi(t) := \xi \int_0^t \frac{[g(s)]^\xi}{s} \, ds, \qquad t \ge 0, \tag{3.8}$$

introduced in the proof of [13, Lemma 5.3]. A direct computation shows that

$$f_\xi(t) \approx [g(t)]^\xi \tag{3.9}$$

and that $G \circ f_\xi^{-1}$ is increasing and convex. We then apply Jensen's inequality and Lemma 3.1.5 to have

$$\begin{aligned}\fint_{B_\rho} f_\xi(|Dv|) \, dx &\le (f_\xi \circ G^{-1}) \left(\fint_{B_\rho} G(|Dv|) \, dx \right) \\ &\le (f_\xi \circ G^{-1}) \left[c \left(\frac{\rho}{R}\right)^{(g_0+1)(\alpha-1)} \fint_{B_{R/2}} G(|Dv|) \, dx \right]\end{aligned}$$

$$
\begin{aligned}
&\le (f_\xi \circ G^{-1}) \left[c \left(\frac{\rho}{R}\right)^{(g_0+1)(\alpha-1)} G\left(\fint_{B_R} |Dv|\,dx \right)\right] \\
&\le c \left(\frac{\rho}{R}\right)^{\xi g_1(\alpha-1)} \fint_{B_R} f_\xi(|Dv|)\,dx,
\end{aligned}
$$

which along with (3.9) completes the proof. □

We now proceed with the additional assumption

$$
\mu \in W^{-1,G'}(\Omega) \cap L^1(\Omega), \qquad u \in W_0^{1,G}(\Omega), \tag{3.10}
$$

which will be eventually removed in Section 3.1.3 below. We fix a ball $B_R \equiv B_R(x_0) \subset \Omega$, and define $v \in u + W_0^{1,G}(B_R)$ as the weak solution to

$$
\begin{cases} -\operatorname{div} A(x, Dv) = 0 & \text{in } B_R, \\ \qquad\qquad\quad v = u & \text{on } \partial B_R. \end{cases} \tag{3.11}
$$

The following comparison estimate is from [13, Lemma 5.3]. In fact, its proof is valid for equations with x-dependence as well as in [74, Proposition 4.2].

Lemma 3.1.7. *Let $u \in W_0^{1,G}(\Omega)$ and $v \in u+W_0^{1,G}(B_R)$ be the weak solutions to* (3.1) *and* (3.11), *respectively, under assumptions* (3.2) *and* (3.3). *Then for every ξ satisfying* (3.7), *we have the estimate*

$$
\fint_{B_R} [g(|Du - Dv|)]^\xi\,dx \le c \left[\frac{|\mu|(B_R)}{R^{n-1}} \right]^\xi
$$

for a constant $c \equiv c(\texttt{data}, \xi)$.

Next, we find a global a priori estimate for (3.1). The idea is based on [13, Lemma 5.3], and we only give a sketch of proof. See also [164, Remark 4.8].

Lemma 3.1.8. *Let $u \in W_0^{1,G}(\Omega)$ be the weak solution to* (3.1) *under assumptions* (3.2) *and* (3.3). *Then for every ξ satisfying* (3.7),

$$
\int_\Omega [g(|Du|)]^\xi\,dx \le c[|\mu|(\Omega)]^\xi \tag{3.12}
$$

holds for a constant $c \equiv c(\texttt{data}, |\Omega|, \xi)$.

Proof. By a proper normalization, we may assume that $|\mu|(\Omega) = 1$. We again recall the auxiliary function $f_\xi(\cdot)$ given in (3.8), with ξ satisfying (3.7); note that $f_\xi(1) \le 1$. Since $g_1 \le n-1$, we are always in the "slow growth case"

$$\int^\infty \left(\frac{s}{G(s)} \right)^{\frac{1}{n-1}} ds = \infty.$$

In order to apply the Sobolev embedding theorem, we modify $f_\xi(\cdot)$ as

$$\mathfrak{f}_\xi(t) := \begin{cases} 0 & \text{if } t = 0, \\ f_\xi(1)t & \text{if } t \in (0,1), \\ f_\xi(t) & \text{if } t \in [1,\infty). \end{cases}$$

For $k \in \mathbb{N}$, we recall the truncation operators $T_k(\cdot)$ and $\mathfrak{T}_k(\cdot)$, and define

$$C_k := \left\{ x \in \Omega : \frac{|u(x)|}{c_S(n)\mathcal{F}} \le k \right\}, \quad D_k := \left\{ x \in \Omega : k < \frac{|u(x)|}{c_S(n)\mathcal{F}} \le k+1 \right\}.$$

By normalization once again, we also assume that

$$\mathcal{F} := \left(\int_\Omega \mathfrak{f}_\xi(|Du|)\, dx \right)^{\frac{1}{n}} \ge 1.$$

Then we test (3.1) with

$$\varphi \equiv T_k \left(\frac{u}{c_S(n)\mathcal{F}} \right) \in W_0^{1,G}(\Omega) \cap L^\infty(\Omega),$$

where $c_S(n)$ is the constant appearing in Lemma 2.2.1, to obtain

$$\int_{C_k} G(|Du|)\, dx \le c\mathcal{F} \int_\Omega A(x,Du) \cdot D\varphi\, dx \le c\mathcal{F} \left| \int_\Omega \varphi\, d\mu \right| \le ck\mathcal{F} \qquad (3.13)$$

for every $k \in \mathbb{N}$, where $c \equiv c(\mathtt{data}, |\Omega|)$. In a similar way, we also have

$$\int_{D_k} G(|Du|)\, dx \le c\mathcal{F}.$$

Now, by a straightforward computation as in [13, Lemma 5.3], the function

$f_\xi \circ G^{-1}$ is increasing and concave. Thus, Jensen's inequality and (3.13) imply

$$\begin{aligned}\int_{C_k} f_\xi(|Du|)\,dx &\le |C_k|(f_\xi \circ G^{-1})\left(\fint_{C_k} G(|Du|)\,dx\right)\\ &\le c|C_k|(f_\xi \circ G^{-1})\left(\frac{k\mathcal{F}}{|C_k|}\right) = ck\mathcal{F}H_\xi^{-1}\left(\frac{|C_k|}{k\mathcal{F}}\right),\end{aligned}$$

where

$$H_\xi^{-1}(t) := t^{1-\xi}\left[G^{-1}\left(\frac{1}{t}\right)\right]^{-\xi} \approx_\xi t f_\xi\left(G^{-1}\left(\frac{1}{t}\right)\right).$$

Also, since $\xi < n/(n-1) \le (g_1+1)/g_1$, a direct computation shows that $H_\xi^{-1}(\cdot)$ is increasing.

For the integrals over D_k, we have

$$\int_{D_k} f_\xi(|Du|)\,dx \le c\mathcal{F}H_\xi^{-1}\left(\frac{|D_k|}{\mathcal{F}}\right).$$

Thus, summing up all the integrals, we have

$$\int_\Omega f_\xi(|Du|)\,dx \le c\mathcal{F}\left[H_\xi^{-1}\left(\frac{|\Omega|}{\mathcal{F}}\right) + \sum_{k=1}^\infty H_\xi^{-1}\left(\frac{|D_k|}{\mathcal{F}}\right)\right].$$

We denote the Sobolev conjugate function of $\mathfrak{f}_\xi$ by $(\mathfrak{f}_\xi)_n := \mathfrak{f}_\xi \circ H_n^{-1}$, where $H_n(\cdot)$ is given by (2.2) with the choice $\Phi \equiv \mathfrak{f}_\xi$. From the definition of D_k, it follows that

$$|D_k| \le \frac{1}{(\mathfrak{f}_\xi)_n(k)}\int_{D_k}(\mathfrak{f}_\xi)_n\left(\frac{|u|}{c_S(n)\mathcal{F}}\right)dx.$$

Note that

$$\int_0\left(\frac{s}{\mathfrak{f}_\xi(s)}\right)^{\frac{1}{n-1}}ds < \infty \quad \text{and} \quad \int^\infty\left(\frac{s}{\mathfrak{f}_\xi(s)}\right)^{\frac{1}{n-1}}ds = \infty.$$

Now, using Young's inequality with $\varepsilon \in (0,1)$ to be chosen, we have

$$\sum_{k=1}^\infty H_\xi^{-1}\left(\frac{|D_k|}{\mathcal{F}}\right) \le \sum_{k=1}^\infty H_\xi^{-1}\left(\frac{1}{\mathcal{F}(\mathfrak{f}_\xi)_n(k)}\int_{D_k}(\mathfrak{f}_\xi)_n\left(\frac{|u|}{c_S(n)\mathcal{F}}\right)dx\right)$$

$$\begin{aligned}&\leq \frac{\varepsilon}{\mathcal{F}}\int_\Omega (\mathfrak{f}_\xi)_n\left(\frac{|u|}{c_S(n)\mathcal{F}}\right)dx + c_\varepsilon \sum_{k=1}^\infty \left(H_\xi^{-1}\circ H_\xi^*\right)\left(\frac{1}{(\mathfrak{f}_\xi)_n(k)}\right)\\ &\leq \frac{\varepsilon}{\mathcal{F}}\int_\Omega \mathfrak{f}_\xi(|Du|)\,dx + c_\varepsilon \sum_{k=1}^\infty \left(H_\xi^{-1}\circ H_\xi^*\right)\left(\frac{1}{(\mathfrak{f}_\xi)_n(k)}\right),\end{aligned}$$

where we have also used Lemma 2.2.1. The last series in the right-hand side can be estimated by using a similar argument as the one after [13, (5.40)]. Consequently, we have

$$\begin{aligned}\int_\Omega f_\xi(|Du|)\,dx \leq \tilde{c}\mathcal{F}H_\xi^{-1}\left(\frac{|\Omega|}{\mathcal{F}}\right) + \varepsilon\tilde{c}\int_\Omega f_\xi(|Du|)\,dx + c\varepsilon\\ + c(n, g_0, g_1, |\Omega|, \varepsilon, \xi)\mathcal{F}.\end{aligned}$$

We then choose $\varepsilon = 1/4\tilde{c}$ to reabsorb the second term to the left-hand side. For the first term, we recall the definition of $H_\xi^{-1}(\cdot)$ and the fact that $\mathcal{F} \geq 1$ to discover

$$\mathcal{F}H_\xi^{-1}\left(\frac{|\Omega|}{\mathcal{F}}\right) \leq c\mathcal{F}^\xi[G^{-1}(\mathcal{F})]^{-\xi} \leq c\mathcal{F}^{\xi\left(1-\frac{1}{1+g_1}\right)} = c\mathcal{F}^{\xi\frac{g_1}{1+g_1}}.$$

Since $\xi < n/(n-1) \leq (g_1+1)/g_1$, we have $\xi g_1/(g_1+1) < 1$. Hence we use once again Young's inequality to complete the proof. □

Remark 3.1.9. *Consider a standard, symmetric and nonnegative mollifier* $\phi \in C_0^\infty(B_1)$ *satisfying* $\|\phi\|_{L^1(\mathbb{R}^n)} = 1$, *and then define* $\mu_k := \mu * \phi_k$ *for* $k \in \mathbb{N}$, *where* $\phi_k(x) := k^n\phi(kx)$. *Then* $\mu_k \in L^\infty(\Omega)$, *and we can find a unique weak solution* $u_k \in W_0^{1,G}(\Omega)$ *to* (3.4). *Once we have established* (3.12) *for* u_k, *the compactness and truncation arguments in [28, 29] imply that there exists a function* $u \in W_0^{1,1}(\Omega)$ *such that*

$$u_k \to u \quad \textit{strongly in} \quad W_0^{1,g^\xi}(\Omega) \quad \textit{for every} \quad \xi \in \left[1, \frac{n}{n-1}\right). \tag{3.14}$$

In particular, as a consequence of (3.14), u *solves* (3.1) *in the sense of Definition 3.1.1 and satisfies the global estimate* (3.12) *as well. This is a complete generalization of [168, Theorem 3.1], which was previously mentioned in [13, Section 7]. We also note that*

$$|\mu_k|(\Omega) \leq |\mu|(\Omega), \qquad \|\mu_k\|_{L^{1,\theta}(\Omega)} \leq \|\mu\|_{L^{1,\theta}(\Omega)}. \tag{3.15}$$

3.1.3 Proof of Theorem 3.1.2

We first establish a Morrey type decay estimate for (3.1).

Lemma 3.1.10. *Let $u \in W_0^{1,G}(\Omega)$ be the weak solution to* (3.1) *under assumptions* (3.2) *and* (3.3)*, and assume that* (1.5) *holds with $g_1 + 1 \leq \theta \leq n$. Then for every ξ satisfying* (3.7)*, there exists a constant $c \equiv c(\mathtt{data}, \xi)$ such that the following inequality holds whenever $B_\rho \subset B_R \subset \Omega$ are concentric balls:*

$$\rho^{\xi(\theta-1)} \fint_{B_\rho} [g(|Du|)]^\xi \, dx \leq c \left\{ R^{\xi(\theta-1)} \fint_{B_R} [g(|Du|)]^\xi \, dx + \|\mu\|^\xi_{L^{1,\theta}(\Omega)} \right\}. \tag{3.16}$$

Moreover, for every ball $B_\rho \Subset \Omega$, we have

$$\rho^{\xi(\theta-1)} \fint_{B_\rho} [g(|Du|)]^\xi \, dx \leq c[|\mu|(\Omega) + \|\mu\|_{L^{1,\theta}(\Omega)}]^\xi \tag{3.17}$$

for a constant $c \equiv c(\mathtt{data}, \xi, \operatorname{dist}(B_\rho, \partial\Omega))$.

Proof. Consider the weak solution $v \in u + W_0^{1,G}(B_R)$ to (3.11). Applying Lemmas 3.1.6 and 3.1.7, we have

$$\begin{aligned}
&\int_{B_\rho} [g(|Du|)]^\xi \, dx \leq c \int_{B_\rho} [g(|Dv|)]^\xi \, dx + c \int_{B_\rho} [g(|Du - Dv|)]^\xi \, dx \\
&\leq c \left(\frac{\rho}{R}\right)^{n+\xi g_1(\alpha-1)} \int_{B_R} [g(|Dv|)]^\xi \, dx + c \int_{B_R} [g(|Du - Dv|)]^\xi \, dx \\
&\leq c \left(\frac{\rho}{R}\right)^{n+\xi g_1(\alpha-1)} \int_{B_R} [g(|Du|)]^\xi \, dx + c\|\mu\|^\xi_{L^{1,\theta}(\Omega)} R^{n-\xi(\theta-1)}.
\end{aligned}$$

We then apply Lemma 2.3.11 with

$$\mathcal{Z}(t) := \int_{B_t} [g(|Du|)]^\xi \, dx$$

and $\gamma = n - \xi(\theta - 1) < n + \xi g_1(\alpha - 1)$, to obtain (3.16). Then we recall Lemma 3.1.8 to obtain (3.17). This completes the proof. □

With this above lemma in hand, we start the proof of Theorem 3.1.2, based on the maximal function free technique developed in [168].

Proof of Theorem 3.1.2. Step 1: Reduction to a priori estimates. We recall that it is sufficient to prove (3.5) under assumption (3.10). Once we have estimate (3.5) for weak solutions u_k to approximating problems (3.4), the limiting procedure in Remark 3.1.9, together with (3.15) and the lower semi-continuity of $\|\cdot\|_{\mathcal{M}^{\theta/(\theta-1)}}$, give the desired estimate for a SOLA u.

Step 2: Rescaling. We first fix ξ and ξ_1 satisfying

$$1 \le \xi < \xi_1 < \frac{n}{n-1}$$

and depending only on n. For any open ball $B_R \Subset \Omega$ considered, we define

$$\bar{u}(y) := \frac{u(x_0 + Ry)}{HR}, \qquad \bar{A}(y,z) := \frac{A(x_0 + Ry, Hz)}{g(H)},$$
$$\bar{\mu}(y) := R\frac{\mu(x_0 + Ry)}{g(H)}, \qquad \bar{g}(t) := \frac{g(Ht)}{g(H)}$$

with the choice

$$H := g^{-1}\left[\left(\fint_{B_R} [g(|Du|)]^{\xi_1}\,dx\right)^{\frac{1}{\xi_1}}\right] + g^{-1}\left(R\fint_{B_R} |\mu|\,dx\right) + g^{-1}\left(R^{1-\theta}\|\mu\|_{L^{1,\theta}(\Omega)}\right). \tag{3.18}$$

We may assume $H > 0$, otherwise there is nothing to prove. Then $\bar{A}(\cdot)$ satisfies (3.2) with $g(\cdot)$ replaced by $\bar{g}(\cdot)$, and $\bar{u}$ is a weak solution to

$$-\mathrm{div}\,\bar{A}(y, D\bar{u}) = \bar{\mu} \quad \text{in } B_1.$$

Moreover, we have

$$\left(\fint_{B_1} [g(|D\bar{u}|)]^{\xi_1}\,dy\right)^{\frac{1}{\xi_1}} + g^{-1}\left(\fint_{B_1} |\bar{\mu}|\,dy\right) \le 1 \tag{3.19}$$

and

$$|\bar{\mu}|(B_\rho) \le \rho^{n-\theta}, \qquad \forall\, B_\rho \subset B_1. \tag{3.20}$$

From now on, we will drop the bar notation for the simplicity, recovering it only at the end of *Step 6*. Then (3.16) implies

$$\int_{B_\rho} [g(|Du|)]^{\xi_1}\,dx \le c\rho^{n-\xi_1(\theta-1)} \quad \text{and} \quad \int_{B_\rho} [g(|Du|)]^{\xi}\,dx \le c\rho^{n-\xi(\theta-1)} \tag{3.21}$$

whenever $B_\rho \subset B_{2/3}$, where $c \equiv c(\texttt{data})$.

Step 3: Calderón-Zygmund type decomposition. With a free parameter $M \ge 1$, whose value will be determined later in the proof, we define a set function $CZ(\cdot)$ as

$$CZ(S) := g^{-1}\left[\left(\fint_S [g(|Du|)]^{\xi}\,dx\right)^{\frac{1}{\xi}}\right] + g^{-1}\left[\left(M\fint_S |\mu|\,dx\right)^{\frac{\theta-1}{\theta}}\right]$$

for each measurable subset $S \subset \Omega$ with $0 < |S| < \infty$. We next fix $1/2 \le t < \rho \le 2/3$ and denote

$$E_\lambda^t := \{x \in B_t : |Du(x)| > \lambda\}, \qquad E_\lambda^\rho := \{x \in B_\rho : |Du(x)| > \lambda\}$$

for $\lambda \ge 0$ and concentric balls $B_t \subset B_\rho \subset B_{2/3}$. We now set

$$\lambda_0 := CZ(B_{2/3}), \tag{3.22}$$

and from now on, we consider λ large enough to have

$$\lambda \ge 8^n(\rho - t)^{-n/\xi g_0}\lambda_0 =: \lambda_1 \tag{3.23}$$

and fix any $x_0 \in E_{8\lambda}^t$. Observe that $B(x_0, \rho - t) \subset B_\rho \subset B_{2/3}$, which implies

$$CZ(B(x_0, \rho - t)) \le 8^n(\rho - t)^{-n/\xi g_0} CZ(B_{2/3}) = \lambda_1 \le \lambda. \tag{3.24}$$

We then define the exit time index by

$$i(x_0) := \min\{i \in \mathbb{N} : CZ(B(x_0, 2^{-i}(\rho - t))) \ge 8\lambda\}.$$

By Lebesgue's differentiation theorem, we have $1 \le i(x_0) < \infty$ for a.e. $x_0 \in E_{8\lambda}^t$, and the family $\{B(x_0, 2^{-i(x_0)}(\rho - t)) : x_0 \in E_{8\lambda}^t\}$ covers $E_{8\lambda}^t$ up to a negligible set. We then apply the Besicovitch covering theorem to extract a finite number $Q(n)$ of possibly countable subfamilies of mutually disjoint balls $\{\mathcal{B}_j\}_{1\le j\le Q(n)}$, $\mathcal{B}_j \equiv \{B_i^j\}_i$, whose union covers $E_{8\lambda}^t$ up to a negligible set.

Renaming all these balls, we have a possibly countable family $\{B_k\}_k$. Here, since $i(x_0) \geq 1$ for a.e. $x_0 \in E^t_{8\lambda}$, the radius of B_k does not exceed $(\rho - t)/2$. Then $2B_k \subset B_\rho$, since the center of B_k is in B_t. Summarizing, we have

$$E^t_{8\lambda} \subset \bigcup_k B_k \cup \text{negligible set}, \quad \sum_k |E^\rho_\lambda \cap B_k| \leq Q(n)|E^\rho_\lambda|, \quad 2B_k \subset B_\rho, \tag{3.25}$$

$$8\lambda \leq CZ(B_k) \qquad \text{and} \qquad CZ(2B_k) < 8\lambda \tag{3.26}$$

for every $k \in \mathbb{N}$.

We next denote by R_k the radius of B_k. Then (3.19)-(3.21) imply

$$CZ(B_k) \leq cg^{-1}\left(R_k^{-(\theta-1)}\right),$$

and moreover, (3.26) implies

$$R_k \leq c[g(\lambda)]^{-\frac{1}{\theta-1}}. \tag{3.27}$$

Step 4: A density estimate. Here we single out a generic ball B_k and observe that, by (3.26), one of the following inequalities must hold:

$$g(4\lambda) \leq \left(\fint_{B_k} [g(|Du|)]^\xi \, dx\right)^{\frac{1}{\xi}} \quad \text{or} \quad g(4\lambda) \leq \left(M \fint_{B_k} |\mu| \, dx\right)^{\frac{\theta-1}{\theta}}. \tag{3.28}$$

Let us first consider the case $(3.28)_1$. Using Hölder's inequality, we have

$$\begin{aligned}
[g(4\lambda)]^\xi |B_k| &\leq \int_{B_k} [g(|Du|)]^\xi \, dx \\
&\leq [g(\lambda)]^\xi |B_k \setminus E^\rho_\lambda| + \int_{E^\rho_\lambda \cap B_k} [g(|Du|)]^\xi \, dx \\
&\leq [g(\lambda)]^\xi |B_k \setminus E^\rho_\lambda| + |E^\rho_\lambda \cap B_k|^{1-\frac{\xi}{\xi_1}} \left(\int_{E^\rho_\lambda \cap B_k} [g(|Du|)]^{\xi_1} \, dx\right)^{\frac{\xi}{\xi_1}}.
\end{aligned}$$

By a straightforward manipulation, it follows that

$$2 \leq \left[\frac{g(4\lambda)}{g(\lambda)}\right]^\xi$$

$$\leq \frac{|B_k \setminus E_\lambda^\rho|}{|B_k|} + \left[\frac{|E_\lambda^\rho \cap B_k|}{|B_k|}\right]^{1-\frac{\xi}{\xi_1}} \frac{1}{[g(\lambda)]^\xi} \left(\fint_{B_k} [g(|Du|)]^{\xi_1}\,dx\right)^{\frac{\xi}{\xi_1}}. \tag{3.29}$$

To estimate the last integral, we define $v_k \in u + W_0^{1,G}(2B_k)$ to be the weak solution to

$$\begin{cases} -\operatorname{div} A(x, Dv_k) = 0 & \text{in } 2B_k, \\ v_k = u & \text{on } \partial(2B_k). \end{cases}$$

Triangle inequality gives

$$\fint_{B_k} [g(|Du|)]^{\xi_1}\,dx \leq c \fint_{B_k} [g(|Du - Dv_k|)]^{\xi_1}\,dx + c \fint_{B_k} [g(|Dv_k|)]^{\xi_1}\,dx. \tag{3.30}$$

Using Lemma 3.1.7 and (1.5), we find

$$\begin{aligned} \int_{2B_k} [g(|Du - Dv_k|)]^{\xi_1}\,dx &\leq c[|\mu|(2B_k)]^{\xi_1} R^{n-\xi_1(n-1)} \\ &\leq c[|\mu|(2B_k)] R_k^{n-\xi_1(n-1)+(n-\theta)(\xi_1-1)}, \end{aligned}$$

which leads to

$$\begin{aligned} \fint_{2B_k} [g(|Du - Dv_k|)]^{\xi_1}\,dx &\leq c R_k^{\theta-\xi_1(\theta-1)} \fint_{2B_k} |\mu|\,dx \\ &\overset{(3.27)}{\leq} c[g(\lambda)]^{-\frac{\theta}{\theta-1}+\xi_1} \fint_{2B_k} |\mu|\,dx \\ &\overset{(3.26)}{\leq} c[g(\lambda)]^{\xi_1}, \end{aligned} \tag{3.31}$$

where we have used the facts that $M \geq 1$ and $\theta - \xi_1(\theta - 1) > 0$.

Likewise, we find

$$\begin{aligned} \fint_{2B_k} [g(|Du - Dv_k|)]^{\xi}\,dx &\leq c R_k^{\theta-\xi(\theta-1)} \fint_{2B_k} |\mu|\,dx \\ &\leq c[g(\lambda)]^{-\frac{\theta}{\theta-1}+\xi} \fint_{2B_k} |\mu|\,dx \\ &\leq c[g(\lambda)]^{\xi}. \end{aligned} \tag{3.32}$$

At this point, we apply Lemma 3.1.5 to v_k in order to discover

$$G^{-1}\left[\left(\fint_{B_k}[G(|Dv_k|)]^{\chi}\,dx\right)^{\frac{1}{\chi}}\right] \le c\fint_{2B_k}|Dv_k|\,dx.$$

Now, using the auxiliary function as in Lemma 3.1.6 and applying Jensen's inequality, it follows that

$$\begin{aligned} &G^{-1}\left[\left(\fint_{B_k}[G(|Dv_k|)]^{\chi}\,dx\right)^{\frac{1}{\chi}}\right] + g^{-1}\left[\left(\fint_{B_k}[g(|Dv_k|)]^{\xi_1}\,dx\right)^{\frac{1}{\xi_1}}\right] \\ &\le cg^{-1}\left[\left(\fint_{2B_k}[g(|Dv_k|)]^{\xi}\,dx\right)^{\frac{1}{\xi}}\right]. \end{aligned} \tag{3.33}$$

We then observe that (3.26) and (3.32) imply

$$\begin{aligned} \fint_{2B_k}[g(|Dv_k|)]^{\xi}\,dx &\le c\fint_{2B_k}[g(|Du-Dv_k|)]^{\xi}\,dx + c\fint_{2B_k}[g(|Du|)]^{\xi}\,dx \\ &\le c[g(\lambda)]^{\xi}. \end{aligned} \tag{3.34}$$

Using (3.34) and (3.33), we have

$$\fint_{2B_k}[g(|Dv_k|)]^{\xi_1}\,dx \le c[g(\lambda)]^{\xi_1}. \tag{3.35}$$

But then, we connect (3.31) and (3.35) to (3.30) to obtain

$$\fint_{B_k}[g(|Du|)]^{\xi_1}\,dx \le c[g(\lambda)]^{\xi_1}.$$

Plugging this estimate into (3.29), we have

$$2 \le \frac{|B_k \setminus E_\lambda^\rho|}{|B_k|} + c_1\left[\frac{|E_\lambda^\rho \cap B_k|}{|B_k|}\right]^{1-\frac{\xi}{\xi_1}},$$

which in turn implies

$$\frac{|E_\lambda^\rho \cap B_k|}{|B_k|} \ge \left(\frac{1}{c_1}\right)^{\frac{\xi_1}{\xi_1-\xi}} > 0.$$

Therefore, taking into account the case $(3.28)_2$ as well, we conclude with the density estimate

$$|2B_k| = 2^n|B_k| \le c|E^\rho_\lambda \cap B_k| + \frac{cM[|\mu|(B_k)]}{[g(\lambda)]^{\frac{\theta}{\theta-1}}}. \tag{3.36}$$

Step 5: Estimates on balls. We take another parameter $H \ge 8$, whose value will be chosen later, so that we have

$$|E^t_{H\lambda}| \le \sum |E^t_{H\lambda} \cap B_k|. \tag{3.37}$$

We split each term in the following way:

$$\begin{aligned} |E^t_{H\lambda} \cap B_k| &\le |\{x \in B_k : |Du(x)| > H\lambda\}| \\ &\le |\{x \in B_k : |Du(x) - Dv_k(x)| > H\lambda/2\}| \\ &\quad + |\{x \in B_k : |Dv_k(x)| > H\lambda/2\}| \\ &=: I_k + II_k. \end{aligned} \tag{3.38}$$

We estimate I_k as

$$\begin{aligned} I_k &\le \frac{c}{[g(H\lambda)]^\xi} \int_{B_k} [g(|Du - Dv_k|)]^\xi \, dx \\ &\overset{(3.32)}{\le} c \left[\frac{g(\lambda)}{g(H\lambda)}\right]^\xi \frac{[|\mu|(2B_k)]}{[g(\lambda)]^{\frac{\theta}{\theta-1}}} \\ &\overset{(3.26)}{\le} c \left[\frac{g(\lambda)}{g(H\lambda)}\right]^\xi \frac{|2B_k|}{M}. \end{aligned}$$

As for II_k, we have

$$\begin{aligned} II_k &\le \frac{c}{[G(H\lambda)]^\chi} \int_{B_k} [G(|Dv_k|)]^\chi \, dx \\ &\overset{(3.33)}{\le} \frac{c|2B_k|}{[G(H\lambda)]^\chi} \left\{ (G \circ g^{-1}) \left[\left(\fint_{2B_k} [g(|Dv_k|)]^\xi \, dx \right)^{\frac{1}{\xi}} \right] \right\}^\chi \\ &\overset{(3.34)}{\le} c \left[\frac{G(\lambda)}{G(H\lambda)}\right]^\chi |2B_k|. \end{aligned}$$

Combining the above two estimates with (3.38), and using (3.36), we see that

$$|E^t_{H\lambda}\cap B_k| \le c_*\left\{\left[\frac{G(\lambda)}{G(H\lambda)}\right]^{\chi}+\frac{1}{M}\left[\frac{g(\lambda)}{g(H\lambda)}\right]^{\xi}\right\}|E^\rho_\lambda\cap B_k| + \frac{cM[|\mu|(B_k)]}{[g(\lambda)]^{\frac{\theta}{\theta-1}}}$$

holds for a constant $c_*\equiv c_*(\texttt{data})$. Summing up on k and using (3.37) and (3.25), we have

$$|E^t_{H\lambda}| \le c_*Q(n)\left\{\left[\frac{G(\lambda)}{G(H\lambda)}\right]^{\chi}+\frac{1}{M}\left[\frac{g(\lambda)}{g(H\lambda)}\right]^{\xi}\right\}|E^\rho_\lambda| + \frac{cM[|\mu|(B_1)]}{[g(\lambda)]^{\frac{\theta}{\theta-1}}}. \tag{3.39}$$

We notice that the parameters M and H are still free; we will determine their values in the next step.

Step 6: Iteration. We define the level function $l(\cdot,\cdot)$ as

$$l(\lambda,\gamma) := [g(\lambda)]^{\frac{\theta}{\theta-1}}|E^\gamma_\lambda| \qquad \text{for every}\quad \gamma\in[1/2,2/3] \quad\text{and}\quad \lambda>0.$$

Then (3.39) becomes

$$\begin{aligned} l(H\lambda,t) &\le c_*Q(n)\left\{\left[\frac{G(\lambda)}{G(H\lambda)}\right]^{\chi}+\frac{1}{M}\left[\frac{g(\lambda)}{g(H\lambda)}\right]^{\xi}\right\}\left[\frac{g(H\lambda)}{g(\lambda)}\right]^{\frac{\theta}{\theta-1}} l(\lambda,\rho) \\ &\quad + c\left[\frac{g(H\lambda)}{g(\lambda)}\right]^{\frac{\theta}{\theta-1}} M \\ &\le c_*Q(n)\left\{\left[\frac{G(\lambda)}{G(H\lambda)}\right]^{\chi}+\frac{1}{M}\left[\frac{g(\lambda)}{g(H\lambda)}\right]^{\xi}\right\}\left[\frac{g(H\lambda)}{g(\lambda)}\right]^{\frac{\theta}{\theta-1}} l(\lambda,\rho) \\ &\quad + cH^{\frac{\theta g_1}{\theta-1}}M. \end{aligned}$$

We now observe

$$\left[\frac{G(\lambda)}{G(H\lambda)}\right]^{\chi}\left[\frac{g(H\lambda)}{g(\lambda)}\right]^{\frac{\theta}{\theta-1}} \le cH^{-\frac{\theta}{\theta-1}}\left[\frac{G(\lambda)}{G(H\lambda)}\right]^{\chi-\frac{\theta}{\theta-1}}$$

and consider the following two cases.

(i) If $\chi \geq \theta/(\theta-1)$, then we have

$$\left[\frac{G(\lambda)}{G(H\lambda)}\right]^{\chi}\left[\frac{g(H\lambda)}{g(\lambda)}\right]^{\frac{\theta}{\theta-1}} \leq cH^{-\frac{\theta}{\theta-1}}.$$

(ii) If $\chi < \theta/(\theta-1)$, then we have

$$\begin{aligned}\left[\frac{G(\lambda)}{G(H\lambda)}\right]^{\chi}\left[\frac{g(H\lambda)}{g(\lambda)}\right]^{\frac{\theta}{\theta-1}} &\leq c[G(\lambda)]^{\chi-\frac{\theta}{\theta-1}}H^{-\frac{\theta}{\theta-1}}[G(H\lambda)]^{\frac{\theta}{\theta-1}-\chi} \\ &\leq cH^{(g_1+1)\left(\frac{\theta}{\theta-1}-\chi\right)-\frac{\theta}{\theta-1}}[G(\lambda)]^{\chi-\frac{\theta}{\theta-1}}.\end{aligned}$$

Since $g_1+1 \leq \theta$, it follows that

$$\frac{\theta}{\theta-1} \leq \frac{g_1+1}{g_1} < \frac{g_1+1}{g_1}\chi \implies (g_1+1)\left(\frac{\theta}{\theta-1}-\chi\right) < \frac{\theta}{\theta-1}.$$

Therefore, in any case, we have

$$\lim_{H\to\infty}\left[\frac{G(\lambda)}{G(H\lambda)}\right]^{\chi}\left[\frac{g(H\lambda)}{g(\lambda)}\right]^{\frac{\theta}{\theta-1}} = 0.$$

Now we take $H \equiv H(\texttt{data})$ so large that

$$c_*Q(n)\left[\frac{G(\lambda)}{G(H\lambda)}\right]^{\chi}\left[\frac{g(H\lambda)}{g(\lambda)}\right]^{\frac{\theta}{\theta-1}} \leq \frac{1}{4},$$

and then we finally choose $M \equiv M(\texttt{data})$ large enough to have

$$\frac{c_*Q(n)}{M}\left[\frac{g(H\lambda)}{g(\lambda)}\right]^{\frac{\theta}{\theta-1}-\xi} \leq \frac{1}{4}.$$

Hence we arrive at

$$l(H\lambda,t) \leq \frac{1}{2}l(\lambda,\rho) + c(\texttt{data})$$

whenever $\lambda \geq \lambda_1$, where λ_1 was defined in (3.23). Recalling (2.7), we have

$$\sup_{\lambda\geq H\lambda_1} l(\lambda,t) \leq \frac{1}{2}\|g(|Du|)\|^{\theta/(\theta-1)}_{\mathcal{M}^{\theta/(\theta-1)}(B_\rho)} + c.$$

Considering also the case $\lambda < H\lambda_1$, with (3.22)-(3.24), we discover

$$\|g(|Du|)\|^{\theta/(\theta-1)}_{\mathcal{M}^{\theta/(\theta-1)}(B_t)} \leq \frac{1}{2}\|g(|Du|)\|^{\theta/(\theta-1)}_{\mathcal{M}^{\theta/(\theta-1)}(B_\rho)} + \frac{c[g(H\lambda_0)]^{\frac{\theta}{\theta-1}}}{(\rho-t)^{n/\xi}} + c.$$

Note that we are assuming (3.10). Then, since $\theta/(\theta-1) \leq g_1/(g_1+1)$, it follows that

$$\|g(|Du|)\|_{\mathcal{M}^{\theta/(\theta-1)}(B_1)} < \infty.$$

We then apply Lemma 2.3.12 with

$$\mathcal{Z}(t) \coloneqq \|g(|Du|)\|_{\mathcal{M}^{\theta/(\theta-1)}(B_t)}$$

and $1/2 \leq t < \rho \leq 2/3$, and use (3.19) and (3.22) to finally obtain

$$\|g(|D\bar{u}|)\|^{\theta/(\theta-1)}_{\mathcal{M}^{\theta/(\theta-1)}(B_{1/2})} \leq c(\mathtt{data}), \tag{3.40}$$

where we have recovered the bar notation introduced in *Step 1*.

Step 7: Scaling back. We now consider the case of a general ball B_R. Scaling back, (3.40) becomes

$$\|g(|Du|)\|^{\theta/(\theta-1)}_{\mathcal{M}^{\theta/(\theta-1)}(B_{R/2})} \leq cR^n[g(H)]^{\frac{\theta}{\theta-1}}.$$

Recalling (3.18), and then using (3.17) and (1.5), we obtain

$$\begin{aligned}
[g(H)]^{\frac{\theta}{\theta-1}} &\leq c\left[\left(\fint_{B_R}[g(|Du|)]^{\xi_1}\,dx\right)^{\frac{1}{\xi_1}}\right]^{\frac{\theta}{\theta-1}} \\
&\quad + c\left(R\fint_{B_R}|\mu|\,dx\right)^{\frac{\theta}{\theta-1}} + cR^{-\theta}\|\mu\|^{\frac{\theta}{\theta-1}}_{L^{1,\theta}(\Omega)} \\
&\leq cR^{-\theta}[|\mu|(\Omega) + \|\mu\|_{L^{1,\theta}(\Omega)}]^{\frac{\theta}{\theta-1}},
\end{aligned}$$

where $c \equiv c(\mathtt{data}, \operatorname{dist}(B_R, \partial\Omega)$. Combining the last two displays and replacing the arbitrarily given R by $2R$, we obtain the desired estimate (3.5). $\square$

3.2 Fractional differentiability for elliptic measure data problems with double phase in the borderline case

3.2.1 Main results

We consider the following Dirichlet problem:

$$\begin{cases} -\operatorname{div} A(x, Du) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.41}$$

where $\mu \in \mathcal{M}_b(\Omega)$. The vector field $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be C^1-regular in the second variable, with $\partial A(x,z) = \partial_z A(x,z)$ being Carathéodory regular. It also satisfies the following growth, ellipticity and continuity assumptions:

$$\begin{cases} |A(x,z)| + |\partial A(x,z)||z| \le L\left[|z|^{p-1} + a(x)|z|^{p-1}\log(e+|z|)\right], \\ \nu\left[|z|^{p-2} + a(x)|z|^{p-2}\log(e+|z|)\right]|\xi|^2 \le \partial A(x,z)\xi\cdot\xi, \\ |A(x,z) - A(y,z)| \le L\omega_a(|x-y|)|z|^{p-1}\log(e+|z|) \end{cases} \tag{3.42}$$

for every $z, \xi \in \mathbb{R}^n$ and $x, y \in \Omega$, and for some constants $0 < \nu \le L < \infty$ and $2 \le p \le n$. Here, the modulating coefficient $a : \Omega \to [0,\infty)$ admits ω_a as its modulus of continuity, i.e.,

$$\omega_a(\rho) \coloneqq \sup\{|a(x) - a(y)| : x, y \in \Omega, |x-y| \le \rho\}.$$

We further assume the Lipschitz continuity on $a(\cdot)$:

$$\omega_a(\rho) \le \rho. \tag{3.43}$$

We define two functions $G, g : \Omega \times [0,\infty) \to \mathbb{R}$ by

$$\begin{aligned} G(x,t) &\coloneqq t^p + a(x)t^p\log(e+t), \\ g(x,t) &\coloneqq t^{p-1} + a(x)t^{p-1}\log(e+t). \end{aligned} \tag{3.44}$$

Then the Musielak-Orlicz spaces $W^{1,G}(\Omega)$ and $W^{1,g}(\Omega)$ are where a weak solution and a very weak solution belong to, respectively.

Definition 3.2.1. *A function* $u \in W_0^{1,1}(\Omega)$ *is a SOLA to* (3.41) *under assumptions* (3.42) *if* $A(\cdot, Du) \in L^1(\Omega;\mathbb{R}^n)$,

$$\int_\Omega A(x,Du)\cdot D\varphi\,dx = \int_\Omega \varphi\,d\mu \qquad \forall\,\varphi \in C_0^\infty(\Omega),$$

and moreover there exists a sequence of weak solutions $\{u_k\} \subset W_0^{1,G}(\Omega)$ *to the problems*

$$\begin{cases} -\mathrm{div}\,A(x,Du_k) = \mu_k & \text{in } \Omega,\\ u_k = 0 & \text{on } \partial\Omega \end{cases}$$

such that $u_k \to u$ *in* $W_0^{1,g}(\Omega)$*, and the sequence* $\{\mu_k\} \subset L^\infty(\Omega)$ *converges to* μ *weakly* in the sense of measures and satisfies*

$$\limsup_{k\to\infty} |\mu_k|(B) \le |\mu|(\bar{B})$$

for every ball $B \subset \mathbb{R}^n$*.*

The existence of a SOLA to (3.41) was proved in [62]. Moreover, with the sequence $\{\mu_k\}$ defined as above, it holds that

$$u_k \to u \;\text{ in }\; W_0^{1,q}(\Omega) \qquad \forall\, q < \frac{n(p-1)}{n-1}. \tag{3.45}$$

Now, defining

$$V(x,Du) \coloneqq (|Du|^{p-2} + a(x)|Du|^{p-2}\log(e+|Du|))^{\frac{1}{2}}Du \tag{3.46}$$

and denoting $\texttt{data} \coloneqq (n,p,\nu,L)$, we state our main result:

Theorem 3.2.2. *Let* $u \in W_0^{1,g}(\Omega)$ *be a SOLA to the problem* (3.41) *under assumptions* (3.42) *and* (3.43)*. Then we have*

$$V(\cdot,Du) \in W_{\mathrm{loc}}^{\frac{p-\varepsilon}{2(p-1)},\frac{2(p-1)}{p}}(\Omega;\mathbb{R}^n) \qquad \forall\,\varepsilon \in (0,1). \tag{3.47}$$

Moreover, for any open subset $\Omega' \Subset \Omega$*, we have*

$$\int_{\Omega'}\int_{\Omega'} \frac{|V(x,Du(x)) - V(y,Du(y))|^{2(p-1)/p}}{|x-y|^{n+1-\varepsilon}}\,dx\,dy \le c|\mu|(\Omega), \tag{3.48}$$

where $c \equiv c(\texttt{data}, \mathrm{dist}(\Omega',\partial\Omega), \Omega, \varepsilon)$*.*

In light of the discussion in [164, Section 11.2], the result in Theorem 3.2.2 is optimal in the sense that we cannot allow $\varepsilon = 0$ in (3.47).

3.2.2 Preliminaries

Here, we remark some properties of the functions $g(\cdot)$ and $G(\cdot)$ in (3.44) for $p \geq 2$, see [41, 62] for the proofs. A direct calculation yields

$$\frac{\partial G}{\partial t}(x,t) \approx g(x,t); \qquad G(x,t) \approx \int_0^t g(x,s)\,ds$$

for every $x \in \Omega$ and $t \geq 0$. Furthermore, there holds

$$p \leq \frac{t\partial_t G(x,t)}{G(x,t)} \leq p+1 \quad \text{and} \quad p-1 \leq \frac{t\partial_t^2 G(x,t)}{\partial_t G(x,t)} \leq p - \frac{1}{4}$$

for every $t > 0$ and $x \in \Omega$; in particular, the map $t \mapsto g(x,t)/t$ is increasing for a.e. $x \in \Omega$. We also have

$$t^p \leq G(x,t). \tag{3.49}$$

For a point x_0, we denote $a_0 := a(x_0)$ and

$$g_0(t) := t^{p-1} + a_0 t^{p-1}\log(e+t), \quad G_0(t) := t^p + a_0 t^p \log(e+t).$$

Then both g_0 and G_0 are N-functions. We recall the following result from [97, Lemma 20].

Lemma 3.2.3. *Let ξ and η be two vectors in $\mathbb{R}^n$ with $|\xi| + |\eta| > 0$. Then there holds*

$$\int_0^1 g_0'(|(1-t)\xi + t\eta|)\,dt \approx g_0'(|\xi| + |\eta|).$$

We next recall a property of the vector field $V(\cdot)$ defined in (3.46) related to the monotonocity of $A(\cdot)$. Namely,

$$\begin{aligned} |V(x,z_1) - V(x,z_2)|^2 &\approx \frac{g(x,|z_1|+|z_2|)}{|z_1|+|z_2|}|z_1 - z_2|^2 \\ &\lesssim (A(x,z_1) - A(x,z_2)) \cdot (z_1 - z_2) \end{aligned}$$

holds for any $x \in \Omega$ and $z_1, z_2 \in \mathbb{R}^n$, see [97, Lemma 3]. Moreover, since $t \mapsto g(x,t)/t$ is increasing for a.e. $x \in \Omega$, we have

$$\begin{aligned} G(x, |z_1 - z_2|) &= \frac{g(x, |z_1 - z_2|)}{|z_1 - z_2|} |z_1 - z_2|^2 \\ &\le \frac{g(x, |z_1| + |z_2|)}{|z_1| + |z_2|} |z_1 - z_2|^2 \lesssim |V(x, z_1) - V(x, z_2)|^2. \end{aligned} \tag{3.50}$$

3.2.3 Regularity for homogeneous problems

In this section, we obtain a differentiability result for the limiting equation, which is an extension of Lemma 2.3.8.

Lemma 3.2.4. *Under assumptions* (3.42)*, let* $v \in W^{1,G_0}_{\mathrm{loc}}(\Omega)$ *be a weak solution to*

$$-\operatorname{div} A(x_0, Dv) = 0 \quad \text{in } \Omega. \tag{3.51}$$

Then $V(x_0, Dv) \in W^{1,2}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$*. Moreover, for every number* $t \in (0,1]$ *there exists a positive constant* $c \equiv c(\mathtt{data}, t)$ *such that*

$$\int_{B_{R/2}} |D(V(x_0, Dv))|^{2t} \, dx \le \frac{c}{R^{2t}} \int_{B_R} |V(x_0, Dv) - z_0|^{2t} \, dx \tag{3.52}$$

for every $z_0 \in \mathbb{R}^n$ *and every ball* $B_R \Subset \Omega$*.*

Proof. Step 1: A preliminary estimate. Let $B_R \Subset \Omega$ be a fixed ball. We first establish a classical differentiability estimate under the present assumption:

$$\int_{B_{R/2}} |D(V(x_0, Dv))|^2 \, dx \le \frac{c}{R^2} \int_{B_{3R/4}} |V(x_0, Dv) - z_0|^2 \, dx, \tag{3.53}$$

where $c \equiv c(\mathtt{data})$. This is (3.52) in the case $t = 1$. The differentiability of $V(x_0, Dv)$ was already proved in [97], but estimate (3.53) was obtained only in the case $z_0 = 0$. In order to consider the general case with any choice of z_0, we follow the proof of [69, Theorem 1.I]; see also [121, Theorem 4.1].

We choose a cut-off function $\phi \in C_0^\infty(B_{5R/8})$ satisfying $0 \le \phi \le 1$ in $B_{5R/8}$, $\phi \equiv 1$ in $B_{R/2}$, and $|D\phi| \le 16/R$. Moreover, we fix $i \in \{1, \dots, n\}$, $h \in \mathbb{R}$ satisfying $0 < |h| \le \min\{R, \operatorname{dist}(B_R, \partial\Omega)\}/100$, and any affine function $P : \Omega \to \mathbb{R}$. In the following, with $\{e_i\}_{1 \le i \le n}$ denoting the standard basis of $\mathbb{R}^n$, we simply denote $\tau_{he_i} f \equiv \tau_{i,h} f$.

Testing (3.51) with $\varphi \equiv \tau_{i,-h}(\phi^2\tau_{i,h}(v-P)) \in W_0^{1,G_0}(\Omega)$, we have

$$\int_\Omega \tau_{i,h}(A(x_0, Dv)) \cdot D(\phi^2\tau_{i,h}(u-P))\,dx = 0, \tag{3.54}$$

where $D(\phi^2\tau_{i,h}(v-P)) = \phi^2\tau_{i,h}Dv + 2\phi D\phi(\tau_{i,h}(v-P))$. Note that the C^1-vector field

$$\tilde{A}(z) \coloneqq \int_0^1 A(x_0, z + t\tau_{i,h}Dv)\,dt$$

satisfies

$$\partial\tilde{A}(z) = \int_0^1 \partial A(x_0, z + t\tau_{i,h}Dv)\,dt$$

and so

$$\tau_{i,h}(A(x_0, Dv)) = \partial\tilde{A}(Dv) \cdot \tau_{i,h}Dv.$$

Using this, (3.54) becomes

$$\begin{aligned} I_1 &\coloneqq \int_{B_{5R/8}} \partial\tilde{A}(Dv)\tau_{i,h}Dv \cdot \phi^2\tau_{i,h}Dv\,dx \\ &= -2\int_{B_{5R/8}} \partial\tilde{A}(Dv)\tau_{i,h}Dv \cdot \phi D\phi\tau_{i,h}(v-P)\,dx \eqqcolon I_2. \end{aligned} \tag{3.55}$$

We then apply Lemma 3.2.3 and assumptions (3.42) with $x = x_0$ to estimate

$$\begin{aligned} |I_2| &\le c\int_{B_{5R/8}} |\partial\tilde{A}(Dv)||\tau_{i,h}Dv|\phi|D\phi||\tau_{i,h}(u-P)|\,dx \\ &\le \frac{c}{R}\int_{B_{5R/8}} \phi g_0'(|Dv| + |\tau_{i,h}Dv|)|\tau_{i,h}Dv||\tau_{i,h}(v-P)|\,dx \end{aligned}$$

and

$$I_1 \ge c\int_{B_{5R/8}} \phi^2 g_0'(|Dv| + |\tau_{i,h}Dv|)|\tau_{i,h}Dv|^2\,dx.$$

Connecting the last two displays to (3.55) gives

$$\begin{aligned} &\int_{B_{5R/8}} \phi^2 g_0'(|Dv| + |\tau_{i,h}Dv|)|\tau_{i,h}Dv|^2\,dx \\ &\le \frac{c}{R}\int_{B_{5R/8}} \phi g_0'(|Dv| + |\tau_{i,h}Dv|)|\tau_{i,h}Dv||\tau_{i,h}(v-P)|\,dx \end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{2}\int_{B_{5R/8}} \phi^2 g_0'(|Dv|+|\tau_{i,h}Dv|)|\tau_{i,h}Dv|^2\,dx \\
&\quad + \frac{c}{R^2}\int_{B_{5R/8}} g_0'(|Dv|+|\tau_{i,h}Dv|)|\tau_{i,h}(v-P)|^2\,dx,
\end{aligned}
$$

where we have also used Young's inequality. Consequently, we arrive at

$$
\begin{aligned}
&\int_{B_{R/2}} g_0'(|Dv|+|\tau_{i,h}Dv|)|\tau_{i,h}Dv|^2\,dx \\
&\le \frac{c}{R^2}\int_{B_{5R/8}} g_0'(|Dv|+|\tau_{i,h}Dv|)|\tau_{i,h}(v-P)|^2\,dx,
\end{aligned}
$$

which further implies

$$
\begin{aligned}
&\int_{B_{R/2}} \left|\frac{\tau_{i,h}(V(x_0,Dv))}{h}\right|^2\,dx \\
&\le c\int_{B_{R/2}} \frac{g_0'(|Dv(\cdot+he_i)|+|Dv|)|\tau_{i,h}Dv|^2}{|h|^2}\,dx \\
&\le c\int_{B_{R/2}} \frac{g_0'(|Dv|+|\tau_{i,h}Dv|)|\tau_{i,h}Dv|^2}{|h|^2}\,dx \\
&\le \frac{c}{R^2}\int_{B_{5R/8}} g_0'(|Dv|+|\tau_{i,h}Dv|)\left|\frac{\tau_{i,h}(v-P)}{h}\right|^2\,dx \\
&\le \frac{c}{R^2}\int_{B_{5R/8}} g_0'(|Dv|)\left|\frac{\tau_{i,h}(v-P)}{h}\right|^2\,dx \\
&\quad + \frac{c}{R^2}\int_{B_{5R/8}} g_0'(|Dv(\cdot+he_i)|)\left|\frac{\tau_{i,h}(v-P)}{h}\right|^2\,dx \\
&\le \frac{c}{R^2}\int_{B_{3R/4}} g_0'(|Dv|)\left|\frac{\tau_{i,h}(v-P)}{h}\right|^2\,dx \\
&\quad + \frac{c}{R^2}\int_{B_{3R/4}} g_0'(|Dv|)\left|\frac{\tau_{i,-h}(v-P)}{h}\right|^2\,dx.
\end{aligned}
$$

Since v is locally Lipschitz (see [13, Lemma 4.1] and [157, Lemma 5.1]), we have

$$g_0'(|Dv(x)|)\left|\frac{\tau_{i,\pm h}(v-P)(x)}{h}\right|^2 = g_0'(|Dv(x)|)\left|\int_0^1 D_i(v-P)(x\pm the_i)\,dt\right|^2$$
$$\le g_0'(|Dv(x)|)\int_0^1 |D_i(v-P)(x\pm the_i)|^2\,dt$$
$$\lesssim g_0'(|Dv(x)|)\left(\sup_{B_R}|Dv|^2+|DP|^2\right)$$

and

$$g_0'(|Dv(x)|)\left|\frac{\tau_{i,\pm h}(v-P)(x)}{h}\right|^2 \to g_0'(|Dv(x)|)|D_i(v-P)(x)|^2 \quad \text{as } h\to 0$$

for a.e. $x \in B_{3R/4}$. Thus, the dominated convergence theorem implies

$$\int_{B_{3R/4}} g_0'(|Dv|)\left|\frac{\tau_{i,\pm h}(v-P)}{h}\right|^2 dx \to \int_{B_{3R/4}} g_0'(|Dv|)|D_i(v-P)|^2\,dx \quad \text{as } h\to 0,$$

In turn, for every h as above we have

$$\int_{B_{R/2}}\left|\frac{\tau_{i,h}(V(x_0,Dv))}{h}\right|^2 dx \le \frac{c}{R^2}\int_{B_{3R/4}} g_0'(|Dv|)|D_i(v-P)|^2\,dx.$$

Therefore, it follows that $V(x_0,Dv)\in W^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ with the estimate

$$\int_{B_{R/2}}|D_i(V(x_0,Dv))|^2\,dx \le \frac{c}{R^2}\int_{B_{3R/4}} g_0'(|Dv|)|D_i(v-P)|^2\,dx.$$

Hence, summing up these inequalities for all $i\in\{1,\dots,n\}$, we conclude with

$$\int_{B_{R/2}}|D(V(x_0,Dv))|^2\,dx \le \frac{c}{R^2}\int_{B_{3R/4}} g_0'(|Dv|)|D(v-P)|^2\,dx$$
$$\le \frac{c}{R^2}\int_{B_{3R/4}}|V(x_0,Dv)-V(x_0,DP)|^2\,dx.$$

Since $V(x_0,\cdot)$ is bijective, we can choose the vector $DP\in\mathbb{R}^n$ so that $V(x_0,DP)=z_0$. Then (3.53) follows.

Step 2: Estimates below the natural growth exponent. Next, we follow the proof of [168, Theorem 9.1]. The Sobolev embedding theorem gives

$$\left(\fint_{B_{R/2}} |V(x_0, Dv) - z_0|^{2\chi}\, dx\right)^{\frac{1}{\chi}} \\ \leq c \fint_{B_{R/2}} |V(x_0, Dv) - z_0|^2\, dx + cR^2 \fint_{B_{R/2}} |D(V(x_0, Dv))|^2\, dx,$$

where $\chi = n/(n-2)$ if $n > 2$ and χ is any number larger than 1 when $n = 2$. Matching the last two estimates now gives the reverse Hölder inequality

$$\left(\fint_{B_{R/2}} |V(x_0, Dv) - z_0|^{2\chi}\, dx\right)^{\frac{1}{\chi}} \leq c \fint_{B_{3R/4}} |V(x_0, Dv) - z_0|^2\, dx.$$

By applying the self-improving property of reverse Hölder inequalities (see [118, Remark 6.12]), we get

$$\left(\fint_{B_{3R/4}} |V(x_0, Dv) - z_0|^{2\chi}\, dx\right)^{\frac{1}{2\chi}} \leq c \left(\fint_{B_R} |V(x_0, Dv) - z_0|^{2t}\, dx\right)^{\frac{1}{2t}}.$$

At this point, (3.52) follow from the last estimate together with (3.53) and Hölder's inequality:

$$\begin{aligned} \fint_{B_{R/2}} |D(V(x_0, Dv))|^{2t}\, dx &\leq \left(\fint_{B_{R/2}} |D(V(x_0, Dv))|^2\, dx\right)^t \\ &\leq c \left(\frac{1}{R^2} \fint_{B_{3R/4}} |V(x_0, Dv) - z_0|^2\, dx\right)^t \\ &\leq \frac{c}{R^{2t}} \fint_{B_R} |V(x_0, Dv) - z_0|^{2t}\, dx. \end{aligned}$$ □

Remark 3.2.5. *The above lemma continues to hold if $g_0 \in C^1(0, \infty)$ is any function satisfying*

$$1 \leq \frac{t g_0'(t)}{g_0(t)} \leq g_2 < \infty$$

and $V(x_0, z)$ is replaced by $V_{g_0}(z) := (g_0(|z|)/|z|)^{1/2} z$.

3.2.4 Comparison estimates

In this section, we establish several comparison estimates between (3.41) and the reference problems. To do this, we additionally assume that

$$\mu \in L^{\infty}(\Omega), \qquad u \in W_0^{1,G}(\Omega). \tag{3.56}$$

This assumption will be removed in Section 3.2.5.

We consider the homogeneous problem

$$\begin{cases} -\operatorname{div} A(x, Dw) = 0 & \text{in } B_{2R}, \\ \qquad\qquad w = u & \text{on } \partial B_{2R}. \end{cases} \tag{3.57}$$

In order to obtain several estimates suitable in the setting of measure data problems, we assume that there exists a positive radius

$$R_1 \equiv R_1(L, |\mu|(\Omega), \|Du\|_{L^1(\Omega)}) \leq \frac{1}{|\mu|(\Omega) + \|Du\|_{L^1(\Omega)} + 1}$$

such that

$$r \log \frac{1}{r} \leq \frac{1}{16npL} \quad \text{for every} \quad 0 < r \leq R_1.$$

In the following, we always assume that every ball has radius less than R_1.

Let us first recall a higher integrability result for (3.57). Note that in [62, Lemma 3.5], the restriction $R \leq R_1$ plays a crucial role in establishing estimates suitable in the setting of measure data problems.

Lemma 3.2.6. *Let $w \in u + W_0^{1,G}(B_{2R})$ be the weak solution to* (3.57) *under assumptions* (3.42). *Then there exists $\sigma_0 \equiv \sigma_0(\mathtt{data})$ such that for any $\theta \in (0,1)$, $\sigma \in [0, \sigma_0]$ and $t \in (0,1]$, there holds*

$$\left(\fint_{B_{\theta\rho}} [G(x, |Dw|)]^{1+\sigma} \, dx \right)^{\frac{1}{1+\sigma}} \leq c \left(\fint_{B_\rho} [G(x, |Dw|)]^{t} \, dx \right)^{\frac{1}{t}} \tag{3.58}$$

for a constant $c \equiv c(\mathtt{data}, \theta, t)$, whenever $B_\rho \subset B_{2R}$.

We now establish comparison estimates. The first one is between (3.41) and (3.57).

Lemma 3.2.7. *Let $u \in W_0^{1,G}(\Omega)$ and $w \in u + W_0^{1,G}(B_{2R})$ be the weak solutions to* (3.41) *and* (3.57)*, respectively, under assumptions* (3.42)*. Then we have the estimate*

$$\fint_{B_{2R}} \left(|Du - Dw|^q + |V(x,Du) - V(x,Dw)|^{\frac{2q}{p}}\right) dx \le c \left[\frac{|\mu|(B_{2R})}{(2R)^{n-1}}\right]^{\frac{q}{p-1}} \tag{3.59}$$

for a constant $c \equiv c(\texttt{data}, q)$, whenever

$$1 \le q < \frac{n(p-1)}{n-1}. \tag{3.60}$$

Proof. We first recall the following estimate obtained in [62, Lemma 3.4]:

$$\int_{B_{2R}} \frac{|V(x,Du) - V(x,Dw)|^2}{(h + |u-w|)^{\xi}}\, dx \le c\frac{h^{1-\xi}}{\xi - 1}|\mu|(B_{2R}) \tag{3.61}$$

for a constant $c \equiv c(\texttt{data})$, whenever $h > 0$ and $\xi > 1$.

Since q satisfies (3.60), we can choose $\xi = n(p-q)/(n-q) > 1$ so that

$$\frac{\xi q}{p-q} = \frac{nq}{n-q} = q^*. \tag{3.62}$$

We are now going to apply (3.61) with this choice of ξ and

$$h = \left(\fint_{B_{2R}} |u-w|^{q^*}\, dx\right)^{\frac{1}{q^*}} \le cR\left(\fint_{B_{2R}} |Du - Dw|^q\, dx\right)^{\frac{1}{q}}.$$

We may assume $h > 0$, otherwise $u \equiv w$ in B_R and the lemma follows trivially. Then we have

$$\begin{aligned}
&\fint_{B_{2R}} |V(x,Du) - V(x,Dw)|^{\frac{2q}{p}}\, dx \\
&= \fint_{B_{2R}} \left(\frac{|V(x,Du) - V(x,Dw)|^2}{(h+|u-w|)^{\xi}}\right)^{\frac{q}{p}} (h + |u-w|)^{\frac{\xi q}{p}}\, dx \\
&\le \left(\fint_{B_{2R}} \frac{|V(x,Du) - V(x,Dw)|^2}{(h+|u-w|)^{\xi}}\, dx\right)^{\frac{q}{p}} \left(\fint_{B_{2R}} (h + |u-w|)^{q^*}\, dx\right)^{\frac{p-q}{p}}
\end{aligned}$$

$$
\begin{aligned}
&\le c\left(\frac{|\mu|(B_{2R})}{R^n}h^{1-\xi}\right)^{\frac{q}{p}} h^{\frac{\xi q}{p}} \\
&\le c\left[\frac{|\mu|(B_{2R})}{(2R)^{n-1}}\right]^{\frac{q}{p}} \left(\fint_{B_{2R}} |Du - Dw|^q\,dx\right)^{\frac{1}{p}}. \qquad (3.63)
\end{aligned}
$$

We observe that (3.49) and (3.50) imply

$$
|Du - Dw|^p \lesssim G(x, |Du - Dw|) \lesssim |V(x, Du) - V(x, Dw)|^2
$$

and so

$$
|Du - Dw|^q \lesssim_q |V(x, Du) - V(x, Dw)|^{\frac{2q}{p}}. \qquad (3.64)
$$

Putting this into (3.63) and then using Young's inequality, we have

$$
\begin{aligned}
&\fint_{B_{2R}} |V(x, Du) - V(x, Dw)|^{\frac{2q}{p}}\,dx \\
&\le c\left[\frac{|\mu|(B_{2R})|}{(2R)^{n-1}}\right]^{\frac{q}{p}} \left(\fint_{B_{2R}} |V(x, Du) - V(x, Dw)|^{\frac{2q}{p}}\,dx\right)^{\frac{1}{p}} \\
&\le c\left[\frac{|\mu|(B_{2R})|}{(2R)^{n-1}}\right]^{\frac{q}{p-1}} + \frac{1}{2}\fint_{B_{2R}} |V(x, Du) - V(x, Dw)|^{\frac{2q}{p}}\,dx.
\end{aligned}
$$

This and (3.64) complete the proof. □

With the same ball $B_{2R} \equiv B_{2R}(x_0)$ as before, let $v \in W_0^{1,G_0}(B_R)$ be the unique weak solution to

$$
\begin{cases} -\operatorname{div} A(x_0, Dv) = 0 & \text{in } B_R, \\ v = w & \text{on } \partial B_R. \end{cases} \qquad (3.65)
$$

Lemma 3.2.8. *Let $w \in u + W_0^{1,G}(B_{2R})$ and $v \in w + W_0^{1,G_0}(B_R)$ be the weak solutions to* (3.57) *and* (3.65)*, respectively, under assumptions* (3.42) *and* (3.43)*. Then for each q satisfying* (3.60)*, there exists a constant $c \equiv c(\mathtt{data}, q)$ such that*

$$
\fint_{B_R} |V(x, Dw) - V(x_0, Dv)|^{\frac{2q}{p}}\,dx \le c\left(R\log\frac{1}{R}\right)^{\frac{2q}{p}} \fint_{B_{2R}} [G(x, |Dw|)]^{\frac{q}{p}}dx. \qquad (3.66)
$$

Proof. As in the proof of [62, Lemma 3.6], we have

$$\fint_{B_R} |V(x_0, Dw) - V(x_0, Dv)|^2 \, dx \le cR^2 \fint_{B_{5R/4}} G(x, |Dw|) \, dx$$
$$\le cR^2 \left(\fint_{B_{2R}} [G(x, |Dw|)]^{\frac{q}{p}} \, dx \right)^{\frac{p}{q}}, \quad (3.67)$$

where we have also used (3.58). Now we use the mean value theorem to have

$$|V(x, z) - V(x_0, z)| = \left| g(x, |z|)^{1/2} - g(x_0, |z|)^{1/2} \right| |z|^{1/2} \lesssim R|z|^{p/2} \log(e + |z|).$$

We estimate exactly as in [62, (3.20)-(3.23)], with the help of (2.3), and then apply Lemma 3.2.6 in order to get

$$\begin{aligned} &\fint_{B_R} |V(x, Dw) - V(x_0, Dw)|^2 \, dx \\ &\le cR^2 \fint_{B_R} |Dw|^p \log^2(e + |Dw|) \, dx \\ &\le cR^2 \fint_{B_{5R/4}} G(x, |Dw|) \, dx + cR^2 \log^2 \frac{1}{R} \fint_{B_{5R/4}} G(x, |Dw|) \, dx \\ &\le cR^2 \log^2 \frac{1}{R} \fint_{B_{5R/4}} G(x, |Dw|) \, dx \\ &\le cR^2 \log^2 \frac{1}{R} \left(\fint_{B_{2R}} [G(x, |Dw|)]^{\frac{q}{p}} \, dx \right)^{\frac{p}{q}}. \end{aligned}$$

Combining this inequality with (3.67) and then applying Hölder's inequality, (3.66) follows. □

From Lemmas 3.2.7 and 3.2.8, we have the following comparison estimate between (3.41) and (3.65).

Lemma 3.2.9. *Let u and v be as in* (3.41) *and* (3.65)*, respectively, under assumptions* (3.42) *and* (3.43)*. Then the estimate*

$$\begin{aligned} &\fint_{B_R} |V(x, Du) - V(x_0, Dv)|^{\frac{2q}{p}} \, dx \\ &\le c \left[\frac{|\mu|(B_{2R})}{(2R)^{n-1}} \right]^{\frac{q}{p-1}} + c \left(R \log \frac{1}{R} \right)^{\frac{2q}{p}} \fint_{B_{2R}} [G(x, |Du|)]^{\frac{q}{p}} dx \end{aligned}$$

holds for a constant $c \equiv c(\mathtt{data}, q)$*, whenever* q *satisfies* (3.60).

Proof. We first observe that

$$\begin{aligned}
&\fint_{B_{2R}} [G(x,|Dw|)]^{\frac{q}{p}}\,dx = \fint_{B_{2R}} |V(x,Dw)|^{\frac{2q}{p}}\,dx\\
&\le c\fint_{B_{2R}} |V(x,Du)|^{\frac{2q}{p}}\,dx + c\fint_{B_{2R}} |V(x,Du)-V(x,Dw)|^{\frac{2q}{p}}\,dx\\
&= c\fint_{B_{2R}} [G(x,|Du|)]^{\frac{q}{p}}\,dx + c\fint_{B_{2R}} |V(x,Du)-V(x,Dw)|^{\frac{2q}{p}}\,dx.
\end{aligned}$$

Now we put this into (3.66) and apply (3.59) twice to obtain the desired estimate. □

We end this section with a global a priori estimate for the gradient of solutions to (3.41), which is more explicit and stronger than what follows from (3.45). The proof is almost the same as that of Lemma 3.2.7.

Lemma 3.2.10. *Let* $u \in W_0^{1,G}(\Omega)$ *be the weak solution to* (3.41) *under assumptions* (3.42)*. Then for every* q *satisfying* (3.60)*, there exists a constant* $c \equiv c(\mathtt{data}, q)$ *such that*

$$\int_\Omega [G(x,|Du|)]^{\frac{q}{p}}\,dx \le c[|\mu|(\Omega)]^{\frac{q}{p-1}}. \tag{3.68}$$

Proof. We test (3.41) with $\varphi = h^{1-\xi} - (h+u_\pm)^{1-\xi}$ and argue in a completely similar way as in the proof of (3.61) in [62, Lemma 3.4], with $u-v$ and B_R replaced by u and Ω, respectively. Then we find that the following estimate

$$\int_\Omega \frac{|V(x,Du)|^2}{(h+|u|)^\xi}\,dx \le c\frac{h^{1-\xi}}{\xi-1}|\mu|(\Omega)$$

holds whenever $h>0$ and $\xi>1$.

Now we choose ξ so that $\xi q/(p-q) = q^*$ as in (3.62) and

$$h = \left(\int_\Omega |u|^{q^*}\,dx\right)^{\frac{1}{q^*}} \le c\left(\int_\Omega |Du|^q\,dx\right)^{\frac{1}{q}},$$

again assuming without loss of generality that $h>0$, and then proceed as in

(3.63) to obtain

$$\begin{aligned}\int_\Omega [G(x,|Du|)]^{\frac{q}{p}}\,dx &= \int_\Omega |V(x,Du)|^{\frac{2q}{p}}\,dx \\ &\le \left(\int_\Omega \frac{|V(x,Du)|^2}{(h+|u|)^\xi}\,dx\right)^{\frac{q}{p}} \left(\int_\Omega (h+|u|)^{q^*}\,dx\right)^{\frac{p-q}{q}} \\ &\le c\left(|\mu|(\Omega)h^{1-\xi}\right)^{\frac{q}{p}} h^{\frac{\xi q}{p}} \\ &\le c[|\mu|(\Omega)]^{\frac{q}{p}} \left(\int_\Omega |Du|^q\,dx\right)^{\frac{1}{p}} \\ &\le c[|\mu|(\Omega)]^{\frac{q}{p}} \left(\int_\Omega [G(x,|Du|)]^{\frac{q}{p}}\,dx\right)^{\frac{1}{p}} \\ &\le c[|\mu|(\Omega)]^{\frac{q}{p-1}} + \frac{1}{2}\int_\Omega [G(x,|Du|)]^{\frac{q}{p}}\,dx.\end{aligned}$$

This finally completes the proof. □

3.2.5 Proof of Theorem 3.2.2

With q satisfying

$$p-1 \le q < \frac{n(p-1)}{n-1},$$

we write

$$\delta \coloneqq \frac{p}{2}\left(\frac{n}{q} - \frac{n-1}{p-1}\right); \quad \gamma(t) \coloneqq \frac{\delta}{\delta+1-t} \quad \text{for every } t \in [0,\delta+1).$$

Lemma 3.2.11. *Let $u \in W_0^{1,G}(\Omega)$ be the weak solution to* (3.41) *under assumptions* (3.42) *and* (3.43)*, and let $q \in [p-1, n(p-1)/(n-1))$. Assume that*

$$V(\cdot,Du) \in W^{t,2q/p}_{\mathrm{loc}}(\Omega;\mathbb{R}^n), \qquad \textit{for some } t \in [0,\delta),$$

and that for every couple of open subsets $\Omega' \Subset \Omega'' \Subset \Omega$ there exists a constant c_1, depending only on $\operatorname{dist}(\Omega',\partial\Omega'')$*, such that*

$$[V(\cdot,Du)]^{2q/p}_{t,2q/p;\Omega'} \le c_1 \int_{\Omega''} [G(x,|Du|)]^{\frac{q}{p}}\,dx + c_1[|\mu|(\Omega'')]^{\frac{q}{p-1}}.$$

Then

$$V(\cdot, Du) \in W^{\tilde{t},2q/p}_{\mathrm{loc}}(\Omega;\mathbb{R}^n) \qquad \forall\, \tilde{t} \in [0,\gamma(t)),$$

and for every couple of open subsets $\Omega' \Subset \Omega'' \Subset \Omega$*, there exists a constant* c*, depending only on* $\texttt{data}, \operatorname{dist}(\Omega',\partial\Omega''), \tilde{t}, c_1$*, such that*

$$[V(\cdot,Du)]^{2q/p}_{\tilde{t},2q/p;\Omega'} \le c\int_{\Omega''}[G(x,|Du|)]^{\frac{2q}{p}}\,dx + c[|\mu|(\Omega'')]^{\frac{q}{p-1}}. \tag{3.69}$$

Moreover, for every vector $h \in \mathbb{R}^n$ *with* $0 < |h| < \operatorname{dist}(\Omega',\partial\Omega'')$*, we have*

$$\sup_h \int_{\Omega'} \frac{|\tau_h(V(\cdot,Du))|^{2q/p}}{|h|^{\gamma(t)2q/p}}\,dx \le c\int_{\Omega''}[G(x,|Du|)]^{\frac{q}{p}}\,dx + c[|\mu|(\Omega'')]^{\frac{q}{p-1}}. \tag{3.70}$$

Proof. We fix arbitrary open subsets $\Omega' \Subset \Omega'' \Subset \Omega$, take $\beta \in (0,1)$ to be chosen later, and let $h \in \mathbb{R}^n$ be a vector satisfying

$$0 < |h| \le \min\left\{\left(\frac{\operatorname{dist}(\Omega',\partial\Omega'')}{10000\sqrt{n}}\right)^{\frac{1}{\beta}}, \left(\frac{1}{10000}\right)^{\frac{1}{1-\beta}}\right\} =: d < \operatorname{dist}(\Omega',\partial\Omega''). \tag{3.71}$$

We take $x_0 \in \Omega'$ and fix a ball $B \equiv B(x_0,|h|^\beta)$. Then we consider the weak solutions $w \in u + W^{1,G}_0(8B)$ and $v \in w + W^{1,G_0}_0(4B)$ to the problems (3.57) and (3.65) with $B_{2R} \equiv 8B$, respectively. Since $B + B_{|h|}(0) \subset 4B$ by (3.71), we have

$$\begin{aligned}
&\int_B |\tau_h(V(\cdot,Du))|^{\frac{2q}{p}}\,dx \\
&\le c\int_B |\tau_h(V(x_0,Dv))|^{\frac{2q}{p}}\,dx + c\int_B |V(x,Du)-V(x_0,Dv)|^{\frac{2q}{p}}\,dx \\
&\quad + c\int_B |V(x+h,Du(x+h)) - V(x_0,Dv(x+h))|^{\frac{2q}{p}}\,dx \\
&\le c\int_B |\tau_h(V(x_0,Dv))|^{\frac{2q}{p}}\,dx + c\int_{4B} |V(x,Du)-V(x_0,Dv)|^{\frac{2q}{p}}\,dx \\
&=: I_1 + I_2.
\end{aligned}$$

In order to estimate I_2, we use Lemma 3.2.9, which gives

$$\begin{aligned} I_2 &\le c|h|^{\beta\delta\frac{2q}{p}}[|\mu|(8B)]^{\frac{q}{p-1}} + c\left(8|h|^\beta \log\frac{1}{8|h|^\beta}\right)^{\frac{2q}{p}} \int_{8B} [G(x,|Du|)]^{\frac{q}{p}}\,dx \\ &\le c|h|^{\beta\delta\frac{2q}{p}}\lambda_0(8B), \end{aligned}$$

where we have set

$$\lambda_0(E) \coloneqq \int_E [G(x,|Du|)]^{\frac{q}{p}}\,dx + [|\mu|(E)]^{\frac{q}{p-1}}.$$

In order to estimate I_1, we apply Lemma 3.2.4 with $t = q/p$ to find

$$\begin{aligned} I_1 &\overset{(2.6)}{\le} c|h|^{\frac{2q}{p}} \int_{2B} |D(V(x_0,Dv))|^{\frac{2q}{p}}\,dx \\ &\le c|h|^{(1-\beta)\frac{2q}{p}} \int_{4B} |V(x_0,Dv) - z_0|^{\frac{2q}{p}}\,dx \\ &\le c|h|^{(1-\beta)\frac{2q}{p}} \int_{4B} |V(x,Du) - z_0|^{\frac{2q}{p}}\,dx \\ &\quad + c|h|^{(1-\beta)\frac{2q}{p}} \int_{4B} |V(x,Du) - V(x_0,Dv)|^{\frac{2q}{p}}\,dx. \end{aligned}$$

We then recall that $|h| < 1$ and $0 < \beta < 1$ to discover

$$\begin{aligned} &\int_B |\tau_h(V(\cdot,Du))|^{\frac{2q}{p}}\,dx \\ &\le c|h|^{\beta\delta\frac{2q}{p}+(1-\beta)\frac{2q}{p}}\lambda_0(8B) + c|h|^{(1-\beta)\frac{2q}{p}} \int_{4B} |V(x,Du) - z_0|^{\frac{2q}{p}}\,dx. \end{aligned} \tag{3.72}$$

We next choose $z_0 \in \mathbb{R}^n$ in the last display. We distinguish two cases.

Case $t = 0$. In this case, we take $z_0 = 0$ in (3.72). Then, since $t < \delta$ and $|h| < 1$, we have

$$\int_B |\tau_h(V(\cdot,Du))|^{\frac{2q}{p}}\,dx \le c\left[|h|^{\beta t\frac{2q}{p}+(1-\beta)\frac{2q}{p}} + |h|^{\beta\delta\frac{2q}{p}}\right]\lambda_0(8B).$$

Case $t > 0$. In this case, we take $z_0 = (V(\cdot,Du))_{8B}$ in (3.72) and then apply (2.4) to $V(\cdot, Du) \in W^{t,2q/p}(8B)$, which gives

$$\begin{aligned}\int_B |\tau_h(V(\cdot, Du))|^{\frac{2q}{p}}\, dx &\le c|h|^{\beta\delta\frac{2q}{p}}\lambda_0(8B) + c|h|^{\beta t\frac{2q}{p}+(1-\beta)\frac{2q}{p}}[V(\cdot, Du)]^{2q/p}_{t,2q/p;8B}\\ &\le c\left[|h|^{\beta\delta\frac{2q}{p}} + |h|^{\beta t\frac{2q}{p}+(1-\beta)\frac{2q}{p}}\right]\lambda_t(8B),\end{aligned}$$

where we have set

$$\lambda_t(S) := \lambda_0(S) + \chi_{\{t>0\}}[V(\cdot, Du)]^{2q/p}_{t,2q/p;S}.$$

Observe that this set function is countably super-additive. Thus, we apply the covering argument from [164, Lemma 6.2] to get, for any $0 \le t < \delta$,

$$\int_{\Omega'} |\tau_h(V(\cdot, Du))|^{\frac{2q}{p}}\, dx \le c\left[|h|^{\beta\delta\frac{2q}{p}} + |h|^{(\beta t+1-\beta)\frac{2q}{p}}\right]\lambda_t(\Omega''). \tag{3.73}$$

Now we take $\beta = \gamma(t)/\delta$ so that $\beta t + 1 - \beta = \beta\delta$. Observe that this is admissible, since $t < \delta$ implies $\gamma(t)/\delta < 1$. In turn, for any h as in (3.71), estimate (3.73) becomes

$$\int_{\Omega'} |\tau_h(V(\cdot, Du))|^{\frac{2q}{p}}\, dx \le c_0|h|^{\gamma(t)\frac{2q}{p}}\lambda_t(\Omega'')$$

for $c_0 \equiv c_0(\texttt{data}, q)$. Considering the remaining case $d < |h| < \operatorname{dist}(\Omega', \partial\Omega'')$ and trivially estimating as

$$\begin{aligned}&\int_{\Omega'} |\tau_h(V(\cdot, Du))|^{\frac{2q}{p}}\, dx\\ &\le \frac{c}{d^{\gamma(t)2q/p}}\int_{\Omega'}\left(|V(x+h, Du(x+h))|^{\frac{2q}{p}} + |V(x, Du)|^{\frac{2q}{p}}\right) dx\\ &\le \frac{c}{d^{\gamma(t)2q/p}}\int_{\Omega''}|V(x, Du)|^{\frac{2q}{p}}\, dx = \frac{c}{d^{\gamma(t)2q/p}}\int_{\Omega''}[G(x, |Du|)]^{\frac{q}{p}}\, dx,\end{aligned}$$

we eventually obtain (3.70), which with Lemma 2.2.7 implies (3.69). Since the open subsets considered are arbitrary, the proof is complete. □

Proof of Theorem 3.2.2. We prove estimate (3.48), which with a standard covering argument gives (3.47). Moreover, in the proof, we argue without loss of generality under the additional regularity assumption (3.56). Indeed, once we establish (3.48) for approximating solutions $\{u_k\}$ for a SOLA u as described in Definition 3.2.1, a standard approximation procedure along with the strong convergence (3.45) will give the same estimate for u, thereby completing the proof.

Now the proof follows by iterating Lemma 3.2.11. The basic strategy is similar to that of [164, Lemma 6.3], so here we give a sketch of the iterating process. We define two sequences $\{s_k\}_{k\geq 1}$ and $\{t_k\}_{k\geq 1}$ inductively by

$$s_1 := \frac{\delta}{4(\delta+1)}, \quad t_1 = 2s_1, \quad s_{k+1} := \gamma(s_k) \quad \text{and} \quad t_{k+1} := \frac{\gamma(s_k)+\gamma(t_k)}{2}.$$

Then it follows that

$$t \in (0,\delta) \implies \gamma(t) \in (t,\delta) \qquad \text{and} \qquad \gamma(\delta) = \delta,$$

and that

$$s_k \nearrow \delta, \qquad s_k < t_k < \delta, \qquad \text{and so} \qquad t_k \nearrow \delta.$$

Therefore, applying Lemma 3.2.11 with the choice $t = t_k$ for each $k \in \mathbb{N}$, we can show that

$$V(\cdot, Du) \in W^{t_k, 2q/p}_{\mathrm{loc}}(\Omega; \mathbb{R}^n) \qquad \forall\, k \in \mathbb{N},$$

which with the convergence $t_k \nearrow \delta$ gives

$$V(\cdot, Du) \in W^{t, 2q/p}_{\mathrm{loc}}(\Omega; \mathbb{R}^n) \qquad \forall\, t \in [0,\delta).$$

Moreover, we have the estimate

$$\begin{aligned}
[V(\cdot, Du)]^{2q/p}_{t,2q/p;\Omega'} &\leq c \int_\Omega G(x,|Du|)^{\frac{q}{p}}\, dx + c[|\mu|(\Omega)]^{\frac{q}{p-1}} \\
&\overset{(3.68)}{\leq} c[|\mu|(\Omega)]^{\frac{q}{p-1}}.
\end{aligned}$$

Finally, taking $q = p-1$ so that $\delta = p/[2(p-1)]$, the desired estimate (3.48) follows. □

3.3 Fractional differentiability for parabolic measure data problems

3.3.1 Main results

In this section, we consider the following Cauchy-Dirichlet problem:

$$\begin{cases} \partial_t u - \operatorname{div} A(x,t,Du) = \mu & \text{in } \Omega_T, \\ u = 0 & \text{on } \partial_p \Omega_T, \end{cases} \tag{3.74}$$

where μ is a signed Borel measure on Ω_T with finite total mass $|\mu|(\Omega_T) < \infty$. The Carathéodory vector field $A : \Omega_T \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to satisfy

$$\begin{cases} (A(x,t,\xi_1) - A(x,t,\xi_2)) \cdot (\xi_1 - \xi_2) \geq \nu |\xi_1 - \xi_2|^2, \\ |A(x,t,\xi_1) - A(x,t,\xi_2)| \leq L|\xi_1 - \xi_2|, \\ |A(x,t,0)| \leq Ls \end{cases} \tag{3.75}$$

for all $\xi_1, \xi_2 \in \mathbb{R}^n, x \in \Omega$ and $t \in (-T, 0)$, where $0 < \nu \leq L$ and $s \geq 0$.

Our main regularity assumption on the coefficient is the following:

$$|A(x_1,t,\xi) - A(x_2,t,\xi)| \leq L|x_1 - x_2|^\alpha \left(\kappa(x_1,t) + \kappa(x_2,t)\right)(s + |\xi|) \tag{3.76}$$

for all $x_1, x_2 \in \Omega, t \in (-T, 0)$ and $\xi \in \mathbb{R}^n$, where $\alpha \in (0,1]$ is fixed and $\kappa : \Omega_T \to \mathbb{R}$ is a nonnegative function satisfying

$$\kappa \in L^\gamma(\Omega_T) \;\text{ for some }\; \gamma \geq \frac{n+2}{\alpha}. \tag{3.77}$$

We note that (3.76) and (3.77) are the parabolic analogs of those in [55], see also Remark 3.3.4 below. In fact, such an assumption, related to Calderón spaces introduced in [92], is another way to measure fractional differentiability of the coefficients; in particular, Hölder continuous coefficients are allowed.

As in the elliptic case, we consider SOLA.

Definition 3.3.1. *A function $u \in L^1(-T, 0; W_0^{1,1}(\Omega))$ is a SOLA to* (3.74) *under assumptions* (3.75) *if $A(\cdot, Du) \in L^1(\Omega_T; \mathbb{R}^n)$,*

$$\int_{\Omega_T} (-u\varphi_t + A(x,t,Du) \cdot D\varphi)\, dz = \int_{\Omega_T} \varphi \, d\mu \qquad \forall\, \varphi \in C_0^\infty(\Omega_T),$$

and moreover there exists a sequence of weak solutions $\{u_k\} \subset C(-T,0;L^2(\Omega)) \cap L^2(-T,0;W_0^{1,2}(\Omega))$ *to the problems*

$$\begin{cases} \partial_t u_k - \operatorname{div} A(x,t,Du_k) = \mu_k & \text{in } \Omega_T, \\ u_k = 0 & \text{on } \partial_p \Omega_T \end{cases}$$

such that $u_k \to u$ *in* $L^1(-T,0;W_0^{1,1}(\Omega))$, *where the sequence* $\{\mu_k\} \subset L^\infty(\Omega_T)$ *converges to* μ *weakly* in the sense of measures and satisfies*

$$\limsup_{k\to\infty} |\mu_k|(Q) \le |\mu|(Q \cup \partial_p Q)$$

for every cylinder $Q = B \times (t_1,t_2) \subset \mathbb{R}^{n+1}$.

In [28], the existence of SOLA was proved for general parabolic p-Laplacian type equations. We remark the following existence result, which is a special case of those in [28].

Proposition 3.3.2. *With the sequence* $\{u_k\}$ *defined as above, there exists a SOLA* $u \in L^1(-T,0;W_0^{1,1}(\Omega))$ *to* (3.74). *Moreover, up to a not relabeled subsequence, it holds that*

$$u_k \to u \quad \text{in} \quad L^q(-T,0;W_0^{1,q}(\Omega) \quad \text{for any} \quad q < 2 - \frac{n}{n+1},$$

together with the following global estimate

$$\|Du\|_{L^q(\Omega_T)} \le c(n,\nu,L,q,|\Omega|,T)\left(|\mu|(\Omega_T) + s\right).$$

Throughout this section, we consider

$$q \in \left[1, \frac{(n+2)\gamma}{(n+1)\gamma+n+2}\right), \quad \delta(q) \coloneqq \frac{n+2}{q} - (n+1). \tag{3.78}$$

Now we state our main result. We denote $\texttt{data} \coloneqq (n,\nu,L,\gamma,\|\kappa\|_{L^\gamma(\Omega_T)})$.

Theorem 3.3.3. *Let* $u \in L^1(-T,0;W_0^{1,1}(\Omega))$ *be a SOLA to* (3.74) *under assumptions* (3.75)-(3.77). *For every* q *satisfying* (3.78), *we have*

$$Du \in W^{\sigma,\sigma/2;q}_{\mathrm{loc}}(\Omega_T;\mathbb{R}^n) \qquad \forall\, \sigma \in [0,\min\{\alpha,\delta(q)\}). \tag{3.79}$$

Moreover, for every parabolic cylinder $Q_R \subset \Omega_T$ with $R \leq 1$, we have

$$\begin{aligned}
&\fint_{I_{R/2}} \fint_{B_{R/2}} \int_{B_{R/2}} \frac{|Du(x,t) - Du(y,t)|^q}{|x-y|^{n+q\sigma}}\, dx\, dy\, dt \\
&\quad + \fint_{B_{R/2}} \fint_{I_{R/2}} \int_{I_{R/2}} \frac{|Du(x,t) - Du(x,s)|^q}{|t-s|^{1+q\sigma/2}}\, dt\, ds\, dx \\
&\leq \frac{c}{R^{q\sigma}} \left(\fint_{Q_R} (|Du| + s)^{\frac{q\gamma}{\gamma-q}}\, dz \right)^{\frac{\gamma-q}{\gamma}} + \frac{c}{R^{q\sigma}} \left[\frac{|\mu|(Q_R)}{R^{n+1}} \right]^q
\end{aligned} \tag{3.80}$$

for a constant $c \equiv c(\mathtt{data}, q, \sigma)$.

Remark 3.3.4. *Consider the following problem:*

$$\begin{cases} \partial_t u - \operatorname{div}\left(\mathfrak{c}(x,t) A(Du)\right) = \mu & \text{in } \Omega_T, \\ u = 0 & \text{on } \partial_p \Omega_T, \end{cases}$$

where the vector field $A(\cdot)$ satisfies (3.75), when obviously recast in the case $A(\cdot)$ is independent of x. Moreover, assume that

$$|\mathfrak{c}(x_1,t) - \mathfrak{c}(x_2,t)| \leq (\kappa(x_1,t) + \kappa(x_2,t))|x_1 - x_2|^\alpha$$

holds for all $x_1, x_2 \in \Omega, t \in (-T, 0)$, with α and κ satisfying (3.77). Then Theorem 3.3.3 holds with Du replaced by $A(Du)$, in the same spirit as in [7, 55]. Moreover, under the assumption $\mu \in L^{1,\theta}(\Omega_T)$ with $2 \leq \theta \leq n$, it is possible to prove fractional Sobolev-Morrey regularity results as in [12].

Assumption (3.76) is similar to the one treated in [156], where κ in (3.76) belongs to $L^{2\chi/(\chi-1)}(\Omega_T)$, with $\chi > 1$ being the higher integrability exponent from Gehring's theory for linear homogeneous systems. The value of χ can be very close to 1, which makes $2\chi/(\chi-1)$ to be potentially very large. On the other hand, our assumption (3.77) is natural and sharp for the desired regularity. Our approach relies on the observation that (3.76)-(3.77) imply the VMO condition, which enables us to apply the known L^p-theory, see Lemma 3.3.6. We also point out that Theorem 3.3.3 is a natural extension of [19, Theorem 1.2], where $\alpha = 1$ and $\gamma = \infty$.

3.3.2 Preliminaries

In this section, we use the following notations for parabolic problems:

- We denote a typical point in $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ by $z = (x, t)$.
- A standard parabolic cylinder is denoted by

$$Q_R(z_0) := B_R(x_0) \times I_R(t_0) := B_R(x_0) \times (t_0 - R^2, t_0 + R^2).$$

  We shall omit the center when it is clear from the context.
- For a function $f : \mathbb{R}^{n+1} \to \mathbb{R}^k$ with $k \geq 1$ and $h \in \mathbb{R}$, we denote $\tau_{i,h} f(x,t) := f(x+he_i, t) - f(x,t)$ and $\tau_{t,h} f(x,t) := f(x, t+h) - f(x,t)$, where $\{e_i\}_{i=1}^n$ is the standard basis for $\mathbb{R}^n$.
- For $E \subset \Omega$ and $(t_1, t_2) \subset (-T, 0)$, the parabolic boundary of $C := E \times (t_1, t_2)$ is denoted by

$$\partial_p C := E \times \{t_1\} \cup \partial E \times (t_1, t_2).$$

- For a measurable set $C \subset \mathbb{R}^{n+1}$ and $f \in L^1(C)$, we denote

$$(f)_C := \fint_C f\,dz := \frac{1}{|C|} \int_C f\,dz.$$

- For a measurable set $E \subset \mathbb{R}^n$ and a function $g : E \times (-T, 0) \to \mathbb{R}^k$ such that $g(\cdot, t) \in L^1(E)$ for each fixed $t \in (-T, 0)$, we denote

$$(g(\cdot,t))_E := \fint_E g(x,t)\,dx := \frac{1}{|E|} \int_E g(x,t)\,dx.$$

Here we briefly recall the definition and basic properties of parabolic fractional Sobolev spaces, see [19, Section 4] for details. Let $\theta \in (0,1)$ and $q \geq 1$. Then we say that

$$g \in W^{\theta,\theta/2;q}(\Omega_T) \text{ if } g \in L^q(-T, 0; W^{\theta,q}(\Omega)) \text{ and } g \in L^q(\Omega; W^{\theta/2,q}(-T, 0)).$$

It is a Banach space endowed with the norm

$$\|g\|_{W^{\theta,\theta/2;q}(\Omega_T)} := \|g\|_{L^q(\Omega_T)} + [g]_{W^{\theta,\theta/2;q}(\Omega_T)},$$

where the seminorm $[\cdot]_{W^{\theta,\theta/2;q}(\Omega_T)}$ is defined by

$$[g]^q_{W^{\theta,\theta/2;q}(\Omega_T)} := \int_{-T}^0 \int_\Omega \int_\Omega \frac{|g(x,t)-g(y,t)|^q}{|x-y|^{n+\theta q}}\,dx\,dy\,dt + \int_\Omega \int_{-T}^0 \int_{-T}^0 \frac{|g(x,t)-g(x,s)|^q}{|t-s|^{1+\theta q/2}}\,dt\,ds\,dx.$$

We also note the following Poincaré type inequality for parabolic fractional Sobolev spaces: if Q_R is a parabolic cylinder and $g \in W^{\theta,\theta/2;q}(Q_R)$, then

$$\int_{Q_R} |g-(g)_{Q_R}|^q\,dz \le c(n,q)R^{\theta q}[g]^q_{W^{\theta,\theta/2;q}(Q_R)} \tag{3.81}$$

holds, see [19, 108]. Also, the following result from [19, Corollary 4.5] shows a difference quotient characterization of parabolic fractional Sobolev spaces.

Lemma 3.3.5. *Let $g \in L^q(\Omega_T)$ for some $q \ge 1$. Assume that for $\bar\theta \in (0,1]$, $S > 0$ and bounded open sets $\Omega_1 \times J_1 \Subset \Omega_2 \times J_2 \Subset \Omega_T$, we have*

$$\|\tau_{t,h^2}g\|_{L^q(\Omega_2\times J_2)} + \sum_{i=1}^n \|\tau_{i,h}g\|_{L^q(\Omega_2\times J_2)} \le Sh^{\bar\theta}$$

for every $1 \le i \le n$ and every $h \in \mathbb{R}$ satisfying $0 < |h| \le d_1$, where

$$d_1 := \min\{1, \operatorname{dist}(\Omega_1,\partial\Omega_2), \operatorname{dist}(\Omega_2,\partial\Omega), \operatorname{dist}(J_1,\partial J_2), \operatorname{dist}(J_2,\partial(-T,0))\}/2.$$

Then $g \in W^{\theta,\theta/2;q}(\Omega_1\times J_1)$ for any $\theta \in (0,\bar\theta)$, with the estimate

$$[g]_{W^{\theta,\theta/2;q}(\Omega_1\times J_1)} \le c\left(S + \|g\|_{L^q(\Omega_2\times J_2)}\right)$$

for some positive constant c depending only on q, $\bar\theta-\theta$, d_1, $|\Omega|$ and T.

3.3.3 Some technical results

In this section, we prove a comparison estimate between (3.74) and a homogeneous frozen problem, by adopting the ideas and techniques from [55]. In view of the Definition 3.3.1 and Proposition 3.3.2, we first assume that

$$\mu \in L^\infty(\Omega_T), \quad u \in L^2(-T,0;W_0^{1,2}(\Omega)). \tag{3.82}$$

These assumptions will be removed in Section 3.3.4

We fix a parabolic cylinder $Q_{2R} \equiv Q_{2R}(z_0) \Subset \Omega_T$ with $R \leq 1$, and consider the following homogeneous problem:

$$\begin{cases} \partial_t w - \operatorname{div} A(x,t,Dw) = 0 & \text{in } Q_{2R}, \\ \qquad\qquad w = u & \text{on } \partial_p Q_{2R}. \end{cases} \tag{3.83}$$

We first observe that

$$\begin{aligned} &\lim_{\rho\to 0}\left(\sup_{z'\in\mathbb{R}^{n+1}} \fint_{Q_\rho(z')} \sup_{\xi\in\mathbb{R}^n\setminus\{0\}} \frac{|A(x,t,\xi)-(A(\cdot,t,\xi))_{B_\rho(x)}|}{s+|\xi|}\,dx\,dt\right) \\ &\overset{(3.76)}{\leq} \lim_{\rho\to 0}\left(\rho^\alpha \sup_{z'\in\mathbb{R}^{n+1}} \fint_{Q_\rho(z')} \fint_{B_\rho(x)} (\kappa(x,t)+\kappa(x',t))\,dx'\,dx\,dt\right) \\ &\leq\ c\lim_{\rho\to 0}\rho^{\alpha-\frac{n+2}{\gamma}} \sup_{z'\in\mathbb{R}^{n+1}} \|\kappa\|_{L^\gamma(Q_{2\rho}(z'))} = 0, \end{aligned}$$

where we have used the fact that $\gamma \geq (n+2)/\alpha$ and $\kappa \in L^\gamma(\Omega_T)$. Namely, assumptions (3.76)-(3.77) imply that $x \mapsto A(x,t,\xi)/(s+|\xi|)$ is VMO-regular. Under the VMO condition, it is well known that Dw belongs to $L^p_{\mathrm{loc}}(Q_{2R})$ for all $p \in (1,\infty)$, see [3, Theorem 1] and [112, Theorem 1.8]. By further using the self-improving property of reverse Hölder's inequalities, [19, Lemma 3.2], we have the following estimate.

Lemma 3.3.6. *Let $w \in u+L^2(I_{2R};W_0^{1,2}(B_{2R}))$ be the weak solution to (3.83). Then for any $p \in [1,\infty)$, there exists a constant $c \equiv c(\mathtt{data},p)$ satisfying*

$$\left(\fint_{Q_R} (|Dw|+s)^p\,dz\right)^{\frac{1}{p}} \leq c\fint_{Q_{2R}} (|Dw|+s)\,dz. \tag{3.84}$$

Next, we define $\tilde{A}(t,\xi) := (A(\cdot,t,\xi))_{B_R}$ and consider the frozen problem:

$$\begin{cases} \partial_t v - \operatorname{div} \tilde{A}(t,Dv) = 0 & \text{in } Q_R, \\ \qquad\qquad v = w & \text{on } \partial_p Q_R. \end{cases} \tag{3.85}$$

In the following lemma, we derive a comparison estimate.

Lemma 3.3.7. *Let u and v be as in* (3.74) *and* (3.85)*, respectively. Then for any q satisfying* $(3.78)_1$*, there exists a constant $c \equiv c(\texttt{data}, q)$ such that*

$$\int_{Q_R} |Du - Dv|^q \, dz \le cR^{q\min\{\alpha,\delta(q)\}} \Bigg([|\mu|(Q_{2R})]^q + K^{-\frac{\gamma}{q}} \int_{Q_{2R}} |\kappa|^\gamma \, dz + K^{\frac{\gamma}{\gamma-q}} \int_{Q_{2R}} (|Du| + s)^{\frac{q\gamma}{\gamma-q}} \, dz \Bigg)$$

holds whenever $K > 0$.

Proof. We first recall the following estimate from [19, Lemma 6.4]:

$$\int_{Q_{2R}} |Du - Dw|^q \, dz \le cR^{q\delta(q)} [|\mu|(Q_{2R})]^q. \tag{3.86}$$

Next, we test (3.83) and (3.85) with $w - v$ in order to get

$$\begin{aligned} 0 &= \int_{Q_R} \partial_t (w - v)(w - v) \, dz + \int_{Q_R} (A(x, t, Dw) - \tilde{A}(t, Dv)) \cdot (Dw - Dv) \, dz \\ &=: I_1 + I_2. \end{aligned}$$

By using Steklov formulation, we see that

$$I_1 \ge 0, \quad \text{and so} \quad I_2 \le 0. \tag{3.87}$$

Now we have

$$\begin{aligned} \nu \fint_{Q_R} |Dw - Dv|^2 \, dz &\overset{(3.75)}{\le} \fint_{Q_R} (\tilde{A}(t, Dw) - \tilde{A}(t, Dv)) \cdot (Dw - Dv) \, dz \\ &\overset{(3.87)}{\le} \fint_{Q_R} (\tilde{A}(t, Dw) - A(x, t, Dw)) \cdot (Dw - Dv) \, dz \\ &= \fint_{Q_R} \fint_{B_R} (A(y, t, Dw) - A(x, t, Dw)) \cdot (Dw - Dv) \, dy \, dz \\ &\overset{(3.76)}{\le} cR^\alpha \fint_{Q_R} (\kappa(x, t) + (\kappa(\cdot, t))_{B_R})(|Dw| + s)|Dw - Dv| \, dz \\ &\le cR^{2\alpha} \fint_{Q_R} (\kappa(x, t) + (\kappa(\cdot, t))_{B_R})^2 (|Dw| + s)^2 \, dz + \frac{\nu}{2} \fint_{Q_R} |Dw - Dv|^2 \, dz, \end{aligned}$$

where we have used Young's inequality in the last line. In turn, we obtain

$$\fint_{Q_R} |Dw - Dv|^2 \, dz \le cR^{2\alpha} \fint_{Q_R} (\kappa(x,t) + (\kappa(\cdot,t))_{B_R})^2 (|Dw| + s)^2 \, dz.$$

Then we use Hölder's inequality to discover

$$\begin{aligned}
\fint_{Q_R} |Dw - Dv|^q \, dz &\le \left(\fint_{Q_R} |Dw - Dv|^2 \, dz \right)^{\frac{q}{2}} \\
&\le cR^{q\alpha} \left(\fint_{Q_R} \left(\kappa(x,t) + (\kappa(\cdot,t))_{B_R}\right)^2 (|Dw| + s)^2 \, dz \right)^{\frac{q}{2}} \\
&\le cR^{q\alpha} \left(\fint_{Q_{2R}} |\kappa|^\gamma \, dz \right)^{\frac{q}{\gamma}} \left(\fint_{Q_R} (|Dw| + s)^{\frac{2\gamma}{\gamma-2}} \, dz \right)^{\frac{q(\gamma-2)}{2\gamma}} \\
&\overset{(3.84)}{\le} cR^{q\alpha} \left(\fint_{Q_{2R}} |\kappa|^\gamma \, dz \right)^{\frac{q}{\gamma}} \left(\fint_{Q_{2R}} (|Dw| + s)^{\frac{q\gamma}{\gamma-q}} \, dz \right)^{\frac{\gamma-q}{\gamma}}.
\end{aligned}$$

Using Young's inequality, we arrive at

$$\begin{aligned}
\int_{Q_R} |Dw - Dv|^q \, dz &\le cR^{q\alpha} \|\kappa\|^q_{L^\gamma(Q_{2R})} \left(\int_{Q_{2R}} (|Dw| + s)^{\frac{q\gamma}{\gamma-q}} \, dz \right)^{\frac{\gamma-q}{\gamma}} \\
&\le cR^{q\alpha} \|\kappa\|^q_{L^\gamma(Q_{2R})} \left(\int_{Q_{2R}} (|Du| + s)^{\frac{q\gamma}{\gamma-q}} \, dz \right)^{\frac{\gamma-q}{\gamma}} \\
&\quad + cR^{q\alpha} \|\kappa\|^q_{L^\gamma(Q_{2R})} \left(\int_{Q_{2R}} |Du - Dw|^{\frac{q\gamma}{\gamma-q}} \, dz \right)^{\frac{\gamma-q}{\gamma}} \\
&\le cR^{q\alpha} \left(K^{-\frac{\gamma}{q}} \|\kappa\|^\gamma_{L^\gamma(Q_{2R})} + K^{\frac{\gamma}{\gamma-q}} \int_{Q_{2R}} (|Du| + s)^{\frac{q\gamma}{\gamma-q}} \, dz \right) \\
&\quad + cR^{q\alpha} \|\kappa\|^q_{L^\gamma(\Omega_T)} \left(\int_{Q_{2R}} |Du - Dw|^{\frac{q\gamma}{\gamma-q}} \, dz \right)^{\frac{\gamma-q}{\gamma}}.
\end{aligned}$$

This estimate and (3.86) imply the desired estimate. □

We now recall a fractional regularity estimate for (3.85), which follows from [19, Lemma 7.1] and Hölder's inequality.

Lemma 3.3.8. *Let $v \in L^2(I_R; W_0^{1,2}(Q_R))$ be the weak solution to* (3.85)*. Then for each q satisfying* (3.78)*, there exists a constant $c \equiv c(\mathtt{data}, q)$ such that*

$$\int_{Q_{R/4}} |D^2 v|^q \, dz + \int_{Q_{R/8}} \frac{|\tau_{t,h} Dv|^q}{|h|^{q/2}} \, dz \leq \frac{c}{R^q} \int_{Q_R} |Dv - z_0|^q \, dz$$

holds for every $z_0 \in \mathbb{R}^n$.

3.3.4 Proof of Theorem 3.3.3

We prove (3.80), which with a standard covering argument gives (3.79). Moreover, without loss of generality, we confine ourselves to the situation when (3.82) holds. Once estimate (3.80) is obtained for regular solutions, a standard approximation process as the one presented in the proof of [19, Theorem 1.2] gives the desired estimate for a SOLA.

We denote $m := \min\{\alpha, \delta(q)\}$, where $\delta(q)$ is given in $(3.78)_2$. Then we define for $\theta \in [0, m+1)$

$$\omega(\theta) := \frac{m}{m+1-\theta}$$

and

$$\begin{aligned} \lambda_{K,\theta}(C) := [|\mu|(C)]^q + K^{-\frac{\gamma}{q}} \int_C |\kappa|^\gamma \, dz + K^{\frac{\gamma}{\gamma-q}} \int_C (|Du| + s)^{\frac{q\gamma}{\gamma-q}} \, dz \\ + \chi_{\{\theta>0\}} [Du]^q_{W^{\theta,\theta/2;q}(C)}. \end{aligned} \tag{3.88}$$

Moreover in the next lemma, we shall deal with fixed open sets

$$\Omega_0 \times J_0 \Subset \Omega_1 \times J_1 \Subset \Omega_2 \times J_2 \Subset \Omega_T \tag{3.89}$$

such that

$$\operatorname{dist}(\Omega_0, \partial\Omega_1) \approx \operatorname{dist}(\Omega_1, \partial\Omega_2) =: d_1, \quad \operatorname{dist}(J_0, \partial J_1) \approx \operatorname{dist}(J_1, \partial J_2) =: d_2.$$

We now conduct a bootstrap argument.

Lemma 3.3.9. *Assume that for open sets as in* (3.89) *it holds that $Du \in W^{\theta,\theta/2;q}(\Omega_2 \times J_2)$ for some $\theta \in [0, m)$, and that*

$$[Du]^q_{W^{\theta,\theta/2;q}(\Omega_1 \times J_1)} \leq c_1 \lambda_{K,0}(\Omega_2 \times J_2). \tag{3.90}$$

Then it follows that $Du \in W^{\tilde{\theta},\tilde{\theta}/2;q}(\Omega_0 \times J_0)$ *for every* $\tilde{\theta} \in (0, \omega(\theta))$, *and moreover there exists a constant* c_2, *depending only on* $\texttt{data}, q, d_1, d_2, \tilde{\theta}, c_1$, *such that*

$$[Du]^q_{W^{\tilde{\theta},\tilde{\theta}/2;q}(\Omega_0\times J_0)} \le c_2 \lambda_{K,0}(\Omega_2 \times J_2). \tag{3.91}$$

Proof. We take $\beta \in (0,1)$ to be chosen later, and let $h \in \mathbb{R}$ be such that

$$0 < |h| < \min\left\{ \left(\frac{d_1}{1000\sqrt{n}}\right)^{\frac{1}{\beta}}, \left(\frac{\sqrt{d_2}}{1000}\right)^{\frac{1}{\beta}}, \frac{1}{1000\sqrt{n}} \right\}. \tag{3.92}$$

We take $z_0 \in \Omega_0 \times J_0$, and fix a parabolic cylinder $Q \equiv Q_{|h|^\beta}(z_0)$. Let w and v be weak solutions to (3.83) and (3.85), respectively, with $Q_{2R} \equiv 16Q$. We then use Lemma 3.3.7, Lemma 3.3.8 and (3.81) to discover that

$$\begin{aligned}
&\sum_{i=1}^n \int_Q |\tau_{i,h} Du|^q \, dz + \int_Q |\tau_{t,h^2} Du|^q \, dz \\
&\le c \sum_{i=1}^n \int_Q |\tau_{i,h} Dv|^q \, dz + c \int_Q |\tau_{t,h^2} Dv|^q \, dz + c \int_{8Q} |Du - Dv|^q \, dz \\
&\le c|h|^q \left(\int_{2Q} |D^2 v|^q \, dz + \int_Q \frac{|\tau_{t,h^2} Dv|^q}{|h|^q} \, dz \right) + \int_{8Q} |Du - Dv|^q \, dz \\
&\le c|h|^{q(1-\beta)} \int_{8Q} |Du - Dv|^q \, dz + c|h|^{q(1-\beta)} \int_{8Q} |Du - (Du)_{8Q}|^q \, dz \\
&\quad + c \int_{8Q} |Du - Dv|^q \, dz \\
&\le c \int_{8Q} |Du - Dv|^q \, dz + c|h|^{q(1-\beta)} \int_{8Q} |Du - (Du)_{8Q}|^q \, dz \\
&\le c|h|^{q\beta m} \lambda_{K,0}(16Q) + c|h|^{q(1-\beta)+q\beta\theta} [Du]_{W^{\theta,\theta/2;q}(8Q)} \\
&\le c \left[|h|^{q\beta m} + |h|^{q(1-\beta+\beta\theta)} \right] \lambda_{K,\theta}(16Q),
\end{aligned} \tag{3.93}$$

where we used the fact that since $|h| < 1$ and $\beta < 1$, $|h|^{q(1-\beta)} \le 1$. Next, we take $\beta = 1/(m+1-\theta)$ so that $\beta m = 1 - \beta + \beta\theta = \omega(\theta)$. This is admissible since $\theta < m$ implies $\beta < 1$. Then (3.93) becomes

$$\sum_{i=1}^n \int_Q |\tau_{i,h} Du|^q \, dz + \int_Q |\tau_{t,h^2} Du|^q \, dz \le c|h|^{q\omega(\theta)} \lambda_{K,\theta}(16Q) \tag{3.94}$$

for $c \equiv c(\texttt{data}, d_1, d_2, c_1)$, where $\lambda_{K,\theta}(\cdot)$ is given in (3.88) and $Q \equiv Q_{|h|^\beta}(z_0)$ with $h \in \mathbb{R}^n$ satisfying (3.92).

Now we proceed with a covering argument similar to those in [19, 164]. Let $\{C_j\}_{j=1}^N$ be a disjoint family of cuboids parallel to the coordinate axis and centers (y_j, s_j), so that the union of $\{C_j\}_{j=1}^N$ covers $\Omega_0 \times J_0$:

$$C_j := \left\{x \in \mathbb{R}^n : \max_{1 \le i \le n} |x_i - (y_j)_i| < \sqrt{n}|h|^\beta \right\} \times (s_j - |h|^{2\beta}, s_j + |h|^{2\beta})$$

and

$$\bigcup_{j=1}^N C_j \supset \Omega_0 \times J_0.$$

For each C_j, we choose the smallest open cylinder Q_j satisfying $C_j \subset Q_j \subset 16Q_j \subset \Omega_2 \times J_2$. By construction, we have that the dilated cylinders $16Q_j$ intersects each other no more than a fixed number, say $\mathcal{H} \equiv \mathcal{H}(n)$. Also note that the set function $\lambda_{K,\theta}(\cdot)$ is countably super-additive, to discover

$$\begin{aligned}
&\sum_{i=1}^n \int_{\Omega_0 \times J_0} |\tau_{i,h} Du|^q \, dz + \int_{\Omega_0 \times J_0} |\tau_{t,h^2} Du|^q \, dz \\
&\le \sum_{j=1}^N \left(\sum_{i=1}^n \int_{Q_j} |\tau_{i,h} Du|^q \, dz + \int_{Q_j} |\tau_{t,h^2} Du|^q \, dz \right) \\
&\overset{(3.94)}{\le} c|h|^{q\omega(\theta)} \sum_{j=1}^N \lambda_{K,\theta}(16Q_j) \\
&\le c\mathcal{H}|h|^{q\omega(\theta)} \lambda_{K,\theta}(\Omega_2 \times J_2) \\
&\overset{(3.90)}{\le} c|h|^{q\omega(\theta)} \lambda_{K,0}(\Omega_2 \times J_2).
\end{aligned}$$

Finally, (3.91) follows from Lemma 3.3.5. □

In particular, choosing

$$K = \left(\int_{\Omega_2 \times J_2} |\kappa|^\gamma \, dz \right)^{\frac{q(\gamma - q)}{\gamma^2}} \left(\int_{\Omega_2 \times J_2} (|Du| + s)^{\frac{q\gamma}{\gamma - q}} \, dz \right)^{-\frac{q(\gamma - q)}{\gamma^2}}$$

in the above lemma gives:

Lemma 3.3.10. *Under the setting of Lemma 3.3.9, assume that*

$$[Du]^q_{W^{\theta,\theta/2;q}(\Omega_1\times J_1)} \le c_1[|\mu|(\Omega_2\times J_2)]^q + c_1\left(\int_{\Omega_2\times J_2}(|Du|+s)^{\frac{q\gamma}{\gamma-q}}\,dz\right)^{\frac{\gamma-q}{\gamma}}$$

for some $\theta\in[0,m)$, then we have

$$[Du]^q_{W^{\tilde\theta,\tilde\theta/2;q}(\Omega_0\times J_0)} \le c_2[|\mu|(\Omega_2\times J_2)]^q + c_2\left(\int_{\Omega_2\times J_2}(|Du|+s)^{\frac{q\gamma}{\gamma-q}}\,dz\right)^{\frac{\gamma-q}{\gamma}}$$

for all $\tilde\theta\in(0,\omega(\theta))$, with the constant c_2 given in Lemma 3.3.9.

We now prove Theorem 3.3.3. It suffices to derive estimate (3.80).

Proof of Theorem 3.3.3. By a standard scaling argument, we may assume that $R=1$ and

$$\left(\int_{Q_1}(|Du|+s)^{\frac{q\gamma}{\gamma-q}}\,dz\right)^{\frac{\gamma-q}{\gamma}} + [|\mu|(Q_1)]^q \le 1.$$

Then we assert that

$$[Du]_{W^{\sigma,\sigma/2;q}(Q_{1/2})} \le c(\mathtt{data},q). \tag{3.95}$$

We consider two sequences $\{t_k\}$ and $\{s_k\}$ defined by

$$t_0 := \omega(0)/2,\quad t_{k+1} := \omega(t_k),\quad s_0 := \omega(0)/4,\quad s_{k+1} := (\omega(t_k)+\omega(s_k))/2,$$

which satisfy

$$t_k \nearrow m,\qquad t_k<s_k<m,\qquad \text{and so}\qquad s_k\nearrow m.$$

Then by iterating Lemma 3.3.10, we conclude that

$$Du\in W^{\sigma,\sigma/2;q}(Q_{1/2})\quad \forall\,\sigma\in[0,m),$$

with estimate (3.95). Scaling back to Q_R, we finish the proof. □

Chapter 4

Elliptic obstacle problems with measure data

In this chapter, we consider obstacle problems related to

$$-\operatorname{div} A(Du) = \mu \quad \text{in } \Omega,$$

where the vector field $A : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be C^1-regular on $\mathbb{R}^n$ for $p \geq 2$ and on $\mathbb{R}^n \setminus \{0\}$ for $p < 2$. It also satisfies the growth and ellipticity assumptions (2.8). We also denote $\texttt{data} = (n, p, \nu, L)$.

For a given boundary data $g \in W^{1,p}(\Omega)$, we set

$$\mathcal{T}_g^{1,p}(\Omega) \coloneqq \left\{ u : \Omega \to \mathbb{R} \mid T_t(u - g) \in W_0^{1,p}(\Omega) \text{ for every } t > 0 \right\}.$$

For any $u \in \mathcal{T}_g^{1,p}(\Omega)$, there exists a unique measurable map $Z_u : \Omega \to \mathbb{R}^n$ such that

$$D[T_t(u)] = \chi_{\{|u|<t\}} Z_u \qquad \text{a.e. in } \Omega$$

for every $t > 0$, see [23, Lemma 2.1]. If $u \in \mathcal{T}_g^{1,p}(\Omega) \cap W^{1,1}(\Omega)$, then Z_u coincides with the weak derivative Du of u. In what follows, we denote Z_u by Du for the simplicity of notation.

In Sections 4.1 and 4.2, we assume (2.8) with

$$p > 2 - \frac{1}{n}, \tag{4.1}$$

which ensures $u \in W^{1,1}(\Omega)$.

4.1 Potential estimates for obstacle problems with measure data

In this section, we consider the obstacle problem $OP(\psi;\mu)$ with the constraint $u \geq \psi$ a.e. in Ω, where $\psi \in W^{1,p}(\Omega)$ is a given obstacle. If $p > n$, then it follows that $\mu \in W^{-1,p'}(\Omega)$ by the Morrey embedding theorem, and $OP(\psi;\mu)$ is characterized by the variational inequality

$$\int_\Omega A(Du) \cdot D(\phi - u)\, dx \geq \int_\Omega (\phi - u)\, d\mu \tag{4.2}$$

for every $\phi \in u + W_0^{1,p}(\Omega)$ with $\phi \geq \psi$ a.e. in Ω. Moreover, the existence of its unique weak solution follows from monotone operator theory [134]. On the other hand, as in the case of obstacle-free problems, when $p \leq n$ such a variational inequality is not available for $OP(\psi;\mu)$. In this case, we adopt the notion of *limits of approximating solutions* introduced in [189].

Definition 4.1.1. *Suppose that an obstacle $\psi \in W^{1,p}(\Omega)$, measure data $\mu \in \mathcal{M}_b(\Omega)$ and boundary data $g \in W^{1,p}(\Omega)$ with $(\psi - g)_+ \in W_0^{1,p}(\Omega)$ are given. We say that a function $u \in \mathcal{T}_g^{1,p}(\Omega)$ with $u \geq \psi$ a.e. in Ω is a limit of approximating solutions to the obstacle problem $OP(\psi;\mu)$ under assumptions* (2.8) *if there exist a sequence of functions $\{\mu_k\} \subset W^{-1,p'}(\Omega) \cap L^1(\Omega)$ with*

$$\begin{cases} \mu_k \overset{*}{\rightharpoonup} \mu \ \ \text{in } \mathcal{M}_b(\Omega), \\ \displaystyle\limsup_{k\to\infty} |\mu_k|(B) \leq |\mu|(\bar{B}) \quad \text{for every ball } B \subset \mathbb{R}^n \end{cases}$$

and weak solutions $u_k \in g + W_0^{1,p}(\Omega)$ with $u_k \geq \psi$ a.e. in Ω to the variational inequalities

$$\int_\Omega A(Du_k) \cdot D(\phi - u_k)\, dx \geq \int_\Omega (\phi - u_k)\, d\mu_k$$

for every $\phi \in u_k + W_0^{1,p}(\Omega)$ with $\phi \geq \psi$ a.e. in Ω, such that

$$\begin{cases} u_k \to u & \text{a.e. in } \Omega, \\ \displaystyle\int_\Omega |u_k - u|^\gamma\, dx \to 0 & \text{for every } 0 < \gamma < \dfrac{n(p-1)}{n-p}, \\ \displaystyle\int_\Omega |Du_k - Du|^q\, dx \to 0 & \text{for every } 0 < q < \dfrac{n(p-1)}{n-1}. \end{cases}$$

4.1.1 Main results

The first result is concerned with the case $2 - 1/n < p \le 2$:

Theorem 4.1.2. *Let $u \in W^{1,1}(\Omega)$ with $u \ge \psi$ a.e. in Ω be a limit of approximating solutions to the problem $OP(\psi;\mu)$ under assumptions* (2.8) *with $2 - 1/n < p \le 2$. If*

$$\lim_{\rho\to 0}\left[\frac{|\mu|(B_\rho(x_0))}{\rho^{n-1}} + \left(\fint_{B_\rho(x_0)} \varphi^*(|A(D\psi) - (A(D\psi))_{B_\rho(x_0)}|)\,dx\right)^{\frac{1}{p'}}\right] = 0 \tag{4.3}$$

holds for a certain $x_0 \in \Omega$, then $A(Du)$ has vanishing mean oscillation at x_0, that is, there holds

$$\lim_{\rho\to 0}\fint_{B_\rho(x_0)} |A(Du) - (A(Du))_{B_\rho(x_0)}|\,dx = 0. \tag{4.4}$$

Moreover, if

$$\mathbf{I}_1^{\mu}(x_0,2R) + \int_0^{2R}\left(\fint_{B_r(x_0)} \varphi^*(|A(D\psi) - (A(D\psi))_{B_r(x_0)}|)\,dx\right)^{\frac{1}{p'}}\frac{dr}{r} < \infty \tag{4.5}$$

holds on a ball $B_{2R}(x_0) \subset \Omega$, then x_0 is a Lebesgue point of $A(Du)$ with the following estimate

$$\begin{aligned}
&|A(Du(x_0)) - (A(Du))_{B_{2R}(x_0)}|\\
&\le c\fint_{B_{2R}(x_0)} |A(Du) - (A(Du))_{B_{2R}(x_0)}|\,dx + c\mathbf{I}_1^{\mu}(x_0,2R)\\
&\quad + c\int_0^{2R}\left(\fint_{B_r(x_0)} \varphi^*(|A(D\psi) - (A(D\psi))_{B_r(x_0)}|)\,dx\right)^{\frac{1}{p'}}\frac{dr}{r},
\end{aligned} \tag{4.6}$$

where $\varphi^*(\cdot)$ is a function defined in Section 2.3.1 and $c \equiv c(\mathtt{data})$.

The second result, concerning the case $p > 2$, is the following:

Theorem 4.1.3. *Let $u \in W^{1,p-1}(\Omega)$ with $u \ge \psi$ a.e. in Ω be a limit of approximating solutions to the problem $OP(\psi;\mu)$ under assumptions* (2.8) *with $p > 2$. If*

$$\mathbf{I}_1^{\mu}(x_0, 2R) + \int_0^{2R} \left(\fint_{B_r(x_0)} \varphi^*(|A(D\psi) - (A(D\psi))_{B_r(x_0)}|)\, dx \right)^{\frac{1}{2}} \frac{dr}{r} < \infty \tag{4.7}$$

holds on a ball $B_{2R}(x_0) \subset \Omega$, then x_0 is a Lebesgue point of $A(Du)$ with the following estimate

$$\begin{aligned} &|A(Du(x_0)) - (A(Du))_{B_{2R}(x_0)}| \\ &\le c \fint_{B_{2R}(x_0)} |A(Du) - (A(Du))_{B_{2R}(x_0)}|\, dx + c\mathbf{I}_1^{\mu}(x_0, 2R) \\ &\quad + c \left[\int_0^{2R} \left(\fint_{B_r(x_0)} \varphi^*(|A(D\psi) - (A(D\psi))_{B_r(x_0)}|)\, dx \right)^{\frac{1}{2}} \frac{dr}{r} \right]^{\frac{2}{p'}} \\ &\quad + c \left(\fint_{B_{2R}(x_0)} (|A(Du)| + s^{p-1})\, dx \right)^{\frac{p-2}{2(p-1)}} \\ &\qquad \cdot \int_0^{2R} \left(\fint_{B_r(x_0)} \varphi^*(|A(D\psi) - (A(D\psi))_{B_r(x_0)}|)\, dx \right)^{\frac{1}{2}} \frac{dr}{r}, \end{aligned} \tag{4.8}$$

where $c \equiv c(\mathtt{data})$.

Remark 4.1.4. *In* (4.6) *and* (4.8)*, the terms related to the obstacle are slightly different from the one in* (1.16)*. This is simply due to the presence of the constant s in* (2.8)*, whose role is to distinguish the degenerate case ($s = 0$) from the non-degenerate one ($s \neq 0$). When $s = 0$, we have $\varphi^*(t) \approx t^{p'}$. Moreover, when $s \neq 0$ and $p \geq 2$, we have $\varphi^*(t) \lesssim t^{p'}$.*

We note that the constant c involved in estimate (4.8) is stable as $p \searrow 2$ (see Remark 4.1.29 below), that is, letting $p \searrow 2$ in (4.8) we obtain the same estimate as the one in (4.6). We conjecture that estimate (4.6) continues to hold when $p > 2$, which is expected to be sharp in view of the results in [35, 145]. The main issue is to handle the obstacle appropriately in the linearization process, see Remark 4.1.25. Also, it would be interesting to extend the range of p in Theorem 4.1.2 to $1 < p \leq 2 - 1/n$ by modifying and developing the approaches in [105, 173, 174, 175]. We further expect that obstacle problems with measure data and Dini continuous coefficients can be further studied by using the techniques developed in [110, 133, 146, 147].

We now give some consequences of the above theorems.

Corollary 4.1.5. *Under the assumptions of Theorems 4.1.2 and 4.1.3, we have the following pointwise estimates.*

(1) If $2-1/n<p\le 2$, *then*

$$|Du(x_0)|^{p-1} \le c \fint_{B_{2R}(x_0)} (|Du|+s)^{p-1}\,dx + c\mathbf{I}_1^{\mu}(x_0,2R) + c\int_0^{2R} \left(\fint_{B_r(x_0)} \varphi^*(|A(D\psi)-(A(D\psi))_{B_r(x_0)}|)\,dx \right)^{\frac{1}{p'}} \frac{dr}{r}.$$

(2) If $p>2$, *then*

$$|Du(x_0)|^{p-1} \le c \fint_{B_{2R}(x_0)} (|Du|+s)^{p-1}\,dx + c\mathbf{I}_1^{\mu}(x_0,2R) + c\left[\int_0^{2R} \left(\fint_{B_r(x_0)} \varphi^*(|A(D\psi)-(A(D\psi))_{B_r(x_0)}|)\,dx \right)^{\frac{1}{2}} \frac{dr}{r} \right]^{\frac{2}{p'}}.$$

Once we have the excess decay estimates in the proof of Theorems 4.1.2 and 4.1.3, we can also obtain the following C^1-regularity criteria by applying the arguments in [109, Theorem 1] and [146, Theorem 4]. We also refer to [147] for a direct proof which does not appeal to potentials.

Corollary 4.1.6. *Under the assumptions of Theorems 4.1.2 and 4.1.3, assume that* μ *satisfies one of the following two conditions:*

(i) $\mu \in L(n,1)$ *locally in* Ω,

(ii) $|\mu|(B_r) \le h(r)r^{n-1}$ *for every ball* $B_r \subset \Omega$, *with* $h:[0,\infty)\to[0,\infty)$ *satisfying*

$$\int_0 h(r)\frac{dr}{r} < \infty,$$

and that $A(D\psi)$ *has Dini mean oscillation, which means that*

$$\int_0 [\omega(r)]^{\frac{1}{\max\{p',2\}}} \frac{dr}{r} < \infty, \quad \text{where } \omega(r) := \sup_{y\in\Omega} \fint_{B_r(y)} \varphi^*(|A(D\psi)-(A(D\psi))_{B_r(y)}|)\,dx.$$

Then Du *is continuous in* Ω.

4.1.2 Reverse Hölder's inequalities for homogeneous obstacle problems

We start this section with the following comparison principle from [189, Lemma 2.1], which enables us to establish comparison estimates between obstacle problems and obstacle-free problems.

Lemma 4.1.7. *Assume that $\mathcal{O} \subset \mathbb{R}^n$ is a bounded domain and that the vector field $A : \mathbb{R}^n \to \mathbb{R}^n$ satisfies* (2.8) *with $p > 1$. Then for functions $w, \psi \in W^{1,p}(\mathcal{O})$ that satisfy*

$$\begin{cases} -\operatorname{div} A(Dw) \geq -\operatorname{div} A(D\psi) & \text{in } \mathcal{O}, \\ \quad w \geq \psi & \text{on } \partial\mathcal{O} \end{cases}$$

in the weak sense, there holds $w \geq \psi$ a.e. in $\mathcal{O}$.

We use the following notations for the admissible sets of $OP(\psi;\mu)$: given an open set $\mathcal{O} \subseteq \Omega$, we denote

$$\mathcal{A}_\psi(\mathcal{O}) \coloneqq \left\{\phi \in W^{1,p}(\mathcal{O}) : \phi \geq \psi \text{ a.e. in } \mathcal{O}\right\}$$

and

$$\mathcal{A}_\psi^g(\mathcal{O}) \coloneqq \left\{\phi \in g + W_0^{1,p}(\mathcal{O}) : \phi \geq \psi \text{ a.e. in } \mathcal{O}\right\} \quad \text{for } g \in \mathcal{A}_\psi(\mathcal{O}).$$

We aim to prove reverse Hölder type inequalities for the following homogeneous obstacle problem:

$$\begin{cases} \displaystyle\int_\Omega A(Dw_1) \cdot D(\phi - w_1)\, dx \geq 0 & \forall\, \phi \in \mathcal{A}_\psi^{w_1}(\Omega), \\ \qquad\qquad w_1 \geq \psi & \text{a.e. in } \Omega. \end{cases} \tag{4.9}$$

In order to establish various estimates suitable in the setting of measure data problems, we need certain reverse Hölder type inequalities. We first recall such results for obstacle-free problems, see [100, Lemma 3.2] and [100, Corollary 3.5] for the proof. Moreover, using the self-improving property of reverse Hölder's inequalities such as the one in [118, Remark 6.12], we state them as follows:

Lemma 4.1.8. *Let $w_2 \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a weak solution to*

$$-\operatorname{div} A(Dw_2) = -\operatorname{div} A(D\psi) \quad \text{in } \Omega, \tag{4.10}$$

under assumptions (2.8) *with $p > 1$. Then for any $\sigma \in (0,1)$ there exists a constant $c \equiv c(\mathtt{data}, \sigma)$ such that*

$$\fint_B |V(Dw_2) - V(z_0)|^2 \, dx \le c \left(\fint_{2B} |V(Dw_2) - V(z_0)|^{2\sigma} \, dx \right)^{\frac{1}{\sigma}} + c \fint_{2B} (\varphi_{|z_0|})^*(|A(D\psi) - A(\xi_0)|) \, dx$$

holds for every $z_0, \xi_0 \in \mathbb{R}^n$, whenever $2B \Subset \Omega$. Moreover, there exists a constant $c \equiv c(\mathtt{data})$ such that

$$\fint_B |V(Dw_2) - V(z_0)|^2 \, dx \le c(\varphi_{|z_0|})^* \left(\fint_{2B} |A(Dw_2) - A(z_0)| \, dx \right) + c \fint_{2B} (\varphi_{|z_0|})^*(|A(D\psi) - A(\xi_0)|) \, dx$$

holds for every $z_0, \xi_0 \in \mathbb{R}^n$, whenever $2B \Subset \Omega$.

In addition to Lemma 4.1.8, we need to establish similar reverse Hölder's inequalities for (4.9) as well. Note that, due to the obstacle constraint, we cannot use the same test functions as in the proof of Lemma 4.1.8 to prove Lemma 4.1.9 below. To overcome this, we first obtain a Caccioppoli type estimate for comparison maps involving (4.10), and then use a comparison estimate in Lemma 4.1.14 between (4.9) and (4.10). We remark that we will prove Lemma 4.1.14 without using any of the lemmas in this section.

Lemma 4.1.9. *Let $w_1 \in \mathcal{A}_\psi(\Omega)$ be a weak solution to* (4.9) *under assumptions* (2.8) *with $p > 1$. Then for any $\sigma \in (0,1)$ there exists a constant $c \equiv c(\mathtt{data}, \sigma)$ such that*

$$\fint_B |V(Dw_1) - V(z_0)|^2 \, dx \le c \left(\fint_{2B} |V(Dw_1) - V(z_0)|^{2\sigma} \, dx \right)^{\frac{1}{\sigma}} + c \fint_{2B} (\varphi_{|z_0|})^*(|A(D\psi) - A(\xi_0)|) \, dx \tag{4.11}$$

holds for every $z_0, \xi_0 \in \mathbb{R}^n$*, whenever* $2B \Subset \Omega$*. Moreover, there exists a constant* $c \equiv c(\texttt{data})$ *such that*

$$\fint_B |V(Dw_1) - V(z_0)|^2 \, dx \le c(\varphi_{|z_0|})^* \left(\fint_{2B} |A(Dw_1) - A(z_0)| \, dx \right) + c \fint_{2B} (\varphi_{|z_0|})^* (|A(D\psi) - A(\xi_0)|) \, dx \tag{4.12}$$

holds for every $z_0, \xi_0 \in \mathbb{R}^n$*, whenever* $2B \Subset \Omega$*.*

Proof. By a standard scaling argument, we may assume that $B = B_{1/2}(0)$. For each $1/2 < r \le 1$ we consider the weak solution $w_{2,r} \in w_1 + W_0^{1,p}(B_r)$ to

$$\begin{cases} -\operatorname{div} A(Dw_{2,r}) = -\operatorname{div} A(D\psi) & \text{in } B_r, \\ w_{2,r} = w_1 & \text{on } \partial B_r. \end{cases} \tag{4.13}$$

We take a number ρ such that $1/2 \le \rho < r$ and a cut-off function $\eta \in C_0^\infty(B_r)$ such that $0 \le \eta \le 1$, $\eta \equiv 1$ in B_ρ and $|D\eta| \le 4/(r-\rho)$. We test (4.13) with

$$(w_{2,r} - (w_{2,r})_{B_r} - z_0 \cdot x)\eta^\ell, \quad \text{where} \quad \ell := \max\{p, 2\},$$

and estimate in a standard way

$$\begin{aligned}
&\int_{B_r} |V(Dw_{2,r}) - V(z_0)|^2 \eta^\ell \, dx \\
&\le c \int_{B_r} (A(Dw_{2,r}) - A(z_0)) \cdot (Dw_{2,r} - z_0) \eta^\ell \, dx \\
&\le c \int_{B_r} |A(Dw_{2,r}) - A(z_0)||D\eta|\eta^{\ell-1} |w_{2,r} - (w_{2,r})_{B_r} - z_0 \cdot x| \, dx \\
&\quad + c \int_{B_r} |A(D\psi) - A(\xi_0)||D[(w_{2,r} - (w_{2,r})_{B_r} - z_0 \cdot x)\eta^\ell]| \, dx \\
&\le c \int_{B_r} |A(Dw_{2,r}) - A(z_0)| \frac{|w_{2,r} - (w_{2,r})_{B_r} - z_0 \cdot x|}{r - \rho} \eta^{\ell-1} \, dx \\
&\quad + c \int_{B_r} |A(D\psi) - A(\xi_0)| \frac{|w_{2,r} - (w_{2,r})_{B_r} - z_0 \cdot x|}{r - \rho} \eta^{\ell-1} \, dx \\
&\quad + c \int_{B_r} |A(D\psi) - A(\xi_0)||Dw_{2,r} - z_0| \eta^\ell \, dx.
\end{aligned}$$

Using Young's inequality (2.15) with the Young function $\varphi_{|z_0|}(\cdot)$ and reabsorbing terms into the left-hand side, we obtain the following Caccioppoli type estimate:

$$\begin{aligned}
&\int_{B_\rho} |V(Dw_{2,r}) - V(z_0)|^2 \, dx \\
&\le c \left(\frac{r}{r-\rho}\right)^{\ell} \int_{B_r} \varphi_{|z_0|}\left(\frac{|w_{2,r} - (w_{2,r})_{B_r} - z_0 \cdot x|}{r}\right) dx \\
&\quad + c \int_{B_r} (\varphi_{|z_0|})^*(|A(D\psi) - A(\xi_0)|) \, dx.
\end{aligned}$$

Now, using the triangle inequality twice, we find

$$\begin{aligned}
&\int_{B_\rho} |V(Dw_1) - V(z_0)|^2 \, dx \\
&\le 2 \int_{B_\rho} |V(Dw_{2,r}) - V(z_0)|^2 \, dx + 2 \int_{B_\rho} |V(Dw_1) - V(Dw_{2,r})|^2 \, dx \\
&\le c \left(\frac{r}{r-\rho}\right)^{\ell} \int_{B_r} \varphi_{|z_0|}\left(\frac{|w_1 - (w_1)_{B_r} - z_0 \cdot x|}{r}\right) dx \\
&\quad + c \left(\frac{r}{r-\rho}\right)^{\ell} \int_{B_r} \varphi_{|z_0|}\left(\frac{|w_1 - w_{2,r} - (w_1)_{B_r} + (w_2)_{B_r}|}{r}\right) dx \\
&\quad + c \int_{B_r} |V(Dw_1) - V(Dw_{2,r})|^2 \, dx + c \int_{B_r} (\varphi_{|z_0|})^*(|A(D\psi) - A(\xi_0)|) \, dx.
\end{aligned}$$

Then we apply Lemma 2.2.2 and Hölder's inequality to obtain

$$\begin{aligned}
&\int_{B_\rho} |V(Dw_1) - V(z_0)|^2 \, dx \\
&\le c \left(\frac{r}{r-\rho}\right)^{\ell} \int_{B_r} \varphi_{|z_0|}\left(\frac{|w_1 - (w_1)_{B_r} - z_0 \cdot x|}{r}\right) dx \\
&\quad + c \left(\frac{r}{r-\rho}\right)^{\ell} \int_{B_r} \varphi_{|z_0|}(|Dw_1 - Dw_{2,r}|) \, dx \\
&\quad + c \int_{B_r} |V(Dw_1) - V(Dw_{2,r})|^2 \, dx + c \int_{B_r} (\varphi_{|z_0|})^*(|A(D\psi) - A(\xi_0)|) \, dx.
\end{aligned} \tag{4.14}$$

Here, denoting $m \coloneqq \min\{p, 2\}$, we have

$$\begin{aligned}\varphi_{|z_0|}(|Dw_1 - Dw_{2,r}|) &\overset{(2.18)}{\leq} c\left(\frac{r}{r-\rho}\right)^{\ell(m'-1)} \varepsilon^{1-m'}\varphi_{|Dw_1|}(|Dw_1 - Dw_{2,r}|) \\ &\quad + \left(\frac{r}{r-\rho}\right)^{-\ell} \varepsilon|V(Dw_1) - V(z_0)|^2 \\ &\overset{(2.17)}{\leq} c\left(\frac{r}{r-\rho}\right)^{\ell(m'-1)} \varepsilon^{1-m'}|V(Dw_1) - V(Dw_{2,r})|^2 \\ &\quad + \left(\frac{r}{r-\rho}\right)^{-\ell} \varepsilon|V(Dw_1) - V(z_0)|^2\end{aligned}$$

for any $\varepsilon \in (0, 1]$. Plugging this into (4.14) yields

$$\begin{aligned}&\int_{B_\rho} |V(Dw_1) - V(z_0)|^2 \, dx \\ &\leq c\left(\frac{r}{r-\rho}\right)^{\ell} \int_{B_r} \varphi_{|z_0|}\left(\frac{|w_1 - (w_1)_{B_r} - z_0 \cdot x|}{r}\right) dx \\ &\quad + c\varepsilon^{1-m'}\left(\frac{r}{r-\rho}\right)^{\ell m'} \int_{B_r} |V(Dw_1) - V(Dw_{2,r})|^2 \, dx \\ &\quad + c\varepsilon \int_{B_r} |V(Dw_1) - V(z_0)|^2 \, dx + c\int_{B_r} (\varphi_{|z_0|})^*(|A(D\psi) - A(\xi_0)|) \, dx.\end{aligned} \tag{4.15}$$

To proceed further, we now use the following comparison estimate from Lemma 4.1.14 below:

$$\begin{aligned}&\int_{B_r} |V(Dw_1) - V(Dw_{2,r})|^2 \, dx \\ &\leq \widetilde{\varepsilon}\int_{B_r} |V(Dw_1) - V(z_0)|^2 \, dx + c\widetilde{\varepsilon}^{1-\ell}\int_{B_r} (\varphi_{|z_0|})^*(|A(D\psi) - A(\xi_0)|) \, dx,\end{aligned}$$

which holds with any $\widetilde{\varepsilon} \in (0, 1]$. Choosing

$$\widetilde{\varepsilon} = \left(\frac{r}{r-\rho}\right)^{-\ell m'} \varepsilon^{m'}$$

for $\varepsilon \in (0,1]$ as in the above, and connecting the resulting estimate to (4.15), we find

$$\begin{aligned}
&\int_{B_\rho} |V(Dw_1) - V(z_0)|^2 \, dx \\
&\le c \left(\frac{r}{r-\rho}\right)^{\ell} \int_{B_r} \varphi_{|z_0|} \left(\frac{|w_1 - (w_1)_{B_r} - z_0 \cdot x|}{r}\right) dx \\
&\quad + c\varepsilon^{1-\ell m'} \left(\frac{r}{r-\rho}\right)^{\ell^2 m'} \int_{B_r} (\varphi_{|z_0|})^*(|A(D\psi) - A(\xi_0)|) \, dx \\
&\quad + 2c\varepsilon \int_{B_r} |V(Dw_1) - V(z_0)|^2 \, dx.
\end{aligned}$$

Now we choose $\varepsilon = 1/4c$ and use Lemma 2.2.2 for the first term in the right-hand side in order to have

$$\begin{aligned}
\int_{B_\rho} |V(Dw_1) - V(z_0)|^2 \, dx \le{}& \frac{1}{2} \int_{B_r} |V(Dw_1) - V(z_0)|^2 \, dx \\
&+ \frac{c}{(r-\rho)^{\ell}} \left(\int_{B_1} |V(Dw_1) - V(z_0)|^{2\vartheta} \, dx \right)^{\frac{1}{\vartheta}} \\
&+ \frac{c}{(r-\rho)^{\ell^2 m'}} \int_{B_1} (\varphi_{|z_0|})^*(|A(D\psi) - A(\xi_0)|) \, dx
\end{aligned} \tag{4.16}$$

for any $1/2 \le \rho < r \le 1$, where $\vartheta = \vartheta(\texttt{data}) \in (0,1)$ is the constant in Lemma 2.2.2 when $\Phi = \varphi_{|z_0|}$. Then Lemma 2.3.12 gives (4.11) in the case $\sigma = \vartheta$. For lower values of σ it again follows from the self-improving property [118, Remark 6.12]. Finally, (4.12) is obtained in the same way as in [100]. □

4.1.3 Basic comparison estimates

In this section we derive several comparison estimates. Here we assume that

$$\mu \in W^{-1,p'}(\Omega) \cap L^1(\Omega), \qquad u \in \mathcal{A}^g_\psi(\Omega). \tag{4.17}$$

This assumption will be removed in Section 4.1.7 below.

For a fixed ball $B_{4R} \Subset \Omega$, we first consider the homogeneous obstacle problem

$$\begin{cases} \displaystyle\int_{B_{4R}} A(Dw_1)\cdot D(\phi - w_1)\,dx \geq 0 \quad \forall\, \phi \in \mathcal{A}^u_\psi(B_{4R}), \\ \qquad\qquad\qquad\qquad w_1 \geq \psi \quad \text{a.e. in } B_{4R}, \\ \qquad\qquad\qquad\qquad w_1 = u \quad \text{on } \partial B_{4R}. \end{cases} \tag{4.18}$$

We start with a weighted type energy estimate.

Lemma 4.1.10. *Let $u \in \mathcal{A}^g_\psi(\Omega)$ be the weak solution to (4.2) under assumptions (2.8) and (4.1), and let $w_1 \in \mathcal{A}^u_\psi(B_{4R})$ be as in (4.18). Then*

$$\int_{B_{4R}} \frac{|V(Du)-V(Dw_1)|^2}{(h+|u-w_1|)^\xi}\,dx \leq c\frac{h^{1-\xi}}{\xi-1}|\mu|(B_{4R}) \tag{4.19}$$

holds whenever $h>0$ and $\xi>1$, where $c \equiv c(n,p,\nu)$.

Proof. Estimate (4.19) is exactly the same as the one in [146, Lemma 1]. However, the test functions used in its proof are not available here, since they are not guaranteed to belong to the admissible set in our obstacle problems. Hence we need to modify the test functions. We consider

$$\eta_\pm := \frac{1}{\xi-1}\left[1-\left(1+\frac{(u-w_1)_\pm}{h}\right)^{1-\xi}\right] \in W^{1,p}_0(B_{4R})\cap L^\infty(B_{4R}).$$

Note in particular that $\eta_\pm \geq 0$. Also, by applying the mean value theorem to the function $t \mapsto t^{1-\xi}/(\xi-1)$, we have

$$\eta_\pm(x) = \frac{(u-w_1)_\pm(x)}{h}(\tilde{h}_\pm(x))^{-\xi} \quad \text{for some} \quad 1 \leq \tilde{h}_\pm(x) \leq 1+\frac{(u-w_1)_\pm(x)}{h}$$

whenever $x \in B_{4R}$. Then it follows that

$$\begin{aligned} u - h\eta_+ = u - (\tilde{h}_+)^{-\xi}(u-w_1)_+ \geq u-(u-w_1)_+ = \min\{u,w_1\} \geq \psi, \\ w_1 - h\eta_- = w_1 - (\tilde{h}_-)^{-\xi}(u-w_1)_- \geq w_1-(u-w_1)_- = \min\{u,w_1\} \geq \psi \end{aligned}$$

a.e. in B_{4R}. Therefore, the functions $u \pm h\eta_\mp$ and $w_1 \pm h\eta_\pm$ belong to the admissible set.

We now test the weak formulations

$$\int_{B_{4R}} A(Du) \cdot D(\phi - u)\, dx \geq \int_{B_{4R}} (\phi - u)\, d\mu$$

and

$$\int_{B_{4R}} A(Dw_1) \cdot D(\phi - w_1)\, dx \geq 0$$

with $\phi \equiv u \pm h\eta_{\mp}$ and $\phi \equiv w_1 \pm h\eta_{\pm}$, respectively. Then it follows that

$$\begin{aligned} &\int_{B_{4R}} \frac{A(Du) \cdot D(u - w_1)_+}{(h + (u - w_1)_+)^{\xi}}\, dx \leq \int_{B_{4R}} h^{1-\xi} \eta_+ \, d\mu, \\ &\int_{B_{4R}} \frac{A(Dw_1) \cdot D(u - w_1)_+}{(h + (u - w)_+)^{\xi}}\, dx \geq 0 \end{aligned} \tag{4.20}$$

and

$$\begin{aligned} &\int_{B_{4R}} \frac{A(Du) \cdot D(u - w_1)_-}{(h + (u - w_1)_-)^{\xi}}\, dx \geq \int_{B_{4R}} h^{1-\xi} \eta_- \, d\mu, \\ &\int_{B_{4R}} \frac{A(Dw_1) \cdot D(u - w_1)_-}{(h + (u - w_1)_-)^{\xi}}\, dx \leq 0. \end{aligned} \tag{4.21}$$

Hence, we estimate the difference of two integrals in each of (4.20) and (4.21) by using (2.11), in order to obtain

$$\int_{B_{4R} \cap \{u \geq w_1\}} \frac{|V(Du) - V(Dw_1)|^2}{(h + |u - w_1|)^{\xi}}\, dx \leq c \left| \int_{B_{4R}} h^{1-\xi} \eta_+ \, d\mu \right| \leq c \frac{h^{1-\xi}}{\xi - 1} |\mu|(B_{4R})$$

and

$$\int_{B_{4R} \cap \{u < w_1\}} \frac{|V(Du) - V(Dw_1)|^2}{(h + |u - w_1|)^{\xi}}\, dx \leq c \left| \int_{B_{4R}} h^{1-\xi} \eta_- \, d\mu \right| \leq c \frac{h^{1-\xi}}{\xi - 1} |\mu|(B_{4R}).$$

Combining the last two estimates finally gives (4.19). □

Once we have estimate (4.19), we can obtain the following comparison estimate between (4.2) and (4.18), which is standard in measure data problems; see for example [144, 146]. We also refer to [189, Lemma 3.5] for another proof.

Lemma 4.1.11. *Let* $u \in \mathcal{A}^g_\psi(\Omega)$ *be the weak solution to* (4.2) *under assumptions* (2.8) *and* (4.1)*, and let* $w_1 \in \mathcal{A}^u_\psi(B_{4R})$ *be as in* (4.18)*. Then for every*

$$1 \le q < \min\left\{p, \frac{n(p-1)}{n-1}\right\},$$

there exists a constant $c \equiv c(\texttt{data}, q)$ *such that*

$$\begin{aligned}&\fint_{B_{4R}} \left(|Du - Dw_1|^q + |V(Du) - V(Dw_1)|^{\frac{2q}{p}}\right) dx \\ &\le c\left[\frac{|\mu|(B_{4R})}{(4R)^{n-1}}\right]^{\frac{q}{p-1}} + c\chi_{\{p<2\}}\left[\frac{|\mu|(B_{4R})}{(4R)^{n-1}}\right]^q \left(\fint_{B_{4R}} (|Du| + s)^q \, dx\right)^{2-p}.\end{aligned} \tag{4.22}$$

In the case $2 - 1/n < p < 2$ we need a modified version of the above lemma, that will be actually used with $q = p - 1 < 1$. For this, we need to establish a reverse Hölder type inequality for (4.2). We proceed with an additional argument based on the proof of [143, Proposition 4.1], alongside Lemma 4.1.9.

Lemma 4.1.12. *Let* $u \in \mathcal{A}^g_\psi(\Omega)$ *be the weak solution to* (4.2) *under assumptions* (2.8) *with* $2 - 1/n < p < 2$*. Then for every* $t \in (0,1)$ *there exists a constant* $c \equiv c(\texttt{data}, t)$ *such that*

$$\begin{aligned}\fint_{B_{\tilde{r}/2}} (|Du| + s)\, dx &\le c\left(\fint_{B_{\tilde{r}}} (|Du| + s)^t \, dx\right)^{\frac{1}{t}} + c\left[\frac{|\mu|(B_{\tilde{r}})}{\tilde{r}^{n-1}}\right]^{\frac{1}{p-1}} \\ &\quad + c\left(\fint_{B_{\tilde{r}}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{\tilde{r}}}|)\, dx\right)^{\frac{1}{p}}\end{aligned} \tag{4.23}$$

holds whenever $B_{\tilde{r}/2} \subset B_{\tilde{r}} \subset \Omega$ *are concentric balls.*

Proof. We may assume that $\tilde{r} = 1$ by the standard scaling argument. For each $1/2 < r \le 1$, we consider the weak solution $w_{1,r} \in \mathcal{A}^u_\psi(B_r)$ to

$$\int_{B_r} A(Dw_{1,r}) \cdot D(\phi - w_{1,r})\, dx \ge 0 \quad \forall\, \phi \in \mathcal{A}^u_\psi(B_r).$$

Using Lemma 4.1.11 and Young's inequality, we have for any $\varepsilon \in (0,1)$

$$\begin{aligned}
&\int_{B_r} |Du - Dw_{1,r}|\,dx \\
&\le cr^{n-\frac{n-1}{p-1}}[|\mu|(B_r)]^{\frac{1}{p-1}} + cr^{1-n(2-p)}[|\mu|(B_r)]\left(\int_{B_r}(|Du|+s)\,dx\right)^{2-p} \\
&\le c\varepsilon^{-\frac{2-p}{p-1}}r^{n-\frac{n-1}{p-1}}[|\mu|(B_r)]^{\frac{1}{p-1}} + \varepsilon\int_{B_r}(|Du|+s)\,dx. \qquad (4.24)
\end{aligned}$$

Now we take ρ such that $1/2 \le \rho < r$. Moreover we recall (4.16), but here with a different choice of parameters $\rho \le \tilde{\rho} < \tilde{r} \le r$ and $z_0 = 0$. By applying Lemma 2.3.12 for these parameters, we have

$$\begin{aligned}
\int_{B_\rho}(|Dw_{1,r}|+s)^p\,dx \le \frac{c}{(r-\rho)^2}&\left(\int_{B_r}(|Dw_{1,r}|+s)^{p\vartheta}\,dx\right)^{\frac{1}{\vartheta}} \\
&+ \frac{c}{(r-\rho)^{4p'}}\int_{B_r}\varphi^*(|A(D\psi)-A(\xi_0)|)\,dx
\end{aligned}$$

for some $\vartheta \in (0,1)$. At this moment, a slightly modified version of the self-improving property given in [142, Lemma 5.1] implies that for any $t > 0$

$$\begin{aligned}
\left(\int_{B_\rho}(|Dw_{1,r}|+s)^p\,dx\right)^{\frac{1}{p}} \le \frac{c}{(r-\rho)^{\xi_1}}&\left(\int_{B_r}(|Dw_{1,r}|+s)^t\,dx\right)^{\frac{1}{t}} \\
&+ \frac{c}{(r-\rho)^{\xi_2}}\left(\int_{B_r}\varphi^*(|A(D\psi)-A(\xi_0)|)\,dx\right)^{\frac{1}{p}},
\end{aligned}$$

where $\xi_1, \xi_2 > 0$ depend only on n, p and t. Using this inequality, we estimate

$$\begin{aligned}
&\int_{B_\rho}(|Du|+s)\,dx \le \int_{B_\rho}(|Dw_{1,r}|+s)\,dx + \int_{B_\rho}|Du-Dw_{1,r}|\,dx \\
&\le \frac{c}{(r-\rho)^{\xi_1}}\left[\left(\int_{B_r}(|Du|+s)^t\,dx\right)^{\frac{1}{t}} + \int_{B_r}|Du-Dw_{1,r}|\,dx\right] \\
&\quad + \frac{c}{(r-\rho)^{\xi_2}}\left(\int_{B_r}\varphi^*(|A(D\psi)-A(\xi_0)|)\,dx\right)^{\frac{1}{p}}
\end{aligned}$$

$$\overset{(4.24)}{\leq} \frac{c}{(r-\rho)^{\xi_1}}\left(\int_{B_1}(|Du|+s)^t\,dx\right)^{\frac{1}{t}} + \frac{c\varepsilon}{(r-\rho)^{\xi_1}}\int_{B_r}(|Du|+s)\,dx + \frac{c\varepsilon^{-\frac{2-p}{p-1}}}{(r-\rho)^{\xi_1}}[|\mu|(B_1)]^{\frac{1}{p-1}} + \frac{c}{(r-\rho)^{\xi_2}}\left(\int_{B_1}\varphi^*(|A(D\psi)-A(\xi_0)|)\,dx\right)^{\frac{1}{p}}.$$

Now we choose $\varepsilon = (r-\rho)^{\xi_1}/2c$ to see that

$$\begin{aligned}&\int_{B_\rho}(|Du|+s)\,dx\\ &\leq \frac{1}{2}\int_{B_r}(|Du|+s)\,dx + \frac{c}{(r-\rho)^{\xi_1}}\left(\int_{B_1}(|Du|+s)^t\,dx\right)^{\frac{1}{t}}\\ &\quad + \frac{c}{(r-\rho)^{\frac{\xi_1}{p-1}}}[|\mu|(B_1)]^{\frac{1}{p-1}} + \frac{c}{(r-\rho)^{\xi_2}}\left(\int_{B_1}\varphi^*(|A(D\psi)-A(\xi_0)|)\,dx\right)^{\frac{1}{p}}\end{aligned}$$

holds whenever $1/2 \leq \rho < r < 1$. Applying Lemma 2.3.12 yields the desired estimate. □

Lemma 4.1.13. *Let $u \in \mathcal{A}^g_\psi(\Omega)$ be the weak solution to (4.2) under assumptions (2.8) with $2-1/n < p < 2$, and let $w_1 \in \mathcal{A}^u_\psi(B_{4R})$ be as in (4.18). If $B_{8R} \Subset \Omega$, then we have*

$$\begin{aligned}&\fint_{B_{4R}}\left(|Du-Dw_1|^q + |V(Du)-V(Dw_1)|^{\frac{2q}{p}}\right)dx\\ &\leq c\left[\frac{|\mu|(B_{8R})}{(8R)^{n-1}}\right]^{\frac{q}{p-1}} + c\left[\frac{|\mu|(B_{8R})}{(8R)^{n-1}}\right]^q\left(\fint_{B_{8R}}(|Du|+s)^t\,dx\right)^{\frac{q(2-p)}{t}}\\ &\quad + c\left[\frac{|\mu|(B_{8R})}{(8R)^{n-1}}\right]^q\left(\fint_{B_{8R}}\varphi^*(|A(D\psi)-(A(D\psi))_{B_{8R}}|)\,dx\right)^{\frac{q(2-p)}{p}}\end{aligned} \tag{4.25}$$

for every $q \in (0,1]$ and $t \in (0,q]$, where $c \equiv c(\texttt{data}, q, t)$.

Proof. It suffices to consider only the case $q=1$, since the estimate for lower values of q follows from Hölder's inequality. We apply Lemma 4.1.12 with $B_{\tilde r} \equiv B_{8R}$ to have

$$\left[\frac{|\mu|(B_{4R})}{(4R)^{n-1}}\right]\left(\fint_{B_{4R}}(|Du|+s)\,dx\right)^{2-p}$$

$$\begin{aligned}
&\overset{(4.23)}{\leq} c\left[\frac{|\mu|(B_{8R})}{(8R)^{n-1}}\right]\left(\fint_{B_{8R}}(|Du|+s)^t\,dx\right)^{\frac{2-p}{t}}+c\left[\frac{|\mu|(B_{8R})}{(8R)^{n-1}}\right]^{1+\frac{2-p}{p-1}}\\
&\qquad+c\left[\frac{|\mu|(B_{8R})}{(8R)^{n-1}}\right]\left(\fint_{B_{8R}}\varphi^*(|A(D\psi)-(A(D\psi))_{B_{8R}}|)\,dx\right)^{\frac{2-p}{p}}\\
&=c\left[\frac{|\mu|(B_{8R})}{(8R)^{n-1}}\right]\left(\fint_{B_{8R}}(|Du|+s)^t\,dx\right)^{\frac{2-p}{t}}+c\left[\frac{|\mu|(B_{8R})}{(8R)^{n-1}}\right]^{\frac{1}{p-1}}\\
&\qquad+c\left[\frac{|\mu|(B_{8R})}{(8R)^{n-1}}\right]\left(\fint_{B_{8R}}\varphi^*(|A(D\psi)-(A(D\psi))_{B_{8R}}|)\,dx\right)^{\frac{2-p}{p}}.
\end{aligned}$$

Plugging this into (4.22), we get the desired estimate. □

Next, we also consider the obstacle-free problem

$$\begin{cases}-\operatorname{div} A(Dw_2)=-\operatorname{div} A(D\psi) & \text{in } B_{2R},\\ \qquad w_2=w_1 & \text{on } \partial B_{2R},\end{cases} \tag{4.26}$$

and the limiting problem

$$\begin{cases}-\operatorname{div} A(Dv)=0 & \text{in } B_R,\\ \qquad v=w_2 & \text{on } \partial B_R.\end{cases} \tag{4.27}$$

Lemma 4.1.14. *Let w_1, w_2, and v be defined as above, under assumptions* (2.8) *with $p>1$. Then both*

$$\begin{aligned}\fint_{B_R}|V(Dw_1)-V(Dv)|^2\,dx&\leq\varepsilon\fint_{B_{2R}}|V(Dw_1)-V(z_0)|^2\,dx\\
&\qquad+c\varepsilon^{1-\max\{p,2\}}\fint_{B_{2R}}(\varphi_{|z_0|})^*(|A(D\psi)-A(\xi_0)|)\,dx\end{aligned}$$

and

$$\begin{aligned}\fint_{B_R}|V(Dw_1)-V(Dv)|^2\,dx&\leq\varepsilon(\varphi_{|z_0|})^*\left(\fint_{B_{4R}}|A(Dw_1)-A(z_0)|\,dx\right)\\
&\qquad+c\varepsilon^{1-\max\{p,2\}}\fint_{B_{4R}}(\varphi_{|z_0|})^*(|A(D\psi)-A(\xi_0)|)\,dx\end{aligned}$$

hold for a constant $c\equiv c(\texttt{data})$, whenever $z_0,\xi_0\in\mathbb{R}^n$ and $\varepsilon\in(0,1)$.

Proof. We only prove the first estimate, which with Lemma 4.1.9 directly implies the second one. We test (4.18) and (4.26) with $w_2 \in \mathcal{A}_{\psi}^{w_1}(B_{2R})$ and $w_2 - w_1 \in W_0^{1,p}(B_{2R})$, respectively, in order to have

$$\begin{aligned}
&\fint_{B_{2R}} |V(Dw_1) - V(Dw_2)|^2 \, dx \\
&\overset{(2.11)}{\leq} c \fint_{B_{2R}} (A(Dw_2) - A(Dw_1)) \cdot (Dw_2 - Dw_1) \, dx \\
&\leq c \fint_{B_{2R}} |A(D\psi) - A(\xi_0)||Dw_1 - Dw_2| \, dx.
\end{aligned}$$

Applying Young's inequality (2.15) with the Young function $\varphi_{|Dw_1|}(\cdot)$ and reabsorbing terms, we obtain

$$\fint_{B_{2R}} |V(Dw_1) - V(Dw_2)|^2 \, dx \leq c \fint_{B_{2R}} (\varphi_{|Dw_1|})^*(|A(D\psi) - A(\xi_0)|) \, dx.$$

In a similar way, as for (4.26) and (4.27), we have

$$\begin{aligned}
&\fint_{B_R} |V(Dw_2) - V(Dv)|^2 \, dx \leq c \fint_{B_R} (\varphi_{|Dw_2|})^*(|A(D\psi) - A(\xi_0)|) \, dx \\
&\overset{(2.18)}{\leq} c \fint_{B_R} (\varphi_{|Dw_1|})^*(|A(D\psi) - A(\xi_0)|) \, dx + c \fint_{B_R} |V(Dw_1) - V(Dw_2)|^2 \, dx.
\end{aligned}$$

Combining the above two displays and applying (2.18) again, we find

$$\begin{aligned}
\fint_{B_R} |V(Dw_1) - V(Dv)|^2 \, dx &\leq c \fint_{B_{2R}} (\varphi_{|Dw_1|})^*(|A(D\psi) - A(\xi_0)|) \, dx \\
&\leq \varepsilon \fint_{B_{2R}} |V(Dw_1) - V(z_0)|^2 \, dx \\
&\quad + c\varepsilon^{1-\max\{p,2\}} \fint_{B_{2R}} (\varphi_{|z_0|})^*(|A(D\psi) - A(\xi_0)|) \, dx
\end{aligned}$$

for any $\varepsilon \in (0, 1]$, as desired. □

We also note another simple comparison estimate in the case $p \geq 2$.

Lemma 4.1.15. *Let* w_1 *and* v *be as in* (4.18) *and* (4.27), *respectively, under assumptions* (2.8) *with* $p \geq 2$. *Then we have*

$$\fint_{B_R} \left(|Dw_1 - Dv|^p + |V(Dw_1) - V(Dv)|^2\right) dx \leq c \fint_{B_{2R}} \varphi^*(|A(D\psi) - A(\xi_0)|)\, dx \tag{4.28}$$

for a constant $c \equiv c(\mathtt{data})$, *whenever* $\xi_0 \in \mathbb{R}^n$.

Proof. We recall the following estimate in the proof of Lemma 4.1.14:

$$\fint_{B_R} |V(Dw_1) - V(Dv)|^2\, dx \leq c \fint_{B_{2R}} (\varphi_{|Dw_1|})^*(|A(D\psi) - A(\xi_0)|)\, dx, \tag{4.29}$$

which holds whenever $p > 1$. Now, for $p \geq 2$ we use (2.10) and the fact that $(\varphi_a)^*(t) \lesssim \varphi^*(t)$ holds for $a, t \geq 0$, to derive (4.28). □

To proceed further, we consider additional comparison maps. In the following, we fix a ball

$$B_{4MR} = B_{4MR}(x_0) \Subset \Omega \quad \text{with} \quad M \geq 8 \quad \text{and} \quad R \leq 1, \tag{4.30}$$

where M is a free parameter whose relevant value will be determined in the next section.

We then define comparison maps. The first one is the weak solution $w_{1,*} \in \mathcal{A}^u_\psi(B_{4MR})$ to

$$\begin{cases} \displaystyle\int_{B_{4MR}} A(Dw_{1,*}) \cdot D(\phi - w_{1,*})\, dx \geq 0 & \forall\, \phi \in \mathcal{A}^{w_{1,*}}_\psi(B_{4MR}), \\ \qquad\qquad\qquad\qquad w_{1,*} \geq \psi & \text{a.e. in } B_{4MR}, \\ \qquad\qquad\qquad\qquad w_{1,*} = u & \text{on } \partial B_{4MR}. \end{cases}$$

Accordingly, $w_{2,*} \in w_{1,*} + W^{1,p}_0(B_{2MR})$ is defined as the weak solution to

$$\begin{cases} -\operatorname{div} A(Dw_{2,*}) = -\operatorname{div} A(D\psi) & \text{in } B_{2MR}, \\ \qquad\qquad w_{2,*} = w_{1,*} & \text{on } \partial B_{2MR}. \end{cases}$$

The last one is $v_* \in w_{2,*} + W^{1,p}_0(B_{MR})$ which is defined as the weak solution to

$$\begin{cases} -\operatorname{div} A(Dv_*) = 0 & \text{in } B_{MR}, \\ v_* = w_{2,*} & \text{on } \partial B_{MR}. \end{cases}$$

The following lemma will play a crucial role in the linearization procedure in the case $p > 2$. We note that, in contrast with the results in [61, 62, 146], the weak solutions $w_{1,*}$ and $w_{2,*}$ do not in general enjoy the C^1-regularity unless $D\psi$ is Dini continuous, see [179]. In this situation, we instead use the C^1-regularity of v_* as in [55, Lemma 4.3].

Lemma 4.1.16. *Let $u \in \mathcal{A}^g_\psi(\Omega)$ be the weak solution to (4.2) under assumptions (2.8) with $p \geq 2$, and let $w_{1,*}$, $w_{2,*}$, v_* be the functions defined in the above display. Suppose further that*

$$\left[\frac{|\mu|(B_{4MR})}{(4MR)^{n-1}}\right]^{\frac{1}{p-1}} + \left(\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|)\, dx\right)^{\frac{1}{p}} \leq H\lambda \tag{4.31}$$

holds for some constants $\lambda > 0$ and $H \geq 1$, together with the bounds

$$\frac{\lambda}{H} \leq |Dv_*| + s \leq H\lambda \quad \text{in } B_{4R}. \tag{4.32}$$

Then there exists a constant $c \equiv c(\texttt{data}, M, H)$ such that

$$\begin{aligned} &\fint_{B_{4R}} |Du - Dw_1|\, dx \\ &\leq c\lambda^{2-p}\left[\frac{|\mu|(B_{4MR})}{(4MR)^{n-1}}\right] + c\lambda^{2-p}\left(\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|)\, dx\right)^{\frac{1}{p'}}. \end{aligned} \tag{4.33}$$

Proof. When $p = 2$, (4.33) is a direct consequence of Lemma 4.1.13. Therefore we will assume $p > 2$ in the rest of the proof. We fix the numbers

$$\gamma := \frac{1}{4(p-1)(n+1)} \quad \text{and} \quad \xi := 1 + 2\gamma.$$

We also set

$$\bar{w}_1 := \frac{w_1}{\lambda}, \quad \bar{w}_{1,*} := \frac{w_{1,*}}{\lambda}, \quad \bar{v} := \frac{v}{\lambda}, \quad \bar{v}_* := \frac{v_*}{\lambda}, \quad \text{and} \quad \bar{s} := \frac{s}{\lambda},$$

and then estimate the left-hand side in (4.33) as follows:

$$\begin{aligned}
\fint_{B_{4R}} |Du - Dw_1|\,dx &\overset{(4.32)}{\leq} H^{(p-2)(1+\gamma)} \fint_{B_{4R}} (|D\bar{v}_*| + \bar{s})^{(p-2)(1+\gamma)} |Du - Dw_1|\,dx \\
&\leq c \fint_{B_{4R}} |D\bar{v}_* - D\bar{w}_1|^{(p-2)(1+\gamma)} |Du - Dw_1|\,dx \\
&\quad + c \fint_{B_{4R}} (|D\bar{w}_1| + \bar{s})^{(p-2)(1+\gamma)} |Du - Dw_1|\,dx \\
&=: I_1 + I_2.
\end{aligned} \tag{4.34}$$

To estimate I_1, we use Lemmas 4.1.15 and 4.1.11 with any $q \leq \xi(p-1)$ in order to infer that

$$\begin{aligned}
&\fint_{B_{4R}} |D\bar{v}_* - D\bar{w}_1|^q\,dx \\
&\leq c\lambda^{-q} \fint_{B_{MR}} |Dv_* - Dw_{1,*}|^q\,dx + c\lambda^{-q} \fint_{B_{4MR}} |Dw_{1,*} - Du|^q\,dx \\
&\quad + c\lambda^{-q} \fint_{B_{4R}} |Du - Dw_1|^q\,dx \\
&\leq c\lambda^{-q} \left(\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|)\,dx \right)^{\frac{q}{p}} + c\lambda^{-q} \left[\frac{|\mu|(B_{4MR})}{(4MR)^{n-1}} \right]^{\frac{q}{p-1}} \\
&\leq c \left[\frac{|\mu|(B_{4MR})}{\lambda^{p-1}(4MR)^{n-1}} \right]^{\frac{q}{p-1}} + \left(\frac{1}{\lambda^p} \fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|)\,dx \right)^{\frac{q}{p}}.
\end{aligned} \tag{4.35}$$

Then, using Hölder's inequality and once again Lemma 4.1.11, it follows that

$$\begin{aligned}
I_1 &\leq c \left(\fint_{B_{4R}} |D\bar{v}_* - D\bar{w}_1|^{(p-1)(1+\gamma)}\,dx \right)^{\frac{p-2}{p-1}} \left(\fint_{B_{4R}} |Du - Dw_1|^{p-1}\,dx \right)^{\frac{1}{p-1}} \\
&\leq c \left[\frac{|\mu|(B_{4MR})}{(4MR)^{n-1}} \right]^{\frac{1}{p-1}} \Bigg\{ \left[\frac{|\mu|(B_{4MR})}{\lambda^{p-1}(4MR)^{n-1}} \right] \\
&\qquad + \left(\frac{1}{\lambda^p} \fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|)\,dx \right)^{\frac{1}{p'}} \Bigg\}^{\frac{(p-2)(1+\gamma)}{p-1}}
\end{aligned}$$

$$
\begin{aligned}
&\leq c\lambda\Bigg\{\left[\frac{|\mu|(B_{4MR})}{\lambda^{p-1}(4MR)^{n-1}}\right] \\
&\qquad + \left(\frac{1}{\lambda^p}\fint_{B_{4MR}}\varphi^*(|A(D\psi)-(A(D\psi))_{B_{4MR}}|)\,dx\right)^{\frac{1}{p'}}\Bigg\}^{\frac{(p-2)(1+\gamma)+1}{p-1}} \\
&\leq c\lambda\left\{\left[\frac{|\mu|(B_{4MR})}{\lambda^{p-1}(4MR)^{n-1}}\right]+\left(\frac{1}{\lambda^p}\fint_{B_{4MR}}\varphi^*(|A(D\psi)-(A(D\psi))_{B_{4MR}}|)\,dx\right)^{\frac{1}{p'}}\right\} \\
&= c\lambda^{2-p}\left\{\left[\frac{|\mu|(B_{4MR})}{(4MR)^{n-1}}\right]+\left(\fint_{B_{4MR}}\varphi^*(|A(D\psi)-(A(D\psi))_{B_{4MR}}|)\,dx\right)^{\frac{1}{p'}}\right\},
\end{aligned}
\tag{4.36}
$$

where we have used (4.31) and the fact that

$$
(1+\gamma)(p-2)+1>p-1.
$$

We next estimate I_2. Applying Hölder's inequality and (4.19), and recalling that $\xi=1+2\gamma$, we have for any $h>0$ that

$$
\begin{aligned}
I_2 &\leq c\fint_{B_{4R}}\left[\lambda^{2-p}\frac{(|Dw_1|+|Du|+s)^{p-2}|Du-Dw_1|^2}{(h+|u-w_1|)^{\xi}}\right]^{\frac{1}{2}} \\
&\qquad\cdot\left[(|D\bar{w}_1|+\bar{s})^{(p-2)(1+2\gamma)}(h+|u-w_1|)^{\xi}\right]^{\frac{1}{2}}\,dx \\
&\leq c\fint_{B_{4R}}\left[\lambda^{2-p}\frac{|V(Du)-V(Dw_1)|^2}{(h+|u-w_1|)^{\xi}}\right]^{\frac{1}{2}} \\
&\qquad\cdot\left[(|D\bar{w}_1|+\bar{s})^{(p-2)\xi}(h+|u-w_1|)^{\xi}\right]^{\frac{1}{2}}\,dx \\
&\leq c\left(\lambda^{2-p}\fint_{B_{4R}}\frac{|V(Du)-V(Dw_1)|^2}{(h+|u-w_1|)^{\xi}}\,dx\right)^{\frac{1}{2}} \\
&\qquad\cdot\left(\fint_{B_{4R}}(|D\bar{w}_1|+\bar{s})^{(p-2)\xi}(h+|u-w_1|)^{\xi}\,dx\right)^{\frac{1}{2}} \\
&\leq c\lambda^{\frac{2-p}{2}}\left[h^{1-\xi}\frac{|\mu|(B_{4R})}{R^n}\right]^{\frac{1}{2}}\left(\fint_{B_{4R}}(|D\bar{w}_1|+\bar{s})^{(p-2)\xi}(h+|u-w_1|)^{\xi}\,dx\right)^{\frac{1}{2}}.
\end{aligned}
\tag{4.37}
$$

Now we choose

$$h := \left(\fint_{B_{4R}} (|D\bar{w}_1| + \bar{s})^{(p-2)\xi} |u - w_1|^{\xi} \, dx \right)^{\frac{1}{\xi}} + \delta \tag{4.38}$$

for $\delta > 0$ sufficiently small, thereby obtaining

$$\begin{aligned} &\left(\fint_{B_{4R}} (|D\bar{w}_1| + \bar{s})^{(p-2)\xi} (h + |u - w_1|)^{\xi} \, dx \right)^{\frac{1}{2}} \\ &\le c h^{\frac{\xi}{2}} \left(\fint_{B_{4R}} (|D\bar{w}_1| + \bar{s})^{(p-2)\xi} \, dx \right)^{\frac{1}{2}} + c h^{\frac{\xi}{2}}. \end{aligned}$$

We note that the role of δ in (4.38) is just to guarantee that $h > 0$; we shall eventually let $\delta \to 0$ at the end of the proof. Also, (4.35) and (4.32) imply

$$\begin{aligned} &\fint_{B_{4R}} (|D\bar{w}_1| + \bar{s})^{(p-2)\xi} \, dx \\ &\le c \fint_{B_{4R}} |D\bar{w}_1 - D\bar{v}_*|^{(p-2)\xi} \, dx + c \fint_{B_{4R}} (|D\bar{v}_*| + \bar{s})^{(p-2)\xi} \, dx \\ &\le c \left(\frac{1}{\lambda^p} \fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|) \, dx \right)^{\frac{(p-2)\xi}{p}} \\ &\quad + c \left[\frac{|\mu|(B_{4MR})}{\lambda^{p-1}(4MR)^{n-1}} \right]^{\frac{(p-2)\xi}{p-1}} + c H^{(p-2)\xi} \\ &\le c, \end{aligned}$$

which gives

$$\left(\fint_{B_{4R}} (|D\bar{w}_1| + \bar{s})^{(p-2)\xi} (h + |u - w_1|)^{\xi} \, dx \right)^{\frac{1}{2}} \le c h^{\frac{\xi}{2}}$$

for a constant $c \equiv c(\texttt{data}, M, H)$. Plugging this into (4.37) and applying Young's inequality, we have

$$I_2 \le c \left(\frac{h}{R} \right)^{\frac{1}{2}} \left[\frac{|\mu|(B_{4R})}{\lambda^{p-2} R^{n-1}} \right]^{\frac{1}{2}} \le \frac{c \lambda^{2-p}}{\varepsilon} \left[\frac{|\mu|(B_{4MR})}{(4MR)^{n-1}} \right] + \frac{\varepsilon h}{R} \tag{4.39}$$

whenever $\varepsilon \in (0,1)$, where $c \equiv c(\texttt{data}, M, H)$.

Finally, it remains to estimate h. We estimate

$$\begin{aligned} h &\le c\left(\fint_{B_{4R}} |D\bar{v}_* - D\bar{w}_1|^{(p-2)\xi} |u-w_1|^{\xi}\,dx\right)^{\frac{1}{\xi}} \\ &\quad + c\left(\fint_{B_{4R}} (|D\bar{v}_*| + \bar{s})^{(p-2)\xi} |u-w_1|^{\xi}\,dx\right)^{\frac{1}{\xi}} + \delta \\ &=: I_3 + I_4 + \delta. \end{aligned} \tag{4.40}$$

Using (4.22) and (4.35), I_3 is estimated as

$$\begin{aligned} I_3 &\le c\left(\fint_{B_{4R}} |D\bar{v}_* - D\bar{w}_1|^{\xi(p-1)}\,dx\right)^{\frac{p-2}{\xi(p-1)}} \left(\fint_{B_{4R}} |u-w_1|^{\xi(p-1)}\,dx\right)^{\frac{1}{\xi(p-1)}} \\ &\le cR\left\{\left[\frac{|\mu|(B_{4MR})}{\lambda^{p-1}(4MR)^{n-1}}\right] + \left(\frac{1}{\lambda^p}\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|)\,dx\right)^{\frac{1}{p'}}\right\}^{\frac{p-2}{p-1}} \\ &\quad \cdot \left(\fint_{B_{4R}} |Du - Dw_1|^{\xi(p-1)}\,dx\right)^{\frac{1}{\xi(p-1)}} \\ &\le cR\left\{\left[\frac{|\mu|(B_{4MR})}{\lambda^{p-1}(4MR)^{n-1}}\right] + \left(\frac{1}{\lambda^p}\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|)\,dx\right)^{\frac{1}{p'}}\right\}^{\frac{p-2}{p-1}} \\ &\quad \cdot \left[\frac{|\mu|(B_{4R})}{(4R)^{n-1}}\right]^{\frac{1}{p-1}} \\ &\le cR\lambda\left\{\left[\frac{|\mu|(B_{4MR})}{\lambda^{p-1}(4MR)^{n-1}}\right] + \left(\frac{1}{\lambda^p}\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|)\,dx\right)^{\frac{1}{p'}}\right\} \\ &= cR\lambda^{2-p}\left\{\left[\frac{|\mu|(B_{4MR})}{(4MR)^{n-1}}\right] + \left(\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|)\,dx\right)^{\frac{1}{p'}}\right\}. \end{aligned} \tag{4.41}$$

We now estimate I_4 by using (4.22) and applying (4.32) repeatedly as follows:

$$I_4 \le cH^{p-2}\left(\fint_{B_{4R}} |u-w_1|^{\xi}\,dx\right)^{\frac{1}{\xi}}$$

$$
\begin{aligned}
&\leq cR \fint_{B_{4R}} |Du - Dw_1| \, dx \\
&\leq cH^{(p-2)(1+\gamma)} R \fint_{B_{4R}} (|D\bar{v}_*| + \bar{s})^{(p-2)(1+\gamma)} |Du - Dw_1| \, dx \\
&\leq cR \fint_{B_{4R}} (|D\bar{w}_1| + \bar{s})^{(p-2)(1+\gamma)} |Du - Dw_1| \, dx \\
&\quad + cR \fint_{B_{4R}} |D\bar{v}_* - D\bar{w}_1|^{(p-2)(1+\gamma)} |Du - Dw_1| \, dx.
\end{aligned}
$$

Recalling (4.34)-(4.36), we have

$$
\begin{aligned}
I_4 &\leq cRI_2 + cR\lambda^{2-p} \left[\frac{|\mu|(B_{4MR})}{(4MR)^{n-1}} \right] \\
&\quad + cR\lambda^{2-p} \left(\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|) \, dx \right)^{\frac{1}{p'}} .
\end{aligned}
$$

Combining this inequality with (4.40) and (4.41) yields

$$
\begin{aligned}
\frac{h}{R} &\leq c_* I_2 + c_* \lambda^{2-p} \left[\frac{|\mu|(B_{4MR})}{(4MR)^{n-1}} \right] \\
&\quad + c_* \lambda^{2-p} \left(\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|) \, dx \right)^{\frac{1}{p'}} + \frac{\delta}{R},
\end{aligned}
$$

where $c_* \equiv c_*(\texttt{data}, H, M)$. Plugging the last inequality into (4.39), choosing $\varepsilon = 1/(2c_*)$ and then reabsorbing terms lead to

$$
\begin{aligned}
I_2 &\leq c\lambda^{2-p} \left[\frac{|\mu|(B_{4MR})}{(4MR)^{n-1}} \right] \\
&\quad + c\lambda^{2-p} \left(\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|) \, dx \right)^{\frac{1}{p'}} + \frac{c\delta}{R}.
\end{aligned}
$$

Merging this inequality with (4.34) and (4.36), and finally letting $\delta \to 0$, (4.33) follows. □

We also establish a linearized comparison estimate between (4.18) and (4.27) in the case $p \leq 2$.

Lemma 4.1.17. *Let w_1 and v be as in* (4.18) *and* (4.27)*, respectively, under assumptions* (2.8) *with $1 < p \leq 2$. Then we have*

$$\fint_{B_R} |A(Dw_1) - A(Dv)|\,dx \leq \varepsilon \fint_{B_{4R}} |A(Dw_1) - (A(Dw_1))_{B_{4R}}|\,dx + c_\varepsilon \left(\fint_{B_{4R}} \varphi^*(|A(D\psi) - A(\xi_0)|)\,dx\right)^{\frac{1}{p'}} \quad (4.42)$$

for every $\varepsilon \in (0,1)$ and $\xi_0 \in \mathbb{R}^n$, where $c_\varepsilon \equiv c_\varepsilon(\mathtt{data}, \varepsilon)$ is proportional to some negative power of ε.

Proof. We denote by W_R the vector satisfying $A(W_R) = (A(Dw_1))_{B_R}$. Then

$$\begin{aligned}
(\varphi_{|W_R|})^*\left(\fint_{B_R} |A(Dw_1) - A(Dv)|\,dx\right) &\leq \fint_{B_R} (\varphi_{|W_R|})^*(|A(Dw_1) - A(Dv)|)\,dx \\
&\overset{(2.18)}{\leq} c\gamma_1^{1-\ell} \fint_{B_R} (\varphi_{|Dw_1|})^*(|A(Dw_1) - A(Dv)|)\,dx \\
&\quad + \gamma_1 \fint_{B_R} |V(Dw_1) - V(W_R)|^2\,dx \\
&\overset{(2.17)}{\leq} c\gamma_1^{1-\ell} \fint_{B_R} |V(Dw_1) - V(Dv)|^2\,dx + \gamma_1 \fint_{B_R} |V(Dw_1) - V(W_R)|^2\,dx
\end{aligned}$$

for any $\gamma_1 \in (0,1)$, where $\ell = \max\{p, 2\}$. We next apply Lemmas 4.1.14 and 4.1.9 to estimate each term in the right-hand side, thereby obtaining

$$\begin{aligned}
&(\varphi_{|W_R|})^*\left(\fint_{B_R} |A(Dw_1) - A(Dv)|\,dx\right) \\
&\leq c\gamma_1^{1-\ell}\gamma_2(\varphi_{|W_R|})^*\left(\fint_{B_{4R}} |A(Dw_1) - (A(Dw_1))_{B_R}|\,dx\right) \\
&\quad + c\gamma_1^{1-\ell}\gamma_2^{1-\ell} \fint_{B_{4R}} (\varphi_{|W_R|})^*(|A(D\psi) - A(\xi_0)|)\,dx \\
&\quad + c\gamma_1(\varphi_{|W_R|})^*\left(\fint_{B_{2R}} |A(Dw_1) - (A(Dw_1))_{B_R}|\,dx\right) \\
&\quad + c\gamma_1 \fint_{B_{2R}} (\varphi_{|W_R|})^*(|A(D\psi) - A(\xi_0)|)\,dx
\end{aligned}$$

for any $\gamma_2 \in (0,1)$, where $c \equiv c(\texttt{data})$. Choosing $\gamma_2 = \gamma_1^{\ell}$, we arrive at

$$
\begin{aligned}
&(\varphi_{|W_R|})^* \left(\fint_{B_R} |A(Dw_1) - A(Dv)| \, dx \right) \\
&\leq c\gamma_1 (\varphi_{|W_R|})^* \left(\fint_{B_{4R}} |A(Dw_1) - (A(Dw_1))_{B_{4R}}| \, dx \right) \\
&\quad + c\gamma_1^{1-\ell^2} \fint_{B_{4R}} (\varphi_{|W_R|})^* (|A(D\psi) - A(\xi_0)|) \, dx,
\end{aligned}
$$

which holds whenever $p > 1$. Finally, when $1 < p \leq 2$, a direct calculation as in [9, Lemma 2.13] gives

$$
\begin{aligned}
(\varphi_a)^* \left(t^{\frac{1}{p'}} \right) &\overset{(2.16)}{\approx} \left((a+s)^{p-1} + t^{\frac{1}{p'}} \right)^{p'-2} t^{\frac{2}{p'}} \\
&\approx \begin{cases} (a+s)^{(p-1)(p'-2)} t^{\frac{2}{p'}} & \text{if } t^{\frac{1}{p'}} \leq (a+s)^{p-1}, \\ t & \text{if } t^{\frac{1}{p'}} > (a+s)^{p-1}, \end{cases}
\end{aligned}
$$

with the last function being concave. Namely, $t \mapsto [((\varphi_a)^*)^{-1}(t)]^{p'}$ is quasi-convex. Hence, choosing γ_1 in a suitable way, applying Jensen's inequality to the last term and then using the fact that $t^{p'} \lesssim \varphi^*(t)$ for $1 < p \leq 2$, (4.42) follows. □

4.1.4 Linearized comparison estimates

In this section, we establish linearized comparison estimates between (4.2) and (4.27). Throughout this section, we again assume (4.17) to ensure the existence of weak solutions to (4.2).

4.1.5 The two-scales degenerate alternative

We first consider the situation when

$$
\fint_{B_{4MR}} |A(Du) - (A(Du))_{B_{4MR}}| \, dx \geq \theta \left[|(A(Du))_{B_{R/M}}| + s^{p-1} \right] \tag{4.43}
$$

holds for another free parameter $\theta \in (0,1)$, where M is as in (4.30). The values of M and θ will be determined in the next section; their specific values do not affect the results in this section.

We start by observing that

$$\begin{aligned}
&\fint_{B_{4R}} (|Du|^2+s^2)^{\frac{p-1}{2}}\,dx \overset{(2.12)}{\leq} c\fint_{B_{4R}} (|A(Du)|+s^{p-1})\,dx \\
&\leq c\fint_{B_{4R}} |A(Du)-(A(Du))_{B_{R/M}}|\,dx + c\left[|(A(Du))_{B_{R/M}}|+s^{p-1}\right] \\
&\leq cM^{2n}\fint_{B_{4MR}} |A(Du)-(A(Du))_{B_{4MR}}|\,dx + c\left[|(A(Du))_{B_{R/M}}|+s^{p-1}\right] \\
&\overset{(4.43)}{\leq} cM^{2n}\left(1+\frac{1}{\theta}\right)\fint_{B_{4MR}} |A(Du)-(A(Du))_{B_{4MR}}|\,dx
\end{aligned} \tag{4.44}$$

holds with $c \equiv c(\texttt{data})$.

Lemma 4.1.18. *Let $\theta \in (0,1)$ be such that (4.43) holds and let $M \geq 8$ be as in (4.30). Then the inequality*

$$\begin{aligned}
&\fint_{B_R} |A(Du)-A(Dv)|\,dx \\
&\leq \varepsilon M^{2n}\left(1+\frac{1}{\theta}\right)\fint_{B_{4MR}} |A(Du)-(A(Du))_{B_{4MR}}|\,dx \\
&\quad + c_\varepsilon\left[\frac{|\mu|(B_{8R})}{(8R)^{n-1}}\right] + c_\varepsilon\left(\fint_{B_{8R}} \varphi^*(|A(D\psi)-(A(D\psi))_{B_{8R}}|)\,dx\right)^{\frac{1}{p'}}
\end{aligned} \tag{4.45}$$

holds for any $\varepsilon \in (0,1]$, where $c_\varepsilon \equiv c_\varepsilon(\texttt{data},\varepsilon)$ is proportional to some negative power of ε.

Proof. When $p=2$, (4.45) follows immediately from (2.12), (4.22) and (4.28).

When $p>2$, we use (2.12) and Young's inequality to estimate

$$\begin{aligned}
&\fint_{B_R} |A(Du)-A(Dv)|\,dx \\
&\leq c\fint_{B_R} (|Du|^2+s^2)^{\frac{p-2}{2}}|Du-Dv|\,dx + c\fint_{B_R} |Du-Dv|^{p-1}\,dx \\
&\leq \varepsilon\fint_{B_R} (|Du|^2+s^2)^{\frac{p-1}{2}}\,dx + c\varepsilon^{2-p}\fint_{B_R} |Du-Dv|^{p-1}\,dx
\end{aligned}$$

for any $\varepsilon \in (0,1]$. Then (4.22), (4.28) and (4.44) yield (4.45).

When $2-1/n<p<2$, we have

$$
\begin{aligned}
&\fint_{B_{4R}} |A(Du)-A(Dw_1)|\,dx \overset{(2.12)}{\leq} c\fint_{B_{4R}} |Du-Dw_1|^{p-1}\,dx \\
&\overset{(4.25)}{\leq} c\left[\frac{|\mu|(B_{8R})}{(8R)^{n-1}}\right] + c\left[\frac{|\mu|(B_{8R})}{(8R)^{n-1}}\right]^{p-1}\left(\fint_{B_{8R}} (|Du|+s)^{p-1}\,dx\right)^{2-p} \\
&\quad + \left[\frac{|\mu|(B_{8R})}{(8R)^{n-1}}\right]^{p-1}\left(\fint_{B_{8R}} \varphi^*(|A(D\psi)-(A(D\psi))_{B_{8R}}|)\,dx\right)^{\frac{2-p}{p'}} \\
&\leq \varepsilon\fint_{B_{8R}} (|Du|^2+s^2)^{\frac{p-1}{2}}\,dx + c\varepsilon^{\frac{p-2}{p-1}}\left[\frac{|\mu|(B_{8R})}{(8R)^{n-1}}\right] \\
&\quad + c\left(\fint_{B_{8R}} \varphi^*(|A(D\psi)-(A(D\psi))_{B_{8R}}|)\,dx\right)^{\frac{1}{p'}}
\end{aligned}
$$

for any $\varepsilon \in (0,1]$, where we have applied Young's inequality in the last line. Combining this estimate with (4.42) and using (4.44), we obtain (4.45). □

4.1.6 The two-scales non-degenerate alternative

Here we consider the situation when

$$\fint_{B_{4MR}} |A(Du)-(A(Du))_{B_{4MR}}|\,dx < \theta\left[|(A(Du))_{B_{R/M}}| + s^{p-1}\right] \tag{4.46}$$

is assumed to hold for a number $\theta \in (0,1)$. In the following, we denote

$$\lambda := \left(\fint_{B_{R/M}} (|Du|^2+s^2)^{\frac{p-1}{2}}\,dx\right)^{\frac{1}{p-1}}. \tag{4.47}$$

Then we have the following lemma.

Lemma 4.1.19. *For every $M \geq 8$ as in* (4.30) *and with λ defined as in* (4.47)*, there exists a number $\theta \equiv \theta(M)$ such that if* (4.46) *is in force, then*

$$\fint_{B_{\kappa R}} (|Du|^2+s^2)^{\frac{p-1}{2}}\,dx \leq c\lambda^{p-1}, \quad \forall\ \kappa \in [1/M, 4M] \tag{4.48}$$

holds for a constant $c \equiv c(\mathtt{data})$.

The proof of Lemma 4.1.19 is essentially the same as that of [7, Lemma 5.3]. We only note that in the proof the constant θ is chosen so small that

$$M^{2n}\theta \leq 1. \tag{4.49}$$

We now prove a non-degenerate counterpart of Lemma 4.1.18.

Lemma 4.1.20. *It is possible to determine θ and M as functions of* $\mathtt{data}$ *such that if* (4.46) *is in force, then there holds*

$$\begin{aligned}
&\fint_{B_{R/M}} |A(Du) - A(Dv)|\,dx \\
&\leq c\left[\frac{|\mu|(B_{4MR})}{(4MR)^{n-1}}\right] + c\left(\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|)\,dx\right)^{\frac{1}{p'}} \\
&\quad + c\chi_{\{p>2\}}\left(\fint_{B_{4MR}} (|A(Du)| + s^{p-1})\,dx\right)^{\frac{p-2}{2(p-1)}} \\
&\qquad \cdot \left(\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|)\,dx\right)^{\frac{1}{2}}
\end{aligned} \tag{4.50}$$

for a constant $c \equiv c(\mathtt{data})$.

In the proof of Lemma 4.1.20, we will distinguish two cases, making use of another free parameter $\sigma_1 \in (0,1)$. The first one is when the following inequality holds:

$$\left[\frac{|\mu|(B_{4MR})}{(4MR)^{n-1}}\right] + \left(\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|)\,dx\right)^{\frac{1}{p'}} \leq \sigma_1\lambda^{p-1}. \tag{4.51}$$

The second one is when the above inequality fails; that is,

$$\lambda^{p-1} < \frac{1}{\sigma_1}\left[\frac{|\mu|(B_{4MR})}{(4MR)^{n-1}}\right] + \frac{1}{\sigma_1}\left(\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|)\,dx\right)^{\frac{1}{p'}}. \tag{4.52}$$

The value of σ_1 will be determined in Lemma 4.1.21 below.

Proof of Lemma 4.1.20 in the first case (4.51) **and determination of** σ_1

Lemma 4.1.21. *There exists a choice of the parameters*

$$M \equiv M(\texttt{data}) \geq 8 \qquad \text{and} \qquad \sigma_1 \equiv \sigma_1(\texttt{data}, M) \in (0,1)$$

such that, if $\theta \equiv \theta(M)$ is the constant determined in Lemma 4.1.19 and (4.46) *is in force, then the following bounds hold:*

$$\frac{\lambda}{\tilde{c}_l} \leq |Dv_*| + s \ \text{ in } B_{4R} \quad \text{and} \quad |Dv_*| + s \leq \tilde{c}_u \lambda \ \text{ in } B_{MR/2} \qquad \text{when } p \geq 2, \tag{4.53}$$

and

$$\frac{\lambda}{c_l} \leq |Dv| + s \ \text{ in } B_{4R/M} \quad \text{and} \quad |Dv| + s \leq c_u \lambda \ \text{ in } B_{R/2}, \tag{4.54}$$

with constants $\tilde{c}_l, c_l, \tilde{c}_u, c_u$ depending only on $\texttt{data}$.

Remark 4.1.22. *For brevity, we state and prove* (4.53) *only for $p \geq 2$. In the case $2 - 1/n < p < 2$, we can also prove* (4.53) *in a very similar way as in the proof of* (4.54).

Proof. Step 1: Proof of (4.53). For the upper bound, we use (2.22), (4.22) and (4.28) to have

$$\begin{aligned}
\sup_{B_{MR/2}} (|Dv_*| + s) &\leq c \left(\fint_{B_{MR}} (|Dv_*|^2 + s^2)^{\frac{p-1}{2}} \, dx \right)^{\frac{1}{p-1}} \\
&\leq c \left(\fint_{B_{MR}} (|Du|^2 + s^2)^{\frac{p-1}{2}} \, dx \right)^{\frac{1}{p-1}} + c \left(\fint_{B_{MR}} |Du - Dv_*|^{p-1} \, dx \right)^{\frac{1}{p-1}} \\
&\leq c\lambda + c \left[\frac{|\mu|(B_{4MR})}{(4MR)^{n-1}} \right]^{\frac{1}{p-1}} + c \left(\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|) \, dx \right)^{\frac{1}{p}} \\
&\overset{(4.51)}{\leq} \tilde{c}_u \lambda.
\end{aligned} \tag{4.55}$$

We now prove the lower bound. By using (4.48), we fix a constant $c_4 \equiv$

$c_4(\mathtt{data}) > 1$ satisfying

$$\frac{\lambda^{p-1}}{c_4} \le (|A(Du)|)_{B_{4R/M}} + s^{p-1} \le c_4\lambda^{p-1} \tag{4.56}$$

to find

$$\begin{aligned}
&(|A(Dv_*)|)_{B_{4R/M}} + s^{p-1} \\
&\ge (|A(Du)|)_{B_{4R/M}} + s^{p-1} - \left|(|A(Dv_*)|)_{B_{4R/M}} - (|A(Du)|)_{B_{4R/M}}\right| \\
&= (|A(Du)|)_{B_{4R/M}} + s^{p-1} - \left|\fint_{B_{4R/M}} (|A(Dv_*)| - |A(Du)|)\,dx\right| \\
&\ge \frac{\lambda^{p-1}}{c_4} - \fint_{B_{4R/M}} |A(Dv_*) - A(Du)|\,dx.
\end{aligned} \tag{4.57}$$

By (4.55), (4.22), (4.28) and (4.51), the last integral in (4.57) is estimated as

$$\begin{aligned}
&\fint_{B_{4R/M}} |A(Dv_*) - A(Du)|\,dx \\
&\le c\fint_{B_{4R/M}} |Du - Dv_*|^{p-1}\,dx + c\fint_{B_{4R/M}} (|Dv_*|^2 + s^2)^{\frac{p-2}{2}}|Du - Dv_*|\,dx \\
&\le cM^{2n}\fint_{B_{MR}} |Du - Dv_*|^{p-1}\,dx + cM^{2n}\lambda^{p-2}\left(\fint_{B_{MR}} |Du - Dv_*|^{p-1}\,dx\right)^{\frac{1}{p-1}} \\
&\le cM^{2n}\left\{\left[\frac{|\mu|(B_{4MR})}{(4MR)^{n-1}}\right] + \left(\fint_{B_{MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{MR}}|)\,dx\right)^{\frac{1}{p'}}\right\} \\
&\quad + cM^{2n}\lambda^{p-2}\left\{\left[\frac{|\mu|(B_{4MR})}{(4MR)^{n-1}}\right] + \left(\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|)\,dx\right)^{\frac{1}{p'}}\right\}^{\frac{1}{p-1}} \\
&\le c_5M^{2n}\left[\sigma_1 + \sigma_1^{1/(p-1)}\right]\lambda^{p-1},
\end{aligned}$$

where $c_5 \equiv c_5(\mathtt{data})$. At this stage, we choose $\sigma_1 \equiv \sigma_1(\mathtt{data}, M)$ so small that

$$c_5M^{2n}\left[\sigma_1 + \sigma_1^{1/(p-1)}\right] \le \frac{1}{2c_4} \tag{4.58}$$

in order to have

$$\fint_{B_{4R/M}} |A(Dv_*) - A(Du)|\,dx \leq \frac{\lambda^{p-1}}{2c_4}$$

and therefore

$$(|A(Dv_*)|)_{B_{4R/M}} + s^{p-1} \geq \frac{\lambda^{p-1}}{2c_4}.$$

In particular, there exists a point $\tilde{x}_0 \in B_{4R/M}$ such that

$$|A(Dv_*(\tilde{x}_0))| + s^{p-1} \geq \frac{\lambda^{p-1}}{2c_4}. \tag{4.59}$$

We then apply Lemma 2.3.7, (2.13) and (4.55) to see that

$$\operatorname*{osc}_{B_{4R}} A(Dv_*) \leq \frac{c}{M^{\alpha_A}} \fint_{B_{MR/2}} |A(Dv_*)|\,dx \leq \frac{c_6}{M^{\alpha_A}} \lambda^{p-1}$$

holds for some $c_6 \equiv c_6(\mathtt{data})$. Now, choosing $M \equiv M(\mathtt{data})$ so large that

$$\frac{c_6}{M^{\alpha_A}} \leq \frac{1}{4c_4} \implies \operatorname*{osc}_{B_{4R}} A(Dv_*) \leq \frac{\lambda^{p-1}}{4c_4}, \tag{4.60}$$

and then combining this with (4.59), we have

$$|A(Dv_*(x))| + s^{p-1} \geq |A(Dv_*(\tilde{x}_0))| + s^{p-1} - \operatorname*{osc}_{B_{4R}} A(Dv_*) \geq \frac{\lambda^{p-1}}{4c_4} \quad \forall\, x \in B_{4R}.$$

We then recall (2.12) and choose $\tilde{c}_l \equiv \tilde{c}_l(\mathtt{data})$ in a suitable way, to prove the lower bound in (4.53).

Step 2: proof of (4.54). We first prove the upper bound. For $p \geq 2$, we get

$$\begin{aligned}
\sup_{B_{R/2}} (|Dv| + s) &\overset{(2.22)}{\leq} c \left(\fint_{B_R} (|Dv|^2 + s^2)^{\frac{p-1}{2}}\,dx \right)^{\frac{1}{p-1}} \\
&\leq c \left(\fint_{B_R} (|Du|^2 + s^2)^{\frac{p-1}{2}}\,dx \right)^{\frac{1}{p-1}} + c \left(\fint_{B_R} |Du - Dv|^{p-1}\,dx \right)^{\frac{1}{p-1}} \\
&\overset{(4.22),(4.28)}{\leq} c\lambda + cM^{\frac{n-1}{p-1}} \left[\frac{|\mu|(B_{4MR})}{(4MR)^{n-1}} \right]^{\frac{1}{p-1}}
\end{aligned}$$

$$
\begin{aligned}
&+ cM^{\frac{n}{p}} \left(\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|)\, dx \right)^{\frac{1}{p}} \\
&\leq c \left[1 + \left(M^{n-1}\sigma_1 \right)^{1/(p-1)} + \left(M^{n/p'}\sigma_1 \right)^{1/(p-1)} \right] \lambda
\end{aligned}
$$

for a constant $c \equiv c(\mathtt{data})$. In the case $2 - 1/n < p < 2$, using Lemma 4.1.17, we have

$$
\begin{aligned}
&\left[\sup_{B_{R/2}} (|Dv| + s) \right]^{p-1} \leq c \fint_{B_R} (|A(Dv)| + s^{p-1})\, dx \\
&\leq c \fint_{B_R} (|A(Dw_1)| + s^{p-1})\, dx + c \fint_{B_R} |A(Dw_1) - A(Dv)|\, dx \\
&\leq c \fint_{B_{4R}} (|A(Dw_1)| + s^{p-1})\, dx \\
&\quad + cM^{\frac{n}{p'}} \left(\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|)\, dx \right)^{\frac{1}{p'}} .
\end{aligned}
$$

We then use (4.48), (4.25) and (4.51) in order to estimate

$$
\begin{aligned}
&\fint_{B_{4R}} (|A(Dw_1)| + s^{p-1})\, dx \leq \fint_{B_{4R}} (|Dw_1| + s)^{p-1}\, dx \\
&\leq c \fint_{B_{4R}} (|Du| + s)^{p-1}\, dx + c \fint_{B_{4R}} |Du - Dw_1|^{p-1}\, dx \\
&\leq c\lambda^{p-1} + cM^{n-1} \left[\frac{|\mu|(B_{4MR})}{(4MR)^{n-1}} \right] \\
&\quad + cM^{(n-1)(p-1)} \left[\frac{|\mu|(B_{4MR})}{(4MR)^{n-1}} \right]^{p-1} \left(\fint_{B_{4R}} (|Du| + s)^{p-1}\, dx \right)^{2-p} \\
&\quad + cM^{\frac{2n-p}{p'}} \left[\frac{|\mu|(B_{4MR})}{(4MR)^{n-1}} \right]^{p-1} \left(\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|)\, dx \right)^{\frac{2-p}{p'}} \\
&\leq c \left[1 + M^{n-1}\sigma_1 + \left(M^{n-1}\sigma_1 \right)^{p-1} + M^{(2n-p)/p'}\sigma_1 \right] \lambda^{p-1}. \qquad (4.61)
\end{aligned}
$$

Combining the above two estimates and using (4.51), we arrive at

$$\begin{aligned}&\left[\sup_{B_{R/2}}(|Dv|+s)\right]^{p-1}\\&\le c\left[1+M^{n-1}\sigma_1+\left(M^{n-1}\sigma_1\right)^{p-1}+M^{(2n-p)/p'}\sigma_1+M^{n/p'}\sigma_1\right]\lambda^{p-1}\end{aligned}$$

for a constant $c\equiv c(\mathtt{data})$. By choosing $\sigma_1\equiv\sigma_1(\mathtt{data},M)$ such that

$$\left(M^{n-1}\sigma_1\right)^{\frac{1}{p-1}}+M^{n-1}\sigma_1+\left(M^{n/p'}\sigma_1\right)^{\frac{1}{p-1}}+\left(M^{(2n-p)/p'}\sigma_1\right)^{\frac{1}{p-1}}\le 1, \tag{4.62}$$

in any case we conclude with

$$\sup_{B_{R/2}}(|Dv|+s)\le c_u\lambda. \tag{4.63}$$

To prove the lower bound, we recall (4.56) and argue as in (4.57), with v_* replaced by v, to find

$$(|A(Dv)|)_{B_{4R/M}}+s^{p-1}\ge\frac{\lambda^{p-1}}{c_4}-\fint_{B_{4R/M}}|A(Dv)-A(Du)|\,dx. \tag{4.64}$$

We need to estimate the last integral. We again distinguish two different cases. When $p\ge 2$, we have

$$\begin{aligned}&\fint_{B_{4R/M}}|A(Dv)-A(Du)|\,dx\\&\overset{(2.12)}{\le} c\fint_{B_{4R/M}}|Du-Dv|^{p-1}\,dx+c\fint_{B_{4R/M}}(|Dv|^2+s^2)^{\frac{p-2}{2}}|Du-Dv|\,dx\\&\overset{(4.55)}{\le} cM^n\fint_{B_R}|Du-Dv|^{p-1}\,dx+cM^n\lambda^{p-2}\left(\fint_{B_R}|Du-Dv|^{p-1}\,dx\right)^{\frac{1}{p-1}}\\&\overset{(4.22),(4.28)}{\le} cM^{2n-1}\left[\frac{|\mu|(B_{4MR})}{(4MR)^{n-1}}\right]\\&\qquad+cM^{n+\frac{n}{p'}}\left(\fint_{B_{4MR}}\varphi^*(|A(D\psi)-(A(D\psi))_{B_{4MR}}|)\,dx\right)^{\frac{1}{p'}}\end{aligned}$$

$$
\begin{aligned}
&+ c\lambda^{p-2} M^{n+\frac{n-1}{p-1}} \left[\frac{|\mu|(B_{4MR})}{(4MR)^{n-1}}\right]^{\frac{1}{p-1}} \\
&+ c\lambda^{p-2} M^{n+\frac{n}{p}} \left(\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|)\,dx\right)^{\frac{1}{p}} \\
\overset{(4.51)}{\le}\ & c_{41}\Big[M^{2n-1}\sigma_1 + M^{n(2p-1)/p}\sigma_1 \\
&\quad + (M^{np-1}\sigma_1)^{1/(p-1)} + (M^{n(p^2-1)/p}\sigma_1)^{1/(p-1)}\Big]\lambda^{p-1}
\end{aligned}
$$

for a constant $c_{41} \equiv c_{41}(\mathtt{data})$. Choosing $\sigma_1 \equiv \sigma_1(\mathtt{data}, M)$ such that

$$
c_{41}\left[M^{2n-1}\sigma_1 + M^{\frac{n(2p-1)}{p}}\sigma_1 + \left(M^{np-1}\sigma_1\right)^{\frac{1}{p-1}} + \left(M^{\frac{n(p^2-1)}{p}}\sigma_1\right)^{\frac{1}{p-1}}\right] \le \frac{1}{2c_4}, \tag{4.65}
$$

we have

$$
\fint_{B_{4R/M}} |A(Dv) - A(Du)|\,dx \le \frac{\lambda^{p-1}}{2c_4}. \tag{4.66}
$$

When $2 - 1/n < p < 2$, we split the integral as

$$
\begin{aligned}
&\fint_{B_{4R/M}} |A(Dv) - A(Du)|\,dx \\
&\le cM^n \fint_{B_R} |A(Dv) - A(Dw_1)|\,dx + cM^n \fint_{B_{4R}} |A(Dw_1) - A(Du)|\,dx \\
&=: I_1 + I_2.
\end{aligned} \tag{4.67}
$$

We estimate I_2 as

$$
\begin{aligned}
I_2 \overset{(2.12)}{\le}\ & cM^n \fint_{B_{4R}} |Dw_1 - Du|^{p-1}\,dx \\
\overset{(4.25)}{\le}\ & cM^{2n-1}\left[\frac{|\mu|(B_{4MR})}{(4MR)^{n-1}}\right] \\
&+ cM^{np-p+1}\left[\frac{|\mu|(B_{4MR})}{(4MR)^{n-1}}\right]^{p-1}\left(\fint_{B_{4R}} (|Du| + s)^{p-1}\,dx\right)^{2-p} \\
&+ cM^{\frac{2n-p}{p'}+1}\left[\frac{|\mu|(B_{4MR})}{(4MR)^{n-1}}\right]^{p-1}\left(\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|)\,dx\right)^{\frac{2-p}{p'}}
\end{aligned}
$$

$$\overset{(4.51)}{\leq} c_{42}\left[M^{2n-1}\sigma_1 + M^{2n-p+1}\sigma_1^{p-1} + M^{(2n-p)/p'+1}\sigma_1\right]\lambda^{p-1}$$

for a constant $c_{42} \equiv c_{42}(\mathtt{data})$. Choosing $\sigma_1 \equiv \sigma_1(\mathtt{data}, M)$ such that

$$c_{42}\left[M^{2n-1}\sigma_1 + M^{2n-p+1}\sigma_1^{p-1} + M^{(2n-p)/p'+1}\sigma_1\right] \leq \frac{1}{4c_4}, \tag{4.68}$$

we arrive at

$$I_2 \leq \frac{\lambda^{p-1}}{4c_4}. \tag{4.69}$$

As for I_1, we have

$$\begin{aligned} I_1 &\overset{(4.42)}{\leq} cM^n\varepsilon \fint_{B_{4R}} |A(Dw_1) - (A(Dw_1))_{B_{4R}}|\,dx \\ &\quad + c_\varepsilon M^n \left(\fint_{B_{4R}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|)\,dx\right)^{\frac{1}{p'}} \\ &\leq cM^n\varepsilon\left(\fint_{B_{4R}} |A(Du) - (A(Du))_{B_{4R}}|\,dx + \fint_{B_{4R}} |A(Du) - A(Dw_1)|\,dx\right) \\ &\quad + c_\varepsilon M^{n\left(1+\frac{1}{p'}\right)}\left(\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|)\,dx\right)^{\frac{1}{p'}} \\ &\leq cM^{2n}\varepsilon \fint_{B_{4MR}} |A(Du) - (A(Du))_{B_{4MR}}|\,dx + c\varepsilon I_2 + c_\varepsilon M^{\frac{n(2p-1)}{p}}\sigma_1\lambda^{p-1} \\ &\overset{(4.46),(4.69)}{\leq} cM^{2n}\theta\varepsilon\lambda^{p-1} + c\varepsilon\lambda^{p-1} + c_\varepsilon M^{\frac{n(2p-1)}{p}}\lambda^{p-1} \\ &\overset{(4.49)}{\leq} c_{43}\left[\varepsilon + c_\varepsilon M^{\frac{n(2p-1)}{p}}\sigma_1\right]\lambda^{p-1} \end{aligned}$$

for constant $c_{43} \equiv c_{43}(\mathtt{data})$ and $c_\varepsilon \equiv c_\varepsilon(\mathtt{data}, \varepsilon)$, whenever $\varepsilon \in (0,1)$. Then, choosing $\varepsilon = 1/(8c_{43}c_4)$ and then $\sigma_1 \equiv \sigma_1(\mathtt{data}, M)$ satisfying

$$c_\varepsilon c_{43} M^{\frac{n(2p-1)}{p}}\sigma_1 \leq \frac{1}{8c_4}, \tag{4.70}$$

it follows that

$$I_1 \leq \frac{\lambda^{p-1}}{4c_4}. \tag{4.71}$$

Connecting the estimates (4.66), (4.67), (4.69) and (4.71) to (4.64), in any case we conclude that

$$(|A(Dv)|)_{B_{4R/M}} + s^{p-1} \geq \frac{\lambda^{p-1}}{2c_4}.$$

We now proceed in a similar way as in *Step 1*. We first choose a point $x_0 \in B_{4R/M}$ satisfying

$$|A(Dv(x_0))| + s^{p-1} \geq \frac{\lambda^{p-1}}{2c_4}. \tag{4.72}$$

Then, again using Lemma 2.3.7, (2.13) and (4.63), we find that

$$\operatorname*{osc}_{B_{4R/M}} A(Dv) \leq \frac{c}{M^{\alpha_A}} \fint_{B_{R/2}} |A(Dv)|\, dx \leq \frac{c_7}{M^{\alpha_A}} \lambda^{p-1}$$

holds for a constant $c_7 \equiv c_7(\texttt{data})$. Choosing M such that

$$\frac{c_7}{M^{\alpha_A}} \leq \frac{1}{4c_4} \tag{4.73}$$

and then combining the resulting inequality with (4.72), we conclude with the lower bound in (4.54) for a suitable constant $c_l \equiv c_l(\texttt{data})$. □

Remark 4.1.23. *We summarize the process of fixing the constants θ, M and σ_1. We first fix $M \equiv M(\texttt{data})$ as in Lemma 4.1.21 satisfying* (4.60) *and* (4.73)*. Then, by Lemma 4.1.19, we determine $\theta \equiv \theta(\texttt{data})$ satisfying* (4.49)*. In a similar way, we finally determine $\sigma_1 \equiv \sigma_1(\texttt{data})$ as in Lemma 4.1.21, by requiring that* (4.58), (4.62), (4.65), (4.68) *and* (4.70) *are satisfied. Consequently, we have chosen θ, M and σ_1 as universal constants for which the assertions of Lemmas 4.1.19 and 4.1.21 hold simultaneously. These values of the parameters will be used in the rest of this section.*

We now prove estimate (4.50). To do this, we first consider the case $p \geq 2$. We estimate

$$\begin{aligned}
&\fint_{B_{R/M}} |A(Du) - A(Dv)|\, dx \\
&\overset{(2.12)}{\leq} c \fint_{B_{R/M}} (|Dv| + s)^{p-2} |Du - Dv|\, dx + c \fint_{B_{R/M}} |Du - Dv|^{p-1}\, dx \\
&\leq c \fint_{B_{R/M}} (|Dv| + s)^{p-2} |Du - Dw_1|\, dx
\end{aligned}$$

$$
\begin{aligned}
&+ c \fint_{B_{R/M}} (|Dv| + s)^{p-2} |Dw_1 - Dv| \, dx + c \fint_{B_{R/M}} |Du - Dv|^{p-1} \, dx \\
&\overset{p \geq 2}{\leq} c \left[\sup_{B_{R/M}} (|Dv| + s) \right]^{p-2} \fint_{B_{R/M}} |Du - Dw_1| \, dx \\
&\quad + c \left[\sup_{B_{R/M}} (|Dv| + s) \right]^{\frac{p-2}{2}} \fint_{B_{R/M}} (|Dw_1| + |Dv| + s)^{\frac{p-2}{2}} |Dw_1 - Dv| \, dx \\
&\quad + c \fint_{B_{R/M}} |Du - Dv|^{p-1} \, dx.
\end{aligned}
$$

Then estimates (4.22) and (4.28), together with the gradient bound $(4.54)_2$, imply

$$
\begin{aligned}
&\fint_{B_{R/M}} |A(Du) - A(Dv)| \, dx \\
&\leq c\lambda^{p-2} \fint_{B_{4R}} |Du - Dw_1| \, dx + c\lambda^{\frac{p-2}{2}} \fint_{B_R} |V(Dw_1) - V(Dv)| \, dx \\
&\quad + c \fint_{B_R} |Du - Dv|^{p-1} \, dx \\
&\leq c\lambda^{p-2} \fint_{B_{4R}} |Du - Dw_1| \, dx \\
&\quad + c\lambda^{\frac{p-2}{2}} \left(\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|) \, dx \right)^{\frac{1}{2}} \\
&\quad + c \left[\frac{|\mu|(B_{4MR})}{(4MR)^{n-1}} \right] + c \left(\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|) \, dx \right)^{\frac{1}{p'}}.
\end{aligned}
$$

Moreover, since we have (4.51) and (4.53), we can apply Lemma 4.1.16 to estimate the first integral in the right-hand side. Recalling the definition of λ in (4.47), we conclude that

$$
\begin{aligned}
&\fint_{B_{R/M}} |A(Du) - A(Dv)| \, dx \\
&\leq c \left[\frac{|\mu|(B_{4MR})}{(4MR)^{n-1}} \right] + c \left(\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|) \, dx \right)^{\frac{1}{p'}}
\end{aligned}
$$

$$
\begin{aligned}
&+ c\left(\fint_{B_{4MR}} (|A(Du)| + s^{p-1})\,dx\right)^{\frac{p-2}{2(p-1)}} \\
&\cdot \left(\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|)\,dx\right)^{\frac{1}{2}}.
\end{aligned}
$$

In the case $2 - 1/n < p < 2$, we have

$$
\begin{aligned}
&\fint_{B_{R/M}} |A(Du) - A(Dv)|\,dx \\
&\overset{(2.12)}{\leq} c\fint_{B_{R/M}} (|Du| + |Dv| + s)^{p-2}|Du - Dv|\,dx \\
&\overset{p<2}{\leq} c\left[\inf_{B_{R/M}} (|Dv| + s)\right]^{p-2} \fint_{B_{R/M}} |Du - Dv|\,dx \\
&\overset{(4.54)}{\leq} c\lambda^{p-2}\left(\fint_{B_{4R}} |Du - Dw_1|\,dx + \fint_{B_R} |Dw_1 - Dv|\,dx\right). \qquad (4.74)
\end{aligned}
$$

We now estimate the two integrals in the right-hand side of (4.74). The first integral is estimated as

$$
\begin{aligned}
&\lambda^{p-2}\fint_{B_{4R}} |Du - Dw_1|\,dx \\
&\overset{(4.25)}{\leq} c\lambda^{p-2}\left[\frac{|\mu|(B_{8R})}{(8R)^{n-1}}\right]^{\frac{1}{p-1}} + c\lambda^{p-2}\left[\frac{|\mu|(B_{8R})}{(8R)^{n-1}}\right]\left(\fint_{B_{8R}} (|Du| + s)^{p-1}\,dx\right)^{\frac{2-p}{p-1}} \\
&\qquad + c\lambda^{p-2}\left[\frac{|\mu|(B_{8R})}{(8R)^{n-1}}\right]\left(\fint_{B_{8R}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{8R}}|)\,dx\right)^{\frac{2-p}{p}} \\
&\overset{(4.51)}{\leq} c\left[\frac{|\mu|(B_{4MR})}{(4MR)^{n-1}}\right]. \qquad (4.75)
\end{aligned}
$$

As for the second one, we recall the following estimate in [168, (9.39)]:

$$
|Dw_1 - Dv| \leq c|V(Dw_1) - V(Dv)|^{\frac{2}{p}} + c|V(Dw_1) - V(Dv)|(|Dv| + s)^{\frac{2-p}{2}}.
$$

Then we use Hölder's inequality to get

$$
\begin{aligned}
&\lambda^{p-2} \fint_{B_R} |Dw_1 - Dv| \, dx \\
&\le c\lambda^{p-2} \left(\fint_{B_R} |V(Dw_1) - V(Dv)|^2 \, dx \right)^{\frac{1}{p}} \\
&\quad + c\lambda^{p-2} \left[\sup_{B_{R/M}} (|Dv| + s) \right]^{\frac{2-p}{2}} \left(\fint_{B_R} |V(Dw_1) - V(Dv)|^2 \, dx \right)^{\frac{1}{2}} \\
&\overset{(4.54)}{\le} c\lambda^{p-2} \left(\fint_{B_R} |V(Dw_1) - V(Dv)|^2 \, dx \right)^{\frac{1}{p}} \\
&\quad + c\lambda^{\frac{p-2}{2}} \left(\fint_{B_R} |V(Dw_1) - V(Dv)|^2 \, dx \right)^{\frac{1}{2}} \\
&=: I_1 + I_2.
\end{aligned} \tag{4.76}
$$

In order to estimate I_1 and I_2, we discover

$$
\begin{aligned}
&\fint_{B_R} |V(Dw_1) - V(Dv)|^2 \, dx \overset{(4.29)}{\le} c \fint_{B_{2R}} (\varphi_{|Dw_1|})^*(|A(D\psi) - (A(D\psi))_{B_{2R}}|) \, dx \\
&\overset{(2.16)}{\le} c \fint_{B_{2R}} |A(D\psi) - (A(D\psi))_{B_{2R}}|^{p'} \, dx \\
&\qquad + c \fint_{B_{2R}} (|Dw_1| + s)^{2-p} |A(D\psi) - (A(D\psi))_{B_{2R}}|^2 \, dx \\
&\le c \fint_{B_{2R}} |A(D\psi) - (A(D\psi))_{B_{2R}}|^{p'} \, dx \\
&\quad + c \left(\fint_{B_{2R}} (|Dw_1| + s)^p \, dx \right)^{\frac{2-p}{p}} \left(\fint_{B_{2R}} |A(D\psi) - (A(D\psi))_{B_{2R}}|^{p'} \, dx \right)^{\frac{2}{p'}} \\
&\le c \left[\left(\fint_{B_{2R}} |A(D\psi) - (A(D\psi))_{B_{2R}}|^{p'} \, dx \right)^{\frac{2-p}{p}} + \left(\fint_{B_{2R}} (|Dw_1| + s)^p \, dx \right)^{\frac{2-p}{p}} \right] \\
&\qquad \cdot \left(\fint_{B_{2R}} |A(D\psi) - (A(D\psi))_{B_{2R}}|^{p'} \, dx \right)^{\frac{2}{p'}} \\
&\le c\lambda^{2-p} \left(\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|) \, dx \right)^{\frac{2}{p'}},
\end{aligned}
$$

where for the last inequality, we have used (4.51), the fact that $t^{p'} \lesssim \varphi^*(t)$ for $p < 2$, and the following estimate from (4.11), (4.51) and(4.61):

$$\begin{aligned}
&\fint_{B_{2R}} (|Dw_1| + s)^p \, dx \le c \fint_{B_{2R}} |V(Dw_1)|^2 \, dx + cs^p \\
&\le c \left(\fint_{B_{4R}} |V(Dw_1)|^{\frac{2}{p'}} \, dx \right)^{p'} + cs^p + c \fint_{B_{4R}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|) \, dx \\
&\le c \left(\fint_{B_{4R}} (|A(Dw_1)| + s^{p-1}) \, dx \right)^{p'} + c \fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|) \, dx \\
&\le c\lambda^p.
\end{aligned}$$

This immediately yields

$$I_2 \le c \left(\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|) \, dx \right)^{\frac{1}{p'}} \tag{4.77}$$

and

$$\begin{aligned}
I_1 &\le c\lambda^{p-2} \cdot \lambda^{\frac{2-p}{p}} \left(\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|) \, dx \right)^{\frac{2}{p'p}} \\
&= c\lambda^{\frac{p-2}{p'}} \left(\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|) \, dx \right)^{\frac{1}{p}\frac{2-p}{p'} + \frac{1}{p'}} \\
&\overset{(4.51)}{\le} c \left(\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|) \, dx \right)^{\frac{1}{p'}}.
\end{aligned} \tag{4.78}$$

Combining all the above estimates (4.74)-(4.78) leads to the desired estimate (4.50).

Proof of Lemma 4.1.20 in the second case (4.52)

We observe that, from (4.48) and (4.52),

$$\begin{aligned}
&\fint_{B_{\kappa R}} (|Du|^2 + s^2)^{\frac{p-1}{2}} \, dx \\
&\le c \left[\frac{|\mu|(B_{4MR})}{(4MR)^{n-1}} \right] + c \left(\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|) \, dx \right)^{\frac{1}{p'}}
\end{aligned} \tag{4.79}$$

holds whenever $\kappa \in [1/M, 4M]$, where $c \equiv c(\texttt{data})$. We recall that $\sigma_1 \equiv \sigma_1(\texttt{data})$ has been determined in Lemma 4.1.21.

Now we prove (4.50). Again, it is straightforward from (4.25) when $p = 2$. If $p > 2$, we estimate

$$\begin{aligned}
&\fint_{B_{R/M}} |A(Du) - A(Dv)|\, dx \\
&\overset{(2.12)}{\leq} c \fint_{B_{R/M}} (|Du|^2 + s^2)^{\frac{p-2}{2}} |Du - Dv|\, dx + c \fint_{B_{R/M}} |Du - Dv|^{p-1}\, dx \\
&\leq c \fint_{B_{R/M}} (|Du|^2 + s^2)^{\frac{p-1}{2}}\, dx + cM^n \fint_{B_R} |Du - Dv|^{p-1}\, dx \\
&\overset{(4.79),(4.22)}{\leq} c \left[\frac{|\mu|(B_{4MR})}{(4MR)^{n-1}}\right] + c \left(\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|)\, dx\right)^{\frac{1}{p'}}.
\end{aligned}$$

If $2 - 1/n < p < 2$, we instead have

$$\begin{aligned}
&\fint_{B_{R/M}} |A(Du) - A(Dw_1)|\, dx \overset{(2.12)}{\leq} cM^n \fint_{B_{4R}} |Du - Dw_1|^{p-1}\, dx \\
&\overset{(4.25)}{\leq} c \left[\frac{|\mu|(B_{8R})}{(8R)^{n-1}}\right] + c \left[\frac{|\mu|(B_{8R})}{(8R)^{n-1}}\right]^{p-1} \left(\fint_{B_{8R}} (|Du|^2 + s^2)^{\frac{p-1}{2}}\, dx\right)^{2-p} \\
&\qquad + c \left[\frac{|\mu|(B_{8R})}{(8R)^{n-1}}\right]^{p-1} \left(\fint_{B_{8R}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{8R}}|)\, dx\right)^{\frac{2-p}{p'}} \\
&\leq c \left[\frac{|\mu|(B_{8R})}{(8R)^{n-1}}\right] + c \fint_{B_{8R}} (|Du|^2 + s^2)^{\frac{p-1}{2}}\, dx \\
&\qquad + c \left(\fint_{B_{8R}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{8R}}|)\, dx\right)^{\frac{1}{p'}} \\
&\overset{(4.79)}{\leq} c \left[\frac{|\mu|(B_{4MR})}{(4MR)^{n-1}}\right] + c \left(\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|)\, dx\right)^{\frac{1}{p'}}.
\end{aligned} \tag{4.80}$$

We then apply Lemma 4.1.17 to discover

$$\fint_{B_{R/M}} |A(Dw_1) - A(Dv)|\, dx$$

$$
\begin{aligned}
&\leq c \fint_{B_{4R}} |A(Dw_1) - (A(Dw_1))_{B_{4R}}|\, dx \\
&\quad + c \left(\fint_{B_{4R}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|)\, dx \right)^{\frac{1}{p'}} \\
&\leq c \fint_{B_{4R}} |A(Du) - (A(Du))_{B_{4R}}|\, dx + c \fint_{B_{4R}} |A(Du) - A(Dw_1)|\, dx \\
&\quad + c \left(\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|)\, dx \right)^{\frac{1}{p'}} .
\end{aligned}
$$

In the above display, the first term in the right-hand side is estimated by using (4.46) and (4.79); the second one is estimated in the same way as in (4.80). Combining the resulting estimate with (4.80) gives (4.50). The proof for the full range $p > 2 - 1/n$ is now complete.

4.1.7 Combining the two alternatives

Taking account of the above two alternatives, we conclude with the following comparison estimate.

Lemma 4.1.24. *Let u and v be the weak solutions to* (4.2) *and* (4.27)*, respectively, under assumptions* (2.8) *and* (4.1)*. Then we have*

$$
\begin{aligned}
&\fint_{B_{R/M}} |A(Du) - A(Dv)|\, dx \\
&\leq \varepsilon \fint_{B_{4MR}} |A(Du) - (A(Du))_{B_{4MR}}|\, dx \\
&\quad + c_\varepsilon \left[\frac{|\mu|(B_{4MR})}{(4MR)^{n-1}} \right] + c_\varepsilon \left(\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|)\, dx \right)^{\frac{1}{p'}} \\
&\quad + c\chi_{\{p>2\}} \left(\fint_{B_{4MR}} (|A(Du)| + s^{p-1})\, dx \right)^{\frac{p-2}{2(p-1)}} \\
&\qquad \cdot \left(\fint_{B_{4MR}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{4MR}}|)\, dx \right)^{\frac{1}{2}} \qquad (4.81)
\end{aligned}
$$

for any $\varepsilon \in (0,1)$, where $c_\varepsilon \equiv c_\varepsilon(\mathtt{data}, \varepsilon)$ is proportional to some negative power of ε, and c depends only on $\mathtt{data}$.

Remark 4.1.25. *The difficulty that prevents us from obtaining* (4.6) *for* $p > 2$ *arises when* (4.46) *and* (4.51) *hold. We could not linearize the last term on the right-hand side of* (4.81) *by adapting the approach in [7]. On the other hand, one might expect to apply the method in [35] to the setting of obstacle problems. Note that the linear Calderón-Zygmund theory [35, Theorem 2.16] is the crucial tool in their approach to handling the non-degenerate alternative case. Indeed, in our setting, we are forced to consider some linear obstacle problems, which makes it hard to apply the linear Calderón-Zygmund theory below the duality exponent.*

Excess decay estimates for limits of approximating solutions

Note that in Sections 4.1.3 and 4.1.4, we have obtained comparison estimates for weak solutions to (4.2) under assumption (4.17), which ensures the existence of weak solutions. In this section, we first prove the following excess decay estimates for weak solutions to (4.2). Note that we have chosen the constant M depending only on `data` in the previous section.

Lemma 4.1.26. *Let* $u \in \mathcal{A}^g_\psi(\Omega)$ *be the weak solution to* (4.2) *under assumptions* (2.8) *and* (4.1)*. Then*

$$
\begin{aligned}
&\fint_{B_\rho} |A(Du) - (A(Du))_{B_\rho}|\,dx \\
&\le c_{\mathrm{ex}} \left(\frac{\rho}{r}\right)^{\alpha_A} \fint_{B_r} |A(Du) - (A(Du))_{B_r}|\,dx + c\left(\frac{r}{\rho}\right)^{n+\gamma} \left[\frac{|\mu|(B_r)}{r^{n-1}}\right] \\
&\quad + c\left(\frac{r}{\rho}\right)^{n+\gamma} \left(\fint_{B_r} \varphi^*(|A(D\psi) - (A(D\psi))_{B_r}|)\,dx\right)^{\frac{1}{p'}} \\
&\quad + c\chi_{\{p>2\}} \left(\frac{r}{\rho}\right)^{n+\gamma} \left(\fint_{B_r} (|A(Du)| + s^{p-1})\,dx\right)^{\frac{p-2}{2(p-1)}} \\
&\qquad \cdot \left(\fint_{B_r} \varphi^*(|A(D\psi) - (A(D\psi))_{B_r}|)\,dx\right)^{\frac{1}{2}}
\end{aligned}
\tag{4.82}
$$

holds whenever $B_\rho \subset B_r \Subset \Omega$ *are concentric balls, where* $c, c_{\mathrm{ex}} \ge 1$ *and* $\gamma \ge 0$ *depend only on* `data`*, and* $\alpha_A \in (0, 1]$ *is as in Theorem 2.3.6.*

Proof. We may assume $\rho \le r/4M^2$ since one can handle the remaining case directly by enlarging the size of the ball. With the comparison map v as in

(4.27) with $R = r/4M$, we apply Lemma 2.3.6 to find

$$\begin{aligned}
&\fint_{B_\rho} |A(Du) - (A(Du))_{B_\rho}|\,dx \le 2 \fint_{B_\rho} |A(Du) - (A(Dv))_{B_\rho}|\,dx \\
&\le 2 \fint_{B_\rho} |A(Dv) - (A(Dv))_{B_\rho}|\,dx + 2 \fint_{B_\rho} |A(Du) - A(Dv)|\,dx \\
&\le c \left(\frac{\rho}{r}\right)^{\alpha_A} \fint_{B_{r/4M^2}} |A(Dv) - (A(Dv))_{B_{r/4M^2}}|\,dx \\
&\quad + c \left(\frac{r}{\rho}\right)^{n} \fint_{B_{r/4M^2}} |A(Du) - A(Dv)|\,dx \\
&\le c \left(\frac{\rho}{r}\right)^{\alpha_A} \fint_{B_{r/4M^2}} |A(Du) - (A(Du))_{B_{r/4M^2}}|\,dx \\
&\quad + c \left(\frac{r}{\rho}\right)^{n} \fint_{B_{r/M^2}} |A(Du) - A(Dv)|\,dx.
\end{aligned}$$

The last integral is estimated by applying Lemma 4.1.24 with the choice $\varepsilon = (\rho/r)^{\alpha_A}$, which implies the desired estimate. □

Now we have to recall Definition 4.1.1 to proceed further. For any limit of approximating solutions u to $OP(\psi;\mu)$ with $\mu \in \mathcal{M}_b(\Omega)$, there exist a sequence of functions $\{\mu_k\} \subset W^{-1,p'}(\Omega) \cap L^1(\Omega)$ and corresponding weak solutions $\{u_k\} \subset \mathcal{A}^g_\psi(\Omega)$ that satisfy the convergence properties described in Definition 4.1.1. In particular, the strong convergence of Du_k implies that (4.82) continues to hold for any limits of approximating solutions to (4.2).

Lemma 4.1.27. *Let $u \in W^{1,\max\{p-1,1\}}(\Omega)$ with $u \ge \psi$ a.e. in Ω be a limit of approximating solutions to $OP(\psi;\mu)$ under assumptions* (2.8) *and* (4.1). *Then* (4.82) *still holds whenever $B_\rho \subset B_r \Subset \Omega$ are concentric balls.*

4.1.8 Proof of Theorem 4.1.2

We fix a ball $B_{2R} = B_{2R}(x_0) \subset \Omega$ as in the statement of Theorem 4.1.2. In the following, all the balls considered will be centered at x_0.

We choose an integer $K = K(\mathtt{data}) \ge 4M$ such that

$$\frac{c_{\mathrm{ex}}}{K^{\alpha_A}} \le \frac{1}{2}.$$

Applying Lemma 4.1.27 on arbitrary balls $B_\rho = B_{r/K} \subset B_r \Subset \Omega$, we have

$$\begin{aligned} &\fint_{B_{r/K}} |A(Du) - (A(Du))_{B_{r/K}}|\,dx \\ &\le \frac{1}{2}\fint_{B_r} |A(Du) - (A(Du))_{B_r}|\,dx \\ &\quad + c\left[\frac{|\mu|(B_r)}{r^{n-1}}\right] + c\left(\fint_{B_r} \varphi^*(|A(D\psi) - (A(D\psi))_{B_r}|)\,dx\right)^{\frac{1}{p'}}. \end{aligned} \tag{4.83}$$

For $i = 0, 1, 2, \ldots$, we define $R_i \coloneqq R/K^i$, $B_i \coloneqq B_{R_i}(x_0)$,

$$k_i \coloneqq (A(Du))_{B_i} \quad \text{and} \quad E_i \coloneqq \fint_{B_i} |A(Du) - (A(Du))_{B_i}|\,dx.$$

Step 1: Proof of (4.4). Let us prove that $A(Du)$ has vanishing mean oscillation at x_0 when (4.3) holds. Applying (4.83) with $r = R_{i-1}$ for every $i \ge 1$ gives

$$E_i \le \frac{1}{2}E_{i-1} + c\left[\frac{|\mu|(B_{i-1})}{R_{i-1}^{n-1}} + \left(\fint_{B_{i-1}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{i-1}}|)\,dx\right)^{\frac{1}{p'}}\right]. \tag{4.84}$$

Iterating the above inequality, we have that for any $k \ge 0$

$$\begin{aligned} E_k &\le \frac{1}{2^k}E_0 + c\sum_{i=1}^{k} \frac{1}{2^{k-i}} \frac{|\mu|(B_{i-1})}{R_{i-1}^{n-1}} \\ &\quad + c\sum_{i=1}^{k} \frac{1}{2^{k-i}} \left(\fint_{B_{i-1}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{i-1}}|)\,dx\right)^{\frac{1}{p'}} \\ &\le \frac{1}{2^k}E_0 + c\sup_{0<\rho\le R}\left[\frac{|\mu|(B_\rho)}{\rho^{n-1}} + \left(\fint_{B_\rho} \varphi^*(|A(D\psi) - (A(D\psi))_{B_\rho}|)\,dx\right)^{\frac{1}{p'}}\right]. \end{aligned}$$

From (4.3), for any $\delta > 0$, we temporarily fix the radius $R \equiv R(\delta) > 0$ in this step to satisfy

$$\sup_{0<\rho\le R}\left[\frac{|\mu|(B_\rho)}{\rho^{n-1}}+\left(\fint_{B_\rho}\varphi^*(|A(D\psi)-(A(D\psi))_{B_\rho}|)\,dx\right)^{\frac{1}{p'}}\right]<\delta.$$

Then there exists $k_0\in\mathbb{N}$ such that

$$\frac{1}{2^{k_0}}E_0\le\delta.$$

Then for any $0<r\le R_{k_0}$, we obtain

$$\begin{aligned}&\fint_{B_r}|A(Du)-(A(Du))_{B_r}|\,dx\\&\le\frac{K^n}{2^{k_0-1}}E_0+c\sup_{0<\rho\le R}\left[\frac{|\mu|(B_\rho)}{\rho^{n-1}}+\left(\fint_{B_\rho}\varphi^*(|A(D\psi)-(A(D\psi))_{B_\rho}|)\,dx\right)^{\frac{1}{p'}}\right]\\&\le c\delta.\end{aligned}$$

Since δ is an arbitrary positive constant, (4.4) follows.

Step 2: Proof of the pointwise estimate (4.6). To prove (4.6), let us first verify that $\{k_i\}$ is a Cauchy sequence in $\mathbb{R}^n$. Take any $m_1<m_2\in\mathbb{N}$. Summing up (4.84) over $i\in\{m_1+1,\dots,m_2\}$, we have

$$\begin{aligned}\sum_{i=m_1+1}^{m_2}E_i\le\frac{1}{2}\sum_{i=m_1}^{m_2-1}E_i+c\sum_{i=m_1}^{m_2-1}\frac{|\mu|(B_i)}{R_i^{n-1}}\\+c\sum_{i=m_1}^{m_2-1}\left(\fint_{B_i}\varphi^*(|A(D\psi)-(A(D\psi))_{B_i}|)\,dx\right)^{\frac{1}{p'}}\end{aligned}$$

and hence

$$\begin{aligned}\sum_{i=m_1}^{m_2}E_i\le 2E_{m_1}+2c\sum_{i=m_1}^{m_2-1}\frac{|\mu|(B_i)}{R_i^{n-1}}\\+2c\sum_{i=m_1}^{m_2-1}\left(\fint_{B_i}\varphi^*(|A(D\psi)-(A(D\psi))_{B_i}|)\,dx\right)^{\frac{1}{p'}}.\end{aligned}\tag{4.85}$$

By the calculations in [146, (115)], we notice

$$\sum_{i=m_1}^{m_2-1}\left[\frac{|\mu|(B_i)}{R_i^{n-1}}\right] \leq c(K)\mathbf{I}_1^{\mu}(x_0, 2R_{m_1}) \tag{4.86}$$

and

$$\begin{aligned}
&\sum_{i=m_1}^{m_2-1}\left(\fint_{B_i}\varphi^*(|A(D\psi)-(A(D\psi))_{B_i}|)\,dx\right)^{\frac{1}{p'}} \\
&\leq c(K)\int_0^{2R_{m_1}}\left(\fint_{B_r(x_0)}\varphi^*(|A(D\psi)-(A(D\psi))_{B_r(x_0)}|)\,dx\right)^{\frac{1}{p'}}\frac{dr}{r}.
\end{aligned} \tag{4.87}$$

Plugging (4.86) and (4.87) in (4.85), we have

$$\begin{aligned}
|k_{m_1}-k_{m_2}| &\leq \sum_{i=m_1}^{m_2-1}|k_i-k_{i+1}| \leq K^n\sum_{i=m_1}^{m_2-1}E_i \\
&\leq cE_{m_1} + c\mathbf{I}_1^{\mu}(x_0, 2R_{m_1}) \\
&\quad + c\int_0^{2R_{m_1}}\left(\fint_{B_r}\varphi^*(|A(D\psi)-(A(D\psi))_{B_r}|)\,dx\right)^{\frac{1}{p'}}\frac{dr}{r}.
\end{aligned} \tag{4.88}$$

Note that (4.5) implies (4.3) and

$$\lim_{\rho\to 0}\left[\mathbf{I}_1^{\mu}(x_0,\rho) + \int_0^{\rho}\left(\fint_{B_r}\varphi^*(|A(D\psi)-(A(D\psi))_{B_r}|)\,dx\right)^{\frac{1}{p'}}\frac{dr}{r}\right] = 0.$$

Therefore, $A(Du)$ has vanishing mean oscillation at x_0. Then for every $\varepsilon > 0$ we can take $N \in \mathbb{N}$ such that

$$E_N + \mathbf{I}_1^{\mu}(x_0, 2R_N) + \int_0^{2R_N}\left(\fint_{B_r}\varphi^*(|A(D\psi)-(A(D\psi))_{B_r}|)\,dx\right)^{\frac{1}{p'}}\frac{dr}{r} < \varepsilon.$$

Hence, we obtain for any $N \leq m_1 < m_2$

$$|k_{m_1}-k_{m_2}| < c\varepsilon,$$

which implies that $\{k_i\}$ is a Cauchy sequence in $\mathbb{R}^n$. As a consequence of the classical Lebesgue measure theory, x_0 is a Lebesgue point of $A(Du)$.

Now we again take an arbitrary small constant $\varepsilon > 0$. Since x_0 is a Lebesgue point of $A(Du)$, we can take $m \in \mathbb{N}$ large enough to satisfy

$$|A(Du(x_0)) - (A(Du))_{B_m}| \le \varepsilon.$$

It then follows from (4.88) that

$$\begin{aligned}
&|A(Du(x_0)) - (A(Du))_{B_0}| \\
&\le |A(Du(x_0)) - (A(Du))_{B_m}| + |(A(Du))_{B_m} - (A(Du))_{B_0}| \\
&\le \varepsilon + cE_0 + c\mathbf{I}_1^{\mu}(x_0, 2R) \\
&\quad + c\int_0^{2R} \left(\fint_{B_r(x_0)} \varphi^*(|A(D\psi) - (A(D\psi))_{B_r(x_0)}|)\, dx \right)^{\frac{1}{p'}} \frac{dr}{r}.
\end{aligned} \tag{4.89}$$

Recalling that ε is an arbitrary positive constant, we conclude that

$$\begin{aligned}
&|A(Du(x_0)) - (A(Du))_{B_{2R}(x_0)}| \\
&\le |A(Du(x_0)) - (A(Du))_{B_0}| + |(A(Du))_{B_0} - (A(Du))_{B_{2R}(x_0)}| \\
&\le c\fint_{B_{2R}(x_0)} |A(Du) - (A(Du))_{B_{2R}(x_0)}|\, dx + c\mathbf{I}_1^{\mu}(x_0, 2R) \\
&\quad + c\int_0^{2R} \left(\fint_{B_r(x_0)} \varphi^*(|A(D\psi) - (A(D\psi))_{B_r(x_0)}|)\, dx \right)^{\frac{1}{p'}} \frac{dr}{r}.
\end{aligned}$$

This completes the proof of Theorem 4.1.2. □

4.1.9 Proof of Theorem 4.1.3

The basic strategy of the proof is similar as in the previous section, but here we need to deal with the last term appearing in the above excess decay estimate (4.82). For this reason, we will first obtain a bound of the integral averages of $A(Du)$ over a sequence of concentric balls. We start with a ball $B_{2R} = B_{2R}(x_0) \subset \Omega$ as in the statement of Theorem 4.1.3, and all the balls considered will be centered at x_0.

We recall (4.7). We choose an integer $K \equiv K(\texttt{data}) \ge 4M$ large enough to have

$$\frac{c_{\mathrm{ex}}}{K^{\alpha_A}} \le \frac{1}{2}. \tag{4.90}$$

Applying Lemma 4.1.27 on arbitrary balls $B_\rho \equiv B_{r/K} \subset B_r \Subset \Omega$, we have

$$\begin{aligned}
&\fint_{B_{r/K}} |A(Du) - (A(Du))_{B_{r/K}}| \, dx \\
&\le \frac{1}{2} \fint_{B_r} |A(Du) - (A(Du))_{B_r}| \, dx \\
&\quad + c \left[\frac{|\mu|(B_r)}{r^{n-1}} + \left(\fint_{B_r} \varphi^*(|A(D\psi) - (A(D\psi))_{B_r}|) \, dx \right)^{\frac{1}{p'}} \right] \\
&\quad + c \left(\fint_{B_r} (|Du| + s)^{p-1} \, dx \right)^{\frac{p-2}{2(p-1)}} \left(\fint_{B_r} \varphi^*(|A(D\psi) - (A(D\psi))_{B_r}|) \, dx \right)^{\frac{1}{2}}.
\end{aligned} \tag{4.91}$$

For $i = 0, 1, 2, \ldots$, we define $R_i := R/K^i$, $B_i := B_{R_i}(x_0)$,

$$h_i := \fint_{B_i} |A(Du)| \, dx, \quad k_i := (A(Du))_{B_i}, \quad E_i := \fint_{B_i} |A(Du) - (A(Du))_{B_i}| \, dx.$$

We apply (4.91) with $r \equiv R_{i-1}$ to find

$$\begin{aligned}
E_i &\le \frac{1}{2} E_{i-1} + c \left[\frac{|\mu|(B_{i-1})}{R_{i-1}^{n-1}} + \left(\fint_{B_{i-1}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{i-1}}|) \, dx \right)^{\frac{1}{p'}} \right] \\
&\quad + c \left(h_{i-1} + s^{p-1} \right)^{\frac{p-2}{2(p-1)}} \left(\fint_{B_{i-1}} \varphi^*(|A(D\psi) - (A(D\psi))_{B_{i-1}}|) \, dx \right)^{\frac{1}{2}},
\end{aligned} \tag{4.92}$$

whenever $i \ge 1$. Summing up this estimate over $i \in \{1, \ldots, m\}$ for any $m \in \mathbb{N}$, we have

$$\begin{aligned}
\sum_{i=1}^{m} E_i &\le \frac{1}{2} \sum_{i=0}^{m-1} E_i + c \sum_{i=0}^{m-1} \left[\frac{|\mu|(B_i)}{R_i^{n-1}} + \left(\fint_{B_i} \varphi^*(|A(D\psi) - (A(D\psi))_{B_i}|) \, dx \right)^{\frac{1}{p'}} \right] \\
&\quad + c \sum_{i=0}^{m-1} \left(h_i + s^{p-1} \right)^{\frac{p-2}{2(p-1)}} \left(\fint_{B_i} \varphi^*(|A(D\psi) - (A(D\psi))_{B_i}|) \, dx \right)^{\frac{1}{2}}
\end{aligned}$$

and therefore

$$\sum_{i=1}^{m} E_i \le E_0 + 2c \sum_{i=0}^{m-1} \left[\frac{|\mu|(B_i)}{R_i^{n-1}} + \left(\fint_{B_i} \varphi^*(|A(D\psi) - (A(D\psi))_{B_i}|)\, dx \right)^{\frac{1}{p'}} \right] + 2c \sum_{i=0}^{m-1} \left(h_i + s^{p-1} \right)^{\frac{p-2}{2(p-1)}} \left(\fint_{B_i} \varphi^*(|A(D\psi) - (A(D\psi))_{B_i}|)\, dx \right)^{\frac{1}{2}}. \tag{4.93}$$

Then it holds that

$$\begin{aligned} |k_{m+1}| &\le \sum_{i=0}^{m} |k_{i+1} - k_i| + |k_0| \\ &\le \sum_{i=0}^{m} \fint_{B_{i+1}} |A(Du) - (A(Du))_{B_i}|\, dx + |k_0| \le K^n \sum_{i=0}^{m} E_i + |k_0|. \end{aligned}$$

Combining this and (4.93) gives

$$\begin{aligned} |k_{m+1}| \le cE_0 + c|k_0| &+ c \sum_{i=0}^{m-1} \frac{|\mu|(B_i)}{R_i^{n-1}} \\ &+ c \sum_{i=0}^{m-1} \left(\fint_{B_i} \varphi^*(|A(D\psi) - (A(D\psi))_{B_i}|)\, dx \right)^{\frac{1}{p'}} \\ &+ c \sum_{i=0}^{m-1} \left(h_i + s^{p-1} \right)^{\frac{p-2}{2(p-1)}} \left(\fint_{B_i} \varphi^*(|A(D\psi) - (A(D\psi))_{B_i}|)\, dx \right)^{\frac{1}{2}}. \end{aligned} \tag{4.94}$$

Moreover, from (4.92) we also have

$$E_m \le \frac{1}{2^m} E_0 + c \sum_{i=0}^{m-1} \left[\frac{|\mu|(B_i)}{R_i^{n-1}} + \left(\fint_{B_i} \varphi^*(|A(D\psi) - (A(D\psi))_{B_i}|)\, dx \right)^{\frac{1}{p'}} \right] + c \sum_{i=0}^{m-1} \left(h_i + s^{p-1} \right)^{\frac{p-2}{2(p-1)}} \left(\fint_{B_i} \varphi^*(|A(D\psi) - (A(D\psi))_{B_i}|)\, dx \right)^{\frac{1}{2}}. \tag{4.95}$$

We recall (4.86) and further note that, since $2/p' > 1$,

$$\begin{aligned}
&\sum_{i=0}^{m-1}\left(\fint_{B_i}\varphi^*(|A(D\psi)-(A(D\psi))_{B_i}|)\,dx\right)^{\frac{1}{p'}}\\
&\le\left[\sum_{i=0}^{m}\left(\fint_{B_i}\varphi^*(|A(D\psi)-(A(D\psi))_{B_i}|)\,dx\right)^{\frac12}\right]^{\frac{2}{p'}}\\
&\le c(K)\left[\int_0^{2R}\left(\fint_{B_r}\varphi^*(|A(D\psi)-(A(D\psi))_{B_r}|)\,dx\right)^{\frac12}\frac{dr}{r}\right]^{\frac{2}{p'}}. \qquad (4.96)
\end{aligned}$$

We can also easily see

$$E_0+|k_0|\le 3\fint_{B_R}|A(Du)|\,dx \quad\text{ and }\quad |k_1|\le K^n\fint_{B_R}|A(Du)|\,dx. \qquad (4.97)$$

Combining (4.94) and (4.95), and then using the fact that $h_i \le |k_i| + E_i$, we discover

$$\begin{aligned}
h_{m+1} &\le |k_{m+1}|+E_{m+1}\\
&\le cE_0+c|k_0|+c\sum_{i=0}^{m}\left[\frac{|\mu|(B_i)}{R_i^{n-1}}+\left(\fint_{B_i}\varphi^*(|A(D\psi)-(A(D\psi))_{B_i}|)\,dx\right)^{\frac{1}{p'}}\right]\\
&\quad+c\sum_{i=0}^{m}\left(h_i+s^{p-1}\right)^{\frac{p-2}{2(p-1)}}\left(\fint_{B_i}\varphi^*(|A(D\psi)-(A(D\psi))_{B_i}|)\,dx\right)^{\frac12}\\
&\le c_8\mathbf{I}_1^{\mu}(x_0,2R)+c_8\left[\int_0^{2R}\left(\fint_{B_r}\varphi^*(|A(D\psi)-(A(D\psi))_{B_r}|)\,dx\right)^{\frac12}\frac{dr}{r}\right]^{\frac{2}{p'}}\\
&\quad+c_8\sum_{i=0}^{m}\left(h_i+s^{p-1}\right)^{\frac{p-2}{2(p-1)}}\left(\fint_{B_i}\varphi^*(|A(D\psi)-(A(D\psi))_{B_i}|)\,dx\right)^{\frac12}\\
&\quad+c_8\fint_{B_R}|A(Du)|\,dx \qquad (4.98)
\end{aligned}$$

for some constant $c_8 \equiv c_8(\texttt{data})$, where we also have used (4.86), (4.96) and (4.97) for the last inequality.

Now we need to establish a uniform estimate for h_m.

Lemma 4.1.28. *There exists a constant* $c_9 \equiv c_9(\texttt{data}) \geq 1$ *such that, for*

$$\Gamma := c_9 \fint_{B_{2R}(x_0)} (|A(Du)| + s^{p-1})\, dx + c_9 \mathbf{I}_1^{\mu}(x_0, 2R) + c_9 \left[\int_0^{2R} \left(\fint_{B_r(x_0)} \varphi^*(|A(D\psi) - (A(D\psi))_{B_r(x_0)}|)\, dx \right)^{\frac{1}{2}} \frac{dr}{r} \right]^{\frac{2}{p'}}, \quad (4.99)$$

there holds

$$h_m \leq \Gamma \quad \textit{for every } m \geq 0. \quad (4.100)$$

Proof. Let us initially consider c_9 as a free parameter to be determined at the end of the proof. We will proceed by strong induction. First, $h_0 \leq \Gamma$ is obvious; if $c_9 \geq K^n$, then $h_1 \leq \Gamma$. We now assume that (4.100) holds for $i = 0, 1, \ldots, m$. Using this in (4.98), we find

$$\begin{aligned} h_{m+1} &\leq \frac{2c_8}{c_9}\Gamma + 2c_8 \Gamma^{\frac{p-2}{2(p-1)}} \sum_{i=0}^{m} \left(\fint_{B_i} \varphi^*(|A(D\psi) - (A(D\psi))_{B_i}|)\, dx \right)^{\frac{1}{2}} \\ &\leq \frac{2c_8}{c_9}\Gamma + 2c_8 \Gamma^{\frac{p-2}{2(p-1)}} c(K) \int_0^{2R} \left(\fint_{B_r} \varphi^*(|A(D\psi) - (A(D\psi))_{B_r}|)\, dx \right)^{\frac{1}{2}} \frac{dr}{r} \\ &\leq \left(\frac{2c_8}{c_9} + \frac{2c_8 c(K)}{c_9^{p'/2}} \right) \Gamma. \end{aligned}$$

Recalling that $K \equiv K(\texttt{data})$ has been fixed in (4.90), we now choose the constant $c_9 \equiv c_9(\texttt{data})$ satisfying

$$c_9 \geq K^n \quad \text{and} \quad \frac{2c_8}{c_9} + \frac{2c_8 c(K)}{c_9^{p'/2}} \leq 1$$

in order to conclude that $h_{m+1} \leq \Gamma$, as desired. □

Connecting (4.86), (4.87), (4.96) and (4.100) to (4.93), we obtain

$$\sum_{i=1}^{m} E_i \leq E_0 + c \sum_{i=0}^{m-1} \left[\frac{|\mu|(B_i)}{R_i^{n-1}} + \left(\fint_{B_i} \varphi^*(|A(D\psi) - (A(D\psi))_{B_i}|)\, dx \right)^{\frac{1}{p'}} \right] + c\Gamma^{\frac{p-2}{2(p-1)}} \sum_{i=0}^{m-1} \left(\fint_{B_i} \varphi^*(|A(D\psi) - (A(D\psi))_{B_i}|)\, dx \right)^{\frac{1}{2}}$$

$$
\begin{aligned}
&\le E_0 + c\mathbf{I}_1^{\mu}(x_0, 2R) \\
&\quad + c\left[\int_0^{2R} \left(\fint_{B_r} \varphi^*(|A(D\psi) - (A(D\psi))_{B_r}|)\,dx\right)^{\frac{1}{2}} \frac{dr}{r}\right]^{\frac{2}{p'}} \\
&\quad + c\Gamma^{\frac{p-2}{2(p-1)}} \int_0^{2R} \left(\fint_{B_r} \varphi^*(|A(D\psi) - (A(D\psi))_{B_r}|)\,dx\right)^{\frac{1}{2}} \frac{dr}{r}.
\end{aligned}
$$

Here, the left-hand side is bounded uniformly with respect to $m \in \mathbb{N}$, and so $E_i \to 0$ as $i \to \infty$. Therefore, $A(Du)$ has vanishing mean oscillation at x_0.

At this stage, one can prove the pointwise bound (4.8) by modifying the calculations in Section 4.1.8. For any $m_1 < m_2 \in \mathbb{N}$, a suitable adaptation of (4.93) gives

$$
\begin{aligned}
|k_{m_1} - k_{m_2}| &\le cE_{m_1} + c\mathbf{I}_1^{\mu}(x_0, 2R_{m_1}) \\
&\quad + c\left[\int_0^{2R_{m_1}} \left(\fint_{B_r} \varphi^*(|A(D\psi) - (A(D\psi))_{B_r}|)\,dx\right)^{\frac{1}{2}} \frac{dr}{r}\right]^{\frac{2}{p'}} \\
&\quad + c\Gamma^{\frac{p-2}{2(p-1)}} \int_0^{2R_{m_1}} \left(\fint_{B_r} \varphi^*(|A(D\psi) - (A(D\psi))_{B_r}|)\,dx\right)^{\frac{1}{2}} \frac{dr}{r}.
\end{aligned}
$$

Taking into account the fact that $A(Du)$ has vanishing mean oscillation, we find that $\{k_i\}$ is a Cauchy sequence. Then a similar calculation as in (4.89) gives

$$
\begin{aligned}
&|A(Du(x_0)) - (A(Du))_{B_{2R}(x_0)}| \\
&\le c\fint_{B_{2R}(x_0)} |A(Du) - (A(Du))_{B_{2R}(x_0)}|\,dx + c\mathbf{I}_1^{\mu}(x_0, 2R) \\
&\quad + c\left[\int_0^{2R} \left(\fint_{B_r(x_0)} \varphi^*(|A(D\psi) - (A(D\psi))_{B_r(x_0)}|)\,dx\right)^{\frac{1}{2}} \frac{dr}{r}\right]^{\frac{2}{p'}} \\
&\quad + c\Gamma^{\frac{p-2}{2(p-1)}} \int_0^{2R} \left(\fint_{B_r(x_0)} \varphi^*(|A(D\psi) - (A(D\psi))_{B_r(x_0)}|)\,dx\right)^{\frac{1}{2}} \frac{dr}{r}.
\end{aligned}
$$

In the above display, we recall the definition of Γ given in (4.99), and then apply Young's inequality with conjugate exponents $2(p-1)/(p-2)$ and $2/p'$. This finally gives (4.8), and the proof of Theorem 4.1.3 is complete. □

Remark 4.1.29. *As mentioned in Section 4.1.1, the constant c in* (4.8) *for $p > 2$ remains bounded when $p \searrow 2$, which gives the same estimate as the one in* (4.6)*. This is due to the stability of the constants in* (2.34) *and* (4.81)*.*

4.2 Fractional differentiability for double obstacle problems with measure data

In this section, we consider the double obstacle problem $OP(\psi_1, \psi_2; \mu)$ with the constraint $\psi_1 \le u \le \psi_2$ a.e. in Ω, where $\psi_1, \psi_2 \in W^{1,p}(\Omega)$ are given obstacles satisfying $\psi_1 \le \psi_2$ a.e. in Ω. Limits of approximating solutions can be defined analogously.

Definition 4.2.1. *Suppose that we are given two obstacles $\psi_1, \psi_2 \in W^{1,p}(\Omega)$ with $\psi_1 \le \psi_2$ a.e. in Ω, measure data $\mu \in \mathcal{M}_b(\Omega)$ and boundary data $g \in W^{1,p}(\Omega)$ with $(\psi_1 - g)_+, (\psi_2 - g)_- \in W_0^{1,p}(\Omega)$. We say that a function $u \in \mathcal{T}_g^{1,p}(\Omega)$ with $\psi_1 \le u \le \psi_2$ a.e. in Ω is a limit of approximating solutions to the obstacle problem $OP(\psi_1, \psi_2; \mu)$ under assumptions* (2.8) *if there exist a sequence of functions $\{\mu_k\} \subset W^{-1,p'}(\Omega) \cap L^1(\Omega)$ with*

$$\begin{cases} \mu_k \overset{*}{\rightharpoonup} \mu \quad \textit{in } \mathcal{M}_b(\Omega), \\ \limsup\limits_{k\to\infty} |\mu_k|(B) \le |\mu|(\bar{B}) \quad \textit{for every ball } B \subset \mathbb{R}^n \end{cases} \tag{4.101}$$

and weak solutions $u_k \in g + W_0^{1,p}(\Omega)$ with $\psi_1 \le u_k \le \psi_2$ a.e. in Ω to the variational inequalities

$$\int_\Omega A(Du_k) \cdot D(\phi - u_k)\, dx \ge \int_\Omega (\phi - u_k)\, d\mu_k$$

for every $\phi \in u_k + W_0^{1,p}(\Omega)$ with $\psi_1 \le \phi \le \psi_2$ a.e. in Ω, such that

$$\begin{cases} u_k \to u & \textit{a.e. in } \Omega, \\ \displaystyle\int_\Omega |u_k - u|^\gamma\, dx \to 0 & \textit{for every } 0 < \gamma < \dfrac{n(p-1)}{n-p}, \\ \displaystyle\int_\Omega |Du_k - Du|^q\, dx \to 0 & \textit{for every } 0 < q < \dfrac{n(p-1)}{n-1}. \end{cases} \tag{4.102}$$

By following the classical approach in [28, 29], the existence of limits of approximating solutions to $OP(\psi_1, \psi_2; \mu)$ was proved in [43].

4.2.1 Main results

The aim of this section is to prove an analog of (1.8) for $OP(\psi_1, \psi_2; \mu)$, under suitable differentiability assumptions on the obstacles. More precisely, we assume that $\psi_1, \psi_2 \in W^{1,p}(\Omega) \cap W^{2,1}(\Omega)$ satisfy

$$\mathcal{D}\Psi_i := \operatorname{div} A(D\psi_i) \in L^1(\Omega), \quad i = 1, 2. \tag{4.103}$$

We assume (4.1). When $p < 2$, we also assume that $\partial A(\cdot)$ is symmetric, i.e.,

$$\partial_i A_j = \partial_j A_i \qquad \forall\, i, j \in \{1, \dots, n\}. \tag{4.104}$$

Theorem 4.2.2. *Let $u \in W^{1,\max\{p-1,1\}}(\Omega)$ with $\psi_1 \le u \le \psi_2$ a.e. in Ω be a limit of approximating solutions to $OP(\psi_1, \psi_2; \mu)$ under assumptions* (2.8) *when $p \ge 2$, and assumptions* (2.8) *and* (4.104) *when $2 - 1/n < p < 2$. Assume that $\psi_1, \psi_2 \in W^{1,p}(\Omega) \cap W^{2,1}(\Omega)$ satisfy* (4.103). *Then*

$$A(Du) \in W^{\sigma,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^n) \qquad \forall\, \sigma \in (0, 1). \tag{4.105}$$

Moreover, for any $\sigma \in (0,1)$, there exists a constant $c \equiv c(\texttt{data}, \sigma)$ such that

$$\begin{aligned} &\fint_{B_{R/2}} \int_{B_{R/2}} \frac{|A(Du(x)) - A(Du(y))|}{|x-y|^{n+\sigma}}\, dx\, dy \\ &\le \frac{c}{R^\sigma} \fint_{B_R} |A(Du)|\, dx + \frac{c}{R^\sigma} \left[\frac{|\mu|(B_R)}{R^{n-1}} \right] \\ &\quad + \frac{c}{R^\sigma} \left[\frac{|\mathcal{D}\Psi_1|(B_R)}{R^{n-1}} \right] + \frac{c}{R^\sigma} \left[\frac{|\mathcal{D}\Psi_2|(B_R)}{R^{n-1}} \right] \end{aligned} \tag{4.106}$$

whenever $B_R \Subset \Omega$ is a ball.

Once we prove the above theorem, we can also obtain the following corollary in the case $p \ge 2$, which corresponds to [7, Theorem 1.3].

Corollary 4.2.3. *Under the assumptions of Theorem 4.2.2 with $p \ge 2$, let $u \in W^{1,p-1}(\Omega)$ with $\psi_1 \le u \le \psi_2$ a.e. in Ω be a limit of approximating solutions to $OP(\psi_1, \psi_2; \mu)$. Then for every $\gamma \in [0, p-2]$, we have*

$$(|Du|^2 + s^2)^{\gamma/2} Du \in W^{\sigma\frac{\gamma+1}{p-1}, \frac{p-1}{\gamma+1}}_{\mathrm{loc}}(\Omega; \mathbb{R}^n) \qquad \forall\, \sigma \in (0,1).$$

Moreover, for any $\sigma \in (0,1)$, there exists a constant $c \equiv c(\texttt{data}, \sigma)$ such that

$$[(|Du|^2+s^2)^{\gamma/2}Du]_{\sigma\frac{\gamma+1}{p-1},\frac{p-1}{\gamma+1};B_{R/2}} \le c\left(\frac{1}{R^\sigma}\int_{B_R}|A(Du)|\,dx\right)^{\frac{\gamma+1}{p-1}} + c\left[R^{1-\sigma}|\mu|(B_R)\right]^{\frac{\gamma+1}{p-1}} + c\left[R^{1-\sigma}|\mathcal{D}\Psi_1|(B_R)\right]^{\frac{\gamma+1}{p-1}} + c\left[R^{1-\sigma}|\mathcal{D}\Psi_2|(B_R)\right]^{\frac{\gamma+1}{p-1}}$$

whenever $B_R \Subset \Omega$ is a ball.

This shows a trade between integrability and differentiability according to the power of nonlinear functions of the gradient. In particular, this gives differentiability results analogous to those in [164]. For instance, in the case $\gamma = 0$, we have

$$Du \in W^{\frac{\sigma}{p-1},p-1}_{\mathrm{loc}}(\Omega;\mathbb{R}^n) \qquad \forall\, \sigma \in (0,1).$$

We also consider the case when the measure satisfies the density condition (1.5) with $p \le \theta \le n$.

Theorem 4.2.4. *Let $u \in W^{1,\max\{p-1,1\}}(\Omega)$ with $\psi_1 \le u \le \psi_2$ a.e. in Ω be a limit of approximating solutions to $OP(\psi_1,\psi_2;\mu)$ under assumptions* (2.8) *when $2 \le p \le n$, and assumptions* (2.8) *and* (4.104) *when $2-1/n < p < 2$. Assume that $\psi_1, \psi_2 \in W^{1,p}(\Omega) \cap W^{2,1}(\Omega)$ satisfy* (4.103) *and*

$$\mu, \mathcal{D}\Psi_1, \mathcal{D}\Psi_2 \in L^{1,\theta}(\Omega) \tag{4.107}$$

with $p \le \theta \le n$. Then

$$A(Du) \in W^{\sigma,1,\theta}_{\mathrm{loc}}(\Omega;\mathbb{R}^n) \qquad \forall\, \sigma \in (0,1).$$

Moreover, for any ball $B_R \Subset \Omega$ and $\sigma \in (0,1)$, we have

$$[A(Du)]_{\sigma,1,\theta;B_{R/3}} \le cR^{\theta-n-\sigma}\int_{B_R}(|Du|+s)^{p-1}\,dx + cR^{1-\sigma}\|\mu\|_{L^{1,\theta}(B_R)} + cR^{1-\sigma}\|\mathcal{D}\Psi_1\|_{L^{1,\theta}(B_R)} + cR^{1-\sigma}\|\mathcal{D}\Psi_2\|_{L^{1,\theta}(B_R)} \tag{4.108}$$

for a constant $c \equiv c(\texttt{data}, \sigma)$.

We also have the counterpart of Corollary 4.2.3, which reads as follows.

Corollary 4.2.5. *Under the assumptions of Theorem 4.2.4 with* $2 \le p \le n$, *let* $u \in W^{1,p-1}(\Omega)$ *with* $\psi_1 \le u \le \psi_2$ *a.e. in* Ω *be a limit of approximating solutions to* $OP(\psi_1, \psi_2; \mu)$. *Then for every* $\gamma \in [0, p-2]$, *we have*

$$(|Du|^2 + s^2)^{\gamma/2} Du \in W^{\sigma\frac{\gamma+1}{p-1}, \frac{p-1}{\gamma+1}, \theta}_{\mathrm{loc}}(\Omega; \mathbb{R}^n) \qquad \forall\ \sigma \in (0,1).$$

Moreover, for any ball $B_R \Subset \Omega$ *and* $\sigma \in (0,1)$, *we have*

$$\begin{aligned} &[(|Du|^2 + s^2)^{\gamma/2} Du]_{\sigma\frac{\gamma+1}{p-1}, \frac{p-1}{\gamma+1}, \theta; B_{R/3}} \\ &\le cR^{(\theta-n-\sigma)\frac{\gamma+1}{p-1}} \left(\int_{B_R} (|Du| + s)^{p-1}\, dx \right)^{\frac{\gamma+1}{p-1}} \\ &\quad + cR^{(1-\sigma)\frac{\gamma+1}{p-1}} \left(\|\mu\|_{L^{1,\theta}(B_R)} + \|\mathcal{D}\Psi_1\|_{L^{1,\theta}(B_R)} + \|\mathcal{D}\Psi_2\|_{L^{1,\theta}(B_R)} \right)^{\frac{\gamma+1}{p-1}} \end{aligned}$$

for a constant $c \equiv c(\mathtt{data}, \sigma)$.

Remark 4.2.6. *Under assumption* (4.103), *the differentiability result given in* (4.105) *is sharp in the sense that we cannot have* $A(Du) \in W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$, *which is already invalid for the case of equation* (1.1). *An interesting question is whether we can obtain lower differentiability results for* $A(Du)$ *under weaker assumptions than* (4.103). *For instance, in view of* (1.10), *it should be interesting that one can prove, for some fixed* $\alpha \in (0,1)$,

$$A(D\psi_1), A(D\psi_2) \in W^{\alpha,1}_{\mathrm{loc}} \implies A(Du) \in W^{\sigma,1}_{\mathrm{loc}} \quad \forall\ \sigma \in (0, \alpha).$$

Note that in the case $\alpha = 1$, *this follows from Theorem 4.2.2. For lower values of* α, *the techniques in [9] are not directly applicable. Indeed, the reason why [9, Theorem 1.1] covers only the p-Laplace operator with* $p \ge 2$ *and* $n = 2$ *is that its proof uses certain regularity results for p-harmonic functions, related to the* $C^{p'}$*-conjecture, in order to obtain* (1.10). *In the case of general structure conditions and higher dimensions, the range of* σ *in* (1.10) *is more restricted and even not clear, as mentioned in [9, Remark 4.2].*

4.2.2 Comparison estimates

We introduce the following notations for the admissible sets of the problem $OP(\psi_1, \psi_2; \mu)$: given an open set $\mathcal{O} \subseteq \Omega$ and a function $g \in W^{1,p}(\mathcal{O})$ with

$\psi_1 \le g \le \psi_2$ a.e. in $\mathcal{O}$, we denote

$$\mathcal{A}^g_{\psi_1,\psi_2}(\mathcal{O}) := \left\{ \varphi \in g + W^{1,p}_0(\mathcal{O}) : \psi_1 \le \varphi \le \psi_2 \text{ a.e. in } \mathcal{O} \right\}.$$

Throughout this section, we recall (4.103) and further assume that

$$\mu \in W^{-1,p'}(\Omega) \cap L^1(\Omega) \qquad \text{and} \qquad u \in \mathcal{A}^g_{\psi_1,\psi_2}(\Omega) \tag{4.109}$$

satisfy

$$\int_\Omega A(Du) \cdot D(\phi - u)\, dx \ge \int_\Omega (\phi - u)\, d\mu \qquad \forall\, \phi \in \mathcal{A}^u_{\psi_1,\psi_2}(\Omega), \tag{4.110}$$

in order to derive several comparison estimates.

With $B_{2R} \Subset \Omega$ being a fixed ball, we first consider the single obstacle problem

$$\begin{cases} \displaystyle\int_{B_{2R}} A(Dw_1) \cdot D(\phi - w_1)\, dx & \\ \displaystyle\ge \int_{B_{2R}} A(D\psi_2) \cdot D(\phi - w_1)\, dx & \forall\, \phi \in \mathcal{A}^u_{\psi_1}(B_{2R}), \\ w_1 \ge \psi_1 & \text{a.e. in } B_{2R}, \\ w_1 = u & \text{on } \partial B_{2R}. \end{cases} \tag{4.111}$$

We then consider the two obstacle-free problems

$$\begin{cases} -\operatorname{div} A(Dw_2) = -\operatorname{div} A(D\psi_1) & \text{in } B_{2R}, \\ w_2 = u & \text{on } \partial B_{2R} \end{cases} \tag{4.112}$$

and

$$\begin{cases} -\operatorname{div} A(Dv) = 0 & \text{in } B_{2R}, \\ v = u & \text{on } \partial B_{2R}. \end{cases} \tag{4.113}$$

We start with a weighted type energy estimate, whose proof is essentially the same as that of (4.19).

Lemma 4.2.7. *Let $u \in \mathcal{A}^g_{\psi_1,\psi_2}(\Omega)$ be the weak solution to* (4.110) *under* *assumptions* (2.8)*, and let* $w_1 \in \mathcal{A}^u_{\psi_1}(B_{2R})$ *be as in* (4.111)*. Then*

$$\int_{B_{2R}} \frac{|V(Du) - V(Dw_1)|^2}{(h + |u - w_1|)^\xi}\, dx \le c\frac{h^{1-\xi}}{\xi - 1}\left(|\mu|(B_{2R}) + |\mathcal{D}\Psi_2|(B_{2R})\right) \tag{4.114}$$

holds whenever $h > 0$ *and* $\xi > 1$*, where* $c \equiv c(\texttt{data})$*.*

Proof. We first show that $w_1 \le \psi_2$ a.e. in B_{2R} by a similar argument as in the proof of Lemma 4.1.7. Testing (4.111) with $\phi = \min\{w_1, \psi_2\}$ and using (2.11), we get

$$\int_{B_{2R}\cap\{w_1\ge\psi_2\}} |V(Dw_1) - V(D\psi_2)|^2\, dx \le 0.$$

Recalling that $w_1 \in u + W^{1,p}_0(B_{2R})$ and $u \le \psi_2$ a.e. in B_{2R}, we obtain $(w_1 - \psi_2)_+ \in W^{1,p}_0(B_{2R})$, which in turn implies $w_1 \le \psi_2$ a.e. in B_{2R}.

We now turn to the proof of (4.114). The estimate is essentially similar to that in [146, Lemma 2], but we need to modify the test functions due to the setting of obstacle problems. We consider

$$\eta_\pm \coloneqq \frac{1}{\xi - 1}\left[1 - \left(1 + \frac{(u - w_1)_\pm}{h}\right)^{1-\xi}\right],$$

which obviously belong to $W^{1,p}_0(B_{2R}) \cap L^\infty(B_{2R})$. Also, by applying the mean value theorem to the function $t \mapsto t^{1-\xi}/(\xi - 1)$, we have

$$\eta_\pm(x) = \frac{(u - w_1)_\pm(x)}{h}(\tilde{h}_\pm(x))^{-\xi} \quad \text{for some} \quad 1 < \tilde{h}_\pm(x) < 1 + \frac{(u - w_1)_\pm(x)}{h}$$

whenever $x \in B_{2R}$. Using this and the fact that $\eta_\pm \ge 0$, we observe that

$$\min\{u, w_1\} \le u - h\eta_+ \le u \le u + h\eta_- \le \max\{u, w_1\} \quad \text{a.e. in } B_{2R}$$

and

$$\min\{u, w_1\} \le w_1 - h\eta_- \le w_1 \le w_1 + h\eta_+ \le \max\{u, w_1\} \quad \text{a.e. in } B_{2R}.$$

Therefore, $u \pm h\eta_\mp$ and $w_1 \pm h\eta_\pm$ belong to the admissible set $\mathcal{A}^u_{\psi_1,\psi_2}(B_{2R})$.

We now test (4.110) with $\phi \equiv u \pm h\eta_{\mp}$ to have

$$\int_{B_{2R}} \frac{A(Du)\cdot D(u-w_1)_+}{(h+(u-w_1)_+)^{\xi}}\,dx \le h^{1-\xi}\int_{B_{2R}} \eta_+\,d\mu$$

and

$$\int_{B_{2R}} \frac{A(Du)\cdot D(u-w_1)_-}{(h+(u-w_1)_-)^{\xi}}\,dx \ge h^{1-\xi}\int_{B_{2R}} \eta_-\,d\mu.$$

Similarly, testing (4.111) with $\phi \equiv w_1 \pm h\eta_{\pm}$, we obtain

$$\int_{B_{2R}} \frac{A(Dw_1)\cdot D(u-w_1)_+}{(h+(u-w)_+)^{\xi}}\,dx \ge -h^{1-\xi}\int_{B_{2R}} \eta_+\,\mathcal{D}\Psi_2\,dx$$

and

$$\int_{B_{2R}} \frac{A(Dw_1)\cdot D(u-w_1)_-}{(h+(u-w_1)_-)^{\xi}}\,dx \le -h^{1-\xi}\int_{B_{2R}} \eta_-\,\mathcal{D}\Psi_2\,dx.$$

Hence, using (2.11) in each case we obtain

$$\begin{aligned}
&\int_{B_{2R}\cap\{u\ge w_1\}} \frac{|V(Du)-V(Dw_1)|^2}{(h+|u-w_1|)^{\xi}}\,dx \\
&\le c\left|\int_{B_{2R}} h^{1-\xi}\eta_+\,d\mu\right| + c\left|\int_{B_{2R}} \mathcal{D}\Psi_2 h^{1-\xi}\eta_+\,dx\right| \\
&\le c\frac{h^{1-\xi}}{\xi-1}\left(|\mu|(B_{2R}) + |\mathcal{D}\Psi_2|(B_{2R})\right)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{B_{2R}\cap\{u< w_1\}} \frac{|V(Du)-V(Dw_1)|^2}{(h+|u-w_1|)^{\xi}}\,dx \\
&\le c\left|\int_{B_{2R}} h^{1-\xi}\eta_-\,d\mu\right| + c\left|\int_{B_{2R}} \mathcal{D}\Psi_2 h^{1-\xi}\eta_-\,dx\right| \\
&\le c\frac{h^{1-\xi}}{\xi-1}\left(|\mu|(B_{2R}) + |\mathcal{D}\Psi_2|(B_{2R})\right).
\end{aligned}$$

Combining the last two estimates finally gives (4.114). □

As in Section 4.1.3, the following comparison estimate between (4.110) and (4.111) follows by the arguments in [144, Lemma 4.3] and [146, Lemma 1]. We also refer to [43, Lemma 4.1] for another proof.

Lemma 4.2.8. *Let $u \in \mathcal{A}^g_{\psi_1,\psi_2}(\Omega)$ be the weak solution to* (4.110) *under assumptions* (2.8)*, and let $w_1 \in \mathcal{A}^u_{\psi_1}(B_{2R})$ be as in* (4.111)*. Then we have*

$$\begin{aligned}
&\fint_{B_{2R}} \left(|Du - Dw_1|^q + |V(Du) - V(Dw_1)|^{\frac{2q}{p}}\right) dx \\
&\le c\left[\frac{|\mu|(B_{2R})}{(2R)^{n-1}}\right]^{\frac{q}{p-1}} + c\left[\frac{|\mathcal{D}\Psi_2|(B_{2R})}{(2R)^{n-1}}\right]^{\frac{q}{p-1}} \\
&\quad + c\chi_{\{p<2\}}\left(\left[\frac{|\mu|(B_R)}{(2R)^{n-1}}\right]^q + \left[\frac{|\mathcal{D}\Psi_2|(B_{2R})}{(2R)^{n-1}}\right]^q\right)\left(\fint_{B_{2R}} (|Du| + s)^q\, dx\right)^{2-p}
\end{aligned} \tag{4.115}$$

for every q satisfying

$$1 < q < \min\left\{p, \frac{n(p-1)}{n-1}\right\}, \tag{4.116}$$

where $c \equiv c(\mathtt{data}, q)$.

In light of Lemma 4.1.7, the following lemmas can be proved in exactly the same way.

Lemma 4.2.9. *Let w_1, w_2 and v be as in* (4.111), (4.112) *and* (4.113)*, respectively. Then*

$$\int_{B_{2R}} \frac{|V(Dw_1) - V(Dw_2)|^2}{(h_1 + |w_1 - w_2|)^{\xi_1}}\, dx \le c\frac{h_1^{1-\xi_1}}{\xi_1 - 1}\left(|\mathcal{D}\Psi_1|(B_{2R}) + |\mathcal{D}\Psi_2|(B_{2R})\right)$$

and

$$\int_{B_{2R}} \frac{|V(Dw_2) - V(Dv)|^2}{(h_2 + |w_2 - v|)^{\xi_2}}\, dx \le c\frac{h_2^{1-\xi_2}}{\xi_2 - 1}|\mathcal{D}\Psi_1|(B_{2R})$$

hold for any $h_1, h_2 > 0$ and $\xi_1, \xi_2 > 1$, where $c \equiv c(\mathtt{data})$.

Lemma 4.2.10. *Let w_1, w_2 and v be as in* (4.111), (4.112) *and* (4.113)*,* *respectively. Then we have*

$$\begin{aligned}&\fint_{B_{2R}} \left(|Dw_1 - Dw_2|^q + |V(Dw_1) - V(Dw_2)|^{\frac{2q}{p}}\right) dx \\ &\le c\left[\frac{|\mathcal{D}\Psi_1|(B_{2R})}{(2R)^{n-1}}\right]^{\frac{q}{p-1}} + c\left[\frac{|\mathcal{D}\Psi_2|(B_{2R})}{(2R)^{n-1}}\right]^{\frac{q}{p-1}} \\ &\quad + c\chi_{\{p<2\}}\left(\left[\frac{|\mathcal{D}\Psi_1|(B_{2R})}{(2R)^{n-1}}\right]^q + \left[\frac{|\mathcal{D}\Psi_2|(B_{2R})}{(2R)^{n-1}}\right]^q\right)\left(\fint_{B_{2R}} (|Dw_1| + s)^q \, dx\right)^{2-p}\end{aligned} \tag{4.117}$$

and

$$\begin{aligned}&\fint_{B_{2R}} \left(|Dw_2 - Dv|^q + |V(Dw_2) - V(Dv)|^{\frac{2q}{p}}\right) dx \\ &\le c\left[\frac{|\mathcal{D}\Psi_1|(B_{2R})}{(2R)^{n-1}}\right]^{\frac{q}{p-1}} + c\chi_{\{p<2\}}\left[\frac{|\mathcal{D}\Psi_1|(B_{2R})}{(2R)^{n-1}}\right]^q \left(\fint_{B_{2R}} (|Dw_2| + s)^q \, dx\right)^{2-p}\end{aligned} \tag{4.118}$$

for every q satisfying (4.116)*, where $c \equiv c(\mathtt{data}, q)$.*

In the following, we consider the positive measure $\Lambda \in \mathcal{M}_b(\Omega)$ defined by

$$\Lambda(S) := |\mu|(S) + |\mathcal{D}\Psi_1|(S) + |\mathcal{D}\Psi_2|(S) \tag{4.119}$$

for each measurable set $S \subseteq \Omega$. Combining Lemmas 4.2.8 and 4.2.10, we establish a comparison estimate between (4.110) and (4.113).

Lemma 4.2.11. *Let u and v be as in* (4.110) *and* (4.113)*, respectively. Then*

$$\begin{aligned}&\fint_{B_{2R}} \left(|Du - Dv|^q + |V(Du) - V(Dv)|^{\frac{2q}{p}}\right) dx \\ &\le c\left[\frac{\Lambda(B_{4R})}{(4R)^{n-1}}\right]^{\frac{q}{p-1}} + c\chi_{\{p<2\}}\left[\frac{\Lambda(B_{4R})}{(4R)^{n-1}}\right]^q \left(\fint_{B_{4R}} (|Du| + s)^t \, dx\right)^{\frac{q(2-p)}{t}}\end{aligned} \tag{4.120}$$

holds for every q satisfying

$$0 < q < \min\left\{p, \frac{n(p-1)}{n-1}\right\} \tag{4.121}$$

and $t \in (0, q]$, *where* $c \equiv c(\texttt{data}, q, t)$.

Proof. Estimate (4.120) in the case $t = q \geq 1$ can be obtained from Lemmas 4.2.8 and 4.2.10; it is immediate when $p \geq 2$. When $2 - 1/n < p < 2$, we observe that

$$\begin{aligned}
&\left(\left[\frac{|\mathcal{D}\Psi_1|(B_{2R})}{(2R)^{n-1})}\right]^q + \left[\frac{|\mathcal{D}\Psi_2|(B_{2R})}{(2R)^{n-1}}\right]^q\right)\left(\fint_{B_{2R}} (|Dw_1| + s)^q \, dx\right)^{2-p} \\
&\leq c\left[\frac{\Lambda(B_{2R})}{(2R)^{n-1}}\right]^q \left(\fint_{B_{2R}} (|Du| + s)^q \, dx + \fint_{B_{2R}} |Du - Dw_1|^q \, dx\right)^{2-p} \\
&\leq c\left[\frac{\Lambda(B_{2R})}{(2R)^{n-1}}\right]^q \left(\fint_{B_{2R}} (|Du| + s)^q \, dx\right)^{2-p} \\
&\quad + c\left[\frac{\Lambda(B_{2R})}{(2R)^{n-1}}\right]^q \left(\left[\frac{|\mu|(B_{2R})}{(2R)^{n-1}}\right]^{\frac{q}{p-1}} + \fint_{B_{2R}} (|Du| + s)^q \, dx\right)^{2-p} \\
&\leq c\left[\frac{\Lambda(B_{2R})}{(2R)^{n-1}}\right]^{\frac{q}{p-1}} + c\left[\frac{\Lambda(B_{2R})}{(2R)^{n-1}}\right]^q \left(\fint_{B_{2R}} (|Du| + s)^q \, dx\right)^{2-p}, \qquad (4.122)
\end{aligned}$$

where we have used (4.115) and Young's inequality. Estimating in a completely similar way, and then using (4.122), we also have

$$\begin{aligned}
&\left[\frac{\mathcal{D}\Psi_1(B_{2R})}{(2R)^{n-1}}\right]^q \left(\fint_{B_{2R}} (|Dw_2| + s)^q \, dx\right)^{2-p} \\
&\leq c\left[\frac{\Lambda(B_{2R})}{(2R)^{n-1}}\right]^{\frac{q}{p-1}} + c\left[\frac{\Lambda(B_{2R})}{(2R)^{n-1}}\right]^q \left(\fint_{B_{2R}} (|Dw_1| + s)^q \, dx\right)^{2-p} \\
&\leq c\left[\frac{\Lambda(B_{2R})}{(2R)^{n-1}}\right]^{\frac{q}{p-1}} + c\left[\frac{\Lambda(B_{2R})}{(2R)^{n-1}}\right]^q \left(\fint_{B_{2R}} (|Du| + s)^q \, dx\right)^{2-p}. \qquad (4.123)
\end{aligned}$$

Combining (4.115), (4.117), (4.118), (4.122) and (4.123), we arrive at

$$\begin{aligned}
&\fint_{B_{2R}} \left(|Du - Dv|^q + |V(Du) - V(Dv)|^{\frac{2q}{p}}\right) dx \\
&\leq c\left[\frac{\Lambda(B_{2R})}{(2R)^{n-1}}\right]^{\frac{q}{p-1}} + c\left[\frac{\Lambda(B_{2R})}{(2R)^{n-1}}\right]^q \left(\fint_{B_{2R}} (|Du| + s)^q \, dx\right)^{2-p} \qquad (4.124)
\end{aligned}$$

whenever q satisfies (4.116). Now, estimate (4.120) for lower values of $q \in$

$(0,1)$ and $t \in (0,q]$ follows from the arguments in [143, Proposition 4.1] and [7, Lemma 3.1]. □

In order to linearize the above comparison estimates, we establish an additional estimate. We fix a ball

$$B_{2MR} \equiv B_{2MR}(x_0) \Subset \Omega \quad \text{with} \quad M \geq 8 \quad \text{and} \quad R \leq 1, \tag{4.125}$$

where M is a free parameter whose value will be determined later in this section.

We then consider the following comparison maps. The first one is $w_{1,*} \in \mathcal{A}^{u}_{\psi_1}(B_{MR})$, which is defined as the weak solution to

$$\begin{cases} \displaystyle\int_{B_{MR}} A(Dw_{1,*}) \cdot D(\phi - w_{1,*})\, dx \\ \displaystyle\geq \int_{B_{MR}} A(D\psi_2) \cdot D(\phi - w_{1,*})\, dx & \forall\ \phi \in \mathcal{A}^{u}_{\psi_1}(B_{MR}), \\ \qquad w_{1,*} \geq \psi_1 & \text{a.e. in } B_{MR}, \\ \qquad w_{1,*} = u & \text{on } \partial B_{MR}. \end{cases}$$

Accordingly, $w_{2,*} \in u + W^{1,p}_0(B_{MR})$ is defined as the weak solution to

$$\begin{cases} -\operatorname{div} A(Dw_{2,*}) = -\operatorname{div} A(D\psi_2) & \text{in } B_{MR}, \\ \qquad w_{2,*} = u & \text{on } \partial B_{MR}. \end{cases}$$

The last one is $v_* \in u + W^{1,p}_0(B_{MR})$ which is defined as the weak solution to

$$\begin{cases} -\operatorname{div} A(Dv_*) = 0 & \text{in } B_{MR}, \\ \qquad v_* = u & \text{on } \partial B_{MR}. \end{cases}$$

The following lemma will be of crucial importance in the linearization procedure for $p \geq 2$.

Lemma 4.2.12. *Let $u \in \mathcal{A}^{g}_{\psi_1,\psi_2}(\Omega)$ be the weak solution to* (4.110) *under assumptions* (2.8) *with $p \geq 2$, and let $w_{1,*}$, $w_{2,*}$, v_* be the functions defined in the above display. Suppose further that*

$$\left[\frac{\Lambda(B_{MR})}{(MR)^{n-1}}\right] \leq \lambda^{p-1} \tag{4.126}$$

holds for some $\lambda > 0$*, together with the bounds*

$$\frac{\lambda}{H} \le |Dv_*| + s \le H\lambda \quad \text{in } B_{2R}, \tag{4.127}$$

where $H \ge 1$ *is a constant. Then there exists a constant* $c \equiv c(\mathtt{data}, M, H)$ *such that*

$$\fint_{B_{2R}} |Du - Dv| \, dx \le c\lambda^{2-p} \left[\frac{\Lambda(B_{MR})}{(MR)^{n-1}} \right]. \tag{4.128}$$

Proof. We basically follow the proof of [146, Lemma 3]. When $p = 2$, (4.128) is a direct consequence of Lemma 4.2.11. Therefore we will assume $p > 2$ in the rest of the proof. We fix two constants

$$\gamma := \frac{1}{4(p-1)(n+1)} \quad \text{and} \quad \xi := 1 + 2\gamma,$$

and moreover set

$$\bar{w}_1 := \frac{w_1}{\lambda}, \quad \bar{w}_{1,*} := \frac{w_{1,*}}{\lambda}, \quad \bar{v} := \frac{v}{\lambda}, \quad \bar{v}_* := \frac{v_*}{\lambda}, \quad \text{and} \quad \bar{s} := \frac{s}{\lambda}.$$

Triangle inequality gives

$$\begin{aligned} \fint_{B_{2R}} |Du - Dv| \, dx \le \fint_{B_{2R}} |Du - Dw_1| \, dx + \fint_{B_{2R}} |Dw_1 - Dw_2| \, dx \\ + \fint_{B_{2R}} |Dw_2 - Dv| \, dx, \end{aligned} \tag{4.129}$$

and we are going to estimate each integral on the right-hand side separately.

We start with the first integral. Using (4.127), we have

$$\begin{aligned} \fint_{B_{2R}} |Du - Dw_1| \, dx &\le H^{(p-2)(1+\gamma)} \fint_{B_{2R}} (|D\bar{v}_*| + \bar{s})^{(p-2)(1+\gamma)} |Du - Dw_1| \, dx \\ &\le c \fint_{B_{2R}} |D\bar{v}_* - D\bar{w}_1|^{(p-2)(1+\gamma)} |Du - Dw_1| \, dx \\ &\quad + c \fint_{B_{2R}} (|D\bar{w}_1| + \bar{s})^{(p-2)(1+\gamma)} |Du - Dw_1| \, dx \end{aligned} \tag{4.130}$$

for a constant $c \equiv c(\mathtt{data}, H)$.

To estimate the second-last integral in (4.130), we observe that (4.115) and (4.124) imply

$$\begin{aligned} \fint_{B_{2R}} |D\bar{v}_* - D\bar{w}_1|^q \, dx &\le c\lambda^{-q} \left(\fint_{B_{MR}} |Dv_* - Du|^q \, dx + \fint_{B_{2R}} |Du - Dw_1|^q \, dx \right) \\ &\le c\lambda^{-q} \left[\frac{\Lambda(B_{MR})}{(MR)^{n-1}} \right]^{\frac{q}{p-1}} \end{aligned} \tag{4.131}$$

for any q satisfying (4.116), where $c \equiv c(\mathtt{data}, M, H, q)$. We then estimate, again using (4.115) and (4.124),

$$\begin{aligned} &\fint_{B_{2R}} |D\bar{v}_* - D\bar{w}_1|^{(p-2)(1+\gamma)} |Du - Dw_1| \, dx \\ &\le c \left(\fint_{B_{2R}} |D\bar{v}_* - D\bar{w}_1|^{(p-1)(1+\gamma)} \, dx \right)^{\frac{p-2}{p-1}} \left(\fint_{B_{2R}} |Du - Dw_1|^{p-1} \, dx \right)^{\frac{1}{p-1}} \\ &\le c \left[\frac{\Lambda(B_{MR})}{\lambda^{p-1}(MR)^{n-1}} \right]^{\frac{(p-2)(1+\gamma)}{p-1}} \left[\frac{|\mu|(B_{MR})}{(MR)^{n-1}} \right]^{\frac{1}{p-1}} \\ &\le c\lambda \left[\frac{\Lambda(B_{MR})}{\lambda^{p-1}(MR)^{n-1}} \right]^{\frac{(p-2)(1+\gamma)+1}{p-1}} \\ &\le c\lambda \left[\frac{\Lambda(B_{MR})}{\lambda^{p-1}(MR)^{n-1}} \right] = c\lambda^{2-p} \left[\frac{\Lambda(B_{MR})}{(MR)^{n-1}} \right]. \end{aligned} \tag{4.132}$$

Here, for the last inequality, we have used (4.126) and the fact that

$$(1+\gamma)(p-2) + 1 > p - 1.$$

Combining (4.132) with (4.130) gives

$$\begin{aligned} &\fint_{B_{2R}} |Du - Dw_1| \, dx \\ &\le c\lambda^{2-p} \left[\frac{\Lambda(B_{MR})}{(MR)^{n-1}} \right] + c \fint_{B_{2R}} (|D\bar{w}_1| + \bar{s})^{(p-2)(1+\gamma)} |Du - Dw_1| \, dx. \end{aligned} \tag{4.133}$$

Estimating similarly as in (4.130) and this time using (4.117), (4.118) and

(4.124), we also have

$$
\begin{aligned}
&\fint_{B_{2R}} |Dw_1 - Dw_2| \, dx \\
&\leq c\lambda^{2-p} \left[\frac{\Lambda(B_{MR})}{(MR)^{n-1}} \right] + c \fint_{B_{2R}} (|D\bar{w}_2| + \bar{s})^{(p-2)(1+\gamma)} |Dw_1 - Dw_2| \, dx \quad (4.134)
\end{aligned}
$$

and

$$
\begin{aligned}
&\fint_{B_{2R}} |Dw_2 - Dv| \, dx \\
&\leq c\lambda^{2-p} \left[\frac{\Lambda(B_{MR})}{(MR)^{n-1}} \right] + c \fint_{B_{2R}} (|D\bar{v}| + \bar{s})^{(p-2)(1+\gamma)} |Dw_2 - Dv| \, dx. \quad (4.135)
\end{aligned}
$$

We now estimate the last integral on the right-hand side in each of (4.133), (4.134) and (4.135). For simplicity, we will again give full details only for (4.133). We apply Hölder's inequality and (4.114), recalling that $\xi = 1 + 2\gamma$, in order to have for any $h > 0$

$$
\begin{aligned}
&\fint_{B_{2R}} (|D\bar{w}_1| + \bar{s})^{(p-2)(1+\gamma)} |Du - Dw_1| \, dx \\
&\leq c \fint_{B_{2R}} \left[\lambda^{2-p} \frac{(|Dw_1| + |Du| + s)^{p-2} |Du - Dw_1|^2}{(h + |u - w_1|)^{\xi}} \right]^{\frac{1}{2}} \\
&\qquad \cdot \left[(|D\bar{w}_1| + \bar{s})^{(p-2)(1+2\gamma)} (h + |u - w_1|)^{\xi} \right]^{\frac{1}{2}} \, dx \\
&\leq c \fint_{B_{2R}} \left[\lambda^{2-p} \frac{|V(Du) - V(Dw_1)|^2}{(h + |u - w_1|)^{\xi}} \right]^{\frac{1}{2}} \\
&\qquad \cdot \left[(|D\bar{w}_1| + \bar{s})^{(p-2)\xi} (h + |u - w_1|)^{\xi} \right]^{\frac{1}{2}} \, dx \\
&\leq c \left(\lambda^{2-p} \fint_{B_{2R}} \frac{|V(Du) - V(Dw_1)|^2}{(h + |u - w_1|)^{\xi}} \, dx \right)^{\frac{1}{2}} \\
&\qquad \cdot \left(\fint_{B_{2R}} (|D\bar{w}_1| + \bar{s})^{(p-2)\xi} (h + |u - w_1|)^{\xi} \, dx \right)^{\frac{1}{2}} \\
&\leq c\lambda^{\frac{2-p}{2}} \left[h^{1-\xi} \frac{\Lambda(B_{2R})}{R^n} \right]^{\frac{1}{2}} \left(\fint_{B_{2R}} (|D\bar{w}_1| + \bar{s})^{(p-2)\xi} (h + |u - w_1|)^{\xi} \, dx \right)^{\frac{1}{2}}. \quad (4.136)
\end{aligned}
$$

We then choose

$$h := \left(\fint_{B_{2R}} (|D\bar{w}_1| + \bar{s})^{(p-2)\xi} |u - w_1|^\xi \, dx \right)^{\frac{1}{\xi}} + \delta \tag{4.137}$$

for $\delta > 0$ sufficiently small, which gives

$$\begin{aligned} &\left(\fint_{B_{2R}} (|D\bar{w}_1| + \bar{s})^{(p-2)\xi} (h + |u - w_1|)^\xi \, dx \right)^{\frac{1}{2}} \\ &\leq c h^{\frac{\xi}{2}} \left(\fint_{B_{2R}} (|D\bar{w}_1| + \bar{s})^{(p-2)\xi} \, dx \right)^{\frac{1}{2}} + c h^{\frac{\xi}{2}}. \end{aligned}$$

We note that the role of δ in (4.137) is just to guarantee that $h > 0$, as we let $\delta \to 0$ at the end of the proof. Also, since (4.131) and (4.127) imply

$$\begin{aligned} &\fint_{B_{2R}} (|D\bar{w}_1| + \bar{s})^{(p-2)\xi} \, dx \\ &\leq c \fint_{B_{2R}} |D\bar{w}_1 - D\bar{v}_*|^{(p-2)\xi} \, dx + c \fint_{B_{2R}} (|D\bar{v}_*| + \bar{s})^{(p-2)\xi} \, dx \\ &\leq c \left[\frac{\Lambda(B_{MR})}{\lambda^{p-1} (MR)^{n-1}} \right]^{\frac{(p-2)\xi}{p-1}} + c H^{(p-2)\xi} \\ &\leq c, \end{aligned}$$

we further have

$$\left(\fint_{B_{2R}} (|D\bar{w}_1| + \bar{s})^{(p-2)\xi} (h + |u - w_1|)^\xi \, dx \right)^{\frac{1}{2}} \leq c h^{\frac{\xi}{2}}.$$

It then follows from (4.136) that

$$\begin{aligned} &\fint_{B_{2R}} (|D\bar{w}_1| + \bar{s})^{(p-2)(1+\gamma)} |Du - Dw_1| \, dx \\ &\leq c \left(\frac{h}{R} \right)^{\frac{1}{2}} \left[\frac{\Lambda(B_{2R})}{\lambda^{p-2} R^{n-1}} \right]^{\frac{1}{2}} \leq \frac{c \lambda^{2-p}}{\varepsilon} \left[\frac{\Lambda(B_{MR})}{(MR)^{n-1}} \right] + \frac{\varepsilon h}{R} \end{aligned} \tag{4.138}$$

holds whenever $\varepsilon \in (0, 1)$.

Finally, it remains to estimate h. We write

$$\begin{aligned} h &\le c\left(\fint_{B_{2R}} |D\bar{v}_* - D\bar{w}_1|^{(p-2)\xi} |u-w_1|^{\xi}\,dx\right)^{\frac{1}{\xi}} \\ &\quad + c\left(\fint_{B_{2R}} (|D\bar{v}_*| + \bar{s})^{(p-2)\xi} |u-w_1|^{\xi}\,dx\right)^{\frac{1}{\xi}} + \delta \\ &=: I_1 + I_2 + \delta. \end{aligned} \tag{4.139}$$

Using (4.117), (4.118) and (4.131), we estimate I_1 as

$$\begin{aligned} I_1 &\le c\left(\fint_{B_{2R}} |D\bar{v}_* - D\bar{w}_1|^{\xi(p-1)}\,dx\right)^{\frac{p-2}{\xi(p-1)}} \left(\fint_{B_{2R}} |u-w_1|^{\xi(p-1)}\,dx\right)^{\frac{1}{\xi(p-1)}} \\ &\le cR\left[\frac{\Lambda(B_{MR})}{\lambda^{p-1}(MR)^{n-1}}\right]^{\frac{p-2}{p-1}} \cdot \left(\fint_{B_{2R}} |Du-Dw_1|^{\xi(p-1)}\,dx\right)^{\frac{1}{\xi(p-1)}} \\ &\le cR\left[\frac{\Lambda(B_{MR})}{\lambda^{p-1}(MR)^{n-1}}\right]^{\frac{p-2}{p-1}} \left[\frac{|\mu|(B_{2R})}{(2R)^{n-1}}\right]^{\frac{1}{p-1}} \\ &\le cR\lambda^{2-p}\left[\frac{\Lambda(B_{MR})}{(MR)^{n-1}}\right]. \end{aligned} \tag{4.140}$$

As for I_2, we apply (4.115) and (4.127) to have

$$\begin{aligned} I_2 &\le cH^{p-2}\left(\fint_{B_{2R}} |u-w_1|^{\xi}\,dx\right)^{\frac{1}{\xi}} \\ &\le cR\fint_{B_{2R}} |Du-Dw_1|\,dx \\ &\le cH^{(p-2)(1+\gamma)}MR\fint_{B_{2R}} (|D\bar{v}_*| + \bar{s})^{(p-2)(1+\gamma)} |Du-Dw_1|\,dx \\ &\le cR\fint_{B_{2R}} (|D\bar{w}_1| + \bar{s})^{(p-2)(1+\gamma)} |Du-Dw_1|\,dx \\ &\quad + cR\fint_{B_{2R}} |D\bar{v}_* - D\bar{w}_1|^{(p-2)(1+\gamma)} |Du-Dw_1|\,dx. \end{aligned}$$

Applying (4.132) to the last integral, we obtain

$$I_2 \le cR \fint_{B_{2R}} (|D\bar{w}_1| + \bar{s})^{(p-2)(1+\gamma)} |Du - Dw_1| \, dx + cR\lambda^{2-p} \left[\frac{\Lambda(B_{MR})}{(MR)^{n-1}} \right].$$

From this inequality together with (4.139) and (4.140), we find

$$\frac{h}{R} \le c_* \fint_{B_{2R}} (|D\bar{w}_1| + \bar{s})^{(p-2)(1+\gamma)} |Du - Dw_1| \, dx + c_*\lambda^{2-p} \left[\frac{\Lambda(B_{MR})}{(MR)^{n-1}} \right] + \frac{\delta}{R},$$

where $c_* \equiv c_*(\mathtt{data}, H, M)$.

Putting this inequality into (4.138), choosing $\varepsilon = 1/(2c_*)$ and then reabsorbing terms, we arrive at

$$\fint_{B_{2R}} (|D\bar{w}_1| + \bar{s})^{(p-2)(1+\gamma)} |Du - Dw_1| \, dx \le c\lambda^{2-p} \left[\frac{\Lambda(B_{MR})}{(MR)^{n-1}} \right] + \frac{c\delta}{R}.$$

We let $\delta \to 0$ and then combine the resulting estimate with (4.133) to have

$$\fint_{B_{2R}} |Du - Dw_1| \, dx \le c\lambda^{2-p} \left[\frac{\Lambda(B_{MR})}{(MR)^{n-1}} \right]. \tag{4.141}$$

In a similar way, using Lemma 4.2.9, we can also deduce

$$\fint_{B_{2R}} (|D\bar{w}_2| + \bar{s})^{(p-2)(1+\gamma)} |Dw_1 - Dw_2| \, dx \le c\lambda^{2-p} \left[\frac{\Lambda(B_{MR})}{(MR)^{n-1}} \right]$$

and

$$\fint_{B_{2R}} (|D\bar{v}| + \bar{s})^{(p-2)(1+\gamma)} |Dw_2 - Dv| \, dx \le c\lambda^{2-p} \left[\frac{\Lambda(B_{MR})}{(MR)^{n-1}} \right].$$

Combining the last two displays with (4.134) and (4.135), we obtain

$$\fint_{B_{2R}} |Dw_1 - Dw_2| \, dx + \fint_{B_{2R}} |Dw_2 - Dv| \, dx \le c\lambda^{2-p} \left[\frac{\Lambda(B_{MR})}{(MR)^{n-1}} \right]. \tag{4.142}$$

Finally, connecting (4.141) and (4.142) to (4.129), the desired estimate (4.128) follows. □

We now establish linearized comparison estimates between (4.110) and (4.113). We notice that, recalling the results in Sections 2.3.2 and 4.2.2, the linearization procedure is basically the same as in [7, Section 5]; the only difference is that μ in the estimates therein is replaced by Λ defined in (4.119). Therefore we just sketch the strategies and omit the details.

With $M \geq 8$ as in (4.125), and another free parameter $\theta_1 \in (0,1)$, we consider the following two alternatives. One is *the two-scales degenerate alternative*

$$\fint_{B_{2MR}} |A(Du) - (A(Du))_{B_{2MR}}|\,dx \geq \theta_1 \left[|(A(Du))_{B_{2R/M}}| + s^{p-1} \right],$$

and the other is *the two-scales non-degenerate alternative*

$$\fint_{B_{2MR}} |A(Du) - (A(Du))_{B_{2MR}}|\,dx < \theta_1 \left[|(A(Du))_{B_{2R/M}}| + s^{p-1} \right].$$

In the non-degenerate alternative, we denote

$$\lambda := \left(\fint_{B_{2R/M}} (|Du|^2 + s^2)^{\frac{p-1}{2}}\,dx \right)^{\frac{1}{p-1}}$$

and further distinguish two cases, making use of an additional free parameter $\sigma_1 \in (0, 1/2^n)$:

$$\frac{\Lambda(B_{2MR})}{(2MR)^{n-1}} \leq \sigma_1 \lambda^{p-1} \quad \text{or} \quad \lambda^{p-1} < \frac{1}{\sigma_1} \left[\frac{\Lambda(B_{2MR})}{(2MR)^{n-1}} \right].$$

Taking account of the above alternatives, and fixing M, θ_1 and σ_1 as universal constants depending only on $\texttt{data}$ as in [7, Remark 4], we let $K = 4M^2$ and then make an elementary modification in order to conclude with the following comparison estimate.

Lemma 4.2.13. *Let* $u \in \mathcal{A}^g_{\psi_1,\psi_2}(\Omega)$ *be the weak solution to* (4.110) *under assumptions* (2.8) *when* $p \geq 2$*, and assumptions* (2.8) *and* (4.104) *when* $2 - 1/n < p < 2$*. There exists a constant* $K \equiv K(\texttt{data}) \geq 128$ *such that if* $B_{2KR} \Subset \Omega$ *is a ball with* $R \leq 1$*, and if* v *is the unique weak solution to*

(4.113) *with* B_{2R} *replaced by* $B_{2\sqrt{K}R}$, *then the following inequalities*

$$\begin{aligned}&\fint_{B_{4R}} |A(Du) - A(Dv)|\, dx\\&\le cR^{\delta m} \fint_{B_{KR}} |A(Du) - (A(Du))_{B_{KR}}|\, dx + c\left[\frac{\Lambda(B_{KR})}{(KR)^{n-1+\delta|p-2|}}\right]\end{aligned}$$

and

$$\begin{aligned}&\fint_{B_{2R}} |D(A(Dv))|\, dx\\&\le \frac{c}{R} \fint_{B_{KR}} |A(Du) - (A(Du))_{B_{KR}}|\, dx + c\left[\frac{\Lambda(B_{KR})}{(KR)^{n+\delta|p-2|}}\right]\end{aligned}$$

hold whenever $\delta \ge 0$, *where* $m = \min\{p-1, 1\}$ *and* $c \equiv c(\mathtt{data})$.

4.2.3 Proof of Theorem 4.2.2

In this section, we prove estimate (4.106), which with a standard covering argument gives (4.105). The proof will be divided into three steps.

Step 1: Reduction to a priori estimates. First of all, we note that we may restrict ourselves to the case when (4.109) holds. In fact, given a limit of approximating solutions u to $OP(\psi_1, \psi_2; \mu)$, let $\{u_k\}$ and $\{\mu_k\}$ be the two sequences as described in Definition 4.2.1. Once we have estimate (4.106) for u_k and μ_k, an elementary manipulation as in [7, Section 6.1], along with (4.101) and (4.102), gives the same estimate for u and μ as well. Note that we do not approximate the obstacles.

Step 2: Rescaling. For the proof of (4.106), we may assume $B_R \equiv B_R(x_0) \equiv B_1 \equiv B_1(0)$ and

$$\int_{B_1} |A(Du)|\, dx + \Lambda(B_1) \le c(n,p),$$

and then prove

$$\int_{B_{1/2}} \int_{B_{1/2}} \frac{|A(Du(x)) - A(Du(y))|}{|x-y|^{n+\sigma}}\, dx\, dy \le c \tag{4.143}$$

for every $\sigma \in (0,1)$, where $c \equiv c(\mathtt{data}, \sigma)$. This can be done by a standard scaling argument as follows. We define

$$H := \left\{ \fint_{B_R} |A(Du)|\, dx + \left[\frac{\Lambda(B_R)}{R^{n-1}} \right] \right\}^{\frac{1}{p-1}}$$

and

$$\begin{aligned} \tilde{u}(y) &:= \frac{u(x_0 + Ry)}{HR}, & \tilde{\psi}_i(y) &:= \frac{\psi_i(x_0 + Ry)}{HR} \quad (i = 1, 2), \\ \tilde{\mu}(y) &:= \frac{R\mu(x_0 + Ry)}{H^{p-1}}, & \tilde{A}(z) &:= \frac{A(Hz)}{H^{p-1}}, \end{aligned} \tag{4.144}$$

for $y \in B_1$ and $z \in \mathbb{R}^n$. We may assume that $H > 0$, otherwise there is nothing to prove.

Step 3: Proof of (4.143) *and conclusion.* In the following, we fix the open sets

$$\Omega' \Subset \Omega'' \subset B_1 \quad \text{with} \quad d := \operatorname{dist}(\Omega', \partial\Omega'').$$

We moreover define

$$\kappa := \frac{m}{m + |p-2|} = \begin{cases} 1/(p-1) & \text{if } p \geq 2, \\ p - 1 & \text{if } p < 2 \end{cases}$$

and

$$\gamma(t) := [1 - \kappa(1-t)][\kappa(1-t) + t], \qquad t \in [0, 1).$$

The main essence here is the following bootstrap lemma, which can be proved in the same way as in [7, Lemma 6.1], again modulo replacing μ by Λ in the estimates therein.

Lemma 4.2.14. *For open sets as above, assume that $A(Du) \in W^{t,1}(\Omega'')$ for some $t \in [0, 1)$ and*

$$[A(Du)]_{t,1;\Omega''} \leq c_1$$

for some $c_1 > 0$, when $t > 0$. Then $A(Du) \in W^{\tilde{t},1}(\Omega')$ for every $\tilde{t} \in [0, \gamma(t))$ with the estimate

$$[A(Du)]_{\tilde{t},1;\Omega'} \leq c_2,$$

where c_2 depends only on $\mathtt{data}, d$ and $\tilde{t}$ if $t = 0$ and also on c_1 if $t > 0$.

Finally, for any vector $h \in \mathbb{R}^n$ *such that* $|h| < \operatorname{dist}(\Omega', \partial B_1)$*, it holds that*

$$\sup_h \int_{\Omega'} \frac{|\tau_h(A(Du))|}{|h|^{\gamma(t)}}\,dx \le c_3,$$

where c_3 *depends only on* $\mathtt{data}$ *and* d *if* $t = 0$ *and also on* c_1 *if* $t > 0$.

Finally, we iterate the above lemma by using exactly the same argument as the one after [7, Lemma 6.1], which implies the following: for every $\sigma, \varepsilon \in (0,1)$, there exists a constant $c \equiv c(\mathtt{data}, \sigma, \varepsilon)$ such that

$$[A(Du)]_{\sigma,1;B_{1-\varepsilon}} + \sup_{0<|h|<\varepsilon} \int_{B_{1-\varepsilon}} \frac{|\tau_h(A(Du))|}{|h|^{\sigma}}\,dx \le c. \tag{4.145}$$

This in particular implies (4.143), and the proof of Theorem 4.2.2 is complete.

Moreover, Corollary 4.2.3 easily follows from (4.106) and the following inequality in [7, Section 7]:

$$|(|z_1|^2+s^2)^{\gamma/2}z_1 - (|z_2|^2+s^2)^{\gamma/2}z_2|^{\frac{p-1}{\gamma+1}} \lesssim |A(z_1)-A(z_2)|, \tag{4.146}$$

valid for every $z_1, z_2 \in \mathbb{R}^n$ and $\gamma \in [0, p-2]$.

Remark 4.2.15. *We note that, considering again the scaling arguments in Step 2,* (4.145) *also implies* $A(Du) \in N^{\sigma,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ *for every* $\sigma \in (0,1)$*. Moreover, the estimate*

$$\sup_{0<|h|<R/2} \int_{B_{R/2}} \frac{|\tau_h(A(Du))|}{|h|^{\sigma}}\,dx \le \frac{c}{R^{\sigma}} \int_{B_R} |A(Du)|\,dx + cR^{1-\sigma}[\Lambda(B_R)]$$

holds for any ball $B_R \Subset \Omega$ *and* $\sigma \in (0,1)$*, where* $c \equiv c(\mathtt{data}, \sigma)$*. For more on such differentiability results and related problems, see [165].*

4.2.4 Proof of Theorem 4.2.4

With (4.107) in force, we first obtain a Morrey type decay estimate.

Lemma 4.2.16. *Let* $u \in W^{1,\max\{p-1,1\}}(\Omega)$ *with* $\psi_1 \le u \le \psi_2$ *a.e. in* Ω *be a limit of approximating solutions to* $OP(\psi_1, \psi_2; \mu)$ *under assumptions* (2.8) *with* $2 - 1/n < p \le n$*. Assume* (4.103) *and* (4.107) *with* $p \le \theta \le n$*. Then for* *every q satisfying (4.121), there exists a constant $c \equiv c(\texttt{data}, q)$ such that*

$$\rho^{\frac{q(\theta-1)}{p-1}} \fint_{B_\rho} (|Du|+s)^q\,dx \le cR^{\frac{q(\theta-1)}{p-1}} \fint_{B_R} (|Du|+s)^q\,dx + c\|\Lambda\|^{q/(p-1)}_{L^{1,\theta}(B_R)}$$

holds whenever $B_\rho \subset B_R \Subset \Omega$ are concentric balls.

Proof. Without loss of generality, we again argue under assumption (4.109); a standard approximation argument will lead to the same result for a limit of approximating solutions as well.

Let $B_\rho \subset B_R$ be concentric balls as in the statement; we may assume $\rho \le R/2$. We consider the weak solution v to (4.113) with the ball B_{2R} replaced by $B_{R/2}$. We then recall the following decay estimate below the natural exponent (see for instance [164, Lemma 3.3]):

$$\int_{B_\rho} (|Dv|+s)^q\,dx \le c\left(\frac{\rho}{R}\right)^{n-q+q\beta} \int_{B_{R/2}} (|Dv|+s)^q\,dx, \tag{4.147}$$

where $c \equiv c(\texttt{data}, q) \ge 1$ and $\beta \equiv \beta(\texttt{data}) \in (0,1]$.

Using (4.147), we proceed as

$$\begin{aligned}
\int_{B_\rho} (|Du|+s)^q\,dx &\le c\int_{B_\rho} (|Dv|+s)^q\,dx + c\int_{B_\rho} |Du-Dv|^q\,dx \\
&\le c\left(\frac{\rho}{R}\right)^{n-q+q\beta} \int_{B_{R/2}} (|Dv|+s)^q\,dx + c\int_{B_{R/2}} |Du-Dv|^q\,dx \\
&\le c\left(\frac{\rho}{R}\right)^{n-q+q\beta} \int_{B_{R/2}} (|Du|+s)^q\,dx + c\int_{B_{R/2}} |Du-Dv|^q\,dx.
\end{aligned} \tag{4.148}$$

To estimate the last integral, we apply the comparison estimate (4.120) along with the density condition $\Lambda \in L^{1,\theta}(\Omega)$. In the case $2-1/n < p < 2$, we further apply Young's inequality to the second term in the right-hand side of (4.120). In turn, we have for $p > 2-1/n$

$$\begin{aligned}
\int_{B_{R/2}} |Du-Dv|^q\,dx &\le c_\varepsilon R^{n-\frac{q(n-1)}{p-1}} [\Lambda(B_R)]^{\frac{q}{p-1}} + \varepsilon \int_{B_R} (|Du|+s)^q\,dx \\
&\le c_\varepsilon R^{n-\frac{q(\theta-1)}{p-1}} \|\Lambda\|^{q/(p-1)}_{L^{1,\theta}(B_R)} + \varepsilon \int_{B_R} (|Du|+s)^q\,dx
\end{aligned} \tag{4.149}$$

whenever $\varepsilon \in (0,1)$, where $c_\varepsilon \equiv c_\varepsilon(\texttt{data}, q, \varepsilon)$. Combining (4.148) and (4.149) gives

$$\int_{B_\rho} (|Du| + s)^q \, dx \leq c \left[\left(\frac{\rho}{R} \right)^{n-q+q\beta} + \varepsilon \right] \int_{B_R} (|Du| + s)^q \, dx + c_\varepsilon R^{n - \frac{q(\theta-1)}{p-1}} \|\Lambda\|_{L^{1,\theta}(B_R)}^{q/(p-1)}$$

for any $\varepsilon \in (0,1)$. Since we are assuming $p \leq \theta$, and ε can be chosen arbitrarily small, we can apply Lemma 2.3.11 with the choice

$$\mathcal{Z}(t) = \int_{B_t} (|Du| + s)^q \, dx$$

and $\gamma = n - q(\theta - 1)/(p - 1) < n - q + q\beta$, which concludes the proof. □

Remark 4.2.17. *We note that Lemma 4.2.16 continues to hold for problems with measurable coefficients. This is due to the validity of* (4.120) *and* (4.147) *for such problems.*

We now prove Theorem 4.2.4. We again confine ourselves to the case when $B_R \equiv B_1 \equiv B_1(0)$ and

$$\fint_{B_1} (|Du| + s)^{p-1} \, dx + \|\Lambda\|_{L^{1,\theta}(B_1)} \leq 1$$

hold, and then prove

$$[A(Du)]_{\sigma,1,\theta;B_{1/3}} \leq c \tag{4.150}$$

for every $\sigma \in (0,1)$, where $c \equiv c(\texttt{data}, \sigma)$. This can be again done by a scaling argument as in (4.144), this time with the choice

$$H := \left\{ \fint_{B_R} (|Du| + s)^{p-1} \, dx + \|\Lambda\|_{L^{1,\theta}(B_R)} R^{1-\theta} \right\}^{\frac{1}{p-1}}.$$

Then we have

$$\Lambda(B_\rho) \leq \rho^{n-\theta} \qquad \forall \; B_\rho \subset B_1. \tag{4.151}$$

Moreover, Lemma 4.2.16 with a simple covering argument implies

$$\rho^{\theta-1} \fint_{B_\rho} |A(Du)| \, dx \leq c \qquad \forall \; B_\rho \subset B_{2/3}. \tag{4.152}$$

Let $B_r \subset B_{1/3}$ be an arbitrary ball. Then (4.106) and the scaling argument imply

$$\int_{B_r}\int_{B_r} \frac{|A(Du(x)) - A(Du(y))|}{|x-y|^{n+\sigma}}\, dxdy \le cr^{-\sigma}\int_{B_{2r}} |A(Du)|\, dx + cr^{1-\sigma}\Lambda(B_{2r}) \le cr^{n-\theta+1-\sigma}, \tag{4.153}$$

where we have also used (4.151) and (4.152) for the last inequality. In particular, since $r \le 1$, we have

$$r^{\theta-n}\int_{B_r}\int_{B_r} \frac{|A(Du(x)) - A(Du(y))|}{|x-y|^{n+\sigma}}\, dxdy \le c.$$

Recalling that $B_r \subset B_{1/3}$ is arbitrary, we conclude with (4.150), which completes the proof of Theorem 4.2.4. Again, Corollary 4.2.5 follows from (4.108) and (4.146).

Remark 4.2.18. *In the above proof, especially looking at* (4.153)*, we can slightly improve Theorem 4.2.4 and Corollary 4.2.5 as follows:*

$$A(Du) \in W^{\sigma,1,\theta-1+\sigma}_{\mathrm{loc}}(\Omega;\mathbb{R}^n) \qquad \forall\, \sigma \in (0,1),$$
$$(|Du|^2+s^2)^{\gamma/2}Du \in W^{\sigma\frac{\gamma+1}{p-1},\frac{p-1}{\gamma+1},\theta-1+\sigma}_{\mathrm{loc}}(\Omega;\mathbb{R}^n) \qquad \forall\, \sigma \in (0,1).$$

These results perfectly fit with the scaling property of the problem; at the same time, they allow to recover the maximal integrability results in [164, 168], in light of the embedding of fractional Sobolev-Morrey spaces. Indeed, we have that $W^{\alpha,q,\theta}$ embeds in L^t for every $t < \theta q/(\theta-\alpha q)$ whenever $\alpha q < \theta$, see for instance [186]. This improvement of Morrey space regularity can be thought as a compensation for the lack of differentiability $\sigma < 1$; in particular, it also arises in the case of general measures. Namely, Theorem 4.2.2 and Corollary 4.2.3 are optimal in the scale of fractional Sobolev spaces, but not in the scale of fractional Sobolev-Morrey spaces.

4.3 Comparison principle for obstacle problems with L^1-data

In this section, we consider obstacle problems related to

$$- \operatorname{div} A(x, Du) = f \quad \text{in } \Omega, \tag{4.154}$$

where $f \in L^1(\Omega)$. The vector field $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be C^1-regular in the second variable, with $\partial_z A(\cdot)$ being Carathéodory regular, and to satisfy the following growth and monotonicity assumptions

$$|A(x, z)| + |z||\partial_z A(x, z)| \leq L|z|^{p-1} \tag{4.155}$$

and

$$0 < (A(x, z_1) - A(x, z_2)) \cdot (z_1 - z_2) \tag{4.156}$$

for every $z, z_1, z_2 \in \mathbb{R}^n$ with $z \neq 0$, $z_1 \neq z_2$ and a.e. $x \in \Omega$, where $L > 0$ and $p > 1$ are fixed constants. At certain stages, in order to obtain several regularity results, we will also consider the following ellipticity assumption:

$$\nu|z|^{p-2}|\xi|^2 \leq \partial_z A(x, z)\xi \cdot \xi \tag{4.157}$$

for some $\nu > 0$ and for every $z \in \mathbb{R}^n \setminus \{0\}$, $\xi \in \mathbb{R}^n$ and a.e. $x \in \Omega$. It is readily seen that (4.157) implies the following monotonicity condition

$$(|z_1| + |z_2|)^{p-2}|z_1 - z_2|^2 \leq c(A(x, z_1) - A(x, z_2)) \cdot (z_1 - z_2) \tag{4.158}$$

for any $z_1, z_2 \in \mathbb{R}^n$.

In this section, we provide a comparison principle for obstacle problems with L^1-data. As a consequence, we show that the solution to a given obstacle problem with zero Dirichlet boundary condition is indeed affected by only the positive part of the obstacle, instead of the whole obstacle.

We first recall the definition of limits of approximating solutions, in a slightly different way.

Definition 4.3.1. *Assume that $\psi, g \in W^{1,p}(\Omega)$ with $(\psi - g)_+ \in W_0^{1,p}(\Omega)$ and $f \in L^1(\Omega)$. We say that a function $u \in \mathcal{T}_g^{1,p}(\Omega)$ with $u \geq \psi$ a.e. in Ω is a limit of approximating solutions to the obstacle problem $OP(\psi; f)$ if there* *are a sequence of functions*

$$\{f_k\} \subset L^\infty(\Omega) \quad \textit{with} \quad f_k \to f \textit{ in } L^1(\Omega) \tag{4.159}$$

and a sequence of solutions $\{u_k\} \subset \mathcal{A}_\psi^g(\Omega)$ *to*

$$\int_\Omega A(x, Du_k) \cdot D(\phi - u_k)\, dx \geq \int_\Omega f_k(\phi - u_k)\, dx \qquad \forall\ \phi \in \mathcal{A}_\psi^g(\Omega)$$

with the following convergence

$$\begin{cases} u_k \to u & \textit{a.e. in } \Omega, \\ \displaystyle\int_\Omega |u_k - u|^r\, dx \to 0 & \textit{for every } 0 < r < \dfrac{n(p-1)}{n-p}, \\ \displaystyle\int_\Omega |Du_k - Du|^q\, dx \to 0 & \textit{for every } 0 < q < \dfrac{n(p-1)}{n-1}. \end{cases} \tag{4.160}$$

We refer to [189, Lemma 3.4] for the proof of the existence of limits of approximating solutions under assumptions (4.155) and (4.158). It is worth mentioning that in [189], $\{f_k\}$ is taken to be a sequence in $W^{-1,p'}(\Omega) \cap L^1(\Omega)$ which is not contained in $L^\infty(\Omega)$ in general. However, if one takes $\{f_k\}$ to be the sequence of mollifications of f as in Remark 3.1.9, then it is a subset of $L^\infty(\Omega)$. Hence, it is not restrictive to take $\{f_k\} \subset L^\infty(\Omega)$ in Definition 4.3.1. Moreover, such a construction gives the strong L^1-convergence (4.159) for L^1-data, while only weak* convergence can be assured for measure data. This will play a crucial role in the proof of uniqueness results in Lemma 4.3.3.

4.3.1 Comparison principles

Let us first consider the comparison principle for weak solutions to obstacle problems. For an obstacle function $\psi \in W^{1,p}(\Omega)$, a Dirichlet boundary data $g \in W^{1,p}(\Omega)$ with $(\psi - g)_+ \in W_0^{1,p}(\Omega)$ and a function $f \in W^{-1,p'}(\Omega) \cap L^1(\Omega)$, the obstacle problem for (4.154) is formulated by the variational inequality

$$\int_\Omega A(x, Du) \cdot D(\phi - u)\, dx \geq \int_\Omega f(\phi - u)\, dx \qquad \forall\ \phi \in \mathcal{A}_\psi^g(\Omega), \tag{4.161}$$

The comparison principle for weak solutions to (4.161) is well-known, which we state as follows:

Lemma 4.3.2. *Let $g, \psi_1, \psi_2 \in W^{1,p}(\Omega)$ satisfy $(\psi_1 - g)_+, (\psi_2 - g)_+ \in W_0^{1,p}(\Omega)$ and $f_1, f_2 \in L^\infty(\Omega)$. Under assumptions* (4.155) *and* (4.156)*, let $u_1 \in \mathcal{A}^g_{\psi_1}(\Omega)$ and $u_2 \in \mathcal{A}^g_{\psi_2}(\Omega)$ be the unique weak solutions to* (4.161) *with $(\psi, f) = (\psi_1, f_1)$ and $(\psi, f) = (\psi_2, f_2)$, respectively. Then*

$$\psi_1 \leq \psi_2,\ f_1 \leq f_2 \quad \textit{implies} \quad u_1 \leq u_2 \quad \textit{a.e. in } \Omega.$$

We refer to [185, Theorem 3.2] for the proof of Lemma 4.3.2, where the authors actually considered inhomogeneous double obstacle problems with nonstandard growth. Its proof works for Lemma 4.3.2 in a similar way, as mentioned in [185, Remark 3.7]. We note that such a comparison principle is obtained in the context of the Lewy-Stampacchia inequalities in an abstract form, see also [184]. For similar results in the setting of nonlocal problems, see [192].

In order to extend Lemma 4.3.2 to any limits of approximating solutions, we need the following uniqueness result.

Lemma 4.3.3. *Let $g, \psi \in W^{1,p}(\Omega)$ satisfy $(\psi - g)_+ \in W_0^{1,p}(\Omega)$ and $f \in L^1(\Omega)$. Under assumptions* (4.155) *and* (4.158)*, there exists a unique limit of approximating solutions $u \in \mathcal{T}^{1,p}_g(\Omega)$ to $OP(\psi; f)$.*

Proof. As mentioned above, the existence of u is proved in [189, Lemma 3.4]. To show the uniqueness, let u and $\bar{u}$ be two limits of approximating solutions to $OP(\psi; f)$. Then there are sequences of functions $\{f_k\}, \{\bar{f}_k\} \subset L^\infty(\Omega)$ with $f_k \to f$ and $\bar{f}_k \to f$ in $L^1(\Omega)$, and corresponding sequences of weak solutions $\{u_k\}, \{\bar{u}_k\} \subset \mathcal{A}^g_\psi(\Omega)$ to (4.161) with the data $\{f_k\}, \{\bar{f}_k\}$, respectively.

We then observe that $u_k + T_t(\bar{u}_k - u_k), \bar{u}_k + T_t(u_k - \bar{u}_k) \in \mathcal{A}^g_\psi(\Omega)$ for each $t > 0$. Testing $u_k + T_t(\bar{u}_k - u_k)$ to (4.161) with (u_k, f_k) and $\bar{u}_k + T_t(u_k - \bar{u}_k)$ to (4.161) with $(\bar{u}_k, \bar{f}_k)$ and subtracting them, we have

$$\begin{aligned} &\int_\Omega \chi_{\{|u_k - \bar{u}_k| \leq t\}} (A(x, Du_k) - A(x, D\bar{u}_k)) \cdot (Du_k - D\bar{u}_k)\, dx \\ &\leq \int_\Omega (f_k - \bar{f}_k) T_t(u_k - \bar{u}_k)\, dx \end{aligned} \tag{4.162}$$

for $k \in \mathbb{N}$. The last convergence in (4.160) implies $Du_k \to Du$ a.e. in Ω, so we apply Fatou's lemma to (4.162) to discover

$$\int_\Omega \chi_{\{|u - \bar{u}| \leq t\}} (A(x, Du) - A(x, D\bar{u})) \cdot (Du - D\bar{u})\, dx = 0,$$

where we have also used (4.156). Then $Du = D\bar{u}$ a.e. in the set $\{|u-\bar{u}| \le t\}$ for every $t > 0$. Taking into account the fact that $u, \bar{u} \in \mathcal{T}_g^{1,p}(\Omega)$, we obtain $T_t(u-\bar{u}) = 0$ for each $t > 0$, from which the desired uniqueness follows. □

Note that if a limit of approximating solutions u to $OP(\psi; f)$ under assumptions (4.155) and (4.158) belongs to the energy space $W^{1,p}(\Omega)$, then Lemma 4.3.3 implies that u is the unique weak solution to (4.161).

Theorem 4.3.4. *Let $g, \psi_1, \psi_2 \in W^{1,p}(\Omega)$ satisfy $(\psi_1 - g)_+, (\psi_2 - g)_+ \in W_0^{1,p}(\Omega)$ and $f_1, f_2 \in L^1(\Omega)$. Under assumptions* (4.155) *and* (4.158)*, let u_1 and u_2 be the limits of approximating solutions to $OP(\psi_1; f_1)$ and $OP(\psi_2; f_2)$, respectively. Then*

$$\psi_1 \le \psi_2,\ f_1 \le f_2 \quad \textit{implies} \quad u_1 \le u_2 \quad \textit{a.e. in } \Omega.$$

Proof. Assume that $\psi_1 \le \psi_2$ and $f_1 \le f_2$. We now extend f_1 and f_2 by zero outside Ω and then take $f_{1,k} = \eta_{1/k} * f_1$ and $f_{2,k} = \eta_{1/k} * f_2$ for each $k \in \mathbb{N}$, where $\eta_{1/k}$ is the standard mollifier. Let $u_{1,k}$ and $u_{2,k}$ be the weak solutions to (4.161) with $(\psi, f) = (\psi_1, f_{1,k})$ and $(\psi, f) = (\psi_2, f_{2,k})$, respectively. Then, since $f_{1,k} \le f_{2,k}$, Lemma 4.3.2 implies that $u_{1,k} \le u_{2,k}$ for every k. From Lemma 4.3.3 and Definition 4.3.1, we conclude that $u_1 \le u_2$ a.e. in Ω. □

We now consider problems with zero Dirichlet boundary condition and nonnegative data. It is readily seen that if $g \equiv 0$ and $0 \le f \in W^{-1,p'}(\Omega)$, then the unique weak solution $u \in \mathcal{A}_\psi^g(\Omega)$ to (4.161) with the obstacle function $\psi \in W^{1,p}(\Omega)$ is a weak supersolution to equation (4.154). Then the maximum principle implies $u \ge 0$ a.e. in Ω; hence, $u \in \mathcal{A}_{\psi_+}^g(\Omega)$ is the weak solution to (4.161) with the obstacle function $\psi_+ \in W^{1,p}(\Omega)$. This fact, together with the approximating procedure and Lemma 4.3.3, yields the following:

Corollary 4.3.5. *Let $g \equiv 0$, $\psi \in W^{1,p}(\Omega)$ satisfy $\psi_+ \in W_0^{1,p}(\Omega)$ and $0 \le f \in L^1(\Omega)$. Under assumptions* (4.155) *and* (4.158)*, the limit of approximating solutions u to $OP(\psi; f)$ is indeed the limit of approximating solutions to $OP(\psi_+; f)$.*

This result continues to hold for problems with nonnegative measure data, under an additional assumption on the approximating sequences.

Corollary 4.3.6. *Let $g \equiv 0$, $\psi \in W^{1,p}(\Omega)$ satisfy $\psi_+ \in W_0^{1,p}(\Omega)$ and $\mu \in \mathcal{M}_b(\Omega)$ be a nonnegative measure. Under assumptions* (4.155) *and* (4.158)*,* *let u be a limit of approximating solutions to $OP(\psi;\mu)$ such that the approximating sequence $\{\mu_k\}$ for μ as described in Definition 4.1.1 is made of nonnegative functions. Then u is indeed a limit of approximating solutions to $OP(\psi_+;\mu)$.*

We note that a limit of approximating solutions to an obstacle problem is equal to the obstacle in a set called the contact set, so the regularity of the solution is at best limited to that of the obstacle. Moreover, Corollary 4.3.5 implies that, in the case of zero Dirichlet boundary condition and nonnegative L^1-data, the contact set is contained in the set $\{\psi \geq 0\}$.

4.3.2 Applications to regularity results

In this section, we apply Corollary 4.3.5 to improve three kinds of regularity results for $OP(\psi;f)$: gradient potential estimates, fractional differentiability, and global Calderón-Zygmund type estimates. In what follows, we assume $g \equiv 0$, $f \geq 0$ and the vector field $A(\cdot)$ satisfies (4.155) and (4.157).

An application to gradient potential estimates

Here we assume that $A(\cdot)$ does not depend on the variable x, and improve the gradient potential estimates in Section 4.1. For brevity, we confine ourselves to report the improvement of Corollary 4.1.5.

Theorem 4.3.7. *Let $u \in W_0^{1,\max\{p-1,1\}}(\Omega)$ with $u \geq \psi$ a.e. in Ω be the limit of approximating solutions to the problem $OP(\psi;f)$ under assumptions* (4.155) *and* (4.157) *with $p > 2 - 1/n$. If*

$$\mathbf{I}_1^f(x_0,R) + \int_0^R \left(\fint_{B_r(x_0)} |A(D\psi_+) - (A(D\psi_+))_{B_r(x_0)}|^{p'}\,dx \right)^{\frac{1}{m}} \frac{dr}{r} < \infty$$

holds on a ball $B_R(x_0) \subset \Omega$, where $m := \max\{p', 2\}$, then x_0 is a Lebesgue point of $A(Du)$. Moreover, there exists a constant $c \equiv c(\mathtt{data})$ such that

$$\begin{aligned}|Du(x_0)|^{p-1} \leq{}& c \fint_{B_R(x_0)} |Du|^{p-1}\,dx + c\mathbf{I}_1^f(x_0,R) \\ &+ c\left[\int_0^R \left(\fint_{B_r(x_0)} |A(D\psi_+) - (A(D\psi_+))_{B_r(x_0)}|^{p'}\,dx \right)^{\frac{1}{m}} \frac{dr}{r} \right]^{\frac{m}{p'}}.\end{aligned}$$

An application to fractional differentiability

Here we assume that $A(\cdot)$ does not depend on the variable x, and improve the fractional differentiability results in Section 4.2. Note that all the results in Section 4.2 hold for single obstacle problems in a similar way.

Theorem 4.3.8. *Let $u \in W_0^{1,\max\{p-1,1\}}(\Omega)$ with $u \geq \psi$ a.e. in Ω be the limit of approximating solutions to the problem $OP(\psi; f)$ under assumptions* (4.155) *and* (4.157) *with $p > 2 - 1/n$. In the case $2 - 1/n < p < 2$, assume further that $\partial A(\cdot)$ is symmetric. If the obstacle $\psi \in W^{1,p}(\Omega)$ satisfies $\psi_+ \in W^{2,1}(\Omega)$ and $\mathcal{D}\Psi_+ := \operatorname{div} A(D\psi_+) \in L^1(\Omega)$, then*

$$A(Du) \in W^{\sigma,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^n) \qquad \forall\, \sigma \in (0,1).$$

Moreover, for any $\sigma \in (0,1)$, there exists a constant $c \equiv c(\mathtt{data}, \sigma)$ such that

$$\begin{aligned}&\int_{B_{R/2}} \int_{B_{R/2}} \frac{|A(Du(x)) - A(Du(y))|}{|x-y|^{n+\sigma}}\, dx\, dy \\ &\leq \frac{c}{R^\sigma} \int_{B_R} |A(Du)|\, dx + \frac{c}{R^{\sigma-1}} \int_{B_R} f\, dx + \frac{c}{R^{\sigma-1}} \int_{B_R} |\mathcal{D}\Psi_+|\, dx\end{aligned}$$

whenever $B_R \Subset \Omega$ is a ball.

An application to global Calderón-Zygmund type estimates

Here, we assume that $f \in L^{q_0}(\Omega)$ for

$$q_0 = \begin{cases} \dfrac{np}{np-n+p} & \text{if } p < n, \\ \dfrac{3}{2} & \text{if } p \geq n. \end{cases}$$

Then $f \in W^{-1,p'}(\Omega)$, and the limit of approximating solutions u to $OP(\psi; f)$ with $\psi \in W^{1,p}(\Omega)$ satisfying $\psi_+ \in W_0^{1,p}(\Omega)$ is the weak solution to (4.161) with the obstacle function ψ_+.

Calderón-Zygmund type estimates for obstacle problems with p-growth were first proved in [31]. Later in [44], such local estimates were extended to global ones under suitable assumptions on the vector field $A(\cdot)$ and the domain Ω, which we state as follows:

Definition 4.3.9. *Let $\delta \in (0,1/8)$ and $R>0$ be given. We say that $(A(\cdot),\Omega)$ is (δ,R)-vanishing if the following two conditions hold:*

(i) Denoting

$$\theta(S)(x) := \sup_{z\in\mathbb{R}^n\setminus\{0\}} \frac{1}{|z|^{p-1}} \left| A(x,z) - \fint_S A(\tilde{x},z)\,d\tilde{x} \right|$$

for any measurable set $S\subset\mathbb{R}^n$ and $x\in S$, we have

$$\sup_{0<r<R}\sup_{y\in\mathbb{R}^n} \fint_{B_r(y)} \theta(B_r(y))(x)\,dx \le \delta.$$

(ii) For each $y\in\partial\Omega$ and $r\in(0,R]$, there exists a coordinate system $\{\tilde{y}_1,\dots,\tilde{y}_n\}$, depending on y and r, such that y is at the origin and

$$B_r(0)\cap\{\tilde{y}_n > \delta r\} \subset B_r(0)\cap\Omega \subset B_r(0)\cap\{\tilde{y}_n > -\delta r\}.$$

A domain satisfying (ii) is called a (δ,R)-Reifenberg flat domain. Note that its definition is motivated from Lipschitz domains with small Lipschitz constant. In particular, a (δ,R)-Reifenberg flat domain satisfies the following measure density conditions (see [60]):

$$\sup_{0<r\le R}\sup_{x\in\Omega}\frac{|B_r(x)|}{|\Omega\cap B_r(x)|} \le \left(\frac{2}{1-\delta}\right)^n \le \left(\frac{16}{7}\right)^n,$$
$$\inf_{0<r\le R}\inf_{x\in\partial\Omega}\frac{|\Omega^c\cap B_r(x)|}{|B_r(x)|} \ge \left(\frac{1-\delta}{2}\right)^n \ge \left(\frac{7}{16}\right)^n.$$

We recall the result in [44] in the following way.

Lemma 4.3.10. *Let $u\in\mathcal{A}^0_\psi(\Omega)$ be the weak solution to*

$$\int_\Omega A(x,Du)\cdot D(\phi-u)\,dx \ge \int_\Omega F\cdot D(\phi-u)\,dx \qquad \forall\ \phi\in\mathcal{A}^0_\psi(\Omega) \tag{4.163}$$

under assumptions (4.155) and (4.157), where $F\in L^{p'}(\Omega;\mathbb{R}^n)$ is a given vector field. Assume that $D\psi\in L^{pq}(\Omega;\mathbb{R}^n)$ and $F\in L^{p'q}(\Omega;\mathbb{R}^n)$ for some $q\in(1,\infty)$. Then there exists a constant $\delta_1\equiv\delta_1(\texttt{data},q)>0$ such that if $(A(\cdot),\Omega)$ is (δ_1,R)-vanishing, then

$$\|Du\|_{L^{pq}(\Omega)} \le c\|D\psi\|_{L^{pq}(\Omega)} + c\|F\|_{L^{p'q}(\Omega)} \tag{4.164}$$

holds for a constant $c\equiv c(\texttt{data},q,R,\Omega)$.

The above result was later extended to several problems with nonstandard growth, see [8, 42] and references therein. We also refer to [132] and [49, 53] for the extensions of (4.164) to obstacle problems with measurable nonlinearities and to double obstacle problems, respectively.

Theorem 4.3.11. *Let* $u \in \mathcal{A}^0_\psi(\Omega)$ *be the weak solution to* (4.161) *under assumptions* (4.155) *and* (4.157). *Assume that*

$$D\psi_+ \in L^{pq}(\Omega;\mathbb{R}^n) \;\; \text{and} \;\; f \in L^{m(q)}(\Omega)$$

for some $q \in (1,\infty)$, *where*

$$m(q) = \max\left\{ \frac{npq}{n(p-1)+pq}, 1 \right\}. \tag{4.165}$$

Then there exists a constant $\delta \equiv \delta(\mathtt{data}, q) > 0$ *such that if* $(A(\cdot),\Omega)$ *is* (δ, R)*-vanishing, then*

$$\|Du\|_{L^{pq}(\Omega)} \le c\|D\psi_+\|_{L^{pq}(\Omega)} + c\|f\|_{L^{m(q)}(\Omega)}$$

holds for a constant $c \equiv c(\mathtt{data}, q, R, \Omega)$.

Proof. We first consider the unique SOLA $v \in W^{1,1}_0(\Omega)$ to

$$\begin{cases} -\triangle v = f & \text{in } \Omega, \\ \quad\;\; v = 0 & \text{on } \partial\Omega, \end{cases}$$

and recall the following Calderón-Zygmund type estimates for elliptic measure data problems (see for instance [40, Theorem 1.2]): for any $\gamma > 0$, there exists a constant $\delta_2 \equiv \delta_2(n,\gamma) > 0$ such that if $M_1(f) \in L^\gamma(\Omega)$ and Ω is (δ_2, R)-Reifenberg flat, then

$$\|Dv\|_{L^\gamma(\Omega)} \le c\|M_1(f)\|_{L^\gamma(\Omega)} \tag{4.166}$$

holds for a constant $c \equiv c(n,\gamma,R,\Omega)$. Here, $M_1(f)$ is the 1-fractional maximal function of f, defined by

$$M_1(f)(x) := \sup_{r>0} \left(r \fint_{B_r(x)} f \, d\tilde{x} \right).$$

Note that for any $q > 1$, the exponent $m(q)$ in (4.165) is chosen to satisfy

$$m(q) = \begin{cases} (p'q)_* & \text{if } q \geq n'/p', \\ 1 & \text{otherwise,} \end{cases}$$

where $(p'q)_*$ is the inverse Sobolev exponent of $p'q$. Thus, the embedding property of fractional maximal operators (see for example [135]) implies

$$\|M_1(f)\|_{L^{p'q}(\Omega)} \leq c\|f\|_{L^{m(q)}(\Omega)}. \tag{4.167}$$

In particular, we have $Dv \in L^{p'q}(\Omega; \mathbb{R}^n)$. It then follows from Corollary 4.3.5 that u is the weak solution to (4.163) with $F = Dv$ and ψ replaced by ψ_+. Finally, after choosing $\delta = \min\{\delta_1, \delta_2\}$, we combine (4.166) and (4.167) with (4.164) in order to obtain the desired estimate. □

Remark 4.3.12. *In Theorem 4.3.11, we considered obstacle problems with a nonnegative function $f \in W^{-1,p'}(\Omega)$ in order to apply estimate* (4.164) *and the comparison principle in Corollary 4.3.5. The related results can be extended to obstacle problems with nonstandard growth conditions. It would be interesting to extend Theorem 4.3.11 to irregular obstacle problems with nonnegative L^1-data.*

Chapter 5

Mixed local and nonlocal equations with measure data

5.1 Main results

We consider the following mixed local and nonlocal elliptic equation

$$- \operatorname{div} A(x, Du) + \mathcal{L}u = \mu \quad \text{in } \Omega, \tag{5.1}$$

where $\mu \in \mathcal{M}_b(\mathbb{R}^n)$. The Carathéodory vector field $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to satisfy the following growth and monotonicity conditions:

$$\begin{cases} |A(x,z)| \leq \Lambda |z|^{p-1}, \\ \Lambda^{-1}(|z_1| + |z_2|)^{p-2}|z_1 - z_2|^2 \leq (A(x,z_1) - A(x,z_2)) \cdot (z_1 - z_2) \end{cases} \tag{5.2}$$

for every $x \in \Omega$ and $z, z_1, z_2 \in \mathbb{R}^n$, with $\Lambda \geq 1$ being a fixed constant. The nonlocal operator $\mathcal{L}$ is defined by

$$\mathcal{L}u(x) \coloneqq \text{P.V.} \int_{\mathbb{R}^n} |u(x) - u(y)|^{p-2}(u(x) - u(y))K(x,y)\, dy, \tag{5.3}$$

where $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a measurable, symmetric kernel satisfying

$$\frac{1}{\Lambda |x-y|^{n+sp}} \leq K(x,y) \leq \frac{\Lambda}{|x-y|^{n+sp}} \quad \text{for a.e. } x, y \in \mathbb{R}^n \text{ with } x \neq y. \tag{5.4}$$

In this chapter, we assume

$$s \in (0,1), \qquad p > 2 - \frac{1}{n} \tag{5.5}$$

and use the abbreviation $\texttt{data} := (n, s, p, \Lambda, \mathrm{diam}(\Omega))$.

We adopt the following definition of nonlocal tails in [116], which is slightly different from the one in [93, 94]. For a measurable function $f : \mathbb{R}^n \to \mathbb{R}$, $x_0 \in \mathbb{R}^n$ and $r > 0$, we define

$$\mathrm{Tail}(f; x_0, r) := \left(r^p \int_{\mathbb{R}^n \setminus B_r(x_0)} \frac{|f(x)|^{p-1}}{|x - x_0|^{n+sp}} \, dx \right)^{\frac{1}{p-1}}.$$

We will omit the point x_0 when it is clear from the context. Accordingly, we define the tail space as

$$L^{p-1}_{sp}(\mathbb{R}^n) := \left\{ f : \mathbb{R}^n \to \mathbb{R} \;\middle|\; \int_{\mathbb{R}^n} \frac{|f(x)|^{p-1}}{(1 + |x|)^{n+sp}} \, dx < \infty \right\}.$$

Observe that $f \in L^{p-1}_{sp}(\mathbb{R}^n)$ if and only if $\mathrm{Tail}(f; x_0, r) < \infty$ for any $x_0 \in \mathbb{R}^n$ and $r > 0$. We also note that $W^{1,p}(\mathbb{R}^n) \subset L^{p-1}_{sp}(\mathbb{R}^n)$.

With the space $\mathcal{X}^{1,p}_0(\Omega)$ to be introduced in the next section, we first define weak solutions. In the following, we denote

$$\Phi_p(t) := |t|^{p-2} t, \qquad t \in \mathbb{R}. \tag{5.6}$$

Definition 5.1.1. *Let $\mu \in W^{-1,p'}(\Omega)$. We say that a function $u \in W^{1,p}(\mathbb{R}^n)$ is a weak solution to the equation* (5.1)*, under assumptions* (5.2)-(5.4) *with $p > 1$ and $s \in (0,1)$, if*

$$\begin{aligned} &\int_\Omega A(x, Du) \cdot D\varphi \, dx \\ &+ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi_p(u(x) - u(y))(\varphi(x) - \varphi(y)) K(x,y) \, dx dy = \langle \mu, \varphi \rangle \end{aligned} \tag{5.7}$$

holds for any $\varphi \in C^\infty_0(\Omega)$. Accordingly, we say that u is a weak subsolution (resp. supersolution) to (5.1) *if* (5.7) *holds with "=" replaced by "≤" (resp. "≥") for every nonnegative $\varphi \in C^\infty_0(\Omega)$. Moreover, given any $g \in W^{1,p}(\mathbb{R}^n)$,* *we say that u is a weak solution to the problem*

$$\begin{cases} -\operatorname{div} A(x, Du) + \mathcal{L}u = \mu & \text{in } \Omega, \\ \qquad\qquad\qquad\qquad u = g & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{5.8}$$

if u is a weak solution to (5.1) *and, in addition, $u - g \in \mathcal{X}_0^{1,p}(\Omega)$.*

The existence and uniqueness of weak solution to (5.8) can be proved by standard monotonicity methods [193], see [57, Appendix] for details.

We next define SOLA.

Definition 5.1.2. *Let $\mu \in \mathcal{M}_b(\mathbb{R}^n)$ and $g \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^{p-1}_{sp}(\mathbb{R}^n)$. We say that a function*

$$u \in W^{1,q}(\Omega) \ \text{ for } \ \max\{p-1, 1\} =: q_0 \le q < \bar{q} := \min\left\{\frac{n(p-1)}{n-1}, p\right\} \tag{5.9}$$

is a SOLA to (5.8)*, under assumptions* (5.2)-(5.5)*, if it is a distributional solution, i.e.,*

$$\int_\Omega A(x, Du) \cdot D\varphi \, dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi_p(u(x) - u(y))(\varphi(x) - \varphi(y)) K(x, y) \, dx dy = \int_{\mathbb{R}^n} \varphi \, d\mu$$

holds for any $\varphi \in C_0^\infty(\Omega)$, and $u = g$ a.e. in $\mathbb{R}^n \setminus \Omega$. Moreover, there exists a sequence of weak solutions $\{u_k\} \subset W^{1,p}(\mathbb{R}^n)$ to the Dirichlet problems

$$\begin{cases} -\operatorname{div} A(x, Du_k) + \mathcal{L}u_k = \mu_k & \text{in } \Omega \\ \qquad\qquad\qquad\qquad u_k = g_k & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

in the sense of Definition 5.1.1, such that u_k converges to u a.e. in $\mathbb{R}^n$ and locally in $L^q(\mathbb{R}^n)$. Here the sequence $\{\mu_k\} \subset C_0^\infty(\mathbb{R}^n)$ converges to μ weakly in the sense of measures in Ω and also satisfies

$$\limsup_{k \to \infty} |\mu_k|(B) \le |\mu|(\overline{B}) \tag{5.10}$$

for every ball $B \subset \mathbb{R}^n$. The sequence $\{g_k\} \subset C_0^\infty(\mathbb{R}^n)$ converges to g in the *following sense: for any ball $B_r \equiv B_r(z)$, it holds that*

$$\lim_{k\to\infty} \|g_k - g\|_{W^{1,p}(B_r)} = 0 \qquad \text{and} \qquad \lim_{k\to\infty} \mathrm{Tail}(g_k - g; z, r) = 0.$$

We now state the first result concerning the existence of SOLA.

Theorem 5.1.3. *Let $\mu \in \mathcal{M}_b(\mathbb{R}^n)$ and $g \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^{p-1}_{sp}(\mathbb{R}^n)$. Under assumptions (5.2)-(5.5), there exists a SOLA u to (5.8) such that $u \in W^{1,q}(\Omega)$ for every q satisfying (5.9).*

The next result is a pointwise upper bound via Wolff potentials.

Theorem 5.1.4. *Let $\mu \in \mathcal{M}_b(\mathbb{R}^n)$ and $g \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^{p-1}_{sp}(\mathbb{R}^n)$. Let u be a SOLA to (5.8) under assumptions (5.2)-(5.5), and assume that the Wolff potential $\mathbf{W}^{\mu}_{1,p}(x_0, r)$ is finite for a ball $B_r(x_0) \subset \Omega$. Then x_0 is a Lebesgue point of u in the sense that there exists the precise representative of u at x_0*

$$u(x_0) \coloneqq \lim_{\rho\to 0} (u)_{B_\rho(x_0)}. \tag{5.11}$$

Moreover, the estimate

$$|u(x_0)| \le c\mathbf{W}^{\mu}_{1,p}(x_0, r) + c\left(\fint_{B_r(x_0)} |u|^{q_0}\,dx\right)^{\frac{1}{q_0}} + c\mathrm{Tail}(u; x_0, r) \tag{5.12}$$

holds for a constant $c \equiv c(\mathtt{data})$, where $q_0 \coloneqq \max\{p-1, 1\}$.

We can also obtain a lower bound when both μ and u are nonnegative, which implies the sharpness of estimate (5.12).

Theorem 5.1.5. *Let $\mu \in \mathcal{M}_b(\mathbb{R}^n)$ be a nonnegative measure and $g \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^{p-1}_{sp}(\mathbb{R}^n)$. Let u be a SOLA to (5.8) under assumptions (5.2)-(5.5) with $p < n$. Assume that u is nonnegative in a ball $B_r(x_0) \subset \Omega$ and that the approximating sequence $\{\mu_k\}$ for μ as described in Definition 5.1.2 is made of nonnegative functions. Then the estimate*

$$\mathbf{W}^{\mu}_{1,p}(x_0, r/8) \le cu(x_0) + c\mathrm{Tail}(u_-; x_0, r/2) \tag{5.13}$$

holds for a constant $c \equiv c(\mathtt{data})$, whenever $\mathbf{W}^{\mu}_{1,p}(x_0, r/8)$ is finite. In this case, according to Theorem 5.1.4, $u(x_0)$ is defined as the precise representative of u at x_0 as in (5.11). Moreover, when $\mathbf{W}^{\mu}_{1,p}(x_0, r/8)$ is infinite, we *have*

$$\lim_{t\to 0}(u)_{B_t(x_0)} = \infty. \tag{5.14}$$

Once we have the potential upper bound in Theorem 5.1.4, the well-known mapping properties of Wolff potentials [76] imply the following local Calderón-Zygmund type estimates.

Corollary 5.1.6. *Let* $\mu \in \mathcal{M}_b(\mathbb{R}^n)$ *and* $g \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^{p-1}_{sp}(\mathbb{R}^n)$. *Let* u *be a SOLA to* (5.8) *under assumptions* (5.2)-(5.5). *Then*

- *If* $p < n$, *then* u *belongs to the Marcinkiewicz space* $\mathcal{M}_{\mathrm{loc}}^{\frac{n(p-1)}{n-p}}(\Omega)$.
- *If* $1 < \gamma < n/p$, *then we have the implication*

$$\mu \in L^{\gamma}_{\mathrm{loc}}(\Omega) \implies u \in L_{\mathrm{loc}}^{\frac{n\gamma(p-1)}{n-p\gamma}}(\Omega).$$

Note that we actually prove Theorem 5.1.4 as a corollary of the following result, which is a global oscillation/excess decay estimate. Unlike the case of local equations, we have to consider an excess functional which also reflects long-range interactions. We define

$$E(v; z, r) \coloneqq \left(\fint_{B_r(z)} |v - (v)_{B_r(z)}|^{q_0}\, dx \right)^{\frac{1}{q_0}} + \mathrm{Tail}(v - (v)_{B_r(z)}; z, r), \tag{5.15}$$

where $q_0 \coloneqq \max\{p-1, 1\}$. We will also omit the point x_0 when it is clear from the context.

Theorem 5.1.7. *Under the assumptions of Theorem 5.1.4, we have the estimate*

$$\int_0^r E(u; x_0, t)\frac{dt}{t} + \left|(u)_{B_r(x_0)} - u(x_0)\right| \le c\mathbf{W}^{\mu}_{1,p}(x_0, r) + cE(u; x_0, r) \tag{5.16}$$

for a constant $c \equiv c(\mathtt{data})$, *whenever* $\mathbf{W}^{\mu}_{1,p}(x_0, r)$ *is finite.*

A notable consequence of the above theorem is the following continuity criterion.

Theorem 5.1.8. *Let $\mu \in \mathcal{M}_b(\mathbb{R}^n)$ and $g \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^{p-1}_{sp}(\mathbb{R}^n)$. Let u be a SOLA to* (5.8) *under assumptions* (5.2)-(5.5)*, and let $\Omega' \Subset \Omega$ be an open subset. If*

$$\lim_{t\to 0} \sup_{x\in\Omega'} \mathbf{W}^{\mu}_{1,p}(x,t) = 0, \tag{5.17}$$

then u is continuous in Ω'.

This theorem gives the following corollary concerning the continuity of solutions in borderline cases.

Corollary 5.1.9. *Let $\mu \in \mathcal{M}_b(\mathbb{R}^n)$ and $g \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^{p-1}_{sp}(\mathbb{R}^n)$. Let u be a SOLA to* (5.8) *under assumptions* (5.2)-(5.5) *with $p < n$. If one of the following two conditions holds:*

(i) $\mu \in L(n,1)$ *locally in* Ω,

(ii) $|\mu|(B_r) \le h(r)r^{n-p}$ *for every ball $B_r \subset \mathbb{R}^n$, with $h : [0,\infty) \to [0,\infty)$ satisfying*

$$\int_0 [h(r)]^{\frac{1}{p-1}} \frac{dr}{r} < \infty,$$

then u is continuous in Ω.

We finally note that if the measure satisfies a better density condition, then we can further improve the regularity of SOLA. In order to describe such phenomena, we recall the definition of fractional (restricted and centered) maximal functions. For $x_0 \in \mathbb{R}^n$ and $r > 0$, we define

$$M^{\mu}_{\beta}(x_0,r) \coloneqq \sup_{0<\rho<r} \frac{|\mu|(B_\rho(x_0))}{\rho^{n-\beta}}, \qquad \beta \in (0,n).$$

From the definitions of M^{μ}_p and $\mathbf{W}^{\mu}_{1,p}$, we have for any $\delta > 0$

$$\left[M^{\mu}_p(x_0,r)\right]^{\frac{1}{p-1}} \le c\mathbf{W}^{\mu}_{1,p}(x_0,2r) \le c(\delta)\left[M^{\mu}_{p-\delta}(x_0,2r)\right]^{\frac{1}{p-1}}.$$

The last theorem shows a Hölder continuity criterion for SOLA to (5.1) in terms of the concentration of the measure, which is analogous to the classical results in [127, 158].

Theorem 5.1.10. *Let $\mu \in \mathcal{M}_b(\mathbb{R}^n)$ and $g \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^{p-1}_{sp}(\mathbb{R}^n)$. Let u be a SOLA to* (5.8) *under assumptions* (5.2)-(5.5)*, and let $\Omega' \Subset \Omega$ be an open set. If*

$$\sup_{x\in\Omega'} \left(E(u;x,r) + \left[M^{\mu}_{p-\delta}(x,r) \right]^{\frac{1}{p-1}} \right) < \infty \tag{5.18}$$

for some $\delta \in (0,p]$ and $r < \mathrm{dist}(\Omega', \partial\Omega)$, then $u \in C^{0,\beta}(\Omega')$ for

$$\beta = \begin{cases} \delta/(p-1) & \text{if } \delta < \alpha(p-1), \\ \text{any number in } (0,\alpha), & \text{otherwise.} \end{cases}$$

Here, $\alpha \in (0,1)$ is the Hölder exponent for weak solutions to the homogeneous equation

$$-\mathrm{div}\, A(x,Dv) + \mathcal{L}v = 0 \quad \text{in } \Omega,$$

see Lemma 5.3.5 below for details.

Remark 5.1.11. *In estimates* (5.12) *and* (5.13)*, the dependence of the constant c on* $\mathrm{diam}(\Omega)$ *can be removed, provided $r \leq 1$.*

Remark 5.1.12. *We can see that, by applying the methods in [152], our results continue to hold for more general equations*

$$-\mathrm{div} A(x,Du) + \mathcal{L}_{\Phi} u = \mu \quad \text{in } \Omega.$$

Here the nonlocal operator $\mathcal{L}_{\Phi}$ is defined by

$$\mathcal{L}_{\Phi} u(x) := \mathrm{P.V.} \int_{\mathbb{R}^n} \Phi(u(x) - u(y)) K(x,y)\, dy,$$

where $K(\cdot,\cdot)$ is a measurable, not necessarily symmetric kernel satisfying (5.4) *and $\Phi : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying*

$$\Lambda^{-1}|t|^p \leq \Phi(t)t \leq \Lambda |t|^p, \qquad \forall\, t \in \mathbb{R}.$$

5.2 Preliminaries

We consider the space $\mathcal{X}^{1,p}_0(\Omega) \subset W^{1,p}(\mathbb{R}^n)$ defined as the closure of $C^\infty_0(\Omega)$ with respect to the norm $\|\cdot\|_{W^{1,p}(\mathbb{R}^n)}$. Of course, we identify each element in $C^\infty_0(\Omega)$ with its zero extension to $\mathbb{R}^n$. Being a closed linear subspace of

$W^{1,p}(\mathbb{R}^n)$, $\mathcal{X}_0^{1,p}(\Omega)$ is a separable, reflexive Banach space. It is in fact characterized as

$$\mathcal{X}_0^{1,p}(\Omega) = \left\{ f \in W^{1,p}(\mathbb{R}^n) : f|_\Omega \in W_0^{1,p}(\Omega),\ f = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \right\}.$$

Moreover, we have

$$\|f\|_{W^{1,p}(\mathbb{R}^n)} = \|f\|_{W^{1,p}(\Omega)} \approx \|Df\|_{L^p(\Omega)} \qquad \forall\ f \in \mathcal{X}_0^{1,p}(\Omega),$$

where the last equivalence follows from Poincaré's inequality.

We also note the following result from [116, Lemma 2.3].

Lemma 5.2.1. *Let $p \geq 1$ and $s \in (0,1)$. There exists a constant $c \equiv c(n,s,p,\Omega)$ such that*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}}\, dxdy \leq c \int_\Omega |Df|^p\, dx$$

for every $f \in \mathcal{X}_0^{1,p}(\Omega)$.

5.3 Regularity for homogeneous equations

Here we collect various local regularity results for the homogeneous equation

$$-\operatorname{div} A(x, Dv) + \mathcal{L}v = 0 \quad \text{in } \Omega. \tag{5.19}$$

We start by recalling the following Caccioppoli estimate with tail, see [116, Lemma 3.1].

Lemma 5.3.1. *Let $v \in W^{1,p}(\mathbb{R}^n)$ be a weak subsolution to* (5.19) *under assumptions* (5.2)-(5.4) *with $p > 1$ and $s \in (0,1)$. Then, for any ball $B_r \equiv B_r(x_0) \subset \Omega$ and nonnegative $\phi \in C_0^\infty(B_r)$, we have*

$$\begin{aligned}
&\int_{B_r} |D(w_+\phi)|^p\, dx + \int_{B_r} \int_{B_r} |w_+(x)\phi(x) - w_+(y)\phi(y)|^p K(x,y)\, dxdy \\
&\leq c \int_{B_r} w_+^p |D\phi|^p\, dx + c \int_{B_r} \int_{B_r} (\max\{w_+(x), w_+(y)\})^p |\phi(x) - \phi(y)|^p K(x,y)\, dxdy \\
&\quad + c \left(\sup_{x \in \operatorname{supp}\phi} \int_{\mathbb{R}^n \setminus B_r} w_+^{p-1}(y) K(x,y)\, dy \right) \int_{B_r} w_+ \phi^p\, dx,
\end{aligned}$$

where $w_+ := (v-k)_+$ *for any* $k \in \mathbb{R}$*, and* $c \equiv c(\texttt{data})$*. If* v *is a weak supersolution to* (5.19)*, the estimate holds with* w_+ *replaced by* $w_- := (v-k)_-$*.*

Combining the above lemma with Sobolev's embedding theorem, and then applying De Giorgi iteration, we obtain the local boundedness of weak subsolutions, see [116, Theorem 4.1].

Lemma 5.3.2. *Let* $v \in W^{1,p}(\mathbb{R}^n)$ *be a weak subsolution to* (5.19) *under assumptions* (5.2)-(5.4) *with* $p > 1$ *and* $s \in (0,1)$*. Then, for any ball* $B_r \equiv B_r(x_0) \subset \Omega$ *and* $k \in \mathbb{R}$*, we have*

$$\sup_{B_{r/2}}(v-k)_+ \le c\left(\fint_{B_r}(v-k)_+^p\,dx\right)^{\frac{1}{p}} + \mathrm{Tail}((v-k)_+;x_0,r/2),$$

where $c \equiv c(\texttt{data})$*. If* v *is a weak supersolution to* (5.19)*, this estimate holds with* $(v-k)_+$ *replaced by* $(v-k)_-$*.*

The estimate in Lemma 5.3.2 can be thought as a kind of reverse Hölder's inequality. By a modification which is completely similar to the one presented in [152, Corollary 2.1], we can also obtain the following:

Lemma 5.3.3. *Let* $v \in W^{1,p}(\mathbb{R}^n)$ *be a weak subsolution to* (5.19) *under assumptions* (5.2)-(5.4) *with* $p > 1$ *and* $s \in (0,1)$*. Then for any ball* $B_r \equiv B_r(x_0) \subset \Omega$ *and* $k \in \mathbb{R}$*, we have*

$$\sup_{B_{\sigma r}}(v-k)_+ \le \frac{c}{(1-\sigma)^{np/(p-1)}}\left[\fint_{B_r}(v-k)_+\,dx + \mathrm{Tail}((v-k)_+;x_0,r/2)\right]$$

whenever $\sigma \in (0,1)$*, where* $c \equiv c(\texttt{data})$*. If* v *is a weak supersolution to* (5.19)*, this estimate holds with* $(v-k)_+$ *replaced by* $(v-k)_-$*.*

Using this lemma, we establish a Caccioppoli type estimate below the natural exponent.

Lemma 5.3.4. *Let* $v \in W^{1,p}(\mathbb{R}^n)$ *be a weak solution to* (5.19) *under assumptions* (5.2)-(5.4) *with* $p > 1$ *and* $s \in (0,1)$*. Then for any ball* $B_r \equiv B_r(x_0) \Subset \Omega$ *and* $k \in \mathbb{R}$*, we have*

$$\fint_{B_{\sigma r}}|Dv|^q\,dx \le \frac{c}{(1-\sigma)^{\theta q}r^q}\left[\fint_{B_r}|v-k|\,dx + \mathrm{Tail}(v-k;x_0,r/2)\right]^q$$

whenever $q \in [1,p]$ *and* $\sigma \in [1/2,1)$*, where* $c \equiv c(\texttt{data})$ *and* $\theta \equiv \theta(n,p)$*.*

Proof. It suffices to consider the case $q = p$ only, as the result for lower values of q follows from Hölder's inequality. We choose a cut-off function $\phi \in C_0^\infty(B_{(1+\sigma)r/2})$ satisfying $0 \le \phi \le 1$, $\phi \equiv 1$ in $B_{\sigma r}$ and $|D\phi| \le 4/[(1-\sigma)r]$. Applying Lemma 5.3.1 with this choice of ϕ, we obtain

$$\begin{aligned}
&\fint_{B_{\sigma r}} |Dv|^p\,dx \\
&\le \frac{c}{[(1-\sigma)r]^p}\fint_{B_{(1+\sigma)r/2}} |v-k|^p\,dx + \frac{c}{[(1-\sigma)r]^{sp}}\fint_{B_{(1+\sigma)r/2}} |v-k|^p\,dx \\
&\quad + \frac{c}{(1-\sigma)^{n+sp}r^p}\fint_{B_{(1+\sigma)r/2}} |v-k|\,dx \left(r^p\int_{\mathbb{R}^n\setminus B_r} \frac{|v(x)-k|^{p-1}}{|x-x_0|^{n+sp}}\,dx\right) \\
&\le \frac{c}{(1-\sigma)^{n+p}r^p}\left[\sup_{B_{(1+\sigma)r/2}} |v-k|^p + [\mathrm{Tail}(v-k;x_0,r/2)]^p\right]
\end{aligned}$$

where we have used the fact that

$$\frac{|x-x_0|}{|x-y|} \le \frac{c(n)}{1-\sigma}$$

for $x \in \mathbb{R}^n \setminus B_r(x_0)$ and $y \in B_{(1+\sigma)r/2}(x_0)$, and then Young's inequality. Also, Lemma 5.3.3 implies

$$\sup_{B_{(1+\sigma)r/2}} |v-k|^p \le \frac{c}{(1-\sigma)^{np^2/(p-1)}}\left[\fint_{B_r} |v-k|\,dx + \mathrm{Tail}(v-k;x_0,r/2)\right]^p.$$

Combining the above two estimates gives the desired result. □

Using Lemma 5.3.1 and a logarithmic lemma [116, Lemma 3.4], we can prove the following oscillation estimate for weak solutions, which in turn yields local Hölder continuity. Here we state it in a slightly different form, by further applying Lemma 5.3.3. Moreover, we can also prove nonlocal weak Harnack and Harnack inequalities, see [116, Sections 5-8] for details.

Lemma 5.3.5. *Let $v \in W^{1,p}(\mathbb{R}^n)$ be a weak solution to* (5.19) *under assumptions* (5.2)-(5.4) *with $p > 1$ and $s \in (0,1)$. Then v is locally Hölder continuous. In particular, there exist constants $\alpha \in (0,1)$ and $c \ge 1$, both* *depending only on* `data`*, such that*

$$\operatorname*{osc}_{B_\rho(x_0)} v \le c\left(\frac{\rho}{r}\right)^{\alpha}\left[\fint_{B_{2r}(x_0)} |v-k|\,dx + \mathrm{Tail}(v-k;x_0,r/2)\right]$$

whenever $0<\rho\le r$ *and* $k\in\mathbb{R}$.

Lemma 5.3.6. *Let* $v\in W^{1,p}(\mathbb{R}^n)$ *be a weak supersolution to* (5.19)*, under assumptions* (5.2)-(5.4) *with* $p>1$ *and* $s\in(0,1)$*, such that* $v\ge 0$ *in a ball* $B_R\equiv B_R(x_0)\subset\Omega$*. Let* $q\in(1,p)$*,* $d>0$ *and define* $w:=(v+d)^{(p-q)/q}$*. Then*

$$\begin{aligned}
&\int_{B_r}\phi^p|Dw|^p\,dx+\int_{B_r}\int_{B_r}(\min\{\phi(x),\phi(y)\})^p|w(x)-w(y)|^pK(x,y)\,dxdy\\
&\le c\int_{B_r}w^p|D\phi|^p\,dx+c\int_{B_r}\int_{B_r}(\max\{w(x),w(y)\})^p|\phi(x)-\phi(y)|^pK(x,y)\,dxdy\\
&\quad+c\left(\sup_{x\in\operatorname{supp}\phi}\int_{\mathbb{R}^n\setminus B_r}K(x,y)\,dy+d^{1-p}R^{-p}[\mathrm{Tail}(u_-;x_0,R)]^{p-1}\right)\int_{B_r}w^p\phi^p\,dx
\end{aligned}$$

holds for any $B_r\equiv B_r(x_0)\subset B_{3R/4}(x_0)$ *and nonnegative* $\phi\in C_0^\infty(B_r)$*, where* $c\equiv c($`data`$)$.

Lemma 5.3.7. *Let* $v\in W^{1,p}(\mathbb{R}^n)$ *be a weak supersolution to* (5.19) *under assumptions* (5.2)-(5.4) *with* $p>1$ *and* $s\in(0,1)$ *such that* $v\ge 0$ *in* $B_R\equiv B_R(x_0)\subset\Omega$*. Let*

$$\bar t := \begin{cases}\dfrac{n(p-1)}{n-p} & \text{if } 1<p<n,\\ \infty & \text{if } p\ge n.\end{cases}$$

Then the following estimate holds for any $B_r\equiv B_r(x_0)\subset B_{R/2}(x_0)$ *and for any* $t<\bar t$*:*

$$\left(\fint_{B_r} v^t\,dx\right)^{\frac{1}{t}} \le c\inf_{B_{2r}} v + c\left(\frac{r}{R}\right)^{\frac{p}{p-1}}\mathrm{Tail}(v_-;x_0,R),$$

where $c\equiv c($`data`$)$.

We moreover prove an excess decay estimate for (5.19), which will play a crucial role in proving Theorem 5.1.7. Recalling the definition of the excess functional in (5.15), we state a few basic properties (see [152, Lemma 2.4] for proof): for any $\eta, \zeta \in L^{q_0}(B_r(z)) \cap L^{p-1}_{sp}(\mathbb{R}^n)$, where $q_0 = \max\{p-1, 1\}$, we have the decay

$$E(\eta; z, \sigma r) \leq c(\sigma, n, s, p) E(\eta; z, r)$$

and the quasi-triangle inequality

$$E(\eta + \zeta; z, r) \leq c(p) \left(E(\eta; z, r) + E(\zeta; z, r) \right).$$

Theorem 5.3.8. *Let $v \in W^{1,p}(\mathbb{R}^n)$ be a weak solution to (5.19) under assumptions (5.2)-(5.4) with $p > 1$ and $s \in (0,1)$. Then we have*

$$E(v; x_0, \rho) \leq c \left(\frac{\rho}{r} \right)^{\alpha} E(v; x_0, r)$$

whenever $0 < \rho \leq r$, where $\alpha \in (0,1)$ is as in Lemma 5.3.5 and $c \equiv c(\mathtt{data})$.

Proof. We may assume that $\rho \leq r/4$. Note that Lemma 5.3.5 and Hölder's inequality imply

$$\operatorname*{osc}_{B_t} v \leq c \left(\frac{t}{r} \right)^{\alpha} E(r) \qquad \forall\, t \in [\rho, r/4].$$

In particular, it follows that

$$\left(\fint_{B_\rho} |v - (v)_{B_\rho}|^{q_0} \, dx \right)^{\frac{1}{q_0}} \leq \operatorname*{osc}_{B_\rho} v \leq c \left(\frac{\rho}{r} \right)^{\alpha} E(r). \tag{5.20}$$

Let us now estimate the tail term appearing in the definition of $E(\rho)$. We start splitting as

$$\begin{aligned} \left[\mathrm{Tail}(v - (v)_{B_\rho}; \rho) \right]^{p-1} &= \rho^p \int_{\mathbb{R}^n \setminus B_\rho} \frac{|v(x) - (v)_{B_\rho}|^{p-1}}{|x - x_0|^{n+sp}} \, dx \\ &= \rho^p \int_{\mathbb{R}^n \setminus B_{r/4}} \frac{|v(x) - (v)_{B_\rho}|^{p-1}}{|x - x_0|^{n+sp}} \, dx \\ &\quad + \rho^p \int_{B_{r/4} \setminus B_\rho} \frac{|v(x) - (v)_{B_\rho}|^{p-1}}{|x - x_0|^{n+sp}} \, dx. \end{aligned} \tag{5.21}$$

Then we note that, by Lemma 5.3.3 and Hölder's inequality,

$$|(v)_{B_\rho} - (v)_{B_r}| \le \sup_{B_{r/2}} |v - (v)_{B_r}| \le cE(r).$$

Hence, again using Hölder's inequality, we estimate the first integral in the right-hand side of (5.21) as

$$\begin{aligned}
&\rho^p \int_{\mathbb{R}^n \setminus B_{r/4}} \frac{|v(x) - (v)_{B_\rho}|^{p-1}}{|x - x_0|^{n+sp}}\, dx \\
&\le c\rho^p \int_{\mathbb{R}^n \setminus B_{r/4}} \frac{|v(x) - (v)_{B_r}|^{p-1}}{|x - x_0|^{n+sp}}\, dx + c\left(\frac{\rho}{r}\right)^p r^{(1-s)p} |(v)_{B_\rho} - (v)_{B_r}|^{p-1} \\
&\le c\rho^p \int_{\mathbb{R}^n \setminus B_r} \frac{|v(x) - (v)_{B_r}|^{p-1}}{|x - x_0|^{n+sp}} + c\left(\frac{\rho}{r}\right)^p r^{(1-s)p} \fint_{B_r} |v - (v)_{B_r}|^{p-1}\, dx \\
&\quad + c\left(\frac{\rho}{r}\right)^p r^{(1-s)p} E(r)^{p-1} \\
&\le c\left(\frac{\rho}{r}\right)^p E(r)^{p-1}. \qquad (5.22)
\end{aligned}$$

As for the second integral, we have

$$\begin{aligned}
\rho^p \int_{B_{r/4} \setminus B_\rho} \frac{|v(x) - (v)_{B_\rho}|^{p-1}}{|x - x_0|^{n+sp}}\, dx &\le cr^{(1-s)p} \int_\rho^{r/4} \left(\frac{\rho}{t}\right)^p \left(\operatorname*{osc}_{B_t} v\right)^{p-1} \frac{dt}{t} \\
&\le cE(r)^{p-1} \int_\rho^{r/4} \left(\frac{\rho}{t}\right)^p \left(\frac{t}{r}\right)^{\alpha(p-1)} \frac{dt}{t} \\
&\le \frac{c}{p - \alpha(p-1)} \left(\frac{\rho}{r}\right)^{\alpha(p-1)} E(r)^{p-1}. \qquad (5.23)
\end{aligned}$$

Combining (5.21), (5.22) with (5.23), we arrive at

$$\mathrm{Tail}(v - (v)_{B_\rho}; \rho) \le c\left(\frac{\rho}{r}\right)^\alpha E(r).$$

This and (5.20) imply the desired result. □

5.4 Comparison estimates

In this section we derive several comparison estimates. Here we assume that

$$\mu \in C_0^\infty(\mathbb{R}^n), \qquad g \in W^{1,p}(\mathbb{R}^n).$$

This a priori assumption will be removed with a proper approximation procedure in Section 5.5 below.

For a fixed ball $B_{2r} \equiv B_{2r}(x_0) \subset \mathbb{R}^n$, we first consider the weak solution $u \in W^{1,p}(\mathbb{R}^n)$ to the Dirichlet problem

$$\begin{cases} -\operatorname{div} A(x, Du) + \mathcal{L}u = \mu & \text{in } B_{2r}, \\ u = g & \text{in } \mathbb{R}^n \setminus B_{2r}. \end{cases} \tag{5.24}$$

Next, we define $v \in W^{1,p}(\mathbb{R}^n)$ to be the weak solution to the homogeneous Dirichlet problem

$$\begin{cases} -\operatorname{div} A(x, Dv) + \mathcal{L}v = 0 & \text{in } B_{r}, \\ v = u & \text{in } \mathbb{R}^n \setminus B_{r}. \end{cases} \tag{5.25}$$

Lemma 5.4.1. *Let u and v be as in (5.24) and (5.25), respectively. Then we have*

$$\int_{B_r} \frac{|V(Du) - V(Dv)|^2}{(d + |u - v|)^\xi}\, dx \le c \frac{d^{1-\xi}}{\xi - 1} |\mu|(B_r)$$

for a constant $c \equiv c(\texttt{data})$, whenever $d > 0$ and $\xi > 1$.

Proof. We test (5.24) and (5.25) with

$$\varphi_\pm := \pm\left(d^{1-\xi} - (d + (u - v)_\pm)^{1-\xi}\right) \in \mathcal{X}_0^{1,p}(B_r) \cap L^\infty(B_r).$$

Recalling the notation (5.6), we obtain

$$\begin{aligned} &I_{1,\pm} + I_{2,\pm} + I_{3,\pm} \\ &:= \int_{B_r}\int_{B_r} \left(\Phi_p(u(x) - u(y)) - \Phi_p(v(x) - v(y))\right)(\varphi_\pm(x) - \varphi_\pm(y)) K(x,y)\, dxdy \\ &\quad + 2\int_{\mathbb{R}^n \setminus B_r}\int_{B_r} \left(\Phi_p(u(x) - u(y)) - \Phi_p(v(x) - v(y))\right)\varphi_\pm(x) K(x,y)\, dxdy \end{aligned}$$

$$
\begin{aligned}
&+\int_{B_r} (A(x,Du)-A(x,Dv))\cdot D\varphi_\pm\,dx \\
&= \int_{B_r} \varphi_\pm\,d\mu. \qquad (5.26)
\end{aligned}
$$

From the definition of $\varphi_\pm$, we immediately have

$$
\left|\int_{B_r} \varphi_\pm\,d\mu\right| \le d^{1-\xi}|\mu|(B_r). \qquad (5.27)
$$

Proceeding as in the proof of [152, Lemma 3.1], we have

$$
I_{1,\pm}, I_{2,\pm} \ge 0.
$$

As for $I_{3,\pm}$, we observe that

$$
\begin{aligned}
I_{3,+} &= (\xi-1)\int_{B_r\cap\{u\ge v\}} \frac{(A(x,Du)-A(x,Dv))\cdot(Du-Dv)}{(d+|u-v|)^\xi}\,dx \\
&\ge \frac{\xi-1}{c}\int_{B_r\cap\{u\ge v\}} \frac{|V(Du)-V(Dv)|^2}{(d+|u-v|)^\xi}\,dx
\end{aligned}
$$

and

$$
\begin{aligned}
I_{3,-} &= (\xi-1)\int_{B_r\cap\{u< v\}} \frac{(A(x,Du)-A(x,Dv))\cdot(Du-Dv)}{(d+|u-v|)^\xi}\,dx \\
&\ge \frac{\xi-1}{c}\int_{B_r\cap\{u< v\}} \frac{|V(Du)-V(Dv)|^2}{(d+|u-v|)^\xi}\,dx.
\end{aligned}
$$

Combining the estimates found for $I_{1,\pm}, I_{2,\pm}, I_{3,\pm}$ with (5.26) and (5.27), the desired estimate follows. □

Once we have the above lemma, we can proceed as in [144, 146] to obtain the following comparison estimate between (5.8) and (5.25).

Lemma 5.4.2. *Let u and v be as in* (5.24) *and* (5.25)*, respectively. Then for every q satisfying*

$$
1 \le q < \min\left\{p, \frac{n(p-1)}{n-1}\right\} = \bar{q}, \qquad (5.28)
$$

we have the estimate

$$\fint_{B_r} |Du - Dv|^q \, dx \le c \left[\frac{|\mu|(B_r)}{r^{n-1}}\right]^{\frac{q}{p-1}} + c\chi_{\{p<2\}} \left[\frac{|\mu|(B_r)}{r^{n-1}}\right]^q \left(\fint_{B_r} |Du|^q \, dx\right)^{2-p} \tag{5.29}$$

for some constant $c \equiv c(\texttt{data}, q)$.

In the case $2 - 1/n < p < 2$, we need to handle the additional quantity appearing in the right-hand side of (5.29). To do this, we proceed with an argument similar to the one in [152, Lemma 3.5], making use of the Caccioppoli estimates established in Section 5.3.

Lemma 5.4.3. *Let* u *and* v *be as above, and assume that* $2 - 1/n < p < 2$. *Then for every* $q \in [1, \bar{q})$, *there exists* $c \equiv c(\texttt{data}, q)$ *such that*

$$\left(\fint_{B_r} |Du - Dv|^q \, dx\right)^{\frac{1}{q}} \le c \left[\frac{|\mu|(B_{2r})}{(2r)^{n-1}}\right]^{\frac{1}{p-1}} + c \left[\frac{E(u; x_0, 2r)}{2r}\right]^{2-p} \left[\frac{|\mu|(B_{2r})}{(2r)^{n-1}}\right]. \tag{5.30}$$

Proof. For each $\varphi \in W^{1,q}(B_t)$ with $q \in [1, \bar{q})$ and $t \in (0, 2r)$, we denote

$$F(\varphi; t) := \left(\fint_{B_t} |D\varphi|^q \, dx\right)^{\frac{1}{q}}.$$

For $1 \le \sigma' < \sigma \le 2$, we define $v_\sigma \in W^{1,p}(\mathbb{R}^n)$ as the weak solution to the problem

$$\begin{cases} -\operatorname{div} A(x, Dv_\sigma) = 0 & \text{in } B_{\sigma r}, \\ v_\sigma = u & \text{in } \mathbb{R}^n \setminus B_{\sigma r}. \end{cases}$$

We start with the obvious estimate

$$F(u; \sigma' r) \le F(v_\sigma; \sigma' r) + F(u - v_\sigma; \sigma' r). \tag{5.31}$$

By Lemma 5.4.2, we have

$$F(u - v_\sigma; \sigma r) \le c \left[\frac{|\mu|(B_{2r})}{(2r)^{n-1}}\right]^{\frac{1}{p-1}} + c \left[\frac{|\mu|(B_{2r})}{(2r)^{n-1}}\right] [F(u; \sigma r)]^{2-p}. \tag{5.32}$$

We then apply Lemma 5.3.4 to $v - (v_\sigma)_{B_\sigma}$, which implies

$$F(v_\sigma; \sigma' r) \leq \frac{c}{(\sigma - \sigma')^\theta r} \left[\fint_{B_{\sigma r}} |v_\sigma - (v_\sigma)_{B_{\sigma r}}| \, dx + \mathrm{Tail}(v_\sigma - (v_\sigma)_{B_\sigma}; \sigma r/2) \right].$$

The first term in the right-hand side is estimated as

$$\begin{aligned} \fint_{B_{\sigma r}} |v_\sigma - (v_\sigma)_{B_{\sigma r}}| \, dx &\leq 2 \fint_{B_{\sigma r}} |u - v_\sigma| \, dx + c \fint_{B_{2r}} |u - (u)_{B_{2r}}| \, dx \\ &\leq 2 \fint_{B_{\sigma r}} |u - v_\sigma| \, dx + cE(u; 2r) \\ &\leq crF(u - v_\sigma; \sigma r) + cE(u; 2r), \end{aligned}$$

where we have also used Poincaré's inequality. We then split the tail term as

$$\begin{aligned} &\mathrm{Tail}(v_\sigma - (v_\sigma)_{B_{\sigma r}}; \sigma r/2) \\ &\leq c\mathrm{Tail}(u - (u)_{B_{\sigma r}}; \sigma r/2) + c\mathrm{Tail}(u - v_\sigma - (u - v_\sigma)_{B_{\sigma r}}; \sigma r/2) \end{aligned} \tag{5.33}$$

to estimate each term separately. The first term is estimated as

$$\begin{aligned} \mathrm{Tail}(u - (u)_{B_{\sigma r}}; \sigma r/2) &\leq c\mathrm{Tail}(u - (u)_{B_{2r}}; \sigma r/2) + cr^{(1-s)p'} |(u)_{B_{\sigma r}} - (u)_{B_{2r}}| \\ &\leq c\mathrm{Tail}(u - (u)_{B_{2r}}; \sigma r/2) + c \fint_{B_{2r}} |u - (u)_{B_{2r}}| \, dx \\ &\leq cE(u; 2r). \end{aligned} \tag{5.34}$$

As for the second term, we have

$$\begin{aligned} &\mathrm{Tail}(u - v_\sigma - (u - v_\sigma)_{B_{\sigma r}}; \sigma r/2) \\ &\leq c\mathrm{Tail}(u - v_\sigma; \sigma r/2) + cr^{(1-s)p'} |(u - v_\sigma)_{B_{\sigma r}}| \\ &\leq c\mathrm{Tail}(u - v_\sigma; \sigma r/2) + c \fint_{B_{\sigma r}} |u - v_\sigma| \, dx \\ &= c \fint_{B_{\sigma r}} |u - v_\sigma| \, dx \leq crF(u - v_\sigma; \sigma r). \end{aligned} \tag{5.35}$$

Combining (5.33), (5.34) and (5.35), we arrive at

$$F(v_\sigma; \sigma' r) \leq \frac{c}{(\sigma - \sigma')^\theta} \left[r^{-1} E(u; 2r) + F(u - v_\sigma; \sigma r) \right]. \tag{5.36}$$

Connecting (5.32) and (5.36) to (5.31) yields

$$F(u;\sigma' r) \le \frac{c}{(\sigma-\sigma')^{\theta}}\left\{r^{-1}E(u;2r) + \left[\frac{|\mu|(B_{2r})}{(2r)^{n-1}}\right]^{\frac{1}{p-1}}\right\} + \frac{c}{(\sigma-\sigma')^{\theta}}\,[F(u;\sigma r)]^{2-p}\left[\frac{|\mu|(B_{2r})}{(2r)^{n-1}}\right].$$

We apply Young's inequality to the last term in the right-hand side, with conjugate exponents $1/(2-p)$ and $1/(p-1)$, in order to see that

$$F(u;\sigma' r) \le \frac{1}{2}F(u;\sigma r) + \frac{c}{(\sigma-\sigma')^{\theta/(p-1)}}\left\{r^{-1}E(u;2r) + \left[\frac{|\mu|(B_{2r})}{(2r)^{n-1}}\right]^{\frac{1}{p-1}}\right\}$$

holds for some $c \equiv c(\texttt{data}, q)$, whenever $1 \le \sigma' \le \sigma \le 2$. Then, applying Lemma 2.3.12, we conclude that

$$\left(\fint_{B_r} |Du|^q\,dx\right)^{\frac{1}{q}} \le c\left\{r^{-1}E(u;2r) + \left[\frac{|\mu|(B_{2r})}{(2r)^{n-1}}\right]^{\frac{1}{p-1}}\right\}.$$

This inequality and (5.29) yield (5.30) after an elementary manipulation. □

The above two lemmas and Sobolev's embedding theorem imply the following comparison estimate.

Lemma 5.4.4. *Let u and v be as in* (5.24) *and* (5.25)*, respectively. Let $\gamma \in [1,\gamma^*)$, with*

$$\gamma^* := \begin{cases} \dfrac{n(p-1)}{n-p} & \text{if } p < n, \\ \infty & \text{if } p \ge n. \end{cases}$$

Then there exists a constant $c \equiv c(\texttt{data}, \gamma)$ such that

$$\left(\fint_{B_r} |u-v|^{\gamma}\,dx\right)^{\frac{1}{\gamma}} \le c\left[\frac{|\mu|(B_{2r})}{(2r)^{n-p}}\right]^{\frac{1}{p-1}} + c\chi_{\{p<2\}}[E(u;x_0,2r)]^{2-p}\left[\frac{|\mu|(B_{2r})}{(2r)^{n-p}}\right].$$

We end this section with the following comparison estimate in Ω instead of balls, whose proof is the same as those of Lemmas 5.4.1 and 5.4.2.

Lemma 5.4.5. *Given $g \in W^{1,p}(\mathbb{R}^n)$, let $u, \tilde{v} \in W^{1,p}(\mathbb{R}^n)$ be weak solutions to the problems*

$$\begin{cases} -\operatorname{div} A(x, Du) + \mathcal{L}u = \mu & \text{in } \Omega, \\ \qquad\qquad\qquad\quad u = g & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

and

$$\begin{cases} -\operatorname{div} A(x, D\tilde{v}) + \mathcal{L}\tilde{v} = 0 & \text{in } \Omega, \\ \qquad\qquad\qquad\quad \tilde{v} = g & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

respectively. Then we have the estimate

$$\int_\Omega |Du - D\tilde{v}|^q \, dx \le c \, [|\mu|(\Omega)]^{\frac{q}{p-1}} + c\chi_{\{p<2\}} \, [|\mu|(\Omega)]^q \left(\int_\Omega |Du|^q \, dx \right)^{2-p}$$

for every q satisfying (5.28), where $c \equiv c(\mathtt{data}, \Omega, q)$.

5.5 Existence of SOLA

Here we prove Theorem 5.1.3. The proof will be divided into three steps.

Step 1: Construction of the approximating problems. We start with the following lemma, whose proof is the same as that of [152, Lemma 4.1].

Lemma 5.5.1. *Fix $g \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^{p-1}_{sp}(\mathbb{R}^n)$ and $z \in \Omega$. There exists a sequence $\{g_k\} \subset C_0^\infty(\mathbb{R}^n)$ such that, for any $R > 0$,*

$$g_k \to g \ \text{ in } \ W^{1,p}(B_R) \quad \text{and} \quad \int_{\mathbb{R}^n \setminus B_R(z)} \frac{|g_k(y) - g(y)|^{p-1}}{|y - z|^{n+sp}} \, dy \to 0 \tag{5.37}$$

as $k \to \infty$. Moreover, for every $\varepsilon > 0$ there exist a radius $\tilde{R} > 0$ and an index $\tilde{k} \in \mathbb{N}$, both depending on ε, such that

$$\int_{\mathbb{R}^n \setminus B_R(z)} \frac{|g(y)|^{p-1} + |g_k(y)|^{p-1}}{|y - z|^{n+sp}} \, dy \le \varepsilon$$

whenever $k \ge \tilde{k}$ and $R \ge \tilde{R}$. Finally, we have for every $R > 0$

$$\sup_k \|g_k\|_{W^{1,p}(B_R)} \le c(R, g(\cdot)).$$

We next construct an approximating sequence $\{u_k\}$ described in Definition 5.1.2. Note that the sequence $\{\mu_k\}$ obtained via convolutions as in Remark 3.1.9 satisfies the convergence property and (5.10). Accordingly, we define the weak solutions $u_k, v_k \in W^{1,p}(\mathbb{R}^n)$ to the Dirichlet problems

$$\begin{cases} -\operatorname{div} A(x, Du_k) + \mathcal{L}u_k = \mu_k & \text{in } \Omega, \\ \qquad\qquad\qquad\qquad u_k = g_k & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{5.38}$$

and

$$\begin{cases} -\operatorname{div} A(x, Dv_k) + \mathcal{L}v_k = 0 & \text{in } \Omega, \\ \qquad\qquad\qquad\qquad v_k = g_k & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{5.39}$$

respectively. In the following lemma, we establish an initial estimate for v_k.

Lemma 5.5.2. *There exist constants $c \equiv c(\texttt{data}) \geq 1$ and $\sigma \equiv \sigma(\texttt{data}) \in (0, 1/4]$ satisfying the following: if $B_R \equiv B_R(z)$ is a ball with center $z \in \Omega$ and radius $R \geq 1$ such that $\Omega \subset B_{\sigma R}(z)$, then*

$$\begin{aligned} \|Dv_k\|_{L^p(\Omega)} + [v_k]_{s,p;B_R} &\leq c\|Dg_k\|_{L^p(\Omega)} + c[g_k]_{s,p;B_R} \\ &\quad + cR^{-s}\|g_k\|_{L^p(B_R)} + cR^{\frac{n}{p}-s}\operatorname{Tail}(g_k; z, R). \end{aligned}$$

Proof. We test (5.39) with $\varphi = v_k - g_k$. Recalling the notation (5.6), we have

$$\begin{aligned} 0 &= \int_\Omega A(x, Dv_k) \cdot (Dv_k - Dg_k)\,dx \\ &\quad + \int_{B_R}\int_{B_R} \Phi_p(v_k(x) - v_k(y))(v_k(x) - v_k(y) - (g_k(x) - g_k(y)))K(x,y)\,dxdy \\ &\quad + 2\int_{\mathbb{R}^n \setminus B_R}\int_{B_R} \Phi_p(v_k(x) - g_k(y))(v_k(x) - g_k(x))K(x,y)\,dxdy \\ &=: I_1 + I_2 + I_3. \end{aligned} \tag{5.40}$$

By using Young's inequality, we directly have

$$I_1 \geq \frac{1}{c}\|Dv_k\|^p_{L^p(\Omega)} - c\|Dg_k\|^p_{L^p(\Omega)} \quad \text{and} \quad I_2 \geq \frac{1}{c}[v_k]^p_{s,p;B_R} - c[g_k]^p_{s,p;B_R}. \tag{5.41}$$

We split I_3 as

$$\begin{aligned} I_3 \geq & -c \int_{\mathbb{R}^n \setminus B_R} \int_{B_R} \frac{|v_k(x)|^{p-1}|v_k(x)-g_k(x)|}{|x-y|^{n+sp}}\,dxdy \\ & - c \int_{\mathbb{R}^n \setminus B_R} \int_{B_R} \frac{|g_k(y)|^{p-1}|v_k(x)-g_k(x)|}{|x-y|^{n+sp}}\,dxdy \\ & =: -I_{3,1} - I_{3,2}. \end{aligned}$$

Using the fact that

$$|x-y| \geq |y-z| - |x-z| \geq \frac{3}{4}|y-z|$$

for $x \in \operatorname{supp}(v_k - g_k) \subseteq B_{R/4}$ and $y \in \mathbb{R}^n \setminus B_R$, along with Lemma 2.2.6 and Young's inequality, we have

$$\begin{aligned} I_{3,1} & \leq c \int_{\mathbb{R}^n \setminus B_R} \int_{B_R} \frac{|v_k(x)|^{p-1}|v_k(x)-g_k(x)|}{|y-z|^{n+sp}}\,dxdy \\ & \leq \frac{c}{R^{sp}} \int_{B_R} |v_k|^{p-1}|v_k - g_k|\,dx \\ & \leq \frac{c_\varepsilon}{R^{sp}} \int_{B_R} |v_k|^p\,dx + \frac{c\varepsilon}{R^{sp}} \int_{B_R} |v_k - g_k|^p\,dx \\ & \leq c_\varepsilon R^{-sp} \|v_k\|^p_{L^p(B_R)} + c\varepsilon [v_k - g_k]^p_{s,p;B_R} \\ & \leq c_\varepsilon R^{-sp} \|v_k\|^p_{L^p(B_R)} + c\varepsilon [v_k]^p_{s,p;B_R} + c\varepsilon [g_k]^p_{s,p;B_R} \end{aligned}$$

for any $\varepsilon \in (0,1)$, where $c_\varepsilon \equiv c_\varepsilon(\texttt{data}, \varepsilon)$. We further observe that, by applying Lemma 2.2.6 to $v_k - g_k$ in the ball $B_{2\sigma R}$,

$$\begin{aligned} \|v_k\|^p_{L^p(B_R)} & \leq c\|v_k - g_k\|^p_{L^p(B_{\sigma R})} + c\|g_k\|^p_{L^p(B_R)} \\ & \leq cR^{sp}\sigma^{sp}[v_k]^p_{s,p;B_R} + cR^{sp}\sigma^{sp}[g_k]_{s,p;B_R} + c\|g_k\|^p_{L^p(B_R)}. \end{aligned}$$

From the last two inequalities, we obtain the estimate for $I_{3,1}$:

$$I_{3,1} \leq c_\varepsilon \left([g_k]^p_{s,p;B_R} + R^{-sp}\|g_k\|^p_{L^p(B_R)} \right) + \left(c_\varepsilon \sigma^{sp} + c\varepsilon \right) [v_k]^p_{s,p;B_R}. \tag{5.42}$$

In a similar manner, we estimate $I_{3,2}$ as

$$\begin{aligned} I_{3,2} &= \int_{\mathbb{R}^n \setminus B_R} \int_{B_R} \frac{|g_k(y)|^{p-1} |v_k(x) - g_k(x)|}{|x-y|^{n+sp}} \, dxdy \\ &\le c \int_{\mathbb{R}^n \setminus B_R} \frac{|g_k(y)|^{p-1}}{|y-z|^{n+sp}} \, dy \int_{B_R} |v_k - g_k| \, dx \\ &\le cR^{-p} [\mathrm{Tail}(g_k; z, R)]^{p-1} \cdot R^{s+\frac{n}{p'}} [v_k - g_k]_{s,p;B_R} \\ &\le c [\mathrm{Tail}(g_k; z, R)]^{p-1} \cdot R^{s(1-p)+\frac{n}{p'}} [v_k - g_k]_{s,p;B_R} \\ &\le \varepsilon [v_k]^p_{s,p;B_R} + \varepsilon [g_k]^p_{s,p;B_R} + c_\varepsilon R^{n-sp} [\mathrm{Tail}(g_k; z, R)]^p. \end{aligned} \tag{5.43}$$

Combining (5.40), (5.41), (5.42) and (5.43), we have

$$\begin{aligned} &\|Dv_k\|^p_{L^p(\Omega)} + [v_k]^p_{s,p;B_R} \\ &\le (c_\varepsilon \sigma^{sp} + c_0 \varepsilon) \, [v_k]^p_{s,p;B_R} + c\|Dg_k\|^p_{L^p(\Omega)} \\ &\quad + c_\varepsilon \left([g_k]^p_{s,p;B_R} + R^{-sp} \|g_k\|^p_{L^p(B_R)} \right) + c_\varepsilon R^{n-sp} [\mathrm{Tail}(g_k; z, R)]^p \end{aligned}$$

for some $c_0 \equiv c_0(\texttt{data})$. Choosing first $\varepsilon = 1/(4c_0)$ and then $\sigma \equiv \sigma(\texttt{data})$ so small that $c_\varepsilon \sigma^{sp} \le 1/4$, the desired estimate follows. □

Step 2: A priori estimates for approximating solutions. We fix a point $z \in \Omega$ satisfying $\Omega \subset B_{\sigma R}(z)$ with $R := \max\{1, 4\sigma^{-1} \operatorname{diam}(\Omega)\}$, where $\sigma \equiv \sigma(\texttt{data})$ is the constant determined in Lemma 5.5.2. Now, with $q \in [1, \bar{q})$, we apply Lemmas 5.5.2 and 5.4.5 in order to have

$$\begin{aligned} \int_\Omega |Du_k|^q \, dx &\le c \int_\Omega |Dv_k|^q \, dx + c \int_\Omega |Du_k - Dv_k|^q \, dx \\ &\le c\|Dg_k\|^q_{L^p(\Omega)} + c[g_k]^q_{s,p;B_R} + c\|g_k\|^q_{L^p(B_R)} + c[\mathrm{Tail}(g_k; z, R)]^q \\ &\quad + c[|\mu_k|(\Omega)]^{\frac{q}{p-1}} + c\chi_{\{p<2\}} [|\mu_k|(\Omega)]^q \left(\int_\Omega |Du_k|^q \, dx \right)^{2-p}. \end{aligned}$$

When $2 - 1/n < p < 2$, we further apply Young's inequality with conjugate exponents $1/(p-1)$ and $1/(2-p)$ and then reabsorb the last term. Consequently, in any case, we have

$$\int_\Omega |Du_k|^q \, dx \le c\|Dg_k\|^q_{L^p(\Omega)} + c\|g_k\|^q_{W^{s,p}(B_R)} + c[\mathrm{Tail}(g_k; z, R)]^q + c[|\mu_k|(\Omega)]^{\frac{q}{p-1}}$$

and therefore, applying Poincaré's inequality to $u_k - g \in W_0^{1,p}(\Omega)$,

$$\begin{aligned}\int_\Omega |u_k|^q\,dx &\le c\int_\Omega |u_k-g_k|^q\,dx + c\int_\Omega |g_k|^q\,dx\\ &\le c\int_\Omega |Du_k-Dg_k|^q\,dx + c\int_\Omega |g_k|^q\,dx\\ &\le c\int_\Omega |Du_k|^q\,dx + c\int_\Omega |Dg_k|^q\,dx + c\int_\Omega |g_k|^q\,dx\\ &\le c\|Dg_k\|^q_{L^p(\Omega)} + c\|g_k\|^q_{W^{s,p}(B_R)} + c[\mathrm{Tail}(g_k;z,R)]^q + c[|\mu_k|(\Omega)]^{\frac{q}{p-1}}.\end{aligned}$$

All in all, using $(3.15)_1$ and (5.37), we have the uniform estimate for u_k:

$$\int_\Omega |u_k|^q\,dx + \int_\Omega |Du_k|^q\,dx \le c(\mathtt{data}, |\Omega|, |\mu|(\Omega), g(\cdot), q). \tag{5.44}$$

In a similar way, we apply Lemma 5.2.1 to $u_k - g \in \mathcal{X}_0^{1,p}(\Omega)$ to discover that for every $h \in (0,1)$,

$$\begin{aligned}[u_k]_{h,q;\Omega} &\le [u_k-g]_{h,q;\Omega} + [g]_{h,q;\Omega}\\ &\le c\|Du_k - Dg\|_{L^q(\Omega)} + [g]_{h,q;\Omega}\\ &\le c(\mathtt{data}, |\Omega|, |\mu|(\Omega), g(\cdot), h, q).\end{aligned} \tag{5.45}$$

Step 3: A limiting process and existence of SOLA. From the results in the previous step, we have the following: there exists $u \in W^{1,q}(\Omega)$ for every $q \in [p-1,\bar{q})$ with $u = g$ in $\mathbb{R}^n \setminus \Omega$ such that, up to a subsequence,

$$\begin{cases} u_k \rightharpoonup u & \text{in } W^{1,q}(\Omega)\\ u_k \to u & \text{in } L^q(\Omega)\\ u_k \to u & \text{a.e. in } \mathbb{R}^n. \end{cases} \tag{5.46}$$

Once we have estimate (5.44), with the same spirit as in [28], we obtain

$$Du_k \to Du \quad \text{in } L^q(\Omega), \qquad \forall\, q \in [1,\bar{q}).$$

In particular, this and Vitali convergence theorem yield

$$A(x,Du_k) \to A(x,Du) \quad \text{in } L^1(\Omega).$$

Finally, it remains to show that u is a distributional solution to (5.8). Testing (5.38) with $\varphi \in C_0^\infty(\Omega)$ and recalling the notation (5.6), we have

$$\begin{aligned}\int_{\mathbb{R}^n} \varphi\, d\mu_k = &\int_\Omega A(x, Du_k) \cdot D\varphi\, dx \\ &+ \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Phi_p(u(x)-u(y))(\varphi(x)-\varphi(y))K(x,y)\,dxdy \\ &+ \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (\Phi_p(u_k(x)-u_k(y)) - \Phi_p(u(x)-u(y))) \\ &\qquad\cdot(\varphi(x)-\varphi(y))K(x,y)\,dxdy. \end{aligned} \tag{5.47}$$

We now recall (5.45), (5.46) and the fact that $u = g$ a.e. in $\mathbb{R}^n \setminus \Omega$. Then, we are in a position to argue in exactly the same way as in [152, Step 3 in Section 4], concluding that the last term in the right-hand side of (5.47) converges to zero as $k \to \infty$. This completes the proof of Theorem 5.1.3.

5.6 Potential estimates

5.6.1 Proof of Theorems 5.1.4 and 5.1.7

We first obtain an excess decay estimate for u.

Lemma 5.6.1. *Let u be a SOLA to* (5.8) *under assumptions* (5.4) *and* (5.5). *Then there exist constants $c \equiv c(\mathtt{data})$ and $\eta \equiv \eta(n,p)$ such that*

$$E(u; x_0, \sigma\rho) \le c\sigma^\alpha E(u; x_0, \rho) + c\sigma^{-\eta}\left[\frac{|\mu|(\overline{B}_\rho(x_0))}{\rho^{n-p}}\right]^{\frac{1}{p-1}} \tag{5.48}$$

for any ball $B_\rho(x_0) \subset \mathbb{R}^n$ and $\sigma \in (0,1)$.

Proof. Let $\{u_k\}$ be an approximating sequence for the SOLA u with measure μ_k and boundary data g_k, as described in Definition 5.1.2. We consider the comparison map v_k defined as the weak solution to

$$\begin{cases} -\operatorname{div} A(x, Dv_k) + \mathcal{L}v_k = 0 & \text{in } B_{\rho/2}(x_0), \\ v_k = u_k & \text{in } \mathbb{R}^n \setminus B_{\rho/2}(x_0). \end{cases}$$

Since $v_k = u_k$ in $\mathbb{R}^n \setminus B_{\rho/2}(x_0)$, we have for any $t < \rho/2$

$$E(u_k - v_k; t) \le c\left(\frac{\rho}{t}\right)^{\frac{n}{q_0}} \left(\fint_{B_\rho} |u_k - v_k|^{q_0}\, dx\right)^{\frac{1}{q_0}}.$$

This and Theorem 5.3.8 imply

$$\begin{aligned} E(u_k; \sigma\rho) &\le cE(v_k; \sigma\rho) + c\sigma^{-\frac{n}{q_0}} \left(\fint_{B_\rho} |u_k - v_k|^{q_0}\, dx\right)^{\frac{1}{q_0}} \\ &\le c\sigma^\alpha E(v_k; \rho) + c\sigma^{-\frac{n}{q_0}} \left(\fint_{B_\rho} |u_k - v_k|^{q_0}\, dx\right)^{\frac{1}{q_0}} \\ &\le c\sigma^\alpha E(u_k; \rho) + c\sigma^{-\frac{n}{q_0}} \left(\fint_{B_\rho} |u_k - v_k|^{q_0}\, dx\right)^{\frac{1}{q_0}}. \end{aligned}$$

We then apply Lemma 5.4.4 to estimate the last term in the right-hand side. When $p \ge 2$, we directly have

$$\left(\fint_{B_\rho} |u_k - v_k|^{q_0}\, dx\right)^{\frac{1}{q_0}} \le c\left[\frac{|\mu_k|(B_\rho)}{\rho^{n-p}}\right]^{\frac{1}{p-1}}.$$

When $2 - 1/n < p < 2$, we further apply Young's inequality with conjugate exponents $1/(2-p)$ and $1/(p-1)$ to have

$$\begin{aligned} \left(\fint_{B_\rho} |u_k - v_k|^{q_0}\, dx\right)^{\frac{1}{q_0}} &\le c\left[\frac{|\mu_k|(B_\rho)}{\rho^{n-p}}\right]^{\frac{1}{p-1}} + c[E(u_k; \rho)]^{2-p}\left[\frac{|\mu_k|(B_\rho)}{\rho^{n-p}}\right] \\ &\le \delta E(u_k; \rho) + c\delta^{\frac{p-2}{p-1}}\left[\frac{|\mu_k|(B_\rho)}{\rho^{n-p}}\right]^{\frac{1}{p-1}} \end{aligned}$$

for any $\delta \in (0,1)$. Choosing $\delta = \sigma^{\alpha + n/q_0}$ and finally letting $k \to \infty$, we obtain (5.48). □

We are now ready to prove Theorem 5.1.7.

Proof of Theorem 5.1.7. We start by integrating (5.48) with respect to Haar measure and then making an elementary manipulation, to get

$$\int_{\rho}^{r} E(u;\sigma t)\frac{dt}{t} \leq c\sigma^{\alpha}\int_{\rho}^{r} E(u;t)\frac{dt}{t} + c\sigma^{-\eta}\int_{\rho}^{r}\left[\frac{|\mu|(B_t)}{t^{n-p}}\right]^{\frac{1}{p-1}}\frac{dt}{t}$$

for any $\rho \in (0,r]$. Thus, choosing $\sigma \equiv \sigma(\texttt{data})$ so small that

$$c\sigma^{\alpha} = \frac{1}{2},$$

changing variables and then reabsorbing terms, we obtain

$$\int_{\sigma\rho}^{r} E(u;\sigma t)\frac{dt}{t} \leq 2\int_{\sigma r}^{r} E(u;t)\frac{dt}{t} + c\int_{\rho}^{r}\left[\frac{|\mu|(B_t)}{t^{n-p}}\right]^{\frac{1}{p-1}}\frac{dt}{t}.$$

We further note the inequality

$$\int_{\sigma r}^{r} E(u;t)\frac{dt}{t} \leq cE(u;r)$$

in order to have

$$\int_{\rho}^{r} E(u;t)\frac{dt}{t} \leq cE(u;r) + c\int_{\rho}^{r}\left[\frac{|\mu|(B_t)}{t^{n-p}}\right]^{\frac{1}{p-1}}\frac{dt}{t}. \tag{5.49}$$

This gives the bound for the first term on the left-hand side of (5.16).

We now prove the bound for the second term, after showing the existence of the limit in (5.11). To this end, let $0 < \tilde{\rho} \leq \rho/2 < r/8$ and choose $m \in \mathbb{N}$ and $\theta \in (1/4, 1/2]$ such that $\tilde{\rho} = \theta^m \rho$. Then

$$\left|(u)_{B_\rho} - (u)_{B_{\tilde{\rho}}}\right| \leq \sum_{i=0}^{m-1}\left|(u)_{B_{\theta^i\rho}} - (u)_{B_{\theta^{i+1}\rho}}\right| \leq \theta^{-\frac{n}{q_0}}\sum_{i=0}^{m-1} E(u;\theta^i\rho).$$

Recalling the elementary inequality (see for instance [189, Lemma 2.3])

$$\begin{aligned}\sum_{i=0}^{m-1} E(u;\theta^i\rho) &= \frac{1}{\log(1/\theta)}\sum_{i=0}^{m-1}\int_{\theta^i\rho}^{\theta^{i-1}} E(u;\theta^i\rho)\frac{dt}{t} \\ &\leq c\sum_{i=0}^{m-1}\int_{\theta^i\rho}^{\theta^{i-1}\rho} E(u;t)\frac{dt}{t} \leq c\int_{\tilde{\rho}}^{\rho/\theta} E(u;t)\frac{dt}{t},\end{aligned}$$

and then using (5.49), we have

$$\left|(u)_{B_\rho} - (u)_{B_{\tilde\rho}}\right| \le c\int_{\tilde\rho}^{\rho/\theta} E(u;t)\frac{dt}{t} \tag{5.50}$$

and therefore

$$\left|(u)_{B_\rho} - (u)_{B_{\tilde\rho}}\right| \le cE(u;r) + c\mathbf{W}^{\mu}_{1,p}(x_0,r). \tag{5.51}$$

Then we note that the finiteness of $\mathbf{W}^{\mu}_{1,p}(x_0,r)$ implies the finiteness of the left-hand side of (5.49). In turn, by the absolute continuity of the integral, (5.50) implies that $\{(u)_{B_\rho}\}$ is a Cauchy net. Consequently, the limit in (5.11) exists and therefore defines the precise representative of u at x_0. Now we let $\tilde\rho \to 0$ in (5.51) and take $\rho = r/4$ to have

$$|(u)_{B_{r/4}} - u(x_0)| \le cE(u;r) + c\mathbf{W}^{\mu}_{1,p}(x_0,r).$$

On the other hand, we trivially have

$$|(u)_{B_r} - (u)_{B_{r/4}}| \le cE(u;r).$$

combining the last two estimates with (5.49) finally gives (5.16). Also, the estimate (5.12) easily follows from (5.16). □

5.6.2 Proof of Theorem 5.1.5

Here we note that if $\mu \in C_0^\infty(\mathbb{R}^n)$ is nonnegative, then every weak solution u to (5.1) is a weak supersolution to the homogeneous equation (5.19).

Lemma 5.6.2. *Let u be the weak solution to* (5.8) *with $\mu \in C_0^\infty(\mathbb{R}^n)$ being nonnegative, such that $u \ge 0$ in $B_{4r} \equiv B_{4r}(x_0) \subset \Omega$. Then the inequality*

$$\begin{aligned}\frac{\mu(B_r)}{r^{n-p}} \le cr^{p-1}\fint_{B_{3r/2}} |Du|^{p-1}\,dx + cr^{p-1}\fint_{B_{3r/2}}\int_{B_{3r/2}} \frac{|u(x)-u(y)|^{p-1}}{|x-y|^{n+sp-1}}\,dxdy \\ + c\left[\inf_{B_r} u + \mathrm{Tail}(u_-;x_0,4r)\right]^{p-1}\end{aligned}$$

holds for a constant $c \equiv c(\mathtt{data})$.

Proof. Let $\phi \in C_0^\infty(B_{5r/4})$ be a cut-off function satisfying $0 \le \phi \le 1$, $\phi \equiv 1$ in B_r and $|D\phi| \le 16/r$. Testing (5.8) with ϕ, we have

$$\begin{aligned}\frac{\mu(B_r)}{r^{n-p}} &\le cr^p \fint_{B_{3r/2}} |Du|^{p-1}|D\phi|\,dx \\ &\quad + cr^p \int_{B_{3r/2}} \fint_{B_{3r/2}} |u(x)-u(y)|^{p-1}|\phi(x)-\phi(y)|K(x,y)\,dxdy \\ &\quad + cr^p \int_{\mathbb{R}^n\setminus B_{3r/2}} \fint_{B_{3r/2}} (u(x)-u(y))_+^{p-1}\phi(x)K(x,y)\,dxdy \\ &\le cr^{p-1} \fint_{B_{3r/2}} |Du|^{p-1}\,dx + cr^{p-1} \fint_{B_{3r/2}} \int_{B_{3r/2}} \frac{|u(x)-u(y)|^{p-1}}{|x-y|^{n+sp-1}}\,dxdy \\ &\quad + cr^p \int_{\mathbb{R}^n\setminus B_{3r/2}} \fint_{B_{3r/2}} (u(x)-u(y))_+^{p-1}\phi(x)K(x,y)\,dxdy.\end{aligned}$$

Using (5.4) and the fact that $|y-x_0| \le 16|x-y|$ for $x \in \operatorname{supp}\phi \subset B_{5r/4}$ and $y \in \mathbb{R}^n \setminus B_{3r/2}$, we estimate the last integral as

$$\begin{aligned}& r^p \int_{\mathbb{R}^n\setminus B_{3r/2}} \fint_{B_{3r/2}} (u(x)-u(y))_+^{p-1}\phi(x)K(x,y)\,dxdy \\ &\le cr^p \int_{\mathbb{R}^n\setminus B_{3r/2}} \fint_{B_{3r/2}} \left([u(x)]^{p-1} + [u_-(y)]^{p-1}\right)\phi(x)\frac{dxdy}{|y-x_0|^{n+sp}} \\ &\le cr^{(1-s)p} \fint_{B_{3r/2}} u^{p-1}\,dx + c[\mathrm{Tail}(u_-;x_0,4r)]^{p-1}.\end{aligned}$$

Applying Lemma 5.3.7 completes the proof. □

Lemma 5.6.3. *Let u be the weak solution to* (5.8) *with $\mu \in C_0^\infty(\mathbb{R}^n)$ being nonnegative, such that $u \ge 0$ in $B_{4r} \equiv B_{4r}(x_0) \subset \Omega$. Let $h \in (0,s)$, $q \in (0,\bar{q})$, where $\bar{q}$ has been defined in* (5.28). *Then we have the estimate*

$$\begin{aligned}&\left(\fint_{B_{3r/2}} |Du|^q\,dx\right)^{\frac{1}{q}} + \left(\fint_{B_{3r/2}} \int_{B_{3r/2}} \frac{|u(x)-u(y)|^q}{|x-y|^{n+hq}}\,dxdy\right)^{\frac{1}{q}} \\ &\le \frac{c}{r}\left[\inf_{B_r} u + \mathrm{Tail}(u_-;x_0,4r)\right] \qquad (5.52)\end{aligned}$$

for a constant $c \equiv c(\mathtt{data}, h, q)$.

Proof. Let

$$d \equiv d_\delta := \inf_{B_r} u + \mathrm{Tail}(u_-; x_0, 4r) + \delta \qquad \text{for } \ \delta > 0.$$

We note that the presence of δ is just to guarantee that $d > 0$; we will eventually let $\delta \to 0$ at the end of the proof. With this choice of d, we set

$$\bar{u} := u + d, \qquad w := \bar{u}^{1-\frac{m}{p}} \qquad \text{for } \ m \in (1, p).$$

We then choose a cut-off function $\phi \in C_0^\infty(B_{7r/4})$ satisfying $0 \le \phi \le 1$, $\phi \equiv 1$ in $B_{3r/2}$ and $|D\phi| \le 16/r$. Applying Lemma 5.3.6 (with $2r$ instead of r and $R = 4r$), we obtain

$$\fint_{B_{3r/2}} |Dw|^p \, dx + \fint_{B_{3r/2}} \int_{B_{3r/2}} \frac{|w(x) - w(y)|^p}{|x-y|^{n+sp}} \, dxdy \le \frac{c}{r^p} \fint_{B_{2r}} w^p \, dx.$$

By Lemma 5.3.7 and the definition of d, the right-hand side is estimated as

$$\fint_{B_{2r}} w^p \, dx \le c d^{p-m}.$$

We then estimate the left-hand side from below. For the first term, we apply Hölder's inequality and Lemma 5.3.7 to have

$$\begin{aligned} \fint_{B_{3r/2}} |Du|^q \, dx &= \left(\frac{p}{p-m}\right)^q \fint_{B_{3r/2}} |\bar{u}|^{\frac{mq}{p}} |Dw|^q \, dx \\ &\le \left(\frac{p}{p-m}\right)^q \left(\fint_{B_{3r/2}} |\bar{u}|^{\frac{mq}{p-q}} \, dx\right)^{\frac{p-q}{p}} \left(\fint_{B_{3r/2}} |Dw|^p \, dx\right)^{\frac{q}{p}} \\ &\le c \left(\frac{p}{p-m}\right)^q d^{\frac{mq}{p}} \left(\fint_{B_{3r/2}} |Dw|^p \, dx\right)^{\frac{q}{p}}, \end{aligned}$$

provided that

$$\frac{mq}{p-q} < \frac{n(p-1)}{n-p}.$$

For the second term, we follow the proof of [154, Lemma 8.6.4], again using

Hölder's inequality and Lemma 5.3.7, to have

$$
\begin{aligned}
&\fint_{B_{3r/2}}\int_{B_{3r/2}} \frac{|u(x)-u(y)|^q}{|x-y|^{n+hq}}\,dxdy \\
&\le \left(\fint_{B_{3r/2}}\int_{B_{3r/2}} \frac{[\bar{u}(x)+\bar{u}(y)]^{mq/(p-q)}}{|x-y|^{n+(h-s)qp/(p-q)}}\,dxdy\right)^{\frac{p-q}{p}} \\
&\quad \cdot \left(\fint_{B_{3r/2}}\int_{B_{3r/2}} \frac{|u(x)-u(y)|^p}{[\bar{u}(x)+\bar{u}(y)]^m}\frac{dxdy}{|x-y|^{n+sp}}\right)^{\frac{q}{p}} \\
&\le c\left(\frac{p}{p-m}\right)^q \left(r^{(s-h)p}d^m\right)^{\frac{q}{p}} \left(\fint_{B_{3r/2}}\int_{B_{3r/2}} \frac{|w(x)-w(y)|^p}{|x-y|^{n+sp}}\,dxdy\right)^{\frac{q}{p}},
\end{aligned}
$$

which also holds provided

$$
\frac{mq}{p-q} < \frac{n(p-1)}{n-p} \qquad \text{and} \qquad h < s.
$$

We can always find $m > 1$ satisfying the above condition, since

$$
\frac{q}{p-q} < \frac{n(p-1)}{n-p} \iff q < \frac{n(p-1)}{n-1}.
$$

Combining the three estimates in the above display, we arrive at

$$
\fint_{B_{3r/2}}\int_{B_{3r/2}} |Du|^q\,dx + \fint_{B_{3r/2}}\int_{B_{3r/2}} \frac{|u(x)-u(y)|^q}{|x-y|^{n+hq}}\,dxdy \le \frac{cd^q}{r^q}.
$$

Recalling the definition of d and letting $\delta \to 0$, (5.52) follows. □

Lemma 5.6.4. *Let u be the weak solution to* (5.8) *with $\mu \in C_0^\infty(\mathbb{R}^n)$ being nonnegative, such that $u \ge 0$ in $B_{4r} \equiv B_{4r}(x_0) \subset \Omega$. Then*

$$
\left[\frac{\mu(B_r)}{r^{n-p}}\right]^{\frac{1}{p-1}} \le c\left[\inf_{B_r} u + \mathrm{Tail}(u_-; x_0, 4r)\right]
$$

holds for a constant $c \equiv c(\texttt{data})$.

Proof. Lemma 5.6.2 implies

$$
\begin{aligned}
\left[\frac{\mu(B_r)}{r^{n-p}}\right]^{\frac{1}{p-1}} \le{}& cr\left(\fint_{B_{3r/2}} |Du|^{p-1}\,dx\right)^{\frac{1}{p-1}} \\
&+ cr\left(\fint_{B_{3r/2}}\int_{B_{3r/2}} \frac{|u(x)-u(y)|^{p-1}}{|x-y|^{n+sp-1}}\,dxdy\right)^{\frac{1}{p-1}} \\
&+ c\left[\inf_{B_r} u + \mathrm{Tail}(u_-;x_0,4r)\right]. \qquad (5.53)
\end{aligned}
$$

We observe that

$$
\left(\frac{|x-y|}{r}\right)^{1-sp} \le c\left(\frac{|x-y|}{r}\right)^{-h(p-1)}
$$

for $x,y \in B_{3r/2}$, provided that

$$
1-sp \ge -h(p-1) \iff h \ge \frac{sp-1}{p-1}.
$$

Since $(sp-1)/(p-1) < s$, we can always find $h \in (0,s)$ satisfying the above condition. With such a choice of h and $q = p-1$, Lemma 5.6.3 gives

$$
\begin{aligned}
&\left(\fint_{B_{3r/2}} |Du|^{p-1}\,dx\right)^{\frac{1}{p-1}} + \left(\fint_{B_{3r/2}}\int_{B_{2r}} \frac{|u(x)-u(y)|^{p-1}}{|x-y|^{n+sp-1}}\,dxdy\right)^{\frac{1}{p-1}} \\
&\le \frac{c}{r}\left[\inf_{B_r} u + \mathrm{Tail}(u_-;x_0,4r)\right].
\end{aligned}
$$

Combining this estimate with (5.53), we finish the proof. □

Proof of Theorem 5.1.5. In what follows, all the balls considered are concentric with center x_0 as in the statement. Let $\{u_k\}$ be an approximating sequence for the SOLA u as described in Definition 5.1.2, with the functions μ_k being nonnegative. Then we apply Lemma 5.3.2 to get

$$
\sup_{B_{r/2}}(u_k)_- \le c\left[\fint_{B_r}(u_k)_-\,dx + \mathrm{Tail}((u_k)_-;x_0,r/2)\right].
$$

Since we have the convergence of u_k and the fact that u is nonnegative in B_r, we have

$$\limsup_{k\to\infty} \sup_{B_{r/2}} (u_k)_- \le c\mathrm{Tail}(u_-; x_0, r/2),$$

for a constant $c \equiv c(\mathtt{data})$.

We next observe that the function

$$\tilde{u}_k := u_k - \inf_{B_{r/2}} u_k$$

is nonnegative in $B_{r/2}$. Then, denoting

$$m_{\rho,k} := \inf_{B_\rho} \tilde{u}_k \qquad \text{and} \qquad T_{\rho,k} := \mathrm{Tail}((\tilde{u}_k - m_{\rho,k}); x_0, \rho)$$

for $\rho \in (0, r/2]$ and applying Lemma 5.6.4, we discover that

$$\left[\frac{\mu_k(B_\rho)}{\rho^{n-p}}\right]^{\frac{1}{p-1}} \le c\,(m_{\rho,k} - m_{4\rho,k} + T_{4\rho,k})$$

holds for any $\rho \in (0, r/8)$, where $c \equiv c(\mathtt{data})$. Moreover, with $M \ge 1$ being a free parameter to be chosen in a few lines, we estimate for any $\rho \in (0, r/2)$

$$\begin{aligned}
T_{\rho,k} &= \rho^{\frac{p}{p-1}} \left[\int_{\mathbb{R}^n\setminus B_\rho} \frac{(\tilde{u}_k(x) - m_{\rho,k})_-^{p-1}}{|x-x_0|^{n+sp}}\,dx\right]^{\frac{1}{p-1}} \\
&\le c\rho^{\frac{p}{p-1}} \left[\int_{\mathbb{R}^n\setminus B_{M\rho}} \frac{(\tilde{u}_k(x) - m_{\rho,k})_-^{p-1}}{|x-x_0|^{n+sp}}\,dx\right]^{\frac{1}{p-1}} \\
&\quad + c\rho^{\frac{p}{p-1}} \left[\int_{B_{M\rho}\setminus B_\rho} \frac{(\tilde{u}_k(x) - m_{\rho,k})_-^{p-1}}{|x-x_0|^{n+sp}}\,dx\right]^{\frac{1}{p-1}} \\
&\le cM^{-\frac{p}{p-1}}(m_{\rho,k} - m_{M\rho,k} + T_{M\rho,k}) + c(m_{\rho,k} - m_{M\rho,k}) \\
&\le cM^{-\frac{p}{p-1}} T_{M\rho,k} + c(m_{\rho,k} - m_{M\rho,k}),
\end{aligned} \tag{5.54}$$

where we have used the inequality

$$(\tilde{u}_k(x) - m_{\rho,k})_- \le (\tilde{u}_k(x) - m_{M\rho,k})_- + m_{\rho,k} - m_{M\rho,k}.$$

Now, with any $t \in (0, r/8M)$ being fixed, we integrate (5.54) and then use change of variables to obtain

$$\begin{aligned}
&\int_t^{r/(8M)} T_{4\rho,k} \frac{d\rho}{\rho} = \int_{4t}^{r/(2M)} T_{\rho,k} \frac{d\rho}{\rho} \\
&\leq cM^{-\frac{p}{p-1}} \int_t^{r/(2M)} T_{M\rho,k} \frac{d\rho}{\rho} + c \int_t^{r/(2M)} (m_{\rho,k} - m_{M\rho,k}) \frac{d\rho}{\rho} \\
&= cM^{-\frac{p}{p-1}} \int_{tM}^{r/2} T_{\rho,k} \frac{d\rho}{\rho} + c \left(\int_t^{Mt} m_{\rho,k} \frac{d\rho}{\rho} + \int_{r/(2M)}^{r/2} m_{\rho,k} \frac{d\rho}{\rho} \right).
\end{aligned}$$

Choosing $M \equiv M(\mathtt{data})$ satisfying

$$cM^{-\frac{p}{p-1}} = \frac{1}{2}$$

and making elementary manipulations as those after [154, (7.8)], we have

$$\int_t^{r/8} T_{4\rho,k} \frac{d\rho}{\rho} \leq cm_{t,k} + cT_{r/2,k}$$

for some $c \equiv c(\mathtt{data})$, whenever $t < r/(8M)$. Once we have the last inequality, we again proceed as in the proof of [154, Theorem 8.14] to obtain

$$\int_t^{r/8} \left[\frac{\mu_k(B_\rho)}{\rho^{n-p}} \right]^{\frac{1}{p-1}} \frac{d\rho}{\rho} \leq cm_{t,k} + cT_{r/2,k} \tag{5.55}$$

and

$$\limsup_{k\to\infty} m_{t,k} + \limsup_{k\to\infty} T_{r/2,k} \leq c(u)_{B_t} + c\mathrm{Tail}(u_-; x_0, r/2).$$

For the left-hand side of (5.55), we can use the convergence $\mu_k \overset{*}{\rightharpoonup} \mu$ and Lebesgue's dominated convergence theorem. Hence, letting $k \to \infty$, we get

$$\int_t^{r/8} \left[\frac{\mu(B_\rho)}{\rho^{n-p}} \right]^{\frac{1}{p-1}} \frac{d\rho}{\rho} \leq c(u)_{B_t} + c\mathrm{Tail}(u_-; x_0, r/2) \qquad \forall\, t \in (0, r/(8M)),$$

where $c \equiv c(\mathtt{data})$. Now, if $\mathbf{W}^{\mu}_{1,p}(x_0, r/8)$ is finite, then Theorem 5.1.4 implies the existence of the precise representative of u at x_0 as described in (5.11). Consequently, letting $t \to 0$ in the above inequality gives (5.13). In a similar way, if $\mathbf{W}^{\mu}_{1,p}(x_0, r/8) = \infty$, then (5.14) immediately follows. □

5.7 Continuity criteria for SOLA

5.7.1 Proof of Theorem 5.1.8

Let $\Omega' \Subset \Omega$ be fixed as in the statement.

Step 1: Local VMO-regularity. We first show that u is locally VMO-regular in Ω', which means that for any $\Omega'' \Subset \Omega'$,

$$\lim_{t\to 0} E(u; x, t) = 0 \qquad \text{uniformly in } x \in \Omega''. \tag{5.56}$$

Observe that the assumption (5.17) in particular implies that

$$\lim_{t\to 0} \frac{|\mu|(B_t(x))}{t^{n-p}} = 0 \qquad \text{uniformly in } x \in \Omega'. \tag{5.57}$$

A straightforward calculation as in the proof of [152, Theorem 1.5] shows that, with $r < \operatorname{dist}(\Omega'', \partial\Omega')/100$,

$$H := \sup_{x\in\Omega''} E(u; x, r) < \infty.$$

This together with (5.48) gives

$$E(u; x, \sigma r) \le c\sigma^{\alpha} H + c\sigma^{-\eta} \left[\frac{|\bar{\mu}|(B_r(x))}{r^{n-p}}\right]^{\frac{1}{p-1}}$$

whenever $x \in \Omega''$ and $r < \operatorname{dist}(\Omega'', \partial\Omega')/100$. Given any $\varepsilon > 0$, we first choose $\tilde{\sigma} > 0$ satisfying

$$c\sigma^{\alpha} H \le \frac{\varepsilon}{4} \qquad \text{for every } \sigma < \tilde{\sigma}.$$

Then, by (5.57), we choose $\tilde{r}_\varepsilon \in (0, \operatorname{dist}(\Omega'', \partial\Omega')/100)$ satisfying

$$c\sigma^{-\eta} \left[\frac{|\bar{\mu}|(B_r(x))}{r^{n-p}}\right]^{\frac{1}{p-1}} \le \frac{\varepsilon}{4} \qquad \text{for every } r \le \tilde{r}_\varepsilon \text{ and } x \in \Omega''.$$

Summarizing, we have proved that for every $\varepsilon > 0$, there exists a radius $r_\varepsilon = \tilde{\sigma}\tilde{r}_\varepsilon$, depending only on `data` and H, such that $E(u; x, r) < \varepsilon$ for $r \le r_\varepsilon$ and $x \in \Omega''$. This proves (5.56).

Step 2: Continuity. We now show that for any $\varepsilon > 0$ and $z \in \Omega'$, there exists small $\delta \equiv \delta(\varepsilon, \mathtt{data}) > 0$ such that

$$\operatorname*{osc}_{B_\delta(z)\cap\Omega} u \le \varepsilon.$$

Let us fix $\varepsilon > 0$ and $z \in \Omega'$. Note that (5.56) in particular implies

$$E(u; z, t) \to 0 \qquad \text{as } \ t \to 0. \tag{5.58}$$

Moreover, by using the triangle inequality, we have for any $\tilde{z} \in B_t(z)$

$$E(u; \tilde{z}, t) \le cE(u; z, 2t).$$

Recalling Theorem 5.1.7, we also have

$$\left|(u)_{B_r(x)} - u(x)\right| \le c\mathbf{W}^\mu_{1,p}(x, r) + cE(u; x, r)$$

for all $x \in \Omega'$ and $r < \min\{1, \operatorname{dist}(\Omega', \partial\Omega)\}$. Therefore, using the above two estimates leads to

$$\begin{aligned} |u(z) - u(\tilde{z})| &\le \left|u(z) - (u)_{B_t(z)}\right| + \left|u(\tilde{z}) - (u)_{B_t(\tilde{z})}\right| + \left|(u)_{B_t(z)} - (u)_{B_t(\tilde{z})}\right| \\ &\le c\left(E(u; z, 2t) + \mathbf{W}^\mu_{1,p}(z, 2t) + \mathbf{W}^\mu_{1,p}(\tilde{z}, 2t)\right) \end{aligned}$$

for any $\tilde{z} \in B_t(z)$. In the last display, by (5.17) and (5.58), we can find a small $t > 0$ for which the right-hand side is less than ε. Taking δ to be this t, the proof is complete.

Remark 5.7.1. *In Step 1, we actually proved that* (5.57) *implies the local VMO-regularity of u in Ω'.*

5.7.2 Proof of Theorem 5.1.10

By Lemma 5.6.1, we have

$$E(u; x, \sigma t) \le c\sigma^\alpha E(u; x, t) + c\sigma^{-\eta}\left[\frac{|\mu|(\overline{B}_t(x))}{t^{n-p}}\right]^{\frac{1}{p-1}}$$

whenever $B_t(x) \subset B_r(x) \subset \Omega$ and $\sigma \in (0,1)$. Multiplying this inequality with $(\sigma t)^{-\beta}$, we find

$$\begin{aligned}(\sigma t)^{-\beta} E(u;x,\sigma t) &\le c\sigma^{\alpha-\beta} t^{-\beta} E(u;x,t) + c\sigma^{-\eta-\beta}\left[\frac{|\mu|(\overline{B}_t(x))}{t^{n-p+\beta(p-1)}}\right]^{\frac{1}{p-1}} \\ &\le c\sigma^{\alpha-\beta} t^{-\beta} E(u;x,t) + c\sigma^{-\eta-\beta}\left[M^{\mu}_{p-\beta(p-1)}(x,r)\right]^{\frac{1}{p-1}}.\end{aligned}$$

We choose $\sigma \equiv \sigma(\mathtt{data})$ so that $c\sigma^{\alpha-\beta} = 1/2$ and then iterate the resulting inequality to have

$$(\sigma^i t)^{-\beta} E(u;x,\sigma^i t) \le ct^{-\beta} E(u;x,t) + c\left[M^{\mu}_{p-\beta(p-1)}(x,r)\right]^{\frac{1}{p-1}} \quad \forall\, i \in \mathbb{N}\cup\{0\}.$$

We now take $t = \sigma r$ in the last inequality. Then for any $\rho \in (0,r)$, we choose $i \in \mathbb{N}\cup\{0\}$ satisfying $\sigma^{i+1} r < \rho \le \sigma^i r$ to get

$$\rho^{-\beta} E(u;x,\rho) \le cr^{-\beta} E(u;x,r) + c\left[M^{\mu}_{p-\beta(p-1)}(x,r)\right]^{\frac{1}{p-1}}.$$

Eventually, we are able to take supremum with respect to ρ, thereby obtaining

$$\sup_{0<\rho<r} \rho^{-\beta} E(u;x,\rho) \le cr^{-\beta} E(u;x,r) + c\left[M^{\mu}_{p-\beta(p-1)}(x,r)\right]^{\frac{1}{p-1}}.$$

Recalling (5.18), the desired Hölder continuity now follows from Campanato's characterization of Hölder spaces [118, Theorem 2.9].

Chapter 6

Nonlocal double phase problems

In this chapter, we study the regularity theory for weak solutions to the following nonlocal equation:

$$\mathcal{L}u = 0 \quad \text{in } \Omega, \tag{6.1}$$

where the integrodifferential operator $\mathcal{L}$ is defined by

$$\begin{aligned}\mathcal{L}u(x) &:= \text{P.V.} \int_{\mathbb{R}^n} |u(x) - u(y)|^{p-2}(u(x) - u(y))K_{sp}(x, y)\, dy \\ &\quad + \text{P.V.} \int_{\mathbb{R}^n} a(x, y)|u(x) - u(y)|^{q-2}(u(x) - u(y))K_{tq}(x, y)\, dy.\end{aligned}$$

Here, $K_{sp}, K_{tq} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ are suitable kernels with orders (s, p) and (t, q), respectively, for some $0 < s \leq t < 1 < p \leq q < \infty$, and $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a nonnegative modulating coefficient.

A prototype of nonlocal double phase problems is the following equation:

$$\begin{aligned}&\text{P.V.} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{n+sp}}\, dy \\ &\quad + \text{P.V.} \int_{\mathbb{R}^n} a(x, y)\frac{|u(x) - u(y)|^{q-2}(u(x) - u(y))}{|x - y|^{n+tq}}\, dy = 0 \quad \text{in } \Omega,\end{aligned} \tag{6.2}$$

which is the case when $K_{sp}(x, y) \equiv |x - y|^{-n-sp}$ and $K_{tq}(x, y) \equiv |x - y|^{-n-tq}$

in (6.1). It is in fact the Euler-Lagrange equation of the functional

$$v \mapsto \iint_{\mathcal{C}_\Omega} \frac{1}{p}\frac{|v(x)-v(y)|^p}{|x-y|^{n+sp}} + a(x,y)\frac{1}{q}\frac{|v(x)-v(y)|^q}{|x-y|^{n+tq}}\,dxdy, \tag{6.3}$$

where

$$\mathcal{C}_\Omega \coloneqq (\mathbb{R}^n \times \mathbb{R}^n) \setminus ((\mathbb{R}^n \setminus \Omega) \times (\mathbb{R}^n \setminus \Omega)). \tag{6.4}$$

The local version corresponding to (6.2) is the double phase equation

$$-\operatorname{div}\left(|Du|^{p-2}Du + a(x)|Du|^{q-2}Du\right) = 0 \quad \text{in } \Omega. \tag{6.5}$$

Starting from [79, 80], the regularity for weak solutions to (6.5) and minimizers of corresponding variational integral has been exhaustively studied, see [18, 50, 81, 86, 90, 181] and references therein. In particular, for local boundedness and Hölder continuity, it is shown that

$$a(\cdot) \in L^\infty_{\mathrm{loc}}(\Omega), \ \begin{cases} p \le q \le \dfrac{np}{n-p} & \text{when } \ p < n, \\ p \le q < \infty & \text{when } \ p \ge n \end{cases} \implies u \in L^\infty_{\mathrm{loc}}(\Omega),$$

$$u \in L^\infty_{\mathrm{loc}}(\Omega),\ a(\cdot) \in C^{0,\alpha}_{\mathrm{loc}}(\Omega),\ q \le p+\alpha \implies u \in C^{0,\gamma}_{\mathrm{loc}}(\Omega),$$

see [16, 79, 83].

6.1 Main results

We say that a function $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is symmetric if $f(x,y) = f(y,x)$ for every $x, y \in \mathbb{R}^n$. The kernels $K_{sp}, K_{tq} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ are measurable, symmetric and satisfy

$$\begin{aligned} \frac{\Lambda^{-1}}{|x-y|^{n+sp}} &\le K_{sp}(x,y) \le \frac{\Lambda}{|x-y|^{n+sp}}, \\ \frac{\Lambda^{-1}}{|x-y|^{n+tq}} &\le K_{tq}(x,y) \le \frac{\Lambda}{|x-y|^{n+tq}} \end{aligned} \tag{6.6}$$

for a.e. $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$, where $\Lambda > 1$ and

$$1 < p \le q < \infty, \qquad 0 < s \le t < 1. \tag{6.7}$$

The modulating coefficient $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is assumed to be nonnegative, measurable, symmetric and bounded:

$$0 \le a(x,y) = a(y,x) \le \|a\|_{L^\infty}, \qquad x, y \in \mathbb{R}^n. \tag{6.8}$$

In addition, in Theorem 6.1.2 and Section 6.5, we also assume that

$$|a(x_1,y_1) - a(x_2,y_2)| \le [a]_\alpha (|x_1 - x_2| + |y_1 - y_2|)^\alpha, \quad \alpha > 0, \tag{6.9}$$

for every $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^n \times \mathbb{R}^n$. Throughout this chapter, we use the abbreviations

$$\begin{cases} \texttt{data} \coloneqq (n,s,t,p,q,\Lambda,\|a\|_{L^\infty}) \\ \texttt{data}_1 \coloneqq (n,s,t,p,q,\Lambda,\|a\|_{L^\infty},\alpha,[a]_\alpha). \end{cases}$$

With the relevant function spaces including $\mathcal{A}(\Omega)$ and $L^{q-1}_{sp}(\mathbb{R}^n)$ to be introduced in the next section, we introduce weak solutions under consideration. We say that $u \in \mathcal{A}(\Omega)$ is a weak solution to (6.1) if

$$\begin{aligned} \iint_{\mathcal{C}_\Omega} \big[&|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))K_{sp}(x,y) \\ &+ a(x,y)|u(x)-u(y)|^{q-2}(u(x)-u(y))(\varphi(x)-\varphi(y))K_{tq}(x,y)\big]\,dxdy = 0 \end{aligned} \tag{6.10}$$

for every $\varphi \in \mathcal{A}(\Omega)$ with $\varphi = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$. In addition, we say that $u \in \mathcal{A}(\Omega)$ is a weak subsolution (resp. supersolution) if (6.10) with "=" replaced by "$\le$ (resp. $\ge$)" holds for every $\varphi \in \mathcal{A}(\Omega)$ satisfying $\varphi \ge 0$ a.e. in $\mathbb{R}^n$ and $\varphi = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$. Existence and uniqueness of weak solutions to (6.1) with a Dirichlet boundary condition will be discussed in Section 6.3.

Now we state our main results. The first one is the local boundedness of weak solutions.

Theorem 6.1.1. *Let $K_{sp}, K_{tq}, a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be symmetric and satisfy* (6.6)-(6.8). *If*

$$\begin{cases} p \le q \le \dfrac{np}{n-sp} & \textit{when} \;\; sp < n, \\ p \le q < \infty & \textit{when} \;\; sp \ge n, \end{cases} \tag{6.11}$$

then every weak solution $u \in \mathcal{A}(\Omega) \cap L^{q-1}_{sp}(\mathbb{R}^n)$ to (6.1) *is locally bounded in Ω.*

The second one is the local Hölder continuity. Here, we assume that $a(\cdot,\cdot)$ is Hölder continuous in $\mathbb{R}^n \times \mathbb{R}^n$ and that u is locally bounded in Ω.

Theorem 6.1.2. *Let $K_{sp}, K_{tq}, a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be symmetric and satisfy* (6.6)-(6.8)*. If $a(\cdot,\cdot)$ satisfies* (6.9) *and*

$$tq \leq sp + \alpha, \tag{6.12}$$

then every weak solution $u \in \mathcal{A}(\Omega) \cap L^{q-1}_{sp}(\mathbb{R}^n)$ to (6.1) *which is locally bounded in Ω is locally Hölder continuous in Ω. More precisely, for every open subset $\Omega' \Subset \Omega$, there exists $\gamma \in (0,1)$, depending only on* $\mathtt{data}_1$ *and $\|u\|_{L^\infty(\Omega')}$, such that $u \in C^{0,\gamma}_{\mathrm{loc}}(\Omega')$.*

Remark 6.1.3. *In view of Theorem 6.1.1, we also see that, under the setting in Theorem 6.1.2, if*

$$\begin{cases} p \leq q \leq \min\left\{\dfrac{np}{n-sp}, \dfrac{sp+\alpha}{t}\right\} & \textit{when} \ \ sp < n, \\ p \leq q \leq \dfrac{sp+\alpha}{t} = \dfrac{n+\alpha}{t} & \textit{when} \ \ sp = n, \end{cases}$$

then every weak solution $u \in \mathcal{A}(\Omega) \cap L^{q-1}_{sp}(\mathbb{R}^n)$ to (6.1) *is locally Hölder continuous.*

Remark 6.1.4. *Here we give a heuristic explanation on the condition* (6.12)*. Under assumptions* (6.7)-(6.9)*, we write the integrand of the energy functional in* (6.3) *as*

$$\left(1 + \frac{p}{q}|u(x)-u(y)|^{q-p} a(x,y)|x-y|^{sp-tq}\right) \frac{1}{p} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}, \quad (x,y) \in \mathcal{C}_\Omega.$$

If $B_r \Subset \Omega$ with $r \in (0,1]$, u is bounded in B_r and $a(x,y) = |x-y|^\alpha$ for $x,y \in B_r$, which is a simple example of $a(\cdot,\cdot)$ satisfying (6.9)*, then* (6.12) *implies*

$$1 + \frac{p}{q}|u(x)-u(y)|^{q-p} a(x,y)|x-y|^{sp-tq} \approx 1 \ \ \textit{for} \ \ (x,y) \in B_r \times B_r,$$

as the second term is bounded by $(2\|u\|_{L^\infty})^{q-p} 2^{\alpha+sp-tq}$. This means that we can control the double phase type energy functional in (6.3) *by means of the $W^{s,p}$-energy in local regions in Ω. This is the exact nonlocal analog of what happens in the local case* (6.5).

In addition, one can expect that Theorems 6.1.1 and 6.1.2, with some minor modifications in their proofs, still hold for the case when (6.12) *is in force with any* $s, t \in (0,1)$, $p, q > 1$ *and* $\alpha \geq 0$. *Note that if* $a(\cdot,\cdot)$ *is bounded only, then we can take* $\alpha = 0$ *and* (6.12) *becomes* $tq \leq sp$.

6.2 Preliminaries

6.2.1 Function spaces

We always assume that s, t, p, and q satisfy (6.7) and that $K_{sp}, K_{tq}, a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfy (6.6) and (6.8). We denote

$$H(x,y,\tau) \coloneqq \frac{\tau^p}{|x-y|^{sp}} + a(x,y)\frac{\tau^q}{|x-y|^{tq}}, \quad x, y \in \mathbb{R}^n \text{ and } \tau \geq 0, \tag{6.13}$$

and

$$\varrho(v;S) \coloneqq \int_S \int_S H(x,y,|v(x)-v(y)|)\,\frac{dxdy}{|x-y|^n} \tag{6.14}$$

for each measurable set $S \subseteq \mathbb{R}^n$ and $v : S \to \mathbb{R}$. Then we define a function space concerned with weak solutions to (6.1) by

$$\mathcal{A}(\Omega) \coloneqq \left\{ v : \mathbb{R}^n \to \mathbb{R} \;\middle|\; v|_\Omega \in L^p(\Omega),\ \iint_{\mathcal{C}_\Omega} H(x,y,|v(x)-v(y)|)\,\frac{dxdy}{|x-y|^n} < \infty \right\},$$

where $\mathcal{C}_\Omega$ is defined in (6.4). Note that $\varrho(v;\Omega) < \infty$ whenever $v \in \mathcal{A}(\Omega)$, which in particular implies

$$\mathcal{A}(\Omega) \subset W^{s,p}(\Omega).$$

We note that if $sp > n$, then every function in $W^{s,p}(\Omega)$ is locally Hölder continuous, see for example [95, Theorem 8.2]. Thus, in this chapter we assume without loss of generality that

$$sp \leq n.$$

Moreover, by Lemma 2.2.4, we have

$$\mathcal{A}(\Omega) \subset L^q_{\mathrm{loc}}(\Omega) \quad \text{if} \quad \begin{cases} p < q \leq \dfrac{np}{n-sp} & \text{when } sp < n, \\ p < q < \infty & \text{when } sp \geq n. \end{cases} \tag{6.15}$$

This will be used in the proof of several results concerning local boundedness.

We next define

$$L^{q-1}_{sp}(\mathbb{R}^n) := \left\{ v : \mathbb{R}^n \to \mathbb{R} \,\Big|\, \int_{\mathbb{R}^n} \frac{|v(x)|^{q-1}}{(1+|x|)^{n+sp}}\,dx < \infty \right\}.$$

Let $m \in \{s,t\}$ and $\ell \in \{p,q\}$. Since we have

$$\frac{1+|x|}{|x-x_0|} \le \frac{1+|x-x_0|+|x_0|}{|x-x_0|} \le 1+\frac{1+|x_0|}{r}, \quad \frac{|v(x)|^{\ell-1}}{(1+|x|)^{n+m\ell}} \le \frac{|v(x)|^{q-1}+1}{(1+|x|)^{n+sp}}$$

for $x \in \mathbb{R}^n \setminus B_r(x_0)$, we see that the nonlocal tail

$$\int_{\mathbb{R}^n\setminus B_r(x_0)} \frac{|v(x)|^{\ell-1}}{|x-x_0|^{n+m\ell}}\,dx$$

is finite whenever $v \in L^{q-1}_{sp}(\mathbb{R}^n)$ and $B_r(x_0) \subset \mathbb{R}^n$.

Remark 6.2.1. *If $v \in L^{q_0}(\mathbb{R}^n)$ for some $q_0 \ge q-1$, or if $v \in L^{q-1}(B_R(0)) \cap L^\infty(\mathbb{R}^n \setminus B_R(0))$ for some $R>0$, then $v \in L^{q-1}_{sp}(\mathbb{R}^n)$. Moreover, we have that*

$$W^{s,p}(\mathbb{R}^n) \subset L^{q-1}_{sp}(\mathbb{R}^n) \quad \textit{if} \;\; q \le p^*_s + 1.$$

6.2.2 Inequalities

The following two lemmas are simple consequences of the fractional Sobolev-Poincaré inequality. They will be used in the proof of Theorems 6.1.1 and 6.1.2, respectively.

Lemma 6.2.2. *Assume that the constants s, t, p and q satisfy* (6.7) *and* (6.11). *Then for every $f \in W^{s,p}(B_r)$ we have*

$$\begin{aligned} \fint_{B_r} \left|\frac{f}{r^s}\right|^p + L_0\left|\frac{f}{r^t}\right|^q dx &\le cL_0 r^{(s-t)q}\left(\fint_{B_r}\int_{B_r} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dxdy\right)^{\frac{q}{p}} \\ &\quad + c\left(\frac{|\operatorname{supp} f|}{|B_r|}\right)^{\frac{sp}{n}} \fint_{B_r}\int_{B_r} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dxdy \\ &\quad + c\left(\frac{|\operatorname{supp} f|}{|B_r|}\right)^{p-1} \fint_{B_r} \left|\frac{f}{r^s}\right|^p + L_0\left|\frac{f}{r^t}\right|^q dx \end{aligned}$$

for a constant $c \equiv c(n, s, t, p, q)$*, where* L_0 *is any positive constant.*

Proof. Applying Hölder's inequality and (2.5), we have

$$\fint_{B_r} \left|\frac{f}{r^t}\right|^q dx \leq c \left(\fint_{B_r} \left| \frac{f - (f)_{B_r}}{r^t} \right|^{p_s^*} dx \right)^{\frac{q}{p_s^*}} + c \left| \frac{(f)_{B_r}}{r^t} \right|^q$$
$$\leq c r^{(s-t)q} \left(\fint_{B_r} \int_{B_r} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \, dxdy \right)^{\frac{q}{p}} + c \left| \frac{(f)_{B_r}}{r^t} \right|^q .$$

Likewise, we obtain

$$\fint_{B_r} \left|\frac{f}{r^s}\right|^p dx \leq c \left(\frac{|\operatorname{supp} f|}{|B_r|} \right)^{\frac{sp}{n}} \left(\fint_{B_r} \left| \frac{f - (f)_{B_r}}{r^s} \right|^{p_s^*} dx \right)^{\frac{p}{p_s^*}} + c \left| \frac{(f)_{B_r}}{r^s} \right|^p$$
$$\leq c \left(\frac{|\operatorname{supp} f|}{|B_r|} \right)^{\frac{sp}{n}} \fint_{B_r} \int_{B_r} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \, dxdy + c \left| \frac{(f)_{B_r}}{r^s} \right|^p .$$

We also have

$$\left| \frac{(f)_{B_r}}{r^s} \right|^p + L_0 \left| \frac{(f)_{B_r}}{r^t} \right|^q$$
$$\leq r^{-sp} \left(\frac{|\operatorname{supp} f|}{|B_r|} \right)^{p-1} \fint_{B_r} |f|^p \, dx + L_0 r^{-tq} \left(\frac{|\operatorname{supp} f|}{|B_r|} \right)^{q-1} \fint_{B_r} |f|^q \, dx$$
$$\leq \left(\frac{|\operatorname{supp} f|}{|B_r|} \right)^{p-1} \fint_{B_r} \left|\frac{f}{r^s}\right|^p + L_0 \left|\frac{f}{r^t}\right|^q dx.$$

We combine the above three displays to complete the proof. □

Lemma 6.2.3. *Assume that the constants* s, t, p *and* q *satisfy* (6.7) *and that the function* $a(\cdot, \cdot)$ *satisfies* (6.9) *with* α *satisfying* (6.12)*. Let* $B_r \subseteq B_R$ *be concentric balls with* $R/2 \leq r \leq R \leq 1$*. Then for any* $f \in L^\infty(B_r)$ *we have*

$$\left[\fint_{B_r} \left(\left|\frac{f}{r^s}\right|^p + a_2 \left|\frac{f}{r^t}\right|^q \right)^\kappa dx \right]^{\frac{1}{\kappa}}$$
$$\leq c \left(1 + \|f\|_{L^\infty(B_r)}^{q-p} \right) \fint_{B_r} \int_{B_r} H(x, y, |f(x) - f(y)|) \frac{dxdy}{|x-y|^n}$$
$$+ c \left(1 + \|f\|_{L^\infty(B_r)}^{q-p} \right) \fint_{B_r} \left|\frac{f}{r^s}\right|^p + a_1 \left|\frac{f}{r^t}\right|^q dx$$

for some $c \equiv c(n,s,t,p,q,[a]_\alpha)$*, whenever the right-hand side is finite, where*

$$\kappa := \min\left\{\frac{p_s^*}{p}, \frac{q_t^*}{q}\right\} > 1, \quad a_1 := \inf_{B_R\times B_R} a(\cdot,\cdot) \quad \text{and} \quad a_2 := \sup_{B_R\times B_R} a(\cdot,\cdot).$$

Proof. Using the assumptions, we estimate

$$\begin{aligned}
&\left[\fint_{B_r}\left(\left|\frac{f}{r^s}\right|^p + a_2\left|\frac{f}{r^t}\right|^q\right)^\kappa dx\right]^{\frac{1}{\kappa}} \\
&\le c\left[\fint_{B_r}\left(\left|\frac{f}{r^s}\right|^p + a_1\left|\frac{f}{r^t}\right|^q\right)^\kappa + \left(r^{\alpha+sp-tq}\|f\|_{L^\infty(B_r)}^{q-p}\left|\frac{f}{r^s}\right|^p\right)^\kappa dx\right]^{\frac{1}{\kappa}} \\
&\le c\left(1+\|f\|_{L^\infty(B_r)}^{q-p}\right)\left[\fint_{B_r}\left(\left|\frac{f}{r^s}\right|^p + a_1\left|\frac{f}{r^t}\right|^q\right)^\kappa dx\right]^{\frac{1}{\kappa}}.
\end{aligned}$$

We next apply (2.5) to see that

$$\begin{aligned}
&\left[\fint_{B_r}\left(\left|\frac{f}{r^s}\right|^p + a_1\left|\frac{f}{r^t}\right|^q\right)^\kappa dx\right]^{\frac{1}{\kappa}} \\
&\le c\left[\fint_{B_r}\left(\left|\frac{f-(f)_{B_r}}{r^s}\right|^p + a_1\left|\frac{f-(f)_{B_r}}{r^t}\right|^q\right)^\kappa dx\right]^{\frac{1}{\kappa}} + c\left|\frac{(f)_{B_r}}{r^s}\right|^p + ca_1\left|\frac{(f)_{B_r}}{r^t}\right|^q \\
&\le c\fint_{B_r}\int_{B_r}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}} + a_1\frac{|f(x)-f(y)|^q}{|x-y|^{n+tq}}\,dxdy + c\fint_{B_r}\left|\frac{f}{r^s}\right|^p + a_1\left|\frac{f}{r^t}\right|^q dx \\
&\le c\fint_{B_r}\int_{B_r}H(x,y,|f(x)-f(y)|)\frac{dxdy}{|x-y|^n} + c\fint_{B_r}\left|\frac{f}{r^s}\right|^p + a_1\left|\frac{f}{r^t}\right|^q dx.
\end{aligned}$$

Then the conclusion follows. □

We also note the following numerical inequalities.

Lemma 6.2.4. *Let* $p \ge 1$ *and* $a, b \ge 0$*. Then we have*

$$a^p - b^p \le pa^{p-1}|a-b|$$

and, for any $\varepsilon \in (0,1)$*,*

$$a^p - b^p \le \varepsilon b^p + c(p)\varepsilon^{1-p}|a-b|^p.$$

Proof. The first display is a direct consequence of Mean Value Theorem; note that we may assume $a \geq b$, otherwise it is obvious. For the proof of the second display, see [94, Lemma 3.1]. □

We end this section with a standard iteration lemma from [118, Lemma 7.1].

Lemma 6.2.5. *Let* $\{y_i\}_{i=0}^{\infty}$ *be a sequence of nonnegative numbers satisfying*

$$y_{i+1} \leq b_1 b_2^i y_i^{1+\beta}, \qquad i = 0, 1, 2, \dots,$$

for some constants $b_1, \beta > 0$ *and* $b_2 > 1$. *If*

$$y_0 \leq b_1^{-1/\beta} b_2^{-1/\beta^2},$$

then $y_i \to 0$ *as* $i \to \infty$.

6.3 Existence of weak solutions

In this section we show the existence of weak solutions to (6.1). By a standard argument, such as the one in the proof of [93, Theorem 2.3], we see that $u \in \mathcal{A}(\Omega)$ is a weak solution to (6.1) if and only if it is a minimizer of the functional

$$\mathcal{E}(v;\Omega) := \iint_{\mathcal{C}_\Omega} \frac{1}{p}|v(x)-v(y)|^p K_{sp}(x,y)+a(x,y)\frac{1}{q}|v(x)-v(y)|^q K_{tq}(x,y)\,dxdy. \tag{6.16}$$

We say that $u \in \mathcal{A}(\Omega)$ is a minimizer of (6.16) if

$$\mathcal{E}(u;\Omega) \leq \mathcal{E}(v;\Omega)$$

for every $v \in \mathcal{A}(\Omega)$ with $v = u$ a.e. in $\mathbb{R}^n \setminus \Omega$. Therefore, we prove the existence and uniqueness of the minimizer of (6.16) with a Dirichlet boundary condition.

Theorem 6.3.1. *Let* Ω *be a bounded domain and* $g \in \mathcal{A}(\Omega)$ *be a given boundary data. Let* $K_{sp}, K_{tq}, a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *be symmetric and satisfy* (6.6)-(6.8). *Then there exists a unique minimizer* $u \in \mathcal{A}(\Omega)$ *of* (6.16) *with* $u = g$ *a.e. in* $\mathbb{R}^n \setminus \Omega$. *Moreover, if* $g \in \mathcal{A}(\Omega) \cap L^{q-1}_{sp}(\mathbb{R}^n)$, *then* $u \in \mathcal{A}(\Omega) \cap L^{q-1}_{sp}(\mathbb{R}^n)$.

Proof. The uniqueness follows directly from the fact that the function $\tau \mapsto \tau^p + a(x,y)\tau^q$ is strictly convex for each fixed (x,y). Now we prove the existence. The admissible set

$$\mathcal{A}_g(\Omega) \coloneqq \{v \in \mathcal{A}(\Omega) : v = g \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}$$

is obviously nonempty, as $g \in \mathcal{A}_g(\Omega)$. Let $\{u_k\} \subset \mathcal{A}_g(\Omega)$ be a minimizing sequence. Then there exists a constant C such that

$$[u_k]^p_{s,p;\Omega} = \int_\Omega \int_\Omega \frac{|u_k(x) - u_k(y)|^p}{|x-y|^{n+sp}}\,dxdy \le \Lambda \mathcal{E}(u_k;\Omega) \le C \qquad \forall\, k \in \mathbb{N}.$$

In particular, Lemma 2.2.3 implies that $\{[u_k]_{s_0,p;\Omega}\}$ is bounded for any $s_0 \in (0,s)$. Then we choose a ball $B_R \equiv B_R(x_0) \supset \Omega$ with $R \ge 1$ and fix $s_0 \in (0, s/2)$ with $np/(n+s_0p) =: p_0 > 1$. Since $u_k - g = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$, the fractional Sobolev embedding [95, Theorem 6.5] implies

$$\begin{aligned}
&\left(\int_{B_R} |u_k - g|^p\,dx\right)^{\frac{p_0}{p}} \le c[u_k - g]^{p_0}_{s_0,p_0;\mathbb{R}^n} \\
&\le c[u_k - g]^{p_0}_{s_0,p_0;B_R} + c\int_{B_R} |u_k(y) - g(y)|^{p_0} \left(\int_{\mathbb{R}^n \setminus B_R} \frac{dx}{|x-y|^{n+s_0p_0}}\right) dy \\
&\le c[u_k - g]^{p_0}_{s_0,p_0;B_R} + c\int_{B_R} |u_k(y) - g(y)|^{p_0} \left(\int_{B_{2R} \setminus B_R} \frac{dx}{|x-y|^{n+s_0p_0}}\right) dy \\
&\le c\int_{B_{2R}} \int_{B_R} \frac{|(u_k - g)(x) - (u_k - g)(y)|^{p_0}}{|x-y|^{n+s_0p_0}}\,dxdy \\
&\le c[u_k - g]^{p_0}_{s_0,p_0;B_{2R}},
\end{aligned} \tag{6.17}$$

where we have used the fact that

$$\int_{\mathbb{R}^n \setminus B_R} \frac{dx}{|x-y|^{n+s_0p_0}} \le (1 + c(n)) \int_{B_{2R} \setminus B_R} \frac{dx}{|x-y|^{n+s_0p_0}} \qquad \forall\, y \in B_R.$$

Applying Lemma 2.2.3 to the right-hand side of (6.17), we have for all $k \in \mathbb{N}$

$$\begin{aligned}
&\left(\int_{B_R} |u_k - g|^p\,dx\right)^{\frac{p_0}{p}} \\
&\le c[u_k - g]^{p_0}_{s_0,p_0;B_{2R}}
\end{aligned}$$

$$
\begin{aligned}
&\leq cR^{sp_0}[u_k - g]^{p_0}_{s,p;B_{2R}} \\
&\leq cR^{sp_0}\left(\iint_{\mathcal{C}_\Omega} \frac{|u_k(x)-u_k(y)|^p}{|x-y|^{n+sp}}\,dxdy + \iint_{\mathcal{C}_\Omega} \frac{|g(x)-g(y)|^p}{|x-y|^{n+sp}}\,dxdy\right)^{\frac{p_0}{p}} \\
&\leq cR^{sp_0}\left(C + \iint_{\mathcal{C}_\Omega} \frac{|g(x)-g(y)|^p}{|x-y|^{n+sp}}\,dxdy\right)^{\frac{p_0}{p}}.
\end{aligned}
$$

This implies that $\{u_k - g\}$ is bounded in $L^p(B_R)$, and hence in $W^{s,p}_0(B_R)$. By the compact embedding theorem for fractional Sobolev spaces [95, Theorem 7.1], there exist a subsequence $\{u_{k_j} - g\}$ and $v \in L^p(B_R)$ such that

$$
\begin{cases} u_{k_j} - g \longrightarrow v & \text{in} \quad L^p(B_R), \\ u_{k_j} - g \longrightarrow v & \text{a.e. in} \quad B_R, \end{cases} \qquad \text{as} \quad j \to \infty.
$$

We extend v to $\mathbb{R}^n$ by letting $v = 0$ on $\mathbb{R}^n \setminus B_R$ and set $u \coloneqq v + g$. Then $u_{k_j} \to u$ a.e. in $\mathbb{R}^n$. Finally, Fatou's lemma implies

$$
\mathcal{E}(u;\Omega) \leq \liminf_{j\to\infty} \mathcal{E}(u_{k_j};\Omega).
$$

This means that $u \in \mathcal{A}_g(\Omega)$ and it is a minimizer of $\mathcal{E}$. □

Remark 6.3.2. *In fact, the above theorem still holds even when $a(\cdot,\cdot) \geq 0$ is not bounded above.*

6.4 Caccioppoli estimates and local boundedness

We start with the following lemma which implies that the multiplication of any function in $\mathcal{A}(\Omega)$ and a cut-off function is also a function in $\mathcal{A}(\Omega)$. We recall the notation (6.14).

Lemma 6.4.1. *Assume that the constants s, t, p and q satisfy* (6.7), *and $\eta \in W^{1,\infty}_0(B_r)$. If one of the following two conditions holds:*

(i) The inequality (6.11) *holds and $v \in L^p(B_{2r})$ satisfies $\varrho(v;B_{2r}) < \infty$;*

(ii) $v \in L^q(B_{2r})$ satisfies $\varrho(v;B_{2r}) < \infty$,

then $\varrho(v\eta;\mathbb{R}^n) < \infty$. In particular, $v\eta \in \mathcal{A}(\Omega)$ whenever $\Omega \supset B_{2r}$.

Proof. We only consider the case (ii), since we also have $v \in L^q(B_{2r})$ in the case (i) by (6.15). We write

$$\varrho(v\eta;\mathbb{R}^n) = \varrho(v\eta;B_{2r}) + 2\int_{\mathbb{R}^n\setminus B_{2r}}\int_{B_{2r}} H(x,y,|v(x)\eta(x)|)\,\frac{dxdy}{|x-y|^n}.$$

The first term on the right-hand side is estimated as

$$\begin{aligned}
\varrho(v\eta;B_{2r}) &\le c\int_{B_{2r}}\int_{B_{2r}} H(x,y,|(v(x)-v(y))\eta(y)|)\,\frac{dxdy}{|x-y|^n} \\
&\quad + c\int_{B_{2r}}\int_{B_{2r}} H(x,y,|v(x)(\eta(x)-\eta(y))|)\,\frac{dxdy}{|x-y|^n} \\
&\le c\left(\|\eta\|_{L^\infty(B_{2r})}+1\right)^q \varrho(v;B_{2r}) \\
&\quad + c\|D\eta\|^p_{L^\infty(B_{2r})}\int_{B_{2r}}|v(x)|^p\int_{B_{4r}(x)}\frac{dy}{|x-y|^{n+(s-1)p}}\,dx \\
&\quad + c\|D\eta\|^q_{L^\infty(B_{2r})}\|a\|_{L^\infty}\int_{B_{2r}}|v(x)|^q\int_{B_{4r}(x)}\frac{dy}{|x-y|^{n+(t-1)q}}\,dx \\
&\le c\left(\|\eta\|_{L^\infty(B_{2r})}+1\right)^q \varrho(v;B_{2r}) + c\|D\eta\|^p_{L^\infty(B_{2r})}r^{(1-s)p}\int_{B_{2r}}|v(x)|^p\,dx \\
&\quad + c\|D\eta\|^q_{L^\infty(B_{2r})}\|a\|_{L^\infty}r^{(1-t)q}\int_{B_{2r}}|v(x)|^q\,dx \\
&< \infty.
\end{aligned}$$

The second term is estimated as

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus B_{2r}}\int_{B_{2r}} H(x,y,|v(x)\eta(x)|)\,\frac{dxdy}{|x-y|^n} \\
&\le \left(\|\eta\|_{L^\infty(B_{2r})}+1\right)^q\int_{\mathbb{R}^n\setminus B_{2r}}\int_{B_r}\frac{|v(x)|^p}{|x-y|^{n+sp}} + \|a\|_{L^\infty}\frac{|v(x)|^q}{|x-y|^{n+tq}}\,dxdy \\
&\le c\left(\|\eta\|_{L^\infty(B_{2r})}+1\right)^q\left(\frac{1}{r^{sp}}\int_{B_r}|v(x)|^p\,dx + \frac{\|a\|_{L^\infty}}{r^{tq}}\int_{B_r}|v(x)|^q\,dx\right) < \infty,
\end{aligned}$$

and the conclusion follows. □

Next, we prove a nonlocal Caccioppoli type estimate. We again recall

(6.14) with (6.13), and further define

$$h(x,y,\tau) := \frac{\tau^{p-1}}{|x-y|^{sp}} + a(x,y)\frac{\tau^{q-1}}{|x-y|^{tq}}, \quad x,y \in \mathbb{R}^n, \tau \geq 0. \tag{6.18}$$

Lemma 6.4.2. *Let $K_{sp}, K_{tq}, a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be symmetric and satisfy* (6.6)-(6.8)*, and $u \in \mathcal{A}(\Omega) \cap L^{q-1}_{sp}(\mathbb{R}^n)$ be a weak solution to* (6.1)*. Let $B_{2r} \equiv B_{2r}(x_0) \Subset \Omega$ be a ball, and assume that* (6.11) *holds or u is bounded in B_{2r}. Then for any $\phi \in C_0^\infty(B_r)$ with $0 \leq \phi \leq 1$, we have*

$$\begin{aligned}
&\int_{B_r}\int_{B_r} H(x,y,|w_\pm(x)-w_\pm(y)|)(\phi^q(x)+\phi^q(y))\,\frac{dxdy}{|x-y|^n} \\
&\leq c\int_{B_r}\int_{B_r} H\big(x,y,|(\phi(x)-\phi(y))(w_\pm(x)+w_\pm(y))|\big)\,\frac{dxdy}{|x-y|^n} \\
&\quad + c\left(\sup_{x\in \operatorname{supp}\phi}\int_{\mathbb{R}^n\setminus B_r} h(x,y,w_\pm(y))\,\frac{dy}{|x-y|^n}\right)\int_{B_r} w_\pm(x)\phi^q(x)\,dx
\end{aligned} \tag{6.19}$$

for some $c \equiv c(n,s,t,p,q,\Lambda)$, where $w_\pm := (u-k)_\pm$ with $k \geq 0$.

Proof. We only prove the estimate for w_+, since the estimate for w_- can be proved similarly. In light of Lemma 6.4.1, we can test the weak formulation (6.10) with $w_+\phi^q \in \mathcal{A}(\Omega)$. Using the short notation

$$\Phi_\ell(\tau) := |\tau|^{\ell-2}\tau \quad \text{for} \quad \ell \in \{p,q\} \quad \text{and} \quad \tau \in \mathbb{R}, \tag{6.20}$$

we have

$$\begin{aligned}
0 &= \int_{B_r}\int_{B_r} [\Phi_p(u(x)-u(y))K_{sp}(x,y) + a(x,y)\Phi_q(u(x)-u(y))K_{tq}(x,y)] \\
&\qquad\qquad \cdot (w_+(x)\phi^q(x) - w_+(y)\phi^q(y))\,dxdy \\
&\quad + 2\int_{\mathbb{R}^n\setminus B_r}\int_{B_r} [\Phi_p(u(x)-u(y))w_+(x)\phi^q(x)K_{sp}(x,y) \\
&\qquad\qquad\qquad + a(x,y)\Phi_q(u(x)-u(y))w_+(x)\phi^q(x)K_{tq}(x,y)]\,dxdy \\
&=: I_1 + I_2.
\end{aligned}$$

We first estimate I_1. Assume that $u(x) \geq u(y)$. Then,

$$\Phi_\ell(u(x)-u(y))(w_+(x)\phi^q(x) - w_+(y)\phi^q(y))$$

$$
\begin{aligned}
&= (u(x)-u(y))^{\ell-1}((u(x)-k)_+\phi^q(x)-(u(y)-k)_+\phi^q(y))\\
&= \begin{cases} (w_+(x)-w_+(y))^{\ell-1}(w_+(x)\phi^q(x)-w_+(y)\phi^q(y)), & u(x)\geq u(y)\geq k\\ (u(x)-u(y))^{\ell-1}w_+(x)\phi^q(x), & u(x)>k\geq u(y)\\ 0, & k\geq u(x)\geq u(y)\end{cases}\\
&\geq (w_+(x)-w_+(y))^{\ell-1}(w_+(x)\phi^q(x)-w_+(y)\phi^q(y))\\
&= \Phi_\ell(w_+(x)-w_+(y))(w_+(x)\phi^q(x)-w_+(y)\phi^q(y)),
\end{aligned}
$$

and hence

$$
\begin{aligned}
I_1 \geq \int_{B_r}\int_{B_r} &\big[\Phi_p(w_+(x)-w_+(y))K_{sp}(x,y)\\
&+a(x,y)\Phi_q(w_+(x)-w_+(y))K_{tq}(x,y)\big]\\
&\cdot(w_+(x)\phi^q(x)-w_+(y)\phi^q(y))\,dxdy. \qquad (6.21)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&w_+(x)\phi^q(x)-w_+(y)\phi^q(y)\\
&=\frac{w_+(x)-w_+(y)}{2}(\phi^q(x)+\phi^q(y))+\frac{w_+(x)+w_+(y)}{2}(\phi^q(x)-\phi^q(y)),
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\Phi_\ell(w_+(x)-w_+(y))(w_+(x)\phi^q(x)-w_+(y)\phi^q(y))\\
&\geq |w_+(x)-w_+(y)|^\ell\frac{\phi^q(x)+\phi^q(y)}{2}\\
&\qquad -|w_+(x)-w_+(y)|^{\ell-1}\frac{w_+(x)+w_+(y)}{2}|\phi^q(x)-\phi^q(y)|.
\end{aligned}
$$

Here, we use Lemma 6.2.4 to see that

$$
\begin{aligned}
|\phi^q(x)-\phi^q(y)| &\leq q(\phi^{q-1}(x)+\phi^{q-1}(y))|\phi(x)-\phi(y)|\\
&\leq c(q)(\phi^q(x)+\phi^q(y))^{(q-1)/q}|\phi(x)-\phi(y)|.
\end{aligned}
$$

Thus, using Young's inequality, we get

$$
\begin{aligned}
&|w_+(x)-w_+(y)|^{\ell-1}(w_+(x)+w_+(y))|\phi^q(x)-\phi^q(y)|\\
&\leq |w_+(x)-w_+(y)|^{\ell-1}(w_+(x)+w_+(y))(\phi^q(x)+\phi^q(y))^{\frac{\ell-1}{\ell}+\frac{q-\ell}{q\ell}}|\phi(x)-\phi(y)|
\end{aligned}
$$

$$\begin{aligned}&\le \varepsilon|w_+(x)-w_+(y)|^\ell(\phi^q(x)+\phi^q(y))\\&\quad + c_\varepsilon(\phi^q(x)+\phi^q(y))^{(q-\ell)/q}|\phi(x)-\phi(y)|^\ell(w_+(x)+w_+(y))^\ell.\end{aligned}$$

Since $0 \le \phi \le 1$ and $(q-\ell)/q \ge 0$, after choosing ε so small, we discover

$$\begin{aligned}&\Phi_\ell(w_+(x)-w_+(y))(w_+(x)\phi^q(x)-w_+(y)\phi^q(y))\\&\ge |w_+(x)-w_+(y)|^\ell\frac{\phi^q(x)+\phi^q(y)}{4} - c|\phi(x)-\phi(y)|^\ell(w_+(x)+w_+(y))^\ell.\end{aligned}$$

We notice that by the symmetry of the above inequality for x and y, we also have the same inequality when $u(x) < u(y)$. Inserting this into (6.21) and using (6.6), we have

$$\begin{aligned}I_1 \ge{}& \frac{1}{4\Lambda}\int_{B_r}\int_{B_r} H(x,y,|w_+(x)-w_+(y)|)(\phi^q(x)+\phi^q(y))\,\frac{dxdy}{|x-y|^n}\\&- c\int_{B_r}\int_{B_r} H(x,y,|\phi(x)-\phi(y)|(w_+(x)+w_+(y)))\,\frac{dxdy}{|x-y|^n}.\end{aligned}$$

For I_2, we observe that

$$\Phi_\ell(u(x)-u(y))w_+(x) \ge -w_+^{\ell-1}(y)w_+(x)$$

and use (6.6) and (6.18), to find

$$\begin{aligned}I_2 &\ge -c\int_{\mathbb{R}^n\setminus B_r}\int_{B_r} h(x,y,w_+(y))w_+(x)\phi^q(x)\,\frac{dxdy}{|x-y|^n}\\&\ge -c\left(\sup_{x\in\operatorname{supp}\phi}\int_{\mathbb{R}^n\setminus B_r} h(x,y,w_+(y))\,\frac{dy}{|x-y|^n}\right)\int_{B_r} w_+(x)\phi^q(x)\,dx.\end{aligned}$$

Combining the above estimates with $I_1 + I_2 = 0$, we obtain (6.19). □

Remark 6.4.3. *In fact, we can obtain* (6.19) *when $q > p_s^*$ and u is not bounded in B_{2r}, by using a truncation argument as in [182, Lemma 4.2] provided the right-hand side of* (6.19) *is finite.*

Now, we are ready to prove the local boundedness of weak solutions to (6.1).

Proof of Theorem 6.1.1. For brevity, we denote

$$H_0(\tau) \coloneqq \tau^p + \|a\|_{L^\infty}\tau^q, \quad \tau \geq 0.$$

In the following, c means a constant depending only on `data`.

Let $B_r \equiv B_r(x_0) \Subset \Omega$ be a fixed ball with $r \leq 1$. For $r/2 \leq \rho < \sigma \leq r$ and $k > 0$, we denote

$$A^+(k,\rho) \coloneqq \{x \in B_\rho : u(x) \geq k\}$$

and apply Lemma 6.2.2 with $f \equiv (u-k)_+$ to get

$$\begin{aligned}
\rho^{-sp} \fint_{B_\rho} H_0(f)\,dx &\leq \fint_{B_\rho} \left(\frac{f}{\rho^s}\right)^p + \|a\|_{L^\infty}\left(\frac{f}{\rho^t}\right)^q dx \\
&\leq c\|a\|_{L^\infty}\rho^{(s-t)q}\left(\fint_{B_\rho}\int_{B_\rho}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dxdy\right)^{\frac{q}{p}} \\
&\quad + c\left(\frac{|A^+(k,\rho)|}{|B_\rho|}\right)^{\frac{sp}{n}}\fint_{B_\rho}\int_{B_\rho}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dxdy \\
&\quad + c\left(\frac{|A^+(k,\rho)|}{|B_\rho|}\right)^{p-1}\fint_{B_\sigma}\left(\frac{f}{\rho^s}\right)^p + \|a\|_{L^\infty}\left(\frac{f}{\rho^t}\right)^q dx. \qquad (6.22)
\end{aligned}$$

We now fix $0 < h < k$ and observe that, for $x \in A^+(k,\rho) \subset A^+(h,\rho)$,

$$\begin{aligned}
&(u(x)-h)_+ = u(x)-h \geq k-h, \\
&(u(x)-h)_+ = u(x)-h \geq u(x)-k = (u(x)-k)_+.
\end{aligned}$$

This implies

$$|A^+(k,\rho)| \leq \int_{A^+(k,\rho)} \frac{(u-h)_+^p}{(k-h)^p}\,dx \leq \frac{1}{(k-h)^p}\int_{A^+(h,\sigma)} H_0((u-h)_+)\,dx \quad (6.23)$$

and

$$\begin{aligned}
\int_{B_\sigma}(u-k)_+\,dx &\leq \int_{B_\sigma}(u-h)_+\left(\frac{(u-h)_+}{k-h}\right)^{p-1} dx \\
&\leq \frac{1}{(k-h)^{p-1}}\int_{B_\sigma} H_0((u-h)_+)\,dx. \qquad (6.24)
\end{aligned}$$

We then choose a cut-off function $\phi \in C_0^\infty(B_{(\rho+\sigma)/2})$ satisfying $0 \le \phi \le 1$, $\phi \equiv 1$ in B_ρ and $|D\phi| \le 4/(\sigma-\rho)$. Denoting the tail by

$$T(v;r) := \int_{\mathbb{R}^n \setminus B_r} \frac{|v(x)|^{p-1}}{|x-x_0|^{n+sp}} + \|a\|_{L^\infty} \frac{|v(x)|^{q-1}}{|x-x_0|^{n+sp}}\,dx,$$

Lemma 6.4.2 gives

$$\begin{aligned}
&\fint_{B_\rho}\int_{B_\rho} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dxdy \\
&\le \frac{c}{(\sigma-\rho)^p}\fint_{B_\sigma}(u(x)-h)_+^p\int_{B_\sigma}\frac{1}{|x-y|^{n+(s-1)p}}\,dydx \\
&\quad + \frac{c\|a\|_{L^\infty}}{(\sigma-\rho)^q}\fint_{B_\sigma}(u(x)-h)_+^q\int_{B_\sigma}\frac{1}{|x-y|^{n+(t-1)q}}\,dydx \\
&\quad + c\left(\sup_{x\in\operatorname{supp}\phi}\int_{\mathbb{R}^n\setminus B_\sigma}\frac{(u(y)-k)_+^{p-1}}{|x-y|^{n+sp}} + \|a\|_{L^\infty}\frac{(u(y)-k)_+^{q-1}}{|x-y|^{n+tq}}\,dy\right)\fint_{B_\sigma}(u-k)_+\,dx \\
&\le \frac{c\rho^{(1-s)p}}{(\sigma-\rho)^p}\fint_{B_\sigma}(u-h)_+^p\,dx + \frac{c\|a\|_{L^\infty}\rho^{(1-t)q}}{(\sigma-\rho)^q}\fint_{B_\sigma}(u-h)_+^q\,dx \\
&\quad + c\left(\frac{\sigma+\rho}{\sigma-\rho}\right)^{n+tq}[T((u-k)_+;\sigma)]\fint_{B_\sigma}(u-k)_+\,dx \\
&\le \frac{cr^{(1-t)p}}{(\sigma-\rho)^q}\fint_{B_\sigma}H_0((u-h)_+)\,dx + \frac{cr^{n+tq}}{(\sigma-\rho)^{n+tq}}[T((u-k)_+;\sigma)]\fint_{B_\sigma}(u-k)_+\,dx,
\end{aligned}$$

where we have used the fact that

$$\frac{|y-x_0|}{|y-x|} \le 1 + \frac{|x-x_0|}{|y-x|} \le 1 + \frac{\sigma+\rho}{\sigma-\rho} \le 2\frac{\sigma+\rho}{\sigma-\rho}$$

for $x \in \operatorname{supp}\phi$ and $y \in \mathbb{R}^n \setminus B_\sigma$. Combining this estimate together with (6.22)-(6.24) implies

$$\begin{aligned}
&\rho^{-sp}\fint_{B_\rho}H_0((u-k)_+)\,dx \\
&\le c\rho^{(s-t)q}\frac{r^{(1-t)q}}{(\sigma-\rho)^{q^2/p}}\left(\fint_{B_\sigma}H_0((u-h)_+)\,dx\right)^{\frac{q}{p}}
\end{aligned}$$

$$
\begin{aligned}
&+ c\rho^{(s-t)q} \frac{r^{(n+tq)q/p}}{(\sigma-\rho)^{(n+tq)q/p}} \frac{[T((u-k)_+;\sigma)]^{q/p}}{(k-h)^{q/p'}} \left(\fint_{B_\sigma} H_0((u-h)_+)\,dx \right)^{\frac{q}{p}} \\
&+ \frac{c}{(k-h)^{sp^2/n}} \frac{r^{(1-t)p}}{(\sigma-\rho)^q} \left(\fint_{B_\sigma} H_0((u-h)_+)\,dx \right)^{1+\frac{sp}{n}} \\
&+ \frac{cr^{n+tq}}{(\sigma-\rho)^{n+tq}} \frac{[T((u-k)_+;\sigma)]}{(k-h)^{sp^2/n+p-1}} \left(\fint_{B_\sigma} H_0((u-h)_+)\,dx \right)^{1+\frac{sp}{n}} \\
&+ \frac{cr^{-tq}}{(k-h)^{p(p-1)}} \left(\fint_{B_\sigma} H_0((u-h)_+)\,dx \right)^{p}.
\end{aligned}
$$

Now, for $i = 0, 1, 2, \ldots$ and $k_0 > 1$, we write

$$
\sigma_i \coloneqq \frac{r}{2}(1+2^{-i}), \quad k_i \coloneqq 2k_0(1-2^{-i-1}) \ \text{ and } \ y_i \coloneqq \int_{A^+(k_i,\sigma_i)} H_0((u-k_i)_+)\,dx.
$$

Since $H_0(u) \in L^1(\Omega)$ from the assumption (6.11), we see that

$$
y_0 = \int_{A^+(k_0,r)} H_0((u-k_0)_+)\,dx \quad \longrightarrow \quad 0 \quad \text{as} \quad k_0 \ \to \ \infty.
$$

First, we consider $k_0 > 1$ so large that

$$
y_i \le y_{i-1} \le \cdots \le y_0 \le 1, \quad i = 1, 2, \ldots.
$$

Then, since

$$
T((u-k_i)_+;\sigma_i) \le T(u;r/2) < \infty,
$$

we have

$$
\begin{aligned}
y_{i+1} &\le \tilde{c} \Big[2^{iq^2/p} y_i^{q/p} + 2^{i[(n+tq)q/p+q/p']} y_i^{q/p} \\
&\qquad + 2^{i(sp^2/n+q)} y_i^{1+sp/n} + 2^{i(n+tq+sp^2/n+p-1)} y_i^{1+sp/n} + 2^{ip(p-1)} y_i^{p} \Big] \\
&\le \tilde{c} 2^{\theta i} y_i^{1+\beta}
\end{aligned}
$$

for some constant $\tilde{c} > 0$ depending on `data`, r and $T(u;r/2)$, where

$$
\theta = \frac{(n+p+q)q}{p} + p^2, \qquad \beta = \min\left\{ \frac{q}{p} - 1, \frac{sp}{n}, p-1 \right\}.
$$

Finally, we can choose k_0 so large that

$$y_0 \le \tilde{c}^{-1/\beta} 2^{-\theta/\beta^2}$$

holds. Then Lemma 6.2.5 implies

$$y_\infty = \int_{A^+(2k_0, r/2)} H_0((u-2k_0)_+)\,dx = 0,$$

which means that $u \le 2k_0$ a.e. in $B_{r/2}$.

Applying the same argument to $-u$, we finally obtain $u \in L^\infty(B_{r/2})$. □

6.5 Hölder continuity

Throughout this section, we assume that the modulating coefficient $a(\cdot,\cdot)$ satisfies (6.8)-(6.9) with α satisfying (6.12), and that a weak solution $u \in \mathcal{A}(\Omega) \cap L^{q-1}_{sp}(\mathbb{R}^n)$ under consideration is locally bounded in Ω. We fix any $\Omega' \Subset \Omega$ and define

$$M \equiv M(\Omega') := 1 + \|u\|^{q-p}_{L^\infty(\Omega')}. \tag{6.25}$$

6.5.1 Logarithmic estimates

We first obtain a logarithmic type estimate. This implies Corollary 6.5.2, which will play a crucial role in the proof of Hölder continuity.

Lemma 6.5.1. *Let $K_{sp}, K_{tq}, a : \mathbb{R}^n \to \mathbb{R}$ be symmetric and satisfy* (6.6)-(6.9) *with α satisfying* (6.12)*. Let $u \in \mathcal{A}(\Omega) \cap L^{q-1}_{sp}(\mathbb{R}^n)$ be a weak supersolution to* (6.1) *such that $u \in L^\infty(\Omega')$ and $u \ge 0$ in a ball $B_R \equiv B_R(x_0) \subset \Omega'$ with $R < 1$. Then the following estimate holds true for any $0 < \rho < R/2$ and $d > 0$:*

$$\begin{aligned}
&\int_{B_\rho}\int_{B_\rho} \left|\log\left(\frac{u(x)+d}{u(y)+d}\right)\right| \frac{dydx}{|x-y|^n} \\
&\le c\widetilde{M}^2 \Bigg(\rho^n + \rho^{n+sp} d^{1-p} \int_{\mathbb{R}^n\setminus B_R} \frac{u_-^{p-1}(y) + u_-^{q-1}(y)}{|y-x_0|^{n+sp}}\,dy \\
&\qquad\qquad + \rho^{n+tq} d^{1-q} \int_{\mathbb{R}^n\setminus B_R(x_0)} \frac{u_-^{q-1}(y)}{|y-x_0|^{n+tq}}\,dy\Bigg)
\end{aligned}$$

for some $c \equiv c(\texttt{data}_1)$*, where* $\widetilde{M} \equiv \widetilde{M}(\Omega') \coloneqq 1 + (\|u\|_{L^\infty(\Omega')} + d)^{q-p}$*.*

Proof. We recall (6.13), (6.18) and (6.20), and further denote

$$\widetilde{H}(x,y,\tau) \coloneqq \frac{\tau^p}{\rho^{sp}} + a(x,y)\frac{\tau^q}{\rho^{tq}}, \quad \widetilde{h}(x,y,\tau) \coloneqq \frac{\tau^{p-1}}{\rho^{sp}} + a(x,y)\frac{\tau^{q-1}}{\rho^{tq}},$$
$$G(\tau) \coloneqq \frac{\tau^p}{\rho^{sp}} + a_2\frac{\tau^q}{\rho^{tq}}, \quad g(\tau) \coloneqq \frac{\tau^{p-1}}{\rho^{sp}} + a_2\frac{\tau^{q-1}}{\rho^{tq}},$$

where $\tau \geq 0$ and

$$a_2 \coloneqq \sup_{B_{2\rho}\times B_{2\rho}} a(\cdot,\cdot).$$

Let $\phi \in C_0^\infty(B_{3\rho/2})$ be a cut-off function satisfying $0 \leq \phi \leq 1$, $\phi \equiv 1$ in B_ρ and $|D\phi| \leq 4/\rho$. Testing (6.10) with $\varphi(x) = \phi^q(x)/g(u(x)+d)$, we have

$$\begin{aligned}
0 \leq & \int_{B_{2\rho}}\int_{B_{2\rho}} [\Phi_p(u(x)-u(y))K_{sp}(x,y) + a(x,y)\Phi_q(u(x)-u(y))K_{tq}(x,y)] \\
& \qquad\qquad \cdot \left(\frac{\phi^q(x)}{g(u(x)+d)} - \frac{\phi^q(y)}{g(u(y)+d)}\right) dxdy \\
& + 2\int_{\mathbb{R}^n\setminus B_{2\rho}}\int_{B_{2\rho}} [\Phi_p(u(x)-u(y))K_{sp}(x,y) \\
& \qquad\qquad + a(x,y)\Phi_q(u(x)-u(y))K_{tq}(x,y)]\,\frac{\phi^q(x)}{g(u(x)+d)}\,dxdy \\
& \eqqcolon I_1 + I_2.
\end{aligned}$$

Moreover in I_1, we denote by $F(x,y)$ the integrand with respect to the measure $dxdy/|x-y|^n$, that is,

$$I_1 = \int_{B_{2\rho}}\int_{B_{2\rho}} F(x,y)\,\frac{dxdy}{|x-y|^n},$$

$$\begin{aligned}
F(x,y) \coloneqq & [\Phi_p(u(x)-u(y))K_{sp}(x,y) + a(x,y)\Phi_q(u(x)-u(y))K_{tq}(x,y)] \\
& \cdot |x-y|^n\left(\frac{\phi^q(x)}{g(u(x)+d)} - \frac{\phi^q(y)}{g(u(y)+d)}\right).
\end{aligned}$$

We also denote $\bar{u}(x) \coloneqq u(x)+d$. Next, we estimate I_1 and I_2 separately. The remaining part of the proof is divided into four steps.

Step 1: Estimate of $F(x,y)$ when $\bar{u}(x) \geq \bar{u}(y) \geq \bar{u}(x)/2$. We observe that

$$\begin{aligned}
&\frac{\phi^q(x)}{g(\bar{u}(x))} - \frac{\phi^q(y)}{g(\bar{u}(y))} \\
&= \frac{\phi^q(x) - \phi^q(y)}{g(\bar{u}(y))} + \phi^q(x)\left(\frac{1}{g(\bar{u}(x))} - \frac{1}{g(\bar{u}(y))}\right) \\
&\leq \frac{q\phi^{q-1}(x)|\phi(x) - \phi(y)|}{g(\bar{u}(y))} + \phi^q(x)\int_0^1 \frac{d}{d\sigma}\left(\frac{1}{g(\sigma\bar{u}(x) + (1-\sigma)\bar{u}(y))}\right) d\sigma.
\end{aligned}$$

To estimate the last integral, note that

$$\frac{d}{d\sigma}\left(\frac{1}{g(\sigma\bar{u}(x) + (1-\sigma)\bar{u}(y))}\right) = -\frac{g'(\sigma\bar{u}(x) + (1-\sigma)\bar{u}(y))}{g^2(\sigma\bar{u}(x) + (1-\sigma)\bar{u}(y))}(\bar{u}(x) - \bar{u}(y)),$$

where a direct calculation shows

$$\frac{g'(\tau)}{g^2(\tau)} = \frac{(p-1)\dfrac{\tau^{p-2}}{\rho^{sp}} + (q-1)a_2\dfrac{\tau^{q-2}}{\rho^{tq}}}{\left(\dfrac{\tau^{p-1}}{\rho^{sp}} + a_2\dfrac{\tau^{q-1}}{\rho^{tq}}\right)^2}, \quad \text{hence } \frac{p-1}{G(\tau)} \leq \frac{g'(\tau)}{g^2(\tau)} \leq \frac{q-1}{G(\tau)}.$$

Thus we have

$$\begin{aligned}
\frac{\phi^q(x)}{g(\bar{u}(x))} - \frac{\phi^q(y)}{g(\bar{u}(y))} &\leq \frac{q\phi^{q-1}(x)|\phi(x) - \phi(y)|}{g(\bar{u}(y))} - (p-1)\frac{\phi^q(x)(\bar{u}(x) - \bar{u}(y))}{G(\bar{u}(x))} \\
&\leq \frac{q\phi^{q-1}(x)|\phi(x) - \phi(y)|}{g(\bar{u}(y))} - \frac{p-1}{2^q}\frac{\phi^q(x)(\bar{u}(x) - \bar{u}(y))}{G(\bar{u}(y))}.
\end{aligned}$$

Applying this inequality to $F(x,y)$ and using (6.6), we have

$$\begin{aligned}
F(x,y) \leq \Lambda q &\frac{h(x,y,\bar{u}(x) - \bar{u}(y))\phi^{q-1}(x)|\phi(x) - \phi(y)|\bar{u}(y)}{G(\bar{u}(y))} \\
&- \frac{p-1}{2^q\Lambda}\frac{H(x,y,\bar{u}(x) - \bar{u}(y))\phi^q(x)}{G(\bar{u}(y))}. \qquad (6.26)
\end{aligned}$$

Let us now estimate the first term in the right-hand side of (6.26). Applying Young's inequality to the numerator, for any small $\varepsilon > 0$ we obtain

$$h(x,y,\bar{u}(x) - \bar{u}(y))\phi^{q-1}(x)|\phi(x) - \phi(y)|\bar{u}(y)$$

$$
\begin{aligned}
&\leq \frac{\varepsilon(\bar{u}(x)-\bar{u}(y))^p \phi^{(q-1)p'}(x) + c_\varepsilon |\phi(x)-\phi(y)|^p \bar{u}^p(y)}{|x-y|^{sp}} \\
&\quad + a(x,y)\frac{\varepsilon(\bar{u}(x)-\bar{u}(y))^q \phi^q(x) + c_\varepsilon |\phi(x)-\phi(y)|^q \bar{u}^q(y)}{|x-y|^{tq}} \\
&\leq \varepsilon \phi^q(x) H(x,y,\bar{u}(x)-\bar{u}(y)) \\
&\quad + c_\varepsilon \left(\frac{|\phi(x)-\phi(y)|^p \rho^{sp}}{|x-y|^{sp}} \frac{\bar{u}^p(y)}{\rho^{sp}} + a_2 \frac{|\phi(x)-\phi(y)|^q \rho^{tq}}{|x-y|^{tq}} \frac{\bar{u}^q(y)}{\rho^{tq}} \right),
\end{aligned}
$$

where for the last inequality we have used the fact that $x, y \in B_{2\rho}$. It then follows that

$$
\begin{aligned}
&\frac{h(x,y,\bar{u}(x)-\bar{u}(y))\phi^{q-1}(x)|\phi(x)-\phi(y)|\bar{u}(y)}{G(\bar{u}(y))} \\
&\leq \varepsilon \phi^q(x) \frac{H(x,y,\bar{u}(x)-\bar{u}(y))}{G(\bar{u}(y))} \\
&\quad + c_\varepsilon \left(\frac{\rho^{sp}}{|x-y|^{sp}} |\phi(x)-\phi(y)|^p + \frac{\rho^{tq}}{|x-y|^{tq}} |\phi(x)-\phi(y)|^q \right)
\end{aligned}
$$

for any small $\varepsilon > 0$. Putting this into (6.26) and choosing

$$
\varepsilon = \frac{p-1}{2^{q+1} q \Lambda^2},
$$

we have

$$
F(x,y) \leq c\frac{|x-y|^{(1-s)p}}{\rho^{(1-s)p}} + c\frac{|x-y|^{(1-t)q}}{\rho^{(1-t)q}} - \frac{p-1}{2^{q+1}\Lambda} \frac{\phi^q(x) H(x,y,\bar{u}(x)-\bar{u}(y))}{G(\bar{u}(y))}.
$$

In order to estimate the last term in the above display, we note that

$$
a_2 = a_2 - a(x,y) + a(x,y) \leq [a]_\alpha 8^\alpha \rho^\alpha + a(x,y), \quad x, y \in B_{2\rho},
$$

to discover

$$
\begin{aligned}
G(\bar{u}(y)) &= \frac{\bar{u}^p(y)}{\rho^{sp}} + a_2 \frac{\bar{u}^q(y)}{\rho^{tq}} \\
&\leq \frac{\bar{u}^p(y)}{\rho^{sp}} + [a]_\alpha 8^\alpha \rho^{\alpha - tq + sp} \|\bar{u}\|_{L^\infty(\Omega')}^{q-p} \frac{\bar{u}^p(y)}{\rho^{sp}} + a(x,y)\frac{\bar{u}^q(y)}{\rho^{tq}} \\
&\leq (1 + 8^\alpha [a]_\alpha) \left(1 + (\|u\|_{L^\infty(\Omega')} + d)^{q-p}\right) \widetilde{H}(x,y,\bar{u}(y)),
\end{aligned} \tag{6.27}
$$

where we have used the inequality in (6.12) with $\rho \le 1$. Then it follows that

$$-\frac{\phi^q(x)H(x,y,\bar{u}(x)-\bar{u}(y))}{G(\bar{u}(y))} \le -\frac{1}{c\widetilde{M}}\frac{\phi^q(x)H(x,y,\bar{u}(x)-\bar{u}(y))}{\widetilde{H}(x,y,\bar{u}(y))}$$

and therefore

$$F(x,y) \le c\frac{|x-y|^{(1-s)p}}{\rho^{(1-s)p}} + c\frac{|x-y|^{(1-t)q}}{\rho^{(1-t)q}} - \frac{1}{c\widetilde{M}}\frac{\phi^q(x)H(x,y,\bar{u}(x)-\bar{u}(y))}{\widetilde{H}(x,y,\bar{u}(y))}. \tag{6.28}$$

We now need to derive an estimate for $\log \bar{u}$. Observe that

$$\begin{aligned}\log \bar{u}(x) - \log \bar{u}(y) &= \int_0^1 \frac{\bar{u}(x)-\bar{u}(y)}{\sigma\bar{u}(x)+(1-\sigma)\bar{u}(y)}\,d\sigma \\ &\le \frac{\bar{u}(x)-\bar{u}(y)}{\bar{u}(y)} = \frac{\dfrac{\bar{u}(x)-\bar{u}(y)}{|x-y|^s}}{\dfrac{\bar{u}(y)}{\rho^s}}\frac{|x-y|^s}{\rho^s}\end{aligned}$$

and that the function

$$\tau \mapsto \frac{\tau^p + a(x,y)\tau^q|x-y|^{-(t-s)q}}{\tau}, \quad \tau \ge 0,$$

is monotone increasing. We thus obtain

$$\begin{aligned}&\log \bar{u}(x) - \log \bar{u}(y) \\ &\le \left[\frac{\left(\dfrac{\bar{u}(x)-\bar{u}(y)}{|x-y|^s}\right)^p + a(x,y)\left(\dfrac{\bar{u}(x)-\bar{u}(y)}{|x-y|^s}\right)^q\dfrac{1}{|x-y|^{(t-s)q}}}{\left(\dfrac{\bar{u}(y)}{\rho^s}\right)^p + a(x,y)\left(\dfrac{\bar{u}(y)}{\rho^s}\right)^q\dfrac{1}{|x-y|^{(t-s)q}}} + 1\right]\frac{|x-y|^s}{\rho^s} \\ &\le c\frac{H(x,y,\bar{u}(x)-\bar{u}(y))}{\widetilde{H}(x,y,\bar{u}(y))} + \frac{|x-y|^s}{\rho^s}.\end{aligned}$$

For the last inequality, we have used the fact that $|x-y| \le 2\rho$. Finally, inserting this into (6.28), we obtain

$$F(x,y) \le c\frac{|x-y|^{(1-s)p}}{\rho^{(1-s)p}} + c\frac{|x-y|^{(1-t)q}}{\rho^{(1-t)q}} + c\frac{|x-y|^s}{\rho^s} - \frac{\phi^q(x)}{c\widetilde{M}}\log\left(\frac{\bar{u}(x)}{\bar{u}(y)}\right).$$

Step 2: Estimate of $F(x,y)$ when $\bar{u}(x) \geq 2\bar{u}(y)$. We first observe from the second inequality in Lemma 6.2.4 with $\varepsilon = (2^{p-1}-1)/2^p$ that

$$\begin{aligned}
\frac{\phi^q(x)}{g(\bar{u}(x))} - \frac{\phi^q(y)}{g(\bar{u}(y))} &= \frac{\phi^q(x)-\phi^q(y)}{g(\bar{u}(x))} + \phi^q(y)\left(\frac{1}{g(\bar{u}(x))} - \frac{1}{g(\bar{u}(y))}\right) \\
&\leq \frac{\phi^q(x)-\phi^q(y)}{g(\bar{u}(x))} + \phi^q(y)\left(\frac{1}{g(2\bar{u}(y))} - \frac{1}{g(\bar{u}(y))}\right) \\
&\leq \frac{\phi^q(x)-\phi^q(y)}{g(\bar{u}(x))} - \left(1 - \frac{1}{2^{p-1}}\right)\frac{\phi^q(y)}{g(\bar{u}(y))} \\
&\leq \frac{\varepsilon\phi^q(y) + c_\varepsilon|\phi(x)-\phi(y)|^q}{g(\bar{u}(x))} - \left(1 - \frac{1}{2^{p-1}}\right)\frac{\phi^q(y)}{g(\bar{u}(y))} \\
&\leq c\frac{|\phi(x)-\phi(y)|^q}{g(\bar{u}(x))} - \frac{2^{p-1}-1}{2^p}\frac{\phi^q(y)}{g(\bar{u}(y))}.
\end{aligned}$$

This implies

$$F(x,y) \leq c\frac{h(x,y,\bar{u}(x)-\bar{u}(y))|\phi(x)-\phi(y)|^q}{g(\bar{u}(x))} - \frac{1}{c}\frac{h(x,y,\bar{u}(x)-\bar{u}(y))\phi^q(y)}{g(\bar{u}(y))}$$

Estimating the right-hand side similarly as in (6.27), we find

$$F(x,y) \leq c\frac{h(x,y,\bar{u}(x)-\bar{u}(y))|\phi(x)-\phi(y)|^q}{g(\bar{u}(x))} - \frac{\phi^q(y)}{c\widetilde{M}}\frac{h(x,y,\bar{u}(x)-\bar{u}(y))}{\widetilde{h}(x,y,\bar{u}(y))}.$$

The first term in the right-hand side is estimated as

$$\begin{aligned}
&\frac{h(x,y,\bar{u}(x)-\bar{u}(y))|\phi(x)-\phi(y)|^q}{g(\bar{u}(x))} \\
&= \frac{\dfrac{\rho^{sp}}{|x-y|^{sp}}\dfrac{|\bar{u}(x)-\bar{u}(y)|^{p-1}}{\rho^{sp}} + a(x,y)\dfrac{\rho^{tq}}{|x-y|^{tq}}\dfrac{|\bar{u}(x)-\bar{u}(y)|^{q-1}}{\rho^{tq}}}{\dfrac{|\bar{u}(x)-\bar{u}(y)|^{p-1}}{\rho^{sp}} + a_2\dfrac{|\bar{u}(x)-\bar{u}(y)|^{q-1}}{\rho^{tq}}}\frac{|x-y|^q}{\rho^q} \\
&\leq c\left(\frac{\rho^{sp}}{|x-y|^{sp}} + \frac{\rho^{tq}}{|x-y|^{tq}}\right)\frac{|x-y|^q}{\rho^q},
\end{aligned}$$

hence we have

$$F(x,y) \leq c\frac{|x-y|^{q-sp}}{\rho^{q-sp}} + c\frac{|x-y|^{(1-t)q}}{\rho^{(1-t)q}} - \frac{\phi^q(y)}{c\widetilde{M}}\frac{h(x,y,\bar{u}(x)-\bar{u}(y))}{\widetilde{h}(x,y,\bar{u}(y))}. \quad (6.29)$$

Furthermore, in this case we observe that

$$\begin{aligned}\log\bar{u}(x) - \log\bar{u}(y) &\leq \log\left(\frac{2(\bar{u}(x)-\bar{u}(y))}{\bar{u}(y)}\right) \leq c\left(\frac{2(\bar{u}(x)-\bar{u}(y))}{\bar{u}(y)}\right)^{p-1} \\ &\leq c\frac{\left(\frac{\bar{u}(x)-\bar{u}(y)}{|x-y|^s}\right)^{p-1}}{\left(\frac{\bar{u}(y)}{\rho^s}\right)^{p-1}}\frac{|x-y|^{s(p-1)}}{\rho^{s(p-1)}}\end{aligned}$$

and that the function

$$\tau \mapsto \frac{\tau^{p-1} + a(x,y)\tau^{q-1}|x-y|^{-(t-s)q}}{\tau^{p-1}}, \quad \tau \geq 0,$$

is monotone increasing. Thus, again using the fact that $|x-y| \leq 2\rho$, we have

$$\begin{aligned}&\log\bar{u}(x) - \log\bar{u}(y) \\ &\leq c\left[\frac{\left(\frac{\bar{u}(x)-\bar{u}(y)}{|x-y|^s}\right)^{p-1} + a(x,y)\left(\frac{\bar{u}(x)-\bar{u}(y)}{|x-y|^s}\right)^{q-1}\frac{1}{|x-y|^{(t-s)q}}}{\left(\frac{\bar{u}(y)}{\rho^s}\right)^{p-1} + a(x,y)\left(\frac{\bar{u}(y)}{\rho^s}\right)^{q-1}\frac{1}{|x-y|^{(t-s)q}}} + 1\right] \\ &\quad\cdot\frac{|x-y|^{s(p-1)}}{\rho^{s(p-1)}} \\ &\leq c\frac{h(x,y,\bar{u}(x)-\bar{u}(y))}{\widetilde{h}(x,y,\bar{u}(y))} + c\frac{|x-y|^{s(p-1)}}{\rho^{s(p-1)}}.\end{aligned}$$

Finally, inserting this into (6.29), we obtain

$$F(x,y) \leq c\frac{|x-y|^{q-sp}}{\rho^{q-sp}} + c\frac{|x-y|^{(1-t)q}}{\rho^{(1-t)q}} + c\frac{|x-y|^{s(p-1)}}{\rho^{s(p-1)}} - \frac{\phi^q(y)}{c\widetilde{M}}\log\left(\frac{\bar{u}(x)}{\bar{u}(y)}\right).$$

Step 3: Estimate of I_1. From *Step 1* and *Step 2*, we have that

$$F(x,y) \le c\frac{|x-y|^{(1-s)p}}{\rho^{(1-s)p}} + c\frac{|x-y|^{(1-t)q}}{\rho^{(1-t)q}} + c\frac{|x-y|^{s}}{\rho^{s}} + c\frac{|x-y|^{s(p-1)}}{\rho^{s(p-1)}} \\ - \frac{(\min\{\phi(x),\phi(y)\})^q}{c\widetilde{M}}\left|\log\left(\frac{\bar{u}(x)}{\bar{u}(y)}\right)\right|,$$

when $\bar{u}(x) \ge \bar{u}(y)$. Moreover, by the symmetry of the above estimate for x and y, the same estimate still holds when $\bar{u}(x) < \bar{u}(y)$. Therefore, I_1 is finally estimated as follows:

$$\begin{aligned} I_1 &\le -\frac{1}{c\widetilde{M}}\int_{B_\rho}\int_{B_\rho}\left|\log\left(\frac{\bar{u}(x)}{\bar{u}(y)}\right)\right|\frac{dydx}{|x-y|^n} \\ &\quad + c\int_{B_{2\rho}}\int_{B_{4\rho}(x)}\left(\frac{|x-y|^{(1-s)p}}{\rho^{(1-s)p}} + \frac{|x-y|^{(1-t)q}}{\rho^{(1-t)q}}\right. \\ &\qquad\qquad \left. + \frac{|x-y|^{s}}{\rho^{s}} + \frac{|x-y|^{s(p-1)}}{\rho^{s(p-1)}}\right)\frac{dydx}{|x-y|^n} \\ &\le -\frac{1}{c\widetilde{M}}\int_{B_\rho}\int_{B_\rho}\left|\log\left(\frac{\bar{u}(x)}{\bar{u}(y)}\right)\right|\frac{dydx}{|x-y|^n} + c\rho^n. \end{aligned} \tag{6.30}$$

Step 4: Estimate of I_2 and Conclusion. We start with the following observation:

(i) If $y \in B_R \setminus B_{2\rho}$, then $u(y) \ge 0$ and $u(x) - u(y) \le u(x)$;

(ii) If $y \in \mathbb{R}^n \setminus B_R$, then $(u(x)-u(y))_+ \le (u(x)+u_-(y))_+ = u(x)+u_-(y)$.

Using this and the fact that $\operatorname{supp}\phi \subset B_{3\rho/2}$, we write

$$\begin{aligned} I_2 &\le 2\int_{B_{3\rho/2}}\int_{\mathbb{R}^n\setminus B_{2\rho}}\frac{h(x,y,u(x)+d)}{g(u(x)+d)}\frac{dydx}{|x-y|^n} \\ &\quad + 2\int_{B_{3\rho/2}}\int_{\mathbb{R}^n\setminus B_R}\frac{h(x,y,u_-(y))}{g(u(x)+d)}\frac{dydx}{|x-y|^n}. \end{aligned} \tag{6.31}$$

Since we are considering integrals over the complement of balls, we cannot directly compare a_2 and $a(x,y)$ there. In order to overcome this difficulty, we observe that (6.8) and (6.12) imply

$$a(x,y) \le a(x,y) - a(x,x) + a_2$$

$$\begin{aligned}&\leq |a(x,y)-a(x,x)|^{\frac{tq-sp}{\alpha}}(2\|a\|_{L^\infty})^{1-\frac{tq-sp}{\alpha}}+a_2\\&\leq c|x-y|^{tq-sp}+a_2,\end{aligned}\tag{6.32}$$

whenever $x \in B_{2\rho}$ and $y \in \mathbb{R}^n$.

For the first integral in (6.31), we use (6.32) and the fact that $|x-y| > \rho/2$ for $x \in B_{3\rho/2}$ and $y \in \mathbb{R}^n \setminus B_{2\rho}$ to find

$$\begin{aligned}\frac{h(x,y,u(x)+d)}{g(u(x)+d)} &= \frac{\dfrac{\bar{u}^{p-1}(x)}{|x-y|^{sp}}+a(x,y)\dfrac{\bar{u}^{q-1}(x)}{|x-y|^{tq}}}{\dfrac{\bar{u}^{p-1}(x)}{\rho^{sp}}+a_2\dfrac{\bar{u}^{q-1}(x)}{\rho^{tq}}}\\&\leq c\frac{\dfrac{\bar{u}^{p-1}(x)+\bar{u}^{q-1}(x)}{|x-y|^{sp}}+a_2\dfrac{\bar{u}^{q-1}(x)}{|x-y|^{tq}}}{\dfrac{\bar{u}^{p-1}(x)}{\rho^{sp}}+a_2\dfrac{\bar{u}^{q-1}(x)}{\rho^{tq}}} \leq c\widetilde{M}\frac{\rho^{sp}}{|x-y|^{sp}},\end{aligned}$$

which gives

$$\int_{B_{3\rho/2}}\int_{\mathbb{R}^n\setminus B_{2\rho}}\frac{h(x,y,u(x)+d)}{g(u(x)+d)}\frac{dydx}{|x-y|^n} \leq c\widetilde{M}\rho^n. \tag{6.33}$$

For the second integral in (6.31), we use (6.32) and the fact that

$$\frac{|y-x_0|}{|y-x|} \leq 1+\frac{|x-x_0|}{|y-x|} \leq 1+\frac{3\rho/2}{\rho/2} = 4$$

for $x \in B_{3\rho/2}$ and $y \in \mathbb{R}^n \setminus B_{2\rho}$ to find

$$\begin{aligned}\frac{h(x,y,u_-(y))}{g(u(x)+d)} &\leq \frac{\dfrac{u_-^{p-1}(y)}{|x-y|^{sp}}+a(x,y)\dfrac{u_-^{q-1}(y)}{|x-y|^{tq}}}{\dfrac{d^{p-1}}{\rho^{sp}}+a_2\dfrac{d^{q-1}}{\rho^{tq}}}\\&\leq c\frac{\dfrac{u_-^{p-1}(y)+u_-^{q-1}(y)}{|x-y|^{sp}}+a_2\dfrac{u_-^{q-1}(y)}{|x-y|^{tq}}}{\dfrac{d^{p-1}}{\rho^{sp}}+a_2\dfrac{d^{q-1}}{\rho^{tq}}}\end{aligned}$$

$$\leq c\rho^{sp} d^{1-p} \frac{u_-^{p-1}(y) + u_-^{q-1}(y)}{|y-x_0|^{sp}} + c\rho^{tq} d^{1-q} \frac{u_-^{q-1}(y)}{|y-x_0|^{tq}}.$$

Consequently, we obtain

$$\begin{aligned} &\int_{B_{3\rho/2}} \int_{\mathbb{R}^n \setminus B_R} \frac{h(x,y,u_-(y))}{g(u(x)+d)} \frac{dydx}{|x-y|^n} \\ &\leq c\rho^{n+sp} d^{1-p} \int_{\mathbb{R}^n \setminus B_R} \frac{u_-^{p-1}(y) + u_-^{q-1}(y)}{|y-x_0|^{n+sp}}\, dy \\ &\quad + c\rho^{n+tq} d^{1-q} \int_{\mathbb{R}^n \setminus B_R} \frac{u_-^{q-1}(y)}{|y-x_0|^{n+tq}}\, dy. \end{aligned} \tag{6.34}$$

Combining (6.30), (6.31), (6.33) and (6.34), the desired estimate follows. □

The preceding lemma directly implies the following corollary.

Corollary 6.5.2. *Under the assumptions of Lemma 6.5.1, let $d, \zeta > 0$, $\xi > 1$ and define*

$$v := \min\{(\log(\zeta + d) - \log(u+d))_+, \log \xi\}.$$

Then we have

$$\begin{aligned} \fint_{B_\rho} |v - (v)_{B_\rho}|\, dx \leq c\widetilde{M}^2 \Bigg(1 + \rho^{sp} d^{1-p} \int_{\mathbb{R}^n \setminus B_R} \frac{u_-^{p-1}(y) + u_-^{q-1}(y)}{|y-x_0|^{n+sp}}\, dy \\ + \rho^{tq} d^{1-q} \int_{\mathbb{R}^n \setminus B_R} \frac{u_-^{q-1}(y)}{|y-x_0|^{n+tq}}\, dy \Bigg) \end{aligned} \tag{6.35}$$

for some $c \equiv c(\mathtt{data}_1)$, where $\widetilde{M} = 1 + (\|u\|_{L^\infty(\Omega')} + d)^{q-p}$.

Proof. It suffices to observe that

$$\begin{aligned} \fint_{B_\rho} |v - (v)_{B_\rho}|\, dx &\leq \fint_{B_\rho} \fint_{B_\rho} |v(x) - v(y)|\, dydx \\ &\leq c\rho^{-n} \int_{B_\rho} \int_{B_\rho} \frac{|\log(u(x)+d) - \log(u(y)+d)|}{|x-y|^n}\, dydx, \end{aligned}$$

as v is a truncation of $\log(u+d)$. Now Lemma 6.5.1 gives the desired result. □

6.5.2 Proof of Theorem 6.1.2

We are now in a position to prove Theorem 6.1.2. We first recall that $\Omega' \Subset \Omega$ has been fixed in the beginning of the section and the constant M was defined in (6.25). We then fix a ball $B_{2r} \equiv B_{2r}(x_0) \subset \Omega'$. Let $\sigma \in (0, 1/4]$ be a constant depending only on $\mathtt{data}_1$ and $\|u\|_{L^\infty(\Omega')}$ that satisfies

$$\sigma \le \min\left\{\frac{1}{4}, 2^{-\frac{2}{sp}}, 6^{-\frac{4(q-1)}{sq}}, \exp\left(-\frac{c_* M^3}{\nu_*}\right)\right\}, \tag{6.36}$$

where the constants $c_* \equiv c_*(\mathtt{data}_1) \ge 1$ and $\nu_* \equiv \nu_*(\mathtt{data}_1, \|u\|_{L^\infty(\Omega')}) \in (0,1)$ are to be determined in (6.52) and (6.58), respectively, and then choose $\gamma \in (0,1)$ depending only on $\mathtt{data}_1$ and $\|u\|_{L^\infty(\Omega')}$ satisfying

$$\gamma \le \min\left\{\log_\sigma\left(\frac{1}{2}\right), \frac{sp}{2(p-1)}, \frac{tq}{2(q-1)}, \log_\sigma\left(1 - \sigma^{\frac{sq}{2(q-1)}}\right)\right\}. \tag{6.37}$$

We define

$$\begin{aligned} \frac{1}{2}K_0 \coloneqq \sup_{B_r}|u| &+ \left[r^{sp}\int_{\mathbb{R}^n\setminus B_r}\frac{|u(x)|^{p-1}+|u(x)|^{q-1}}{|x-x_0|^{n+sp}}\,dx\right]^{\frac{1}{p-1}} \\ &+ \left[r^{tq}\int_{\mathbb{R}^n\setminus B_r}\frac{|u(x)|^{q-1}}{|x-x_0|^{n+tq}}\,dx\right]^{\frac{1}{q-1}} \end{aligned} \tag{6.38}$$

and, for $j \in \mathbb{N}\cup\{0\}$, we write

$$r_j \coloneqq \sigma^j r, \quad B_j \coloneqq B_{r_j}(x_0) \quad \text{and} \quad K_j \coloneqq \sigma^{\gamma j}K_0.$$

Now, we are going to prove the following oscillation lemma, which implies $u \in C^{0,\gamma}(B_r)$.

Lemma 6.5.3. *Under the assumptions of Theorem 6.1.2, let u be a weak solution to* (6.1)*. Then we have for every $j \in \mathbb{N}\cup\{0\}$*

$$\omega(r_j) \coloneqq \operatorname*{osc}_{B_j} u \le K_j. \tag{6.39}$$

Proof. Step 1: Induction. The proof goes by induction on j. For $j = 0$ it is obvious from the definition of K_0. Now we assume that (6.39) holds for all $i \in \{0, \dots, j\}$ with some $j \ge 0$ and show that it holds also for $j+1$. That is, we will show that

$$\omega(r_{j+1}) \leq K_{j+1}. \tag{6.40}$$

Without loss of generality, we assume that

$$\omega(r_{j+1}) \geq \frac{1}{2}K_{j+1}.$$

Then, this together with the fact that $\sigma^\gamma \geq 1/2$ from (6.37) implies that

$$\omega(r_j) \geq \omega(r_{j+1}) \geq \frac{1}{2}K_{j+1} = \frac{1}{2}\sigma^\gamma K_j \geq \frac{1}{4}K_j. \tag{6.41}$$

We note that either

$$\frac{|2B_{j+1} \cap \{u \geq \inf_{B_j} u + \omega(r_j)/2\}|}{|2B_{j+1}|} \geq \frac{1}{2} \tag{6.42}$$

or

$$\frac{|2B_{j+1} \cap \{u \leq \inf_{B_j} u + \omega(r_j)/2\}|}{|2B_{j+1}|} \geq \frac{1}{2} \tag{6.43}$$

must hold. We accordingly define

$$u_j := \begin{cases} u - \inf\limits_{B_j} u & \text{if (6.42) holds,} \\ \sup\limits_{B_j} u - u & \text{if (6.43) holds.} \end{cases}$$

Then we have

$$u_j \geq 0 \ \text{ in } \ B_j \qquad \text{and} \qquad \frac{|2B_{j+1} \cap \{u_j \geq \omega(r_j)/2\}|}{|2B_{j+1}|} \geq \frac{1}{2}. \tag{6.44}$$

Moreover, u_j is a weak solution to (6.1) satisfying

$$\sup_{B_i} |u_j| \leq \omega(r_i) \leq K_i \qquad \forall\, i \in \{0, \dots, j\}. \tag{6.45}$$

Step 2: Tail estimates. We first show that

$$r_j^{sp} \int_{\mathbb{R}^n \setminus B_j} \frac{|u_j(x)|^{p-1} + |u_j(x)|^{q-1}}{|x - x_0|^{n+sp}}\, dx \leq cM\sigma^{-\gamma(p-1)} K_j^{p-1} \tag{6.46}$$

and

$$r_j^{tq}\int_{\mathbb{R}^n\setminus B_j}\frac{|u_j(x)|^{q-1}}{|x-x_0|^{n+tq}}\,dx\le c\sigma^{-\gamma(q-1)}K_j^{q-1} \tag{6.47}$$

for a constant $c\equiv c(\mathtt{data}_1)$. We will only give the proof of (6.46), since (6.47) can be proved in almost the same way with s and p replaced by t and q, respectively. From (6.45), (6.38) and (6.25), we have

$$\begin{aligned}
& r_j^{sp}\int_{\mathbb{R}^n\setminus B_j}\frac{|u_j(x)|^{p-1}+|u_j(x)|^{q-1}}{|x-x_0|^{n+sp}}\,dx \\
&= r_j^{sp}\sum_{i=1}^{j}\int_{B_{i-1}\setminus B_i}\frac{|u_j(x)|^{p-1}+|u_j(x)|^{q-1}}{|x-x_0|^{n+sp}}\,dx \\
&\quad + r_j^{sp}\int_{\mathbb{R}^n\setminus B_0}\frac{|u_j(x)|^{p-1}+|u_j(x)|^{q-1}}{|x-x_0|^{n+sp}}\,dx \\
&\le \sum_{i=1}^{j}\left(\frac{r_j}{r_i}\right)^{sp}\left[\left(\sup_{B_{i-1}}|u_j|\right)^{p-1}+\left(\sup_{B_{i-1}}|u_j|\right)^{q-1}\right]+cM\left(\frac{r_j}{r_1}\right)^{sp}K_0^{p-1} \\
&\le cM\sum_{i=1}^{j}\left(\frac{r_j}{r_i}\right)^{sp}K_{i-1}^{p-1},
\end{aligned} \tag{6.48}$$

where for the first inequality we have used

$$\begin{aligned}
& r_j^{sp}\int_{\mathbb{R}^n\setminus B_0}\frac{|u_j(x)|^{p-1}+|u_j(x)|^{q-1}}{|x-x_0|^{n+sp}}\,dx \\
&\le c\left(\frac{r_j}{r_0}\right)^{sp}\left[\left(\sup_{B_0}|u|\right)^{p-1}+\left(\sup_{B_0}|u|\right)^{q-1}\right] \\
&\quad + cr_j^{sp}\int_{\mathbb{R}^n\setminus B_0}\frac{|u(x)|^{p-1}+|u(x)|^{q-1}}{|x-x_0|^{n+sp}}\,dx \\
&\le cM\left(\frac{r_j}{r_1}\right)^{sp}K_0^{p-1}.
\end{aligned}$$

Now the sum appearing in (6.48) is estimated as

$$\sum_{i=1}^{j}\left(\frac{r_j}{r_i}\right)^{sp}K_{i-1}^{p-1}=K_0^{p-1}\left(\frac{r_j}{r_0}\right)^{\gamma(p-1)}\sum_{i=1}^{j}\left(\frac{r_{i-1}}{r_i}\right)^{\gamma(p-1)}\left(\frac{r_j}{r_i}\right)^{sp-\gamma(p-1)}$$

$$
\begin{aligned}
&= K_j^{p-1}\sigma^{-\gamma(p-1)}\sum_{i=0}^{j-1}\sigma^{i(sp-\gamma(p-1))} \\
&\leq \frac{\sigma^{-\gamma(p-1)}}{1-\sigma^{sp-\gamma(p-1)}}K_j^{p-1} \leq \frac{\sigma^{-\gamma(p-1)}}{1-\sigma^{sp/2}}K_j^{p-1} \leq 2\sigma^{-\gamma(p-1)}K_j^{p-1},
\end{aligned}
$$

where we have used the facts that $sp-\gamma(p-1)\geq sp/2$ and $\sigma^{sp/2}\leq 1/2$ from (6.36).

Step 3: A density estimate. We next apply Corollary 6.5.2 to the function

$$
v := \min\left\{\left[\log\left(\frac{\omega(r_j)/2+d_j}{u_j+d_j}\right)\right]_+, k\right\},
$$

where $k>0$ is to be chosen and

$$
d_j := \varepsilon K_j \qquad \text{with} \qquad \varepsilon := \sigma^{\frac{sq}{2(q-1)}} \geq \max\left\{\sigma^{\frac{sp}{2(p-1)}}, \sigma^{\frac{tq}{2(q-1)}}\right\}. \tag{6.49}
$$

Note that by (6.41) we see that

$$
d_j \leq 4\omega(r_j) \leq 8\|u\|_{L^\infty(\Omega')}, \quad \text{hence} \quad \widetilde{M}\leq cM. \tag{6.50}
$$

Combining the resulting estimate (6.35) with (6.46)-(6.47), and then using (6.36), (6.49) and (6.50), we obtain

$$
\begin{aligned}
&\fint_{2B_{j+1}} |v-(v)_{2B_{j+1}}|\,dx \\
&\leq cM^2\left[1+Md_j^{1-p}\sigma^{sp-\gamma(p-1)}K_j^{p-1}+d_j^{1-q}\sigma^{tq-\gamma(q-1)}K_j^{q-1}\right] \\
&\leq cM^2\left[1+Md_j^{1-p}\sigma^{sp/2}K_j^{p-1}+d_j^{1-q}\sigma^{tq/2}K_j^{q-1}\right] \\
&\leq cM^3\left[1+\left(d_j^{-1}\sigma^{\frac{sp}{2(p-1)}}K_j\right)^{p-1}+\left(d_j^{-1}\sigma^{\frac{tq}{2(q-1)}}K_j\right)^{q-1}\right] \\
&\leq cM^3
\end{aligned} \tag{6.51}
$$

for a constant $c\equiv c(\mathtt{data}_1)$. In addition, we have from (6.44) that

$$
\begin{aligned}
k &= \frac{1}{|2B_{j+1}\cap\{u_j\geq\omega(r_j)/2\}|}\int_{2B_{j+1}\cap\{v=0\}}(k-v)\,dx \\
&\leq 2\fint_{2B_{j+1}}(k-v)\,dx = 2(k-(v)_{2B_{j+1}}).
\end{aligned}
$$

This inequality and (6.51) imply

$$\begin{aligned}\frac{|2B_{j+1}\cap\{v=k\}|}{|2B_{j+1}|} &\le \frac{2}{k|2B_{j+1}|}\int_{2B_{j+1}\cap\{v=k\}}(k-(v)_{2B_{j+1}})\,dx \\ &\le \frac{2}{k}\fint_{2B_{j+1}}|v-(v)_{2B_{j+1}}|\,dx \le \frac{cM^3}{k}.\end{aligned}$$

At this moment, we choose

$$\begin{aligned}k = \log\left(\frac{\omega(r_j)/2+\varepsilon\omega(r_j)}{3\varepsilon\omega(r_j)}\right) &= \log\left(\frac{1/2+\varepsilon}{3\varepsilon}\right) \\ &\ge \log\left(\frac{1}{6\varepsilon}\right) \ge \log\left(\frac{1}{\sqrt{\varepsilon}}\right) = \frac{sq}{4(q-1)}\log\frac{1}{\sigma},\end{aligned}$$

where we have used the fact that $\sqrt{\varepsilon}=\sigma^{sq/4(q-1)}\le 1/6$ from (6.36). Then it follows that

$$\frac{|2B_{j+1}\cap\{u_j\le d_j\}|}{|2B_{j+1}|}\le\frac{cM^3}{k}\le\frac{c_*M^3}{\log(1/\sigma)} \tag{6.52}$$

for a constant $c_*\equiv c_*(\texttt{data}_1)$.

Step 4: Iteration. Now we proceed with an iteration argument. For $i=0,1,2,\dots$ and for fixed j, we set

$$\rho_i=(1+2^{-i})r_{j+1},\qquad \tilde\rho_i=\frac{\rho_i+\rho_{i+1}}{2},\qquad B^i=B_{\rho_i},\qquad \tilde B^i=B_{\tilde\rho_i}$$

and choose corresponding cut-off functions satisfying

$$\phi_i\in C_0^\infty(\tilde B^i),\quad 0\le\phi_i\le 1,\quad \phi_i\equiv 1 \text{ on } B^{i+1},\quad\text{and}\quad |D\phi_i|\le 2^{i+2}r_{j+1}^{-1}.$$

Furthermore, we set

$$k_i=(1+2^{-i})d_j,\qquad w_i=(k_i-u_j)_+$$

and

$$A_i=\frac{|B^i\cap\{u_j<k_i\}|}{|B^i|}=\frac{|B^i\cap\{w_i>0\}|}{|B^i|}.$$

Note that

$$r_{j+1}<\rho_{i+1}\le\rho_i\le 2r_{j+1},\quad d_j\le k_{i+1}\le k_i\le 2d_j\quad\text{and}\quad 0\le w_i\le k_i\le 2d_j.$$

We then denote

$$a_{1,j} \coloneqq \inf_{B_{2r_{j+1}} \times B_{2r_{j+1}}} a(\cdot,\cdot), \qquad a_{2,j} \coloneqq \sup_{B_{2r_{j+1}} \times B_{2r_{j+1}}} a(\cdot,\cdot)$$

and

$$G(\tau) \coloneqq \frac{\tau^p}{r_{j+1}^{sp}} + a_{2,j}\frac{\tau^q}{r_{j+1}^{tq}}.$$

Using the fact that $r_{j+1} < \rho_{i+1} \le 2r_{j+1}$, and applying Lemma 6.2.3 with $f \equiv w_i$, we obtain

$$\begin{aligned}
&A_{i+1}^{1/\kappa} G(k_i - k_{i+1}) \\
&= \left[\frac{1}{|B^{i+1}|}\int_{B^{i+1}\cap\{u_j \le k_{i+1}\}} [G(k_i - k_{i+1})]^{\kappa}\,dx\right]^{\frac{1}{\kappa}} \\
&\le \left[\fint_{B^{i+1}} [G(w_i)]^{\kappa}\,dx\right]^{\frac{1}{\kappa}} \\
&\le cM \fint_{B^{i+1}}\int_{B^{i+1}} H(x,y,|w_i(x)-w_i(y)|)\frac{dxdy}{|x-y|^n} + cMG(d_j)A_i, \qquad (6.53)
\end{aligned}$$

where for the last inequality we have also used the following estimate:

$$\begin{aligned}
\fint_{B^i} \left|\frac{w_i}{\rho_{i+1}^s}\right|^p + a_{1,j}\left|\frac{w_i}{\rho_{i+1}^t}\right|^q dx &\le c\left[\left(\frac{d_j}{r_{j+1}^s}\right)^p + a_{2,j}\left(\frac{d_j}{r_{j+1}^t}\right)^q\right]\frac{|B^i \cap \{u_j \le k_i\}|}{|B^i|} \\
&= cG(d_j)A_i,
\end{aligned}$$

which is immediate from the definitions of w_i, ρ_i and A_i. In order to estimate the integral on the right-hand side of (6.56), we apply Lemma 6.4.2 to w_i and ϕ_i in the ball B^i. Moreover, we estimate the tail term in the right-hand side by using (6.32):

$$\begin{aligned}
&\fint_{B^{i+1}}\int_{B^{i+1}} H(x,y,|w_i(x)-w_i(y)|)\,\frac{dxdy}{|x-y|^n} \\
&\le c\fint_{B^i}\int_{B^i} H(x,y,(w_i(x)+w_i(y))|\phi_i(x)-\phi_i(y)|)\,\frac{dxdy}{|x-y|^n} \\
&\quad + c\left(\sup_{y\in\tilde{B}^i}\int_{\mathbb{R}^n\setminus B^i}\frac{w_i^{p-1}(x)}{|x-y|^{n+sp}} + a(x,y)\frac{w_i^{q-1}(x)}{|x-y|^{n+tq}}\,dx\right)\fint_{B^i} w_i\phi_i^q\,dx
\end{aligned}$$

$$\leq c \fint_{B^i} \int_{B^i} H(x, y, (w_i(x) + w_i(y))|\phi_i(x) - \phi_i(y)|) \frac{dxdy}{|x-y|^n}$$
$$+ c \left(\sup_{y \in \tilde{B}^i} \int_{\mathbb{R}^n \setminus B^i} \frac{w_i^{p-1}(x) + w_i^{q-1}(x)}{|x-y|^{n+sp}} + a_{2,j} \frac{w_i^{q-1}(x)}{|x-y|^{n+tq}} \, dx \right) \fint_{B^i} w_i \phi_i^q \, dx. \tag{6.54}$$

We estimate the terms in the right-hand side of (6.54) separately. By the definitions of w_i and ϕ_i, we have

$$\begin{aligned}
&\fint_{B^i} \int_{B^i} H(x, y, (w_i(x) + w_i(y))|\phi_i(x) - \phi_i(y)|) \frac{dxdy}{|x-y|^n} \\
&\leq c 2^{ip} r_{j+1}^{-p} k_i^p \frac{1}{|B^i|} \int_{B^i \cap \{u_j \leq k_i\}} \int_{B^i} \frac{1}{|x-y|^{n+(s-1)p}} \, dydx \\
&\quad + c 2^{iq} a_{2,j} r_{j+1}^{-q} k_i^q \frac{1}{|B^i|} \int_{B^i \cap \{u_j \leq k_i\}} \int_{B^i} \frac{1}{|x-y|^{n+(t-1)q}} \, dydx \\
&\leq c 2^{iq} \frac{|B^i \cap \{u_j \leq k_i\}|}{|B^i|} \left(\frac{d_j^p}{r_{j+1}^{sp}} + a_{2,j} \frac{d_j^q}{r_{j+1}^{tq}} \right) \\
&= c 2^{iq} G(d_j) A_i,
\end{aligned} \tag{6.55}$$

and

$$\fint_{B^i} w_i(x) \phi_i^q(x) \, dx \leq c d_j A_i. \tag{6.56}$$

As for the tail term, we first observe the following facts:

$$\frac{|x - x_0|}{|x-y|} \leq 1 + \frac{|y - x_0|}{|x-y|} \leq 1 + \frac{2r_{j+1}}{2^{-(i+1)} r_{j+1}} \leq 2^{i+3}$$

for $x \in \mathbb{R}^n \setminus B^i$ and $y \in \tilde{B}^i$; $w_i \leq k_i \leq 2d_j$ in B_j; and $w_i \leq k_i + |u_j| \leq 2d_j + |u_j|$ in $\mathbb{R}^n \setminus B_j$. Using these facts, (6.46), (6.47) and (6.49), we see that

$$\begin{aligned}
&\sup_{y \in \tilde{B}^i} \int_{\mathbb{R}^n \setminus B^i} \frac{w_i^{p-1}(x) + w_i^{q-1}(x)}{|x-y|^{n+sp}} + a_{2,j} \frac{w_i^{q-1}(x)}{|x-y|^{n+tq}} \, dx \\
&\leq c 2^{i(n+tq)} \int_{\mathbb{R}^n \setminus B_{j+1}} \frac{w_i^{p-1}(x) + w_i^{q-1}(x)}{|x-x_0|^{n+sp}} + a_{2,j} \frac{w_i^{q-1}(x)}{|x-x_0|^{n+tq}} \, dx
\end{aligned}$$

$$
\begin{aligned}
&\leq c2^{i(n+tq)} \int_{\mathbb{R}^n \setminus B_{j+1}} \frac{d_j^{p-1} + d_j^{q-1}}{|x-x_0|^{n+sp}} + a_{2,j} \frac{d_j^{q-1}}{|x-x_0|^{n+tq}} \, dx \\
&\quad + c2^{i(n+tq)} \int_{\mathbb{R}^n \setminus B_j} \frac{|u_j(x)|^{p-1} + |u_j(x)|^{q-1}}{|x-x_0|^{n+sp}} + a_{2,j} \frac{|u_j(x)|^{q-1}}{|x-x_0|^{n+tq}} \, dx \\
&\leq c2^{i(n+tq)} M \left(\frac{d_j^{p-1}}{r_{j+1}^{sp}} + a_{2,j} \frac{d_j^{q-1}}{r_{j+1}^{tq}} \right) + c2^{i(n+tq)} M \left(\frac{\sigma^{\frac{sp}{2}}}{\varepsilon^{p-1}} \frac{d_j^{p-1}}{r_{j+1}^{sp}} + a_{2,j} \frac{\sigma^{\frac{tq}{2}}}{\varepsilon^{q-1}} \frac{d_j^{q-1}}{r_{j+1}^{tq}} \right) \\
&\leq c2^{i(n+tq)} M \frac{G(d_j)}{d_j}. \qquad (6.57)
\end{aligned}
$$

Therefore, combining (6.53), (6.54), (6.55), (6.56) and (6.57), we arrive at

$$
A_{i+1}^{1/\kappa} G(2^{-i-1} d_j) = A_{i+1}^{1/\kappa} G(k_i - k_{i+1}) \leq c2^{i(n+tq+q)} M^2 G(d_j) A_i,
$$

which implies

$$
A_{i+1} \leq c_0 2^{i\kappa(n+tq+2q)} M^{2\kappa} A_i^{\kappa}
$$

for a constant $c_0 \equiv c_0(\mathtt{data}_1) \geq 1$. In order to apply Lemma 6.2.5, it should be guaranteed that

$$
A_0 \leq (c_0 M^{2\kappa})^{-1/(\kappa-1)} 2^{-(n+tq+2q)\kappa/(\kappa-1)^2} =: \nu_*. \qquad (6.58)
$$

This inequality holds by (6.52) and (6.36). More precisely, we have

$$
A_0 = \frac{|2B_{j+1} \cap \{u_j \leq 2d_j\}|}{|2B_{j+1}|} \leq \frac{c_* M^3}{\log(1/\sigma)} \leq \nu_*.
$$

Hence it follows that $A_i \to 0$ as $i \to \infty$, which means that

$$
u_j \geq d_j = \varepsilon K_j \qquad \text{a.e. in } \ B_{j+1}.
$$

From this with (6.45) and (6.37), we finally obtain (6.40) as follows:

$$
\omega(r_{j+1}) = \sup_{B_{j+1}} u_j - \inf_{B_{j+1}} u_j \leq (1-\varepsilon) K_j = \left(1 - \sigma^{\frac{tq}{2(q-1)}}\right) \sigma^{-\gamma} K_{j+1} \leq K_{j+1}.
$$

□

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국문초록

이 학위논문에서는 비선형 측도 데이터 문제들에 대하여 다양한 정칙성 결과들을 얻는다. 해당 결과들은 비선형 칼데론-지그문트 이론 및 비선형 퍼텐셜 이론을 다루는 과정의 일부이다.

첫 번째로, 오를리츠 성장조건 및 경계선 이중위상 성장조건을 가지는 타원형 측도 데이터 문제에 대하여 각각 최대 적분성 및 분수차수 미분성 결과를 얻는다. 또한 포물형 측도 데이터 문제에 대하여 분수차수 미분성을 계수에 대한 최소한의 가정 하에서 증명한다.

두 번째로, 측도 데이터를 가지는 타원형 장애물 문제에 대하여 선형화 기법을 이용함으로써 그레이디언트 퍼텐셜 가늠 및 분수차수 미분성을 증명한다. 특히 비정칙 장애물 문제에 대해 퍼텐셜 가늠을 얻기 위한 새로운 방법을 개발한다. 더 나아가, L^1 데이터를 가지는 단일 장애물 문제에 대하여는 해의 유일성 및 비교 원리를 증명하여 이러한 정칙성 결과들을 개선한다.

마지막으로, 측도 데이터를 가지는 국소 및 비국소 혼합 방정식에 대하여 해의 존재성, 정칙성 및 퍼텐셜 가늠을 증명한다. 또한, 비표준 성장조건을 가지는 비등방적 비국소 문제에 대한 정칙성 이론의 첫걸음으로서, 비국소 이중위상 문제에 대한 횔더 정칙성을 국소 이중위상 문제의 경우과 유사한 최적의 조건 하에서 증명한다.

주요어휘: 측도 데이터, 칼데론-지그문트 이론, 퍼텐셜 이론, 비표준 성장조건, 장애물 문제, 비국소 작용소
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